Homework 5
MAD6206
Carson Mulvey

NB 1. Prove that every connected graph has a spanning tree.

Solution. Consider the following process for a connected graph G:

(1) If G is acyclic, then we are done.

(2) Otherwise, pick an arbitrary cycle in G, $(v_0, v_1, \ldots, v_{i-1}, v_i = v_0)$. We remove $e = (v_0, v_{i-1})$. The path $(v_0, v_1, \ldots, v_{i-1})$ still exists, so any two vertices that were connected by a path containing e are still connected. Thus, $G - e$ is still connected.

(3) Repeat (2) until the graph has no cycles left. This process must terminate as G is finite.

The result of this process will be an acyclic and connected graph. This is equivalent to a tree, as established in Homework 1, and hence is a spanning tree of G. Through this process, we can find a spanning tree for any connected graph.

NB 2. Provide an algorithm whose input is a graph G and whose output is "yes" if G is connected and "no" if G is not connected.

Solution. Let G have vertex set V. Consider the following algorithm:

(1) Pick an arbitary $v \in V$. Let $S_0 = \varnothing$ and $S_1 = \{v\}$.

(2) Recursively calculate $S_{i+1} = S_i \cup N(S_i \setminus S_{i-1})$.
(In other terms, for each new element in S_i, find neighboring vertices through an adjacency list or matrix, adding those that aren't already in S_i to to S_{i+1}.)

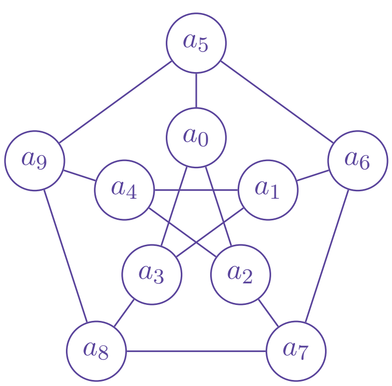
(3) Continue this recursion until $S_n = S_{n+1}$ for some n.

(4) If $S_n = V$, then output "yes". Otherwise, output "no".

This algorithm computes a 'closure' for some vertex v, with S_n being the set of vertices connected to v. A graph G is connected iff all vertices are connected, i.e. if this closure is the vertex set itself for arbitrary v.

NB 3. Prove that the Petersen graph (P) is (1) not Hamiltonian, (2) not planar.

Solution. Enumerate the vertices as follows:



(1) We will show P is not Hamiltonian by way of contradiction. Supposing a Hamiltonian cycle exists, we note that the number of edges in the cycle connecting the inner 5 vertices to the outer 5 vertices must be even, since we must both start and end on the same vertex. This gives two cases:

i. Two connecting edges are in the cycle: The two connecting edges are formed by two inner vertices, say b_0 and b_1, and two outer vertices, say c_0 and c_1. To form the cycle, b_0 and b_1 must be connected by a path of length 4. Because of this, b_0 and b_1 must be adjacent. Similarly, c_0 and c_1 must form a path of length 4 in the cycle, making them adjacent. However, these vertices then form a 4-cycle, which is not possible in the Petersen graph.

ii. Four connecting edges are in the cycle: WLOG let (a_0, a_5) be the edge that is *not* in the cycle, forcing the other edges containing a_0 and a_5, i.e. (a_5, a_6), (a_5, a_9), (a_0, a_2), and (a_0, a_3), to be in the cycle. Additionally, because two edges already contain a_6, the edge (a_6, a_7) *cannot* be in the cycle. This forces the other edge containing a_7, i.e. (a_7, a_8), to be in the cycle. However, we now have a 5-cycle between a_0, a_2, a_7, a_8, and a_3 which is not possible within our Hamiltonian cycle.

(2) To show P is not planar, construct a minor by contracting the edges (a_0, a_5), (a_1, a_6), (a_2, a_7), (a_3, a_8), and (a_4, a_9), resulting in K_5. By Kuratowski's Theorem, P is not planar.

NB 4. Show that the following are equivalent for a graph G: (1) G is bipartite, (2) G is 2-colorable, (3) G contains no odd cycle.

Solution.

(1) $\Rightarrow$ (2) For bipartite $G = (A \sqcup B, V)$, we can color vertices in A and B with our two respective colors.

(2) $\Rightarrow$ (3) We will prove by contradiction. Let G be a 2-colorable graph with an odd cycle. We can color an arbitrary vertex 1, which will force its neighbors to be colored 2. This process can continue, alternating between 1 and 2, but because the cycle is odd, the last two adjacent vertices must be the same color, forming a contradiction.

11.13.12. Let G be a graph with adjacency matrix A. Prove that the (i, j) entry of A^d is equal to the number of walks of length d from i to j

Solution. Let G have degree n. We will prove by induction on d. For $d = 1$, the entry (i, j) of A, $A_{i,j}$, is the number of edges between i and j, which coincides with the number of walks of length 1.

Suppose that for some k, the entry (i, j) of A^k equal the number of walks of length k from i to j. Now consider some (s, t) entry of $A^{k+1} = A^k \cdot A$. By the matrix multiplication formula, we have

$$(A^{k+1})_{s,t} = \sum_{i=1}^{n} (A^k)_{s,i} \cdot A_{i,t}.$$

By our inductive hypothesis, $(A^k)_{s,i}$ is the number of walks of length k from s to i. Additionally, $A_{i,t}$ is the number of edges connecting i and t. Thus, $(A^k)_{s,i} \cdot A_{i,t}$ is the number of walks of length $k+1$ from s to t that passes through i. When summed for each disjoint i, we have the total number of walks of length $k+1$ from s to t. Since this is equal to $(A^{k+1})_{s,t}$, our inductive step is complete.

NB 5. Prove that the chromatic polynomial of a tree of order n is $\chi(x) = x(x-1)^{n-1}$.

Solution. A vertex in a tree will only neighbor its parent and children, if they exist. This makes a coloring proper if and only if every vertex is a different color from its parent (besides the root). We have x options for the root, and coloring the other $n-1$ vertices layer by layer, we have $x-1$ options for each vertex, subtracting 1 for the color of each parent vertex. Multiplying these gives $\chi(x) = x(x-1)^{n-1}$, as desired.