Homework 5
MAD4204
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1. Let $P = [2] \times [3]$, where we view $[n]$ as a chain.

   (a) Draw the poset $J(P)$ and find its join irreducibles.

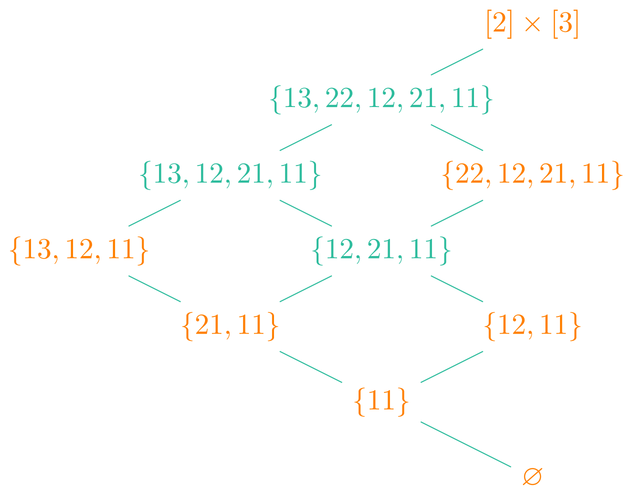
   (b) Show that P and $J(P)$ are ranked and find their rank generating functions.

   (c) Find all the linear extensions of P (Bonus: do this for $J(P)$ too!).

   (d) Compute $\mu(\hat{0}, \hat{1})$ for $J(P)$.

   Solution.

   (a) Denoting (a, b) as ab for shorthand, we see that $J(P) =$



   with orange sets being its join irreducibles.

   (b) Since P essentially creates a 1x2 block-walking grid, P has 3 maximal chains, all of length 3. In $J(P)$, a maximal chain is a path from $\varnothing$ to $[2] \times [3]$. Since an element is added at each step in the path, all maximal chains have length 6. Thus, both P and $J(P)$ are ranked. In particular,

$$F_P(q) = 1 + 2q + 2q^2 + q^3,$$
$$F_{J(P)}(q) = 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6.$$

   (c) For any linear extension L, clearly $L(11) = 1$ and $L(23) = 6$. We can casework by $L(12) \in \{2, 3\}$ to get all linear extensions as follows:

i	$L_i(11)$	$L_i(12)$	$L_i(21)$	$L_i(13)$	$L_i(22)$	$L_i(23)$
1	1	2	3	5	4	6
2	1	2	3	4	5	6
3	1	2	4	3	5	6
4	1	3	2	4	5	6
5	1	3	2	5	4	6

(d) We have $\mu(\hat{0}, \hat{0}) = 1$, so $\mu(\hat{0}, \{11\}) = -\mu(\hat{0}, \hat{0}) = -1$. Then

$$\begin{aligned}\mu(\hat{0}, \{21, 11\}) &= \mu(\hat{0}, \{12, 11\}) \\ &= \mu(\hat{0}, \hat{0}) + \mu(\hat{0}, \{11\}) \\ &= 0.\end{aligned}$$

Continuing this process recursively, we see that $\mu(\hat{0}, p) = 0$ for any $p \in P$ with rank greater than 1. Thus $\mu(\hat{0}, \hat{1}) = 0$.

2. Let L be a finite lattice. Show $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in L$ if and only if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $x, y, z \in L$.

   (A lattice satisfying either of these properties is called a *distributive lattice*.)

   Solution. ($\Longrightarrow$) Assume that $\wedge$ distributes over $\vee$. Then

$$\begin{aligned}(x \vee y) \wedge (x \vee z) &= ((x \vee y) \wedge x) \vee ((x \vee y) \wedge z) \\ &= x \vee ((x \vee y) \wedge z) \\ &= x \vee (z \wedge (x \vee y)) \\ &= x \vee ((z \wedge x) \vee (z \wedge y)) \\ &= (x \vee (z \wedge x)) \vee (z \wedge y) \\ &= x \vee (z \wedge y)\end{aligned}$$

   as desired.

   ($\Longleftarrow$) Now assume that $\vee$ distributes over $\wedge$. We let $\tilde{L}$, the *dual* of L, be the lattice where $p \leq_L q \iff q \leq_{\tilde{L}} p$. We see that $\tilde{L}$ is indeed a lattice, since $\vee_L = \wedge_{\tilde{L}}$ and $\wedge_L = \vee_{\tilde{L}}$. Using this duality, ($\Longleftarrow$) for L is equivalent to ($\Longrightarrow$) for $\tilde{L}$, so we are done. □

3. Let L be a finite distributive lattice. For $t \in L$, let $K_t = \{p \in \mathrm{Irr}(L) : p \leq t\}$. Show

$$t = \bigvee_{p \in K_t} p.$$

   Solution. Since $p \leq t$ for all $p \in K_t$, by Proposition 16.29, $\bigvee_{p \in K_t} p \leq t$. However, since all of p are join irreducible, $t \leq \bigvee_{p \in K_t} p$. Thus

$$t = \bigvee_{p \in K_t} p.$$

   □

4. In class we introduced Young's lattice Y, which is equivalent to the subposet of finite ideals in $J(\mathbb{N} \times \mathbb{N})$ or partitions ordered under containment of Young diagrams.

   (a) Show Y is a distributive lattice and describe $\mathrm{Irr}(Y)$ (Hint: what are $\wedge$ and $\vee$?).

   (b) Let $\lambda = \mu^1 \vee \cdots \vee \mu^k$ where $\{\mu^1, \ldots, \mu^k\} \subset \mathrm{Irr}(Y)$ is an antichain. Give a combinatorial interpretation of k in terms of properties of λ.

Solution.

(a) Because $\wedge$ and $\vee$ are the intersection and union of Young diagrams, respectively, and these operations are distributive, Y must be a distributive lattice.

Since join irreducibles must cover at most one element, $\mathrm{Irr}(Y)$ will contain the empty set, as well as for any integer $n > 0$, partitions of singleton n, as well as the partition $\underbrace{1 + \cdots + 1}_{n \text{ times}}$.

(b) Since partitions of singleton n as described above contain one another, at most one can be chosen to form an antichain. Similarly, partitions of form $\underbrace{1 + \cdots + 1}_{n \text{ times}}$ must contain one another, so at most one can be chosen for an antichain. Then k is at most 2, and λ takes form $n + \underbrace{1 + \cdots + 1}_{m \text{ times}}$ for $n, m \in \mathbb{N}$.