Homework 4
MAD6206
Carson Mulvey

12.10.8. Calculate the Möbius functions of the posets whose Hasse diagrams appear in Fig. 12.1.

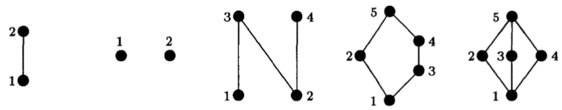

Fig. 12.1. Some Hasse diagrams

Solution.

(a) $\mu(1,1) = \mu(2,2) = 1$, $\mu(1,2) = -1$, $\mu(2,1) = 0$

(b) $\mu(1,1) = \mu(2,2) = 1$, $\mu(1,2) = \mu(2,1) = 0$

(c)

$$[\mu] = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(d)

$$[\mu] = \begin{bmatrix} 1 & -1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(e)

$$[\mu] = \begin{bmatrix} 1 & -1 & -1 & -1 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

12.10.10. Let a, b be elements of a poset P. Prove that $\mu(a,b) = \sum_{i \geq 0} (-1)^i c_i$, where c_i is the number of chains

$$a = x_0 < \cdots < x_i = b.$$

Solution. We can replace c_i with the notation $c(r, i)$, which represents the number of chains

$$a = x_0 < \cdots < x_i = r,$$

for any element $a \leq r \leq b$.

Lemma. For elements a, b, with $i > 0$,

$$c(b, i+1) = \sum_{a \le r < b} c(r, i).$$

Proof. Consider a chain from a to b where

$$a = x_0 < \cdots < x_i < x_{i+1} = b.$$

Removing b from this chain produces a chain

$$a = x_0 < \cdots < x_i = r.$$

Conversely, a chain from a to r of length $i+1$ can be extended to b, creating a chain of length $i+2$. Thus, we have a bijection between chains of length $i+2$ from a to b (LHS), and chains from a to r of length $i+1$ for all possible r (RHS). (end of Lemma)

For two elements a, b, let H be the maximal i such that $c(b, i)$ is nonzero. We will prove by strong induction on H. For $H = 0$, we have $a = x_0 = b$, and $\mu(a, b) = \sum_{i \ge 0} (-1)^i c(b, i) = 1$, as desired. Now consider elements $a \ne b$ where $H = K > 0$. For some element r with $a \le r < b$, we have $H < K$ between a and r, as all chains are smaller than the maximal chain from a to b. By our inductive hypothesis, we have

$$\mu(a, r) = \sum_{i \ge 0} (-1)^i c(r, i). \tag{$*$}$$

By the recursive formula for the Möbius function, we have

$$\mu(a, b) = -\sum_{a \le r < b} \mu(a, r).$$

But $(*)$ implies

$$\begin{aligned}
\mu(a, b) &= \sum_{a \le r < b} \sum_{i \ge 0} (-1)^{i+1} c(r, i) \\
&= \sum_{i \ge 0} \sum_{a \le r < b} (-1)^{i+1} c(r, i) \\
&= \sum_{i \ge 0} (-1)^{i+1} \sum_{a \le r < b} c(r, i) \\
&= \sum_{i \ge 0} (-1)^{i+1} c(b, i+1) \qquad \text{(by Lemma)} \\
&= \sum_{i \ge 1} (-1)^i c(b, i).
\end{aligned}$$

But $(-1)^0 c(b, 0) = 0$ when $a \ne b$, so

$$\mu(a, b) = \sum_{i \ge 0} (-1)^i c(b, i),$$

as desired. □

Non-book 1. Prove that a poset of size $mn+1$ has a chain of size greater than m or an antichain of size greater than n.

Solution. Let poset P have no antichain with greater than n elements. Then let the maximum antichain size (i.e. the width) of P be $a \leq n$. By Dilworth's Theorem, the number of elements in a minimal chain cover is also a. Then by the Pigeonhole Principle, at least one chain in any chain cover must contain at least $\lceil \frac{mn+1}{a} \rceil$ elements. But since $a \leq n$, we have

$$\begin{aligned}\left\lceil \frac{mn+1}{a} \right\rceil &\geq \left\lceil \frac{mn+1}{n} \right\rceil \\ &= \left\lceil m + \frac{1}{n} \right\rceil \\ &= m+1.\end{aligned}$$

Thus, P contains a chain with greater than m elements. □

Non-book 2. Prove that I is an ideal if and only if it is an initial segment of some linear extension of P.

Solution. ($\Rightarrow$) The case of $I = P$ is clear, so consider $I \neq P$. We can create linear extensions $f_1 \colon I \to [|I|]$ on I as a subposet and $f_2 \colon P \backslash I \to \{|I|+1, \ldots, |P|\}$ on $P \backslash I$ as a subposet. Such linear extensions must exist as both subposets are nonempty. Then define $f \colon P \to [|P|]$ by mapping $x \in P$ to $f_1(x)$ if $x \in I$ and $f_2(x)$ otherwise. We then claim that f is a valid linear extension of P. To see this, we take elements x, y in P with $x \leq y$. If $x, y \in I$ or $x, y \notin I$, then $f(x) \leq f(y)$ by definition of f_1 and f_2, respectively. If $x \in I$ and $y \notin I$, then $f(x) \leq f(y)$ is clear by comparing the codomains of f_1 and f_2. Finally, $y \in I$ and $x \notin I$ is not possible, as $x \leq y$ implies $x \in I$ by definition of an ideal. Thus, f is a linear extension with I as its initial segment.

($\Leftarrow$) Let I be the initial segment of a linear extension f on P. Let $x \in I$. If $y \leq x$, then $f(y) \leq f(x)$ by definition of a linear extension. This implies that y is also in the initial segment of f, i.e. $y \in I$. Thus, I is an ideal. □

Non-book 3. Let L be a finite poset with a maximum element $\hat{1}$ such that every two elements have a meet. Prove that L is a lattice.

Solution. Let $x, y \in L$. Define $\mathcal{B} = \{b \in L \colon x, y \leq b\}$. We note that $\hat{1} \in \mathcal{B}$, so $\mathcal{B}$ is nonempty. We enumerate $\mathcal{B} = \{b_1, b_2, \ldots, b_k\}$. We then consider the meet of all b_i,

$$m = \bigwedge_{i=1}^{k} b_i.$$

We claim that the join of x and y exists, namely $x \vee y = m$. Since $x, y \leq b_i$ for all i, x and y are at most as large as the common lower bound of all b_i. Thus, $x, y \leq m$, and $m \in \mathcal{B}$. Moreover, for all $i \in [k]$, we have $m \leq b_i$, which implies $m = \min \mathcal{B}$. Thus $x \vee y = m$, as desired. □