STA260: PROBABILITY AND STATISTICS II
SPRING 2021

TUTORIAL 10 (TUT9101)

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YUEHUAN HE

UNIVERSITY OF TORONTO

Practice Question 1 (9.104)

Suppose that $Y_1, Y_2, \ldots, Y_n$ constitute a random sample from the density function

$$f(y \mid \theta) = \begin{cases} e^{-(y-\theta)}, & y > \theta, \\ 0, & \text{elsewhere} \end{cases}$$

where θ is an unknown, positive constant.

a Find an estimator $\hat{\theta}_1$ for θ by the method of moments.

b Find an estimator $\hat{\theta}_2$ for θ by the method of maximum likelihood.

c Adjust $\hat{\theta}_1$ and $\hat{\theta}_2$ so that they are unbiased. Find the efficiency of the adjusted $\hat{\theta}_1$ relative to the adjusted $\hat{\theta}_2$.

a $\mu_1' = E(Y) = \int_\theta^{+\infty} e^{-(y-\theta)} y \, dy = \int_0^{+\infty} e^{-t} (t+\theta) \, dt = \int_0^{+\infty} e^{-t} t \, dt + \theta \int_0^{+\infty} e^{-t} dt$

$= [-e^{-t} t]\Big|_0^{+\infty} - \int_0^{+\infty} (-e^{-t}) \, dt + \theta \int_0^{+\infty} e^{-t} dt = \theta + 1$

$m_1' = \bar{Y}$ method of moment estimator $\hat{\theta}_1 = \bar{Y} - 1$

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b. likelihood function $\mathcal{L}(\theta) = \begin{cases} e^{-\sum_{i=1}^{n} y_i + n\theta}, & \theta < y_1, y_2, \ldots, y_n \\ 0, & \text{elsewhere} \end{cases}$

$\theta \uparrow$, $n\theta \uparrow$ $(\theta > 0)$, $e^{n\theta} \uparrow$, $e^{-\sum_{i=1}^{n} y_i} \cdot e^{n\theta} \uparrow$ larger θ. larger $\mathcal{L}(\theta)$

$\therefore$ largest θ satisfy all constraints gives $\hat{\theta}_2 = Y_{(1)} = \min\{Y_1 \cdots Y_n\}$

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c. $E(\hat{\theta}_1) = E(\bar{Y}) - 1 = (\theta + 1) - 1 = \theta$ unbiased.

$E(\hat{\theta}_2) = E(Y_{(1)}) = \int_{\theta}^{+\infty} n e^{-n(y-\theta)} y \, dy = \theta + \frac{1}{n}$ adjust it to be $\hat{\theta}_2' = Y_{(1)} - \frac{1}{n}$

$V(\hat{\theta}_1) = V(\bar{Y}) = \frac{1}{n} V(Y) = \frac{1}{n}\left[\underbrace{\int_{\theta}^{+\infty} e^{-(y-\theta)} y^2 \, dy}_{(\theta^2 + 2\theta + 2)} - \underbrace{E(\bar{Y})^2}_{(\theta+1)^2}\right] = \frac{1}{n}$

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$$V(\hat{\theta}_2') = V(Y_{(1)} - \tfrac{1}{n}) = V(Y_{(1)}) = \int_{\theta}^{+\infty} n e^{-n(y-\theta)} y^2 \, dy - E(Y_{(1)})^2 = \frac{1}{n^2}.$$

$$\text{efficiency}(\hat{\theta}_1, \hat{\theta}_2') = \frac{V(\hat{\theta}_2')}{V(\hat{\theta}_1)} = \frac{\frac{1}{n^2}}{\frac{1}{n}} = \frac{1}{n}.$$

Practice Question 2 (9.106)

Suppose that $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a Poisson distribution with mean λ. Find the MVUE of $P(Y_i = 0) = e^{-\lambda}$. [*Hint:* Make use of the Rao–Blackwell theorem.]

Using Rao-Blackwell Thoerem, find $\hat{\theta}^* = E(\hat{\theta} \mid U)$ where $\hat{\theta}$ is an unbiased estimator

U is a sufficient statistic

note that $P(Y_i = 0) = e^{-\lambda}$, define $T = \begin{cases} 1, & Y_1 = 0 \\ 0, & \text{otherwise} \end{cases}$

then $\hat{\theta} = T$, $E(\hat{\theta}) = E(T) = 1 \cdot P(Y_1 = 0) = e^{-\lambda}$

$\hat{\theta}$ is an unbiased estimator for $e^{-\lambda}$

from problem 9.5.4 in notes, $U = \sum_{i=1}^{n} Y_i$ is sufficient

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$$E(\hat{\theta} \mid U = u) = P(T = 1 \mid \textstyle\sum_{i=1}^{n} Y_i = u) = P(Y_1 = 0 \mid \sum_{i=1}^{n} Y_i = u)$$

$$= \frac{P(Y_1 = 0, \sum_{i=1}^{n} Y_i = u)}{P(\sum_{i=1}^{n} Y_i = u)} = \frac{P(Y_1 = 0)\, P(\sum_{i=2}^{n} Y_i = u)}{P(\sum_{i=1}^{n} Y_i = u)}$$

$U = \sum_{i=1}^{n} Y_i \sim \text{Poisson}(n\lambda)$

$\sum_{i=2}^{n} Y_i \sim \text{Poisson}((n-1)\lambda)$

$$= \frac{e^{-\lambda} \cdot e^{-(n-1)\lambda} \frac{[(n-1)\lambda]^u}{u!}}{e^{-n\lambda} \frac{(n\lambda)^u}{u!}} = \frac{(n-1)^u}{n^u} = \left(\frac{n-1}{n}\right)^u$$

$\therefore \hat{\theta}^* = E(\hat{\theta} \mid U) = \left(\frac{n-1}{n}\right)^{\sum_{i=1}^{n} Y_i}$ is the MVUE of $P(Y_i = 0) = e^{-\lambda}$.

Practice Question 3 (10.101a)

Suppose that $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a population having an exponential distribution with mean θ.

Derive the most powerful test for $H_0 : \theta = \theta_0$ against $H_a : \theta = \theta_a$, where $\theta_a < \theta_0$.

by Neyman-Pearson lemma, the best critical region is

$$\frac{\mathcal{L}(\theta_0)}{\mathcal{L}(\theta_a)} \leq k$$

since $0 < \theta_a < \theta_0$

$$\frac{1}{\theta_0} < \frac{1}{\theta_a}$$

$$\mathcal{L}(\theta) = \mathcal{L}(\theta : y) = \prod_{i=1}^{n} f(y_i \mid \theta) = \frac{1}{\theta^n} e^{-\frac{1}{\theta}\sum_{i=1}^n y_i}$$

$$\left(\frac{1}{\theta_0} - \frac{1}{\theta_a}\right) < 0$$

$$\therefore \frac{\mathcal{L}(\theta_0)}{\mathcal{L}(\theta_a)} = \frac{\theta_a^n}{\theta_0^n} e^{-\left(\frac{1}{\theta_0} - \frac{1}{\theta_a}\right)\sum_{i=1}^n y_i} \leq k$$

$$\left(\frac{1}{\theta_a} - \frac{1}{\theta_0}\right) > 0$$

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Derive the most powerful test for $H_0 : \theta = \theta_0$ against $H_a : \theta = \theta_a$, where $\theta_a < \theta_0$.

$$\frac{\theta_a^n}{\theta_0^n} e^{-\left(\frac{1}{\theta_0} - \frac{1}{\theta_a}\right)\sum_{i=1}^n y_i} \leq k \quad \Rightarrow \quad \sum_{i=1}^n y_i \leq k^* = \left(\frac{1}{\theta_a} - \frac{1}{\theta_0}\right)^{-1} \ln\left(k \cdot \frac{\theta_0^n}{\theta_a^n}\right)$$

$$= \left(\frac{1}{\theta_a} - \frac{1}{\theta_0}\right) > 0$$

under H_0, $Y_i \overset{iid}{\sim}$ exponential (θ_0), i.e. $Y_i \overset{iid}{\sim}$ Gamma $(1, \theta_0)$ $\therefore \sum_{i=1}^n Y_i \sim$ Gamma (n, θ_0)

$$\frac{2}{\theta_0}\sum_{i=1}^n Y_i \sim \text{Gamma}(n, 2), \text{ i.e. } \frac{2}{\theta_0}\sum_{i=1}^n Y_i \sim \chi^2(2n)$$

$$\therefore P\left(\frac{2}{\theta_0}\sum_{i=1}^n Y_i \leq \chi^2_{1-\alpha}(2n)\right) = \alpha$$

$$P\left(\sum_{i=1}^n Y_i \leq \frac{\theta_0}{2}\chi^2_{1-\alpha}(2n)\right) = \alpha$$

$\therefore$ test statistic: $\sum_{i=1}^n Y_i$

rejection region: $\sum_{i=1}^n Y_i \leq \frac{\theta_0}{2}\chi^2_{1-\alpha}(2n)$

THANK YOU FOR ATTENDING TODAY'S TUTORIAL

Feedbacks: yuehuan.he@utoronto.ca