STA260: PROBABILITY AND STATISTICS II
SPRING 2021

TUTORIAL 9 (TUT9101)

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Practice Problem 1 (Exercise 10.3)

10.3 An experimenter has prepared a drug dosage level that she claims will induce sleep for 80% of people suffering from insomnia. After examining the dosage, we feel that her claims regarding the effectiveness of the dosage are inflated. In an attempt to disprove her claim, we administer her prescribed dosage to 20 insomniacs and we observe Y, the number for whom the drug dose induces sleep. We wish to test the hypothesis $H_0 : p = .8$ versus the alternative, $H_a : p < .8$.

a Find the rejection region of the form $\{y \le c\}$ so that $\alpha \approx .01$.

b For the rejection region in part (a), find β when $p = .6$.

c For the rejection region in part (a), find β when $p = .4$.

Table 1 Binomial Probabilities

Tabulated values are $P(Y \le a) = \sum_{y=0}^{a} p(y)$. (Computations are rounded at third decimal place.)

(d) $n = 20$

	p													
a	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99	a
0	.818	.358	.122	.012	.001	.000	.000	.000	.000	.000	.000	.000	.000	0
1	.983	.736	.392	.069	.008	.001	.000	.000	.000	.000	.000	.000	.000	1
2	.999	.925	.677	.206	.035	.004	.000	.000	.000	.000	.000	.000	.000	2
3	1.000	.984	.867	.411	.107	.016	.001	.000	.000	.000	.000	.000	.000	3
4	1.000	.997	.957	.630	.238	.051	.006	.000	.000	.000	.000	.000	.000	4
5	1.000	1.000	.989	.804	.416	.126	.021	.002	.000	.000	.000	.000	.000	5
6	1.000	1.000	.998	.913	.608	.250	.058	.006	.000	.000	.000	.000	.000	6
7	1.000	1.000	1.000	.968	.772	.416	.132	.021	.001	.000	.000	.000	.000	7
8	1.000	1.000	1.000	.990	.887	.596	.252	.057	.005	.000	.000	.000	.000	8
9	1.000	1.000	1.000	.997	.952	.755	.412	.128	.017	.001	.000	.000	.000	9
10	1.000	1.000	1.000	.999	.983	.872	.588	.245	.048	.003	.000	.000	.000	10
11	1.000	1.000	1.000	1.000	.995	.943	.748	.404	.113	.010	.000	.000	.000	11
12	1.000	1.000	1.000	1.000	.999	.979	.868	.584	.228	.032	.000	.000	.000	12
13	1.000	1.000	1.000	1.000	1.000	.994	.942	.750	.392	.087	.002	.000	.000	13
14	1.000	1.000	1.000	1.000	1.000	.998	.979	.874	.584	.196	.011	.000	.000	14
15	1.000	1.000	1.000	1.000	1.000	1.000	.994	.949	.762	.370	.043	.003	.000	15
16	1.000	1.000	1.000	1.000	1.000	1.000	.999	.984	.893	.589	.133	.016	.000	16
17	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.996	.965	.794	.323	.075	.001	17
18	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.999	.992	.931	.608	.264	.017	18
19	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.999	.988	.878	.642	.182	19

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a Find the rejection region of the form $\{y \le c\}$ so that $\alpha \approx .01$.

$\alpha = P_{H_0}\{y \le c\}$ i.e. $P\{y \le c \mid p = 0.8\} = \alpha$

from table 1 (binomial probabilities) $P\{y \le 11 \mid p = 0.8\} \approx 0.1$

$\therefore$ rejection region $\{y \le 11\}$

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b For the rejection region in part (a), find β when $p = .6$.

$$\beta_{p=0.6} = P\{\text{retain } H_0 \text{ when } H_0 \text{ is false}, p=0.6\}$$

$$= P\{\underbrace{y > 11}_{\text{retain } H_0} \mid p=0.6\}$$

$$= 1 - \underbrace{P\{y \le 11 \mid p=0.6\}}_{0.404 \text{ (from table 1)}} = 0.596$$

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c For the rejection region in part (a), find β when $p = .4$.

$$\beta_{p=0.4} = \{ y > 11 \mid p = 0.4 \}$$

$$= 1 - P\{ \underbrace{y \leq 11 \mid p = 0.4}_{0.943} \}$$

$$= 0.057$$

Practice Problem 2

10.6 We are interested in testing whether or not a coin is balanced based on the number of heads Y on 36 tosses of the coin. ($H_0 : p = .5$ versus $H_a : p \neq .5$). If we use the rejection region $|y - 18| \geq 4$, what is

a the value of α?

b the value of β if $p = .7$?

$y \leq 14$ or $y \geq 22$

$\uparrow$

$$\alpha = P\{\text{reject } H_0 \mid H_0 \text{ is true}\} = P\{|y-18| \geq 4 \mid p = 0.5\}$$

$$= 1 - P\{14 < y < 22 \mid p = 0.5\}$$

$$= 1 - \sum_{i=15}^{21} C_{36}^{i} \left(\tfrac{1}{2}\right)^{36} = 0.243$$

$$\beta_{p=0.7} = P\{\text{retain } H_0 \mid H_0 \text{ is false},\ p = 0.7\}$$

$$= P\{|y-18| < 3 \mid p = 0.7\} = \sum_{i=15}^{21} C_{36}^{i} (0.7)^{i} (0.3)^{36-i} = 0.092$$

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Practice Problem 3

10.5 Let Y_1 and Y_2 be independent and identically distributed with a uniform distribution over the interval $(\theta,\ \theta+1)$. For testing $H_0 : \theta = 0$ versus $H_a : \theta > 0$, we have two competing tests:

Test 1: Reject H_0 if $Y_1 > .95$.

10.89 Refer to Exercise 10.5. Find the power of test 1 for each alternative in (a)–(e).

a $\theta = .1.$

b $\theta = .4.$

c $\theta = .7.$

d $\theta = 1.$

power $= 1-\beta = P\{\text{reject } H_0 \mid H_0 \text{ is false}\}$

$= P\{y_1 > 0.95 \mid \theta\}$

a. $P\{y_1 > 0.95 \mid \theta = 0.1\}$

$= \int_{0.95}^{1.1} 1\, dy$

$= 0.15$

b. $P\{y_1 > 0.95 \mid \theta = 0.4\}$

$= \int_{0.95}^{1.4} 1\, dy$

$= 0.45$

similarly

c power $= 0.75$

d. power $= 1$

$n=4$

10.95 Suppose that we have a random sample of four observations from the density function

$$f(y \mid \theta) = \begin{cases} \left(\dfrac{1}{2\theta^3}\right) y^2 e^{-y/\theta}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find the rejection region for the most powerful test of $H_0 : \theta = \theta_0$ against $H_a : \theta = \theta_a$, assuming that $\theta_a > \theta_0$. [*Hint:* Make use of the χ^2 distribution.]

$$L(\theta : y) = \frac{1}{2^n \theta^{3n}} \left(\prod_{i=1}^{n} y_i^2\right) e^{-\frac{1}{\theta}\sum_{i=1}^{n} y_i} \quad \text{for } y>0$$

$$\frac{L(\theta_0 ; y)}{L(\theta_a ; y)} = \frac{\theta_a^{3n}}{\theta_0^{3n}} e^{-\left(\frac{1}{\theta_0} - \frac{1}{\theta_a}\right)\sum_{i=1}^{n} y_i} \overset{(n=4)}{=} \frac{\theta_a^{12}}{\theta_0^{12}} e^{-\left(\frac{1}{\theta_0} - \frac{1}{\theta_a}\right)\sum_{i=1}^{4} y_i}$$

use theorem 10.2.2 to find critical region $\dfrac{\theta_a^{12}}{\theta_0^{12}} e^{-\left(\frac{1}{\theta_0} - \frac{1}{\theta_a}\right)\sum_{i=1}^{4} y_i} \le k$

Practice Problem 4

10.95 Suppose that we have a random sample of four observations from the density function

$$f(y \mid \theta) = \begin{cases} \left(\frac{1}{2\theta^3}\right) y^2 e^{-y/\theta}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find the rejection region for the most powerful test of $H_0 : \theta = \theta_0$ against $H_a : \theta = \theta_a$, assuming that $\theta_a > \theta_0$. [*Hint:* Make use of the χ^2 distribution.]

$$\frac{\theta_a^{12}}{\theta_0^{12}} e^{-\left(\frac{1}{\theta_0} - \frac{1}{\theta_a}\right)\sum_{i=1}^4 y_i} \le k \Rightarrow \sum_{i=1}^4 y_i \geq k^*, \quad k^* = \frac{\ln\left(\frac{\theta_0^{12}}{\theta_a^{12}} k\right)}{\frac{1}{\theta_a} - \frac{1}{\theta_0}}$$

under H_0, $Y_1, \cdots, Y_4 \overset{iid}{\sim} \text{Gamma}(3, \theta_0)$

$\frac{2}{\theta_0}\left(\sum_{i=1}^4 y_i\right) \sim \text{Gamma}(12, 2)$ i.e. $\frac{2}{\theta_0}\left(\sum_{i=1}^4 y_i\right) \sim \chi^2(24)$

the rejection region for the most powerful test is $\left\{\frac{2\left(\sum_{i=1}^4 y_i\right)}{\theta_0} \geq \chi^2_\alpha\right\}$

THANK YOU FOR ATTENDING TODAY’S TUTORIAL

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