Quantum signal processing without angle finding

Abhijeet Alase

Centre for Engineered Quantum Systems, School of Physics,
The University of Sydney, Sydney, New South Wales 2006, Australia[*]
(Dated: January 14, 2025)

Quantum signal processing (QSP) has emerged as a unifying subroutine in quantum algorithms. In QSP, we are given a function f and a unitary black-box U, and the goal is to construct a quantum circuit for implementing $f(U)$ to a given precision. The existing approaches to performing QSP require a classical preprocessing step to compute rotation angle parameters for quantum circuits that implement f approximately. However, this classical computation often becomes a bottleneck, limiting the scalability and practicality of QSP. In this work, we propose a novel approach to QSP that bypasses the computationally intensive angle-finding step. Our method leverages a quantum circuit for implementing a diagonal operator that encodes f, which can be constructed from a classical circuit for evaluating f. This approach to QSP simplifies the circuit design significantly while enabling nearly optimal implementation of functions of block-encoded Hermitian matrices for black-box functions. Our circuit closely resembles the phase estimation-based circuit for function implementation, challenging conventional skepticism about its efficiency. By reducing classical overhead, our work significantly broadens the applicability of QSP in quantum computing.

I. INTRODUCTION

Quantum signal processing (QSP) subroutine [1–3] underpins the state-of-the-art algorithms for performing many quantum computational tasks, including Hamiltonian simulation, solving a system of linear equations, solving differential equations and optimization [4]. In the block encoding framework [3, 5], QSP leads to an efficient strategy for implementing functions of Hermitian matrices and for performing quantum singular value transformation (QSVT). QSVT powered by QSP has proved crucial in developing quantum algorithms with optimal or exponentially improved dependence on precision for the aforementioned tasks. In QSP, we are given a function f and a black-box unitary U, and the goal is to construct a quantum circuit to implement $f(U)$ up to a given precision ϵ. The quantum circuits proposed in the literature [2, 5, 6] for QSP are extremely efficient in their use of the quantum resources, namely queries to U, $U^\dagger$ and additional one- and two-qubit gates. However, in these approaches, one first needs to classically calculate certain rotation angles that parametrize the quantum circuit for implementing a Laurent polynomial approximation to f [7, 8]. This classical step has been identified as a bottleneck in the existing approaches to QSP, and may limit the utility of QSP in practice [7–11]. In this paper, we construct a circuit for QSP that bypasses the polynomial approximation and angle-finding steps, thereby significantly simplifying the design of the circuit and reducing the classical cost of QSP.

Let us begin by reviewing one of the existing approaches for QSP [6, 12]. The function $f : S^1 \to \mathbb{C}$ on the unit circle is either fixed, or it is assumed to be provided via coefficients of some series expansion, such

as Taylor series or Chebyshev series. Using these coefficients, one classically computes a Laurent polynomial $f_d \in \mathbb{C}_d[z, z^{-1}]$ of degree d approximating f on the unit circle. The degree d is chosen such that the approximation error

$$\|f(z) - f_d(z)\|_\infty := \max_{z \in S^1} |f(z) - f_d(z)| \qquad (1)$$

is within the desired precision ϵ. In the next step, an alternating sequence of the directionally controlled operation $W = |0\rangle\langle 0| \otimes U + |1\rangle\langle 1| \otimes U^\dagger$ and single-qubit rotations on the first (control) qubit is employed to implement $f_d(U)$. The angles specifying single-qubit rotations around the x- and the z-axes are the parameters of this quantum circuit. Assuming $\|f_d\|_\infty \leq 1$, this can be achieved with d uses of W and d composite single-qubit rotations.

Calculation of the single-qubit rotation angles parametrizing the QSP circuit is known to be challenging and has attracted significant attention over the last few years [5, 7–11]. While these angles can be computed classically by some deterministic algorithms, their complexity scales unfavourably as $\mathbf{O}(d^2)$ or worse [5, 9]. Apart from their high cost, these deterministic algorithms are known to be numerically unstable [9]. Alternatively, optimization algorithms can be used to compute these angles approximately [8–11], but a provable bound on the asymptotic complexity of such algorithms is not known to our knowledge.

Apart from angle-finding, the classical cost of computing a Laurent polynomial approximation f_d itself can be comparable or higher than the cost of computing the rotation angles [5, 13, 14]. For achieving the best query complexity, one aims to obtain the best Laurent polynomial approximation $f_{d,*}(x)$ to $f(x)$. While some iterative algorithms are known for computing the best Laurent polynomial approximation [15], the convergence rate of such algorithms for general f is not known, thereby obstructing a rigorous upper bound on the classical cost of

computing such an approximation. Another approach is to use truncated or interpolated Fourier series to obtain a Laurent polynomial approximation, which could result in a suboptimal dependence of the query complexity on precision in general [16].

In this paper, we use ideas from the theory of interpolation [17] to construct a new circuit for QSP. The design of this circuit is extremely simple and bypasses both the classical steps required by the existing approaches to QSP, namely finding a Laurent polynomial approximation as well as finding rotation angles to implement that polynomial. For this purpose, we assume access to a diagonal unitary encoding of f [18]. A similar encoding was used for implementing matrix inverse in the seminal HHL algorithm [19]. Such an encoding can always be constructed from a classical circuit for computing f (see Sec. III for further discussion). Several methods for constructing such an encoding are known [18]. Our circuit for QSP, which uses this encoding of f, can be constructed exponentially faster on a classical computer, and has the same query complexity as the existing approaches up to a constant prefactor. Interestingly, our circuit makes only one use of the operator encoding f. We also show the application of our circuit for QSP to implement functions of Hermitian matrices and implementing quantum singular value transformation (QSVT) [5].

There are two main implications of our work. First, our circuit for QSP, and therefore for implementing functions of Hermitian matrices, can be constructed efficiently and stably on a classical computer for any efficiently computable f. This overcomes the key limitation of the existing approaches to QSP, without compromising the scaling of the query complexity. Second, our circuit for QSP is remarkably similar to the phase estimation-based circuit, for instance the one used in HHL algorithm for matrix inversion. Prior to our work, this approach was considered expensive due to unfavourable complexity of the phase estimation subroutine. Our work, in effect, provides a different way of analyzing errors in the phase estimation-based approach for implementing functions of matrices. Thus, it is seen that the accuracy of the implemented function is often, to our surprise, significantly better than the accuracy to which the eigenphases are estimated in this approach.

The organization of the rest of the paper is as follows. In Sec. II, we provide an overview of the main results. In Sec. III, we prove the correctness and complexity of our circuit for QSP. In Sec. IV, we discuss the application of this circuit for performing functions of block-encoded Hermitian matrices and singular value transformation of arbitrary block-encoded matrices. Finally, in Sec. V, we summarize the results and discuss open problems.

II. RESULTS AND DISCUSSION

To state the main results of the paper, we need to introduce three definitions. First, let us define the diagonal encoding of a given function f.

Definition 1 (Diagonal encoding of a function). Let $f : S^1 \to \mathbb{C}$ be such that $\|f\|_\infty \leq 1$. For $d = 2^m$ with $m \in \mathbb{Z}^+$, we say that an $(m+3)$-qubit unitary operator $U_{f,4d}$ encodes f if

$$\langle j'| \langle 0| U_{f,4d} |j\rangle |0\rangle = \delta_{jj'} f\left(e^{i2\pi j/4d}\right), \quad j, j' \in [4d], \quad (2)$$

where $[4d] := \{0, 1, \ldots, 4d-1\}$.

Next, we define block encoding of a square matrix.

Definition 2 (Block encoding [5]). For a $2^n \times 2^n$ matrix A and $\alpha_A, \epsilon_A \in \mathbb{R}^+$, we say that an $(n+a_A)$-qubit unitary operator U_A is a $(\alpha_A, a_A, \epsilon_A)$-block encoding of A if

$$|A - \alpha_A \langle 0^a| U_A |0^a\rangle| \leq \epsilon_A. \quad (3)$$

Finally, we define the best Laurent polynomial approximation to a function.

Definition 3 (Best Laurent polynomial approximation [17]). For a function $f : S^1 \to \mathbb{C}$ on the unit circle, we say that $f_{d,*} \in \mathbb{C}_d[z, z^{-1}]$ is the best degree-d Laurent polynomial approximation to f if

$$\|f - f_{d,*}\|_\infty = \min_{g \in \mathbb{C}_d[z, z^{-1}]} \|f - g\|_\infty. \quad (4)$$

The best approximation error is defined to be

$$E_d(f) := \|f - f_{d,*}\|_\infty. \quad (5)$$

Note that $E_d(f)$ can also be interpreted as the error in the best trigonometric polynomial approximation to the function $\tilde{f} : \mathbb{R} \to \mathbb{C}$ defined by $\tilde{f}(\theta) = f(e^{i\theta})$. This follows from the fact that $\tilde{f}_{d,*} : \mathbb{R} \to \mathbb{C}$ defined by $\tilde{f}_{d,*}(\theta) = f_{d,*}(e^{i\theta})$ is a degree-d trigonometric polynomial in θ, and therefore it is the best trigonometric polynomial approximation to $\tilde{f}(\theta)$.

We are now ready to state the main result of this paper.

Theorem 4 (QSP). *Given an n-qubit black-box unitary U and a unitary $U_{f,4d}$ for $d = 2^m$, $m \in \mathbb{Z}^+$ encoding a function $f : S^1 \to \mathbb{C}$ with $\|f\|_\infty \leq 1$, we can construct a $(\sqrt{2}, m+3, (1+\sqrt{2})E_d(f))$-block encoding of $f(U)$ by making $4d-1$ uses of c-U and c-$U^\dagger$ each, one use of $U_{f,4d}$, and $\mathbf{O}(m^2)$ additional single- and two-qubit gates. Moreover, the circuit description for this block encoding can be computed in poly(m) time on a classical computer.*

The proof of Theorem 4 is included in Sec. III. Although d is just a parameter in Theroem 4, it is convenient to think of d as the degree of the approximating polynomial for the purpose of comparison to the conventional QSP circuit. Notice that the classical cost in Theorem 4 scales as $\mathbf{O}(m^2) \in \mathbf{O}(\log^2 d)$. If the classical cost of constructing a circuit for $U_{f,4d}$ is in polylog(d), then the overall classical cost of our algorithm is also polylog(d). In this case, our circuit provides an exponential improvement in the classical cost over the circuits proposed for

QSP in the literature, which require $\mathbf{O}(d^3)$ classical operations for angle-finding alone [11], and more for computing a degree-d polynomial approximation for f [14]. More importantly, our circuit for QSP is extremely simple and is uniquely determined by the parameter d, and therefore overcomes the issue of numerical instabilities encountered by the conventional approaches to QSP [11].

If one uses the best Laurent polynomial approximation of degree d, which is demanding to compute, then the error for conventional QSP scales as $E_d(f)$. In comparison, the error for our circuit for QSP scales as $(1+\sqrt{2})E_d(f)$, which is worse only by a constant factor. However, the conventional approach to QSP makes many queries to the (classical) function oracle for the purpose of computing the best Laurent polynomial approximation, and the queries grow with the required precision. Our circuit makes only one query to $U_{f,4d}$, which in turn can be constructed using two queries to the binary quantum oracle for f as we discuss in Sec. III. These desirable features of our new circuit for QSP come at the cost of only a constant overhead for queries to U compared to the conventional QSP circuit ($\mathbf{O}(d)$ instead of d). Therefore, we achieve a circuit for QSP that makes nearly optimal uses of U, $U_{f,4d}$ and additional two-qubit gates.

In Sec. IV, we study the application of our new circuit for QSP for implementing functions of Hermitian matrices (FHM) [5, 14, 20, 21]. Our main result is the following theorem.

Theorem 5 (FHM). *Let $g : [-1, 1] \to \mathbb{C}$ with $|g(x)| \leq 1$ and let $f : S^1 \to \mathbb{C}$ be defined by $f(e^{i\theta}) = g(\cos\theta)$ for $\theta \in \mathbb{R}$. Given an $(n + a)$-qubit block encoding U_H for an n-qubit Hamiltonian H and a unitary $U_{f,4d}$ for $d = 2^m$, $m \in \mathbb{Z}^+$, we can construct a $(\sqrt{2}, a + m + 3, (1+\sqrt{2})E_d(f))$-block encoding $U_{g(H)}$ of $g(H)$ by making $4d-1$ uses of c-U_H and c-$U_H^\dagger$ each, one use of $U_{f,4d}$, and $\mathbf{O}(da)$ additional two-qubit gates. Moreover, a circuit description of $U_{g(H)}$ can be computed using poly(m) classical operations.*

Similar to QSP, the query complexity of FHM stated in Theorem 5 is equivalent to the best results in the literature up to a constant prefactor, but the classical cost is exponentially smaller. Moreover, we discuss how Jackson's inequalities can be used to bound the error $(1 + \sqrt{2})E_d(f)$ in the resulting block encoding. As an application, we show that for a given non-integer positive number c, a block encoding of $|H|^c$ to precision ϵ can be constructed using $\mathbf{O}(1/\epsilon^{1/c})$ queries to U_H and $U_H^\dagger$. We also discuss how analytic and entire functions can be handled within this approach. Finally, we discuss how Theorem 5 can be used to perform quantum singular value transformation (QSVT), achieving equivalent query complexity as the best results in the literature but with a circuit that is easy to construct classically.

III. QUANTUM SIGNAL PROCESSING BY POLYNOMIAL INTERPOLATION

In this section, we prove Theorem 4 by constructing a block encoding of $f(U)$. Let us first address the special case when f is a degree-d Laurent polynomial. Our strategy is to make use of interpolation by Laurent polynomial. We use of the following expression for $f(z)$.

Lemma 6. *Let $f \in \mathbb{C}[z, z^{-1}]$ be a degree-d Laurent polynomial and $d_1 > 2d$ an integer. Then for $\{z_k = \exp(2\pi i k/d_1), \ k = 0, \ldots, d_1 - 1\}$,*

$$f(z) = z^{-d}\frac{1}{d_1}\sum_{j,k=0}^{d_1-1} f(z_k)z_k^d\left(\frac{z}{z_k}\right)^j, \qquad (6)$$

Proof. Being a degree-d Laurent polynomial, $f(z)$ can be expressed as

$$f(z) = \sum_{j=-d}^{d} \beta_j z^j. \qquad (7)$$

By choosing the interpolation points to be $\{z_k\}$, we can express the coefficients of f as $\beta_j = (\sum_{k=0}^{d_1-1} f(z_k)z_k^{-j})/d_1$. Substituting this expression for the coefficients in Eq. (7) yields Eq. (6). $\square$

It is instructive to consider a block encoding of $f(U)$ based on this formula.

Lemma 7. *Let $f \in \mathbb{C}[z, z^{-1}]$ be a degree-d Laurent polynomial and $d_1 > 2d$ an integer. Let U be an n-qubit unitary operator and*

$$\mathcal{W}_U = \sum_{j=0}^{d_1-1} U^j \otimes |j\rangle\langle j|, \quad |+_{d_1}\rangle = \frac{1}{\sqrt{d_1}}\sum_{j=0}^{d_1-1}|j\rangle,$$

$$V_{d_1} = \sum_{k=0}^{d_1-1} z_k |\varphi_k\rangle\langle\varphi_k|, \quad |\varphi_k\rangle = \frac{1}{\sqrt{d_1}}\sum_{j=0}^{d_1-1} z_k^j |j\rangle. \quad (8)$$

Then

$$\frac{f(U)}{\sqrt{d_1}} = (U^\dagger)^d \langle 0|[\mathbb{1}\otimes f(V_{d_1})V_{d_1}^d]\mathcal{W}_U|+_{d_1}\rangle. \qquad (9)$$

Proof. It suffices to prove that Eq. (9) holds in each eigenspace of U, i.e.

$$\frac{f(z)}{\sqrt{d_1}} = z^{-d}\langle 0|f(V_{d_1})V_{d_1}^d\mathcal{W}_z|+_{d_1}\rangle \qquad (10)$$

where $\mathcal{W}_z = \sum_{j=0}^{d_1-1} z^j |j\rangle\langle j|$ and $|z| = 1$. The statement of the lemma then follows by linearity. Eq. (10) follows
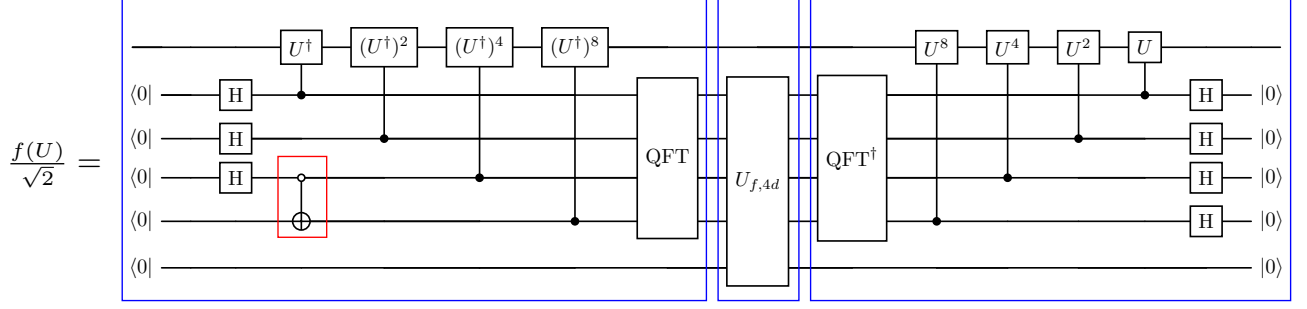
FIG. 1. Circuit diagram (in the Kitaev notation) for block encoding of $f(U)$, where f is a degree-d Laurent polynomial (here $d = 4$). The circuit acts on the signal register (the top wire) and $3 + \lceil \log_2(d) \rceil = 5$ ancilla qubits (all except the top wire). The three parts of the circuit outlined by the three blue boxes can be identified as, from right to left, the phase estimation circuit, controlled rotation specified by f, and a modified inverse of the phase estimation circuit, respectively. The modification refers to the replacement of a Hadamard gate by a zero-controlled NOT gate, outlined by the red box.

from

$$z^{-d} \langle 0|f(V_{d_1})V_{d_1}^d \mathcal{W}_z |+_{d_1}\rangle$$

$$= \frac{z^{-d}}{\sqrt{d_1}} \sum_{j=0}^{d_1-1} \langle 0|f(V_{d_1})V_{d_1}^d z^j |j\rangle$$

$$= \frac{z^{-d}}{\sqrt{d_1}} \sum_{j=0}^{d_1-1} \sum_{k=0}^{d_1-1} \langle 0| f(z_k)z_k^d |\varphi_k\rangle \langle\varphi_k| z^j |j\rangle$$

$$= \frac{z^{-d}}{d_1\sqrt{d_1}} \sum_{j=0}^{d_1-1} \sum_{k=0}^{d_1-1} f(z_k)z_k^d \left(\frac{z}{z_k}\right)^j$$

$$= \frac{f(z)}{\sqrt{d_1}}, \tag{11}$$

where in the last line we used Lemma 6. $\qquad\square$

A block encoding for $f(U)$ in terms of $\mathcal{W}_U$ can then be obtained easily. For simplicity, we set $d_1 = 4d = 2^{m+2}$. Then the block encoding of $f(V_{4d})$ can be obtained using quantum Fourier transform (QFT) [22], which follows from

$$f(V_{4d}) = \sum_{k'=0}^{4d-1} |\varphi_{k'}\rangle \langle k'| \sum_{l=0}^{4d-1} f(z_l) |l\rangle \langle l| \sum_{k=0}^{4d-1} |k\rangle \langle\varphi_k|$$
$$= \text{QFT} \left(\langle 0|U_{f,4d}|0\rangle\right) \text{QFT}^\dagger. \tag{12}$$

As $\mathcal{W}_U$ can be implemented by $4d - 1$ queries to c-U, this block encoding uses $\mathbf{O}(d)$ queries to c-U, which is encouraging.

However, the scaling factor for this block encoding of $f(U)$ is $\sqrt{4d}$, which is suboptimal. This scaling factor can be improved to a constant by using a different interpolating polynomial given in the next lemma.

Lemma 8. *Let $f \in \mathbb{C}[z, z^{-1}]$ be a degree-d Laurent polynomial and $\{z_k = \exp(2\pi ik/4d), \ k = 0, \ldots, 4d - 1\}$. Then*

$$f(z) = \frac{1}{8d^2} \sum_{j'=d}^{3d-1} \sum_{j,k=0}^{4d-1} f(z_k) \left(\frac{z}{z_k}\right)^{j-j'}. \tag{13}$$

Proof. Observe that for all $j' \in \{d, d+1, \ldots, 3d-1\}$, we have

$$f(z) = \frac{1}{4d} \sum_{j,k=0}^{4d-1} f(z_k) \left(\frac{z}{z_k}\right)^{j-j'}. \tag{14}$$

We obtain Eq. (13) by averaging Eq. (14) over $j' \in \{d, d+1, \ldots, 3d-1\}$. $\qquad\square$

As a corollary of Lemma 8, we have

$$f(U) = \frac{1}{8d^2} \sum_{j'=d}^{3d-1} \sum_{j,k=0}^{4d-1} f(z_k) \left(\frac{U}{z_k}\right)^{j-j'}, \tag{15}$$

which follows by linearity.

Based on this interpolation, we can now construct a block encoding of $f(U)$ with constant scaling factor.

Lemma 9. *Let $f \in \mathbb{C}[z, z^{-1}]$ be a degree-d Laurent polynomial and*

$$|+_{2d}\rangle = \frac{1}{\sqrt{2d}} \sum_{j=d}^{3d-1} |j\rangle, \quad |+_{4d}\rangle = \frac{1}{\sqrt{4d}} \sum_{j=0}^{4d-1} |j\rangle. \tag{16}$$

Let U, $\mathcal{W}_U$ and V_{4d} be as defined in Lemma 7. Then

$$f(U) = \sqrt{2} \langle +_{2d}|\mathcal{W}_U^\dagger [\mathbb{1} \otimes f(V_{4d})]\mathcal{W}_U|+_{4d}\rangle. \tag{17}$$

Proof. A similar calculation to the one in the proof of Lemma 7 yields

$$\sqrt{2} \langle +_{2d}|\mathcal{W}_z^\dagger f(V_{4d})\mathcal{W}_z |+_{4d}\rangle$$

$$= \frac{1}{8d^2} \sum_{j'=d}^{3d-1} \sum_{j,k=0}^{4d-1} f(z_k) \left(\frac{z}{z_k}\right)^{j-j'}$$

$$= f(z). \tag{18}$$

By linearity, we have

$$\sqrt{2}\,\langle +_{2d}|\mathcal{W}_U^\dagger f(V_{4d})\mathcal{W}_U\,|+_{4d}\rangle$$
$$= \frac{1}{8d^2}\sum_{j'=d}^{3d-1}\sum_{j,k=0}^{4d-1} f(z_k)\left(\frac{U}{z_k}\right)^{j-j'}$$
$$= f(U). \tag{19}$$

QED. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

A unitary circuit for block encoding of $f(U)$ is shown in Fig. 1. Interestingly, this circuit is remarkably similar to the circuit for function implementation based on quantum phase estimation, such as the one used in HHL algorithm for matrix inversion. The circuit constructed above can be divided into three parts, the first part being the standard quantum phase estimation (QPE), the second part being the function implementation using $U_{f,4d}$, and the third part resembling uncomputing QPE with one of the Hadamard gates replaced by a zero-controlled Not gate. This change of a single gate has a huge impact on the performance of the overall circuit! The analysis in Ref. [19] showed that in the absence of this small replacement, the error in implementation scales as $\mathrm{poly}(1/d)$. This is because the first part of the circuit, which performs phase estimation, records phases of U to an accuracy $\mathbf{O}(1/d)$. Yet, our analysis shows that as long as f is a degree-d Laurent polynomial, the circuit in Fig. 1 block encodes $f(U)$ exactly!

So far, we have constructed an exact block encoding of $f(U)$ assuming f to be a degree-d Laurent polynomial. Our strategy to address the case of general f, and thereby to prove Theorem 4, is to use the exact same circuit to block encode $f(U)$. Of course, this block encoding is not exact, and in order to prove Theorem 4, we need to bound the error.

Proof of Theorem 4. We first prove the bound on the error. The operator exactly block encoded by our circuit is

$$\sqrt{2}\,\langle +_{2d}|\mathcal{W}_U^\dagger f(V_{4d})\mathcal{W}_U\,|+_{4d}\rangle$$
$$= \frac{1}{8d^2}\sum_{j'=d}^{3d-1}\sum_{j,k=0}^{4d-1} f(z_k)\left(\frac{U}{z_k}\right)^{j-j'}$$
$$= f_d(U), \tag{20}$$

where we introduced

$$f_d(z) = \frac{1}{8d^2}\sum_{j'=d}^{3d-1}\sum_{j,k=0}^{4d-1} f(z_k)\left(\frac{z}{z_k}\right)^{j-j'}. \tag{21}$$

Note that f_d is a degree-$(3d-1)$ Laurent polynomial. Our task now is to establish an upper bound on $\|f(U) - f_d(U)\|$.

Using subadditivity of the operator norm, we have

$$\|f(U) - f_d(U)\| \le$$
$$\|f(U) - f_{d,*}(U)\| + \|f_{d,*}(U) - f_d(U)\|, \tag{22}$$

where $f_{d,*}$ is the best degree-d Laurent polynomial approximation to f as before. A bound on the first term on the right-hand side is obtained using

$$\|f(U) - f_{d,*}(U)\| \le \|f - f_{d,*}\|_\infty = E_d(f). \tag{23}$$

To bound $\|f_d(U) - f_{d,*}(U)\|$, observe that

$$f_d(z) - f_{d,*}(z) = \sqrt{2}\,\langle +_{2d}|\mathcal{W}_z^\dagger r_d(V_{4d})\mathcal{W}_z|+_{4d}\rangle, \tag{24}$$

with $r_d = f - f_{d,*}$. Since $\mathcal{W}_z$ is unitary for $|z| = 1$, we have $\|\mathcal{W}_z\| = 1$, which leads to

$$|f_d(z) - f_{d,*}(z)| \le \sqrt{2}\,|f(z) - f_{d,*}(z)|$$
$$\implies \|f(U) - f_d(U)\| \le \|f_{d,*} - f_d\|_\infty \le \sqrt{2}E_d(f). \tag{25}$$

Now by Eq. (22), we have

$$\|f(U) - f_d(U)\| \le (1 + \sqrt{2})E_d(f), \tag{26}$$

as stated in the theorem.

We now count the resources. The circuit in Fig. 1 uses $m + 3$ ancilla qubits and makes $4d - 1$ uses of c-U and c-$U^\dagger$ each and one use of $U_{f,4d}$. QFT acting on $m + 2$ qubits requires $\mathbf{O}(m^2)$ additional single- and two-qubit gates [22]. Finally, generating a description of the circuit using $\mathrm{poly}(m)$ classical operations is straightforward, which completes the proof. $\qquad\square$

It is worth remarking that some simple strategies can be employed to improve the prefactor for the uses of c-U and c-$U^\dagger$. For instance, instead of constructing a block encoding of $f(U)$ based on Eq. (20), one may use the equivalent expression

$$f_d(U) = \sqrt{2}\,\langle +_{2d}|(\mathcal{W}_U'')^\dagger f(V_{4d})\mathcal{W}_U'|+_{4d}\rangle \tag{27}$$

as the starting point, where

$$\mathcal{W}_U' = \sum_{j=-2d}^{2d-1} U^j\otimes|j\rangle\,\langle j|, \quad \mathcal{W}_U'' = \sum_{j=-d}^{d-1} U^j\otimes|j\rangle\,\langle j|. \tag{28}$$

Then $\mathcal{W}_U'$ ($\mathcal{W}_U''$) can be implemented with $2d$ (d) uses of c-U and c-$U^\dagger$ each. As a result, the block encoding of $f(U)$ makes $3d$ uses of c-U and c-$U^\dagger$ each. One can achieve further savings by using directionally-controlled operation $W = |0\rangle\,\langle 0|\otimes U + |1\rangle\,\langle 1|\otimes U^\dagger$, which roughly halves the query complexity. It is also possible to choose a smaller value for d_1 instead of $d_1 = 4d$ used above, at the cost of a larger scaling constant for the block encoding of $f(U)$.

To perform QSP according to the circuit constructed above, the unitary $U_{f,4d}$ needs to be constructed starting from a description of f. There are several ways to construct such a unitary [18]. One approach [19] starts with an efficient reversible classical circuit O_f for computing f to b bits of precision. A quantum circuit for computing f can then be constructed by replacing each

classical gate by its quantum counterpart. Now an approximation $\tilde{U}_{f,4d}$ to $U_{f,4d}$ is obtained by first computing f using $O_{f,4d}$ and recording the outcome in the b-bit register, then applying controlled-rotation by an angle written in this register and finally uncomputing the b-bit register by applying $O_f^\dagger$. In effect, $\tilde{U}_{f,4d}$ makes one use of O_f and $O_f^\dagger$ each and $\mathbf{O}(b)$ two-qubit gates. The error $\delta = \left\| \tilde{U}_{f,4d} - U_{f,4d} \right\|$ depends on the binary representation of complex numbers being used, but for commonly used representation, this error scales as $\delta \in \mathbf{O}(2^{-b})$. Finally $U_{f,4d}$ in Theorem 4 can be replaced by $\tilde{U}_{f,4d}$, leading to a total error bounded by $(1+\sqrt{2})E_d(f) + \sqrt{2}\delta$.

Finally, observe that in the process of proving Theorem 4, we have discovered an interesting Laurent polynomial approximation to a given function on the unit circle, as we state in the next theorem.

Theorem 10. *For any $f : S^1 \to \mathbb{C}$, the polynomial approximation $f_d : S^1 \to \mathbb{C}$ defined in Eq. (21) satisfies*

$$\|f - f_d\|_\infty \le (1+\sqrt{2})E_d(f). \tag{29}$$

Quite interestingly, the approximation error is bounded by a constant times the approximation error of the best Laurent polynomial approximation. However, we emphasize that we are comparing the approximation error of a higher degree polynomial with that of the best approximation error achievable by lower-degree polynomials; f_d is a degree-$(3d-1)$ Laurent polynomial and achieves an approximation error of the order of $E_d(f)$. Nevertheless, this polynomial approximation could be useful in situations where one requires error scaling to be provably equal to the error scaling of the best polynomial approximation.

This concludes our discussion of the new circuit for QSP. In the next section, we explore its applications to implementing functions of a block-encoded Hermitian matrix and singular value transformation of an arbitrary block-encoded matrix.

IV. APPLICATIONS TO FUNCTIONS OF A HERMITIAN MATRIX AND SINGULAR VALUE TRANSFORMATION

As in the case of conventional QSP, this new circuit for QSP can be used for implementing a given function $h : [-1, 1] \to \mathbb{C}$ of a Hermitian matrix H block-encoded in a unitary U_H. We first recall a known lemma and include its proof for completeness [5].

Lemma 11. *Let U_H be an $(n+a)$-qubit self-inverse block encoding of an n-qubit Hamiltonian H, and let G be the Hermitian matrix with spectrum in $[-\pi, \pi)$ satisfying*

$$e^{iG} = (2\Pi - \mathbb{1})U_H, \quad \Pi = |0^a\rangle \langle 0^a|. \tag{30}$$

Then for any function $g : [-1, 1] \to \mathbb{C}$,

$$g(H) = \langle 0^a | g(\cos G) | 0^a \rangle. \tag{31}$$

Proof. It suffices to show that for any eigenstate $|\lambda\rangle$ of H, we have $\langle 0^a | g(\cos G) | 0^a \rangle |\lambda\rangle = g(\lambda) |\lambda\rangle$. To prove this equality, let

$$|\psi\rangle = |0^a\rangle |\lambda\rangle, \quad |\psi^\perp\rangle = (\mathbb{1} - \Pi)U_H |0^a\rangle |\lambda\rangle / \sqrt{1 - \lambda^2}. \tag{32}$$

Then

$$U_H |\psi\rangle = \lambda |\psi\rangle + \sqrt{1 - \lambda^2} |\psi^\perp\rangle,$$
$$U_H |\psi^\perp\rangle = \sqrt{1 - \lambda^2} |\psi\rangle - \lambda |\psi^\perp\rangle. \tag{33}$$

Using

$$(2\Pi - \mathbb{1}) |\psi\rangle = |\psi\rangle, \quad (2\Pi - \mathbb{1}) |\psi^\perp\rangle = - |\psi^\perp\rangle, \tag{34}$$

we get

$$(2\Pi - \mathbb{1})U_H |\psi\rangle = \lambda |\psi\rangle - \sqrt{1 - \lambda^2} |\psi^\perp\rangle,$$
$$(2\Pi - \mathbb{1})U_H |\psi^\perp\rangle = \sqrt{1 - \lambda^2} |\psi\rangle + \lambda |\psi^\perp\rangle. \tag{35}$$

Therefore, the space $\mathcal{V} = \text{span}\{|\psi\rangle, |\psi^\perp\rangle\}$ is invariant under the action of $(2\Pi - \mathbb{1})U_H$, and furthermore, $(2\Pi - \mathbb{1})U_H$ acts as

$$[(2\Pi - \mathbb{1})U_H]_\mathcal{V} = \begin{bmatrix} \lambda & -\sqrt{1-\lambda^2} \\ \sqrt{1-\lambda^2} & \lambda \end{bmatrix}$$
$$= \exp(-i \arccos(\lambda)\sigma_y) \tag{36}$$

in this basis. Then G acts on $\mathcal{V}$ as

$$[G]_\mathcal{V} = -\arccos(\lambda)\sigma_y. \tag{37}$$

Consequntly,

$$g(\cos [G])_\mathcal{V} = g(\cos(\arccos(\lambda))\mathbb{1}) = g(\lambda)\mathbb{1}, \tag{38}$$

which proves $g(\cos [G]) |\psi\rangle = g(\lambda) |\psi\rangle$ as desired. $\square$

The proof of Theorem 5 is now straightforward.

Proof of Theorem 5. Notice that $(2\Pi - \mathbb{1})$ can be implemented using $\mathbf{O}(a)$ two-qubit gates. Since $f(e^{i\theta}) = g(\cos\theta)$, we have

$$f((2\Pi - \mathbb{1})U_H) = f(e^{iG}) = g(\cos G), \tag{39}$$

which is a block encoding of $g(H)$. Using Theorem 4, a $(\sqrt{2}, a + m + 3, (1 + \sqrt{2})E_d(f))$-block encoding of $f((2\Pi - \mathbb{1})U_H)$ can be constructed using $U_{f,4d}$. The number of queries to U_H and $U_H^\dagger$ as well as the number of classical operations follow from Theorem 4. $\square$

In comparison, the conventional QSP can achieve precision up to $E_d(f)$ with d queries to U, but requires $\text{poly}(d) = \exp(m)$ classical operations for finding the best polynomial approximation and rotation angles to implement the same. In this sense, our algorithm achieves nearly optimal queries with exponentially smaller classical cost.

Before moving to the applications of Theorem 5, let us look more closely at the circuit for FHM. This circuit constructs a block encoding for an approximation of $f(e^{iG})$, namely $f_d(e^{iG})$, thereby implementing the approximation $g_d(\cos G) = f_d(e^{iG})$ to $g(\cos G)$. It is insightful to obtain an explicit expression for g_d Using $f(e^{i\theta}) = g(\cos\theta)$ in Eq. (21), we have

$$f_d(z) = \frac{1}{8d^2} \sum_{j'=d}^{3d-1} \sum_{j,k=0}^{4d-1} f(z_k) \left(\frac{z}{z_k}\right)^{j-j'}$$
$$= \frac{1}{8d^2} \sum_{j'=d}^{3d-1} \sum_{j,k=0}^{4d-1} g(x_k) \left(\frac{z}{z_k}\right)^{j-j'}, \qquad (40)$$

where we defined $x_k = \cos\theta_k$ and $\theta_k = \arg z_k$. We therefore have

$$f_d(e^{i\theta}) = \frac{1}{8d^2} \sum_{j'=d}^{3d-1} \sum_{j,k=0}^{4d-1} g(x_k) \left(\frac{e^{i\theta}}{z_k}\right)^{j-j'}. \qquad (41)$$

By changing the variables according $j \mapsto 4d-1-j$, $j' \mapsto 4d-1-j'$ and $k \mapsto 4d-k \mod 4d$, we obtain

$$f_d(e^{i\theta}) = \frac{1}{8d^2} \sum_{j'=d}^{3d-1} \sum_{j,k=0}^{4d-1} g(x_k) \left(\frac{e^{-i\theta}}{z_k}\right)^{j-j'} = f_d(e^{-i\theta}). \qquad (42)$$

Since we have discovered that $f_d(e^{i\theta}) = f_d(e^{-i\theta})$, we now define a new function $g_d : [-1,1] \to \mathbb{C}$ by $g_d(\cos\theta) = f_d(e^{i\theta})$. By averaging Eqs. (41) and (42), we obtain

$$g_d(\cos\theta) = \frac{1}{8d^2} \sum_{j'=d}^{3d-1} \sum_{j,k=0}^{4d-1} g(x_k) \left(\frac{e^{i(j-j')\theta} + e^{-i(j-j')\theta}}{2z_k^{j-j'}}\right)$$
$$= \frac{1}{8d^2} \sum_{j'=d}^{3d-1} \sum_{j,k=0}^{4d-1} g(x_k) \frac{\cos((j-j')\theta)}{z_k^{j-j'}}$$
$$= \frac{1}{8d^2} \sum_{j'=d}^{3d-1} \sum_{j,k=0}^{4d-1} g(x_k) \frac{T_{|j-j'|}(\cos\theta)}{z_k^{j-j'}}, \qquad (43)$$

where T_j denotes Chebyshev polynomial of the first kind of order j. We have therefore revealed that our circuit for FHM implements the degree-$(3d-1)$ polynomial approximation to $g(x)$ given by

$$g_d(x) = \frac{1}{8d^2} \sum_{j'=d}^{3d-1} \sum_{j,k=0}^{4d-1} g(x_k) \frac{T_{|j-j'|}(x)}{z_k^{j-j'}}. \qquad (44)$$

With further algebra, $g_d(x)$ can also be expressed as

$$g_d(x) = \sum_{r=0}^{3d-1} \beta_r T_r(x), \qquad (45)$$

where

$$\beta_r = \begin{cases} \frac{1}{4d} \sum_{k=0}^{4d-1} g(x_k) & \text{if } r = 0 \\ \frac{2}{4d} \sum_{k=0}^{4d-1} g(x_k) T_r(x_k) & \text{if } 1 \leq r \leq d \\ \frac{2(3d-r)}{8d^2} \sum_{k=0}^{4d-1} g(x_k) T_r(x_k) & \text{if } d < r \leq 3d-1. \end{cases} \qquad (46)$$

Theorem 5 also bounds the approximation error by $(1 + \sqrt{2})E_d(f)$, where $f(e^{i\theta}) = g(\cos\theta)$ or equivalently $f = g \circ \cos \circ \arg$. We have therefore obtained a result on polynomial approximation of a function, which we now state in the form of a theorem.

Theorem 12. *Let $g : [-1,1] \to \mathbb{C}$ be a function and let $g_d : [-1,1] \to \mathbb{C}$ be defined by Eq. (44). Then for any $x \in [-1,1]$,*

$$|g(x) - g_d(x)| \leq (1 + \sqrt{2})E_d(g \circ \cos \circ \arg). \qquad (47)$$

Theorem 12 can also be proved directly using Theorem 10. The polynomial approximation g_d can be used for performing FHM using conventional QSP circuit based on alternating projections. Such an approach would guarantee an error scaling proportional to the best polynomial approximation. Interestingly, calculation of g_d does not require any iterative procedure like for Remez exchange algorithm. Nevertheless, the calculation of the coefficients of g_d requires $\text{poly}(d)$ classical operations, which is not desirable. A bigger disadvantage of this approach is that we need a degree-$(3d-1)$ polynomial g_d to obtain an error of the order of $E_d(f)$. In practice, a degree-$(3d-1)$ polynomial approximation obtained by truncating the Chebyshev series of g fares significantly better than g_d for most functions.

Returning to the applications of Theorem 5, let us consider the task of Hamiltonian simulation. We are given a blcok encoding U_H of a Hamiltonian H, a time $t \in \mathbb{R}^+$, a precision ϵ, and the task is to construct a block encoding of e^{iHt}. Clearly, we have $g(x) = e^{itx}$ and therefore $f(\theta) = e^{it\cos\theta}$. In Ref. [5], it was shown that there exists a degree-d trigonometric polynomial approximating $f(\cos\theta)$ such that the approximation error $\varepsilon_d(f)$ is bounded from above by

$$\varepsilon_d(f) \leq \frac{5}{4}\left(\frac{e|t|}{2d}\right)^d, \quad d \geq |t| - 1. \qquad (48)$$

Since $E_d(f)$ is the approximation error for the best trigonometric polynomial, we have $E_d(f) \leq \varepsilon_d(f)$. Consequently, our circuit generates a block encoding of e^{iHt} with error

$$\epsilon \leq \frac{(1+\sqrt{2})5}{4}\left(\frac{e|t|}{2d}\right)^d \qquad (49)$$

using $\mathbf{O}(d)$ queries to U_H and $U_H^\dagger$. It is easy to verify following the steps in Ref. [5] that this error scaling leads to an optimal algorithm for Hamiltonian simulation. Similar arguments could be extended for implementation of other functions of Hermitian matrices.

Theorem 5 provides an error bound $\epsilon = (1+\sqrt{2})E_d(f)$ for a given d. This result can be bolstered with a general, tight upper bound on $E_d(f)$ that depends on some properties of f. Fortunately, such tight bounds on $E_d(f)$ are known in the form of Jackson's inequalities [17] and their generalizations for a large class of functions. These inequalities bound $E_d(f)$ in terms of some properties of the

function $\tilde{f} : \mathbb{R} \to \mathbb{C}$ defined by $\tilde{f}(\theta) = f(e^{i\theta}) = g(\cos\theta)$. For instance, if $\tilde{f}$ is r times differentiable, then Jackson's inequalities bound $E_d(f)$ in terms of the modulus of continuity of the rth derivative of $\tilde{f}$.

Definition 13. For a function $\tilde{f} : \mathbb{R} \to \mathbb{C}$ and a $\Delta \in \mathbb{R}^+$, the modulus of continuity $\omega(\Delta, \tilde{f})$ is defined to be

$$\omega(\Delta, \tilde{f}) = \sup_{|x_1 - x_2| \leq \Delta} \left\{ \left| \tilde{f}(x_1) - \tilde{f}(x_2) \right| \right\}. \quad (50)$$

Jackson's inequality gives us the following bound on the best approximation error $E_d(f)$.

Lemma 14. Let $f : S^1 \to \mathbb{C}$ be a function such that $\tilde{f}(\theta) = f(e^{i\theta})$ is r times differentiable. Then

$$E_d(f) \leq \frac{C(r)\omega(1/d, \tilde{f}^{(r)})}{d^r}, \quad (51)$$

where

$$C(r) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k(r+1)}}{(2k+1)^{r+1}} \quad (52)$$

is a constant independent of f and d.

The constants $\{C(r)\}$ are known as Akhiezer-Krein-Favard constants. For $r = 0$ and $r = 1$, we have $C(0) = 1$ and $C(1) = \pi/2$ respectively.

As a specific application, we now derive the complexity of implementing $|H|^c$ for $c \in \mathbb{R}^+$. It suffices to derive a bound on $E_d(f)$ for the corresponding function f.

Theorem 15. Let c be a non-integer positive constant, i.e. $c \in \mathbb{R}^+$, $c \notin \mathbb{Z}^+$. Let $g(x) = |x|^c$ and $f(e^{i\theta}) = g(\cos\theta) = |\cos\theta|^c$. Then

$$E_d(f) \in \mathbf{O}\left(\frac{1}{d^c}\right). \quad (53)$$

Proof. Note that f is $\lfloor c \rfloor$ times differentiable for all $\theta \in \mathbb{R}$, whereas its $(\lfloor c \rfloor + 1)$th derivative diverges at the values of θ for which $\cos(\theta) = 0$. To apply Jackson's inequality in Lemma 14, we need to bound $\omega(1/d, f^{\lfloor c \rfloor})$. Suppose first that $\lfloor c \rfloor$ is an even integer. Then we can express $f^{\lfloor c \rfloor}(\theta)$ as

$$f^{\lfloor c \rfloor}(\theta) = \sum_{j=0}^{\lfloor c \rfloor/2} \alpha_j(c) |\cos\theta|^{c-2j} \quad (54)$$

for some constants $\alpha_j(c) \in \mathbb{R}$. Then

$$\omega(1/d, f^{\lfloor c \rfloor}) \leq \omega(1/d, \sum_{j=0}^{\lfloor c \rfloor/2-1} \alpha_j(c) |\cos\theta|^{c-2j}) +$$
$$\omega(1/d, \alpha_{\lfloor c \rfloor/2}(c) |\cos\theta|^{c-\lfloor c \rfloor}). \quad (55)$$

Out of the two terms on the right-hand-side, the first is the modulus of continuity of a function with derivative of bounded from above, say by $K_{c,1}$. Therefore

$$\omega(1/d, \sum_{j=0}^{\lfloor c \rfloor/2-1} \alpha_j(c) |\cos\theta|^{c-2j}) \leq K_c/d$$
$$\leq K_c/d^{c-\lfloor c \rfloor}. \quad (56)$$

Now let us consider the second term. The derivative of $|\cos\theta|^{c-\lfloor c \rfloor}$ diverges at the values of θ for which $\cos(\theta) = 0$. To bound the modulus of continuity, we use

$$\omega(1/d, \alpha_{\lfloor c \rfloor/2}(c) |\cos\theta|^{c-\lfloor c \rfloor}) = \alpha_{\lfloor c \rfloor/2}(c) \left| \sin\frac{1}{d} \right|^{c-\lfloor c \rfloor}$$
$$\leq \frac{\alpha_{\lfloor c \rfloor/2}(c)}{d^{c-\lfloor c \rfloor}}. \quad (57)$$

Putting together the two terms, we get

$$\omega(1/d, f^{\lfloor c \rfloor}) \leq \frac{K_c'}{d^{c-\lfloor c \rfloor}}, \quad K_c' = K_c + \alpha_{\lfloor c \rfloor/2}(c). \quad (58)$$

For the case $\lfloor c \rfloor$ odd, we can use the expansion

$$f^{\lfloor c \rfloor}(\theta) = \sum_{j=0}^{(\lfloor c \rfloor-1)/2} \alpha_j(c) |\cos\theta|^{c-2j+1} \operatorname{sign}(\cos\theta) \sin\theta, \quad (59)$$

to show that

$$\omega(1/d, f^{\lfloor c \rfloor}) \leq \frac{K_c''}{d^{c-\lfloor c \rfloor}} \quad (60)$$

for a c-dependent constant K_c''. Now using Lemma 14, we obtain

$$E_d(f) \leq \frac{\omega(1/d, f^{\lfloor c \rfloor})}{d^{\lfloor c \rfloor}} \leq \frac{K}{d^c}, \quad (61)$$

where $K = K_c'$ for $\lfloor c \rfloor$ even and $K = K_c''$ for $\lfloor c \rfloor$ odd. QED. $\qquad \square$

Observe that $E_d(f)$ for the functions considered above does not decay exponentially in d. This is not a limitation of our approach, rather an unavoidable consequence of the fact that $g(\theta) = |\cos\theta|^c$ is not analytic at $\theta = \pm\pi/2$ when c is not an integer. The query complexity for implementing $|H|^c$ using our approach is optimal up to the prefactor and scales as $\mathbf{O}(1/\epsilon^{1/c})$. As a corollary, a block encoding of $\sqrt{|H|}$ can be constructed using $\mathbf{O}(1/\epsilon^2)$ queries to U_H and $U_H^\dagger$.

Stronger bounds on the scaling of $E_d(f)$ are known for classes of functions with a higher degree of smoothness [17]. Such classes include infinitely differentiable functions, functions analytic in a neighborhood of the real line and for functions that are analytic everywhere, i.e. entire functions. These bounds are stated in terms of (ψ, β)-derivative framework [23] used for classification of smooth functions. Qualitatively, these results show

that $E_d(f)$ decays exponentially in d if $\tilde{f}(\theta) = f(e^{i\theta})$ is analytic in a neighborhood of the real line. Moreover, if $\tilde{f}(\theta)$ is entire, then $E_d(f)$ scales super-exponentially with d, i.e. for any $\gamma > 0$, there exists a $d_0 \in \mathbb{Z}^+$ such that $E_d(f) \leq \exp(-\gamma d)$ for $d > d_0$ [17, 24]. Interestingly, analyticity of g implies that of $\tilde{f} = g(\cos\theta)$. Therefore, our circuit for implementing $g(H)$ achieves super-exponential accuracy for any entire function g. Such results were previously proved in the literature for some specific functions on a case-by-case basis [5, 13, 14]. The examples include $g(x) = e^{ixt}$ appearing in Hamiltonian simulation [25, 26], $g(x) = (1 - e^{ixt})/x$ appearing in differential-equation solvers [27, 28] and $g(x) = e^{-\beta x}$ appearing in Gibbs sampling [29, 30] and SDP solvers [14].

Theorem 5 can be further used for performing quantum singular value transformation (QSVT) using known techniques. In QSVT, we are given a block encoding of an arbitrary matrix A with singular value decomposition $A = V_1^\dagger S V_2$ and a function g of definite parity, and the goal is to construct a block encoding of $g^{\text{SV}}(A)$, where $g^{\text{SV}}(A) = V_1^\dagger g(S) V_2$ if g is an odd function and $g^{\text{SV}}(A) = V_2^\dagger g(S) V_2$ if g is an even function. To perform QSVT, we first encode A in the off-digonal block of a Hermitian matrix

$$H_A = \begin{bmatrix} 0 & A^\dagger \\ A & 0 \end{bmatrix}. \tag{62}$$

A block encoding of H_A can be constructed by making one use of the block encoding of A and its inverse respectively. Then we construct a block encoding of $g(H_A)$ to desired accuracy as described in Theroem 5, We then invoke the fact that [5]

$$g^{\text{SV}}(A) = \begin{cases} \langle 0|g(H_A)|0\rangle & \text{if } p = 0 \\ \langle 1|g(H_A)|0\rangle & \text{if } p = 1 \end{cases}. \tag{63}$$

Therefore, our new circuit for QSP enables us to achieve nearly optimal dependence on precision for both FHM and QSVT for arbitrary g, without expensive or unstable classical preprocesing.

V. CONCLUSION

In conclusion, we have constructed a quantum circuit for QSP based on interpolation by Laurent polynomi-als. This circuit overcomes the bottleneck of the existing approaches, namely expensive and unstable classical pre-processing. The circuit is quite simple and universal with only one tunable parameter d, which determines the precision. Our circuit for QSP enables achieving nearly optimal scaling for implementing functions of block-encoded Hermitian matrices and for performing singular value transformation of arbitrary block-encoded matrices with circuits that are easy to construct classically. It is particularly valuable when one is interested in implementing black-box functions, as only a constant number of queries to the function oracle are required.

Interestingly, our circuit is very similar to the phase-estimation based circuit for function implementation, and therefore uncovers a new connection between quantum phase estimation and QSP. Prior to this work, phase estimation-based approaches for quantum signal processing and for implementing functions of Hermitian matrices were considered ineffective, as they were assumed to lead to polynomial precision, in contrast to exponential precision offered by alternating projection methods [2, 5]. We show that an exponential precision can be achieved by phase estimation-based approaches by replacing only one Hadamard gate in the circuit by a two-qubit gate, as explained in Sec. III.

In the future, we shall investigate whether our techniques can be applied for quantum eigenvalue transformation for non-normal matrices [31, 32]. Building on the insights from this work, it would be interesting to see whether the techniques for phase estimation for non-normal matrices [31, 33, 34] prove useful for this task.

[1] G. H. Low, T. J. Yoder, and I. L. Chuang, Methodology of resonant equiangular composite quantum gates, Phys. Rev. X **6**, 041067 (2016).

[2] G. H. Low and I. L. Chuang, Optimal hamiltonian simulation by quantum signal processing, Phys. Rev. Lett. **118**, 010501 (2017).

[3] G. H. Low and I. L. Chuang, Hamiltonian simulation by qubitization, Quantum **3**, 163 (2019).

[4] J. M. Martyn, Z. M. Rossi, A. K. Tan, and I. L. Chuang, Grand unification of quantum algorithms, PRX Quantum **2**, 040203 (2021).

[5] A. Gilyén, Y. Su, G. H. Low, and N. Wiebe, Quantum singular value transformation and beyond: exponential improvements for quantum matrix arithmetics, in *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing*, STOC 2019 (Association for

Computing Machinery, New York, NY, USA, 2019) p. 193–204.

[6] D. Motlagh and N. Wiebe, Generalized quantum signal processing, PRX Quantum **5**, 020368 (2024).

[7] J. Haah, Product Decomposition of Periodic Functions in Quantum Signal Processing, Quantum **3**, 190 (2019).

[8] R. Chao, D. Ding, A. Gilyen, C. Huang, and M. Szegedy, Finding angles for quantum signal processing with machine precision, arXiv preprint arXiv:2003.02831 (2020).

[9] Y. Dong, X. Meng, K. B. Whaley, and L. Lin, Efficient phase-factor evaluation in quantum signal processing, Phys. Rev. A **103**, 042419 (2021).

[10] S. Yamamoto and N. Yoshioka, Robust angle finding for generalized quantum signal processing, arXiv preprint arXiv:2402.03016 (2024).

[11] Y. Dong, L. Lin, H. Ni, and J. Wang, Infinite quantum signal processing, Quantum **8**, 1558 (2024).

[12] D. W. Berry, D. Motlagh, G. Pantaleoni, and N. Wiebe, Doubling the efficiency of hamiltonian simulation via generalized quantum signal processing, Phys. Rev. A **110**, 012612 (2024).

[13] G. H. Low and I. L. Chuang, Hamiltonian simulation by uniform spectral amplification, arXiv preprint arXiv:1707.05391 (2017).

[14] J. van Apeldoorn, A. Gilyén, S. Gribling, and R. de Wolf, Quantum SDP-Solvers: Better upper and lower bounds, Quantum **4**, 230 (2020).

[15] W. Fraser, A survey of methods of computing minimax and near-minimax polynomial approximations for functions of a single independent variable, J. ACM **12**, 295–314 (1965).

[16] H. Wang, How much faster does the best polynomial approximation converge than legendre projection?, Numer. Math. **147**, 481–503 (2021).

[17] A. I. Stepanets, *Methods of Approximation Theory* (De Gruyter, Berlin, Boston, 2005).

[18] J. Zylberman, U. Nzongani, A. Simonetto, and F. Debbasch, Efficient quantum circuits for non-unitary and unitary diagonal operators with space-time-accuracy tradeoffs, arXiv preprint arXiv:2404.02819 (2024).

[19] A. W. Harrow, A. Hassidim, and S. Lloyd, Quantum algorithm for linear systems of equations, Phys. Rev. Lett. **103**, 150502 (2009).

[20] S. Chakraborty, A. Gilyén, and S. Jeffery, The Power of Block-Encoded Matrix Powers: Improved Regression Techniques via Faster Hamiltonian Simulation, in *46th International Colloquium on Automata, Languages, and Programming (ICALP 2019)*, Leibniz International Proceedings in Informatics (LIPIcs), Vol. 132, edited

by C. Baier, I. Chatzigiannakis, P. Flocchini, and S. Leonardi (Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl, Germany, 2019) pp. 33:1–33:14.

[21] S. Subramanian, S. Brierley, and R. Jozsa, Implementing smooth functions of a hermitian matrix on a quantum computer, Journal of Physics Communications **3**, 065002 (2019).

[22] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition* (Cambridge University Press, Cambridge, 2010).

[23] A. I. Stepanets, Classification of periodic functions and the rate of convergence of their fourier series, Math. USSR-Izv. **28**, 99 (1987).

[24] A. I. Stepanets, A. S. Serdyuk, and A. L. Shidlich, Classification of infinitely differentiable periodic functions, Ukrainian Mathematical Journal **60**, 1982 (2008).

[25] D. Aharonov and A. Ta-Shma, Adiabatic quantum state generation and statistical zero knowledge, in *Proceedings of the Thirty-Fifth Annual ACM Symposium on Theory of Computing*, STOC '03 (Association for Computing Machinery, New York, NY, USA, 2003) p. 20–29.

[26] D. W. Berry, A. M. Childs, and R. Kothari, Hamiltonian simulation with nearly optimal dependence on all parameters, in *2015 IEEE 56th Annual Symposium on Foundations of Computer Science* (2015) pp. 792–809.

[27] D. W. Berry, High-order quantum algorithm for solving linear differential equations, J. Phys. A Math. Theor. **47**, 105301 (2014).

[28] H. Krovi, Improved quantum algorithms for linear and nonlinear differential equations, Quantum **7**, 913 (2023).

[29] D. Poulin and P. Wocjan, Sampling from the thermal quantum gibbs state and evaluating partition functions with a quantum computer, Phys. Rev. Lett. **103**, 220502 (2009).

[30] Z. Holmes, G. Muraleedharan, R. D. Somma, Y. Subasi, and B. Şahinoğlu, Quantum algorithms from fluctuation theorems: Thermal-state preparation, Quantum **6**, 825 (2022).

[31] G. H. Low and Y. Su, Quantum eigenvalue processing, arXiv preprint arXiv:2401.06240 (2024).

[32] D. An, A. M. Childs, L. Lin, and L. Ying, Laplace transform based quantum eigenvalue transformation via linear combination of hamiltonian simulation, arXiv preprint arXiv:2411.04010 (2024).

[33] C. Shao, Computing eigenvalues of diagonalizable matrices on a quantum computer, ACM Transactions on Quantum Computing **3** (2022).

[34] A. Alase and S. Karuvade, Resolvent-based quantum phase estimation: Towards estimation of parametrized eigenvalues, arXiv preprint arXiv:2410.04837 (2024).