Subspaces

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Vector space, subspace, $W_1 \cup W_2 \longrightarrow$ is not a subspace (in general)

$\rightarrow$ Suppose $(V, +, \cdot)$ is a vector space over field $\mathbb{F}$ and W_1 and W_2 are the subspaces of V,

Sum of subspaces: $W_1 + W_2 = \{ x + y \mid x \in W_1, y \in W_2 \}$

(i) $W_1 + W_2$ is a subspace of V over $\mathbb{F}$.

To show: using subspace test, $W_1 + W_2$ is subspace of V.

1) As W_1 is a subspace of V & W_2 is subspace of V

$$0 \in W_1 \quad \& \quad 0 \in W_2$$

so, $0 + 0 = 0 \in W_1 + W_2$

2) Suppose u and v are two vectors in $W_1 + W_2$

✓ $u = x_1 + y_1$ where $x_1 \in W_1$ and $y_1 \in W_2$

✓ $v = x_2 + y_2$ where $x_2 \in W_1$ and $y_2 \in W_2$

✓ $$u + v = (x_1 + y_1) + (x_2 + y_2) = \underbrace{(\underset{\in W_1}{x_1} + \underset{\in W_1}{x_2})}_{\in W_1} + \underbrace{(\underset{\in W_2}{y_1} + \underset{\in W_2}{y_2})}_{\in W_2}$$

$$\in W_1 + W_2$$

ie. $u + v \in W_1 + W_2 \quad \forall\, u, v \in W_1 + W_2$

3) Suppose $u \in W_1 + W_2$ so, $u = \underline{x_1 + y_1}$ where $x_1 \in W_1$, $y_1 \in W_2$ and say $\alpha \in \mathbb{F}$ be a scaler

So, $\underline{\alpha \cdot u} = \alpha \cdot (x_1 + y_1)$

$= \alpha \cdot x_1 + \alpha \cdot y_1$ {By Distributive law}

$= \underbrace{\alpha \cdot \underset{\in W_1}{x_1}}_{\in W_1} + \underbrace{\alpha \cdot \underset{\in W_2}{y_1}}_{\in W_2}$

$\in W_1 + W_2$

ie. $u \in W_1 + W_2$ and α is any scaler, so $\alpha \cdot u \in W_1 + W_2$

So, By subspace test, $W_1 + W_2$ is a subspace of V.

Subspace test. $W \subseteq V$ ✓ one-step test

Subspace test. $W \subseteq V$

Two step subspace test:
- $0 \in W$
- $x+y \in W$
- $\alpha x \in W$

$\equiv$ one-step test:
$$\begin{cases} 0 \in W \\ \alpha x + \beta y \in W \end{cases} \quad \alpha, \beta \in \mathbb{F}$$

$\begin{cases} \alpha x \in W \\ \beta y \in W \\ \alpha x + \beta y \in W \end{cases}$ α is scalar,

suppose $\alpha = \beta = 1 \Rightarrow x+y \in W$
$\beta = 0 \Rightarrow \alpha x \in W$

Direct sum: W_1 is a subspace of V, W_2 is subspace of V.

Def

Now, $W_1 + W_2$ (sum of W_1 & W_2) is also a subspace of V.

We say the sum $W_1 + W_2$ is direct sum $\oplus$ $\{W_1 \oplus W_2\}$ if

- $W_1 \cap W_2 = \{0\}$ $\quad W_1 \oplus W_2$

Def^n Span of a finite set.

Let S be a finite set, $S = \{v_1, v_2, ---, v_n\}$ then span(S) is the linear combination of the all the vectors in S.

$$\text{span}(S) = \{ \underbrace{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n}_{\text{Linear combination of } v_1, v_2, ---, v_n} \mid \alpha_i \text{ are scalers} \}$$

$\rightarrow$ Suppose V is a vector space over field $\mathbb{F}$ and assume that set S (finite) is finite subset of V, then span(S) forms a vector subspace of V.

Proof: By subspace test. (one-step test).

$S = \{v_1, v_2, ---, v_n\}$ where $v_i \in V$ for each $i = 1, 2, --, n$

$\rightarrow$ $\text{span}(S) = \{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \mid \alpha_i \text{ are scalars}\}$

① so if we put $\alpha_1 = 0, \alpha_2 = 0, ---, \alpha_n = 0$

$0 \cdot v_1 + 0 \cdot v_2 + --- + 0 \cdot v_n = 0 \in \text{span}(S)$

② say $x \in \text{span}(S)$ and $y \in \text{span}(S)$

$\hookrightarrow$ $\exists$ scalars $\alpha_1, \alpha_2, ---, \alpha_n$ such that $\quad$ $\exists$ scalers $\beta_1, --, \beta_n$

↳ $\exists$ scalers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

$$y = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$$

$\exists$ scalers $\beta_1, \ldots, \beta_n$ such that

Now, for any two scalers α and β

$$\alpha x + \beta y = \alpha[\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n] + \beta[\beta_1 v_1 + \cdots + \beta_n v_n]$$

$$= [\alpha\alpha_1 v_1 + \alpha\alpha_2 v_2 + \cdots + \alpha\alpha_n v_n] + [\beta\beta_1 v_1 + \cdots + \beta\beta_n v_n]$$

$$[\underbrace{(\alpha\alpha_1 + \beta\beta_1)}_{\in \mathbb{F} \text{ Scaler}} v_1 + \underbrace{(\alpha\alpha_2 + \beta\beta_2)}_{\in \mathbb{F}} v_2 + \cdots + \underbrace{(\alpha\alpha_n + \beta\beta_n)}_{\in \mathbb{F} \text{ Scaler}} v_n]$$

$\in span(S)$

So, $\alpha x + \beta y \in span(S) \quad \forall\, x, y \in span(S)$ & $\alpha, \beta \in \mathbb{F}$

$\Rightarrow$ $span(S)$ is a subspace of V.

$\rightarrow$ is $span(S)$ the smallest subspace of V containing S ??

Suppose K is any subspace of V containg S.

$S = \{v_1, \ldots, v_n\}$

$S \subseteq K$

↳ subspace of V

Now, $v_i \in S \subseteq K$ for every $i = 1, 2, \ldots, n$ $\quad \{ v_i \in S \subseteq K$

and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are scalers from field $\mathbb{F}$

$$span(S) = \{\underbrace{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n}_{\text{linear combination of vectors of } S} \in K$$

for every $\alpha_i \in \mathbb{F}$

$\Rightarrow$ $span(S) \subseteq K$.

i.e. $S \subseteq span(S) \subseteq K$

$\Rightarrow$ $span(S)$ is smallest subspace of V containing S.

$\rightarrow$ V is a vector space over $\mathbb{F}$ and W_1, W_2 are two subspaces of V.

$W_1 + W_2$ forms a subspace of V

... containing W_1 and W_2 ??

$W_1 + W_2$ forms a subspace of V

[is. $W_1 + W_2$ smallest subspace containing W_1 and W_2 ??]

$W_1 + W_2 = \{x + y \mid x \in W_1, y \in W_2\}$.

$0 \in W_2$ {As W_2 is subspace}

$\Rightarrow \quad W_1 \subseteq W_1 + W_2$ and $W_2 \subseteq W_1 + W_2$

Is there any subspace smaller than $W_1 + W_2$ which contains both W_1 & W_2 ??

Think. No. $W_1 + W_2$ is smallest subspace.

Do your self.

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linearly Dependent Set.

let's say S is a finite subset of V containing vectors

$$S = \{v_1, v_2, \ldots, v_n\}$$

Then, set S is said to be linearly dependent set / vectors $v_1, v_2, \ldots, v_n$ are linearly dependent vectors if

- $\exists$ scalers $\alpha_1, \alpha_2, \ldots, \alpha_n$ from field $\mathbb{F}$ (Not all scalers are zero)

such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

where $\alpha_i \neq 0$ for some $i \in \{1, 2, \ldots, n\}$

otherwise, vectors $v_1, v_2, \ldots, v_n$ are said to be linearly independent vectors

(set S is linearly independent.

Ex. $V = \mathbb{R}^3$ — set of all 3-tuples $\quad \mathbb{F} = \mathbb{R}$

$= \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$.

Now, $S = \{\underbrace{(1,2,3)}_{v_1}, \overbrace{(2,3,4)}^{v_2}, \overbrace{(3,5,7)}^{v_3}\} \subseteq V$

finite subset of V.

Question $\rightarrow$ is set S a linearly dependent set or linearly independent set ?

$v_1 = (1,2,3)$ ✓

$v_2 = (2,3,4)$ ✓

$v_3 = (3,5,7)$

$v_1 + v_2 = (1,2,3) + (2,3,4)$
$= (3,5,7)$
$= v_3$

By observation

$v_1 + v_2 - v_3 = 0$

$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

$\alpha_1 = 1, \quad \alpha_2 = 1, \quad \alpha_3 = -1$

So, we have scalers, $\alpha_1 = 1 \neq 0$, α_2, α_3 such that

$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

$\Rightarrow$ v_1, v_2, v_3 are linearly dependent vectors

or S is linearly dependent set.

$V = \mathbb{R}^3$.

$\rightarrow$ $S = \{(1,0,0), (0,1,0), (0,0,1)\} \subseteq \mathbb{R}^3$

$\hookrightarrow$ is set S a linearly independent or dependent set ?

$v_1 = (1,0,0)$

$v_2 = (0,1,0)$

$v_3 = (0,0,1)$

$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \underbrace{\alpha_1 (1,0,0)}_{(\alpha_1,0,0)} + \underbrace{\alpha_2 (0,1,0)}_{(0,\alpha_2,0)} + \underbrace{\alpha_3 (0,0,1)}_{(0,0,\alpha_3)}$

$= \vec{0}$

$(\alpha_1, \alpha_2, \alpha_3) = \vec{0} = (0,0,0)$

$\Rightarrow (\alpha_1, \alpha_2, \alpha_3) = (0,0,0)$

$\Rightarrow \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$

ie. Set S is linearly independent set.

Remarks:

(1) Any set containing zero vector is linearly dependent set.

V, $\quad S = \{\vec{0}, v_1, \dots, v_n\}$

So, we have scalers, $\alpha_1 = 1, \alpha_2 = 0, \dots, \alpha_{n+1} = 0$

$\alpha_1 0 + \alpha_2 v_1 + \alpha_3 v_2 + \dots + \alpha_{n+1} v_n = 1 \cdot 0 + 0 \cdot v_1 + 0 v_2 + \dots + 0 v_n =$

$= 0$

But $\alpha_1 = 1 \neq 0$, By def^n set S has to been linearly dependent set

for example.

$V = M_{2\times 2}(\mathbb{R})$ $\qquad \mathbb{F} = \mathbb{R}$.

↳ set of all 2×2 matrices where entries are real.

$$S = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \right\}$$

↓

0 vector of $M_{2\times 2}(\mathbb{R})$

i.e. set S is linearly dependent set as it has zero vector of $M_{2\times 2}(\mathbb{R})$