

# The Qudit Arthurs-Kelly Measurement... for SIC's!

Matthew B. Weiss

UMB: QBism Group

9/12/25

From *Bell System Technical Journal*:

## **B.S.T.J. BRIEFS**

### **On the Simultaneous Measurement of a Pair of Conjugate Observables**

**By E. ARTHURS and J. L. KELLY, JR.**

(Manuscript received December 16, 1964)

A precise theory of the simultaneous measurement of a pair of conjugate observables is necessary for obtaining the classical limit from the quantum theory, for determining the limitations of coherent quantum mechanical amplifiers, etc. The uncertainty principle, of course, does

# Arthurs-Kelly procedure

- ▶ Prepare two ancillas in a special starting state.
- ▶ Coherently shift ancilla 1 conditional on the position of the system.
- ▶ Coherently shift ancilla 2 conditional on the momentum of the system.
- ▶ Measure the ancillas.
  - ▶ Realizes a “Weyl-Heisenberg covariant POVM” on the system.
  - ▶ E.g., coherent state measurement, or “heterodyne” measurement in optics.
  - ▶ Can be adapted to qudits: in particular, for a SIC measurement.

# Weyl-Heisenberg Group and SIC's

Let  $\mathcal{H}_d$  be a finite dimensional Hilbert space, and let  $\omega = e^{2\pi i/d}$ :

$$Z|m\rangle = \omega^m|m\rangle \quad X|m\rangle = |m+1\rangle \quad D_{a,b} = X^a Z^b. \quad (1)$$

Let  $\Pi_{a,b} = D_{a,b}^\dagger \Pi D_{a,b}$  for fiducial  $\Pi = |\phi\rangle\langle\phi|$ :

$$\text{tr}(\Pi_{a,b}\Pi_{a',b'}) = \frac{d\delta_{aa'}\delta_{bb'} + 1}{d+1} \implies \text{SIC}. \quad (2)$$

A SIC set: regular simplex inscribed in quantum state-space!

## Qudit Arthurs-Kelly

Let  $|\cdot\rangle$  denote discrete position states and  $|\cdot\rangle_p$  denote discrete momentum states.

$$U = \left( \sum_m I \otimes X^{-m} \otimes |m\rangle_p \langle m|_p \right) \left( \sum_k X^{-k} \otimes I \otimes |k\rangle \langle k| \right) \quad (3)$$

Kraus operators: given computational basis outcomes  $(x, y)$  on the ancillas, we update the system via

$$K_{xy} = \frac{1}{\sqrt{d}} \sum_{km} \omega^{-km} \langle x+k, y+m | \gamma \rangle |m\rangle_p \langle k|. \quad (4)$$

$|\gamma\rangle$  is the initial state of the ancillas,

$$\langle k, m | \gamma \rangle = \omega^{km} \langle m | F^\dagger \Pi | k \rangle, \quad (5)$$

where  $F$  is the Fourier transform operator, and  $\Pi$  is the fiducial.

## Preparing the ancillas...

...from the fiducial:

$$\langle k, m | \gamma \rangle = \omega^{km} \langle m | F^\dagger \Pi | k \rangle \quad (6)$$

$$= \langle k, m | \left( \sum_j |j\rangle \langle j| \otimes Z^j \right) (I \otimes F^\dagger) (|\phi^*\rangle \otimes |\phi\rangle). \quad (7)$$

## $d = 4$ SIC fiducial

Exploit the “monomial representation” to write

$$|\phi\rangle = (H \otimes I) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i\pi(-1/4)} & 0 & 0 \\ 0 & 0 & e^{i\pi(1/4)} & 0 \\ 0 & 0 & 0 & e^{i\pi(1/2)} \end{pmatrix} \frac{1}{\sqrt{5 + \sqrt{5}}} \begin{pmatrix} \sqrt{2 + \sqrt{5}} \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad (8)$$

where  $H$  is the Hadamard gate.

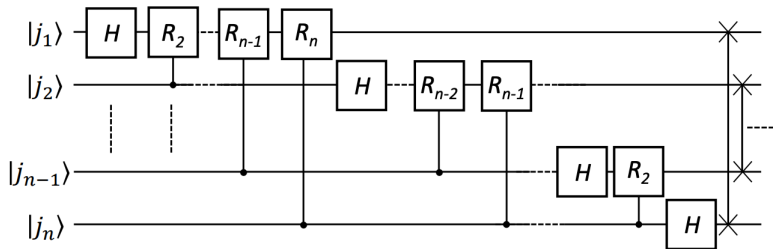
# Qubit implementation

$$\text{Let } R(k) = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{pmatrix}.$$

$$Z = \bigotimes_{j=0}^{n-1} R(j+1)$$

$$X = F^\dagger Z F. \quad (9)$$

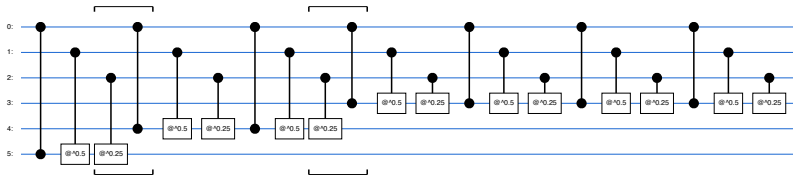
Qudit Fourier transform:





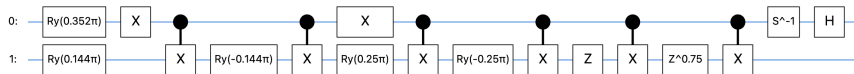
## CZ &amp; CX

CZ can be built out of a series of qudit-controlled Z's, and CX obtained by Fourier transforming. Below: first three qubits target, second three qubits control:



$$\begin{aligned} & \left( I \otimes |0\rangle\langle 0| \otimes I \otimes I + Z^4 \otimes |1\rangle\langle 1| \otimes I \otimes I \right) \\ & \left( I \otimes I \otimes |0\rangle\langle 0| \otimes I + Z^2 \otimes I \otimes |1\rangle\langle 1| \otimes I \right) \\ & \left( I \otimes I \otimes I \otimes |0\rangle\langle 0| + Z \otimes I \otimes I \otimes |1\rangle\langle 1| \right) \\ &= \sum_{m=0}^7 Z^m \otimes |m\rangle\langle m| \end{aligned}$$

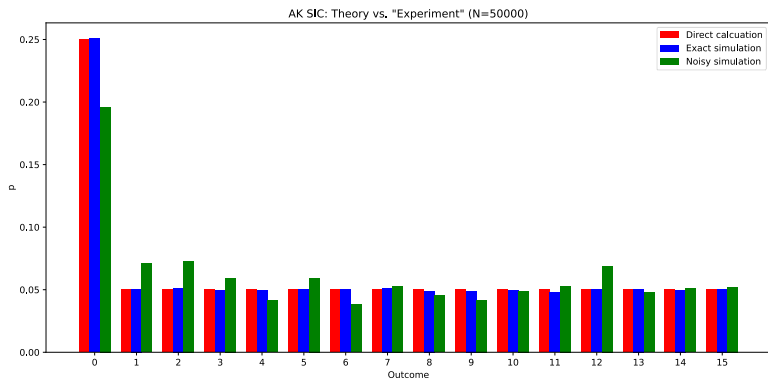
# Preparing the $d = 4$ fiducial



# Gate counts

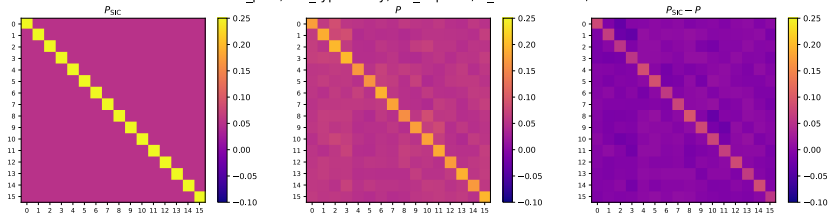
<b>Fiducial preparation</b>		<b>Ancilla preparation</b>		<b>Arthurs-Kelly unitary</b>	
Gate type	Count	Gate type	Count	Gate type	Count
Ry	5	SwapPowGate	1	HPowGate	12
PauliX	2	HPowGate	2	CZPowGate	18
CXPowGate	6	CZPowGate	7	SwapPowGate	6
ZPowGate	3				
HPowGate	1				

# On Willow: SIC measurement on SIC fiducial (AK)



Gate Type	Count
PhasedXZGate	67
CZPowGate	84
PhasedXPowGate	99
ZPowGate	10
MeasurementGate	1

willow\_pink, run\_type: noisy, wh\_impl: ak, n\_shots: 100000, d: 4



$$\|P_{\text{SIC}} - P\| \approx 0.3225.$$

## A simplification

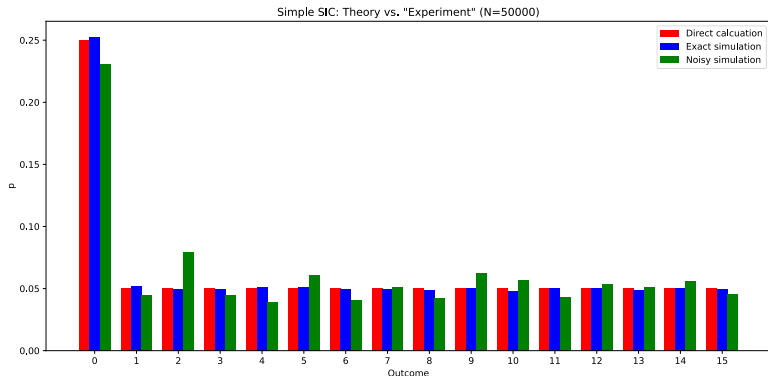
Let  $|\psi\rangle$  an arbitrary state, and  $|\phi\rangle$  be the fiducial.

$$\begin{aligned} & \left( \langle a| \otimes \langle b| \right) (I \otimes F^\dagger) \left( \sum_j X^{-j} \otimes |j\rangle \langle j| \right) (|\psi\rangle \otimes |\phi^*\rangle) \\ &= \frac{1}{\sqrt{d}} \langle \phi | D_{a,b}^\dagger | \psi \rangle = \frac{1}{\sqrt{d}} \langle \phi_{a,b} | \psi \rangle : \end{aligned} \tag{10}$$

The probabilities for a computational basis measurement are precisely the SIC probabilities. Just one ancilla!

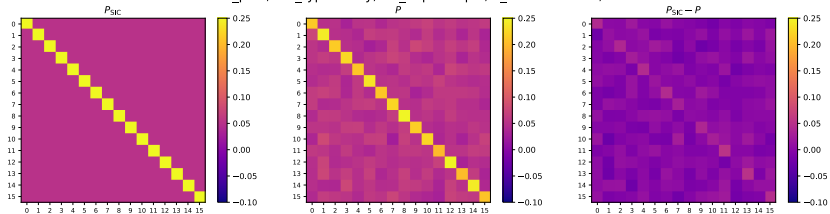
Gate Type	Count
HPowGate	6
CZPowGate	9
SwapPowGate	3

# On Willow: SIC measurement on SIC fiducial (simple)



Gate Type	Count
PhasedXPowGate	30
CZPowGate	25
PhasedXZGate	20
MeasurementGate	1

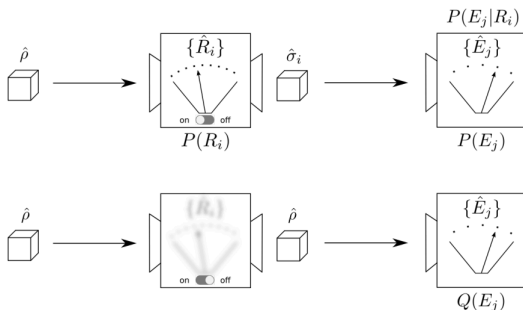
willow\_pink, run\_type: noisy, wh\_impl: simple, n\_shots: 100000, d: 4



$$\|P_{\text{SIC}} - P\| = 0.2126.$$



# The Born Rule



$$P(E_j) = \sum_i P(E_j|R_i)P(R_i) \quad (11)$$

$$Q(E_j) = \sum_{ik} P(E_j|R_i)\Phi_{ik}P(R_k) \quad (12)$$

$$= \sum_i P(E_j|R_i) \left[ (d+1)P(R_i) - \frac{1}{d} \right] \quad (13)$$

where  $\Phi = P(R|R)^{-1}$ .

# Consistency

Let us consider the following set of experiments:

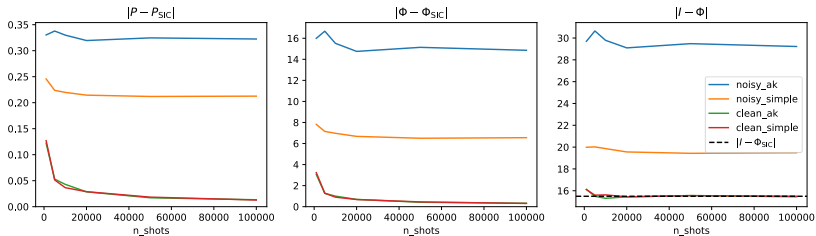
1. Preparing the SIC states in turn, and then a SIC measurement. Yields matrix of frequencies  $P_{ij} = f(R_i|R_j)$ .
2. Prepare the computational basis states  $\{\Pi_i = |i\rangle\langle i|\}$ , and then performing a SIC. Yields frequencies  $p_{ij} = f(R_j|\Pi_i)$ .
3. Prepare the SIC states, and a computational basis measurement:  $C_{ij} = f(\Pi_i|R_j)$ .
4. Prepare the computational basis states, and then a computational basis measurement:  $q_{ij} = f(\Pi_i|\Pi_j)$ .

Letting  $\Phi = P^{-1}$ , the Born rule then demands the following consistency criterion,

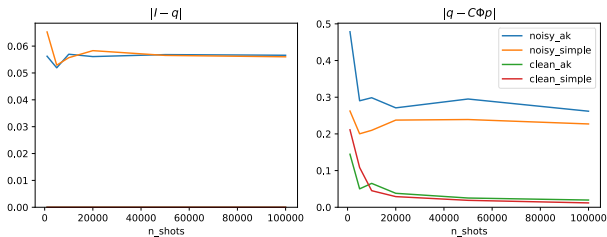
$$q = C\Phi p. \quad (14)$$

# Errors

SIC error, willow\_pink, d: 4

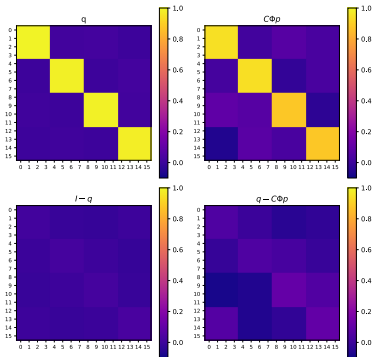


Sky/ground error, willow\_pink, d: 4

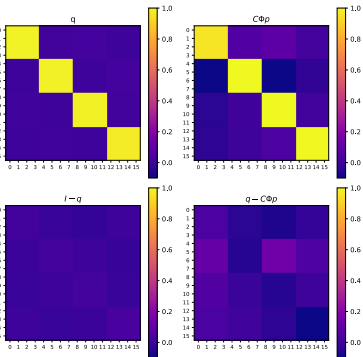


# Errors

willow\_pink, run\_type: noisy, wh\_impl: simple, n\_shots: 100000, d: 4



willow\_pink, run\_type: noisy, wh\_impl: ak, n\_shots: 100000, d: 4



$$\|I - q\| \approx 0.056$$

$$\|q - C\Phi p\|_{\text{simple}} \approx 0.2271$$

$$\|q - C\Phi p\|_{\text{ak}} \approx 0.2618$$