STA260: PROBABILITY AND STATISTICS II
SPRING 2021

TUTORIAL 12 (TUT9101)

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Practice Question 1 (11.1)

If $\hat{\beta}_0$ and $\hat{\beta}_1$ are the least-squares estimates for the intercept and slope in a simple linear regression model, show that the least-squares equation $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$ always goes through the point $(\overline{x}, \overline{y})$. [*Hint:* Substitute $\overline{x}$ for x in the least-squares equation and use the fact that $\hat{\beta}_0 = \overline{y} - \hat{\beta}_1 \overline{x}$.]

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$$

since $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \Rightarrow \bar{y} = \hat{\beta}_0 + \hat{\beta}_1 \bar{x}$

$\therefore (\bar{x}, \bar{y})$ is on the line $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$.

Practice Question 2 (11.10)

Suppose that we have postulated the model

$$Y_i = \beta_1 x_i + \varepsilon_i \qquad i = 1, 2, \dots, n,$$

where the ε_i's are independent and identically distributed random variables with $E(\varepsilon_i) = 0$. Then $\hat{y}_i = \hat{\beta}_1 x_i$ is the predicted value of y when $x = x_i$ and SSE $= \sum_{i=1}^{n}[y_i - \hat{\beta}_1 x_i]^2$. Find the least-squares estimator of β_1. (Notice that the equation $y = \beta x$ describes a straight line passing through the origin. The model just described often is called the *no-intercept* model.)

$$\frac{\partial}{\partial \hat{\beta}_1} SSE = \sum_{i=1}^{n} (2(y_i - \hat{\beta}_1 x_i)(-x_i)) = 2\sum_{i=1}^{n} (\hat{\beta}_1 x_i^2 - x_i y_i)$$

$$\frac{\partial SSE}{\partial \hat{\beta}_1} = 0 \Rightarrow \hat{\beta}_1 \left(\sum_{i=1}^{n} x_i^2\right) = \sum_{i=1}^{n} x_i y_i \qquad \hat{\beta}_1 = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$

Practice Question 3 (11.15)

a Derive the following identity:

$$\begin{aligned}\text{SSE} &= \sum_{i=1}^{n}(y_i - \hat{y}_i)^2 = \sum_{i=1}^{n}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 \\ &= \sum_{i=1}^{n}(y_i - \overline{y})^2 - \hat{\beta}_1 \sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y}) = S_{yy} - \hat{\beta}_1 S_{xy}.\end{aligned}$$

Notice that this provides an easier computational method of finding SSE.

b Use the computational formula for SSE derived in part (a) to prove that SSE $\leq S_{yy}$. [*Hint:* $\hat{\beta}_1 = S_{xy}/S_{xx}$.]

a $\text{SSE} = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2 \overset{\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i}{=\!=} \sum_{i=1}^{n}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$

$\overset{\hat{\beta}_0 = \overline{y} - \hat{\beta}_1 \overline{x}}{=\!=} \sum_{i=1}^{n}(y_i - \overline{y} - \hat{\beta}_1(x_i - \overline{x}))^2$

$= \sum_{i=1}^{n}(y_i - \overline{y})^2 - 2\hat{\beta}_1 \sum_{i=1}^{n}(y_i - \overline{y})(x_i - \overline{x}) + \hat{\beta}_1^2 \sum_{i=1}^{n}(x_i - \overline{x})^2$

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Notice that this provides an easier computational method of finding SSE.

b Use the computational formula for SSE derived in part (a) to prove that $\text{SSE} \leq S_{yy}$. [*Hint:* $\hat{\beta}_1 = S_{xy}/S_{xx}$.]

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

$$\sum_{i=1}^{n}(y_i - \bar{y})^2 - 2\hat{\beta}_1 \sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x}) + \hat{\beta}_1^2 \sum_{i=1}^{n}(x_i - \bar{x})^2$$

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Notice that this provides an easier computational method of finding SSE.

b Use the computational formula for SSE derived in part (a) to prove that SSE $\leq S_{yy}$. [*Hint:* $\hat{\beta}_1 = S_{xy}/S_{xx}$.]

$S_{xy} = \sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})$, $S_{yy} = \sum_{i=1}^{n}(y_i - \overline{y})^2$, $S_{xx} = \sum_{i=1}^{n}(x_i - \overline{x})^2$

then $\hat{\beta}_1 = \dfrac{S_{xy}}{S_{xx}}$, $\hat{\beta}_1 S_{xx} = S_{xy}$.

$$\text{SSE} = S_{yy} - 2\hat{\beta}_1 S_{xy} + \underbrace{\hat{\beta}_1^2 S_{xx}}_{\hat{\beta}_1(\hat{\beta}_1 S_{xx})} = S_{yy} - \hat{\beta}_1 S_{xy}$$

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b from (a) $SSE = S_{yy} - \hat{\beta}_1 S_{xy} = S_{yy} - \frac{S_{xy}}{S_{xx}} \cdot S_{xy} = S_{yy} - \frac{(S_{xy})^2}{S_{xx}} \leq S_{yy}$

$S_{xy} \geq 0, \quad S_{xx} > 0.$

$\therefore SSE \leq S_{yy}$

Practice Question 4 (11.21)

Under the assumptions of Exercise 11.20, find $\text{Cov}(\hat{\beta}_0, \hat{\beta}_1)$. Use this answer to show that $\hat{\beta}_0$ and $\hat{\beta}_1$ are independent if $\sum_{i=1}^{n} x_i = 0$. [*Hint:* $\text{Cov}(\hat{\beta}_0, \hat{\beta}_1) = \text{Cov}(\overline{Y} - \hat{\beta}_1\overline{x}, \hat{\beta}_1)$. Use Theorem 5.12 and the results of this section.]

$$\text{Cov}(\hat{\beta}_0, \hat{\beta}_1) = \text{Cov}(\overline{Y} - \hat{\beta}_1 \overline{x}, \hat{\beta}_1)$$

$$= \text{Cov}(\overline{Y}, \hat{\beta}_1) - \text{Cov}(\hat{\beta}_1 \overline{x}, \hat{\beta}_1)$$

$$= 0 - \overline{x}\, V(\hat{\beta}_1)$$

$$= 0 - \overline{x}\, \frac{\sigma^2}{S_{xx}}$$

$$\sum_{i=1}^{n} x_i = 0 \Rightarrow \overline{x} = 0 \Rightarrow \text{Cov}(\hat{\beta}_0, \hat{\beta}_1) = 0.$$

Practice Question 5 (10.107)

Let S_1^2 and S_2^2 denote, respectively, the variances of independent random samples of sizes n and m selected from normal distributions with means μ_1 and μ_2 and common variance σ^2. If μ_1 and μ_2 are unknown, construct a likelihood ratio test of $H_0 : \sigma^2 = \sigma_0^2$ against $H_a : \sigma^2 = \sigma_a^2$, assuming that $\sigma_a^2 > \sigma_0^2$.

$$L(\theta \mid x_1 \cdots x_n, y_1 \cdots y_m) = \left(\frac{1}{\sqrt{2\pi}}\right)^{m+n} \left(\frac{1}{\sigma}\right)^{m+n} \exp\left\{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^n x_i - \mu_1\right)^2 - \frac{1}{2\sigma^2}\left(\sum_{i=1}^m y_i - \mu_2\right)^2\right\}$$

$\Omega_0 = \{\sigma^2 : \sigma^2 = \sigma_0^2\}$, $\hat{\mu}_1 = \bar{x}$, $\hat{\mu}_2 = \bar{y}$, $\hat{\sigma}^2 = \sigma_0^2$ maximizes $L(\theta)$

$\Omega = \Omega_0 \cup \Omega_a = \{\sigma_0^2, \sigma_a^2\}$ $\quad$ $\hat{\mu}_1 = \bar{x}$, $\hat{\mu}_2 = \bar{y}$, $\hat{\sigma}^2$ is either σ_0^2 or σ_a^2

$$\lambda(x, y) = \frac{L(\hat{\Omega}_0)}{L(\hat{\Omega})} = \left(\frac{\hat{\sigma}_\Omega^2}{\sigma_0^2}\right)^{\frac{m+n}{2}} \exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\hat{\sigma}_\Omega^2}\right)\sum_{i=1}^n (x_i - \bar{x})^2 - \frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\hat{\sigma}_\Omega^2}\right)\sum_{i=1}^m (y_i - \bar{y})^2\right\}$$

$$= \left(\frac{\hat{\sigma}_\Omega^2}{\sigma_0^2}\right)^{\frac{m+n}{2}} \exp\left\{-\frac{1}{2} \cdot \frac{1}{\sigma_0^2}\left[\sum_{i=1}^n (x_i - \bar{x})^2 + \sum_{i=1}^m (y_i - \bar{y})^2\right] + \frac{1}{2} \cdot \frac{1}{\hat{\sigma}_\Omega^2}\left[\sum_{i=1}^n (x_i - \bar{x})^2 + \sum_{i=1}^m (y_i - \bar{y})^2\right]\right\}$$

Let S_1^2 and S_2^2 denote, respectively, the variances of independent random samples of sizes n and m selected from normal distributions with means μ_1 and μ_2 and common variance σ^2. If μ_1 and μ_2 are unknown, construct a likelihood ratio test of $H_0 : \sigma^2 = \sigma_0^2$ against $H_a : \sigma^2 = \sigma_a^2$, assuming that $\sigma_a^2 > \sigma_0^2$.

let $V = \frac{1}{\sigma_0^2}\left(\sum_{i=1}^{n}(x_i - \bar{x})^2 + \sum_{i=1}^{m}(y_i - \bar{y})^2\right) = \frac{(n-1)S_1^2 + (m-1)S_2^2}{\sigma_0^2}$

$\lambda(x,y) = \left(\frac{\hat{\sigma}_\Omega^2}{\sigma_0^2}\right)^{\frac{m+n}{2}} \exp\left\{-\frac{1}{2}V + \frac{1}{2}\frac{\sigma_0^2}{\hat{\sigma}_\Omega^2}V\right\} = \left(\frac{\hat{\sigma}_\Omega^2}{\sigma_0^2}\right)\exp\left\{-\frac{1}{2}V\left(1 - \frac{\sigma_0^2}{\hat{\sigma}_\Omega^2}\right)\right\}$

$\lambda(x,y) = \begin{cases} \left(\frac{\sigma_a^2}{\sigma_0^2}\right)^{\frac{m+n}{2}} \exp\left\{-\frac{1}{2}\left(1 - \frac{\sigma_0^2}{\sigma_a^2}\right)V\right\} & \hat{\sigma}_\Omega = \sigma_a \\ 1 & \hat{\sigma}_\Omega = \sigma_0 \end{cases}$

rejection region $\{\lambda(x,y) \leq k\}$ $\quad \lambda(x,y) \leq 1 \quad \therefore$ for $\alpha > 0$, $\hat{\sigma}_\Omega = \sigma_a$, $k < 1$

$\left\{\left(\frac{\sigma_a^2}{\sigma_0^2}\right)^{\frac{m+n}{2}} \exp\left\{-\frac{1}{2}\left(1 - \frac{\sigma_0^2}{\sigma_a^2}\right)V\right\} < k\right\}$

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$$\left\{ \left(\frac{\sigma_a^2}{\sigma_0^2}\right)^{\frac{m+n}{2}} \exp\left\{ -\frac{1}{2}\left(1-\frac{\sigma_0^2}{\sigma_a^2}\right) V \right\} < k \right\} \iff \{ V > k^* \} \qquad k^* = \frac{2}{\frac{\sigma_0^2}{\sigma_a^2}-1}\left[\ln k + (m+n)(\ln\sigma_0 - \ln\sigma_a)\right]$$

$\because \sigma_a^2 > \sigma_0^2 \quad \therefore \frac{\sigma_0^2}{\sigma_a^2} < 1$

$\frac{\sigma_0^2}{\sigma_a^2} - 1 < 0$

under H_0,

$$V = \frac{1}{\sigma_0^2}\left(\sum_{i=1}^n (x_i-\bar{x})^2 + \sum_{i=1}^m (y_i-\bar{y})^2\right) = \frac{(n-1)S_1^2}{\sigma_0^2} + \frac{(m-1)S_2^2}{\sigma_0^2}$$

$V \sim \chi^2(m+n-2)$

$\therefore$ rejection region $\{ V > \chi^2_\alpha(m+n-2) \}$

Practice Question6 (10.113)

Suppose that independent random samples of sizes n_1 and n_2 are to be selected from normal populations with means μ_1 and μ_2, respectively, and common variance σ^2.

Refer to Exercise 10.112. Show that in testing of $H_0: \mu_1 = \mu_2$ versus $H_a: \mu_1 \neq \mu_2$ (σ^2 unknown) the likelihood ratio test reduces to the two-sample t test.

$$\mathcal{L}(\theta) = \left(\frac{1}{\sqrt{2\pi}}\right)^{n_1+n_2} \left(\frac{1}{\sigma^2}\right)^{\frac{n_1+n_2}{2}} \exp\left\{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n_1}(x_i-\mu_1)^2 + \sum_{i=1}^{n_2}(y_i-\mu_2)^2\right)\right\}$$

$$\hat{\mu} = \frac{n_1\bar{X}+n_2\bar{Y}}{n_1+n_2}.$$

$\Omega_0 = \{(\mu_1,\mu_2) : \mu_1=\mu_2\}$

$$\hat{\mu}_1 = \hat{\mu}_2 = \frac{n_1\bar{X}+n_2\bar{Y}}{n_1+n_2} \qquad \hat{\sigma}^2 = \frac{1}{n}\left(\sum_{i=1}^{n_1}(X_i-\hat{\mu})^2 + \sum_{i=1}^{n_2}(Y_i-\hat{\mu})^2\right)$$

Ω unrestricted

$$\mu_1 = \bar{X}, \quad \mu_2 = \bar{Y} \qquad \hat{\sigma}^2 = \frac{1}{n}\left(\sum_{i=1}^{n_1}(X_i-\bar{X})^2 + \sum_{i=1}^{n_2}(Y_i-\bar{Y})^2\right)$$

Practice Question6 (10.113)

Suppose that independent random samples of sizes n_1 and n_2 are to be selected from normal populations with means μ_1 and μ_2, respectively, and common variance σ^2.

Refer to Exercise 10.112. Show that in testing of $H_0: \mu_1 = \mu_2$ versus $H_a: \mu_1 \neq \mu_2$ (σ^2 unknown) the likelihood ratio test reduces to the two-sample t test.

$$\lambda(x,y) = \left(\frac{\hat{\sigma}^2_{\Omega}}{\hat{\sigma}^2_{\Omega_0}} \right)^{\frac{n_1+n_2}{2}} \leq k \iff \frac{\hat{\sigma}^2_{\Omega_0}}{\hat{\sigma}^2_{\Omega}} \geq k'$$

$$\hat{\mu} = \frac{n_1\bar{X} + n_2\bar{Y}}{n_1+n_2} \qquad \frac{n_1 n_2}{n_1+n_2}(\bar{X}-\bar{Y})^2$$

$$\frac{\hat{\sigma}^2_{\Omega_0}}{\hat{\sigma}^2_{\Omega}} = \frac{\sum_{i=1}^{n_1}(X_i-\hat{\mu})^2 + \sum_{i=1}^{n_2}(Y_i-\hat{\mu})^2}{\sum_{i=1}^{n_1}(X_i-\bar{X})^2 + \sum_{i=1}^{n_2}(Y_i-\bar{Y})^2} = \frac{\sum_{i=1}^{n_1}(X_i-\bar{X})^2 + \sum_{i=1}^{n_2}(Y_i-\bar{Y})^2 + n_1(\bar{X}-\hat{\mu})^2 + n_2(\bar{Y}-\hat{\mu})^2}{\sum_{i=1}^{n_1}(X_i-\bar{X})^2 + \sum_{i=1}^{n_2}(Y_i-\bar{Y})^2}$$

$$= 1 + \frac{n_1 n_2}{n_1+n_2}\left(\frac{(\bar{X}-\bar{Y})^2}{\sum_{i=1}^{n_1}(X_i-\bar{X})^2 + \sum_{i=1}^{n_2}(Y_i-\bar{Y})^2} \right)$$

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Refer to Exercise 10.112. Show that in testing of $H_0: \mu_1 = \mu_2$ versus $H_a: \mu_1 \neq \mu_2$ (σ^2 unknown) the likelihood ratio test reduces to the two-sample t test.

let $t^* = \dfrac{\bar{X} - \bar{Y}}{\sqrt{\sum_{i=1}^{n_1} (X_i - \bar{X})^2 + \sum_{i=1}^{n_2} (Y_i - \bar{Y})^2}}$

then $\lambda(x, y) = 1 + \dfrac{n_1 n_2}{n_1 + n_2} t^{*2}$

then $\{\lambda(x, y) \leq k\} \iff \{|t^*| \leq k'\} \iff \{|t| \leq k^*\}$

$$t = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \cdot \sqrt{\frac{\sum_{i=1}^{n} (X_i - \bar{X})^2 + \sum_{i=1}^{4} (Y_i - \bar{Y})^2}{n_1 + n_2 - 2}}}$$

THANK YOU FOR ATTENDING TODAY'S TUTORIAL

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