Question 1 (6 points) Use Taylor's method of order 2 to approximate the solutions for the initial value problem

$$\frac{dy}{dt} = -y, \;\; 0 \leqslant t \leqslant 1, \;\; y(0) = 1, \;\; \text{with } h = \frac{1}{2}.$$

Solution

First, recall we derive Taylor's method by using the Taylor series expansion

$$y(t+h) = y(t) + y'(t)h + \frac{y''(t)}{2}h^2 + \mathcal{O}(h^3)$$

which gives the stepping formula (where we make use of the fact that $y' = f(y,t)$)

$$y_{j+1} = y_j + hf(y_j, t_j) + \frac{h^2}{2}\left(\partial_t f(y_j, t_j) + \partial_y f(y_j, t_j) \cdot f(y_j, t_j)\right)$$

Substituting in $f(y,t) = -y$, $\partial_y f(y,t) = -1$, $\partial_t f(y,t) = 0$ gives

$$y_{j+1} = y_j - hy_j + \frac{h^2}{2}y_j$$

We are giving the initial condition $y(0) = 1$. In order to estimate $y(1)$ with a step size of $h = 1/2$, we must take 2 steps.

$$\begin{aligned} y_1 &= y_0 - hy_0 + \frac{h^2}{2}y_0 = 1 - \frac{1}{2}\cdot 1 + \frac{(1/2)^2}{2}\cdot 1 = \boxed{\frac{5}{8}} \\ y_2 &= y_1 - hy_1 + \frac{h^2}{2}y_1 = \frac{5}{8} - \frac{1}{2}\cdot\frac{5}{8} + \frac{(1/2)^2}{2}\cdot\frac{5}{8} = \boxed{\frac{25}{64}} \end{aligned}$$

Question 2 (4 points) Determine the local truncation error of Euler's method

$$y_{n+1} = y_n + h\, f(y_n, t_n)$$

and express your answer in the format $\mathcal{O}(h^p)$ for some positive integer p.

Proof

The local truncation error is defined as

$$\tau = \frac{y(t_{n+1}) - y(t_n)}{h} - f(y(t_n), t_n).$$

Using the Taylor series expansion, we have that

$$y(t_{n+1}) - y(t_n) = y(t_n + h) - y(t_n) = [y(t_n) + hy^{'}(t_n) + \mathcal{O}(h^2)] - y(t_n) = hy^{'}(t_n) + \mathcal{O}(h^2).$$

Also, we know that

$$f(y(t_n), t_n) = y^{'}(t_n).$$

Combining these two gives

$$\tau = \frac{hy^{'}(t_n) + \mathcal{O}(h^2)}{h} - y^{'}(t_n) = \mathcal{O}(h), \;\text{with } p = \boxed{1}.$$