

# Homework 7

Put your name and student ID here

2021-04-12

**Q1(多选):** Let  $X_1, \dots, X_n$  be an iid sample of Poisson distribution with parameter  $\lambda$ . Which of the following are unbiased estimates of  $\lambda$ ? ( )

- A.  $\bar{X}$
- B.  $S_n^{*2}$
- C.  $(\bar{X} + S_n^{*2})/2$
- D.  $S_n^2$

**Q2:** Let  $X_1, \dots, X_n$  be an iid sample of  $N(\mu, \sigma^2)$ , where  $\mu, \sigma$  are unknown parameters. Let  $T_k = k \sum_{i=1}^n (X_i - \bar{X})^2$  be an estimator of  $\sigma^2$ . Particularly, when  $k = 1/n$ ,  $T_k = S_n^2$ , and when  $k = 1/(n-1)$ ,  $T_k = S_n^{*2}$ . Find a value of  $k$  such that  $T_k$  is the most efficient one by taking account of MSE.

**Q3:** Let  $X_1, \dots, X_n$  be a simple random sample of the population  $X$  with  $\mu_k = E[(X - E[X])^k]$ . Prove that

$$\text{Var}[S_n^{*2}] = \frac{\mu_4}{n} - \frac{(n-3)\mu_2^2}{n(n-1)}.$$

Hint: Replace  $X_i$  with  $Y_i = X_i - E[X]$  in  $S_n^{*2}$  and then do the calculation using moments of  $Y_i$ .

**Q4:** Let  $X_1, \dots, X_n$  be a simple random sample taken from the density

$$f(x; \theta) = \frac{2x}{\theta^2}, \quad 0 \leq x \leq \theta.$$

1. Find an expression for  $\hat{\theta}_L$ , the maximum likelihood estimator (MLE) for  $\theta$ .
2. Find an expression for  $\hat{\theta}_M$ , the method of moments estimator for  $\theta$ .
3. For the two estimators  $\hat{\theta}_L$  and  $\hat{\theta}_M$ , which one is more efficient in terms of MSE?
4. Inspecting the consistency of the estimators  $\hat{\theta}_L$  and  $\hat{\theta}_M$ .

**Q4:** Suppose that  $X_1, \dots, X_n$  is an iid sample from the exponential density,  $f(x) = \lambda e^{-\lambda x}$ ,  $x > 0$ , and  $\lambda > 0$ .

- (a) Show that  $\hat{\lambda}_n = X_1$  is not consistent for  $\lambda$ .
- (b) Show that  $\hat{\lambda}_n = X_1 + \dots + X_n$  is not consistent for  $\lambda$ .

**Q5:** Suppose that the population  $X$  has a density  $f(x; \theta)$ , where  $\theta \in (a, b)$ . The Fisher information is defined by  $I(\theta) = E[(\frac{d}{d\theta} \ln f(X; \theta))^2]$ . If  $\int_R \frac{d^2}{d\theta^2} f(x; \theta) dx = \frac{d^2}{d\theta^2} \int_R f(x; \theta) dx$  holds, prove that

$$I(\theta) = -E \left[ \frac{d^2}{d\theta^2} \ln f(X; \theta) \right].$$

**Q6:** Suppose that the population  $X$  follows a Poisson distribution with parameter  $\lambda > 0$ , and  $X_1, \dots, X_n$  is iid sample of  $X$ .

1. Use the two formulas in Q5 to compute the Fisher information  $I(\lambda)$  (where  $f(x; \theta)$  refers to the probability mass function (PMF) of the Poisson distribution  $X$  for this case) and see whether they are the same.
2. Prove that the sample mean  $\bar{X}$  as an unbiased estimator of  $\lambda$  attains the Cramer-Rao lower bound. Based on this result, could you find the best unbiased estimator in Q1?