Assignment 6
MAA4211
Carson Mulvey

(Graded) 2.7.3. For both proofs, it suffices to show (i), as (ii) is immediately implied by the contrapositive.

(a) Suppose $\sum_{k=1}^{\infty} b_k$ converges. Let $\epsilon > 0$. Then there exists and $N \in \mathbb{N}$ such that for $n > m \geq N$, we have
$$|b_{m+1} + b_{m+2} + \cdots + b_n| < \epsilon.$$
However, because $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$, we can sum these inequalities to show that $a_{m+1} + a_{m+2} + \cdots + a_n \leq b_{m+1} + b_{m+2} + \cdots + b_n$. Because these quantities are positive, this implies
$$|a_{m+1} + a_{m+2} + \cdots + a_n| \leq |b_{m+1} + b_{m+2} + \cdots + b_n| < \epsilon.$$
Thus, taking the same N as picked for (b_n), we can apply the Cauchy Criterion for Series again, showing that $\sum_{k=1}^{\infty} a_k$ converges.

(b) Let (s_k) and (t_k) be the sequences of partial sums of (a_k) and (b_k), respectively. Because all terms of (a_k) and (b_k) are positive, both (s_k) and (t_k) are increasing sequences.
Now suppose $\sum_{k=1}^{\infty} b_k$ converges to t. Then, since (t_k) is increasing, (t_k) is bounded by t. However, since $a_k \leq b_k$ for all $k \in \mathbb{N}$, we also have $s_k \leq t_k$ for all $k \in \mathbb{N}$. This implies that (s_k) is also bounded by t. Thus, (s_k) is monotonic and bounded. By the Monotone Convergence Theorem, it must converge, and hence $\sum_{k=1}^{\infty} a_k$ also converges.