Homework 4

MAS4301

Carson Mulvey

NB 4.1 We claim that ϕ is an isomorphism. It suffices to show that ϕ is one-to-one, onto, and operation-preserving.

(One-to-one) Let $\pi_a, \pi_b \in \text{Perm}(A)$ satisfy $\phi(\pi_a) = \phi(\pi_b)$. Then

$$\begin{aligned} f^{-1} \circ \pi_a \circ f &= f^{-1} \circ \pi_b \circ f \\ \implies (f \circ f^{-1}) \circ \pi_a \circ (f \circ f^{-1}) &= (f \circ f^{-1}) \circ \pi_b \circ (f \circ f^{-1}) \\ \implies \pi_a &= \pi_b, \end{aligned}$$

so ϕ is one-to-one.

(Onto) Now let $\pi' \in S_n$. We must find a $\pi \in \text{Perm}(A)$ such that $\phi(\pi) = \pi'$. If such a π were to exist, then

$$\begin{aligned} f^{-1} \circ \pi \circ f &= \pi' \\ \implies (f \circ f^{-1}) \circ \pi \circ (f \circ f^{-1}) &= f \circ \pi' \circ f^{-1} \\ \implies \pi &= f \circ \pi' \circ f^{-1}. \end{aligned}$$

But we see that since f maps from $\{1, 2, \ldots, n\}$ to A, it follows that $f \circ \pi' \circ f^{-1} \in \text{Perm}(A)$. Thus, taking $\pi = f \circ \pi' \circ f^{-1}$ as solved, we see that such a π always exists.

(Operation-preserving) Let π_a and π_b be in $\text{Perm}(A)$. Then

$$\begin{aligned} \phi(\pi_a \circ \pi_b) &= f^{-1} \circ \pi_a \circ \pi_b \circ f \\ &= f^{-1} \circ \pi_a \circ (f \circ f^{-1}) \circ \pi_b \circ f \\ &= (f^{-1} \circ \pi_a \circ f) \circ (f^{-1} \circ \pi_b \circ f) \\ &= \phi(\pi_a) \circ \phi(\pi_b), \end{aligned}$$

so ϕ is operation-preserving. $\square$

6.4 We have $U(8) = \{1, 3, 5, 7\}$ and $U(10) = \{1, 3, 7, 9\}$. Noting that $U(10) = \{3^0, 3^1, 3^3, 3^2\} = \langle 3 \rangle$, we see that $U(10)$ is cyclic. However, note that for $U(8)$,

$$\begin{aligned}
\langle 1 \rangle &= \{1\}, \\
\langle 3 \rangle &= \{1, 3\}, \\
\langle 5 \rangle &= \{1, 5\}, \\
\langle 7 \rangle &= \{1, 7\}.
\end{aligned}$$

Since none of the elements of $U(8)$ are generators, $U(8)$ is not cyclic. By Theorem 6.3, a cyclic and noncyclic group cannot be isomorphic, so $U(10)$ and $U(8)$ cannot be isomorphic.

6.12 Suppose that $\alpha(g) = g^{-1}$ for all $g \in G$ is an automorphism. Then for $f, g \in G$,

$$\begin{aligned}
fg &= \alpha((fg)^{-1}) \\
&= \alpha(g^{-1}f^{-1}) \\
&= \alpha(g^{-1})\alpha(f^{-1}) \\
&= gf,
\end{aligned}$$

so G is abelian.

Now suppose that G is abelian. Since α^2 is an identity mapping, we have $\alpha^{-1} = \alpha$, so α is a bijection. Also, for f and g in G, we have

$$\begin{aligned}
\alpha(fg) &= (fg)^{-1} \\
&= g^{-1}f^{-1} \\
&= f^{-1}g^{-1} \\
&= \alpha(f)\alpha(g).
\end{aligned}$$

We see that α is operation-preserving, so α is an automorphism. □

6.32 If $\phi(x) = 9x \pmod{50}$, then ϕ is an automorphism. Note that $\phi(7) = 63 \pmod{50} = 13$ holds.

6.34 We see that $\phi(k_1)$ and $\phi(k_2)$ are in $\phi(K)$ if $k_1, k_2 \in K$. Then, by operation preservation of an isomorphism, we have $\phi(k_1)\phi(k_2)^{-1} = \phi(k_1k_2^{-1})$. Since K is a subgroup, K is closed under both inverses and the group operation in G, so $k_1k_2^{-1} \in K$. Thus $\phi(k_1)\phi(k_2)^{-1}$ is in $\bar{G}$. Then since $\phi(k_1)$ and $\phi(k_2)$ being in $\bar{G}$ implies that $\phi(k_1)\phi(k_2)^{-1}$ is in $\bar{G}$, by the One-Step Subgroup Test, $\phi(K)$ is a subgroup of $\bar{G}$. □

6.42 Because ϕ^2 is an identity mapping, we have $\phi = \phi^{-1}$, so ϕ is a bijection. Also, for $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ in $\mathbb{R}^n$, we have

$$\begin{aligned}
\phi((a_1, \ldots, a_n) + (b_1, \ldots, b_n)) &= \phi((a_1 + b_1, \ldots, a_n + b_n) \\
&= (-a_1 - b_1, \ldots, -a_n - b_n) \\
&= (-a_1, \ldots, -a_n) + (-b_1, \ldots, -b_n) \\
&= \phi((a_1, a_2, \ldots, a_n)) + \phi((b_1, b_2, \ldots, b_n)).
\end{aligned}$$

We see that ϕ preserves componentwise addition, so ϕ is an automorphism. Geometrically, ϕ represents a reflection over the origin. In $\mathbb{R}^2$ particularly, that is equivalent to a rotation by π radians. □