Homework 4 Revisions
MAD4204
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3. For P a finite poset, show the number of elements in a maximum chain equals the number of antichains in the smallest antichain cover.

   Solution. Let the *height* of $y \in P$ be the maximum length of a saturated chain

   $$x \lessdot p_1 \lessdot p_2 \lessdot \cdots \lessdot y,$$

   with x a minimal element of P. Then let A_i be the set of elements of height i. We claim that for any $1 \leq i \leq h$, A_i is an antichain. Indeed, note that if $x, y \in A_i$ have $x < y$, then we can extend the chain of length i ending at x instead to end at y. This means that the height of y must be greater than i, so $y \notin A_i$, forming a contradiction.

   Now let C be a maximal chain of size h. Then since h is also the largest height of any element in P, we see that $\bigcup_{i=1}^{h} A_h$ is an antichain cover of size n. Additionally, if an antichain cover of size $g < h$ were to exist, then some antichain must contain at least 2 distict elements in C, since $|C| = h$. However, these two elements would be comparable, forming a contradiction. Thus, the smallest antichain cover has size h.