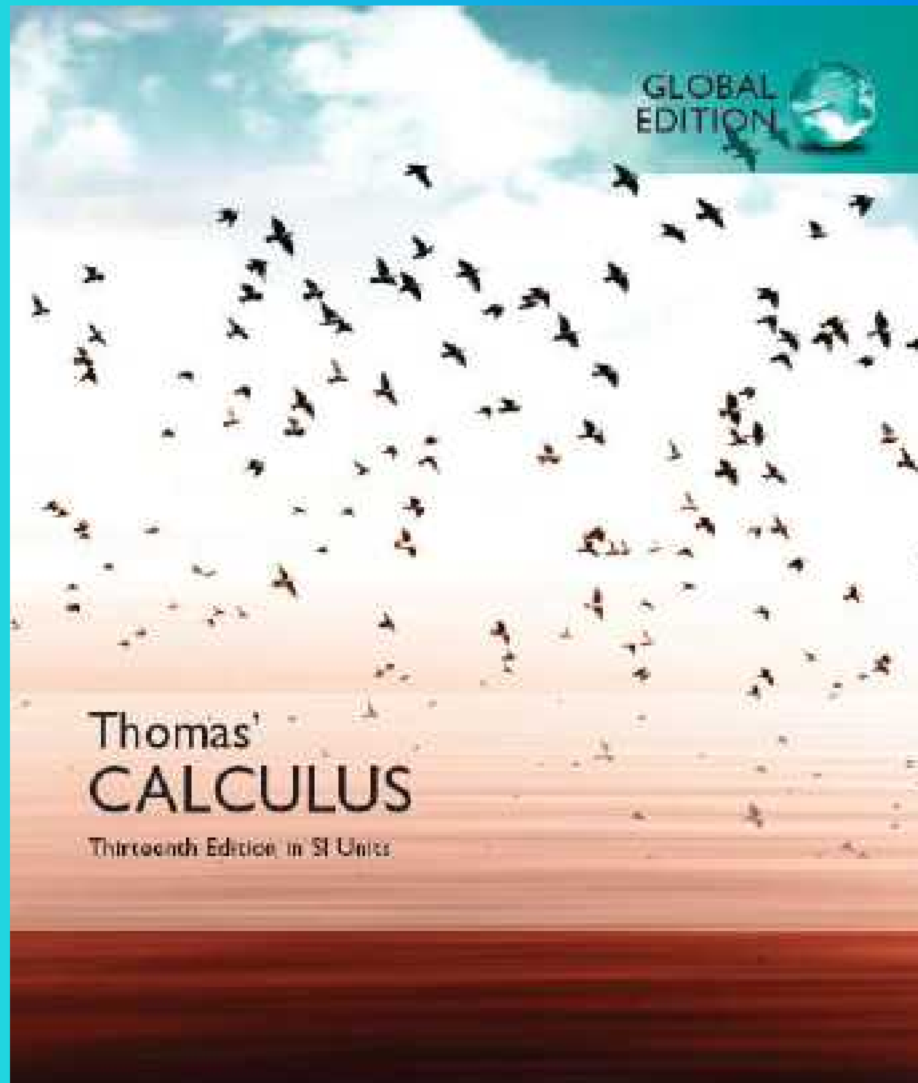

GLOBAL EDITION
Thomas'
CALCULUS
Thirteenth Edition in SI Units

Chapter 16

Integrals and Vector Fields

向量场的积分

16.1

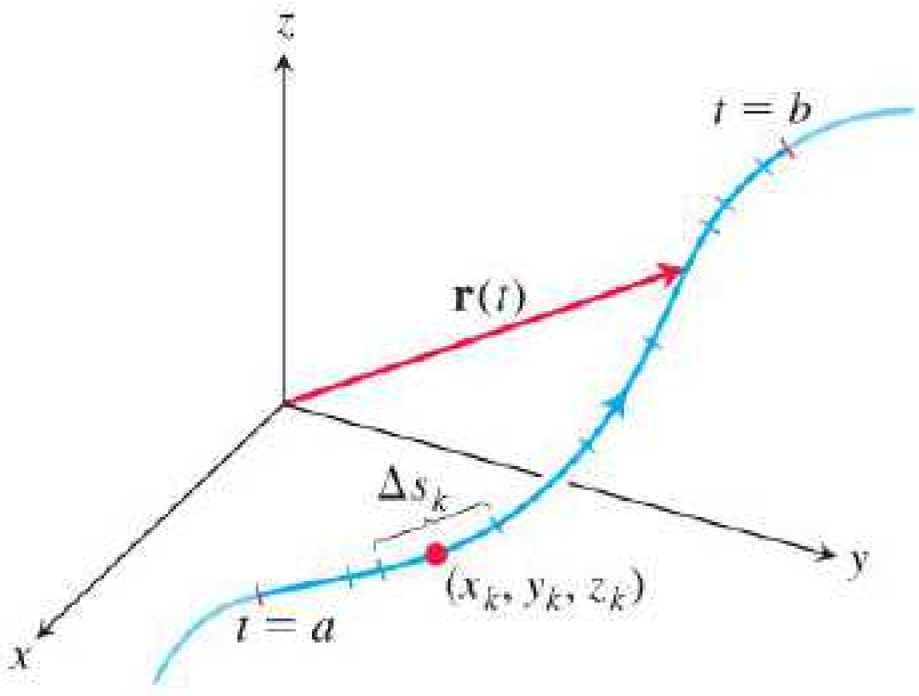
Line Integrals

线积分

To calculate the total mass of a wire lying along a curve in space,

$$S_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta s_k, \qquad \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta s_k,$$

Suppose that $f(x, y, z)$ is a real-valued function over the curve C



DEFINITION If f is defined on a curve C given parametrically by $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$, $a \le t \le b$, then the **line integral of f over C** is

$$\int_C f(x, y, z)\, ds = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\, \Delta s_k,$$

provided this limit exists.

If the curve C is smooth for $a \le t \le b$

function f is continuous on C, then the limit can be shown to exist.

$$s(t) = \int_a^t |\mathbf{v}(\tau)|\, d\tau, \qquad ds = |\mathbf{v}(t)|\, dt$$

$$\int_C f(x, y, z)\, ds = \int_a^b f(g(t), h(t), k(t))\, |\mathbf{v}(t)|\, dt.$$

How to Evaluate a Line Integral

To integrate a continuous function $f(x, y, z)$ over a curve C:

1. Find a smooth parametrization of C,

$$\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}, \qquad a \le t \le b.$$

2. Evaluate the integral as

$$\int_C f(x, y, z)\, ds = \int_a^b f(g(t), h(t), k(t))\left|\mathbf{v}(t)\right|\, dt.$$

$$= \int_a^b f(g(t), h(t), k(t))\sqrt{g'(t)^2 + h'(t)^2 + k'(t)^2}\, dt$$

$$\int_C f\, ds = \int_{C_1} f\, ds + \int_{C_2} f\, ds + \cdots + \int_{C_n} f\, ds.$$

EXAMPLE 1

Integrate $f(x, y, z) = x - 3y^2 + z$ over the line segment C joining the origin to the point $(1, 1, 1)$

Solution $\quad \mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, \qquad 0 \le t \le 1.$

$$|\mathbf{v}(t)| = |\mathbf{i} + \mathbf{j} + \mathbf{k}| = \sqrt{3}$$

$$\int_C f(x, y, z)\, ds = \int_0^1 f(t, t, t)\left(\sqrt{3}\right) dt$$

$$= \int_0^1 (t - 3t^2 + t)\sqrt{3}\, dt = \sqrt{3}\int_0^1 (2t - 3t^2)\, dt = 0.$$

EXAMPLE 2

Figure 16.3 shows another path from the origin to (1, 1, 1), the union of line segments C_1 and C_2. Integrate $f(x, y, z) = x - 3y^2 + z$ over $C_1 \cup C_2$.

Solution

$$C_1: \quad \mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}, \quad 0 \le t \le 1; \quad |\mathbf{v}| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

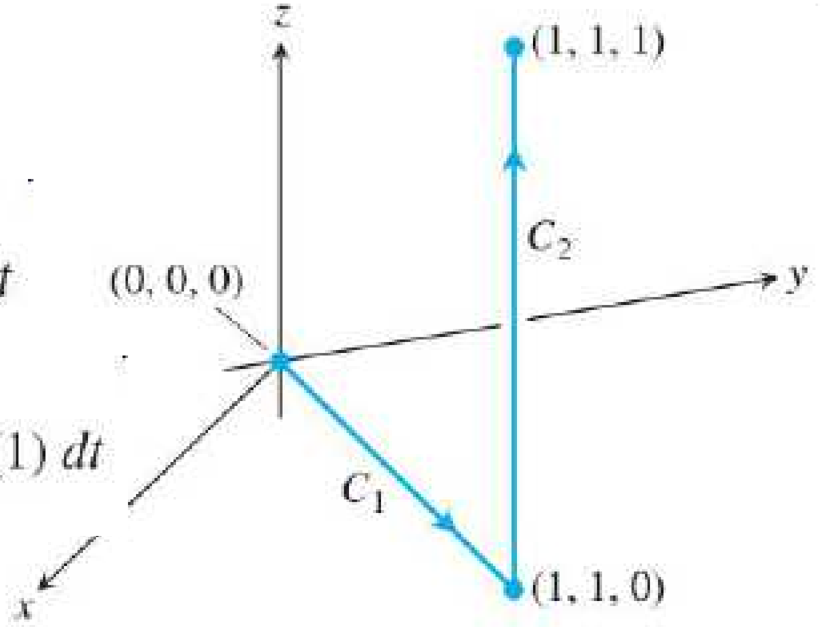
$$C_2: \quad \mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 1; \quad |\mathbf{v}| = \sqrt{0^2 + 0^2 + 1^2} = 1.$$

$$\int_{C_1 \cup C_2} f(x, y, z)\, ds =$$

$$= \int_0^1 f(t, t, 0)\sqrt{2}\, dt + \int_0^1 f(1, 1, t)(1)\, dt$$

$$= \int_0^1 (t - 3t^2 + 0)\sqrt{2}\, dt + \int_0^1 (1 - 3 + t)(1)\, dt$$

$$= -\frac{\sqrt{2}}{2} - \frac{3}{2}.$$



EXAMPLE 3

Find the line integral of $f(x, y, z) = 2xy + \sqrt{z}$ over the helix $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$, $0 \le t \le \pi$.

Solution $\mathbf{v}(t) = \mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$ $\quad |v(t)| = \sqrt{2}$

$$f(\mathbf{r}(t)) = f(\cos t, \sin t, t) = 2\cos t \sin t + \sqrt{t} = \sin 2t + \sqrt{t}.$$

$$\int_C f(x, y, z)\, ds = \int_0^{\pi} \left(\sin 2t + \sqrt{t}\right)\sqrt{2}\, dt$$

$$= \sqrt{2}\left[-\frac{1}{2}\cos 2t + \frac{2}{3}t^{3/2}\right]_0^{\pi} = \frac{2\sqrt{2}}{3}\pi^{3/2} \approx 5.25.$$

Mass and Moment Calculations

C is parametrized by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},\ a \le t \le b,$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

$$M = \int_a^b \delta(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

$$M_{yz} = \int_C x\,\delta\, ds, \qquad M_{xz} = \int_C y\,\delta\, ds, \qquad M_{xy} = \int_C z\,\delta\, ds$$

$$\bar{x} = M_{yz}/M, \qquad \bar{y} = M_{xz}/M, \qquad \bar{z} = M_{xy}/M$$

EXAMPLE 4

A slender metal arch, denser at the bottom than top, lies along the semicircle $y^2 + z^2 = 1, z \geq 0$, in the yz-plan Find the center of the arch's mass if the density at the point (x, y, z) on the arch is $\delta(x, y, z) = 2 - z$.

Solution We know that $\bar{x} = 0$ and $\bar{y} = 0$

$$\mathbf{r}(t) = (\cos t)\mathbf{j} + (\sin t)\mathbf{k}, \qquad 0 \leq t \leq \pi.$$

$$|\mathbf{v}(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = 1,$$

$$M = \int_C \delta\, ds = \int_C (2 - z)\, ds = \int_0^{\pi} (2 - \sin t)\, dt = 2\pi - 2$$

$$M_{xy} = \int_C z\delta\, ds = \int_C z(2 - z)\, ds = \int_0^{\pi} (\sin t)(2 - \sin t)\, dt = \frac{8 - \pi}{2}$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{8 - \pi}{2} \cdot \frac{1}{2\pi - 2} = \frac{8 - \pi}{4\pi - 4} \approx 0.57.$$

the center of mass is $(0, 0, 0.57)$.

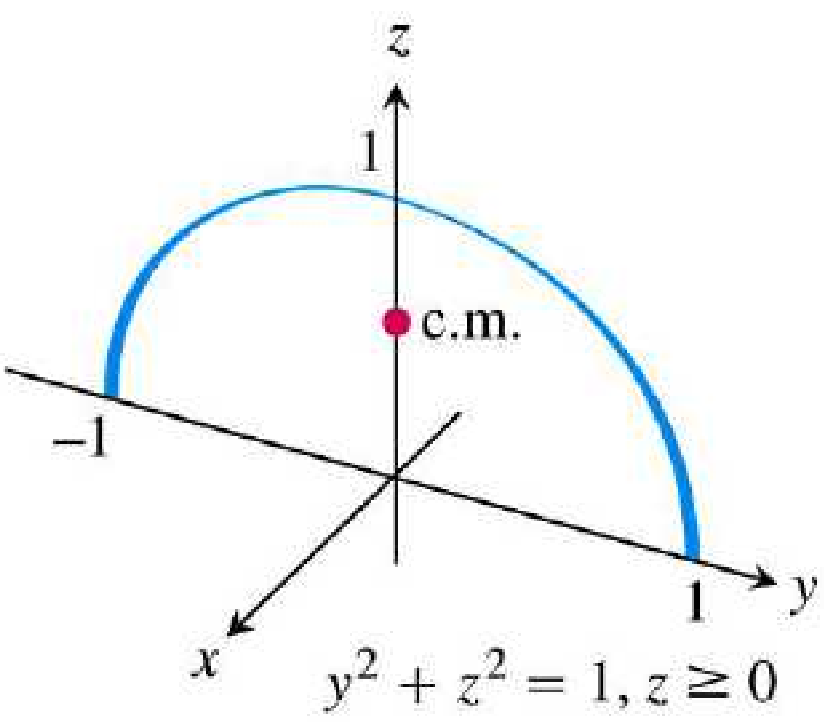
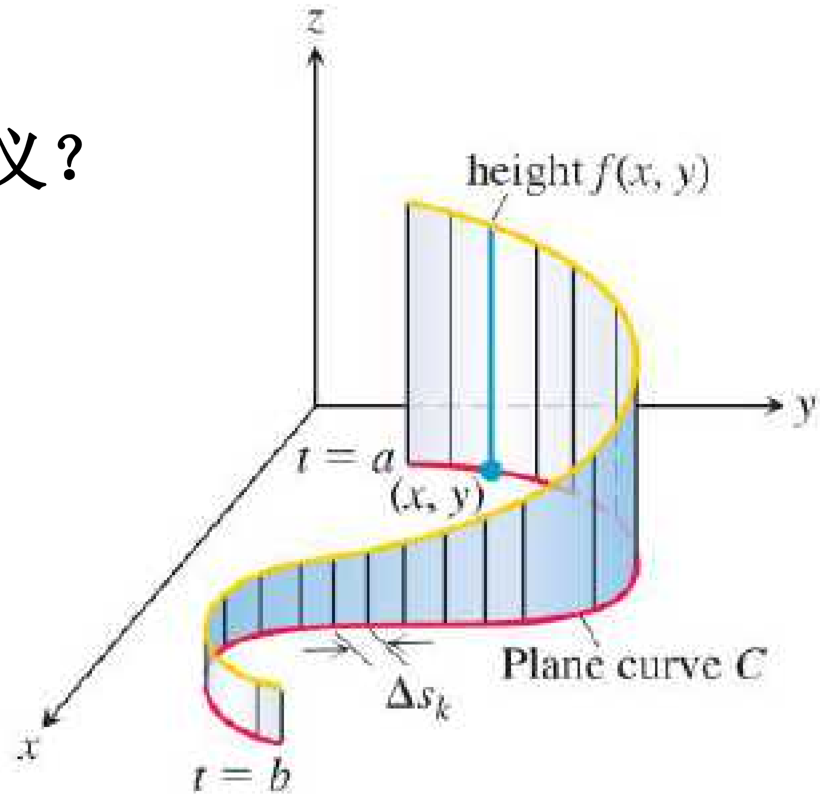


FIGURE 16.4 Example 3 shows how to find the center of mass of a circular arch of variable density.

Line Integrals in the Plane

$$\int_C f\, ds = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k)\, \Delta s_k,$$

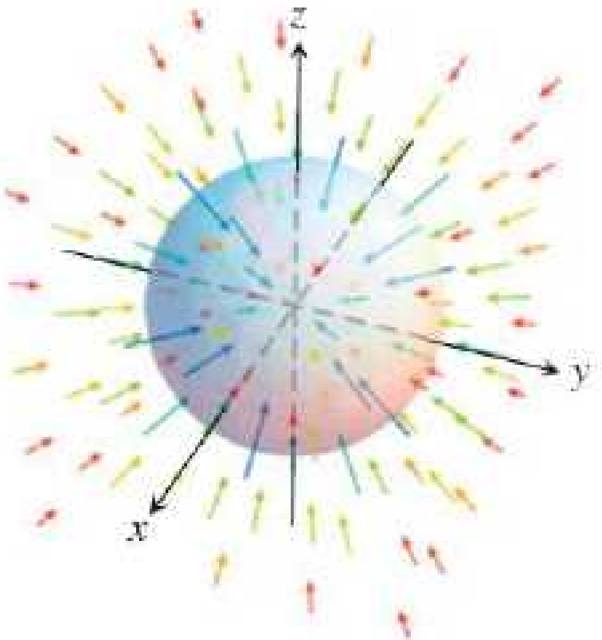
$\int_{\substack{x+y=1 \\ 0 \le x \le 1}} (x^2 + y^2)\, ds$的几何意义？

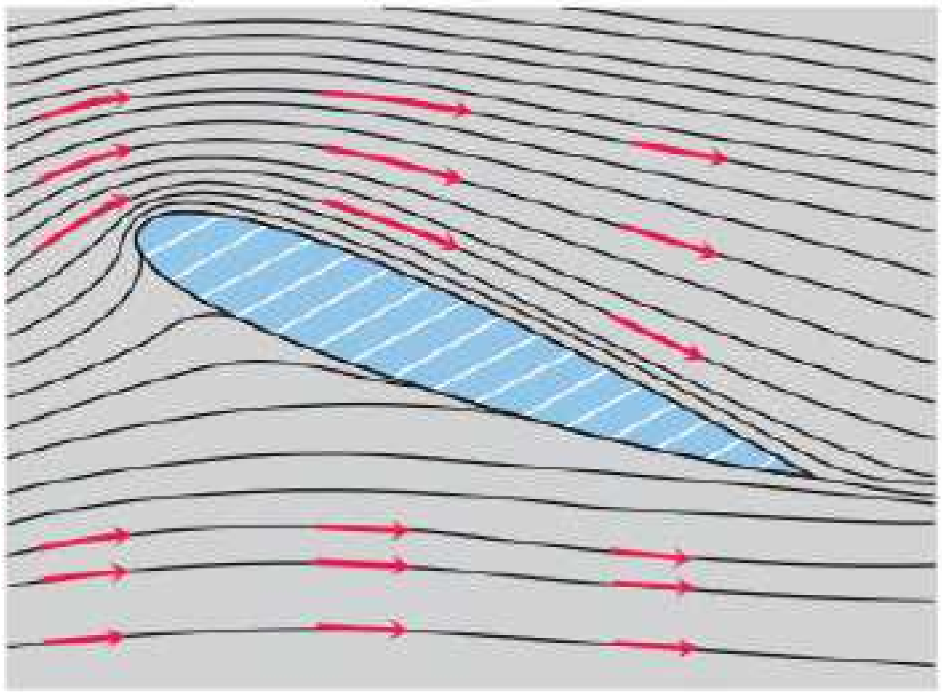


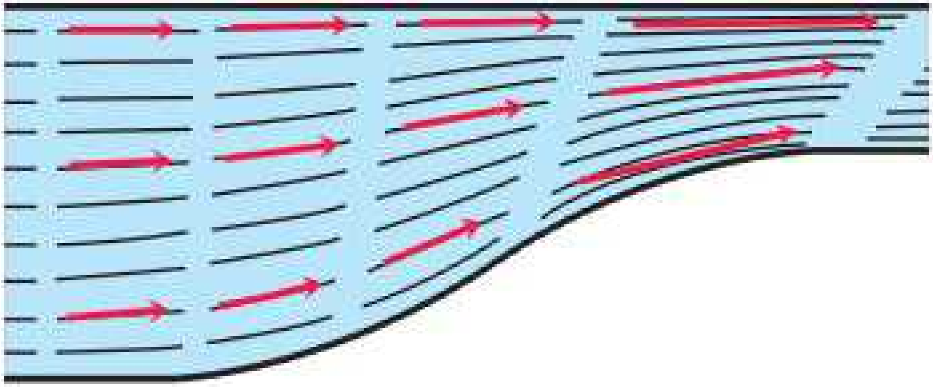
16.2

Vector Fields and Line Integrals: Work, Circulation, and Flux
向量场和线积分，功，环流量

Vector Fields







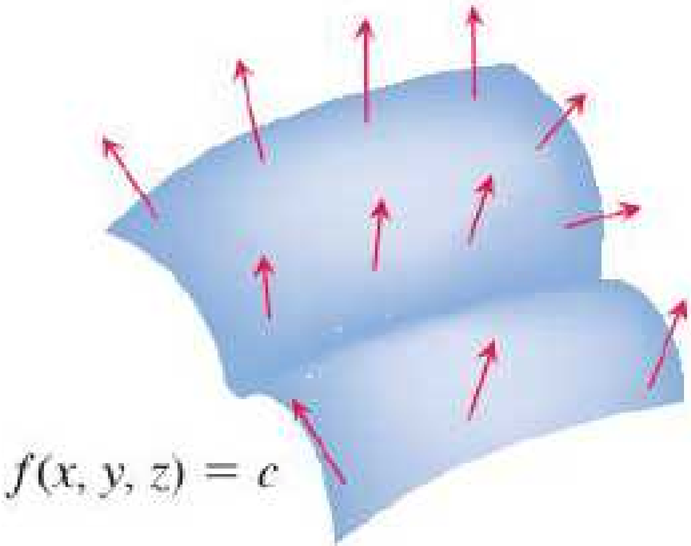
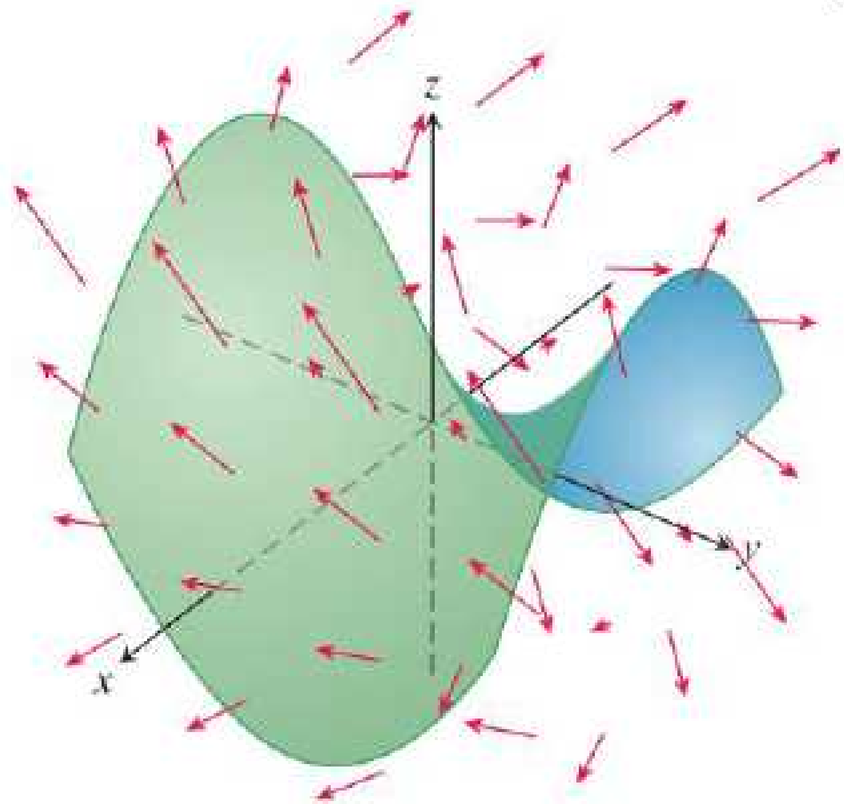
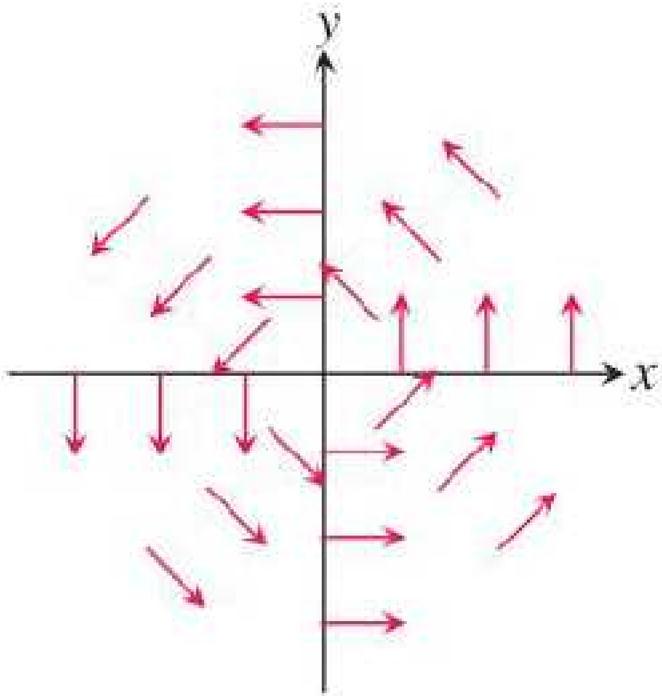
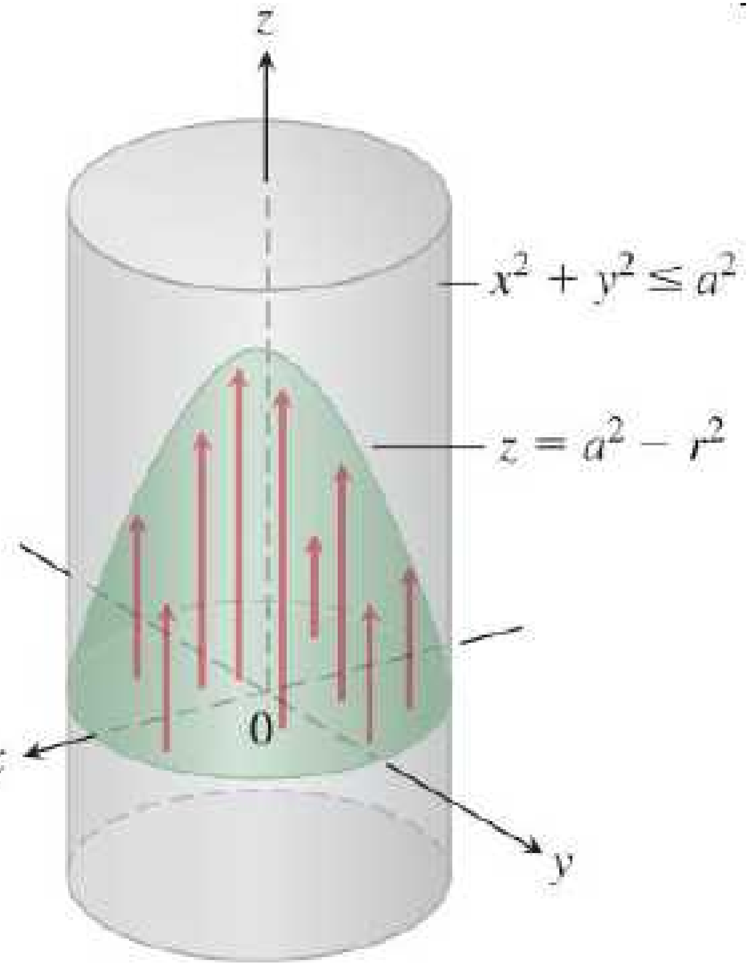


FIGURE 16.10 The field of gradient vectors ∇f on a surface $f(x, y, z) = c$.

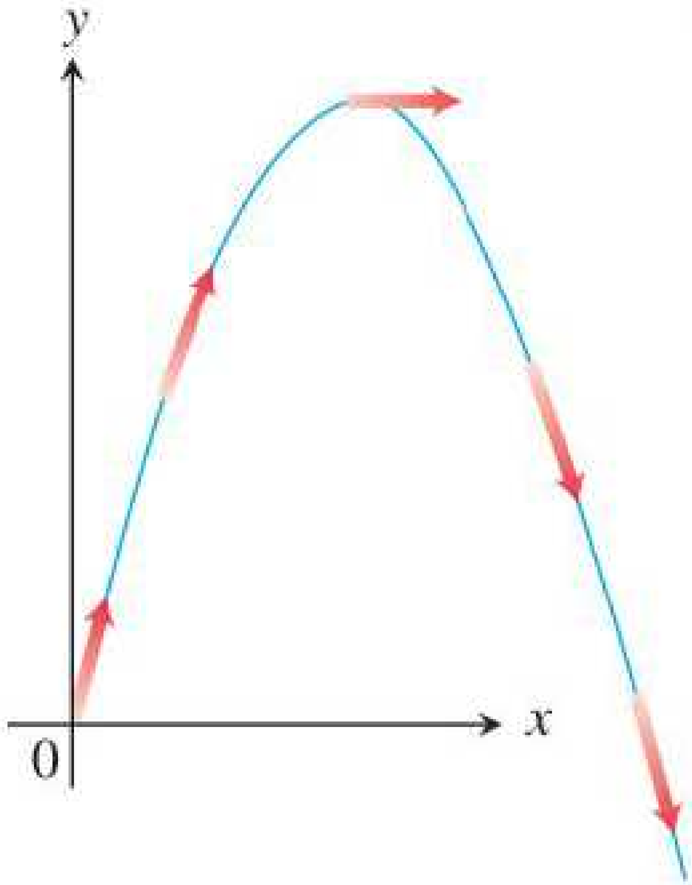




$\mathbf{F} = (-y\mathbf{i} + x\mathbf{j})/(x^2 + y^2)^{1/2}$



$\mathbf{v} = (a^2 - r^2)\mathbf{k}$


y
x
0

Gradient Fields

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

EXAMPLE 1

Suppose that the temperature T at each point (x, y, z) in a region of space is given by $T = 100 - x^2 - y^2 - z^2$,

and that $\mathbf{F}(x, y, z)$ is defined to be the gradient of T. Find the vector field $\mathbf{F}$.

Solution $\mathbf{F} = \nabla T = -2x\mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k}.$

Line Integrals of Vector Fields

Assume that the vector field $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$

represents a force

C is parametrized by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, $a \le t \le b$,

Find the work done by the force $\mathbf{F}$ over C

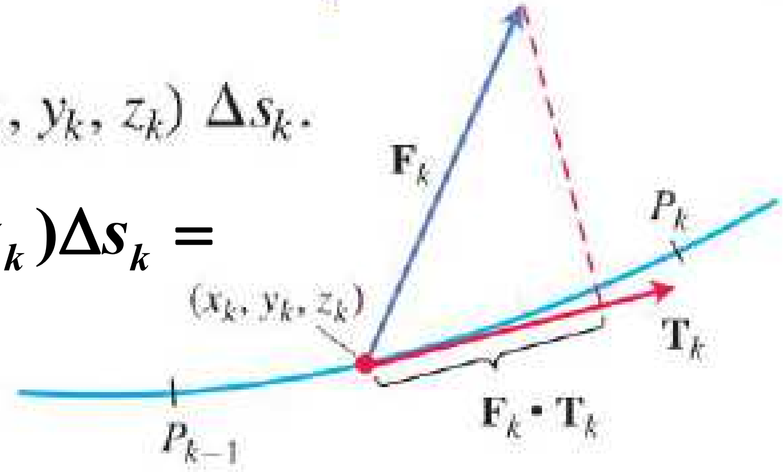
We divide C into n subarcs $P_{k-1}P_k$ with lengths Δs_k,

choose any point (x_k, y_k, z_k) in the subarc $P_{k-1}P_k$

let $\mathbf{T}(x_k, y_k, z_k)$ be the unit tangent

$$W \approx \sum_{k=1}^{n} W_k \approx \sum_{k=1}^{n} \mathbf{F}(x_k, y_k, z_k) \cdot \mathbf{T}(x_k, y_k, z_k)\, \Delta s_k .$$

$$W = \lim_{\|P\| \to 0} \sum_{k=1}^{n} F(x_k, y_k, z_k) \cdot T(x_k, y_k, z_k) \Delta s_k = \int_C \mathbf{F} \cdot \mathbf{T}\, ds.$$



DEFINITION Let $\mathbf{F}$ be a vector field with continuous components defined along a smooth curve C parametrized by $\mathbf{r}(t)$, $a \le t \le b$. Then the **line integral of F along C** is

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{ds}\right) ds = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

$$\frac{d\mathbf{r}}{ds} ds = d\mathbf{r}$$

Evaluating the Line Integral of $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ Along C: $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt.$$

Evaluating the Line Integral of $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ Along C: $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt.$$

EXAMPLE 2

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = z\mathbf{i} + xy\mathbf{j} - y^2\mathbf{k}$ along the curve C given by $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + \sqrt{t}\,\mathbf{k}$, $0 \le t \le 1$.

Solution $\mathbf{F}(\mathbf{r}(t)) = \sqrt{t}\,\mathbf{i} + t^3\mathbf{j} - t^2\mathbf{k}$ $\quad \dfrac{d\mathbf{r}}{dt} = 2t\mathbf{i} + \mathbf{j} + \dfrac{1}{2\sqrt{t}}\mathbf{k}.$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_0^1 \left(2t^{3/2} + t^3 - \frac{1}{2}t^{3/2}\right) dt = \frac{17}{20}.$$

C is parametrized by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, $a \le t \le b$,

$$dr = (dx)\vec{i} + (dy)\vec{j} + (dz)\vec{k}$$

Let $\vec{F}(x, y, z) = M(x, y, z)\vec{i} + N(x, y, z)\vec{j} + P(x, y, z)\vec{k}$

then $\int_C F \cdot dr = \int_C M(x, y, z)dx + N(x, y, z)dy + P(x, y, z)dz$

$$= \int_C M(x, y, z)dx + \int_C N(x, y, z)dy + \int_C P(x, y, z)dz$$

$$\int_C M(x, y, z)dx = \int_a^b M(x(t), y(t), z(t))x'(t)dt$$

$$\int_C N(x, y, z)dy = \int_a^b N(x(t), y(t), z(t))y'(t)dt$$

$$\int_C P(x, y, z)dz = \int_a^b P(x(t), y(t), z(t))z'(t)dt$$

EXAMPLE 3

Evaluate the line integral $\int_C -y\,dx + z\,dy + 2x\,dz$, where C is the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, $0 \le t \le 2\pi$.

Solution

$$\int_C -y\,dx + z\,dy + 2x\,dz = \int_0^{2\pi} \left[(-\sin t)(-\sin t) + t\cos t + 2\cos t\right] dt$$

$$= \int_0^{2\pi} \left[2\cos t + t\cos t + \sin^2 t\right] dt$$

$$= \pi.$$

DEFINITION Let C be a smooth curve parametrized by $\mathbf{r}(t)$, $a \le t \le b$, and $\mathbf{F}$ be a continuous force field over a region containing C. Then the **work** done in moving an object from the point $A = \mathbf{r}(a)$ to the point $B = \mathbf{r}(b)$ along C is

$$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt.$$

EXAMPLE 4

Find the work done by the force field $\mathbf{F} = (y - x^2)\mathbf{i} + (z - y^2)\mathbf{j} + (x - z^2)\mathbf{k}$ along the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $0 \le t \le 1$, from $(0, 0, 0)$ to $(1, 1, 1)$

Solution

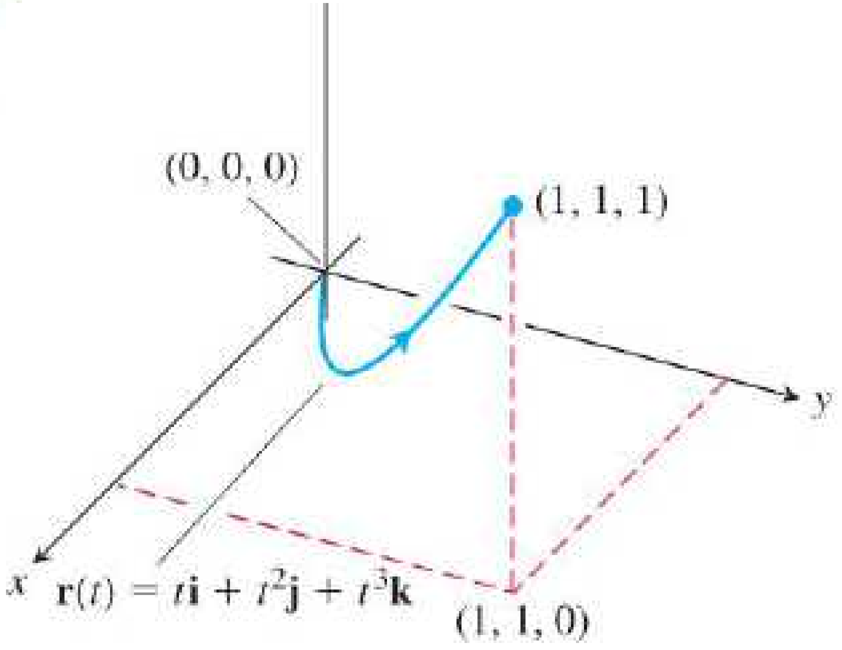
$$\begin{aligned}\mathbf{F} &= (y - x^2)\mathbf{i} + (z - y^2)\mathbf{j} + (x - z^2)\mathbf{k} \\ &= (t^2 - t^2)\mathbf{i} + (t^3 - t^4)\mathbf{j} + (t - t^6)\mathbf{k}.\end{aligned}$$

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}(t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}.$$

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 2t^4 - 2t^5 + 3t^3 - 3t^8.$$

$$\text{Work} = \int_0^1 (2t^4 - 2t^5 + 3t^3 - 3t^8)\, dt$$

$$= \left[\frac{2}{5}t^5 - \frac{2}{6}t^6 + \frac{3}{4}t^4 - \frac{3}{9}t^9\right]_0^1 = \frac{29}{60}.$$



EXAMPLE 5

Find the work done by the force field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in moving an object along the curve C parametrized by $\mathbf{r}(t) = \cos(\pi t)\mathbf{i} + t^2\mathbf{j} + \sin(\pi t)\mathbf{k}, \quad 0 \le t \le 1.$

Solution

$$\mathbf{F}(\mathbf{r}(t)) = \cos(\pi t)\mathbf{i} + t^2\mathbf{j} + \sin(\pi t)\mathbf{k}.$$

$$\frac{d\mathbf{r}}{dt} = -\pi \sin(\pi t)\mathbf{i} + 2t\mathbf{j} + \pi\cos(\pi t)\mathbf{k}.$$

$$\mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} = -\pi \sin(\pi t)\cos(\pi t) + 2t^3 + \pi \sin(\pi t)\cos(\pi t) = 2t^3.$$

$$\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_0^1 2t^3\, dt = \frac{t^4}{2}\Big]_0^1 = \frac{1}{2}.$$

Flow Integrals and Circulation for Velocity Fields

DEFINITIONS If $\mathbf{r}(t)$ parametrizes a smooth curve C in the domain of a continuous velocity field $\mathbf{F}$, the **flow** along the curve from $A = \mathbf{r}(a)$ to $B = \mathbf{r}(b)$ is

$$\text{Flow} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds.$$

EXAMPLE 6

A fluid's velocity field is $\mathbf{F} = x\mathbf{i} + z\mathbf{j} + y\mathbf{k}$. Find the flow along the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}, 0 \le t \le \pi/2$.

Solution

$$\mathbf{F} = x\mathbf{i} + z\mathbf{j} + y\mathbf{k} = (\cos t)\mathbf{i} + t\mathbf{j} + (\sin t)\mathbf{k}$$

$$\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}.$$

$$\begin{aligned} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} &= (\cos t)(-\sin t) + (t)(\cos t) + (\sin t)(1) \\ &= -\sin t \cos t + t \cos t + \sin t. \end{aligned}$$

$$\text{Flow} = \int_{t=a}^{t=b} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_0^{\pi/2} (-\sin t \cos t + t \cos t + \sin t)\, dt$$

$$= \frac{\pi}{2} - \frac{1}{2}.$$

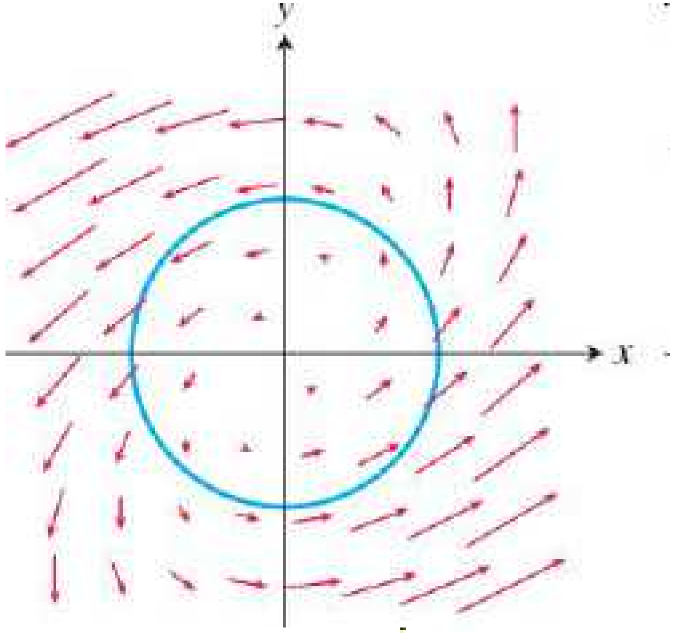
EXAMPLE 7

Find the circulation of the field $\mathbf{F} = (x - y)\mathbf{i} + x\mathbf{j}$ around the circle $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j},\ 0 \leq t \leq 2\pi$

Solution $\mathbf{F} = (x - y)\mathbf{i} + x\mathbf{j} = (\cos t - \sin t)\mathbf{i} + (\cos t)\mathbf{j}.$

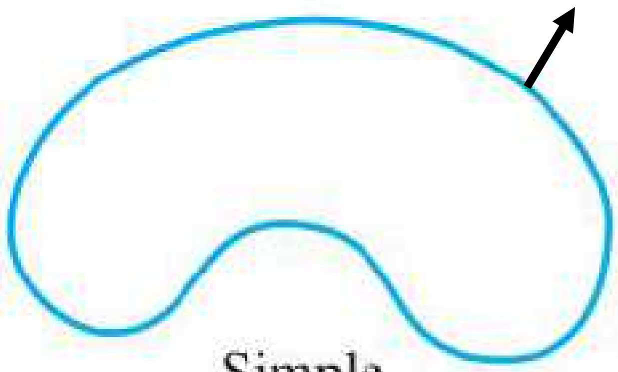
$$\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}.$$

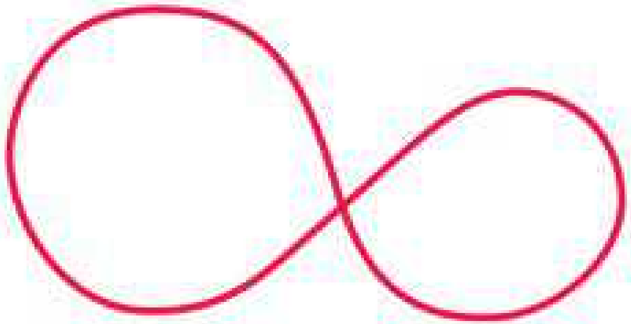
$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin t \cos t + \underbrace{\sin^2 t + \cos^2 t}_{1}$$



$$\text{Circulation} = \int_0^{2\pi} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\,dt = \int_0^{2\pi} (1 - \sin t \cos t)\,dt = 2\pi.$$

Flux Across a Simple Closed Plane Curve





To find the rate at which a fluid is entering or leaving a region enclosed by a smooth simple closed curve C in the xy- plane, we calculate the line integral over C of $\mathbf{F} \cdot \mathbf{n}$,

$\mathbf{n}$ the direction of the curve's outward-pointing normal vector.

DEFINITION If C is a smooth simple closed curve in the domain of a continuous vector field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in the plane, and if $\mathbf{n}$ is the outward-pointing unit normal vector on C, the **flux** of $\mathbf{F}$ across C is

$$\text{Flux of } \mathbf{F} \text{ across C} = \int_C \mathbf{F} \cdot \mathbf{n}\, ds.$$

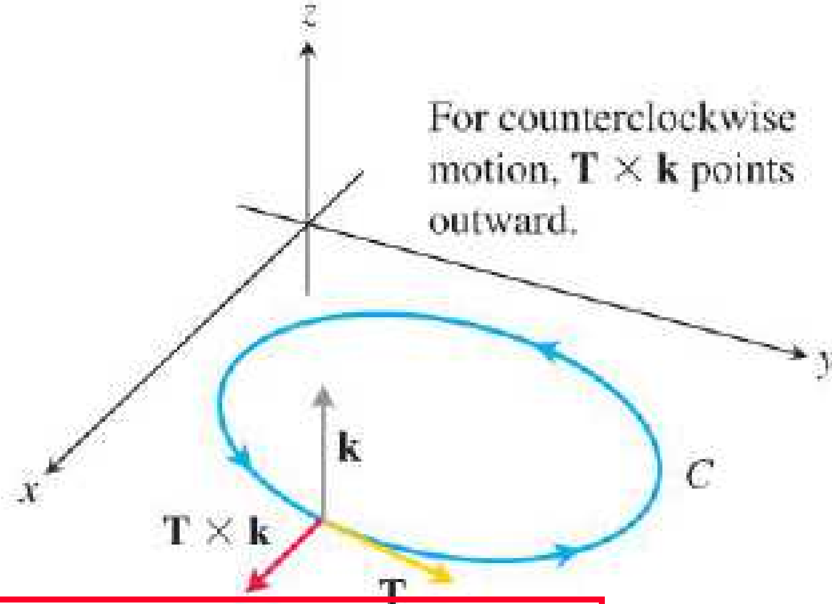
Calculating Flux Across a Smooth Closed Plane Curve

$$x = g(t), \qquad y = h(t), \qquad a \le t \le b, \ C \text{ counterclockwise}$$

$$\mathbf{n} = \mathbf{T} \times \mathbf{k} = \left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}\right) \times \mathbf{k} = \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}.$$

$$\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

$$\mathbf{F} \cdot \mathbf{n} = M(x, y)\frac{dy}{ds} - N(x, y)\frac{dx}{ds}.$$



For counterclockwise motion, $\mathbf{T} \times \mathbf{k}$ points outward.

$$\boxed{\textbf{Flux} = \int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right) ds = \oint_C M\, dy - N\, dx}$$

EXAMPLE 8

Find the flux of $\mathbf{F} = (x - y)\mathbf{i} + x\mathbf{j}$ across the circle $x^2 + y^2 = 1$ in the xy-plane.

Solution $\text{Flux} = \oint_C M dy - N dx$ counterclockwise

$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j},\ 0 \le t \le 2\pi,$

$$M = x - y = \cos t - \sin t, \qquad dy = d(\sin t) = \cos t\, dt$$

$$N = x = \cos t, \qquad dx = d(\cos t) = -\sin t\, dt,$$

$$\text{Flux} = \oint_C M\,dy - N\,dx = \int_0^{2\pi} (\cos^2 t - \sin t \cos t + \cos t \sin t)\,dt$$

$$= \int_0^{2\pi} \cos^2 t\,dt = \int_0^{2\pi} \frac{1 + \cos 2t}{2}\,dt = \left[\frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{2\pi} = \pi.$$

16.3

Path Independence, Conservative Fields, and Potential Functions

路径无关，保守场和势函数

例.Find the work by the force $\vec{F}=2xy\vec{i}+x^2\vec{j}$ on the curve:

(1) $\vec{r}(t)=t\vec{i}+t^2\vec{j},\quad 0\le t\le 1;$

(2) $\vec{r}(t)=t\vec{i}+t\vec{j},\quad 0\le t\le 1.$

(3) $\vec{r}(t)=t\vec{i}+t^3\vec{j},\quad W=1!$

Path Independence

Solution (1) $\vec{F}=2t^3\vec{i}+t^2\vec{j}\qquad \vec{r}'(t)=\vec{i}+2t\vec{j},$

$$W=\int_0^1 \vec{F}\cdot\vec{r}'(t)dt=\int_0^1(4t^3)dt=1$$

Solution (2) $\vec{F}=2t^2\vec{i}+t^2\vec{j}\qquad \vec{r}'(t)=\vec{i}+\vec{j},$

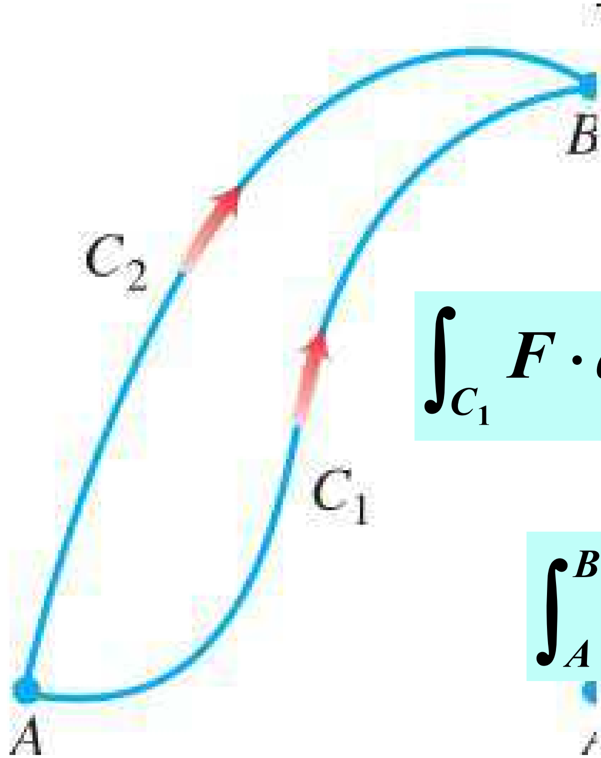
$$W=\int_0^1 \vec{F}\cdot\vec{r}'(t)dt=\int_0^1(3t^2)dt=1$$

In gravitational and electric fields,

the amount of work it takes to move a mass or charge from one point to another depends on the initial and final positions of the object-not on which path is taken.

Path Independence

DEFINITIONS Let $\mathbf{F}$ be a vector field defined on an open region D in space, and suppose that for any two points A and B in D the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ along a path C from A to B in D is the same over all paths from A to B. Then the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **path independent in D** and the field $\mathbf{F}$ is **conservative on D**.



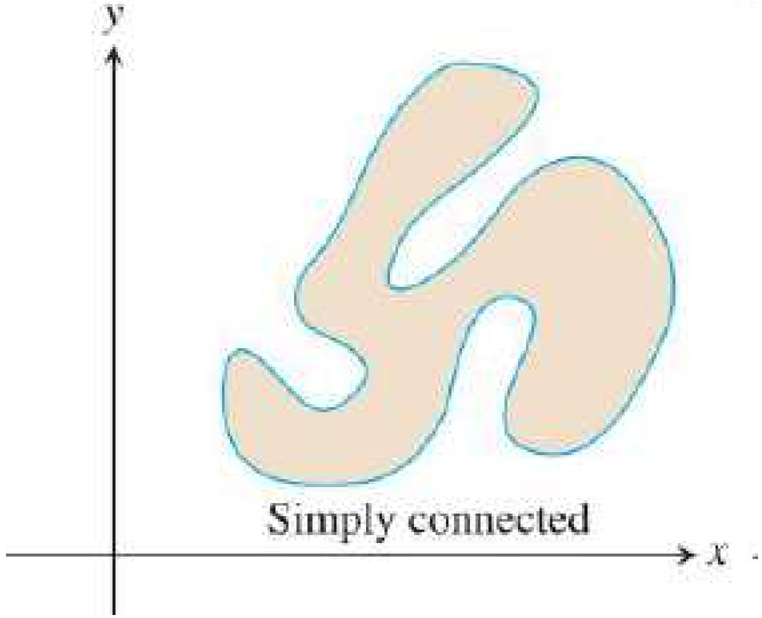
$$\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr$$

$$\int_A^B F \cdot dr$$

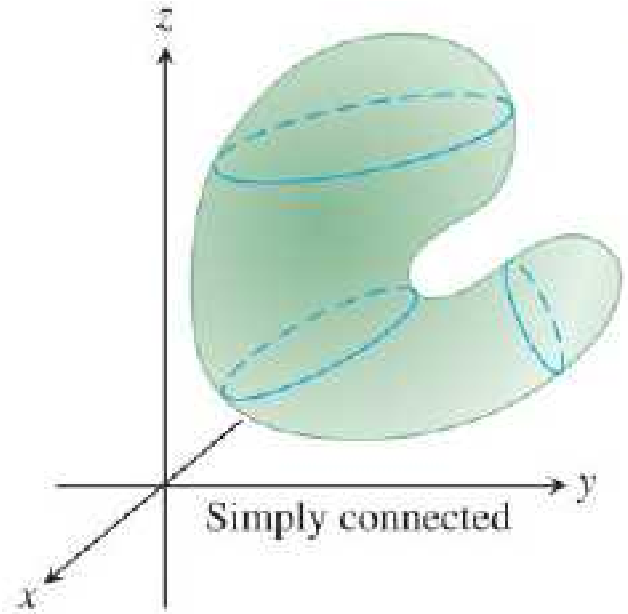
$\int_C \mathbf{F} \cdot d\mathbf{r}$ **is path independent in** D and the field $\mathbf{F}$ **is conservative on** D.

DEFINITION If $\mathbf{F}$ is a vector field defined on D and $\mathbf{F} = \nabla f$ for some scalar function f on D, then f is called a **potential function for F**.

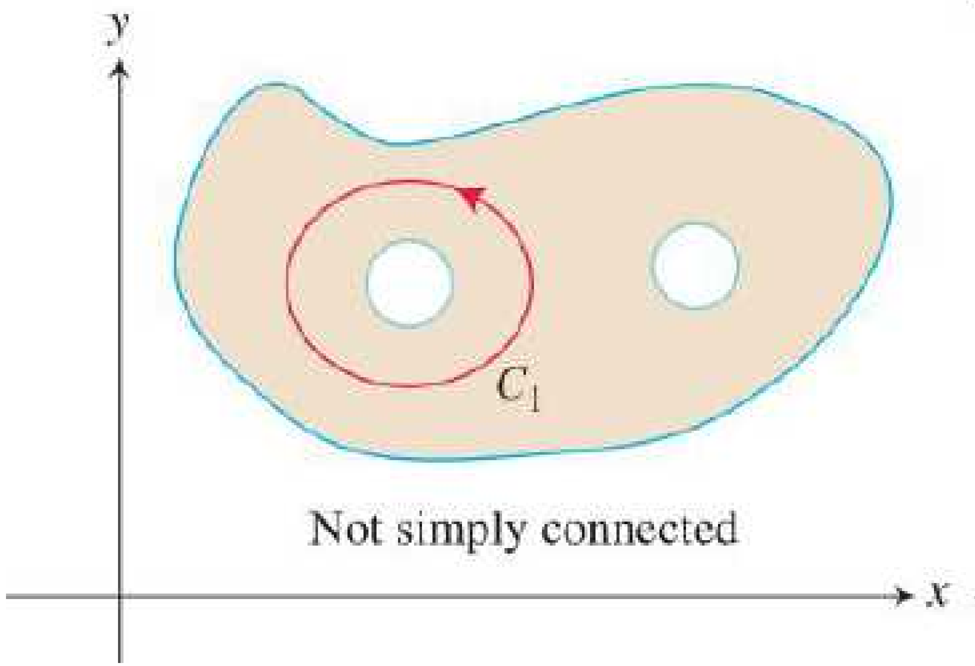
The curves we consider are **piecewise smooth**.
fields $\mathbf{F}$ whose components have continuous first partial derivatives.
The domains D we consider are **connected**.

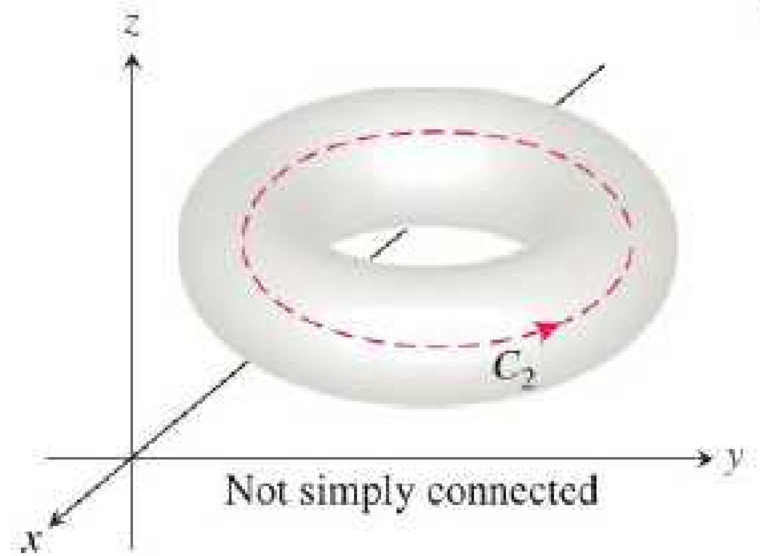


Simply connected



Simply connected


y
C_1
Not simply connected
x


z
C_2
y
Not simply connected
x

Line Integrals in Conservative Fields

THEOREM 1—Fundamental Theorem of Line Integrals Let C be a smooth curve joining the point A to the point B in the plane or in space and parametrized by $\mathbf{r}(t)$. Let f be a differentiable function with a continuous gradient vector $\mathbf{F} = \nabla f$ on a domain D containing C. Then $\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$.

Proof of Theorem 1

Suppose that A and B are two points in region D and that

C: $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$, $a \le t \le b$, joining A to B.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t=a}^{t=b} \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))\, dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = f(B) - f(A).$$

EXAMPLE 1

Suppose the force field $\mathbf{F} = \nabla f$ is the gradient of the function

$$f(x, y, z) = -\frac{1}{x^2 + y^2 + z^2}.$$

Find the work done by $\mathbf{F}$ in moving an object along a smooth curve C joining $(1, 0, 0)$ to $(0, 0, 2)$ that does not pass through the origin.

Solution

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(0, 0, 2) - f(1, 0, 0) = -\frac{1}{4} - (-1) = \frac{3}{4}.$$

THEOREM 2—Conservative Fields are Gradient Fields Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field whose components are continuous throughout an open connected region D in space. Then $\mathbf{F}$ is conservative if and only if $\mathbf{F}$ is a gradient field ∇f for a differentiable function f.

Proof of Theorem 2 $\Leftarrow$ If $\mathbf{F}$ is a gradient field, then $\mathbf{F} = \nabla f$

Theorem 1 shows that $\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$.

$\mathbf{F}$ satisfies the definition of a conservative field.

$\Rightarrow$ suppose that $\mathbf{F}$ is a conservative vector field.

$$\forall B(x, y, z) \in D, A(x_0, y_0, z_0) \in D,$$

$$\int_A^B F \cdot d\mathbf{r} = \int_{(x_0, y_0, z_0)}^{(x, y, z)} F \cdot d\mathbf{r} \quad \text{denoted by} \quad f(x, y, z)$$

$$f(x,y,z)=\int_{(x_0,y_0,z_0)}^{(x,y,z)} F\cdot d\mathrm{r} \quad =\int_{(x_0,y_0,z_0)}^{(x_0,y,z)} F\cdot d\mathrm{r}+\int_{(x_0,y,z)}^{(x,y,z)} F\cdot d\mathrm{r}$$

$$\int_{(x_0,y,z)}^{(x,y,z)} F\cdot d\mathrm{r} \qquad\qquad r(t)=t\mathrm{i}+y\mathrm{j}+z\mathrm{k},\ x_0\le t\le x,$$

$$=\int_{x_0}^{x}(M(t,y,z)\mathrm{i}+N(t,y,z)\mathrm{j}+P(t,y,z)\mathrm{k})\cdot((\mathrm{d}t)\mathrm{i}+0\mathrm{j}+0\mathrm{k})$$

$$=\int_{x_0}^{x} M(t,y,z)\mathrm{d}t \qquad\qquad \frac{\partial f}{\partial x}=M(x,y,z)$$

similarly, $\dfrac{\partial f}{\partial y}=N(x,y,z) \qquad \dfrac{\partial f}{\partial z}=P(x,y,z)$

$$\mathbf{F}=\nabla f. \qquad\qquad \int_{(x_0,y_0,z_0)}^{(x,y,z)} F\cdot d\mathrm{r}=\int_{(x_0,y_0,z_0)}^{(x,y,z)} Mdx+Ndy+Pdz$$

$$d\left(\int_{(x_0,y_0,z_0)}^{(x,y,z)} Mdx+Ndy+Pdz\right)=Mdx+Ndy+Pdz$$

EXAMPLE 2 Find the work done by the conservative field

$$\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \nabla f, \quad \text{where} \quad f(x, y, z) = xyz,$$

along any smooth curve C joining the point $A(-1, 3, 9)$ to $B(1, 6, -4)$.

Solution

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_A^B \nabla f \cdot d\mathbf{r} = f(B) - f(A)$$

$$= xyz\big|_{(1,6,-4)} - xyz\big|_{(-1,3,9)}$$

$$= (1)(6)(-4) - (-1)(3)(9)$$

$$= -24 + 27 = 3.$$

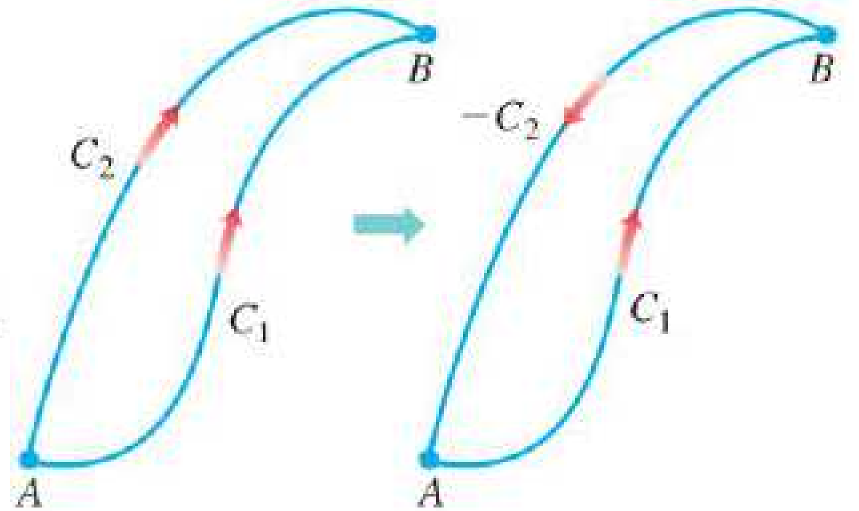
THEOREM 3—Loop Property of Conservative Fields The following statements are equivalent.

1. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ around every loop (that is, closed curve C) in D.
2. The field $\mathbf{F}$ is conservative on D.

Proof that Part 1 ⇒ Part 2

: for any two points A and B in D,

any two paths C_1 and C_2 from A to B.



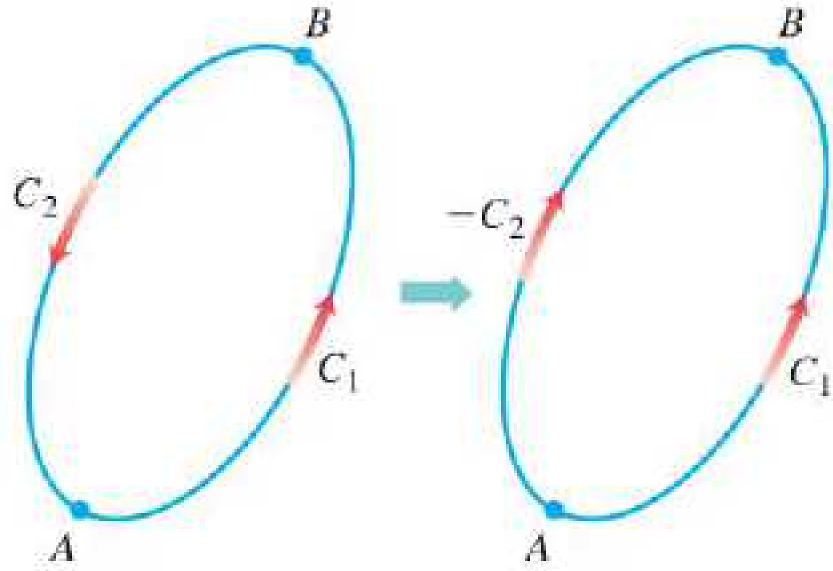
$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

Proof that Part 2 $\Rightarrow$ Part 1

over any closed loop C. We pick two points A and B on C

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_A^B \mathbf{F}\cdot d\mathbf{r} - \int_A^B \mathbf{F}\cdot d\mathbf{r} = 0.$$



Theorem 2 | Theorem 3

$\mathbf{F} = \nabla f$ on D $\Leftrightarrow$ $\mathbf{F}$ conservative on D $\Leftrightarrow$ $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ over any loop in D

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B \nabla f \cdot d\mathbf{r} = f(B) - f(A).$$

Two questions arise:

1. How do we know whether a given vector field $\mathbf{F}$ is conservative?
2. If $\mathbf{F}$ is in fact conservative, how do we find a potential function f

Component Test for Conservative Fields

Let $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ be a field on an open simply connected domain whose component functions have continuous first partial derivatives. Then, $\mathbf{F}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \qquad \text{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}. \tag{2}$$

$\Rightarrow$ **Proof that Equations (2) hold if F is conservative**

$$\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}. \qquad M = \frac{\partial f}{\partial x}, N = \frac{\partial f}{\partial y}, P = \frac{\partial f}{\partial z}$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial y\,\partial z} = \frac{\partial^2 f}{\partial z\,\partial y} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial N}{\partial z}.$$

The others in Equations (2) are proved similarly.

$\Leftarrow$ later on.

如何求势函数?

Once we know that **F** is conservative,

$$\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$$

① $$\frac{\partial f}{\partial x} = M, \qquad \frac{\partial f}{\partial y} = N, \qquad \frac{\partial f}{\partial z} = P,$$

We accomplish this by integrating the three equations

② $$f(x,y,z) = \int_{(x_0,y_0,z_0)}^{(x,y,z)} F \cdot dr = \int_{(x_0,y_0,z_0)}^{(x,y,z)} Mdx + Ndy + Pdz$$

EXAMPLE 3

Show that $\mathbf{F} = (e^x \cos y + yz)\mathbf{i} + (xz - e^x \sin y)\mathbf{j} + (xy + z)\mathbf{k}$ is conservative over its natural domain and find a potential function for it.

Solution

$$M = e^x \cos y + yz, \qquad N = xz - e^x \sin y, \qquad P = xy + z$$

$$\frac{\partial P}{\partial y} = x = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x}, \qquad \frac{\partial N}{\partial x} = -e^x \sin y + z = \frac{\partial M}{\partial y}.$$

$\mathbf{F}$ is conservative.

$$\frac{\partial f}{\partial x} = e^x \cos y + yz, \qquad \frac{\partial f}{\partial y} = xz - e^x \sin y, \qquad \frac{\partial f}{\partial z} = xy + z.$$

$$f(x, y, z) = e^x \cos y + xyz + g(y, z).$$

$$\frac{\partial f}{\partial x} = e^x \cos y + yz, \qquad \frac{\partial f}{\partial y} = xz - e^x \sin y, \qquad \frac{\partial f}{\partial z} = xy + z.$$

$$f(x, y, z) = e^x \cos y + xyz + g(y, z).$$

$$-e^x \sin y + xz + \frac{\partial g}{\partial y} = xz - e^x \sin y,$$

so $\partial g/\partial y = 0$. Therefore, g is a function of z alone,

$$f(x, y, z) = e^x \cos y + xyz + h(z).$$

$$xy + \frac{dh}{dz} = xy + z, \quad \text{or} \qquad \frac{dh}{dz} = z, \quad h(z) = \frac{z^2}{2} + C.$$

$$f(x, y, z) = e^x \cos y + xyz + \frac{z^2}{2} + C.$$

EXAMPLE 4

Show that $\mathbf{F} = (2x - 3)\mathbf{i} - z\mathbf{j} + (\cos z)\mathbf{k}$ is not conservative.

Solution $\dfrac{\partial P}{\partial y} = \dfrac{\partial}{\partial y}(\cos z) = 0, \qquad \dfrac{\partial N}{\partial z} = \dfrac{\partial}{\partial z}(-z) = -1.$

so $\mathbf{F}$ is not conservative.

EXAMPLE 5 Show that the vector field $\mathbf{F} = \dfrac{-y}{x^2 + y^2}\mathbf{i} + \dfrac{x}{x^2 + y^2}\mathbf{j}$

satisfies the equations in the Component Test, but is not conservative over its natural domain. Explain why

Solution $M = -y/(x^2 + y^2), N = x/(x^2 + y^2)$, and $P = 0$.

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \qquad \frac{\partial P}{\partial x} = 0 = \frac{\partial M}{\partial z}, \qquad \frac{\partial M}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial N}{\partial x}.$$

the unit circle C in the xy-plane.

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j},\ 0 \le t \le 2\pi.$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_0^{2\pi} \left(\sin^2 t + \cos^2 t\right) dt = 2\pi.$$

Since the line integral of **F** around the loop C is not zero,

the field **F** is not conservative, by Theorem 3.

the test assumes

that the domain of **F** is simply connected, which is not the case here.

the ball of radius 1 centered at the point (2, 2, 2), this new domain D

the field **F** in Example 5 is conservative on D.

Exact Differential Forms

work and circulation integrals in the differential form ;

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\,dx + N\,dy + P\,dz$$

if $M\,dx + N\,dy + P\,dz$ is the total differential of a function f

then
$$\int_C M\,dx + N\,dy + P\,dz = \int_C \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$= \int_A^B \nabla f \cdot d\mathbf{r} = f(B) - f(A).$$

$$\Updownarrow$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_A^B df = f(B) - f(A),$$

F - -conservative

DEFINITIONS Any expression $M(x, y, z)\,dx + N(x, y, z)\,dy + P(x, y, z)\,dz$ is a **differential form**. A differential form is **exact** on a domain D in space if

$$M\,dx + N\,dy + P\,dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = df$$

for some scalar function f throughout D. 恰当微分形式

Component Test for Exactness of $M\,dx + N\,dy + P\,dz$

The differential form $M\,dx + N\,dy + P\,dz$ is exact on an open simply connected domain if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \qquad \text{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

This is equivalent to saying that the field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative.

Theorem 2 Theorem 3

$\mathbf{F} = \nabla f$ on D $\Leftrightarrow$ $\mathbf{F}$ conservative on D $\Leftrightarrow$ $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ over any loop in D

simply connected $\Updownarrow$

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

$\Updownarrow$ simply connected

$$M\,dx + N\,dy + P\,dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = df$$

The curves we consider are **piecewise smooth.**
fields $\mathbf{F}$ whose components have continuous first partial derivatives.
The domains D we consider are **connected.**

EXAMPLE 6

Show that $y\,dx + x\,dy + 4\,dz$ is exact and evaluate the integral

$$\int_{(1,1,1)}^{(2,3,-1)} y\,dx + x\,dy + 4\,dz$$

over any path from $(1, 1, 1)$ to $(2, 3, -1)$.

Solution let $M = y, N = x, P = 4$:

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \qquad \frac{\partial N}{\partial x} = 1 = \frac{\partial M}{\partial y}.$$

$y\,dx + x\,dy + 4\,dz$ is exact, so $y\,dx + x\,dy + 4\,dz = df$

$$\frac{\partial f}{\partial x} = y, \qquad \frac{\partial f}{\partial y} = x, \qquad \frac{\partial f}{\partial z} = 4. \qquad f(x, y, z) = xy + 4z + C.$$

$$\int_{(1,1,1)}^{(2,3,-1)} y\,dx + x\,dy + 4\,dz = f(2, 3, -1) - f(1, 1, 1) = -3.$$

16.4

Green’s Theorem in the Plane

平面上的格林定理

$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ is the velocity field of a fluid flowing in the plane and that the first partial derivatives of M and N are continuous

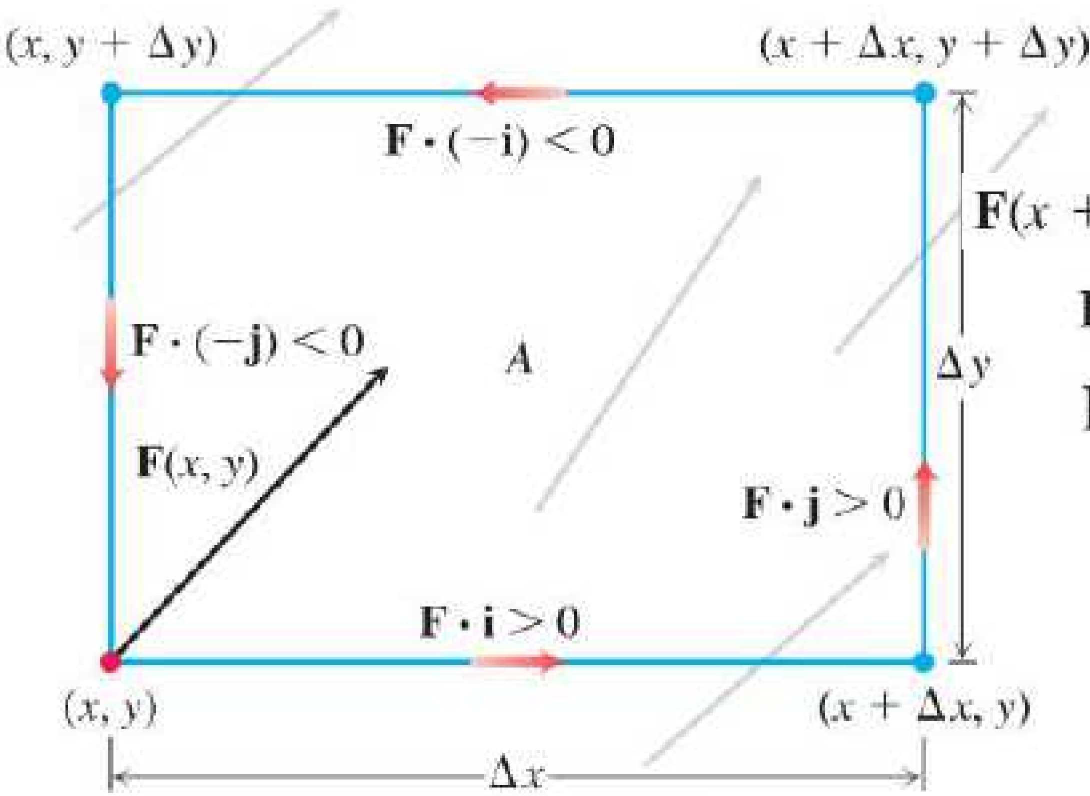
The **circulation** $\oint_C \mathbf{F}(x, y) \cdot d\mathbf{r}$

若F连续，直觉感到C上的环流量与C内每个点附近的环流量有关的！

来看C内每个点附近的环流量：

circulation density

$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in a domain containing the rectangle A,



$$\mathbf{F}(x, y) \cdot \mathbf{i}\, \Delta x = M(x, y)\Delta x.$$

$$\mathbf{F}(x + \Delta x, y) \cdot \mathbf{j}\, \Delta y = N(x + \Delta x, y)\Delta y$$

$$\mathbf{F}(x, y) \cdot (-\mathbf{j})\, \Delta y = -N(x, y)\Delta y$$

$$\mathbf{F}(x, y + \Delta y) \cdot (-\mathbf{i})\, \Delta x$$

$$= -M(x, y + \Delta y)\Delta x$$

We sum Top and bottom: $-(M(x, y + \Delta y) - M(x, y))\Delta x \approx -\left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x$

Right and left: $(N(x + \Delta x, y) - N(x, y))\Delta y \approx \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y.$

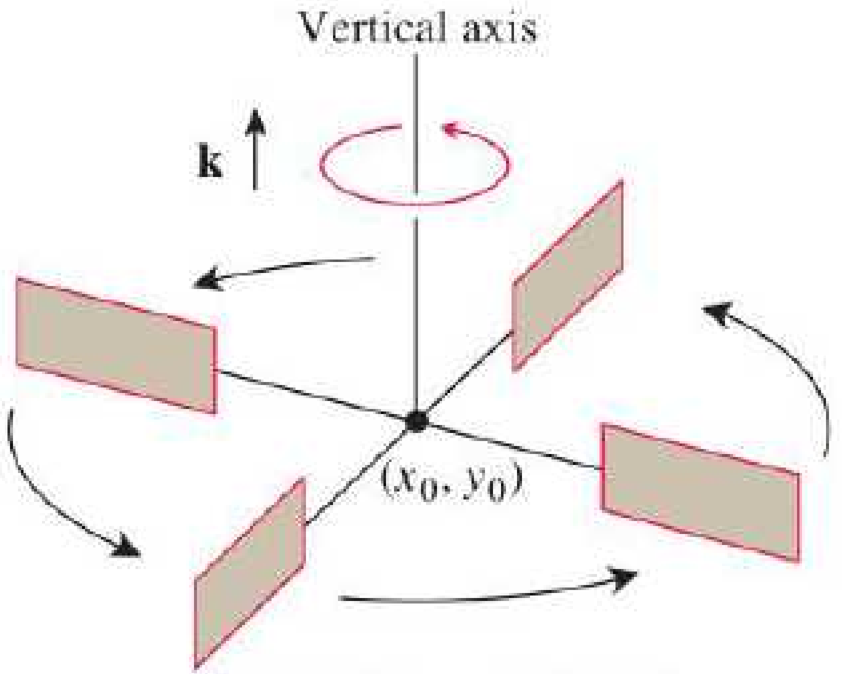
$$\text{Circulation rate around rectangle} \approx \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Delta x\, \Delta y.$$

$$\frac{\text{Circulation around rectangle}}{\text{rectangle area}} \approx \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

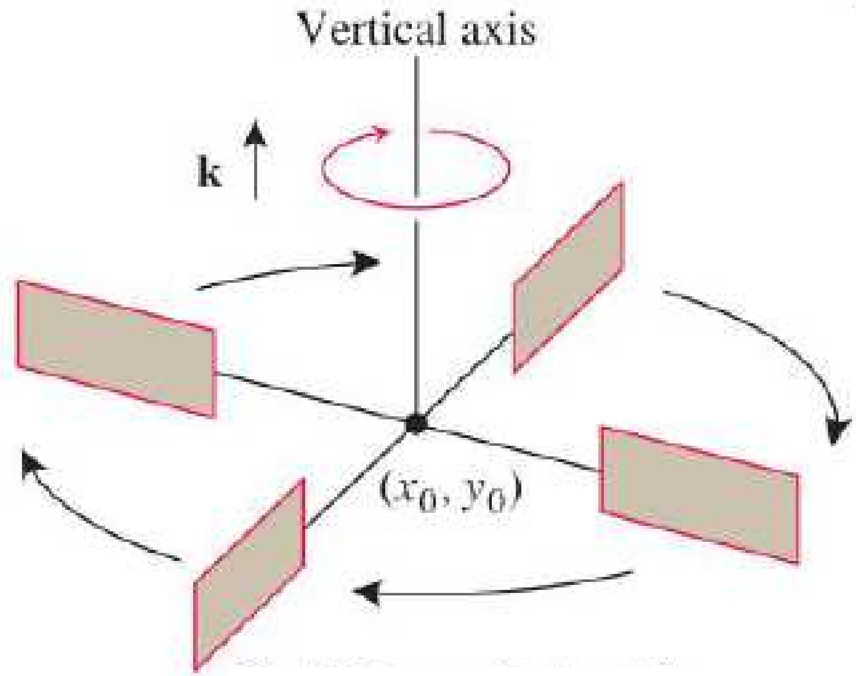
let Δx and Δy approach zero

DEFINITION The **circulation density** of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is the scalar expression $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$.

This expression is also called **the k-component of the curl**, denoted by $(\text{curl } \mathbf{F}) \cdot \mathbf{k}$.



$$\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)_{(x_0, y_0)} > 0$$



$$\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)_{(x_0, y_0)} < 0$$

EXAMPLE 1

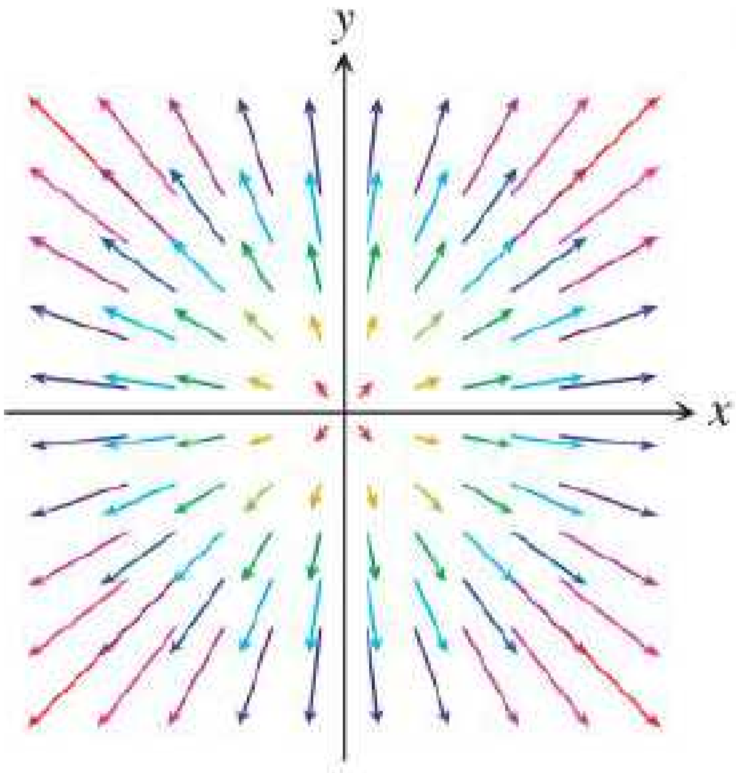
The following vector fields represent the velocity of a gas flowing in the xy-plane. Find the circulation density of each vector field and interpret its physical meaning

(a) *Uniform expansion or compression:* $\mathbf{F}(x, y) = cx\mathbf{i} + cy\mathbf{j}$

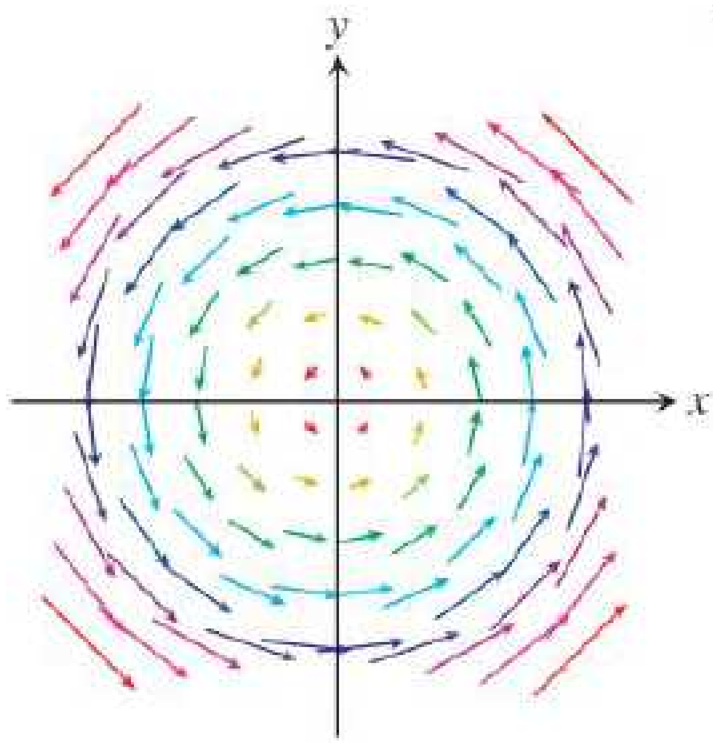
(b) *Uniform rotation:* $\mathbf{F}(x, y) = -cy\mathbf{i} + cx\mathbf{j}$

(c) *Shearing flow:* $\mathbf{F}(x, y) = y\mathbf{i}$

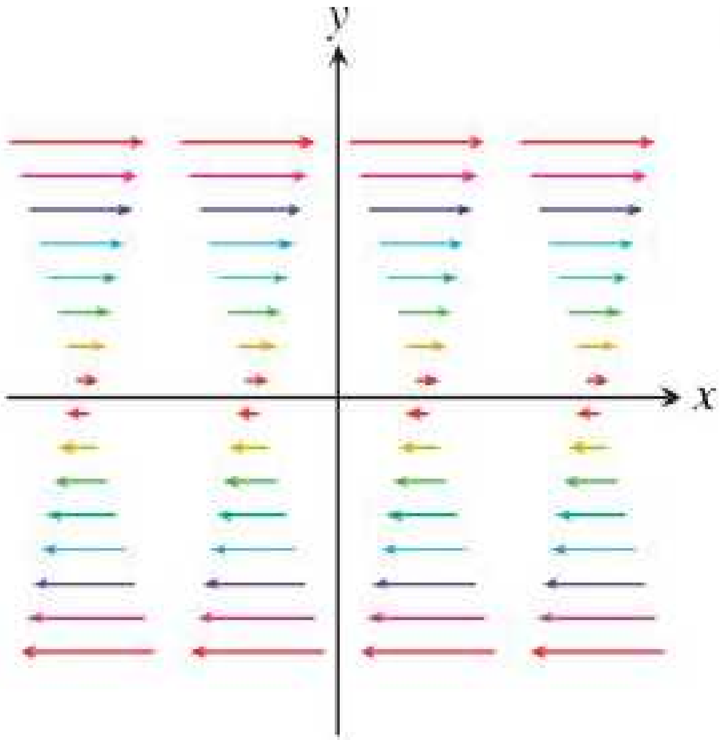
(d) *Whirlpool effect:* $\mathbf{F}(x, y) = \dfrac{-y}{x^2 + y^2}\mathbf{i} + \dfrac{x}{x^2 + y^2}\mathbf{j}$



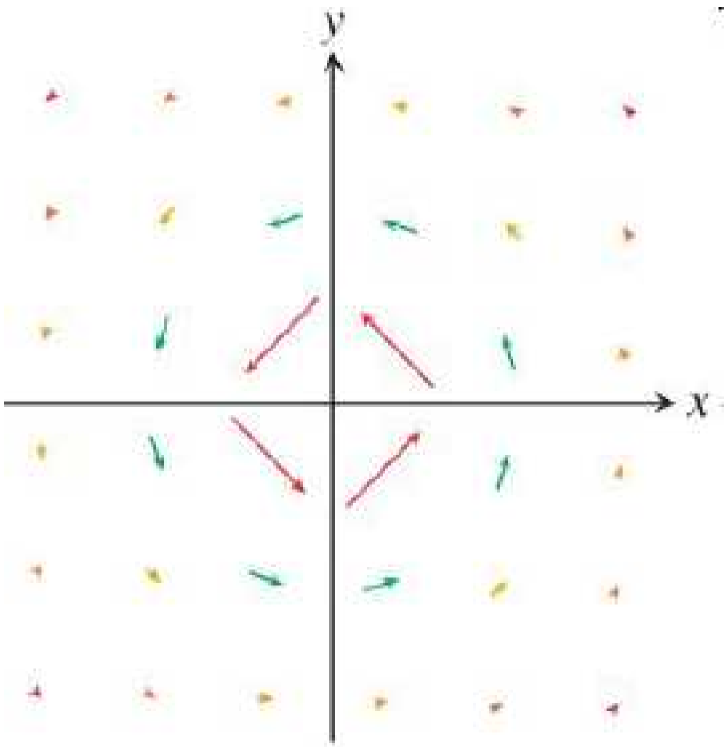
(a)



(b)



(c)



(d)

Solution (a) $(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \dfrac{\partial(cy)}{\partial x} - \dfrac{\partial(cx)}{\partial y} = 0$

(b) $(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \dfrac{\partial}{\partial x}(cx) - \dfrac{\partial}{\partial y}(-cy) = 2c.$

If $c > 0$, the rotation is counterclockwise;
if $c < 0$, the rotation is clockwise.

(c) $(\text{curl } \mathbf{F}) \cdot \mathbf{k} = -\dfrac{\partial}{\partial y}(y) = -1.$

$$(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2 + y^2}\right) = \frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2} = 0.$$

(a) *Uniform expansion or compression:* $\mathbf{F}(x, y) = cx\mathbf{i} + cy\mathbf{j}$

(b) *Uniform rotation:* $\mathbf{F}(x, y) = -cy\mathbf{i} + cx\mathbf{j}$

(c) *Shearing flow:* $\mathbf{F}(x, y) = y\mathbf{i}$

(d) *Whirlpool effect:* $\mathbf{F}(x, y) = \dfrac{-y}{x^2 + y^2}\mathbf{i} + \dfrac{x}{x^2 + y^2}\mathbf{j}$

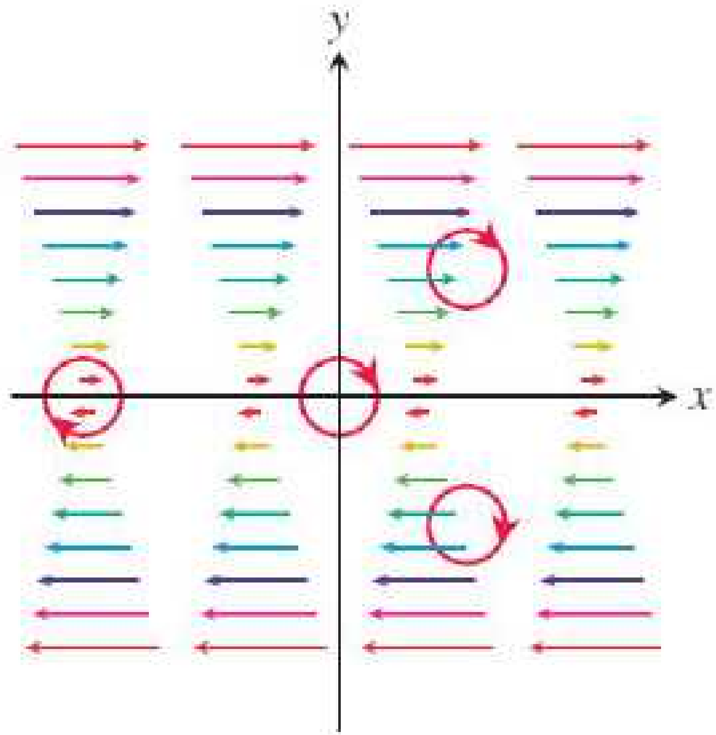
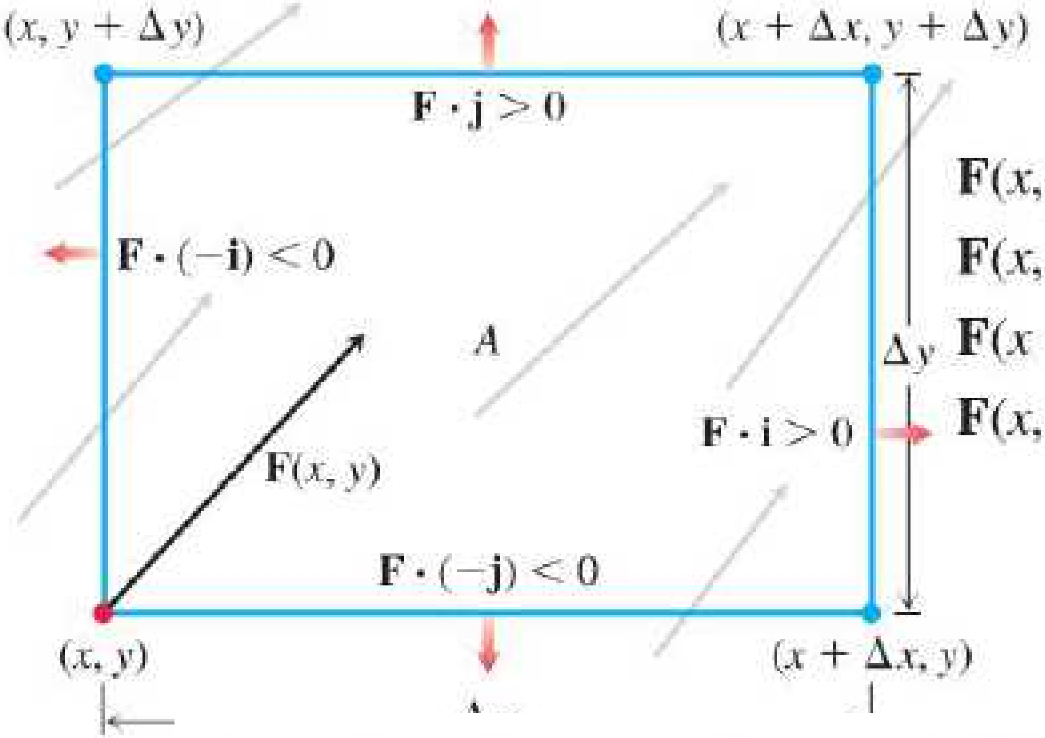


FIGURE 16.29 A shearing flow pushes the fluid clockwise around each point (Example 1c).

Divergence

$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in a domain containing the rectangle A,



$$\mathbf{F}(x, y + \Delta y) \cdot \mathbf{j}\, \Delta x = N(x, y + \Delta y)\Delta x$$
$$\mathbf{F}(x, y) \cdot (-\mathbf{j})\, \Delta x = -N(x, y)\Delta x$$
$$\mathbf{F}(x + \Delta x, y) \cdot \mathbf{i}\, \Delta y = M(x + \Delta x, y)\Delta y$$
$$\mathbf{F}(x, y) \cdot (-\mathbf{i})\, \Delta y = -M(x, y)\Delta y$$

Top and bottom: $$(N(x, y + \Delta y) - N(x, y))\Delta x \approx \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x$$

Right and left: $$(M(x + \Delta x, y) - M(x, y))\Delta y \approx \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y.$$

$$\text{Flux across rectangle boundary} \approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x \Delta y.$$

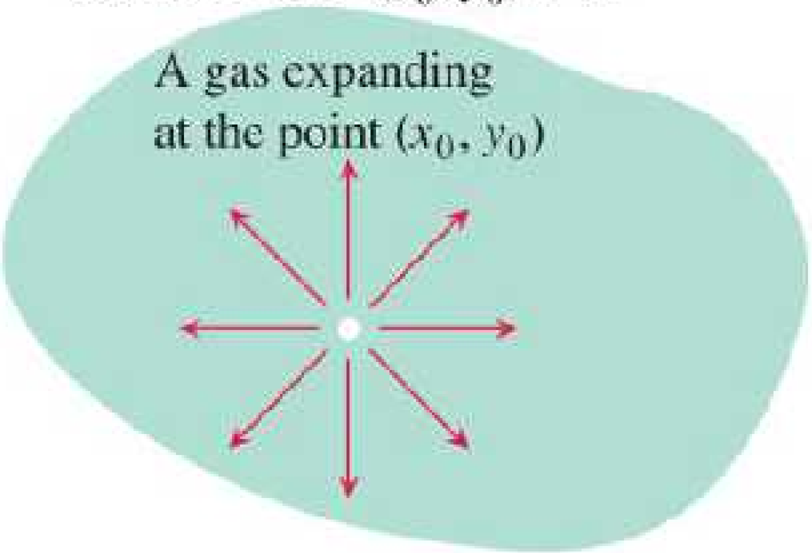
$$\frac{\text{Flux across rectangle boundary}}{\text{rectangle area}} \approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right).$$

let Δx and Δy approach zero: the flux density of $\mathbf{F}$ at the point (x, y).

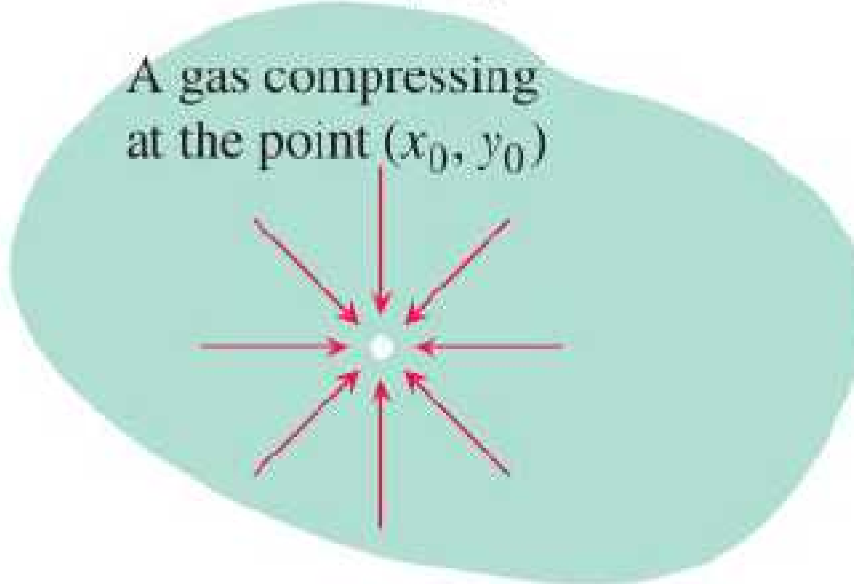
DEFINITION The **divergence (flux density)** of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is

$$\operatorname{div} \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.$$

Source: div $\mathbf{F}(x_0, y_0) > 0$



Sink: div $\mathbf{F}(x_0, y_0) < 0$



EXAMPLE 2

Find the divergence, and interpret what it means, for each vector field in Example 1 representing the velocity of a gas flowing in the xy-plane.

(a) *Uniform expansion or compression:* $\mathbf{F}(x, y) = cx\mathbf{i} + cy\mathbf{j}$

(b) *Uniform rotation:* $\mathbf{F}(x, y) = -cy\mathbf{i} + cx\mathbf{j}$

(c) *Shearing flow:* $\mathbf{F}(x, y) = y\mathbf{i}$

(d) *Whirlpool effect:* $\mathbf{F}(x, y) = \dfrac{-y}{x^2 + y^2}\mathbf{i} + \dfrac{x}{x^2 + y^2}\mathbf{j}$

Solution

(a) $\operatorname{div} \mathbf{F} = \dfrac{\partial}{\partial x}(cx) + \dfrac{\partial}{\partial y}(cy) = 2c$:

(b) $\operatorname{div} \mathbf{F} = \dfrac{\partial}{\partial x}(-cy) + \dfrac{\partial}{\partial y}(cx) = 0$:

(c) $\operatorname{div} \mathbf{F} = \dfrac{\partial}{\partial x}(y) = 0$:

(d) $\operatorname{div} \mathbf{F} = \dfrac{\partial}{\partial x}\left(\dfrac{-y}{x^2 + y^2}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{x}{x^2 + y^2}\right) = \dfrac{2xy}{(x^2 + y^2)^2} - \dfrac{2xy}{(x^2 + y^2)^2} = 0$:

Two Forms for Green's Theorem

THEOREM 4—Green's Theorem (Circulation-Curl or Tangential Form) Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the counterclockwise circulation of $\mathbf{F}$ around C equals the double integral of $(\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}$ over R.

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \oint_C M\, dx + N\, dy}_{\text{Counterclockwise circulation}} = \underbrace{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy}_{\text{Curl integral}}$$

THEOREM 5—Green's Theorem (Flux-Divergence or Normal Form) Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the outward flux of $\mathbf{F}$ across C equals the double integral of div $\mathbf{F}$ over the region R enclosed by C.

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{n}\, ds}_{\text{Outward flux}} = \oint_C M\, dy - N\, dx = \underbrace{\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\, dy}_{\text{Divergence integral}}$$

EXAMPLE 3 Verify both forms of Green's Theorem for the vector field

$$\mathbf{F}(x, y) = (x - y)\mathbf{i} + x\mathbf{j}$$

and the region R bounded by the unit circle

$$C: \quad \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \quad 0 \le t \le 2\pi.$$

Solution

$$M = \cos t - \sin t, \qquad dx = d(\cos t) = -\sin t\, dt,$$

$$N = \cos t, \qquad dy = d(\sin t) = \cos t\, dt,$$

$$\oint_C M\, dx + N\, dy = \int_{t=0}^{t=2\pi} (\cos t - \sin t)(-\sin t\, dt) + (\cos t)(\cos t\, dt)$$

$$= \int_0^{2\pi} (-\sin t \cos t + 1)\, dt = 2\pi$$

$$\frac{\partial M}{\partial x} = 1, \qquad \frac{\partial M}{\partial y} = -1, \qquad \frac{\partial N}{\partial x} = 1, \qquad \frac{\partial N}{\partial y} = 0.$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (1 - (-1))\,dx\,dy = 2\pi.$$

$$\oint_C M\,dy - N\,dx = \int_{t=0}^{t=2\pi} (\cos t - \sin t)(\cos t\,dt) - (\cos t)(-\sin t\,dt)$$

$$= \int_0^{2\pi} \cos^2 t\,dt = \pi$$

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \iint_R (1 + 0)\,dx\,dy = \iint_R dx\,dy = \pi.$$

EXAMPLE 4 Evaluate the line integral $\oint_C xy\,dy - y^2\,dx,$

C is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$.

Solution

$$\oint_C -y^2\,dx + xy\,dy = \iint_R (y - (-2y))\,dx\,dy = \int_0^1 \int_0^1 3y\,dx\,dy$$

$$= \int_0^1 3y\,dy = \frac{3}{2}y^2\bigg]_0^1 = \frac{3}{2}.$$

$$\underbrace{\oint_C \mathbf{F}\cdot\mathbf{T}\,ds}_{\text{Counterclockwise circulation}} = \oint_C M\,dx + N\,dy = \underbrace{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy}_{\text{Curl integral}}$$

EXAMPLE 5

Calculate the outward flux of the vector field $\mathbf{F}(x, y) = 2e^{xy}\mathbf{i} + y^3\mathbf{j}$ across the square bounded by the lines $x = \pm 1$ and $y = \pm 1$.

Solution

$$\text{Flux} = \oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \oint_C M\, dy - N\, dx$$

$$= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\, dy = \int_{-1}^{1} \int_{-1}^{1} (2ye^{xy} + 3y^2)\, dx\, dy$$

$$= \int_{-1}^{1} (2e^y + 6y^2 - 2e^{-y})\, dy = \Big[2e^y + 2y^3 + 2e^{-y} \Big]_{-1}^{1} = 4.$$

Proof of Green's Theorem for Special Regions

M, N, and their first partial derivatives are continuous at every point

C be a smooth simple closed curve in the xy-plane

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

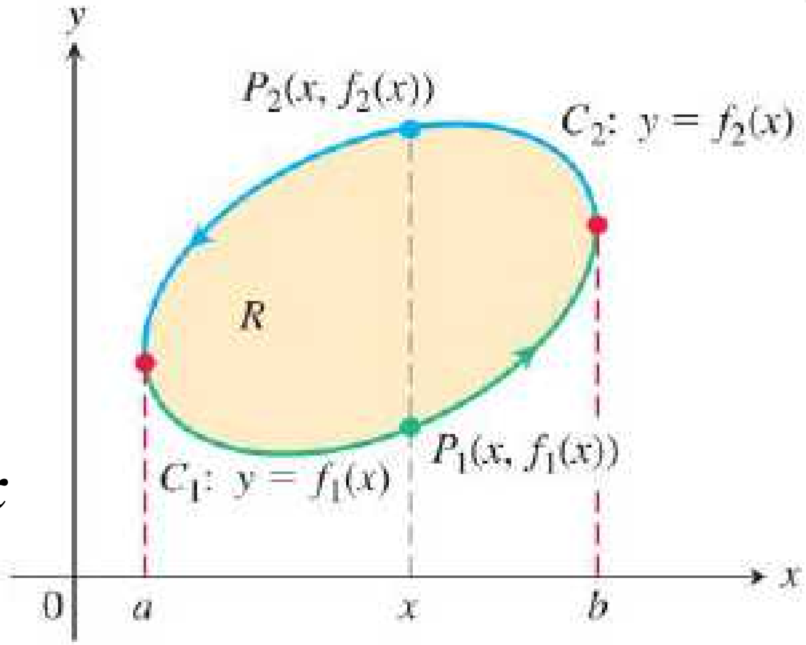
R be the region enclosed by C

$$\oint_C M dx = \int_{C_1} M dx + \int_{C_2} M dx$$

$$= \int_a^b M(x, f_1(x))dx + \int_b^a M(x, f_2(x))dx$$

$$= \int_a^b M(x, f_1(x))dx - \int_a^b M(x, f_2(x))dx$$

$$\iint_R \frac{\partial M}{\partial y} dxdy = \int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y} dydx$$



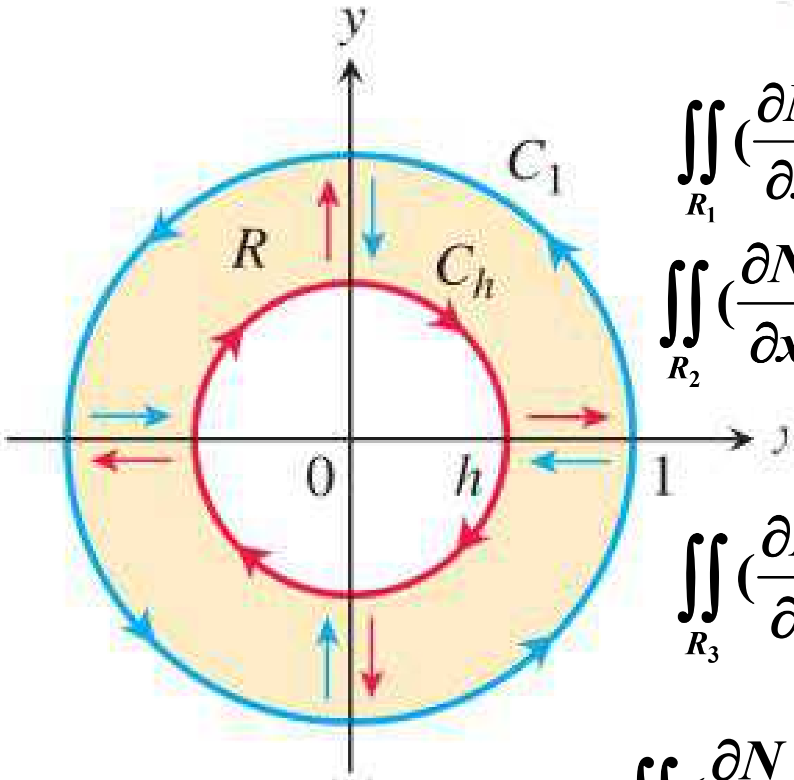
$$\int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\, dy = M(x, y)\Big]_{y=f_1(x)}^{y=f_2(x)} = M(x, f_2(x)) - M(x, f_1(x)).$$

$$\int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\, dy\, dx = \int_a^b \left[M(x, f_2(x)) - M(x, f_1(x)) \right] dx$$

$$= -\oint_C M\, dx.$$

$$\oint_C M\, dx = \iint_R \left(-\frac{\partial M}{\partial y} \right) dx\, dy.$$

$$\oint_C N\, dy = \iint_R \frac{\partial N}{\partial x}\, dx\, dy.$$



$$\iint_{R_1} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_{C_{11}+C_{h1}} Mdx + Ndy + s_{11} + s_{12}$$

$$\iint_{R_2} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_{C_{12}+C_{h2}} Mdx + Ndy + s_{21} + s_{22}$$

$$\iint_{R_3} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_{C_{13}+C_{h3}} Mdx + Ndy + s_{31} + s_{32}$$

$$\iint_{R_4} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_{C_{14}+C_{h4}} Mdx + Ndy + s_{41} + s_{42}$$

$$\iint_{R} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_{C_1+C_h} Mdx + Ndy$$

16.5

Surfaces and Area

曲面与曲面的面积

Parametrizations of Surfaces

We have defined curves in the plane in three different ways:

Explicit form: $y = f(x)$

Implicit form: $F(x, y) = 0$

Parametric vector form: $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}, \quad a \le t \le b.$

We have analogous definitions of surfaces in space:

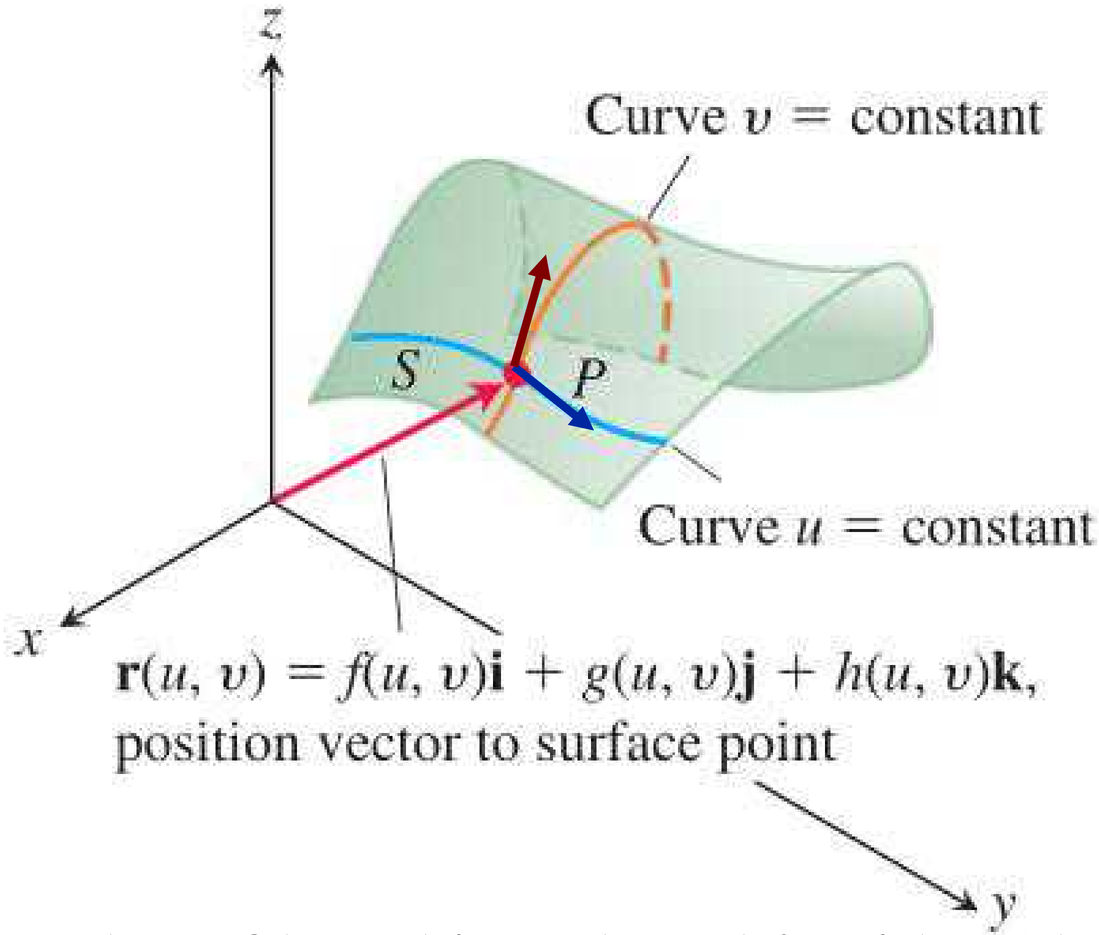
Explicit form: $z = f(x, y)$

Implicit form: $F(x, y, z) = 0.$

Parametric vector form: $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$

$$x = f(u, v), \qquad y = g(u, v), \qquad z = h(u, v).$$

$$(u, v) \in R$$



$$\mathbf{r}(u, v_0) = f(u, v_0)\mathbf{i} + g(u, v_0)\mathbf{j} + h(u, v_0)\mathbf{k}$$

$$\mathbf{r}(u_0, v) = f(u_0, v)\mathbf{i} + g(u_0, v)\mathbf{j} + h(u_0, v)\mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial u}(u_0, v_0)$$

$$\frac{\partial \mathbf{r}}{\partial v}(u_0, v_0)$$

EXAMPLE 1 Find a parametrization of the cone

$$z = \sqrt{x^2 + y^2}, \qquad 0 \le z \le 1.$$

Solution

$x = r\cos\theta$, $y = r\sin\theta$, and $z = \sqrt{x^2 + y^2} = r$,

$0 \le r \le 1$ and $0 \le \theta \le 2\pi$.

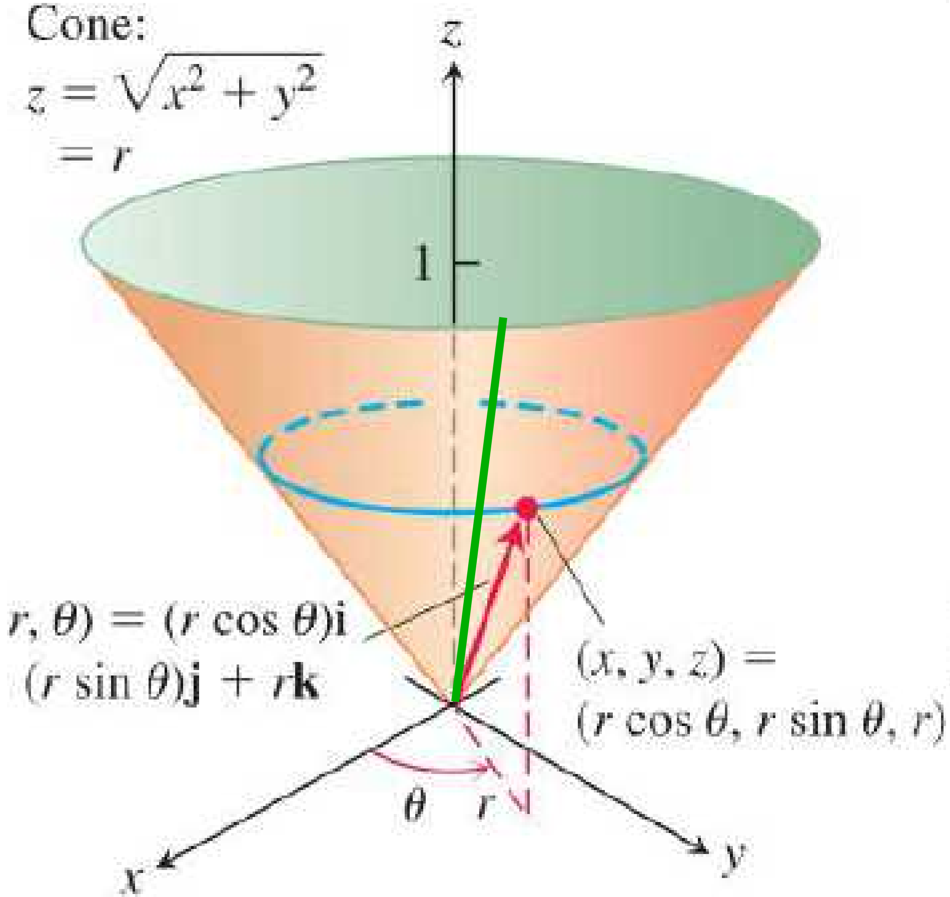
$$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k},$$

$$0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$

$$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k},$$

$$\mathbf{r}(r_0, \theta) = (r_0\cos\theta)\mathbf{i} + (r_0\sin\theta)\mathbf{j} + r_0\mathbf{k}$$

$$\mathbf{r}(r, \frac{\pi}{4}) = (\frac{\sqrt{2}}{2}r)\mathbf{i} + (\frac{\sqrt{2}}{2}r)\mathbf{j} + r\,\mathbf{k}$$



EXAMPLE 2 Find a parametrization of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution $x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta, z = \rho \cos\phi, \quad \rho = a,$

$$x = a \sin\phi \cos\theta, \quad y = a \sin\phi \sin\theta, \quad \text{and} \quad z = a \cos\phi,$$

$$\mathbf{r}(\phi, \theta) = (a \sin\phi \cos\theta)\mathbf{i} + (a \sin\phi \sin\theta)\mathbf{j} + (a \cos\phi)\mathbf{k},$$
$$0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi.$$

EXAMPLE 3 Find a parametrization of the cylinder

$$x^2 + (y - 3)^2 = 9, \qquad 0 \le z \le 5.$$

Solution

$$x = r\cos\theta, y = r\sin\theta, \text{ and } \quad z = z.$$

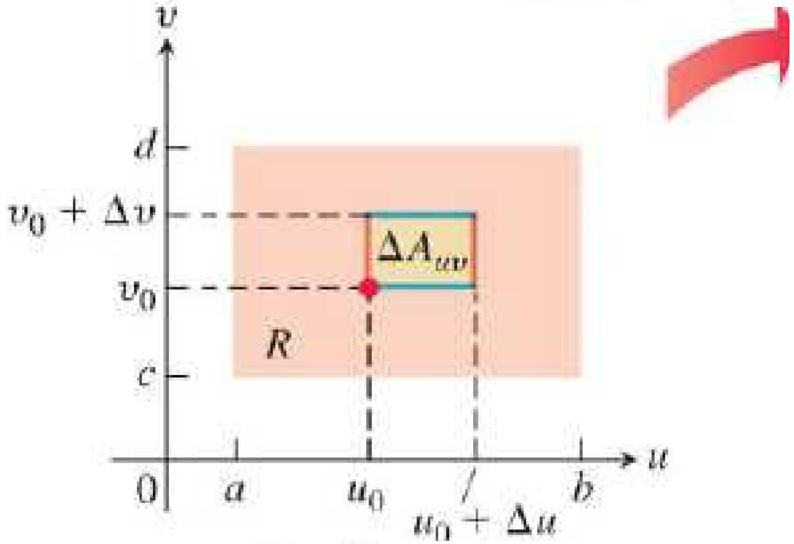
$r = 6\sin\theta$

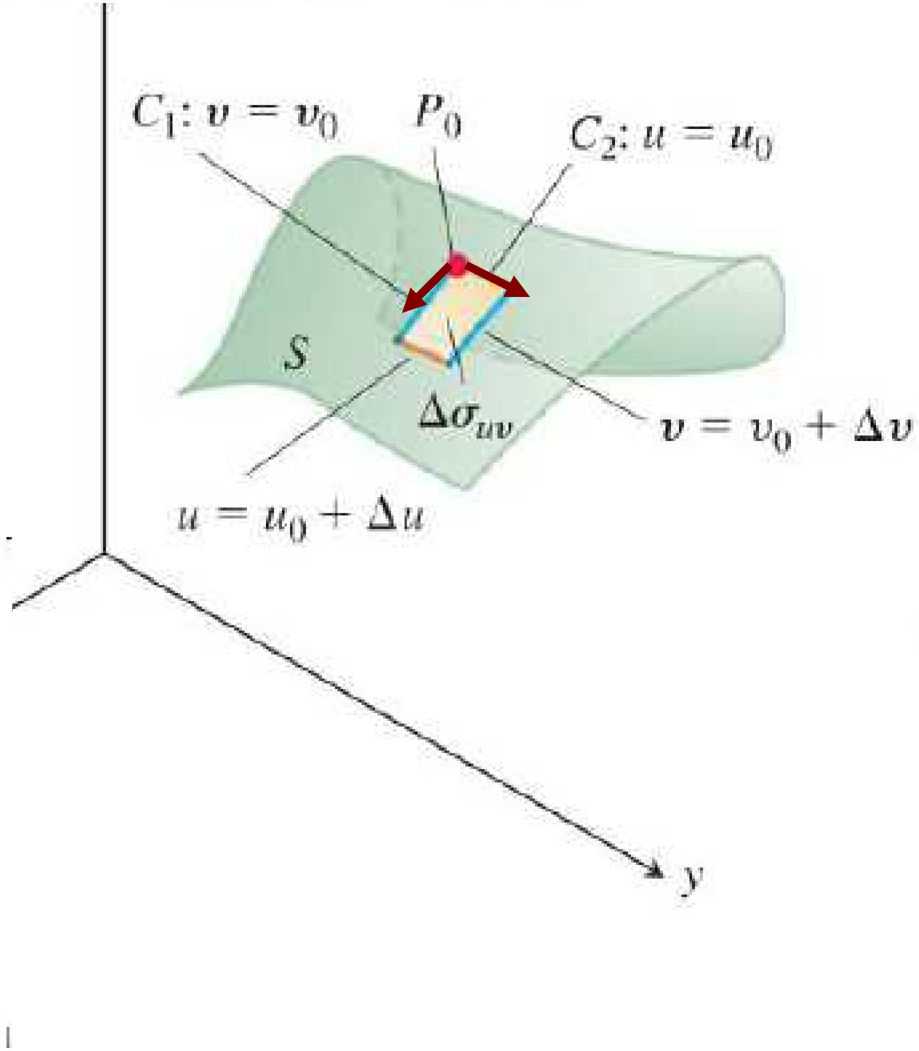
$$x = r\cos\theta = 6\sin\theta\cos\theta = 3\sin 2\theta$$
$$y = r\sin\theta = 6\sin^2\theta$$
$$z = z.$$

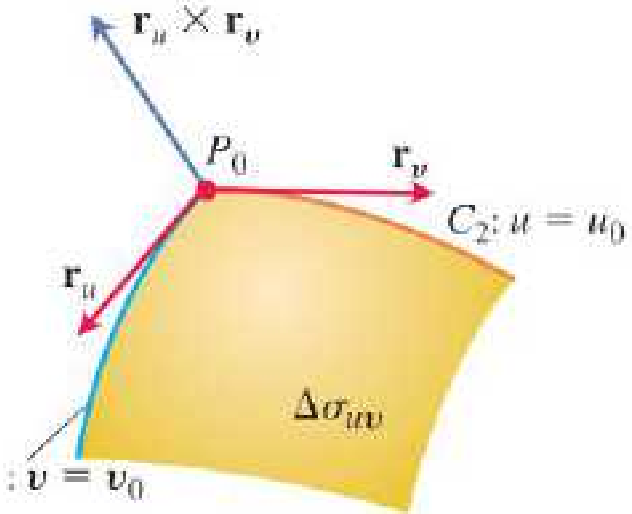
$$\mathbf{r}(\theta, z) = (3\sin 2\theta)\mathbf{i} + (6\sin^2\theta)\mathbf{j} + z\mathbf{k}, \quad 0 \le \theta \le \pi, \quad 0 \le z \le 5.$$

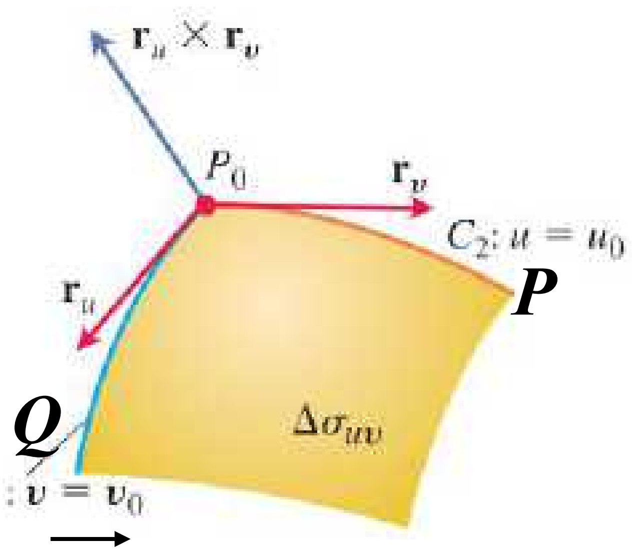
Surface Area

$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$$









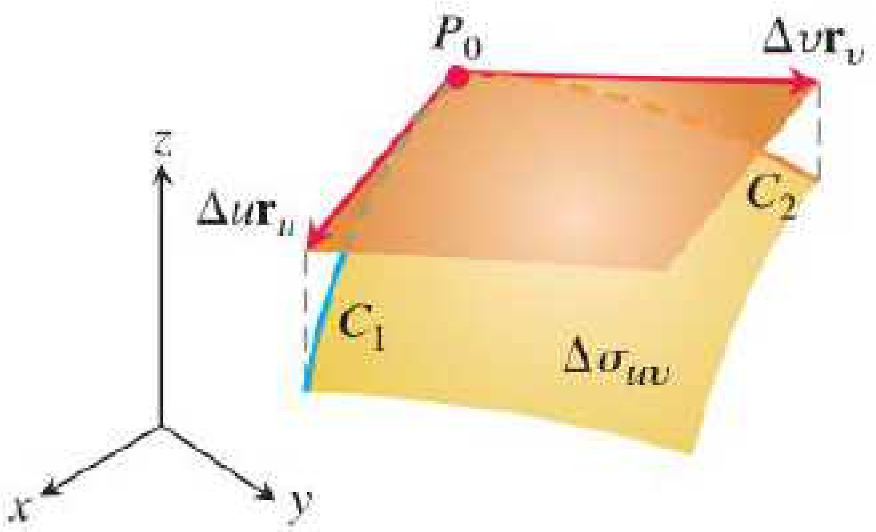
$$\overrightarrow{P_0P} = \mathbf{r}(P) - \mathbf{r}(P_0)$$
$$= \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0)$$
$$= \frac{\mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0)}{\Delta v}\Delta v$$
$$\approx \frac{\partial \mathbf{r}(u_0, v_0)}{\partial v}\Delta v$$

$$\overrightarrow{P_0Q} = \mathbf{r}(Q) - \mathbf{r}(P_0)$$
$$= \mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0)$$
$$\approx \frac{\partial \mathbf{r}(u_0, v_0)}{\partial u}\Delta u$$

$$\Delta\sigma_{uv} \approx \left|\Delta u \mathbf{r}_u \times \Delta v \mathbf{r}_v\right| = \left|\mathbf{r}_u \times \mathbf{r}_v\right| \Delta u \, \Delta v.$$

$$A \approx \sum_{n} |\mathbf{r}_u \times \mathbf{r}_v| \, \Delta u \, \Delta v.$$ As Δu and Δv approach zero

$$A = \int_c^d \int_a^b |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv.$$



DEFINITION The **area** of the smooth surface

$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}, \qquad a \le u \le b, \quad c \le v \le d$$

is

$$A = \iint_R |\mathbf{r}_u \times \mathbf{r}_v| \, dA = \int_c^d \int_a^b |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv.$$

Surface Area Differential for a Parametrized Surface

$$d\sigma = |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv \qquad \iint_S d\sigma$$

DEFINITION A parametrized surface $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$ is **smooth** if $\mathbf{r}_u$ and $\mathbf{r}_v$ are continuous and $\mathbf{r}_u \times \mathbf{r}_v$ is never zero on the interior of the parameter domain.

EXAMPLE 4 Find the surface area of the cone in Example 1 (Figure 16.37).

Solution $z = \sqrt{x^2 + y^2}, \quad 0 \le z \le 1.$

$$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}, \qquad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= -(r\cos\theta)\mathbf{i} - (r\sin\theta)\mathbf{j} + \underbrace{(r\cos^2\theta + r\sin^2\theta)}\mathbf{k}.$$

$$|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{2r^2} = \sqrt{2}r.$$

$$A = \int_0^{2\pi}\int_0^1 |\mathbf{r}_r \times \mathbf{r}_\theta|\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{2}r\, dr\, d\theta = \int_0^{2\pi} \frac{\sqrt{2}}{2} d\theta = \frac{\sqrt{2}}{2}(2\pi) = \pi\sqrt{2}$$

EXAMPLE 5 Find the surface area of a sphere of radius a.

Solution $\mathbf{r}(\phi, \theta) = (a \sin \phi \cos \theta)\mathbf{i} + (a \sin \phi \sin \theta)\mathbf{j} + (a \cos \phi)\mathbf{k},$

$$0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi.$$

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= (a^2 \sin^2 \phi \cos \theta)\mathbf{i} + (a^2 \sin^2 \phi \sin \theta)\mathbf{j} + (a^2 \sin \phi \cos \phi)\mathbf{k}.$$

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{a^4 \sin^4 \phi \cos^2 \theta + a^4 \sin^4 \phi \sin^2 \theta + a^4 \sin^2 \phi \cos^2 \phi}$$

$$= a^2 \sin \phi,$$

$$A = \int_0^{2\pi} \int_0^{\pi} a^2 \sin \phi \, d\phi \, d\theta = 4\pi a^2 \quad \text{units squared.}$$

EXAMPLE 6

Let S be the "football" surface formed by rotating the curve $x = \cos z$, $y = 0, -\pi/2 \le z \le \pi/2$ around the z-axis. Find a parametrization for S and compute its surface area.

Solution We let (x, y, z) be an arbitrary point on this circle,

$$\sqrt{x^2 + y^2} = \cos z \qquad x = r\cos\theta, y = r\sin\theta, z = z$$

$$r = \cos z, \qquad \therefore\ x = \cos z\cos\theta, y = \cos z\sin\theta, z = z$$

$$\therefore\ \mathbf{r}(r,\theta) = (\cos z\cos\theta)\mathbf{i} + (\cos z\sin\theta)\mathbf{j} + z\mathbf{k}$$

$$-\frac{\pi}{2} \le z \le \frac{\pi}{2}, 0 \le \theta \le 2\pi.$$

$$\therefore\ \mathbf{r}(u,v) = (\cos u\cos v)\mathbf{i} + (\cos u\sin v)\mathbf{j} + u\mathbf{k}$$

$$\mathbf{r}_u = -\sin u \cos v \, \mathbf{i} - \sin u \sin v \, \mathbf{j} + \mathbf{k}$$

$$\mathbf{r}_v = -\cos u \sin v \, \mathbf{i} + \cos u \cos v \, \mathbf{j}.$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin u \cos v & -\sin u \sin v & 1 \\ -\cos u \sin v & \cos u \cos v & 0 \end{vmatrix}$$

$$= -\cos u \cos v \, \mathbf{i} - \cos u \sin v \, \mathbf{j}$$

$$- \sin u \cos u \; \mathbf{k}$$

$$|\mathbf{r}_u \times \mathbf{r}_v| = \cos u \sqrt{1 + \sin^2 u}.$$

$$A = \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \cos u \sqrt{1 + \sin^2 u} \, du \, dv.$$

$$= 2 \int_0^{2\pi} \int_0^1 \sqrt{1 + w^2} \; dw \, dv \quad = 2\pi \left[\sqrt{2} + \ln\left(1 + \sqrt{2}\right) \right].$$

EXAMPLE 8

Derive the surface area differential $d\sigma$ of the surface $z = f(x, y)$ over a region R in the xy-plane **(a)** parametrically using Equation (5),

Solution $d\sigma = |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv$ $\qquad$ $x = u, y = v, z = f(u, v),$

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + f(u, v)\mathbf{k}.$$

$\mathbf{r}_u = \mathbf{i} + f_u\mathbf{k}, \mathbf{r}_v = \mathbf{j} + f_v\mathbf{k}$ $\qquad$ $\mathbf{r}_u \times \mathbf{r}_v = -f_u\mathbf{i} - f_v\mathbf{j} + \mathbf{k}.$

$$|\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv = \sqrt{f_u^{\,2} + f_v^{\,2} + 1} \, du \, dv.$$

$$d\sigma = \sqrt{f_x^{\,2} + f_y^{\,2} + 1} \, dx \, dy.$$

Formula for the Surface Area of a Graph $z = f(x, y)$

For a graph $z = f(x, y)$ over a region R in the xy-plane, the surface area formula is

$$A = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy. \tag{8}$$

EXAMPLE 8

Find the area of the surface $x^2 - 2\ln x + \sqrt{15}y - z = 0$ above the square R: $1 \le x \le 2$, $0 \le y \le 1$, in the xy-plane.

Solution

$$A = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dxdy$$

$$= \int_1^2 \int_0^1 2\left(x + \frac{1}{x}\right) dydx \quad = 3 + 2\ln 2$$

S: $z=\sqrt{x^2+y^2}, x^2+y^2 \le 1$. **Find the area.**

$$A=\iint_R \sqrt{1+f_x^2+f_y^2}\,dxdy$$

$$=\iint_R \sqrt{2}\,dxdy = \sqrt{2}\pi.$$

16.6

Surface Integrals
面积分

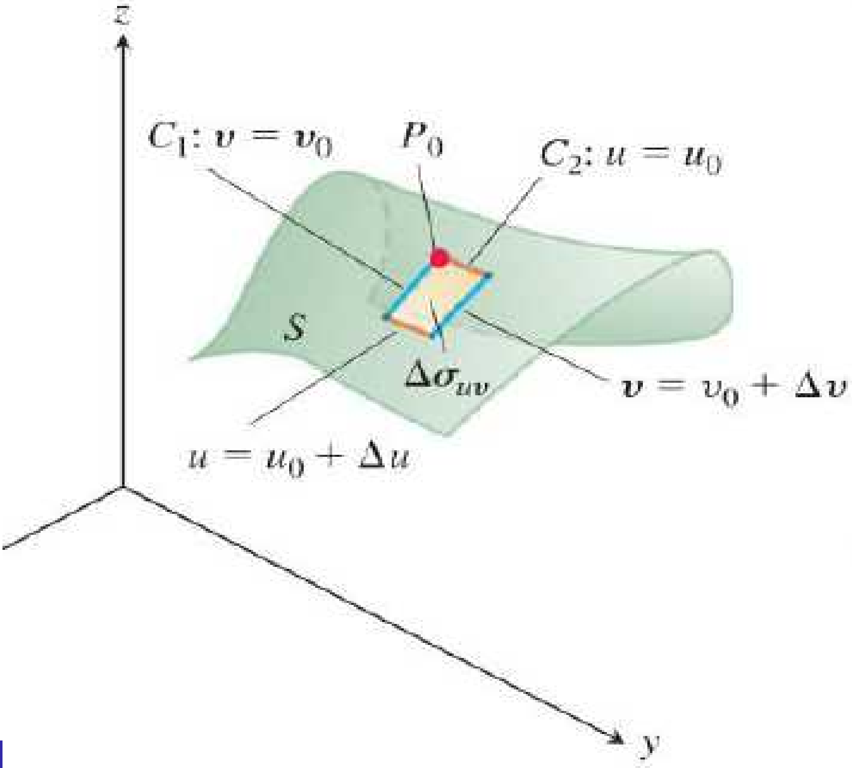
Surface Integrals

we can calculate the total mass of S

$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}, \qquad (u, v) \in R.$$

Suppose that the function $G(x, y, z)$ gives the *mass density*

subdivision of R



$$G(x_k, y_k, z_k)\Delta\sigma_k$$

$$\lim_{\|T\|\to 0} \sum_{k=1}^{n} G(x_k, y_k, z_k)\Delta\sigma_k$$

$$= \iint_S G(x, y, z)\, d\sigma$$

$$\iint_S G(x, y, z)\, d\sigma = \lim_{n \to \infty} \sum_{k=1}^{n} G(x_k, y_k, z_k)\, \Delta\sigma_k.$$

Formulas for a Surface Integral of a Scalar Function

1. For a smooth surface S defined **parametrically** as $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$, $(u, v) \in R$, and a continuous function $G(x, y, z)$ defined on S, the surface integral of G over S is given by the double integral over R,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(f(u, v), g(u, v), h(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv.$$

3. For a surface S given **explicitly** as the graph of $z = f(x, y)$, where f is a continuously differentiable function over a region R in the xy-plane, the surface integral of the continuous function G over S is given by the double integral over R,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(x, y, f(x, y)) \sqrt{f_x{}^2 + f_y{}^2 + 1}\, dx\, dy.$$

if $S = S_1 + S_2 + \cdots + S_n$**, then**

$$\iint_S G\, d\sigma = \iint_{S_1} G\, d\sigma + \iint_{S_2} G\, d\sigma + \cdots + \iint_{S_n} G\, d\sigma.$$

EXAMPLE 1

Integrate $G(x, y, z) = x^2$ over the cone $z = \sqrt{x^2 + y^2}$, $0 \le z \le 1$.

Solution $\iint_S x^2 d\sigma$ $\quad S : \mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + r\mathbf{k}$,

$0 \le r \le 1, 0 \le \theta \le 2\pi$,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(f(u, v), g(u, v), h(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv.$$

$\mathbf{r}_r = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} + \mathbf{k}$, $\quad \mathbf{r}_\theta = (-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j}$,

$\mathbf{r}_r \times \mathbf{r}_\theta = (-r \cos \theta)\mathbf{i} + (-r \sin \theta)\mathbf{j} + r\mathbf{k}$, $\quad |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{2}r$,

$$\iint_S x^2\, d\sigma = \int_0^{2\pi} \int_0^1 \left(r^2 \cos^2 \theta\right)\left(\sqrt{2}r\right) dr\, d\theta$$

$$= \sqrt{2} \int_0^{2\pi} \int_0^1 r^3 \cos^2 \theta\, dr\, d\theta = \frac{\sqrt{2}}{4} \int_0^{2\pi} \cos^2 \theta\, d\theta = \frac{\pi\sqrt{2}}{4}.$$

EXAMPLE 2

Integrate $G(x, y, z) = xyz$ over the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$

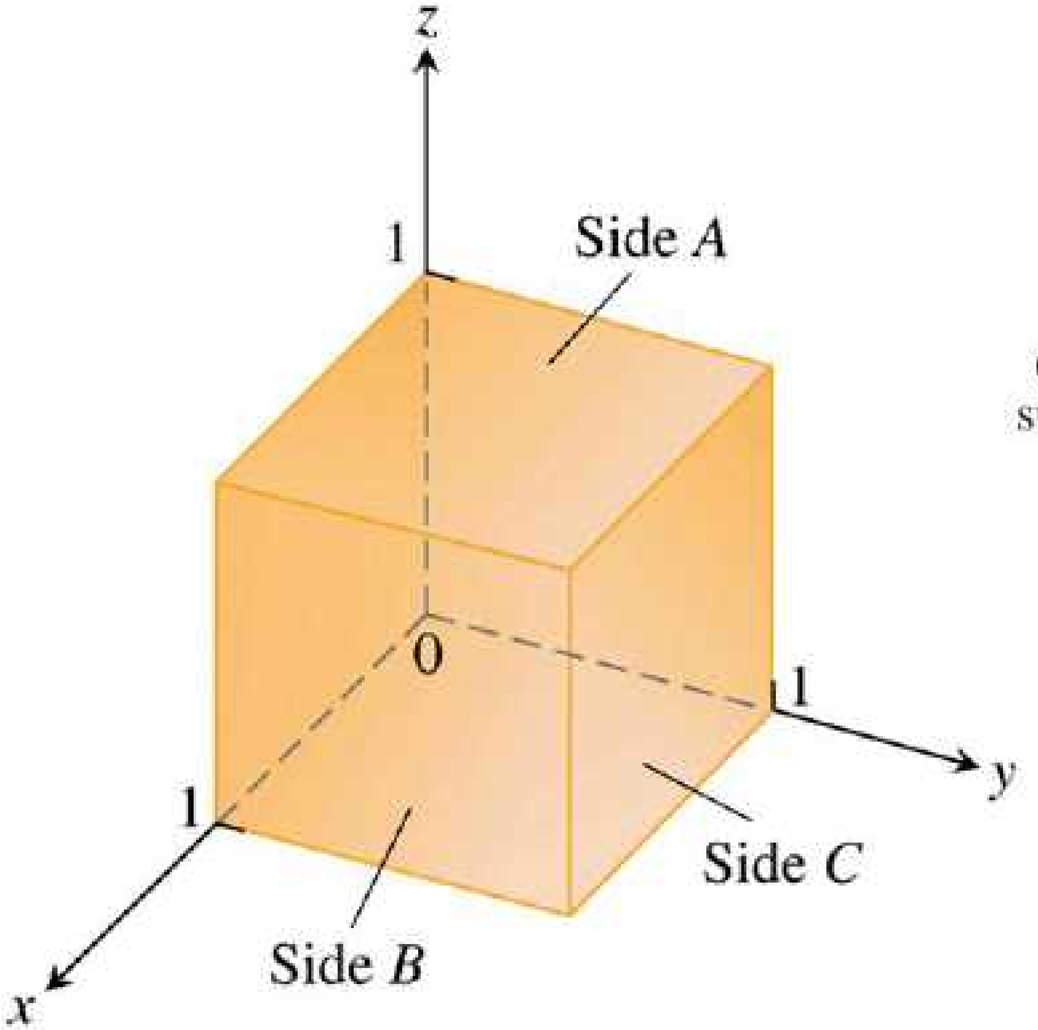
Solution
$$\iint_{\substack{\text{Cube}\\\text{surface}}} xyz\,d\sigma = \iint_{\text{Side } A} xyz\,d\sigma + \iint_{\text{Side } B} xyz\,d\sigma + \iint_{\text{Side } C} xyz\,d\sigma.$$

$$\iint_S G(x, y, z)\,d\sigma = \iint_R G(x, y, f(x, y))\sqrt{f_x{}^2 + f_y{}^2 + 1}\,dx\,dy.$$

Side A $\quad z = 1$

$$\iint_{\text{Side } A} xyz\,d\sigma = \iint_{R_{xy}} xy\,dx\,dy = \int_0^1\int_0^1 xy\,dx\,dy = \int_0^1 \frac{y}{2}dy = \frac{1}{4}.$$

Symmetry tells us that the integrals of xyz over sides B and C are also $1/4$.



$$\iint\limits_{\substack{\text{Cube}\\\text{surface}}} xyz\, d\sigma = \frac{3}{4}.$$

FIGURE 16.47 The cube in Example 2.

EXAMPLE 3

Integrate $G(x, y, z) = \sqrt{1 - x^2 - y^2}$ over the "football" surface S formed by rotating the curve $x = \cos z, y = 0, -\pi/2 \leq z \leq \pi/2$, around the z-axis.

Solution $\displaystyle\iint_S \sqrt{1 - x^2 - y^2}\, d\sigma$

$$\mathbf{r}(u, v) = (\cos u \cos v)\mathbf{i} + (\cos u \sin v)\mathbf{j} + u\mathbf{k}$$

$$x = \cos u \cos v, \quad y = \cos u \sin v, \quad z = u, \quad -\frac{\pi}{2} \leq u \leq \frac{\pi}{2}, 0 \leq v \leq 2\pi.$$

$$\sqrt{1 - x^2 - y^2} = |\sin u|. \qquad d\sigma = |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv$$

$$\mathbf{r}_u = (-\sin u \cos v)\mathbf{i} + (-\sin u \sin v)\mathbf{j} + \mathbf{k},$$

$$\mathbf{r}_v = (-\cos u \sin v)\mathbf{i} + (\cos u \cos v)\mathbf{j},$$

$$\mathbf{r}_u \times \mathbf{r}_v = (-\cos u \cos v)\mathbf{i} + (-\cos u \sin v)\mathbf{j} + (-\sin u \sin v)\mathbf{k},$$

$$|\mathbf{r}_u \times \mathbf{r}_v| = |\cos u| \sqrt{1 + \sin^2 u},$$

$$\iint_{S} \sqrt{1 - x^2 - y^2}\, d\sigma = \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} |\sin u| \cos u \sqrt{1 + \sin^2 u}\, du\, dv$$

$$= 2\int_0^{2\pi} \int_0^{\pi/2} \sin u \cos u \sqrt{1 + \sin^2 u}\, du\, dv$$

$$= \int_0^{2\pi} \int_1^{2} \sqrt{w}\, dw\, dv = 2\pi \cdot \frac{2}{3} w^{3/2} \Big]_1^2 = \frac{4\pi}{3}\left(2\sqrt{2} - 1\right).$$

EXAMPLE 4

Evaluate $\iint_S \sqrt{x(1 + 2z)}\, d\sigma$ on the portion of the cylinder $z = y^2/2$ over the triangular region R: $x \geq 0$, $y \geq 0$, $x + y \leq 1$ in the xy-plane

Solution $G(x, y, z) = \sqrt{x(1 + 2z)} = \sqrt{x}\sqrt{1 + y^2}$.

$$\iint\limits_S G(x, y, z)\, d\sigma = \iint\limits_R G(x, y, f(x, y)) \sqrt{f_x{}^2 + f_y{}^2 + 1}\, dx\, dy.$$

$$z = f(x, y) = y^2/2,$$

$$d\sigma = \sqrt{f_x{}^2 + f_y{}^2 + 1}\, dx\, dy = \sqrt{0 + y^2 + 1}\, dx\, dy$$

$$\iint\limits_S G(x, y, z)\, d\sigma = \iint\limits_R \left(\sqrt{x}\sqrt{1 + y^2}\right)\sqrt{1 + y^2}\, dx\, dy$$

$$= \int_0^1 \int_0^{1-x} \sqrt{x}(1 + y^2)\, dy\, dx$$

$$= \int_0^1 \left(\frac{4}{3}x^{1/2} - 2x^{3/2} + x^{5/2} - \frac{1}{3}x^{7/2} \right) dx$$

$$= \frac{8}{9} - \frac{4}{5} + \frac{2}{7} - \frac{2}{27} = \frac{284}{945}$$

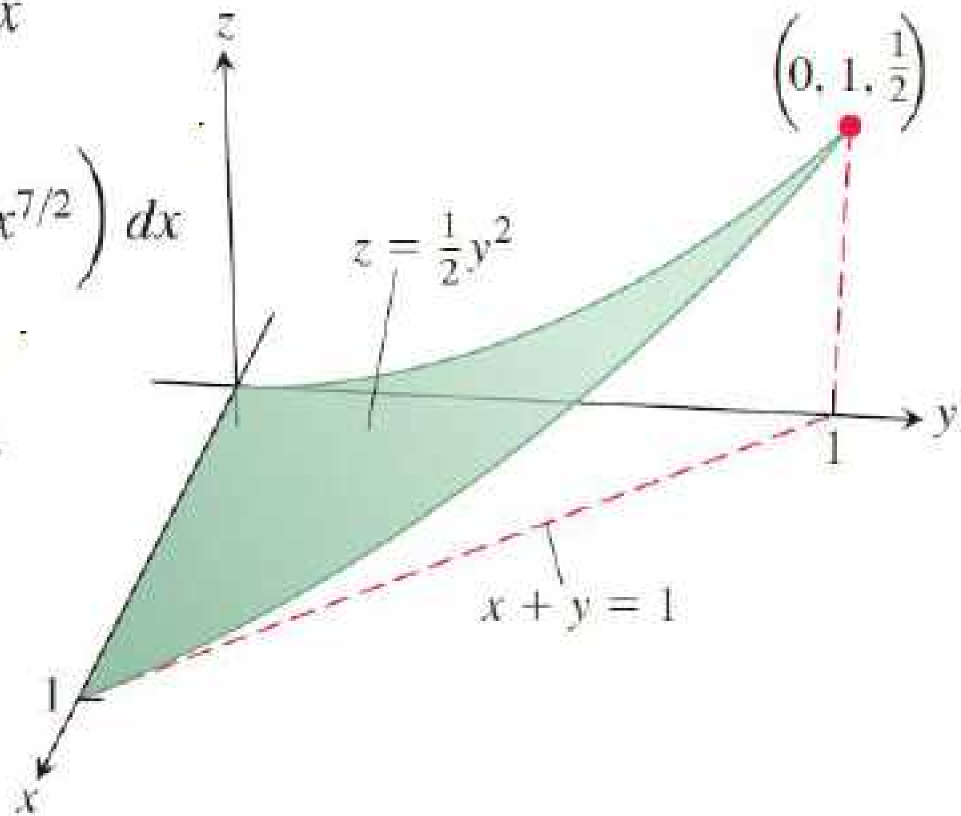


FIGURE 16.48 The surface S in Example 4.

Moments and Masses of Thin Shells

Mass: $M = \iint_S \delta \, d\sigma$

$$M_{yz} = \iint_S x \, \delta \, d\sigma, \qquad M_{xz} = \iint_S y \, \delta \, d\sigma, \qquad M_{xy} = \iint_S z \, \delta \, d\sigma$$

$$\bar{x} = M_{yz}/M, \qquad \bar{y} = M_{xz}/M, \qquad \bar{z} = M_{xy}/M$$

EXAMPLE 8

Find the center of mass of a thin shell of density $\delta = 1/z^2$ cut from the cone $z = \sqrt{x^2 + y^2}$ by the planes $z = 1$ and $z = 2$

Solution $\bar{x} = \bar{y} = 0.$ $\bar{z} = M_{xy}/M.$

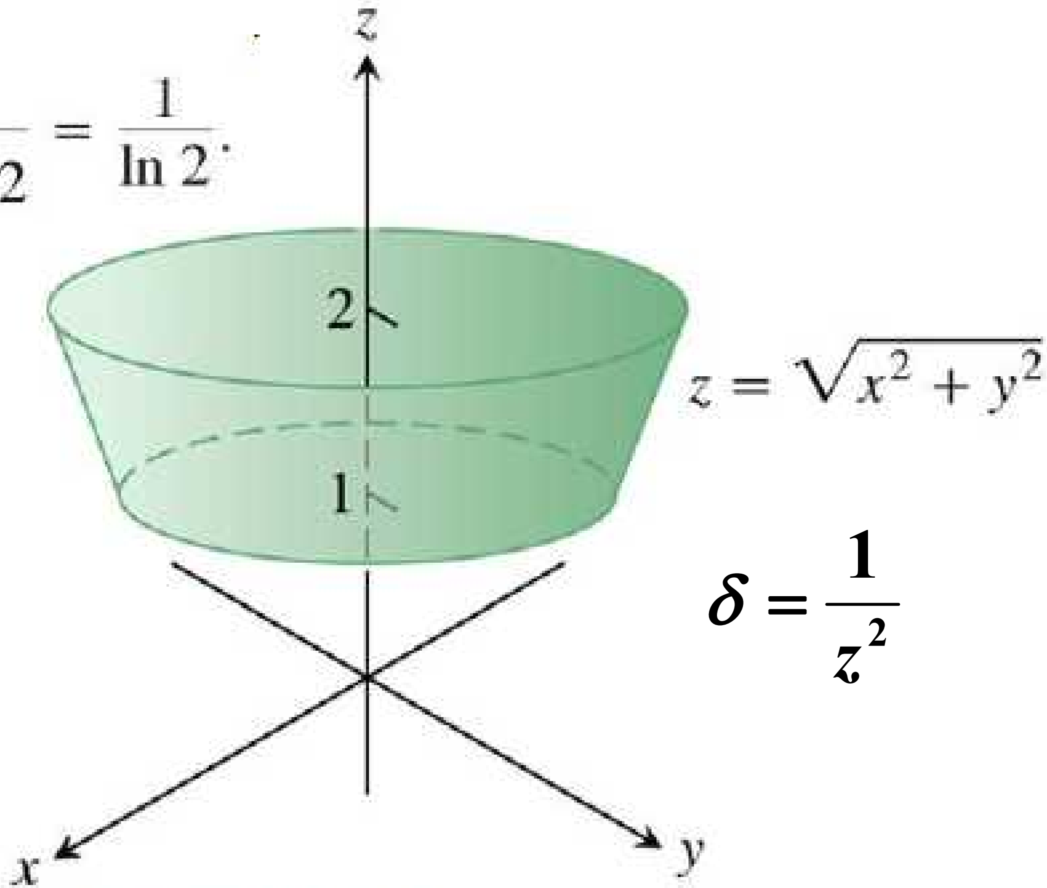
$$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}, \quad 1 \le r \le 2, \quad 0 \le \theta \le 2\pi,$$

$$|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{2}r. \qquad M = \iint_S \delta\, d\sigma = \int_0^{2\pi}\int_1^2 \frac{1}{r^2}\sqrt{2}r\, dr\, d\theta$$

$$= \sqrt{2}\int_0^{2\pi} \Big[\ln r\Big]_1^2 d\theta = \sqrt{2}\int_0^{2\pi} \ln 2\, d\theta = 2\pi\sqrt{2}\ln 2,$$

$$M_{xy} = \iint_S \delta z\, d\sigma = \int_0^{2\pi}\int_1^2 \frac{1}{r^2} r\sqrt{2}r\, dr\, d\theta = 2\pi\sqrt{2},$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{2\pi\sqrt{2}}{2\pi\sqrt{2}\ln 2} = \frac{1}{\ln 2}.$$



$$\delta = \frac{1}{z^2}$$

FIGURE 16.54 The cone frustum formed when the cone $z = \sqrt{x^2 + y^2}$ is cut by the planes $z = 1$ and $z = 2$ (Example 8).

Orientation of a Surface

The curve C in a line integral i $\mathbf{r}(t)$

$\vec{r}'(t)$ **points in the forward direction.**

For a surface S, $\mathbf{r}(u, v)$

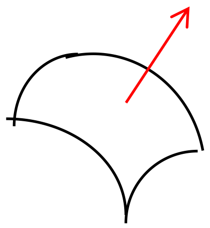
$\mathbf{r}_u \times \mathbf{r}_v$ that is normal to the surface,

$-(\mathbf{r}_u \times \mathbf{r}_v)$ is also normal to the surface,

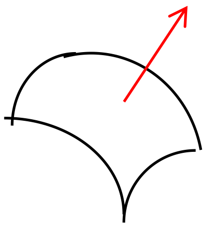
we specify which of these normals we are going to use

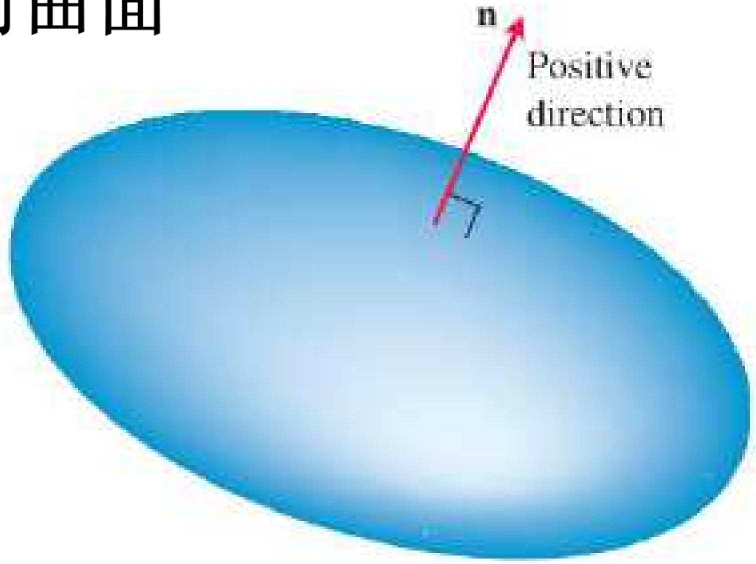
Once $\mathbf{n}$ has been chosen,

we have **oriented** the surface,



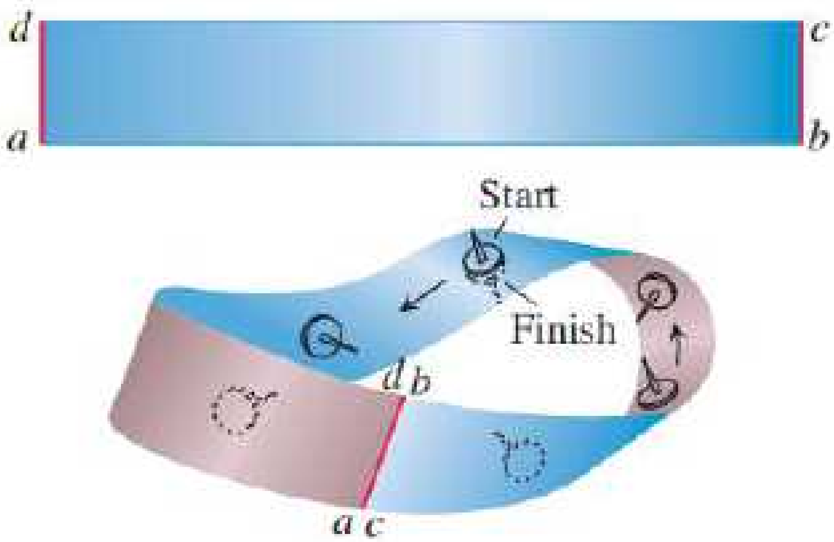
an **oriented surface**. 有向曲面





双侧曲面

The Möbius band



单侧曲面

Surface Integrals of Vector Fields

DEFINITION Let $\mathbf{F}$ be a vector field in three-dimensional space with continuous components defined over a smooth surface S having a chosen field of normal unit vectors $\mathbf{n}$ orienting S. Then the **surface integral of F over *S*** is

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma.$$

It is also called the flux of the vector field F through the oriented surface *S*.

EXAMPLE 5

Find the flux of $\mathbf{F} = yz\mathbf{i} + x\mathbf{j} - z^2\mathbf{k}$ through the parabolic cylinder $y = x^2$, $0 \le x \le 1$, $0 \le z \le 4$, in the direction $\mathbf{n}$ indicated in Figure

Solution $$\textbf{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

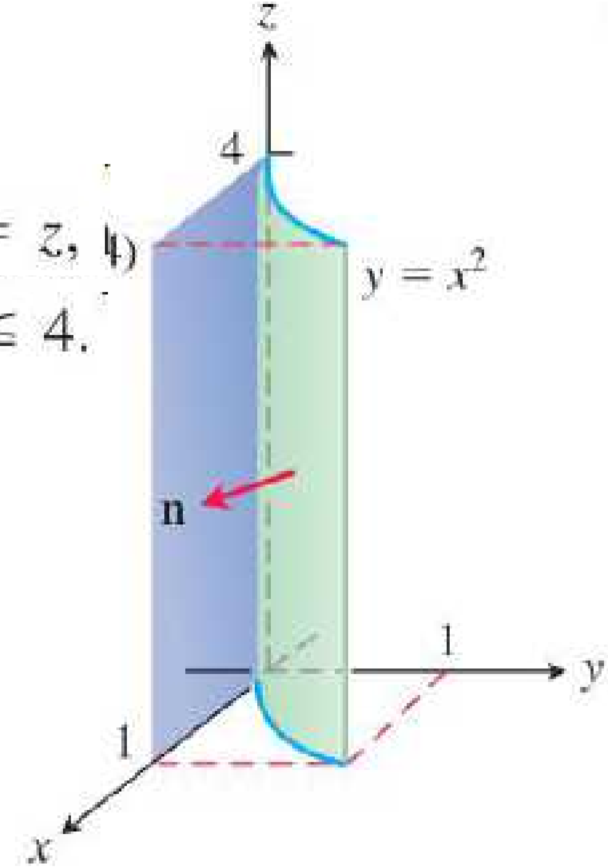
the surface we have $x = x$, $y = x^2$, and $z = z$,

$\mathbf{r}(x, z) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}$, $0 \le x \le 1$, $0 \le z \le 4$.

$\mathbf{r}_z \times \mathbf{r}_x$?

$$\mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2x\mathbf{i} - \mathbf{j}.$$

$$\mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_z}{|\mathbf{r}_x \times \mathbf{r}_z|} = \frac{2x\mathbf{i} - \mathbf{j}}{\sqrt{4x^2 + 1}}.$$



$$\mathbf{F} = yz\mathbf{i} + x\mathbf{j} - z^2\mathbf{k} = x^2z\mathbf{i} + x\mathbf{j} - z^2\mathbf{k}.$$

$$\mathbf{F}\cdot\mathbf{n} = \frac{1}{\sqrt{4x^2+1}}\left((x^2z)(2x) + (x)(-1) + (-z^2)(0)\right)$$

$$= \frac{2x^3z - x}{\sqrt{4x^2+1}}.$$

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^4\int_0^1 \frac{2x^3z - x}{\sqrt{4x^2+1}}\left|\mathbf{r}_x \times \mathbf{r}_z\right|\,dx\,dz$$

$$= \int_0^4\int_0^1 \frac{2x^3z - x}{\sqrt{4x^2+1}}\sqrt{4x^2+1}\,dx\,dz = \int_0^4\int_0^1 (2x^3z - x)\,dx\,dz$$

$$= \int_0^4 \frac{1}{2}(z-1)\,dz = 2.$$

$$S: \quad \mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}, \qquad (u, v) \in R.$$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \pm \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv$$

$$\boxed{\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \pm \iint_R \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv}$$

$$S: \quad z = f(x, y) \quad f(x, y) - z = 0 \quad n = \pm \frac{f_x \mathbf{i} + f_y \mathbf{j} - \mathbf{k}}{\sqrt{1 + f_x^2 + f_y^2}}$$

$$\boxed{\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \pm \iint_{R_{xy}} \mathbf{F}(x, y, f(x, y)) \cdot (f_x \mathbf{i} + f_y \mathbf{j} - \mathbf{k})\, dx\, dy}$$

EXAMPLE 6

Find the flux of $\mathbf{F} = yz\mathbf{j} + z^2\mathbf{k}$ outward through the surface S cut from the cylinder $y^2 + z^2 = 1$, $z \geq 0$, by the planes $x = 0$ and $x = 1$.

Solution

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \pm\iint_{R_{xy}} \mathbf{F}\cdot(f_x\mathbf{i} + f_y\mathbf{j} - \mathbf{k})\,dx\,dy$$

$$= -\iint_{R_{xy}} \mathbf{F}\cdot(f_x\mathbf{i} + f_y\mathbf{j} - \mathbf{k})\,dx\,dy$$

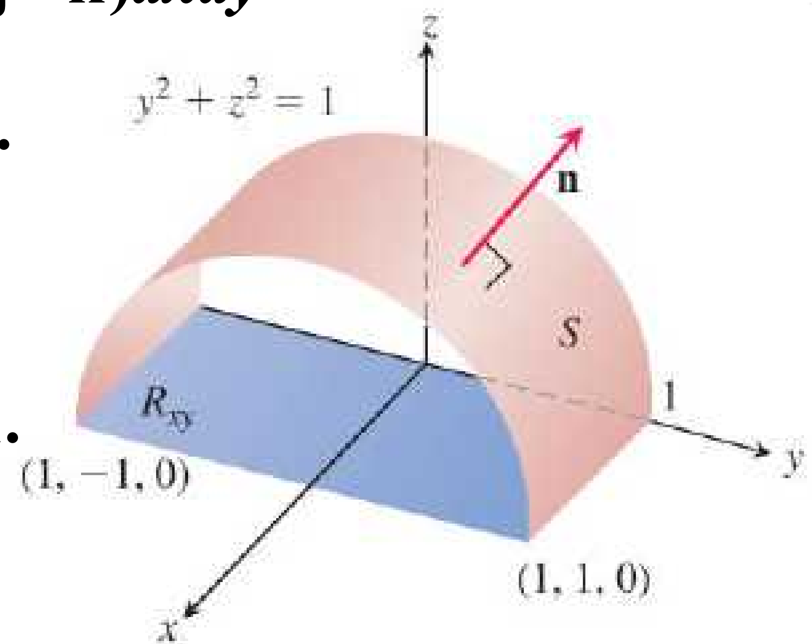
$$S: z = \sqrt{1-y^2},\ 0 \leq x \leq 1,\ -1 \leq y \leq 1.$$

$$f_x\mathbf{i} + f_y\mathbf{j} - \mathbf{k} = -\frac{y}{\sqrt{1-y^2}}\mathbf{j} - \mathbf{k}.$$

$$\mathbf{F} = y\sqrt{1-y^2}\,\mathbf{j} + (1-y^2)\mathbf{k}.$$

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = -\iint_{R_{xy}} -1\,dx\,dy$$

$$= \text{area}(R_{xy}) = 2.$$



Find the flux of the field $\mathbf{F}(x, y, z) = 4x\mathbf{i} + 4y\mathbf{j} + 2\mathbf{k}$ outward (away from the z-axis) through the surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 1$.

$$\text{Flux} = \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = +\iint_{R_{xy}} \mathbf{F}\cdot(f_x\mathbf{i} + f_y\mathbf{j} - \mathbf{k})\,dx\,dy$$

$$S: z = x^2 + y^2,\ R_{xy}: x^2 + y^2 \le 1, \quad f_x\mathbf{i} + f_y\mathbf{j} - \mathbf{k} = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k},$$

$$\mathbf{F} = 4x\mathbf{i} + 4y\mathbf{j} + 2\mathbf{k}.$$

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_{R_{xy}} (8x^2 + 8y^2 - 2)\,dx\,dy$$

$$= 8\iint_{R_{xy}} (x^2 + y^2)\,dx\,dy - \iint_{R_{xy}} 2\,dx\,dy = 8\int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta - 2\pi = 2\pi$$

16.7

Stokes' Theorem

斯托克斯定理

two-dimensional vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$

$(\partial N/\partial x - \partial M/\partial y)$. the $\mathbf{k}$-component of a *curl vector* field,

measures the rate of rotation of $\mathbf{F}$ around an axis parallel to $\mathbf{k}$.

For $\mathbf{F} = N\mathbf{j} + P\mathbf{k}$ $\qquad (\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z})\mathbf{i}$

For $\mathbf{F} = P\mathbf{k} + M\mathbf{i}$ $\qquad (\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x})\mathbf{j}$

$\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ $\qquad 2\mathbf{F} = (M\mathbf{i} + N\mathbf{j}) + (P\mathbf{k} + M\mathbf{i}) + (N\mathbf{j} + P\mathbf{k})$

$$(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\mathbf{k} + (\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x})\mathbf{j} + (\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z})\mathbf{i}$$

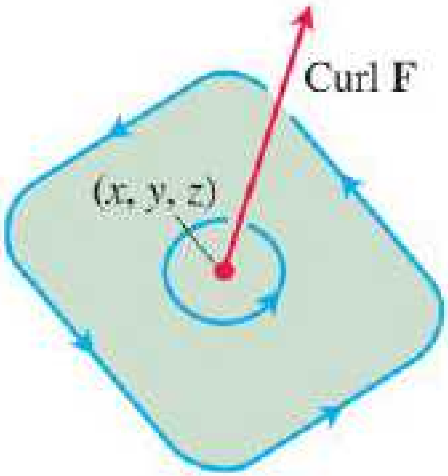
Suppose $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$

is the velocity field of a fluid flowing in space.

$$\operatorname{curl} \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}.$$

is called the **curl vector**, the vector measures the rate of rotation

Notice that $(\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} = (\partial N/\partial x - \partial M/\partial y)$



$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

∇ is pronounced "del," as well as "grad f."

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}.$$

$$\boxed{\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} \qquad (3)}$$

EXAMPLE 1 Find the curl of $\mathbf{F} = (x^2 - z)\mathbf{i} + xe^z\mathbf{j} + xy\mathbf{k}$.

Solution

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - z & xe^z & xy \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(xe^z)\right)\mathbf{i} - \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(x^2 - z)\right)\mathbf{j} + \left(\frac{\partial}{\partial x}(xe^z) - \frac{\partial}{\partial y}(x^2 - z)\right)\mathbf{k}$$

$$= x(1 - e^z)\mathbf{i} - (y + 1)\mathbf{j} + e^z\mathbf{k}.$$

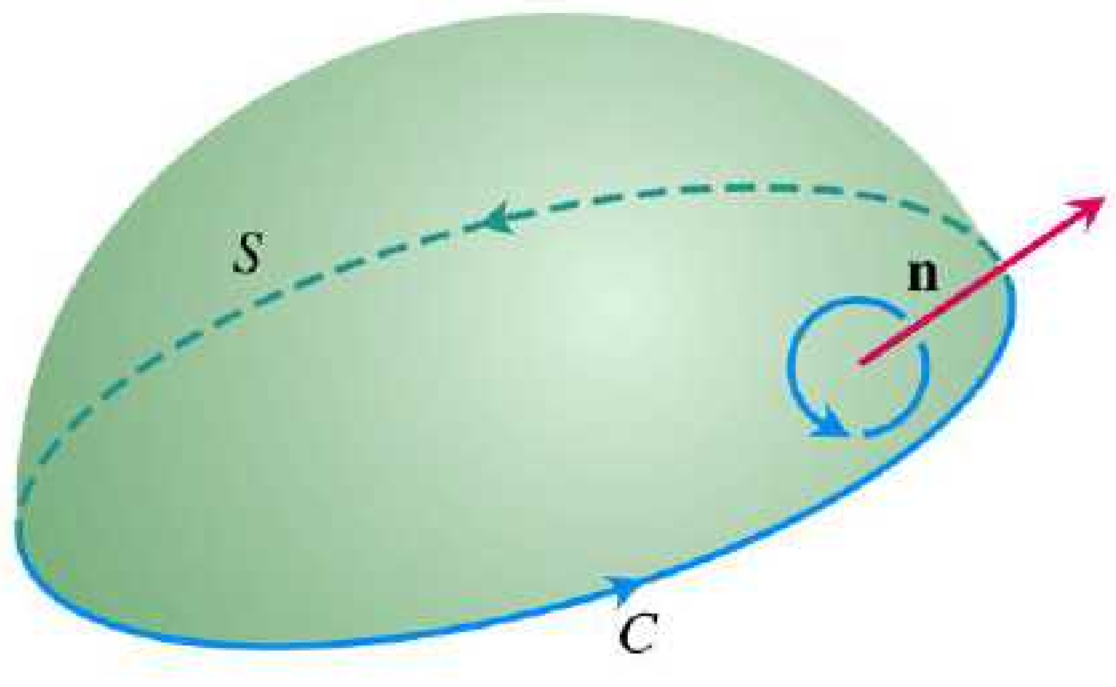


FIGURE 16.56 The orientation of the bounding curve C gives it a right-handed relation to the normal field $\mathbf{n}$. If the thumb of a right hand points along $\mathbf{n}$, the fingers curl in the direction of C.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R \operatorname{curl}\mathbf{F} \cdot \mathbf{k}\, d\sigma = \iint_S \operatorname{curl}\mathbf{F} \cdot \mathbf{n}\, d\sigma$$

Stokes' Theorem

THEOREM 6—Stokes' Theorem Let S be a piecewise smooth oriented surface having a piecewise smooth boundary curve C. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field whose components have continuous first partial derivatives on an open region containing S. Then the circulation of $\mathbf{F}$ around C in the direction counterclockwise with respect to the surface's unit normal vector $\mathbf{n}$ equals the integral of the curl vector field $\nabla \times \mathbf{F}$ over S:

$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{Counterclockwise circulation}} = \underbrace{\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma}_{\text{Curl integral}}$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma$$

Counterclockwise circulation

Curl integral

if two different oriented surfaces S_1 and S_2 have the same boundary C,

$$\iint_{S_1} \nabla \times \mathbf{F} \cdot \mathbf{n}_1 \, d\sigma = \iint_{S_2} \nabla \times \mathbf{F} \cdot \mathbf{n}_2 \, d\sigma.$$

EXAMPLE 4

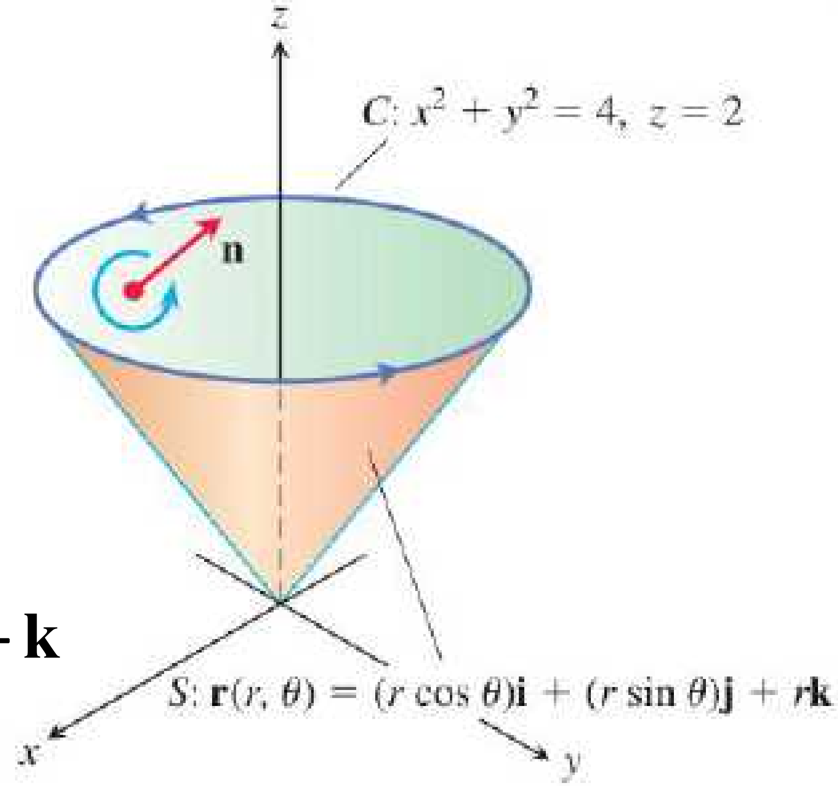
Find the circulation of the field $\mathbf{F} = (x^2 - y)\mathbf{i} + 4z\mathbf{j} + x^2\mathbf{k}$ around the curve C in which the plane $z = 2$ meets the cone $z = \sqrt{x^2 + y^2}$, counterclockwise as viewed from above

Solution $S\text{:}\quad z = 2, (x, y) \in R_{xy}$

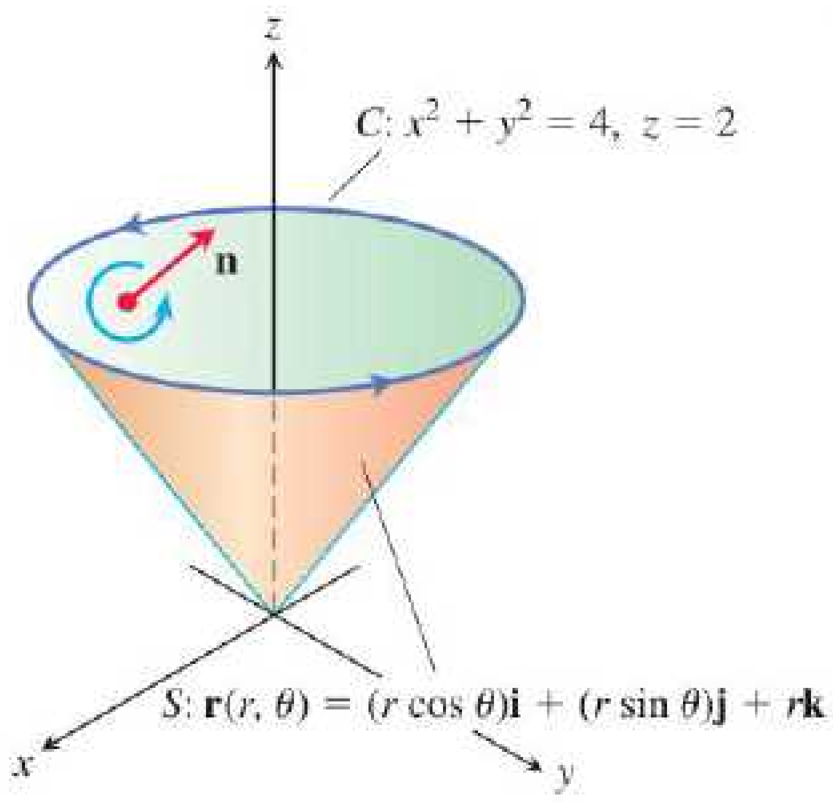
$$\mathbf{n} = \mathbf{k}$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = (-4)\mathbf{i} + (-2x)\mathbf{j} + \mathbf{k}$$



$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_S d\sigma = 4\pi.$$



EXAMPLE 6

Find a parametrization for the surface S formed by the part of the hyperbolic paraboloid $z = y^2 - x^2$ lying inside the cylinder of radius one around the z-axis and for the boundary curve C of S.

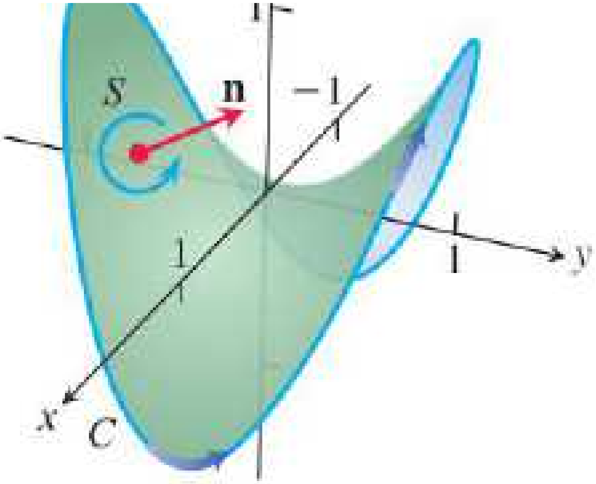
Then verify Stokes' Theorem for S using the normal having positive $\mathbf{k}$-component and the vector field $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + x^2\mathbf{k}$.

Solution

$$0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$

$$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r^2(\sin^2\theta - \cos^2\theta)\mathbf{k},$$

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F}\cdot\mathbf{n}\, d\sigma$$



$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin^2 t - \cos^2 t)\mathbf{k}, \quad 0 \le t \le 2\pi$$

$$\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (4 \sin t \cos t)\mathbf{k}, \ 0 \le t \le 2\pi.$$

$$\mathbf{F} = (\sin t)\mathbf{i} - (\cos t)\mathbf{j} + (\cos^2 t)\mathbf{k}.$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_0^{2\pi} \left(-\sin^2 t - \cos^2 t + 4 \sin t \cos^3 t\right) dt$$

$$= \int_0^{2\pi} \left(4 \sin t \cos^3 t - 1\right) dt = \Big[-\cos^4 t - t\Big]_0^{2\pi} = -2\pi.$$

$$\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + r^2(\sin^2 \theta - \cos^2 \theta)\mathbf{k},$$

$$0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -x & x^2 \end{vmatrix} = -2x\mathbf{j} - 2\mathbf{k} = -(2r\cos\theta)\mathbf{j} - 2\mathbf{k}$$

$$\mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + 2r(\sin^2\theta - \cos^2\theta)\mathbf{k}$$

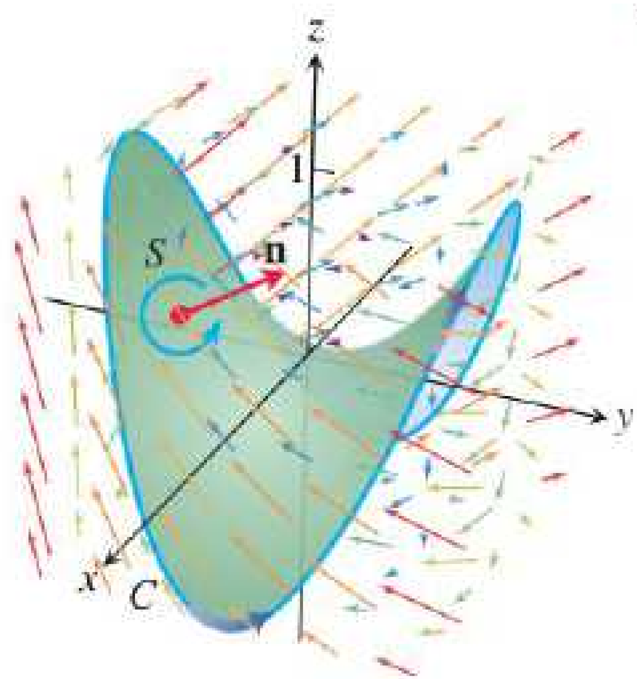
$$\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} + 4r^2(\sin\theta\cos\theta)\mathbf{k}$$

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 2r(\sin^2\theta - \cos^2\theta) \\ -r\sin\theta & r\cos\theta & 4r^2(\sin\theta\cos\theta) \end{vmatrix}$$

$$= 2r^2(2\sin^2\theta\cos\theta - \sin^2\theta\cos\theta + \cos^3\theta)\mathbf{i}$$

$$-2r^2(2\sin\theta\cos^2\theta + \sin^3\theta + \sin\theta\cos^2\theta)\mathbf{j} + r\mathbf{k}.$$

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = + \iint_R \nabla \times \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv$$



$$= \int_0^{2\pi} \int_0^1 \nabla \times \mathbf{F} \cdot (\mathbf{r}_r \times \mathbf{r}_\theta)\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \left[4r^3 \left(2 \sin\theta \cos^3\theta + \sin^3\theta \cos\theta + \sin\theta \cos^3\theta \right) - 2r \right] dr\, d\theta$$

$$= \int_0^{2\pi} (3 \sin\theta \cos^3\theta + \sin^3\theta \cos\theta - 1)\, d\theta = -2\pi.$$

$$S : z = y^2 - x^2, \qquad R_{xy} : \quad x^2 + y^2 \le 1.$$

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = -\iint_{R_{xy}} \nabla \times \mathbf{F} \cdot (f_x \mathbf{i} + f_y \mathbf{j} - k)\, dx\, dy$$

$$f_x \mathbf{i} + f_y \mathbf{j} - \mathbf{k} = -2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -x & x^2 \end{vmatrix} = -2x\mathbf{j} - 2\mathbf{k}$$

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = -\iint_{R_{xy}} (-4xy + 2)\, dx\, dy$$

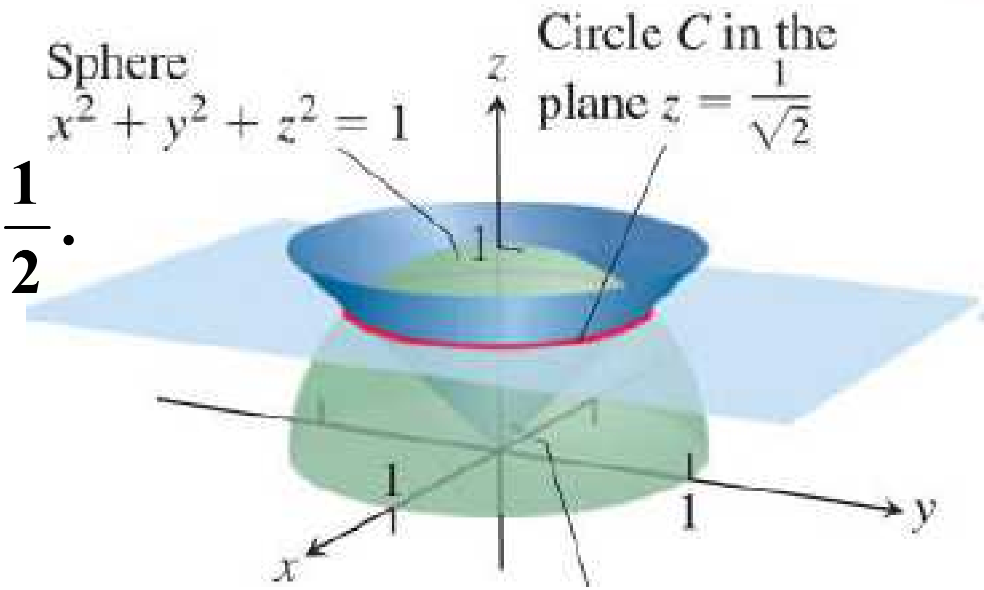
$$= -\iint_{R_{xy}} 2\, dx\, dy = -2\pi$$

EXAMPLE 7 Calculate the circulation of the vector field

$$\mathbf{F} = (x^2 + z)\mathbf{i} + (y^2 + 2x)\mathbf{j} + (z^2 - y)\mathbf{k}$$

along the curve of intersection of the sphere $x^2 + y^2 + z^2 = 1$ with the cone $z = \sqrt{x^2 + y^2}$ traversed in the counterclockwise direction around the z-axis when viewed from above.

Solution $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$



$$S: z = \frac{1}{\sqrt{2}}, \quad R_{xy}: x^2 + y^2 \le \frac{1}{2}.$$

$$\mathbf{n} = \mathbf{k}$$

$$\mathbf{n} = \mathbf{k}$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 + z & y^2 + 2x & z^2 - y \end{vmatrix} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k},$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{k}\, d\sigma = \iint_S 2\, d\sigma$$

$$= 2 \cdot \pi \left(\frac{1}{\sqrt{2}} \right)^2 = \pi.$$

EXAMPLE 9

Use Stokes' Theorem to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, if $\mathbf{F} = xz\mathbf{i} + xy\mathbf{j} + 3xz\mathbf{k}$

C is the boundary of the portion of the plane $2x + y + z = 2$ in the first octant,

counterclockwise as viewed from above

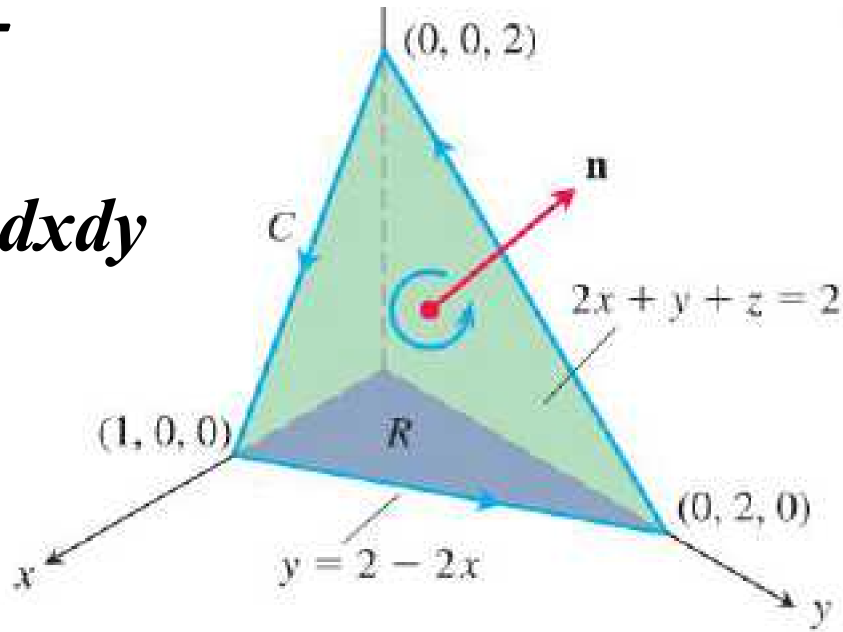
Solution

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma$$

$$= \iint_{R_{xy}} \nabla\times\mathbf{F}\cdot(f_x\mathbf{i} + f_y\mathbf{j} - \mathbf{k})\,dxdy$$

$$z = 2 - 2x - y$$

$$f_x\mathbf{i} + f_y\mathbf{j} - \mathbf{k} = -2\mathbf{i} - \mathbf{j} - \mathbf{k}$$



$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz & xy & 3xz \end{vmatrix} = (x - 3z)\mathbf{j} + y\mathbf{k}.$$

$$= (7x + 3y - 6)\mathbf{j} + y\mathbf{k} \qquad f_x\mathbf{i} + f_y\mathbf{j} - \mathbf{k} = -2\mathbf{i} - \mathbf{j} - \mathbf{k}$$

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = -\iint_{R_{xy}} \nabla\times\mathbf{F}\cdot(f_x\mathbf{i} + f_y\mathbf{j} - \mathbf{k})\,dxdy$$

$$= -\iint_{R_{xy}} (-7x - 4y + 6)\,dxdy$$

$$= \int_0^1\int_0^{2-2x} (7x + 4y - 6)\,dydx = -1.$$

EXAMPLE 10

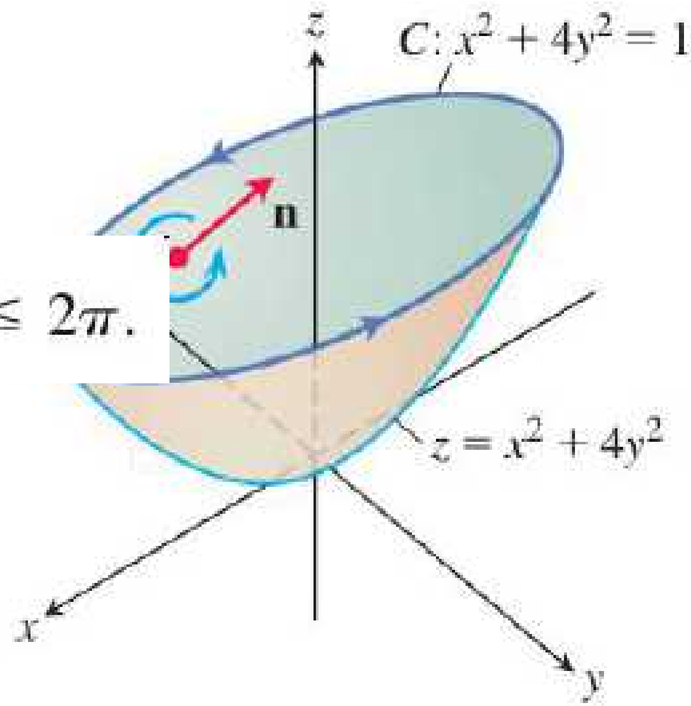
Let the surface S be the elliptical paraboloid $z = x^2 + 4y^2$ lying beneath the plane $z = 1$ We define the orientation of S by taking the *inner* normal vector $\mathbf{n}$ to the surface, having a positive $\mathbf{k}$-component. Find the flux of $\nabla \times \mathbf{F}$ across S in the direction $\mathbf{n}$ for the vector field $\mathbf{F} = y\mathbf{i} - xz\mathbf{j} + xz^2\mathbf{k}$. **Solution**

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + \frac{1}{2}(\sin t)\mathbf{j} + \mathbf{k}, \qquad 0 \le t \le 2\pi.$$

$$\mathbf{F}(\mathbf{r}(t)) = \frac{1}{2}(\sin t)\mathbf{i} - (\cos t)\mathbf{j} + (\cos t)\mathbf{k}$$

$$\frac{d\mathbf{r}}{dt} = -(\sin t)\mathbf{i} + \frac{1}{2}(\cos t)\mathbf{j}.$$



$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot\frac{d\mathbf{r}}{dt}\,dt = \int_0^{2\pi}\left(-\frac{1}{2}\sin^2 t - \frac{1}{2}\cos^2 t\right)dt$$

$$= -\frac{1}{2}\int_0^{2\pi} dt = -\pi. \qquad \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = -\pi.$$

$$\iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma$$

$$S:\ z=1,\quad R_{xy}: x^2+4y^2\le 1,\quad \mathbf{n}=\mathbf{k}$$

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -xz & xz^2 \end{vmatrix} = x\mathbf{i} - z^2\mathbf{j} - (z+1)\mathbf{k} = x\mathbf{i} - \mathbf{j} - 2\mathbf{k}$$

$$\iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_S -2\,d\sigma = -2\pi\cdot 1\cdot\frac{1}{2} = -\pi.$$

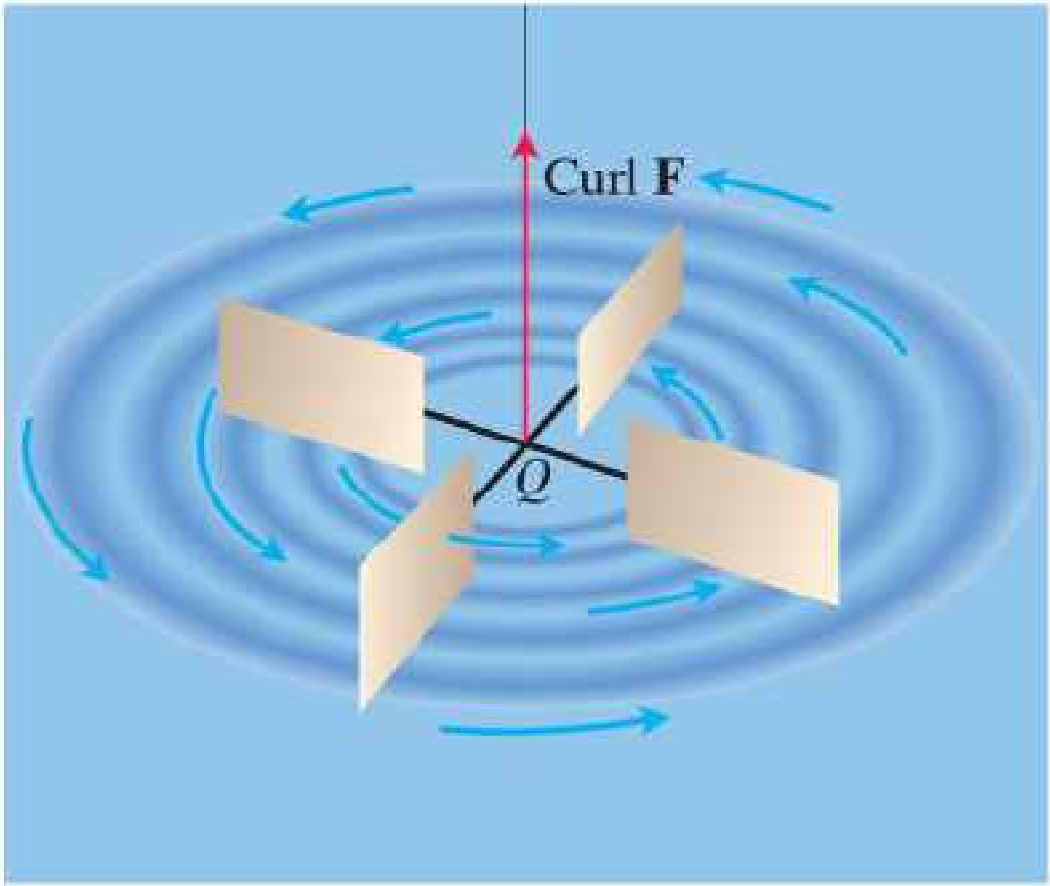


FIGURE 16.62 A small paddle wheel in a fluid spins fastest at point Q when its axle points in the direction of curl **F**.

THEOREM 7—Curl F = 0 Related to the Closed-Loop Property If $\nabla \times \mathbf{F} = \mathbf{0}$ at every point of a simply connected open region D in space, then on any piecewise-smooth closed path C in D,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

$$\nabla \times \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} = 0$$

F是保守场

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

An Important Identity

f有连续二阶偏导,

$$\operatorname{curl} \boxed{\operatorname{grad} f} = \mathbf{0} \qquad \text{or} \qquad \nabla \times \nabla f = \mathbf{0}$$

F

$$\nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix} = (f_{zy} - f_{yz})\mathbf{i} - (f_{zx} - f_{xz})\mathbf{j} + (f_{yx} - f_{xy})\mathbf{k}.$$

$$= \mathbf{0}$$

若F在单连通域内处处一阶偏导连续，则

F是保守场当且仅当curlF=0

16.8

The Divergence Theorem and a Unified Theory

散度定理和统一理论

Divergence in Three Dimensions

The **divergence** of a vector field $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$

EXAMPLE 1

The following vector fields represent the velocity of a gas flowing in space. Find the divergence of each vector field and interpret its physical meaning.

(a) Expansion: $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

(b) Compression: $\mathbf{F}(x, y, z) = -x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$

(c) Rotation about the z-axis: $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j}$

(d) Shearing along parallel horizontal planes: $\mathbf{F}(x, y, z) = z\mathbf{j}$

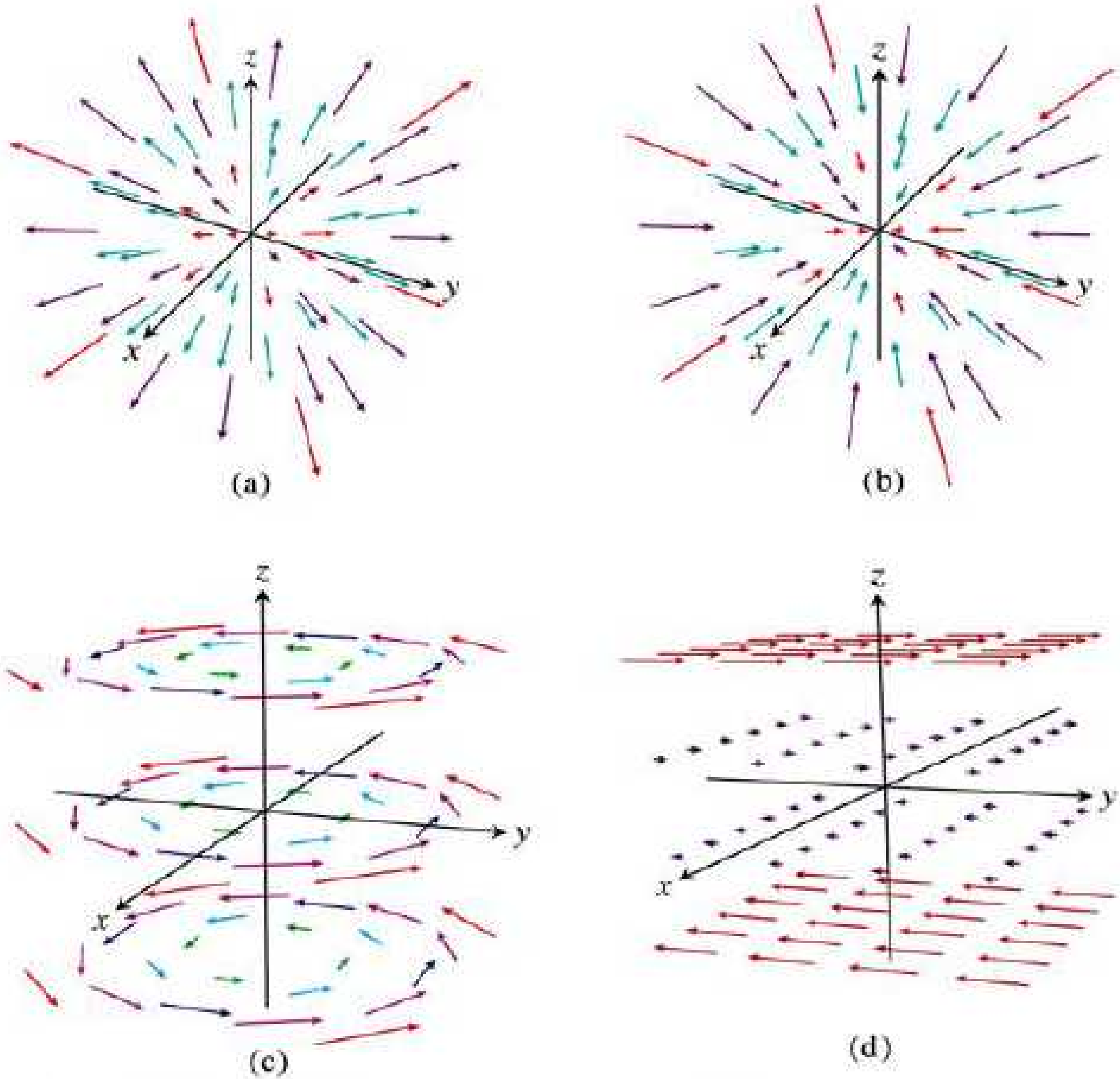


FIGURE 16.69 Velocity fields of a gas flowing in space (Example 1).

Solution

(a) $\operatorname{div}\mathbf{F} = \dfrac{\partial}{\partial x}(x) + \dfrac{\partial}{\partial y}(y) + \dfrac{\partial}{\partial z}(z) = 3$: expansion at all points.

(b) $\operatorname{div}\mathbf{F} = \dfrac{\partial}{\partial x}(-x) + \dfrac{\partial}{\partial y}(-y) + \dfrac{\partial}{\partial z}(-z) = -3$:

compression at all points.

(c) $\operatorname{div}\mathbf{F} = \dfrac{\partial}{\partial x}(-y) + \dfrac{\partial}{\partial y}(x) = 0$:

neither expanding nor compressing at any point.

(d) $\operatorname{div}\mathbf{F} = \dfrac{\partial}{\partial y}(z) = 0$:

neither expanding nor compressing at any point.

Divergence Theorem

THEOREM 8—Divergence Theorem Let $\mathbf{F}$ be a vector field whose components have continuous first partial derivatives, and let S be a piecewise smooth oriented closed surface. The flux of $\mathbf{F}$ across S in the direction of the surface's outward unit normal field $\mathbf{n}$ equals the triple integral of the divergence $\nabla \cdot \mathbf{F}$ over the region D enclosed by the surface:

$$\underbrace{\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma}_{\text{Outward flux}} = \underbrace{\iiint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right) dx\,dy\,dz}_{\text{Divergence integral}}$$

EXAMPLE 2

Evaluate both sides of Equation (2) for the expanding vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ over the sphere $x^2 + y^2 + z^2 = a^2$

Solution Let $f(x, y, z) = x^2 + y^2 + z^2 - a^2$

The outer unit normal to S, calculated from the gradient of $f(x, y, z)$

$$\mathbf{n} = \frac{2(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{\sqrt{4(x^2 + y^2 + z^2)}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}.$$

$$\mathbf{F} \cdot \mathbf{n}\, d\sigma = \frac{x^2 + y^2 + z^2}{a}\, d\sigma = \frac{a^2}{a}\, d\sigma = a\, d\sigma.$$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_S a\, d\sigma = a \iint_S d\sigma = a(4\pi a^2) = 4\pi a^3.$$

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = \iiint_D 3\, dV = 3\left(\frac{4}{3}\pi a^3\right) = 4\pi a^3.$$

COROLLARY The outward flux across a piecewise smooth oriented closed surface S is zero for any vector field $\mathbf{F}$ having zero divergence at every point of the region enclosed by the surface.

$$\underbrace{\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma}_{\text{Outward flux}} = \underbrace{}_{\text{Divergence integral}} = 0$$

EXAMPLE 3

Find the flux of $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$ outward through the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$.

Solution

$$\text{Flux} = \iint\limits_{\substack{\text{Cube}\\ \text{surface}}} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint\limits_{\substack{\text{Cube}\\ \text{interior}}} \nabla \cdot \mathbf{F}\, dV$$

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(xz) = y + z + x$$

$$\text{Flux} = \int_0^1 \int_0^1 \int_0^1 (x + y + z)\, dx\, dy\, dz = \frac{3}{2}.$$

EXAMPLE 4 求向量场 $F = xy\mathbf{i} + y^3\mathbf{j} - yz\mathbf{k}$ 向外通过曲面 S 的通量，其中 S 是由 $z = x^2 + y^2$，圆柱面 $x^2 + y^2 = 1$ 及 $z = 0$ 围成立体的表面．

Solution

$$\oiint_S F \cdot n\,d\sigma = \iiint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right) dxdydz$$

$$= \iiint_D (3y^2)\,dxdydz = \int_0^{2\pi}\int_0^1\int_0^{r^2} 3r^2 \sin^2\theta\, r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi} \sin^2\theta \int_0^1 3r^5\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi} \sin^2\theta\,d\theta$$

$$= \frac{1}{4}\int_0^{2\pi} (1 - \cos 2\theta)\,d\theta = \frac{\pi}{2}.$$

THEOREM 9 If $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a vector field with continuous second partial derivatives, then $\operatorname{div}(\operatorname{curl}\mathbf{F}) = \nabla \cdot (\nabla \times \mathbf{F}) = 0.$

Proof $\operatorname{div}(\operatorname{curl}\mathbf{F}) = \nabla \cdot (\nabla \times \mathbf{F})$

$$= \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$= \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} + \frac{\partial^2 M}{\partial y \partial z} - \frac{\partial^2 P}{\partial y \partial x} + \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y}$$

$$= 0,$$

$\mathbf{G} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, **G**是旋度场?

Unifying the Integral Theorems

a two-dimensional field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$

$$= M(x,y)\mathbf{i} + N(x,y)\mathbf{j} + 0\mathbf{k}$$

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \iint_R \nabla\cdot\mathbf{F}\,dA.$$

$$\iint_S F\cdot n\,d\sigma = \iiint_D \nabla\cdot F\,dV$$

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R \nabla\times\mathbf{F}\cdot\mathbf{n}\,dA.$$

$$\oint_C F\cdot T\,ds = \iint_S \nabla\times F\cdot n\,d\sigma$$

Green's Theorem and Its Generalization to Three Dimensions

Tangential form of Green's Theorem: $$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{n}\, dA$$

Stokes' Theorem: $$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

Normal form of Green's Theorem: $$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \nabla \cdot \mathbf{F}\, dA$$

Divergence Theorem: $$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV$$

Thanks!