STA260: PROBABILITY AND STATISTICS II
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Practice Question 1 (10.101b)

Suppose that $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a population having an exponential distribution with mean θ.

a Derive the most powerful test for $H_0 : \theta = \theta_0$ against $H_a : \theta = \theta_a$, where $\theta_a < \theta_0$.

b Is the test derived in part (a) uniformly most powerful for testing $H_0 : \theta = \theta_0$ against $H_a : \theta < \theta_0$?

$\frac{1}{\theta_0} \sum_{i=1}^{n} Y_i \sim \text{Gamma}(n, 2)$, i.e. $\frac{2}{\theta_0} \sum_{i=1}^{n} Y_i \sim \chi^2(2n)$

$\therefore P\left(\frac{2}{\theta_0} \sum_{i=1}^{n} Y_i \le \chi^2_{1-\alpha}(2n)\right) = \alpha$

$P\left(\sum_{i=1}^{n} Y_i \le \frac{\theta_0}{2} \chi^2_{1-\alpha}(2n)\right) = \alpha$

$\therefore$ test statistic : $\sum_{i=1}^{n} Y_i$

rejection region : $\sum_{i=1}^{n} Y_i \le \frac{\theta_0}{2} \chi^2_{1-\alpha}(2n)$

Practice Question 1 (10.101b)

Suppose that $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a population having an exponential distribution with mean θ.

a Derive the most powerful test for $H_0 : \theta = \theta_0$ against $H_a : \theta = \theta_a$, where $\theta_a < \theta_0$.

b Is the test derived in part (a) uniformly most powerful for testing $H_0 : \theta = \theta_0$ against $H_a : \theta < \theta_0$?

b since the rejection region in (a) does not depend on θ_a (when $\theta_a < \theta_0$)

it's the uniformly most powerful test

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Practice Question 2 (10.95b)

Suppose that we have a random sample of four observations from the density function

$$f(y \mid \theta) = \begin{cases} \left(\dfrac{1}{2\theta^3}\right) y^2 e^{-y/\theta}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find the rejection region for the most powerful test of $H_0 : \theta = \theta_0$ against $H_a : \theta = \theta_a$, assuming that $\theta_a > \theta_0$. [*Hint:* Make use of the χ^2 distribution.]

b Is the test given in part (a) uniformly most powerful for the alternative $\theta > \theta_0$?

$\frac{2}{\theta_0}(\sum_{i=1}^{4} y_i) \sim \text{Gamma}(12, 2)$ i.e. $\frac{2}{\theta_0}(\sum_{i=1}^{4} y_i) \sim \chi^2(24)$

the rejection region for the most powerful test is $\left\{ \frac{2(\sum_{i=1}^{4} y_i)}{\theta_0} \geq \chi^2_\alpha \right\}$

Practice Question 2 (10.95b)

Suppose that we have a random sample of four observations from the density function

$$f(y \mid \theta) = \begin{cases} \left(\dfrac{1}{2\theta^3}\right) y^2 e^{-y/\theta}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find the rejection region for the most powerful test of $H_0 : \theta = \theta_0$ against $H_a : \theta = \theta_a$, assuming that $\theta_a > \theta_0$. [*Hint:* Make use of the χ^2 distribution.]

b Is the test given in part (a) uniformly most powerful for the alternative $\theta > \theta_0$?

b since the rejection region in (a) does not depend on θ_a (when $\theta_a < \theta_0$)

it's the uniformly most powerful test

Practice Question 3 (10.99)

Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a population having a Poisson distribution with mean λ.

a Find the form of the rejection region for a most powerful test of $H_0 : \lambda = \lambda_0$ against $H_a : \lambda = \lambda_a$, where $\lambda_a > \lambda_0$.

b Recall that $\sum_{i=1}^{n} Y_i$ has a Poisson distribution with mean $n\lambda$. Indicate how this information can be used to find any constants associated with the rejection region derived in part (a).

a $\quad \mathcal{L}(\lambda) = \mathcal{L}(\lambda; y_1 \cdots y_n) = \prod_{i=1}^{n} p(y_i \mid \lambda) = \prod_{i=1}^{n} \frac{\lambda^{y_i} e^{-\lambda}}{y_i!} = \frac{\lambda^{\sum_{i=1}^{n} y_i} e^{-n\lambda}}{\prod_{i=1}^{n} y_i!}$

using Neyman-Pearson lemma

$$\frac{\mathcal{L}(\lambda_0)}{\mathcal{L}(\lambda_a)} = \left(\frac{\lambda_0}{\lambda_a}\right)^{\sum_{i=1}^{n} y_i} e^{-n(\lambda_0 - \lambda_a)} < k$$

$$\left(\sum_{i=1}^{n} y_i\right) \ln\left(\frac{\lambda_0}{\lambda_a}\right) < \ln k + n(\lambda_0 - \lambda_a) \quad \Leftrightarrow$$

$$\because \lambda_a > \lambda_0 \quad \therefore \frac{\lambda_0}{\lambda_a} < 1 \quad \ln\left(\frac{\lambda_0}{\lambda_a}\right) < 0 \quad \therefore \sum_{i=1}^{k} y_i > k^*$$

$$k^* = \left[\ln\left(\frac{\lambda_0}{\lambda_a}\right)\right]^{-1} \left(\ln k + n(\lambda_0 - \lambda_a)\right)$$

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Practice Question 3 (10.99)

Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a population having a Poisson distribution with mean λ.

a Find the form of the rejection region for a most powerful test of $H_0 : \lambda = \lambda_0$ against $H_a : \lambda = \lambda_a$, where $\lambda_a > \lambda_0$.

b Recall that $\sum_{i=1}^{n} Y_i$ has a Poisson distribution with mean $n\lambda$. Indicate how this information can be used to find any constants associated with the rejection region derived in part (a).

$\therefore$ form of rejection region $\quad \{\sum_{i=1}^{n} y_i > k^*\}$

b. $\sum_{i=1}^{n} Y_i \sim \text{Poisson}(n\lambda)$

under H_0, $\quad \sum_{i=1}^{n} Y_i \sim \text{Poisson}(n\lambda_0)$

$$P\left(\sum_{i=1}^{n} y_i > k^*\right) = \alpha$$

Practice Question 3 (10.99)

Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a population having a Poisson distribution with mean λ.

c Is the test derived in part (a) uniformly most powerful for testing $H_0 : \lambda = \lambda_0$ against $H_a : \lambda > \lambda_0$? Why?

d Find the form of the rejection region for a most powerful test of $H_0 : \lambda = \lambda_0$ against $H_a : \lambda = \lambda_a$, where $\lambda_a < \lambda_0$.

c. since the rejection region in (a) does not depend on λ_a (when $\lambda_a > \lambda_0$)

it's the uniformly most powerful test

d. using Neyman-Pearson lemma

$$\frac{\mathcal{L}(\lambda_0)}{\mathcal{L}(\lambda_a)} = \left(\frac{\lambda_0}{\lambda_a}\right)^{\sum_{i=1}^n y_i} e^{-n(\lambda_0-\lambda_a)} < k$$

$$\left(\sum_{i=1}^n y_i\right) \boxed{\ln\left(\frac{\lambda_0}{\lambda_a}\right)} < \ln k + n(\lambda_0 - \lambda_a)$$

$\because \lambda_a < \lambda_0$

$\therefore \frac{\lambda_0}{\lambda_a} > 1 \quad \ln\left(\frac{\lambda_0}{\lambda_a}\right) > 0$

$\therefore$ form of RR

$\{\sum_{i=1}^n y_i < k^*\}$

Practice Question 3 (10.99)

Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a population having a Poisson distribution with mean λ.

e Is the test derived in part (a) uniformly most powerful for testing $H_0 : \lambda = \lambda_0$ against $H_\alpha : \lambda \neq \lambda_0$

e not. since rejection region depends on $\lambda_a \in \Omega_a$. ($\lambda_a > \lambda_0$ or $\lambda_a < \lambda_0$).

Practice Question 4 (10.105)

Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a normal distribution with mean μ (unknown) and variance σ^2. For testing $H_0 : \sigma^2 = \sigma_0^2$ against $H_a : \sigma^2 > \sigma_0^2$, find the likelihood ratio test.

$\Omega_0 = \{\sigma^2 : \sigma^2 = \sigma_0^2\}$ $\Omega_a = \{\sigma^2 : \sigma^2 > \sigma_0^2\}$ $\Omega = \Omega_0 \cup \Omega_a = \{\sigma^2 : \sigma^2 \geq \sigma_0^2\}$

$$L(\theta|y) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} \exp\left\{-\sum_{i=1}^n \frac{(y_i-\mu)^2}{2\sigma^2}\right\}$$

For Ω_0, $\hat{\mu} = \bar{y}$, $\hat{\sigma}^2 = \sigma_0^2$ (by analyzing $\frac{\partial L}{\partial \mu}$)

For Ω, $\hat{\mu} = \bar{y}$, $\hat{\sigma}^2_\Omega = \begin{cases} \frac{1}{n}\sum_{i=1}^n (y_i - \bar{y})^2, & \text{if } \frac{1}{n}\sum_{i=1}^n (y_i-\bar{y})^2 > \sigma_0^2 \\ \sigma_0^2, & \text{otherwise} \end{cases}$

$$\lambda(y) = \frac{\sup_{\theta\in\Omega_0} L(\theta|y)}{\sup_{\theta\in\Omega} L(\theta|y)} = \left(\frac{\hat{\sigma}^2_\Omega}{\sigma_0^2}\right)^{\frac{n}{2}} \exp\left\{-\left(\frac{1}{2\sigma_0^2} - \frac{1}{2\hat{\sigma}^2_\Omega}\right)\sum_{i=1}^n (y_i - \bar{y})^2\right\}$$

Practice Question 4 (10.105)

Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a normal distribution with mean μ (unknown) and variance σ^2. For testing $H_0 : \sigma^2 = \sigma_0^2$ against $H_a : \sigma^2 > \sigma_0^2$, find the likelihood ratio test.

$$\therefore \lambda(y) = \begin{cases} \left(\dfrac{\sum_{i=1}^n (y_i - \bar{y})^2}{n\sigma_0^2}\right)^{\frac{n}{2}} \exp\left\{-\dfrac{\sum_{i=1}^n (y_i-\bar{y})^2}{2\sigma_0^2} + \dfrac{n}{2}\right\} & \text{if } \frac{1}{n}\sum_{i=1}^n (y_i - \bar{y})^2 > \sigma_0^2 \\ 1 & \text{otherwise} \end{cases}$$

rejection region $\{\lambda(y) \le k\}$

let $\chi^2 = \dfrac{(n-1)S^2}{\sigma_0^2}$, i.e. $\chi^2 = \dfrac{\sum_{i=1}^n (y_i - \bar{y})^2}{n\sigma_0^2}$ then $\lambda(y) = (\chi^2)^{\frac{n}{2}} \exp\left\{-\frac{n}{2}\chi^2 + \frac{n}{2}\right\}$ if $\chi^2 > 1$

$\dfrac{\partial \lambda}{\partial \chi^2} < 0$ $\quad \therefore \lambda(y) \le k \iff \chi^2 \ge k^*$ $\quad$ i.e. rejection region

$P\{\chi^2 \ge k^*\} = \alpha$ $\quad \left\{\dfrac{(n-1)S^2}{\sigma_0^2} \ge \chi^2_\alpha\right\}$