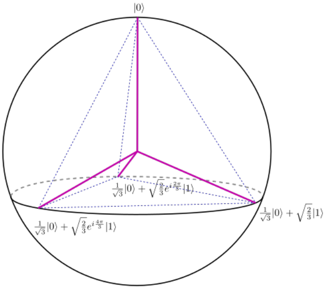
# A ququart SIC for ion trap quantum computers

Matthew B. Weiss

UMB: QBism Group

9/12/25

# Symmetric informationally complete sets




Fix a Hilbert space $\mathcal{H}_d$. If you can find a set of $d^2$ rank-1 projectors $\{\Pi_i\}$ satisfying

$$\mathrm{tr}(\Pi_i \Pi_j) = \frac{d\delta_{i,j} + 1}{d + 1}, \tag{1}$$

then you've found regular simplex inscribed in quantum state-space: a SIC!

# SIC's

- SIC states (with a group covariance) have *maximal magic*.
- The corresponding POVM $\{E_i = \frac{1}{d}\Pi_i\}$ is *optimal for linear quantum state tomography*.
- A SIC-based generalization of the double-split experiment demonstrates the *inherent contextuality* of quantum theory.
- The numbers involved in constructing a SIC have *deep algebraic number theoretic significance*.
- The question of whether SIC's exist in all dimensions directly relates to Hilbert's 12 problem about abelian extensions of the rationals!

# The Weyl-Heisenberg Group

Fixing $\mathcal{H}_d$, let $\omega = e^{2\pi i/d}$, and define clock and shift operators,

$$Z|m\rangle = \omega^m|m\rangle \qquad X|m\rangle = |m+1\rangle \qquad X = F^\dagger Z F, \tag{2}$$

where $F$ is the DFT, as well as Weyl-Heisenberg displacement operators,

$$D_{a,b} = X^a Z^b. \tag{3}$$

Save one sporadic case, the orbit of a special "fiducial" state $\Pi$ under the WH group gives the $d^2$ SIC states,

$$\Pi_{a,b} = D_{a,b}\Pi D_{a,b}^\dagger. \tag{4}$$

WH group covariance means:

- We can find fiducials more easily classically.
- We can implement SICs more easily quantum mechanically.

# Historical flavor

From *Bell System Technical Journal*:

## B.S.T.J. BRIEFS

### On the Simultaneous Measurement of a Pair of Conjugate Observables

**By E. ARTHURS and J. L. KELLY, JR.**

(Manuscript received December 16, 1964)

A precise theory of the simultaneous measurement of a pair of conjugate observables is necessary for obtaining the classical limit from the quantum theory, for determining the limitations of coherent quantum mechanical amplifiers, etc. The uncertainty principle, of course, does

# Arthurs-Kelly procedure

- Prepare two ancillas in a special starting state.
- Coherently shift ancilla 1 conditional on the position of the system.
- Coherently shift ancilla 2 conditional on the momentum of the system.
- Measure the ancillas.

Realizes a "Weyl-Heisenberg covariant POVM" on the system.

- E.g., a coherent state measurement, or "heterodyne" measurement in optics.
- Can be adapted to qudits: in particular, for a SIC measurement.

# Qudit Arthurs-Kelly

Let $|\cdot\rangle$ denote discrete position states and $|\cdot\rangle_p$ denote discrete momentum states.

$$U_{AK} = \left(\sum_m I \otimes X^{-m} \otimes |m\rangle_p \langle m|_p\right) \left(\sum_k X^{-k} \otimes I \otimes |k\rangle\langle k|\right), \tag{5}$$

where $|m\rangle_p = F|m\rangle$.

# Qudit Arthurs-Kelly

- To prepare the staring state of the two ancillas,

$$U_{PA} = \left( \sum_j |j\rangle\langle j| \otimes Z^j \otimes I \right) \left( I \otimes F^\dagger \otimes I \right). \quad (6)$$

- Let $|\phi\rangle$ be a SIC fiducial, and $|\psi\rangle$ the system state, then

$$|\text{post-interaction}\rangle = U_{AK} U_{PA} \Big( |\phi^*\rangle \otimes |\phi\rangle \otimes |\psi\rangle \Big). \quad (7)$$

- Measuring the two ancillas gives $d^2$ possible outcomes with

$$P(a, b) = \frac{1}{d} \text{tr}(\Pi_{a,b} \rho). \quad (8)$$

- Afterwards, the system is projected into the SIC state $\Pi_{a,b}$.

# A simplification

- Let $|\psi\rangle$ an arbitrary state, and $|\phi\rangle$ be the fiducial.

$$\Big(\langle a| \otimes \langle b|\Big)(I \otimes F^\dagger) \left(\sum_j X^{-j} \otimes |j\rangle\langle j|\right) \Big(|\psi\rangle \otimes |\phi^*\rangle\Big)$$
$$= \frac{1}{\sqrt{d}} \langle \phi | D^\dagger_{a,b} | \psi \rangle \tag{9}$$

- The probabilities for a computational basis measurement are precisely the SIC probabilities.
- Just one ancilla!
- But afterwards, the state is projected into a computational basis state.

# A $d = 4$ fiducial

Exploit the "monomial representation" to write

$$|\phi\rangle = (H \otimes I) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i\pi(-1/4)} & 0 & 0 \\ 0 & 0 & e^{i\pi(1/4)} & 0 \\ 0 & 0 & 0 & e^{i\pi(1/2)} \end{pmatrix} \frac{1}{\sqrt{5+\sqrt{5}}} \begin{pmatrix} \sqrt{2+\sqrt{5}} \\ 1 \\ 1 \\ 1 \end{pmatrix}, \tag{10}$$

where $H$ is the qubit Hadamard gate.

# Ion traps

- *Experimental single-setting quantum state tomography*: you implement a qubit SIC using a native quqart.
- Can we realize a ququart SIC using two (or three) native ququarts?
- *A universal qudit quantum processor with trapped ions*: working with $d = 10$ qudits, you show how to implement a controlled exchange gate $C_{\mathrm{EX}}$ from which one can implement $C_{\mathrm{SUM}}$, that is, calling the controlled shift.
- How hard is it to do the discrete Fourier transform?
- Compare performance to tensor products of qubit SIC's in estimating observables, etc.

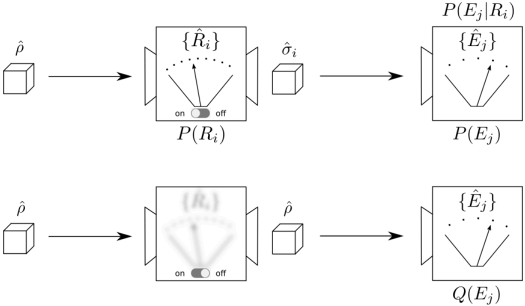
# From tomography to the Born rule

- Reconstructing density matrices from SIC probabilities:

$$\rho = \sum_i \left[ (d+1)P(R_i) - \frac{1}{d} \right] \Pi_i, \tag{11}$$

  where $P(R_i) = \frac{1}{d}\mathrm{tr}(\Pi_i \rho)$.
- Leads to an elegant reformulation of the Born rule as a consistency rule for probability assignments...

# The Born Rule




$$P(E_j) = \sum_i P(E_j|R_i)P(R_i) \tag{12}$$

$$Q(E_j) = \sum_{ik} P(E_j|R_i)\Phi_{ik}P(R_k) \tag{13}$$

where $\Phi = P(R|R)^{-1}$ for any IC $\{R_i\}$ and $\{\sigma_i\}$.

# The Born Rule

- In particular, for a SIC:

$$Q(E_j) = \sum_i P(E_j|R_i)\left[(d+1)P(R_i) - \frac{1}{d}\right]. \qquad (14)$$

- Among all IC-POVMs with $d^2$ elements, $\|I - \Phi\|$ achieves its minimum only for SIC's.
- Other measurements artificially exaggerate the difference between classical and quantum!
- Let $W(E_j) = (d+1)P(R_i) - \frac{1}{d}$: these are quasiprobabilities.
- Their negativity diagnoses contextuality.
- The more noise, the less negativity.

# Consistency

Let us consider the following set of experiments:

1. Prepare the SIC states in turn, and then a SIC measurement. Yields matrix of probabilities $P_{ij} = P(R_i|R_j)$.
2. Prepare the computational basis states $\{\Pi_i = |i\rangle\langle i|\}$, and then performing a SIC. Yields probabilities $p_{ij} = P(R_j|\Pi_i)$.
3. Prepare the SIC states, and a computational basis measurement: $C_{ij} = P(\Pi_i|R_j)$.
4. Prepare the computational basis states, and then a computational basis measurement: $q_{ij} = P(\Pi_i|\Pi_j)$.

Letting $\Phi = P^{-1}$, the Born rule then demands the following consistency criterion,

$$q = C\Phi p. \tag{15}$$

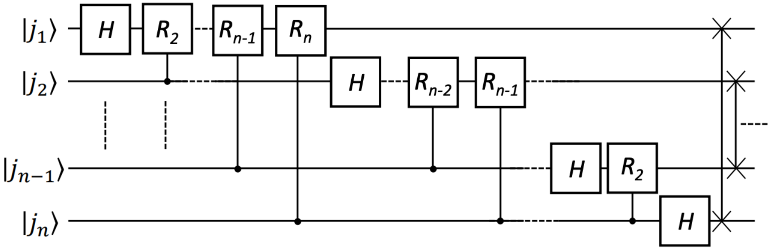
Can we achieve this in the presence of noise?

# For reference, a qubit implementation

Let $\mathtt{R}(k) = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{pmatrix}$.

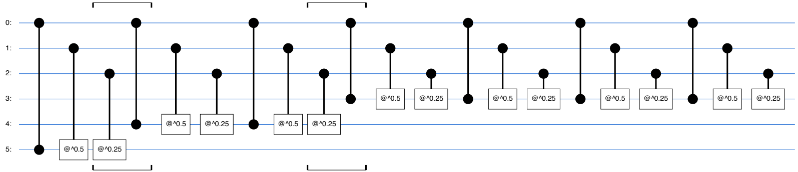
$$\mathtt{Z} = \bigotimes_{j=0}^{n-1} \mathtt{R}(j+1) \qquad\qquad \mathtt{X} = \mathtt{F}^{\dagger}\mathtt{ZF}. \tag{16}$$

Qudit Fourier transform:



# CZ & CX

CZ can be built out of a series of qudit-controlled Z's, and CX obtained by Fourier transforming. Below: first three qubits target, second three qubits control:




$$\left(I \otimes |0\rangle\langle 0| \otimes I \otimes I + Z^4 \otimes |1\rangle\langle 1| \otimes I \otimes I\right)$$
$$\left(I \otimes I \otimes |0\rangle\langle 0| \otimes I + Z^2 \otimes I \otimes |1\rangle\langle 1| \otimes I\right)$$
$$\left(I \otimes I \otimes I \otimes |0\rangle\langle 0| + Z \otimes I \otimes I \otimes |1\rangle\langle 1|\right)$$
$$= \sum_{m=0}^{7} Z^m \otimes |m\rangle\langle m|$$

# Preparing a $d = 4$ fiducial

As a circuit: