COL861: Special Topics in Hardware Systems

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**Content:** Dense coding, Teleportation, Non-local games

# Dense Coding

Alice wants to share **two classical bits** with Bob by sending a **single qubit**. Alice and Bob have already shared an **EPR pair**.

## Protocol

The initial state of Alice and Bob's qubit pair, where the first qubit as read is Alice's qubit,

$$|\psi_0\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

.
Alice prepares one of the ***Bell States*** based on what classical bits she has by doing certain operations as listed below on her qubit of ***EPR Pair***.

| Classical Bits (b1 b0) | U | $\lvert\psi_1\rangle = U\lvert\psi_0\rangle$ |
|---|---|---|
| 00 | $I \otimes I$ | $\frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 11\rangle)$ |
| 01 | $X \otimes I$ | $\frac{1}{\sqrt{2}}(\lvert 10\rangle + \lvert 01\rangle)$ |
| 10 | $Z \otimes I$ | $\frac{1}{\sqrt{2}}(\lvert 00\rangle - \lvert 11\rangle)$ |
| 11 | $ZX \otimes I$ | $\frac{1}{\sqrt{2}}(\lvert 01\rangle - \lvert 10\rangle)$ |

Then, Alice sends her qubit to Bob, and Bob does the measurement of the combined qubit system in ***Bell basis*** by doing ***Controlled Not*** on his qubit controlled by Alice's qubit, Hadamard on Alice's qubit and then measurement in standard basis giving out the initial classical bits, as the table shows.

| Classical Bits (b1 b0) | $\lvert\psi_2\rangle = CX\lvert\psi_1\rangle$ | $\lvert\psi_3\rangle = (H \otimes I)\lvert\psi_2\rangle$ |
|---|---|---|
| 00 | $\frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 10\rangle) = \lvert +\rangle \otimes \lvert 0\rangle$ | $\lvert 00\rangle$ |
| 01 | $\frac{1}{\sqrt{2}}(\lvert 11\rangle + \lvert 01\rangle) = \lvert +\rangle \otimes \lvert 1\rangle$ | $\lvert 01\rangle$ |
| 10 | $\frac{1}{\sqrt{2}}(\lvert 00\rangle - \lvert 10\rangle) = \lvert -\rangle \otimes \lvert 0\rangle$ | $\lvert 10\rangle$ |
| 11 | $\frac{1}{\sqrt{2}}(\lvert 01\rangle - \lvert 11\rangle) = \lvert -\rangle \otimes \lvert 1\rangle$ | $\lvert 11\rangle$ |

Thus, after measurement, Bob knows the initial classical bits.

# Quantum Teleportation

Alice has a qubit $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$ that she wants to send to Bob by sending two classical bits of information. Alice and Bob have already shared an ***EPR Pair***. So, the initial state of the system of qubits is read so that the qubit to be sent is first, followed by Alice's EPR part and then Bob's.

$$|\Psi_0\rangle = |\phi\rangle \otimes (\frac{|00\rangle + |11\rangle}{\sqrt{2}}) = (\alpha|0\rangle + \beta|1\rangle) \otimes (\frac{|00\rangle + |11\rangle}{\sqrt{2}})$$

$$= \frac{\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle}{\sqrt{2}}$$

$$= \frac{(\alpha|00\rangle + \beta|10\rangle) \otimes |0\rangle + (\alpha|01\rangle + \beta|11\rangle) \otimes |1\rangle}{\sqrt{2}}$$

## Protocol

First, Alice entangles her both qubits, by *controlled Not* on ***EPR*** qubit and ***Hadamard*** on qubit to be sent.

$$|\Psi_1\rangle = (((H \otimes I)CX) \otimes I)|\psi_0\rangle$$

$$= \frac{(\alpha|{+0}\rangle + \beta|{-1}\rangle) \otimes |0\rangle + (\alpha|{+1}\rangle + \beta|{-0}\rangle) \otimes |1\rangle}{\sqrt{2}}$$

$$= \frac{(\alpha|00\rangle + \alpha|10\rangle + \beta|01\rangle - \beta|11\rangle) \otimes |0\rangle + (\alpha|01\rangle + \alpha|11\rangle + \beta|00\rangle - \beta|10\rangle) \otimes |1\rangle}{2}$$

$$= \frac{|00\rangle \otimes (\alpha|0\rangle + \beta|1\rangle) + |01\rangle \otimes (\alpha|1\rangle + \beta|0\rangle) + |10\rangle \otimes (\alpha|0\rangle - \beta|1\rangle) + |11\rangle \otimes (\alpha|1\rangle - \beta|0\rangle)}{2}$$

Then, Alice measures her two qubits and sends these outcomes of measurement to Bob via the classical channel.

| Measurement outcomes (b1 b0) | State after measurement ($|\psi_2\rangle$ ) |
|---|---|
| 00 | $|00\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)$ |
| 01 | $|01\rangle \otimes (\alpha|1\rangle + \beta|0\rangle)$ |
| 10 | $|10\rangle \otimes (\alpha|0\rangle - \beta|1\rangle)$ |
| 11 | $|11\rangle \otimes (\alpha|1\rangle - \beta|0\rangle)$ |

Then, based on the values of Classical bits, Bob does the following operations on his qubit. This step is solely planned so as to transform Bob's qubit to the qubit to be sent.

| Classical Bits (b1 b0) | U | $|\psi_3\rangle = U|\psi_2\rangle$ |
|---|---|---|
| 00 | $I \otimes I \otimes I$ | $|00\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)$ |
| 01 | $I \otimes I \otimes X$ | $|01\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)$ |
| 10 | $I \otimes I \otimes Z$ | $|10\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)$ |
| 11 | $I \otimes I \otimes ZX$ | $|11\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)$ |

Thus, Bob now has the qubit identical to what Alice had wanted to send. Also, it can be seen that the initial qubit is not cloned at Bob's end because the initial qubit was destroyed (by measurement) before sending classical bits to Bob.

# Non-Local Games

## CHSH game

### Description

The participants are Alice and Bob, as well as a referee. Before the game starts, Alice and Bob can prepare any kind of strategy, but after the game starts, they can't communicate with each other. As the game starts, the referee gives bit $x$ to Alice and bit $y$ to Bob. Then, Alice has to give another bit $a$ back to the referee, and Bob gives bit $b$ to the referee. Then referee checks if

$$a \oplus b = x \wedge y$$

It is proved that any classical deterministic or probabilistic strategy can't promise more than 75% chances of winning.

### Strategy based on Quantum Computation

Both Alice and Bob share parts of an ***EPR Pair*** before the game starts. The following notation is going to be used a lot.

$$|\Psi_\theta\rangle = cos\theta\,|0\rangle + sin\theta\,|1\rangle$$

A Fact about the ***EPR Pair*** ($|\phi^+\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$). For any orthogonal basis of form $\{|\Psi_\theta\rangle, |\Psi_{\theta+\frac{\pi}{2}}\rangle\}$,

$$|\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{|\Psi_\theta\Psi_\theta\rangle + |\Psi_{\theta+\frac{\pi}{2}}\Psi_{\theta+\frac{\pi}{2}}\rangle}{\sqrt{2}}$$

. **Proof:** Note that $|0\rangle = cos\theta\,|\Psi_\theta\rangle - sin\theta\,|\Psi_{\theta+\frac{\pi}{2}}\rangle$ and $|1\rangle = sin\theta\,|\Psi_\theta\rangle + cos\theta\,|\Psi_{\theta+\frac{\pi}{2}}\rangle$. Then, if these are substituted in $|\phi^+\rangle$, then the result straightaway follows. Q.E.D.

Also note,

$$U_\theta = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$

is rotation anticlockwise by angle $\theta$. Now, as Alice sees bit $\boldsymbol{x}$, she does following with her qubit,

| Bit $X$ | Possible Actions |
|---|---|
| 0 | Measure in $\{|\Psi_0\rangle, |\Psi_{\frac{\pi}{2}}\rangle\}$ basis / Apply $U_0$ on qubit and measure in $\{|0\rangle, |1\rangle\}$ basis |
| 1 | Measure in $\{|\Psi_{\frac{\pi}{4}}\rangle, |\Psi_{\frac{3\pi}{4}}\rangle\}$ basis / Apply $U_{-\frac{\pi}{4}}$ on qubit and measure in $\{|0\rangle, |1\rangle\}$ basis |

And when Bob sees bit $\boldsymbol{y}$, he does following with his qubit,

| Bit $Y$ | Possible Actions |
|---|---|
| 0 | Measure in $\{|\Psi_{\frac{\pi}{8}}\rangle, |\Psi_{\frac{5\pi}{8}}\rangle\}$ basis / Apply $U_{-\frac{\pi}{8}}$ on qubit and measure in $\{|0\rangle, |1\rangle\}$ basis |
| 1 | Measure in $\{|\Psi_{-\frac{\pi}{8}}\rangle, |\Psi_{\frac{3\pi}{8}}\rangle\}$ basis / Apply $U_{\frac{\pi}{8}}$ on qubit and measure in $\{|0\rangle, |1\rangle\}$ basis |

. Then, Alice and Bob share their measurement outcome as bits $\boldsymbol{a}$ and $\boldsymbol{b}$ with the referee. This strategy guarantees 85% chances of winning. The order of actions of Alice and Bob doesn't matter.

**Mathematical reasoning**

let's enumerate first the winning action for each possible combination of pair ***(x,y)***.

| Bits *(X, Y)* | Winning combinations of *(a,b)* |
|---|---|
| 00 | 00 , 11 |
| 01 | 00 , 11 |
| 10 | 00 , 11 |
| 11 | 01 , 10 |

Doing the analysis for each case.
First, consider when ***(x,y)*** $= (0, 0)$. Since the order of actions doesn't matter, let's assume Alice acted first. Initial state was,

$$|\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

and as earlier stated, and based on her $\boldsymbol{x}$ bit she does measurement on standard basis. Then, the new state is

| Outcome of measurement | New state $|\Psi_2\rangle$ |
|---|---|
| 0 | $|00\rangle$ |
| 1 | $|11\rangle$ |

Then, as Bob does his action based on his bit $\boldsymbol{y}$,

| $|\psi_2\rangle$ | Before Bob's measurement $|\Psi_3\rangle$ | Measurement outcome |
|---|---|---|
| $|00\rangle$ | $(I \otimes U_{\frac{-\pi}{8}})\,|00\rangle = |0\rangle \otimes (cos(-\frac{\pi}{8})\,|0\rangle + sin(-\frac{\pi}{8})\,|1\rangle)$ | 0 (chances $cos^2(-\frac{\pi}{8}) \approx 85\%$) else 1 |
| $|11\rangle$ | $(I \otimes U_{\frac{-\pi}{8}})\,|11\rangle = |1\rangle \otimes (-sin(-\frac{\pi}{8})\,|0\rangle + cos(-\frac{\pi}{8})\,|1\rangle)$ | 1 (chances $cos^2(-\frac{\pi}{8}) \approx 85\%$) else 0 |

Thus, it can be seen that with 85% chances, they measure either $(0, 0)$ or $(1, 1)$ (since Alice measures 0 with 50% chances and Bob measures 0 given Alice measured 0 with 85% chances, and similarly Alice measures 1 with 50% chances and Bob measures 1 given Alice measured 1 with 85% chances, so in total, chances they both measured same value $= 0.5 \times 85\% + 0.5 \times 85\% = 85\%$), where both combinations lead to win.

Case when ***(x,y)*** $= (0, 1)$. Initial state was,

$$|\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

and based on her $\boldsymbol{x}$ bit she does measurement on standard basis. Then, the new state is

| Outcome of measurement | New state $|\Psi_2\rangle$ |
|---|---|
| 0 | $|00\rangle$ |
| 1 | $|11\rangle$ |

Then, as Bob does his action based on his bit $\boldsymbol{y}$,

| $|\psi_2\rangle$ | Before Bob's measurement $|\Psi_3\rangle$ | Measurement outcome |
|---|---|---|
| $|00\rangle$ | $(I \otimes U_{\frac{\pi}{8}})\,|00\rangle = |0\rangle \otimes (cos(\frac{\pi}{8})\,|0\rangle + sin(\frac{\pi}{8})\,|1\rangle)$ | 0 (chances $cos^2(\frac{\pi}{8}) \approx 85\%$) else 1 |
| $|11\rangle$ | $(I \otimes U_{\frac{\pi}{8}})\,|11\rangle = |1\rangle \otimes (-sin(\frac{\pi}{8})\,|0\rangle + cos(\frac{\pi}{8})\,|1\rangle)$ | 1 (chances $cos^2(\frac{\pi}{8}) \approx 85\%$) else 0 |

Thus, it can be seen that with 85% chances, they measure either $(0, 0)$ or $(1, 1)$, both of which lead to a win.

Case when ***x,y*** $= (1, 0)$.
Initial state was,

$$|\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

and based on her ***x*** bit she applies $U_{-\frac{\pi}{4}}$ before measuring in standard basis. Then,

$$|\Psi_1\rangle = (U_{-\frac{\pi}{4}} \otimes I)\,|\phi^+\rangle$$

, and as stated and proved earlier that,

$$|\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{|\Psi_\theta\Psi_\theta\rangle + |\Psi_{\theta+\frac{\pi}{2}}\Psi_{\theta+\frac{\pi}{2}}\rangle}{\sqrt{2}}, \forall\theta \in [0, 2\pi]$$

. Thus,

$$|\Psi_1\rangle = (U_{-\frac{\pi}{4}} \otimes I)(\frac{|\Psi_{\frac{\pi}{4}}\Psi_{\frac{\pi}{4}}\rangle + |\Psi_{\frac{3\pi}{4}}\Psi_{\frac{3\pi}{4}}\rangle}{\sqrt{2}})$$
$$= \frac{|\Psi_0\Psi_{\frac{\pi}{4}}\rangle + |\Psi_{\frac{\pi}{2}}\Psi_{\frac{3\pi}{4}}\rangle}{\sqrt{2}} = \frac{|0\rangle\,|\Psi_{\frac{\pi}{4}}\rangle + |1\rangle\,|\Psi_{\frac{3\pi}{4}}\rangle}{\sqrt{2}}$$

Then, Alice measures her qubit in standard basis.

| Outcome of measurement | New state $|\Psi_2\rangle$ |
|---|---|
| 0 | $|0\rangle\,|\psi_{\frac{\pi}{4}}\rangle$ |
| 1 | $|1\rangle\,|\psi_{\frac{3\pi}{4}}\rangle$ |

Then, as Bob does his action based on his bit ***y***,

| $|\psi_2\rangle$ | Before Bob's measurement $|\Psi_3\rangle$ | Measurement outcome |
|---|---|---|
| $|0\rangle\,|\psi_{\frac{\pi}{4}}\rangle$ | $(I \otimes U_{-\frac{\pi}{8}})\,|0\rangle\,|\psi_{\frac{\pi}{4}}\rangle = |0\rangle \otimes (cos(\frac{\pi}{8})\,|0\rangle + sin(\frac{\pi}{8})\,|1\rangle)$ | 0 (chances $cos^2(\frac{\pi}{8}) \approx 85\%$) else 1 |
| $|1\rangle\,|\psi_{\frac{3\pi}{4}}\rangle$ | $(I \otimes U_{-\frac{\pi}{8}})\,|1\rangle\,|\psi_{\frac{3\pi}{4}}\rangle = |1\rangle \otimes (cos(\frac{5\pi}{8})\,|0\rangle + sin(\frac{5\pi}{8})\,|1\rangle)$ | 1 (chances $sin^2(\frac{5\pi}{8}) \approx 85\%$) else 0 |

Thus, it can be seen that with 85% chances, they measure either $(0, 0)$ or $(1, 1)$, both of which lead to a win.

Case when ***x,y*** $= (1, 1)$.
Initial state was,

$$|\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

and based on her ***x*** bit she applies $U_{-\frac{\pi}{4}}$ before measuring in standard basis. Then,

$$|\Psi_1\rangle = (U_{-\frac{\pi}{4}} \otimes I)\,|\phi^+\rangle$$

, and as stated and proved earlier that,

$$|\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{|\Psi_\theta\Psi_\theta\rangle + |\Psi_{\theta+\frac{\pi}{2}}\Psi_{\theta+\frac{\pi}{2}}\rangle}{\sqrt{2}}, \forall\theta \in [0, 2\pi]$$

. Thus,

$$|\Psi_1\rangle = (U_{-\frac{\pi}{4}} \otimes I)(\frac{|\Psi_{\frac{\pi}{4}}\Psi_{\frac{\pi}{4}}\rangle + |\Psi_{\frac{3\pi}{4}}\Psi_{\frac{3\pi}{4}}\rangle}{\sqrt{2}})$$
$$= \frac{|\Psi_0\Psi_{\frac{\pi}{4}}\rangle + |\Psi_{\frac{\pi}{2}}\Psi_{\frac{3\pi}{4}}\rangle}{\sqrt{2}} = \frac{|0\rangle\,|\Psi_{\frac{\pi}{4}}\rangle + |1\rangle\,|\Psi_{\frac{3\pi}{4}}\rangle}{\sqrt{2}}$$

Then, Alice measures her qubit in standard basis.

| Outcome of measurement | New state $\|\Psi_2\rangle$ |
|---|---|
| 0 | $\|0\rangle\|\psi_{\frac{\pi}{4}}\rangle$ |
| 1 | $\|1\rangle\|\psi_{\frac{3\pi}{4}}\rangle$ |

Then, as Bob does his action based on his bit $\boldsymbol{y}$,

| $\|\psi_2\rangle$ | Before Bob's measurement $\|\Psi_3\rangle$ | Measurement outcome |
|---|---|---|
| $\|0\rangle\|\psi_{\frac{\pi}{4}}\rangle$ | $(I \otimes U_{\frac{\pi}{8}})\|0\rangle\|\psi_{\frac{\pi}{4}}\rangle = \|0\rangle \otimes (cos(\frac{3\pi}{8})\|0\rangle + sin(\frac{3\pi}{8})\|1\rangle)$ | 1 (chances $sin^2(\frac{3\pi}{8}) \approx 85\%$) else 0 |
| $\|1\rangle\|\psi_{\frac{3\pi}{4}}\rangle$ | $(I \otimes U_{\frac{\pi}{8}})\|1\rangle\|\psi_{\frac{3\pi}{4}}\rangle = \|1\rangle \otimes (cos(\frac{7\pi}{8})\|0\rangle + sin(\frac{7\pi}{8})\|1\rangle)$ | 0 (chances $cos^2(\frac{7\pi}{8}) \approx 85\%$) else 1 |

Thus, it can be seen that with 85% chances, they measure either $(0,1)$ or $(1,0)$, both of which lead to a win.

### Conclusion

In all cases, it was seen that the strategy guarantees approx. 85% chances of winning, which is more than any classical strategy.

## GHZ game

### Description

Participants are Alice, Bob, and Charles, as well as a referee. Before the game starts, Alice, Bob and Charles can prepare any kind of strategy, but after the game starts, they can't communicate with each other. As the game starts, the referee gives bit $x$ to Alice and bit $y$ to Bob and bit $z$ to Charles. Then, Alice has to give another bit $a$ back to the referee and Bob gives bit $b$ to the referee, and Charles gives bit $c$ to the referee. Given that it is promised

$$x \oplus y \oplus z = 0$$

Then to win it is required that

$$a \oplus b \oplus c = x \vee y \vee z$$

It is proved that any classical deterministic or probabilistic strategy can't promise more than 75% chances of winning.

### Strategy based on Quantum Computation

Before the game starts, each holds a qubit of ***GHZ triplet***, where $|GHZ\rangle = \frac{|000\rangle+|111\rangle}{\sqrt{2}}$. Note the basis,

$$B_1 = \{\frac{|0\rangle + |1\rangle}{\sqrt{2}}, \frac{|0\rangle - |1\rangle}{\sqrt{2}}\}$$

$$B_2 = \{\frac{|0\rangle + i\,|1\rangle}{\sqrt{2}}, \frac{|0\rangle - i\,|1\rangle}{\sqrt{2}}\}$$

. All of them act on their qubit based on the following rules:

| x/y/z | Possible actions |
|---|---|
| 0 | Measure in $B_1$ basis / apply $H$ on qubit and measure in standard basis |
| 1 | Measure in $B_2$ basis / apply $HS^\dagger$ on qubit and measure in standard basis |

where $HS^\dagger$ is represented in standard basis as

$$HS^\dagger = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}$$

They then share their measurement outcomes as $(a, b, c)$. The order of participants' actions doesn't matter. This strategy guarantees 100% chances of winning.

### Mathematical Reasoning

First, Listed below are the possible combinations of $(x, y, z)$ and the corresponding winning $(a, b, c)$ combination.

| $(x, y, z)$ | Winning combinations $(a, b, c)$ |
|---|---|
| $(0, 0, 0)$ | $(0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0)$ |
| $(0, 1, 1)$ | $(0, 0, 1), (0, 1, 0), (1, 0, 0), (1, 1, 1)$ |
| $(1, 0, 1)$ | $(0, 0, 1), (0, 1, 0), (1, 0, 0), (1, 1, 1)$ |
| $(1, 1, 0)$ | $(0, 0, 1), (0, 1, 0), (1, 0, 0), (1, 1, 1)$ |

.

Now, let's analyze for each combination of $(x, y, z)$ separately.
**When** $(x, y, z) = (0, 0, 0)$,
The initial state is $|GHZ\rangle$. $(H \otimes H \otimes H)$ is supposed to be applied to the combined state based on the case considered.

$$|\Psi_1\rangle = (H \otimes H \otimes H)\,|GHZ\rangle = (H \otimes H \otimes H)(\frac{|000\rangle + |111\rangle}{\sqrt{2}})$$

$$= \frac{(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) + (|0\rangle - |1\rangle) \otimes (|0\rangle - |1\rangle) \otimes (|0\rangle - |1\rangle)}{4}$$

$$= \frac{|000\rangle + |011\rangle + |101\rangle + |110\rangle}{2}$$

Measurement of these three qubits always gives a winning combination, as listed in the earlier table. Thus, the chances of winning is 100% for this combination of $(x, y, z) = (0, 0, 0)$

**When** $(x, y, z) = (0, 1, 1)$,
The initial state is $|GHZ\rangle$. $(H \otimes HS^\dagger \otimes HS^\dagger)$ is supposed to be applied to the combined state based on the case considered.

$$|\Psi_1\rangle = (H \otimes HS^\dagger \otimes HS^\dagger)\,|GHZ\rangle = (H \otimes HS^\dagger \otimes HS^\dagger)(\frac{|000\rangle + |111\rangle}{\sqrt{2}})$$

$$= \frac{(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) + (|0\rangle - |1\rangle) \otimes (-i\,|0\rangle + i\,|1\rangle) \otimes (-i\,|0\rangle + i\,|1\rangle)}{4}$$

$$= \frac{(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) - (|0\rangle - |1\rangle) \otimes (|0\rangle - |1\rangle) \otimes (|0\rangle - |1\rangle)}{4}$$

$$= \frac{|001\rangle + |100\rangle + |010\rangle + |111\rangle}{2}$$

Measurement of these three qubits always gives a winning combination, as listed in the earlier table.

Thus, the chances of winning is 100% for this combination of $(x, y, z) = (0, 1, 1)$

**When** $(x, y, z) = (1, 0, 1)$,
The initial state is $|GHZ\rangle$. $(HS^\dagger \otimes H \otimes HS^\dagger)$ is supposed to be applied to the combined state based on the case considered.

$$|\Psi_1\rangle = (HS^\dagger \otimes H \otimes HS^\dagger)\,|GHZ\rangle = (HS^\dagger \otimes H \otimes HS^\dagger)(\frac{|000\rangle + |111\rangle}{\sqrt{2}})$$

$$= \frac{(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) + (-i\,|0\rangle + i\,|1\rangle) \otimes (|0\rangle - |1\rangle) \otimes (-i\,|0\rangle + i\,|1\rangle)}{4}$$

$$= \frac{(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) - (|0\rangle - |1\rangle) \otimes (|0\rangle - |1\rangle) \otimes (|0\rangle - |1\rangle)}{4}$$

$$= \frac{|001\rangle + |100\rangle + |010\rangle + |111\rangle}{2}$$

Measurement of these three qubits always gives a winning combination, as listed in the earlier table. Thus, the chances of winning is 100% for this combination of $(x, y, z) = (1, 0, 1)$

**When** $(x, y, z) = (1, 1, 0)$,
The initial state is $|GHZ\rangle$. $(HS^\dagger \otimes HS^\dagger \otimes H)$ is supposed to be applied to the combined state based on the case considered.

$$|\Psi_1\rangle = (HS^\dagger \otimes HS^\dagger \otimes H)\,|GHZ\rangle = (HS^\dagger \otimes HS^\dagger \otimes H)(\frac{|000\rangle + |111\rangle}{\sqrt{2}})$$

$$= \frac{(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) + (-i\,|0\rangle + i\,|1\rangle) \otimes (-i\,|0\rangle + i\,|1\rangle) \otimes (|0\rangle - |1\rangle)}{4}$$

$$= \frac{(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) - (|0\rangle - |1\rangle) \otimes (|0\rangle - |1\rangle) \otimes (|0\rangle - |1\rangle)}{4}$$

$$= \frac{|001\rangle + |100\rangle + |010\rangle + |111\rangle}{2}$$

Measurement of these three qubits always gives a winning combination, as listed in the earlier table. Thus, the chances of winning is 100% for this combination of $(x, y, z) = (1, 1, 0)$

## Conclusion

Thus, in all possible and allowed combinations of $(x, y, z)$, the winning combination of $(a, b, c)$ is guaranteed with 100% chances. This is commendable given that any classical strategy could not guarantee more than 75% chances of winning.