Homework 2

MAD4204

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2. (a) A graph $G = (V, E)$ is *factor critical* if $G - v$ has a perfect matching for every $v \in V$. Prove no bipartite graph is factor critical.

   Solution. We will prove by contradiction. Suppose that a bipartite graph $G = (A \sqcup B, E)$ is factor critical. Then, for arbitrary $a \in A$ and $b \in B$, $G - a$ and $G - b$ each have perfect matchings. Since a bipartite graph with a perfect matching must have equal cardinality between its disjoint sets, $G - a$ and $G - b$ give $|A| - 1 = |B|$ and $|A| = |B| - 1$, respectively, which together form a contradiction. Thus, no bipartite graph is factor critical. $\square$

   (b) Let $G = (A \sqcup B, E)$ be a bipartite graph with $|A| = 10, |B| = 12$, $d(a) \leq 4$ for all $a \in A$ and $d(b) = 3$ for all $b \in B$. Prove that G has a matching of size at least 9.

3. Fix $k, n \in \mathbb{N}$ with $k < n/2$. Let $G = (A \sqcup B, E)$ be the bipartite graph with

$$A = \binom{[n]}{k}, \quad B = \binom{[n]}{k+1}, \quad E = \{(X, Y) : X \in A, Y \in B, X \subseteq Y\}.$$

Prove that G contains a matching of size $\binom{n}{k} = |A|$.

Bonus point: Construct a perfect matching on G.

Solution. Using the result from Problem 1, we have a matching of $|A|$ elements iff $|N(S)| \geq |S| + |A| - |A| = |S|$ for all subsets $S \subseteq A$. An element $a \in A$ has $n - k$ elements to add to make an element in B, while an element $b \in B$ has $k + 1$ elements to remove to make an element in A. However, $k < n/2$ implies $n - k \geq k + 1$, so the degree of a vertex in A is always greater than or equal to one in B. Thus $|N(S)| < |S|$ is impossible for any subset $S \subseteq A$, so we must have $|N(S)| \geq |S|$ as desired.