Assignment 8

MAA4211

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(Graded) 4.4.9. (a) Let $\epsilon > 0$. Since f is Lipschitz, there exists M such that

$$\frac{|f(x) - f(y)|}{|x - y|} \leq M$$
$$\implies |f(x) - f(y)| \leq M|x - y|$$

for all $x \neq y \in A$. Choosing $\delta = \epsilon/M$, we see that $|x - y| < \delta$ implies

$$|f(x) - f(y)| \leq M|x - y| < M \cdot \frac{\epsilon}{M} = \epsilon$$

for $x \neq y \in A$. When $x = y$, $|x - y| = 0 < \delta$ clearly implies $|f(x) - f(y)| = 0 < \epsilon$. Thus, f is uniformly continuous on A.

(b) The converse is false. Consider $f(x) = \sqrt{x}$ over the domain $[0, \infty)$ (Exercise 4.4.7.). We see that f is continuous on $[0, 1]$, a compact set, so $f|_{[0,1]}$ is uniformly continuous. Additionally, for all $x \neq y \in [1, \infty)$, we have

$$\frac{|f(x) - f(y)|}{|x - y|} = \frac{|\sqrt{x} - \sqrt{y}|}{|(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})|} = \frac{1}{\sqrt{x} + \sqrt{y}} \leq \frac{1}{2},$$

making $f|_{[1,\infty)}$ Lipschitz, and hence uniformly continuous by (a). Thus, f is uniformly continuous.

Now let $M > 0$. Taking $x = \frac{1}{4M^2}$, $y = \frac{1}{16M^2}$, we have $x \neq y \in [0, \infty)$ and

$$\frac{|f(x) - f(y)|}{|x - y|} = \frac{\left|\frac{1}{2M} - \frac{1}{4M}\right|}{\left|\frac{1}{4M^2} - \frac{1}{16M^2}\right|} = \frac{\left|\frac{1}{4M}\right|}{\left|\frac{3}{16M^2}\right|} = \frac{4}{3}|M| > M.$$

Thus, no upper bound M for the slope of f can exist, making f not Lipshitz. Thus, f is a counterexample to the converse of (a).