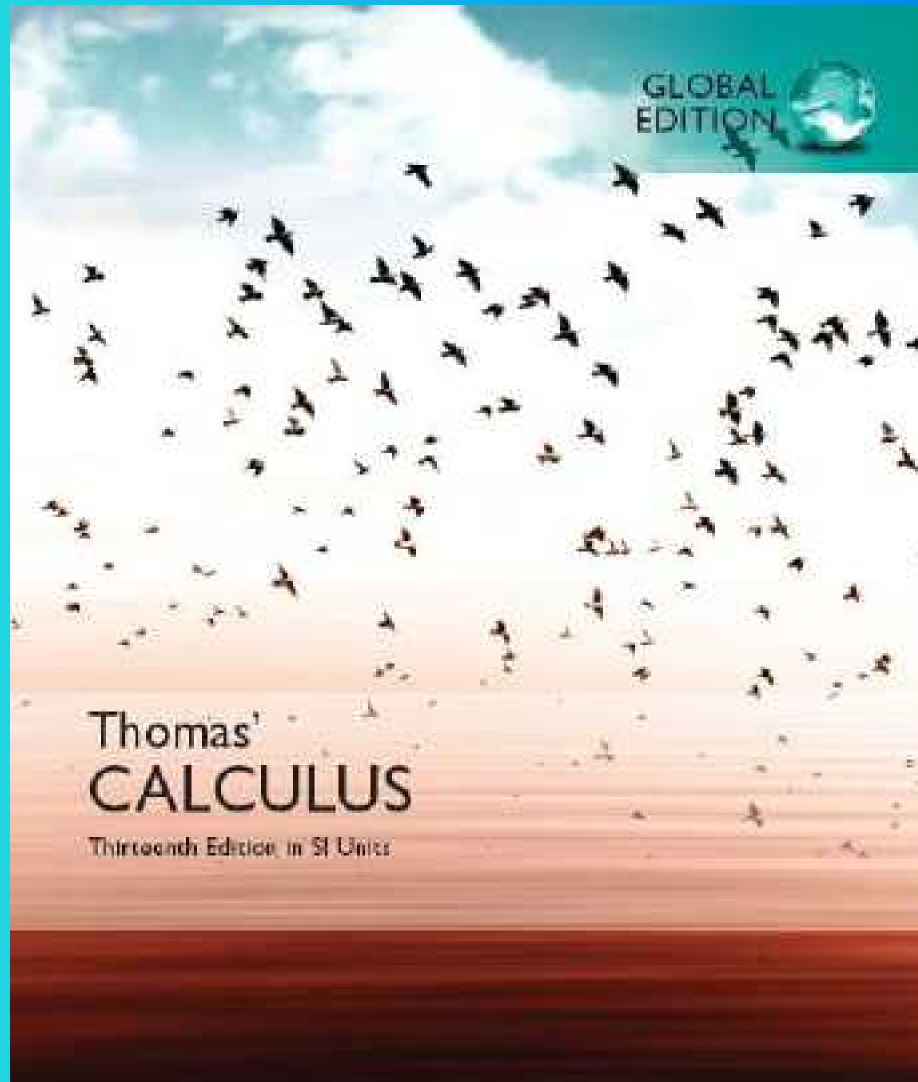
GLOBAL
EDITION
Thomas'
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Thirteenth Edition in SI Units

Chapter 10

Infinite Sequences and Series
无穷数列和级数

10.1

Sequences

数列

割圆术：

正六边形的面积 A_1

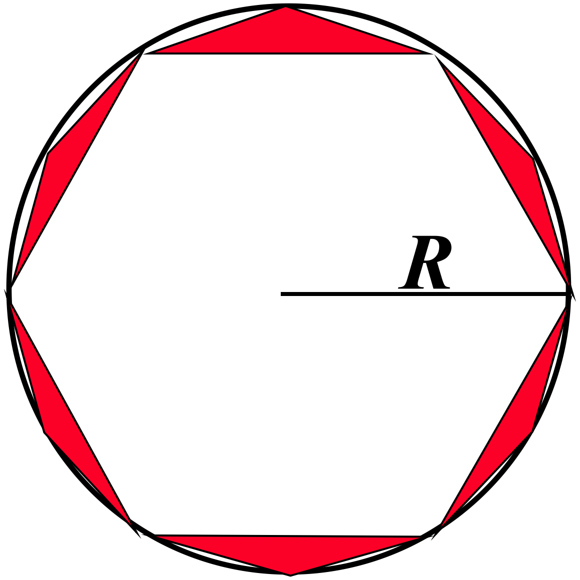
正十二边形的面积 A_2

…… ……

正$6\times 2^{n-1}$形的面积 A_n

$A_1, A_2, A_3, \cdots, A_n, \cdots$

$\Rightarrow S$



Sequences

A sequence is a list of numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$

in a given order.

For example,

$2, 4, 6, 8, 10, 12, \ldots, 2n, \ldots$ $\quad a_n = 2n.$ $\quad$ **index**

$$a_n = \sqrt{n}, \quad b_n = (-1)^{n+1}\frac{1}{n}, \quad c_n = \frac{n-1}{n}, \quad d_n = (-1)^{n+1},$$

$$\{a_n\} = \{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\}$$

$$\{b_n\} = \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1}\frac{1}{n}, \ldots\right\}$$

$$\{c_n\} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n-1}{n}, \ldots\right\}$$

$$\{d_n\} = \{1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}.$$

We also sometimes write a sequence $\{a_n\} = \{\sqrt{n}\}_{n=1}^{\infty}$.

$$a_n = f(n)$$

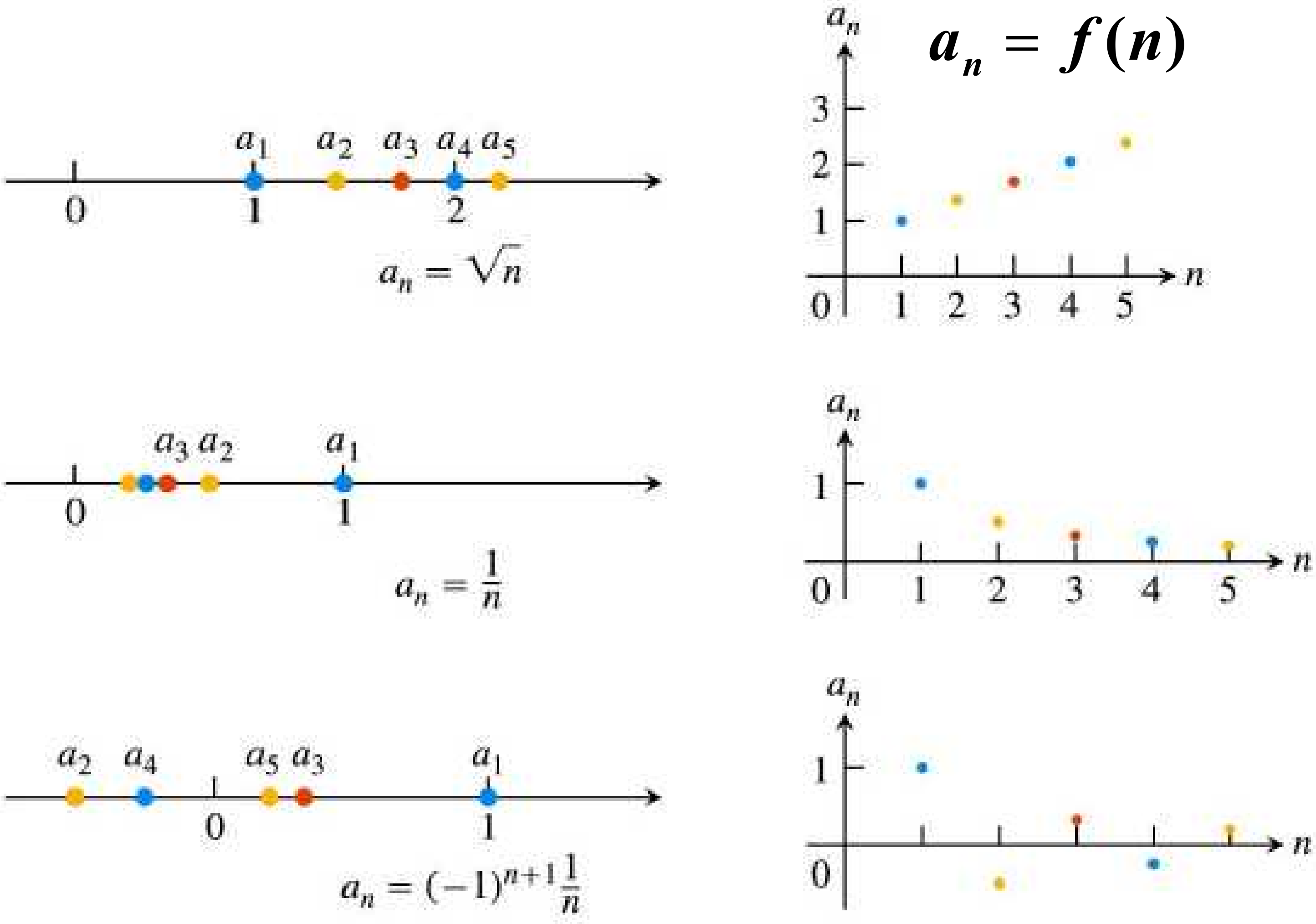


FIGURE 10.1 Sequences can be represented as points on the real line or as points in the plane where the horizontal axis n is the index number of the term and the vertical axis a_n is its value.

Convergence and Divergence

Sometimes the numbers in a sequence approach a single value as the index n increases.

$$\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, 1 - \frac{1}{n}, \ldots\right\}$$

$$\lim_{n\to\infty} a_n = L$$

$$\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\}$$

$$\{1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}$$

$$\lim_{x\to\infty} f(x) = L:$$

$$\forall \varepsilon > 0, \exists B > 0, \text{such that for all } x > B, |f(x) - L| < \varepsilon.$$

$$\lim_{n\to\infty} a_n = L:$$

$$\forall \varepsilon > 0, \exists N > 0, \text{such that for all } n > N, |a_n - L| < \varepsilon.$$

DEFINITIONS The sequence $\{a_n\}$ **converges** to the number L if for every positive number ϵ there corresponds an integer N such that for all n,

$$n > N \quad \Rightarrow \quad |a_n - L| < \epsilon.$$

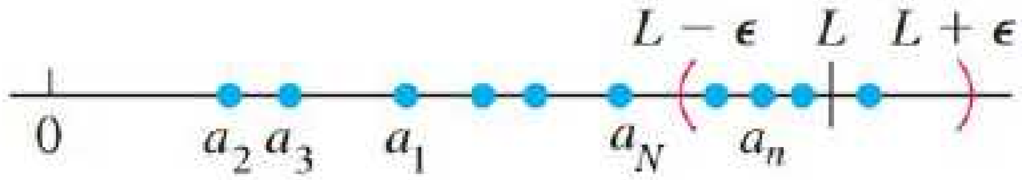
If no such number L exists, we say that $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence (Figure 10.2).

$$\lim_{n\to\infty} a_n = L:$$

$$\forall \varepsilon > 0, \exists N > 0, \text{such that for all } n > N, |a_n - L| < \varepsilon.$$

$$\text{for all } n > N, L - \varepsilon < a_n < L + \varepsilon.$$



EXAMPLE 1 Show that

(a) $\lim_{n\to\infty} \frac{1}{n} = 0$ **(b)** $\lim_{n\to\infty} k = k$ (any constant k)

Solution **(a)** Let $\epsilon > 0$ be given. such that for all n,

$$n > N \quad \Rightarrow \quad \left|\frac{1}{n} - 0\right| < \epsilon.$$

if $(1/n) < \epsilon$ or $n > 1/\epsilon$. If N is any integer greater than $1/\epsilon$, the implication will hold for all $n > N$. **Let** $N = [\frac{1}{\varepsilon}] + 1$

This proves that $\lim_{n\to\infty}(1/n) = 0$.

(b) Let $\epsilon > 0$ be given. an integer N such that for all n,

$$n > N \quad \Rightarrow \quad |k - k| < \epsilon.$$

Since $k - k = 0$, we can use any positive integer for N

This proves that $\lim_{n\to\infty} k = k$ for any constant k.

EXAMPLE 2 Show that the sequence $\{1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}$ diverges.

Solution Suppose the sequence converges to some number L.

By choosing $\epsilon = 1/2$

Since the number 1 appears repeatedly as every other term

$|L - 1| < 1/2$, or equivalently, $1/2 < L < 3/2$.

Likewise, the number -1 appears repeatedly in the sequence

So we must also have

$|L - (-1)| < 1/2$, or equivalently, $-3/2 < L < -1/2$.

Therefore, no such limit L exists and so the sequence diverges.

补充 证明 $\lim_{n\to\infty} q^n = 0$，其中 $|q| < 1$.

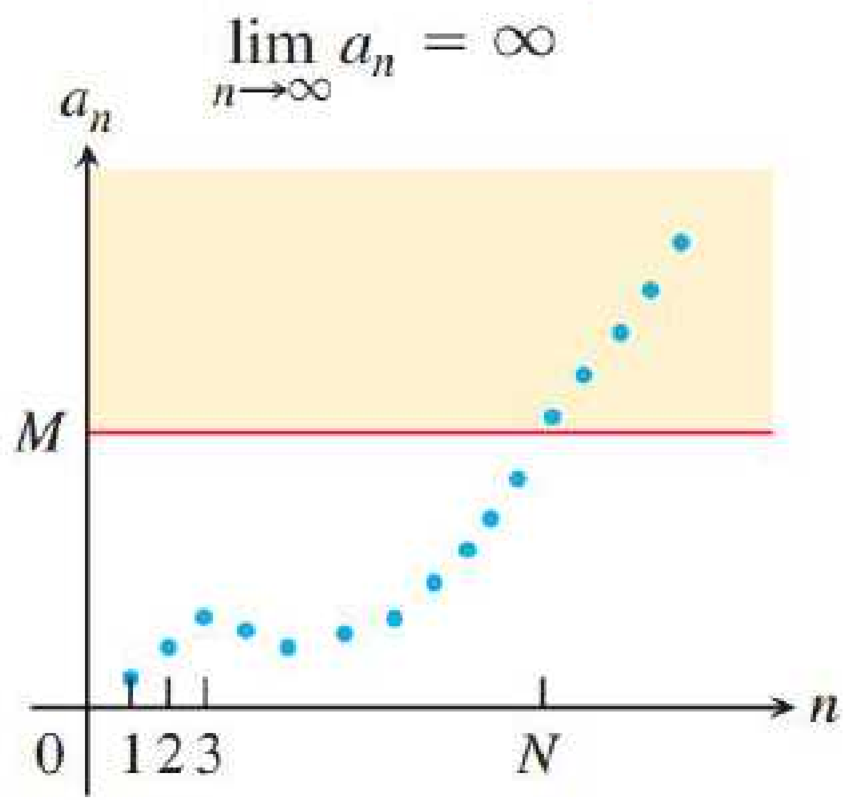
证 任给 $\varepsilon > 0$, 若 $q = 0$, 则 $\lim_{n\to\infty} q^n = \lim_{n\to\infty} 0 = 0$;

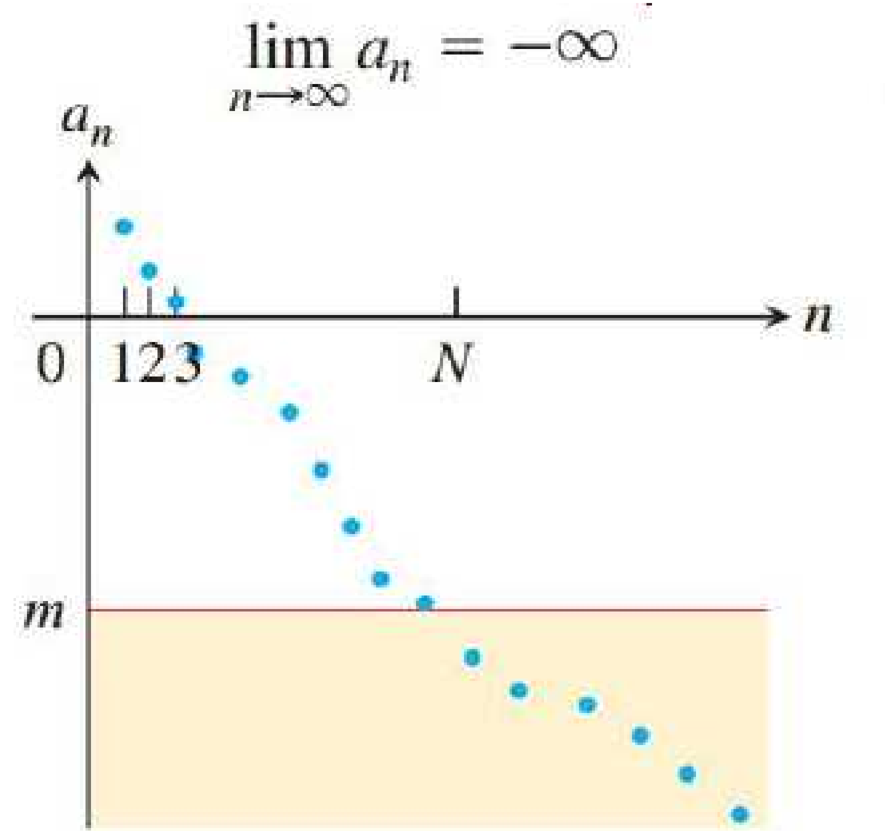
若 $0 < |q| < 1$, **要使** $|q^n - 0| < \varepsilon$, $n \ln|q| < \ln \varepsilon$,

只须 $n > \dfrac{\ln \varepsilon}{\ln|q|}$, **取** $N = [\dfrac{\ln \varepsilon}{\ln|q|}] + 1$, 则当 $n > N$ 时,

就有 $|q^n - 0| < \varepsilon$, $\therefore \lim_{n\to\infty} q^n = 0.$

$$\lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0, \quad \lim_{n\to\infty} \left(-\frac{1}{\sqrt{2}}\right)^n = 0,$$





$$\lim_{n\to\infty} a_n = \infty :$$

$$\forall M > 0, \exists N > 0, \text{such that for all } n > N, a_n > M.$$

DEFINITION The sequence $\{a_n\}$ **diverges to infinity** if for every number M there is an integer N such that for all n larger than N, $a_n > M$. If this condition holds we write

$$\lim_{n \to \infty} a_n = \infty \qquad \text{or} \qquad a_n \to \infty.$$

Similarly if for every number m there is an integer N such that for all $n > N$ we have $a_n < m$, then we say $\{a_n\}$ **diverges to negative infinity** and write

$$\lim_{n \to \infty} a_n = -\infty \qquad \text{or} \qquad a_n \to -\infty.$$

$\{1, -2, 3, -4, 5, -6, 7, -8, \ldots\}$ $\qquad$ $\{1, 0, 2, 0, 3, 0, \ldots\}$

Calculating Limits of Sequences

THEOREM 1 Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (any number k)
4. *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
5. *Quotient Rule:* $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$ if $B \neq 0$

EXAMPLE 3

(a) $\lim_{n\to\infty}\left(-\frac{1}{n}\right) = -1 \cdot \lim_{n\to\infty}\frac{1}{n} = -1 \cdot 0 = 0$

(b) $\lim_{n\to\infty}\left(\frac{n-1}{n}\right) = \lim_{n\to\infty}\left(1-\frac{1}{n}\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty}\frac{1}{n} = 1$

(c) $\lim_{n\to\infty}\frac{5}{n^2} = 5 \cdot \lim_{n\to\infty}\frac{1}{n} \cdot \lim_{n\to\infty}\frac{1}{n} = 5 \cdot 0 \cdot 0 = 0$

(d) $\lim_{n\to\infty}\frac{4-7n^6}{n^6+3} = \lim_{n\to\infty}\frac{(4/n^6)-7}{1+(3/n^6)} = \frac{0-7}{1+0} = -7.$

$\{a_n\} = \{1, 2, 3, \ldots\}$ and $\{b_n\} = \{-1, -2, -3, \ldots\}$

$\{a_n + b_n\} = \{0, 0, 0, \ldots\}$ Be cautious

THEOREM 2—The Sandwich Theorem for Sequences

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers.

If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N, and if

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

if $|b_n| \leq c_n$ and $c_n \to 0$, then $b_n \to 0$ because $-c_n \leq b_n \leq c_n$.

EXAMPLE 4 Since $1/n \to 0$, we know that

(a) $\dfrac{\cos n}{n} \to 0$ because $-\dfrac{1}{n} \leq \dfrac{\cos n}{n} \leq \dfrac{1}{n}$;

(b) $\frac{1}{2^n} \to 0$ because $0 \le \frac{1}{2^n} \le \frac{1}{n}$;

(c) $(-1)^n \frac{1}{n} \to 0$ because $-\frac{1}{n} \le (-1)^n \frac{1}{n} \le \frac{1}{n}$.

THEOREM 3—The Continuous Function Theorem for Sequences

If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

EXAMPLE 5 Show that $\sqrt{(n+1)/n} \to 1$.

Solution We know that $(n+1)/n \to 1$. $\sqrt{x}$ is continuous at 1.

$$\sqrt{(n+1)/n} \to \sqrt{1} = 1.$$

EXAMPLE 6 The sequence $\{1/n\}$ converges to 0.

$f(x) = 2^x$, $2^{1/n} = f(1/n) \to f(L) = 2^0 = 1.$

$$\lim_{n\to\infty} x^{\frac{1}{n}} = 1 \, (x > 0).$$

THEOREM 4

Suppose that $f(x)$ is a function defined for all $x \geq n_0$

$a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x\to\infty} f(x) = L \qquad \Rightarrow \qquad \lim_{n\to\infty} a_n = L.$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e \Rightarrow \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2} \Rightarrow \lim_{n\to\infty} \tan^{-1} n = \frac{\pi}{2}$$

Proof for each positive number ϵ there is a

number M such that for all x, $x > M \Rightarrow |f(x) - L| < \epsilon$.

Let N be an integer greater than M Then

$$n > N \Rightarrow |a_n - L| = |f(n) - L| < \epsilon \qquad \lim_{n\to\infty} a_n = L.$$

Using L'Hôpital's Rule

EXAMPLE 7 Show that $\lim_{n\to\infty} \frac{\ln n}{n} = 0.$

Solution $$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \frac{0}{1} = 0.$$

We conclude that $\lim_{n\to\infty} \frac{\ln n}{n} = 0.$

EXAMPLE 8 Does the sequence whose nth term is

$$a_n = \left(\frac{n+1}{n-1}\right)^n$$

converge? If so, find $\lim_{n\to\infty} a_n$.

Solution The limit leads to the indeterminate form 1^∞.

$$\ln a_n = \ln\left(\frac{n+1}{n-1}\right)^n = n\ln\left(\frac{n+1}{n-1}\right).$$

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} n\ln\left(\frac{n+1}{n-1}\right) = \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{1/n}$$

$$= \lim_{n\to\infty} \frac{-2/(n^2-1)}{-1/n^2} = \lim_{n\to\infty} \frac{2n^2}{n^2-1} = 2.$$

$$a_n = e^{\ln a_n} \to e^2.$$

Commonly Occurring Limits

THEOREM 5

The following six sequences converge to the limits listed below:

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0 \qquad \lim_{n\to\infty} \sqrt[n]{n} = 1$$

$$\lim_{n\to\infty} x^{1/n} = 1 \quad (x > 0) \qquad \lim_{n\to\infty} x^n = 0 \quad (|x| < 1)$$

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\text{any } x) \qquad \lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad (\text{any } x)$$

$$(1+\frac{x}{n})^n = [(1+\frac{x}{n})^{\frac{n}{x}}]^x \to e^x \qquad 0 < \frac{2^n}{n!} \le \frac{4}{n} \to 0$$

EXAMPLE 9

(a) $\dfrac{\ln(n^2)}{n} = \dfrac{2\ln n}{n} \to 2 \cdot 0 = 0$

(b) $\sqrt[n]{n^2} = n^{2/n} = (n^{1/n})^2 \to (1)^2 = 1$

(c) $\sqrt[n]{3n} = 3^{1/n}(n^{1/n}) \to 1 \cdot 1 = 1$

(d) $\left(-\dfrac{1}{2}\right)^n \to 0$

(e) $\left(\dfrac{n-2}{n}\right)^n = \left(1 + \dfrac{-2}{n}\right)^n \to e^{-2}$

(f) $\dfrac{100^n}{n!} \to 0$

Recursive Definitions

sequences are often defined **recursively** by giving

1. The value(s) of the initial term or terms, and
2. A rule, called a **recursion formula**,

for calculating any later term from terms that precede it.

EXAMPLE 10

(a) The statements $a_1 = 1$ and $a_n = a_{n-1} + 1$ for $n > 1$

we have $a_2 = a_1 + 1 = 2$, $a_3 = a_2 + 1 = 3$, and so on.

(b) The statements $a_1 = 1$ and $a_n = n \cdot a_{n-1}$ for $n > 1$ we have $a_2 = 2 \cdot a_1 = 2$, $a_3 = 3 \cdot a_2 = 6$, $a_4 = 24$, and so on.

(c) The statements $a_1 = 1$, $a_2 = 1$, and $a_{n+1} = a_n + a_{n-1}$ for $n > 2$ we have 1, 1, 2, 3, 5, . . . of **Fibonacci numbers**.

Bounded Monotonic Sequences

DEFINITIONS A sequence $\{a_n\}$ is **bounded from above** if there exists a number M such that $a_n \leq M$ for all n. The number M is an **upper bound** for $\{a_n\}$

the **least upper bound** for $\{a_n\}$ **supremum**

no number less than M is an upper bound for $\{a_n\}$

A sequence $\{a_n\}$ is **bounded from below** if there exists a number m such that $a_n \geq m$ for all n. The number m is a **lower bound** for $\{a_n\}$.

the **greatest lower bound** for $\{a_n\}$ **infimum**

no number greater than m is a lower bound for $\{a_n\}$

EXAMPLE 11

(a) The sequence $1, 2, 3, \ldots, n, \ldots$ has no upper bound

it is bounded below by every real number less than or equal to 1

The number $m = 1$ is the greatest lower bound

(b) The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$ is bounded above by

every real number greater than or equal to 1.

$M = 1$ is the least upper bound

also bounded below by every number less than or equal to $\frac{1}{2}$,

$\frac{1}{2}$ is its greatest lower bound.

If a sequence $\{a_n\}$ converges to the number L, then

$|a_n - L| < 1$ if $n > N$. $L - 1 < a_n < L + 1 \quad$ for $n > N$.

If M is a number larger than $L + 1$ and all of $a_1, a_2, \ldots, a_N$,

then for every index n we have $a_n \le M$ is bounded from above.

Similarly, $\{a_n\}$ is **bounded from below**

Therefore, all convergent sequences are bounded.

$\{(-1)^{n+1}\}$ $\{\sin n\}$ 收敛的数列必有界。

DEFINITIONS

单调不减

A sequence $\{a_n\}$ is **nondecreasing** if $a_n \leq a_{n+1}$ for all n.

单调不增

The sequence is **nonincreasing** if $a_n \geq a_{n+1}$ for all n.

The sequence $\{a_n\}$ is **monotonic** if it is either nondecreasing or nonincreasing.

EXAMPLE 12

(a) The sequence $1, 2, 3, \ldots, n, \ldots$ is nondecreasing.

(b) The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$ is nondecreasing.

(c) The sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots$ is nonincreasing.

(d) The constant sequence 3, 3, 3, . . . , 3, . . . is both nondecreasing and nonincreasing.

(e) The sequence 1, −1, 1, −1, 1, −1, . . . is not monotonic.

THEOREM 6—The Monotonic Sequence Theorem

If a sequence $\{a_n\}$ is both bounded and monotonic, then the sequence converges.

A nondecreasing sequence that is bounded from above always has a least upper bound.
a nonincreasing sequence bounded from below always has a greatest lower bound.

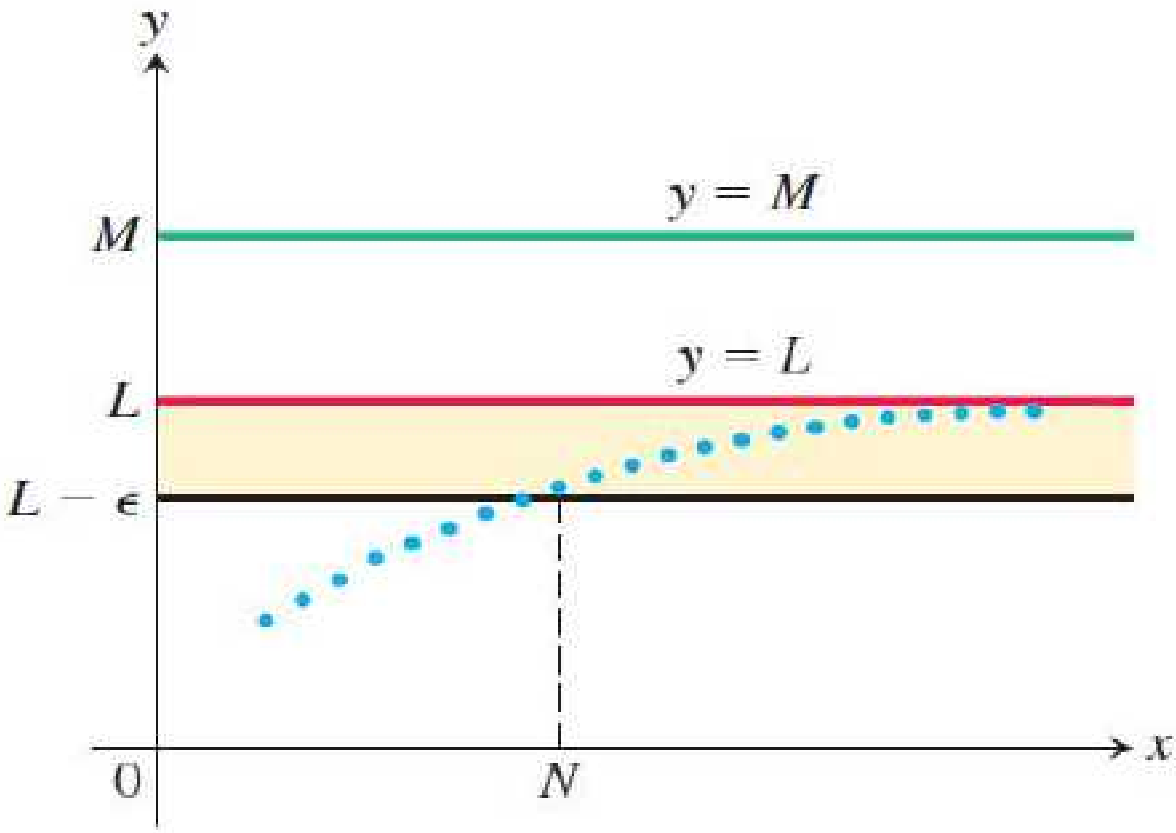


FIGURE 10.7 If the terms of a nondecreasing sequence have an upper bound M, they have a limit $L \leq M$.

Proof Suppose $\{a_n\}$ is nondecreasing, L is its least upper bound,

if ϵ is a positive number.

a. $a_n \le L$ for *all* values of n, and $a_n < L + \varepsilon$

b. given any $\epsilon > 0$, there exists at least one integer N

for which $a_N > L - \epsilon$.

The fact that $\{a_n\}$ is nondecreasing tells us further that

$$a_n \ge a_N > L - \epsilon \qquad \text{for all } n \ge N. \quad |a_n - L| < \epsilon.$$

The sequence converges to L

The proof for nonincreasing sequences bounded from below is similar.

$\{(-1)^{n+1}/n\}$ converges and is bounded, but it is not monotonic

Ex.13 Find the limit for the sequence $\{a_n\}$:

$$a_1 = \sqrt{1}, \quad a_{n+1} = \sqrt{1 + a_n}, n = 1,2,\cdots.$$

Solution: $a_1 = \sqrt{1}, \quad a_{n+1} = \sqrt{1 + a_n}, n = 1,2,\cdots.$

$$a_{n+1} - a_n = \sqrt{1 + a_n} - \sqrt{1 + a_{n-1}} = \frac{a_n - a_{n-1}}{\sqrt{1 + a_n} + \sqrt{1 + a_{n-1}}}$$

$\because a_2 > a_1$, 设$a_n > a_{n-1}$ 则$a_{n+1} > a_n$ **increasing**

$\because a_1 < 2$, 设$a_n < 2$ 则$a_{n+1} < 2$ **bounded from above**

increasing and bounded from above $\Rightarrow$ **converge**

Let $\lim\limits_{n\to\infty} a_n = l$, $\quad l = \sqrt{1 + l}$, $\quad l = \dfrac{1+\sqrt{5}}{2}$.

10.2

Infinite Series
无穷级数

An *infinite series* is the sum of an infinite sequence

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

The sum of the first n terms $s_n = a_1 + a_2 + a_3 + \cdots + a_n$

the *nth partial sum*

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

$$s_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} \qquad s_n = 2 - \frac{1}{2^{n-1}}.$$

$$\lim_{n\to\infty} s_n = 2 \qquad 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \lim_{n\to\infty} s_n = 2$$

DEFINITIONS Given a sequence of numbers $\{a_n\}$

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$ is an **infinite series.**

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

is the **sequence of partial sums** of the series,

If the sequence of partial sums converges to a limit L, $\lim_{n\to\infty} s_n = L$

we say that the series **converges** and that its **sum** is L.

we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums does not converge,

we say that the series **diverges.**

$$\sum_{n=1}^{\infty} a_n, \qquad \sum_{k=1}^{\infty} a_k, \qquad \text{or} \qquad \sum a_n$$

Geometric Series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

a and r are fixed real numbers and $a \neq 0$.

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots ,$$

$$1 - \frac{1}{3} + \frac{1}{9} - \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots .$$

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$rs_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

$$s_n = \frac{a(1 - r^n)}{1 - r}, \qquad (r \neq 1).$$

If $|r| < 1$, $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges to $a/(1 - r)$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \qquad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

EXAMPLE 1

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots = \sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{1}{3}\right)^{n-1} = \frac{1/9}{1-(1/3)} = \frac{1}{6}.$$

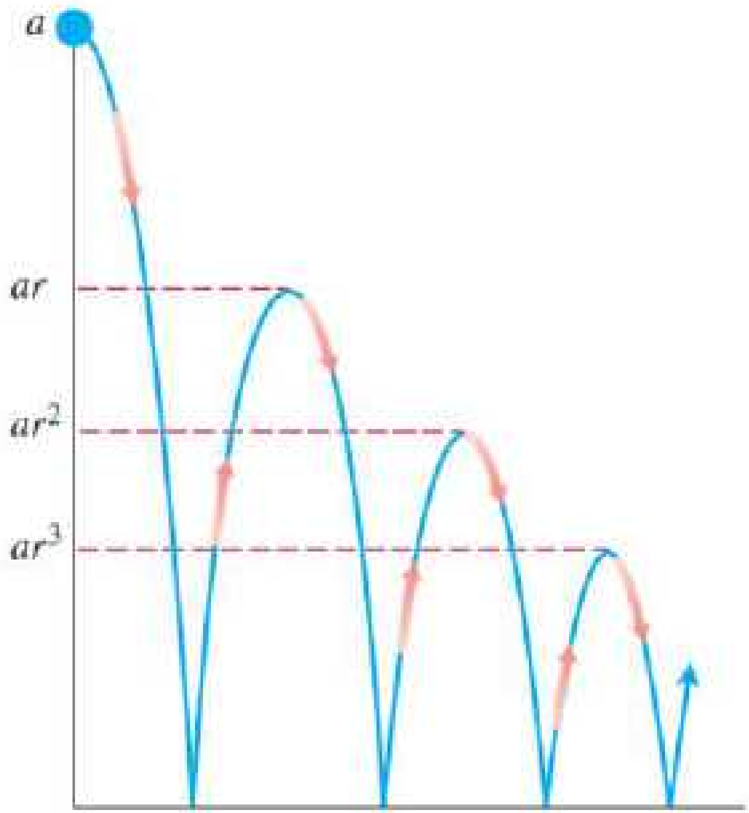
EXAMPLE 2 $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = \frac{5}{1+(1/4)} = 4$

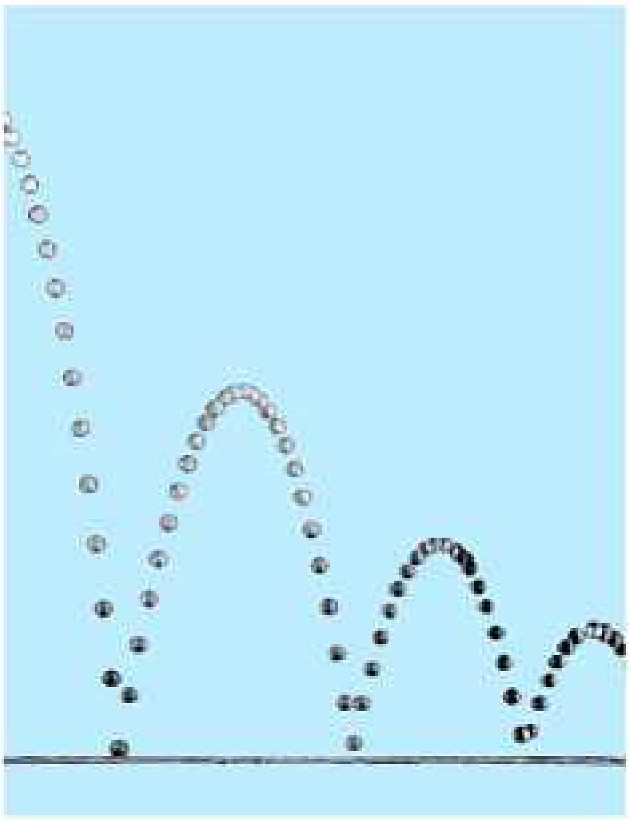
EXAMPLE 3 You drop a ball from a meters above a flat surface. Each time the ball hits the surface after falling a distance h, it rebounds a distance rh, where r is positive but less than 1. Find the total distance the ball travels up and down

Solution The total distance is

$$s = a + 2ar + 2ar^2 + 2ar^3 + \cdots$$

$$= a + \frac{2ar}{1 - r} = a\frac{1 + r}{1 - r}.$$





(b)

FIGURE 10.9 (a) Example 3 shows how to use a geometric series to calculate the total vertical distance traveled by a bouncing ball if the height of each rebound is reduced by the factor r. (b) A stroboscopic photo of a bouncing ball. (*Source: PSSC Physics*, 2nd ed., Reprinted by permission of Educational Development Center, Inc.)

EXAMPLE 4 Express the repeating decimal 5.232323 . . . as the ratio of two integers.

Solution

$$5.232323\ldots = 5 + \frac{23}{100} + \frac{23}{(100)^2} + \frac{23}{(100)^3} + \cdots$$

$$= 5 + \frac{23}{100}\left(1 + \frac{1}{100} + \left(\frac{1}{100}\right)^2 + \cdots\right)$$

$$= 5 + \frac{23}{100}\left(\frac{1}{0.99}\right) = 5 + \frac{23}{99} = \frac{518}{99}$$

EXAMPLE 5 Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Solution $$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

$$\sum_{n=1}^{k} \frac{1}{n(n+1)} = \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$s_k = \left(\frac{1}{1} - \cancel{\frac{1}{2}} \right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} \right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} \right) + \cdots + \left(\cancel{\frac{1}{k}} - \frac{1}{k+1} \right)$$

$$s_k = 1 - \frac{1}{k+1}. \rightarrow 1 \text{ as } k \rightarrow \infty.$$

The series converges, and its sum is 1: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$

The nth-Term Test for a Divergent Series

THEOREM 7 If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

if the series $\sum_{n=1}^{\infty} a_n$ converges

let S represent the series' sum and $s_n = a_1 + a_2 + \cdots + a_n$

both s_n and s_{n-1} are close to S,

$$a_n = s_n - s_{n-1} \quad \to \quad S - S = 0.$$

EXAMPLE 6 The series $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because

$$\frac{n+1}{n} \to 1$$

EXAMPLE 7

(a) $\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \to \infty$.

(b) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\frac{n+1}{n} \to 1$.

(c) $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n\to\infty} (-1)^{n+1}$ does not exist.

(d) $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges because $\lim_{n\to\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0$.

EXAMPLE 8

$$1 + \underbrace{\frac{1}{2} + \frac{1}{2}}_{2 \text{ terms}} + \underbrace{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}}_{4 \text{ terms}} + \cdots + \underbrace{\frac{1}{2^n} + \frac{1}{2^n} + \cdots + \frac{1}{2^n}}_{2^n \text{ terms}} + \cdots$$

$$1 + 1 + 1 + \cdots + 1 + \cdots \qquad \text{diverges}$$

Combining Series

THEOREM 8 If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. *Sum Rule:* $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. *Difference Rule:* $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. *Constant Multiple Rule* $\sum ka_n = k\sum a_n = kA$ (any number k).

Proof $A_n = a_1 + a_2 + \cdots + a_n, \quad B_n = b_1 + b_2 + \cdots + b_n.$

$$\begin{aligned} s_n &= (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n) \\ &= A_n + B_n. \end{aligned}$$

Since $A_n \to A$ and $B_n \to B$, we have $s_n \to A + B$

As corollaries of Theorem 8,

1. Every nonzero constant multiple of a divergent series diverges.
2. If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ both diverge.

EXAMPLE 9 Find the sums of the following series.

(a) $$\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}} \right) = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}}$$

$$= \frac{1}{1-(1/2)} - \frac{1}{1-(1/6)} = \frac{4}{5}$$

(b) $$\sum_{n=0}^{\infty} \frac{4}{2^n} = 4\sum_{n=0}^{\infty} \frac{1}{2^n} = 4\left(\frac{1}{1-(1/2)} \right) = 8$$

Adding or Deleting Terms

We can add a finite number of terms to a series or delete a finite number of terms without altering the series' convergence

The convergence or divergence of a series is not affected by its first few terms.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_{k-1} + \sum_{n=k}^{\infty} a_n.$$

Only the "tail" of the series influences whether it converges or diverges.

判断 $\sum_{n=1}^{\infty}\frac{9}{2^{n+10}}$ 的敛散性.

判断 $\sum_{n=-2}^{\infty}\left(\frac{(-1)^n}{2^{n+1}}+\frac{1}{3^n}\right)$ 的敛散性.

判断 $\sum_{n=10}^{\infty}\frac{1}{(1+\frac{1}{n})^n}$ 的敛散性.

EXAMPLE 10 We can write the geometric series

$$\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}=1+\frac{1}{2}+\frac{1}{4}+\cdots$$

as $\sum_{n=0}^{\infty}\frac{1}{2^n}$, $\sum_{n=5}^{\infty}\frac{1}{2^{n-5}}$, or even $\sum_{n=-4}^{\infty}\frac{1}{2^{n+4}}$.

10.3

The Integral Test
积分判别法

Nondecreasing Partial Sums

Suppose that $\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n \geq 0$ for all n.

$$s_{n+1} = s_n + a_n, \text{ so}$$

$$s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_n \leq s_{n+1} \leq \cdots.$$

COROLLARY OF THEOREM 6

A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums are bounded from above.

EXAMPLE 1 the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

Although the nth term $1/n$ does go to zero, the series diverges

$$1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{>\frac{2}{4}=\frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{>\frac{4}{8}=\frac{1}{2}} + \underbrace{\left(\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right)}_{>\frac{8}{16}=\frac{1}{2}} + \cdots$$

$$s_{2^n} > 1 + \frac{n}{2}$$

so the sequence of partial sums is not bounded from above. The harmonic series diverges.

The Integral Test

EXAMPLE 2 Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots$$

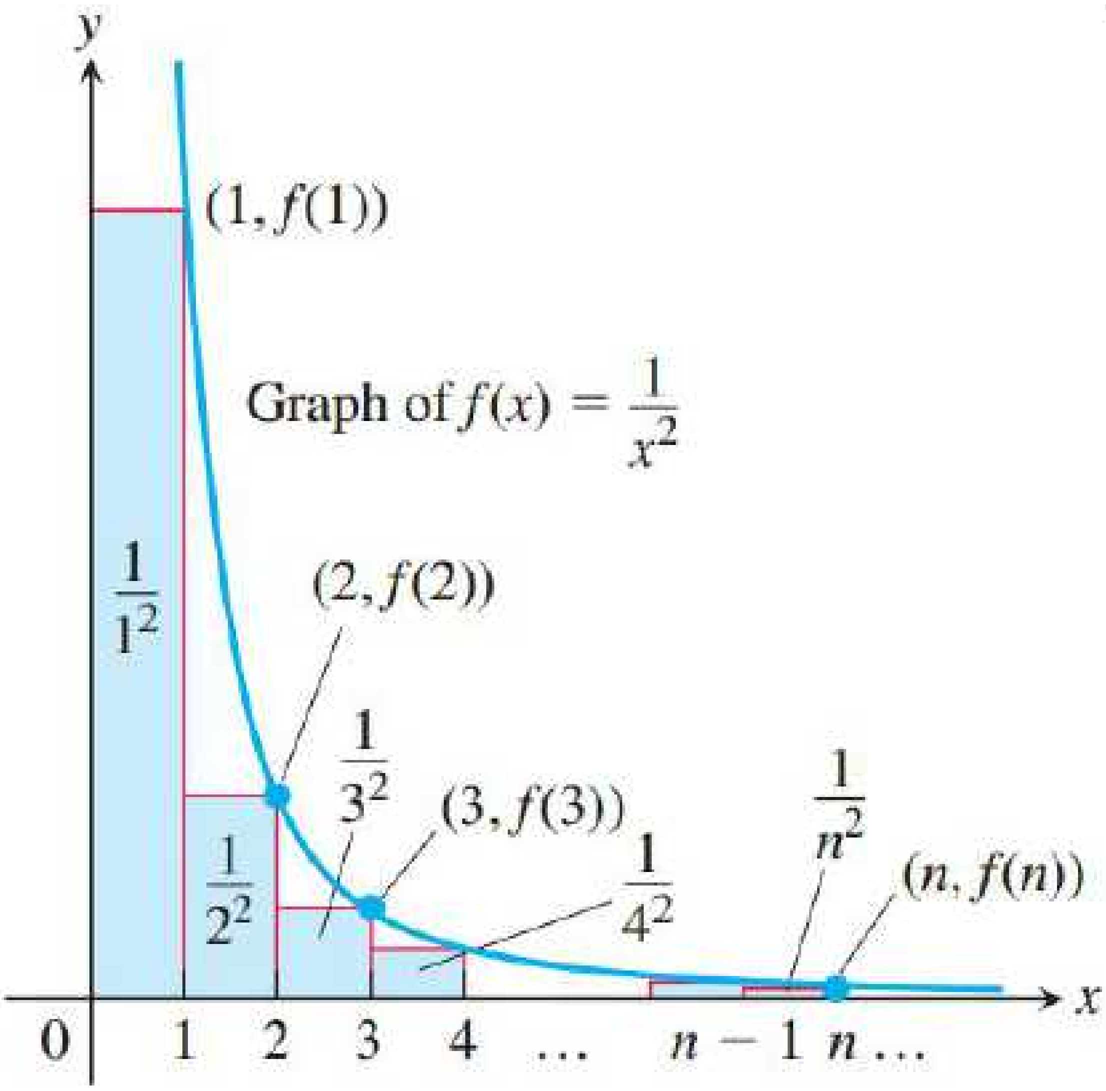
Solution by comparing it with $\int_1^{\infty} (1/x^2)\, dx$.

$$s_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$$

$$< 1 + \int_1^{\infty} \frac{1}{x^2} dx = 2.$$

s_n are bounded from above (by 2)

the series converges.



y
(1, f(1))
Graph of $f(x) = \frac{1}{x^2}$
$\frac{1}{1^2}$
(2, f(2))
$\frac{1}{3^2}$
(3, f(3))
$\frac{1}{n^2}$
$\frac{1}{2^2}$
$\frac{1}{4^2}$
(n, f(n))
x
0 1 2 3 4 … n − 1 n …

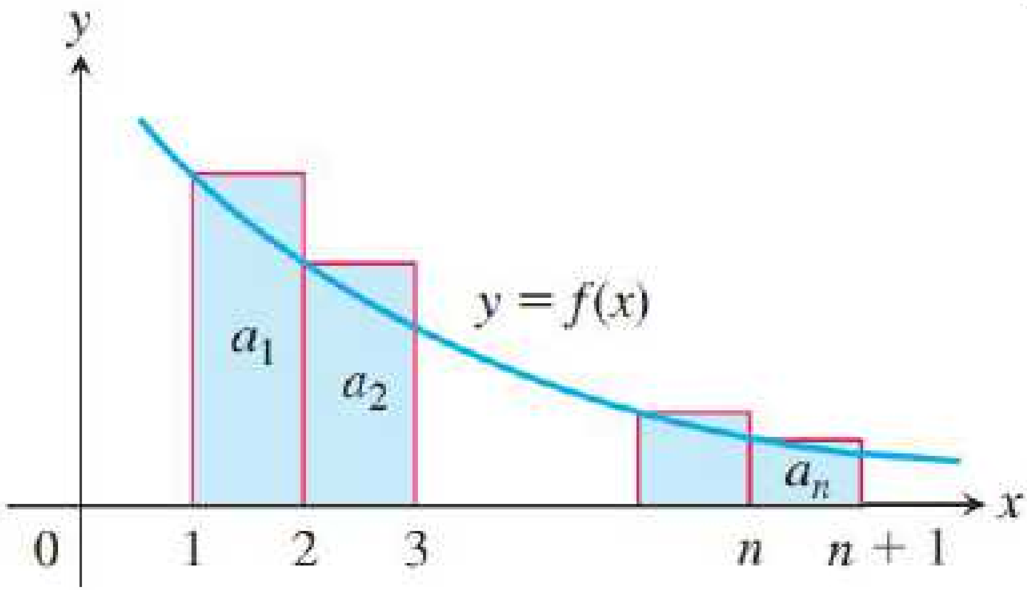
THEOREM 9—The Integral Test Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

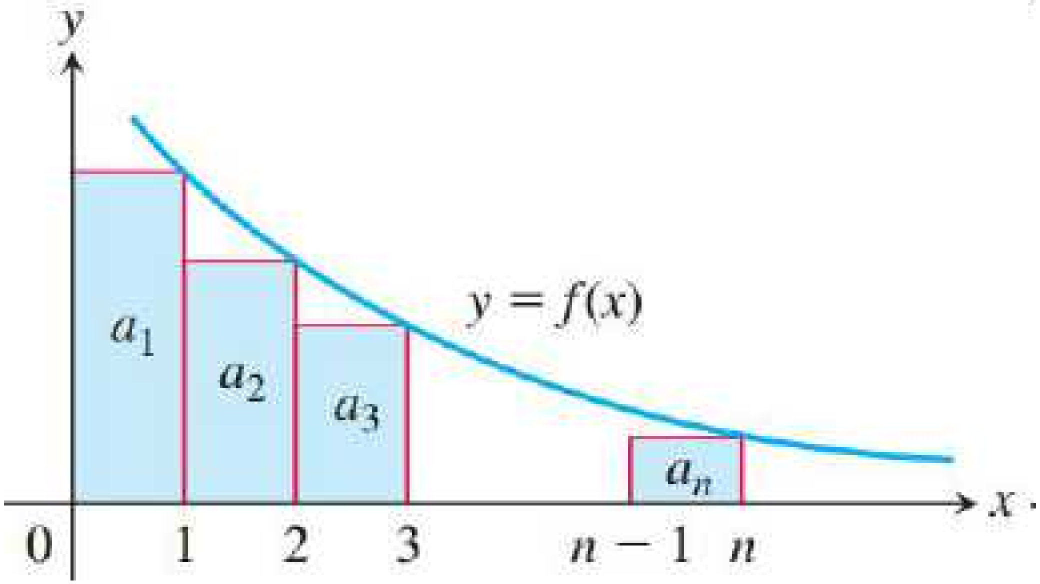
Proof

$$\int_1^{n+1} f(x)\,dx \leq a_1 + a_2 + \cdots + a_n.$$

$$a_2 + a_3 + \cdots + a_n \leq \int_1^{n} f(x)\,dx.$$

$$\int_1^{n+1} f(x)\,dx \leq a_1 + a_2 + \cdots + a_n \leq a_1 + \int_1^{n} f(x)\,dx.$$


y
y = f(x)
a1
a2
an
x
0
1
2
3
n
n + 1


y
y = f(x)
a1
a2
a3
an
x
0
1
2
3
n − 1
n

EXAMPLE 3 Show that the ***p*-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

(p a real constant) converges if $p > 1$, and diverges if $p \leq 1$.

Solution

If $p > 1$, then $f(x) = 1/x^p$ is a positive decreasing function

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \int_1^{\infty} x^{-p}\,dx = \lim_{b\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^b = \frac{1}{p-1},$$

the series converges

If $p \leq 0$, the series diverges by the nth-term test.

If $0 < p < 1$, then $1 - p > 0$ and

$$\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{1-p}\lim_{b\to\infty}(b^{1-p} - 1) = \infty.$$

The series diverges

If $p = 1$, we have the (divergent) harmonic series

The p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$

converges if $p > 1$, diverges if $p \le 1$.

EXAMPLE 4 The series $\sum_{n=1}^{\infty}(1/(n^2+1))$

$$\int_1^{\infty}\frac{1}{x^2+1}\,dx = \lim_{b\to\infty}\Big[\arctan x\Big]_1^b$$
$$= \lim_{b\to\infty}\Big[\arctan b - \arctan 1\Big] = \frac{\pi}{4}.$$

the series converges.

EXAMPLE 5 Determine the convergence or divergence of the series.

(a) $\sum_{n=1}^{\infty} ne^{-n^2}$ **(b)** $\sum_{n=1}^{\infty}\frac{1}{2^{\ln n}}$

Solutions

$$\int_1^{\infty}\frac{x}{e^{x^2}}\,dx = \frac{1}{2}\int_1^{\infty}\frac{du}{e^u} = \frac{1}{2e}.$$

$$\int_1^{\infty}\frac{dx}{2^{\ln x}} = \int_0^{\infty}\frac{e^u\,du}{2^u} = \int_0^{\infty}\left(\frac{e}{2}\right)^u du = \infty.$$

Error Estimation

$$\sum_{n=1}^{+\infty} a_n = S$$

the **remainder** R_n $R_n = S - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots.$

Bounds for the Remainder in the Integral Test

Suppose $\{a_k\}$ is a sequence of positive terms with $a_k = f(k)$, where f is a continuous positive decreasing function and that Σa_n converges. Then the remainder $R_n = S - s_n$

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx.$$

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \le S \le s_n + \int_{n}^{\infty} f(x)\,dx$$

EXAMPLE 6 Estimate the sum of the series $\Sigma(1/n^2)$ using the inequalities in (2) and $n = 10$.

Solution We have that

$$\int_n^{\infty} \frac{1}{x^2}dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_n^b = \lim_{b\to\infty}\left(-\frac{1}{b}+\frac{1}{n}\right) = \frac{1}{n}.$$

$$s_{10} + \frac{1}{11} \le S \le s_{10} + \frac{1}{10}.$$

$$s_{10} = 1 + (1/4) + (1/9) + (1/16) + \cdots + (1/100) \approx 1.54977,$$

$$1.64068 \le S \le 1.64977.$$ $$\sum_{n=1}^{\infty}\frac{1}{n^2} \approx 1.6452.$$ the midpoint

10.4

Comparison Tests
比较判别法

THEOREM 10—The Comparison Test

Let $\sum a_n$, $\sum c_n$, and $\sum d_n$ be series with nonnegative terms. Suppose that for some integer N

$$d_n \le a_n \le c_n \qquad \text{for all} \qquad n > N.$$

(a) If $\sum c_n$ converges, then $\sum a_n$ also converges.

(b) If $\sum d_n$ diverges, then $\sum a_n$ also diverges.

Proof In Part (a), the partial sums of $\sum a_n$ are bounded above by

$$S_n = a_1 + a_2 + \cdots + a_n$$

$$\le c_1 + c_2 + \cdots + c_n \le M$$

EXAMPLE 1 We apply Theorem 10 to several series.

$$\sum_{n=1}^{\infty} \frac{5}{5n-1} \qquad \frac{5}{5n-1} = \frac{1}{n - \frac{1}{5}} > \frac{1}{n} \qquad \text{diverges}$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

all positive and less than or equal to the corresponding terms of

$$1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots . = 1 + \frac{1}{1-(1/2)} = 3.$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}} \qquad \frac{1}{\sqrt{n(n+1)(n+2)}} < \frac{1}{n^{3/2}} \qquad \text{converges}$$

converges

$$5 + \frac{2}{3} + \frac{1}{7} + 1 + \frac{1}{2 + \sqrt{1}} + \frac{1}{4 + \sqrt{2}} + \frac{1}{8 + \sqrt{3}} + \cdots + \frac{1}{2^n + \sqrt{n}} + \cdots$$

$$\leq 5 + \frac{2}{3} + \frac{1}{7} + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots. \qquad \text{converges.}$$

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right) \qquad \sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

The Limit Comparison Test

THEOREM 11—Limit Comparison Test Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer).

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Proof We will prove Part 1.

$$n > N \implies \left|\frac{a_n}{b_n} - c\right| < \frac{c}{2}. \qquad \left(\frac{c}{2}\right)b_n < a_n < \left(\frac{3c}{2}\right)b_n.$$

EXAMPLE 2 Which of the following series converge, diverge?

(a) $\frac{3}{4}+\frac{5}{9}+\frac{7}{16}+\frac{9}{25}+\cdots = \sum_{n=1}^{\infty}\frac{2n+1}{n^2+2n+1}$

(b) $\frac{1}{1}+\frac{1}{3}+\frac{1}{7}+\frac{1}{15}+\cdots = \sum_{n=1}^{\infty}\frac{1}{2^n-1}$

(c) $\frac{1+2\ln 2}{9}+\frac{1+3\ln 3}{14}+\frac{1+4\ln 4}{21}+\cdots = \sum_{n=2}^{\infty}\frac{1+n\ln n}{n^2+5}$

Solution (a) $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n}$ diverges

(b) $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{2^n}$

converges

(c) $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty}\frac{\ln n}{n}$

diverges

EXAMPLE 3 Does $\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$ converge?

Solution $\frac{\ln n}{n^{6/4}} = \frac{\ln n}{n^{1/4}} \cdot \frac{1}{n^{5/4}} < \frac{1}{n^{5/4}}$ it converges.

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^{1.05}} \qquad \sum_{n=1}^{\infty} \frac{\ln^2 n}{n^{1.05}} \qquad \sum_{n=1}^{\infty} \frac{\ln^q n}{n^{1.05}}$$

$$\sum_{n=1}^{\infty} \frac{\ln^q n}{n^p}, p > 1$$

10.5

Absolute Convergence; The Ratio and Root Tests

绝对收敛；比值法和根值法

DEFINITION A series $\sum a_n$ **converges absolutely** if $\sum |a_n|$ converges.

THEOREM 12—The Absolute Convergence Test

$$\text{If } \sum_{n=1}^{\infty} |a_n| \text{ converges, then } \sum_{n=1}^{\infty} a_n \text{ converges.}$$

Proof For each n, $-|a_n| \le a_n \le |a_n|$,

$0 \le a_n + |a_n| \le 2|a_n|$. $\sum_{n=1}^{\infty}(a_n + |a_n|)$ converges

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(a_n + |a_n|) - \sum_{n=1}^{\infty}|a_n|.$$

Therefore, $\sum_{n=1}^{\infty} a_n$ converges.

EXAMPLE 1 This example gives two series that converge absolutely.

(a) For $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots,$

$$\left|(-1)^{n+1}\frac{1}{n^2}\right| = \frac{1}{n^2}$$ it converges absolutely.

(b) For $\sum_{n=1}^{\infty}\frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \cdots,$

$$\left|\frac{\sin n}{n^2}\right| \le \frac{1}{n^2}$$ it converges absolutely.

The Ratio Test

THEOREM 13—The Ratio Test Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho.$$

Then **(a)** the series *converges absolutely* if $\rho < 1$, **(b)** the series *diverges* if $\rho > 1$ or ρ is infinite, **(c)** the test is *inconclusive* if $\rho = 1$.

Proof (a) $\boldsymbol{\rho < 1}$**.** Let r be a number between ρ and 1. Since

$$\left|\frac{a_{n+1}}{a_n}\right| \to \rho, \qquad \left|\frac{a_{n+1}}{a_n}\right| < \rho + \epsilon = r, \qquad \text{when } n \geq N.$$

That is,

$$\begin{aligned}
|a_{N+1}| &< r|a_N|, \\
|a_{N+2}| &< r|a_{N+1}| < r^2|a_N|, \\
|a_{N+3}| &< r|a_{N+2}| < r^3|a_N|, \\
&\vdots \\
|a_{N+m}| &< r|a_{N+m-1}| < r^m|a_N|.
\end{aligned}$$

Therefore.

$$\sum_{m=N}^{\infty} |a_m| = \sum_{m=0}^{\infty} |a_{N+m}| \leq \sum_{m=0}^{\infty} |a_N|\, r^m = |a_N| \sum_{m=0}^{\infty} r^m.$$

That is, the series $\sum a_n$ is absolutely convergent.

(b) $1 < \rho \leq \infty$. From some index M on,

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 \qquad \text{and} \qquad |a_M| < |a_{M+1}| < |a_{M+2}| < \cdots.$$

The terms of the series do not approach zero as n becomes infinite,

and the series diverges

(c) $\rho = 1$. The two series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

For $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$: $\displaystyle\left|\frac{a_{n+1}}{a_n}\right| = \frac{1/(n+1)}{1/n} = \frac{n}{n+1} \rightarrow 1.$

For $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$: $\displaystyle\left|\frac{a_{n+1}}{a_n}\right| = \frac{1/(n+1)^2}{1/n^2} = \left(\frac{n}{n+1}\right)^2 \rightarrow 1^2 = 1.$

EXAMPLE 2 Investigate the convergence of the following series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{4^n n!n!}{(2n)!}$

Solution We apply the Ratio Test to each series.

convergent

(a) $$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2^{n+1} + 5)/3^{n+1}}{(2^n + 5)/3^n} = \frac{1}{3} \cdot \frac{2^{n+1} + 5}{2^n + 5} \rightarrow \frac{1}{3} \cdot \frac{2}{1} = \frac{2}{3}.$$

diverges

(b) $$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n!n!(2n + 2)(2n + 1)(2n)!}{(n + 1)!(n + 1)!(2n)!} = \frac{4n + 2}{n + 1} \rightarrow 4.$$

(c) $$\left|\frac{a_{n+1}}{a_n}\right| = \frac{4^{n+1}(n + 1)!(n + 1)!}{(2n + 2)(2n + 1)(2n)!} \cdot \frac{(2n)!}{4^n n!n!} = \frac{2(n + 1)}{2n + 1} \rightarrow 1.$$

a_{n+1} is always greater than a_n diverges

The Root Test

$$a_n = \begin{cases} n/2^n, & n \text{ odd} \\ 1/2^n, & n \text{ even.} \end{cases}$$

$$\sum_{n=1}^{\infty} a_n = \frac{1}{2^1} + \frac{1}{2^2} + \frac{3}{2^3} + \frac{1}{2^4} + \frac{5}{2^5} + \frac{1}{2^6} + \frac{7}{2^7} + \cdots$$

$$\frac{a_{n+1}}{a_n} : \frac{1}{2}, \frac{3}{2}, \frac{1}{2\cdot 3}, \frac{5}{2}, \frac{1}{2\cdot 5}, \frac{7}{2}, \frac{1}{2\cdot 7}, \frac{9}{2}, \frac{1}{2\cdot 9}, \ldots$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} \dfrac{1}{2n}, & n \text{ odd} \\[2ex] \dfrac{n+1}{2}, & n \text{ even} \end{cases}$$

As $n \to \infty$, the ratio is alternately small and large and has no limit.

THEOREM 14—The Root Test Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty} = \sqrt[n]{|a_n|} = \rho.$$

Then **(a)** the series *converges absolutely* if $\rho < 1$, **(b)** the series *diverges* if $\rho > 1$ or ρ is infinite, **(c)** the test is *inconclusive* if $\rho = 1$.

Proof (a) $\rho < 1$. Choose an $\epsilon > 0$ so small that $\rho + \epsilon < 1$.

So there exists an index M such that $\sqrt[n]{|a_n|} < \rho + \epsilon$ when $n \geq M$.

$|a_n| < (\rho + \epsilon)^n$ for $n \geq M$. $\sum a_n$ converges absolutely.

(b) $1 < \rho \leq \infty$. we have $\sqrt[n]{|a_n|} > 1$, $|a_n| > 1$ for $n > M$.

diverges

(c) $\rho = 1$. The series $\sum_{n=1}^{\infty}(1/n)$ and $\sum_{n=1}^{\infty}(1/n^2)$

show that the test is not conclusive

EXAMPLE 3 Consider again the series Σa_n converge?

$$a_n = \begin{cases} n/2^n, & n \text{ odd} \\ 1/2^n, & n \text{ even.} \end{cases}$$

Solution We apply the Root Test, finding that

$$\sqrt[n]{|a_n|} = \begin{cases} \sqrt[n]{n}/2, & n \text{ odd} \\ 1/2, & n \text{ even.} \end{cases}$$

Therefore,

$$\frac{1}{2} \leq \sqrt[n]{|a_n|} \leq \frac{\sqrt[n]{n}}{2}.$$

we have $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1/2$

so the series converges absolutely by the Root Test.

EXAMPLE 4 Which of the following series converge, and which diverge?

$$\text{(a)}\ \sum_{n=1}^{\infty}\frac{n^2}{2^n} \qquad \text{(b)}\ \sum_{n=1}^{\infty}\frac{2^n}{n^3} \qquad \text{(c)}\ \sum_{n=1}^{\infty}\left(\frac{1}{1+n}\right)^n$$

Solution

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{2^n}$ converges because $\displaystyle\sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}} = \frac{\left(\sqrt[n]{n}\right)^2}{2} \to \frac{1^2}{2} < 1.$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{n^3}$ diverges because $\displaystyle\sqrt[n]{\frac{2^n}{n^3}} = \frac{2}{\left(\sqrt[n]{n}\right)^3} \to \frac{2}{1^3} > 1.$

(c) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{1+n}\right)^n$ converges because $\displaystyle\sqrt[n]{\left(\frac{1}{1+n}\right)^n} = \frac{1}{1+n} \to 0 < 1.$

10.6

Alternating Series and Conditional Convergence

交错级数和条件收敛

A series in which the terms are alternately positive and negative

an **alternating series**

alternating harmonic series,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots \qquad (1)$$

$$-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \frac{(-1)^n 4}{2^n} + \cdots \qquad (2)$$

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots + (-1)^{n+1} n + \cdots \qquad (3)$$

$$a_n = (-1)^{n+1} u_n \qquad \text{or} \qquad a_n = (-1)^n u_n$$

where $u_n = |a_n|$ is a positive number.

THEOREM 15—The Alternating Series Test The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if all three of the following conditions are satisfied:

1. The u_n's are all positive.
2. The positive u_n's are (eventually) nonincreasing: $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.
3. $u_n \to 0$.

Proof Assume $N = 1$. If n is an even integer, say $n = 2m$,

$$\begin{aligned} s_{2m} &= (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2m-1} - u_{2m}) \\ &= u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots - (u_{2m-2} - u_{2m-1}) - u_{2m}. \end{aligned}$$

Hence $s_{2m+2} \geq s_{2m}$, $\quad s_{2m} \leq u_1$.

Since $\{s_{2m}\}$ is nondecreasing and bounded from above,

it has a limit, say

$$\lim_{m\to\infty} s_{2m} = L.$$

$\lim_{n\to\infty} s_n = L$

$$s_{2m+1} = s_{2m} + u_{2m+1} \to L + 0 = L.$$

EXAMPLE 1 The alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

it therefore converges

EXAMPLE 2 Consider the sequence where $u_n = 10n/(n^2 + 16)$.

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n$$

Define $f(x) = 10x/(x^2 + 16)$.

$$f'(x) = \frac{10(16 - x^2)}{(x^2 + 16)^2} \leq 0 \qquad \text{whenever } x \geq 4.$$

It follows that $u_n \geq u_{n+1}$ for $n \geq 4$.

$u_n = 10n/(n^2 + 16). \to 0.$

$\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ it therefore converges

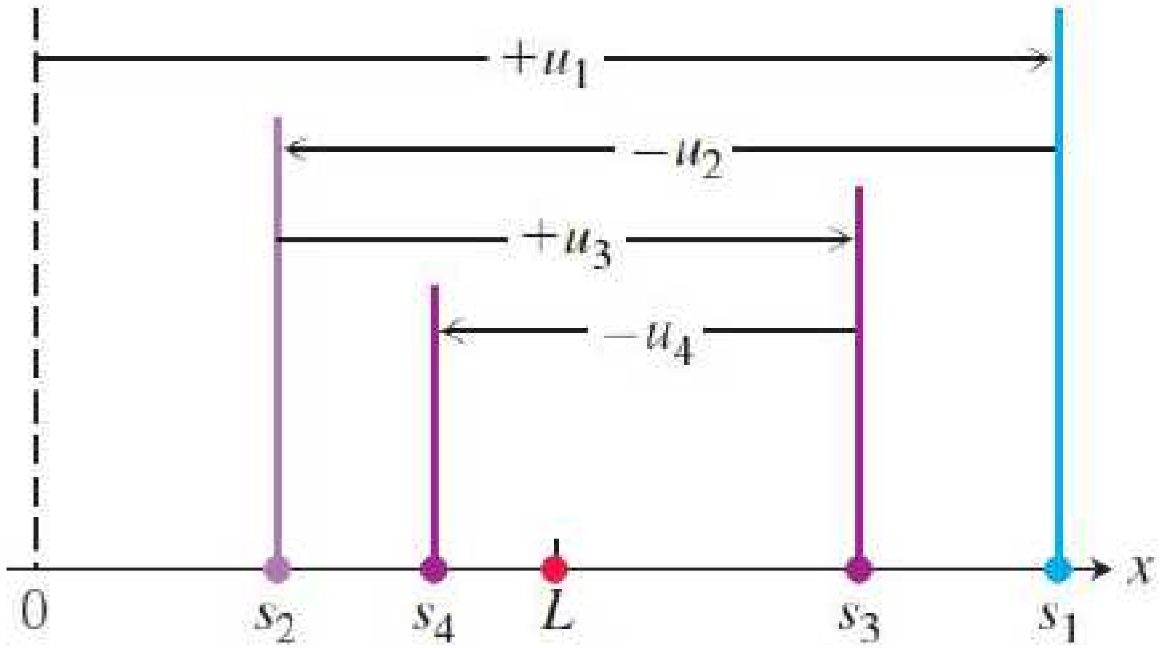


FIGURE 10.13 The partial sums of an alternating series that satisfies the hypotheses of Theorem 15 for $N = 1$ straddle the limit from the beginning.

THEOREM 16—The Alternating Series Estimation Theorem If the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the three conditions of Theorem 15, then for $n \geq N$,

$$s_n = u_1 - u_2 + \cdots + (-1)^{n+1}u_n$$

approximates the sum L of the series with an error whose absolute value is less than u_{n+1}, the absolute value of the first unused term. Furthermore, the sum L lies between any two successive partial sums s_n and s_{n+1}, and the remainder, $L - s_n$, has the same sign as the first unused term.

$$\begin{aligned}\left|\sum_{n=1}^{\infty}(-1)^{n+1}u_n - s_n\right| &= u_{n+1} - u_{n+2} + u_{n+3} - \cdots \\ &= u_{n+1} - (u_{n+2} - u_{n+3}) - (u_{n+4} - u_{n+5})\cdots \\ &\leq u_{n+1}\end{aligned}$$

EXAMPLE 3 We try Theorem 16 on a series whose sum we know:

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} + \frac{1}{256} - \cdots$$

$$\frac{1}{1-(-1/2)} = \frac{1}{3/2} = \frac{2}{3}, \quad s_8 = 0.6640625 \quad s_9 = 0.66796875.$$

$$0.6640625 < (2/3) < 0.66796875.$$

$$(2/3) - 0.6640625 = 0.0026041666\ldots$$

is less than $(1/256) = 0.00390625$.

Conditional Convergence

DEFINITION A convergent series that is not absolutely convergent is **conditionally convergent**.

The alternating harmonic series is conditionally convergent

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$$

EXAMPLE 4 If p is a positive constant, the alternating p-series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots, \quad p > 0$$

converges.

If $p > 1$, the series converges absolutely

If $0 < p \leq 1$, the series converges conditionally

Absolute convergence $(p = 3/2)$: $1 - \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} - \frac{1}{4^{3/2}} + \cdots$

Conditional convergence $(p = 1/2)$: $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$

Rearranging Series

THEOREM 17—The Rearrangement Theorem for Absolutely Convergent Series If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, \ldots, b_n, \ldots$ is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n.$$

EXAMPLE 5 $\sum_{n=1}^{\infty}(-1)^{n+1}/n = L$

$$2L = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$$

$$= 2\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \cdots\right)$$

$$= 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \frac{2}{11} - \cdots.$$

$$(2-1) - \frac{1}{2} + \left(\frac{2}{3} - \frac{1}{3}\right) - \frac{1}{4} + \left(\frac{2}{5} - \frac{1}{5}\right) - \frac{1}{6} + \left(\frac{2}{7} - \frac{1}{7}\right) - \frac{1}{8} + \cdots$$

$$= \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \cdots\right)$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = L.$$

?

Summary of Tests

1. **The *n*th-Term Test:** If it is not true that $a_n \to 0$, then the series diverges.
2. **Geometric series:** $\sum ar^n$ converges if $|r| < 1$; otherwise it diverges.
3. ***p*-series:** $\sum 1/n^p$ converges if $p > 1$; otherwise it diverges.
4. **Series with nonnegative terms:** Try the Integral Test or try comparing to a known series with the Comparison Test or the Limit Comparison Test. Try the Ratio or Root Test.
5. **Series with some negative terms:** Does $\sum |a_n|$ converge by the Ratio or Root Test, or by another of the tests listed above? Remember, absolute convergence implies convergence.
6. **Alternating series:** $\sum a_n$ converges if the series satisfies the conditions of the Alternating Series Test.

10.7

Power Series

幂级数

Power Series and Convergence

DEFINITIONS **A power series about $x = 0$** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots .$$

A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

in which the **center** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

EXAMPLE 1

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots .$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \qquad -1 < x < 1.$$

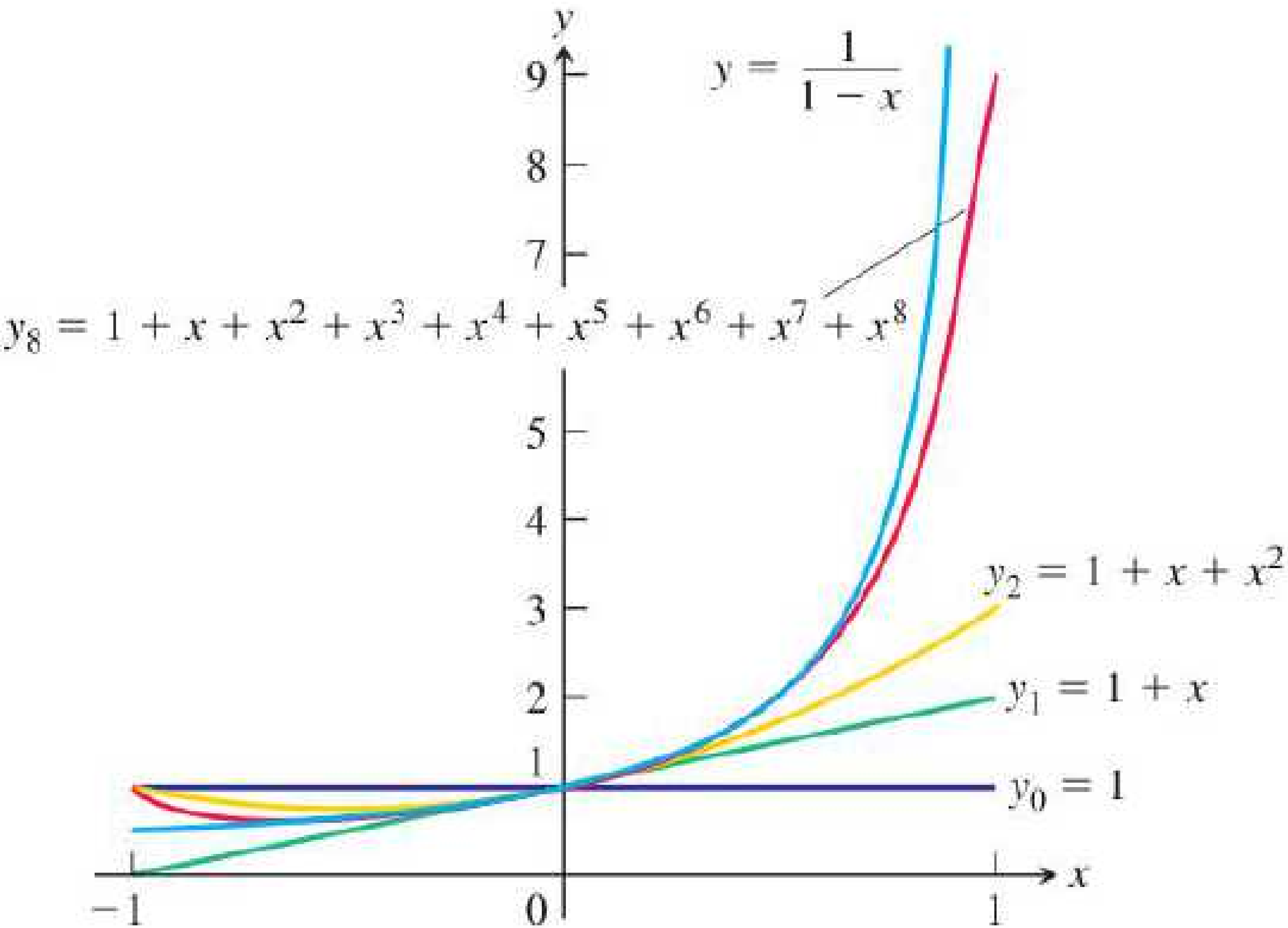


FIGURE 10.14 The graphs of $f(x) = 1/(1-x)$ in Example 1 and four of its polynomial approximations.

EXAMPLE 2 $1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots$

The series converges for $\left|\frac{x-2}{2}\right| < 1$ or $0 < x < 4$.

$$\frac{1}{1-r} = \frac{1}{1 + \frac{x-2}{2}} = \frac{2}{x},$$

$$\frac{2}{x} = 1 - \frac{(x-2)}{2} + \frac{(x-2)^2}{4} - \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots,$$

$$0 < x < 4.$$

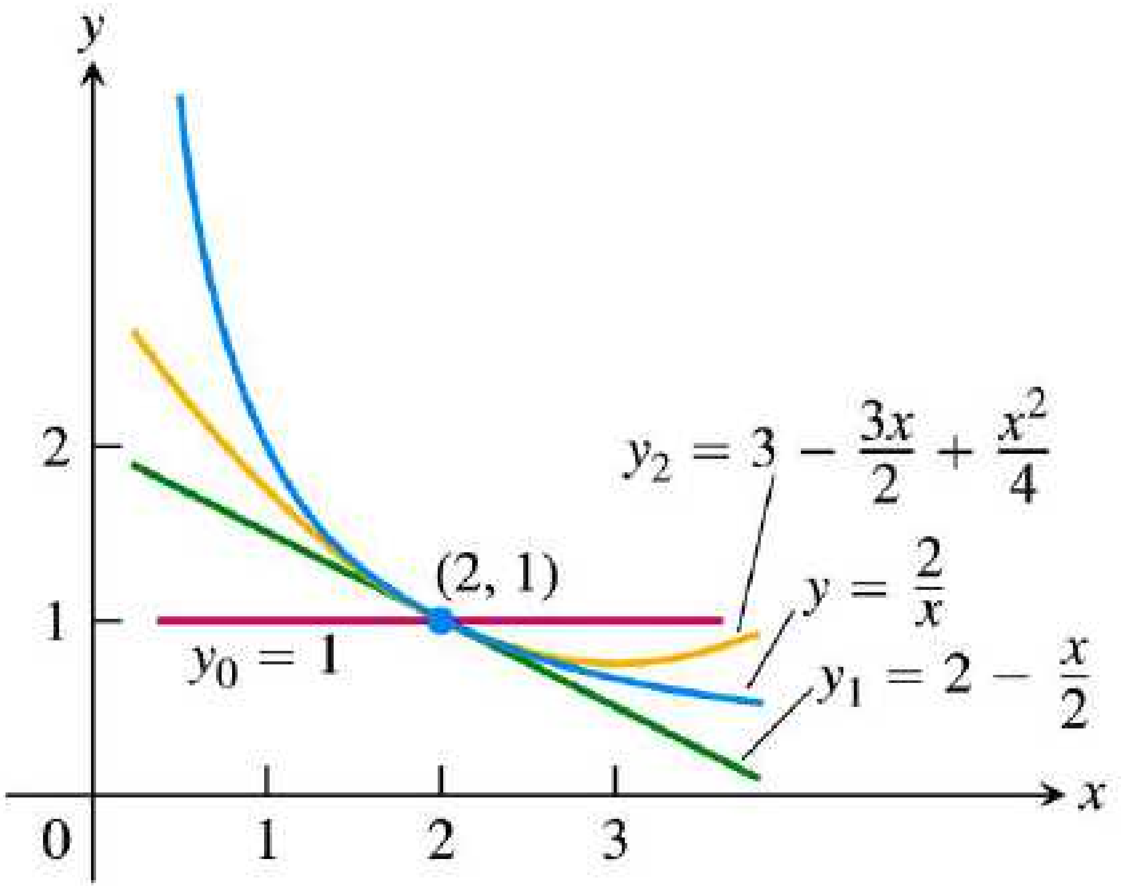


FIGURE 10.15 The graphs of $f(x) = 2/x$ and its first three polynomial approximations (Example 2).

EXAMPLE 3 For what values of x do the following power series converge?

(a) $$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

(b) $$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

(c) $$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

(d) $$\sum_{n=0}^{\infty} n!x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots$$

Solution Apply the Ratio Test to the series $\sum |u_n|$,

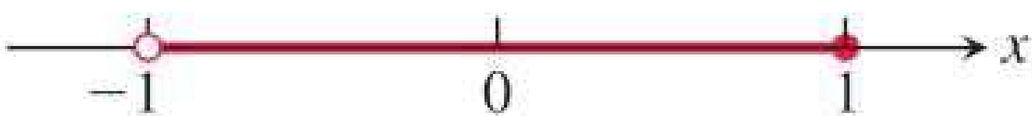
$$\text{(a)}\quad \left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x}\right| = \frac{n}{n+1}|x| \rightarrow |x|.$$

The series converges absolutely for $|x| < 1$. It diverges if $|x| > 1$

At $x = 1$, we get the alternating harmonic series converges.

At $x = -1$, we get $-1 - 1/\,1/3 - 1/4 - \cdots$ it diverges.

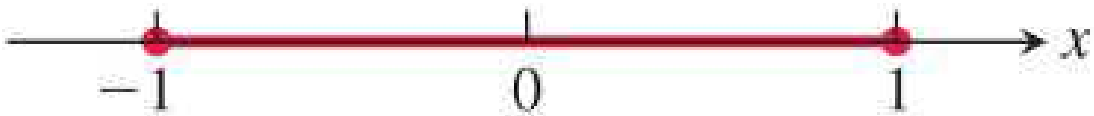
Series (a) converges for $-1 < x \leq 1$ and diverges elsewhere.



(b) $$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{2n+1}}{2n+1}\cdot\frac{2n-1}{x^{2n-1}}\right| = \frac{2n-1}{2n+1}x^2 \to x^2.$$

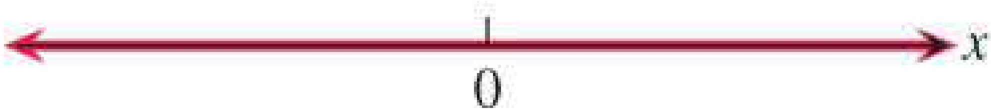
The series converges absolutely for $x^2 < 1$. It diverges for $x^2 > 1$

converges for $-1 \le x \le 1$ and diverges elsewhere.



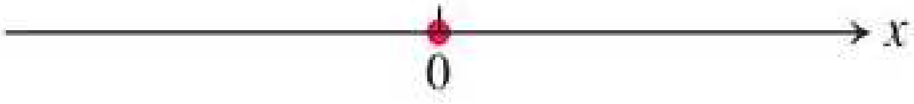
(c) $$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0 \text{ for every } x.$$

The series converges absolutely for all x.

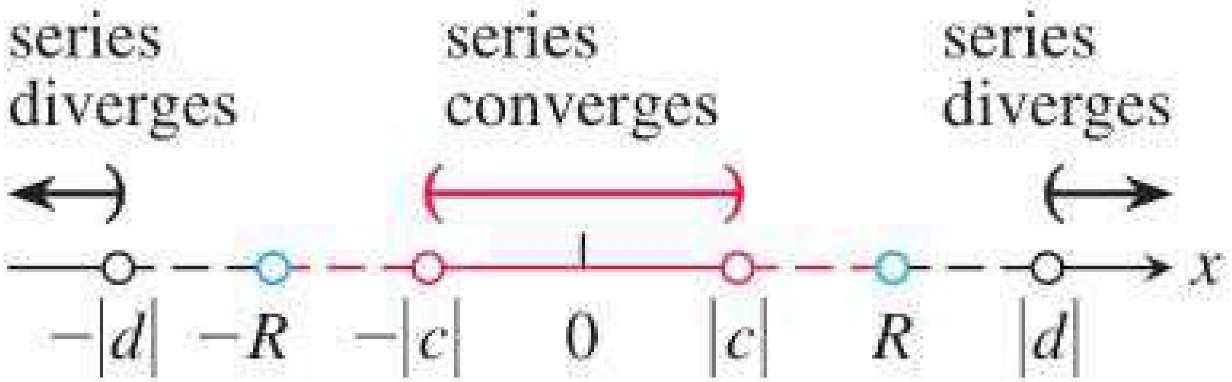


(d) $$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = (n+1)|x| \to \infty \text{ unless } x = 0.$$

The series diverges for all values of x except $x = 0$.



THEOREM 18—The Convergence Theorem for Power Series If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ converges at $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$.



Proof

Suppose the series $\sum_{n=0}^{\infty} a_n c^n$ converges. Then $\lim_{n\to\infty} a_n c^n = 0$

$|a_n c^n| < 1$ for all $n > N$, so that $|a_n| < \dfrac{1}{|c|^n}$ for $n > N$.

Now take any x such that $|x| < |c|$, so that $|x|/|c| < 1$.

$$|a_n|\,|x|^n < \frac{|x|^n}{|c|^n} \qquad \text{for } n > N.$$

$\sum_{n=0}^{\infty} |x/c|^n$ converges. $\sum_{n=0}^{\infty} |a_n|\,|x^n|$ converges,

$\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $-|c| < x < |c|$

suppose that the series $\sum_{n=0}^{\infty} a_n x^n$ diverges at $x = d$.

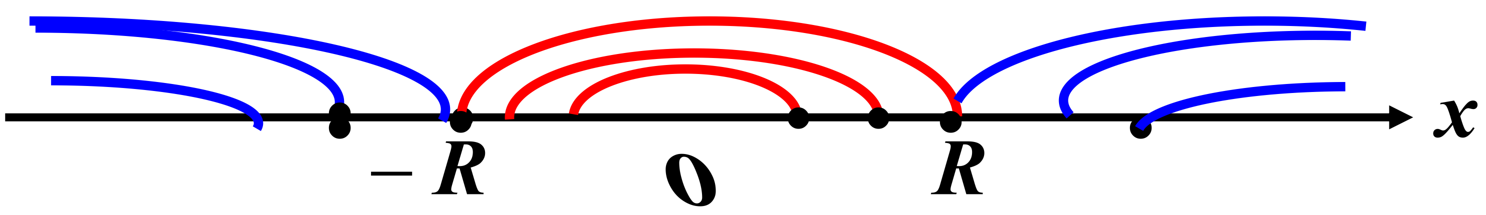
$|x| > |d|$ and the series converges at x,

series also converges at d, contrary to our assumption.

So the series diverges for all x w $|x| > |d|$.

The Radius of Convergence of a Power Series

R is called the **radius of convergence** of the power series,



COROLLARY TO THEOREM 18 The convergence of the series $\sum c_n(x-a)^n$ is described by one of the following three cases:

1. There is a positive number R such that the series diverges for x with $|x-a| > R$ but converges absolutely for x with $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges absolutely for every x ($R = \infty$).
3. The series converges at $x = a$ and diverges elsewhere ($R = 0$).

R is called the **radius of convergence** of the power series,

How to Test a Power Series for Convergence

1. *Use the Ratio Test (or Root Test) to find the interval where the series converges absolutely.* Ordinarily, this is an open interval

$$|x - a| < R \qquad \text{or} \qquad a - R < x < a + R.$$

2. *If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint*, as in Examples 3a and b. Use a Comparison Test, the Integral Test, or the Alternating Series Test.

3. *If the interval of absolute convergence is $a - R < x < a + R$, the series diverges for $|x - a| > R$* (it does not even converge conditionally) because the nth term does not approach zero for those values of x.

例 求下列幂级数的收敛域:

(1) $\sum_{n=1}^{\infty}(-1)^n\frac{2^n}{\sqrt{n}}(x-\frac{1}{2})^n$. (2) $\sum_{n=1}^{\infty}\frac{x^{2n-1}}{2^n}$

解(1) $\rho=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{2\sqrt{n}}{\sqrt{n+1}}|x-\frac{1}{2}|=2|x-\frac{1}{2}|$

即 $\left|x-\frac{1}{2}\right|<\frac{1}{2}$收敛, $\left|x-\frac{1}{2}\right|>\frac{1}{2}$发散.

$x\in(0,1)$收敛, $x\in(-\infty,0)\cup(1,+\infty)$发散.

当$x=0$时, $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$, 发散, 当$x=1$时, $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$, 收敛

故收敛域为$(0,1]$.

解(2) $$\lim_{n\to\infty}\left|\frac{u_{n+1}(x)}{u_n(x)}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+1}\big/2^{n+1}}{x^{2n-1}\big/2^{n}}\right| = \frac{1}{2}|x|^2,$$

当 $\frac{1}{2}x^2 < 1$, 即 $|x| < \sqrt{2}$时, 级数收敛,

当 $\frac{1}{2}x^2 > 1$, 即 $|x| > \sqrt{2}$时, 级数发散,

当$x = \sqrt{2}$时，级数为$\sum_{n=1}^{\infty}\frac{1}{\sqrt{2}}$, 级数发散,

当$x = -\sqrt{2}$时，级数为$\sum_{n=1}^{\infty}\frac{-1}{\sqrt{2}}$, 级数发散,

原级数的收敛域为 $(-\sqrt{2}, \sqrt{2})$.

Operations on Power Series

THEOREM 19—The Series Multiplication Theorem for Power Series If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n.$$

$$\sum_{n=1}^{\infty} (a_n \pm b_n) x^n = A(x) \pm B(x).$$

$$\left(\sum_{n=0}^{\infty} x^n\right)\cdot\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}\right)$$

$$= (1 + x + x^2 + \cdots)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right)$$

$$= x \quad + (1-\frac{1}{2})x^2 \quad + (1-\frac{1}{2}+\frac{1}{3})x^3 \quad + (1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4})x^4$$

$$+ (1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5})x^5 + \cdots$$

$$= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}\right)x^n \qquad = x + \frac{x^2}{2} + \frac{5x^3}{6} - \frac{x^4}{6}\cdots.$$

$$-1 < x < 1$$

THEOREM 20 If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$, then $\sum_{n=0}^{\infty} a_n (f(x))^n$ converges absolutely for any continuous function f on $|f(x)| < R$.

$1/(1 - x) = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$

$1/(1 - 4x^2) = \sum_{n=0}^{\infty} (4x^2)^n$ for $|4x^2| < 1$ or $|x| < 1/2$.

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-)^n x^{2n} \qquad -1 < x < 1$$

THEOREM 21—The Term-by-Term Differentiation Theorem If $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, it defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad \text{on the interval} \qquad a - R < x < a + R.$$

This function f has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1},$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2},$$

$f(x)$在收敛点处连续.

and so on. Each of these derived series converges at every point of the interval $a - R < x < a + R$.

Caution Term-by-term differentiation might not work

$$\sum_{n=1}^{\infty} \frac{\sin(n!x)}{n^2} \quad \text{converges for all } x.$$

$$\sum_{n=1}^{\infty} \frac{n! \cos(n!x)}{n^2}, \quad \text{diverges for all } x.$$

$$\forall x = \frac{p}{q}, \quad |\cos(n!x)| = |\cos(n!\frac{p}{q})| \ge c > 0, \qquad n > q$$

$\forall x$是无理数，取有理数列a_k使$a_k \to x$

$$|\cos(n!a_k)| \ge c > 0 \quad k \to \infty \qquad |\cos(n!x)| \ge c > 0$$

EXAMPLE 4

Find series for $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$$

$$= \sum_{n=0}^{\infty} x^n, \qquad -1 < x < 1.$$

Solution

$$f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots$$

$$= \sum_{n=1}^{\infty} nx^{n-1}, \qquad -1 < x < 1;$$

$$\sum_{n=1}^{\infty} n\left(\frac{1}{3}\right)^n = \frac{3}{4}.$$

$$f''(x) = \frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + \cdots + n(n-1)x^{n-2} + \cdots$$

$$= \sum_{n=2}^{\infty} n(n-1)x^{n-2}, \qquad -1 < x < 1.$$

求$\sum_{n=1}^{\infty}(n+1)n(\frac{1}{3})^n$的和.

$$\sum_{n=1}^{\infty}(n+1)n(\frac{1}{3})^n = \frac{1}{3}\sum_{n=1}^{\infty}(n+1)n(\frac{1}{3})^{n\text{-}1}.$$

$$= \frac{1}{3}\frac{2}{(1-\frac{1}{3})^3} = \frac{9}{4}$$

THEOREM 22—The Term-by-Term Integration Theorem

Suppose that
$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n$$
converges for $a - R < x < a + R$ $(R > 0)$. Then
$$\sum_{n=0}^{\infty} c_n \frac{(x - a)^{n+1}}{n + 1}$$
converges for $a - R < x < a + R$ and
$$\int f(x)\, dx = \sum_{n=0}^{\infty} c_n \frac{(x - a)^{n+1}}{n + 1} + C$$
for $a - R < x < a + R$.

EXAMPLE 5 Identify the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, -1 \le x \le 1.$$

Solution $f'(x) = 1 - x^2 + x^4 - x^6 + \cdots, \qquad -1 < x < 1.$

$$f'(x) = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}.$$

$$f(x) = \int f'(x)\,dx = \int \frac{dx}{1+x^2} = \tan^{-1} x + C.$$

The series for $f(x)$ is zero when $x = 0$, so $C = 0$. Hence

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \tan^{-1} x, \qquad -1 < x < 1.$$

EXAMPLE 6 $$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

converges on the open interval $-1 < t < 1$. Therefore,

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \Big]_0^x$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \qquad -1 < x < 1.$$

Alternating Harmonic Series Sum

$$\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

10.8

Taylor and Maclaurin Series
泰勒级数和麦克劳林级数

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \qquad -1 < x < 1.$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$$

系数如何得到?

if we assume that $f(x)$ is the sum of a power series about $x = a$,

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \qquad f(a) = a_0 \quad f'(a) = a_1 \quad \frac{f^{(n)}(a)}{n!} = a_n$$

$$= a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots$$

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots + na_n(x-a)^{n-1} + \cdots,$$

$$f''(x) = 1 \cdot 2a_2 + 2 \cdot 3a_3(x-a) + 3 \cdot 4a_4(x-a)^2 + \cdots,$$

$$f'''(x) = 1 \cdot 2 \cdot 3a_3 + 2 \cdot 3 \cdot 4a_4(x-a) + 3 \cdot 4 \cdot 5a_5(x-a)^2 + \cdots,$$

$f^{(n)}(x) = n!a_n +$ a sum of terms with $(x-a)$ as a factor.

$$f'(a) = a_1, \qquad f''(a) = 1 \cdot 2a_2, \qquad f'''(a) = 1 \cdot 2 \cdot 3a_3,$$

$$f^{(n)}(a) = n!a_n. \qquad a_n = \frac{f^{(n)}(a)}{n!}.$$

$$f(x) \overset{?}{=} f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2$$

$$+ \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \cdots.$$

Taylor and Maclaurin Series

DEFINITIONS Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots .$$

The **Maclaurin series of f** is the Taylor series generated by f at $x = 0$, or

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots .$$

EXAMPLE 1 Find the Taylor series generated by $f(x) = 1/x$ at $a = 2$.

Solution We need to find $f(2), f'(2), f''(2), \ldots$.

$$f(x) = x^{-1}, \quad f'(x) = -x^{-2}, \quad f''(x) = 2!x^{-3}, \quad \cdots, \quad f^{(n)}(x) = (-1)^n n! x^{-(n+1)},$$

$$f(2) = 2^{-1} = \frac{1}{2}, \quad f'(2) = -\frac{1}{2^2}, \quad \frac{f''(2)}{2!} = 2^{-3} = \frac{1}{2^3}, \quad \cdots, \quad \frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{2^{n+1}}.$$

The Taylor series is

$$f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \cdots$$

$$\frac{1}{x} = \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \cdots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \cdots.$$

This is a geometric series

$0 < x < 4.$

its sum is

$$\frac{1/2}{1 + (x-2)/2} = \frac{1}{2 + (x-2)} = \frac{1}{x}.$$

Taylor Polynomials

$$P_1(x) = f(a) + f'(a)(x - a). \approx f(x)$$

$$f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 \approx f(x)$$

DEFINITION Let f be a function with derivatives of order k for $k = 1, 2, \ldots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the **Taylor polynomial of order n** generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x - a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

EXAMPLE 2

Find the Taylor series and the Taylor polynomials generated by $f(x) = e^x$ at $x = 0$.

Solution

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$= 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

$$P_n(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}.$$

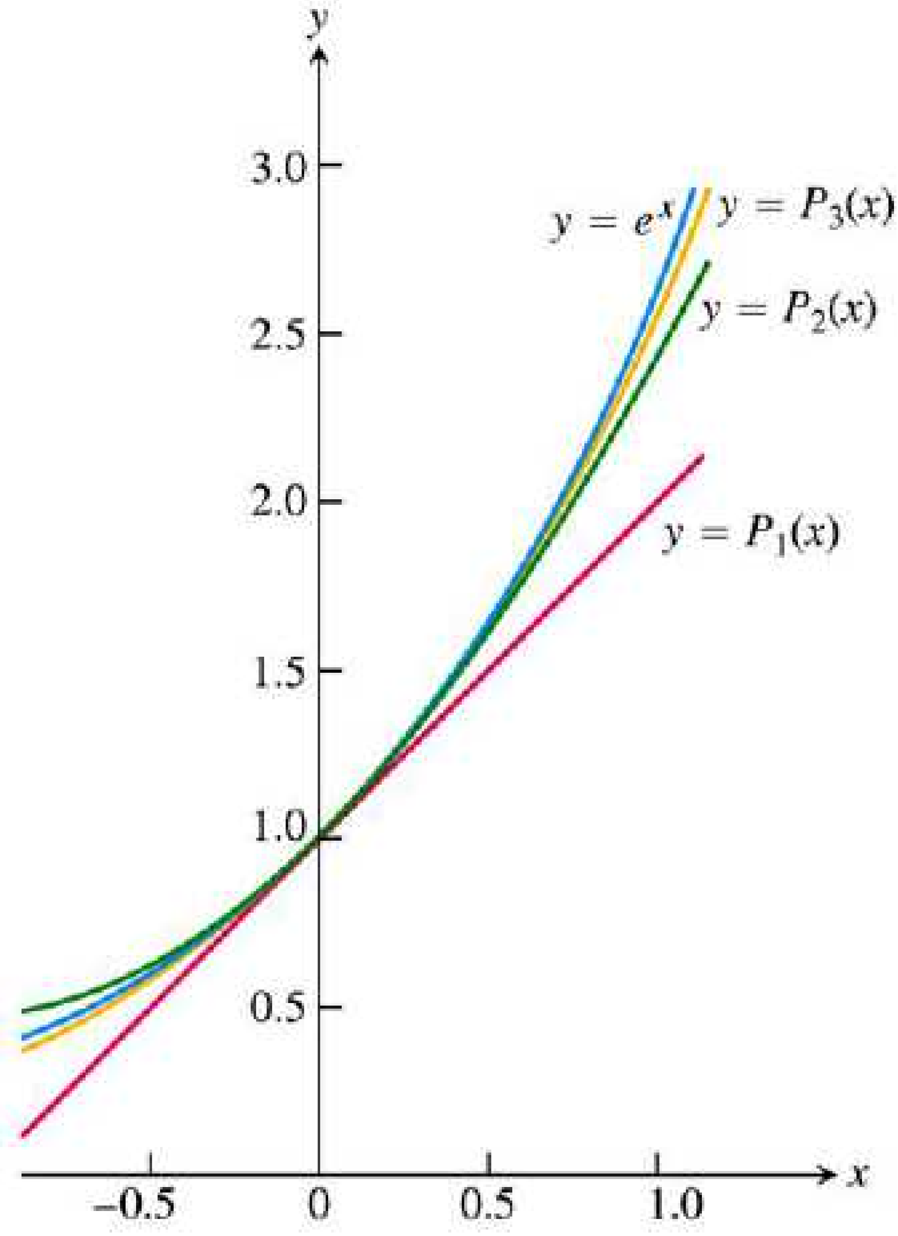


FIGURE 10.17 The graph of $f(x) = e^x$ and its Taylor polynomials

$$P_1(x) = 1 + x$$
$$P_2(x) = 1 + x + (x^2/2!)$$
$$P_3(x) = 1 + x + (x^2/2!) + (x^3/3!).$$

Notice the very close agreement near the center $x = 0$ (Example 2).

EXAMPLE 3

Find the Taylor series and Taylor polynomials generated by $f(x) = \cos x$ at $x = 0$.

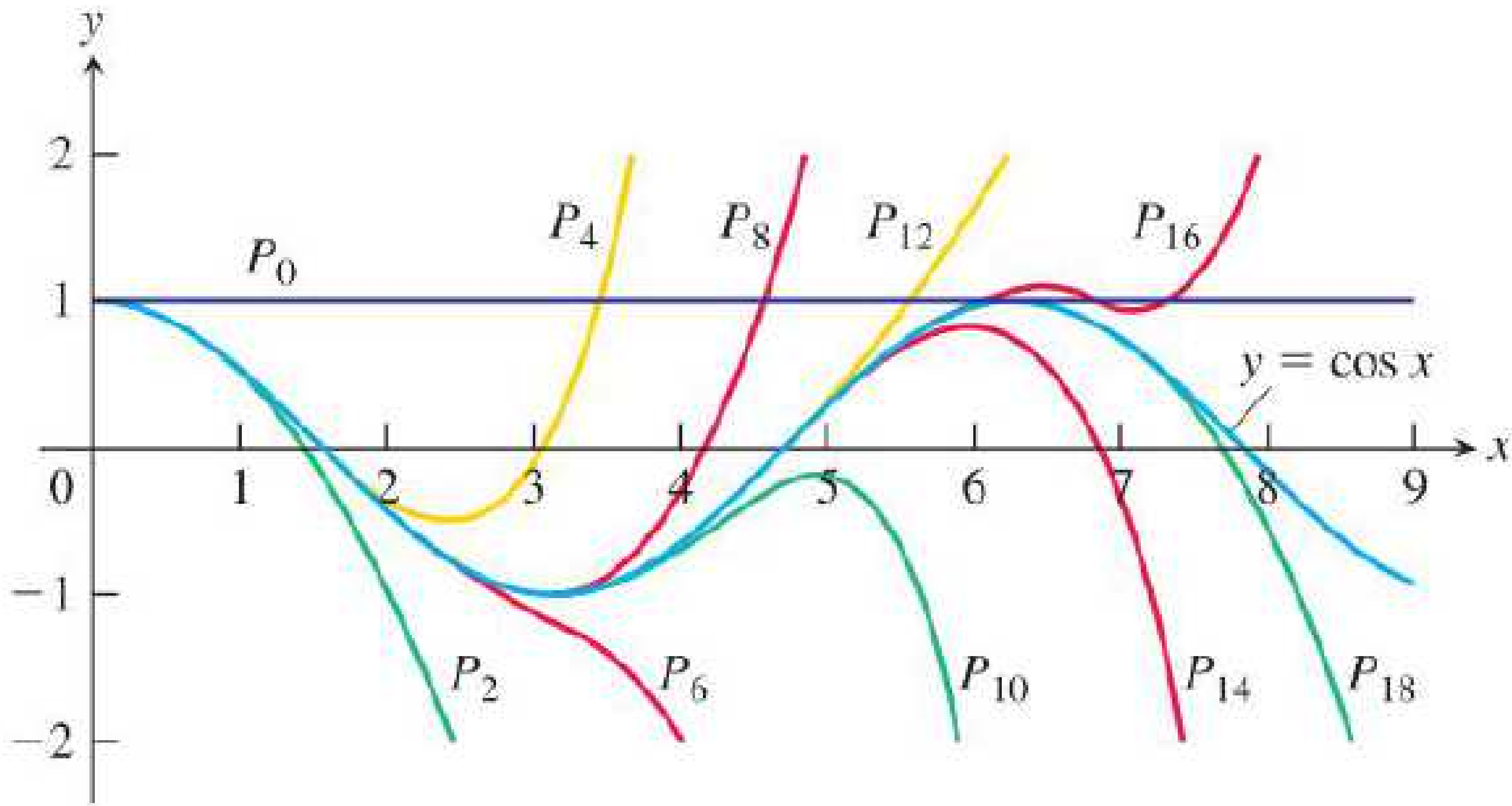
Solution

$$\begin{aligned} f(x) &= \cos x, & f'(x) &= -\sin x, \\ f''(x) &= -\cos x, & f^{(3)}(x) &= \sin x, \\ &\vdots & &\vdots \\ f^{(2n)}(x) &= (-1)^n \cos x, & f^{(2n+1)}(x) &= (-1)^{n+1} \sin x. \end{aligned}$$

$$f^{(2n)}(0) = (-1)^n, \qquad f^{(2n+1)}(0) = 0.$$

$$\begin{aligned} &f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ &= 1 + 0 \cdot x - \frac{x^2}{2!} + 0 \cdot x^3 + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}. \end{aligned}$$

$$P_{2n}(x) = P_{2n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!}.$$

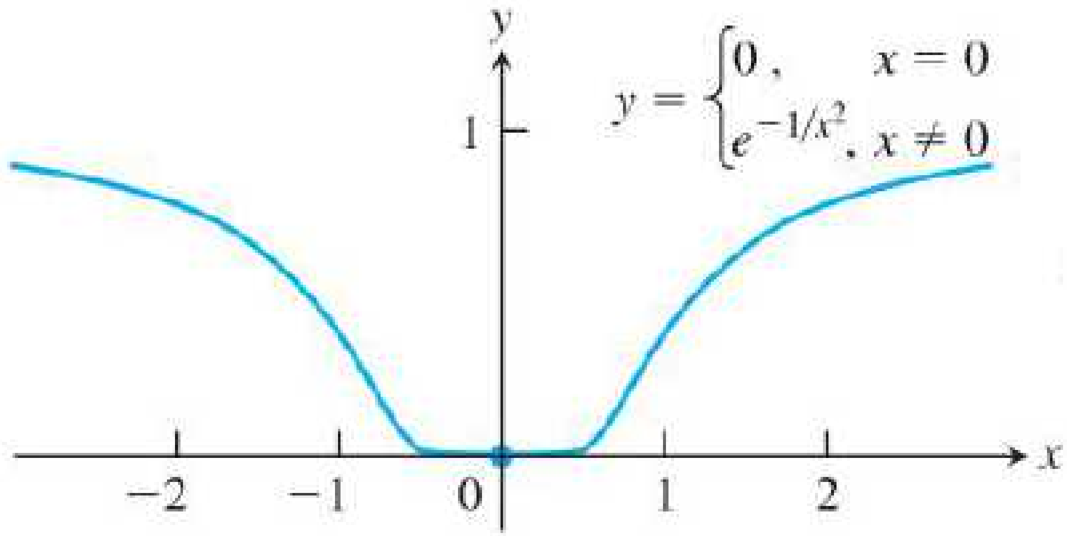


EXAMPLE 4 $f(x) = \begin{cases} 0, & x = 0 \\ e^{-1/x^2}, & x \neq 0 \end{cases}$ $\quad f^{(n)}(0) = 0$ for all n.

the Taylor series generated by f at $x = 0$ is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \neq \boldsymbol{f(x)}$$

$$= 0 + 0 + \cdots + 0 + \cdots.$$



$$f(x) \stackrel{?}{=} f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$+ \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

Two questions still remain.

1. For what values of x can we normally expect a Taylor series to converge to its generating function?
2. How accurately do a function's Taylor polynomials approximate the function on a given interval?

The answers are provided by a theorem of Taylor in the next section.

10.9

Convergence of Taylor Series
泰勒级数的收敛性

THEOREM 23—Taylor's Theorem If f and its first n derivatives $f', f'', \ldots, f^{(n)}$ are continuous on the closed interval between a and b, and $f^{(n)}$ is differentiable on the open interval between a and b, then there exists a number c between a and b such that

$$f(b) = f(a) + f'(a)(b - a) + \frac{f''(a)}{2!}(b - a)^2 + \cdots$$

$$+ \frac{f^{(n)}(a)}{n!}(b - a)^n + \frac{f^{(n+1)}(c)}{(n + 1)!}(b - a)^{n+1}.$$

Taylor's Formula

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots$$
$$+ \frac{f^{(n)}(a)}{n!}(x - a)^n + R_n(x), \qquad (1)$$

where **remainder**

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n + 1)!}(x - a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x. \qquad (2)$$

for each $x \in I$, $f(x) = P_n(x) + R_n(x)$.

$$f(x) = P_n(x) + R_n(x)$$

$$\lim_{n\to\infty} P_n(x) = f(x) - \lim_{n\to\infty} R_n(x) \qquad \lim_{n\to\infty} P_n(x) = f(x) \Leftrightarrow \lim_{n\to\infty} R_n(x) = 0$$

If $R_n(x) \to 0$ as $n \to \infty$ for all $x \in I$, we say that the Taylor series generated by f at $x = a$ **converges** to f on I, and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

EXAMPLE 1

Show that the Taylor series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for every real value of x.

Solution The function has derivatives of all orders throughout $(-\infty, \infty)$.

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x)$$

$$R_n(x) = \frac{e^c}{(n+1)!}x^{n+1} \qquad \text{for some } c \text{ between } 0 \text{ and } x.$$

$$|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!} \qquad \text{when } x \le 0, \qquad \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$$

$$|R_n(x)| < e^x \frac{x^{n+1}}{(n+1)!} \qquad \text{when } x > 0. \qquad \lim_{n\to\infty} R_n(x) = 0,$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots. \qquad \text{for every } x,$$

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n(1),$$

$$R_n(1) = e^c \frac{1}{(n+1)!} < \frac{3}{(n+1)!}.$$

let $n = 10 \quad e \approx 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{10!}$

$$|R_n| < \frac{e}{(n+1)!} < \frac{3}{11!} < 10^{-7}$$

THEOREM 24—The Remainder Estimation Theorem If there is a positive constant M such that $\left|f^{(n+1)}(t)\right| \leq M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$\left|R_n(x)\right| \leq M \frac{|x-a|^{n+1}}{(n+1)!}.$$

If this inequality holds for every n and the other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$.

EXAMPLE 2

Show that the Taylor series for $\sin x$ at $x = 0$ converges for all x.

Solution

$$f(x) = \sin x, \qquad f'(x) = \cos x,$$

$$f''(x) = -\sin x, \qquad f'''(x) = -\cos x,$$

$$\vdots \qquad\qquad \vdots$$

$$f^{(2k)}(x) = (-1)^k \sin x, \qquad f^{(2k+1)}(x) = (-1)^k \cos x,$$

$$f^{(2k)}(0) = 0 \quad \text{and} \quad f^{(2k+1)}(0) = (-1)^k.$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots$$

for all x.

$$|R_{2k+1}(x)| \le 1 \cdot \frac{|x|^{2k+2}}{(2k+2)!} \to 0 \text{ as } k \to \infty,$$

EXAMPLE 3

Show that the Taylor series for $\cos x$ at $x = 0$ converges to $\cos x$

Solution

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + R_{2k}(x).$$

$$|R_{2k}(x)| \le 1 \cdot \frac{|x|^{2k+1}}{(2k+1)!} \to 0 \text{ as } k \to \infty.$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

for all x.

EXAMPLE 4 Using Taylor Series

Using known series, find the first few terms of the Taylor series for the given function using power series operations.

(a) $\frac{1}{3}(2x + x\cos x)$ **(b)** $e^x \cos x$

Solution

(a) $$\frac{1}{3}(2x + x\cos x) = \frac{2}{3}x + \frac{1}{3}x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k\frac{x^{2k}}{(2k)!} + \cdots\right)$$

$$= \frac{2}{3}x + \frac{1}{3}x - \frac{x^3}{3!} + \frac{x^5}{3\cdot 4!} - \cdots = x - \frac{x^3}{6} + \frac{x^5}{72} - \cdots$$

(b) $$e^x \cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\cdot\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$$

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$$

$$\cos 2x = \sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k)!} = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots$$

$$= 1 - \frac{2^2 x^2}{2!} + \frac{2^4 x^4}{4!} - \frac{2^6 x^6}{6!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} x^{2k}}{(2k)!}.$$

EXAMPLE 5

For what values of x can we replace $\sin x$ by $x - (x^3/3!)$ with an error of magnitude no greater than 3×10^{-4}?

Solution

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

after $(x^3/3!)$ is no greater than $\left|\frac{x^5}{5!}\right| = \frac{|x|^5}{120}$.

$$\frac{|x|^5}{120} < 3 \times 10^{-4} \qquad |x| < \sqrt[5]{360 \times 10^{-4}} \approx 0.514.$$

when x is positive,

the estimate $x - (x^3/3!)$ for $\sin x$ is an under estimate

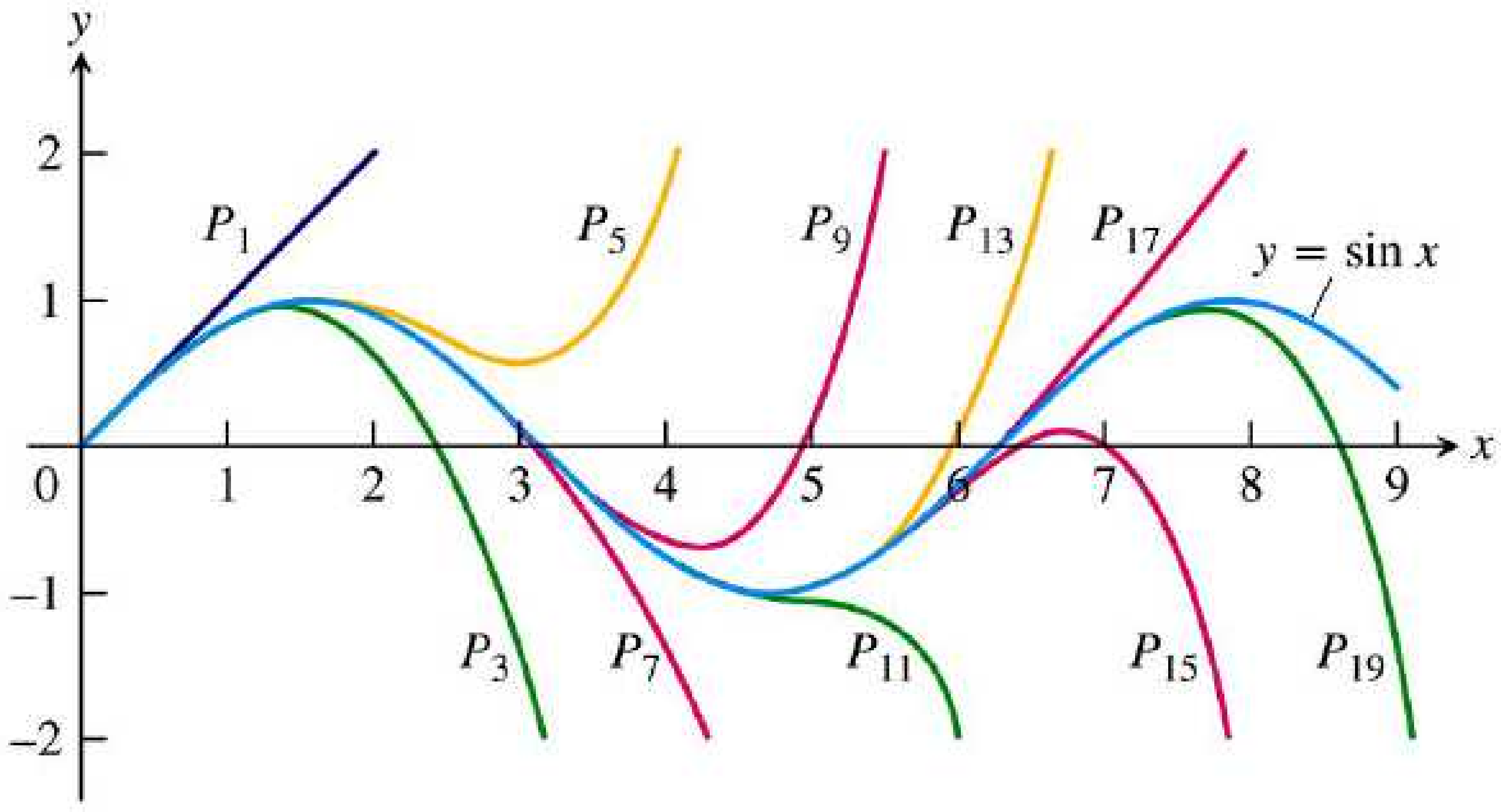


FIGURE 10.20 The polynomials

$$P_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

converge to $\sin x$ as $n \to \infty$. Notice how closely $P_3(x)$ approximates the sine curve for $x \le 1$ (Example 5).

A Proof of Taylor's Theorem

We prove Taylor's theorem assuming $a < b$.

$$P_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

$$\phi_n(x) = P_n(x) + K(x - a)^{n+1} \quad \phi^{(k)}(a) = f^{(k)}(a), k = 0, 1, \cdots, n$$

We now choose the particular value of K that makes the curve $y = \phi_n(x)$ agree with the original curve $y = f(x)$ at $x = b$. In symbols,

$$f(b) = P_n(b) + K(b - a)^{n+1}, \qquad \text{or} \qquad K = \frac{f(b) - P_n(b)}{(b - a)^{n+1}}.$$

$F(x) = f(x) - \phi_n(x)$ $\quad$ F and F' are continuous on $[a, b]$,

$F(a) = F(b) = 0$ $\quad$ $F'(c_1) = 0$ $\quad$ for some c_1 in (a, b).

$F'(a) = F'(c_1) = 0$ and both F' and F'' are continuous on $[a, c_1]$,

$F''(c_2) = 0$ $\quad$ for some c_2 in (a, c_1).

$$c_3 \quad \text{in } (a, c_2) \qquad \text{such that } F'''(c_3) = 0,$$

$$c_4 \quad \text{in } (a, c_3) \qquad \text{such that } F^{(4)}(c_4) = 0,$$

$$\vdots$$

$$c_n \quad \text{in } (a, c_{n-1}) \qquad \text{such that } F^{(n)}(c_n) = 0.$$

$F^{(n)}$ is continuous on $[a, c_n]$ and differentiable on (a, c_n),

$F^{(n)}(a) = F^{(n)}(c_n) = 0,$ there is a number c_{n+1} in (a, c_n)

$$F^{(n+1)}(c_{n+1}) = 0.$$

$$F^{(n+1)}(x) = f^{(n+1)}(x) - 0 - (n + 1)!K. \qquad K = \frac{f^{(n+1)}(c)}{(n + 1)!}$$

$$f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n + 1)!}(b - a)^{n+1}. \qquad c = c_{n+1} \text{ in } (a, b).$$

10.10

The Binomial Series and Applications of Taylor Series
二项式级数和泰勒级数的应用

The Binomial Series for Powers and Roots

$$f(x) = (1 + x)^m, \qquad m \text{ is constant,}$$

$$\begin{aligned}
f(x) &= (1 + x)^m \\
f'(x) &= m(1 + x)^{m-1} \\
f''(x) &= m(m - 1)(1 + x)^{m-2} \\
f'''(x) &= m(m - 1)(m - 2)(1 + x)^{m-3} \\
&\ \vdots \\
f^{(k)}(x) &= m(m - 1)(m - 2)\cdots(m - k + 1)(1 + x)^{m-k}.
\end{aligned}$$

$$f^{(k)}(0) = m(m-1)\cdots(m-k+1), k = 0, 1, \cdots$$

$$(1 + x)^m =$$

called the **binomial series,**

$$1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots$$

$$+ \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}x^k + \cdots.$$

$$\left|\frac{u_{k+1}}{u_k}\right| = \left|\frac{m-k}{k+1}x\right| \to |x| \quad \text{as } k \to \infty.$$

converges absolutely for $|x| < 1$.

The Binomial Series

For $-1 < x < 1$,

$$(1 + x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k,$$

where we define

$$\binom{m}{1} = m, \qquad \binom{m}{2} = \frac{m(m-1)}{2!},$$

and

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!} \qquad \text{for } k \geq 3.$$

EXAMPLE 1 If $m = -1$,

$$\binom{-1}{1} = -1, \qquad \binom{-1}{2} = \frac{-1(-2)}{2!} = 1,$$

$$\binom{-1}{k} = \frac{-1(-2)(-3)\cdots(-1-k+1)}{k!} = (-1)^k\left(\frac{k!}{k!}\right) = (-1)^k.$$

$$(1+x)^{-1} = 1 + \sum_{k=1}^{\infty}(-1)^k x^k = 1 - x + x^2 - x^3 + \cdots + (-1)^k x^k + \cdots.$$

$$-1 < x < 1$$

EXAMPLE 2

$$(1+x)^{1/2} = 1 + \frac{x}{2} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3$$

$$+ \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}x^4 + \cdots$$

$$(1+x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots,$$

$$-1 < x < 1$$

$$\sqrt{1+x} \approx 1 + (x/2) \quad \text{for } |x| \text{ small.}$$

$$\sqrt{1-x^2} \approx 1 - \frac{x^2}{2} - \frac{x^4}{8} \quad \text{for } |x^2| \text{ small}$$

$$\sqrt{1-\frac{1}{x}} \approx 1 - \frac{1}{2x} - \frac{1}{8x^2} \quad \text{for } \left|\frac{1}{x}\right| \text{ small, that is, } |x| \text{ large.}$$

Evaluating Nonelementary Integrals

EXAMPLE 3 Express $\int \sin x^2\, dx$ as a power series.

Solution From the series for $\sin x$ we substitute x^2 for x to obtain

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{18}}{9!} - \cdots.$$

Therefore,

$$\int \sin x^2\, dx = C + \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \frac{x^{10}}{19 \cdot 9!} - \cdots.$$

EXAMPLE 4 Estimate $\int_0^1 \sin x^2\,dx$ with an error of less than 0.001.

Solution

$$\int_0^1 \sin x^2\,dx = \frac{1}{3} - \frac{1}{7\cdot 3!} + \frac{1}{11\cdot 5!} - \frac{1}{15\cdot 7!} + \frac{1}{19\cdot 9!} - \cdots.$$

$$\frac{1}{11\cdot 5!} \approx 0.00076 \qquad \int_0^1 \sin x^2\,dx \approx \frac{1}{3} - \frac{1}{42} \approx 0.310.$$

$$\int_0^1 \sin x^2\,dx \approx \frac{1}{3} - \frac{1}{7\cdot 3!} + \frac{1}{11\cdot 5!} - \frac{1}{15\cdot 7!} \approx 0.310268$$

with an error of less than 10^{-6}.

$$\int_0^1 \sin x^2\,dx \approx \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \frac{1}{75600} + \frac{1}{6894720} \approx 0.310268303,$$

with an error of about 1.08×10^{-9}.

Arctangents

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n\frac{x^{2n+1}}{2n+1} + R_n(x),$$

If $|x| \le 1$,

$$|R_n(x)| \le \frac{|x|^{2n+3}}{2n+3}. \to 0 \text{ as } n \to \infty.$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots,$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad |x| \leq 1.$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots + \frac{(-1)^n}{2n+1} + \cdots.$$

Leibniz's formula:

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots,$$

$$\tan^{-1}\frac{1}{2} = \frac{1}{2} - \frac{1}{2^3 \cdot 3} + \frac{1}{2^5 \cdot 5} - \frac{1}{2^7 \cdot 7} + \cdots$$

$$\tan^{-1}\frac{1}{3} = \frac{1}{3} - \frac{1}{3^3 \cdot 3} + \frac{1}{3^5 \cdot 5} - \frac{1}{3^7 \cdot 7} + \cdots$$

$$\tan\alpha = \frac{1}{2}, \tan\beta = \frac{1}{3},$$

$$\tan(\alpha + \beta) = 1,$$

$$\alpha + \beta = \frac{\pi}{4},$$

取前四项计算，误差不超过10^{-4}

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots + \frac{(-1)^n}{2n+1} + \cdots.$$

若要误差不超过10^{-4}，需要5000项！

Evaluating Indeterminate Forms

EXAMPLE 5 Evaluate $\displaystyle\lim_{x\to 1}\frac{\ln x}{x-1}$.

Solution $$\frac{1}{x}=\frac{1}{1-(1-x)}=1+(1-x)+(1-x)^2+\cdots+(1-x)^n+\cdots$$

$$=1-(x-1)+(x-1)^2+\cdots+(-1)^n(x-1)^n+\cdots$$

$$\ln x=(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}+\cdots+\frac{(-1)^{n-1}(x-1)^n}{n}+\cdots$$

$$|x-1|<1$$

$$\lim_{x\to 1}\frac{\ln x}{x-1}=\lim_{x\to 1}\left(1-\frac{1}{2}(x-1)+\cdots\right)=1.$$

EXAMPLE 6 Evaluate $\lim_{x\to 0} \frac{\sin x - \tan x}{x^3}$.

Solution $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \qquad \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots.$

$$\sin x - \tan x = -\frac{x^3}{2} - \frac{x^5}{8} - \cdots = x^3\left(-\frac{1}{2} - \frac{x^2}{8} - \cdots\right)$$

$$\lim_{x\to 0} \frac{\sin x - \tan x}{x^3} = \lim_{x\to 0}\left(-\frac{1}{2} - \frac{x^2}{8} - \cdots\right) = -\frac{1}{2}.$$

EXAMPLE 7 Find $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$.

Solution

$$\frac{1}{\sin x}-\frac{1}{x}=\frac{x-\sin x}{x\sin x}=\frac{x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)}{x\cdot\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)}$$

$$=\frac{x^3\left(\frac{1}{3!}-\frac{x^2}{5!}+\cdots\right)}{x^2\left(1-\frac{x^2}{3!}+\cdots\right)}=x\cdot\frac{\frac{1}{3!}-\frac{x^2}{5!}+\cdots}{1-\frac{x^2}{3!}+\cdots}.$$

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to 0}\left(x\cdot\frac{\frac{1}{3!}-\frac{x^2}{5!}+\cdots}{1-\frac{x^2}{3!}+\cdots}\right)=0.$$

Euler's Identity

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad (-\infty, +\infty)$$

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots$$

$$i^2 = -1, \qquad i^3 = i^2 i = -i, \qquad i^4 = i^2 i^2 = 1, \qquad i^5 = i^4 i = i,$$

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \frac{i^6\theta^6}{6!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) = \cos\theta + i\sin\theta.$$

DEFINITION For any real number θ, $e^{i\theta} = \cos\theta + i\sin\theta$.

Euler's identity.

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \qquad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \qquad -1 < x \leq 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad |x| \leq 1$$