Homework 1
MAD4204
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1. Let $G = ([n], E)$ be a graph and let $\overline{G} = ([n], \binom{[n]}{2} \setminus E)$ be its complement. Prove for n sufficiently large that at least one of G and $\overline{G}$ contains a cycle.

   Your proof should include a value n that guarantees this property.

   Solution. Test

2. Let $G = ([n], E)$ be a finite simple graph. Let M be a maximal matching in G and M' be a maximum matching in G. Prove that $|M'| \leq 2|M|$.

3. For $\mathcal{M}$ a matroid on ground set S defined in terms of bases, we say $I \subseteq S$ is *independent* if there exists $B \in \mathcal{M}$ so that $I \subseteq B$. An alternate definition of a matroid $\mathcal{M}_I$ on ground set S in terms of independent sets is that $\mathcal{M}_I \subseteq 2^S$ so that:

   - (hereditary property) if $A \subseteq B \in \mathcal{M}_I$, then $A \in \mathcal{M}_I$,
   - (augmentation property) for $A, B \in \mathcal{M}_I$ with $|A| < |B|$ there exists $b \in B$ so that $A \cup \{b\} \in \mathcal{M}_I$.

   We show these definitions are equivalent by solving:

   (a) For $\mathcal{M}_I$ a matroid defined in terms of its independent sets, we say $B \in \mathcal{M}_I$ is a *basis* if B is maximal in $\mathcal{M}_I$. Prove two bases in $\mathcal{M}_I$ satisfy the exchange property.

   (b) For $\mathcal{M}_B$ a matroid defined in terms of its bases, show that two independent sets A, B of $\mathcal{M}$ satisfy the augmentation property.

   (Compare this with Lemma 10.10 in the course text)

4. For $\mathcal{M}$ a matroid on ground set S (from Problem 3, we can define it in terms of bases or independent sets, whichever is more convenient), let $w : S \to \mathbb{R}_{\geq 0}$. For B a basis in $\mathcal{M}$, define $w(B) = \sum_{b \in B} w(b)$.

   Describe an algorithm for finding the basis B minimizing $w(B)$ and prove that it is optimal.

   (Hint: compare to the greedy algorithm for finding the minimum weighted spanning tree)