MAD 4204, Homework 4

Problems to turn in:

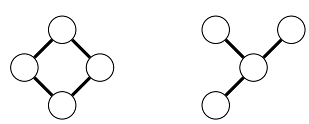
1. For P a finite poset, let $J(P)$ be the set of ideals in P and $A(P)$ be the set of antichains.
   (a) Find $\#J(P)$ and $\#A(P)$ for a chain. For an antichain.
   (b) Find $\#J(P)$ and $\#A(P)$ for B_3.
   (c) Must $\#J(P) = \#A(P)$? Why or why not? Explain.
2. (a) For P a poset with n elements, prove P contains a chain with at least $\sqrt{n}$ elements or an antichain with at least $\sqrt{n}$ elements.
   (b) Prove Hall's theorem using Dilworth's theorem.
3. For P a finite poset, show the number of elements in a maximum chain equals the number of antichains in the smallest antichain cover.
4. Let $M(n,k)$ be the multiset consisting of k copies of each element in $[n]$. Let $P(n,k)$ be the poset on submultisets of $M(n,k)$ ordered by containment, e.g.

$$\{\{1,1,4\}\} \subseteq \{\{1,1,1,3,3,4,5,5\}\} \quad \text{but} \quad \{\{1,1,4\}\} \not\subseteq \{\{1,3,3,4,4\}\}.$$

   Find a general formula for $\mu_{P(n,k)}(x,y)$, and explain how it relates to Example 16.20.

Recommended problems:

1. For P a poset, confirm that $(J(P), \subseteq)$ is a partial order. If $P = [n]$, what is it?
2. For what poset P is $(J(P), \subseteq)$ isomorphic to B_n?
3. For P a poset, a P-partition of height k is an order-preserving function $f : P \to [k]$. Let $\mathcal{O}_P(k)$ be the number of P-partitions of height k. Find a formula for $\mathcal{O}_P(k)$ where P is
   (a) an antichain with n elements
   (b) a chain with n elements
   (c) the two posets below



   (d) Show $\mathcal{O}_P(k)$ is a polynomial. What is its degree? Find the leading coefficient.
4. Find the number of linear extensions for the poset $[2] \times [n]$.
5. Find the number of two-element antichains and the number of two-element chains in B_n.