MAD 4204, Homework 2

Problems to turn in:

1. Prove that a bipartite graph $G = (A \sqcup B, E)$ has a matching of k elements if and only if for all subsets $S \subseteq A$, we have

$$|N(S)| \geq |S| + k - |A| = k - |A - S|.$$

(Note: the second equality is immediate)

2. (a) A graph $G = (V, E)$ is *factor critical* if $G - v$ has a perfect matching for every $v \in V$. Prove no bipartite graph is factor critical.

   (b) Let $G = (A \sqcup B, E)$ be a bipartite graph with $|A| = 10, |B| = 12$, $d(a) \leq 4$ for all $a \in A$ and $d(b) = 3$ for all $b \in B$. Prove that G has a matching of size at least 9.

3. Fix $k, n \in \mathbb{N}$ with $k < n/2$. Let $G = (A \sqcup B, E)$ be the bipartite graph with

$$A = \binom{[n]}{k}, \quad B = \binom{[n]}{k+1}, \quad E = \{(X, Y) : X \in A, Y \in B, X \subseteq Y\}.$$

Prove that G contains a matching of size $\binom{n}{k} = |A|$.

Bonus point: Construct a perfect matching on G.

4. Let G be a graph (not necessarily bipartite!) with $2n$ vertices so that every vertex has degree at least n. Prove that G has a perfect matching.

Recommended problems:

1. Do all the Quick Check problems in Chapter 11.

2. Is there a bipartite graph with degree sequence $(3, 3, 3, 3, 3, 3, 3, 3, 3, 5, 6, 6)$?

3. A round robin football tournament has $2n$ teams ($n > 2$). Two rounds have been played so far so that no team has played another twice. Prove we can split the teams into two groups of n teams so that no teams of the same group have played each other yet.

4. Come up with a proof of 3. that extends to graphs $G = (V, E)$ with $2n$ vertices so that for $a \neq b$ with $(a, b) \notin E$, we have $d(a) + d(b) \geq 2n$.

5. An *independent set* in $G = (V, E)$ is a set $I \subset V$ so that for $u, v \in I$ we have $(u, v) \notin E$. Let $\alpha(G)$ be the size of a maximum independent set in G and $\nu(G)$ be the size of a maximum matching. For G bipartite, prove that $\alpha(G) + \nu(G) = |V|$.

(Hint: you may also want to solve Exercise 11.30 in the text)