Assignment 1 - Homework Exercises on Approximation Algorithms

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A.I-1

We will show that the approximation ratio of the $GreedySchedulingAlgorithm$ is at least $2 - \frac{1}{m}$ by showing an example as follow.

Let's consider this setting:

- 3 machines: M1, M2, M3
- 7 jobs: 1, 1, 1, 1, 1, 1, 3

$GreedySchedulingAlgorithm$ will come up with this scheduling:

- M1: 1 1 3
- M2: 1 1
- M3: 1 1

Thus $ALG = makespan = 1 + 1 + 3 = 5$
We know that:

$$OPT \geq max(Average_{load}, max(tj)) \tag{1}$$

Where

$$Average_{load} = \frac{1}{m} \sum_{i=1}^{n} j_i = \frac{9}{3} = 3$$

And

$$max(tj) = 3$$

Here we can find a solution with $makespan = 3$. That is:

- M1: 3
- M2: 1 1 1
- M3: 1 1 1

Therefore, $OPT = 3$
Thus, the approximation ratio is:

$$\rho = \frac{ALG}{OPT} = \frac{5}{3} \tag{2}$$

According to the theorem, the estimated ratio is:

$$\rho_{estimated} = 2 - \frac{1}{m} = 2 - \frac{1}{3} = \frac{5}{3} \tag{3}$$

From 2 and 3, we have $\rho_{estimated} = \rho$. Therefore, this bound is tight.

A.I-2

From the question, we know that

$$\begin{aligned} m &= 10 \\ \sum_{j=1}^{n} t_j &\geq 1000 \\ t_j &\in [1, 20]\,; for\, all\, i \leq j \leq n \end{aligned}$$

Let T'_{i^*} denote the load of M_i before t^*_j, last job, is assigned to the machine. Thus T^*_i, which represents makespan of the assignment, equals to

$$T^*_i = T'_{i^*} + t^*_j$$

Because T'_{i^*} is the minimum load among all machines, it is less than the average of machine loads (excluding t^*_j).

$$T'_{i^*} \leq \frac{1}{m}\left[\sum_{j=1}^{n} t_j - t_j^*\right]$$

Then we can derive

$$\begin{aligned} T_i^* &= T'_{i^*} + t_j^* \\ &\leq \frac{1}{m}\left[\sum_{j=1}^{n} t_j - t_j^*\right] + t_j^* \\ &\leq \frac{1}{m}\sum_{j=1}^{n} t_j + (1 - \frac{1}{m})t_j^* \\ &\leq 100 + (1 - \frac{1}{10})20 \\ &\leq 118 \end{aligned}$$

According *Algorithm Greedy Scheduling* and the question, we know

$$max\left(\frac{1}{m}\sum_{j=1}^{n} t_j \, , \max_{1\leq j\leq n}(t_j)\right) \leq LB \leq OPT$$

$$\max_{1\leq j\leq n}(t_j) = 20$$

Then we can derive

$$max\left(\frac{1}{m}\sum_{j=1}^{n} t_j \, , 20\right) \leq LB$$

Since $\frac{1}{m}\sum_{j=1}^{n} t_j \geq 1000$, thus

$$\begin{aligned} 100 &\leq LB \\ &\leq OPT \end{aligned}$$

Therefore, approximation-ratio(ρ) equals to

$$\begin{aligned} T_i^* &\leq \rho\, OPT \\ \frac{118}{100} &\leq \rho \\ 1.18 &\leq \rho \end{aligned}$$

For this particular setting, *Algorithm Greedy Scheduling* is 1.18 approximation algorithm.

A.I-3

(i)

Assume we have the optimal solution, which has n squares. That is:

$$n \leq LB \leq OPT$$

Now we put our set of points into a grid with unit-size cells. Each unit square can overlap at most 4 cells in such a grid, then our optimal solution can be split into at most $4n$ squares. Therefore,

$$ALG \leq 4n = 4.OPT$$

So this algorithm is 4-approximation.

(ii)

We propose algorithm 1.

Algorithm 1 Finding minimum row square cover

Require: Set of Points P
Ensure: Min Square Cover min
Operation:

```
QuickSortAscending(S)
set currentCoveringPosition = p_1.x − 1
for all Point p in P do
    if p.x > currentCoveringPosition then
        create square s = (p.x, 1, p.x + 1, 0);
        add s to S
        set currentCoveringPosition = p.x + 1
    end if
    set min = sizeofS
     return min
end for
```

This algorithm is correct because:

- Every point in p will be covered by a square
- There are no intersections between the squares because we traverse in one direction

This algorithm consists of 2 parts: QuickSort and Traversing the Point to create squares. Let t be the run time of this algorithm, $t_{quicksort}$ be the time for quick-sort, and t_{assign} be the time for creating the squares. We have:

$$t = t_{quicksort} + t_{assign} \leq n \log n + n = O(n \log n) \tag{4}$$

Thus the runtime of this algorithm is $O(n \log n)$.

(iii)

The idea of our algorithm is that, we put all the points in to a coordinate system, then we divide the coordinate system into a set of unit rows (i.e. rows with height 1). For each row, we use algorithm 1 to find the minimum size square-cover. The global min-square-cover is the sum of all row-square-cover.

Algorithm 2 Finding global minimum square cover

Input: Set of points P
Output: Min Square Cover min
Operation:
```
currentMin = 0;
for all Row r in the space do
    currentMin += FindRowMinSquare()
end for
set min = currentMin
return min
```

Theorem. FindingGlobalMinimumSC is $2 - approximation$

Proof. We prove this Theorem by induction.

If the optimal solution consists of only 1 square, then $OPT_1 = 1$. After applying our algorithm, the square can be split into at most 2 squares.

This is true because if the algorithm returns more than 2 squares, then there is a row which consists of more than 1 square. It means the margin of our points is larger than 1, then the optimal must have more than 1 squares to fit them. It contradicts with our assumption that $OPT_1 = 1$.

So $ALG_1 \leq 2 = 2OPT_1$

Suppose when $n = k$, the algorithm is true, that is $ALG_k \leq 2OPT_k$

Now we add some addtional points which insert another square into the optimal solution. Now $n = k + 1$.

We have $OPT_{k+1} = OPT_k + 1$

Applying our algorithm, the final result is the sum of the original input ($n = k$) and the new input ($n = 1$). We know that the Theorem holds for both of them. We have:

$$\begin{aligned} ALG_{k+1} &= ALG_k + ALG_1 \\ &\leq 2OPT_k + 2 \\ &= 2(OPT_k + 1) \\ &= 2OPT_{k+1} \end{aligned}$$

Thus the Theorem also holds for $n = k + 1$. Therefore, it holds for all values of n.

In conclusion, this algorithm is $2 - approximation$.

□

AII.1

(i)

We prove this statement by contradiction.

- Suppose that $V \setminus C$ is not an independent set of G. Then there exists a pair of vertices (u, v) in $V \setminus C$ which are connected by an edge $e \in E$. Thus, both u and v are not in C. Therefore, C is not the vertex cover of G anymore.
- Suppose C is not the vertex cover of G, then there exists a pair of vertices (u, v) that are connected by an edge $e \in E$ but are not in C. Thus, $u \in (V \setminus C)$ and $u \in (V \setminus C)$. Therefore, $(V \setminus C)$ is not the vertex cover of G anymore.

From the reasoning above, we can state that: C is the vertex cover of G if and only if $V \setminus C$ is an independent set of G.

(ii)

We prove that $ApproxMaxIndependentSet$ is not a 2-approximation algorithm by showing a counter example. That is, consider a complete graph, for example, a graph $G = (V, E)$ where $V = x_1, x_2$ and $E = (x_1, x_2)$.

Applying the $ApproxMinVertexCover(G)$, we get $C = x_1, x_2$ (picking both vertices from the edge).

Now we take the approx max independent set $ALG = V \setminus C = \emptyset$.

The optimal solution now is $OPT = 1$ (picking x_1 or x_2).

The approximation ratio is $\rho = \frac{OPT}{ALG} = \infty \neq 2$.

So the approximation ratio is not 2.

AII.2

(i)

The best case of the solution is when there is a common variable in every clause, then the optimal solution is that variable. Thus:

$$OPT \geq 1 = LB$$

In this problem, we are not to sure whether the duplication of variables in a DNF clause is allowed or not. If it is allowed, we can define a CNF like this:

$$(a \vee b \vee c) \wedge (a \vee b \vee b) \wedge (a \vee a \vee a)$$

In the worst case, the algorithm picks c, b, a sequentially. So $ALG = 3 = n$. Thus the approximation ratio is $\frac{n}{LB} = n$.

This result is not very interesting because n is also the upper bound of the solution. Indeed, $(a \vee a)$ can be reduced to only a, thus the clause is not in the 3-CNF form anymore.

Assume that such duplication is not allowed. Then in the worst case, $OPT = 1$ as we explained before, while the algorithm can pick at most $n - 2$ variables. It is because if all of n variables are used, then the last 2 ones are in some DNF claus(es) which are covered by the previous variables. If not all of them are used, then $ALG < n - 2$.

The case when $ALG = n - 2$ is as follow, we have n variables and 2 common ones in all clauses.

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge (x_1 \vee x_2 \vee x_5) \wedge ... \wedge (x_1 \vee x_2 \vee x_n)$$

There are 2 common variables and $n-2$ other different variables distributed among the clauses. In the worst case, the algorithm returns $ALG = n - 2$. Then the approximation ratio is

$$\rho = \frac{n-2}{OPT} = n - 2$$

(ii)

For the algorithm to become a 3-approximation algorithm it should be modified so that in each iteration it chooses all three elements in a clause and eliminate all the clauses in CNF that contain any of these three elements.

Proof

Let D^* be a subset of D that only contains clauses that do not share any variable.

The optimal solution to the problem OPT contains at least one variable from each clause in D^* therefore :

$$OPT \geq |D^*|$$

In our algorithm after the modification, we select three variables in each clause in D^*, because there are no clauses that share common variables with them. The other clauses which have common variables as any clause in D^* are deleted. Thus:

$$\begin{aligned} ALG &= 3|D^*| \\ &\leq 3OPT \end{aligned}$$

Therefore it is a 3-approximation algorithm.

AII.3

(i)

Suppose a d-hypergraph $G = (V, E)$ which every edge $e \in E$ is incident to d vertices in V. To formulate 0/1 linear programming, we introduce $X = \{ x_i,$

x_2, ... , x_n } which x_i represents $v_i \in V$ in a linear programming. If $x_i = 1$, it means we pick v_i to the set of double vertex cover, $C \subset V$, and otherwise $x_i = 0$. For this solution, we want to find a minimum double vertex cover which requires at least 2 vertices from each edge are in C. Then, we can derive a constraint for 0/1 linear programming

.

$$\sum_{v_i \in e} x_i \geq 2 \text{ for all } e \in E \text{ ; at least 2 vertices are selected.}$$

Thus, we then formulate the linear programming.

$$\begin{aligned} &\text{Minimize} && \sum_{x=1}^{n} x_i \\ &\text{Subject to} \\ &\qquad \sum_{v_i \in e} x_i \geq 2 && \text{for all } e \in E \\ &\qquad x_i = \{0,1\} && \text{for all } x_i \in X \end{aligned}$$

(ii)

Because we can not solve 0/1 linear program in polynomial time, what we have to do next is to relax the program to be a normal linear program by replacing {0,1} constraint with $0 \leq x \leq 1$

Thus, the linear program is

$$\begin{aligned} &\text{Minimize} && \sum_{x=1}^{n} x_i \\ &\text{Subject to} \\ &\qquad \sum_{v_i \in e} x_i \geq 2 && \text{for all } e \in E \\ &\qquad 0 \leq x_i \leq 1 && \text{for all } x_i \in X \end{aligned}$$

Let τ denote the rounding threshold such that

$$x_i = \begin{cases} 1, & \text{if } x_i \geq \tau \\ 0, & \text{otherwise} \end{cases}$$

Algorithm 3 Finding double vertex cover

Input: V, E

Output: A minimum double vertex cover

Operation:

Solve the relaxed linear program corresponding to the given problem.

Minimize $\sum_{x=1}^{n} x_i$

Subject to

$\sum_{v_i \in e} x_i \geq$ for all $e \in E$

$0 \leq x_i \leq 1$ for all $x_i \in X$

C $\leftarrow \{ v_i \in V : x_i \geq \tau \}$

return C

The next step is to derive τ such that all constraints are satisfied and the algorithm always return a valid solution. Let denote x^* to be an ideal value of any x_i such that it satisfies all constraints.

$$\sum_{x_i \in e} x_i \geq 2 \, \forall e \in E$$

$$\sum_{i=1}^{d} x_i \geq 2$$

$$x_1 + \sum_{i=1}^{d-1} \geq 2$$

We would like to find a threshold τ that is small enough so that at least

2 vertices are selected. The extreme case is when $x_1 = 1$, then we have:

$$\begin{aligned} 1 + \sum_{i=1}^{d-1} x_i &\geq 2 \\ (d-1)x_i &\geq 1 \\ x_i &\geq \frac{1}{d-1} \\ \therefore \tau &= \frac{1}{d-1} \end{aligned}$$

Let W denote the value of an optimal to the relaxed linear program and OPT denote the minimum number of double vertex cover. Then $OPT \geq W$.

Now we can derive,

$$\begin{aligned} |C| &= \sum_{v_i \in C} 1 \\ &\leq \sum_{v_i \in C} (d-1)x_i \\ &\leq (d-1) \sum_{v_i \in C} x_i \\ &\leq (d-1)W \\ &\leq (d-1)OPT \end{aligned}$$

(iii)

Lets take an example of a complete 3-hypergraph, where the optimal double vertex cover is $|V|-1$ to make sure every edge has at least 2 vertices selected. So the result of the 0/1-LP is $|V| - 1$.

The relaxed-LP formulation is as follow:

- Miminize $\sum_{i=1}^{n} x_i$
- Subject to: $\sum_{x_j \in e} x_j \geq 2$ for all edge e AND $0 \leq x_i \leq 1$

We run the algorithm by performing that relaxed-LP on the complete 3-hypergraph, and then round the result following the condition $x \geq \frac{1}{2}$.

For the complete 3-hypergraph, the relaxed-LP will return $x_i = \frac{2}{3}$ for all i so that each sum of vertices in an edge is 2.

Then the algorithm will pick all of the vertices because they satisfy the condition. The result is:

$$ALG = \frac{2}{3}|V|$$

The integrality gap, denoted by IG, is:

$$\begin{aligned} IG &= \frac{|V| - 1}{\frac{2|V|}{3}} \\ &= \frac{3}{2} - \frac{3}{2|V|} \end{aligned}$$

A.III-1

(i)

Let $T = \sum_{j=1}^{n} t_i$ be the total time of all the jobs. Since we define large job as having time $t \geq \epsilon T$, the maximum number of large jobs is:

$$n_{max} = \frac{T}{\epsilon T} = \frac{1}{\epsilon}$$

For each large job, there are two possible ways of assigning it to a machine. Therefore the possible ways that $\frac{1}{\epsilon}$ jobs can be scheduled into 2 machine is:

$$NumberOfWays = 2^{\frac{1}{\epsilon}}$$

Since machine 1 and machine 2 are identical (e.g. $0 - n$ and $n - 0$ are equal), we remove the duplicates leaving the total number of schedules at :

$$\begin{aligned} NumberOfWays &= \frac{2^{\frac{1}{\epsilon}}}{2} \\ &= 2^{\frac{1}{\epsilon}-1} \end{aligned}$$

(ii)

We have

$$OPT \geq max(\frac{T}{2}, t_{max})$$

where t_{max} is the maximum job size.
To obtain PTAS, We split the jobs up into two types :

$$\text{Job is } \begin{cases} \text{Large if } t_j \geq \epsilon T \\ \text{Small otherwise} \end{cases}$$

The PTAS algorithm runs as follow.

Algorithm 4 Load Balancing PTAS

Require: Set J of n jobs with running time t_j, ϵ
Ensure: Minimum max(makespan) between two machines.
Operation:
Define set of large jobs L where $t_j \geq \epsilon T \, \forall t_j \in L$
Define set of small jobs $S := J \setminus L$
List all possible combinations of $t_j \in L$ into 2 machines
Select the combination that has the lowest max(makespan)
for all t_j in S **do**
    Schedule t_j to the machine which has the lower current makespan
end for
return The max(makespan) between 2 machines

Proof
Let denote:

- ALG as a makespan generated by our solution
- T' as the makespan in the selected machine before assigning the last job
- t_{last} as the size of the last scheduled job

We come up with:

$$\begin{aligned} ALG &= T' + t_{last} \\ &\leq \frac{T - t_{last}}{2} + t_{last} \\ &\leq \frac{T}{2} + \frac{\epsilon T}{2} \\ &\leq (1+\epsilon)\frac{T}{2} \\ &\leq (1+\epsilon)LB \\ &\leq (1+\epsilon)OPT \end{aligned}$$

When ϵ is small enough, ALG will return OPT solution due to the brute-force approach for large jobs. Then the approximation ratio still holds as $(1+\epsilon)$ where ϵ is very small ($\epsilon \to 0$).

Running Time Since there can be only at most $2^{\frac{1}{\epsilon}-1}$ possible large jobs, then the brute-force part of the algorithm will have a running time of $O(2^{\frac{1}{\epsilon}-1})$. The greedy scheduling of the large jobs is faster with linear time $O(n)$ for n jobs. Leaving a total running time - $O(2^{\frac{1}{\epsilon}-1} + n)$, which is polynomial in term of n. So this algorithm satisfies the condition of a PTAS.

AIII.2

(i)

Let d denote the distance between 2 arbitrary vertices corresponding to P and d^* denote the distance after rounding $p_{i,x}, p_{i,y}$ where $p_{i,x}$ and $p_{i,y}$ denote the x- and y-coordinate of $p_i \in P$, by Δ.

$$\begin{aligned} d &= \sqrt{(p_{i,x} - p_{j,x})^2 + (p_{i,y} - p_{j,y})^2} \\ d^* &= \sqrt{(p_{i,x^*} - p_{j,x^*})^2 + (p_{i,y^*} - p_{j,y^*})^2} \end{aligned}$$

We know that the range of p_x^* is

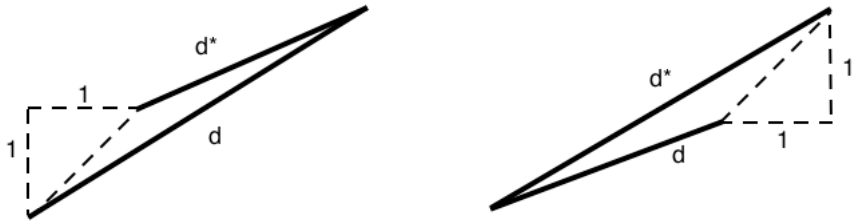
$$\frac{px}{\Delta} \leq p_x^* \leq \frac{px}{\Delta} + 1$$

Then we derive the range of d^*

$$\sqrt{(\frac{p_{i,x}}{\Delta} - (\frac{p_{j,x}}{\Delta} + 1))^2 + (\frac{p_{i,y}}{\Delta} - (\frac{p_{j,y}}{\Delta} + 1)^2} \leq d^* \leq \sqrt{(\frac{p_{i,x}}{\Delta} + 1 - \frac{p_{j,x}}{\Delta})^2 + (\frac{p_{i,y}}{\Delta} + 1 - \frac{p_{j,y}}{\Delta})^2}$$

From the triangle inequality property, such that a, b and c are the lengh of the triangle edges

$$c \leq a + b$$



We can simplify the range of d^* to

$$\sqrt{(\frac{p_{i,x}}{\Delta} - \frac{p_{j,x}}{\Delta})^2 + (\frac{p_{i,y}}{\Delta} - \frac{p_{j,y}}{\Delta})^2} - \sqrt{2} \leq d^* \leq \sqrt{(\frac{p_{i,x}}{\Delta} - \frac{p_{j,x}}{\Delta})^2 + (\frac{p_{i,y}}{\Delta} - \frac{p_{j,y}}{\Delta})^2} + \sqrt{2}$$

$$\frac{d}{\Delta} - \sqrt{2} \leq d^* \leq \frac{d}{\Delta} + \sqrt{2}$$

Hence, the error of d^* is $2\sqrt{2}$ at most.

The rounded integers are indeed the multiple of Δ. Thus the maximum error of a single length is $2\Delta\sqrt{2}$.

Here we have n lengths in the path. Therefore, the total error is:

$$2n\sqrt{2}\Delta = \varepsilon OPT$$

$$\Delta = \frac{\varepsilon OPT}{2n\sqrt{2}}$$

(ii)

Let P and P^* denote set of edges from the optimal solution and the PTAS algorithm respectively and we know that $length^*(P) \geq length^*(P^*)$, then we have

$$\sum_{p_i,p_j \in P^*} d^*_{ij} \leq \sum_{p_i,p_j \in P} d^*_{ij}$$

Thus, we can derive

$$\begin{aligned}
length(P^*) &= \sum_{p_i,p_j \in P^*} d_{ij} \\
&\leq \sum_{p_i,p_j \in P^*} \Delta(d^*_{ij} + \sqrt{2}) \\
&\leq \Delta \sum_{p_i,p_j \in P^*} (d^*_{ij} + \sqrt{2}) \\
&\leq \Delta(\sum_{p_i,p_j \in P^*} d^*_{ij} + |P^*|\sqrt{2}) \\
&\leq \Delta(\sum_{p_i,p_j \in P} d^*_{ij} + |P^*|\sqrt{2}) \\
&\leq \Delta(\sum_{p_i,p_j \in P} (\frac{d_{ij}}{\Delta} + \sqrt{2}) + n\sqrt{2}) \\
&\leq \Delta(\sum_{p_i,p_j \in P} \frac{d_{ij}}{\Delta} + 2n\sqrt{2}) \\
&\leq \sum_{p_i,p_j \in P} d_{ij} + \Delta 2n\sqrt{2} \\
&\leq length\,(P) + \Delta 2n\sqrt{2} \\
&\leq OPT + \left(\frac{\varepsilon OPT}{2n\sqrt{2}}\right) 2n\sqrt{2} \\
&\leq (1+\epsilon)OPT
\end{aligned}$$

(iii)

Let m^* denote the new boundary of the coordinate after rounding p_x, p_y to p_x^*, p_y^* and we also know that

$$\begin{aligned} m &= max(p_x, p_y) \\ OPT &\geq 2m \end{aligned}$$

Thus

$$\frac{m}{\Delta} \leq m^* \leq \frac{m}{\Delta} + 1$$

Then, we can derive the running time

$$\begin{aligned} m* &\leq \frac{m}{\Delta} + 1 \\ &\leq \frac{m2n\sqrt{2}}{\epsilon OPT} + 1 \\ &\leq \frac{m2n\sqrt{2}}{\epsilon 2m} + 1 \\ &\leq \frac{n\sqrt{2}}{\epsilon} + 1 \end{aligned}$$

Therefore, the running time is

$$\begin{aligned} O\left(nm^*\right) &= O\Big(n\frac{n\sqrt{2}}{\epsilon} + n\Big) \\ &= O\Big(\frac{n^2\sqrt{2}}{\epsilon}\Big) \end{aligned}$$

AIII.3

(i)

Suppose that we have such $ALG(G, \epsilon)$ that can return a $(1-\epsilon)-approximation$ solution to the problem. We can define an FPTAS as follow:

- Set ϵ to some value

- Return $ALG(G, \epsilon)$

The above algorithm runs in polynomial time.

Because we know that $ALG(G, \epsilon) \in \mathbb{N}$, so that if we can find such ϵ that the algorithm yields

$$OPT - 1 < ALG\,(G, \epsilon) \leq OPT$$

Then, we can get OPT in polynomial time.

In order to get such ϵ, we need:

$$\begin{aligned} ALG\,(G, \epsilon) &> OPT - 1 \\ &> \left(1 - \frac{1}{OPT}\right) OPT \end{aligned}$$

This holds when $\epsilon < \frac{1}{OPT}$.

Indeed, we do not know the exact value of OPT. But the above property implies that there exists some values of ϵ that can help $ALG(G, \epsilon)$ return OPT. Because our algorithm uses $ALG(G, \epsilon)$ as the main routine, so it also runs in polynomial time.

Now we have an algorithm that can return the optimal solution in polynomial time. Then our problem is not NP-Hard anymore. This contradicts with the definition.

Therefore, there is no FPTAS for the problem.

(ii)

The proof above indeed implies that there exists some values of ϵ that can help any $(1 - \epsilon) - approx$ algorithm return OPT.

Therefore, if there exists a PTAS for this problem, it also returns OPT with such ϵ because PTAS also produces $(1 - \epsilon) - approx$ solutions. This conflicts with the NP-Hard property of the problem as being explained above.

Therefore there is no PTAS for the problem.