The Qudit Arthurs-Kelly Measurement... for SIC's!

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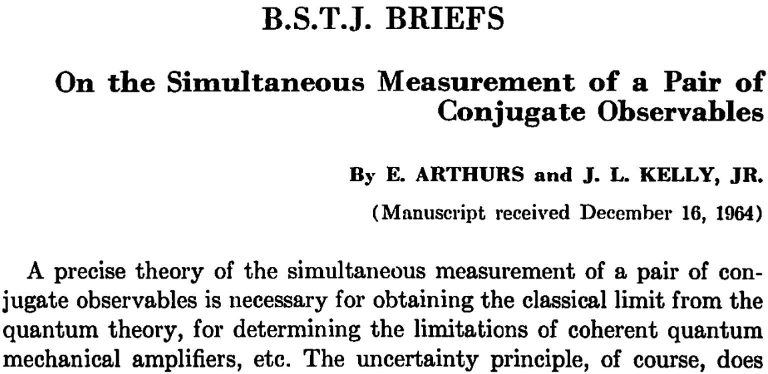
UMB: QBism Group

9/12/25

Historical flavor

From *Bell System Technical Journal*:



B.S.T.J. BRIEFS

On the Simultaneous Measurement of a Pair of Conjugate Observables

By E. ARTHURS and J. L. KELLY, JR.

(Manuscript received December 16, 1964)

A precise theory of the simultaneous measurement of a pair of conjugate observables is necessary for obtaining the classical limit from the quantum theory, for determining the limitations of coherent quantum mechanical amplifiers, etc. The uncertainty principle, of course, does

Arthurs-Kelly procedure

- Prepare two ancillas in a special starting state.
- Coherently shift ancilla 1 conditional on the position of the system.
- Coherently shift ancilla 2 conditional on the momentum of the system.
- Measure the ancillas.
  - Realizes a "Weyl-Heisenberg covariant POVM" on the system.
  - E.g., coherent state measurement, or "heterodyne" measurement in optics.
  - Can be adapted to qudits: in particular, for a SIC measurement.

Weyl-Heisenberg Group and SIC's

Let $\mathcal{H}_d$ be a finite dimensional Hilbert space, and let $\omega = e^{2\pi i/d}$:

$$Z|m\rangle = \omega^m|m\rangle \qquad X|m\rangle = |m+1\rangle \qquad D_{a,b} = X^a Z^b. \tag{1}$$

Let $\Pi_{a,b} = D^\dagger_{a,b}\Pi D_{a,b}$ for fiducial $\Pi = |\phi\rangle\langle\phi|$:

$$\mathrm{tr}(\Pi_{a,b}\Pi_{a',b'}) = \frac{d\delta_{aa'}\delta_{bb'} + 1}{d+1} \Longrightarrow \text{SIC}. \tag{2}$$

A SIC set: regular simplex inscribed in quantum state-space!

Qudit Arthurs-Kelly

Let $|\cdot\rangle$ denote discrete position states and $|\cdot\rangle_p$ denote discrete momentum states.

$$U = \left(\sum_m I \otimes X^{-m} \otimes |m\rangle_p\langle m|_p\right)\left(\sum_k X^{-k} \otimes I \otimes |k\rangle\langle k|\right) \quad (3)$$

Kraus operators: given computational basis outcomes (x, y) on the ancillas, we update the system via

$$K_{xy} = \frac{1}{\sqrt{d}} \sum_{km} \omega^{-km} \langle x+k, y+m|\gamma\rangle \; |m\rangle_p\langle k|. \quad (4)$$

$|\gamma\rangle$ is the initial state of the ancillas,

$$\langle k, m|\gamma\rangle = \omega^{km}\langle m|F^\dagger \Pi|k\rangle, \quad (5)$$

where F is the Fourier transform operator, and Π is the fiducial.

Preparing the ancillas...

...from the fiducial:

$$\langle k, m|\gamma\rangle = \omega^{km}\langle m|F^\dagger \Pi|k\rangle \tag{6}$$

$$= \langle k, m| \left(\sum_j |j\rangle\langle j| \otimes Z^j\right) \left(I \otimes F^\dagger\right)\left(|\phi^*\rangle \otimes |\phi\rangle\right). \tag{7}$$

$d = 4$ SIC fiducial

Exploit the "monomial representation" to write

$$|\phi\rangle =$$

$$(H \otimes I) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i\pi(-1/4)} & 0 & 0 \\ 0 & 0 & e^{i\pi(1/4)} & 0 \\ 0 & 0 & 0 & e^{i\pi(1/2)} \end{pmatrix} \frac{1}{\sqrt{5+\sqrt{5}}} \begin{pmatrix} \sqrt{2+\sqrt{5}} \\ 1 \\ 1 \\ 1 \end{pmatrix}, \tag{8}$$

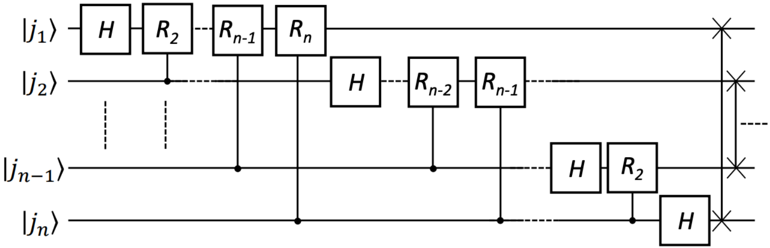
where H is the Hadamard gate.

Qubit implementation

Let $\mathtt{R}(k) = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{pmatrix}$.

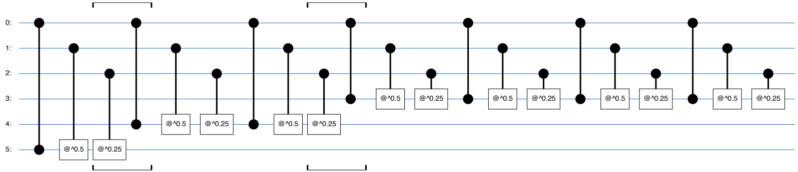
$$\mathtt{Z} = \bigotimes_{j=0}^{n-1} \mathtt{R}(j+1) \qquad\qquad \mathtt{X} = \mathtt{F}^\dagger \mathtt{Z} \mathtt{F}. \tag{9}$$

Qudit Fourier transform:



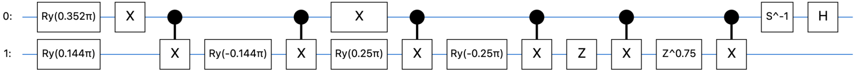
CZ & CX

CZ can be built out of a series of qudit-controlled Z's, and CX obtained by Fourier transforming. Below: first three qubits target, second three qubits control:



$$\left(I \otimes |0\rangle\langle 0| \otimes I \otimes I + Z^4 \otimes |1\rangle\langle 1| \otimes I \otimes I\right)$$
$$\left(I \otimes I \otimes |0\rangle\langle 0| \otimes I + Z^2 \otimes I \otimes |1\rangle\langle 1| \otimes I\right)$$
$$\left(I \otimes I \otimes I \otimes |0\rangle\langle 0| + Z \otimes I \otimes I \otimes |1\rangle\langle 1|\right)$$
$$= \sum_{m=0}^{7} Z^m \otimes |m\rangle\langle m|$$

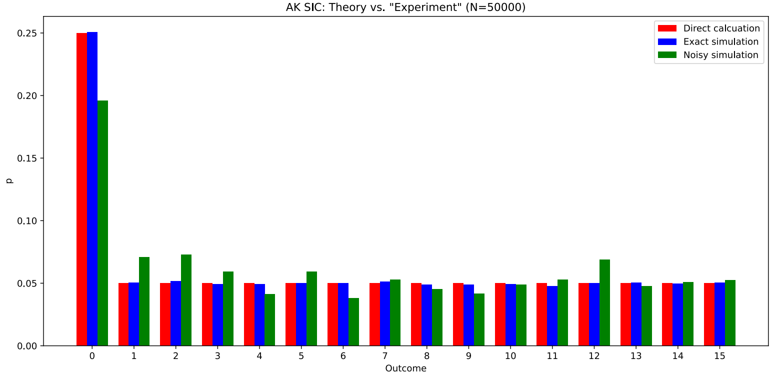
Preparing the $d = 4$ fiducial



Gate counts

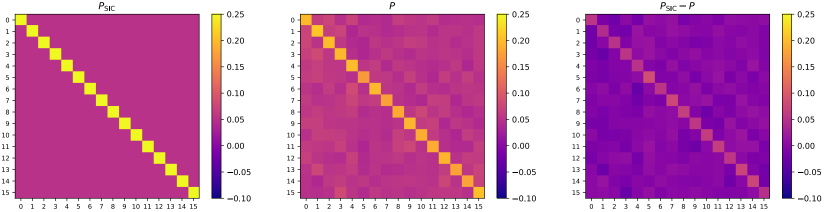
Fiducial preparation		Ancilla preparation		Arthurs-Kelly unitary	
Gate type	Count	Gate type	Count	Gate type	Count
Ry	5	SwapPowGate	1	HPowGate	12
PauliX	2	HPowGate	2	CZPowGate	18
CXPowGate	6	CZPowGate	7	SwapPowGate	6
ZPowGate	3				
HPowGate	1				

On Willow: SIC measurement on SIC fiducial (AK)



Gate Type	Count
PhasedXZGate	67
CZPowGate	84
PhasedXPowGate	99
ZPowGate	10
MeasurementGate	1

d: 4, d4_monomial, noisy, ak, n_shots: 100000



$$\|P_{\mathrm{SIC}} - P\| \approx 0.3225.$$

A simplification

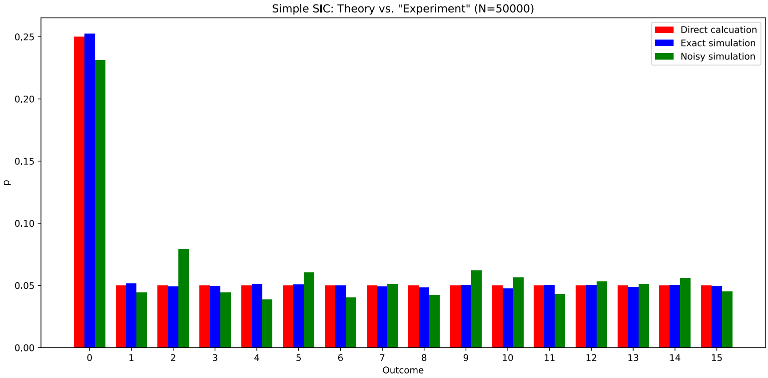
Let $|\psi\rangle$ an arbitrary state, and $|\phi\rangle$ be the fiducial.

$$\Big(\langle a| \otimes \langle b|\Big)(I \otimes F^\dagger)\left(\sum_j X^{-j} \otimes |j\rangle\langle j|\right)\Big(|\psi\rangle \otimes |\phi^*\rangle\Big)$$
$$= \frac{1}{\sqrt{d}}\langle\phi|D^\dagger_{a,b}|\psi\rangle = \frac{1}{\sqrt{d}}\langle\phi_{a,b}|\psi\rangle : \qquad (10)$$

The probabilities for a computational basis measurement are precisely the SIC probabilities. Just one ancilla!

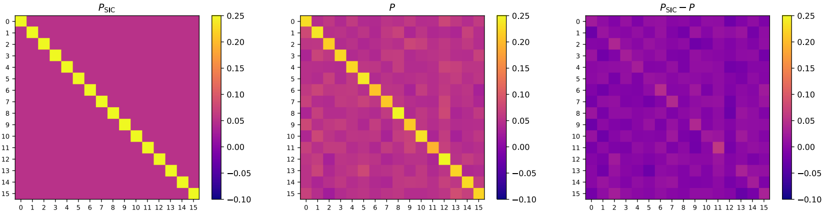
Gate Type	**Count**
HPowGate	6
CZPowGate	9
SwapPowGate	3

On Willow: SIC measurement on SIC fiducial (simple)



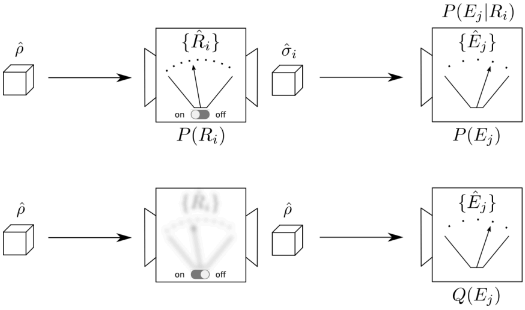
Gate Type	Count
PhasedXPowGate	30
CZPowGate	25
PhasedXZGate	20
MeasurementGate	1

d: 4, d4_monomial, noisy, simple, n_shots: 100000



$$\| P_{\mathrm{SIC}} - P \| = 0.2126.$$

The Born Rule



$$P(E_j) = \sum_i P(E_j|R_i)P(R_i) \tag{11}$$

$$Q(E_j) = \sum_{ik} P(E_j|R_i)\Phi_{ik}P(R_k) \tag{12}$$

$$= \sum_i P(E_j|R_i)\left[(d+1)P(R_i) - \frac{1}{d}\right] \tag{13}$$

where $\Phi = P(R|R)^{-1}$.

Consistency

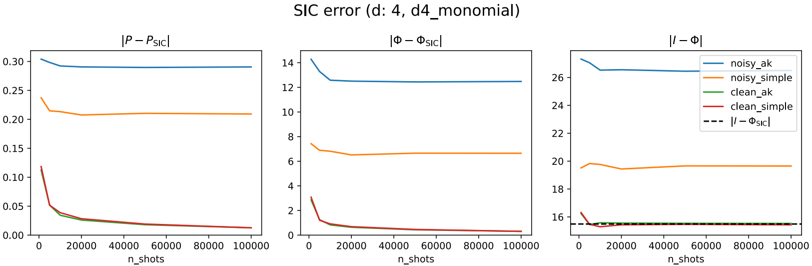
Let us consider the following set of experiments:

1. Prepare the SIC states in turn, and then a SIC measurement. Yields matrix of probabilities $P_{ij} = P(R_i|R_j)$.
2. Prepare the computational basis states $\{\Pi_i = |i\rangle\langle i|\}$, and then performing a SIC. Yields probabilities $p_{ij} = P(R_j|\Pi_i)$.
3. Prepare the SIC states, and a computational basis measurement: $C_{ij} = P(\Pi_i|R_j)$.
4. Prepare the computational basis states, and then a computational basis measurement: $q_{ij} = P(\Pi_i|\Pi_j)$.

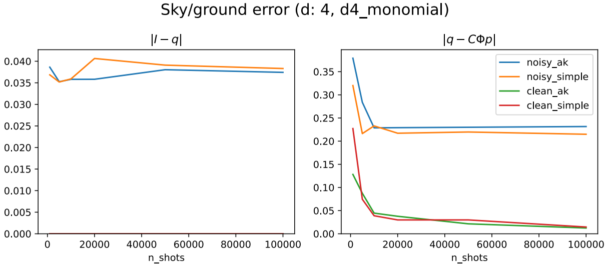
Letting $\Phi = P^{-1}$, the Born rule then demands the following consistency criterion,

$$q = C\Phi p. \tag{14}$$

Errors

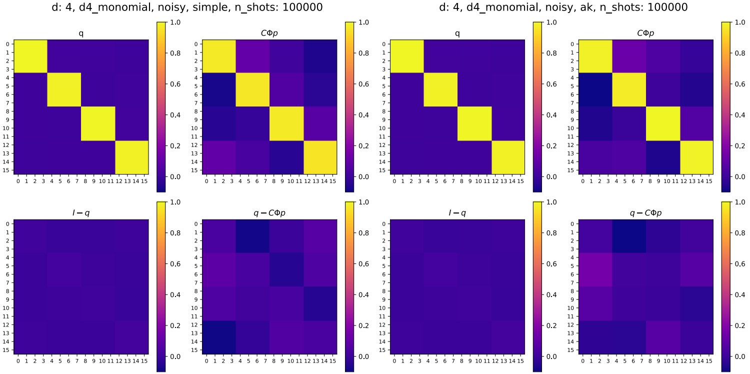


SIC error (d: 4, d4_monomial)
$|P - P_{SIC}|$
$|\Phi - \Phi_{SIC}|$
$|I - \Phi|$
noisy_ak
noisy_simple
clean_ak
clean_simple
$|I - \Phi_{SIC}|$
n_shots



Sky/ground error (d: 4, d4_monomial)
$|I - q|$
$|q - C\Phi p|$
noisy_ak
noisy_simple
clean_ak
clean_simple
n_shots

Errors

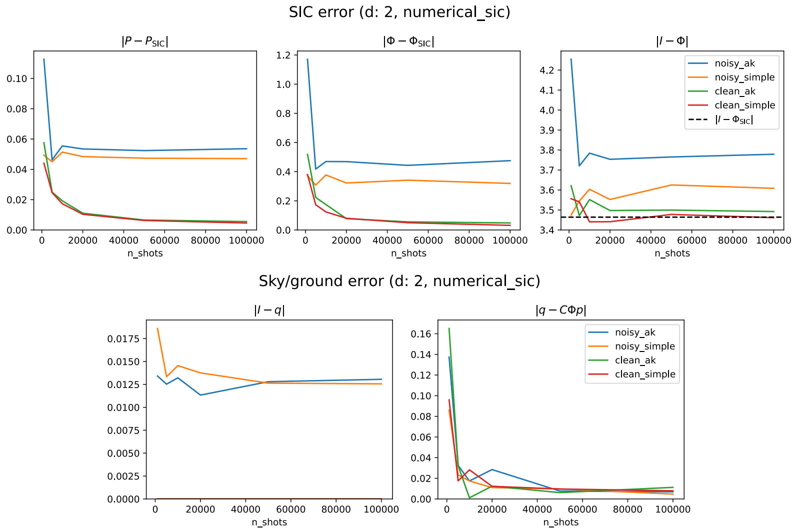


$$\|I - q\| \approx 0.056$$
$$\|q - C\Phi p\|_{\text{simple}} \approx 0.2271$$
$$\|q - C\Phi p\|_{\text{ak}} \approx 0.2618$$

Higher dimensions

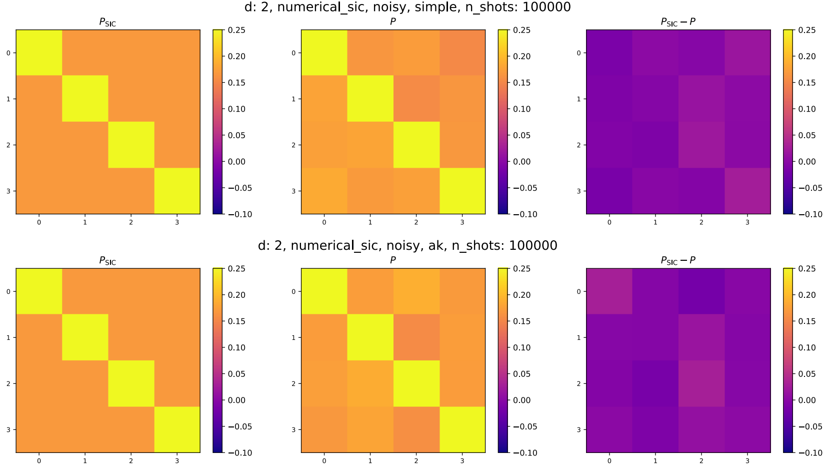
Using an arbitrary state preparation ansatz (which allows for easy complex conjugation of the state), we can extend our results to any power of 2 dimension using numerical SIC solutions.

$d = 2$



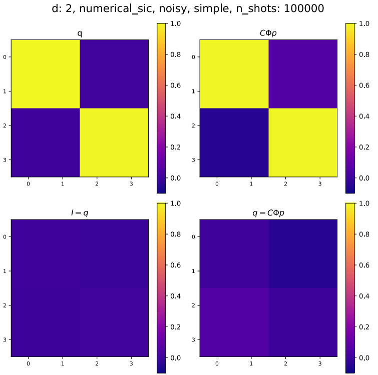
SIC error (d: 2, numerical_sic)
$|P - P_{\mathrm{SIC}}|$
$|\Phi - \Phi_{\mathrm{SIC}}|$
$|I - \Phi|$
noisy_ak
noisy_simple
clean_ak
clean_simple
$|I - \Phi_{\mathrm{SIC}}|$
n_shots
Sky/ground error (d: 2, numerical_sic)
$|I - q|$
$|q - C\Phi p|$
noisy_ak
noisy_simple
clean_ak
clean_simple
n_shots

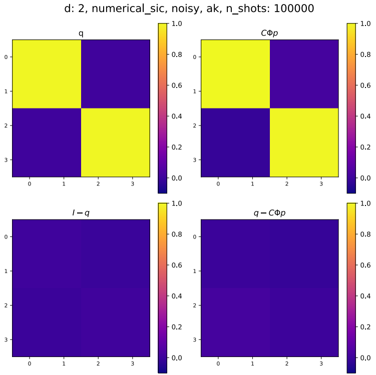
$d = 2$



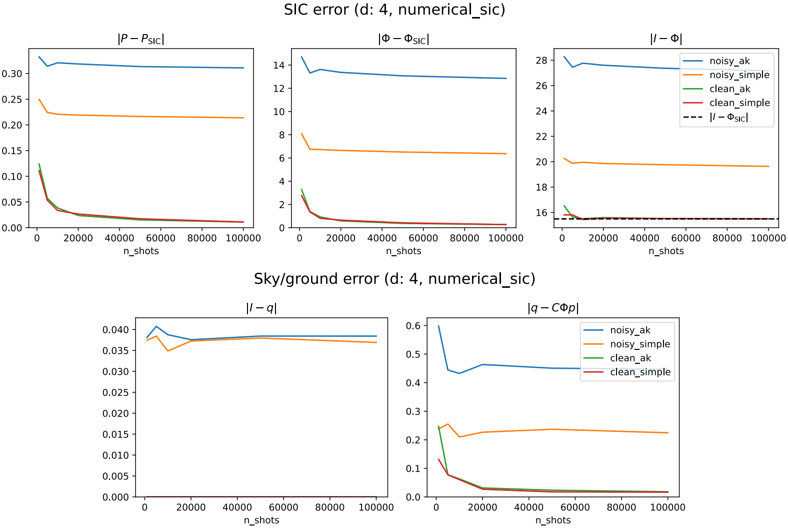
d: 2, numerical_sic, noisy, simple, n_shots: 100000
P_{SIC}
P
$P_{\mathrm{SIC}} - P$
d: 2, numerical_sic, noisy, ak, n_shots: 100000
P_{SIC}
P
$P_{\mathrm{SIC}} - P$
0.25
0.20
0.15
0.10
0.05
0.00
−0.05
−0.10

$d = 2$

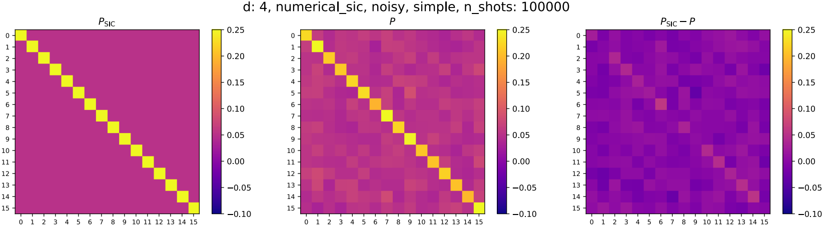

d: 2, numerical_sic, noisy, simple, n_shots: 100000
q
CΦp
I − q
q − CΦp

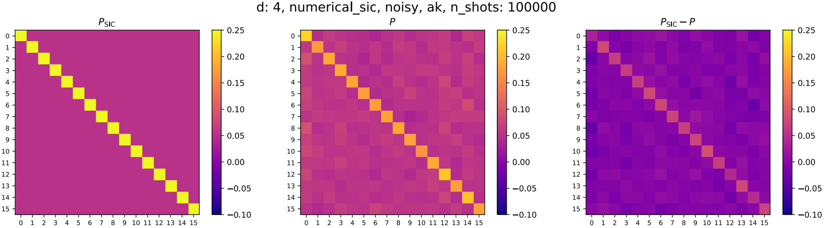

d: 2, numerical_sic, noisy, ak, n_shots: 100000
q
CΦp
I − q
q − CΦp

$d = 4$

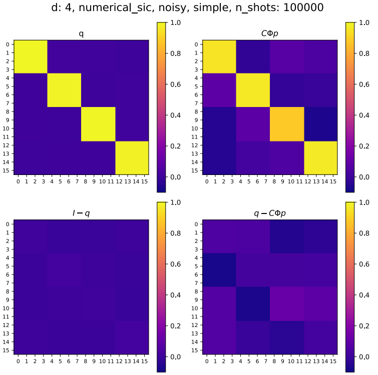

SIC error (d: 4, numerical_sic)
$|P - P_{\mathrm{SIC}}|$
$|\Phi - \Phi_{\mathrm{SIC}}|$
$|I - \Phi|$
noisy_ak
noisy_simple
clean_ak
clean_simple
$|I - \Phi_{\mathrm{SIC}}|$
n_shots
Sky/ground error (d: 4, numerical_sic)
$|I - q|$
$|q - C\Phi p|$
noisy_ak
noisy_simple
clean_ak
clean_simple
n_shots

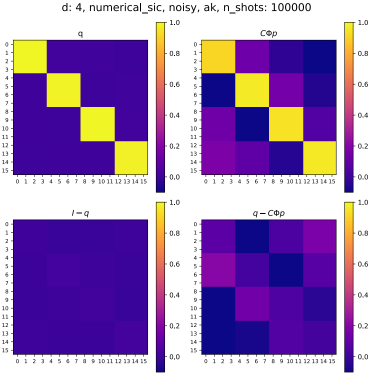
$d = 4$


d: 4, numerical_sic, noisy, simple, n_shots: 100000
P_{SIC}
P
$P_{\mathrm{SIC}} - P$
0.25
0.20
0.15
0.10
0.05
0.00
−0.05
−0.10

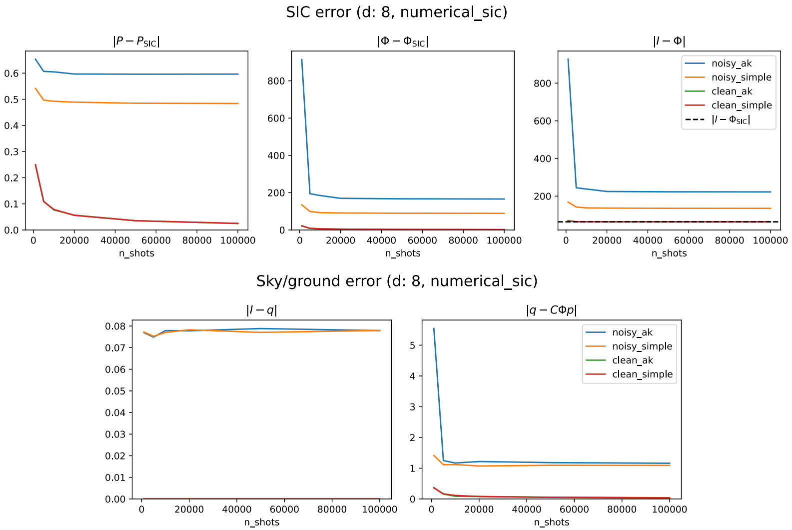

d: 4, numerical_sic, noisy, ak, n_shots: 100000
P_{SIC}
P
$P_{\mathrm{SIC}} - P$
0.25
0.20
0.15
0.10
0.05
0.00
−0.05
−0.10

$d = 4$


d: 4, numerical_sic, noisy, simple, n_shots: 100000
q
$C\Phi p$
$I - q$
$q - C\Phi p$

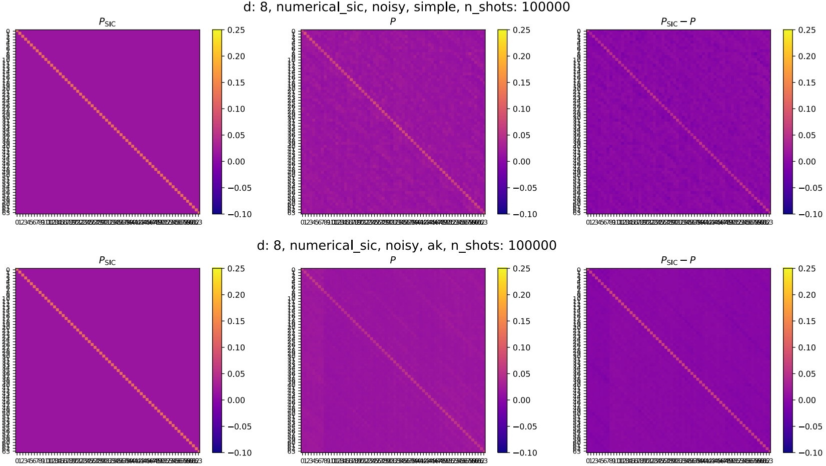

d: 4, numerical_sic, noisy, ak, n_shots: 100000
q
$C\Phi p$
$I - q$
$q - C\Phi p$

$d = 8$



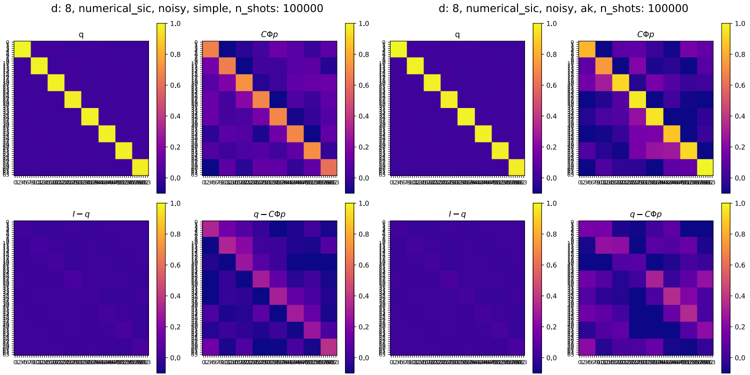
SIC error (d: 8, numerical_sic)
$|P - P_{\mathrm{SIC}}|$
$|\Phi - \Phi_{\mathrm{SIC}}|$
$|I - \Phi|$
noisy_ak
noisy_simple
clean_ak
clean_simple
$|I - \Phi_{\mathrm{SIC}}|$
n_shots
Sky/ground error (d: 8, numerical_sic)
$|I - q|$
$|q - C\Phi p|$
noisy_ak
noisy_simple
clean_ak
clean_simple
n_shots

$d = 8$



d: 8, numerical_sic, noisy, simple, n_shots: 100000
P_{SIC}
P
$P_{\mathrm{SIC}} - P$
0.25
0.20
0.15
0.10
0.05
0.00
−0.05
−0.10
d: 8, numerical_sic, noisy, ak, n_shots: 100000
P_{SIC}
P
$P_{\mathrm{SIC}} - P$

$d = 8$


d: 8, numerical_sic, noisy, simple, n_shots: 100000
d: 8, numerical_sic, noisy, ak, n_shots: 100000
q
CΦp
I − q
q − CΦp
1.0
0.8
0.6
0.4
0.2
0.0

Effect of depolarization on Φ

If one depolarizes the SIC states by a parameter α, then

$$\Phi = \left(\frac{d+1}{1-\alpha}\right) I + \left(1 - \frac{d+1}{1-\alpha}\right) \frac{1}{d^2} J, \tag{15}$$

where J is the matrix of all 1's.

- In the limit of maximum depolarization, Φ ceases to be invertible.
- As you approach this limit, the diagonal entries diverge in the positive direction, and the off-diagonal in the negative direction.

On the other hand...

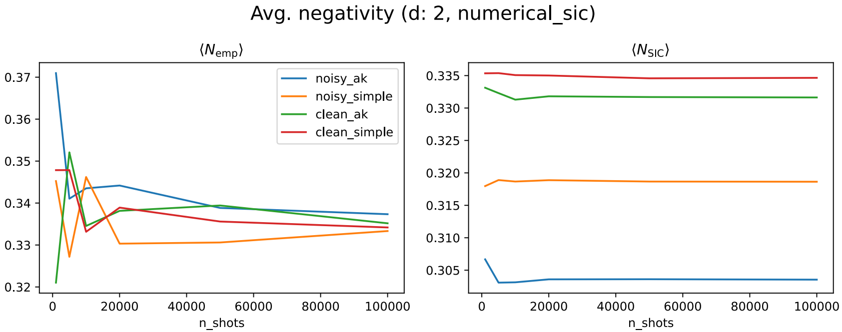
- Since $\Phi = P^{-1}$, Φ sends the probability vectors for the reference states to the vertices of the probability simplex.
- It follows that the subset of states for which the action of Φ yields valid probabilities is just the polytope formed by the reference distributions.
- Euclidean volume of the probability simplex: $\frac{d}{(d^2-1)!}$.
- The Euclidean volume of the SIC simplex: $\frac{d}{(d^2-1)!}\left(\frac{1}{d+1}\right)^{d^2-1}$.
- Volume ratio:

$$R(d) = \left(\frac{1}{d+1}\right)^{d^2-1}. \tag{16}$$

- As $d \to \infty$, $R(d) \to 0$.

Average negativity

- A non-negative quasi-probability distribution $\rightarrow$ distribution over hidden variables. So how "quantum" are we on average?
- Prepare 1000 random pure states, perform the SIC, and consider the average sum of negative entries of their quasi-probabilities.



On the left: use the empirical Φ. On the right: use Φ_{SIC}.