The Qudit Arthurs-Kelly Measurement... for SIC's!

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UMB: QBism Group

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Historical flavor

From *Bell System Technical Journal*:

B.S.T.J. BRIEFS

On the Simultaneous Measurement of a Pair of Conjugate Observables

By E. ARTHURS and J. L. KELLY, JR.

(Manuscript received December 16, 1964)

A precise theory of the simultaneous measurement of a pair of conjugate observables is necessary for obtaining the classical limit from the quantum theory, for determining the limitations of coherent quantum mechanical amplifiers, etc. The uncertainty principle, of course, does

Arthurs-Kelly procedure

- Prepare two ancillas in a special starting state.
- Coherently shift ancilla 1 conditional on the position of the system.
- Coherently shift ancilla 2 conditional on the momentum of the system.
- Measure the ancillas.
  - Realizes a “Weyl-Heisenberg covariant POVM” on the system.
  - E.g., coherent state measurement, or “heterodyne” measurement in optics.
  - Can be adapted to qudits: in particular, for a SIC measurement.

Weyl-Heisenberg Group and SIC's

Let $\mathcal{H}_d$ be a finite dimensional Hilbert space, and let $\omega = e^{2\pi i/d}$:

$$Z|m\rangle = \omega^m |m\rangle \qquad X|m\rangle = |m+1\rangle \qquad D_{a,b} = X^a Z^b. \tag{1}$$

Let $\Pi_{a,b} = D^\dagger_{a,b} \Pi D_{a,b}$ for fiducial $\Pi = |\phi\rangle\langle\phi|$:

$$\mathrm{tr}(\Pi_{a,b}\Pi_{a',b'}) = \frac{d\delta_{aa'}\delta_{bb'} + 1}{d+1} \Longrightarrow \text{SIC}. \tag{2}$$

A SIC set: regular simplex inscribed in quantum state-space!

Qudit Arthurs-Kelly

Let $|\cdot\rangle$ denote discrete position states and $|\cdot\rangle_p$ denote discrete momentum states.

$$U = \left(\sum_m I \otimes X^{-m} \otimes |m\rangle_p \langle m|_p\right)\left(\sum_k X^{-k} \otimes I \otimes |k\rangle\langle k|\right) \quad (3)$$

Kraus operators: given computational basis outcomes (x, y) on the ancillas, we update the system via

$$K_{xy} = \frac{1}{\sqrt{d}} \sum_{km} \omega^{-km} \langle x+k, y+m|\gamma\rangle \; |m\rangle_p \langle k|. \quad (4)$$

$|\gamma\rangle$ is the initial state of the ancillas,

$$\langle k, m|\gamma\rangle = \omega^{km} \langle m|F^\dagger \Pi|k\rangle, \quad (5)$$

where F is the Fourier transform operator, and Π is the fiducial.

Preparing the ancillas...

...from the fiducial:

$$\begin{aligned} \langle k, m|\gamma\rangle &= \omega^{km}\langle m|F^\dagger \Pi|k\rangle && (6)\\ &= \langle k, m| \left(\sum_j |j\rangle\langle j| \otimes Z^j\right) \left(I \otimes F^\dagger\right)\left(|\phi^*\rangle \otimes |\phi\rangle\right). && (7) \end{aligned}$$

$d = 4$ SIC fiducial

Exploit the "monomial representation" to write

$$|\phi\rangle = (H \otimes I) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i\pi(-1/4)} & 0 & 0 \\ 0 & 0 & e^{i\pi(1/4)} & 0 \\ 0 & 0 & 0 & e^{i\pi(1/2)} \end{pmatrix} \frac{1}{\sqrt{5+\sqrt{5}}} \begin{pmatrix} \sqrt{2+\sqrt{5}} \\ 1 \\ 1 \\ 1 \end{pmatrix}, \tag{8}$$

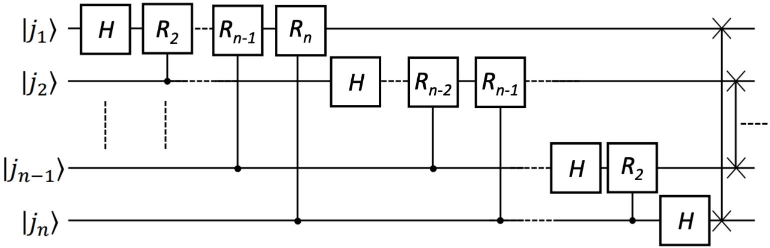
where H is the Hadamard gate.

Qubit implementation

Let $\mathtt{R}(k) = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{pmatrix}$.

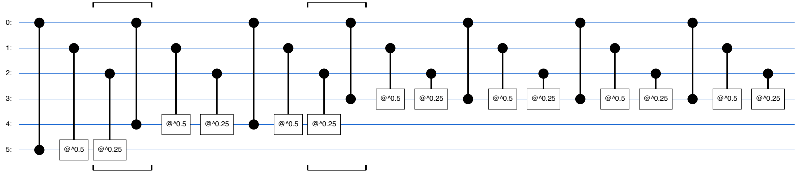
$$\mathtt{Z} = \bigotimes_{j=0}^{n-1} \mathtt{R}(j+1) \qquad \mathtt{X} = \mathtt{F}^{\dagger}\mathtt{Z}\mathtt{F}. \tag{9}$$

Qudit Fourier transform:



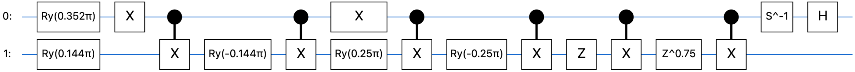
CZ & CX

CZ can be built out of a series of qudit-controlled Z's, and CX obtained by Fourier transforming. Below: first three qubits target, second three qubits control:



$$\left(I \otimes |0\rangle\langle 0| \otimes I \otimes I + Z^4 \otimes |1\rangle\langle 1| \otimes I \otimes I\right)$$
$$\left(I \otimes I \otimes |0\rangle\langle 0| \otimes I + Z^2 \otimes I \otimes |1\rangle\langle 1| \otimes I\right)$$
$$\left(I \otimes I \otimes I \otimes |0\rangle\langle 0| + Z \otimes I \otimes I \otimes |1\rangle\langle 1|\right)$$
$$= \sum_{m=0}^{7} Z^m \otimes |m\rangle\langle m|$$

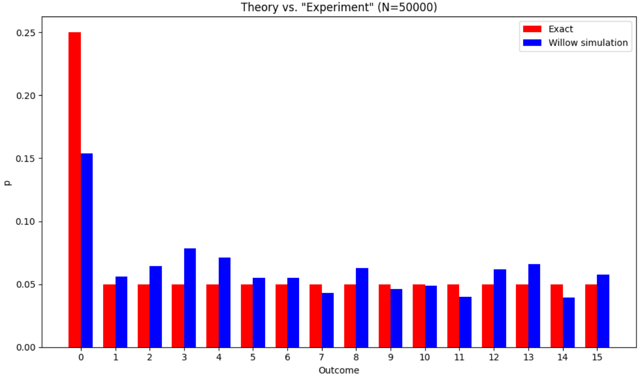
Preparing the $d = 4$ fiducial



Gate counts

Fiducial preparation		Ancilla preparation		Arthurs-Kelly unitary	
Gate type	Count	Gate type	Count	Gate type	Count
Ry	5	SwapPowGate	1	HPowGate	12
PauliX	2	HPowGate	2	CZPowGate	18
CXPowGate	6	CZPowGate	7	SwapPowGate	6
ZPowGate	3				
HPowGate	1				

On Willow: SIC measurement on SIC fiducial



Gate Type	Count
PhasedXZGate	124
CZPowGate	127
PhasedXPowGate	159
ZPowGate	9
MeasurementGate	1

A simplification

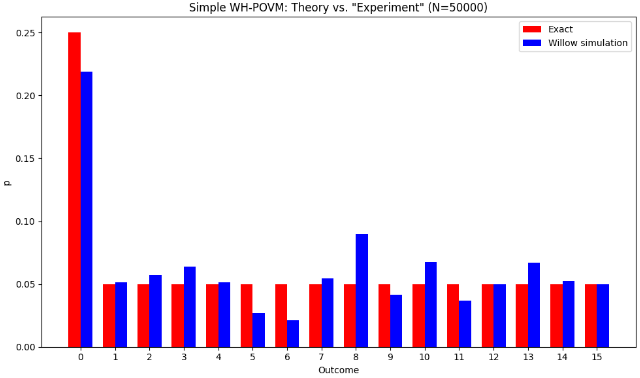
Let $|\psi\rangle$ an arbitrary state, and $|\phi\rangle$ be the fiducial.

$$\Big(\langle a| \otimes \langle b|\Big)(I \otimes F^\dagger)\left(\sum_j X^{-j} \otimes |j\rangle\langle j|\right)\Big(|\psi\rangle \otimes |\phi^*\rangle\Big)$$
$$= \frac{1}{\sqrt{d}}\langle\phi|D^\dagger_{a,b}|\psi\rangle = \frac{1}{\sqrt{d}}\langle\phi_{a,b}|\psi\rangle : \tag{10}$$

The probabilities for a computational basis measurement are precisely the SIC probabilities. Just one ancilla!

Gate Type	**Count**
HPowGate	6
CZPowGate	9
SwapPowGate	3

On Willow II: SIC measurement on SIC fiducial

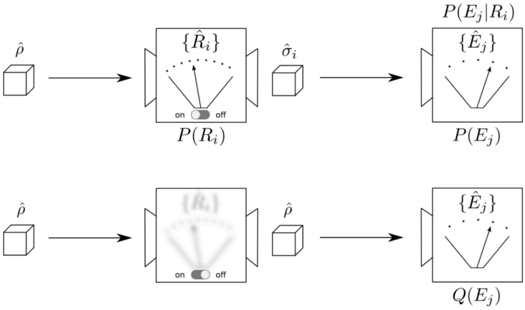


Gate Type	Count
PhasedXPowGate	31
CZPowGate	23
ZPowGate	3
PhasedXZGate	21
MeasurementGate	1

Outlook

- Circuit simplifications?
- Handle grid structure natively?
- Qubit picking? Calibration?
- Alternative methods entirely.
- The Born rule...

The Born Rule



$$P(E_j) = \sum_i P(E_j|R_i)P(R_i) \tag{11}$$

$$Q(E_j) = \sum_{ik} P(E_j|R_i)\Phi_{ik}P(R_k) \tag{12}$$

$$= \sum_i P(E_j|R_i)\left[(d+1)P(R_i) - \frac{1}{d}\right] \tag{13}$$

where $\Phi = P(R|R)^{-1}$.