Representation of Quantum Circuits with Clifford and $\pi/8$ Gates

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Abstract. In this paper, we introduce the notion of a normal form of one qubit quantum circuits over the basis $\{H, P, T\}$, where H, P and T denote the Hadamard, Phase and $\pi/8$ gates, respectively. This basis is known as the *standard set* and its universality has been shown by Boykin et al. [FOCS '99]. Our normal form has several nice properties: (i) Every circuit over this basis can easily be transformed into a normal form, and (ii) Every two normal form circuits compute same unitary matrix if and only if both circuits are identical. We also show that the number of unitary operations that can be represented by a circuit over this basis that contains at most n T-gates is exactly $192 \cdot (3 \cdot 2^n - 2)$.

Keywords: clifford group, representation of universal set, normal form

1 Introduction and results

Quantum computing is a very active area of research because of its ability to efficiently solve problems for which no efficient classical algorithms are known. For example, it is possible for a quantum computer to solve integer factorization in polynomial time with Shor's algorithms [7]. However, it is not yet known whether quantum computers are strictly more powerful than classical computers.

Quantum algorithms are realized by a quantum circuit consisting of basic gates corresponding to unitary matrices. In other words, the design of quantum algorithms can be seen as a decomposition of a unitary matrix into a product of matrices chosen from a basic set. A discrete set of quantum gates is called *universal* if any unitary transformation can be approximated with an arbitrary precision by a circuit involving those gates only. For example, Boykin et al. [2] proved that the basis $\{H, T, CNOT\}$ is universal, where HCT, and $CNOT$ are called the Hadamard gate, the $\pi/8$ gate, and the controlled-NOT gate, respectively, and given by

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \ , \quad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \ , \quad CNOT = \begin{pmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,0\,1 \\ 0\,0\,1\,0 \end{pmatrix}.$$

The basis $\{H, T, CNOT\}$ is called the *standard set* [6, pp. 195] and plays a fundamental role in the theory of quantum computing as the classical universal set $\{AND, NOT\}$ plays in the theory of classical computing.

The Solovey-Kitaev theorem (see [4] or [6, Appendix 3]) says that polynomial size quantum circuits over this standard set can solve all the problems in **BQP**, where **BQP** is the class of problems that can be solved efficiently by quantum computers.

The situation is dramatically changed if we replace the T-gate by the T^2-gate in this basis. The gate that performs the unitary operation $P = T^2$ is known as the Phase gate.

Quantum circuits over the basis $\{H, P, CNOT\}$ is usually called *stabilizer circuits* or *clifford circuits*. The Gottesman-Knill theorem says that circuits over this basis $\{H, P, CNOT\}$ are not more powerful than classical computers (see e.g., [6, Chap. 10.5.4]). A stronger limitation of clifford circuits has also been derived [1, 3]. Recently, Buhrman et al. [3] showed that every Boolean function that can be represented by a clifford circuit is written as the parity of a subset of input variables or its negation.

These give an insight that the T-gate is the root of the power of quantum computing. It may be natural to expect that the research on the effect of the T-gate may lead to better understanding of why a quantum computer can efficiently compute some hard problems.

In this paper, we concentrate on *one qubit* circuits over the standard set, i.e., $\{H, T\}$ and analyze the properties of them. It seems difficult to give an efficient representation for a given unitary matrix with elements of such a discrete universal set, because a relation between a quantum circuit and the corresponding unitary matrix is not clear. However, if a good representation is found, it will be useful for designing an efficient quantum circuit.

The main contribution of this paper is as follows: We introduce a representation named *normal form* for one qubit circuits over the universal basis $\{H, T\}$. Let $\mathcal{C}_1$ be the set of 2×2 unitary matrices that can be represented by a circuit over the clifford basis $\{H, P\}$. The set $\mathcal{C}_1$ forms a group known as *Clifford group* and has order 192. Our normal form is defined recursively as follows.

(a) For each $D \in \mathcal{C}_1$, a shortest circuit over $\{H, P\}$ that represents D is a normal form (we break ties arbitrarily).
(b) If C is a normal form whose leftmost (closest to the output) gate is not T, then each of TC, HTC, and $PHTC$ is a normal form.

Equivalently, a normal form circuit is of the form $W_n T W_{n-1} T \cdots T W_1 T W_0$ for some $n \geq 0$ where $W_n \in \{I, H, PH\}$, $W_i \in \{H, PH\}$ for $i = 1, \ldots, n-1$ and $W_0 \in \mathcal{C}_1$.

Our normal form has several good properties :

(1) a normal form circuit has high regularity,
(2) every one qubit circuit over $\{H, T\}$ (or $\{H, P, T\}$) can easily be transformed into an equivalent normal form circuit, and
(3) two normal form circuits perform same computation if and only if both circuits are identical.

(3) is a surprising property. This enables us to decide whether two normal form circuits perform same computation without calculating the matrix product. This can also be used to estimate the number of 2×2 unitary matrices represented by a circuit over $\{H, P, T\}$ with at most n T-gates. The number is exactly $192 \cdot (3 \cdot 2^n - 2)$.

This paper is organized as follows. In Section 2, we introduce our definitions and notations. In Section 3, we define the normal form and discuss its properties. Section 4 is devoted to the proof of our main result. In Section 5, we discuss the number of matrices that can be represented by a circuit over $\{H, P, T\}$ with a limited number of T-gates.

2 Preliminaries

In this section, we introduce the definitions and notations needed to understand the normal form of circuits.

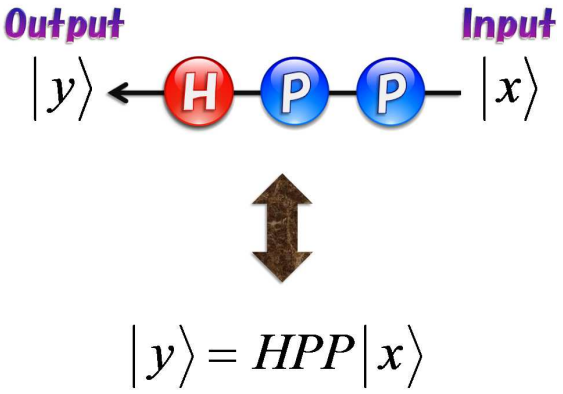

Fig. 1. A circuit and the corresponding computation.

Throughout the paper, we concentrate on *one qubit* quantum circuits. A one qubit quantum circuit can be represented by a string consisting of symbols each of which represents a gate. An operation performed by a gate is represented by a unitary matrix of degree two. For example, $HPP = HP^2$ expresses a circuit which performs operations P, P and H in this order from the input side. By convention, when we draw a circuit, the input is on the right side and the computation proceeds from right to left (see Figure 1). We usually distinguish a circuit from the matrix computed by the circuit, because different circuits may yield same computation. For example, two circuits HHP and PHH perform the same computation since $H \cdot H \cdot P = P \cdot H \cdot H = P$. We define the quantum circuit family as follows.

Definition 1. *Let* $\mathbf{X} = \{G_1, G_2, \cdots, G_m\}$ *be the set of symbols, or "gates". The set of all strings on* $\mathbf{X}$ *is denoted by* $F(\mathbf{X})$*. A map* f_u *from* $F(\mathbf{X})$ *to a set of unitary matrices of degree two is defined as follows : For a gate* $G \in \mathbf{X}$*, let* $f_u(G)$ *be a unitary matrix representing an operation performed by* G*. For* $A_1, \ldots, A_n \in \mathbf{X}$*,*

$$f_u(A_n A_{n-1} \cdots A_1) \;=\; f_u(A_n) \cdot f_u(A_{n-1}) \cdot \cdots \cdot f_u(A_1).$$

In what follows, we say that a circuit C *computes the matrix* $f_u(C)$*. A set of all unitary matrices computed by a circuit over* $\mathbf{X}$ *is given by*

$$f_u(F(\mathbf{X})) \;:=\; \{f_u(x) \mid x \in F(\mathbf{X})\}.$$

The equivalence relation on $F(\mathbf{X})$ *is defined to be*

$$a \sim b \overset{\text{def}}{\Longleftrightarrow} f_u(a) = f_u(b) \quad , \quad a, b \in F(\mathbf{X}).$$

The quantum circuits family on $\mathbf{X}$*, denoted by* $C(\mathbf{X})$*, is defined as*

$$C(\mathbf{X}) \;:=\; F(\mathbf{X}) / \sim \;=\; \left\{ [a] \;\middle|\; a \in F(\mathbf{X}) \right\},$$

where

$$[a] \;:=\; \left\{ b \in F(\mathbf{X}) \middle| \, a \sim b \right\}.$$

Here $[a]$ is the equivalence class consisting of all circuits that computes the same matrix as a. Therefore $C(\mathbf{X})$ is a family of equivalence classes of strings, or circuits.

Throughout the paper, we usually distinguish a circuit from the corresponding matrix. However, when there is no danger of confusion, we will simply denote A instead of $f_u(A)$.

In this paper, we mainly consider quantum circuits over two sets of basis $\{H, P\}$ and $\{H, P, T\}$, where H, P and T are the Hadamard, phase and $\pi/8$ gates, respectively.

Definition 2. *Let $\mathbf{X_c} := \{H, P\}$ and we call $\mathbf{X_c}$ the clifford basis. The clifford circuit family is defined as $C(\mathbf{X_c})$, where*

$$f_u(H) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad , \quad f_u(P) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

Definition 3. *Let $\mathbf{X_s} := \{H, P, T\}$, and we call $\mathbf{X_s}$ the standard basis. The standard circuit family is defined as $C(\mathbf{X_s})$, where*

$$f_u(T) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}.$$

Readers may wonder why the symbol P appears in our standard basis $\mathbf{X_s}$, because

$$f_u(TT) = f_u(P).$$

We include it in $\mathbf{X_s}$ in order to make the standard circuit family be strictly more powerful than the clifford circuit family. The complexity of a given unitary matrix is usually defined as the minimum number of basic gates needed to compute it. If we don't include P in $\mathbf{X_s}$, then we need two gates to compute P on the standard basis whereas it can be computed by a single gate on the clifford basis.

Unitary groups corresponding to the clifford and standard circuit families play a fundamental role in the theory of quantum computing.

Definition 4. *The clifford group $\mathcal{C}_1$ on one qubit is defined as*

$$\langle H, P \rangle := f_u\Big(F(\mathbf{X_c})\Big).$$

In other words, $\mathcal{C}_1 = \langle H, P \rangle$ is the set of all 2×2 unitary matrices that can be computed by a circuit over $\{H, P\}$. The order of $\mathcal{C}_1$ is known to be 192. Note also that there is a trivial bijection from the clifford circuit family to $\mathcal{C}_1$.

Definition 5. *The standard group on one qubit is defined as*

$$\langle H, P, T \rangle := f_u\Big(F(\mathbf{X_s})\Big).$$

In other words, $\langle H, P, T \rangle$ is the set of all 2×2 unitary matrices that can be computed by a circuit over $\{H, P, T\}$. It is known that $\langle H, P, T \rangle$ is infinite group and is universal [2] in a sense that any 2×2 unitary matrix can be approximated by a matrix in this group with an arbitrary precision.

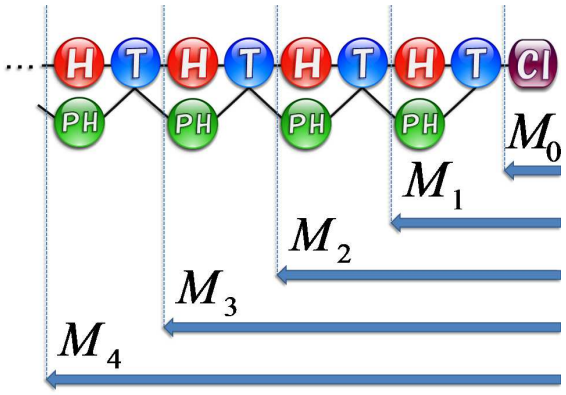

Fig. 2. The normal form

3 Representative of the standard circuit family

In this section, we introduce the notion of a normal form circuit, which can be used as a representative of classes of the standard circuit family.

We first define the representative of the clifford circuit family as follows.

Definition 6. *Let* $[a] \in C(\mathbf{X_c})$. *A representative of* $[a]$ *is defined to be a shortest string in* $[a]$ *(we break ties arbitrarily). A representative of the clifford circuit family is called a clifford circuit.*

This seems to be a natural definition, and in fact, the way of selection will not largely affect the analysis of $\mathcal{C}_1$, because the order of $\mathcal{C}_1$ is relatively small, say 192.

It seems to be difficult to find representatives of the standard circuit family because it is an infinite group. Hence we first generate all the circuits consisting of relatively small number of gates by using a computer, and then analyze them in order to find a "pattern". This leads to the following definition of our "normal form".

Definition 7. *A normal form circuit is a circuit over* $\{H, P, T\}$ *and is defined recursively as follows :*

(a) Every clifford circuit is a normal form.
(b) If C *is a normal form whose leftmost gate (closest to the output) is not* T*, then each of* TC*,* HTC*, and* $PHTC$ *is a normal form.*

For example, if D is a clifford circuit, then TD and $PHTHTD$ are normal form whereas $THPHD$ and TTD are not. Equivalently, a normal form circuit is of the form $W_nTW_{n-1}T\cdots TW_1TW_0$ for some $n \geq 0$ where $W_n \in \{I, H, PH\}$, $W_i \in \{H, PH\}$ for $i = 1, \ldots, n-1$, W_0 is a clifford circuit, and I is the 2×2 identity matrix (see Figures 2 and 3). The set $\mathbf{M}_n$ in Figure 2 is defined as the set of all matrices that can be computed by a circuit over $\{H, P, T\}$ that contains at most n T-gates. Note that $\mathbf{M}_0 = \mathcal{C}_1$.

Our normal form representation is very powerful and appealing, because it has nice properties as follows:

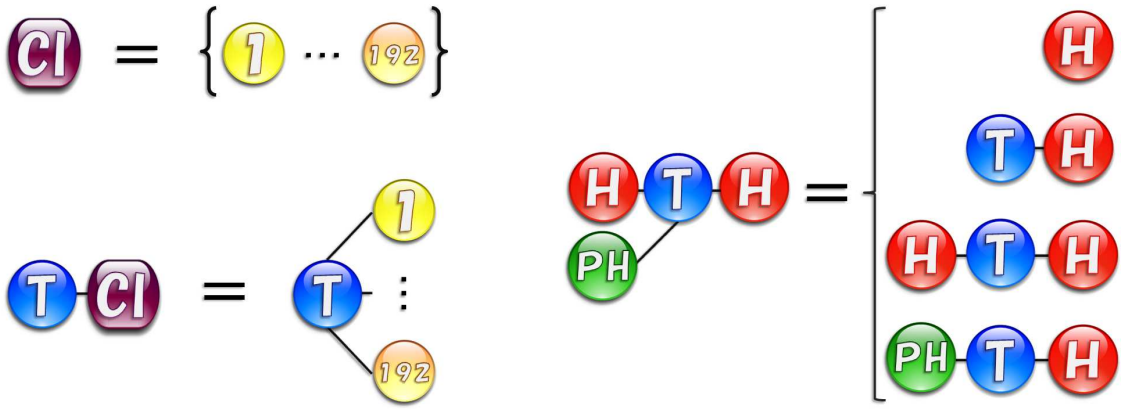

Fig. 3. (Left) Cl in Figure 2 denotes the set of shortest circuits over $\{H, P\}$ for each matrix in $\mathcal{C}_1$, these are denoted by $1 \sim 192$ $(= |\mathcal{C}_1|)$. (Right) A normal form circuit is corresponding to a path from an arbitrary chosen gate to one of the rightmost gates in Figure 2.

(1) a normal form circuit has high regularity,
(2) every one qubit circuit over $\{H, P, T\}$ can easily be transformed into an equivalent normal form circuit,
(3) two normal form circuits compute same matrix if and only if both circuits are identical (comparison can be made as a string).

Remark 1. In this paper, we concentrate on circuits over the basis $\{H, P, T\}$. However, we can also define a normal form for circuits over other bases. For example, for circuits over the basis $\{R, P, T\}$, where

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

we can show that if we replace H with R in the definition of our normal form, then the modified normal form satisfies all the above properties.

We now state our main theorem.

Theorem 1. *The number of normal forms in each equivalence class of* $C(\mathbf{X_s})$ *is exactly one.* □

We prove this theorem in the next section. Theorem 1 can also be used to derive the order of $\mathbf{M_n}$, which will be described in Section 5.

4 The proof of Theorem 1

We divide the proof of Theorem 1 into two parts :

(I) Every quantum circuit over $\{H, P, T\}$ can be transformed into an equivalent normal form circuit (we call this operation "normalization").
(II) For every two distinct normal form circuits C_1 and C_2, $f_u(C_1) \neq f_u(C_2)$ holds.

The statement (I) guarantees that each equivalence class contains at least one normal form circuit, and the statement (II) asserts the uniqueness.

4.1 The proof of (I)

In order to show Statement (I), we describe the normalization procedure. We first give a useful property of the clifford group $\mathcal{C}_1$. Let $C_T(\mathcal{C}_1)$ be a subgroup of $\mathcal{C}_1$ defined as

$$C_T(\mathcal{C}_1) := \left\{ TgT^{-1} \middle| \, g, TgT^{-1} \in \mathcal{C}_1 \right\}.$$

Note that $g \in C_T(\mathcal{C}_1)$ if and only if $TgT^{-1} \in C_T(\mathcal{C}_1)$, and that a generating set of $C_T(\mathcal{C}_1)$ is $\left\{ P, HP^2H, (HP)^3 \right\}$, namely

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, e^{\frac{\pi}{8}i} \cdot I \right\}.$$

Note also that there exists an isomorphism mapping $C_T(\mathcal{C}_1)/\mathcal{K}$ into D_4, namely

$$C_T(\mathcal{C}_1)/\mathcal{K} \simeq D_4,$$

where

$$\mathcal{K} = \left\{ e^{\frac{i\pi}{8}k} \cdot I \;\middle|\; k \in \{0, 1, \ldots, 7\} \right\}$$

and D_4 is the dihedral group of degree four. The following fact is easily verified by a direct calculation.

Fact 1 *The clifford group $\mathcal{C}_1$ can be represented as follows:*

$$\mathcal{C}_1 = C_T(\mathcal{C}_1) + HC_T(\mathcal{C}_1) + PHC_T(\mathcal{C}_1),$$

where $HC_T(\mathcal{C}_1) := \{Hh | \, h \in C_T(\mathcal{C}_1)\}$, and $PHC_T(\mathcal{C}_1) := \{PHh | \, h \in C_T(\mathcal{C}_1)\}$. □

Note that $C_T(\mathcal{C}_1)$, $HC_T(\mathcal{C}_1)$ and $PHC_T(\mathcal{C}_1)$ in the above fact are the residue classes of $\mathcal{C}_1$ and have order 64. Fact 1 guarantees that for every W_0 in $\mathcal{C}_1$, $W_0T = S_0TW_1$ for some $S_0 \in \{I, H, PH\}$ and some clifford circuit W_1. This gives a set of basic transformation rules of our normalization. The correctness follows from the definition of $C_T(\mathcal{C}_1)$: (i) if $W_0 \in C_T(\mathcal{C}_1)$ then $W_0 = TW_1T^{-1}$ for some $W_1 \in \mathcal{C}_1$, (ii) if $W_0 \in HC_T(\mathcal{C}_1)$ then $W_0 = HTW_1T^{-1}$ for some $W_1 \in \mathcal{C}_1$, and (iii) if $W_0 \in PHC_T(\mathcal{C}_1)$ then $W_0 = PHTW_1T^{-1}$ for some $W_1 \in \mathcal{C}_1$.

A complete table of our basic transformation rules, which contains $|\mathcal{C}_1| = 192$ rules and categorized into three groups, is given in Appendix.

Roughly speaking, the normalization is to apply this transformation rule to a given circuit from left to right. For example, when the initial circuit is given by $W_3TW_2TW_1TW_0$, where W_i, $i = 0, 1, 2, 3$ is a clifford circuit, the normalization proceeds as described below. Here we use symbols W_i to denote a clifford circuit, and S_i to denote a circuit in $\{I, H, PH\}$.

$$\begin{aligned}
W_3\ T\ W_2\ T\ W_1\ T\ W_0 &= \underline{W_3\ T}\ W_2\ T\ W_1\ T\ W_0 \\
&= \underline{S_0\ T\ W_4}\ W_2\ T\ W_1\ T\ W_0 && \text{(apply the rule to } W_3T\text{)} \\
&= S_0\ T\ W_5\ T\ W_1\ T\ W_0 && (W_5 := W_4W_2) \\
&= S_0\ T\ \underline{W_5\ T}\ W_1\ T\ W_0 \\
&= S_0\ T\ \underline{S_1\ T\ W_6}\ W_1\ T\ W_0 && \text{(apply the rule to } W_5T\text{)} \\
&= S_0\ T\ S_1\ T\ W_7\ T\ W_0 && (W_7 := W_6W_1) \\
&= S_0\ T\ S_1\ T\ \underline{W_7\ T}\ W_0 \\
&= S_0\ T\ S_1\ T\ \underline{S_2\ T\ W_8}\ W_0 && \text{(apply the rule to } W_7T\text{)} \\
&= S_0\ T\ S_1\ T\ S_2\ T\ W_9 && (W_9 := W_8W_0).
\end{aligned}$$

If $S_i = I$ at some step of the process, then we "merge" two T-gates into a P gate by applying the identity $P = TT$, and continue the process. For example, if $S_2 = I$ in the above equation, then we further transform the last circuit to

$$S_0\ T\ S_1\ T\ S_2\ T\ W_9 = S_0\ T\ S_1\ P\ W_9 = S_0\ T\ W_{10}.$$

It is obvious that, for every input circuit over $\{H, P, T\}$, the resulting circuit of this normalization process is a normal form circuit. This established Statement (I). □

It should be noted that the normalization can be performed in time linear in the number of gates in an initial circuit.

4.2 The proof of (II)

The proof of Statement (II) is divided into two subproofs:

(II-A) If (II) is false, i.e., there are two distinct normal form circuits C_1 and C_2 with $f_u(C_1) = f_u(C_2)$, then there is a normal form circuit C containing one or more T-gates such that $f_u(C) = I$.

(II-B) For every normal form circuit C containing one or more T-gates, $f_u(C) \neq I$ holds.

The meaning of the symbols appeared in the proof are as follows.

- $W_a, W_b, \ldots, W_z$, and W_j $(j = 0, 1, 2, \cdots)$ denote a clifford circuit,
- A_j, B_j, C_j, D_j , $j = 1, 2, \cdots$ denote H or PH,
- A', B', C', D' , $j = 0, 1, 2, \cdots$ denote I or H or PH.

Before we proceed to the proof of Statement (II), we give the following simple lemma that says the set $\mathbf{M_{=n}}$ is closed under the inverse operation, where $\mathbf{M_{=n}} := \mathbf{M_n} \backslash \mathbf{M_{n-1}}$. Recall that $\mathbf{M_n}$ denotes the set of all matrices that can be computed by a circuit over $\{H, P, T\}$ with at most n T-gates.

Lemma 1. *Let C be the normal form circuit containing n T-gates. If $f_u(C) \in \mathbf{M_{=n}}$, then $f_u(C)^{-1} \in \mathbf{M_{=n}}$.*

Proof. It is trivial for $n = 0$. Let $n \geq 1$ and suppose that C satisfies $f_u(C) \in \mathbf{M_{=n}}$. Then C can be written as

$$A'\ T\ A_{n-1}\ T\ \cdots\ T\ A_2\ T\ A_1\ T\ W_a. \tag{1}$$

The inverse matrix of $f_u(C)$ is given by

$$W_a^{-1}\ T^{-1}\ A_1^{-1}\ T^{-1}\ A_2^{-1}\ \cdots\ \left(A'\right)^{-1},$$

and is represented as

$$W_n\ T\ W_{n-1}\ T\ \cdots\ T\ W_1\ T\ W_0 \tag{2}$$

since $T^{-1} = T^7 = TP^3$. This implies $f_u(C)^{-1} \in \mathbf{M_n}$.

Suppose that the lemma is false; $f_u(C)^{-1} \in \mathbf{M_{n-1}}$. The above argument gives $(f_u(C)^{-1})^{-1} = f_u(C) \in \mathbf{M_{n-1}}$, which contradicts the assumption that $f_u(C) \in \mathbf{M_{=n}}$. This completes the proof of the lemma. □

The proof of (II-A) Suppose that (II) is false, i.e., there are two distinct normal form circuits U_a and U_b such that $f_u(U_a) = f_u(U_b)$. Fix an arbitrary such pair (U_a, U_b) that minimizes $t(U_a) + t(U_b)$, where $t(C)$ denotes the number of occurrences of T in C. Put $m = t(U_a)$ and $n = t(U_b)$. Without loss of generality we assume that $m \geq n$. We write U_a and U_b as

$$U_a := A'TA_{m-1} \cdots A_2TA_1TW_a,$$
$$U_b := B'TB_{n-1} \cdots B_2TB_1TW_b,$$

respectively. We can also assume that $A' \neq B'$ since otherwise a subcircuit of U_a starting at A_{m-1} and a subcircuit of U_b starting at B_{n-1} compute same matrix.

For a while we identify a circuit with the corresponding matrix. Then we can write

$$A'TA_{m-1} \cdots A_2TA_1TW_a \;=\; B'TB_{n-1} \cdots B_2TB_1TW_b. \tag{3}$$

The inverse matrix of $f_u(U_b)$ is

$$W_b^{-1}T^{-1}B_1^{-1}T^{-1}B_2^{-1} \cdots B_n^{-1}T^{-1}(B')^{-1}, \tag{4}$$

and this can also be represented by a normal form with n T-gates by Lemma 1, which we write as

$$C'TC_{n-1} \cdots C_2TC_1TW_c. \tag{5}$$

Here we divide the proof into two cases.
(Case 1) $m > n$.

By multiplying Eq. (5) from the right to both sides of Eq. (3), we have

$$A'TA_{m-1} \cdots A_2TA_1T\underline{W_a\ C'T}C_{n-1}T \cdots TC_2TC_1TW_c = I. \tag{6}$$

We consider the normalization of the LHS of Eq. (6). By applying the basic normalization rule to $W_a\ C'T$ at the underlined part in Eq. (6), we obtain

$$A'TA_{m-1} \cdots A_2TA_1T\ \underset{\sim}{S\ T}\underset{\sim}{W_0}C_{n-1}T \cdots TC_2TC_1TW_c = I, \tag{7}$$

where $S = I$, H or PH. If $S = H$ or PH, then the leftmost T never disappeared during the normalization process, and hence a normalized circuit for the LHS of Eq. (6) contains at least one T and computes the identity matrix.

We now assume $S = I$. Then Eq. (7) is written as

$$A'TA_{m-1} \cdots A_2T\underline{A_1\ P\ W_0C_{n-1}T} \cdots TC_2TC_1TW_c = I.$$

Note that the left and right T-gates of S in Eq. (7) are disappeared by applying the identity $P = T^2$. By applying the basic transformation rule again to $A_1PW_0C_{n-1}T$ at the underlined part in the above equation, we obtain

$$A'TA_{m-1} \cdots A_2T\underset{\sim}{S}\underset{\sim}{T\ :}\ \cdots TC_2TC_1TW_c = I. \tag{8}$$

If $S = H$ or PH, then Statement (II-A) is established by the same argument as above. If $S = I$ for every step of the normalization, then the normalized circuit for the LHS of Eq. (6) has $m - n$ T-gates. These complete the proof of Case 1.

(Case 2) $m = n$.

For $n = m \leq 1$, we can check the statement by a direct computation.

Let $n = m > 1$. By multiplying Eq. (4) from the right to both sides of Eq. (3), we have

$$A'TA_{n-1}\cdots A_2TA_1T\underline{W_aW_b^{-1}TP^3}B_1^{-1}T^{-1}\cdots T^{-1}(B')^{-1} = I.$$

By applying the basic transformation rule to the underlined part in the above equation, we obtain

$$A'TA_{n-1}\cdots A_2TA_1T\underbrace{STW_z}P^3B_1^{-1}T^{-1}\cdots T^{-1}(B')^{-1} = I. \tag{9}$$

If $S = H$ or $S = PH$, then we can see that the normalized circuit of the LHS of Eq. (9) contains at least one T by the same argument to the proof of Case 1.

Assume that $S = I$, i.e.,

$$A'TA_{n-1}\cdots A_2TA_1PW_zP^3B_1^{-1}T^{-1}\cdots T^{-1}(B')^{-1} = I.$$

This implies

$$A'TA_{n-1}\cdots A_2TA_1 = B'T\cdots T(PW_zP^3B_1^{-1})^{-1}.$$

If we replace the rightmost term by an equivalent clifford circuit, then both sides in the above equation is a normal form circuit that contains $n-1$ T-gates. In addition, since $A' \neq B'$, these are different circuits. This contradicts our choice of U_a and U_b. These complete the proof of Case 2, and so the proof of (II-A). □

The proof of (II-B) The idea of the proof is borrowed from the stabilizer formalism [6, p.454] (or see also [1, 5]). Let $|\psi\rangle$ denote a one qubit state :

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

where $|\alpha|^2 + |\beta|^2 = 1$. Let

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Definition 8. *For a tuple of three real numbers $(x, y, z) \in \mathbb{R}^3$, the matrix $M_{(x,y,z)}$ is defined as*

$$M_{(x,y,z)} := xX + yY + zZ.$$

We say that (x, y, z) stabilizes $|\psi\rangle$ if

$$M_{(x,y,z)}|\psi\rangle = |\psi\rangle.$$

The following two facts are easily verified.

Fact 2 *If (x, y, z) stabilizes $|0\rangle$, then $(x, y, z) = (0, 0, 1)$.*

Proof. Since (x, y, z) stabilizes $|0\rangle$, we have

$$M_{(x,y,z)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} z & x - yi \\ x + yi & -z \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} z \\ x + yi \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

This implies $(x, y, z) = (0, 0, 1)$ since x, y and z are real. □

Fact 3 *Suppose that (x, y, z) stabilizes $|\psi\rangle$. Then $T|\psi\rangle$ is stabilized by $\frac{1}{\sqrt{2}} \cdot (x-y, x+y, \sqrt{2}z)$, $HT|\psi\rangle$ is stabilized by $\frac{1}{\sqrt{2}} \cdot (\sqrt{2}z, -x-y, x-y)$, and $PHT|\psi\rangle$ is stabilized by $\frac{1}{\sqrt{2}} \cdot (x + y, \sqrt{2}z, x - y)$.*

Proof. Let U be an arbitrary unitary matrix of degree two, and suppose that matrix M of degree two stabilizes $|\psi\rangle$. Since

$$UMU^\dagger U|\psi\rangle = UM|\psi\rangle = U|\psi\rangle,$$

$UMU^\dagger$ stabilizes $U|\phi\rangle$. Therefore transitions of each stabilizer matrix is given by

$$HXH^\dagger = Z, \quad HYH^\dagger = -Y, \quad HZH^\dagger = X, \tag{10}$$

$$PXP^\dagger = Y, \quad PYP^\dagger = -X, \quad PZP^\dagger = Z, \tag{11}$$

$$TXT^\dagger = \frac{X+Y}{\sqrt{2}}, \quad TYT^\dagger = \frac{Y-X}{\sqrt{2}}, \quad TZT^\dagger = Z. \tag{12}$$

Suppose that $|\psi\rangle$ is stabilized by (x, y, z). Eq . (12) gives the stabilizer matrix of $T|\psi\rangle$:

$$T(xX + yY + zZ)T^\dagger = \frac{(x-y)X}{\sqrt{2}} + \frac{(x+y)Y}{\sqrt{2}} + zZ. \tag{13}$$

Eq. (10) and Eq. (13) give the stabilizer matrix of $HT|\psi\rangle$:

$$H\left(\frac{(x-y)X}{\sqrt{2}} + \frac{(x+y)Y}{\sqrt{2}} + zZ\right)H^\dagger = zX - \frac{(x+y)Y}{\sqrt{2}} + \frac{(x-y)Z}{\sqrt{2}}. \tag{14}$$

Eq. (11) and Eq. (14) give the stabilizer matrix of $PHT|\psi\rangle$:

$$P\left(zX - \frac{(x+y)Y}{\sqrt{2}} + \frac{(x-y)Z}{\sqrt{2}}\right)P^\dagger = \frac{(x+y)X}{\sqrt{2}} + zY + \frac{(x-y)Z}{\sqrt{2}}.$$

□

If the number of T in C is one, then we can see that Statement (II-B) is true by a direct computation. Hence we only need to consider normal form circuits with at least two T-gates.

Let C be a normal form circuit containing $k \geq 2$ T-gates:

$$C = C_k T C_{k-1} T \cdots T C_0,$$

where $C_k \in \{I, H, PH\}$, $C_i \in \{H, PH\}$ for $1 \leq i < k$, and C_0 is a clifford circuit. For $\ell \leq k$, let $C_{(\ell)}$ be a subcircuit of C defined as

$$C_{(\ell)} = C_\ell T C_{\ell-1} T \cdots T C_0.$$

In order to show Statement (II-B), it is sufficient to show that the stabilizer matrix of $C|0\rangle$ is *not* $(0,0,1)$ (by Fact 2). To see this, we observe the transition of the stabilizer matrices of $C_{(\ell)}|0\rangle$ for $\ell = 0, \ldots, k$.

Since C_0 contains only H and P, $C_0|0\rangle$ is stabilized by $(x,y,z) \in \{(0,0,\pm 1),(0,\pm 1,0),(\pm 1,0,0)\}$ (by Eqs. (10) and (11)). From Fact 3, $C_{(\ell)}|0\rangle$ is stabilized by a matrix of the form

$$\frac{1}{\sqrt{2}^{\ell}}(x_a + x_b\sqrt{2}, y_a + y_b\sqrt{2}, z_a + z_b\sqrt{2}), \tag{15}$$

where $x_a, x_b, y_a, y_b, z_a, z_b \in \mathbb{Z}$. From Eq. (15) and Fact 3, $TC_{(\ell)}|0\rangle$, $HTC_{(\ell)}|0\rangle$, and $PHTC_{(\ell)}|0\rangle$ are stabilized by

$$\frac{1}{\sqrt{2}^{\ell+1}}\left((x_a - y_a) + (x_b - y_b)\sqrt{2},\ (x_a + y_a) + (x_b + y_b)\sqrt{2}, 2z_b + z_a\sqrt{2}\right), \tag{16}$$

$$\frac{1}{\sqrt{2}^{\ell+1}}\left(2z_b + z_a\sqrt{2}, -(x_a + y_a) - (x_b + y_b)\sqrt{2},\ (x_a - y_a) + (x_b - y_b)\sqrt{2}\right), \tag{17}$$

$$\frac{1}{\sqrt{2}^{\ell+1}}\left((x_a + y_a) + (x_b + y_b)\sqrt{2}, 2z_b + z_a\sqrt{2},\ (x_a - y_a) + (x_b - y_b)\sqrt{2}\right), \tag{18}$$

respectively.

Consider a circuit C that contains ℓ T-gates and $C|0\rangle$ is stabilized by a matrix of the form Eq. (15). We define nine classes of circuits depending on the parities of x_a, x_b, y_a, y_b, z_a and z_b in Eq. (15).

T1: x_b and z_b are **odd** numbers, and other four are **even** numbers.
T2: y_b and z_b are **odd** numbers, and other four are **even** numbers.
T3: x_b and y_b are **odd** numbers, and other four are **even** numbers.
T4: x_b, y_a and z_a are **odd** numbers, and x_a, y_b and z_b are **even** numbers.
T5: x_a, y_b and z_a are **odd** numbers, and x_b, y_a and z_b are **even** numbers.
T6: x_a, y_a and z_b are **odd** numbers, and x_b, y_b and z_a are **even** numbers.
T7: x_a is **even** number, and the other five are **odd** numbers.
T8: y_a is **even** number, and the other five are **odd** numbers.
T9: z_a is **even** number, and the other five are **odd** numbers.

Note that every circuit C with $C|0\rangle = |0\rangle$ does not belong to every class since $|0\rangle$ is stabilized by $(0,0,1)$, i.e., all of x_a, x_b, y_a and y_b must be even. We are now ready to finish the proof of Statement (II-B).

When $k = 2$, we can confirm that $C\,|0\rangle$ is not stabilized by $(0,0,1)$ by computing all patterns directly, and thus $C|0\rangle \neq |0\rangle$. We now assume $k \geq 3$. We divide the proof into two cases depending on the stabilizer matrix of $C_0\,|0\rangle$.

(Case 1) $C_0|0\rangle$ is stabilized by $(0,0,\pm 1)$.

We can easily check that $C_{(2)}|0\rangle = C_2TC_1TC_0|0\rangle$ is stabilized by $(x,y,z) = 1/2 \cdot (0,\pm\sqrt{2},\pm\sqrt{2})$ or $1/2 \cdot (\pm\sqrt{2},0,\pm\sqrt{2})$ using Fact 3. Namely, $C_{(2)}$ belongs to T1 or T2. In addition, it is easy to confirm that Eqs. (16), (17) and (18) give the following two facts.

Fact 4 *If $C_{(\ell)}$ belongs to T1 or T2, then $HTC_{(\ell)}$ belongs to T2, and $PHTC_{(\ell)}$ belongs to T1.* □

Fact 5 *If $C_{(\ell)}$ belongs to T1 or T2, then $TC_{(\ell)}$ belongs to T3.* □

By Facts 4 and 5, we can conclude that C belongs to T1, T2 or T3. This implies that the stabilizer matrix of $C|0\rangle$ is not $(0,0,1)$, and hence $f_u(C) \neq I$.

(Case 2) $C_0|0\rangle$ is stabilized by $(0,\pm 1,0)$ or $(\pm 1,0,0)$.

The proof is analogous to the proof of Case 1.

We can easily verify that $C_{(2)}|0\rangle$ is stabilized by $(x,y,z) = 1/2 \cdot (\pm\sqrt{2},\pm 1,\pm 1)$ or $1/2 \cdot (\pm 1,\pm\sqrt{2},\pm 1)$. Namely, $C_{(2)}$ belongs to T4 or T5. In addition, it is easy to verify that Eqs. (16), (17) and (18) give the following fact.

Fact 6 *All of the following is true:*

(i) If $C_{(\ell)}$ belongs to T4 or T5, then $HTC_{(\ell)}$ belongs to T7, and $PHTC_{(\ell)}$ belongs to T8.
(ii) If $C_{(\ell)}$ belongs to T7 or T8, then $HTC_{(\ell)}$ belongs to T4, and $PHTC_{(\ell)}$ belongs to T5.
(iii) If $C_{(\ell)}$ belongs to T4 or T5, then $TC_{(\ell)}$ belongs to T9.
(iv) If $C_{(\ell)}$ belongs to T7 or T8, then $TC_{(\ell)}$ belongs to T6.

□

By the above fact, we can show that C belongs to T4, T5, T6, T7, T8 or T9. This implies that the stabilizer matrix of $C|0\rangle$ is not $(0,0,1)$, and hence $f_u(C) \neq I$. This completes the proof of Case 2, and of Statement (II-B). □

5 The number of normal form circuits

Our main theorem can also be used to derive the number of 2×2 matrices computed by a circuit over the standard basis $\{H,P,T\}$ using at most n T-gates.

Corollary 1. *For all nonnegative integers n, $|\mathbf{M_n}| = |\mathcal{C}_1| \cdot (3 \cdot 2^n - 2) = 192 \cdot (3 \cdot 2^n - 2)$.*

Proof. The definition of the normal form and Theorem 1 gives

$$|\mathbf{M}_n| = \begin{cases} \alpha/2\ , & n = 0, \\ 2\,|\mathbf{M}_{n-1}| + \alpha\ , & n > 0, \end{cases}$$

where $\alpha = 384$. The corollary follows from this recurrence formula. □

Corollary 2. *For all positive integers n, $|\mathbf{M_{=n}}| = 576 \cdot 2^{n-1}$.*

6 Concluding remarks

In this paper, we introduce the notion of a normal form of one qubit quantum circuits over the standard basis $\{H,T,P\}$. In addition, we prove that the number of 2×2 unitary matrices computed by a circuit over $\{H,T,P\}$ that contains at most n T-gates is exactly $192 \cdot (3 \cdot 2^n - 2)$. Obviously, it is a challenging future work to extend our result to circuits with multiple qubits. In other words, our next goal is to give "n-qubit normal form".

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Appendix

$$IT = TI$$
$$PT = TP$$
$$PPT = TPP$$
$$PPPT = TPPP$$
$$HPPPHPHT = THPPH$$
$$HPHPHT = THPHPH$$
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Table 1. The transformation rules of the form $W_0T = TW_1$.

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\end{aligned}
$$

Table 2. The transformation rules of the form $W_0T = HTW_1$. There is a rule $HW_0T = HTW_1$ in this table if and only if there is a rule $W_0T = TW_1$ in the table for $S_0 = I$ (Table 1).

$$PHT = PHT$$
$$PHPT = PHTP$$
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Table 3. The transformation rules of the form $W_0T = PHTW_1$. There is a rule $PHW_0T = PHTW_1$ in this table if and only if there is a rule $W_0T = TW_1$ in the table for $S_0 = I$ (Table 1).