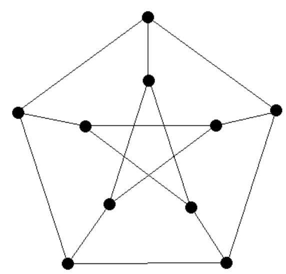
MAD 4204, Homework 3

For $G = (V, E)$ an (undirected) graph and $k \in \mathbb{N}$, the *chromatic polynomial* $p_G(k)$ is the number of proper $[k]$–colorings of G.

For $G = (V, E)$ an (undirected) graph, an **orientation** of G is a directed graph $\rho = (V, E')$ where for each $(u, v) \in E$, exactly one of $u \to v$ and $v \to u$ appears in E'. We say an orientation ρ and a coloring c are **compatible** if $u \to v$ in ρ implies $c(u) \leq c(v)$.

Problems to turn in:

1. 

   (a) Show the Petersen graph is not planar

   (b) Find the Petersen graph's chromatic number.

   (c) Let $G = (\binom{[5]}{2}, E)$ where $(A, B) \in E$ if and only if $A \cap B \neq \varnothing$. Determine the chromatic number of G. For example, $(\{1, 2\}, \{2, 4\}) \in E$ but $(\{1, 2\}, \{3, 4\}) \notin E$. Is G planar?

   (**Hint:** Another way to think about this is to color the **edges** of K_5 so that edges sharing a vertex have different colors)

2. Prove for all connected simple planar graphs $G = (V, E)$ with $|V| \geq 3$ that $|E| \leq 3 \cdot |V| - 6$. As a consequence, show that all simple planar graphs have a vertex of degree at most 5.

3. The *dual* of a matroid $\mathcal{M}_B$ on ground set S defined in terms of bases is

$$\mathcal{M}_B^* = \{S \setminus B : B \in \mathcal{M}_B\}.$$

   (a) Prove that $\mathcal{M}_B^*$ is a matroid defined in terms of bases.

   (b) Recall for G a connected simple graph that $\mathcal{M}_B(G) = \{T \text{ a spanning tree of } G\}$ is a matroid. For G planar with dual graph G^*, prove $\mathcal{M}_B^*(G) = \mathcal{M}_B(G^*)$.

4. (Problems (8)–(12) from the worksheet. You may assume problems (1)–(7))

   Let $k \in \mathbb{N}$ and define $\hat{p}_G(k) = (-1)^k p_G(k)$. Prove that $\hat{p}_G(k)$ is the number of compatible pairs (ρ, c) where c is a (not necessarily proper) $[k]$–coloring of G and ρ is an acyclic orientation. In particular, show $\hat{p}_G(1)$ is the number of acyclic orientations of G.

   Hint: Your proof will likely want to use the result

   Lemma: Every closed walk $v_0 \to v_1 \to \cdots \to v_n = v_0$ contains a directed cycle.

Recommended problems:
Let $\mathcal{M}_I$ be a matroid defined in terms of its independent sets on finite ground set S. For $T \subseteq S$, the *rank* of T, denoted $r(T)$, is the maximum size of an independent set $I \in \mathcal{M}_I$ so that $I \subseteq T$. We say $T \subseteq S$ is a k–*flat* of $\mathcal{M}_I$ if it is a maximal set of rank k, that is $r(T) = k$ and $r(T') > k$ whenever $T \subset T'$ with $T \neq T'$. Let $\mathcal{M}_F$ be the set of flats of $\mathcal{M}$.
(In a linear algebra context, a flat is a maximal set of rank k)

Let $\mathcal{M}_I$ be a matroid on ground set S defined in terms of bases. For $A \subseteq S$, the *restriction* of $\mathcal{M}_I$ to A is

$$\mathcal{M}_I \mid_A = \{I \in \mathcal{M}_I : I \in A\}.$$

Now assume $A \in \mathcal{M}_I$ and let $B = S \setminus A$. The *contraction* of $\mathcal{M}_I$ to A is

$$\mathcal{M}_I/A = \{I \subseteq A : I \cup B \text{ is independent}\}.$$

1. Let $\mathcal{M}$ be a matroid on finite ground set S. Show that $\mathcal{M}_F$ satisfies:

   (F1) $S \in \mathcal{M}_F$.

   (F2) If X and Y are flats, then $X \cap Y$ is a flat.

   (F3) For X a flat and $y \in S \setminus X$, there is a unique flat Y with $X \cup \{y\} \subseteq Y$ so that $X \subseteq Y$ and if U is a flat with $X \subseteq U \subseteq Y$ then $U = X$ or $U = Y$.

   (W will return to this in a few weeks)

2. For $\mathcal{M}$ a matroid on ground set S with flats $\mathcal{M}_F$, the *characteristic polynomial* of $\mathcal{M}$ is

   $$\chi_{\mathcal{M}}(x) = \sum_{F \in \mathcal{M}_F} x^{r(F)}.$$

   Prove for $s \in S$ that the characteristic polynomial satisfies the deletion-contraction relation

   $$\chi_{\mathcal{M}}(x) = \chi_{\mathcal{M}|_{S\setminus\{s\}}}(x) - \chi_{\mathcal{M}/\{s\}}(x).$$

   Bonus (get an A in the class): show that the absolute value of the coefficients is unimodal.

3. Exercise 12.25 from the text.

4. Prove every polyhedron has two faces that have the same number of vertices.

5. Let P be a convex polyhedron with no triangular face. Show $|E| \leq 2 \cdot |V| - 4$.

6. Show if every subgraph of G has a vertex with degree at most d then G is $(d + 1)$–colorable.