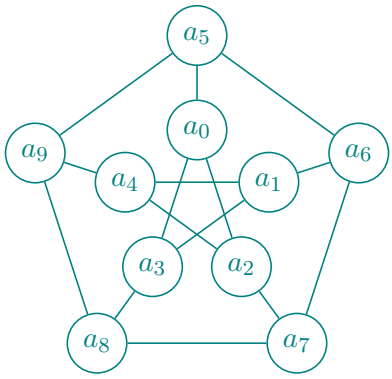
Homework 2
MAD4204
Carson Mulvey

1. (a) Show the Petersen graph is not planar.

Solution. Enumerate the vertices as follows:



Constructing a minor by contracting the edges (a_0, a_5), (a_1, a_6), (a_2, a_7), (a_3, a_8), and (a_4, a_9) results in K_5. Thus, by Kuratowski's Theorem, the Petersen graph is not planar.

(b) Find the Petersen graph's chromatic number.

Solution. Clearly a proper 1-coloring is not possible. Using the same enumeration as in (a), we attempt to construct a proper 2-coloring $\omega : \{a_i\}_{i=0}^{9} \to \{1, 2\}$. WLOG, let $\omega(a_0) = 1$. Then we must have $\omega(a_2) = \omega(a_3) = 2$. This forces $\omega(a_1) = \omega(a_4) = 1$. But since a_1 and a_4 are adjacent, this coloring would not be proper.

We can, however, find a proper 3-coloring $\omega : \{a_i\}_{i=0}^{9} \to \{1, 2, 3\}$. One example is

$$\begin{aligned} \omega(a_0) = \omega(a_6) = \omega(a_8) = \omega(a_9) &= 1 \\ \omega(a_3) = \omega(a_4) = \omega(a_5) = \omega(a_7) &= 2 \\ \omega(a_1) = \omega(a_2) &= 3. \end{aligned}$$

Thus, the Petersen graph has chromatic number 3.

(c) Let $G = (\binom{[5]}{2}, E)$ where $(A, B) \in E$ if and only if $A \cap B \neq \varnothing$. Determine the chromatic number of G. For example, $(\{1, 2\}, \{2, 4\}) \in E$ but $(\{1, 2\}, \{3, 4\}) \notin E$. Is G planar?

Solution. We will prove that G is not planar by contradiction. Assume that G is planar, and let $V = \binom{[5]}{2}$ so that $|V| = \binom{5}{2} = 10$. Note that for any particular integer from 1 to 5, we have 4 elements of V containing that integer. Hence, for $\{a, b\} \in V$, a is shared by 3 other elements of V, and b is shared by 3 elements of V (distinct from those sharing a). Thus for all $v \in V$, we have $\deg(v) = 6$. An edge is shared by two vertices, so $|E| = \frac{1}{2} \cdot \deg(v) \cdot |V| = 30$. By Euler's Theorem on planar graphs, we have $|F| = |E| - |V| + 2 = 22$.

However, we also know that a face is composed of at least 3 edges, and an edge is shared by two faces. Thus $|E| \geq \frac{3}{2}|F|$. But $\frac{3}{2}|F| = 33 > 30$. We have a contradiction, so G cannot be planar.

2. Prove for all connected simple planar graphs $G = (V, E)$ that $|E| \leq 3 \cdot |V| - 6$. As a consequence, show that all simple planar graphs have a vertex of degree at most 5.

   Solution. Since each face of a planar graph is comprised of at least 3 edges, and an edge is shared by two faces, we have $|F| \leq \frac{2}{3}|E|$. By Euler's Theorem, we have

$$\begin{aligned} |E| + 2 &= |F| + |V| \\ &\leq \frac{2}{3}|E| + |V| \\ \Rightarrow \frac{1}{3}|E| &\leq |V| - 2 \\ \Rightarrow |E| &\leq 3|V| - 6. \end{aligned}$$

   Assume a simple planar graph has vertices all with degree at least 6. Then using the above inequality, we have

$$\begin{aligned} 6|V| &\leq \sum_{v \in V} \deg(v) \\ &= 2|E| \\ &\leq 6|V| - 12 \end{aligned}$$

   which forms a contradiction. Thus, all simple planar graphs must have a vertex of degree at most 5.

3. The *dual* of a matroid $\mathcal{M}_B$ on ground set S defined in terms of bases is

$$\mathcal{M}_B^* = \{S \setminus B : B \in \mathcal{M}_B\}.$$

   (a) Prove that $\mathcal{M}_B^*$ is a matroid defined in terms of bases.

   Solution. Let there be $A, B \in \mathcal{M}_B^*$ and some $a \in A$. We need to show the exchange property, or the existence of some $b \in B$ where $(A \setminus \{a\}) \cup \{b\} \in \mathcal{M}_B^*$. Let $A' = S \setminus A$ and $B' = S \setminus B$. Then $A', B' \in \mathcal{M}_B$. We then pick a particular $b \in A'$ so that $(A' \setminus \{b\}) \cup \{a\} \in \mathcal{M}_B$. Since $S \setminus ((A' \setminus \{b\}) \cup \{a\})$ is equivalent to $(A \setminus \{a\}) \cup \{b\}$, we have $(A \setminus \{a\}) \cup \{b\} \in \mathcal{M}_B^*$.

   (b) Recall for G a connected simple graph that $\mathcal{M}_B(G) = \{T \text{ a spanning tree of } G\}$ is a matroid. For G planar with dual graph G^*, prove $\mathcal{M}_B^*(G) = \mathcal{M}_B(G^*)$.

4. Let $k \in \mathbb{N}$ and define $\hat{p}_G(k) = (-1)^k p_G(k)$. Prove that $\hat{p}_G(k)$ is the number of compatible pairs (ρ, c) where c is a (not necessarily proper) $[k]$–coloring of G and ρ is an acyclic orientation. In particular, show $\hat{p}_G(1)$ is the number of acyclic orientations of G.

   Solution. Using Equation (1) from the handout, we get

$$\begin{aligned} p_G(-k) &= p_{G-e}(-k) - p_{G/e}(-k) \\ \Rightarrow (-1)^n p_G(-k) &= (-1)^n p_{G-e}(-k) - (-1)^n p_{G/e}(-k) \\ \Rightarrow (-1)^n p_G(-k) &= (-1)^n p_{G-e}(-k) + (-1)^{n-1} p_{G/e}(-k) \\ \Rightarrow \hat{p}_G(k) &= \hat{p}_{G-e}(k) + \hat{p}_{G/e}(k), \end{aligned}$$

where $(-1)^{n-1}p_{G/e}(-k) = \hat{p}_{G/e}(k)$ because G/e has $n-1$ vertices.

We describe a process for showing that a compatible pair (ρ, c) follows the above relation with $\hat{p}_G(k)$. For a particular edge e, we have two cases. In one, e completes a cycle in G, so $G - e$ adds a compatible pair. In the other, G/e forms a cycle, which would remove a compatible pair.

We have a bijection between compatible pairs in G and in either $G - e$ or G/e. This follows the same relation as $\hat{p}_G(k)$, so $\hat{p}_G(k)$ would find the number of compatible pairs for a proper $[k]$-coloring.