Assignment 4
MAA4211
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(Graded) 2.2.6. Let (a_n) be a sequence that converges to a and converges to b. Now, let $\epsilon > 0$. There must exist an N_1 such that for all $n \geq N_1$, we have $|a_n - a| < \epsilon/2$. There must also exist an N_2 such that for all $n \geq N_2$, we have $|a_n - b| < \epsilon/2$. Then, for all $n \geq \max\{N_1, N_2\}$,

$$\begin{aligned} |a-b| &= |a - b + a_n - a_n| \\ &\leq |a_n - a| + |a_n - b| \qquad \text{(triangle inequality)} \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

However, this implies that $|a - b| < \epsilon$ for all $\epsilon > 0$. By Theorem 1.2.6 from the book, this shows that $a = b$. $\square$

2.5.1. (a) Let (a_n) be a sequence with a bounded subsequence $(a_{n_1}, a_{n_2}, \ldots)$. By the Bolzano–Weierstrass Theorem, $(a_{n_1}, a_{n_2}, \ldots)$ must itself have a convergent subsequence. However, this convergent subsequence is transitively also a subsequence of (a_n), hence making (a) impossible.

(b) Let (a_n) be a sequence such that

$$a_k = (-1)^k \left(\frac{k-1}{2k}\right) + \frac{1}{2}.$$

Then the subsequence $(a_1, a_3, a_5, \ldots)$ converges to 0, while $(a_2, a_4, a_6, \ldots)$ converges to 1. Additionally, all a_k are in the open interval $(0, 1)$, so this sequence satisfies (b).

(c) Let (a_n) be a sequence such that

$$a_k = \prod_{i=1}^{\infty} \frac{p_i^{m_i - 1} - 1}{p_i^{m_i}},$$

where $p_1^{m_1} p_2^{m_2} \cdots$ is the unique prime factorization of k. We note that $(p^{i(m-1)} - 1)/p^{im}$ as a sequence indexed by i converges to $1/p^m$.

(d) Let (a_n) have subsequences converging to each element of $\{1/i : i \in \mathbb{N}\}$. We claim that such an (a_n) must converge to 0. Note: I could not find justification. Below is my attempted proof.

Let $\epsilon > \epsilon' > 0$. For a subsequence converging to some $1/i$, we know that there exists an N_i such that all natural numbers $n \geq N_i$ *whose indices are in the subsequence* must satisfy $|a_n - 1/i| < \epsilon' = \epsilon + 1/k$. We now pick large enough k so that $\max\{N_1, N_2, \ldots, N_k\}$... :(