STA260: PROBABILITY AND STATISTICS II
SPRING 2021

TUTORIAL 8 (TUT9101)

JUN 2, 2021

YUEHUAN HE

UNIVERSITY OF
TORONTO

***9.93** Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a population with density function

$(\theta > 0)$

$$f(y \mid \theta) = \begin{cases} \dfrac{2\theta^2}{y^3}, & \theta < y < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

In Exercise 9.53, you showed that $Y_{(1)} = \min(Y_1, Y_2, \ldots, Y_n)$ is sufficient for θ.

a Find the MLE for θ. [*Hint:* See Example 9.16.]

$$\mathcal{L}(\theta) = \prod_{i=1}^{n} f(y_i \mid \theta) = \begin{cases} \dfrac{2^n \theta^{2n}}{\prod_{i=1}^{n} y_i^3}, & \boxed{\theta < y_1, \ldots, y_n < \infty} \\ 0, & \text{otherwise} \end{cases} \Rightarrow \theta < y_{(1)}$$

$\theta \uparrow \quad \theta^{2n} \uparrow \quad \mathcal{L}(\theta) \uparrow$ largest θ maximize $\mathcal{L}(\theta)$

MLE for θ: $\hat{\theta} = Y_{(1)}$.

9.83 Suppose that $Y_1, Y_2, \ldots, Y_n$ constitute a random sample from a uniform distribution with probability density function

$$f(y \mid \theta) = \begin{cases} \dfrac{1}{2\theta + 1}, & 0 \le y \le 2\theta + 1, \\ 0, & \text{otherwise.} \end{cases}$$

a Obtain the MLE of θ.

b Obtain the MLE for the *variance* of the underlying distribution.

a $$L(\theta) = \prod_{i=1}^{n} f(y_i \mid \theta) = \begin{cases} \dfrac{1}{(2\theta+1)^n}, & 0 \le y_1, \cdots, y_n \le 2\theta+1, \quad i = 1, 2, \cdots, n \\ 0, & \text{otherwise} \end{cases}$$

$\theta \uparrow \quad (2\theta+1) \uparrow \quad (2\theta+1)^n \uparrow \quad \dfrac{1}{(2\theta+1)^n} \downarrow$ find smallest θ to maximize $\mathcal{L}(\theta)$

lower bound of θ : $Y_{(n)} = \max\{Y_1, \cdots, Y_n\}$, $2\hat{\theta}+1 = Y_{(n)}$ maximizes $\mathcal{L}(\theta)$.

MLE of $\hat{\theta} = \frac{1}{2}(Y_{(n)} - 1)$

9.83 Suppose that $Y_1, Y_2, \ldots, Y_n$ constitute a random sample from a uniform distribution with probability density function

$$f(y \mid \theta) = \begin{cases} \frac{1}{2\theta + 1}, & 0 \le y \le 2\theta + 1, \\ 0, & \text{otherwise.} \end{cases}$$

a Obtain the MLE of θ.

b Obtain the MLE for the *variance* of the underlying distribution.

b $E(Y) = \int_0^{2\theta+1} \frac{1}{2\theta+1} t \, dt = \frac{1}{2\theta+1} \cdot \frac{1}{2}(2\theta+1)^2 = \frac{1}{2}(2\theta+1)$

$E(Y^2) = \int_0^{2\theta+1} \frac{1}{2\theta+1} t^2 \, dt = \frac{1}{2\theta+1} \cdot \frac{1}{3}(2\theta+1)^3 = \frac{1}{3}(2\theta+1)^2$

$V(Y) = E(Y^2) - (E(Y))^2 = \frac{1}{3}(2\theta+1)^2 - \frac{1}{4}(2\theta+1)^2 = \frac{1}{12}(2\theta+1)^2$

by invariance property $g(x) = \frac{1}{12}(2\theta+1)^2$ $g(\hat{\theta})$ is the MLE of $\frac{1}{12}(2\hat{\theta}+1)^2$

$\frac{1}{12} Y_{(n)}^2$ is the MLE of $V(Y)$.

UNIVERSITY OF TORONTO

9.97 The geometric probability mass function is given by

$$p(y \mid p) = p(1-p)^{y-1}, \qquad y = 1, 2, 3, \ldots.$$

A random sample of size n is taken from a population with a geometric distribution.

a Find the method-of-moments estimator for p.

b Find the MLE for p.

a $\mu_1' = E(Y) = \sum_{y=1}^{\infty} p(1-p)^{y-1} \cdot y = p + \sum_{y=1}^{\infty} p(1-p)^{y}(y+1)$.

$(1-p)\mu_1' = \sum_{y=1}^{\infty} p(1-p)^{y} \cdot y$

$\therefore p \cdot \mu_1' = p + \sum_{y=1}^{\infty} p(1-p)^{y} = p + (1-p) = 1 \Rightarrow \mu_1' = \frac{1}{p}$

$m_1' = \bar{Y}$, $\bar{Y} = \frac{1}{\hat{p}}$, method of moments estimator $\hat{p} = \frac{1}{\bar{Y}}$

9.97 The geometric probability mass function is given by

$$p(y \mid p) = p(1-p)^{y-1}, \qquad y = 1, 2, 3, \ldots .$$

A random sample of size n is taken from a population with a geometric distribution.

a Find the method-of-moments estimator for p.

b Find the MLE for p.

b. $\mathcal{L}(p) = \prod_{i=1}^{n} p(y_i \mid p) = p^n (1-p)^{y_1 + \cdots + y_n - n}$

$\ell(p) = \ln(\mathcal{L}(p)) = n \ln p + (\sum_{i=1}^{n} y_i - n) \ln(1-p)$

$\ell'(p) = \frac{n}{p} - \frac{\sum_{i=1}^{n} y_i - n}{1-p}$

$\ell'(p) = 0 \Rightarrow (1-p) n - p(\sum_{i=1}^{n} y_i - n) = 0$

$\Rightarrow (\sum_{i=1}^{n} y_i) p = n \Rightarrow p = \frac{n}{\sum_{i=1}^{n} y_i}$

UNIVERSITY OF TORONTO

9.107 Suppose that a random sample of length-of-life measurements, $Y_1, Y_2, \ldots, Y_n$, is to be taken of components whose length of life has an exponential distribution with mean θ. It is frequently of interest to estimate

$$\overline{F}(t) = 1 - F(t) = e^{-t/\theta},$$

the *reliability* at time t of such a component. For any fixed value of t, find the MLE of $\overline{F}(t)$.

prob 9.8.13 in notes

$Y_1 \ldots Y_n \overset{iid}{\sim} \text{Gamma}(\alpha, \theta)$

$\hat{\theta} = \frac{\overline{Y}}{\alpha}$ is the MLE of θ

similarly, exponential distribution with mean θ (let $\alpha = 1$)

$\hat{\theta} = \overline{Y}$ is the MLE of θ

9.107 Suppose that a random sample of length-of-life measurements, $Y_1, Y_2, \ldots, Y_n$, is to be taken of components whose length of life has an exponential distribution with mean θ. It is frequently of interest to estimate

$$\overline{F}(t) = 1 - F(t) = e^{-t/\theta},$$

the *reliability* at time t of such a component. For any fixed value of t, find the MLE of $\overline{F}(t)$.

following invariance principal $\quad f(x) = e^{-\frac{t}{x}}$

$f(\bar{Y})$ is the MLE of $f(\theta)$

i.e. $e^{-\frac{t}{\bar{Y}}}$ is the MLE of $\overline{F}(t) = e^{-\frac{t}{\theta}}$.

THANK YOU FOR ATTENDING TODAY'S TUTORIAL

Feedbacks: yuehuan.he@utoronto.ca