Homework 1 Revisions
MAD4204
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2. Let $G = ([n], E)$ be a finite simple graph. Let M be a maximal matching in G and M' be a maximum matching in G. Prove that $|M'| \leq 2|M|$.

   Solution. Let $V(S)$ denote the set of vertices in at least one edge of set $S \subseteq E$. That is, $V(S) = \{v \in V : \exists w \in V, (v, w) \in E\}$. Then we have $|V(M)| = 2|M|$ and $|V(M')| = 2|M'|$, since vertices cannot be shared between edges in a matching.

   Suppose that an edge $(v, w) \in E$ had $v, w \notin V(M)$. Then (v, w) can be added to M, which would make M not maximal. Thus, every edge in E has at least one vertex in $V(M)$. But since $M' \subseteq E$, every edge in M' must also have at least one vertex in $V(M)$, which would then also be in $V(M') \cup V(M)$. It follows that

   $$\begin{aligned} |M'| &\leq |V(M') \cup V(M)| \\ &\leq |V(M)| \\ &= 2|M| \end{aligned}$$

   as desired. □

4. For $\mathcal{M}$ a matroid on ground set S (from Problem 3, we can define it in terms of bases or independent sets, whichever is more convenient), let $w : S \to \mathbb{R}_{\geq 0}$. For B a basis in $\mathcal{M}$, define $w(B) = \sum_{b \in B} w(b)$.

   Describe an algorithm for finding the basis B minimizing $w(B)$ and prove that it is optimal.

   (Hint: compare to the greedy algorithm for finding the minimum weighted spanning tree)

   Solution. We will define a matroid in terms of its independent sets, $\mathcal{M}_I$. We can define a greedy algorithm that finds a basis B minimizing $w(B)$ by starting with $B = \varnothing$. We then construct B by continually considering an element $a \in S$ with $a \notin B$ that minimizes $w(S)$. If $B \cup \{a\} \in \mathcal{M}_I$, then we add a to B, and otherwise, we don't add a and continue the algorithm. We claim that this is an optimal algorithm.

   We will prove this by contradiction. Assume that the greedy algorithm gives a basis B, while the matroid has a basis A, where $w(B) > w(A)$. In question 3, it was proven that all matroid bases have the same cardinality, so let $|B| = |A| = n$. Enumerate the elements of B as $b_1, b_2, \ldots, b_n$ and the elements of A as $a_1, a_2, \ldots, a_n$ such that $w(b_1) \leq w(b_1) \leq \cdots \leq w(b_n)$ and $w(a_1) \leq w(a_1) \leq \cdots \leq w(a_n)$.

   Since A must eventually reach a smaller weight than B, let i be the smallest integer where $\sum_{j=1}^{i} w(a_j) < \sum_{j=1}^{i} w(b_j)$. We have $i > 1$ since $w(b_1)$ is minimal over the whole matroid.

   Since A "takes over" at i, we have $w(a_i) < w(b_i)$. Let B_{i-1} be the independent set formed in the first $i - 1$ steps and A_i be the independent set formed in the first i steps. By the augmentation property, since $|B_{i-1}| < |A_i|$, we we have some $a_j \in A$ such that $B_{i-1} \cup \{a_j\}$ is an independent set. However, since $j \leq i$, $w(a_j) < w(b_i)$, so b_i can't be added at the ith step of the algorithm, forming a contradiction. Thus, the algorithm produces a minimal basis.