Revisions for Problem 7.2.3

MAA4211

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(a) ($\Leftarrow$) Suppose there exists a sequence of partitions $(P_n)_{n=1}^{\infty}$ where

$$\lim_{n\to\infty}[U(f,P_n)-L(f,P_n)]=0.$$

Let $\epsilon>0$. By our limit, there must exist some sufficiently large n such that

$$U(f)-L(f)\le U(f,P_n)-L(f,P_n)<\epsilon.$$

Because ϵ is arbitrary, we have $U(f)=L(f)$, making f integrable.

($\Rightarrow$) Now suppose that f is integrable on $[a,b]$. Let k be a positive integer. By the Integrability Criterion, there must exist a partition P_k of $[a,b]$ such that

$$U(f,P_k)-L(f,P_k)<\frac{1}{k}.$$

Consider the sequence $(P_n)_{n=1}^{\infty}$ formed by taking the partition described above for each positive integer. Let $\epsilon>0$. We can pick sufficiently large n such that

$$U(f,P_n)-L(f,P_n)<\frac{1}{n}<\epsilon.$$

Thus, for $(P_n)_{n=1}^{\infty}$, we have

$$\lim_{n\to\infty}[U(f,P_n)-L(f,P_n)]=0,$$

as desired.

(b) Because P_n divides $[0,1]$ into n equal subintervals, we have $x_k=\frac{k}{n}$. Because $f(x)=x$, we have

$$m_k=\inf[x_{k-1},x_k]=x_{k-1}=\frac{k-1}{n}$$

and

$$M_k=\sup[x_{k-1},x_k]=x_k=\frac{k}{n}.$$

This gives

$$\begin{aligned}
U(f,P_n)&=\sum_{k=1}^{n}M_k(x_k-x_{k-1})\\
&=\sum_{k=1}^{n}\frac{k}{n}\left(\frac{k}{n}-\frac{k-1}{n}\right)\\
&=\frac{1}{n^2}\sum_{k=1}^{n}k\\
&=\frac{1}{n^2}\cdot\frac{n(n+1)}{2}\\
&=\frac{n+1}{2n}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
L(f, P_n) &= \sum_{k=1}^{n} m_k (x_k - x_{k-1}) \\
&= \sum_{k=1}^{n} \frac{k-1}{n} \left(\frac{k}{n} - \frac{k-1}{n} \right) \\
&= \frac{1}{n^2} \sum_{k=1}^{n} (k-1) \\
&= \frac{1}{n^2} \cdot \frac{(n-1) \cdot n}{2} \\
&= \frac{n-1}{2n}.
\end{aligned}$$

(c) Using the formulas from (b), we have

$$\begin{aligned}
\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] &= \lim_{n\to\infty} \left[\frac{n+1}{2n} - \frac{n-1}{2n} \right] \\
&= \lim_{n\to\infty} \frac{1}{n} \\
&= 0.
\end{aligned}$$

Hence, by part (a), we know that f is integrable on $[0, 1]$. Then,

$$\begin{aligned}
\int_0^1 f &= \lim_{n\to\infty} U(f, P_n) \\
&= \lim_{n\to\infty} \frac{n+1}{2n} \\
&= \frac{1}{2}.
\end{aligned}$$