MAD 4204, Final

For G a simple graph, let $\mathcal{M}(G)$ be the matroid whose bases are the spanning trees of G.
Problems to turn in:

1. Projects here: `https://ufl.instructure.com/courses/393318/pages/projects`
   Read two projects and write a two paragraph response summarizing the main results and what you learned. Some of these are quite long, but try to read at least one of them in full detail. Please make sure to identify two aspects of each project that you liked!

2. Let G be a simple planar graph with $n \geq 3$ vertices. Prove that G has at most $3n-6$ edges.

   Solution. Since each face of a planar graph is comprised of at least 3 edges, and an edge is shared by two faces, we have $|F| \leq \frac{2}{3}|E|$. By Euler's Theorem, we have

   $$\begin{aligned} |E|+2 &= |F|+|V| \\ &\leq \frac{2}{3}|E|+|V| \\ \implies \frac{1}{3}|E| &\leq |V|-2 \\ \implies |E| &\leq 3|V|-6. \end{aligned}$$

   Thus, for $|V|=n$, G has at most $3n-6$ edges, as desired. $\square$

3. For P a poset, let $\mathbb{C}[P]$ be the vector space over $\mathbb{C}$ whose basis is the elements of P. Define the linear transformations $U_P, D_P : \mathbb{C}[P] \to \mathbb{C}[P]$ by, for $y \in P$,

   $$U_P(y) = \sum_{y \lessdot z} z \quad \text{and} \quad D_P(y) = \sum_{x \lessdot y} x.$$

   For example, in the Boolean lattice B_3 we have $U_{B_3}(\{1\}+\{2\}) = 2\cdot\{1,2\}+\{1,3\}+\{2,3\}$. Also, let I_P be the identity map.

   Recall Young's lattice $\mathcal{Y}$ is the poset of integer partitions ordered by containment. In the remainder of the problem, define $U := U_{\mathcal{Y}}$, $D := D_{\mathcal{Y}}$ and $I := I_{\mathcal{Y}}$.

   (a) Prove that $DU - UD = I$ as linear transformations.

   (b) For $\lambda \in \mathcal{Y}$, let f^λ be the number of saturated chains from $\varnothing$ to λ in $\mathcal{Y}$. Explain why

   $$D^n U^n(\varnothing) = \left(\sum_{\lambda \vdash n} (f^\lambda)^2\right) \varnothing.$$

   Note we are first applying U n times, then D n times.

   (c) Using (a), prove that $D^n U^n(\varnothing) = n! \cdot \varnothing$. Conclude from (b) that $n! = \sum_{\lambda \vdash n} (f^\lambda)^2$. (Hint: Observe that $D^2U^2 = D(I+UD)U = DU + DUDU$. Try to generalize this.)

   Solution. We first define $u(\lambda)$ to be the number of unique parts of a partition λ. For instance, $\lambda = 3+2+2+1+1$ has 3 unique parts (3, 2, and 1), so $u(\lambda) = 3$.

(a) It suffices to show that $(DU - UD)(\lambda) = I(\lambda) = \lambda$ for all $\lambda \in \mathcal{Y}$. If $\lambda =$, Now let λ be an arbitrary nonempty partition.

4. The pair $(X, \mathcal{B})$ is a $t-(v,k,\ell)$ design if X is a finite set and $\mathcal{B} \subseteq \binom{X}{k}$ so that every t–element set $T \subset X$ is a subset of exactly ℓ elements of $\mathcal{B}$.

   (a) For $s \in [t-1]$, show that $(X, \mathcal{B})$ is an $s-(v,k,\rho)$ design for some $\rho \in \mathbb{N}$ and find a formula for ρ in terms of t, v, k and ℓ.

   (b) Show the following quantity is a positive integer for $0 \leq i < t$:

$$\ell \binom{v-i}{t-i} \Big/ \binom{k-i}{t-i}.$$

Solution.

(a)

Extra credit

1. (3 points) Complete Problem 5 on the Schubert Decomposition worksheet.

2. (2 points) Prove Theorem 1 from the Schubert Decomposition worksheet.

3. (8 points) Construct $\mathcal{B} \subseteq \binom{[9]}{4}^2$ so that:

   - for $(A, B) \in \mathcal{B}$, we have $A \cap B = \varnothing$;
   - for every pair $\{a, b\} \subseteq [9]$, there are exactly three pairs $(A, B) \in \mathcal{B}$ so that $\{a, b\} \subseteq A$ or $\{a, b\} \subseteq B$;
   - for every triple $\{a, b, c\} \subseteq [9]$ there is exactly one pair $(A, B) \in \mathcal{B}$ so that $\{a, b, c\} \subseteq A$ or $\{a, b, c\} \subseteq B$;

Solution. We claim that such a $\mathcal{B}$ *cannot* exist [1].

For $\mathcal{B}$ to satisfy the third condition, there are $\binom{9}{3} = 84$ triples that must be a subset of A or B for exactly one $(A, B) \in \mathcal{B}$. However, taking arbitrary $(A, B) \in \mathcal{B}$, we see that there are $\binom{4}{3} = 4$ distinct triples in each of A and B. Since A and B are disjoint, we must have 8 distinct triples in any element of $\mathcal{B}$. Because triples must also be distinct between elements of $\mathcal{B}$ to follow the third condition, the total number of triples in $\mathcal{B}$ would be $8 \cdot \#\mathcal{B}$. However, $8 \nmid 84$, so we have a contradiction. □

[1] I understand that it's a stretch to prove nonexistence for a question asking to "construct", but I came up with this counterargument and have yet to find fault in it.