Homework 2

MAD6206

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Non-book. The number of ways to roll k dice summing to s is the number of weak compositions of s into k parts, but with a maximum of 6 per part. We can count the number of ways *with* going over 6 for i parts by picking those i parts and removing 6 from each of them. Thus, by inclusion-exclusion (PIE), we have

$$(\#\text{ ways}) = \sum_{i=0}^{k}(-1)^i\binom{k}{i}\binom{s-6i-1}{k-1}.$$

In particular, for $k=2$, we have

$$\begin{aligned}(\#\text{ ways}) &= \binom{s-1}{1} - 2\binom{s-6-1}{1} + \binom{s-12-1}{1} \\ &= \begin{cases} 0 & x<2 \\ s-1 & 2\le x\le 7 \\ 13-s & 7<x\le 12 \\ 0 & x>12 \end{cases}.\end{aligned}$$

3.13.3. (b) Let m be the number of males and n the number of females in a club, with k members being picked to form a committee. Then the LHS counts, for each possible amount of male students picked, i $(0\le i\le m)$, the number of ways to pick the rest $(k-i)$ female. The RHS counts the number of ways to pick a commmitte k from all male and female members in one step. Because LHS and RHS count the same committee-forming, they are equal.

(c) Shift the index of n down by 2. Then it suffices to show

$$\sum_{i=0}^{k}\binom{n-2+k+i}{n-2} = \binom{n+k-1}{n-1}.$$

The RHS counts the number of weak compositions of k into n parts. The LHS counts the number of weak compositions of all integers from 0 to k into $n-1$ parts. These are equivalent, as we can 'separate' the first part of the composition, and consider the rest of the composition depending on how much of k is summed towards by the first part.

(d) Let

$$\begin{aligned}f(x) &= (x+1)^n \\ &= \sum_{k=0}^{n}\binom{n}{k}x^k.\end{aligned}$$

Then

$$\begin{aligned}f'(x) &= n(x+1)^{n-1} \\ &= \sum_{k=0}^{n}k\binom{n}{k}x^{k-1}.\end{aligned}$$

Plugging in $x = 1$ gives the desired identity. Note that the $k = 0$ term does not contribute to the sum.

3.13.15. (a) We can find the number of ways to choose elements from X two at a time without replacement, and then divide by the $k!$ ways to permutate the k parts. This gives

$$\begin{aligned}(\#\text{ factors}) &= \frac{1}{k!}\prod_{i=1}^{k}\binom{2i}{2} \\ &= \frac{1}{k!}\prod_{i=1}^{k} i(2i-1) \\ &= \frac{(1\cdot 2\cdot \dots \cdot k)(1\cdot 3\cdot \dots \cdot (2k-1))}{k!} \\ &= (2k-1)!!,\end{aligned}$$

as desired.

(b) ($\Rightarrow$) Suppose that a permutation X were to interchange a k-subset with its complement. Then we can express the k subset as $a_1, a_2, \dots, a_k$, and the complement of a_i as $\bar{a}_i$. Then our permutation can be expressed in cycle notation as

$$(a_1\bar{a}_1)(a_2\bar{a}_2)\cdots(a_k\bar{a}_k),$$

which has only even cycles.

($\Leftarrow$) Now suppose that a permutation consisted only of even cycles. Express the permutation in cycle notation, and let X_e and X_o partition X such that X_e contains all elements of X which have an even index within its cycle, and X_o contains all elements of X with an odd index within its cycle. Then our permutation must interchange X_e and X_o, as an even cycle will always map an even indexed element to an odd indexed element, and vice versa. We see that X_e satisfies the k-subset we are looking for.

4.8.1. (a) We will prove by strong induction on n. Our base cases are clear, as $F_1 = 1$ counts the one way of choosing 0 seats, and $F_2 = 2$ counts the empty and singleton set. Now assume the inductive hypothesis for seat amounts from 0 to k. Then, since $F_{k+2} = F_{k+1} + F_k$, we see that the number of ways of choosing a subset with $k + 1$ seats is the sum of the ways for k and $k - 1$ seats. Indeed, we note that if we add the most recently added (final) chair in the line to the subset, we cannot add the second-to-last chair, giving $k - 1$ seats left to choose from. However, if we do not add the final chair, we have the same situation as having k seats to begin with.

(b) For our small cases, we note that $F_2 + F_0 = 3$ counts one chair, the other chair, or neither, and $F_3 + F_1 = 4$ counts any of the three chairs or none of them.

Now for the general case of $k > 3$, we see that if we add a new chair to the circle (k) and include it in the subset, then neither neighboring chair can be selected, giving $k - 3$ chairs to work with. If we do not include the new chair in the subset, we have $k - 1$ chairs to work with. In either case, we no longer have potential neighbors on either end of the circle, so by part (a), the $k - 3$ case has F_k subsets, and the $k - 1$ case has F_{k-2} subsets, as desired.

4.8.9. (iii) We have

$$f(n+1) = 1 + \sum_{i=0}^{n-1} f(i) \tag{1}$$

$$f(n) = 1 + \sum_{i=0}^{n-2} f(i). \tag{2}$$

Subtracting (2) from (1) gives

$$f(n+1) - f(n) = f(n-1) \implies f(n+1) = f(n) + f(n-1).$$

Using the initial condition, we see that $f(0) = f(1) = 1$, and so we conclude that $f(n)$ maps to the nth Fibonacci number.

4.8.16. This problem is equivalent to the probability that a lattice path from $(0,0)$ to (n,n) does not cross the diagonal, where picking a red and blue ball is equivalent to moving vertically and horizontally on the lattice path, respectively. The number of paths not crossing the diagonal is $C_n = \frac{1}{n+1}\binom{2n}{n}$, and when divided by the total number of paths gives

$$\frac{\frac{1}{n+1}\binom{2n}{n}}{\binom{2n}{n}} = \frac{1}{n+1}.$$