STA260: PROBABILITY AND STATISTICS II
SPRING 2021

TUTORIAL 5 (TUT9101)

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NOTES POSTING PAGE

https://https://heyuehuan.github.io/sta260/

UNIVERSITY OF TORONTO STA260 (TUT9101) Tutorial Notes Join Tutorial Piazza Quercus

STA260: Probability and Statistics II

Tutorial Session TUT9101

Summer I 2021

Tutorial Information Tutorials will start on the first week of classes and run through the last week of classes. Suggested problems from the textbook will serve as an appropriate preparation for the tutorial discussions.

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Tutorial Time Wednesday 11:00 AM - 12:00 Noon; Friday 11:00 AM - 12:00 Noon.

Tutorial Zoom Room Zoom Room ID: 81038277811. Please find the **password** in the **course outline posted on Quercus**.

Join Zoom Meeting | Piazza Forum | Quercus

Tutorial Notes

Notes are posted every Friday evening.

- **Tutorial 1 (May 5)**

  Tutorial Note
- **Tutorial 2 (May 7)**

  Tutorial Note
- **Tutorial 3 (May 12)**

  Tutorial Note
- **Tutorial 4 (May 14)**

  Tutorial Note

8.39 Suppose that the random variable Y has a gamma distribution with parameters $\alpha = 2$ and an unknown β. In Exercise 6.46, you used the method of moment-generating functions to prove a general result implying that $2Y/\beta$ has a χ^2 distribution with 4 degrees of freedom (df). Using $2Y/\beta$ as a pivotal quantity, derive a 90% confidence interval for β.

Problem 1

8.39 Suppose that the random variable Y has a gamma distribution with parameters $\alpha = 2$ and an unknown β. In Exercise 6.46, you used the method of moment-generating functions to prove a general result implying that $2Y/\beta$ has a χ^2 distribution with 4 degrees of freedom (df). Using $2Y/\beta$ as a pivotal quantity, derive a 90% confidence interval for β.

$\because \quad 2\frac{Y}{\beta} \sim \chi^2(4)$

$\therefore \quad P(0.7107 < 2\frac{Y}{\beta} < 9.4877) = 0.9$

$\Rightarrow P\left(\frac{2Y}{9.4877} < \beta < \frac{2Y}{0.7107}\right) = 0.9$

$\Rightarrow$ 90% CI for β : $\left(\frac{Y}{4.7439}, \frac{Y}{0.3554}\right)$

Problem 2

8.41 Suppose that Y is normally distributed with mean 0 and unknown variance σ^2. Then Y^2/σ^2 has a χ^2 distribution with 1 df. Use the pivotal quantity Y^2/σ^2 to find a

a 95% confidence interval for σ^2.

b 95% upper confidence limit for σ^2.

c 95% lower confidence limit for σ^2.

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b 95% upper confidence limit for σ^2.

c 95% lower confidence limit for σ^2.

$$\frac{Y^2}{\sigma^2} \sim \chi^2(1)$$

a. $$P\left(0.0009821 < \frac{Y^2}{\sigma^2} < 5.02389\right) = 0.95$$

$$\Rightarrow P\left(\frac{Y^2}{5.02389} < \sigma^2 < \frac{Y^2}{0.0009821}\right) = 0.95$$

$$\left(\frac{Y^2}{5.02389}, \frac{Y^2}{0.0009821}\right)$$

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a 95% confidence interval for σ^2.

b 95% upper confidence limit for σ^2.

c 95% lower confidence limit for σ^2.

b. $P\left(0.0039321 < \frac{Y^2}{\sigma^2}\right) = 0.95 \Rightarrow P\left(\sigma^2 < \frac{Y^2}{0.0039321}\right) = 0.95$

$\left(-\infty, \frac{Y^2}{0.0039321}\right)$

c. $P\left(\frac{Y^2}{\sigma^2} < 3.84146\right) = 0.95 \Rightarrow P\left(\frac{Y^2}{3.84146} < \sigma^2\right) = 0.95$

$\left(\frac{Y^2}{3.84146}, +\infty\right)$

Problem 3

8.43 Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample of size n from a population with a uniform distribution on the interval $(0, \theta)$. Let $Y_{(n)} = \max(Y_1, Y_2, \ldots, Y_n)$ and $U = (1/\theta)Y_{(n)}$.

a Show that U has distribution function

$$F_U(u) = \begin{cases} 0, & u < 0, \\ u^n, & 0 \le u \le 1, \\ 1, & u > 1. \end{cases}$$

b Because the distribution of U does not depend on θ, U is a pivotal quantity. Find a 95% lower confidence bound for θ.

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b Because the distribution of U does not depend on θ, U is a pivotal quantity. Find a 95% lower confidence bound for θ.

a. for $u \in [0,1]$

$$F_U(u) = P(U \le u) = P\left(\frac{1}{\theta}\max\{Y_1, \ldots Y_n\} \le u\right)$$

$$= P(Y_1 \le \theta u, \ldots, Y_n \le \theta u)$$

$$= [P(Y_i \le \theta u)]^n$$

$$= u^n$$

for $u < 0$

$F_U(u) = 0$

for $u > 1$

$F_U(u) = 1$

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Problem 3

a Show that U has distribution function

$$F_U(u) = \begin{cases} 0, & u < 0, \\ u^n, & 0 \le u \le 1, \\ 1, & u > 1. \end{cases}$$

b Because the distribution of U does not depend on θ, U is a pivotal quantity. Find a 95% lower confidence bound for θ.

b. $P(U \le a) = 0.95$ i.e. $F_U(a) = 0.95 \Rightarrow a = (0.95)^{\frac{1}{n}}$

$\therefore P\left(\frac{Y_{(n)}}{\theta} \le (0.95)^{\frac{1}{n}}\right) = 0.95 \Rightarrow P(\theta \ge Y_{(n)}(0.95)^{-\frac{1}{n}}) = 0.95$

$\therefore [Y_{(n)}(0.95)^{-\frac{1}{n}}, +\infty).$

8.44 Let Y have probability density function

$$f_Y(y) = \begin{cases} \dfrac{2(\theta - y)}{\theta^2}, & 0 < y < \theta, \\ 0, & \text{elsewhere.} \end{cases}$$

a Show that Y has distribution function

$$F_Y(y) = \begin{cases} 0, & y \le 0, \\ \dfrac{2y}{\theta} - \dfrac{y^2}{\theta^2}, & 0 < y < \theta, \\ 1, & y \ge \theta. \end{cases}$$

b Show that Y/θ is a pivotal quantity.

c Use the pivotal quantity from part (b) to find a 90% lower confidence limit for θ.

8.44 Let Y have probability density function

Problem 4

$$f_Y(y) = \begin{cases} \dfrac{2(\theta - y)}{\theta^2}, & 0 < y < \theta, \\ 0, & \text{elsewhere.} \end{cases}$$

a Show that Y has distribution function

$$F_Y(y) = \begin{cases} 0, & y \leq 0, \\ \dfrac{2y}{\theta} - \dfrac{y^2}{\theta^2}, & 0 < y < \theta, \\ 1, & y \geq \theta. \end{cases}$$

b Show that Y/θ is a pivotal quantity.

c Use the pivotal quantity from part (b) to find a 90% lower confidence limit for θ.

a. for $y \in (0, \theta)$

$$F_Y(y) = P(Y \leq y) = \int_0^y \frac{2}{\theta^2}(\theta - t)\,dt = \frac{2}{\theta^2}\left(\theta t - \frac{1}{2}t^2\right)\Big|_0^y = \frac{2y}{\theta} - \frac{y^2}{\theta^2}$$

for $y < 0$ $F_Y(y) = 0$ for $y > \theta$. $F_Y(y) = 1$

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c Use the pivotal quantity from part (b) to find a 90% lower confidence limit for θ.

b. $F_{\frac{Y}{\theta}}(y) = \begin{cases} 0 & y \le 0 \\ 2y - y^2 & y \in (0, 1) \\ 1 & y \ge 1 \end{cases}$ independent of θ

$\therefore \frac{Y}{\theta}$ is a pivot quality

8.44 Let Y have probability density function

$$f_Y(y) = \begin{cases} \dfrac{2(\theta - y)}{\theta^2}, & 0 < y < \theta, \\ 0, & \text{elsewhere.} \end{cases}$$

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b Show that Y/θ is a pivotal quantity.

c Use the pivotal quantity from part (b) to find a 90% lower confidence limit for θ.

Problem 4

$P(\frac{Y}{\theta} < b) = 0.9$ $\quad$ $\underbrace{F_{\frac{Y}{\theta}}(b) = 0.9}$ $\Rightarrow 2b - b^2 = 0.9 \Rightarrow b = \frac{2 \pm \sqrt{0.4}}{2}$

$\Rightarrow b < 1$ $\quad$ $\therefore b = 1 - \sqrt{0.1}$

$P(\frac{Y}{\theta} < 1 - \sqrt{0.1}) = 0.9 \Rightarrow P(\theta > \frac{Y}{1-\sqrt{0.1}}) = 0.9 \Rightarrow (\frac{Y}{1-\sqrt{0.1}}, \infty)$

Problem 6

8.57 Refer to Exercise 8.29. According to the result given there, 51% of the $n = 1001$ adults polled in November 2003 claimed to be baseball fans. Construct a 99% confidence interval for the proportion of adults who professed to be baseball fans in November 2003 (after the World Series). Interpret this interval.

* 8.47, 8.48

see notes.

Problem 6

8.57 Refer to Exercise 8.29. According to the result given there, 51% of the $n = 1001$ adults polled in November 2003 claimed to be baseball fans. Construct a 99% confidence interval for the proportion of adults who professed to be baseball fans in November 2003 (after the World Series). Interpret this interval.

$\alpha = 0.01 \qquad \frac{\alpha}{2} = 0.005$

from Problem 8.4.12 in notes

$$\theta \in \left(\hat{\theta} - z_{\frac{\alpha}{2}} \cdot \sigma_{\hat{\theta}}, \ \hat{\theta} + z_{\frac{\alpha}{2}} \sigma_{\hat{\theta}}\right)$$

$$\hat{\theta} = 0.51, \quad z_{0.005} = 2.576, \quad \sigma_{\hat{\theta}} \approx \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.51 \times 0.49}{1001}} \approx 0.0158$$

$$\therefore \theta \in (0.51 - 2.576 \times 0.0158, \ 0.51 + 2.576 \times 0.0158)$$

$$\theta \in (0.4693, \ 0.5507)$$

8.58 The administrators for a hospital wished to estimate the average number of days required for inpatient treatment of patients between the ages of 25 and 34. A random sample of 500 hospital patients between these ages produced a mean and standard deviation equal to 5.4 and 3.1 days, respectively. Construct a 95% confidence interval for the mean length of stay for the population of patients from which the sample was drawn.

Problem 7

8.58 The administrators for a hospital wished to estimate the average number of days required for inpatient treatment of patients between the ages of 25 and 34. A random sample of 500 hospital patients between these ages produced a mean and standard deviation equal to 5.4 and 3.1 days, respectively. Construct a 95% confidence interval for the mean length of stay for the population of patients from which the sample was drawn.

large-sample

θ

$\alpha = 5\%$, $\frac{\alpha}{2} = 0.025$, $Z_{\frac{\alpha}{2}} = 1.96$

Use Large Sample Confidence Interval

$\hat{\theta} = 5.4$

$\sigma_{\hat{\theta}} \approx \frac{3.1}{\sqrt{500}}$

$$\theta \in \left(\hat{\theta} - Z_{\frac{\alpha}{2}} \sigma_{\hat{\theta}},\ \hat{\theta} + Z_{\frac{\alpha}{2}} \sigma_{\hat{\theta}}\right)$$

$(5.128, 5.672)$

8.59 When it comes to advertising, "'tweens" are not ready for the hard-line messages that advertisers often use to reach teenagers. The Geppeto Group study[6] found that 78% of 'tweens understand and enjoy ads that are silly in nature. Suppose that the study involved $n = 1030$ 'tweens.

a Construct a 90% confidence interval for the proportion of 'tweens who understand and enjoy ads that are silly in nature.

b Do you think that "more than 75%" of all 'tweens enjoy ads that are silly in nature? Why?

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Problem 8

large sample

a Construct a 90% confidence interval for the proportion of 'tweens who understand and enjoy ads that are silly in nature.

b Do you think that "more than 75%" of all 'tweens enjoy ads that are silly in nature? Why?

a. $\alpha = 0.1$ $\quad \frac{\alpha}{2} = 0.05$ $\quad Z_{\frac{\alpha}{2}} = 1.645$

$\hat{\theta} = 0.78$

$\theta \in (\hat{\theta} - Z_{\frac{\alpha}{2}} \sigma_{\hat{\theta}}, \ \hat{\theta} + Z_{\frac{\alpha}{2}} \sigma_{\hat{\theta}})$

$\sigma_{\hat{\theta}} \approx \sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}} = \sqrt{\frac{0.78 \times 0.22}{1030}}$

$= 0.0129$

$(0.759, 0.801)$

b. $75\% \notin CI$, likely to be true.

THANK YOU FOR ATTENDING TODAY'S TUTORIAL

Feedbacks: yuehuan.he@utoronto.ca