Elementary Matrices

$$I \xrightarrow[\text{}]{e \;(\text{one elementary operation})} e(I)$$

I ↳ identity matrix

$e(I)$ ↳ Elementary Matrix.

1) Elementary Matrix is invertible.

if A is any matrix, A is invertible $\Longleftrightarrow$ $e(A)$ is invertible

In particular, if $A = I$

As I is invertible $\{ I^{-1} = I \}$

$\Rightarrow$ $e(I)$ is invertible Elementary matrix is invertible

2) Suppose A is any matrix and e be the elementary row operation

$$A \xrightarrow{e} e(A) \qquad\qquad I \xrightarrow{e} e(I) = E$$

$$e(A) = \boxed{EA}$$

↳ Multiplying elementary matrix.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow[R_2 \to R_2 + R_1]{e} \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix} = e(A) = B$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow[R_2 \to R_2 + R_1]{e} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = e(I) = E$$

$$B = e(A) = EA$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix} = B$$

$$A \xrightarrow{e} e(A)$$

e → elementary row operation

$e(A) = EA$ where E is elementary matrix corresponding to e.

3) if A is any matrix

3) if A is any matrix

$$A \xrightarrow[\text{elementary column operation}]{f} f(A) \qquad f(A) = AF$$

where F is the elementary matrix corresponding to f.

4) Inverse of elementary matrix is elementary matrix.

Suppose that E is elementary matrix, so by def^n, E can be obtained by elementary row operation on I.

$$\boxed{E = e(I)}$$ where e is elementary row operation.

So, By Existence of the inverse elementary row operation

so, $\exists$ an elementary row operation (e') {same type as e} such that

$$ee'(A) = e'e(A) = A \quad \text{for any matrix } A.$$

In particular, $A = I$. {Assume $e'(I) = E'$}.

$$ee'(I) = e'e(I) = I$$

$$e(E') = e'(E) = I$$

$$EE' = E'E = I$$

$$\Rightarrow E' = E^{-1}$$

So, inverse of elementary matrix E is E', where $E' = e'(I)$ is an elementary matrix.

5) Any invertible matrix can be written as product of elementary matrices.

Lemma: Any invertible matrix is row equivalent to identity matrix.

Proof: "Probably in next class" Echelon form & RREF.

Proof (5)

let us assume A is invertible matrix, By lemma, $A \sim I$ (row equivalence)

that means, I can be obtained by a finite number of elementary row operations on A.

$$A \xrightarrow{e_1} e_1(A) \xrightarrow{e_2} e_2 e_1(A) - - - - \xrightarrow{e_n} I$$

Thus,

$$I = \underbrace{e_n e_{n-1} e_{n-2} \cdots e_3 e_2 e_1}_{\text{n elementary row operations}}(A)$$

Let us assume that, E_k is elementary matrix corresponding to each e_k

$E_k = e_k(I)$ for $k = 1, 2, \ldots, n$.

$$I = \underbrace{E_n E_{n-1} E_{n-2} E_{n-3} \cdots E_3 E_2 E_1}A$$

so, $A^{-1} = E_n E_{n-1} E_{n-2} \cdots E_3 E_2 E_1$

Taking inverse both side

$$A = (A^{-1})^{-1} = (E_n E_{n-1} E_{n-2} \cdots E_3 E_2 E_1)^{-1}$$

$$\boxed{A = E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_{n-1}^{-1} E_n^{-1}}$$

Here, E_k^{-1} is elementary matrix as inverse of elementary matrix is again an elementary matrix. $\forall\, k = 1, 2, 3, \ldots, n$

Thus, Every invertible matrix can be written as product of elementary matrices.

Que.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad$ $\det(A) = 4 - 6 = -2 \neq 0$

ie. A is invertible

So, A can be written as product of elementary matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow[e_1]{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \xrightarrow[e_2]{R_2 \to \frac{-1}{2} \cdot R_2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow[e_3]{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

↳ identity

$I = e_3 e_2 e_1 (A)$

$e_1 :\ R_2 \to R_2 - 3R_1$

$e_2 :\ R_2 \to \frac{-1}{2} \cdot R_2$

$e_3 :\ R_1 \to R_1 - 2R_2$

Inverse of these elementary row operation

$e_1^{-1} :\ R_2 \to R_2 + 3R_1$

$e_2^{-1} :\ R_2 \to \frac{1}{-1/2} R_2 = -2R_2$

$e_3^{-1} :\ R_1 \to R_1 + 2R_2$

$E_1^{-1} = e_1^{-1}(I) =$

$$\begin{bmatrix} 1 & 0 \\ \ & \ \end{bmatrix} \xrightarrow{R_2 \to R_2 + 3R_1} \begin{bmatrix} 1 & 0 \\ \ & \ \end{bmatrix} = E_1^{-1} = e_1^{-1}(I)$$

$E_1 = e_1(I)$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow[e_1^{-1}]{R_2 \to R_2 + 3R_1} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = E_1^{-1} = e_1^{-1}(I)$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow[e_2^{-1}]{R_2 \to -2R_2} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} = E_2^{-1}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow[e_3^{-1}]{R_1 \to R_1 + 2R_2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = E_3^{-1}$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} \Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

6) Suppose A is row equivalent to B, if and only if $\exists$ a matrix P (invertible) such that $B = PA$.

Proof: Forward case) $\Rightarrow$

Suppose A is row equivalent to B.

By defn. $\exists$ a sequence of elementary row operations $e_1, e_2, \dots, e_n$ such that

$$B = e_n e_{n-1} e_{n-2} \cdots e_3 e_2 e_1 (A)$$

and let E_k be the corresponding elementary matrices to elementary row operations e_k $\{k = 1, 2, \dots n\}$

$$B = \underbrace{(E_n E_{n-1} \cdots E_3 E_2 E_1)}_{\text{invertible}} A$$

Consider the matrix $P = E_n E_{n-1} \cdots E_3 E_2 E_1$ is an invertible matrix.

$\boxed{B = PA}$ where P is invertible matrix

Backward part) Suppose that $B = PA$ where P is invertible matrix.

To prove: B is row equivalent to A.

As, P is invertible, so it can be written as the product of elementary matrices.

let us assume, $P = E_n E_{n-1} \cdots E_3 E_2 E_1$

$$\Rightarrow B = (E_n E_{n-1} \cdots E_2 E_1) A$$

Now, for every matrix A,

$$e_K(A) = E_K(A) \quad \text{for every } K = 1, 2, \ldots, n.$$

$$\Rightarrow \quad B = e_n e_{n-1} \cdots e_2 e_1 (A)$$

where each e_K is the elementary row operation corresponding to E_K for $K = 1, 2, \ldots$

ie. B can be obtained by a finite sequence of elementary row operations on A.

$\underline{\underline{A \sim B}}$.

Echelon Form.

Def^n: A matrix is said to be in echelon form if

- All rows consisting of zeroes are at bottom.
- The leading coefficients of a non-zero row is always strictly right to the leading non-zero coefficient of row above it.

pivots   Pivots

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

zero rows

$\hookrightarrow$ Echelon form

$$B = \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ * & 0 & * & * \end{bmatrix} \begin{matrix} \\ \leftarrow R_2 \\ \leftarrow R_3 \end{matrix}$$

is matrix B in echelon form?

Not Echelon form

$\hookrightarrow$ Pivot in R_3 is not strictly right to the the leading coefficient of R_2.

$$C = \begin{bmatrix} 0 & 0 & 0 \\ * & * & * \\ 0 & * & 0 \end{bmatrix}$$

zero row is not at bottom

so, C is not in echelon form

R_1

$$\begin{bmatrix} 0 & * & 0 \end{bmatrix} \quad so, -$$

$$D = \begin{array}{c} \begin{bmatrix} * & * & * & * \\ * & * & * & \times \\ * & * & * & * \end{bmatrix} \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

is this in echelon form or not?

Not.

As leading coefficient in R_2 is not strictly right to the leading coefficient in R_1

Pivot

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}$$

This matrix is in Echelon form.

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow$$ This matrix is in Echelon form.

No pivot.

Next class -

- Echelon Form
- Row-Reduced Echelon Form
- Proof of the lemma
- Rank of the matrix

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