MAD 4204, Homework 1

Standard notation: Recall from class that $[n] := \{1, 2, \dots, n\}$ and that for S a set and $k \in \mathbb{N}$ that $\binom{S}{k}$ is the set of k–element subsets of S. Also, let 2^S denote the power set of S. For $\mathcal{F} \subseteq 2^S$, we say $F \in \mathcal{F}$ is *maximal* if $F \subseteq F' \in \mathcal{F}$ implies $F' = F$ and *maximum* if $|F'| \leq |F|$ for all $F' \in \mathcal{F}$.

A *matroid* $\mathcal{M}_B$ on the finite ground set S defined in terms of bases is a set $\mathcal{M}_B \subseteq \binom{S}{k}$ satisfying the *exchange property*: for all $A, B \in \mathcal{M}_B$ and $a \in A$ there exists some $b \in B$ so that the set $(A \setminus \{a\}) \cup \{b\}$ is in $\mathcal{M}_B$. Here, each $B \in \mathcal{M}$ is a *basis*.

Problems to turn in:

1. Let $G = ([n], E)$ be a graph and let $\overline{G} = ([n], \binom{[n]}{2} \setminus E)$ be its complement. Prove for n sufficiently large that at least one of G and $\overline{G}$ contains a cycle.

   Your proof should include a value n that guarantees this property.

2. Let $G = ([n], E)$ be a finite simple graph. Let M be a maximal matching in G and M' be a maximum matching in G. Prove that $|M'| \leq 2|M|$.

3. For $\mathcal{M}$ a matroid on ground set S defined in terms of bases, we say $I \subseteq S$ is *independent* if there exists $B \in \mathcal{M}$ so that $I \subseteq B$. An alternate definition of a matroid $\mathcal{M}_I$ on ground set S in terms of independent sets is that $\mathcal{M}_I \subseteq 2^S$ so that:

   - (hereditary property) if $A \subseteq B \in \mathcal{M}_I$, then $A \in \mathcal{M}_I$,
   - (augmentation property) for $A, B \in \mathcal{M}_I$ with $|A| < |B|$ there exists $b \in B$ so that $A \cup \{b\} \in \mathcal{M}_I$.

   We show these definitions are equivalent by solving:

   (a) For $\mathcal{M}_I$ a matroid defined in terms of its independent sets, we say $B \in \mathcal{M}_I$ is a *basis* if B is maximal in $\mathcal{M}_I$. Prove two bases in $\mathcal{M}_I$ satisfy the exchange property.

   (b) For $\mathcal{M}_B$ a matroid defined in terms of its bases, show that two independent sets A, B of $\mathcal{M}$ satisfy the augmentation property.
   (Compare this with Lemma 10.10 in the course text)

4. For $\mathcal{M}$ a matroid on ground set S (from Problem 3, we can define it in terms of bases or independent sets, whichever is more convenient), let $w : S \to \mathbb{R}_{\geq 0}$. For B a basis in $\mathcal{M}$, define $w(B) = \sum_{b \in B} w(b)$.

   Describe an algorithm for finding the basis B minimizing $w(B)$ and prove that it is optimal.

   (Hint: compare to the greedy algorithm for finding the minimum weighted spanning tree)

Recommended problems:

1. Read Chapter 3 if you haven't already.

2. Do all the Quick Check problems in Chapters 9 and 10.1.

3. For $G = (V, E)$ a finite simple graph, the *neighborhood* of $v \in V$ is the set $N(v) := \{u \in V : (u, v) \in E\}$ and the *degree* of v is $d(v) := |N(v)|$.

   Is there a simple graph G on six vertices whose degrees are 4,4,4,2,1,1? This is called the *degree sequence* of G.

4. Find two trees with the same degree sequence that are not isomorphic.

5. Exercise 10.10: For G a finite simple graph, show at least one of G and $\overline{G}$ is connected.