Vector Spaces

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Def^n: A vector space $\overset{\downarrow}{V}$ over a field $\overset{\text{Notation}}{\overset{\downarrow}{(\mathbb{F})}}$ is a set structure with two binary operation

$$(\underbrace{\overset{\text{vectors}}{V}}_{\text{set}}, \overset{\text{vector addition}}{+}, \underbrace{\cdot}_{\text{Scalar multiplication}})$$

which satisfy following properties.

$(V,+)$ Abelian Group

1) Let x & y be elements of V i.e. x & $y \in V$

then $x+y = y+x$] commutative

Vector addition is commutative

2) Let $x, y, z \in V$

$x + (y+z) = (x+y) + z$]

Vector addition is assosiativite.

3) Additive identity.

$\forall x \in V$, $\exists$ an element $0 \in V$ such that

$x + 0 = 0 + x = x$

Existence of Additive identity

4) Additive inverse

$\forall x \in V$, $\exists$ an element $y \in V$ such that

$x+y = y+x = \underset{\text{additive identity}}{0}$

y is additive inverse of x

Existence of Additive inverse

5) Multiplicative identity. (Field consist of scalar)

$\forall x \in V$, $\exists$ $1 \in \mathbb{F}$

such that $1 \cdot x = x \cdot 1 = x$

1 is multiplicative identity

6) Multiplication is assosiative

(Say $\alpha, \beta \in \mathbb{F}$) and $x \in V$

$(\alpha \cdot \beta) \cdot x = \alpha \cdot (\beta \cdot x)$ $\quad \forall x \in V$, $\forall \alpha, \beta \in \mathbb{F}$

7) Distributive property.

Suppose $\alpha \in \mathbb{F}$ be scalar & $x, y \in V$

So, $\alpha \cdot (x+y) = \alpha \cdot x + \alpha \cdot y$

Scalar multiplication is distributive over vector addition

So, $\alpha \cdot (x+y) = \alpha \cdot x + \alpha \cdot y$

8) Distributive property.

Suppose $\alpha, \beta \in \mathbb{F}$ and $x \in V$

$$(\alpha+\beta) \cdot x = \alpha \cdot x + \beta \cdot x \quad \forall \alpha, \beta \in \mathbb{F}, \ \forall x \in V$$

Addition of scalars is distributive.

Then, we say V is a vector space over field $\mathbb{F}$.

Example.

$$\mathbb{F}^n = \underbrace{\mathbb{F} \times \mathbb{F} \times \mathbb{F} \times \mathbb{F} \times \cdots \times \mathbb{F}}_{n\text{-times}}$$

is n-tuple.

$$= \{ (x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{F} \}$$

$$\mathbb{F}^2 = \{ (x, y) \mid x \in \mathbb{F}, y \in \mathbb{F} \}$$

$V = \mathbb{F}^n$ ✓, $\mathbb{F} = \mathbb{F}$

$n=2$ fix.

$\mathbb{F} = \mathbb{R}$ → itself a field, ↳ set of real numbers.

$V = \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ ✓

We claim: V is a vector space

1) Addition is commutative.

Say $x \in V$ & $y \in V$

So, $x = (x_1, x_2, x_3, \ldots, x_n)$ where each $x_i \in \mathbb{F}$

and $y = (y_1, y_2, y_3, \ldots, y_n)$ where each $y_i \in \mathbb{F}$

$$\begin{aligned}
x+y &= (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) \\
&= (x_1+y_1, x_2+y_2, \ldots, x_n+y_n) \ ✓ \\
&= (y_1+x_1, y_2+x_2, \ldots, y_n+x_n) \ ✓ \\
&= (y_1, y_2, \ldots, y_n) + (x_1, x_2, \ldots, x_n) \\
&= y + x
\end{aligned}$$

As $x_i \in \mathbb{F}$ $\forall i$, $y_i \in \mathbb{F}$ $\Rightarrow x_i + y_i \in \mathbb{F}$

(2) Addition is associative

Say, x, y & $z \in V = \mathbb{F}^n$

$x = (x_1, x_2, \ldots, x_n)$ where each $x_i, y_i, z_i \in \mathbb{F}$

$y = (y_1, y_2, \ldots, y_n)$

$z = (z_1, z_2, \ldots, z_n)$

$\underbrace{x}+\underbrace{(y+z)} = (x+y)+z$] : To show

$$x+\underbrace{(y+z)} = x+(y_1+z_1, y_2+z_2, y_3+z_3, \ldots, y_n+z_n)$$
$$= (x_1, x_2, \ldots, x_n) + (y_1+z_1, y_2+z_2, \ldots, y_n+z_n)$$
$$= (x_1+y_1+z_1, x_2+y_2+z_2, \ldots, x_n+y_n+z_n) \in \mathbb{F}^n$$
$$= ((x_1+y_1)+z_1, (x_2+y_2)+z_2, \ldots, (x_n+y_n)+z_n)$$
$$= [(x_1, x_2, x_3, \ldots, x_n) + (y_1, y_2, \ldots, y_n)] + (z_1, z_2, \ldots, z_n)$$
$$= (x+y)+z$$

③ Existence of additive identity $(\mathbb{F} — 0_{\mathbb{F}}$

$V = \mathbb{F}^n$ $\vec{0} = (0_F, 0_F, 0_F, \ldots, 0_F) \in \mathbb{F}^n$

let say $x \in V$, we have $\vec{0} \in \mathbb{F}^n$ such that

$$x+0 = (x_1, x_2, \ldots, x_n) + (0, 0, \ldots, 0)$$
$$= (x_1+0, x_2+0, \ldots, x_n+0) = (x_1, x_2, \ldots, x_n) = x$$

④ Existence of additive inverse.

lets say $x \in V$, $x = (x_1, x_2, \ldots, x_n)$ where each $x_i \in \mathbb{F}$

so, for every $x_i \in \mathbb{F}$, $i = 1, 2, \ldots, n$, we have $y_i \in \mathbb{F}$ { By existence of additive inverse in field

such that $x_i + y_i = 0_F$ define $y = (y_1, y_2, \ldots, y_n)$

where $y_i + x_i = x_i + y_i = 0$

$\exists$ $y \in V$ such that

$$x+y = (x_1, x_2, x_3, \ldots, x_n) + (y_1, y_2, \ldots, y_n)$$
$$= (x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$$
$$= (0_F, 0_{\mathbb{F}}, \ldots, 0_{\mathbb{F}}) = 0 \in V = \mathbb{F}^n$$

⑤ Multiplicative identity in $\mathbb{F}$

Suppose $x \in V = \mathbb{F}^n$

$x = (x_1, \ldots, x_n)$ where $x_i \in \mathbb{F}$

As $\mathbb{F}$ is field, it has multiplicative identity

so, for every x_i, $\exists$ $1 \in \mathbb{F}$ such that

$$1 \cdot x_i = x_i \quad \forall\ i = 1, 2, \ldots, n$$

So, $1 \cdot x = 1 \cdot (x_1, x_2, \ldots, x_n)$

$$= (1 \cdot x_1, 1 \cdot x_2, \ldots, 1 \cdot x_n) = (x_1, x_2, \ldots, x_n) = x$$

⑥ Scalr. Multiplication is assosiative,

say $\alpha, \beta \in \mathbb{F}$ for any $x \in V = \mathbb{F}^n$

$$(\alpha \cdot \beta) \cdot x = \alpha \cdot (\beta \cdot x)$$] to show/verify

L.H.S:

$$(\alpha \cdot \beta) \cdot x = (\alpha \cdot \beta)(x_1, x_2, \ldots, x_n)$$

$$= ((\alpha \cdot \beta) x_1, (\alpha \cdot \beta) x_2, \ldots, (\alpha \cdot \beta) x_n)$$

R.H.S:

$$\alpha \cdot (\beta \cdot x) = \alpha \cdot (\beta (x_1, x_2, \ldots, x_n))$$

$$= \alpha \cdot (\beta x_1, \beta x_2, \ldots, \beta x_n)$$

$$= (\alpha \cdot \beta x_1, \alpha \cdot \beta x_2, \ldots, \alpha \cdot \beta x_n)$$

LHS = RHS.

⑦ Distributive property.

lets say $\alpha \in \mathbb{F}$ and $x, y \in V = \mathbb{F}^n$

$x = (x_1, x_2, \ldots, x_n)$ where each $x_i \in \mathbb{F}$

$y = (y_1, y_2, \ldots, y_n)$ and each $y_i \in \mathbb{F}$

To verify: [$\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$

L.H.S:

$$\alpha \cdot (x + y) = \alpha \cdot (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

$$= (\alpha \cdot (x_1 + y_1), \alpha \cdot (x_2 + y_2), \ldots, \alpha \cdot (x_n + y_n))$$

$$= (\alpha \cdot x_1 + \alpha \cdot y_1, \alpha \cdot x_2 + \alpha \cdot y_2, \ldots, \alpha \cdot x_n + \alpha \cdot y_n)$$

{ As $x_i, y_i \in \mathbb{F}$, $x_i + y_i \in \mathbb{F}$, $\alpha \in \mathbb{F}$, $\alpha(x_i + y_i) = \alpha \cdot x_i + \alpha \cdot y_i$

R.H.S:

$$\alpha \cdot x + \alpha \cdot y$$

$$= \alpha \cdot (x_1, x_2, \ldots, x_n) + \alpha \cdot (y_1, y_2, \ldots, y_n)$$

$$= (\alpha \cdot x_1, \alpha \cdot x_2, \ldots, \alpha \cdot x_n) + (\alpha \cdot y_1, \alpha \cdot y_2, \ldots, \alpha \cdot y_n)$$

$$= (\alpha \cdot x_1 + \alpha \cdot y_1, \alpha \cdot x_2 + \alpha \cdot y_2, \ldots, \alpha \cdot x_n + \alpha \cdot y_n) = \alpha \cdot (x + y)$$

{ From LHS }

(8) Let us say $\alpha, \beta \in \mathbb{F}$ and $x \in V = \mathbb{F}^n$

$x = (x_1, x_2, \ldots, x_n) \in \mathbb{F}^n$

To verify. $(\alpha+\beta)\cdot x = \alpha \cdot x + \beta \cdot x$

L.H.S. $(\alpha+\beta)\cdot x = (\alpha+\beta)\cdot(x_1, x_2, x_3, \ldots, x_n)$

$= ((\alpha+\beta)\cdot x_1, (\alpha+\beta)\cdot x_2, \ldots, (\alpha+\beta)\cdot x_n)$

$= (\alpha\cdot x_1 + \beta\cdot x_1, \alpha\cdot x_2 + \beta\cdot x_2, \ldots, \alpha\cdot x_n + \beta\cdot x_n)$

$= (\alpha\cdot x_1, \alpha\cdot x_2, \ldots, \alpha\cdot x_n) + (\beta\cdot x_1, \beta\cdot x_2, \ldots, \beta\cdot x_n)$

$= \alpha\cdot(x_1, x_2, \ldots, x_n) + \beta(x_1, x_2, \ldots, x_n)$

$= \alpha\cdot x + \beta\cdot x = \text{R.H.S.}$

$V = \mathbb{F}^n$ — set of n-tuples satisfies all 8 properties

So, V is a vector space over field $\mathbb{F}$.

→ $V = M_{2\times2}(\mathbb{F}) = \{$ set of all 2x2 matrices where each entry is from field $\}$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times2}$ and $a, b, c, d \in \mathbb{F}$ ↳ field.

Suppose, $A, B \in V = M_{2\times2}(\mathbb{F})$

$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ where each element $a_i \in \mathbb{F}$ $\forall\ i = 1, 2, \ldots, 4$

$B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ where each element $b_i \in \mathbb{F}$ $\forall\ i = 1, 2, 3, 4$

$A + B = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{bmatrix} \in M_{2\times2}(\mathbb{F})$

So, say $\alpha \in \mathbb{F}$

$\alpha\cdot A = \alpha\cdot\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} \alpha\cdot a_1 & \alpha\cdot a_2 \\ \alpha\cdot a_3 & \alpha\cdot a_4 \end{bmatrix}$

So, $V = M_{2\times 2}(\mathbb{F})$ satisfies all 8 properties of vector space over $\mathbb{F}$.

→ $V = M_{m\times n}(\mathbb{F}) = \{$ Matrices of size $m \times n$, where each entry of matrix is from field $\mathbb{F}$ $\}$

→ **Def^n.** Field extension,

Let $\mathbb{F}$ be any field, E is called field extension of $\mathbb{F}$ if

(1) E itself is a field

(2) $\mathbb{F} \subseteq E$

Suppose. $\mathbb{F} = \mathbb{Q}$ is field, $\mathbb{Q}$ = set of rational numbers.

$E = \mathbb{Q}\sqrt{2} = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$

$\mathbb{Q} \subseteq E$ $\qquad$ $\boxed{b=0}$ $\quad = \{a + 0\sqrt{2}\} \mid a \in \mathbb{Q}\}$

$= \mathbb{Q}$

$\mathbb{Q} \subseteq \mathbb{Q}\sqrt{2}$ and $\mathbb{Q}\sqrt{2}$ itself a field,

So, we say $\mathbb{Q}\sqrt{2}$ is field extension of $\mathbb{Q}$.

→ Field extension over Field forms a vector space

Let E be field extension of F $\qquad$ $E(\mathbb{F})$ will be vector space.

Trivial Vector spaces.

Let V be any vector space over field $\mathbb{F}$.
$\{0\} \neq$

then $\{0\}$ and V are called trivial vector spaces.

V

$\{0\}$

— Trivial vector space
↓
subspaces of V

Subspace.

Let $(V, +, \cdot)$ be the vector space over field $\mathbb{F}$ then a subset W of V $(W \subseteq V)$

is said to be subspace of V if W itself a vector space over $\mathbb{F}$.

$(W, +, \cdot)$ — $\mathbb{F}$ (Field) & $W \subseteq V$.

↳ Vector space over $\mathbb{F}$ $\Rightarrow$ W is called vector subspace of V over $\mathbb{F}$.

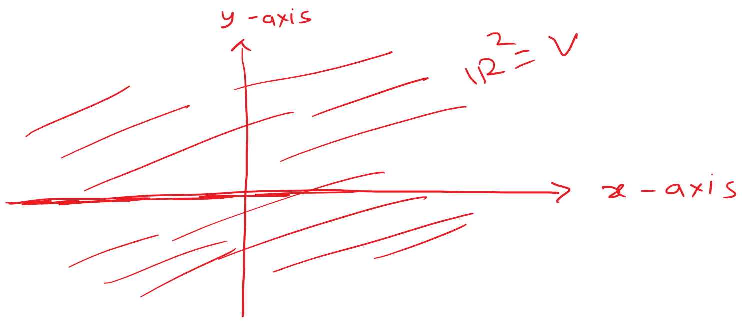
Example.

(1)

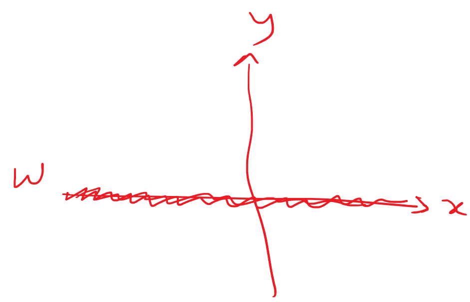
$\mathbb{R} \longrightarrow$ field, $n = 2$ ✓

$V = \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}$. is V a vector space over $\mathbb{R}$?

Yes. $\mathbb{F}^n$ over $\mathbb{F}$ forms a vector space



$W = \{(x, y) \mid y = 0\} = \{(x, 0)\} \subseteq \mathbb{R}^2$



Is W a vector space over $\mathbb{R}$?

Yes, as it satisfies all 8 properties of vector space.

and $W \subsetneq \mathbb{R}^2 \Rightarrow W$ is a subspace of $\mathbb{R}^2$

and $W \neq \{0\}$, $W \neq \mathbb{R}^2 \longrightarrow$ Non-trivial subspace.

$\{0\}$ & $\mathbb{R}^2 \longrightarrow$ trivial subspace

Subspace Test.

$(V, +, \cdot)$ be a vector space over a field $\mathbb{F}$ and $W \subseteq V$ then we

say W is subspace of V over $\mathbb{F}$ if it satisfies

(1) $0 \in W$

(2) $x + y \in W \quad \forall\, x, y \in W$

(3) $\alpha \cdot x \in W \quad \forall\, x \in W \;\&\; \alpha \in \mathbb{F}$

these 3 properties are enough to say that W itself a vector space

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$V = \mathbb{R}^2 = \{(x,y) \mid x \in \mathbb{R},\, y \in \mathbb{R}\}$ $\qquad \mathbb{F} = \mathbb{R} \to$ set of real numbers

$W = \{(x,0) \mid x \in \mathbb{R}\}$

To check : W is a subspace using subspace test

(1) It must contain zero vector

$V = \mathbb{R}^2, \qquad \vec{0} = (0,0)$

does $\vec{0} \in W$?

$(0,0) \in W \Rightarrow \vec{0} \in W.$

(2) Say, $\vec{x} \in W$ and $\vec{y} \in W$.

$\vec{x} = (x, 0) \in W, \qquad \vec{y} = (y, 0) \in W$

$$\vec{x} + \vec{y} = (x,0) + (y,0)$$
$$= (x+y,\, 0+0) = (x+y,\, 0)$$
$$\in W. \qquad \forall\, x, y \in W$$

(3) Say $\alpha \in \mathbb{R}$ be scalar and $\vec{x} \in W$.

$\vec{x} = (x, 0)$

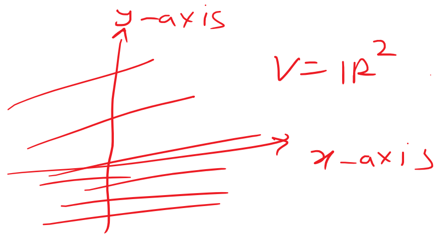
$$\alpha \cdot \vec{x} = \alpha \cdot (x, 0)$$
$$= (\alpha \cdot x,\, \alpha \cdot 0)$$
$$= (\alpha \cdot x,\, 0) \in W$$

ie. $W = \{(x,0) \mid x \in \mathbb{R}\}$ is subspace of $\mathbb{R}^2$ over $\mathbb{R}$.

$\rightarrow \qquad V = \mathbb{R}^2$

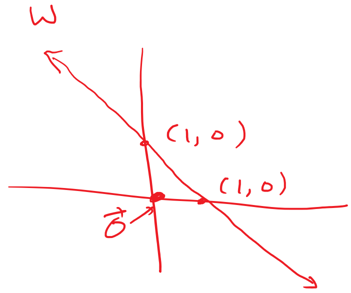
$V = \mathbb{R}^2$

$\rightarrow \quad V = \mathbb{R}^2$



$W = \{ (x, y) \mid x + y = 1 \}$

is a line in $\mathbb{R}^2$



$W \subseteq \mathbb{R}^2$
$\hookrightarrow$ Not a subspace.

To check: W is a subspace or not? { By subspace test }

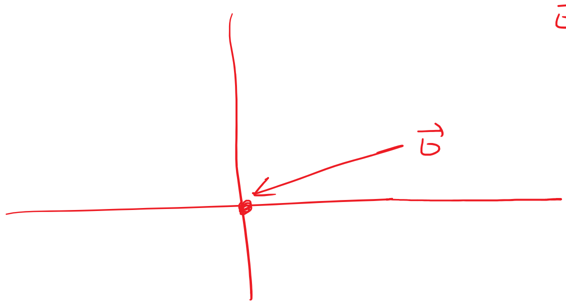
But W does not contain zero vector (Not a subspace)

$\vec{0} = (0, 0)$
$\hookrightarrow$ origin in $\mathbb{R}^2$

$\vec{0} \notin W$

$\rightarrow \quad W_1 = \{ (x, y) \mid x^2 - y^2 = 1 \} \subseteq \mathbb{R}^2$ over field $\mathbb{R}$.
$\hookrightarrow$ vector space

Is W_1 a vector subspace of $\mathbb{R}^2$?

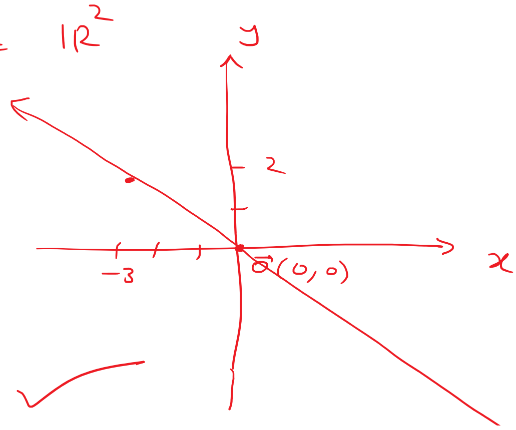


$\vec{0} = (0, 0) \notin W_1$

As, $0^2 - 0^2 = 0 \neq 1$.

$\rightarrow \quad W_2 = \{ (x, y) \mid 2x + 3y = 0 \} \subseteq \mathbb{R}^2$

$(-3, 2)$



Is W_2 a subspace of $\mathbb{R}^2$?

(1) $\vec{0} = (0, 0) \in W_2$

$\vec{0} \in W_2$ ✓

$x = 0, y = 0$

(1) $0 = (0,0) \in W_2$

$x=0, y=0,$ $\qquad \vec{0} \in W_2$

W_2

(2) Say $\vec{x} \in W_2$, $\vec{y} \in W_2$

So, $\vec{x} = (x_1, x_2)$ and $2x_1 + 3x_2 = 0$

$\vec{y} = (y_1, y_2)$ and $2y_1 + 3y_2 = 0$

$\vec{x}+\vec{y} = (x_1, x_2) + (y_1, y_2) = (x_1+y_1, x_2+y_2)$

$2(x_1+y_1) + 3(x_2+y_2)$

$= 2x_1 + 2y_1 + 3x_2 + 3y_2$

$= (2x_1+3x_2) + (2y_1+3y_2) = \vec{0}+\vec{0}$

$= \vec{0}$

So, $\vec{x}+\vec{y} \in W_2$

(3) $\alpha \in \mathbb{R}$, $\vec{x} \in W_2$

$\vec{x} = (x_1, x_2)$ and $2x_1 + 3x_2 = 0$

$\alpha \cdot \vec{x} = (\alpha x_1, \alpha x_2) \in W_2 \iff 2(\alpha x_1) + 3(\alpha x_2) = 0$

$\in W_2$ $\qquad \iff \alpha(2x_1 + 3x_2) = \alpha(0) = 0$

Results.

(1) $(V, +, \cdot)$ be the vector space over field $\mathbb{F}$, W_1 and W_2 are the subspaces of V

$W_1 \cap W_2$ is a subspace or not?

Proof. By subspace test.

(i) As W_1 is a subspace $\vec{0} \in W_1$
W_2 is a subspace $\vec{0} \in W_2$ $\Big\} \Rightarrow \vec{0} \in W_1 \cap W_2$

(ii) Let x and y be two vectors in $W_1 \cap W_2$

$x \in W_1 \cap W_2$, $y \in W_1 \cap W_2$

$\hookrightarrow x \in W_1$, $x \in W_2$, $\quad y \in W_1$, $y \in W_2$

and W_1 is a subspace, and $x, y \in W_1$

$x+y \in W_1$, similarly, $x \in W_2, y \in W_2 \Rightarrow x+y \in W_2$

$\Rightarrow x+y \in W_1 \cap W_2$

(iii) let α be any scalar, ($\alpha \in \mathbb{F}$) and $x \in W_1 \cap W_2$

i.e. $x \in W_1$ & $x \in W_2$

$\alpha \cdot x \in W_1$ {As $\alpha \in \mathbb{F}$ and $x \in W_1$ and W_1 is a subspace

Similarly, $\alpha \cdot x \in W_2$ {As $\alpha \in \mathbb{F}$ and $x \in W_2$ and W_2 is a subspace

$\alpha \cdot x \in W_1 \cap W_2$

By subspace test, $W_1 \cap W_2$ is a subspace of V.

→ Is this true for union as well ?

$W_1 \to$ subspace of V

$W_2 \to$ subspace of V

$W_1 \cup W_2 \longrightarrow$ Is this a subspace of V ?

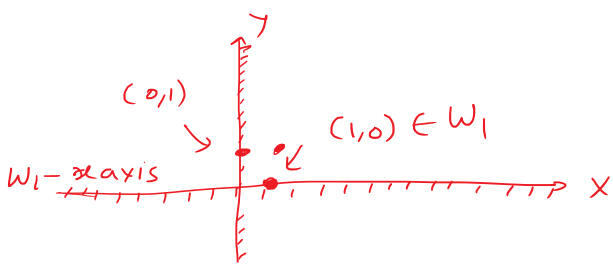
↳ Not true in general (try to find some counter examples)

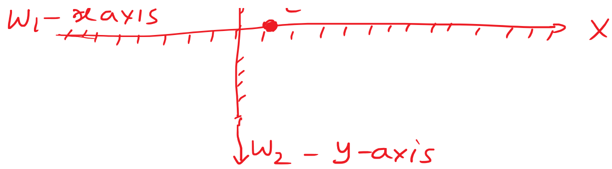
✓ $V = \mathbb{R}^2 = \{(x,y) \mid x, y \in \mathbb{R}\}$ over $\mathbb{F} = \mathbb{R}$.

$W_1 = \{(x,0) \mid x \in \mathbb{R}\} \longrightarrow W_1$ is subspace (check yourself)

$W_2 = \{(0,y) \mid y \in \mathbb{R}\} \longrightarrow W_2$ is subspace of $\mathbb{R}^2$ (")

$W_1 \cup W_2 = \{(x,0) \mid x \in \mathbb{R}\} \cup \{(0,y) \mid y \in \mathbb{R}\}$





$\vec{x} = (1,0) \in W_1$, $\vec{y} = (0,1) \in W_2$

$\vec{x} + \vec{y} = (1,0) + (0,1) = (1,1) \notin W_1 \cup W_2$

Because $(1,1) \notin W_1$, $(1,1) \notin W_2$

$\longrightarrow$ $W_1 \cup W_2$ will be subspace $\iff$ $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

$\downarrow$

$W_1 \cup W_2 = W_2$ or $W_2 \cup W_1 = W_1$

Next class.

- Span(S) forms sub
- Fundamental subspace $\rightarrow$
  - Column space
  - Null space
  - Row space
  - left Null space.
- Linearly independent & Dependent sets
- Basis & Dimension of Subspace.