Homework 4
MAD4204
Carson Mulvey

1. For P a finite poset, let $J(P)$ be the set of ideals in P and $A(P)$ be the set of antichains.
   (a) Find $\#J(P)$ and $\#A(P)$ for a chain. For an antichain.
   (b) Find $\#J(P)$ and $\#A(P)$ for B_3.
   (c) Must $\#J(P) = \#A(P)$? Why or why not? Explain.

   Solution.

   (a) For chain P, each element generates a unique ideal. Conversely, all ideals in P can be traced to a unique maximum element. Thus, including the empty ideal, $\#J(P) = \#P + 1$. Also, all pairs of elements are comparable, so only singleton antichains exist (plus the empty antichain). Thus $\#A(P) = \#P + 1$.
   For an antichain P, all pairs of elements are incomparable, so any subset of P is another antichain. Since no element is strictly less than another, all subsets of P are also ideals. Conversely, ideals and antichains of P must be subsets of P. Thus $\#J(P) = \#A(P) = 2^{\#P}$.
   (b) For $P = B_3$, we look at antichains $\{\{1\},\{2\},\{3\}\}$ and $\{\{1,2\},\{2,3\},\{3,1\}\}$, each of which have 8 antichain subsets. Since the empty set is counted twice, this gives 15 antichains. Besides this, we have $\{\varnothing\}$, $\{\{1,2,3\}\}$, $\{\{1\},\{2,3\}\}$, $\{\{2\},\{3,1\}\}$, and $\{\{3\},\{1,2\}\}$, for a total of $\#A(P) = 20$. All ideals come from extending these antichains to include all subsets of its elements, so $\#J(P) = 20$.
   (c) Yes, $\#J(P) = \#A(P)$ must hold! We will describe a process that creates a bijection between ideals and antichains.
   For any ideal $I \subseteq P$, denote $\tilde{I}$ as the set of maximal elements of I. We note that for any pair $x, y \in \tilde{I}$, x and y are incomparable, since either $x > y$ or $y > x$ would make one of x and y not maximal. Thus $\tilde{I}$ is an antichain.
   Conversely, let A be an antichain. Define $\tilde{A}$ to be the set where $x \in \tilde{A}$ if $x \le a$ for some $a \in A$. If $y \le x$, then by transitivity, $y \le a$ for some $a \in A$, so $y \in \tilde{A}$. This makes $\tilde{A}$ an ideal by definition.

2. (a) For P a poset with n elements, prove P contains a chain with at least $\sqrt{n}$ elements or an antichain with at least $\sqrt{n}$ elements.
   (b) Prove Hall's theorem using Dilworth's theorem.

   Solution.

   (a) Let poset P have no antichain with at least $\sqrt{n}$ elements. Then let the width of P be $a < \sqrt{n}$. By Dilworth's Theorem, the number of elements in a minimal chain cover is also a. Then by the Pigeonhole Principle, at least one chain in any chain cover must contain at least $\lceil n/a \rceil$ elements. But since $a < \sqrt{n}$, we have

$$\begin{aligned} \lceil n/a \rceil &\ge \lceil \sqrt{n} \rceil \\ &\ge \sqrt{n}. \end{aligned}$$

Thus P contains a chain with at least $\sqrt{n}$ elements. $\square$

4. Let $M(n,k)$ be the multiset consisting of k copies of each element in $[n]$. Let $P(n,k)$ be the poset on submultisets of $M(n,k)$ ordered by containment, e.g.

$$\{\{1,1,4\}\} \subseteq \{\{1,1,1,3,3,4,5,5\}\} \quad \text{but} \quad \{\{1,1,4\}\} \not\subseteq \{\{1,3,3,4,4\}\}.$$

Find a general formula for $\mu_{P(n,k)}(x,y)$, and explain how it relates to Example 16.20.

Solution. We know that $a|b$ iff for any prime p, the exponent of p in the prime factorization in a is less than or equal to that of b. Thus, taking the number of copies of some i in a multiset as the power of prime p_i in an integer, we have a mapping between $P(n,k)$ and $(\mathbb{N},|)$, where n,k are arbitrarily large as needed. Thus, $\mu_{P(n,k)}(x,y) = (-1)^n$ if there is exactly 1 more copy of each element in y than that in x, and $\mu_{P(n,k)}(x,y) = 0$ otherwise.