STA260: PROBABILITY AND STATISTICS II
SPRING 2021

TUTORIAL 7 (TUT9101)

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YUEHUAN HE

UNIVERSITY OF TORONTO

9.69 Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from the probability density function

$$f(y\,|\,\theta) = \begin{cases} (\theta+1)y^{\theta}, & 0 < y < 1;\ \theta > -1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find an estimator for θ by the method of moments. Show that the estimator is consistent. Is the estimator a function of the sufficient statistic $-\sum_{i=1}^{n} \ln(Y_i)$ that we can obtain from the factorization criterion? What implications does this have?

$$\mu_1' = E(Y) = \int_0^1 (\theta+1)t^{\theta}\,t\,dt = \frac{\theta+1}{\theta+2}$$

$$m_1' = \frac{1}{n}\sum_{i=1}^{n} Y_i = \bar{Y}$$

$$\bar{Y} = \frac{\hat{\theta}+1}{\hat{\theta}+2} \Rightarrow \bar{Y}\hat{\theta} + 2\bar{Y} = \hat{\theta}+1 \Rightarrow \hat{\theta} = \frac{1-2\bar{Y}}{\bar{Y}-1}$$

9.69 Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from the probability density function

$$f(y \mid \theta) = \begin{cases} (\theta+1)y^{\theta}, & 0 < y < 1;\ \theta > -1, \\ 0, & \text{elsewhere.} \end{cases}$$

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$\bar{Y}$ is a consistent estimator of $\mu = E(Y) = \frac{\theta+1}{\theta+2}$. (problem 9.2.3)

by continous mapping theorem for convergence in probability

$g(x) = \frac{1-2x}{x-1}$, $g(x)$ is continous at $\frac{\theta+1}{\theta+2}$ since $\theta > -1$, $\frac{\theta+1}{\theta+2} \neq 0$.

$\bar{Y} \xrightarrow{P} \mu$ $\quad \therefore g(\bar{Y}) \xrightarrow{P} g(\mu)$ $\quad \hat{\theta} = g(\bar{Y})$, $\quad \dfrac{1 - 2\cdot\frac{\theta+1}{\theta+2}}{\frac{\theta+1}{\theta+2} - 1} = \dfrac{\theta+2-2\theta-2}{-1}$

$= \theta$

$\therefore \hat{\theta} \xrightarrow{P} \theta$ consistent

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Find an estimator for θ by the method of moments. Show that the estimator is consistent. Is the estimator a function of the sufficient statistic $-\sum_{i=1}^{n} \ln(Y_i)$ that we can obtain from the factorization criterion? What implications does this have?

$\hat{\theta} = \dfrac{1-2\bar{Y}}{\bar{Y}-1}$ is not a function of the given sufficient statistics $-\sum_{i=1}^{n} \ln(Y_i)$

$\therefore f(\hat{\theta})$ is not a MVUE.

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9.70 Suppose that $Y_1, Y_2, \ldots, Y_n$ constitute a random sample from a Poisson distribution with mean λ. Find the method-of-moments estimator of λ.

$\mu_1' = \lambda$ $\qquad \bar{Y} = \hat{\lambda}$ $\qquad \Rightarrow$ method-of moments estimator

$m_1' = \bar{Y}$ $\qquad \hat{\lambda} = \bar{Y}$

9.71 If $Y_1, Y_2, \ldots, Y_n$ denote a random sample from the normal distribution with known mean $\mu = 0$ and unknown variance σ^2, find the method-of-moments estimator of σ^2.

$\mu_1' = E(Y) = \mu = 0$

$\mu_2' = E(Y^2) = V(Y) + (E(Y))^2 = \sigma^2$

$m_2' = \frac{1}{n}\sum_{i=1}^{n} Y_i^2 = \hat{\sigma}^2$

method-of-moments estimator $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} Y_i^2$

9.77 Let $Y_1, Y_2, \ldots, Y_n$ denote independent and identically distributed uniform random variables on the interval $(0, 3\theta)$. Derive the method-of-moments estimator for θ.

$$\mu_1' = E(Y) = \int_0^{3\theta} \frac{1}{3\theta} \cdot t \, dt = \frac{1}{3\theta} \cdot \frac{1}{2} \cdot (3\theta)^2 = \frac{3}{2}\theta .$$

$$m_1' = \frac{1}{n} \sum_{i=1}^{n} Y_i = \bar{Y}$$

$$\frac{3}{2}\hat{\theta} = \bar{Y} \quad \Rightarrow \text{method-of-moments estimator } \hat{\theta} = \frac{2}{3}\bar{Y} .$$

9.78 Let $Y_1, Y_2, \ldots, Y_n$ denote independent and identically distributed random variables from a power family distribution with parameters α and $\theta = 3$. Then, as in Exercise 9.43, if $\alpha > 0$,

$$f(y|\alpha) = \begin{cases} \alpha y^{\alpha-1}/3^{\alpha}, & 0 \le y \le 3, \\ 0, & \text{elsewhere.} \end{cases}$$

Show that $E(Y_1) = 3\alpha/(\alpha+1)$ and derive the method-of-moments estimator for α.

$$\mu_1' = E(Y_1) = \int_0^3 \frac{\alpha}{3^\alpha} t^{\alpha-1} \cdot t \, dt = \int_0^3 \frac{\alpha}{3^\alpha} t^\alpha dt = \frac{\alpha}{\alpha+1} \cdot \frac{1}{3^\alpha} 3^{\alpha+1} = \frac{3\alpha}{\alpha+1}$$

$$m_1' = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

method-of-moments estimator.

$$\frac{3\hat{\alpha}}{\hat{\alpha}+1} = \bar{Y} \Rightarrow 3 - \frac{3}{\hat{\alpha}+1} = \bar{Y} \Rightarrow \hat{\alpha}+1 = \frac{3}{3-\bar{Y}} \Rightarrow \hat{\alpha} = \frac{\bar{Y}}{3-\bar{Y}}.$$

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Feedbacks: yuehuan.he@utoronto.ca