→ $W_1 + W_2$ is also a subspace,
↳ Sum of subspaces W_1, W_2

→ span(S) $S = \{v_1, v_2, \ldots, v_n\}$
↳ finite set containing n-vectors

so, $span(S) = \{ \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \mid$ where α_i's are scalers from field $\}$

if $S \subseteq V$ then span(S) is a subspace of V
↳ Proof using subsp

→ span(S) is smallest subspace of V containing S.

so, if any subspace K contain S then $span(S) \subseteq K$.

→ The subspace $W_1 + W_2$ is the smallest subspace containing W_1 and W_2.

ie. if any subspace K contains W_1 & W_2 then K contains $W_1 + W_2$.

Linearly Dependent & Independent Sets:

Let $S = \{v_1, v_2, \ldots, v_n\}$ be a finite set, we say set S is linearly dependent if $\exists$ scalers $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ (Not all scalers are zero) such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

Then, set S is said to be linearly dependent set.
or, $v_1, v_2, \ldots, v_n$ are said to be linearly dependent vector.

Otherwise Set S is called linearly independent set.

Ex. $V = \mathbb{R}^3$ $\mathbb{F} = \mathbb{R}$

$= \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$

Consider $S = \{(1,0,0), (1,1,0), (1,1,1)\} \subseteq V$

↳ finite subset of $\mathbb{R}^3$

Question: Is set S linearly independent or dependent set?

Assume that there are scalars $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$\alpha_1(1,0,0) + \alpha_2(1,1,0) + \alpha_3(1,1,1) = (0,0,0)$$

$$(\alpha_1, 0, 0) + (\alpha_2, \alpha_2, 0) + (\alpha_3, \alpha_3, \alpha_3) = (0,0,0)$$

$$\Rightarrow (\alpha_1 + \alpha_2 + \alpha_3, \ \alpha_2 + \alpha_3, \ \alpha_3) = (0,0,0)$$

$$\Rightarrow \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 0 \\ \alpha_2 + \alpha_3 = 0 \\ \boxed{\alpha_3 = 0} \end{cases} \quad \Rightarrow \alpha_1 = 0, \quad \alpha_2 + 0 = 0 \Rightarrow \boxed{\alpha_2 = 0}$$

i.e. $\alpha_1 = 0, \ \alpha_2 = 0, \ \alpha_3 = 0$

$\Rightarrow$ The vectors v_1, v_2, v_3 of set S are linearly independent

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Remarks:

1) Any set S that contains zero vector is linearly dependent set.

$S = \{v_1, v_2, \ldots, v_n, 0\}$. S has zero vector

Now, we can always choose scalars,

$$\alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_n = 0, \quad \alpha_{n+1} = 1 \neq 0$$

If, we take linear combination of vectors of S with above choice of scalars

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_{n+1} v_{n+1} = 0 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_n + 1 \cdot 0 = 0$$

2. Suppose S is a linearly dependent set (L.D.) set then any superset of S is also linearly dependent.

$S = \{v_1, v_2, \ldots, v_n\}$

say, T is superset of S containing a few extra elements $(u_1, u_2, \ldots, u_m)$

$$S \subseteq T = \{\underbrace{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_m}_{n+m \text{ elements}}\}$$

As S is L.D.

i.e. $\exists$ scalers say $\alpha_1, \alpha_2, \ldots, \alpha_n$ (Not all α_i's are zero)

such that $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$

To show: T is also L.D.

So, we need to find $n+m$ scalers such that linear combination of vectors of T with the choice of scalers is a zero vector.

Define $\beta_1, \beta_2, \ldots, \beta_{n+m}$ scalers such that

$$\beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n + \beta_{n+1} u_1 + \beta_{n+2} u_2 + \cdots + \beta_{n+m} u_m = 0$$

$\beta_1 = \alpha_1$ $\quad$ $\beta_{n+1} = 0$

$\beta_2 = \alpha_2$ $\quad$ $\beta_{n+2} = 0$

$\vdots$ $\quad$ $\vdots$

$\beta_n = \alpha_n$ $\quad$ $\beta_{n+m} = 0$

$=$

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

So, Not all α_i's are zero scalers.

i.e. Not all β_i's are zero

i.e. set T is linearly dependent set

Superset of a linearly dependent set is linearly dependent set.

$\rightarrow$ Subset of a linearly independent set is linearly independent set.

(proof) Do your self. (Similar to above remark).

→ Two vectors in a set S are said to linearly dependent if and only if one vector is multiple of other.

$S = \{v_1, v_2\}$ is L.D. $\Longleftrightarrow$ $v_1 = k \cdot v_2$ where k is any scaler.

Proof: → that means, $\exists\ \alpha_1, \alpha_2$ (Not all α_1, α_2 are zero) such that

$$\alpha_1 v_1 + \alpha_2 v_2 = 0$$

Here:, W.L.O.G Assume that $\alpha_1 \neq 0$.

$$\alpha_1 v_1 = -\alpha_2 v_2$$

$$v_1 = \left(-\frac{\alpha_2}{\alpha_1}\right) \cdot v_2 \qquad \{\alpha_1 \neq 0\}$$

$$v_1 = k \cdot v_2 \qquad \text{where } k = -\frac{\alpha_2}{\alpha_1} \in \mathbb{F}$$

$$\alpha_1 = 1 \cdot v_1 - k v_2 = 0$$

$V = \mathbb{R}^3$, $\mathbb{F} = \mathbb{R}$

$$S = \{\underbrace{(1,1,0)}_{v_1}, \underbrace{(1,3,2)}_{v_2}, \underbrace{(4,9,5)}_{v_3}\}.$$

↳ How will you find that set S is linearly independent or linearly dependent?

$$\rightarrow \quad A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 2 \\ 4 & 9 & 5 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 4 & 9 & 5 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 4R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 5 & 5 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - \frac{5}{2} R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Echelon form of A.

$\underbrace{\text{Row 1, Row 2}}$

↳ has pivot,

V_3 vector is linearly dependent on vectors V_1 & V_2.

$\rightarrow$ $S = \{ (1,0,0), (1,1,0), (1,1,1) \}$

$$\begin{array}{ccc} (1,0,0) & (1,1,0) & (1,1,1) \\ \uparrow & \uparrow & \uparrow \\ v_1 & v_2 & v_3 \end{array}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

Here, $\{v_1, v_2, v_3\}$ is linearly independent set.

v_1, v_2, v_3 are L.I. vectors.

Basis & Dimension of Vector Space

Assume that V is a vector space over field $\mathbb{F}$, A set $B (\subseteq V)$ is said to be basis of V if

1) B is linearly independent set.
2) $\text{span}(B) = V$ $\hookrightarrow$ B generates space V.

and, the number of elements in the basis B is called dimension of vector space V.

Example:

$V = \mathbb{R}^4$, $\mathbb{F} = \mathbb{R}$

$= \{ (x_1, x_2, x_3, x_4) \mid x_i \in \mathbb{R} \}$.

$$B = \{ \underset{\substack{\uparrow \\ v_1}}{(1,0,0,0)}, \underset{\substack{\uparrow \\ v_2}}{(0,1,0,0)}, \underset{\substack{\uparrow \\ v_3}}{(0,0,1,0)}, \underset{\substack{\uparrow \\ v_4}}{(0,0,0,1)} \}$$

is B a basis of $\mathbb{R}^4$?

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1 Is B a linearly independent set?

If we take scalers, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$

$$\alpha_1(1,0,0,0) + \alpha_2(0,1,0,0) + \alpha_3(0,0,1,0) + \alpha_4(0,0,0,1) = (0,0,0,0)$$

$$\Rightarrow (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0,0,0,0) \Rightarrow \begin{matrix} \alpha_1 = 0, & \alpha_3 = 0 \\ \alpha_2 = 0, & \alpha_4 = 0 \end{matrix}$$

Set B is L.I set.

(2) $\text{span}(B) = V = \mathbb{R}^4$.

$B \subseteq \mathbb{R}^4$ ↳ vector space and $\text{span}(B)$ is the smallest subspace containing B

ie. $B \subseteq \text{span}(B) \subseteq \mathbb{R}^4$, ✓

$$\left\{ \begin{matrix} A = B \\ \Leftrightarrow \quad A \subseteq B \text{ and } B \subseteq A \end{matrix} \right\}$$

To show: $\mathbb{R}^4 \subseteq \text{span}(B)$ ✓

Let's choose any vector $\vec{v} \in \mathbb{R}^4$

$\vec{v} = (x_1, x_2, x_3, x_4)$ where $x_i \in \mathbb{R}$.

Now,

$$\boxed{\underbrace{(x_1, x_2, x_3, x_4)}_{\in \mathbb{R}^4} = x_1(1,0,0,0) + x_2(0,1,0,0) + x_3(0,0,1,0) + x_4(0,0,0,1)} \in \text{span}(B)$$

ie. $\mathbb{R}^4 \subseteq \text{span}(B)$ ✓

$\text{span}(B) = \mathbb{R}^4$.

ie. $B = \{\underbrace{(1,0,0,0)}_{e_1}, \underbrace{(0,1,0,0)}_{e_2}, \underbrace{(0,0,1,0)}_{e_3}, \underbrace{(0,0,0,1)}_{e_4}\}$ is basis of $\mathbb{R}^4$

↓ Standard Basis / Canonical Basis.

→ Canonical Basis of $\mathbb{R}^n$ over $\mathbb{R}$.

$e_1 = (1, 0, \ldots, 0)$

$e_2 = (0, 1, 0, \ldots 0)$

$B = \{e_1, e_2, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

$e_2 = (0, 1, 0, \cdots 0)$

$\vdots$

$e_n = (0, 0, \cdots 0, 1)$

$\uparrow$ nth place

$B = \{e_1, e_2, \cdots$

standard basis of $\mathbb{R}$

Dimension of $\mathbb{R}^n$ = No. of elements in Basis

$= n$

$\boxed{\text{Dim}(\mathbb{R}^n) = n}$

↳ Notation for dimension.

→ <u>Question</u> : Does there exist other basis as well?

$V = \mathbb{R}^3$, $\mathbb{F} = \mathbb{R}$.

$v_1 = (2, 0, 0)$

$v_2 = (0, 2, 0)$

$v_3 = (0, 0, 2)$

$S = \{v_1, v_2, v_3\}$

↳ is set S a basis of $\mathbb{R}^3$?

L.I set?

span(S) = $\mathbb{R}^3$.

Now, $(x, y, z) \in \mathbb{R}^3$

↳ $(x, y, z) = \frac{x}{2}(2, 0, 0) + \frac{y}{2}(0, 2, 0) + \frac{z}{2}(0, 0, 2)$

$\in S$ $\in S$ $\in S$

$\in$ span(S)

$\mathbb{R}^3 \subseteq$ span(S) ✓

span(S) $= \alpha_1(2, 0, 0) + \alpha_2(0, 2, 0) + \alpha_3(0, 0, 2)$

$= (2\alpha_1, 2\alpha_2, 2\alpha_3)$

$= (x, y, z) \in \mathbb{R}^3$

$S = \{(3, 0, 0), (0, 3, 0), (0, 0, 3)\}$.

$B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

S is basis

$\times 3$

B was standard basis

S is basis

B was standard basis

×3

$S = \{(1,1,0), (1,0,1), (0,1,1)\}$ Basis

$S = \{$

Next class

↳ Linear Transformation By Jyoti Chahal.

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