Homework 1
MAD6206
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1. Find the first 5 terms in the power series expansion of $\frac{1}{\sqrt{1+x}}$.

   Solution. Let $f(x) = (1+x)^{-1/2} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$. We have

   $$\begin{aligned} f^{(1)}(x) &= -\frac{1}{2}(1+x)^{-3/2}, \\ f^{(2)}(x) &= \frac{3}{4}(1+x)^{-5/2}, \\ f^{(3)}(x) &= -\frac{15}{8}(1+x)^{-7/2}, \\ f^{(4)}(x) &= \frac{105}{16}(1+x)^{-9/2}. \end{aligned}$$

   Then,

   $$\begin{aligned} f(x) &= \frac{f^{(0)}(0)}{0!} + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{2!}x^4 + \dots \\ &= \boxed{1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4} + \dots \end{aligned}$$

2. Prove the recurrence given in class for the Stirling numbers of the second kind.

   Solution. The construction of a partition of $[n]$ into k parts can be split into two cases:

   Case 1: 1 is in its own part. If 1 is in its own part, we only need to partition 2 through n into $k-1$ parts. By definition of Stirling numbers of the second kind, this can be done in $S(n-1, k-1)$ ways.

   Case 2: 1 is not in its own part. We can first partition 2 through n into k parts, which can be done in $S(n-1,k)$ ways. Then, since 1 is not in its own part, we can add it to any of the k distinct parts. Thus, there are $k \cdot S(n-1,k)$ partitions in this case.

   Because these cases are exhaustive and disjoint, we conclude that

   $$S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k).$$

   □

3. Prove the formulas given in class (1) for the number of set partitions of type $(a_1, a_2, \dots, a_k) \vdash n$ and (2) for the number of partitions of cycle type $(a_1, a_2, \dots, a_k) \vdash n$.

4. Using the recurrence, find the exponential generating function for the Bell numbers.

   Solution. Let $I(x) = \sum_{n \geq 0} I_n \frac{x^n}{n!}$, with $I_k = 1$ for all k. Then $I(x) = e^x$.

Also, let $B(x) = \sum_{n \geq 0} B_n \frac{x^n}{n!}$ be the generating function for the Bell numbers. Then

$$\begin{aligned}
B'(x) &= \frac{d}{dx} \left[\sum_{n \geq 0} B_{n+1} \frac{x^{n+1}}{(n+1)!} \right] \\
&= \sum_{n \geq 0} B_{n+1} \frac{x^n}{n!} \\
&= \sum_{n \geq 0} \left[\sum_{k=0}^{n} \binom{n}{k} B_k \right] \frac{x^n}{n!} \\
&= \sum_{n \geq 0} \left[\sum_{k=0}^{n} \binom{n}{k} B_k \cdot I_{n-k} \right] \frac{x^n}{n!} \\
&= B(x) I(x) \\
&= e^x B(x).
\end{aligned}$$

We can solve this differential equation by suggesting that $B(x) = ce^{e^x}$ for some c. Indeed, see that $B'(x) = ce^x e^{e^x} = e^x B(x)$ in this case. Because $B_0 = 1$, we have $B(0) = 1$, which gives $c = e^{-1}$. Thus

$$B(x) = e^{e^x - 1}.$$

5. The following four statements are equivalent: (1) T is a tree; (2) there is a unique path joining every two vertices of T; (3) T is connected and $|E(T)| = n - 1$; (4) T is acyclic and $|E(T)| = n - 1$.

   Solution. (3) $\Rightarrow$ (4) Assume that T is connected and $|E(T)| = n - 1$. Then

   (3) $\Rightarrow$ (4) Assume that T is acyclic and $|E(T)| = n - 1$. Now suppose that T is *not* connected. Then there must exist

   (1) $\Rightarrow$ (2) Assume that T is a tree, i.e. that it is acyclic and connected. Then suppose that there is *not* a unique path between every two vertices of T. Clearly, because T is connected, at least one path must exist between every two vertices. Thus, there must exist vertices n and m between which multiple distinct paths exist, namely

$$\begin{aligned}
p_1 &= \\
p_2 &= .
\end{aligned}$$

$\square$