Assignment 1 - Homework Exercises on Approximation Algorithms

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A.I-1

We will show that the approximation ratio of the $GreedySchedulingAlgorithm$ is at least $2 - \frac{1}{m}$ by showing an example as follow.

Let's consider this setting:

- 3 machines: M1, M2, M3
- 7 jobs: 1, 1, 1, 2, 1, 1, 2

$GreedySchedulingAlgorithm$ will come up with this scheduling:

- M1: 1 2 2
- M2: 1 1
- M3: 1 1

Thus $ALG = makespan = 1 + 2 + 2 = 5$

We know that:

$$OPT \geq Average_{load} \tag{1}$$

Where $Average_{load} = \frac{1}{m} \sum_{i=1}^{n} j_i = \frac{9}{3} = 3$

Here we can find a solution with $makespan = 3$. That is:

- M1: 1 2
- M2: 1 2

- M3: 1 1 1

Therefore, $OPT = 3$
Thus, the approximation ratio is:

$$\rho = \frac{ALG}{OPT} = \frac{5}{3} \tag{2}$$

According to the theorem, the estimated ratio is:

$$\rho_{estimated} = 2 - \frac{1}{m} = 2 - \frac{1}{3} = \frac{5}{3} \tag{3}$$

From 2 and 3, we have $\rho_{estimated} = \rho$. Therefore, this bound is tight.

A.I-2

From the question, we know that

$$\begin{aligned} m &= 10 \\ \sum_{j=1}^{n} t_j &\geq 1000 \\ t_j &\in [1, 20]\,; for\, all\, i \leq j \leq n \end{aligned}$$

Let T'_{i^*} denote the load of M_i before t_j^*, last job, is assigned to the machine. Thus T_i^*, which represents makespan of the assignment, equals to

$$T_i^* = T'_{i^*} + t_j^*$$

Because T'_{i^*} is the minimum load among all machines, so that we can derive

$$T'_{i^*} \leq \frac{1}{m}\sum_{i=1}^{m} T'_i = \sum_{j=1}^{j^*} t_j \leq \frac{1}{m}\left[\sum_{j=1}^{n} t_j - t_j^*\right] \leq LB$$

Then we can derive

$$
\begin{aligned}
T_i^* &= T'_{i^*} + t_j^* \\
&\leq \frac{1}{m}\left[\sum_{j=1}^{n} t_j - t_j^*\right] + t_j^* \\
&\leq \frac{1}{m}\sum_{j=1}^{n} t_j + (1 - \frac{1}{m})t_j^* \\
&\leq 100 + (1 - \frac{1}{10})20 \\
&\leq 118
\end{aligned}
$$

According *Algorithm Greedy Scheduling* and the question, we know

$$
max\left(\frac{1}{m}\sum_{j=1}^{n} t_j\,, \underset{1\leq j\leq n}{max}(t_j)\right) \leq LB \leq OPT
$$

$$
\underset{1\leq j\leq n}{max}(t_j) = 20
$$

Then we can derive

$$
max\left(\frac{1}{m}\sum_{j=1}^{n} t_j\,, 20\right) \leq LB
$$

Since $\frac{1}{m}\sum\limits_{j=1}^{n} t_j \geq 1000$, thus

$$
\begin{aligned}
100 &\leq LB \\
&\leq OPT
\end{aligned}
$$

Therefore, approximation-ratio(ρ) equals to

$$
\begin{aligned}
T_i^* &\leq \rho\, OPT \\
\frac{118}{100} &\leq \rho \\
1.18 &\leq \rho
\end{aligned}
$$

For this particular setting, *Algorithm Greedy Scheduling* is 1.18 approximation algorithm.

AI-3-i)

Assume we have the optimal solution, which has n squares

$$n \leq LB \leq OPT$$

Lemma 1. *Each unit square in the grid can overlap at most 4 cells. Let n_s be the number of square in the integer grid solution. Thus*

$$n_s \leq 4n \leq 4OPT$$

A.I-3-ii

We propose the algorithm as follow.

Algorithm 1 Finding minimum row square cover

Require: Set of Points P
Ensure: Min Square Cover min
Operation:
  set $currentCoveringPosition = 0$
  QuickSortAscending(S)
  for all Point p in P **do**
    if $p.x \leq currentCoveringPosition$ **then**
      create square $s = (p.x, 1, p.x + 1, 0)$;
      add s to S
      set $currentCoveringPosition = p.x + 1$
    end if
    set $min = sizeofS$
    return min
  end for

This algorithm is correct because:

- Every point in p will be covered by a square
- There are no intersections between the squares because we traverse in one direction

This algorithm consists of 2 parts: QuickSort and Traversing the Point to create squares. Let t be the run time of this algorithm, $t_{quicksort}$ be the time for quick-sort, and t_{assign} be the time for creating the squares. We have:

$$t = t_{quicksort} + t_{assign} \leq n \log n + n = O(n \log n) \tag{4}$$

Thus the runtime of this algorithm is $O(n \log n)$.

A.I.3.iii

The idea of our algorithm is that, we put all the points in to a coordinate system, then we divide the coordinate system into a set of unit rows (i.e. rows with height 1). For each row, we use algorithm 1 to find the minimum size square-cover. The global min-square-cover is the sum of all row-square-cover.

Algorithm 2 Finding global minimum square cover

Input: Set of points P
Output: Min Square Cover min
Operation:

```
currentMin = 0;
for all Row r in the space do
    currentMin += FindRowMinSquare()
end for
set min = currentMin
      return min
```

Theorem. FindingGlobalMinimumSC is $2 - approximation$

Proof. We prove this Theorem by induction.

If the optimal solution consists of only 1 square, then $OPT_1 = 1$. After applying our algorithm, the square can be split into at most 2 squares.

This is true because if the algorithm returns more than 2 squares, then there is a row which consists of more than 1 square. It means the margin of our points is larger than 1, then the optimal must have more than 1 squares to fit them. It contradicts with our assumption that $OPT_1 = 1$.

So $ALG_1 \leq 2 = 2OPT_1$

Suppose when $n = k$, the algorithm is true, that is $ALG_k \leq 2OPT_k$

Now we add some addtional points which insert another square into the optimal solution. Now $n = k + 1$.

We have $OPT_{k+1} = OPT_k + 1$

Applying our algorithm, the final result is the sum of the original input $(n = k)$ and the new input $(n = 1)$. We know that the Theorem holds for both of them. We have:

$$ALG_{k+1} = ALG_k + ALG_1 \leq 2OPT_k + 2 = 2(OPT_k + 1) = 2OPT_{k+1} \quad (5)$$

Thus the Theorem also holds for $n = k + 1$. Therefore, it holds for all values of n.

In conclusion, this algorithm is $2 - \textit{approximation}$.

□

AII.1

(i)

We prove this statement by contradiction.

- Suppose that $V \setminus C$ is not an independent set of G. Then there exists a pair of vertices (u, v) in $V \setminus CC$ which are connected by an edge $e \in E$. Thus, both u and v are not in C. Therefore, C is not the vertex cover of G anymore.

- Suppose C is not the vertex cover of G, then there exists a pair of vertices (u, v) that are connected by an edge $e \in E$ but are not in C. Thus, $u \in (V \setminus C)$ and $u \in (V \setminus C)$. Therefore, $(V \setminus C)$ is not the vertex cover of G anymore.

From the reasoning above, we can state that: C is the vertex cover of G if and only if $V \setminus C$ is an independent set of G.

(ii)

We prove that $ApproxMaxIndependentSet$ is not a 2-approximation algorithm by showing a counter example. That is, consider a complete graph, for example, a graph $G = (V, E)$ where $V = x_1, x_2$ and $E = (x_1, x_2)$.

Applying the $ApproxMinVertexCover(G)$, we get $C = x_1, x_2$ (picking both vertices from the edge).

Now we take the approx max independent set $ALG = V \setminus C = \emptyset$.

The optimal solution now is $OPT = 1$ (picking x_1 or x_2).

The approximation ratio is $\rho = \frac{OPT}{ALG} = \infty \neq 2$.

So the approximation ratio is not 2.

AII.2

(i)

Considering the input we can define the best possible scenario for the solution as being a CNF formula where a single element picked from the first clause can be found in all other clasues. The algorithm then would remove all other clauses. This leads us to the assumption that the optimal value OPT in our case is equal to $OPT = 1$

Since no assigned method is specified as to which element of the clause we should choose, we assume a random pick. This leads us to the worst possible choice scenario of a variable existing in all the clauses of the CNF and algorithm always picking the wrong one for the search.

$$(x_1 \vee x_4 \vee x_5) \wedge (x_3 \vee x_1 \vee x_6) \wedge (x_2 \vee x_1 \vee x_7)$$

If the algorithm picked in the first clause x_4 as the variable to look for and x_3 too look for in the second clause it would return a value of one element in each clause until the last clause leaving us with an $ALG = \frac{n}{3} - 1$. Since all elements are different but they share one element in all clauses we pick 1 element in each clause meaning $\frac{1}{3}$ of all used elements with exclusion of the first clause. This would however vary greatly from the most optimal solution of $OPT = 1$ as described previously.

That leads to the approximation ratio of :

$$1 \leq \rho \leq \frac{n}{3} - 1$$

AII.3

(i)

Suppose a d-hypergraph $G = (V, E)$ which every edge $e \in E$ incidenst to d vertices in V. To formulate 0/1 linear programming, we introduce $X = \{\ x_i, x_2, \ldots\ , x_n\ \}$ which x_i represents $v_i \in V$ in a linear programming. If $x_i = 1$, it means we pick v_i to the set of double vertex cover, $C \subset V$, and otherwise $x_i = 0$. For this solution, we want to find a minimum double vertex cover which requires at least 2 vertices from each edge are in C. Then, we can derive a constraint for 0/1 linear programming

.

$$\sum_{v_i \in e} x_i \geq 2 \text{ for all } e \in E \ ;$$

Thus, we then formulate the linear programming.

Minimize $\quad \sum_{x=1}^{n} x_i$

Subject to

$\sum_{v_i \in e} x_i \geq 2 \qquad$ for all $e \in E$; at least 2 vertices are selected.

$x_i = \{0{,}1\} \qquad$ for all $x_i \in X$

(ii)

Because we can not solve 0/1 linear program in polynomial time, what we have to do next is to relax the program to be a normal linear program by replacing {0,1} constraint with $0 \leq x \leq 1$

Thus, the linear program is

Minimize $\quad \sum_{x=1}^{n} x_i$

Subject to

$$\sum_{v_i \in e} x_i \geq 2 \qquad \text{for all } e \in E \text{ ; at least 2 vertices are selected.}$$
$$0 \leq x_i \leq 1 \qquad \text{for all } x_i \in X$$

Let τ denote the rounding threshold such that

$$x_i = \begin{cases} 1, & \text{if } x_i \geq \tau \\ 0, & \text{otherwise} \end{cases}$$

Algorithm 3 Finding double vertex cover

Input: V, E

Output: A minimum double vertex cover

Operation:

Solve the relaxed linear program corresponding to the given problem.

Minimize $\sum_{x=1}^{n} x_i$

Subject to

$\sum_{v_i \in e} x_i \geq$ for all $e \in E$

$0 \leq x_i \leq 1$ for all $x_i \in X$

C $\leftarrow \{ v_i \in V : x_i \geq \tau \}$

return C

The next step is to derive τ such that all constraints are satisfied and the algorithm always return a valid solution. Let denote x^* to be an ideal value of any x_i such that it satisfies all constraints.

$$\begin{aligned}
\sum_{v_i \in e} x_i &\geq 2 \\
\sum_{i=1}^{d} x^* &\geq 2 \\
1 + \sum_{i=1}^{d-1} x^* &\geq 2 \\
(d-1)x^* &\geq 1 \\
x^* &\geq \frac{1}{d-1} \\
\therefore \tau &= \frac{1}{d-1}
\end{aligned}$$

Let denote W denote the value of an optimal to the relaxed linear program and OPT denote the minium number of double vertex cover. Then $OPT \geq W$.

Now we can derive,

$$\begin{aligned}
|C| &= \sum_{v_i \in C} 1 \\
&\leq \sum_{v_i \in C} (d-1)x_i \\
&\leq (d-1) \leq \sum_{v_i \in C} x_i \\
&\leq (d-1)W \\
&\leq (d-1)OPT
\end{aligned}$$

(iii)

Lets take an example of a complete 3-hypergraph, where the optimal double vertex cover is $|V|-1$ to make sure every edge has at least 2 vertices selected. So the result of the 0/1-LP is $|V|-1$.

The relaxed-LP formulation is as follow:

- Miminize $\sum_{i=1}^{n} x_i$
- Subject to: $\sum_{x_j \in e} x_j \geq 2$ for all edge e AND $0 \leq x_i \leq 1$

We run the algorithm by performing that relaxed-LP on the complete 3-hypergraph, and then round the result following the condition $x \geq \frac{1}{2}$.

For the complete 3-hypergraph, the relaxed-LP will return $x_i = \frac{2}{3}$ for all i so that each sum of vertices in an edge is 2.

Then the algorithm will pick all of the vertices because they satisfy the condition. The result is:

$$ALG = \frac{2}{3}|V|$$

The integrality gap is:

$$integrality = \frac{|V| - 1}{\frac{2|V|l}{3}} = \frac{3}{2} - \frac{3}{2|V|}$$

AIII.2

(i)

Let d denote the distance between 2 arbitrary vertices corresponding to P and d^* denote the distance after rounding $p_{i,x}, p_{i,y}$ where $p_{i,x}$ and $p_{i,y}$ denote the x- and y-coordinate of $p_i \in P$, by Δ.

$$d = \sqrt{(p_{i,x} - p_{j,x})^2 + (p_{i,y} - p_{j,y})^2}$$
$$d^* = \sqrt{(p_{i,x^*} - p_{j,x^*})^2 + (p_{i,y^*} - p_{j,y^*})^2}$$

We know that the range of p_x^* is

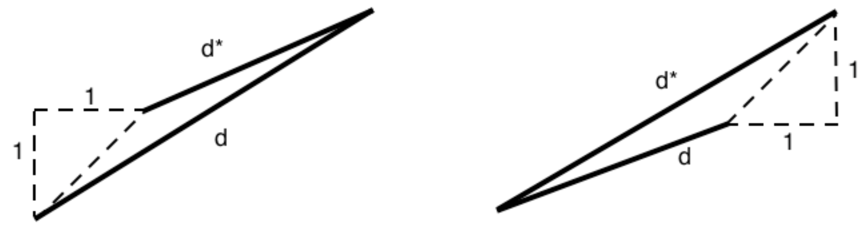
$$\frac{px}{\Delta} \leq p_x^* \leq \frac{px}{\Delta} + 1$$

Then we derive the range of d^*

$$\sqrt{(\frac{p_{i,x}}{\Delta} - (\frac{p_{j,x}}{\Delta} + 1))^2 + (\frac{p_{i,y}}{\Delta} - (\frac{p_{j,y}}{\Delta} + 1)^2} \leq d^* \leq \sqrt{(\frac{p_{i,x}}{\Delta} + 1 - \frac{p_{j,x}}{\Delta})^2 + (\frac{p_{i,y}}{\Delta} + 1 - \frac{p_{j,y}}{\Delta})^2}$$

From the triangle inequality property, such that a, b and c are the lengh of the triangle edges

$$c \leq a + b$$



We can simplify the range of d^* to

$$\sqrt{(\frac{p_{i,x}}{\Delta} - \frac{p_{j,x}}{\Delta})^2 + (\frac{p_{i,y}}{\Delta} - \frac{p_{j,y}}{\Delta})^2} - \sqrt{2} \leq d^* \leq \sqrt{(\frac{p_{i,x}}{\Delta} - \frac{p_{j,x}}{\Delta})^2 + (\frac{p_{i,y}}{\Delta} - \frac{p_{j,y}}{\Delta})^2} + \sqrt{2}$$
$$\frac{d}{\Delta} - \sqrt{2} \leq d^* \leq \frac{d}{\Delta} + \sqrt{2}$$

Hence, the error of d^* is $2\sqrt{2}$ at most.
Therefore,

$$2n\sqrt{2}\Delta = \varepsilon OPT$$
$$\Delta = \frac{\varepsilon OPT}{2n\sqrt{2}}$$

(ii)

Let P and P^* denote set of edges from the optimal solution and the PTAS algorithm respectively and we know that $length^*(T) \geq length^*(T^*)$, then we have

$$\sum_{p_i, p_j \in P^*} d^*_{ij} \leq \sum_{p_i, p_j \in P} d^*_{ij}$$

Thus, we can derive

$$
\begin{aligned}
length(T^*) &= \sum_{p_i, p_j \in P^*} d_{ij} \\
&\leq \sum_{p_i, p_j \in P^*} \Delta(d^*_{ij} + \sqrt{2}) \\
&\leq \Delta \sum_{p_i, p_j \in P^*} (d^*_{ij} + \sqrt{2}) \\
&\leq \Delta \sum_{p_i, p_j \in P} (d^*_{ij} + \sqrt{2}) \\
&\leq \Delta \sum_{p_i, p_j \in P} (\frac{d_{ij}}{\Delta} + 2\sqrt{2}) \\
&\leq \sum_{p_i, p_j \in P} d_{ij} + \Delta \sum_{p_i, p_j \in P} 2\sqrt{2} \\
&\leq length\,(T) + \Delta 2|P|\sqrt{2} \\
&\leq OPT + \left(\frac{\varepsilon OPT}{2n\sqrt{2}}\right) 2n\sqrt{2} \\
&\leq (1+\epsilon) OPT
\end{aligned}
$$

(iii)

Let m^* denote the new boundary of the coordinate after rounding p_x, p_y to p^*_x, p^*_y and we also know that

$$
\begin{aligned}
m &= max(p_x, p_y) \\
OPT &\geq m
\end{aligned}
$$

Thus

$$
\frac{m}{\Delta} \leq m^* \leq \frac{m}{\Delta} + 1
$$

Then, we can derive the running time

$$
\begin{aligned}
m* &\leq \frac{m}{\Delta} + 1 \\
&\leq \frac{m2n\sqrt{2}}{\epsilon OPT} + 1 \\
&\leq \frac{m2n\sqrt{2}}{\epsilon m} + 1 \\
&\leq \frac{2n\sqrt{2}}{\epsilon} + 1
\end{aligned}
$$

Therefore, the running time is

$$
\begin{aligned}
O\left(nm^*\right) &= O\Big(n\frac{2n\sqrt{2}}{\epsilon} + 1\Big) \\
&= O\Big(\frac{2n^2\sqrt{2}}{\epsilon}\Big)
\end{aligned}
$$