Assignment 1 - Homework Exercises on Approximation Algorithms

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A.I-1

We will show that the approximation ratio of the $GreedySchedulingAlgorithm$ is at least $2 - \frac{1}{m}$ by showing an example as follow.

Let's consider this setting:

- 3 machines: M1, M2, M3
- 7 jobs: 1, 1, 1, 2, 1, 1, 2

$GreedySchedulingAlgorithm$ will come up with this scheduling:

- M1: 1 2 2
- M2: 1 1
- M3: 1 1

Thus $ALG = makespan = 1 + 2 + 2 = 5$
We know that:

$$OPT \geq Average_{load} \tag{1}$$

Where $Average_{load} = \frac{1}{m} \sum_{i=1}^{n} j_i = \frac{9}{3} = 3$
Here we can find a solution with $makespan = 3$. That is:

- M1: 1 2
- M2: 1 2

- M3: 1 1 1

Therefore, $OPT = 3$
Thus, the approximation ratio is:

$$\rho = \frac{ALG}{OPT} = \frac{5}{3} \tag{2}$$

According to the theorem, the estimated ratio is:

$$\rho_{estimated} = 2 - \frac{1}{m} = 2 - \frac{1}{3} = \frac{5}{3} \tag{3}$$

From 2 and 3, we have $\rho_{estimated} = \rho$. Therefore, this bound is tight.

A.I-2

From the question, we know that

$$\begin{aligned} m &= 10 \\ \sum_{j=1}^{n} t_j &\geq 1000 \\ t_j &\in [1, 20]\,; for\, all\, i \leq j \leq n \end{aligned}$$

Let T'_{i^*} denote the load of M_i before t^*_j, last job, is assigned to the machine. Thus T^*_i, which represents makespan of the assignment, equals to

$$T^*_i = T'_{i^*} + t^*_j$$

Because T'_{i^*} is the minimum load among all machines, so that we can derive

$$T'_{i^*} \leq \frac{1}{m}\sum_{i=1}^{m} T'_i = \sum_{j=1}^{j^*} t_j \leq \frac{1}{m}\left[\sum_{j=1}^{n} t_j - t^*_j\right] \leq LB$$

Then we can derive

$$\begin{aligned} T_i^* &= T'_{i^*} + t_j^* \\ &\leq \frac{1}{m}\left[\sum_{j=1}^{n} t_j - t_j^*\right] + t_j^* \\ &\leq \frac{1}{m}\sum_{j=1}^{n} t_j + (1 - \frac{1}{m})t_j^* \\ &\leq 100 + (1 - \frac{1}{10})20 \\ &\leq 118 \end{aligned}$$

According *Algorithm Greedy Scheduling* and the question, we know

$$max\left(\frac{1}{m}\sum_{j=1}^{n} t_j\,, \underset{1\leq j\leq n}{max}(t_j)\right) \leq LB \leq OPT$$
$$\underset{1\leq j\leq n}{max}(t_j) = 20$$

Then we can derive

$$max\left(\frac{1}{m}\sum_{j=1}^{n} t_j\,, 20\right) \leq LB$$

Since $\frac{1}{m}\sum_{j=1}^{n} t_j \geq 1000$, thus

$$\begin{aligned} 100 &\leq LB \\ &\leq OPT \end{aligned}$$

Therefore, approximation-ratio(ρ) equals to

$$\begin{aligned} T_i^* &\leq \rho\, OPT \\ \frac{118}{100} &\leq \rho \\ 1.18 &\leq \rho \end{aligned}$$

For this particular setting, *Algorithm Greedy Scheduling* is 1.18 approximation algorithm.

A.I-3

(i)

Assume we have the optimal solution, which has n squares. That is:

$$n \leq LB \leq OPT$$

Now we put our set of points into a grid with unit-size cells. Each unit square can overlap at most 4 cells in such a grid, then our optimal solution can be split into at most $4n$ squares. Therefore,

$$ALG \leq 4n = 4.OPT$$

So this algorithm is 4-approximation.

(ii)

We propose the algorithm as follow.

Algorithm 1 Finding minimum row square cover

Require: Set of Points P
Ensure: Min Square Cover min
Operation:

```
set currentCoveringPosition = −1
QuickSortAscending(S)
for all Point p in P do
    if p.x > currentCoveringPosition then
        create square s = (p.x, 1, p.x + 1, 0);
        add s to S
        set currentCoveringPosition = p.x + 1
    end if
    set min = sizeofS
     return min
end for
```

This algorithm is correct because:

- Every point in p will be covered by a square

- There are no intersections between the squares because we traverse in one direction

This algorithm consists of 2 parts: QuickSort and Traversing the Point to create squares. Let t be the run time of this algorithm, $t_{quicksort}$ be the time for quick-sort, and t_{assign} be the time for creating the squares. We have:

$$t = t_{quicksort} + t_{assign} \leq n \log n + n = O(n \log n) \tag{4}$$

Thus the runtime of this algorithm is $O(n \log n)$.

(iii)

The idea of our algorithm is that, we put all the points in to a coordinate system, then we divide the coordinate system into a set of unit rows (i.e. rows with height 1). For each row, we use algorithm 1 to find the minimum size square-cover. The global min-square-cover is the sum of all row-square-cover.

Algorithm 2 Finding global minimum square cover

Input: Set of points P
Output: Min Square Cover *min*
Operation:

```
currentMin = 0;
for all Row r in the space do
    currentMin += FindRowMinSquare()
end for
set min = currentMin
    return min
```

Theorem. FindingGlobalMinimumSC is $2 - approximation$

Proof. We prove this Theorem by induction.

If the optimal solution consists of only 1 square, then $OPT_1 = 1$. After applying our algorithm, the square can be split into at most 2 squares.

This is true because if the algorithm returns more than 2 squares, then there is a row which consists of more than 1 square. It means the margin of our points is larger than 1, then the optimal must have more than 1 squares to fit them. It contradicts with our assumption that $OPT_1 = 1$.

So $ALG_1 \leq 2 = 2OPT_1$

Suppose when $n = k$, the algorithm is true, that is $ALG_k \leq 2OPT_k$

Now we add some addtional points which insert another square into the optimal solution. Now $n = k + 1$.

We have $OPT_{k+1} = OPT_k + 1$

Applying our algorithm, the final result is the sum of the original input ($n = k$) and the new input ($n = 1$). We know that the Theorem holds for both of them. We have:

$$ALG_{k+1} = ALG_k + ALG_1 \leq 2OPT_k + 2 = 2(OPT_k + 1) = 2OPT_{k+1} \quad (5)$$

Thus the Theorem also holds for $n = k + 1$. Therefore, it holds for all values of n.

In conclusion, this algorithm is $2 - \textit{approximation}$.

□

AII.1

(i)

We prove this statement by contradiction.

- Suppose that $V \setminus C$ is not an independent set of G. Then there exists a pair of vertices (u, v) in $V \setminus C$ which are connected by an edge $e \in E$. Thus, both u and v are not in C. Therefore, C is not the vertex cover of G anymore.

- Suppose C is not the vertex cover of G, then there exists a pair of vertices (u, v) that are connected by an edge $e \in E$ but are not in C. Thus, $u \in (V \setminus C)$ and $u \in (V \setminus C)$. Therefore, $(V \setminus C)$ is not the vertex cover of G anymore.

From the reasoning above, we can state that: C is the vertex cover of G if and only if $V \setminus C$ is an independent set of G.

(ii)

We prove that $ApproxMaxIndependentSet$ is not a 2-approximation algorithm by showing a counter example. That is, consider a complete graph, for example, a graph $G = (V, E)$ where $V = x_1, x_2$ and $E = (x_1, x_2)$.

Applying the $ApproxMinVertexCover(G)$, we get $C = x_1, x_2$ (picking both vertices from the edge).

Now we take the approx max independent set $ALG = V \setminus C = \emptyset$.

The optimal solution now is $OPT = 1$ (picking x_1 or x_2).

The approximation ratio is $\rho = \frac{OPT}{ALG} = \infty \neq 2$.

So the approximation ratio is not 2.

AII.2

(i)

The best possible scenario in the presented case is similar to following:

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_4 \vee x_5) \wedge (x_1 \vee x_6 \vee x_7)$$

It can be seen that in this case x_1 is present in all the clauses of the CNF therefore we can eliminate all the clauses of the equation in the first run. This gives us a following lower bound:

$$LB = 1$$

We deduce then:

$$OPT \geq 1$$

We define approximation ratio ρ as:

$$\rho = \frac{ALG}{OPT}$$

Since we don't know if duplication of elements in a clause is allowed or not we will examine two possible scenarios.

Duplication allowed:

Consider the case when:

$$(x_1 \vee x_2 \vee x_2) \wedge (x_1 \vee x_3 \vee x_3) \wedge (x_1 \vee x_4 \vee x_4) ... \wedge (x_1 \vee x_n \vee x_n)$$

We choose in each iteration elements unique for the clause like $x_2, x_3, x_4 ... x_n$. So we end up having chosen n elements, excluding the one that was common in all the clauses. That gives us:

$$\begin{aligned} ALG &= (n-1) \\ \rho &= (n-1) \end{aligned}$$

Duplication disallowed:

If duplication is not allowed as in the following CNF:

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge (x_1 \vee x_2 \vee x_5) ... \wedge (x_1 \vee x_2 \vee x_n)$$

We can see that the algorithm chooses $n-2$ resulting in :

$$\begin{aligned} ALG &= (n-2) \\ \rho &= (n-2) \end{aligned}$$

(ii)

For the algorithm to become a 3-approximation algorithm it should be modified so that in each iteration it chooses all three elements in a clause and eliminate all the clauses in CNF that contain any of these three elements.

Proof

Let D^* be a subset of D that only contains clauses that don't share any variables.

The optimal solution to the problem OPT contains at least one variable from each clause therefore :

$$OPT \geq |D^*|$$

In our algorithm after the modification, we select three variables in each clause in D^*, because there are no clauses that share common variables with them. The other clauses which have common variables as any clause in D^* are deleted. Thus:

$$
\begin{aligned}
ALG &= 3|D^*| \\
&\leq 3OPT
\end{aligned}
$$

Therefore it is a 3-approximation algorithm.

AII.3

(i)

Suppose a d-hypergraph $G = (V, E)$ which every edge $e \in E$ incidenst to d vertices in V. To formulate 0/1 linear programming, we introduce $X = \{ x_i, x_2, \ldots , x_n \}$ which x_i represents $v_i \in V$ in a linear programming. If $x_i = 1$, it means we pick v_i to the set of double vertex cover, $C \subset V$, and otherwise $x_i = 0$. For this solution, we want to find a minimum double vertex cover which requires at least 2 vertices from each edge are in C. Then, we can derive a constraint for 0/1 linear programming

.

$$\sum_{v_i \in e} x_i \geq 2 \text{ for all } e \in E \text{ ;}$$

Thus, we then formulate the linear programming.

Minimize $\qquad \sum_{x=1}^{n} x_i$

Subject to

$$
\begin{aligned}
&\sum_{v_i \in e} x_i \geq 2 && \text{for all } e \in E \text{ ; at least 2 vertices are selected.} \\
&x_i = \{0,1\} && \text{for all } x_i \in X
\end{aligned}
$$

(ii)

Because we can not solve 0/1 linear program in polynomial time, what we have to do next is to relax the program to be a normal linear program by replacing {0,1} constraint with $0 \leq x \leq 1$

Thus, the linear program is

$$\begin{array}{ll} \text{Minimize} & \sum_{x=1}^{n} x_i \\ \text{Subject to} & \\ \quad \sum_{v_i \in e} x_i \geq 2 & \text{for all } e \in E \text{ ; at least 2 vertices are selected.} \\ \quad 0 \leq x_i \leq 1 & \text{for all } x_i \in X \end{array}$$

Let τ denote the rounding threshold such that

$$x_i = \begin{cases} 1, & \text{if} x_i \geq \tau \\ 0, & \text{otherwise} \end{cases}$$

Algorithm 3 Finding double vertex cover

Input: V, E
Output: A minimum double vertex cover
Operation:

Solve the relaxed linear program corresponding to the given problem.

Minimize $\sum_{x=1}^{n} x_i$

Subject to

$\sum_{v_i \in e} x_i \geq$ for all $e \in E$

$0 \leq x_i \leq 1$ for all $x_i \in X$

C $\leftarrow$ { $v_i \in V$: $x_i \geq \tau$ }

return C

The next step is to derive τ such that all constraints are satisfied and the algorithm always return a valid solution. Let denote x^* to be an ideal value of any x_i such that it satisfies all constraints.

$$
\begin{aligned}
\sum_{v_i \in e} x_i &\geq 2 \\
\sum_{i=1}^{d} x_i &\geq 2 \\
1 + \sum_{i=1}^{d-1} x^i &\geq 2 \\
(d-1)x^i &\geq 1 \\
x_i &\geq \frac{1}{d-1} \\
\therefore \tau &= \frac{1}{d-1}
\end{aligned}
$$

Let denote W denote the value of an optimal to the relaxed linear program and OPT denote the minium number of double vertex cover. Then $OPT \geq W$.

Now we can derive,

$$
\begin{aligned}
|C| &= \sum_{v_i \in C} 1 \\
&\leq \sum_{v_i \in C} (d-1)x_i \\
&\leq (d-1) \sum_{v_i \in C} x_i \\
&\leq (d-1)W \\
&\leq (d-1)OPT
\end{aligned}
$$

(iii)

Lets take an example of a complete 3-hypergraph, where the optimal double vertex cover is $|V|-1$ to make sure every edge has at least 2 vertices selected. So the result of the 0/1-LP is $|V|-1$.

The relaxed-LP formulation is as follow:

- Miminize $\sum_{i=1}^{n} x_i$
- Subject to: $\sum_{x_j \in e} x_j \geq 2$ for all edge e AND $0 \leq x_i \leq 1$

We run the algorithm by performing that relaxed-LP on the complete 3-hypergraph, and then round the result following the condition $x \geq \frac{1}{2}$.

For the complete 3-hypergraph, the relaxed-LP will return $x_i = \frac{2}{3}$ for all i so that each sum of vertices in an edge is 2.

Then the algorithm will pick all of the vertices because they satisfy the condition. The result is:

$$ALG = \frac{2}{3}|V|$$

The integrality gap, denoted by IG, is:

$$\begin{aligned} IG &= \frac{|V| - 1}{\frac{2|V|}{3}} \\ &= \frac{3}{2} - \frac{3}{2|V|} \end{aligned}$$

AIII-1-i)

We assume total time $T = \sum_{j=1}^{n} t_i$ to be the total time of all the jobs. Since we define large job as having time $t \geq \epsilon T$, we can deduce that if all runing jobs are large jobs, the maximal number of those jobs is equal to :

$$n_{max} = \frac{T}{\epsilon T} = \frac{1}{\epsilon}$$

For each job we take into account there are two possible ways of assigning it to a machine. Therefore the possible ways jobs can be scheduled is:

$$2 * 2 * ... * 2 = 2^{\frac{1}{\epsilon}}$$

Since we don't distinguish between the machines, we remove the duplicates leaving the total number of schedules at :

$$\frac{2^{\frac{1}{\epsilon}}}{2} = 2^{\frac{1}{\epsilon} - 1}$$

AIII-1-ii)

Let's denote a total sum of the running time of all jobs as T and running time of the longest job as t_{max}. So $OPT \geq max(\frac{T}{2}, t_{max})$, then $LB = max(\frac{T}{2}, t_{max})$.

To obtain PTAS, first we classify the jobs in the schedule according to problem description We split them up into two sizes :

$$\text{Job is } \begin{cases} \text{Large if } t_j \geq \epsilon T \\ \text{Small if } t_j < \epsilon T \end{cases}$$

Furthermore we can solve the problem for the large jobs by taking all of them and finding a solution using a brute-force method (first generate all possible schedules and then find the makespan for all of them, choose the one with smallest). This leaves us with small jobs to schedule. We can do this using a greedy approach. Simply we take all the small jobs and for each of them we assign it to the machine with the lowest load.

Algorithm 4 Load Balancing PTAS

Require: Set of j jobs with running time t_j

Ensure: Scheduling scenario with a lowest maximum running time of the two machine

Operation:

Split the input into set of small jobs S and a set of large jobs L based on the definition.

Generate a list of all possible scheduling scenarios for large jobs $LargeSchedules$.

Establish $MinMakeSpan := \infty$ and $BestSchedule := null$

for all Scenario $Scen$ in $LargeSchedules$ **do**

  Calculate minimal makespan of the two machines l_{min}

  if $l_{min} \leq MinMakeSpan$ **then**

    $MinMakeSpan := s_{min}$ and $BestSchedule := Scen$

  end if

end for

for all Small Job s in S **do**

  Schedule s to the machine with the lower current load

end for

return BestSchedule

Proof

To prove that our algorithm is indeed a PTAS we need to show that it runs in polynomial time, and the following statement is true :

$$ALG \leq (1+\epsilon)OPT$$

Since part one of the algorithm will always return an optimal solution (it checks all the possible solutions) we can skip it and concentrate instead on the greedy solution to scheduling the small jobs.

If we denote

- M as a makespan generated by our solution
- T total running time of all jobs.
- T' as the running time in our solution before assigning the last job
- t_{last} as the running time of the last scheduled job

We come up with :

$$M = T' + t_{last} \leq \frac{T - t_{last}}{2} + t_{last} \leq \frac{T}{2} + \frac{\epsilon T}{2} \leq (1+\epsilon)\frac{T}{2} \leq (1+\epsilon)LB \leq (1+\epsilon)OPT$$

When ϵ is small enough, there are no small jobs in the schedule, we fallback to the brute-force solution for Large Jobs, therefore the solution is always OPT for that case. Then the approximation ratio still holds as $(1+\epsilon)$ where ϵ is very small ($\epsilon \to 0$).

Running Time Since there can be only at most $2^{\frac{1}{\epsilon}-1}$ possible large jobs, then the brute-force part of the algorithm will have a running time of $O(2^{\frac{1}{\epsilon}-1})$. The greedy scheduling of the large jobs is faster with linear time $O(n)$ for n jobs. Leaving a total running time - $O(2^{\frac{1}{\epsilon}-1} + n)$, which is polynomial in term of n. So this algorithm satisfies the condition of a PTAS.

AIII.2

(i)

Let d denote the distance between 2 arbitrary vertices corresponding to P and d^* denote the distance after rounding $p_{i,x}, p_{i,y}$ where $p_{i,x}$ and $p_{i,y}$ denote the x- and y-coordinate of $p_i \in P$, by Δ.

$$d = \sqrt{(p_{i,x} - p_{j,x})^2 + (p_{i,y} - p_{j,y})^2}$$
$$d^* = \sqrt{(p_{i,x^*} - p_{j,x^*})^2 + (p_{i,y^*} - p_{j,y^*})^2}$$

We know that the range of p_x^* is

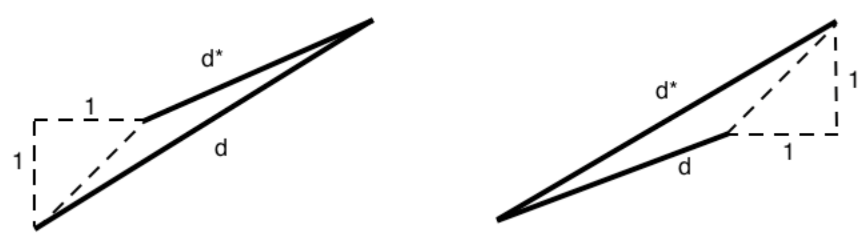
$$\frac{px}{\Delta} \leq p_x^* \leq \frac{px}{\Delta} + 1$$

Then we derive the range of d^*

$$\sqrt{(\frac{p_{i,x}}{\Delta} - (\frac{p_{j,x}}{\Delta} + 1))^2 + (\frac{p_{i,y}}{\Delta} - (\frac{p_{j,y}}{\Delta} + 1)^2} \leq d^* \leq \sqrt{(\frac{p_{i,x}}{\Delta} + 1 - \frac{p_{j,x}}{\Delta})^2 + (\frac{p_{i,y}}{\Delta} + 1 - \frac{p_{j,y}}{\Delta})^2}$$

From the triangle inequality property, such that a, b and c are the lengh of the triangle edges

$$c \leq a + b$$



We can simplify the range of d^* to

$$\sqrt{(\frac{p_{i,x}}{\Delta} - \frac{p_{j,x}}{\Delta})^2 + (\frac{p_{i,y}}{\Delta} - \frac{p_{j,y}}{\Delta})^2} - \sqrt{2} \leq d^* \leq \sqrt{(\frac{p_{i,x}}{\Delta} - \frac{p_{j,x}}{\Delta})^2 + (\frac{p_{i,y}}{\Delta} - \frac{p_{j,y}}{\Delta})^2} + \sqrt{2}$$
$$\frac{d}{\Delta} - \sqrt{2} \leq d^* \leq \frac{d}{\Delta} + \sqrt{2}$$

Hence, the error of d^* is $2\sqrt{2}$ at most.
Therefore,

$$
\begin{aligned}
2n\sqrt{2}\Delta &= \varepsilon OPT \\
\Delta &= \frac{\varepsilon OPT}{2n\sqrt{2}}
\end{aligned}
$$

(ii)

Let P and P^* denote set of edges from the optimal solution and the PTAS algorithm respectively and we know that $length^*(T) \geq length^*(T^*)$, then we have

$$
\sum_{p_i, p_j \in P^*} d^*_{ij} \leq \sum_{p_i, p_j \in P} d^*_{ij}
$$

Thus, we can derive

$$
\begin{aligned}
length(T^*) &= \sum_{p_i, p_j \in P^*} d_{ij} \\
&\leq \sum_{p_i, p_j \in P^*} \Delta(d^*_{ij} + \sqrt{2}) \\
&\leq \Delta \sum_{p_i, p_j \in P^*} (d^*_{ij} + \sqrt{2}) \\
&\leq \Delta \sum_{p_i, p_j \in P} (d^*_{ij} + \sqrt{2}) \\
&\leq \Delta \sum_{p_i, p_j \in P} (\frac{d_{ij}}{\Delta} + 2\sqrt{2}) \\
&\leq \sum_{p_i, p_j \in P} d_{ij} + \Delta \sum_{p_i, p_j \in P} 2\sqrt{2} \\
&\leq length\,(T) + \Delta 2|P|\sqrt{2} \\
&\leq OPT + \left(\frac{\varepsilon OPT}{2n\sqrt{2}}\right) 2n\sqrt{2} \\
&\leq (1 + \epsilon) OPT
\end{aligned}
$$

(iii)

Let m^* denote the new boundary of the coordinate after rounding p_x, p_y to p_x^*, p_y^* and we also know that

$$\begin{aligned} m &= max(p_x, p_y) \\ OPT &\geq 2m \end{aligned}$$

Thus

$$\frac{m}{\Delta} \leq m^* \leq \frac{m}{\Delta} + 1$$

Then, we can derive the running time

$$\begin{aligned} m* &\leq \frac{m}{\Delta} + 1 \\ &\leq \frac{m2n\sqrt{2}}{\epsilon OPT} + 1 \\ &\leq \frac{m2n\sqrt{2}}{\epsilon 2m} + 1 \\ &\leq \frac{n\sqrt{2}}{\epsilon} + 1 \end{aligned}$$

Therefore, the running time is

$$\begin{aligned} O\left(nm^*\right) &= O\Big(n\frac{n\sqrt{2}}{\epsilon} + 1\Big) \\ &= O\Big(\frac{n^2\sqrt{2}}{\epsilon}\Big) \end{aligned}$$

AIII.3

(i)

Because we know that $ALG(G, \epsilon) \in \mathbb{N}$, so that if we can find such ϵ that the algorithm yields

$$OPT - 1 < ALG\left(G, \epsilon\right) \leq OPT$$

Then, we can get OPT in polynomial time.

In order to get such ϵ, we will derive

$$\begin{aligned} ALG\left(G,\epsilon\right) &> OPT - 1 \\ &> \left(1 - \frac{1}{OPT}\right) OPT \end{aligned}$$

Hence we can get OPT if we choose $\epsilon < \frac{1}{OPT}$ and we also know that the algorithm uses $ALG(G, \epsilon)$ as a subroutine.

Therefore, there is no such FPTAS exist.

(ii)

The proof above indeed implies that there is no PTAS such a problem because we know that a PTAS algorithm also computes a $(1-\epsilon)$-approximation for the problem and if we choose $\epsilon > \frac{1}{OPT}$ as the proof above then, the PTAS algorithm will yield OPT in polynomial time of n.

Therefore there is no PTAS exist anymore.