Large-sample properties

Zhijian He

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Consistency

Let $T_n = T(X_1, \ldots, X_n)$ be an estimator of $g(\theta)$.

- T_n is said to be **(weakly) consistent** iff for any $\epsilon > 0$,

$$\lim_{n\to\infty} P(|T_n - g(\theta)| \geq \epsilon) = 0.$$

  Or, $T_n \xrightarrow{p} g(\theta)$.

- T_n is said to be **strongly consistent** iff

$$P\left(\lim_{n\to\infty} T_n = g(\theta)\right) = 1.$$

  Or, $T_n \xrightarrow{w.p.1} g(\theta)$.

Example

Let $X_1, \ldots, X_n$ be iid sample of $U[0, \theta]$, where $\theta > 0$. Is the MLE of θ, $\hat{\theta}_n = X_{(n)}$, consistent?

Note that for any $0 < \epsilon < \theta$,

$$P\left(|\hat{\theta}_n - \theta| < \epsilon\right) = P(\theta - \epsilon < \hat{\theta}_n < \theta) = 1 - \left(\frac{\theta - \epsilon}{\theta}\right)^n \to 1.$$

So $\hat{\theta}_n$ is consistent for θ.

Figure 5.7.1 illustrates the convergence of $\hat{\theta}_n$. As n increases, the shape of $f_{\hat{\theta}_n}(y)$ changes in such a way that the pdf becomes increasingly concentrated in an ε-neighborhood of θ. For any $n > n(\varepsilon, \delta)$, $P(|\hat{\theta}_n - \theta| < \varepsilon) > 1 - \delta$.

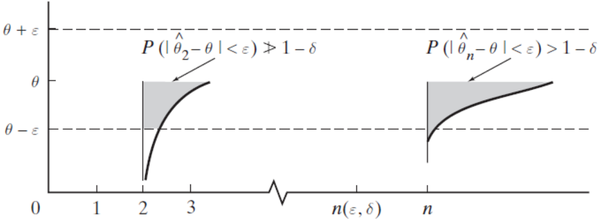

Figure 5.7.1

Continuous Mapping Theorem

Theorem: Assume that $T_n^{(k)} \xrightarrow{p} \mu_k$, as $n \to \infty$ for $k = 1, \ldots, m$. If the function $\phi(t_1, \ldots, t_m)$ is continuous at the point $(\mu_1, \ldots, \mu_m)$, then

$$\phi(T_n^{(1)}, \ldots, T_n^{(m)}) \xrightarrow{p} \phi(\mu_1, \ldots, \mu_m).$$

Consistency of MOM estimators: Take $T_n^{(k)} = M_k = \frac{1}{n}\sum_{i=1}^n X_i^k$ and $\mu_k = \mathbb{E}[X^k]$. By LLN, $T_n^{(k)} \xrightarrow{p} \mu_k$. So if

$$\theta = \phi(\mu_1, \ldots, \mu_m)$$

is continuous, then the MOM estimator

$$\hat{\theta} = \phi(M_1, \ldots, M_m)$$

is consistent.

Examples

Let $X_1, \ldots, X_n$ be iid sample of the population X with mean μ and variance σ^2. Then

1. $\bar{X}$ is consistent for μ;
2. S_n^2 and S_n^{*2} are consistent for σ^2.

Sufficient conditions

Theorem: Let $T_n = T(X_1, \ldots, X_n)$ be an estimator of $g(\theta)$. If

$$\lim_{n\to\infty} \mathbb{E}[T_n] = g(\theta),$$

$$\lim_{n\to\infty} \mathrm{Var}(T_n) = 0,$$

then T_n is consistent for $g(\theta)$.

Remark: The two conditions are equivalent to

$$\lim_{n\to\infty} \mathrm{MSE}(T_n) = 0.$$

Application

Let $X_1, \ldots, X_n$ be iid sample of $U[0, \theta]$, where $\theta > 0$. Is the MLE of θ, $\hat{\theta}_n = X_{(n)}$, consistent?

The PDF of $\hat{\theta}_n$ is given by

$$f(y) = \frac{ny^{n-1}}{\theta^n} 1\{0 \leq y \leq \theta\}.$$

$$\mathrm{E}[\hat{\theta}_n] = \int_0^\theta \frac{ny^n}{\theta^n} dy = \frac{n}{n+1}\theta \to \theta$$

$$\mathrm{E}[\hat{\theta}_n^2] = \int_0^\theta \frac{ny^{n+1}}{\theta^n} dy = \frac{n}{n+2}\theta^2$$

$$\mathrm{Var}[\hat{\theta}_n] = \frac{n}{n+2}\theta^2 - \frac{n^2}{(n+1)^2}\theta^2 = \frac{n\theta^2}{(n+1)^2(n+2)} \to 0$$

Asymptotic Normality

Definition: Let T_n be an estimator of $g(\theta)$. If there exists $\sigma_n(\theta) > 0$ such that

$$\frac{T_n - g(\theta)}{\sigma_n(\theta)} \xrightarrow{d} N(0, 1),$$

then T_n is an asymptotically normal estimator for $g(\theta)$. The $\sigma_n(\theta)^2$ is called the asymptotic variance.

Example: By CLT, $\bar{X}$ is an asymptotically normal estimator for $\mu = \mathrm{E}[X]$.

Large-sample properties for MLE

Theorem: Let $X_1, \ldots, X_n$ be iid sample of a population with PDF $f(x;\theta)$, where $\theta \in \Theta \subset \mathbb{R}$. Under appropriate conditions on $f(x;\theta)$, we have

$$\hat{\theta}_{\mathrm{MLE}} \xrightarrow{p} \theta,$$

$$\sqrt{nI(\theta)}(\hat{\theta}_{\mathrm{MLE}} - \theta) \xrightarrow{d} N(0,1).$$

Remark: MLE's asymptotic variance attains the CR inequality lower bound, i.e.,

$$\mathrm{Var}(\hat{\theta}_{\mathrm{MLE}}) \approx \frac{1}{nI(\theta)}.$$