Homework 1
MAD4204
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1. Let $G = ([n], E)$ be a graph and let $\overline{G} = ([n], \binom{[n]}{2} \setminus E)$ be its complement. Prove for n sufficiently large that at least one of G and $\overline{G}$ contains a cycle.

   Your proof should include a value n that guarantees this property.

   Solution. Let $n = 5$. We first show that if neither G nor $\overline{G}$ contain a cycle, then G must be connected.

   Suppose G is disconnected. In one case, G contains a vertex with no neighbors. Without loss of generality, let $|N(1)| = 0$. If the other four vertices form a complete subgraph, then a cycle trivially exists. Hence, there must exist two vertices a and b where $(a, b) \notin E$. Then $\{1, a, b\}$ forms a cycle in $\overline{G}$.

   The other possible disconnected graph has two connected subgraphs, one connecting 2 vertices, the other connecting 3 vertices. Since the subgraph of cardinality 3 cannot contain a cycle, there must be two disjoint vertices, say a and b. Then let an arbitrary vertex in the subgraph of cardinality 2 be c. Since the two subgraphs are not connected, $\{a, b, c\}$ must form a cycle in $\overline{G}$. Thus G is connected. If G contains no cycles and is connected, then it must be a tree.

   Let $S = \{s \in [n] : N(s) = 1\}$. If $|S| = 2$, then G is isomorphic to $H = ([5], \{(1, 2), (2, 3), (3, 4), (4, 5)\})$. The equivalent to $\{1, 3, 5\}$ would form a cycle in $\overline{G}$. Otherwise, $|S| \geq 3$. Since $|N(s)| = 1$ for all $s \in S$, an edge connecting two elements of S would make the graph disconnected. Thus, no two elements of S can be connected by an edge. Thus 3 arbitrary elements in S form a cycle in $\overline{G}$. Thus for $n = 5$, at least one of G and $\overline{G}$ must contain a cycle.

   For any graph with $n > 5$, any induced subgraph of 5 vertices must follow the same property. Thus for all $n \geq 5$, the property holds.

2. Let $G = ([n], E)$ be a finite simple graph. Let M be a maximal matching in G and M' be a maximum matching in G. Prove that $|M'| \leq 2|M|$.

3. For $\mathcal{M}$ a matroid on ground set S defined in terms of bases, we say $I \subseteq S$ is *independent* if there exists $B \in \mathcal{M}$ so that $I \subseteq B$. An alternate definition of a matroid $\mathcal{M}_I$ on ground set S in terms of independent sets is that $\mathcal{M}_I \subseteq 2^S$ so that:

   - (hereditary property) if $A \subseteq B \in \mathcal{M}_I$, then $A \in \mathcal{M}_I$,
   - (augmentation property) for $A, B \in \mathcal{M}_I$ with $|A| < |B|$ there exists $b \in B$ so that $A \cup \{b\} \in \mathcal{M}_I$.

   We show these definitions are equivalent by solving:

   (a) For $\mathcal{M}_I$ a matroid defined in terms of its independent sets, we say $B \in \mathcal{M}_I$ is a *basis* if B is maximal in $\mathcal{M}_I$. Prove two bases in $\mathcal{M}_I$ satisfy the exchange property.

   Solution. We first show that elements of $\mathcal{M}_I$ are bases iff they have the same cardinality as other bases. Let there be two bases $A, B \in \mathcal{M}_I$ such that, without loss of generality, $|A| < |B|$. Then by the augmentation property, there exists $b \in B \setminus A$ such that

$A \cup \{b\} \in \mathcal{M}_I$. However, since A is maximal, $A \subseteq A \cup \{b\}$ implies $A = A \cup \{b\}$, which can't be true as $b \notin A$. Therefore, all bases in $\mathcal{M}_I$ must have the same cardinality.

For the other direction, let there be basis $A \in \mathcal{M}_I$ and non-basis $B \in \mathcal{M}_I$ so that $|A| = |B|$. Then since B is not maximal, there exists a set C such that $B \cup C \in \mathcal{M}_I$ would be a basis. However, $|A| < |B \cup C|$ contradicts all bases having the same cardinality. Thus, an independent set with the same cardinality of a basis is another basis.

Now let $A, B \in \mathcal{M}_I$ be bases, and $a \in A$. By the hereditary property, $A \setminus \{a\} \in \mathcal{M}_I$. Then, since all bases have the same cardinality, $|A \setminus \{a\}| = |B| - 1$, so $|A| < |B|$. Thus, by the augmentation property, there exists $b \in B$ so that $(A \setminus \{a\}) \cup \{b\} \in \mathcal{M}_I$. Since one element was removed and a different element was added to A, we have $|A| = |(A \setminus \{a\}) \cup \{b\}|$. Thus $(A \setminus \{a\}) \cup \{b\}$ is a basis, and so the exchange property holds.

(b) For $\mathcal{M}_B$ a matroid defined in terms of its bases, show that two independent sets A, B of $\mathcal{M}$ satisfy the augmentation property.

(Compare this with Lemma 10.10 in the course text) *Solution.* Let A, B be two independent sets of $\mathcal{M}$ with $|A| < |B|$. Then let $X_A, X_B \in \mathcal{M}_B$ such that $A \subseteq X_A$ and $B \subseteq X_B$. Then by the exchange property, for some $a \in A$, we have $(X_A \setminus \{a\}) \cup \{b\}$ for some $b \in X_B$. Assuming the hereditary property, since $A \cup \{b\} \subseteq (X_A \setminus \{a\}) \cup \{b\}$, we have $A \cup \{b\}$ an independent set, so the augmentation property holds.

4. For $\mathcal{M}$ a matroid on ground set S (from Problem 3, we can define it in terms of bases or independent sets, whichever is more convenient), let $w : S \to \mathbb{R}_{\geq 0}$. For B a basis in $\mathcal{M}$, define $w(B) = \sum_{b \in B} w(b)$.

   Describe an algorithm for finding the basis B minimizing $w(B)$ and prove that it is optimal.

   (Hint: compare to the greedy algorithm for finding the minimum weighted spanning tree)