Homework 6
MAD4204
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1. Let $S = \{\vec{v}_1, \dots \vec{v}_k\} \subseteq \mathbb{F}_q^n$. Show that $\vec{x} \in \overline{S}$, the affine closure of S, if and only if

$$\vec{x} = \sum_{i=1}^{k} a_i \vec{v}_i \quad \text{with} \quad \sum_{i=1}^{k} a_i = 1.$$

2. (a) Prove the following identity stated in class:

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

(b) Using (a), give an inductive proof that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial in the variable q.

Solution.

(a) Using the fact that $[n]_q! = [n]_q \cdot [n-1]_q!$, we have

$$\begin{aligned}
\begin{bmatrix} n \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{[n]_q!}{[k-1]_q![n-k+1]_q!} + \frac{q^k [n]_q!}{[k]_q![n-k]_q!} \\
&= [n]_q! \left(\frac{[k]_q}{[k]_q![n-k+1]_q!} + \frac{q^k[n-k+1]_q}{[k]_q![n-k+1]_q!} \right) \\
&= [n]_q! \left(\frac{1+q+\dots+q^{k-1}}{[k]_q![n-k+1]_q!} + \frac{q^k(1+q+\dots+q^{n-k}))}{[k]_q![n-k+1]_q!} \right) \\
&= [n]_q! \left(\frac{1+q+\dots+q^{k-1}+q^k+\dots+q^n}{[k]_q![n-k+1]_q!} \right) \\
&= [n]_q! \left(\frac{[n+1]_q}{[k]_q![n-k+1]_q!} \right) \\
&= \frac{[n+1]_q!}{[k]_q![n-k+1]_q!} \\
&= \begin{bmatrix} n+1 \\ k \end{bmatrix}_q
\end{aligned}$$

as desired. □

(b) We will prove by induction on n. For $n = 0$, we either have $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_q = 1$ for $k = 0$ or $\begin{bmatrix} 0 \\ k \end{bmatrix}_q = 0$ for $k > 0$.

Now assume for some $n \in \mathbb{Z}^+$ that for all $k \in \mathbb{Z}^+$, $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial in q. We need to show that $\begin{bmatrix} n+1 \\ k \end{bmatrix}_q$ is also a polynomial in q. But by (a), this is a linear combination of $\begin{bmatrix} n \\ k-1 \end{bmatrix}_q$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q$, which are each polynomials in q by our inductive hypothesis. Thus $\begin{bmatrix} n+1 \\ k \end{bmatrix}_q$ is also a polynomial in q for all $k \in \mathbb{Z}^+$, and our inductive step is complete. □

3. Let H be a hyperplane in $PG(n,q)$. Prove that $PG(n,q) - H$ is isomorphic to $AG(n,q)$. (This is Proposition 5.23 (b) from the alternate text states.)

   Solution. We can algebraically map from $AG(n,q)$ to $PG(n,q) - H$ by adding another coordinate to all points with a value of 1, i.e. mapping

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

   $PG(n,q)$ consists of these new points along with some "hyperplane at infinity" H. These hyperplane points will have a value of 0 first the first coordinate, so $PG(n,q) - H$ will leave precisely the points from $AG(n,q)$ with the new coordinate. This process by simply removing the first coordinate from all points in $PG(n,q) - H$, retrieving $AG(n,q)$. Since all points can be mapped, the point incidences, line incidences, etc. will also be isomorphic. □

2. Prove there are unique projective planes or orders two and three (up to relabeling X).

   Solution. Let $(X, \mathcal{B})$ be a projective plane of order two that isn't the Fano Plane. We claim that Y is a relabeling of a Fano Plane, showing that it is the unique projective plane of order 2. Representing the points of the Fano Plane in $\mathbb{F}_2^3$, we can label our Y similarly

Challenge Problem (8 points extra credit)

1. Define

$$\begin{bmatrix} x \\ k \end{bmatrix}_q = \frac{x \cdot (x - [1]_q) \cdot \ldots \cdot (x - [k-1]_q)}{q^{\binom{k}{2}} [k]!_q}.$$

   Show that

$$\begin{bmatrix} x \\ m \end{bmatrix}_q \cdot \begin{bmatrix} x \\ n \end{bmatrix}_q = \sum_{k=\max m,n}^{m+n} \frac{q^{(k-m)(k-n)}[k]!_q}{[k-m]!_q [k-n]!_q [m+n-k]!_q} \begin{bmatrix} x \\ k \end{bmatrix}_q.$$

   Solution. WLOG assume that $m \geq n$. We will prove by induction on m. For $m = 0$, clearly

$$\begin{bmatrix} x \\ 0 \end{bmatrix}_q \cdot \begin{bmatrix} x \\ 0 \end{bmatrix}_q = \frac{q^{(0)(0)}[0]!_q}{[0]!_q [0]!_q [0]!_q} \begin{bmatrix} x \\ 0 \end{bmatrix}_q = 1.$$

   We now assume for some m that for all $n \leq m$, the identity above holds. Then

$$\sum_{k=\max m+1,n}^{m+n+1} \frac{q^{(k-m+1)(k-n)}[k]_q!}{[k-m+1]_q! [k-n]_q! [m+n-k+1]_q!} \begin{bmatrix} x \\ k \end{bmatrix}_q$$
$$= \sum_{k=m+1}^{m+n+1} \frac{q^{(k-m+1)(k-n)}[k]_q!}{[k-m+1]_q! [k-n]_q! [m+n-k+1]_q!} \begin{bmatrix} x \\ k \end{bmatrix}_q$$