

# Assignment 1 - Homework Exercises on Approximation Algorithms

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## A.I-1

We will show that the approximation ratio of the *GreedySchedulingAlgorithm* is at least  $2 - \frac{1}{m}$  by showing an example as follow.

Let's consider this setting:

- 3 machines: M1, M2, M3
- 7 jobs: 1, 1, 1, 1, 1, 1, 3

*GreedySchedulingAlgorithm* will come up with this scheduling:

- M1: 1 1 3
- M2: 1 1
- M3: 1 1

Thus  $ALG = makespan = 1 + 1 + 3 = 5$

We know that:

$$OPT \geq \max(Average_{load}, \max(tj)) \quad (1)$$

Where

$$Average_{load} = \frac{1}{m} \sum_{i=1}^n j_i = \frac{9}{3} = 3$$

And

$$\max(t_j) = 3$$

Here we can find a solution with  $makespan = 3$ . That is:

- M1: 3
- M2: 1 1 1
- M3: 1 1 1

Therefore,  $OPT = 3$

Thus, the approximation ratio is:

$$\rho = \frac{ALG}{OPT} = \frac{5}{3} \quad (2)$$

According to the theorem, the estimated ratio is:

$$\rho_{estimated} = 2 - \frac{1}{m} = 2 - \frac{1}{3} = \frac{5}{3} \quad (3)$$

From 2 and 3, we have  $\rho_{estimated} = \rho$ . Therefore, this bound is tight.

## A.I-2

From the question, we know that

$$\begin{aligned} m &= 10 \\ \sum_{j=1}^n t_j &\geq 1000 \\ t_j &\in [1, 20] ; \text{ for all } i \leq j \leq n \end{aligned}$$

Let  $T'_{i^*}$  denote the load of  $M_i$  before  $t_j^*$ , last job, is assigned to the machine. Thus  $T_i^*$ , which represents makespan of the assignment, equals to

$$T_i^* = T'_{i^*} + t_j^*$$

Because  $T'_{i^*}$  is the minimum load among all machines, so that we can derive

$$T'_{i^*} \leq \frac{1}{m} \sum_{i=1}^m T'_i = \sum_{j=1}^{j^*} t_j \leq \frac{1}{m} \left[ \sum_{j=1}^n t_j - t_j^* \right] \leq LB$$

Then we can derive

$$\begin{aligned} T_i^* &= T'_{i^*} + t_j^* \\ &\leq \frac{1}{m} \left[ \sum_{j=1}^n t_j - t_j^* \right] + t_j^* \\ &\leq \frac{1}{m} \sum_{j=1}^n t_j + \left(1 - \frac{1}{m}\right) t_j^* \\ &\leq 100 + \left(1 - \frac{1}{10}\right) 20 \\ &\leq 118 \end{aligned}$$

According *Algorithm Greedy Scheduling* and the question, we know

$$\begin{aligned} \max \left( \frac{1}{m} \sum_{j=1}^n t_j, \max_{1 \leq j \leq n} (t_j) \right) &\leq LB \leq OPT \\ \max_{1 \leq j \leq n} (t_j) &= 20 \end{aligned}$$

Then we can derive

$$\max \left( \frac{1}{m} \sum_{j=1}^n t_j, 20 \right) \leq LB$$

Since  $\frac{1}{m} \sum_{j=1}^n t_j \geq 1000$ , thus

$$\begin{aligned} 100 &\leq LB \\ &\leq OPT \end{aligned}$$

Therefore, approximation-ratio( $\rho$ ) equals to

$$\begin{aligned} T_i^* &\leq \rho OPT \\ \frac{118}{100} &\leq \rho \\ 1.18 &\leq \rho \end{aligned}$$

For this particular setting, *Algorithm Greedy Scheduling* is 1.18 approximation algorithm.

### A.I-3

(i)

Assume we have the optimal solution, which has  $n$  squares. That is:

$$n \leq LB \leq OPT$$

Now we put our set of points into a grid with unit-size cells. Each unit square can overlap at most 4 cells in such a grid, then our optimal solution can be split into at most  $4n$  squares. Therefore,

$$ALG \leq 4n = 4.OPT$$

So this algorithm is 4-approximation.

(ii)

We propose algorithm 1.

This algorithm is correct because:

- Every point in  $p$  will be covered by a square
- There are no intersections between the squares because we traverse in one direction

This algorithm consists of 2 parts: QuickSort and Traversing the Point to create squares. Let  $t$  be the run time of this algorithm,  $t_{quicksort}$  be the time for quick-sort, and  $t_{assign}$  be the time for creating the squares. We have:

$$t = t_{quicksort} + t_{assign} \leq n \log n + n = O(n \log n) \quad (4)$$

Thus the runtime of this algorithm is  $O(n \log n)$ .

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**Algorithm 1** Finding minimum row square cover

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**Require:** Set of Points  $P$ **Ensure:** Min Square Cover  $min$ **Operation:**

```
QuickSortAscending( $S$ )
set  $currentCoveringPosition = p_1.x - 1$ 
for all Point  $p$  in  $P$  do
    if  $p.x > currentCoveringPosition$  then
        create square  $s = (p.x, 1, p.x + 1, 0)$ ;
        add  $s$  to  $S$ 
        set  $currentCoveringPosition = p.x + 1$ 
    end if
set  $min = sizeof S$ 
return  $min$ 
end for
```

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(iii)

The idea of our algorithm is that, we put all the points in to a coordinate system, then we divide the coordinate system into a set of unit rows (i.e. rows with height 1). For each row, we use algorithm 1 to find the minimum size square-cover. The global min-square-cover is the sum of all row-square-cover.

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**Algorithm 2** Finding global minimum square cover

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**Input:** Set of points  $P$ **Output:** Min Square Cover  $min$ **Operation:**

```
currentMin = 0;
for all Row  $r$  in the space do
    currentMin += FindRowMinSquare()
end for
set  $min = currentMin$ 
return  $min$ 
```

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**Theorem.** FindingGlobalMinimumSC is 2 – approximation

*Proof.* We prove this Theorem by induction.

If the optimal solution consists of only 1 square, then  $OPT_1 = 1$ . After applying our algorithm, the square can be split into at most 2 squares.

This is true because if the algorithm returns more than 2 squares, then there is a row which consists of more than 1 square. It means the margin of our points is larger than 1, then the optimal must have more than 1 squares to fit them. It contradicts with our assumption that  $OPT_1 = 1$ .

So  $ALG_1 \leq 2 = 2OPT_1$

Suppose when  $n = k$ , the algorithm is true, that is  $ALG_k \leq 2OPT_k$

Now we add some additional points which insert another square into the optimal solution. Now  $n = k + 1$ .

We have  $OPT_{k+1} = OPT_k + 1$

Applying our algorithm, the final result is the sum of the original input ( $n = k$ ) and the new input ( $n = 1$ ). We know that the Theorem holds for both of them. We have:

$$\begin{aligned} ALG_{k+1} &= ALG_k + ALG_1 \\ &\leq 2OPT_k + 2 \\ &= 2(OPT_k + 1) \\ &= 2OPT_{k+1} \end{aligned}$$

Thus the Theorem also holds for  $n = k + 1$ . Therefore, it holds for all values of  $n$ .

In conclusion, this algorithm is 2 - *approximation*.

□

## AII.1

(i)

We prove this statement by contradiction.

- Suppose that  $V \setminus C$  is not an independent set of  $G$ . Then there exists a pair of vertices  $(u, v)$  in  $V \setminus C$  which are connected by an edge  $e \in E$ . Thus, both  $u$  and  $v$  are not in  $C$ . Therefore,  $C$  is not the vertex cover of  $G$  anymore.

- Suppose  $C$  is not the vertex cover of  $G$ , then there exists a pair of vertices  $(u, v)$  that are connected by an edge  $e \in E$  but are not in  $C$ . Thus,  $u \in (V \setminus C)$  and  $v \in (V \setminus C)$ . Therefore,  $(V \setminus C)$  is not the vertex cover of  $G$  anymore.

From the reasoning above, we can state that:  $C$  is the vertex cover of  $G$  if and only if  $V \setminus C$  is an independent set of  $G$ .

(ii)

We prove that *ApproxMaxIndependentSet* is not a 2-approximation algorithm by showing a counter example. That is, consider a complete graph, for example, a graph  $G = (V, E)$  where  $V = x_1, x_2$  and  $E = (x_1, x_2)$ .

Applying the *ApproxMinVertexCover*( $G$ ), we get  $C = x_1, x_2$  (picking both vertices from the edge).

Now we take the approx max independent set  $ALG = V \setminus C = \emptyset$ .

The optimal solution now is  $OPT = 1$  (picking  $x_1$  or  $x_2$ ).

The approximation ratio is  $\rho = \frac{OPT}{ALG} = \infty \neq 2$ .

So the approximation ratio is not 2.

## AII.2

(i)

The best case of the solution is when there is a common variable in every clause, then the optimal solution is that variable. Thus:

$$OPT \geq 1 = LB$$

In this problem, we are not to sure whether the duplication of variables in a DNF clause is allowed or not. If it is allowed, we can define a CNF like this:

$$(a \vee b \vee c) \wedge (a \vee b \vee b) \wedge (a \vee a \vee a)$$

In the worst case, the algorithm picks  $c, b, a$  sequentially. So  $ALG = 3 = n$ . Thus the approximation ratio is  $\frac{n}{LB} = n$ .

This result is not very interesting because  $n$  is also the upper bound of the solution. Indeed,  $(a \vee a)$  can be reduced to only  $a$ , thus the clause is not in the 3-CNF form anymore.

Assume that such duplication is not allowed. Then in the worst case,  $OPT = 1$  as we explained before, while the algorithm can pick at most  $n - 2$  variables. It is because if all of  $n$  variables are used, then the last 2 ones are in some DNF claus(es) which are covered by the previous variables. If not all of them are used, then  $ALG < n - 2$ .

The case when  $ALG = n - 2$  is as follow, we have  $n$  variables and 2 common ones in all clauses.

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge (x_1 \vee x_2 \vee x_5) \wedge \dots \wedge (x_1 \vee x_2 \vee x_n)$$

There are 2 common variables and  $n - 2$  other different variables distributed among the clauses. In the worst case, the algorithm returns  $ALG = n - 2$ . Then the approximation ratio is

$$\rho = \frac{n - 2}{OPT} = n - 2$$

(ii)

For the algorithm to become a 3-approximation algorithm it should be modified so that in each iteration it chooses all three elements in a clause and eliminate all the clauses in CNF that contain any of these three elements.

**Proof**

Let  $D^*$  be a subset of  $D$  that only contains clauses that do not share any variable.

The optimal solution to the problem  $OPT$  contains at least one variable from each clause in  $D^*$  therefore :

$$OPT \geq |D^*|$$

In our algorithm after the modification, we select three variables in each clause in  $D^*$ , because there are no clauses that share common variables with them. The other clauses which have common variables as any clause in  $D^*$



are deleted. Thus:

$$\begin{aligned} ALG &= 3|D^*| \\ &\leq 3OPT \end{aligned}$$

Therefore it is a 3-approximation algorithm.

## AII.3

(i)

Suppose a  $d$ -hypergraph  $G = (V, E)$  which every edge  $e \in E$  is incident to  $d$  vertices in  $V$ . To formulate 0/1 linear programming, we introduce  $X = \{ x_i, x_2, \dots, x_n \}$  which  $x_i$  represents  $v_i \in V$  in a linear programming. If  $x_i = 1$ , it means we pick  $v_i$  to the set of double vertex cover,  $C \subset V$ , and otherwise  $x_i = 0$ . For this solution, we want to find a minimum double vertex cover which requires at least 2 vertices from each edge are in  $C$ . Then, we can derive a constraint for 0/1 linear programming

$$\sum_{v_i \in e} x_i \geq 2 \text{ for all } e \in E ; \text{ at least 2 vertices are selected.}$$

Thus, we then formulate the linear programming.

$$\begin{aligned} &\text{Minimize} && \sum_{i=1}^n x_i \\ &\text{Subject to} && \sum_{v_i \in e} x_i \geq 2 \quad \text{for all } e \in E \\ &&& x_i = \{0,1\} \quad \text{for all } x_i \in X \end{aligned}$$

(ii)

Because we can not solve 0/1 linear program in polynomial time, what we have to do next is to relax the program to be a normal linear program by replacing  $\{0,1\}$  constraint with  $0 \leq x \leq 1$

Thus, the linear program is

$$\begin{array}{ll}
\text{Minimize} & \sum_{x=1}^n x_i \\
\\
\text{Subject to} & \\
& \sum_{v_i \in e} x_i \geq 2 \quad \text{for all } e \in E \\
& 0 \leq x_i \leq 1 \quad \text{for all } x_i \in X
\end{array}$$

Let  $\tau$  denote the rounding threshold such that

$$x_i = \begin{cases} 1, & \text{if } x_i \geq \tau \\ 0, & \text{otherwise} \end{cases}$$

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**Algorithm 3** Finding double vertex cover

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**Input:**  $V, E$

**Output:** A minimum double vertex cover

**Operation:**

Solve the relaxed linear program corresponding to the given problem.

$$\text{Minimize} \quad \sum_{x=1}^n x_i$$

Subject to

$$\sum_{v_i \in e} x_i \geq 2 \quad \text{for all } e \in E$$

$$0 \leq x_i \leq 1 \quad \text{for all } x_i \in X$$

$$C \leftarrow \{ v_i \in V : x_i \geq \tau \}$$

**return** C

---

The next step is to derive  $\tau$  such that all constraints are satisfied and the algorithm always return a valid solution. Let denote  $x^*$  to be an ideal value

of any  $x_i$  such that it satisfies all constraints.

$$\begin{aligned}\sum_{x_i \in e} x_i &\geq 2 \forall e \in E \\ \sum_{i=1}^d x_i &\geq 2 \\ x_1 + \sum_{i=1}^{d-1} x_i &\geq 2\end{aligned}$$

We would like to find a threshold  $\tau$  that is small enough so that at least 2 vertices are selected. The extreme case is when  $x_1 = 1$ , then we have:

$$\begin{aligned}1 + \sum_{i=1}^{d-1} x_i &\geq 2 \\ (d-1)x_i &\geq 1 \\ x_i &\geq \frac{1}{d-1} \\ \therefore \tau &= \frac{1}{d-1}\end{aligned}$$

Let  $W$  denote the value of an optimal to the relaxed linear program and  $OPT$  denote the minimum number of double vertex cover. Then  $OPT \geq W$ .

Now we can derive,

$$\begin{aligned}|C| &= \sum_{v_i \in C} 1 \\ &\leq \sum_{v_i \in C} (d-1)x_i \\ &\leq (d-1) \sum_{v_i \in C} x_i \\ &\leq (d-1)W \\ &\leq (d-1)OPT\end{aligned}$$

(iii)

Lets take an example of a complete 3-hypergraph, where the optimal double vertex cover is  $|V| - 1$  to make sure every edge has at least 2 vertices selected. So the result of the 0/1-LP is  $|V| - 1$ .

The relaxed-LP formulation is as follow:

- Mimize  $\sum_{i=1}^n x_i$
- Subject to:  $\sum_{x_j \in e} x_j \geq 2$  for all edge  $e$  AND  $0 \leq x_i \leq 1$

We run the algorithm by performing that relaxed-LP on the complete 3-hypergraph, and then round the result following the condition  $x \geq \frac{1}{2}$ .

For the complete 3-hypergraph, the relaxed-LP will return  $x_i = \frac{2}{3}$  for all  $i$  so that each sum of vertices in an edge is 2.

Then the algorithm will pick all of the vertices because they satisfy the condition. The result is:

$$ALG = \frac{2}{3}|V|$$

The integrality gap, denoted by  $IG$ , is:

$$\begin{aligned} IG &= \frac{|V| - 1}{\frac{2|V|}{3}} \\ &= \frac{3}{2} - \frac{3}{2|V|} \end{aligned}$$

## A.III-1

(i)

Let  $T = \sum_{j=1}^n t_j$  be the total time of all the jobs. Since we define large job as having time  $t \geq \epsilon T$ , the maximum number of large jobs is:

$$n_{max} = \frac{T}{\epsilon T} = \frac{1}{\epsilon}$$

For each large job, there are two possible ways of assigning it to a machine. Therefore the possible ways that  $\frac{1}{\epsilon}$  jobs can be scheduled into 2 machine is:

$$NumberOfWays = 2^{\frac{1}{\epsilon}}$$

Since machine 1 and machine 2 are identical (e.g.  $0 - n$  and  $n - 0$  are equal), we remove the duplicates leaving the total number of schedules at :

$$\begin{aligned} NumberOfWays &= \frac{2^{\frac{1}{\epsilon}}}{2} \\ &= 2^{\frac{1}{\epsilon}-1} \end{aligned}$$

(ii)

We have

$$OPT \geq \max\left(\frac{T}{2}, t_{max}\right)$$

where  $t_{max}$  is the maximum job size.

To obtain PTAS, We split the jobs up into two types :

$$\text{Job is } \begin{cases} \text{Large if } t_j \geq \epsilon T \\ \text{Small otherwise} \end{cases}$$

The PTAS algorithm runs as follow.

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**Algorithm 4** Load Balancing PTAS

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**Require:** Set  $J$  of  $n$  jobs with running time  $t_j$ ,  $\epsilon$

**Ensure:** Minimum max(makespan) between two machines.

**Operation:**

Define set of large jobs  $L$  where  $t_j \geq \epsilon T \forall t_j \in L$

Define set of small jobs  $S := J \setminus L$

List all possible combinations of  $t_j \in L$  into 2 machines

Select the combination that has the lowest max(makespan)

**for all**  $t_j$  in  $S$  **do**

    Schedule  $t_j$  to the machine which has the lower current makespan

**end for**

**return** The max(makespan) between 2 machines

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**Proof**

Let denote:

- $ALG$  as a makespan generated by our solution
- $T'$  as the makespan in the selected machine before assigning the last job
- $t_{last}$  as the size of the last scheduled job

We come up with:

$$\begin{aligned}
ALG &= T' + t_{last} \\
&\leq \frac{T - t_{last}}{2} + t_{last} \\
&\leq \frac{T}{2} + \frac{\epsilon T}{2} \\
&\leq (1 + \epsilon) \frac{T}{2} \\
&\leq (1 + \epsilon) LB \\
&\leq (1 + \epsilon) OPT
\end{aligned}$$

When  $\epsilon$  is small enough,  $ALG$  will return  $OPT$  solution due to the brute-force approach for large jobs. Then the approximation ratio still holds as  $(1 + \epsilon)$  where  $\epsilon$  is very small ( $\epsilon \rightarrow 0$ ).

**Running Time** Since there can be only at most  $2^{\frac{1}{\epsilon}-1}$  possible large jobs, then the brute-force part of the algorithm will have a running time of  $O(2^{\frac{1}{\epsilon}-1})$ . The greedy scheduling of the large jobs is faster with linear time  $O(n)$  for  $n$  jobs. Leaving a total running time -  $O(2^{\frac{1}{\epsilon}-1} + n)$ , which is polynomial in term of  $n$ . So this algorithm satisfies the condition of a PTAS.

**AIII.2****(i)**

Let  $d$  denote the distance between 2 arbitrary vertices corresponding to  $P$  and  $d^*$  denote the distance after rounding  $p_{i,x}, p_{i,y}$  where  $p_{i,x}$  and  $p_{i,y}$  denote the x- and y-coordinate of  $p_i \in P$ , by  $\Delta$ .

$$d = \sqrt{(p_{i,x} - p_{j,x})^2 + (p_{i,y} - p_{j,y})^2}$$

$$d^* = \sqrt{(p_{i,x^*} - p_{j,x^*})^2 + (p_{i,y^*} - p_{j,y^*})^2}$$

We know that the range of  $p_x^*$  is

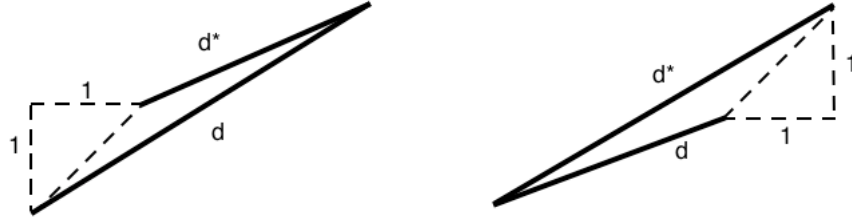
$$\frac{px}{\Delta} \leq p_x^* \leq \frac{px}{\Delta} + 1$$

Then we derive the range of  $d^*$

$$\sqrt{(\frac{p_{i,x}}{\Delta} - (\frac{p_{j,x}}{\Delta} + 1))^2 + (\frac{p_{i,y}}{\Delta} - (\frac{p_{j,y}}{\Delta} + 1))^2} \leq d^* \leq \sqrt{(\frac{p_{i,x}}{\Delta} + 1 - \frac{p_{j,x}}{\Delta})^2 + (\frac{p_{i,y}}{\Delta} + 1 - \frac{p_{j,y}}{\Delta})^2}$$

From the triangle inequality property, such that a, b and c are the length of the triangle edges

$$c \leq a + b$$



We can simplify the range of  $d^*$  to

$$\sqrt{(\frac{p_{i,x}}{\Delta} - \frac{p_{j,x}}{\Delta})^2 + (\frac{p_{i,y}}{\Delta} - \frac{p_{j,y}}{\Delta})^2} - \sqrt{2} \leq d^* \leq \sqrt{(\frac{p_{i,x}}{\Delta} - \frac{p_{j,x}}{\Delta})^2 + (\frac{p_{i,y}}{\Delta} - \frac{p_{j,y}}{\Delta})^2} + \sqrt{2}$$

$$\frac{d}{\Delta} - \sqrt{2} \leq d^* \leq \frac{d}{\Delta} + \sqrt{2}$$

Hence, the error of  $d^*$  is  $2\sqrt{2}$  at most.

The rounded integers are indeed the multiple of  $\Delta$ . Thus the maximum error of a single length is  $2\Delta\sqrt{2}$ .

Here we have  $n$  lengths in the path. Therefore, the total error is:

$$\begin{aligned} 2n\sqrt{2}\Delta &= \varepsilon OPT \\ \Delta &= \frac{\varepsilon OPT}{2n\sqrt{2}} \end{aligned}$$

(ii)

Let  $P$  and  $P^*$  denote set of edges from the optimal solution and the PTAS algorithm respectively and we know that  $length^*(P) \geq length^*(P^*)$ , then we have

$$\sum_{p_i, p_j \in P^*} d_{ij}^* \leq \sum_{p_i, p_j \in P} d_{ij}^*$$

Thus, we can derive



$$\begin{aligned}
length(P^*) &= \sum_{p_i, p_j \in P^*} d_{ij} \\
&\leq \sum_{p_i, p_j \in P^*} \Delta(d_{ij}^* + \sqrt{2}) \\
&\leq \Delta \sum_{p_i, p_j \in P^*} (d_{ij}^* + \sqrt{2}) \\
&\leq \Delta \left( \sum_{p_i, p_j \in P^*} d_{ij}^* + |P^*| \sqrt{2} \right) \\
&\leq \Delta \left( \sum_{p_i, p_j \in P} d_{ij}^* + |P^*| \sqrt{2} \right) \\
&\leq \Delta \left( \sum_{p_i, p_j \in P} \left( \frac{d_{ij}}{\Delta} + \sqrt{2} \right) + n \sqrt{2} \right) \\
&\leq \Delta \left( \sum_{p_i, p_j \in P} \frac{d_{ij}}{\Delta} + 2n \sqrt{2} \right) \\
&\leq \sum_{p_i, p_j \in P} d_{ij} + \Delta 2n \sqrt{2} \\
&\leq length(P) + \Delta 2n \sqrt{2} \\
&\leq OPT + \left( \frac{\varepsilon OPT}{2n \sqrt{2}} \right) 2n \sqrt{2} \\
&\leq (1 + \epsilon) OPT
\end{aligned}$$

(iii)

Let  $m^*$  denote the new boundary of the coordinate after rounding  $p_x, p_y$  to  $p_x^*, p_y^*$  and we also know that

$$\begin{aligned}
m &= \max(p_x, p_y) \\
OPT &\geq 2m
\end{aligned}$$

Thus

$$\frac{m}{\Delta} \leq m^* \leq \frac{m}{\Delta} + 1$$

Then, we can derive the running time

$$\begin{aligned}
m^* &\leq \frac{m}{\Delta} + 1 \\
&\leq \frac{m2n\sqrt{2}}{\epsilon OPT} + 1 \\
&\leq \frac{m2n\sqrt{2}}{\epsilon 2m} + 1 \\
&\leq \frac{n\sqrt{2}}{\epsilon} + 1
\end{aligned}$$

Therefore, the running time is

$$\begin{aligned}
O(nm^*) &= O\left(n \frac{n\sqrt{2}}{\epsilon} + 1\right) \\
&= O\left(\frac{n^2\sqrt{2}}{\epsilon}\right)
\end{aligned}$$

### AIII.3

(i)

Suppose that we have such  $ALG(G, \epsilon)$  that can return a  $(1-\epsilon)$ -approximation solution to the problem. We can define an FPTAS as follow:

- Set  $\epsilon$  to some value
- Return  $ALG(G, \epsilon)$

The above algorithm runs in polynomial time.

Because we know that  $ALG(G, \epsilon) \in \mathbb{N}$ , so that if we can find such  $\epsilon$  that the algorithm yields

$$OPT - 1 < ALG(G, \epsilon) \leq OPT$$

Then, we can get  $OPT$  in polynomial time.

In order to get such  $\epsilon$ , we need:

$$\begin{aligned} \text{ALG}(G, \epsilon) &> \text{OPT} - 1 \\ &> \left(1 - \frac{1}{\text{OPT}}\right) \text{OPT} \end{aligned}$$

This holds when  $\epsilon < \frac{1}{\text{OPT}}$ .

Indeed, we do not know the exact value of  $\text{OPT}$ . But the above property implies that there exists some values of  $\epsilon$  that can help  $\text{ALG}(G, \epsilon)$  return  $\text{OPT}$ . Because our algorithm uses  $\text{ALG}(G, \epsilon)$  as the main routine, so it also runs in polynomial time.

Now we have an algorithm that can return the optimal solution in polynomial time. Then our problem is not NP-Hard anymore. This contradicts with the definition.

Therefore, there is no FPTAS for the problem.

**(ii)**

The proof above indeed implies that there exists some values of  $\epsilon$  that can help any  $(1 - \epsilon) - \text{approx}$  algorithm return  $\text{OPT}$ .

Therefore, if there exists a PTAS for this problem, it also returns  $\text{OPT}$  with such  $\epsilon$  because PTAS also produces  $(1 - \epsilon) - \text{approx}$  solutions. This conflicts with the NP-Hard property of the problem as being explained above.

Therefore there is no PTAS for the problem.