Take-home Final
MAD4204
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1. Read two projects and write a two paragraph response summarizing the main results and what you learned. Some of these are quite long, but try to read at least one of them in full detail. Please make sure to identify two aspects of each project that you liked!

   (omitted for privacy)

2. Let G be a simple planar graph with $n \geq 3$ vertices. Prove that G has at most $3n-6$ edges.

   Solution. Since each face of a planar graph is comprised of at least 3 edges, and an edge is shared by two faces, we have $|F| \leq \frac{2}{3}|E|$. By Euler's Theorem, we have

$$\begin{aligned} |E| + 2 &= |F| + |V| \\ &\leq \frac{2}{3}|E| + |V| \\ \implies \frac{1}{3}|E| &\leq |V| - 2 \\ \implies |E| &\leq 3|V| - 6. \end{aligned}$$

   Thus, for $|V| = n$, G has at most $3n-6$ edges. □

3. For P a poset, let $\mathbb{C}[P]$ be the vector space over $\mathbb{C}$ whose basis is the elements of P. Define the linear transformations $U_P, D_P : \mathbb{C}[P] \to \mathbb{C}[P]$ by, for $y \in P$,

$$U_P(y) = \sum_{y \lessdot z} z \quad \text{and} \quad D_P(y) = \sum_{x \lessdot y} x.$$

   For example, in the Boolean lattice B_3 we have $U_{B_3}(\{1\} + \{2\}) = 2 \cdot \{1,2\} + \{1,3\} + \{2,3\}$. Also, let I_P be the identity map.

   Recall Young's lattice $\mathcal{Y}$ is the poset of integer partitions ordered by containment. In the remainder of the problem, define $U := U_{\mathcal{Y}}$, $D := D_{\mathcal{Y}}$ and $I := I_{\mathcal{Y}}$.

   (a) Prove that $DU - UD = I$ as linear transformations.

   (b) For $\lambda \in \mathcal{Y}$, let f^λ be the number of saturated chains from $\varnothing$ to λ in $\mathcal{Y}$. Explain why

$$D^n U^n(\varnothing) = \left(\sum_{\lambda \vdash n} (f^\lambda)^2 \right) \varnothing.$$

   Note we are first applying U n times, then D n times.

   (c) Using (a), prove that $D^n U^n(\varnothing) = n! \cdot \varnothing$. Conclude from (b) that $n! = \sum_{\lambda \vdash n} (f^\lambda)^2$. (Hint: Observe that $D^2U^2 = D(I + UD)U = DU + DUDU$. Try to generalize this.)

   Solution.

(a) First, we define $u(\lambda)$ to be the number of uniquely valued parts of some λ. For instance, $\alpha = (3, 2, 2, 1)$ has 3 unique parts (namely 3, 2, and 1), so $u(\alpha) = 3$.
Since all partitions in $\mathcal{Y}$ form the basis of $\mathbb{C}[\mathcal{Y}]$, it suffices to show that $(DU - UD)(\lambda) = I(\lambda) = \lambda$ for all $\lambda \in \mathcal{Y}$. If $\lambda = \varnothing$, then

$$\begin{aligned}(DU - UD)(\varnothing) &= D(U(\varnothing) - U(D(\varnothing)) \\ &= D((1)) - U(0) \\ &= \varnothing - 0 = \varnothing.\end{aligned}$$

Now let λ be an arbitrary nonempty partition of rank $n > 0$. We will first find the coefficient of the λ term in $DU(\lambda)$ and $UD(\lambda)$.
We see that $U(\lambda)$ outputs the sum of partitions covering λ, i.e the results of either adding 1 to a part of λ or adding a new part of value 1. Because parts of the same value are indistinguishable, our number of choices for parts of λ to add to is the amount of distinct values within the parts, or $u(\lambda)$. We can also add a new part of value 1 to λ, giving a total of $u(\lambda) + 1$ partitions covering λ. $DU(\lambda)$ will then have a coefficient of $u(\lambda) + 1$ for λ, since each of the $u(\lambda) + 1$ partitions contribute 1 to the coefficient.
Similarly, $D(\lambda)$ returns all partitions resulting from removing 1 from one of the parts of λ. The number of ways to do this is the number of unique parts of λ, or $u(\lambda)$. Since each of the $u(\lambda)$ partitions covered by λ contribute 1 to the coefficient of λ in $UD(\lambda)$, we get a coefficient of $u(\lambda)$.
We will now show that all other terms of $(DU - UD)(\lambda)$ will cancel out to 0.

Claim: For any partition $\alpha \neq \lambda$, $UD(\lambda)$ and $DU(\lambda)$ have the same coefficient for α.

Proof of Claim: If some α has a non-zero coefficient in $UD(\lambda)$, then it is the output of subtracting 1 from a part of λ followed by adding 1 to a part of λ (or creating a new part of 1). Note that this coefficient is 1, as the intermediary poset with one part subtracted from is unique.
On the other hand, the same α can be constructed in $UD(\lambda)$ as described if and only if it can be constructed in $DU(\lambda)$. This is done by *first* adding 1 to a part of λ (or creating a new part of 1), *then* subtracting 1 from another part. There is a one-to-one correspondence between these two constructions, so the coefficient of α is the same (either 0 or 1) between $UD(\lambda)$ and $DU(\lambda)$, as desired.

Now let $A(\lambda)$ denote all non-λ terms output from $DU(\lambda)$. By our claim, the non-λ terms of $UD(\lambda)$ is also $A(\lambda)$. Then we have

$$\begin{aligned}(DU - UD)(\lambda) &= ((u(\lambda) + 1)\lambda + A(\lambda)) - (u(\lambda)\lambda + A(\lambda)) \\ &= \lambda,\end{aligned}$$

which concludes our proof. $\square$

(b) A saturated chain can be thought of as a path to or from $\varnothing$ to some partition λ. For each application of U, at any partition λ, we are summing the number of paths of all permutations that λ covers, which gives the number of paths of λ itself. This creates a 'block walking' scenario where for partitions of rank $0 \leq i \leq n$, the coefficient of each permutation in $U^i(\varnothing)$ is the number of saturated chains to that permutation. Thus, when $i = n$, we have

$$U^n(\varnothing) = \sum_{\lambda \vdash n} (f^\lambda)\lambda.$$

Then, for each application of D, we are 'walking backwards' one step, summing the paths to a permutation α from the permutations covering α. Because of this, for partitions of rank $0 \le j \le n$, the coefficient of some α in $D^{n-j}U^n(\varnothing)$ is the total number of paths from $\varnothing$ to some $\lambda \vdash n$, then back down to α. When $j = 0$, we are then counting the number of paths from $\varnothing$ to some $\lambda \vdash n$, then back to $\varnothing$. Because the saturated chains from $\varnothing$ to λ and back to $\varnothing$ can be chosen independently, we now get our final result of

$$D^n U^n(\varnothing) = \left(\sum_{\lambda \vdash n} (f^\lambda)^2 \right) \varnothing.$$

$\square$

4. The pair $(X, \mathcal{B})$ is a $t-(v,k,\ell)$ design if X is a finite set and $\mathcal{B} \subseteq \binom{X}{k}$ so that every t–element set $T \subset X$ is a subset of exactly ℓ elements of $\mathcal{B}$.

   (a) For $s \in [t-1]$, show that $(X, \mathcal{B})$ is an $s-(v,k,\rho)$ design for some $\rho \in \mathbb{N}$ and find a formula for ρ in terms of t, v, k and ℓ.

   (b) Show the following quantity is a positive integer for $0 \le i < t$:

$$\ell \binom{v-i}{t-i} \Big/ \binom{k-i}{t-i}.$$

Solution.

   (a) We claim that $\rho = \ell\binom{v-s}{t-s}/\binom{k-s}{t-s}$ gives an $s-(v,k,\rho)$ design. It suffices to show that every s–element set $S \subset X$ is a subset of exactly ℓ elements of $\mathcal{B}$. To do this, we will pick an element B along with the elements *not* in S but in T, i.e. $T \setminus S$. One way we can do this is by picking our elements of $T \setminus S$, which can be done in $\binom{v-s}{t-s}$ ways, then picking from the ℓ elements of $\mathcal{B}$. We can also pick B with $S \subset B$ in ρ ways, then pick the elements of $T \setminus S$ in $\binom{k-s}{t-s}$ ways. This equality can be rearranged to lead to the formula in our claim.

   (b) Taking the $s-(v,k,\rho)$ design from (a) with $s = i$, we see that $\rho = \ell\binom{v-i}{t-i}/\binom{k-i}{t-i}$ is a positive integer, so (b) follows immediately.

Extra credit

3 (8 points) Construct $\mathcal{B} \subseteq \binom{[9]}{4}^2$ so that:

- for $(A, B) \in \mathcal{B}$, we have $A \cap B = \varnothing$;
- for every pair $\{a, b\} \subseteq [9]$, there are exactly three pairs $(A, B) \in \mathcal{B}$ so that $\{a, b\} \subseteq A$ or $\{a, b\} \subseteq B$;
- for every triple $\{a, b, c\} \subseteq [9]$ there is exactly one pair $(A, B) \in \mathcal{B}$ so that $\{a, b, c\} \subseteq A$ or $\{a, b, c\} \subseteq B$;

Solution. We claim that such a $\mathcal{B}$ *cannot* exist[1].

For $\mathcal{B}$ to satisfy the third condition, there are $\binom{9}{3} = 84$ triples that must be a subset of A or B for exactly one $(A, B) \in \mathcal{B}$. However, taking arbitrary $(A, B) \in \mathcal{B}$, we see that there are $\binom{4}{3} = 4$ distinct triples in each of A and B. Since A and B are disjoint, we must have 8 distinct triples in any element of $\mathcal{B}$. Because triples must also be distinct between elements of $\mathcal{B}$ to follow the third condition, the total number of triples in $\mathcal{B}$ would be $8 \cdot \#\mathcal{B}$. However, $8 \nmid 84$, so we have a contradiction. $\square$

[1] I know it's a stretch to prove nonexistence for a question asking to "construct", but I have yet to find fault in the contradiction that follows.