Homework 6

MAS4301

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NB 6.1 (c) For $\mathbb{Z}_4 \oplus \mathbb{Z}_2$, there is only one element of order 2, namely $(2, 0)$. However, D, D', H, V, and R_{180} are all elements of order 2 in D_4. The number of elements of any particular order is preserved by all isomorphisms, so D_4 *cannot* be isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_2$.

(d) In the case of $p = 2$, we know that D_4 has order $p^3 = 8$, but D_4 is clearly not abelian (for instance, $R_{90}H \neq HR_{90}$).

9.6 Let $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in GL(2, \mathbb{R})$ and $B = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$, with $a, b, d \in \mathbb{R}$ and $ad \neq 0$, be an element of H. By the 2x2 matrix inversion formula, we have $A^{-1} = \frac{1}{0-1}\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then

$$\begin{aligned} ABA^{-1} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & d \\ a & b \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} d & 0 \\ b & a \end{bmatrix}, \end{aligned}$$

but when $b \neq 0$, we see that $ABA^{-1} \notin H$. Thus it is *false* that $AHA^{-1} \subseteq H$ for all $A \in GL(2, \mathbb{R})$, so H is *not* a normal subgroup of $A \in GL(2, \mathbb{R})$.

9.12 Let G be an abelian group, and H be a normal subgroup of G. Let x and y be elements of G/H. We know that $x = aH$ and $y = bH$ for some $a, b \in G$. Then

$$\begin{aligned} xy &= (aH)(bH) \\ &= abH \\ &= baH && \text{(since } G \text{ is abelian)} \\ &= (bH)(aH) \\ &= yx, \end{aligned}$$

so G/H is clearly also abelian.

9.14 $14 + \langle 8 \rangle$ has order 4.

9.18 $\mathbb{Z}_{60}/\langle 15 \rangle$ has order 15.

9.34 Since 5 and 7 are relatively prime, we know that there exists integers s and t such that $5s + 7t = 1$. Let s and t be as such. Then for any $n \in \mathbb{Z}$, we have

$$\begin{aligned} n \cdot 1 &= n(5s + 7t) \\ &= 5ns + 7nt. \end{aligned}$$

Moreover, $5ns \in H$ and $7nt \in K$, so $n = 5ns + 7nt \in HK$. Thus $\mathbb{Z} \subseteq HK$. Also, $HK \subseteq \mathbb{Z}$ is clear, since an element in HK is the sum of two integers. Thus $\mathbb{Z} = HK$.

Also, since $\mathbb{Z}$ is abelian, H and K are normal subgroups. Additionally, $H \cap K = \{0\}$. Thus, by definition, $Z = H \times K$.