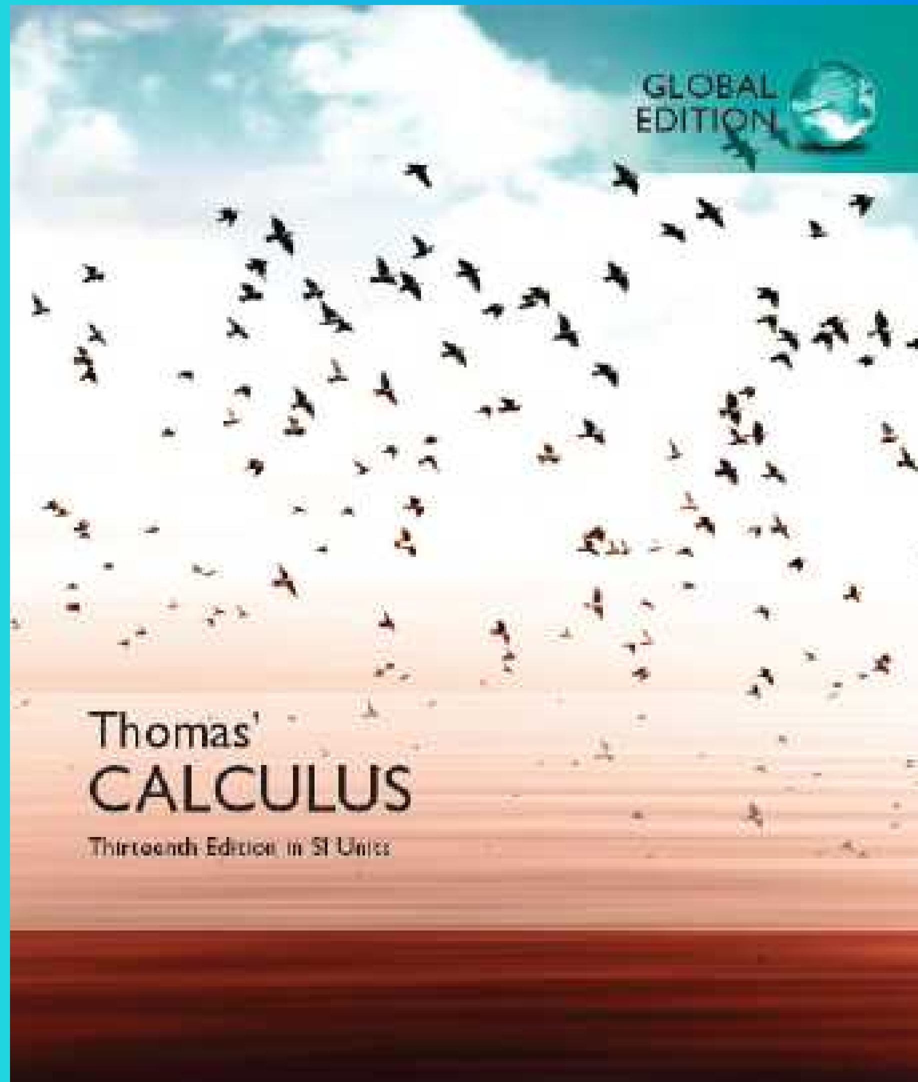

GLOBAL EDITION
Thomas'
CALCULUS
Thirteenth Edition in SI Units

Chapter 14

Partial Derivatives

偏导数

14.1

Functions of Several Variables
多变量函数

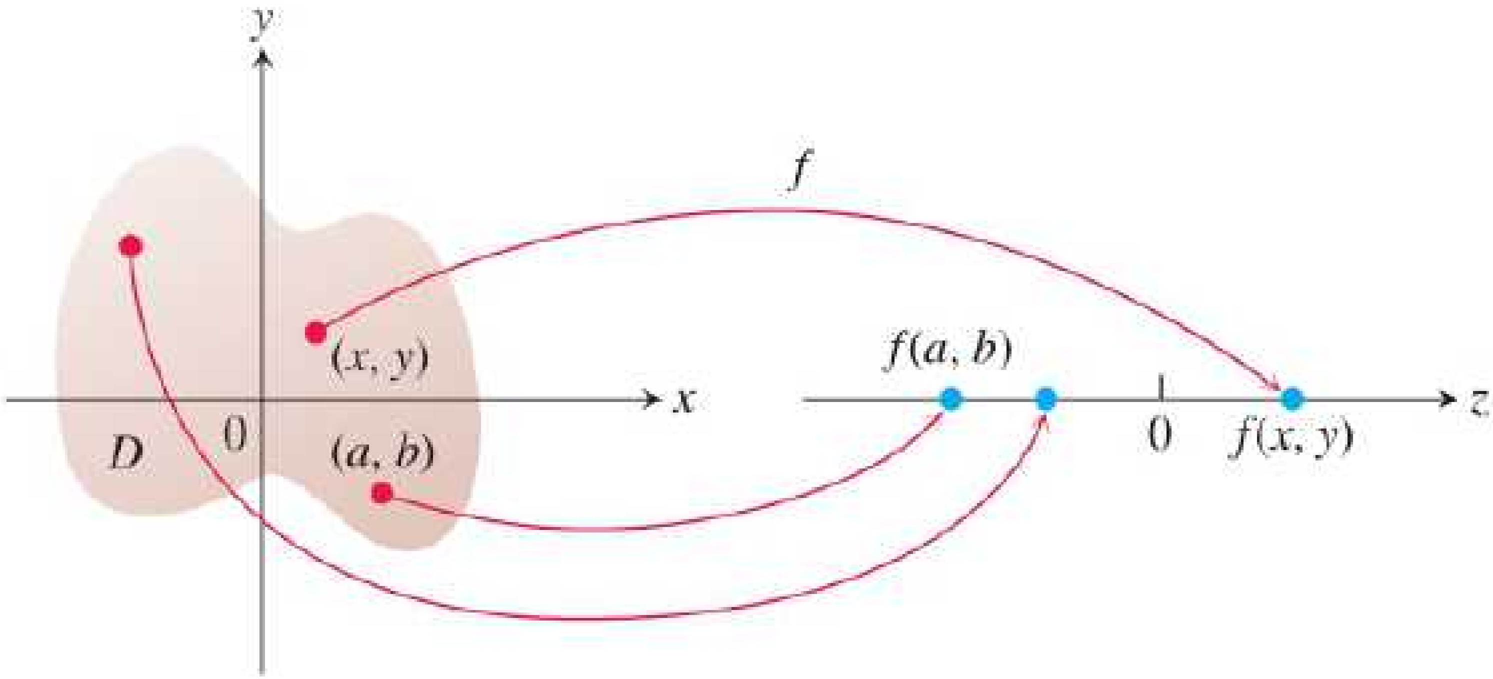
the volume of a right circular cylinder $V = f(r, h).$

DEFINITIONS Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A **real-valued function** f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, \ldots, x_n)$$

to each element in D. The set D is the function's **domain**. The set of w-values taken on by f is the function's **range**. The symbol w is the **dependent variable** of f, and f is said to be a function of the n **independent variables** x_1 to x_n. We also call the x_j's the function's **input variables** and call w the function's **output variable**.

the domain D as a region in xy-plane



$$V = \pi r^2 h.$$

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

$$f(3, 0, 4) = \sqrt{(3)^2 + (0)^2 + (4)^2} = \sqrt{25} = 5.$$

Domains and Ranges

EXAMPLE 1

(a) These are functions of two variables. Note the restrictions

Function	Domain	Range
$z = \sqrt{y - x^2}$	$y \geq x^2$	$[0, \infty)$
$z = \dfrac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
$z = \sin xy$	Entire plane	$[-1, 1]$

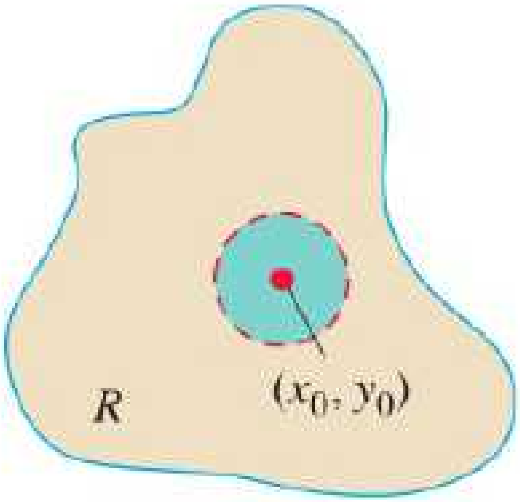
(b) These are functions of three variables with restrictions

Function	**Domain**	**Range**
$w = \sqrt{x^2 + y^2 + z^2}$	Entire space	$[0, \infty)$
$w = \dfrac{1}{x^2 + y^2 + z^2}$	$(x, y, z) \neq (0, 0, 0)$	$(0, \infty)$
$w = xy \ln z$	Half-space $z > 0$	$(-\infty, \infty)$

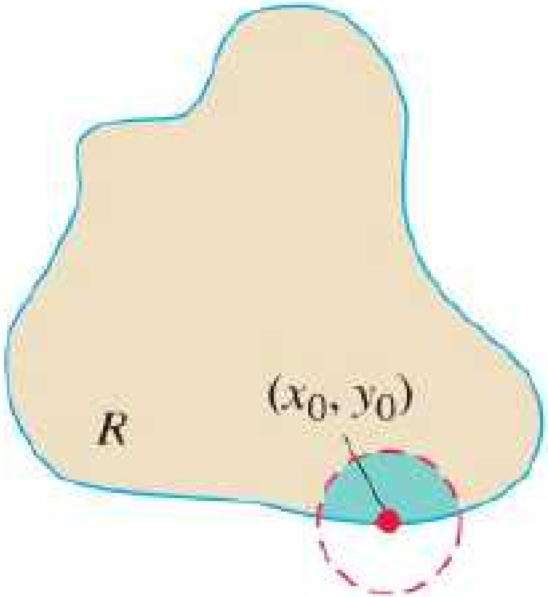
Functions of Two Variables

DEFINITIONS A point (x_0, y_0) in a region (set) R in the xy-plane is an **interior point** of R if it is the center of a disk of positive radius that lies entirely in R (Figure 14.2). A point (x_0, y_0) is a **boundary point** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. (The boundary point itself need not belong to R.)

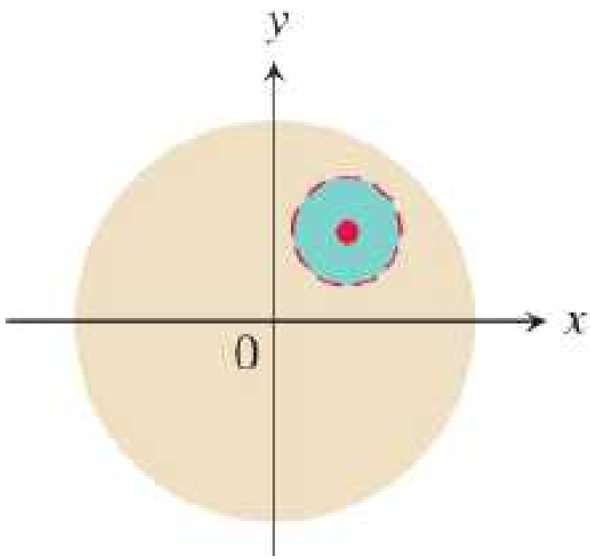
The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its **boundary**. A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points (Figure 14.3).



(a) Interior point

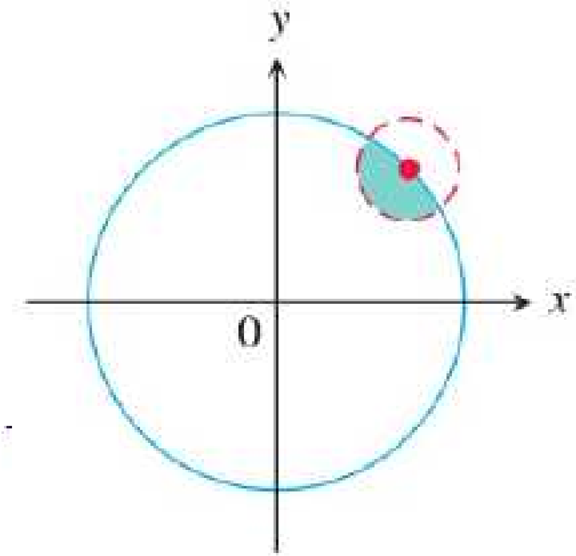


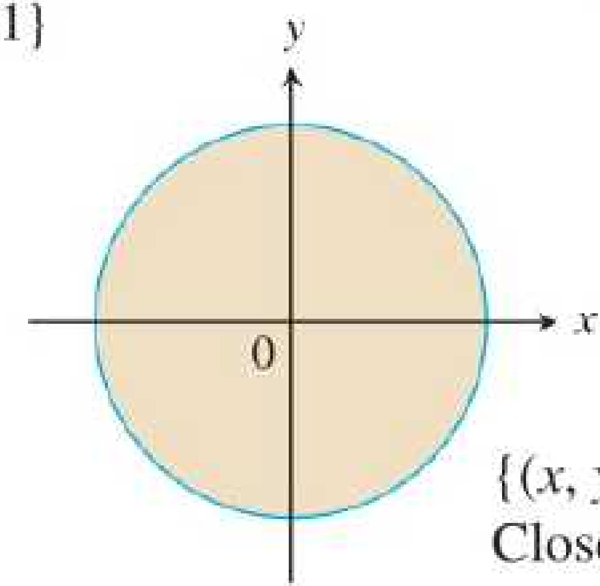
(b) Boundary point



$\{(x, y) \mid x^2 + y^2 < 1\}$
Open unit disk.

$\{(x, y) \mid x^2 + y^2 = 1\}$
Boundary of unit disk. (The unit





$\{(x, y) \mid x^2 + y^2 \leq 1\}$
Closed unit disk.

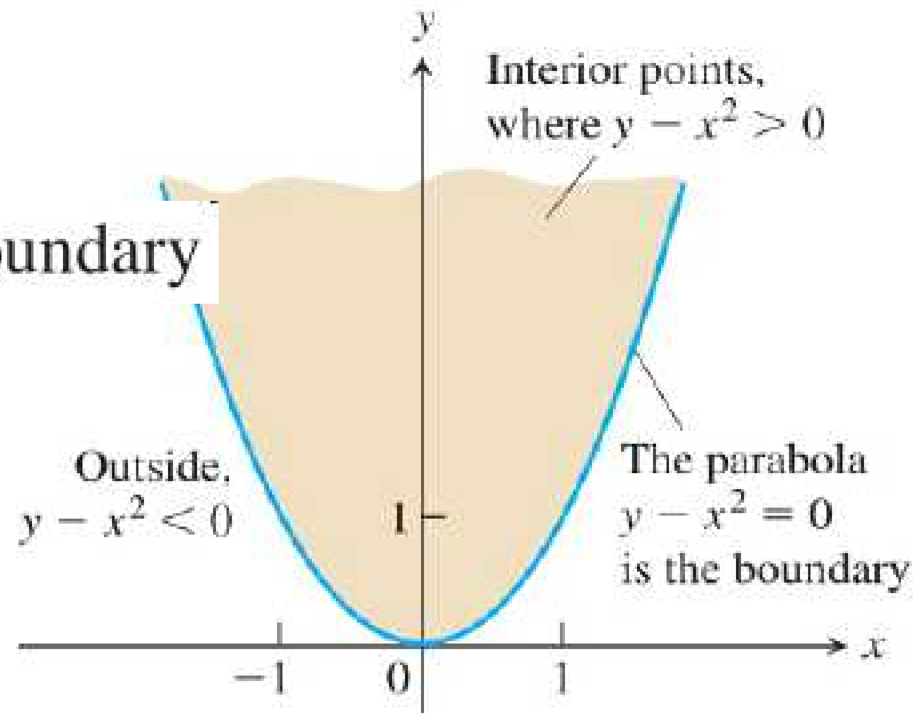
DEFINITIONS A region in the plane is **bounded** if it lies inside a disk of finite radius. A region is **unbounded** if it is not bounded.

EXAMPLE 2 Describe the domain of the function $f(x, y) = \sqrt{y - x^2}$.

Solution Since f is defined only where $y - x^2 \geq 0$,

the domain is the closed, unbounded region

The parabola $y = x^2$ is the boundary



Graphs, Level Curves, and Contours of Functions of Two Variables

There are two standard ways to picture the values of a function $f(x, y)$.

DEFINITIONS The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a **level curve** of f. The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called the **graph** of f. The graph of f is also called the **surface** $z = f(x, y)$.

EXAMPLE 3

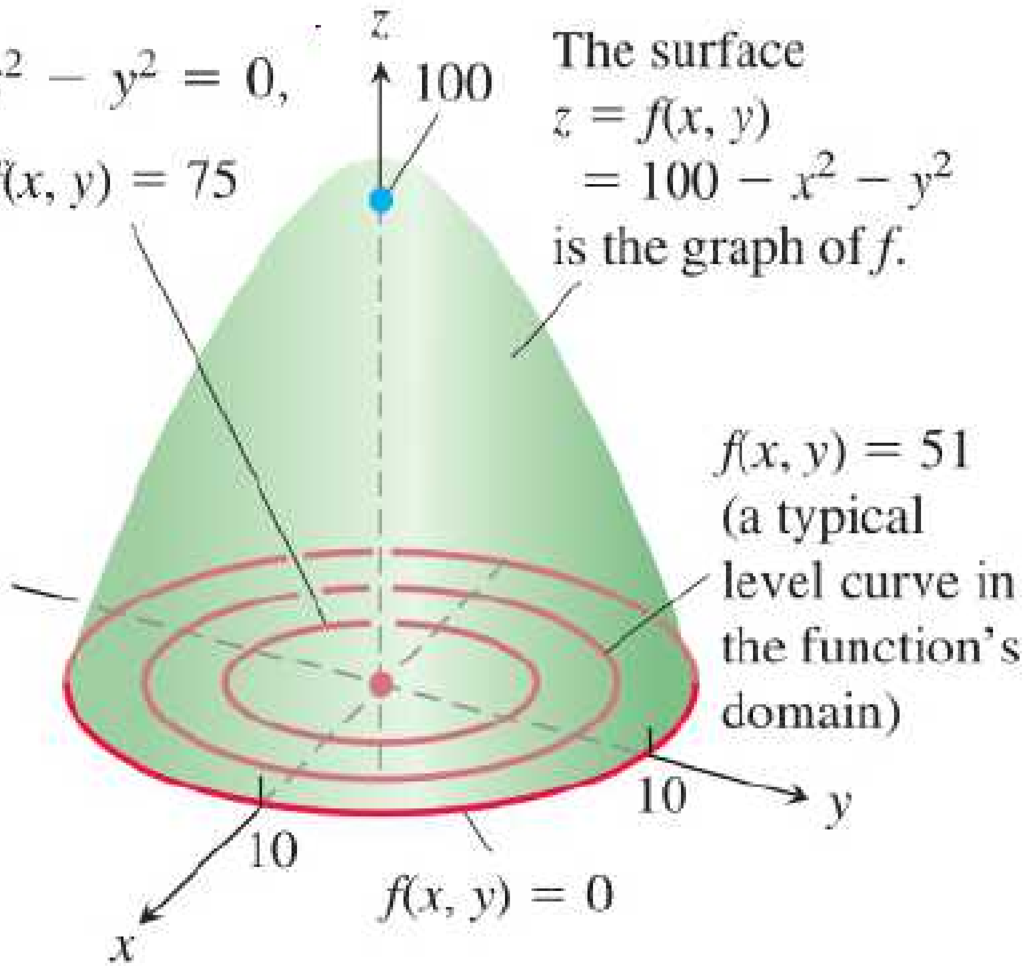
Graph $f(x, y) = 100 - x^2 - y^2$ and plot the level curves $f(x, y) = 0$, $f(x, y) = 51$, and $f(x, y) = 75$ in the domain of f in the plane.

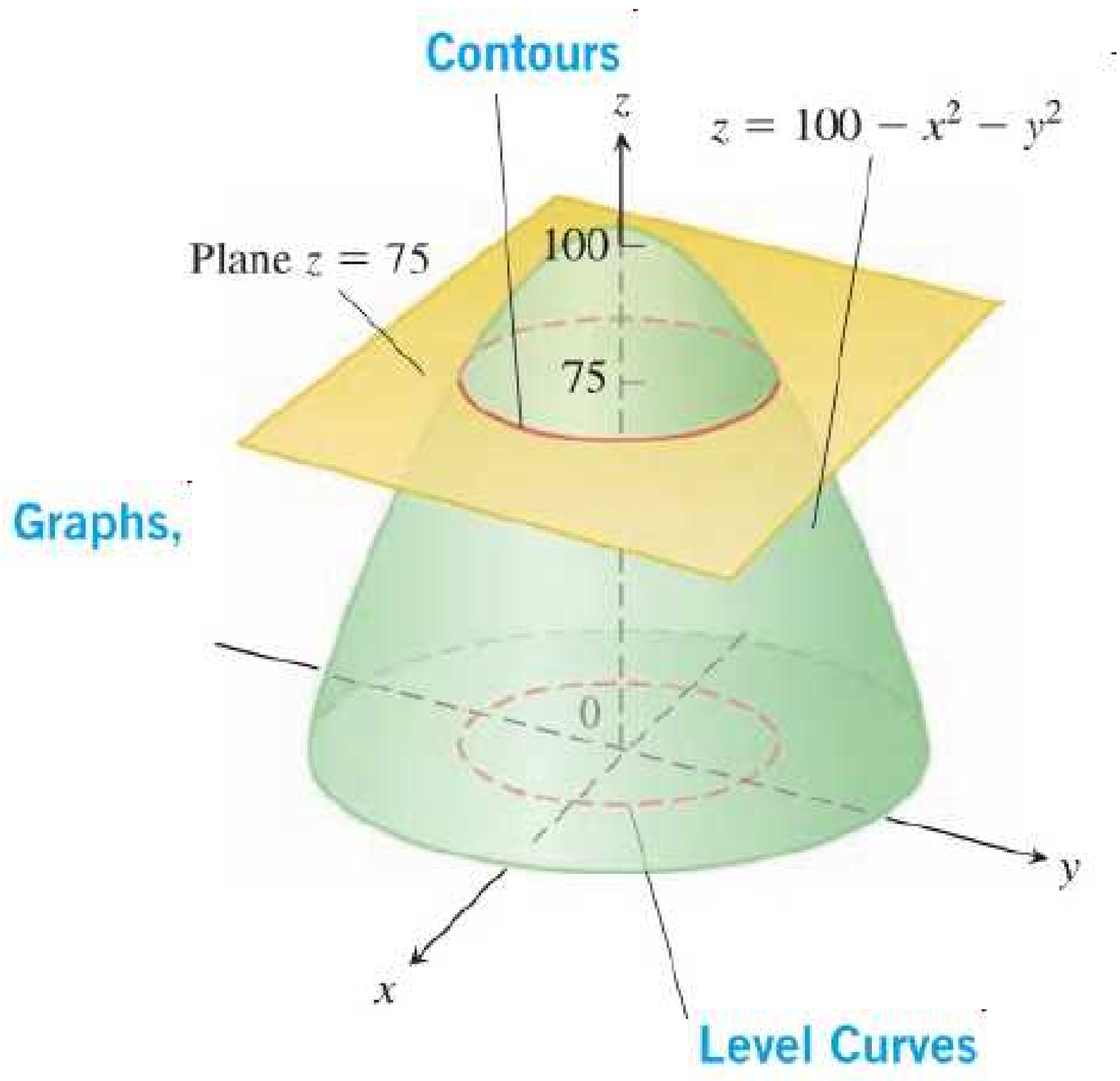
Solution

$f(x, y) = 100 - x^2 - y^2 = 0,$

$f(x, y) = 100 - x^2 - y^2 = 51,$

$f(x, y) = 100 - x^2 - y^2 = 75,$




Contours
$z = 100 - x^2 - y^2$
Plane $z = 75$
100
75
Graphs,
0
x
y
z
Level Curves

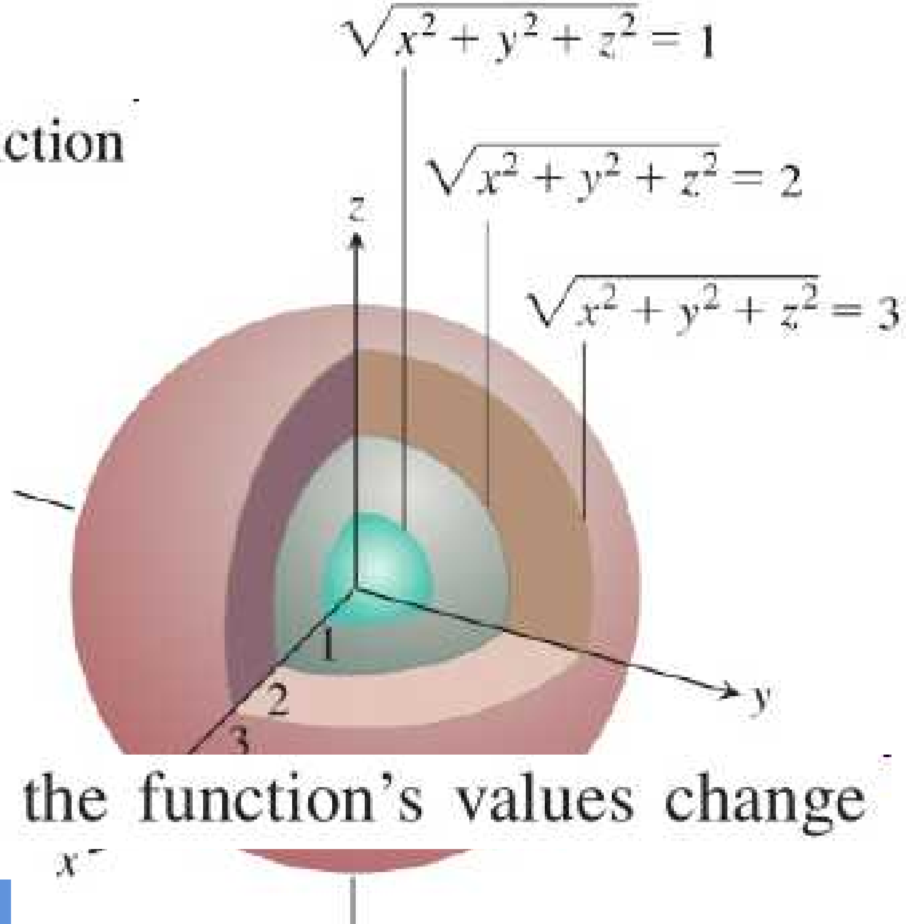
Functions of Three Variables

DEFINITION The set of points (x, y, z) in space where a function of three independent variables has a constant value $f(x, y, z) = c$ is called a **level surface** of f.

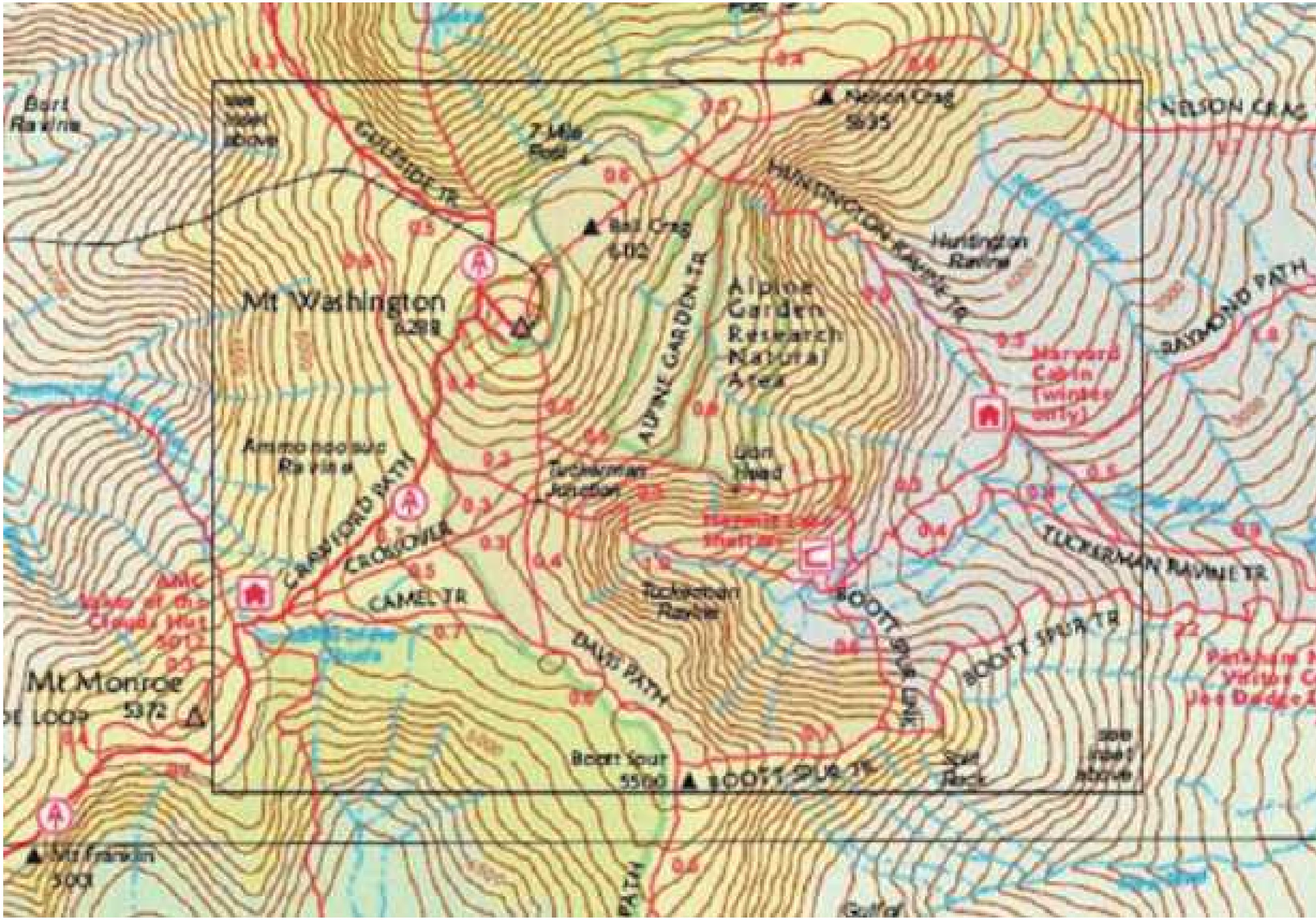
EXAMPLE 4

Describe the level surfaces of the function

$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.

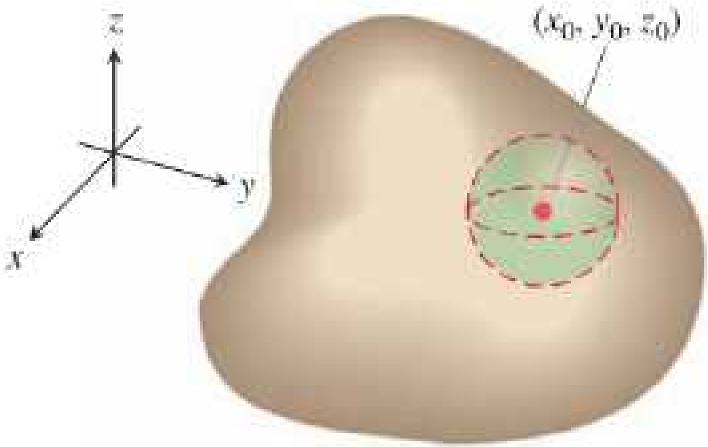


The level surfaces show how the function's values change

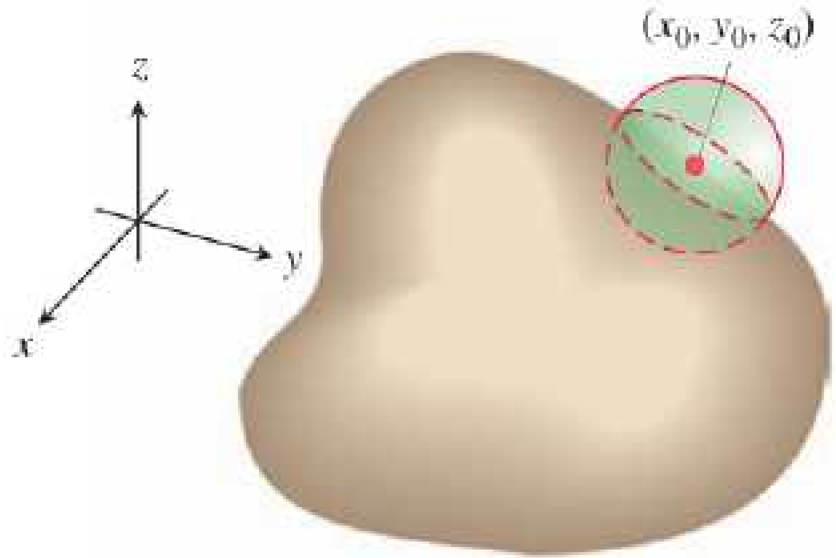

Mt Washington
6288
Mt Monroe
5372
GULFSIDE TR
7 Mile Post
Ball Crag
HUNTINGTON RAVINE TR
Huntington Ravine
ALPINE GARDEN TR
Alpine Garden Research Natural Area
Ammonoosuc Ravine
CRAWFORD PATH
CROSSOVER
CAMEL TR
DAVIS PATH
Tuckerman Junction
Lion Head
Tuckerman Ravine
TUCKERMAN RAVINE TR
BOOTT SPUR TR
BOOTT SPUR LINK
Boott Spur
5500
RAYMOND PATH
Harvard Cabin (winter only)
Nelson Crag
NELSON CRAG T
Mt Franklin

DEFINITIONS A point (x_0, y_0, z_0) in a region R in space is an **interior point** of R if it is the center of a solid ball that lies entirely in R (Figure 14.9a). A point (x_0, y_0, z_0) is a **boundary point** of R if every solid ball centered at (x_0, y_0, z_0) contains points that lie outside of R as well as points that lie inside R (Figure 14.9b). The **interior** of R is the set of interior points of R. The **boundary** of R is the set of boundary points of R.

A region is **open** if it consists entirely of interior points. A region is **closed** if it contains its entire boundary.



(a) Interior point



(b) Boundary point

Examples of *open* sets in space

the interior of a sphere, the open half-space $z > 0$,

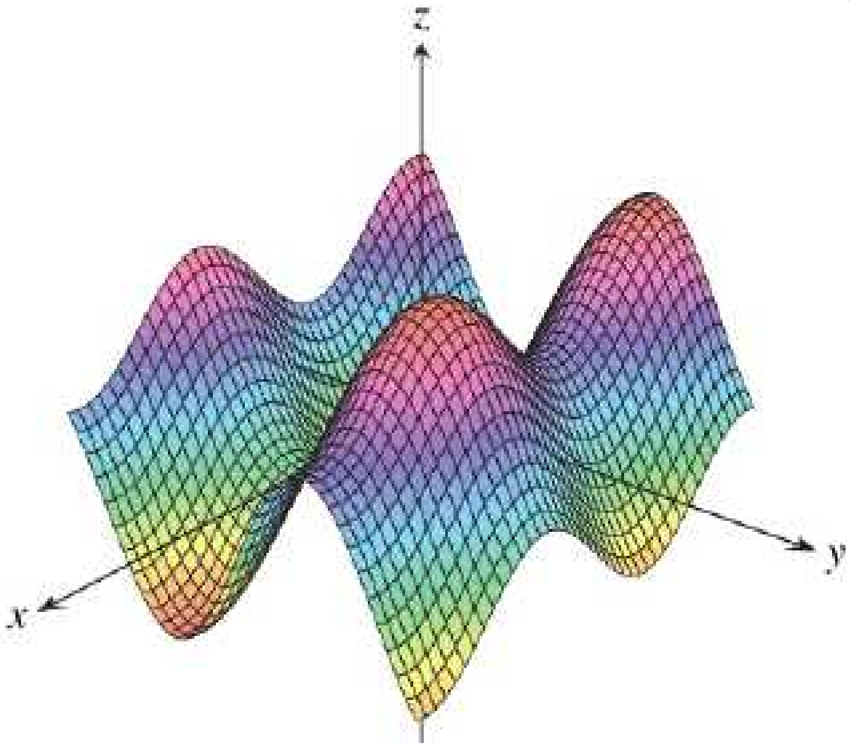
the first octant (where x, y, and z are all positive),

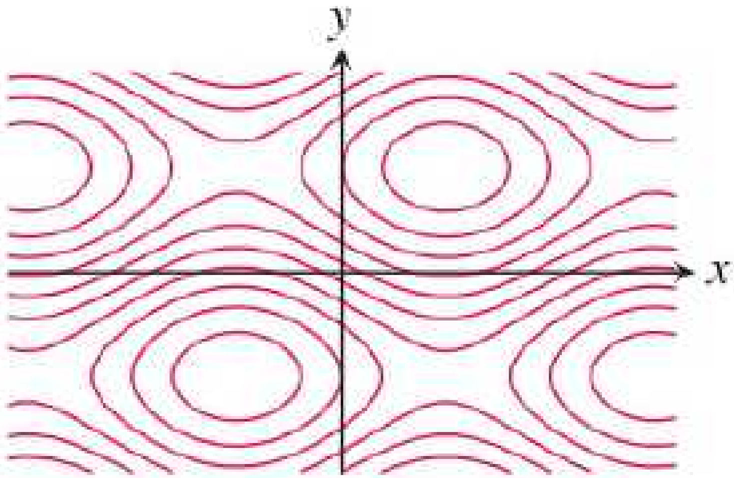
space itself.

a solid cube with a missing face, edge, or corner

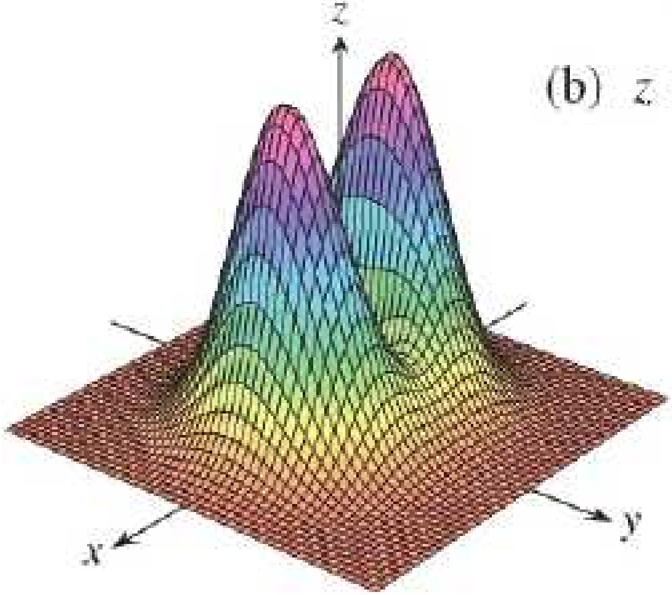
point is *neither open nor closed*.

Computer Graphing

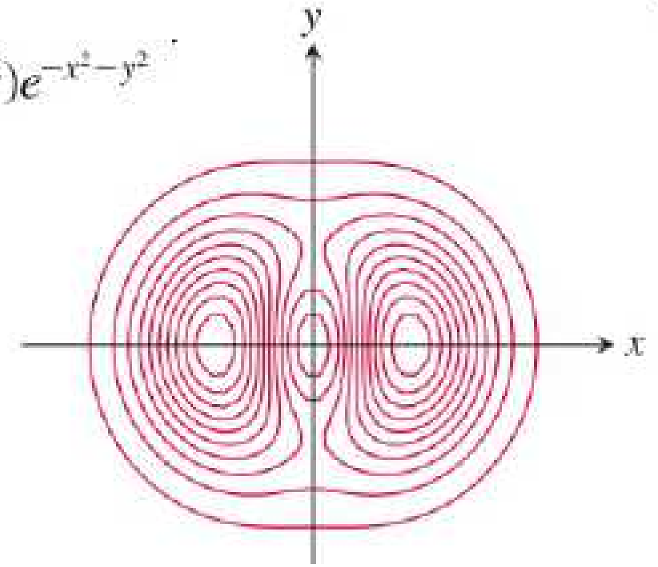


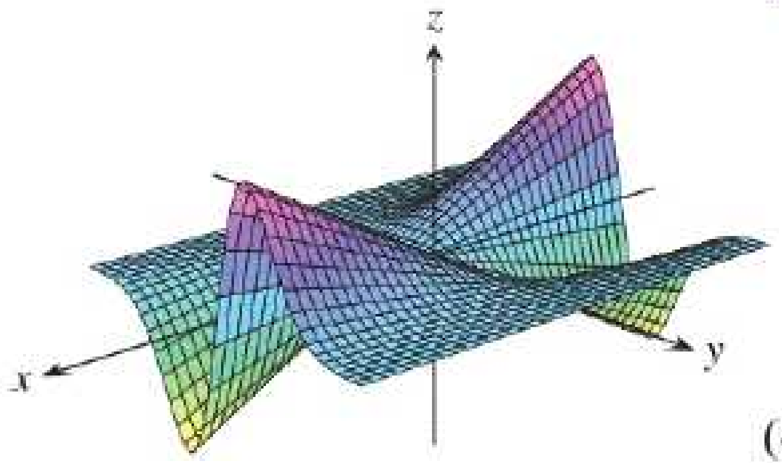


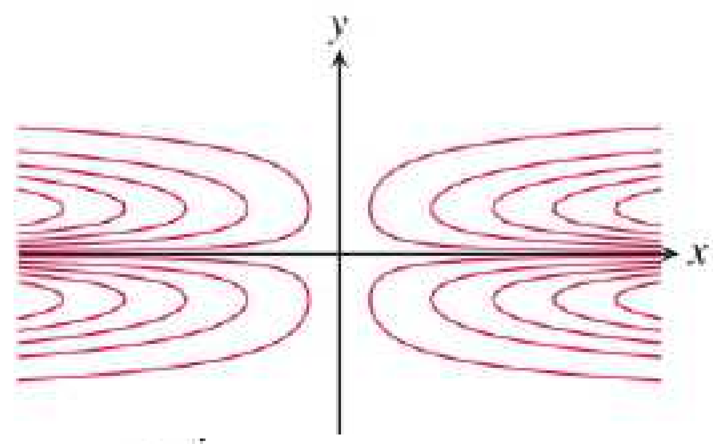
(a) $z = \sin x + 2 \sin y$



(b) $z = (4x^2 + y^2)e^{-x^2-y^2}$







(c) $z = xye^{-y^2}$

14.2

Limits and Continuity in Higher Dimensions
高维的极限与连续

Limits for Functions of Two Variables

DEFINITION We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

$$\lim_{(x,y)\to(x_0,y_0)} x = x_0$$

$$\lim_{(x,y)\to(x_0,y_0)} y = y_0$$

$$\lim_{(x,y)\to(x_0,y_0)} k = k$$

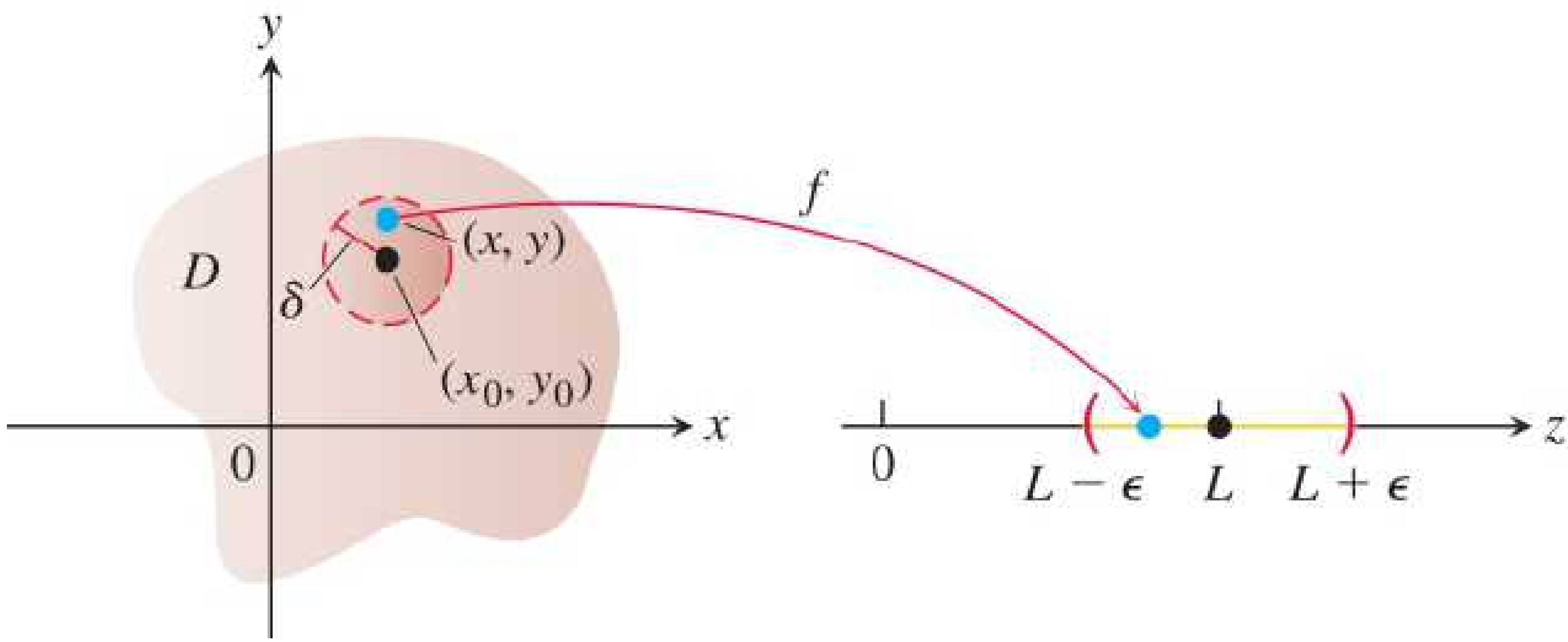
$\forall \varepsilon > 0$, 取 $\delta = \varepsilon$, 当 $0 < |\sqrt{(x-x_0)^2+(y-y_0)^2}| < \delta$

而 $|x-x_0| < |\sqrt{(x-x_0)^2+(y-y_0)^2}| < \delta$

所以 $|x-x_0| < \varepsilon$

$$\forall \varepsilon > 0, \exists \delta > 0, \quad 当0 < \left| \sqrt{(x-x_0)^2 + (y-y_0)^2} \right| < \delta$$

$$有 \quad L-\varepsilon < f(x,y) < L+\varepsilon$$



Let $p(x,y), p_0(x_0,y_0)$, then $f(x,y)=f(p), f(x_0,y_0)=f(p_0)$

$$\lim_{p\to p_0} f(p)=L:\forall\varepsilon>0,\exists\delta>0,\text{s.t}$$

$$0<|pp_0|<\delta\Rightarrow|f(p)-L|<\varepsilon.$$

多元函数极限定义

$$\lim_{x\to x_0} f(x)=L:\forall\varepsilon>0,\exists\delta>0,\text{s.t}$$

$$0<|x-x_0|<\delta\Rightarrow|f(p)-L|<\varepsilon.$$

THEOREM 1—Properties of Limits of Functions of Two Variables The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

$$\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$$

$$\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$$

$$\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL \qquad \text{(any number } k\text{)}$$

$$\lim_{(x,y)\to(x_0,y_0)} (f(x,y)\cdot g(x,y)) = L\cdot M$$

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}, \qquad M \neq 0$$

$$\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n, \ n \text{ a positive integer}$$

$$\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n},$$ n a positive integer, and if n is even, we assume that $L > 0$.

三明治定理：**If**

(1) $g(p) \le f(p) \le h(p)$,

(2) $\lim_{p\to p_0} g(p) = \lim_{p\to p_0} h(p) = L$,

Then $\lim_{p\to p_0} f(p) = L$.

$$\lim_{(x,y)\to(x_0,y_0)} x = x_0, \quad \lim_{(x,y)\to(x_0,y_0)} y = y_0, \quad \lim_{(x,y)\to(x_0,y_0)} k = k.$$

EXAMPLE 1

(a) $$\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \frac{0 - (0)(1) + 3}{(0)^2(1) + 5(0)(1) - (1)^3} = -3$$

(b) $$\lim_{(x,y)\to(3,-4)} \sqrt{x^2 + y^2} = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$$

EXAMPLE 2 Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$.

$$= \lim_{(x,y)\to(0,0)} \frac{\left(x^2 - xy\right)\left(\sqrt{x} + \sqrt{y}\right)}{\left(\sqrt{x} - \sqrt{y}\right)\left(\sqrt{x} + \sqrt{y}\right)} = \lim_{(x,y)\to(0,0)} x\left(\sqrt{x} + \sqrt{y}\right)$$

$$= 0\left(\sqrt{0} + \sqrt{0}\right) = 0$$

EXAMPLE 3 Find $\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2}$ if it exists.

Solution

$$0 \le \left| \frac{4xy^2}{x^2+y^2} \right| \le 4|x| \to 0$$

$$((x,y) \to (0,0))$$

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0.$$

换元法：$x = r\cos\theta, y = r\sin\theta$

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = \lim_{r\to 0} \frac{4r^3\cos\theta\sin^2\theta}{r^2}$$

$$= \lim_{r\to 0} 4r\cos\theta\sin^2\theta = 0$$

$$0 \le |4r\cos\theta\sin^2\theta| \le 4|r|$$

$$\lim_{p\to p_0} f(p) = L : \forall\varepsilon > 0, \exists\delta > 0, \text{s.t}$$

$$0 < |pp_0| < \delta \Rightarrow |f(p) - L| < \varepsilon.$$

注意：定义中$(x, y) \to (x_0, y_0)$的方式是任意的！

EXAMPLE 4 If $f(x, y) = \frac{y}{x}$, does $\lim_{(x, y)\to(0, 0)} f(x, y)$ exist?

Solution

$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx}} \frac{kx}{x} = k$$

$$\lim_{(x,y)\to(0,0)} \frac{y}{x} \quad \text{does not exist.}$$

Two-Path Test for Nonexistence of a Limit

If a function $f(x, y)$ has different limits along two different paths in the domain of f as (x, y) approaches (x_0, y_0), then $\lim_{(x, y)\to(x_0, y_0)} f(x, y)$ does not exist.

Does $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ **exist?** $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ y=kx}} \frac{xy}{x^2+y^2} = \frac{k}{1+k^2},$

Does'nt exist.

EXAMPLE 6 Show that the function $$f(x, y) = \frac{2x^2y}{x^4+y^2}$$

has no limit as (x, y) approaches $(0, 0)$.

Solution $$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=kx^2}} f(x, y) = \lim_{(x,y)\to(0,0)} \left[f(x, y)\Big|_{y=kx^2} \right] = \frac{2k}{1+k^2}.$$

Having the same limit along all straight lines approaching (x_0, y_0) does not imply a limit exists at (x_0, y_0).

Continuity

DEFINITION A function $f(x, y)$ is **continuous at the point (x_0, y_0)** if

1. f is defined at (x_0, y_0),
2. $\lim_{(x, y) \to (x_0, y_0)} f(x, y)$ exists,
3. $\lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0)$.

$$\lim_{p \to p_0} f(p) = f(p_0)$$

A function is **continuous** if it is continuous at every point of its domain.

EXAMPLE 5 Show that

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous at every point except the origin

Solution if $(x_0, y_0) \neq (0,0)$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \lim_{(x,y)\to(x_0,y_0)} \frac{2xy}{x^2+y^2} = \frac{2x_0y_0}{x_0^2+y_0^2}$$

At $(0, 0)$,

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=mx}} f(x, y) = \lim_{(x,y)\to(0,0)} \left[f(x, y) \Big|_{y=mx} \right] = \frac{2m}{1 + m^2}.$$

The limit fails to exist, is not continuous.

Continuity of Composites

If f is continuous at (x_0, y_0) and g is a single-variable function continuous at $f(x_0, y_0)$, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is continuous at (x_0, y_0).

For example, e^{x-y}, $\cos \dfrac{xy}{x^2 + 1}$, $\ln (1 + x^2y^2)$

are continuous at every point (x, y).

Functions of More Than Two Variables

the conclusions about limits and continuity for functions of two variables all extend to functions of three or more variables.

$$\ln(x + y + z) \qquad \text{and} \qquad \frac{y \sin z}{x - 1}$$

are continuous throughout their domains,

$$\lim_{P \to (1,0,-1)} \frac{e^{x+z}}{z^2 + \cos\sqrt{xy}} = \frac{e^{1-1}}{(-1)^2 + \cos 0} = \frac{1}{2},$$

Extreme Values of Continuous Functions on Closed, Bounded Sets

The same

holds true of a function $z = f(x, y)$ that is continuous on a closed, bounded set R like a line segment, a disk, or a filled-in triangle

有界闭集合上的连续函数一定在该集合上取得最大值最小值

14.3

Partial Derivatives
偏导数

Partial Derivatives of a Function of Two Variables

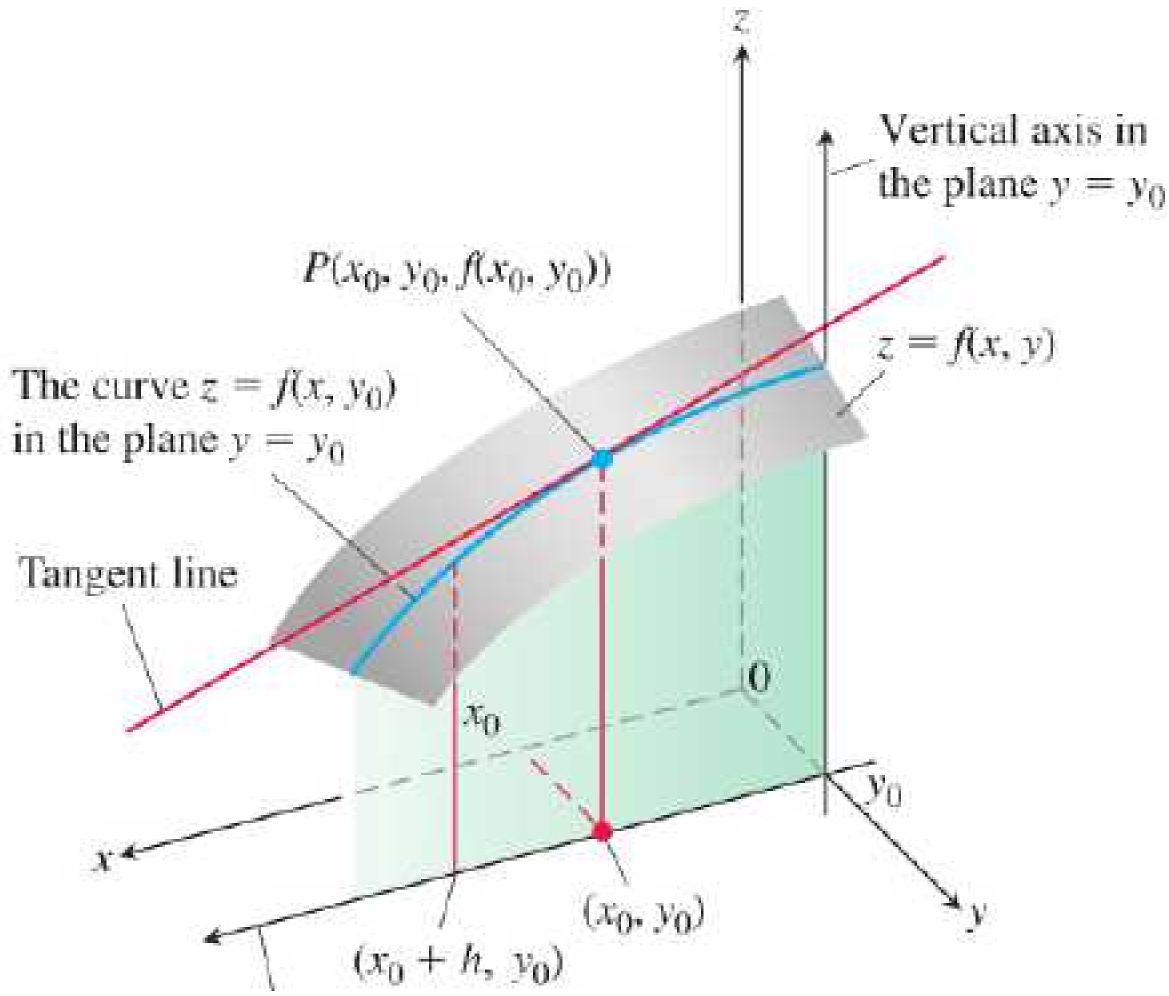
$$z = f(x, y)$$

$$y = y_0$$

$$z = f(x, y_0)$$

$$\left.\frac{dz}{dx}\right|_{x=x_0}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=x_0}$$



DEFINITION The **partial derivative of $f(x, y)$ with respect to x at the point (x_0, y_0)** is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

provided the limit exists.

$$\frac{\partial f}{\partial x}(x_0, y_0) \text{ or } f_x(x_0, y_0), \qquad \left.\frac{\partial z}{\partial x}\right|_{(x_0, y_0)}, \qquad \text{and} \qquad f_x, \ \frac{\partial f}{\partial x}, \ z_x, \text{ or } \frac{\partial z}{\partial x}.$$

An equivalent expression for the partial derivative is

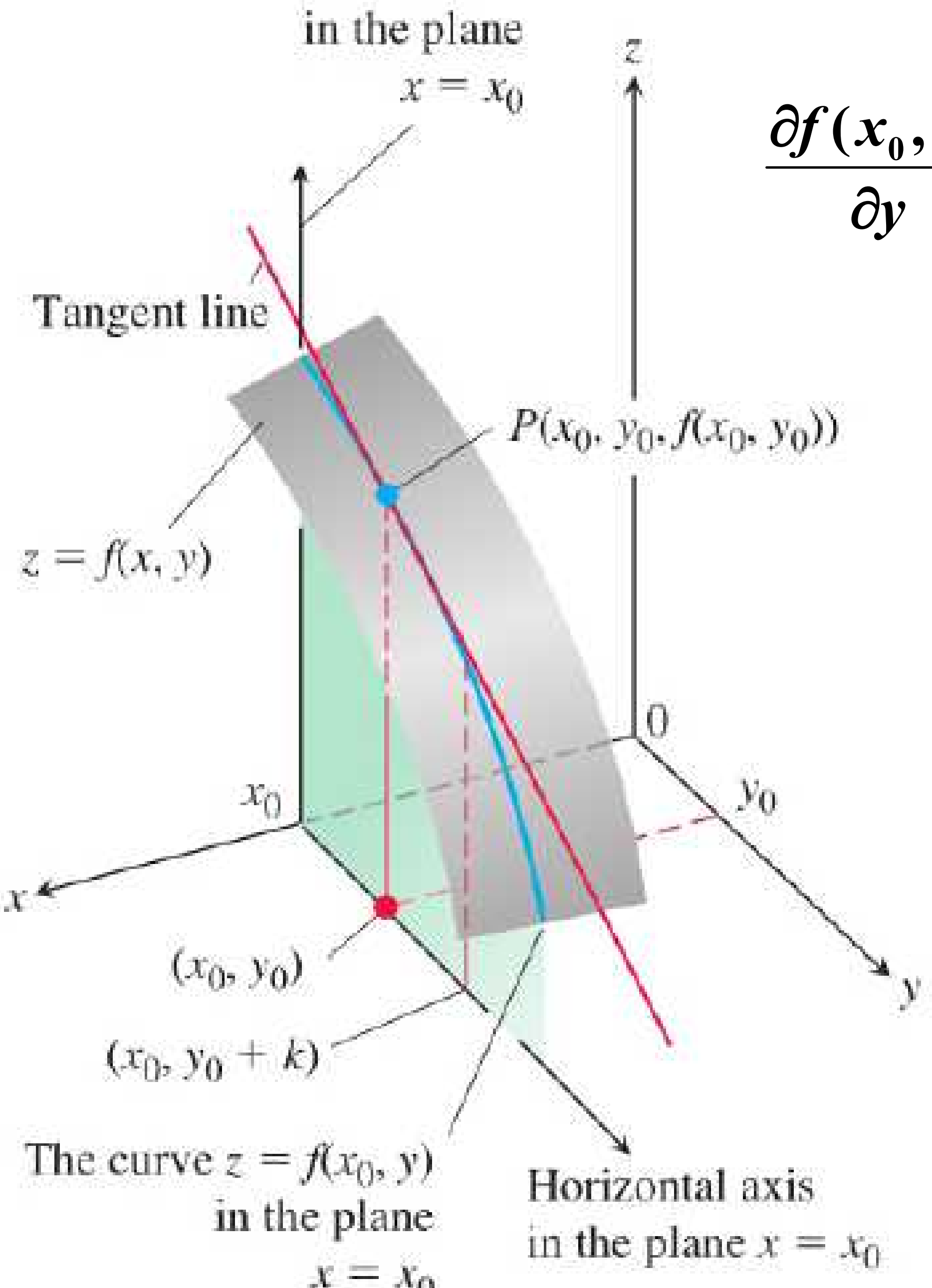
$$\left.\frac{d}{dx} f(x, y_0)\right|_{x = x_0}.$$

DEFINITION **The partial derivative of $f(x, y)$ with respect to y at the point (x_0, y_0) is**

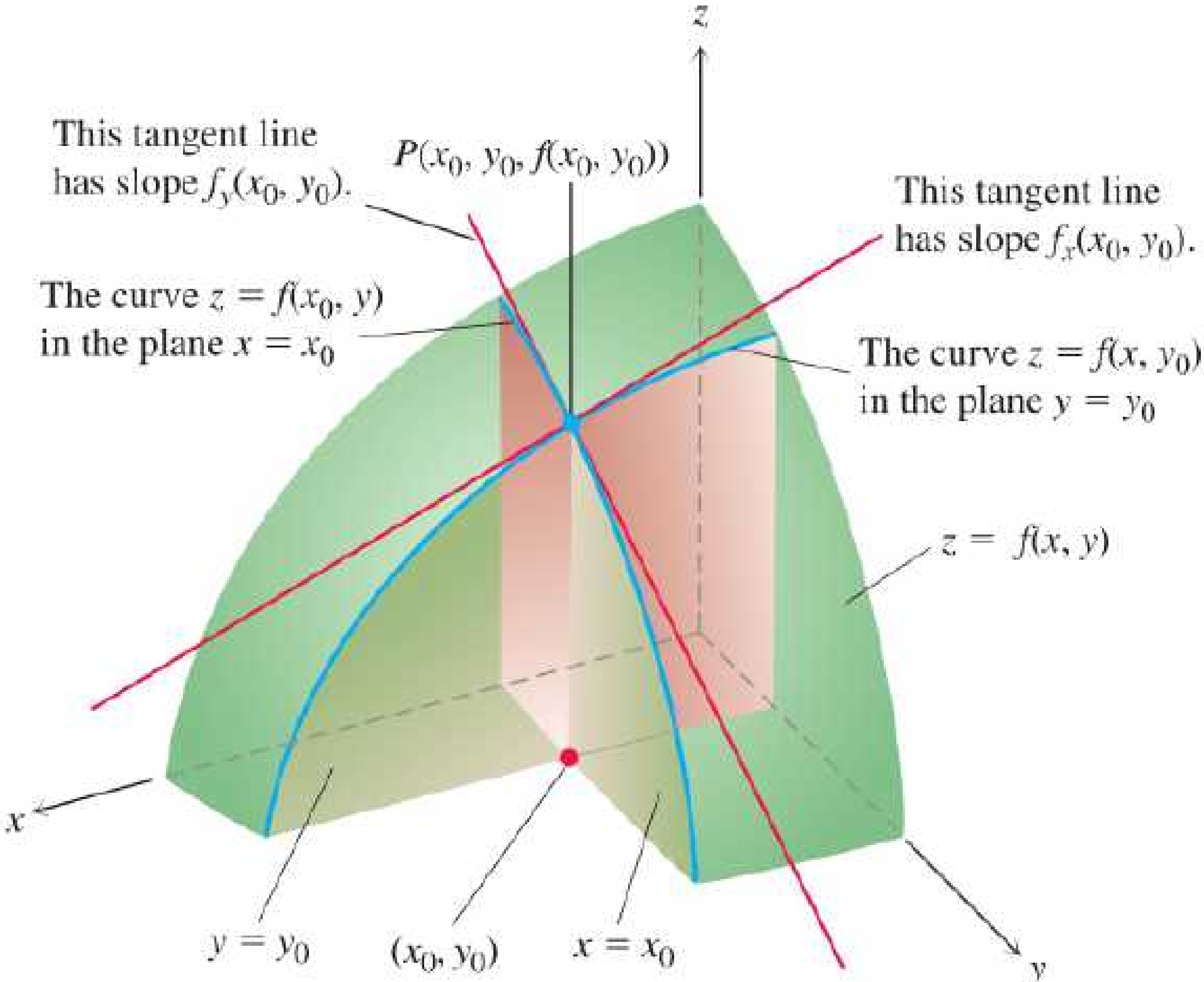
$$\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = \left.\frac{d}{dy} f(x_0, y)\right|_{y=y_0} = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

provided the limit exists.

$$\frac{\partial f}{\partial y}(x_0, y_0), \qquad f_y(x_0, y_0), \qquad \frac{\partial f}{\partial y}, \qquad f_y.$$



$$\left.\frac{\partial f(x_0, y)}{\partial y}\right|_{y=y_0} = \left.\frac{\partial f(x, y)}{\partial y}\right|_{\substack{x=x_0 \\ y=y_0}}$$


z
This tangent line
has slope $f_y(x_0, y_0)$.
$P(x_0, y_0, f(x_0, y_0))$
This tangent line
has slope $f_x(x_0, y_0)$.
The curve $z = f(x_0, y)$
in the plane $x = x_0$
The curve $z = f(x, y_0)$
in the plane $y = y_0$
$z = f(x, y)$
x
$y = y_0$
(x_0, y_0)
$x = x_0$
y

Calculations

EXAMPLE 1 Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(4, -5)$ if

$$f(x, y) = x^2 + 3xy + y - 1.$$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1) = 2x + 3 \cdot 1 \cdot y + 0 - 0 = 2x + 3y.$$

$\partial f/\partial x$ at $(4, -5)$ is $2(4) + 3(-5) = -7$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + y - 1) = 0 + 3 \cdot x \cdot 1 + 1 - 0 = 3x + 1.$$

$\partial f/\partial y$ at $(4, -5)$ is $3(4) + 1 = 13$.

EXAMPLE 2 Find $\partial f/\partial y$ as a function if $f(x, y) = y \sin xy$.

Solution

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y \sin xy) = y\frac{\partial}{\partial y}\sin xy + (\sin xy)\frac{\partial}{\partial y}(y)$$

$$= (y \cos xy)\frac{\partial}{\partial y}(xy) + \sin xy = xy \cos xy + \sin xy.$$

EXAMPLE 3 Find f_x and f_y as functions if $f(x, y) = \dfrac{2y}{y + \cos x}$.

Solution

$$f_x = \frac{\partial}{\partial x}\left(\frac{2y}{y + \cos x}\right) = \frac{2y \sin x}{(y + \cos x)^2}.$$

$$f_y = \frac{\partial}{\partial y}\left(\frac{2y}{y + \cos x}\right) = \frac{(y + \cos x)(2) - 2y(1)}{(y + \cos x)^2} = \frac{2 \cos x}{(y + \cos x)^2}.$$

EXAMPLE 4 Find $\partial z/\partial x$ if the equation $yz - \ln z = x + y$ defines z as a function of the two independent variables x and y.

Solution We differentiate both sides of the equation with respect to x,

$$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}\ln z = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x}$$

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1 + 0$$

$$\frac{\partial z}{\partial x} = \frac{z}{yz - 1}. \qquad \left.\frac{\partial z}{\partial x}\right|_{(1,0)} = -e^{-1}.$$

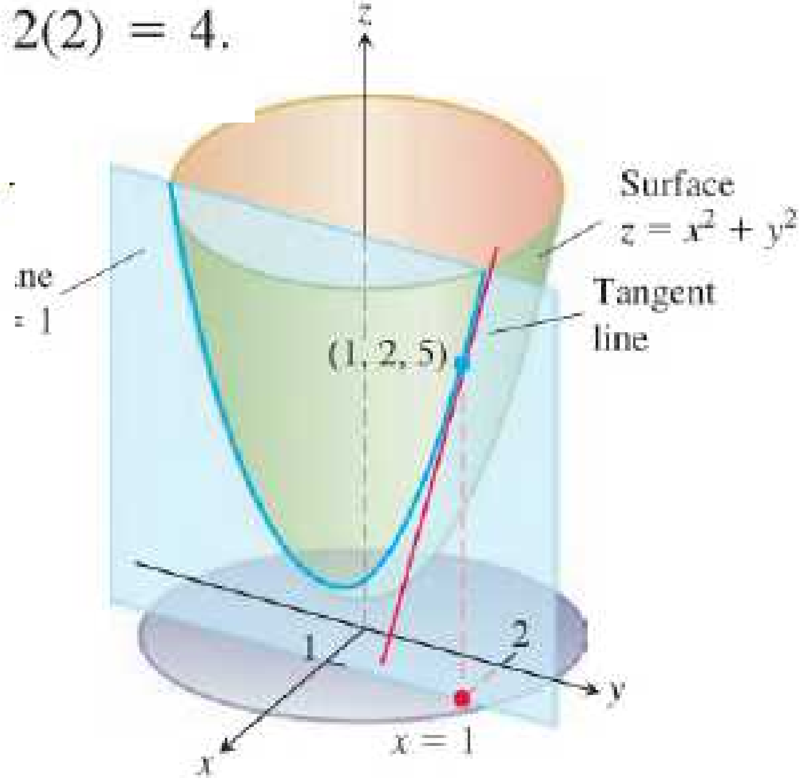
EXAMPLE 5

The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at (1, 2, 5)

Solution The slope is the value of the partial derivative $\partial z/\partial y$ at (1, 2):

$$\left.\frac{\partial z}{\partial y}\right|_{(1,2)} = \left.\frac{\partial}{\partial y}(x^2 + y^2)\right|_{(1,2)} = \left.2y\right|_{(1,2)} = 2(2) = 4.$$

$$\left.\frac{dz}{dy}\right|_{y=2} = \left.\frac{d}{dy}(1 + y^2)\right|_{y=2} = \left.2y\right|_{y=2} = 4.$$



例.讨论函数$z=\sqrt{x^2+y^2}$在(0,0)点的偏导数。

$$\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{|h|}{h}$$

$z_x(0,0)$不存在！　　同理$z_y(0,0)$不存在.

连续但偏导不存在。

Functions of More Than Two Variables

EXAMPLE 6 If x, y, and z are independent variables and

$$f(x, y, z) = x \sin(y + 3z),$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}[x \sin(y + 3z)] = x \frac{\partial}{\partial z} \sin(y + 3z)$$

$$= x \cos(y + 3z) \frac{\partial}{\partial z}(y + 3z) = 3x \cos(y + 3z).$$

EXAMPLE 7

If resistors of R_1, R_2, and R_3 ohms are connected in parallel to make an R-ohm resistor, the value of R can be found from the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Find the value of $\partial R / \partial R_2$ when $R_1 = 30$, $R_2 = 45$, and $R_3 = 90$ ohms.

Solution

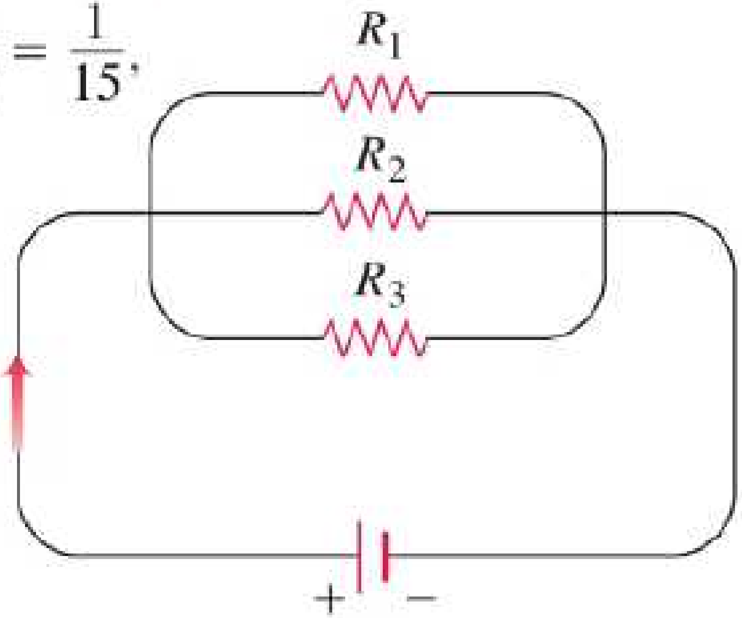
$$\frac{\partial}{\partial R_2}\left(\frac{1}{R}\right) = \frac{\partial}{\partial R_2}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

$$-\frac{1}{R^2}\frac{\partial R}{\partial R_2} = 0 - \frac{1}{R_2{}^2} + 0 \qquad \frac{\partial R}{\partial R_2} = \frac{R^2}{R_2{}^2} = \left(\frac{R}{R_2}\right)^2.$$

When $R_1 = 30$, $R_2 = 45$, and $R_3 = 90$,

$$\frac{1}{R} = \frac{1}{30} + \frac{1}{45} + \frac{1}{90} = \frac{3+2+1}{90} = \frac{6}{90} = \frac{1}{15},$$

$$\frac{\partial R}{\partial R_2} = \left(\frac{15}{45}\right)^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}.$$



Partial Derivatives and Continuity 偏导数存在但是不连续 ！

EXAMPLE 8 $f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$

(a) Find the limit of f as (x, y) approaches $(0, 0)$ along the line $y = x$.

(b) Prove that f is not continuous at the origin.

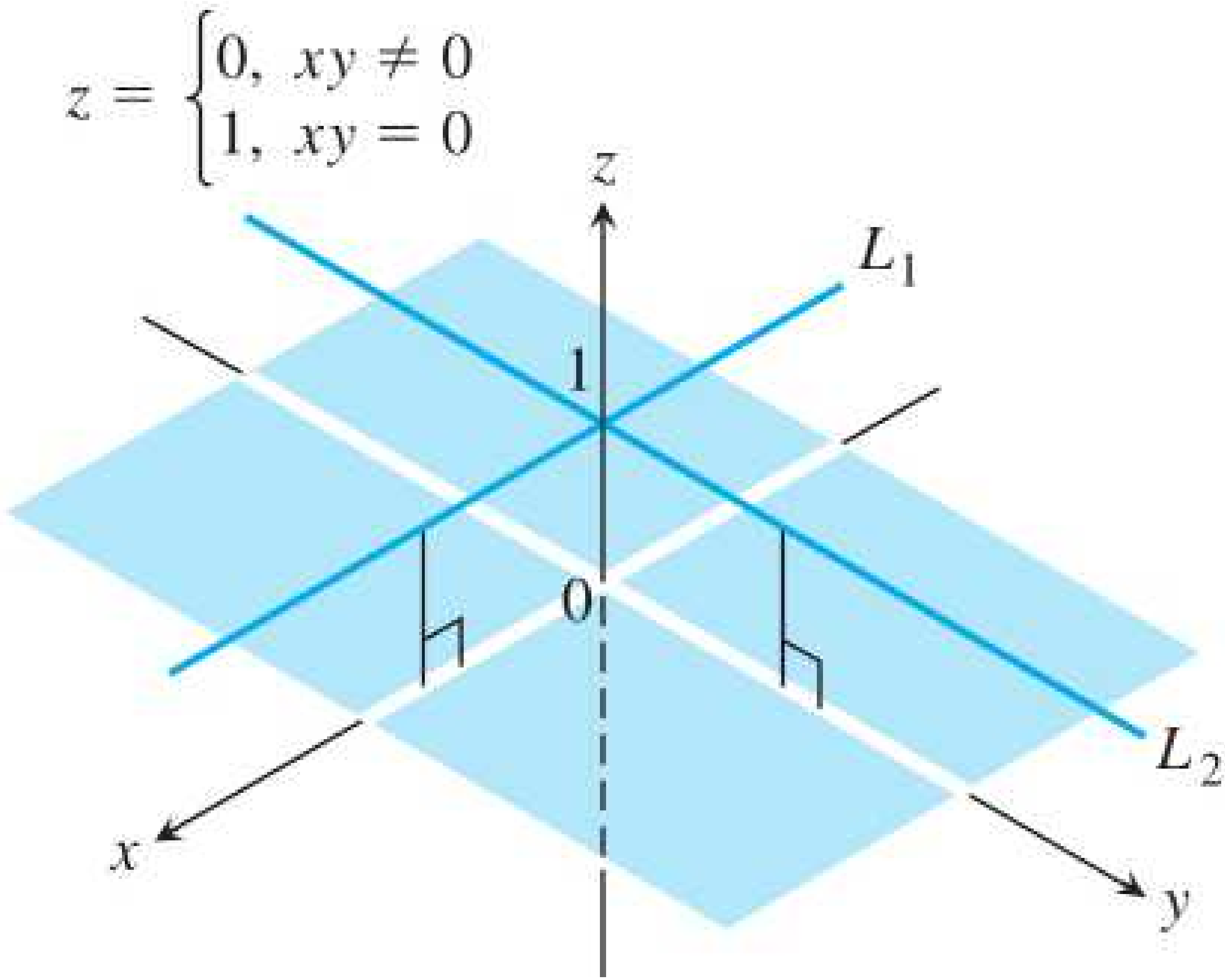
(c) Show that both partial derivatives $\partial f / \partial x$ and $\partial f / \partial y$ exist at the origin.

Solution (a) $$\lim_{(x,y)\to(0,0)} f(x, y)\Big|_{y=x} = \lim_{(x,y)\to(0,0)} 0 = 0.$$

(b) Since $f(0, 0) = 1$, $\lim_{(x,y)\to(0,0)} f(x, y) \neq f(0,0)$

(c) To find $\partial f / \partial x$ at $(0, 0)$, we hold y fixed at $y = 0$.

Then $f(x, y) = 1$ for all x, $\partial f / \partial x = 0$. $\partial f / \partial y = 0$


$z = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$
z
L_1
1
0
L_2
x
y

Second-Order Partial Derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$ the mixed partial derivatives

Differentiate first with respect to y, then with respect to x.

$$\frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx}, \qquad \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy}, \qquad f_{yx} = (f_y)_x$$

$$\frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{yx}, \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{xy}.$$

EXAMPLE 9 If $f(x, y) = x \cos y + ye^x$, find the second-order derivatives

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}.$$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x \cos y + ye^x) = \cos y + ye^x \qquad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x \cos y + ye^x) = -x \sin y + e^x$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x. \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x \cos y.$$

The Mixed Derivative Theorem

THEOREM 2—The Mixed Derivative Theorem If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

EXAMPLE 10 Find $\partial^2 w/\partial x \partial y$ if $w = xy + \dfrac{e^y}{y^2 + 1}$.

Solution $\dfrac{\partial w}{\partial x} = y$ and $\dfrac{\partial^2 w}{\partial y \partial x} = 1.$

$\partial^2 w/\partial x \partial y = 1$ as well.

Let $f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, & \text{if } (x, y) \neq 0, \\ 0, & \text{if } (x, y) = 0. \end{cases}$

a. Show that $\dfrac{\partial f}{\partial y}(x, 0) = x$ for all x, and $\dfrac{\partial f}{\partial x}(0, y) = -y$ for all y.

b. Show that $\dfrac{\partial^2 f}{\partial y \partial x}(0, 0) \neq \dfrac{\partial^2 f}{\partial x \partial y}(0, 0)$.

$$\frac{\partial f}{\partial y}(x,0) = \lim_{h\to 0}\frac{f(x,h) - f(x,0)}{h} = \lim_{h\to 0}\frac{xh\dfrac{x^2 - h^2}{x^2 + h^2}}{h} = x$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \frac{\partial}{\partial y}(f_x(0,y))\Big|_{y=0} \qquad \frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial}{\partial x}(f_y(x,0))\Big|_{x=0}$$

$$= -1 \qquad\qquad = 1$$

Partial Derivatives of Still Higher Order

$$\frac{\partial^3 f}{\partial x \partial y^2} = f_{yyx}, \qquad \frac{\partial^4 f}{\partial x^2 \partial y^2} = f_{yyxx},$$

EXAMPLE 11 Find f_{yxyz} if $f(x, y, z) = 1 - 2xy^2z + x^2y$.

Solution

$$f_y = -4xyz + x^2$$

$$f_{yx} = -4yz + 2x$$

$$f_{yxy} = -4z$$

$$f_{yxyz} = -4.$$

Differentiability

$y = f(x)$ is differentiable at $x = x_0$, then the change Δy

$$\Delta y = f'(x_0)\Delta x + \epsilon \Delta x \qquad \epsilon \to 0 \text{ as } \Delta x \to 0.$$

$$\Delta y = f(x_0 + \Delta x) - f(x_0)$$

DEFINITION A function $z = f(x, y)$ is **differentiable at** (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and Δz satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

in which each of $\epsilon_1, \epsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$. We call f **differentiable** if it is differentiable at every point in its domain, and say that its graph is a **smooth surface**.

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

THEOREM 3—The Increment Theorem for Functions of Two Variables Suppose that the first partial derivatives of $f(x, y)$ are defined throughout an open region R containing the point (x_0, y_0) and that f_x and f_y are continuous at (x_0, y_0). Then the change

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

in the value of f that results from moving from (x_0, y_0) to another point $(x_0 + \Delta x, y_0 + \Delta y)$ in R satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

in which each of $\epsilon_1, \epsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$.

COROLLARY OF THEOREM 3 If the partial derivatives f_x and f_y of a function $f(x, y)$ are continuous throughout an open region R, then f is differentiable at every point of R.

偏导连续必可微

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$$
$$= [f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)]$$
$$+ [f(x, y+\Delta y) - f(x,y)],$$
$$\because f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)$$
$$= f_x(x+\theta_1\Delta x, y+\Delta y)\Delta x \qquad (0<\theta_1<1)$$
$$= f_x(x,y)\Delta x + \varepsilon_1\Delta x$$

其中ε_1为Δx，Δy的函数，

且当$\Delta x \to 0, \Delta y \to 0$时，$\varepsilon_1 \to 0.$

$$f(x, y+\Delta y) - f(x,y) = f_y(x,y)\Delta y + \varepsilon_2\Delta y,$$

当$\Delta y \to 0$时，$\varepsilon_2 \to 0,$

$$\therefore \Delta z = f_x(x,y)\Delta x + \varepsilon_1\Delta x + f_y(x,y)\Delta y + \varepsilon_2\Delta y$$

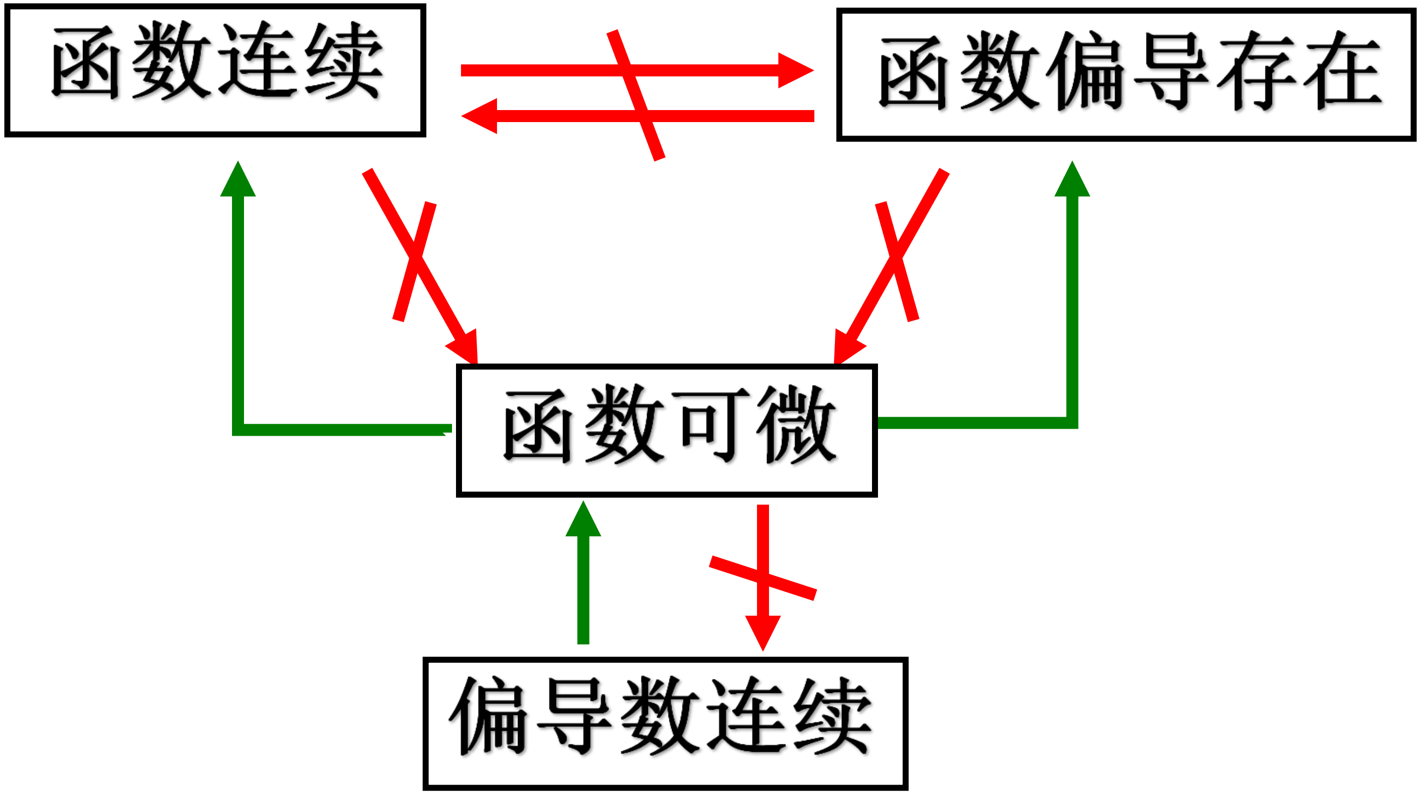
THEOREM 4—Differentiability Implies Continuity If a function $f(x, y)$ is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

$$f(x_0+\Delta x, y_0+\Delta y)-f(x_0,y_0)$$
$$=f_x(x_0,y_0)\Delta x+f(x_0,y_0)\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y$$

$$f(x,y)-f(x_0,y_0)$$
$$=f_x(x_0,y_0)(x-x_0)+f(x_0,y_0)(y-y_0)+\varepsilon_1(x-x_0)+\varepsilon_2(y-y_0)$$

$$\lim_{(x,y)\to(x_0,y_0)}[f(x,y)-f(x_0,y_0)]=0,$$

可微必连续

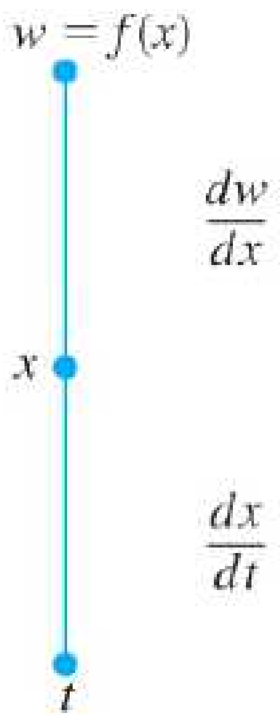

函数连续
函数偏导存在
函数可微
偏导数连续

14.4

The Chain Rule

链式法则

Chain Rule



Dependent variable

Intermediate variable

Independent variable

$$w = f(x)$$

$$x = g(t)$$

$$w(t) = f(g(t)),$$

$$\left.\frac{dw}{dt}\right|_{t=t_0} = \left.\frac{dw}{dx}\right|_{x=g(t_0)} \cdot \left.\frac{dx}{dt}\right|_{t=t_0}$$

$$\left.\frac{dw}{dt}\right|_{t=t_0} = f'(g(t_0)) \cdot g(t_0)$$

$$w = f(x, y) \qquad x = x(t) \qquad y = y(t)$$

THEOREM 5—Chain Rule For Functions of One Independent Variable and Two Intermediate Variables If $w = f(x, y)$ is differentiable and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the composite $w = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dw}{dt} = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t),$$

or

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Proof

Let Δx, Δy, and Δw be the increments that result from changing t from t_0 to $t_0 + \Delta t$.

$$\Delta w = f(x(t_0+\Delta t), y(t_0+\Delta t)) - f(x(t_0), y(t_0))$$

$$= f_x(x(t_0), y(t_0))(x(t_0+\Delta t) - x(t_0))$$

$$+ f_y(x(t_0), y(t_0))(y(t_0+\Delta t) - y(t_0))$$

$$+ \varepsilon_1(x(t_0+\Delta t) - x(t_0)) + \varepsilon_2(y(t_0+\Delta t) - y(t_0))$$

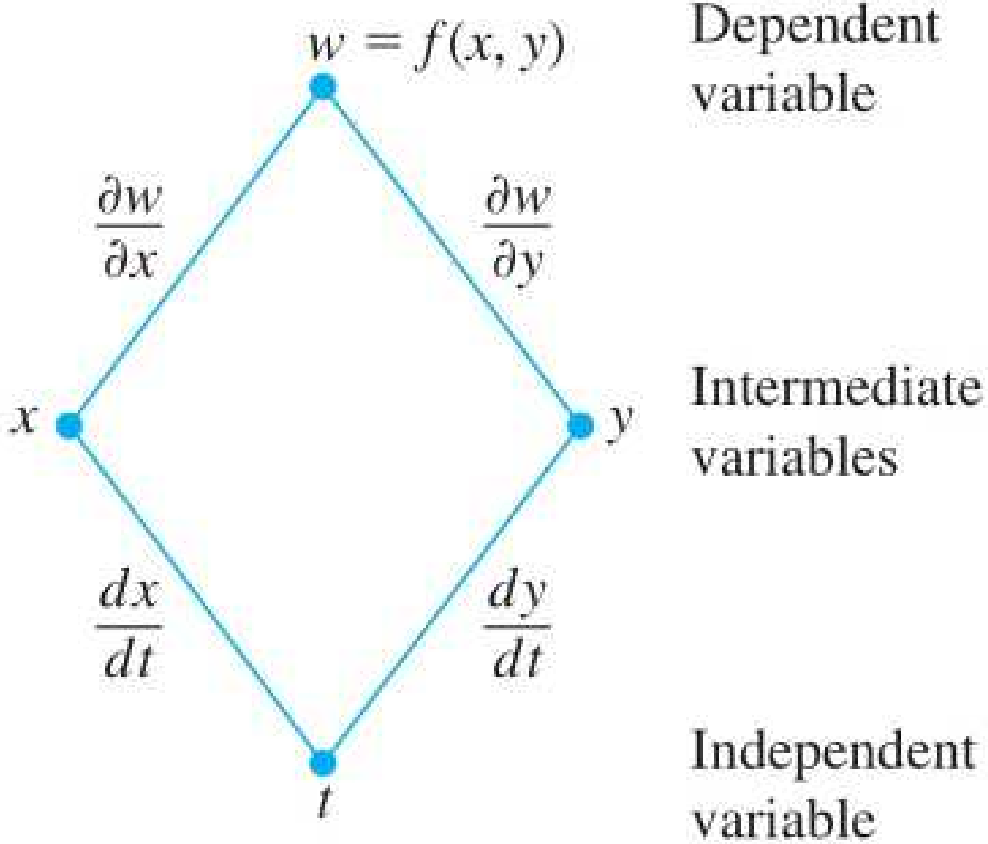
divided with Δt, Letting Δt approach zero

$$\left(\frac{dw}{dt}\right)_{t_0} = \lim_{\Delta t \to 0} \frac{\Delta w}{\Delta t}$$

$$= \left(\frac{\partial w}{\partial x}\right)_{P_0}\left(\frac{dx}{dt}\right)_{t_0} + \left(\frac{\partial w}{\partial y}\right)_{P_0}\left(\frac{dy}{dt}\right)_{t_0} + 0 \cdot \left(\frac{dx}{dt}\right)_{t_0} + 0 \cdot \left(\frac{dy}{dt}\right)_{t_0}.$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

Chain Rule



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

EXAMPLE 1 Use the Chain Rule to find the derivative of

$w = xy$ with respect to t along the path $x = \cos t, y = \sin t$.

What is the derivative's value at $t = \pi/2$?

Solution

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= (y)(-\sin t) + (x)(\cos t) = \cos 2t.$$

$$w = xy = \cos t \sin t = \frac{1}{2}\sin 2t,$$

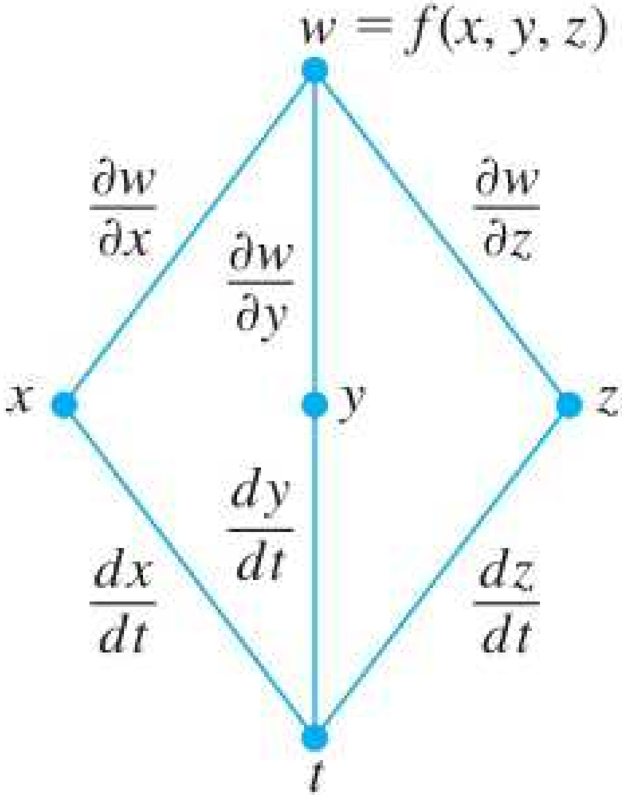
$$\frac{dw}{dt} = \frac{d}{dt}\left(\frac{1}{2}\sin 2t\right) = \frac{1}{2}\cdot 2\cos 2t = \cos 2t.$$

$$\left(\frac{dw}{dt}\right)_{t=\pi/2} = \cos\left(2\cdot\frac{\pi}{2}\right) = \cos \pi = -1.$$

Functions of Three Variables

THEOREM 6—Chain Rule for Functions of One Independent Variable and Three Intermediate Variables If $w = f(x, y, z)$ is differentiable and x, y, and z are differentiable functions of t, then w is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$



EXAMPLE 2 Find dw/dt if

$$w = xy + z, \qquad x = \cos t, \qquad y = \sin t, \qquad z = t.$$

What is the derivative's value at $t = 0$?

Solution

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

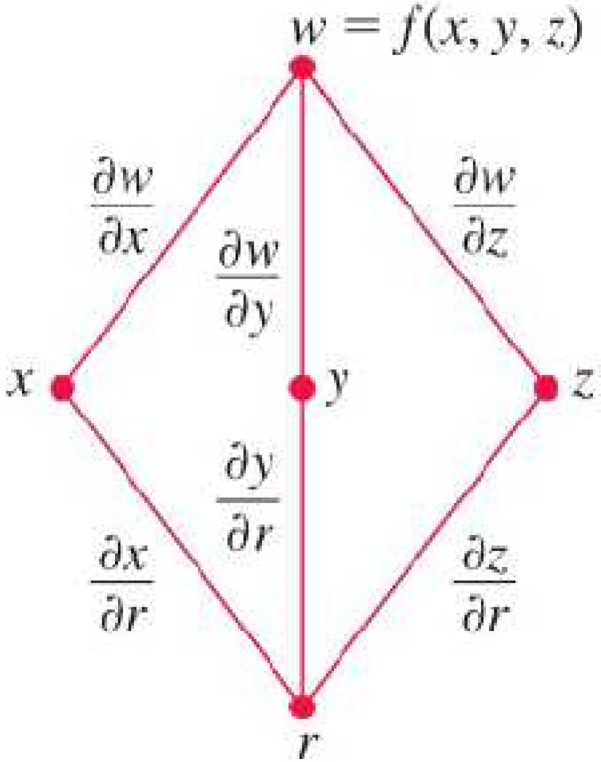
$$= (y)(-\sin t) + (x)(\cos t) + (1)(1)$$

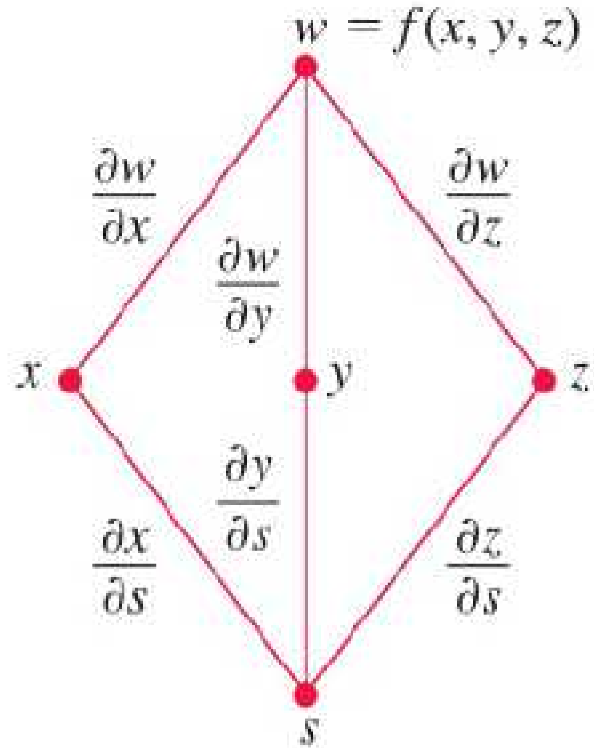
$$= 1 + \cos 2t, \qquad \left(\frac{dw}{dt}\right)_{t=0} = 1 + \cos(0) = 2.$$

$$w = \sin t \cos t + t = \frac{1}{2}\sin 2t + t \qquad \frac{dw}{dt} = \cos 2t + 1$$

THEOREM 7—Chain Rule for Two Independent Variables and Three Intermediate Variables Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$. If all four functions are differentiable, then w has partial derivatives with respect to r and s, given by the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \qquad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$





EXAMPLE 3 Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x + 2y + z^2, \qquad x = \frac{r}{s}, \qquad y = r^2 + \ln s, \qquad z = 2r.$$

Solution

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$= (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2)$$

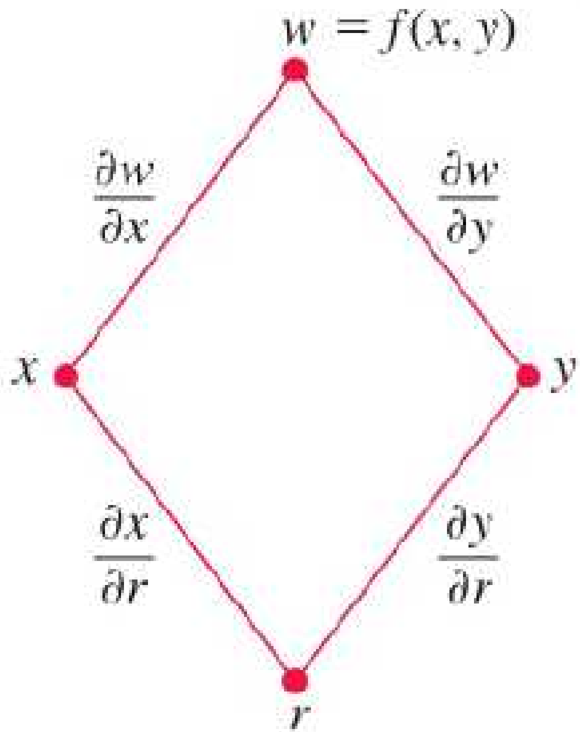
$$= \frac{1}{s} + 4r + (4r)(2) = \frac{1}{s} + 12r$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0) = \frac{2}{s} - \frac{r}{s^2}.$$

If $w = f(x, y)$, $x = g(r, s)$, and $y = h(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}.$$



EXAMPLE 4 Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x^2 + y^2, \qquad x = r - s, \qquad y = r + s.$$

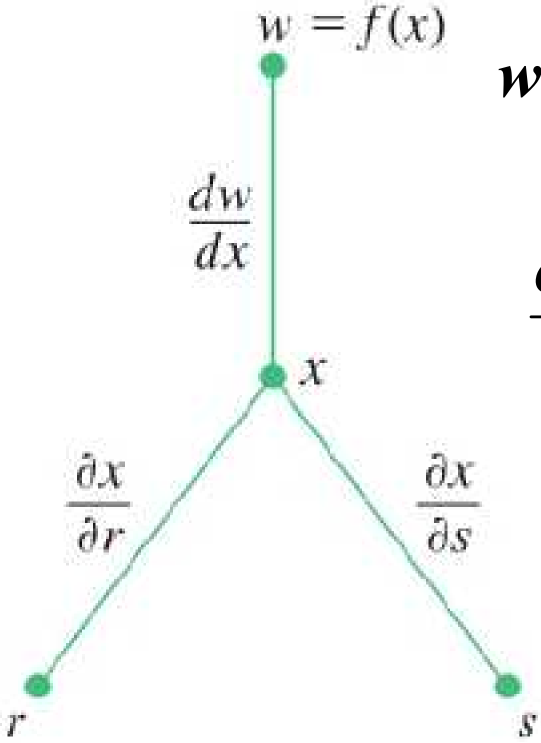
Solution

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \\ &= (2x)(1) + (2y)(1) \\ &= 2(r - s) + 2(r + s) \\ &= 4r\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \\ &= (2x)(-1) + (2y)(1) \\ &= -2(r - s) + 2(r + s) \\ &= 4s\end{aligned}$$

If $w = f(x)$ and $x = g(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}.$$



$w = f(\frac{x}{y})$, **Find** $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$, f **is differential.**

$$\frac{\partial w}{\partial x} = \frac{1}{y} f'(\frac{x}{y}), \qquad \frac{\partial w}{\partial y} = -\frac{x}{y^2} f'(\frac{x}{y}),$$

Implicit Differentiation Revisited

Suppose that

1. The function $F(x, y)$ is differentiable and
2. The equation $F(x, y) = 0$ defines y implicitly as a differentiable function $y = h(x)$.

$$F_x \frac{dx}{dx} + F_y \frac{dy}{dx} = 0,$$

$$F_x \cdot 1 + F_y \cdot \frac{dy}{dx} = 0,$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}. \qquad \text{If } F_y = \partial w/\partial y \neq 0,$$

THEOREM 8—A Formula for Implicit Differentiation Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$ defines y as a differentiable function of x. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

EXAMPLE 5 Use Theorem 8 to find dy/dx if $y^2 - x^2 - \sin xy = 0$.

Solution Take $F(x, y) = y^2 - x^2 - \sin xy$. Then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-2x - y\cos xy}{2y - x\cos xy} = \frac{2x + y\cos xy}{2y - x\cos xy}.$$

多元隐函数的求偏导

$F(x, y, z) = 0$ defines the variable z implicitly as a function $z = f(x, y)$.

$$F_x + F_z \frac{\partial z}{\partial x} = 0 \qquad F_y + F_z \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}. \quad \text{Whenever } F_z \neq 0,$$

EXAMPLE 6

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at $(0, 0, 0)$ if $x^3 + z^2 + ye^{xz} + z\cos y = 0$.

Solution Let $F(x, y, z) = x^3 + z^2 + ye^{xz} + z\cos y$. Then

$$F_x = 3x^2 + zye^{xz}, \qquad F_y = e^{xz} - z\sin y, \qquad F_z = 2z + xye^{xz} + \cos y.$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + zye^{xz}}{2z + xye^{xz} + \cos y} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{e^{xz} - z\sin y}{2z + xye^{xz} + \cos y}.$$

At $(0, 0, 0)$ we find

$$\frac{\partial z}{\partial x} = -\frac{0}{1} = 0 \qquad \text{and} \qquad \frac{\partial z}{\partial y} = -\frac{1}{1} = -1.$$

Functions of Many Variables

suppose that $w = f(x, y, \ldots, v)$ is a differentiable

and the $x, y, \ldots, v$ are differentiable functions of $p, q, \ldots, t$

Then w is a differentiable

$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p} + \cdots + \frac{\partial w}{\partial v}\frac{\partial v}{\partial p}.$$

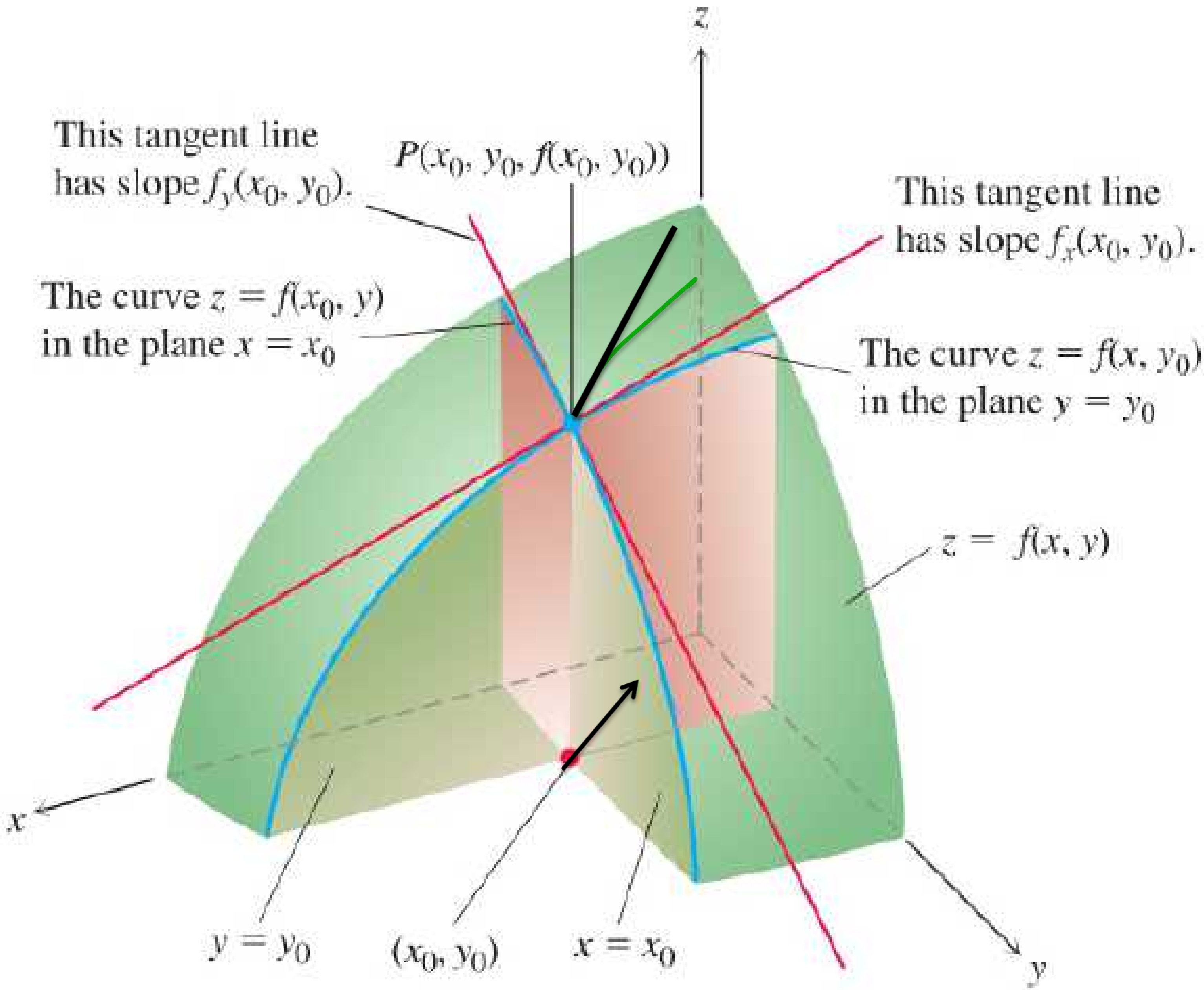
The other equations are obtained by replacing p by $q, \ldots, t$, one at a time.

$$\frac{\partial w}{\partial p} = \left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \ldots, \frac{\partial w}{\partial v}\right) \cdot \left(\frac{\partial x}{\partial p}, \frac{\partial y}{\partial p}, \ldots, \frac{\partial v}{\partial p}\right).$$

14.5

Directional Derivatives and Gradient Vectors

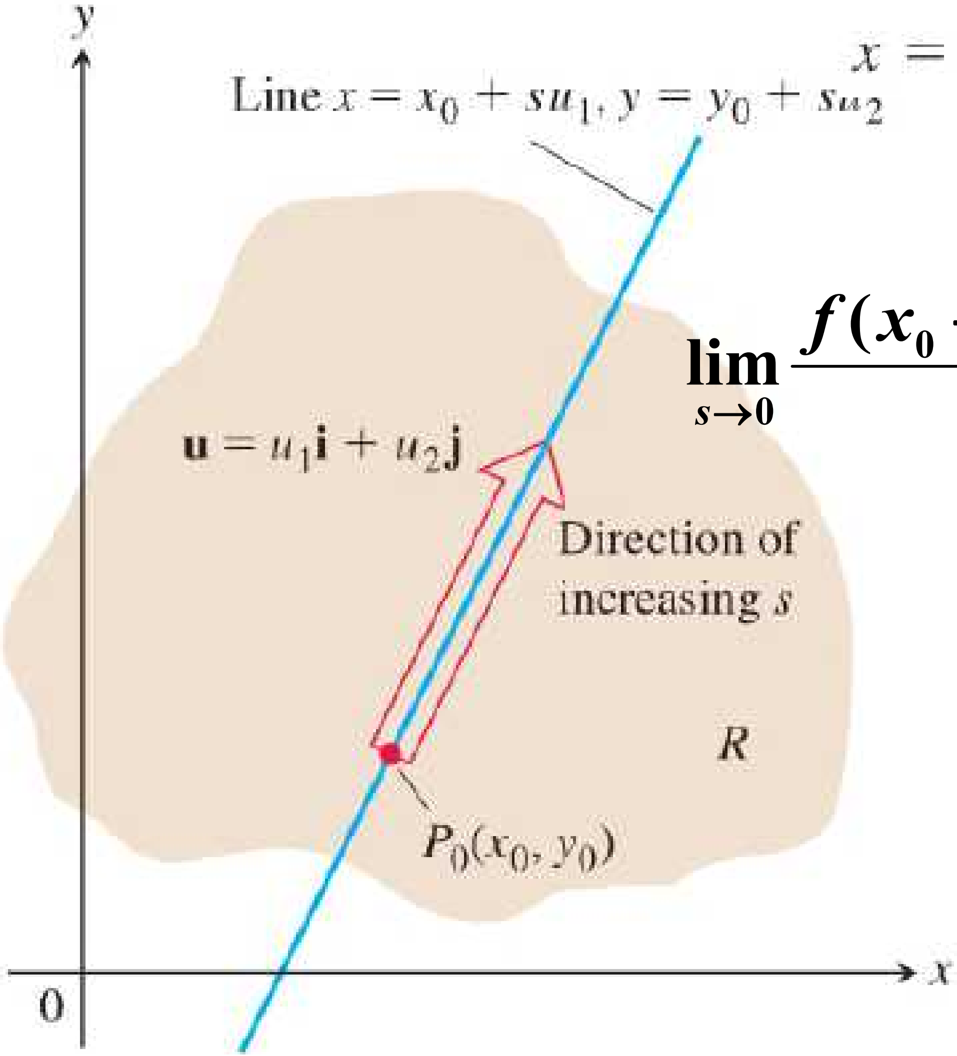
方向导数和梯度



z
This tangent line
has slope $f_y(x_0, y_0)$.
$P(x_0, y_0, f(x_0, y_0))$
This tangent line
has slope $f_x(x_0, y_0)$.
The curve $z = f(x_0, y)$
in the plane $x = x_0$
The curve $z = f(x, y_0)$
in the plane $y = y_0$
$z = f(x, y)$
x
$y = y_0$
(x_0, y_0)
$x = x_0$
y

$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector.

$$x = x_0 + su_1, \qquad y = y_0 + su_2$$



$$\lim_{s\to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

DEFINITION The **derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$** is the number

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}, \tag{1}$$

provided the limit exists.

$$(D_{\mathbf{u}} f)_{P_0}.$$

EXAMPLE 1 Using the definition, find the derivative of

$$f(x, y) = x^2 + xy$$

at $P_0(1, 2)$ in the direction of the unit vector $\mathbf{u} = (1/\sqrt{2})\mathbf{i} + (1/\sqrt{2})\mathbf{j}$.

Solution Applying the definition

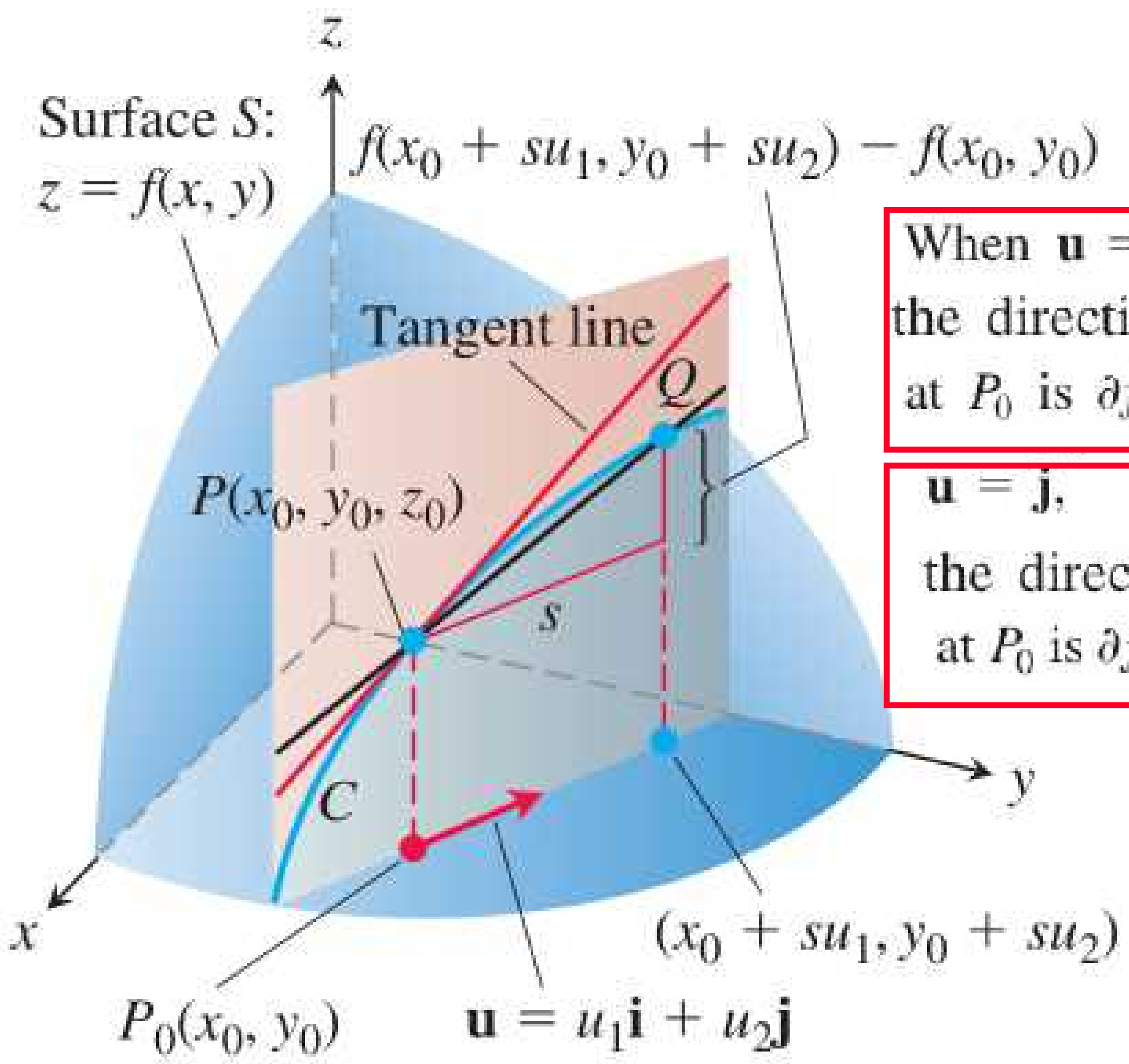
$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

$$= \lim_{s \to 0} \frac{\left(1 + \frac{s}{\sqrt{2}}\right)^2 + \left(1 + \frac{s}{\sqrt{2}}\right)\left(2 + \frac{s}{\sqrt{2}}\right) - (1^2 + 1 \cdot 2)}{s}$$

$$= \lim_{s \to 0} \frac{\frac{5s}{\sqrt{2}} + s^2}{s} = \lim_{s \to 0} \left(\frac{5}{\sqrt{2}} + s\right) = \frac{5}{\sqrt{2}}.$$

The rate of change of $f(x, y) = x^2 + xy$ at $P_0(1, 2)$ in the direction $\mathbf{u}$ is $5/\sqrt{2}$.

Interpretation of the Directional Derivative



When $\mathbf{u} = \mathbf{i}$,
the directional derivative
at P_0 is $\partial f / \partial x$ evaluated at (x_0, y_0).

$\mathbf{u} = \mathbf{j}$,
the directional derivative
at P_0 is $\partial f / \partial y$ evaluated at (x_0, y_0).

Calculation and Gradients

$f(x, y)$ **differentiable** the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$.

$$\left(\frac{df}{ds}\right)_{u,P_0} = \lim_{s\to 0}\frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

$$g(s) = f(x_0 + su_1, y_0 + su_2) \qquad g(0) = f(x_0, y_0)$$

$$\left(\frac{df}{ds}\right)_{u,P_0} = \lim_{s\to 0}\frac{g(s) - g(0)}{s} = g'(0)$$

$$g'(s) = f_x(x_0 + su_1, y_0 + su_2)u_1 + f_y(x_0 + su_1, y_0 + su_2)u_2$$

$$g'(0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$$

$$\left(\frac{df}{ds}\right)_{u,P_0} = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$$

DEFINITION The **gradient vector (gradient)** of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

obtained by evaluating the partial derivatives of f at P_0.

THEOREM 9—The Directional Derivative Is a Dot Product If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = (\nabla f)_{P_0} \cdot \mathbf{u}, \tag{4}$$

the dot product of the gradient ∇f at P_0 and $\mathbf{u}$. In brief, $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$.

EXAMPLE 2

Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point (2, 0) in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

Solution $$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

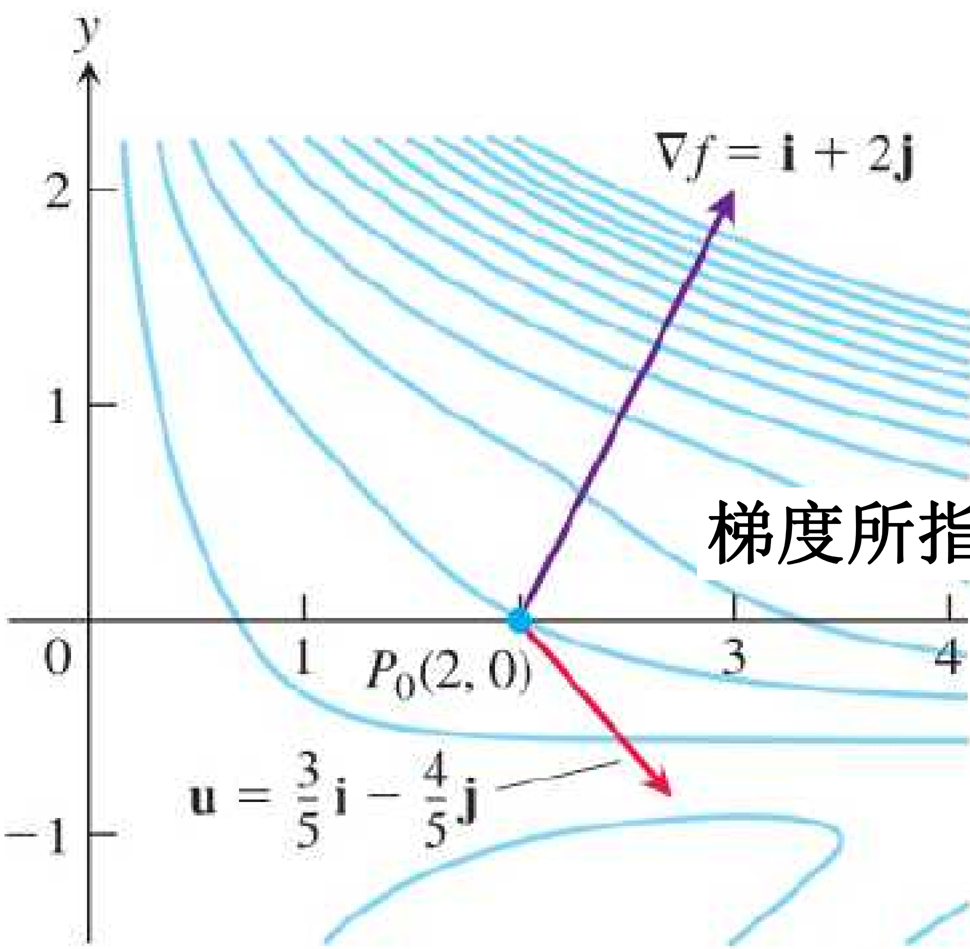
$$f_x(2, 0) = (e^y - y\sin(xy))_{(2,0)} = e^0 - 0 = 1$$

$$f_y(2, 0) = (xe^y - x\sin(xy))_{(2,0)} = 2e^0 - 2 \cdot 0 = 2.$$

$$\nabla f\big|_{(2,0)} = f_x(2, 0)\mathbf{i} + f_y(2, 0)\mathbf{j} = \mathbf{i} + 2\mathbf{j}$$

$$(D_{\mathbf{u}}f)_{(2,0)} = \nabla f\big|_{(2,0)} \cdot \mathbf{u} = (\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1.$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}| \cos\theta = |\nabla f| \cos\theta,$$



函数 f 在P_0点的方向导数的最大值在什么方向达到？

$\theta = 0$时方向导数最大

梯度所指方向是方向导数最大的方向

梯度方向的方向导数是 $|\nabla f|$

Properties of the Directional Derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos\theta$

1. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is $|\nabla f|$.

2. Similarly, f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $-|\nabla f|$.

3. Any direction $\mathbf{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f

As we discuss later, these properties hold in three dimensions

EXAMPLE 3 Find the directions in which $f(x, y) = (x^2/2) + (y^2/2)$

(a) increases most rapidly at the point (1, 1), and

(b) decreases most rapidly at (1, 1).

(c) What are the directions of zero change in f at (1, 1)?

Solution **(a)**

$$(\nabla f)_{(1,1)} = (x\mathbf{i} + y\mathbf{j})_{(1,1)} = \mathbf{i} + \mathbf{j}.$$

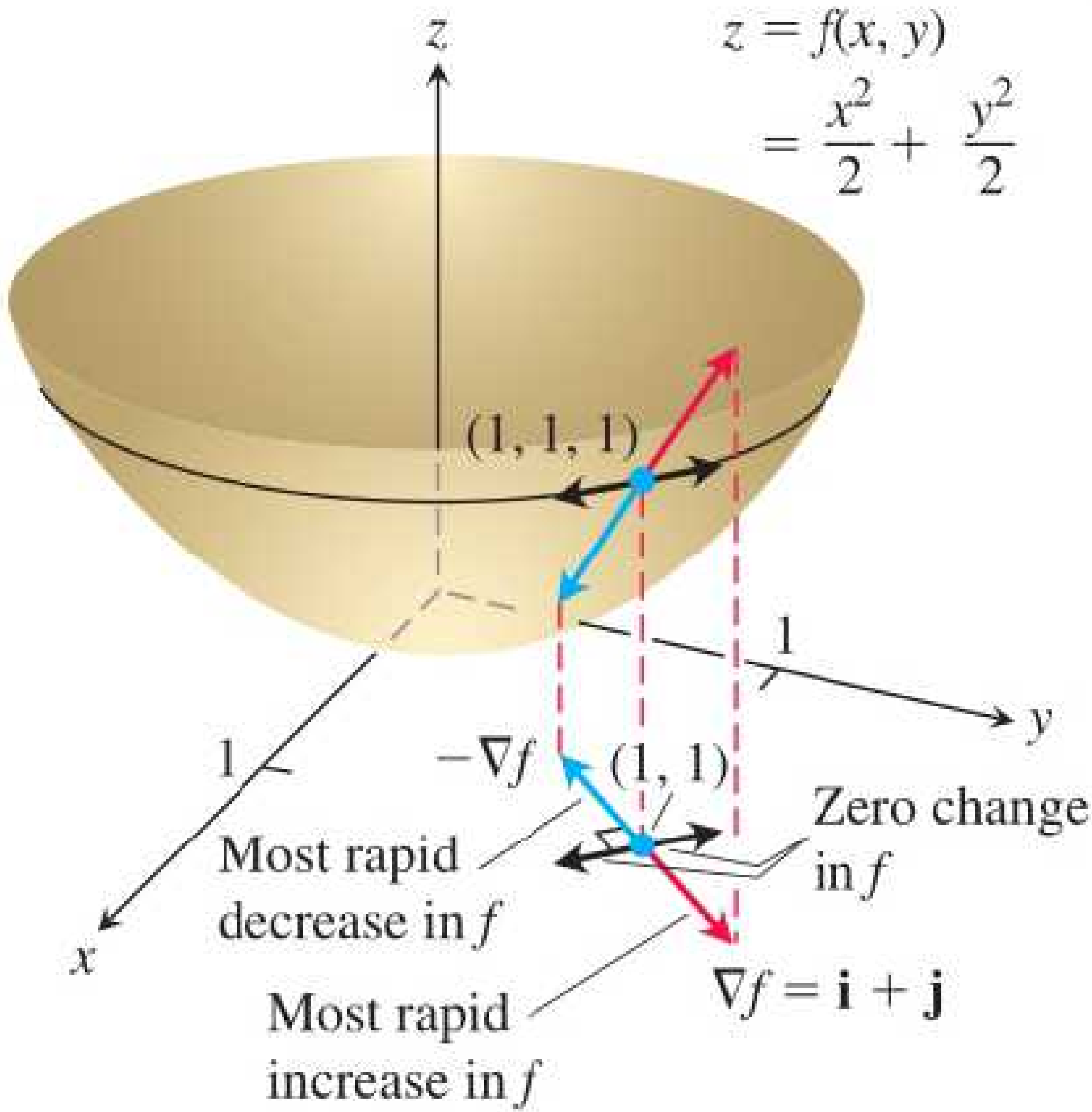
$$\mathbf{u} = \frac{\mathbf{i} + \mathbf{j}}{|\mathbf{i} + \mathbf{j}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{(1)^2 + (1)^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

(b)

$$-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

(c)

$$\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \qquad -\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$



z
$z = f(x, y)$
$= \frac{x^2}{2} + \frac{y^2}{2}$
(1, 1, 1)
1
y
1
$-\nabla f$
(1, 1)
Zero change in f
Most rapid decrease in f
x
$\nabla f = \mathbf{i} + \mathbf{j}$
Most rapid increase in f

Gradients and Tangents to Level Curves

function $f(x, y)$ has a constant value c along a smooth curve

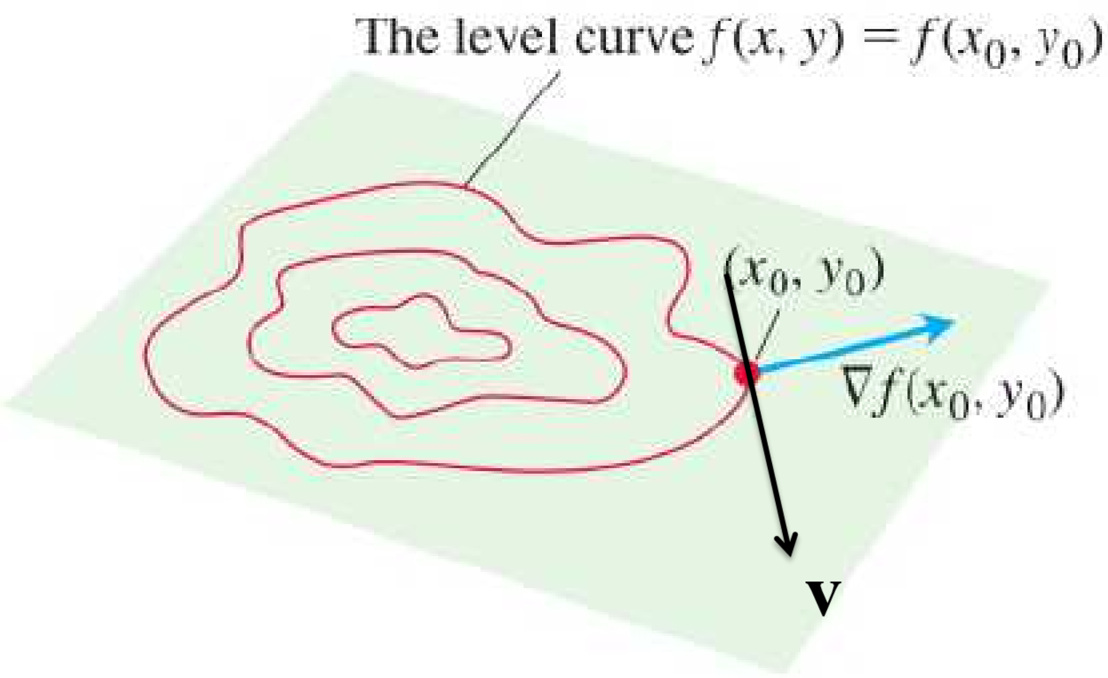
$\mathbf{r} = g(t)\mathbf{i} + h(t)\mathbf{j}$ then $f(g(t), h(t)) = c$.

$$\frac{d}{dt} f(g(t), h(t)) = \frac{d}{dt}(c) \qquad \frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} = 0$$

$$\underbrace{\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dg}{dt}\mathbf{i} + \frac{dh}{dt}\mathbf{j}\right)}_{\frac{d\mathbf{r}}{dt}} = 0.$$

∇f is normal to the tangent vector $d\mathbf{r}/dt$, so it is normal to the curve.

At every point (x_0, y_0) in the domain of a differentiable function $f(x, y)$, the gradient of f is normal to the level curve through (x_0, y_0) (Figure 14.31).



The line through a point $P_0(x_0, y_0)$ normal to a vector $\mathbf{N} = A\mathbf{i} + B\mathbf{j}$ has the equation

$$A(x - x_0) + B(y - y_0) = 0$$

If $\mathbf{N}$ is the gradient $(\nabla f)_{(x_0, y_0)} = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}$,

Tangent Line to a Level Curve

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0$$

EXAMPLE 4 Find an equation for the tangent to the ellipse

$$\frac{x^2}{4} + y^2 = 2 \quad \text{at the point } (-2, 1).$$

Solution The ellipse is a level curve of the function

$$f(x, y) = \frac{x^2}{4} + y^2.$$

$$\nabla f\Big|_{(-2,1)} = \left(\frac{x}{2}\mathbf{i} + 2y\mathbf{j}\right)_{(-2,1)} = -\mathbf{i} + 2\mathbf{j}.$$

The tangent to the ellipse at $(-2, 1)$ is the line

$$(-1)(x + 2) + (2)(y - 1) = 0 \qquad x - 2y = -4.$$

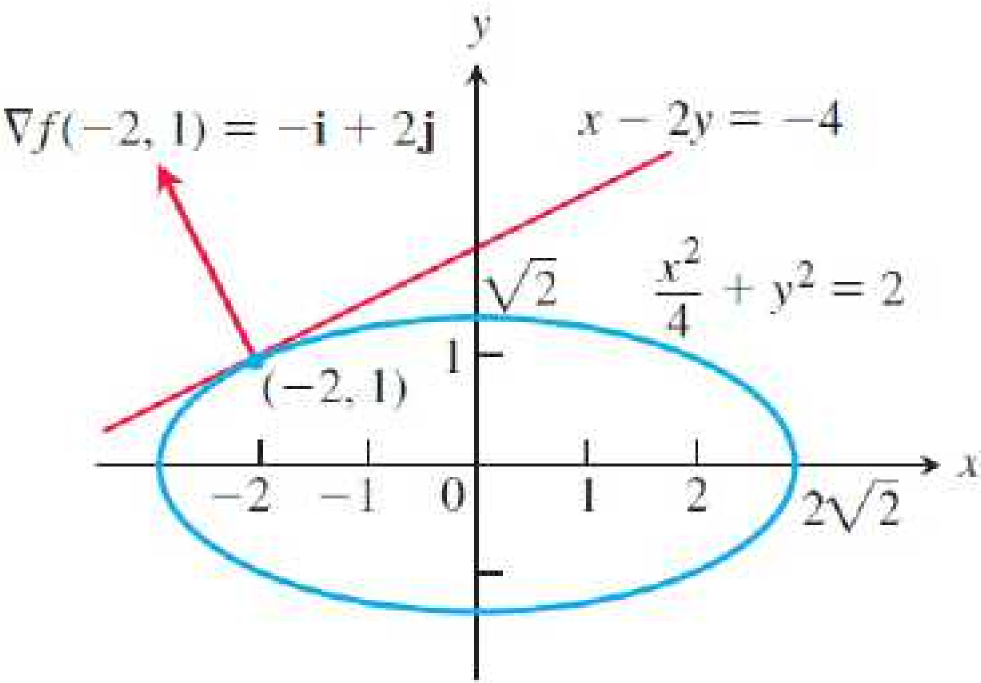


FIGURE 14.32 We can find the tangent to the ellipse $(x^2/4) + y^2 = 2$ by treating the ellipse as a level curve of the function $f(x, y) = (x^2/4) + y^2$ (Example 4).

Algebra Rules for Gradients

1. *Sum Rule:* $\nabla(f+g) = \nabla f + \nabla g$
2. *Difference Rule:* $\nabla(f-g) = \nabla f - \nabla g$
3. *Constant Multiple Rule:* $\nabla(kf) = k\nabla f$ (any number k)
4. *Product Rule:* $\nabla(fg) = f\nabla g + g\nabla f$
5. *Quotient Rule:* $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

EXAMPLE 5 We illustrate two of the rules with

$$f(x, y) = x - y \qquad g(x, y) = 3y$$
$$\nabla f = \mathbf{i} - \mathbf{j} \qquad \nabla g = 3\mathbf{j}.$$

We have

1. $\nabla(f - g) = \nabla(x - 4y) = \mathbf{i} - 4\mathbf{j} = \nabla f - \nabla g$ Rule 2

2. $\nabla(fg) = \nabla(3xy - 3y^2) = 3y\mathbf{i} + (3x - 6y)\mathbf{j}$

$= 3y(\mathbf{i} - \mathbf{j}) + 3y\mathbf{j} + (3x - 6y)\mathbf{j}$ $g\nabla f$ plus terms .

$= 3y(\mathbf{i} - \mathbf{j}) + (3x - 3y)\mathbf{j}$ simplified.

$= 3y(\mathbf{i} - \mathbf{j}) + (x - y)3\mathbf{j} = g\nabla f + f\nabla g$ Rule 4

Functions of Three Variables

a differentiable function $f(x, y, z)$ and a unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 + \frac{\partial f}{\partial z}u_3.$$

$$= |\nabla f||u|\cos\theta = |\nabla f|\cos\theta,$$

EXAMPLE 6

(a) Find the derivative of $f(x, y, z) = x^3 - xy^2 - z$ at $P_0(1, 1, 0)$ in the direction of $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

(b) In what directions does f change most rapidly at P_0, and what are the rates of change in these directions?

Solution **(a)** $|\mathbf{v}| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = \sqrt{49} = 7$

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}.$$

$$f_x = (3x^2 - y^2)_{(1,1,0)} = 2, \qquad f_y = -2xy\big|_{(1,1,0)} = -2, \qquad f_z = -1\big|_{(1,1,0)} = -1.$$

$$(D_{\mathbf{u}}f)_{(1,1,0)} = \nabla f\big|_{(1,1,0)} \cdot \mathbf{u} = (2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \cdot \left(\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right) = \frac{4}{7}.$$

(b) ∇f $\quad -\nabla f$. The rates of change $\quad |\nabla f| \quad -|\nabla f|$

The derivative of $f(x, y, z)$ at a point P is greatest in the direction of $\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$. In this direction, the value of the derivative is $2\sqrt{3}$.

a. What is ∇f at P? Give reasons for your answer.

b. What is the derivative of f at P in the direction of $\mathbf{i} + \mathbf{j}$?

(a) $\nabla f = a(\mathbf{i}+\mathbf{j}-\mathbf{k}) \qquad |\nabla f| = a\sqrt{3} = 2\sqrt{3} \qquad a = 2$

$\nabla f = 2(\mathbf{i}+\mathbf{j}-\mathbf{k})$

(b) $\nabla f \cdot \frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) = (2\mathbf{i}+2\mathbf{j}-2\mathbf{k}) \cdot \frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) = \frac{4}{\sqrt{2}} = 2\sqrt{2}.$

The Chain Rule for Paths in plane

if $f(x, y)$ is differentiable, then the rate at which f changes along a differentiable curve $x = g(t), y = h(t)$ is

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Let the curve $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j}$

$$\frac{df(\mathbf{r}(t))}{dt} = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t).$$

the rate of change along r(t)

The Chain Rule for Paths in space

$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is a smooth path C,

$w = f(\mathbf{r}(t))$ is a scalar function evaluated along C,

$$w = f(x(t), y(t), z(t))$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}. = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t).$$

$$\frac{d}{dt} f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t).$$

the rate of change along r(t) .

14.6

Tangent Planes and Differentials
切平面与微分

If $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is a smooth curve on the level surface $f(x, y, z) = c$ of a differentiable function f,

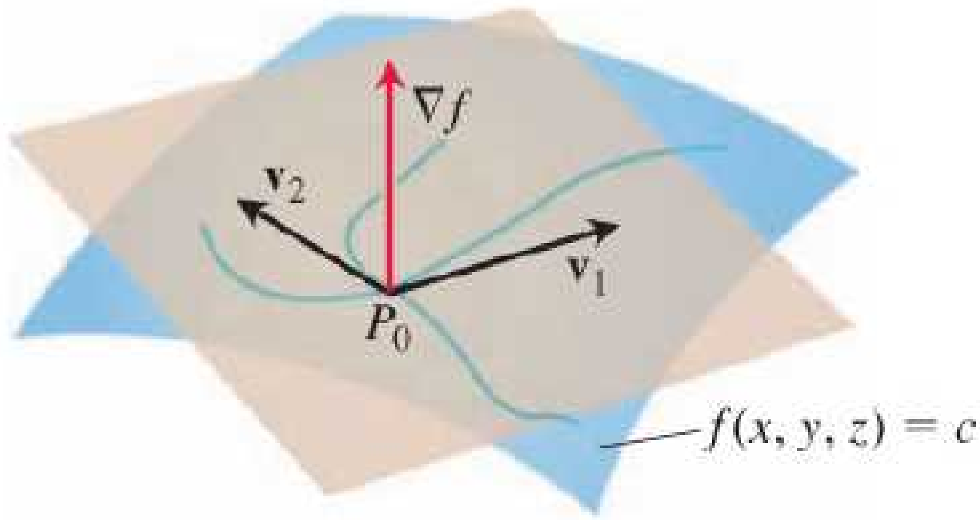
$$f(x(t), y(t), z(t)) = c$$

$$\frac{df(x(t), y(t), z(t))}{dt} = 0,$$

$$\nabla f(r(t)) \cdot r'(t) = 0$$

$$\nabla f(r(t_0)) \cdot r'(t_0) = 0$$

$$\nabla f(P_0) \cdot r'(t_0) = 0$$



the curves' tangent lines all lie in the plane through P_0 normal to ∇f.

DEFINITIONS The **tangent plane** at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ of a differentiable function f is the plane through P_0 normal to $\nabla f|_{P_0}$.

The **normal line** of the surface at P_0 is the line through P_0 parallel to $\nabla f|_{P_0}$.

$$\mathbf{n} = \Delta f\Big|_{P_0}$$

既是切平面的法向量，
又是法线的方向向量

Tangent Plane to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$

Normal Line to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$x = x_0 + f_x(P_0)t, \qquad y = y_0 + f_y(P_0)t, \qquad z = z_0 + f_z(P_0)t$$

EXAMPLE 1 Find the tangent plane and normal line of the level surface

$$f(x, y, z) = x^2 + y^2 + z - 9 = 0 \quad \text{at the point } P_0(1, 2, 4).$$

Solution

$$\nabla f|_{P_0} = (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})_{(1,2,4)} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}.$$

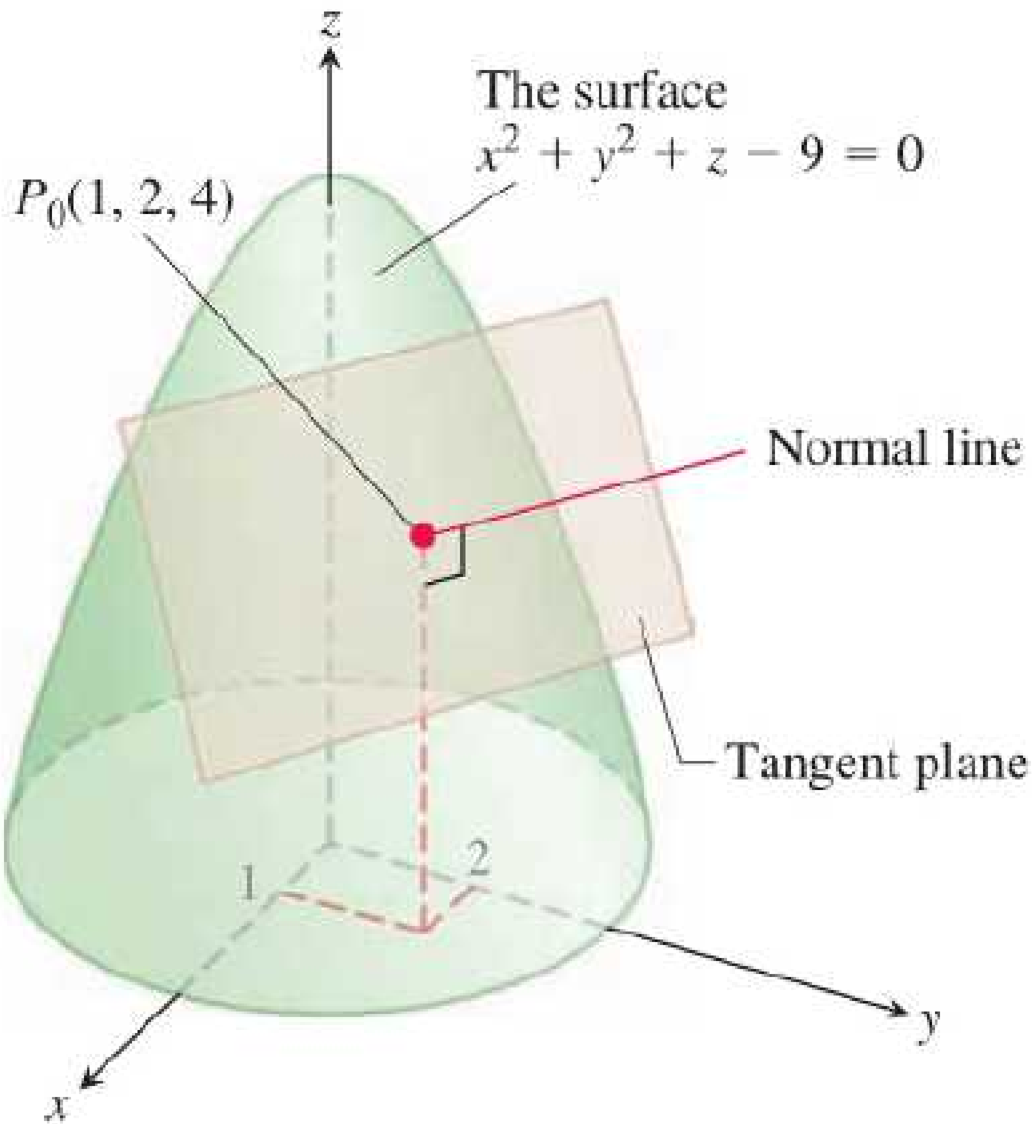
The tangent plane is therefore the plane

$$2(x - 1) + 4(y - 2) + (z - 4) = 0,$$

$$2x + 4y + z = 14.$$

The line normal to the surface at P_0 is

$$x = 1 + 2t, \qquad y = 2 + 4t, \qquad z = 4 + t.$$



z
The surface
$x^2 + y^2 + z - 9 = 0$
$P_0(1, 2, 4)$
Normal line
Tangent plane
1
2
x
y

to a smooth surface $z = f(x, y)$

$$f(x, y) - z = 0.$$

Plane Tangent to a Surface $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$

The plane tangent to the surface $z = f(x, y)$ of a differentiable function f at the point $P_0(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

EXAMPLE 2

Find the plane tangent to the surface $z = x\cos y - ye^x$ at $(0, 0, 0)$.

Solution

$$f_x(0, 0) = (\cos y - ye^x)_{(0,0)} = 1 - 0 \cdot 1 = 1$$

$$f_y(0, 0) = (-x\sin y - e^x)_{(0,0)} = 0 - 1 = -1.$$

The tangent plane is therefore

$$1 \cdot (x - 0) - 1 \cdot (y - 0) - (z - 0) = 0,$$

$$x - y - z = 0.$$

EXAMPLE 3 The surfaces $f(x, y, z) = x^2 + y^2 - 2 = 0$ and $g(x, y, z) = x + z - 4 = 0$ meet in an ellipse E. Find parametric equations for the line tangent to E at the point $P_0(1, 1, 3)$.

Solution The tangent line is parallel to $\mathbf{v} = \nabla f \times \nabla g$.

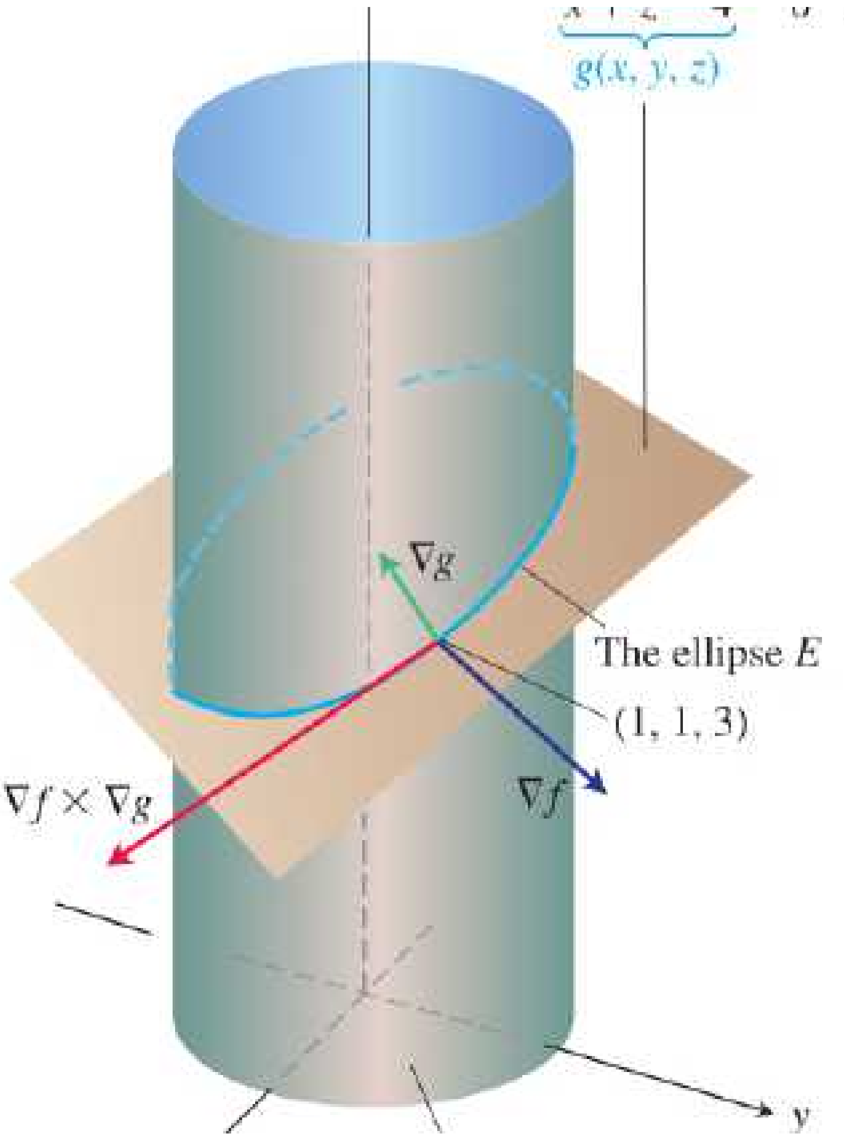
$$\nabla f\big|_{(1,1,3)} = (2x\mathbf{i} + 2y\mathbf{j})_{(1,1,3)} = 2\mathbf{i} + 2\mathbf{j}$$

$$\nabla g\big|_{(1,1,3)} = (\mathbf{i} + \mathbf{k})_{(1,1,3)} = \mathbf{i} + \mathbf{k}$$

$$\mathbf{v} = (2\mathbf{i} + 2\mathbf{j}) \times (\mathbf{i} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}.$$

The tangent line to the ellipse of intersection is

$$x = 1 + 2t, \qquad y = 1 - 2t, \qquad z = 3 - 2t.$$



g(x, y, z)
∇g
The ellipse E
(1, 1, 3)
∇f × ∇g
∇f
y

Estimating Change in a Specific Direction

If f were a function of a single variable, $df = f'(P_0)\,ds.$

For a function of two or more variables, $df = (\nabla f|_{P_0} \cdot \mathbf{u})\,ds,$

Estimating the Change in f in a Direction $\mathbf{u}$

To estimate the change in the value of a differentiable function f when we move a small distance ds from a point P_0 in a particular direction $\mathbf{u}$, use the formula

$$df = \underbrace{(\nabla f|_{P_0} \cdot \mathbf{u})}_{\substack{\text{Directional}\\ \text{derivative}}} \underbrace{ds}_{\substack{\text{Distance}\\ \text{increment}}}$$

EXAMPLE 4 Estimate how much the value of $f(x, y, z) = y \sin x + 2yz$ will change if the point $P(x, y, z)$ moves 0.1 unit from $P_0(0, 1, 0)$ straight toward $P_1(2, 2, -2)$.

Solution $\overrightarrow{P_0P_1} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. $\mathbf{u} = \dfrac{\overrightarrow{P_0P_1}}{|\overrightarrow{P_0P_1}|} = \dfrac{\overrightarrow{P_0P_1}}{3} = \dfrac{2}{3}\mathbf{i} + \dfrac{1}{3}\mathbf{j} - \dfrac{2}{3}\mathbf{k}$.

$$\nabla f|_{(0,1,0)} = ((y\cos x)\mathbf{i} + (\sin x + 2z)\mathbf{j} + 2y\mathbf{k})_{(0,1,0)} = \mathbf{i} + 2\mathbf{k}.$$

$$\nabla f|_{P_0} \cdot \mathbf{u} = (\mathbf{i} + 2\mathbf{k}) \cdot \left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right) = \frac{2}{3} - \frac{4}{3} = -\frac{2}{3}.$$

$$df = (\nabla f|_{P_0} \cdot \mathbf{u})(ds) = \left(-\frac{2}{3}\right)(0.1) \approx -0.067 \text{ unit.}$$

How to Linearize a Function of Two Variables

$z = f(x, y)$ a point (x_0, y_0) f is differentiable.

$$f(x, y) - f(x_0, y_0) = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

$$f(x, y) \approx \underbrace{f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}_{L(x, y)}.$$

DEFINITIONS The **linearization** of a function $f(x, y)$ at a point (x_0, y_0) where f is differentiable is the function

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

$$f(x, y) \approx L(x, y)$$

is the **standard linear approximation** of f at (x_0, y_0).

EXAMPLE 5 Find the linearization of $f(x, y) = x^2 - xy + \frac{1}{2}y^2 + 3$ at the point (3, 2).

Solution

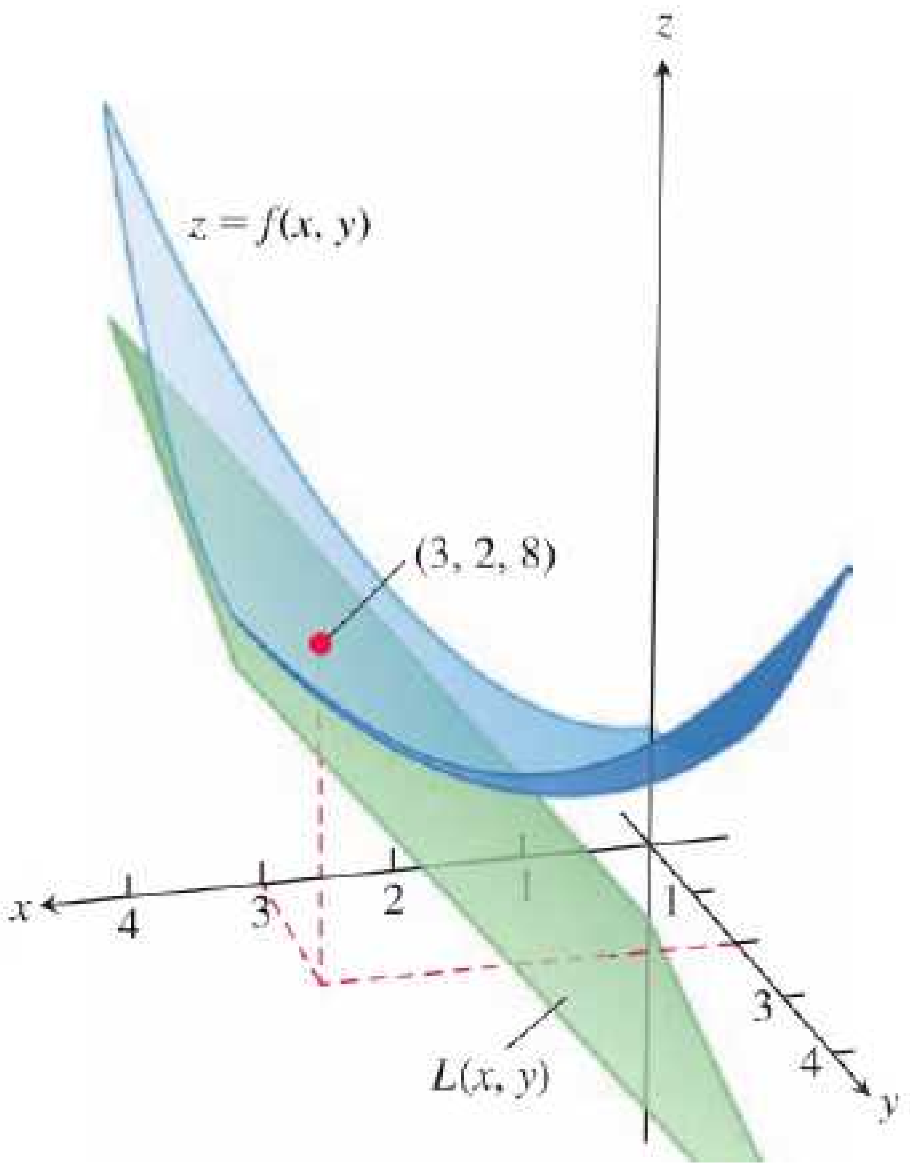
$$f(3, 2) = \left(x^2 - xy + \frac{1}{2}y^2 + 3\right)_{(3,2)} = 8$$

$$f_x(3, 2) = \frac{\partial}{\partial x}\left(x^2 - xy + \frac{1}{2}y^2 + 3\right)_{(3,2)} = (2x - y)_{(3,2)} = 4$$

$$f_y(3, 2) = \frac{\partial}{\partial y}\left(x^2 - xy + \frac{1}{2}y^2 + 3\right)_{(3,2)} = (-x + y)_{(3,2)} = -1,$$

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$= 8 + (4)(x - 3) + (-1)(y - 2) = 4x - y - 2.$$


z
z = f(x, y)
(3, 2, 8)
x
4
3
2
1
1
3
4
y
L(x, y)

The Error in the Standard Linear Approximation

If f has continuous first and second partial derivatives throughout an open set containing a rectangle R centered at (x_0, y_0) and if M is any upper bound for the values of $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ on R, then the error $E(x, y)$ incurred in replacing $f(x, y)$ on R by its linearization

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

satisfies the inequality

$$|E(x, y)| \le \frac{1}{2} M(|x - x_0| + |y - y_0|)^2.$$

Differentials

differentiable

$$f(x_0+\Delta x, y_0+\Delta y)-f(x_0, y_0)$$
$$= f_x(x_0, y_0)\Delta x + f(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

$$df = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

$$\Delta f = df + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

DEFINITION If we move from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, the resulting change

$$df = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy \approx \Delta f$$

in the linearization of f is called the **total differential of f**.

EXAMPLE 6 a cylindrical can is designed to have a radius of 1 in. and a height of 5 in., but that the radius and height are off by the amounts $dr = +0.03$ and $dh = -0.1$. Estimate the change in the volume

Solution $V = \pi r^2 h$, $\quad \Delta V \approx dV = V_r(r_0, h_0)\,dr + V_h(r_0, h_0)\,dh.$

$$dV = 2\pi r_0 h_0\,dr + \pi r_0{}^2\,dh = 2\pi(1)(5)(0.03) + \pi(1)^2(-0.1)$$
$$= 0.3\pi - 0.1\pi = 0.2\pi \approx 0.63 \text{ in}^3$$

EXAMPLE 7

right circular cylindrical tanks that are 25 ft high with a radius of 5 ft.

How sensitive are the tanks'

volumes to small variations in height and radius?

Solution With $V = \pi r^2 h$,

$$\begin{aligned} dV &= V_r(5, 25)\, dr + V_h(5, 25)\, dh \\ &= (2\pi r h)_{(5,25)}\, dr + \left(\pi r^2\right)_{(5,25)}\, dh \\ &= 250\pi\, dr + 25\pi\, dh. \end{aligned}$$

1-unit change in r will change V by about 250π units.

1-unit change in h will change V by about 25π units.

Functions of More Than Two Variables

1. The **linearization** of $f(x, y, z)$ at a point $P_0(x_0, y_0, z_0)$ is

$$L(x, y, z) = f(P_0) + f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0).$$

2. R is a closed rectangular solid centered at P_0 and on which the second partial derivatives of f are continuous.

$|f_{xx}|, |f_{yy}|, |f_{zz}|, |f_{xy}|, |f_{xz}|$, and $|f_{yz}|$ are all less than or equal to M

$$|E| \le \frac{1}{2} M(|x - x_0| + |y - y_0| + |z - z_0|)^2.$$

3. the **total differential** $df = f_x(P_0)\,dx + f_y(P_0)\,dy + f_z(P_0)\,dz$

EXAMPLE 8 Find the linearization $L(x, y, z)$ of $f(x, y, z) = x^2 - xy + 3\sin z$ at the point $(x_0, y_0, z_0) = (2, 1, 0)$.

Find an upper bound for the error incurred in replacing f by L on the rectangular region

$$R: \quad |x - 2| \le 0.01, \qquad |y - 1| \le 0.02, \qquad |z| \le 0.01.$$

Solution

$$f(2, 1, 0) = 2, \qquad f_x(2, 1, 0) = 3, \qquad f_y(2, 1, 0) = -2, \qquad f_z(2, 1, 0) = 3.$$

$$L(x, y, z) = 2 + 3(x - 2) + (-2)(y - 1) + 3(z - 0) = 3x - 2y + 3z - 2.$$

$$f_{xx} = 2, \qquad f_{yy} = 0, \qquad f_{zz} = -3\sin z, \qquad f_{xy} = -1, \qquad f_{xz} = 0, \qquad f_{yz} = 0,$$

we may take $M = 2$ $\quad |E| \le \frac{1}{2}(2)(0.01 + 0.02 + 0.01)^2 = 0.0016.$

14.7

Extreme Values and Saddle Points

极值与鞍点

DEFINITIONS Let $f(x, y)$ be defined on a region R containing the point (a, b). Then

1. $f(a, b)$ is a **local maximum** value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).
2. $f(a, b)$ is a **local minimum** value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).

relative extrema.

THEOREM 10—First Derivative Test for Local Extreme Values If $f(x, y)$ has a local maximum or minimum value at an interior point (a, b) of its domain and if the first partial derivatives exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

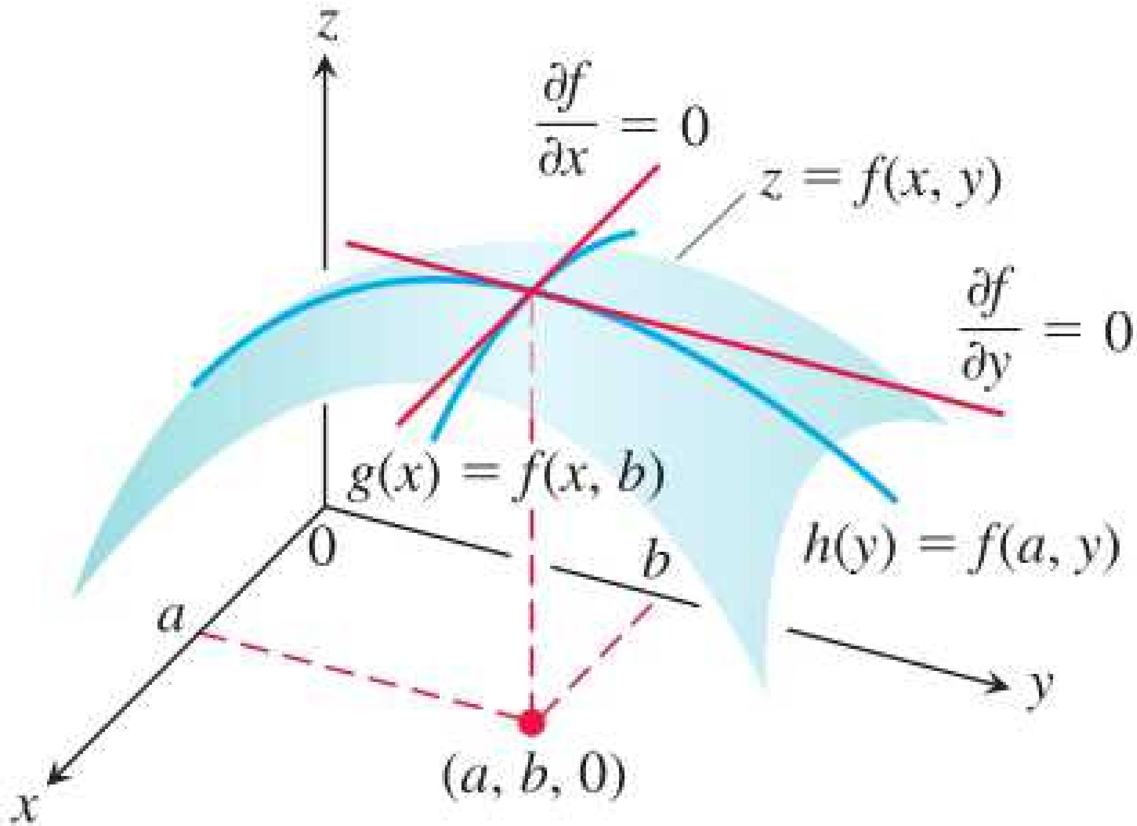
Proof

If f has a local extremum at (a, b), then the function $g(x) = f(x, b)$ has a local extremum at $x = a$ Therefore, $g'(a) = 0$ so $f_x(a, b) = 0$. A similar argument shows that $f_y(a, b) = 0$.

the tangent plane to the surface $z = f(x, y)$ at (a, b),

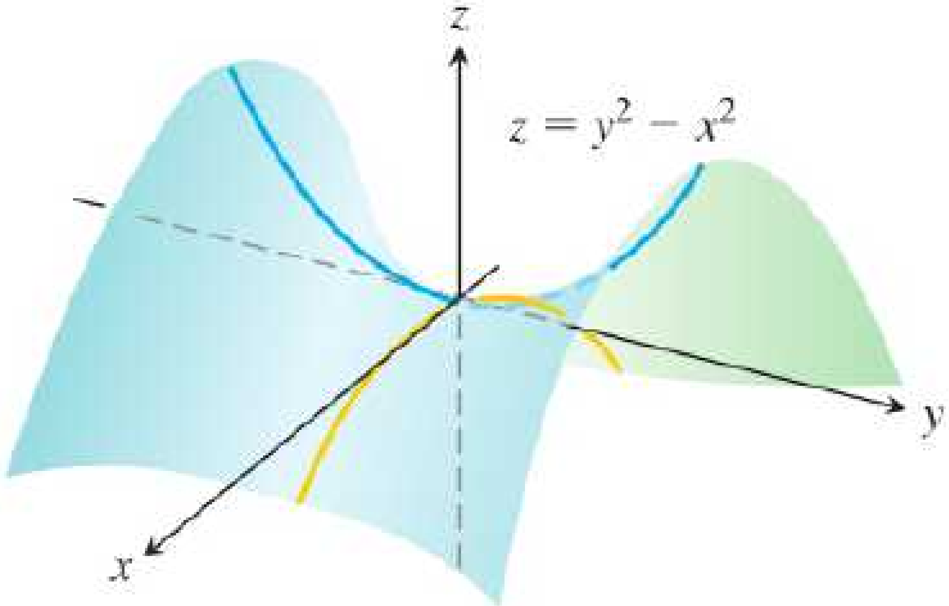
$$f_x(a, b)(x - a) + f_y(a, b)(y - b) - (z - f(a, b)) = 0$$

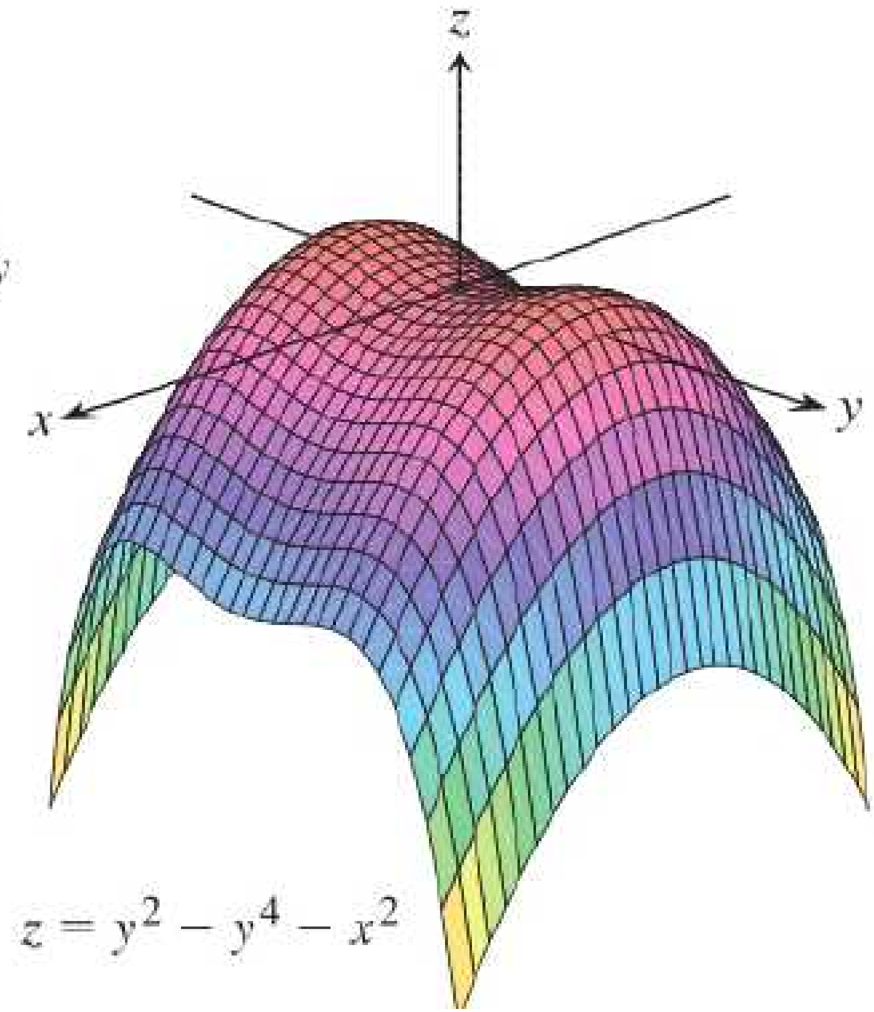
$z = f(a, b)$. a horizontal tangent plane


z
$\frac{\partial f}{\partial x} = 0$
$z = f(x, y)$
$\frac{\partial f}{\partial y} = 0$
$g(x) = f(x, b)$
$h(y) = f(a, y)$
0
a
b
y
$(a, b, 0)$
x

DEFINITION An interior point of the domain of a function $f(x, y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a **critical point** of f.

DEFINITION A differentiable function $f(x, y)$ has a **saddle point** at a critical point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points (x, y) where $f(x, y) < f(a, b)$. The corresponding point $(a, b, f(a, b))$ on the surface $z = f(x, y)$ is called a saddle point of the surface (Figure 14.45).


z
$z = y^2 - x^2$
y
x


z
x
y
$z = y^2 - y^4 - x^2$

EXAMPLE 1 Find the local extreme values of $f(x, y) = x^2 + y^2 - 4y + 9$.

Solution $f_x = 2x$ and $f_y = 2y - 4$

$$f_x = 2x = 0 \quad \text{and} \quad f_y = 2y - 4 = 0.$$

The only possibility is the point $(0, 2)$,

$$f(x, y) = x^2 + (y - 2)^2 + 5 \geq 5 = f(0,2)$$

the critical point $(0, 2)$ gives a local minimum

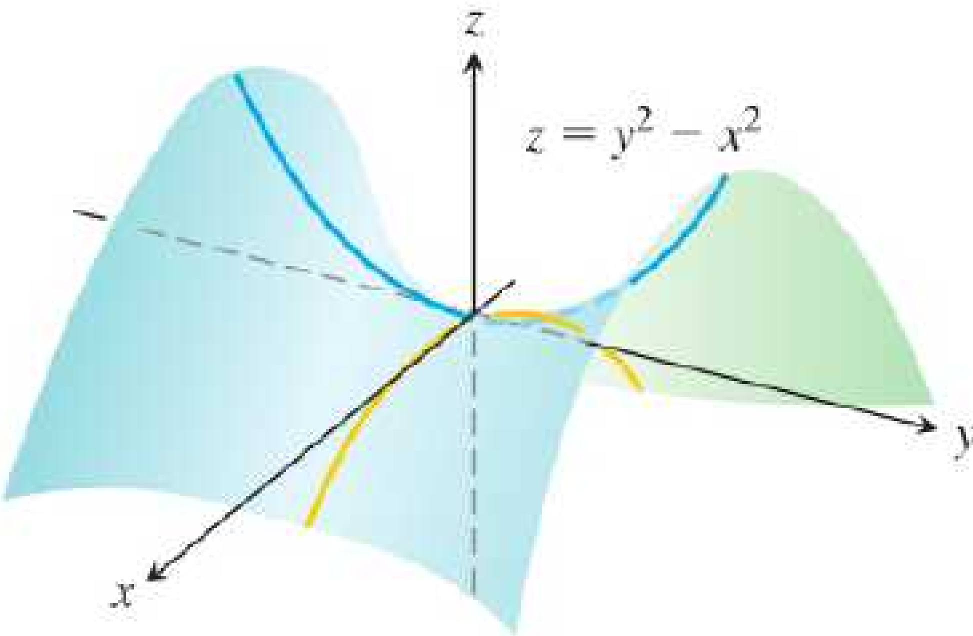
EXAMPLE 2 Find the local extreme values (if any) of $f(x, y) = y^2 - x^2$.

Solution $f_x = -2x$ and $f_y = 2y$

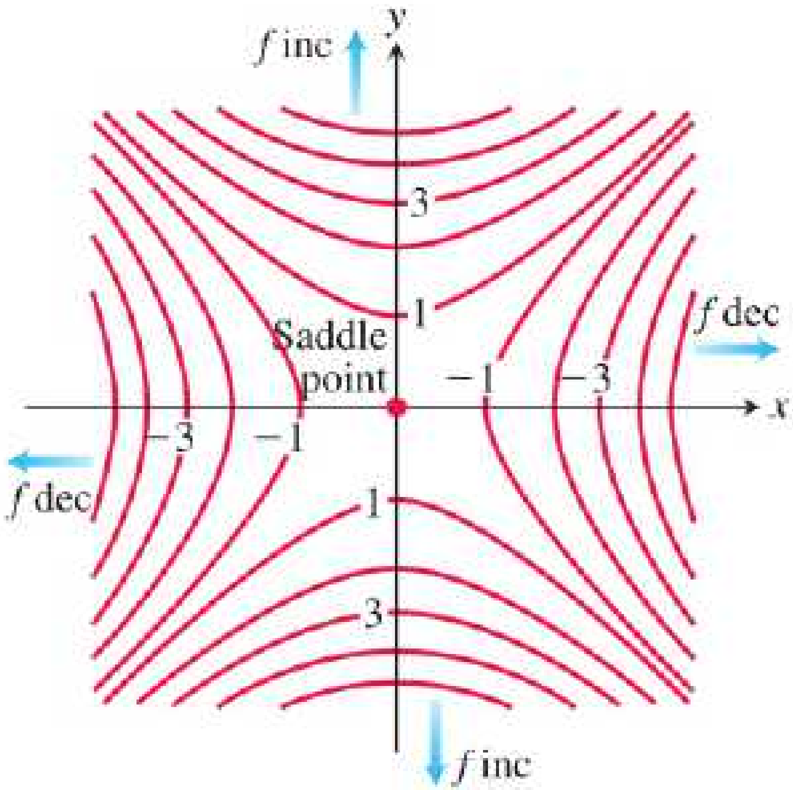
only at the origin $(0, 0)$ where $f_x = 0$ and $f_y = 0$.

$f(0,0) = 0.$ $(0,0,0)$

saddle point at the origin and no local extreme values



z
$z = y^2 - x^2$
y
x



y
f inc
3
1
Saddle point
−1
−3
x
f dec
−3
−1
f dec
1
3
f inc

THEOREM 11—Second Derivative Test for Local Extreme Values Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

i) f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).

ii) f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).

iii) f has a **saddle point** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b).

iv) **the test is inconclusive** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b). In this case, we must find some other way to determine the behavior of f at (a, b).

$f_{xx}f_{yy} - f_{xy}^2$ is called the **discriminant** or **Hessian** of f.

$$AC - B^2 \qquad f_{xx}f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}.$$

EXAMPLE 3 Find the local extreme values of the function

$$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4.$$

Solution $f_x = y - 2x - 2 = 0, \qquad f_y = x - 2y - 2 = 0,$

the point $(-2, -2)$ is the only point where f may take on an extreme value.

$$f_{xx} = -2, \qquad f_{yy} = -2, \qquad f_{xy} = 1.$$

$$f_{xx}f_{yy} - f_{xy}{}^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3.$$

$$f_{xx} < 0 \qquad \text{and} \qquad f_{xx}f_{yy} - f_{xy}{}^2 > 0$$

tells us that f has a local maximum at $(-2, -2)$.

$$f(-2, -2) = 8.$$

EXAMPLE 4

Find the local extreme values of $f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$.

Solution

$$f_x = 6y - 6x = 0 \quad \text{and} \quad f_y = 6y - 6y^2 + 6x = 0.$$

The two critical points are therefore (0, 0) and (2, 2).

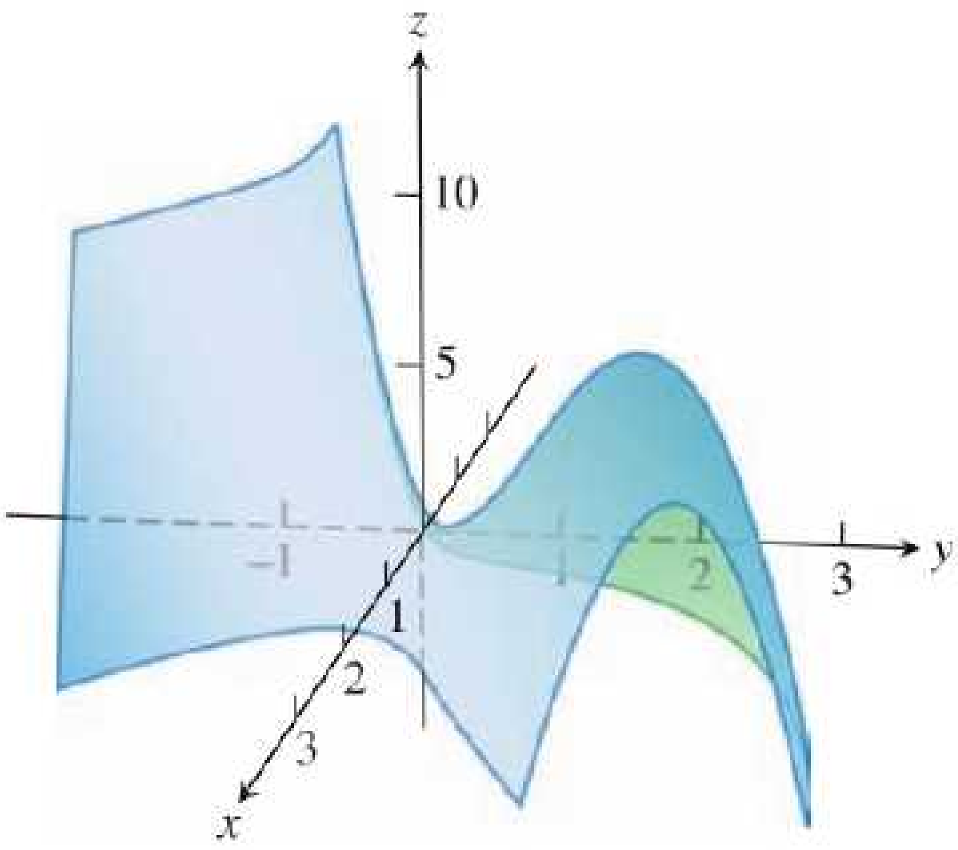
$$f_{xx} = -6, \qquad f_{yy} = 6 - 12y, \qquad f_{xy} = 6.$$

$$f_{xx}f_{yy} - f_{xy}^2 = (-36 + 72y) - 36 = 72(y - 1).$$

At the critical point (0, 0) $\quad f_{xx}f_{yy} - f_{xy}^2 = -72,$
so the function has a saddle point at the origin.

saddle point (0,0,0)

At the critical point (2, 2) $\quad f_{xx}f_{yy} - f_{xy}^2 = 72. \quad f_{xx} = -6,$
the critical point (2, 2) gives a local maximum value of $f(2, 2) = 8$.


z
10
5
−1
1
2
3
y
1
2
3
x

EXAMPLE 5

Find the critical points of the function $f(x, y) = 10xye^{-(x^2+y^2)}$ and use the Second Derivative Test to classify each point

Solution

$$f_x = 10ye^{-(x^2+y^2)} - 20x^2ye^{-(x^2+y^2)} = 10y(1 - 2x^2)e^{-(x^2+y^2)} = 0$$

$$f_y = 10xe^{-(x^2+y^2)} - 20xy^2e^{-(x^2+y^2)} = 10x(1 - 2y^2)e^{-(x^2+y^2)} = 0$$

$$y = 0 \text{ or } 1 - 2x^2 = 0, \qquad x = 0 \text{ or } 1 - 2y^2 = 0.$$

the only critical points are

$$(0, 0), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \text{ and } \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

$$f_{xx} = -20xy(1 - 2x^2)e^{-(x^2+y^2)} - 40xye^{-(x^2+y^2)} = -20xy(3 - 2x^2)e^{-(x^2+y^2)},$$

$$f_{xy} = f_{yx} = 10(1 - 2x^2)e^{-(x^2+y^2)} - 20y^2(1 - 2x^2)e^{-(x^2+y^2)} = 10(1 - 2x^2)(1 - 2y^2)e^{-(x^2+y^2)}$$

$$f_{yy} = -20xy(1 - 2y^2)e^{-(x^2+y^2)} - 40xye^{-(x^2+y^2)} = -20xy(3 - 2y^2)e^{-(x^2+y^2)}.$$

Critical Point	f_{xx}	f_{xy}	f_{yy}	Discriminant D	
$(0, 0)$	0	10	0	-100	saddle point
$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$-\frac{20}{e}$	0	$-\frac{20}{e}$	$\frac{400}{e^2}$	local maximum
$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\frac{20}{e}$	0	$\frac{20}{e}$	$\frac{400}{e^2}$	local minimum;
$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$\frac{20}{e}$	0	$\frac{20}{e}$	$\frac{400}{e^2}$	local minimum;
$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$-\frac{20}{e}$	0	$-\frac{20}{e}$	$\frac{400}{e^2}$	local maximum

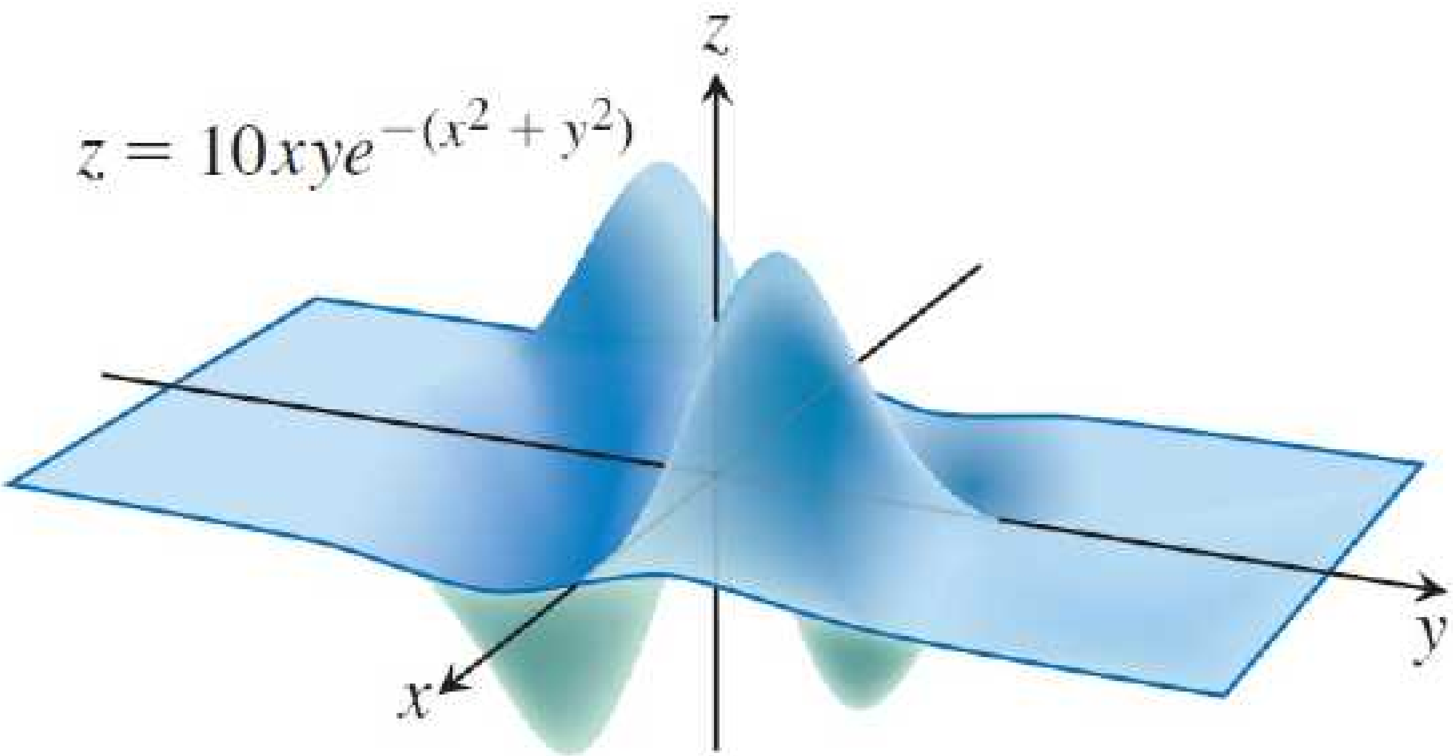


FIGURE 14.49 A graph of the function in Example 5.

Absolute Maxima and Minima on Closed Bounded Regions

1. *List the interior points of R* where f may have local maxima and minima

   These are the critical points of f.

2. *List the boundary points of R* where f has local maxima and minima

3. *Look through the lists* for the maximum and minimum values of f.

EXAMPLE 6 Find the absolute maximum and minimum values of

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$, and $y = 9 - x$.

Solution **(a) Interior points.** $f_x = 2 - 2x = 0$, $\quad f_y = 4 - 2y = 0$,

the single point $(x, y) = (1, 2)$. $\quad f(1, 2) = 7$.

(b) Boundary points.

i) On the segment OA, $y = 0$. $f(x, y) = f(x, 0) = 2 + 2x - x^2 \quad [0, 9]$.

$f'(x, 0) = 2 - 2x = 0$. $x = 1$. $f(1, 0) = 3$.

at the endpoints $\quad f(0, 0) = 2 \quad f(9, 0) = 2 + 18 - 81 = -61$

ii) On the segment OB, $x = 0$ $f(x, y) = f(0, y) = 2 + 4y - y^2$. $[0, 9]$.

$f'(0, y) = 4 - 2y = 0$. the only interior point $(0, 2)$,

$f(0, 2) = 6$.

at the endpoints $f(0, 9) = -43$,

iii) On the segment AB With $y = 9 - x$,

$f(x, y) = 2 + 2x + 4(9 - x) - x^2 - (9 - x)^2 = -43 + 16x - 2x^2$.

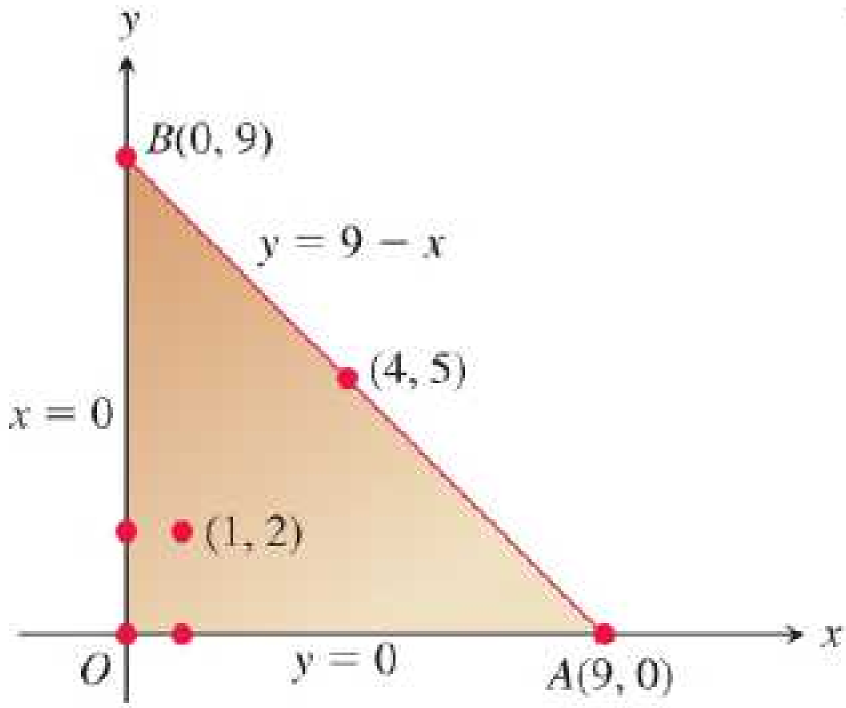
$f'(x, 9 - x) = 16 - 4x = 0$ gives $x = 4$. $f(4, 5) = -11$.

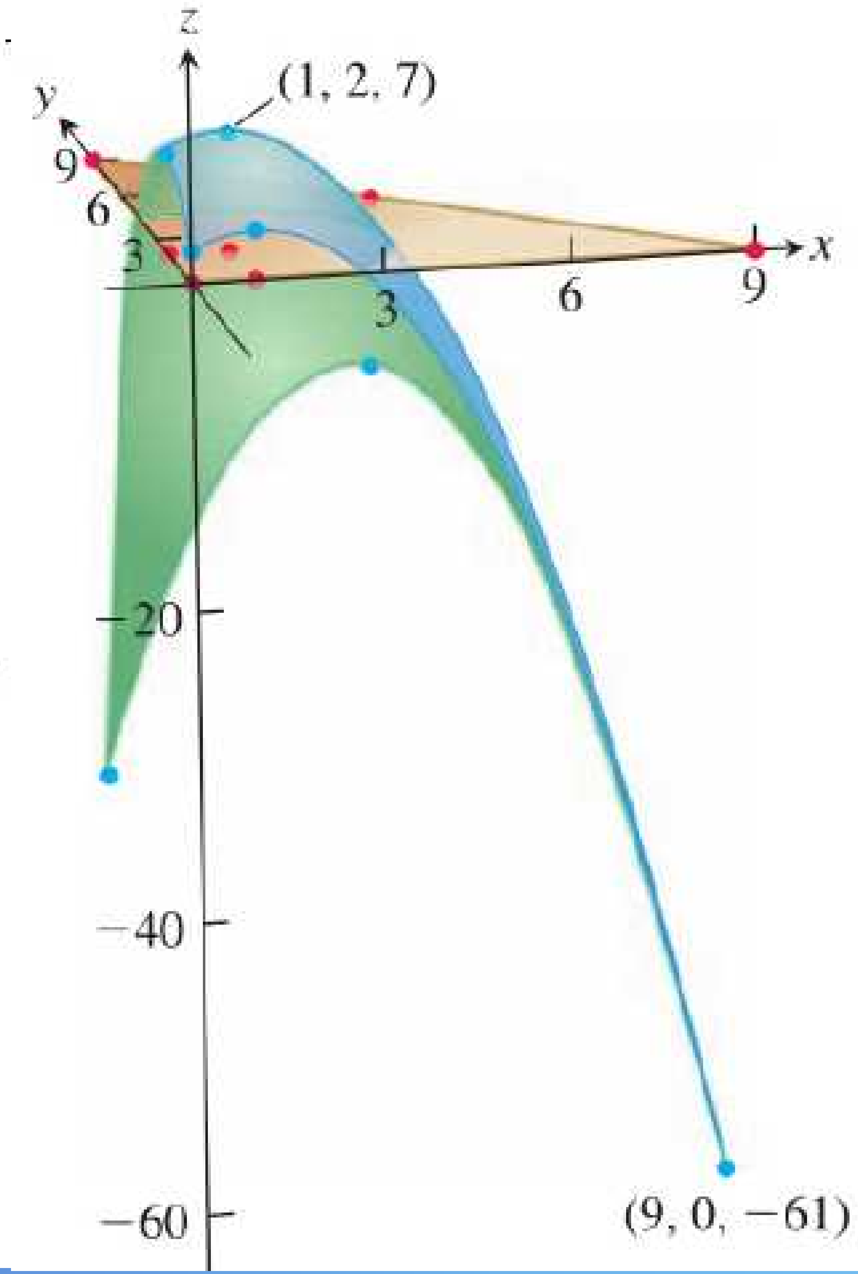
Summary $\boxed{f(1, 2) = 7.}$

$f(0, 0) = 2$ $f(1, 0) = 3$. $\boxed{f(9, 0) = 2 + 18 - 81 = -61}$

$f(0, 9) = -43$, $f(0, 2) = 6$.

$f(4, 5) = -11$.


y
B(0, 9)
y = 9 − x
(4, 5)
x = 0
(1, 2)
O
y = 0
A(9, 0)
x

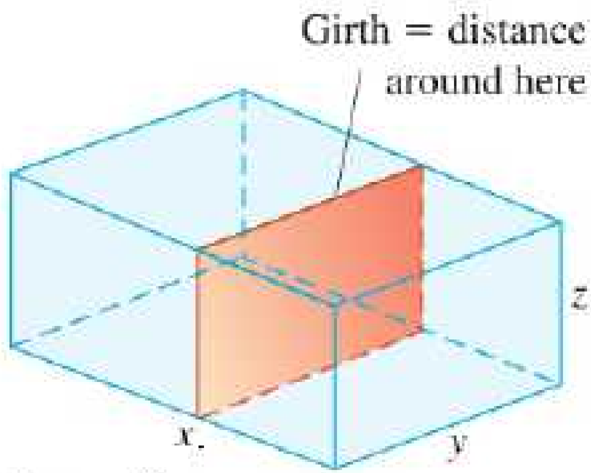

z
(1, 2, 7)
y
9
6
3
x
3
6
9
−20
−40
−60
(9, 0, −61)

EXAMPLE 7

A delivery company accepts only rectangular boxes the sum of whose length and girth (perimeter of a cross-section) does not exceed 108 in. Find the dimensions of an acceptable box of largest volume.

Solution

Let x, y, and z represent the length, width, and height of the rectangular box. Then the girth is $2y + 2z$.



$$V = xyz \quad (\quad x + 2y + 2z = 108$$

$$V(y, z) = (108 - 2y - 2z)yz$$

maximize the volume

$$V_y(y, z) = 108z - 4yz - 2z^2 = (108 - 4y - 2z)z = 0$$

$$V_z(y, z) = 108y - 2y^2 - 4yz = (108 - 2y - 4z)y = 0,$$

gives the critical points $(0, 0)$, $(0, 54)$, $(54, 0)$, and $(18, 18)$.

(0, 0), (0, 54), (54, 0). The volume is zero which are not maximum values.

At the point (18, 18),

$$V_{yy} = -4z, \qquad V_{zz} = -4y, \qquad V_{yz} = 108 - 4y - 4z.$$

$$V_{yy}V_{zz} - V_{yz}^{\ 2} = 16yz - 16(27 - y - z)^2.$$

$$\left[V_{yy}V_{zz} - V_{yz}^{\ 2}\right]_{(18,18)} = 16(18)(18) - 16(-9)^2 > 0$$

$$V_{yy}(18, 18) = -4(18) < 0$$

(18, 18) gives a maximum volume.

$y = 18$ in., and $z = 18$ in. $x = 108 - 2(18) - 2(18) = 36$ in.

The maximum volume is $V = (36)(18)(18) = 11{,}664 \text{ in}^3$.

Summary of Max-Min Tests

The extreme values of $f(x, y)$ can occur only at

i. **boundary points** of the domain of f

ii. **critical points** (interior points where $f_x = f_y = 0$ or points where f_x or f_y fail to exist).

If the first- and second-order partial derivatives of f are continuous throughout a disk centered at a point (a, b) and $f_x(a, b) = f_y(a, b) = 0$, the nature of $f(a, b)$ can be tested with the **Second Derivative Test**:

i. $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) $\Rightarrow$ **local maximum**

ii. $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) $\Rightarrow$ **local minimum**

iii. $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b) $\Rightarrow$ **saddle point**

iv. $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b) $\Rightarrow$ **test is inconclusive.**

14.8

Lagrange Multipliers
拉格朗日乘子法

EXAMPLE 1

Find the point $p(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

Solution

$$\left|\overrightarrow{OP}\right| = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{x^2 + y^2 + z^2}$$

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to the constraint that $\quad 2x + y - z - 5 = 0.$

$$h(x, y) = f(x, y, 2x + y - 5) = x^2 + y^2 + (2x + y - 5)^2$$

$$h_x = 2x + 2(2x + y - 5)(2) = 0, \qquad h_y = 2y + 2(2x + y - 5) = 0.$$

$$10x + 4y = 20, \qquad 4x + 4y = 10,$$

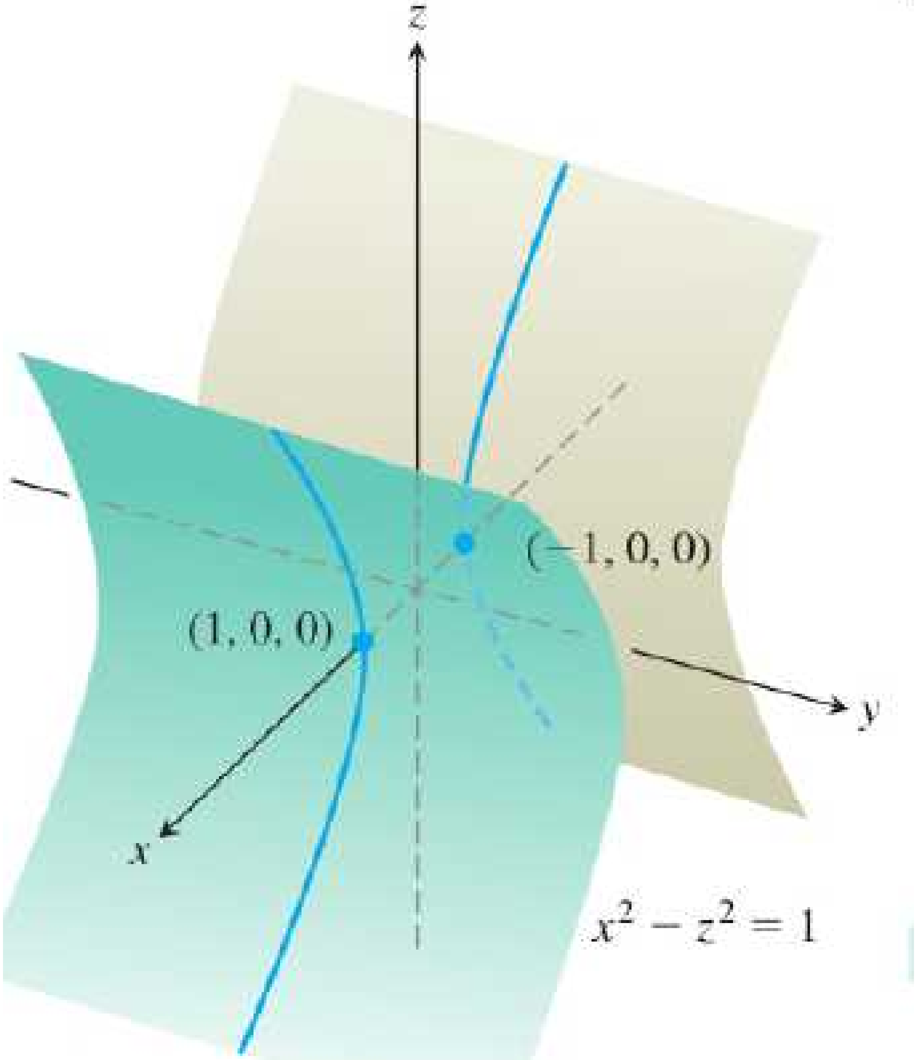
the solution $\quad x = \dfrac{5}{3}, \quad y = \dfrac{5}{6}.\quad$ Closest point: $\quad P\left(\dfrac{5}{3}, \dfrac{5}{6}, -\dfrac{5}{6}\right).$

The distance from P to the origin is $5/\sqrt{6} \approx 2.04$.

EXAMPLE 2

Find the points on the hyperbolic cylinder $x^2 - z^2 - 1 = 0$ that are closest to the origin.

Solution 1



$f(x, y, z) = x^2 + y^2 + z^2$ $x^2 - z^2 - 1 = 0.$ $|x| \geq 1$

minimize the value of the function

$$h(x, y) = x^2 + y^2 + (x^2 - 1) = 2x^2 + y^2 - 1.$$

$$h_x = 4x = 0 \quad \text{and} \quad h_y = 2y = 0,$$

that is, at the point (0, 0). But there are no points on the cylinder

What went wrong?

$x^2 = z^2 + 1.$ $f(x, y, z) = x^2 + y^2 + z^2$ becomes

$$k(y, z) = (z^2 + 1) + y^2 + z^2 = 1 + y^2 + 2z^2$$

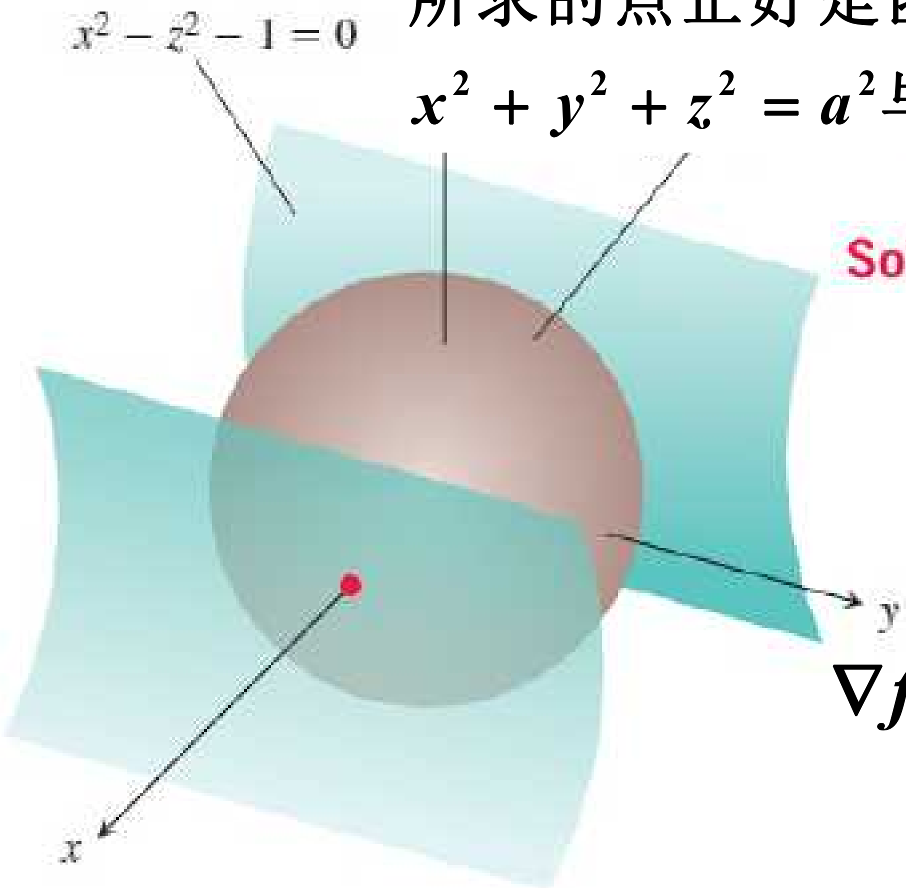
$$k_y = 2y = 0 \quad \text{and} \quad k_z = 4z = 0, \qquad y = z = 0.$$

$$x^2 = z^2 + 1 = 1, \qquad x = \pm 1.$$

The corresponding points on the cylinder are $(\pm 1, 0, 0)$.

$$k(y, z) = 1 + y^2 + 2z^2 \geq 1$$

所求的点正好是函数$f = x^2 + y^2 + z^2$的等值面$x^2 + y^2 + z^2 = a^2$与条件曲面的相切的点！



Solution 2

$$f = x^2 + y^2 + z^2$$

$$g = x^2 - z^2 - 1$$

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

FIGURE 14.54 A sphere expanding like a soap bubble centered at the origin until it just touches the hyperbolic cylinder $x^2 - z^2 - 1 = 0$ (Example 2).

Solution 2 $\nabla f = \lambda \nabla g,$

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(2x\mathbf{i} - 2z\mathbf{k}).$$

$$2x = 2\lambda x, \qquad 2y = 0, \qquad 2z = -2\lambda z.$$

$$x^2 - z^2 - 1 = 0 \qquad x \neq 0.$$

$2x = 2\lambda x$ only if $2 = 2\lambda$, or $\lambda = 1.$

$y = 0, z = 0$ the points $(\pm 1, 0, 0)$.

问题：求$f(x,y,z)$在$g(x,y,z)=0$条件下的极值，是否必有$\nabla f = \lambda \nabla g$?

The Method of Lagrange Multipliers

THEOREM 12—The Orthogonal Gradient Theorem Suppose that $f(x, y, z)$ is differentiable in a region whose interior contains a smooth curve

$$C: \quad \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

If P_0 is a point on C where f has a local maximum or minimum relative to its values on C, then ∇f is orthogonal to C at P_0.

Proof f on C are given by the composite $f(x(t), y(t), z(t))$,

At any point P_0 where f has a local maximum or minimum

$$\left.\frac{df}{dt}\right|_{P_0} = 0 \qquad \left.\frac{df}{dt}\right|_{P_0} = \left.\left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right)\right|_{P_0} = 0$$

$$\left.\nabla f\right|_{P_0} \cdot \mathbf{r}'(P_0) = 0$$

COROLLARY At the points on a smooth curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ where a differentiable function $f(x, y)$ takes on its local maxima and minima relative to its values on the curve, $\nabla f \cdot \mathbf{r}' = 0$.

The Method of Lagrange Multipliers

Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $\nabla g \neq \mathbf{0}$ when $g(x, y, z) = 0$. To find the local maximum and minimum values of f subject to the constraint $g(x, y, z) = 0$ (if these exist), find the values of x, y, z, and λ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x, y, z) = 0. \qquad (\lambda \neq 0)$$

For functions of two independent variables, the condition is similar, but without the variable z.

证明：

若f在条件$g(x,y,z)=0$下在P_0处取得局部极值，

过P_0做曲面$g(x,y,z)=0$上的任意光滑曲线 C：$\mathbf{r}(t)$，

则$f(x,y,z)$在C上点P_0处取得局部极值，

由定理12知 $\nabla f\big|_{P_0}\cdot \mathbf{r}'(P_0)=0$，

由C的任意性可知$\nabla f\big|_{P_0}$是曲面$g(x,y,z)=0$的法向量，

又知∇g是曲面$g(x,y,z)=0$上P_0处的法向量，

从而 $\nabla f\big|_{P_0}=\lambda\nabla g\big|_{P_0}$.

EXAMPLE 3 Find the greatest and smallest values that the function

$f(x, y) = xy$ takes on the ellipse (Figure 14.55) $\dfrac{x^2}{8} + \dfrac{y^2}{2} = 1.$

Solution $$g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0.$$

$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g(x, y) = 0.$$

$$y\mathbf{i} + x\mathbf{j} = \frac{\lambda}{4} x\mathbf{i} + \lambda y\mathbf{j},$$

$$y = \frac{\lambda}{4} x, \qquad x = \lambda y, \qquad \text{and} \qquad y = \frac{\lambda}{4}(\lambda y) = \frac{\lambda^2}{4} y,$$

so that $y = 0$ or $\lambda = \pm 2$.

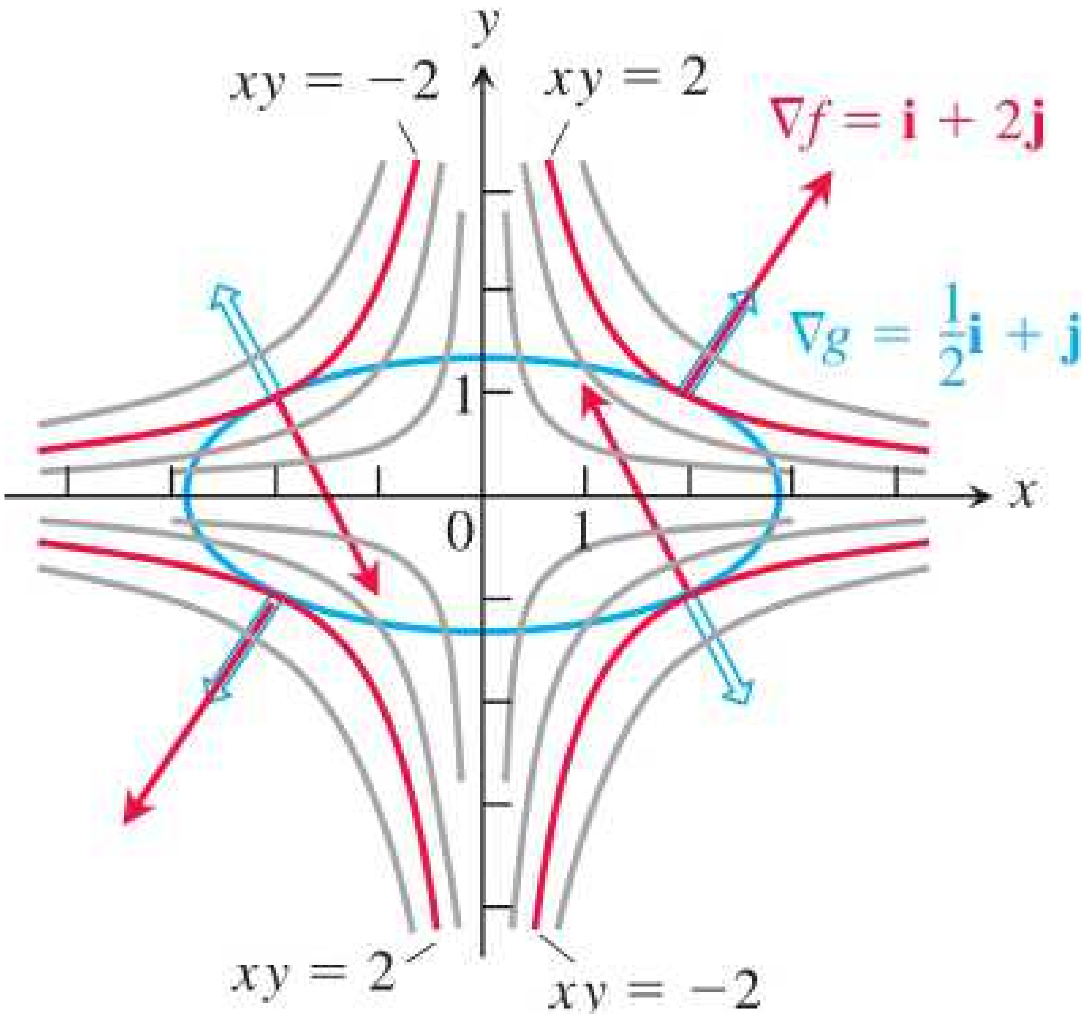
If $y = 0$, then $x = y = 0$. But $(0, 0)$ is not on the ellipse. Hence, $y \neq 0$.

If $y \neq 0$, then $\lambda = \pm 2$ and $x = \pm 2y$.

$$4y^2 + 4y^2 = 8 \qquad \text{and} \qquad y = \pm 1. \qquad (\pm 2, 1), (\pm 2, -1).$$

$xy = 2$ and $xy = -2$.

The Geometry of the Solution



EXAMPLE 4

Find the maximum and minimum values of the function $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

Solution $f(x, y) = 3x + 4y, \qquad g(x, y) = x^2 + y^2 - 1$

$$\nabla f = \lambda \nabla g: \quad 3\mathbf{i} + 4\mathbf{j} = 2x\lambda\mathbf{i} + 2y\lambda\mathbf{j}$$

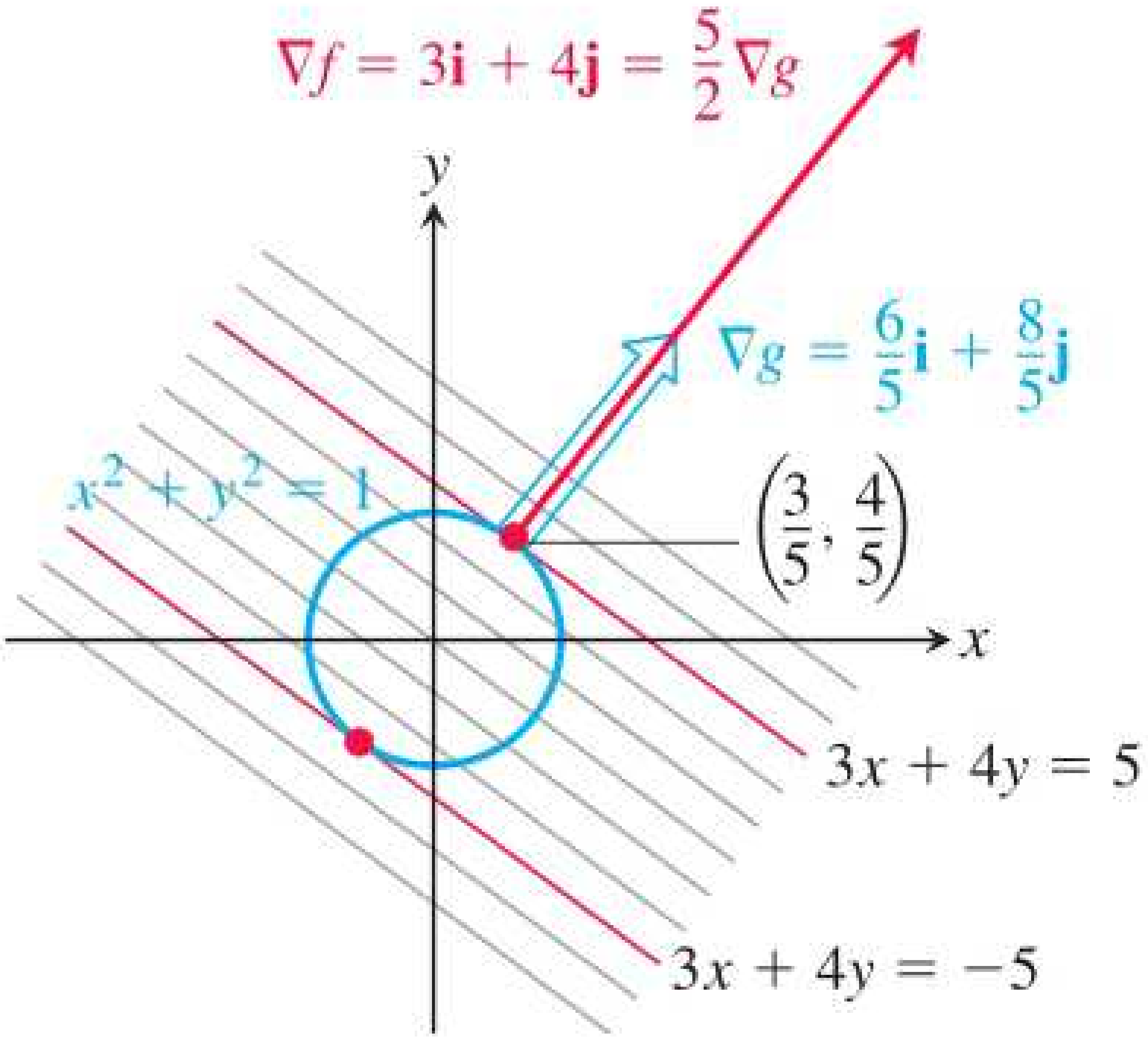
$$\lambda \neq 0; \qquad x = \frac{3}{2\lambda}, \qquad y = \frac{2}{\lambda}.$$

$$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{2}{\lambda}\right)^2 - 1 = 0, \qquad \lambda = \pm\frac{5}{2}.$$

$$x = \frac{3}{2\lambda} = \pm\frac{3}{5}, \qquad y = \frac{2}{\lambda} = \pm\frac{4}{5}, \quad (x, y) = \pm(3/5, 4/5).$$

$$f\left(\frac{3}{5}, \frac{4}{5}\right) = 5, \; f\left(\frac{-3}{5}, \frac{-4}{5}\right) = -5$$

The Geometry of the Solution



Lagrange Multipliers with Two Constraints

find the extreme values of a differentiable function $f(x, y, z)$ whose variables are subject to two constraints. If the constraints are

$$g_1(x, y, z) = 0 \quad \text{and} \quad g_2(x, y, z) = 0$$

g_1 and g_2 are differentiable, with ∇g_1 not parallel to ∇g_2,

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2, \qquad g_1(x, y, z) = 0, \qquad g_2(x, y, z) = 0$$

设$g_1(x,y,z)=0$与$g_1(x,y,z)=0$交线C：$\mathbf{r}(t)$.

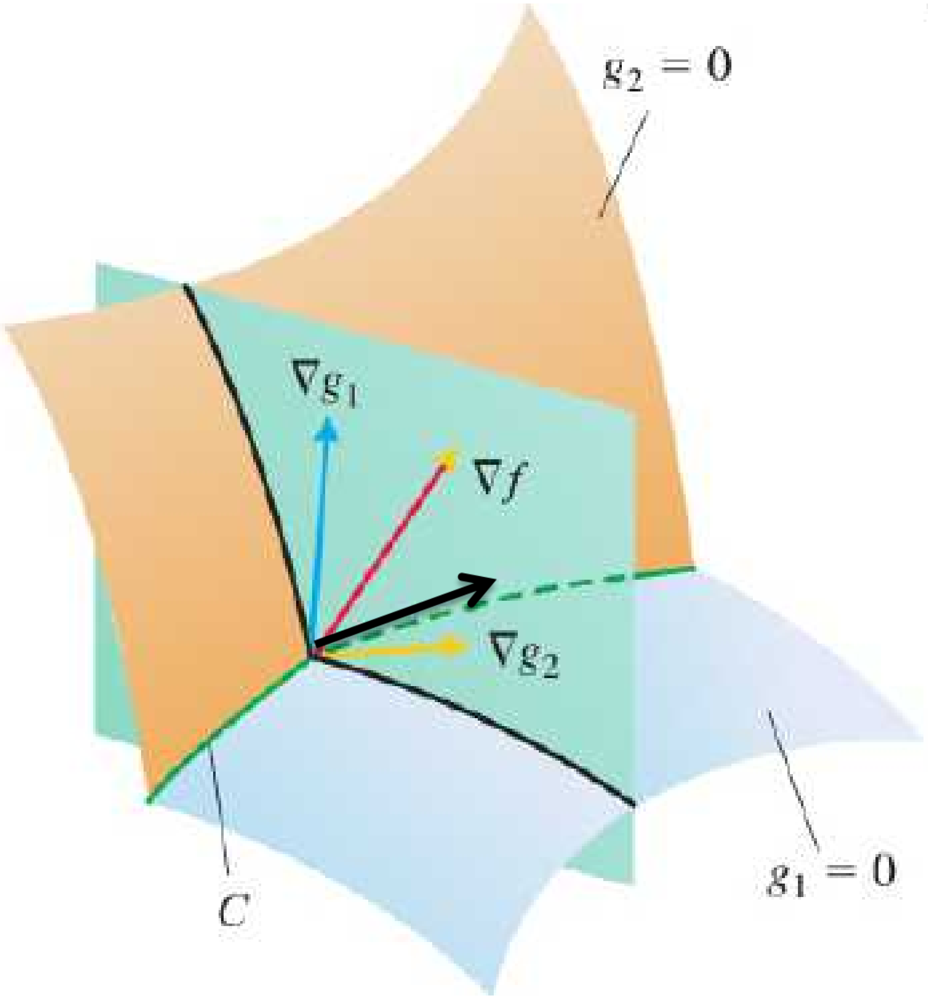
若f在C上P_0处取得局部极值， 则$\nabla f\big|_{P_0}\cdot\mathbf{r}'(P_0)=0$

由于$r(t)$在曲面$g_1(x,y,z)=0$上，则$\nabla g_1\big|_{P_0}\cdot\mathbf{r}'(P_0)=0$

由于$r(t)$在曲面$g_2(x,y,z)=0$上，则$\nabla g_2\big|_{P_0}\cdot\mathbf{r}'(P_0)=0$

∇f lies in the plane determined by ∇g_1 and ∇g_2,

$$\nabla f\Big|_{p_0} = \lambda \nabla g_1\Big|_{p_0} + \mu \nabla g_1\Big|_{p_0}$$



EXAMPLE 5

The plane $x + y + z = 1$ cuts the cylinder $x^2 + y^2 = 1$ in an ellipse Find the points on the ellipse that lie closest to and farthest from the origin.

Solution $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints

$g_1(x, y, z) = x^2 + y^2 - 1 = 0$ $g_2(x, y, z) = x + y + z - 1 = 0.$

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$$

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) + \mu(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = (2\lambda x + \mu)\mathbf{i} + (2\lambda y + \mu)\mathbf{j} + \mu\mathbf{k}$$

$$2x = 2\lambda x + \mu, \qquad 2y = 2\lambda y + \mu, \qquad 2z = \mu.$$

$$2x = 2\lambda x + 2z \Rightarrow (1 - \lambda)x = z,$$

$$2y = 2\lambda y + 2z \Rightarrow (1 - \lambda)y = z.$$

either $\lambda = 1$ and $z = 0$ or $\lambda \neq 1$ and $x = y = z/(1 - \lambda).$

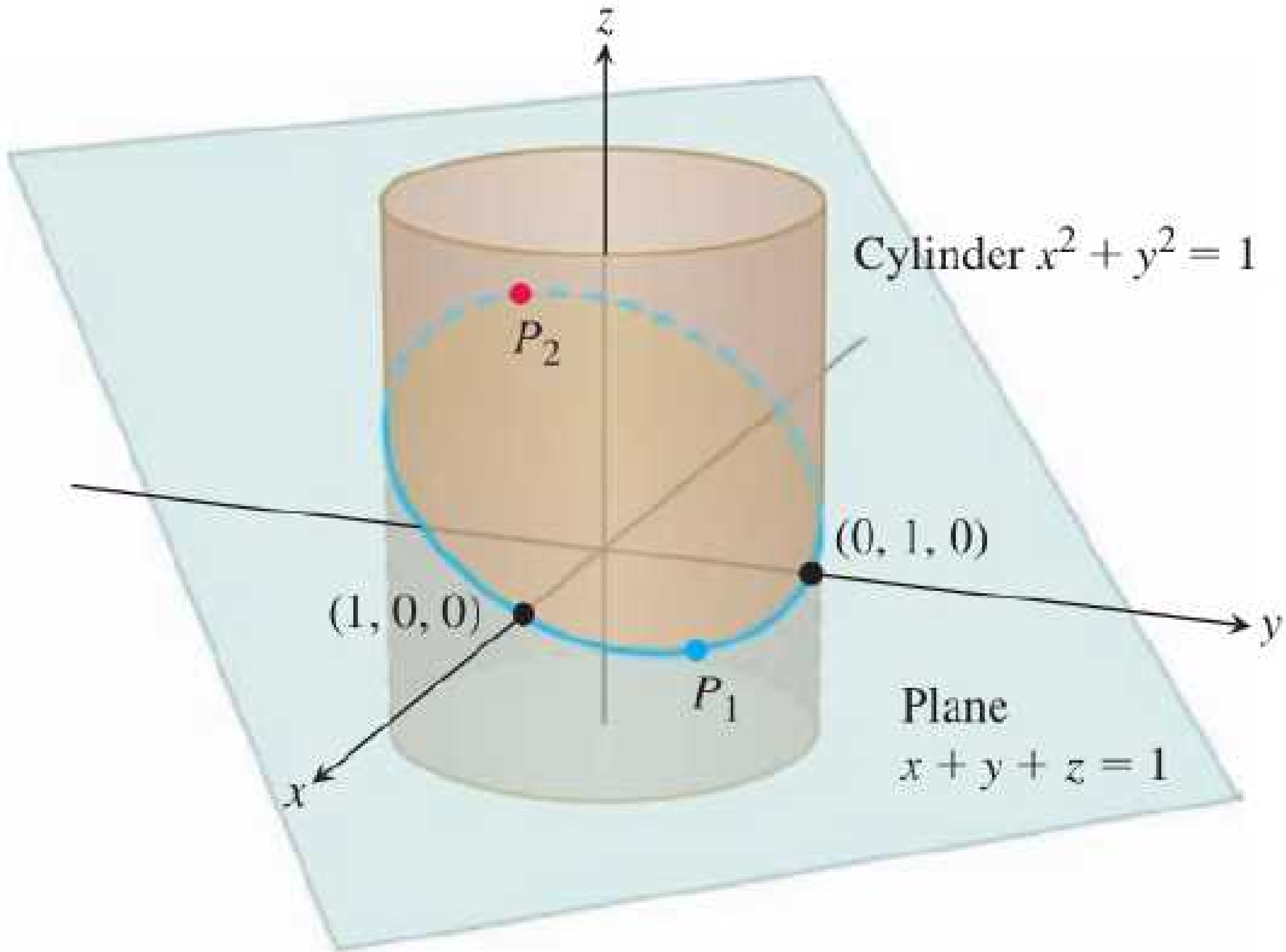
If $z = 0$, the ellipse gives the two points (1, 0, 0) and (0, 1, 0).

$$\boxed{f(1,0.0) = 1, f(0.1.0) = 1}$$

If $x = y$, then The corresponding points on the ellipse are

$$P_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1 - \sqrt{2}\right) \qquad P_2 = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1 + \sqrt{2}\right).$$

$$f(p_1) = 4 - 2\sqrt{2} \qquad \boxed{f(p_2) = 4 + 2\sqrt{2}}$$


z
Cylinder $x^2 + y^2 = 1$
P_2
(0, 1, 0)
(1, 0, 0)
y
P_1
Plane
$x + y + z = 1$
x

14.9

Taylor's Formula for Two Variables

二元函数的泰勒公式

Derivation of the Second Derivative Test

$f(x, y)$ have continuous partial derivatives in an open region R

containing a point $P(a, b)$

where $f_x = f_y = 0$

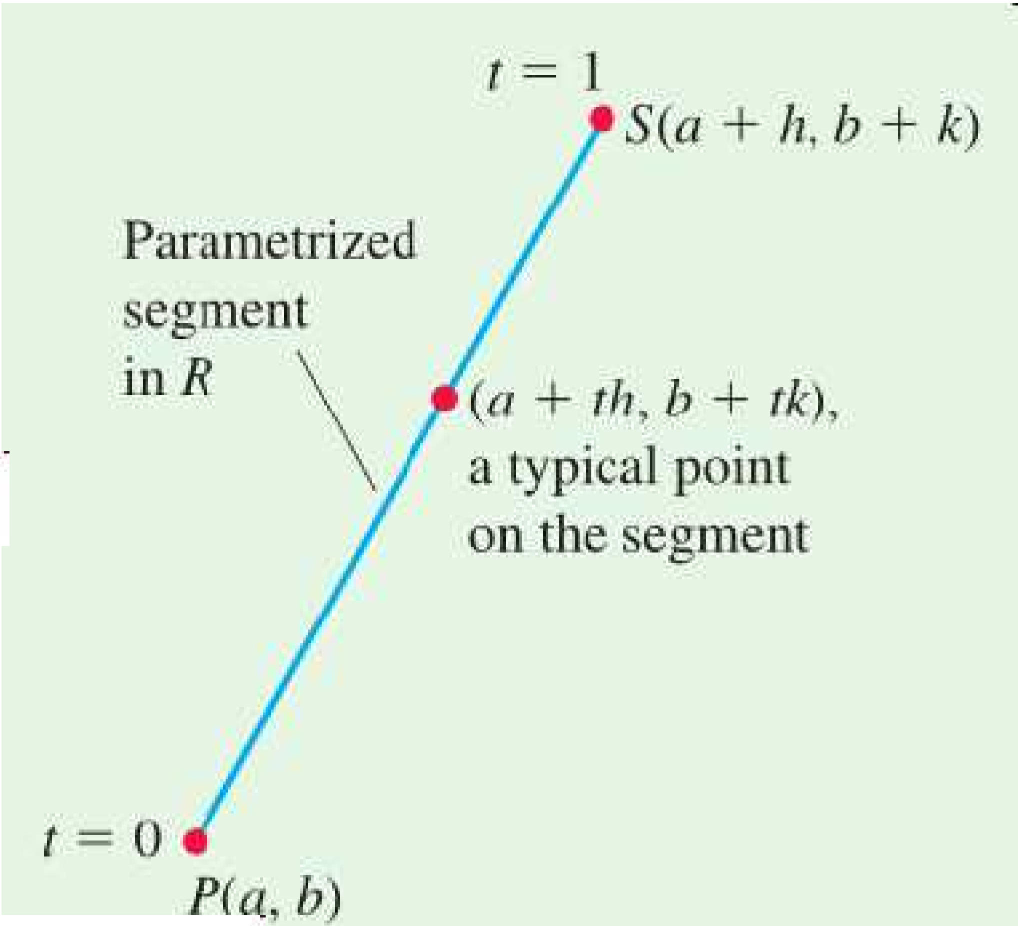
$$f(a+h, b+k) - f(a, b)$$

>0?　　<0?　　符号不定?

point $S(a + h, b + k)$ and the line

segment joining it to P inside R.

segment PS as



$$x = a + th, \qquad y = b + tk, \qquad 0 \le t \le 1.$$

Part of open region R

$$x = a + th, \qquad y = b + tk, \qquad 0 \le t \le 1.$$

$$F(t) = f(a + th, b + tk). \quad f(a+h,b+k) - f(a,b) = F(1) - F(0)$$

$$F(1) = F(0) + F'(0)(1 - 0) + F''(c)\frac{(1-0)^2}{2}$$

$$= F(0) + F'(0) + \frac{1}{2}F''(c) \qquad \text{for some } c \text{ between 0 and 1.}$$

$$F'(t) = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = hf_x + kf_y.$$

$$= hf_x(a+th, b+tk) + kf_y(a+th, b+tk)$$

$$F''(t) = h\frac{df_x(a+th,b+tk)}{dt} + k\frac{df_y(a+th,b+tk)}{dt}$$

$$= h[f_{xx}h + f_{xy}k] + k[f_{yx}h + f_{yy}k]$$

$$= h^2 f_{xx}(a+th,b+tk) + 2khf_{xy}(a+th,b+tk) + k^2 f_{yy}(a+th,b+tk)$$

$$F'(0) = hf_x(a,b) + kf_y(a,b)$$

$$F''(c) = h^2 f_{xx}(a+ch,b+ck) + 2hkf_{xy}(a+ch,b+ck) + k^2 f_{yy}(a+ch,b+ck)$$

$$f(a+h, b+k) = f(a,b) + hf_x(a,b) + kf_y(a,b)$$

二元泰勒公式

$$+ \frac{1}{2}\left(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right)\Big|_{(a+ch,\, b+ck)}.$$

若$f_x(a,b) = f_y(a,b) = 0$,

$$f(a+h, b+k) - f(a,b) = \frac{1}{2}\left(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right)\Big|_{(a+ch,\, b+ck)}.$$

$$Q(c) = (h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy})\big|_{(a+ch,\, b+ck)}.$$

if $Q(0) \neq 0$, the sign of $Q(c)$ will be the same as the sign of $Q(0)$

$$Q(0) = h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)$$

$$Q(0) = k^2\left[f_{xx}(a,b)\left(\frac{h}{k}\right)^2 + 2f_{xy}\frac{h}{k} + f_{yy}\right]$$

$$Q(0) = k^2[f_{xx}(a,b)(\frac{h}{k})^2 + 2f_{xy}\frac{h}{k} + f_{yy}]$$

1. If $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}{}^2 > 0$ at (a, b), then $Q(0) < 0$

   f has a *local maximum* value at (a, b).

2. If $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}{}^2 > 0$ at (a, b), then $Q(0) > 0$

   f has a *local minimum* value at (a, b).

3. If $f_{xx}f_{yy} - f_{xy}{}^2 < 0$ at (a, b),

   对于某些h,k, $Q(0) > 0$, 对于某些h,k, $Q(0) < 0$.

   so f has a *saddle point* at (a, b).

4. If $f_{xx}f_{yy} - f_{xy}{}^2 = 0$, another test is needed.

$$f(x, y) = \underbrace{f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}_{\text{linearization } L(x, y)}$$

$$\underbrace{+ \frac{1}{2}\left((x - x_0)^2 f_{xx} + 2(x - x_0)(y - y_0) f_{xy} + (y - y_0)^2 f_{yy}\right)\Big|_{(x_0 + c(x - x_0),\, y_0 + c(y - y_0))}}_{\text{error } E(x, y)}$$

$$|E| \le \frac{1}{2}\left(|x - x_0|^2 |f_{xx}| + 2|x - x_0||y - y_0||f_{xy}| + |y - y_0|^2 |f_{yy}|\right).$$

if M is an upper bound for the values of $|f_{xx}|$, $|f_{xy}|$, and $|f_{yy}|$ on R,

$$|E| \le \frac{1}{2}\left(|x - x_0|^2 M + 2|x - x_0||y - y_0| M + |y - y_0|^2 M\right)$$

$$= \frac{1}{2} M(|x - x_0| + |y - y_0|)^2.$$

Taylor's Formula for Functions of Two Variables

$$F(t) = f(a + th, b + tk),$$

$$F(1) = F(0) + F'(0) + \frac{F''(0)}{2!} + \frac{F'''(0)}{3!} + \cdots + \frac{F^{(n)}(0)}{n!} + \frac{F^{(n+1)}(c)}{(n+1)!}$$

$$F'(t) = hf_x(a+th, b+tk) + kf_y(a+th, b+tk)$$

$$F'(0) = hf_x(a,b) + kf_y(a,b) = (h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}) f(a,b)$$ operators

$$F''(t) = h^2 f_{xx}(a+th, b+tk) + 2khf_{xy}(a+th, b+tk) + k^2 f_{yy}(a+th, b+tk)$$

$$F''(0) = h^2 f_{xx}(a,b) + 2khf_{xy}(a,b) + k^2 f_{yy}(a,b) = (h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})^2 f(a,b)$$

$$F'''(0) = (h^3 f_{xxx} + 3h^2 k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy})_{(a,b)}$$

$$= (h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})^3 f(a,b)$$

$$F(t) = f(a + th, b + tk),$$

$$F^{(n)}(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a+ht, b+kt)$$

$$F^{(n)}(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a,b)$$

$$F(1) = F(0) + F'(0) + \frac{F''(0)}{2!} + \cdots + \frac{F^{(n)}(0)}{n!} + \text{remainder}.$$

$$f(a+h, b+k) = f(a,b) + (hf_x + kf_y)\big|_{(a,b)} + \frac{1}{2!}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy})\big|_{(a,b)}$$

$$+ \frac{1}{3!}(h^3 f_{xxx} + 3h^2 k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy})\big|_{(a,b)} + \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f\Bigg|_{(a,b)}$$

二元函数的 n阶泰勒公式 $+$ **remainder**

Taylor's Formula for $f(x, y)$ at the Point (a, b)

$f(x, y)$ and its partial derivatives through order $n + 1$ are continuous throughout an open rectangular region R centered at a point (a, b). Then, throughout R,

$$f(a+h, b+k) = f(a, b) + (hf_x + kf_y)\big|_{(a,b)} + \frac{1}{2!}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy})\big|_{(a,b)}$$

$$+ \frac{1}{3!}(h^3 f_{xxx} + 3h^2 k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy})\big|_{(a,b)} + \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f\bigg|_{(a,b)}$$

$$+ \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f\bigg|_{(a+ch,\, b+ck)}.$$

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$+\frac{1}{2}[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2]$$

$$+\frac{1}{3!}[f_{xxx}(x-a)^3 + 3f_{xxy}(x-a)^2(y-b) + 3f_{xyy}(x-a)(y-b)^2 + f_{yyy}(y-b)^3]$$

$$+\cdots\cdots$$

Taylor's Formula for $f(x, y)$ at the Origin

$$f(x,y) = f(0,0) + xf_x + yf_y + \frac{1}{2!}(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy})$$

$$+ \frac{1}{3!}(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}) + \cdots$$

$$\frac{1}{n!}\left(x^n \frac{\partial^n f}{\partial x^n} + nx^{n-1}y \frac{\partial^n f}{\partial x^{n-1}\partial y} + \cdots + y^n \frac{\partial^n f}{\partial y^n}\right)$$

$$+ \frac{1}{(n+1)!}\left(x^{n+1}\frac{\partial^{n+1} f}{\partial x^{n+1}} + (n+1)x^n y \frac{\partial^{n+1} f}{\partial x^n \partial y} + \cdots + y^{n+1}\frac{\partial^{n+1} f}{\partial y^{n+1}}\right)\Bigg|_{(cx,\, cy)}$$

EXAMPLE 1

Find a quadratic approximation to $f(x, y) = \sin x \sin y$ near the origin. How accurate is the approximation if $|x| \le 0.1$ and $|y| \le 0.1$?

Solution We take $n = 2$

$$f(x, y) = f(0, 0) + (xf_x + yf_y) + \frac{1}{2}(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy})$$

$$+ \frac{1}{6}\left(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}\right)_{(cx,\, cy)}.$$

$$f(0, 0) = \sin x \sin y\big|_{(0,0)} = 0, \qquad f_{xx}(0, 0) = -\sin x \sin y\big|_{(0,0)} = 0,$$

$$f_x(0, 0) = \cos x \sin y\big|_{(0,0)} = 0, \qquad f_{xy}(0, 0) = \cos x \cos y\big|_{(0,0)} = 1,$$

$$f_y(0, 0) = \sin x \cos y\big|_{(0,0)} = 0, \qquad f_{yy}(0, 0) = -\sin x \sin y\big|_{(0,0)} = 0,$$

$$\sin x \sin y \approx 0 + 0 + 0 + \frac{1}{2}\left(x^2(0) + 2xy(1) + y^2(0)\right),$$

$$\sin x \sin y \approx xy.$$

$$f_{xxx} = -\cos x \sin y \quad f_{xxy} = -\sin x \cos y \quad f_{xyy} = -\cos x \sin y$$

$$f_{yyy} = -\sin x \cos y$$

$$E(x, y) = \frac{1}{6}\left(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}\right)\Big|_{(cx,\, cy)}.$$

$$|E(x, y)| \le \frac{1}{6}\left((0.1)^3 + 3(0.1)^3 + 3(0.1)^3 + (0.1)^3\right) = \frac{8}{6}(0.1)^3 \le 0.00134$$

14.10

Partial Derivatives with Constrained Variables

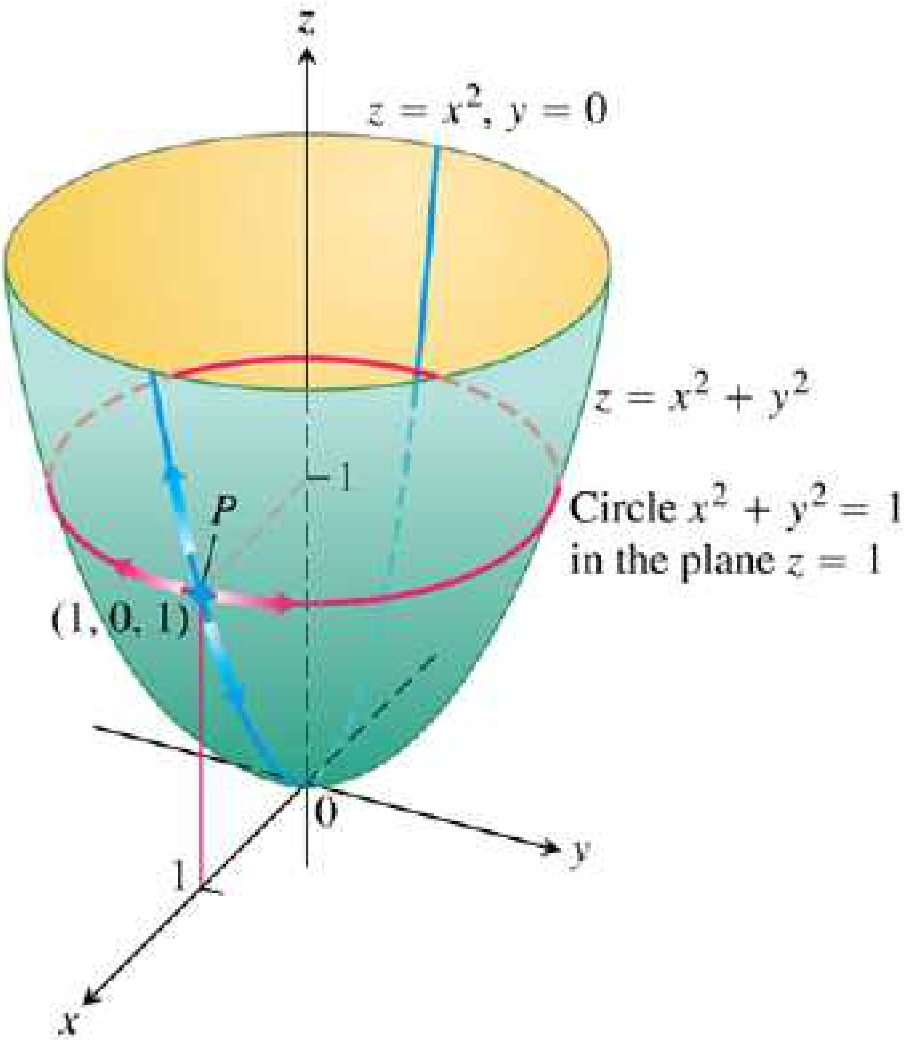


FIGURE 14.61 If P is constrained to lie on the paraboloid $z = x^2 + y^2$, the value of the partial derivative of $w = x^2 + y^2 + z^2$ with respect to x at P depends on the direction of motion (Example 1). (1) As x changes, with $y = 0$, P moves up or down the surface on the parabola $z = x^2$ in the xz-plane with $\partial w/\partial x = 2x + 4x^3$. (2) As x changes, with $z = 1$, P moves on the circle $x^2 + y^2 = 1$, $z = 1$, and $\partial w/\partial x = 0$.

How to Find $\partial w/\partial x$ When the Variables in $w = f(x, y, z)$ Are Constrained by Another Equation

As we saw in Example 1, a typical routine for finding $\partial w/\partial x$ when the variables in the function $w = f(x, y, z)$ are related by another equation has three steps. These steps apply to finding $\partial w/\partial y$ and $\partial w/\partial z$ as well.

1. *Decide* which variables are to be dependent and which are to be independent. (In practice, the decision is based on the physical or theoretical context of our work. In the exercises at the end of this section, we say which variables are which.)
2. *Eliminate* the other dependent variable(s) in the expression for w.
3. *Differentiate* as usual.

EXAMPLE 2 Find $\partial w/\partial x$ at the point $(x, y, z) = (2, -1, 1)$ if

$$w = x^2 + y^2 + z^2, \qquad z^3 - xy + yz + y^3 = 1,$$

and x and y are the independent variables.

Solution It is not convenient to eliminate z in the expression for w. We therefore differentiate both equations implicitly with respect to x, treating x and y as independent variables and w and z as dependent variables. This gives

$$\frac{\partial w}{\partial x} = 2x + 2z\frac{\partial z}{\partial x} \tag{3}$$

and

$$3z^2\frac{\partial z}{\partial x} - y + y\frac{\partial z}{\partial x} + 0 = 0. \tag{4}$$

These equations may now be combined to express $\partial w/\partial x$ in terms of x, y, and z. We solve Equation (4) for $\partial z/\partial x$ to get

$$\frac{\partial z}{\partial x} = \frac{y}{y + 3z^2}$$

and substitute into Equation (3) to get

$$\frac{\partial w}{\partial x} = 2x + \frac{2yz}{y + 3z^2}.$$

The value of this derivative at $(x, y, z) = (2, -1, 1)$ is

$$\left(\frac{\partial w}{\partial x}\right)_{(2,-1,1)} = 2(2) + \frac{2(-1)(1)}{-1 + 3(1)^2} = 4 + \frac{-2}{2} = 3.$$