Homework 2

Put your name and student ID here

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Q1: Assume that $X_1 \sim Gamma(\alpha_1, \lambda)$, $X_2 \sim Gamma(\alpha_2, \lambda)$, and they are independent. Prove that:

1. $Y_1 = X_1 + X_2 \sim Gamma(\alpha_1 + \alpha_2, \lambda)$,
2. $Y_2 = X_1/(X_1 + X_2) \sim Beta(\alpha_1, \alpha_2)$,
3. Y_1 and Y_2 are independent.

Q2: Let $X_1, \ldots, X_n$ be iid random variables whose CDF $F(x)$ is continuous and strictly increasing. Show that

$$T = -2\sum_{i=1}^{n} \ln F(X_i) \sim Gamma(n, 1/2).$$

Q3: Let $Z_1, \ldots, Z_n \overset{iid}{\sim} N(0,1)$. Show that $Z_1^2 \sim Gamma(1/2, 1/2)$ and $\sum_{i=1}^n Z_i^2 \sim Gamma(n/2, 1/2)$.

Q4: Prove that $Var(X) = E[Var(X|Y)] + Var(E[X|Y])$.

Q5: In the context of Bayesian statistics, the parameter θ is treated as a random number. Suppose that $\theta \sim Beta(\alpha_1, \alpha_2)$ as a prior distribution, and the conditional distribution of X given θ is $B(n, \theta)$, where n is a known positive integer, that is $X|\theta \sim B(n, \theta)$. Show that the posterior distribution of θ (given $X = x$) is

$$\theta|X = x \sim Beta(\alpha_1 + x, \alpha_2 + n - x).$$

(Hints: read pages 94-95 of the English textbook)