Elementary Operations

There are three types of elementary operations on matrices.

1) Interchange of two rows/column

for Ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow[e]{R_1 \leftrightarrow R_3} \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix} = e(A)$$

2) Multiplying a row/column by non-zero constant ($k \neq 0$)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow[e]{R_3 \to 2R_2} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 14 & 16 & 18 \end{bmatrix}$$

↳ elementary row operation

3) Adding constant times of a row/colum to another row/column

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 5 \end{bmatrix} \xrightarrow[e]{C_2 \to C_2 + 2C_1} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 7 & 5 \end{bmatrix}$$

↳ elementary column operation

Does elementary operation change the determinant of matrix?

$$A \xrightarrow{e} e(A)$$

e	
$R_i \leftrightarrow R_j$	$\det(e(A)) = -\det(A)$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow[e]{R_1 \leftrightarrow R_2} \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} = e(A)$$

↳ $\det(e(A)) = 2$

$\det(A) = -2$

$\det(e(A)) =$

II) $R_i \to kR_i \quad (k \neq 0)$

$\det(e(A)) = k\det(A)$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow[e]{R_1 \to 3R_1} \begin{bmatrix} 3 & 6 \\ 3 & 4 \end{bmatrix} = e(A)$$

$(\times 3)$

↳ $\det(A) = -2$ ↳ $\det(e(A)) = -6$

III) $R_i \to R_i + cR_j$ ✓

$\det(e(A)) = \det(A)$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow[e]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} = e(A)$$

$\hookrightarrow \det(A) = -2$ $\qquad\qquad$ $\hookrightarrow \det(A) = -2$

e	Effect on determinant	
$R_i \leftrightarrow R_j$	$\det(e(A)) = -\det(A)$	
$R_i \to kR_j$ $(k \neq 0)$	$\det(e(A)) = k\det(A)$	$(k \neq 0)$
$R_i \to R_i + cR_j$	$\det(e(A)) = \det(A)$	(No change)

Suppose that
$\det(A) \neq 0$, $\quad \det(e(A)) \neq 0$

1. A is invertible then $e(A)$ is also invertible.

2.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow[e]{R_1 \to R_1 + R_2} \begin{bmatrix} 4 & 6 \\ 3 & 4 \end{bmatrix} = e(A)$$

$$\xleftarrow[e'?]{R_1 \to R_1 - R_2}$$

$$e(A) = \begin{bmatrix} 4 & 6 \\ 3 & 4 \end{bmatrix} \xrightarrow[e']{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A$$

Thm: Let e be any elementary row operation, then there exists an elementary operation (e') {same type} such that

$$e(e'(A)) = e'(e(A)) = A \quad \text{for any matrix } A$$

Further, e' is called inverse elementary row operation of e.

Proof. Case-I) Suppose e is the elementary row operation that interchange row i by row j

$$e : (R_i \leftrightarrow R_j)$$

e' can be considered same as e. $\quad (e' = e)$

So that, $ee'(A) = e'e(A)$ $\qquad\qquad$ $e'(e(A)) = A$

e' can be considered same as e. $(e' = e)$

So that, $ee'(A) = e'e(A)$

$e' = e$ { Self invertible row elementary inverse }

$e'(e(A)) = A$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \xrightarrow[e]{R_1 \leftrightarrow R_2} \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix} = e(A)$$

$$\begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow[e']{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = A$$

Case II) Suppose e is the elementary row operation which multiples ith row by a non-zero constant $(k \neq 0)$

$e := (R_i \longrightarrow kR_i)$

So, e' can be considered as row operation which multiplies ith row by $\frac{1}{k}$ $\{k \neq 0\}$

$e' := (R_i \longrightarrow \frac{1}{k} \cdot R_i)$

Thus, $ee'(A) = e'e(A) = A$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow[e]{R_3 \rightarrow 2R_3} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 14 & 16 & 18 \end{bmatrix} = e(A)$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 14 & 16 & 18 \end{bmatrix} \xrightarrow{R_3 \rightarrow \frac{1}{2} \cdot R_3} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = A$$

Case-III, Let e be elementary row operation which adds c times row j to row i.

$e : (R_i \longrightarrow R_i + cR_j)$

so, e' can be considered as elementary operation which subtract c times row j from i

$e' : (R_i \longrightarrow R_i - cR_j)$

So, that $ee'(A) = e'e(A) = A$

e	$e^{-1} = e$
$R_i \longleftrightarrow R_j$	$R_i \longleftrightarrow R_j$
$R_i \longrightarrow kR_j$ $(k \neq 0)$	$R_i \longrightarrow \frac{1}{k} \cdot R_j$
$R_i \rightarrow R_i + cR_j$	$R_i \longrightarrow R_i - cR_j$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \xrightarrow[e]{R_2 \rightarrow R_2 - R_1} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix} = e(A$$

$c = -1$

$R_i = \boxed{c}R_j$

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow[e']{R_2 \rightarrow R_2 + R_1} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = A$$

$c = -1$

Def^n. Row equivalence.

A matrix A is said to be row equivalent to matrix B if matrix B can be obtained by a finite number of elementary row operation on A.

$$A \xrightarrow{e_1} e_1(A) \xrightarrow{e_2} e_2(e_1(A)) \longrightarrow \cdots \xrightarrow{e_n} e_n(e_{n-1}(\cdots e_3(e_2(e_1(A)))) = B$$

In other words.

B can written as

$$B = \underbrace{e_n e_{n-1} e_{n-2} \cdots e_3 e_2 e_1}_{n \text{ times elementary operation}}(A)$$

$$\boxed{A \sim B}$$

Prop Is $\sim$ (row equivalence) an equivalence relation?

Yes,

1) Reflexive

for any matrix A, $A \sim A$.

As $\boxed{e_1 : R_1 \longrightarrow R_1 + 0R_2}$ $e_2 : R_1 \to 1 \cdot R_1$ $k=1$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow[e_1]{R_1 \to R_1 + 0R_2} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = e_1(A)$$

e_1 ↳ elementary row operation

$A \sim A$ so, $\sim$ is reflexive,

2) Symmetric,

Suppose A is row equivalent to B ie. $A \sim B$

We need to show: $B \sim A$.

So, B can be obtained by finite (n numbers) of elementary row operation on A.

ie. $\boxed{B = e_n e_{n-1} e_{n-2} \cdots e_3 e_2 e_1(A)}$ —— ①:

$$A \xrightarrow{e_1} e_1(A) \xrightarrow{e_2} e_2(e_1(A)) \longrightarrow \cdots \xrightarrow{e_n} B.$$

Now, To show: A can be obtained by applying elementary row operation on B.

So, Consider e_k' be the inverse elementary row operation of each e_k for $k=1,2,3,\dots,n$.

Eq(1)-

$$B = e_n e_{n-1} e_{n-2} \cdots e_3 e_2 e_1 (A)$$

Applying e_n'

$$e_n'(B) = (e_n'(e_n) e_{n-1} e_{n-2} \cdots e_3 e_2 e_1 (A))$$

$$e_n'(B) = e_{n-1} e_{n-2} \cdots e_3 e_2 e_1 (A)$$

Again applying e_{n-1}'

$$e_{n-1}'(e_n' B) = (e_{n-1}'(e_{n-1}) \cdots e_3 e_2 e_1 (A))$$

Similarly,

$$\underbrace{e_1' e_2' e_3' \cdots e_{n-1}' e_n'}_{\text{finite (n) elementary operation}} (B) = A \qquad B \sim A$$

So, A can be obtained by finite number of elementary operations on B.

So, $B \sim A$.

This mean, $\boxed{\sim}$ is symmetric relation.

3) Transitivity

Suppose A, B and C are three matrices such that

$A \sim B$, $B \sim C$

To show: $A \sim C$

$$B = \underbrace{e_n e_{n-1} e_{n-2} \cdots e_3 e_2 e_1}_{\text{n (finite) number of elementary operations on A}} (A) \quad \text{——— (1)}$$

$$C = \underbrace{f_m f_{m-1} f_{m-2} \cdots f_3 f_2 f_1}_{\text{m (finite) number of elementary operations on B}} (B) \quad \text{——— (2)}$$

Substitue value of B from eq (1) to eq (2).

$$C = \underbrace{f_m f_{m-1} f_{m-2} \cdots f_3 f_2 f_1 (e_n e_{n-1} e_{n-2} \cdots e_3 e_2 e_1}_{(m+n) \text{ finite number of elementary operations on A}}(A))$$

$\boxed{A \sim C}$ $\quad \sim$ is a transitive relation.

$\boxed{A \sim C}$ $\sim$ is a transitive relation.

As, $\sim$ is reflexive, symmetric and transitive

Thus, $\sim$ is an equivalence relation.

Elementary Matrices

<u>Def</u>

Def^n: An Elementary Matrix is a square matrix obtained by applying a single elementary row operation on identity matrix.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[e]{R_2 \to 2 \cdot R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = e(I_3)$$

e → elementary row operation

$e(I_3)$ → Elementary Matrix corresponding to elementary operation e

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow[e]{R_2 \to R_2 + 2R_1} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = e(I_2)$$

e → elementary row operation

$e(I_2)$ → Elementary matrix

Notation: $e(I)$ → where e denotes elementary operation & I is the identity matrix.

Properties

1) Elementary matrix is invertible.

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \xrightarrow[e]{} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \quad e(I)$$

$\det(I_n) = 1$ $\qquad$ $\det(e(I))$

Either $\det(e(I)) = -\det(I_n) = -1$

or $\det(e(I)) = k \cdot \det(I_n) = k \neq 0$

In each of the case, $\det(e(I)) \neq 0$

$\det(e(I)) \neq 0 \Rightarrow e(I)$ Elementary matrix is invertible.

Thm: Inverse of the elementary matrix is an elementary matrix.

Proof.