Echelon Form

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A matrix is said to be in Echelon form if

- All rows consisting ~~of~~ of zeroes are at bottom.
- The pivot (leading non-zero entry in non-zero row) is strictly right to pivot in any non-zero row above it.

pivot

1) $\begin{bmatrix} \boxed{1} & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ← zero row at bottom

Echelon form.

Gaussian-Elimination Method to reduce any matrix to echelon form.

$$A = \begin{bmatrix} \boxed{2} & -2 & 2 & 1 \\ -3 & 6 & 0 & 1 \\ 1 & -7 & 10 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 + \frac{3}{2}R_2} \begin{bmatrix} 2 & -2 & 2 & 1 \\ 0 & 3 & 3 & \frac{5}{2} \\ 1 & -7 & 10 & 2 \end{bmatrix}$$

$\frac{3}{2} + 5$

$$\xrightarrow{R_3 \to R_3 - \frac{1}{2}R_1} \begin{bmatrix} \boxed{2} & -2 & 2 & 1 \\ 0 & \boxed{3} & 3 & \frac{5}{2} \\ 0 & -6 & 9 & \frac{3}{2} \end{bmatrix} \xrightarrow{R_3 \to R_3 + 2R_2} \begin{bmatrix} 2 & -2 & 2 & 1 \\ 0 & 3 & 3 & \frac{5}{2} \\ 0 & 0 & 15 & \frac{13}{2} \end{bmatrix}$$

Echelon form

Three pivots

$\rho(A) = 3 = \#$ of pivots

Def^n Rank of a matrix.

Notation of rank of A

Let A be any matrix, rank of matrix A ($\rho(A)$) is defined as the total number of pivots in echelon form of matrix A.

Ex.
$$A = \begin{bmatrix} 2 & 2 & -1 & 6 & 4 \\ 4 & 4 & 1 & 10 & 13 \\ 8 & 8 & -1 & 26 & 23 \end{bmatrix}$$

To find: Rank of matrix A ($\rho(A)$)

By Gaussian-Elimination method, we find the echelon form of matrix of A.

$$\begin{bmatrix} 2 & 2 & -1 & 6 & 4 \\ 4 & 4 & 1 & 10 & 13 \\ 8 & 8 & -1 & 26 & 23 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 2 & 2 & -1 & 6 & 4 \\ 0 & 0 & 3 & -2 & 5 \\ 8 & 8 & -1 & 26 & 23 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 4R_1} \begin{bmatrix} 2 & 2 & -1 & 6 & 4 \\ 0 & 0 & 3 & -2 & 5 \\ 0 & 0 & 3 & 2 & 7 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 2 & 2 & -1 & 6 & 4 \\ 0 & 0 & 3 & -2 & 5 \\ 0 & 0 & 0 & 4 & 2 \end{bmatrix}$$

Echelon form of A .

Rank of matrix A $= \rho(A) =$ # of pivots in echelon form of matrix A

$= 3$

Row Reduced Echelon Form(RREF)

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Def.

A matrix is said to be in row reduced echelon form (rref) / row canonical form, if

→ The matrix is in echelon form

→ Pivot element in each row must be equal to 1.

→ The pivot in each row must be the only non-zero number in it's column.

Ex.

$$\begin{bmatrix} 5 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

zero row is not at bottom

so. it is not in echelon form.

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Pivot element is not 1.

zero row at bottom.

Echelon form

Not RREF.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

is this matrix RREF?

Yes, RREF.

pivot

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Is this matrix in echelon form?

Yes, in echelon form

Is this matrix in RREF?

Yes, in RREF.

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is this in echelon form?

is this in rref?

Yes, RREF.

Gauss-Jordan Method. : Any matrix can be reduced to it's rref.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$\det(A) = 4 - 6 = -2 \neq 0$. so A is invertible.

$$\Big\downarrow R_2 \to R_2 - 3R_1$$

$$\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \longrightarrow \text{echelon form of } A.$$

$$\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \longrightarrow \text{echelon form}$$

$$\Big\downarrow R_2 \to R_2\left(-\tfrac{1}{2}\right)$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \longrightarrow \text{is this in rref ?}$$

$$\Big\downarrow R_1 \to R_1 - 2R_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \text{RREF}$$

Question : The rref of every invertible matrix is identity matrix ??

$$A \xrightarrow{e_1} e_1(A) \xrightarrow{e_2} e_2(e_1(A)) \longrightarrow \; - - - - \xrightarrow{e_n} R$$

A ↓ invertible matrix

R ↳ RREF(A)

$$\boxed{A \sim R}$$

$$R = e_n e_{n-1} \cdots e_3 e_2 e_1(A)$$

$e_1 \longrightarrow E_1 = e_1(I)$

$e_2 \longrightarrow E_2 = e_2(I)$

$\vdots$

$e_n \longrightarrow E_n = e_n(I)$

Let's say E_k be the corresponding elementary matrix of e_k $(k=1,2,\dots n)$

$$R = \underbrace{(E_n E_{n-1} E_{n-2} \cdots E_3 E_2 E_1)}_{\text{product of invertible matrices}} A$$

↳ product of invertible matrices.

$\Rightarrow$ R is invertible being product of invertible matrices.

Also, R is in rref.

$$R = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ 0 & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & & \vdots \\ \vdots & \vdots & & & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

that means, R has to be identity matrix.

$$\text{RREF}(A) = I$$

for A being invertible.

→ Every invertible matrix has rref equals to identity matrix

Lemma

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Any invertible matrix is row equivalent to the identity matrix.

was used to prove that every invertible matrix can be written as product of elementary matrices.

Proof:

Suppose A is invertible matrix, Also assume that R be the rref of matrix A.

i.e. we have applied a finite sequence of elementary operations on A to get matrix R.

$$A \xrightarrow{e_1} e_1(A) \xrightarrow{e_2} e_2(A) \longrightarrow \cdots \xrightarrow{e_n} R$$

$$\Rightarrow \quad R = \underbrace{E_n E_{n-1} \cdots E_3 E_2 E_1 A}_{\text{product of invertible matrix}} \qquad \text{where} \quad E_k = e_k(I)$$

for $k = 1, 2, \ldots, n$

So, R is rref(A) and invertible that means, $R = I$.

$$I = E_n E_{n-1} \cdots E_3 E_2 E_1 A.$$

$\boxed{A \sim I}$ i.e. A is row equivalent to identity matrix as $R = I$ is obtained by a finite sequence of elementary operation on A.

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1) Let A be any matrix, R be the row reduced echelon form of matrix A.

$$\rho(A) = \rho(R)$$

2) Rank of matrix = Total number of pivots in echelon form of matrix A.

and any row can have atmost one pivot, similarly any column can have atmost one pivot.

Let A be $m \times n$ matrix.

$$\left.\begin{array}{l} \rho(A) \le m \\ \rho(A) \le n \end{array}\right\} \Rightarrow \boxed{\rho(A) \le \min\{m, n\}}$$

Ex.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 & 11 & 12 \end{bmatrix}$$

$\rho(A) \le 6$

$\boxed{\rho(A) \le \min\{m, n\}}$

$$\begin{bmatrix} 7 & 8 & 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 & 17 & 18 \end{bmatrix} \quad 3\times 6 \text{ matrix.}$$

$$\boxed{\rho(A) \le \min\{m,n\}.}$$

$\rho(A) \le 3$ $\quad$ $\left.\begin{matrix}\rho(A) \le 3 \\ \rho(A) \le 6\end{matrix}\right\}$ — $\rho(A) \le 3$

Nullity of a matrix. $\quad$ $(\eta(A)) \leftarrow$ Notation

Nullity of a matrix is defined as the number of the free variables in echelon form of matrix A.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$$

$$\downarrow R_2 \to R_2 - 4R_1$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix} \longrightarrow \text{echelon form}$$

free variable $\rightarrow$ any column not having pivot in echelon form

pivots — 2 $\leftarrow$ Rank(A) $\rho(A)$

free variable — 1 $\leftarrow$ Nullity(A) $\leftarrow \eta(A)$

Rank-Nullity Theorem. $\quad$ Let A be any $m\times n$ matrix then

$$\rho(A) + \eta(A) = \# \text{ of columns of } A = n$$

$$\boxed{\rho(A) + \eta(A) = n} \qquad \boxed{\eta(A) = n - \rho(A)}$$

$\leftarrow$ Statement only

Proof to be done with discussion on fundamental subspaces

Column space, Null space.

Next class. —

- $\rightarrow$ Vector space. "examples" $\quad$ focus on n-tuples $\quad \mathbb{R}^n(\mathbb{R}) \quad \mathbb{F}^n(\mathbb{F})$
- $\rightarrow$ Subspaces