Assignment 1
MAA4211
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1.3.1. (a) A real number s is the *greatest lower bound*, or *infimum*, for a set $A \subseteq \mathbb{R}$ if:

i. s is a lower bound for A;

ii. if b is a lower bound for A, then $b \leq s$.

(b) **Lemma.** *Assume $s \in \mathbb{R}$ is an lower bound for a set $A \subseteq \mathbb{R}$. Then, $s = \inf A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s + \epsilon > a$.*

Proof. ($\Rightarrow$) Assume that $s = \inf A$. Let $\epsilon > 0$, and consider $s + \epsilon$. Because $s + \epsilon > s$, we know that $s + \epsilon$ is not a lower bound for A. This means that there exists an element $a \in A$ such that $s + \epsilon > a$.

($\Leftarrow$) Now assume s is a lower bound such that for all $\epsilon > 0$, there is $a \in A$ satisfying $s + \epsilon > a$. This means that $s + \epsilon$ cannot a lower bound for A for any $\epsilon > 0$. Now let b be an arbitrary real number greater than s. Taking $\epsilon = b - s > 0$, we see that $s + \epsilon = s + (b - s) = b$. Thus, any b greater than s cannot be a lower bound for A. Hence, s is the infimum of A. □

1.3.4. (a) When taking the union of two nonempty sets, each bounded above, the supremum will be the largest of the two supremums. In other terms,

$$\sup(A_1 \cup A_2) = \max\{\sup A_1, \sup A_2\}.$$

Similarly, for $n \in \mathbb{N}$,

$$\sup\left(\bigcup_{k=1}^{n} A_k\right) = \max\left(\bigcup_{k=1}^{n} \{\sup A_k\}\right).$$

(b) Not always. Consider a collection $\{A_k\}_{k=1}^{\infty}$, where $\sup A_k = k$. Now take some $a \in \mathbb{R}$. We can choose an integer $k > a$, and because $\sup A_k = k$, there must exist an element $b \in A_k$ such that $a < b \leq k$. Thus, when considering

$$\sup\left(\bigcup_{k=1}^{\infty} A_k\right),$$

no real number a can be an upper bound, as a larger element b will always exist, and hence the supremum does not exist. This gives a counterexample where the formula in (a) does not work in the infinite case.

1.4.2. We first need to show that s is an upper bound of A. Consider arbitrary $a > s$. We can choose a sufficiently large integer n such that $s < s + \frac{1}{n} < a$. We are given that $s + \frac{1}{n}$ is an upper bound of A, but that makes a greater than an upper bound of A, so $a \notin A$. Because no $a > s$ can be in A, s is an upper bound of A.

We now need to show that there is no upper bound less than s. Consider arbitrary $b < s$. Then, we can pick a sufficiently large integer n such that $b < s - \frac{1}{n} < s$. However, we are given that $s - \frac{1}{n}$ is not an upper bound. Thus, there must exist some element $c \in A$ that is greater than $s - \frac{1}{n}$, which implies $c > b$. Hence b cannot be an upper bound. Because s is an upper bound of A and no $b < s$ can be an upper bound of A, we have shown that $\sup A = s$. $\square$

1.4.8. (b) Consider the sequence $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$, where $J_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$. We note that $0 \in J_i$ for all $i \in \mathbb{N}$. Then

$$\bigcap_{n=1}^{\infty} J_n = \{0\}.$$

This is because for any positive a, we can pick a sufficiently large n for which $a > \frac{1}{n}$, which implies that $a \notin J_n$. For any negative a, we can pick n where $a < -\frac{1}{n}$, in which case $a \notin J_n$.

(c) Consider the sequence $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$, where $L_n = [n, \infty)$. Then

$$\bigcap_{n=1}^{\infty} L_n = \varnothing,$$

since for any real a, we can pick an integer $n > a$, in which case $a \notin L_n$.