Homework 7
MAS4301
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10.14 Under $\varphi\colon \mathbb{Z}_{12} \to \mathbb{Z}_{10}$ mapping $x \mapsto 3x \pmod{10}$, we have

$$\varphi(4) + \varphi(8) = 2 + 4 = 6,$$

but

$$\varphi(4+8) = \varphi(0) = 0.$$

Since $\varphi(4+8) \neq \varphi(4) + \varphi(8)$, φ does not preserve group operations, and therefore cannot be a homomorphism.

10.16 Suppose such a homomorphism φ from $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ onto $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ were to exist. Then since $|\mathbb{Z}_8 \oplus \mathbb{Z}_2| = |\mathbb{Z}_4 \oplus \mathbb{Z}_4| = 16$, φ has a domain and codomain of equal cardinality, so φ must also be injective. Hence such a φ would be an isomorphism. However, $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ and $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ are clearly not isomorphic, since $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ has an element with order 8, (1,0), whereas $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ has a maximal order of 4 for any element. Thus, we have a contradiction, so φ cannot exist.

10.20 0 maps onto; 4 maps to

10.24
(a) $\varphi(x) = 3x$
(b) $\operatorname{im}(\varphi) = \{0, 3, 6, 9, 12\}$
(c) $\ker(\varphi) = \{0, 5, 10, 15, 20, 25, 30, 35, 40, 45\}$
(d) $\varphi^{-1}(3) = \{1, 6, 11, 16, 21, 26, 31, 36, 41, 46\}$