Homework 7

Put your name and student ID here

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Q1(多选): Let $X_1, \dots, X_n$ be an iid sample of Poisson distribution with parameter λ. Which of the following are unbiased estimates of λ? ( )

A. $\bar{X}$

B. S_n^{*2}

C. $(\bar{X} + S_n^{*2})/2$

D. S_n^2

Q2: Let $X_1, \dots, X_n$ be an iid sample of $N(\mu, \sigma^2)$, where μ, σ are unknown parameters. Let $T_k = k \sum_{i=1}^{n} (X_i - \bar{X})^2$ be an estimator of σ^2. Particularly, when $k = 1/n$, $T_k = S_n^2$, and when $k = 1/(n-1)$, $T_k = S_n^{*2}$. Find a value of k such that T_k is the most efficient one by taking account of MSE.

Q3: Let $X_1, \dots, X_n$ be a simple random sample of the population X with $\mu_k = E[(X - E[X])^k]$. Prove that

$$\mathrm{Var}[S_n^{*2}] = \frac{\mu_4}{n} - \frac{(n-3)\mu_2^2}{n(n-1)}.$$

Hint: Replace X_i with $Y_i = X_i - E[X_i]$ in S_n^{*2} and then do the calculation using moments of Y_i.

Q4: Let $X_1, \dots, X_n$ be a simple random sample taken from the density

$$f(x; \theta) = \frac{2x}{\theta^2}, \quad 0 \le x \le \theta.$$

1. Find an expression for $\hat{\theta}_L$, the maximum likelihood estimator (MLE) for θ.
2. Find an expression for $\hat{\theta}_M$, the method of moments estimator for θ.
3. For the two estimators $\hat{\theta}_L$ and $\hat{\theta}_M$, which one is more efficient in terms of MSE?
4. Inspecting the consistency of the estimators $\hat{\theta}_L$ and $\hat{\theta}_M$.

Q5: Suppose that $X_1, \dots, X_n$ is an iid sample from the exponential density, $f(x) = \lambda e^{-\lambda x}, x > 0$, and $\lambda > 0$.

(a) Show that $\hat{\lambda}_n = X_1$ is not consistent for λ.

(b) Show that $\hat{\lambda}_n = X_1 + \cdots + X_n$ is not consistent for λ.

Q6: Suppose that the population X has a density $f(x; \theta)$, where $\theta \in (a, b)$. The Fisher information is defined by $I(\theta) = E[(\frac{d}{d\theta} \ln f(X; \theta))^2]$. If $\int_R \frac{d^2}{d\theta^2} f(x; \theta) dx = \frac{d^2}{d\theta^2} \int_R f(x; \theta) dx$ holds, prove that

$$I(\theta) = -E\left[\frac{d^2}{d\theta^2} \ln f(X; \theta)\right].$$

Q7: Suppose that the population X follows a Poisson distribution with parameter $\lambda > 0$, and $X_1, \dots, X_n$ is iid sample of X.

1. Use the two formulas in Q6 to compute the Fisher information $I(\lambda)$ (where $f(x; \theta)$ refers to the probability mass function (PMF) of the Poisson distribution X for this case) and see whether they are the same.
2. Prove that the sample mean $\bar{X}$ as an unbiased estimator of λ attains the Cramer-Rao lower bound. Based on this result, could you find the best unbiased estimator in Q1?