**ME 614 Spring 2017-Homework 1**

Spatial Discretization

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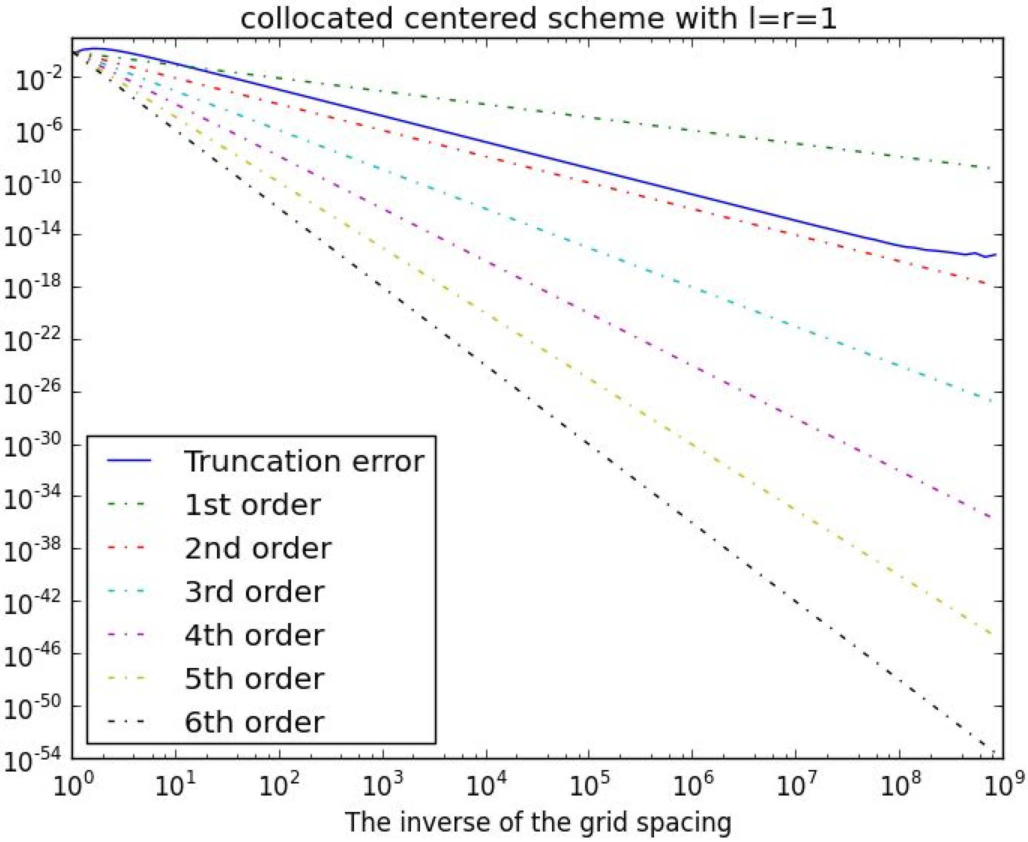
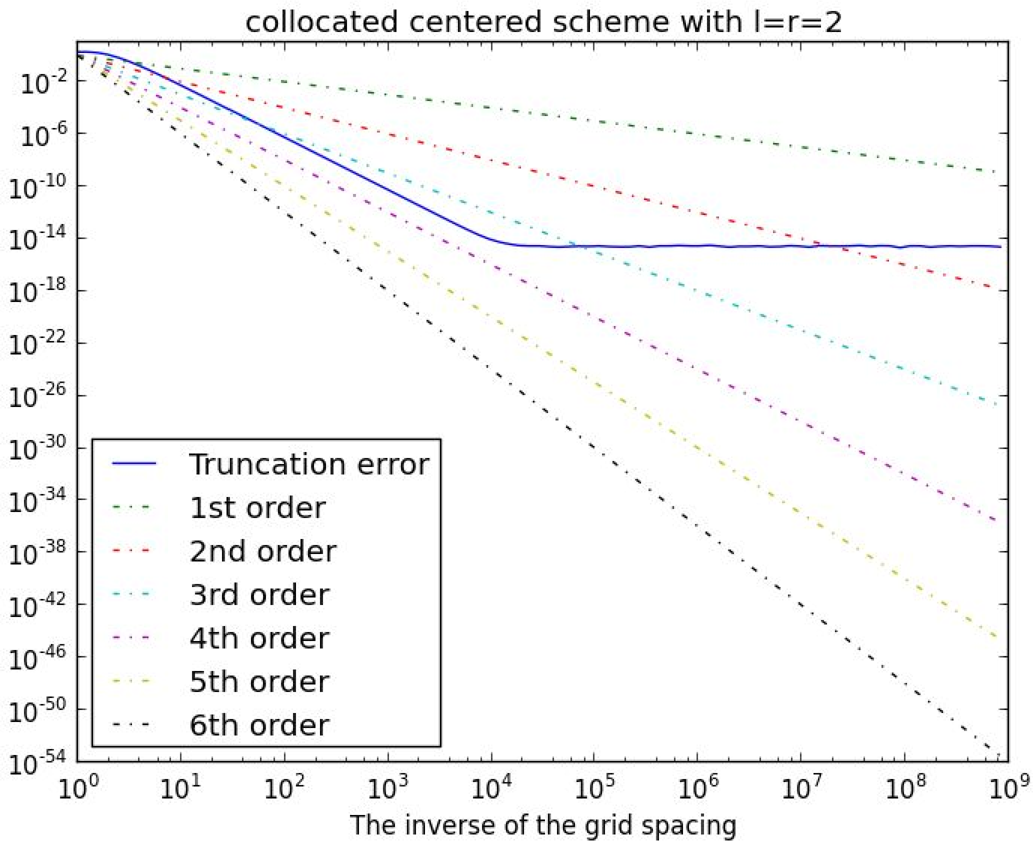
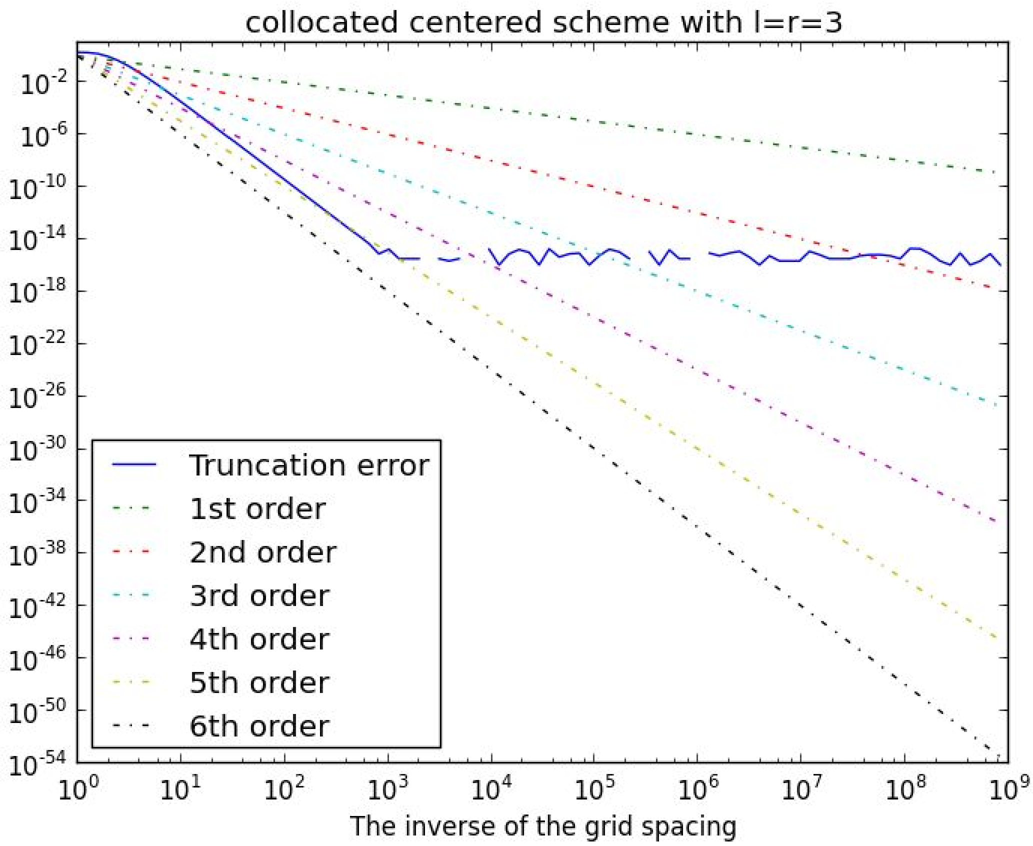
**Problem 1**

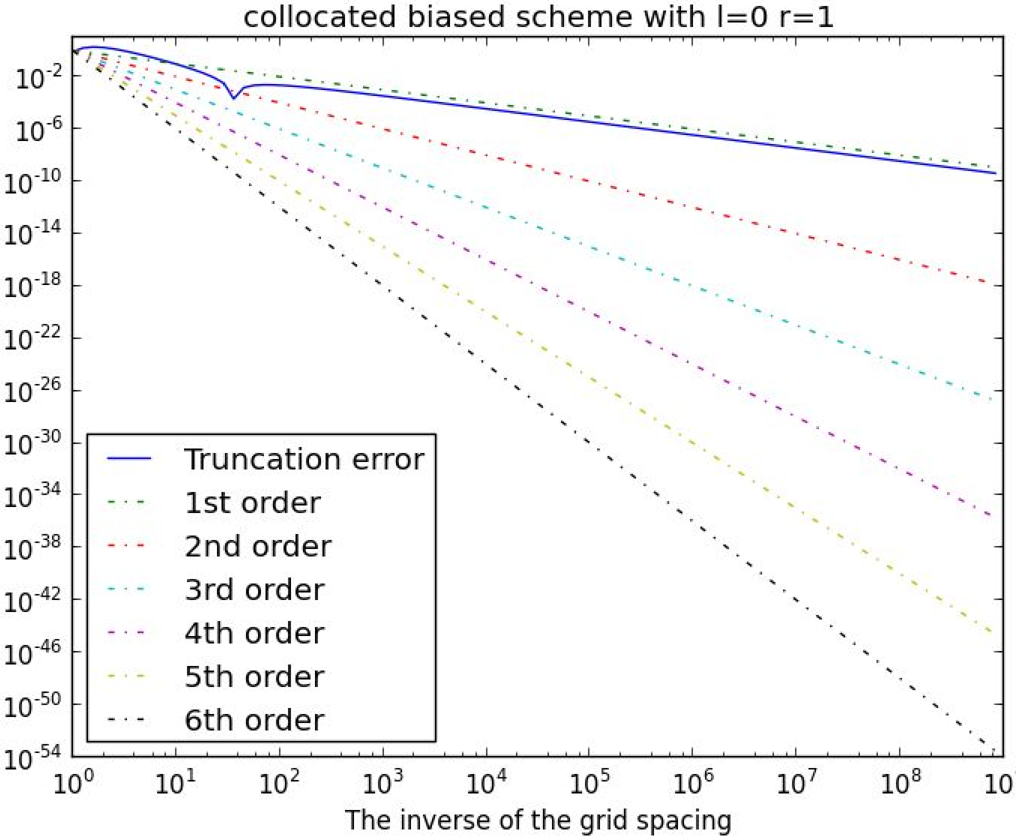
**(a)**

Plot in log-log scale the absolute value of the truncation error versus the inverse of the grid spacing. Plots for different schemes are listed below along with reference truncation error. (1st order, 2nd order, 3rd order, 4th order, 5th order, 6th order)

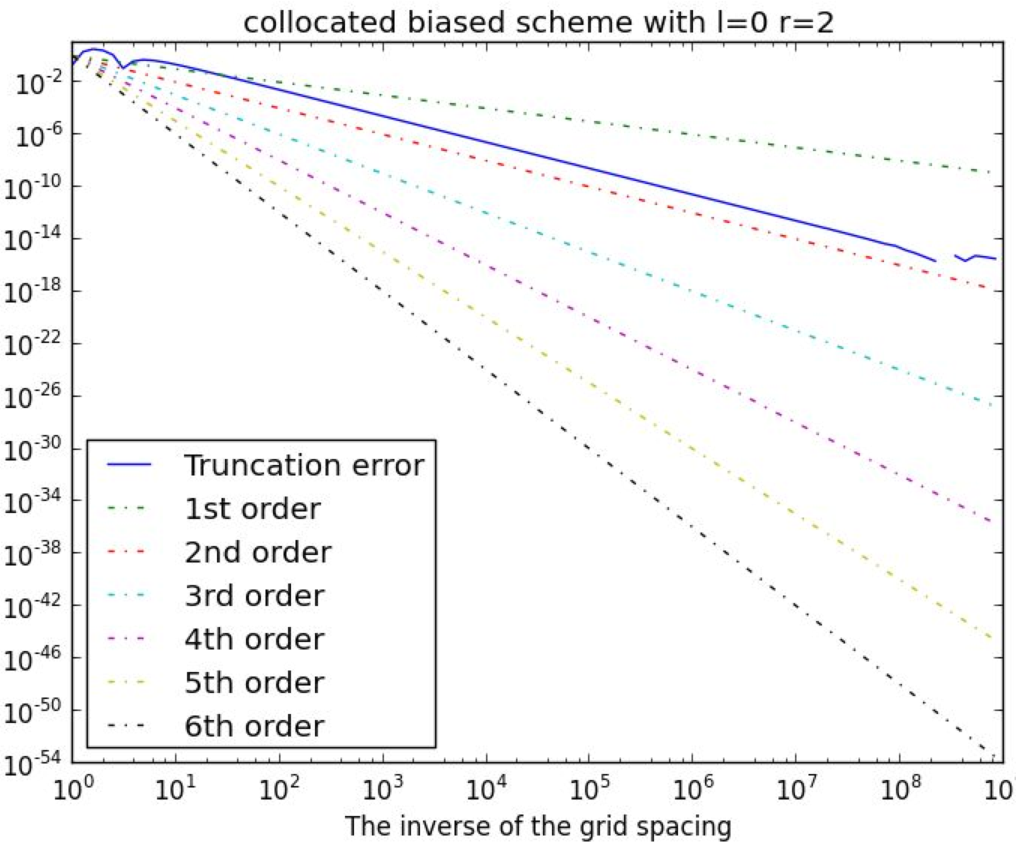



collocated centered scheme with l=r=2
Truncation error
1st order
2nd order
3rd order
4th order
5th order
6th order
10^-2
10^-6
10^-10
10^-14
10^-18
10^-22
10^-26
10^-30
10^-34
10^-38
10^-42
10^-46
10^-50
10^-54
10^0
10^1
10^2
10^3
10^4
10^5
10^6
10^7
10^8
10^9
The inverse of the grid spacing

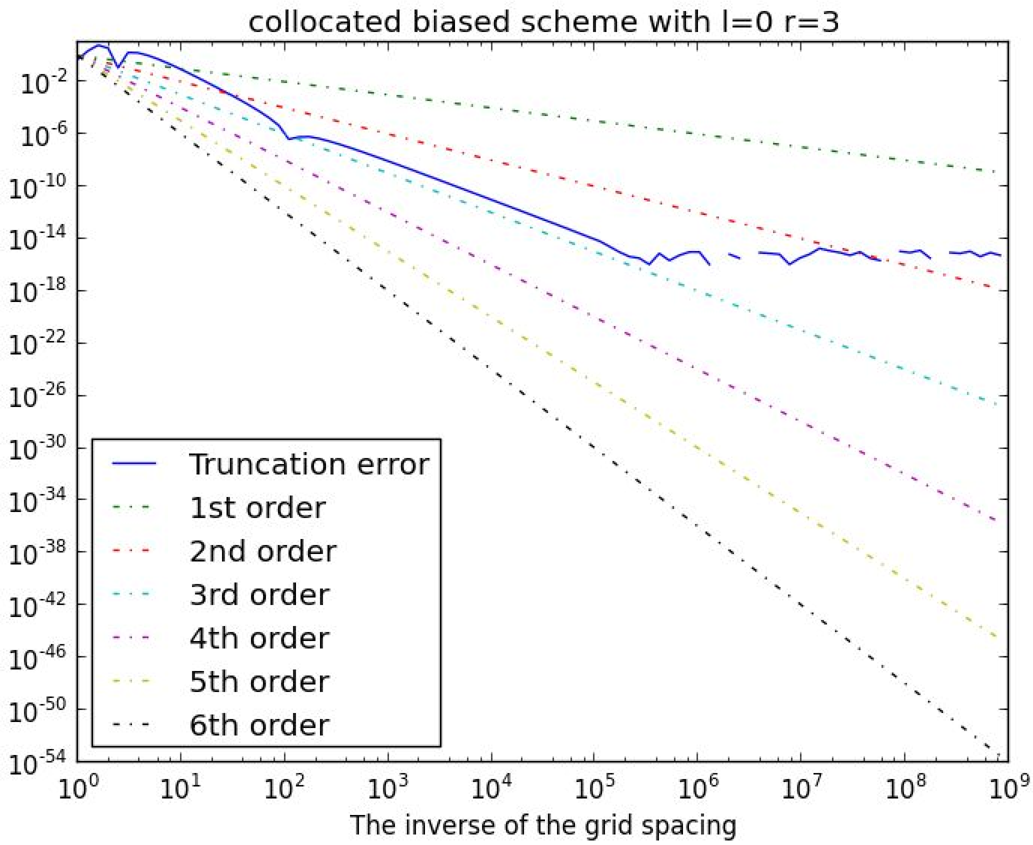

collocated centered scheme with l=r=3
Truncation error
1st order
2nd order
3rd order
4th order
5th order
6th order
10^-2
10^-6
10^-10
10^-14
10^-18
10^-22
10^-26
10^-30
10^-34
10^-38
10^-42
10^-46
10^-50
10^-54
10^0
10^1
10^2
10^3
10^4
10^5
10^6
10^7
10^8
10^9
The inverse of the grid spacing



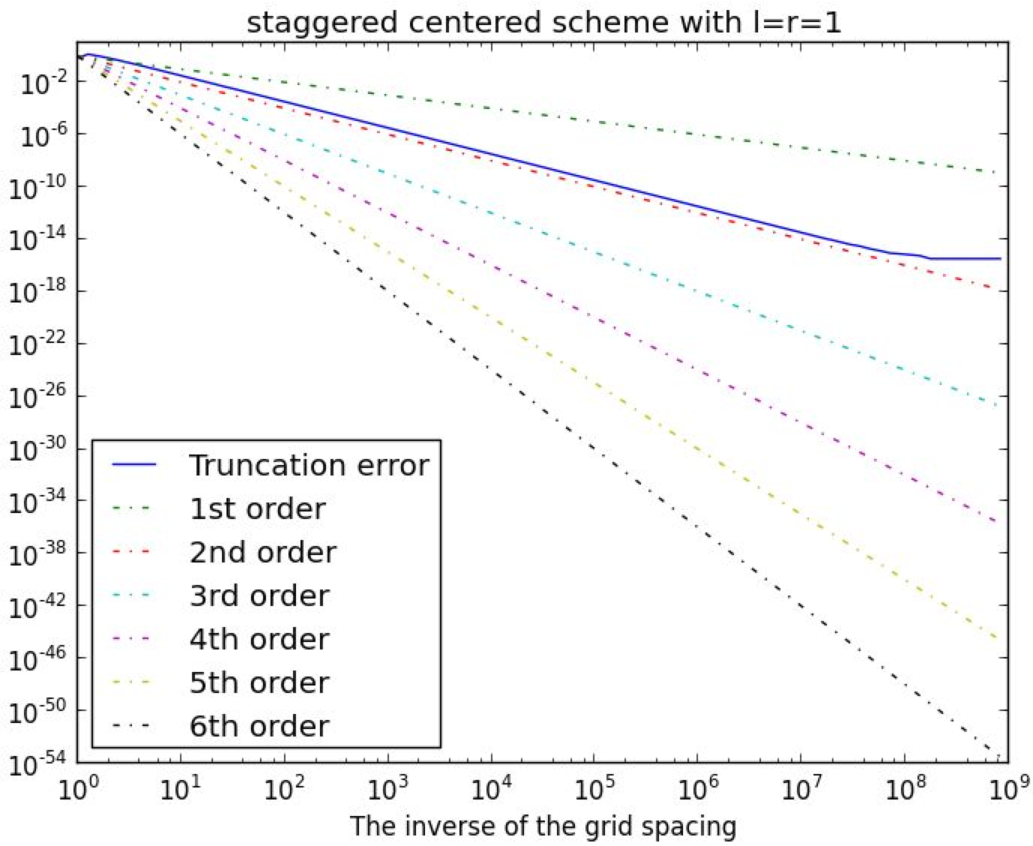
collocated biased scheme with l=0 r=1
Truncation error
1st order
2nd order
3rd order
4th order
5th order
6th order
The inverse of the grid spacing

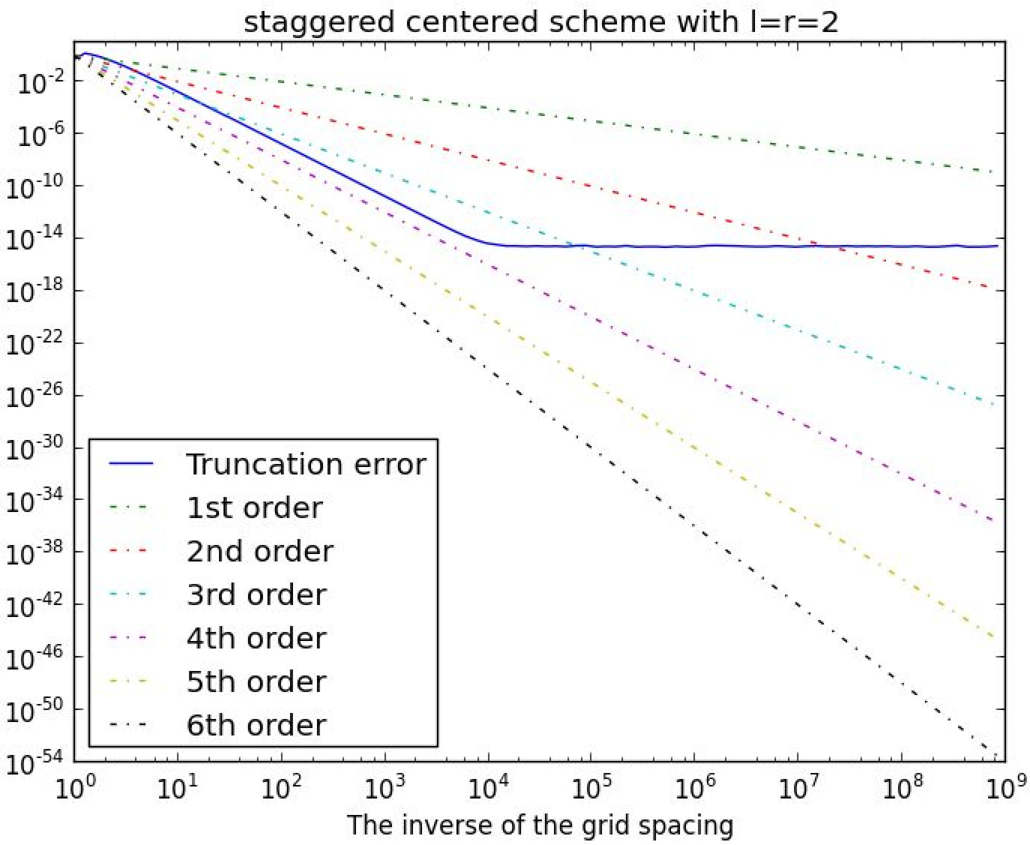
collocated biased scheme with l=0 r=2
Truncation error
1st order
2nd order
3rd order
4th order
5th order
6th order
The inverse of the grid spacing

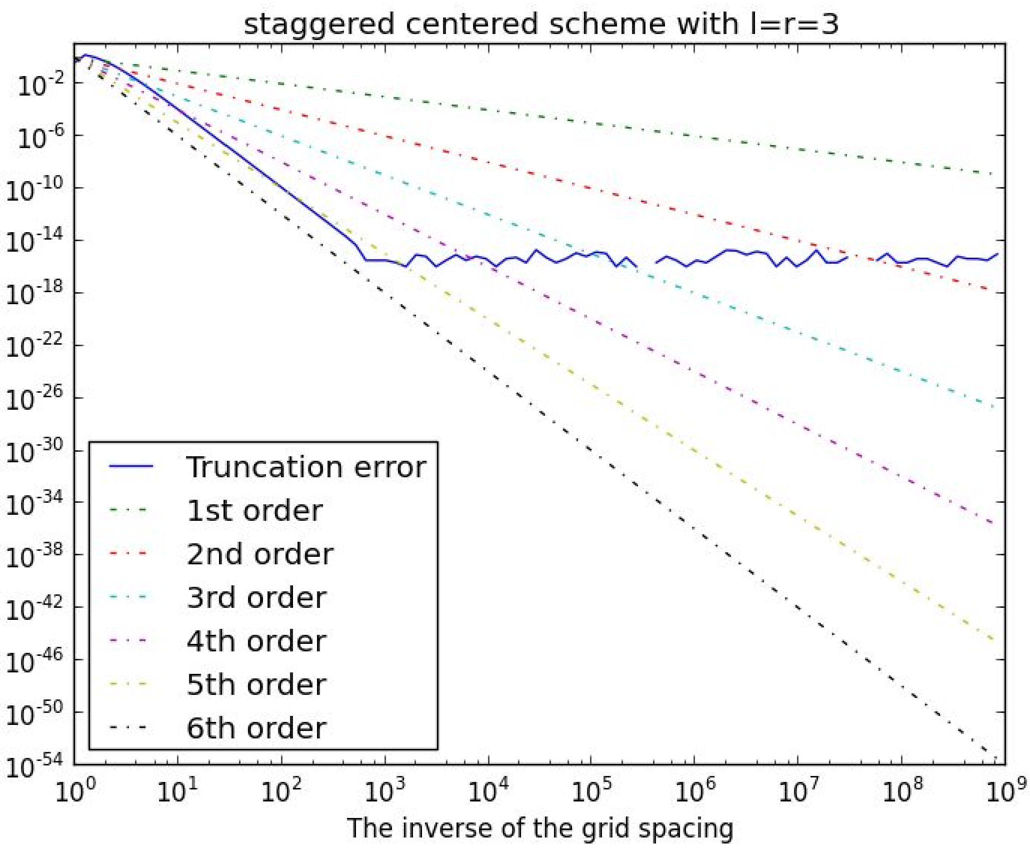


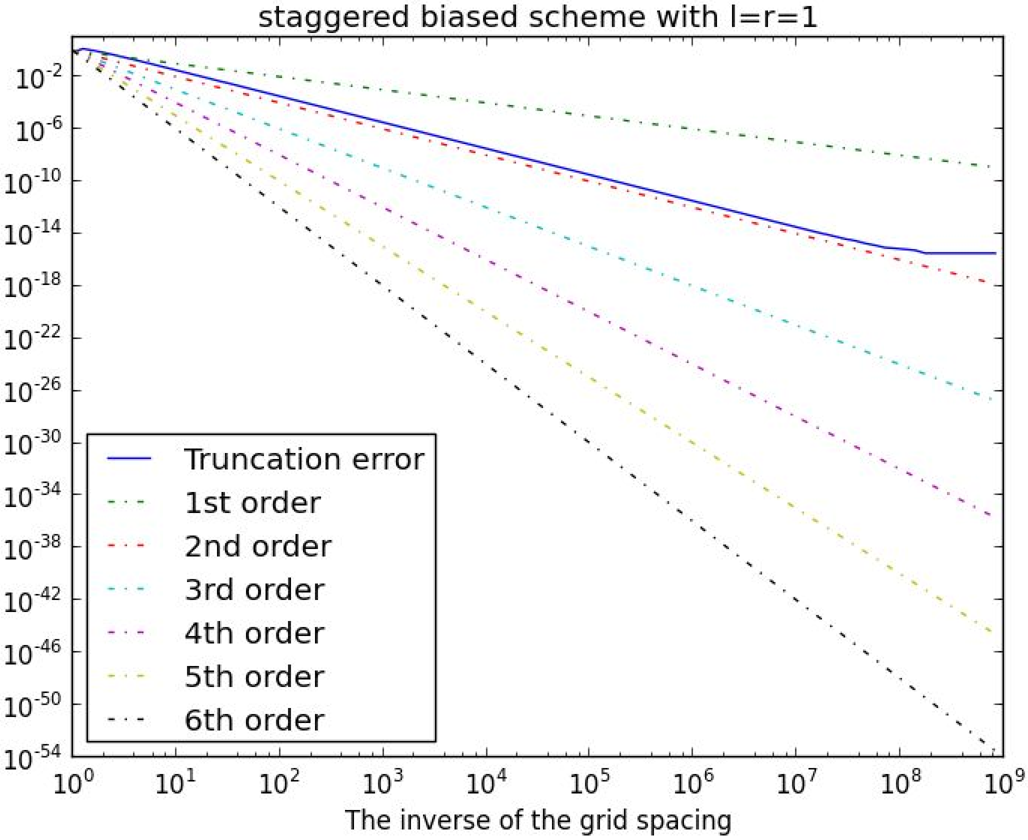
collocated biased scheme with l=0 r=3
Truncation error
1st order
2nd order
3rd order
4th order
5th order
6th order
The inverse of the grid spacing

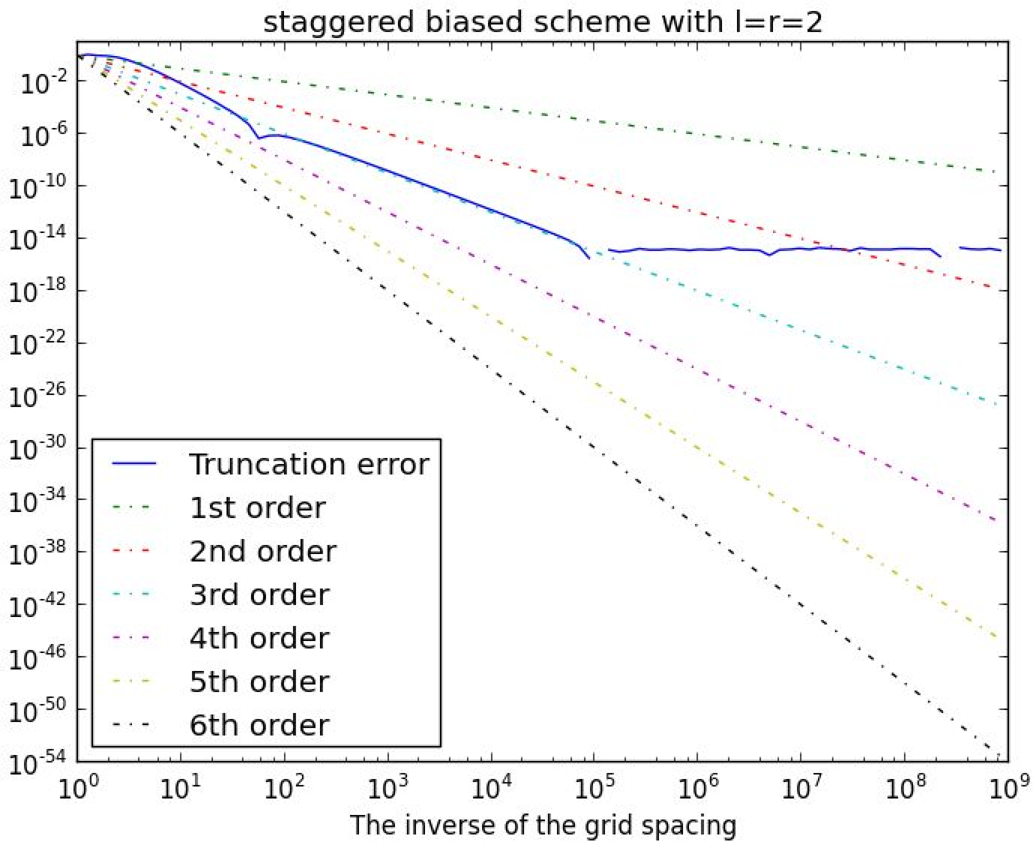
staggered centered scheme with l=r=1
Truncation error
1st order
2nd order
3rd order
4th order
5th order
6th order
The inverse of the grid spacing

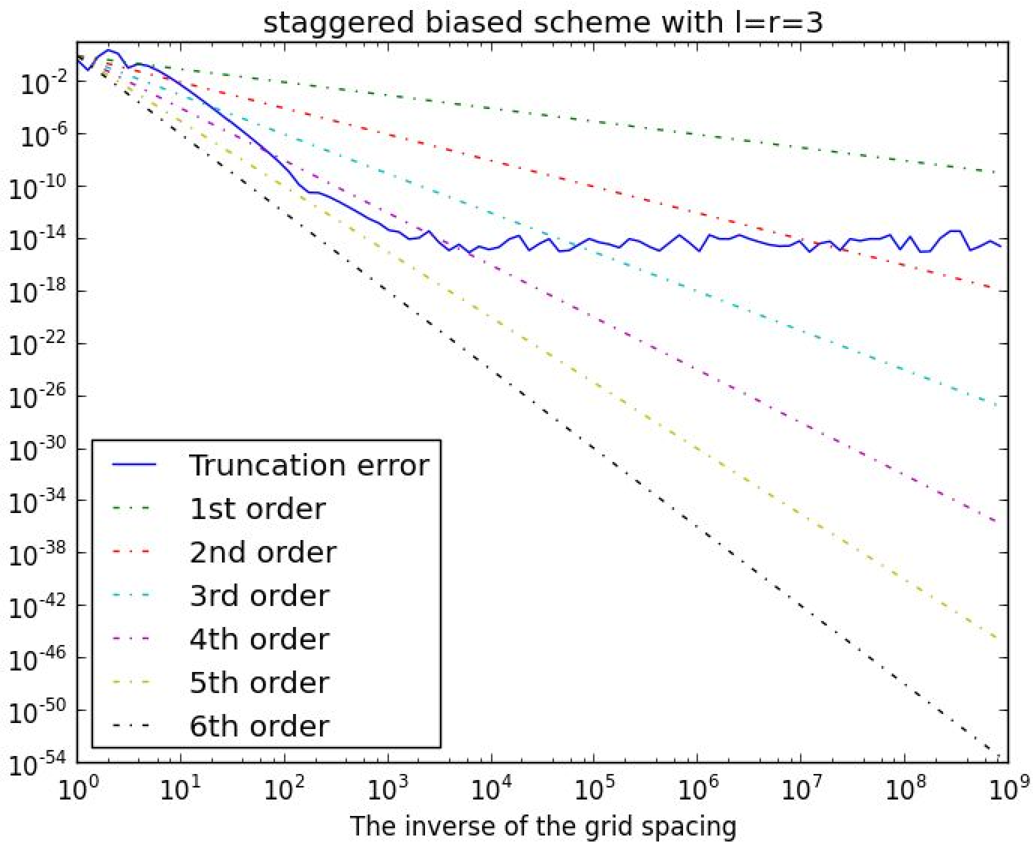

staggered centered scheme with l=r=2
Truncation error
1st order
2nd order
3rd order
4th order
5th order
6th order
The inverse of the grid spacing


staggered centered scheme with l=r=3
Truncation error
1st order
2nd order
3rd order
4th order
5th order
6th order
The inverse of the grid spacing



staggered biased scheme with l=r=1
Truncation error
1st order
2nd order
3rd order
4th order
5th order
6th order
The inverse of the grid spacing


staggered biased scheme with l=r=2
Truncation error
1st order
2nd order
3rd order
4th order
5th order
6th order
The inverse of the grid spacing


staggered biased scheme with l=r=3
Truncation error
1st order
2nd order
3rd order
4th order
5th order
6th order
The inverse of the grid spacing

**(b)**

In the same scheme(for example: collocated centered scheme), the more points to be fitted(the number of points can be expressed as N = l+r), the faster the truncation error decreases. In an area, the order of accuracy always correspond to the order of the polynomial interpolant(For example, a scheme with N=6 has 6th-order of accuracy). If the spacing is too small or too high, the order of accuracy will not correspond to the order of the polynomial interpolant anymore. This feature can be observed in followed figures.

If the order of the polynomial interpolant is high. There will be vibrations in terms of truncation error. This phenomenon is due to the limited length of value the computer can compute. There is another issue related to higher order of polynomial: when the spacing is too large, truncation error will increase as spacing getting smaller.

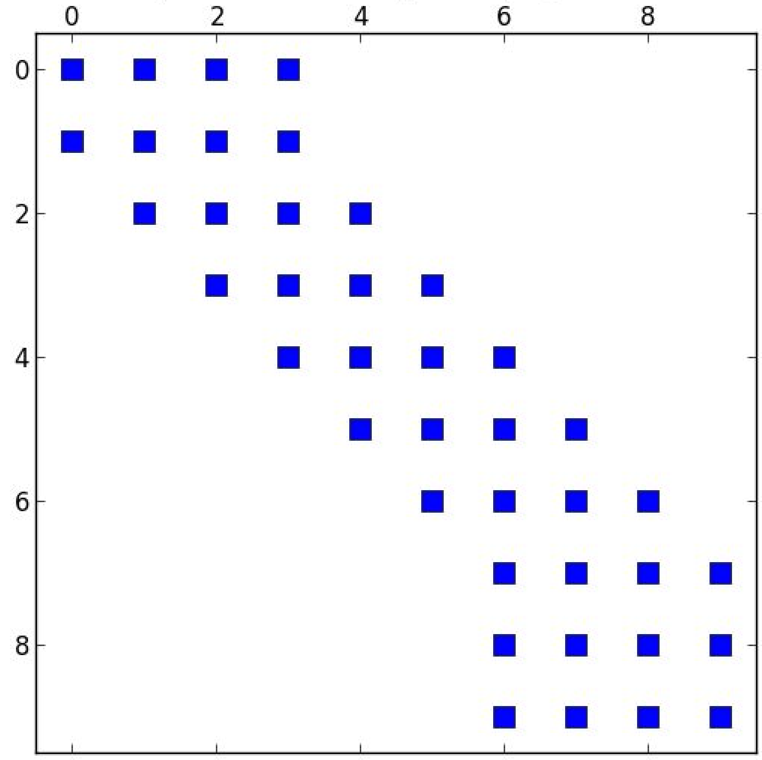
The order of accuracy always corresponding to the order of the polynomial interpolant. For a given order, p, the minimum and maximum order of accuracy we can achieve will be **1 and p** respectively.

## **<u>Problem 2</u>**

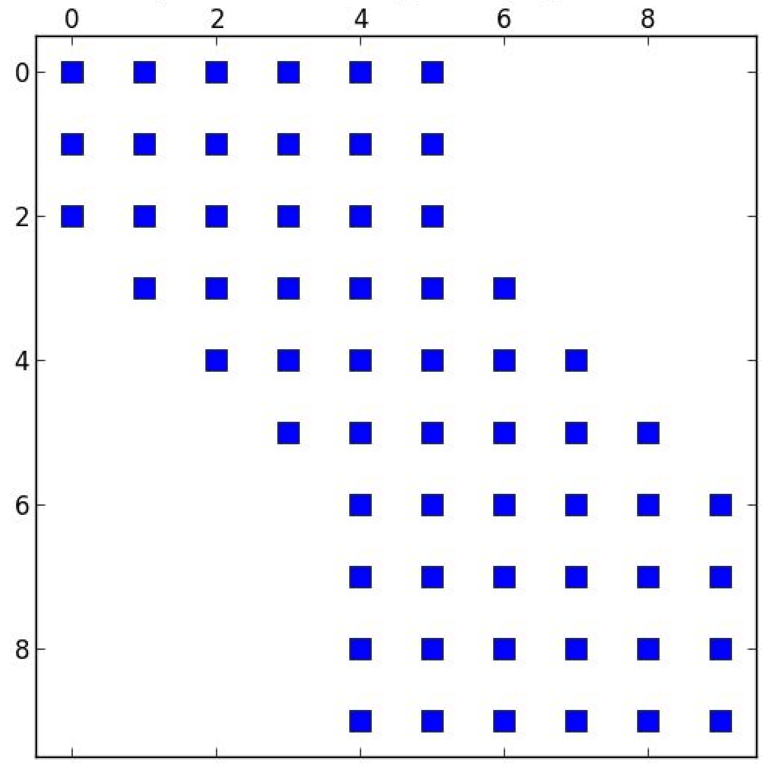
### **<u>(a)</u>**

Generate discrete numerical operators in the form of sparse matrix corresponding to the first and third derivatives with third-order and fifth-order polynomial reconstructions(N=10). The resulting matrix patterns are listed below.

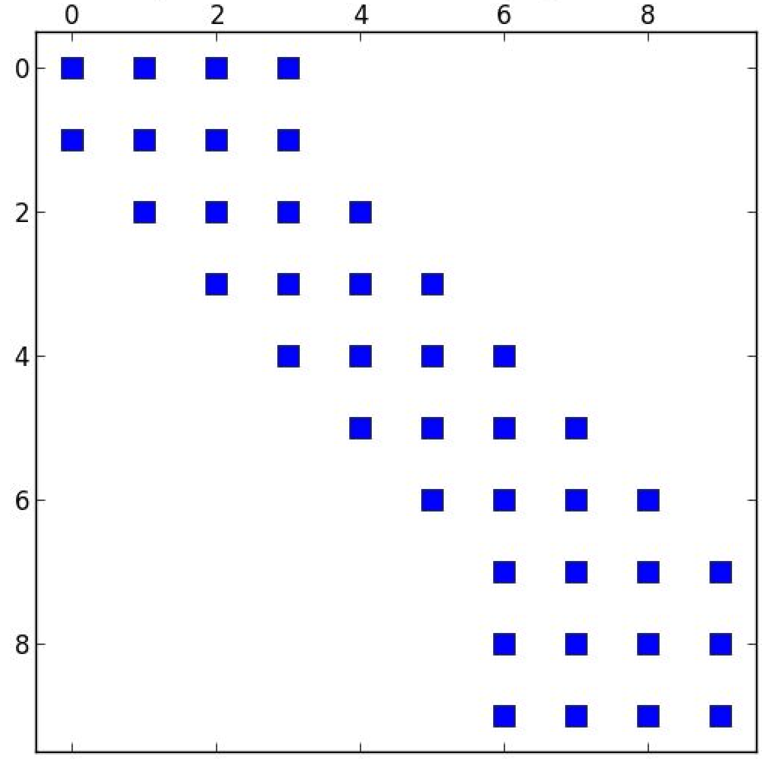
1st derivatives operator with 3rd_order polynomial reconstruction



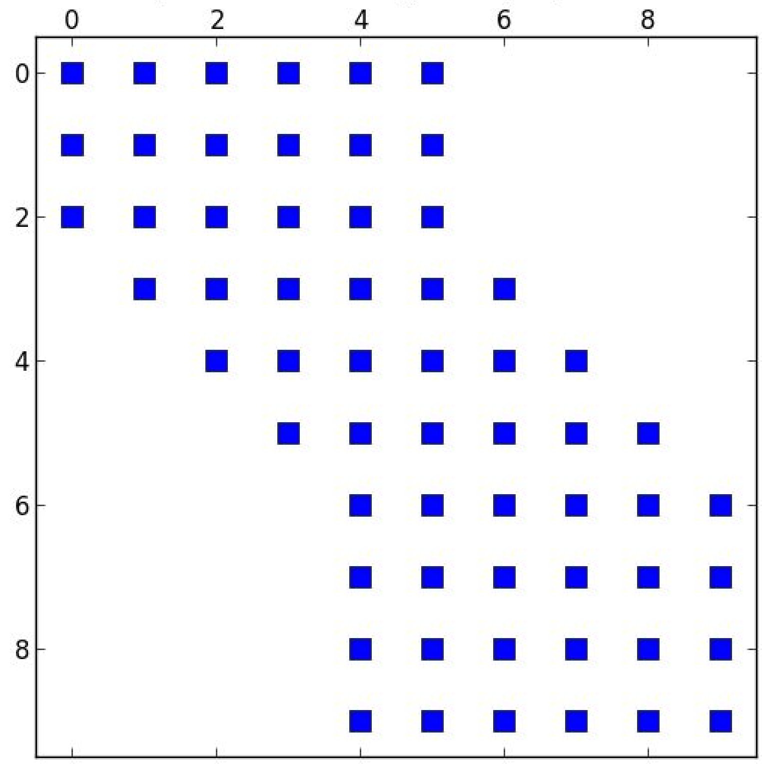

1st derivatives operator with 5th_order polynomial reconstruction



3rd derivatives operator with 3rd-order polynomial reconstruction

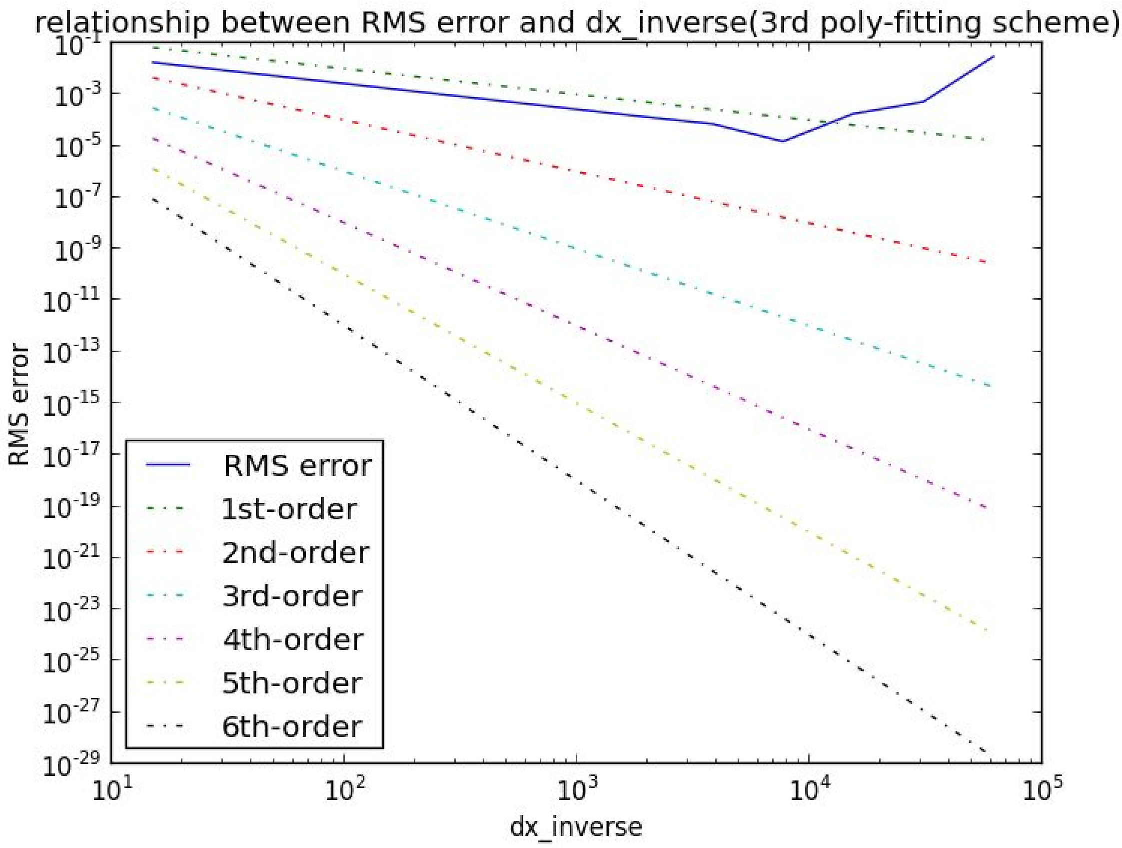


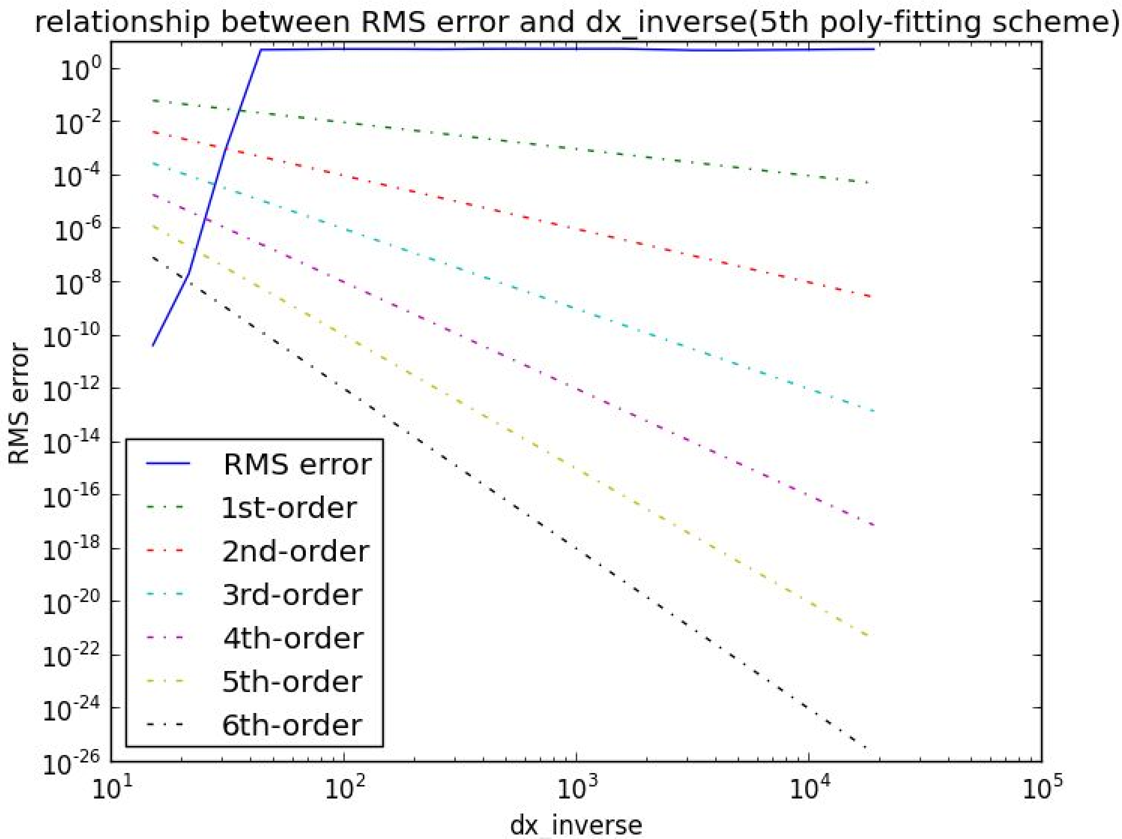

3rd derivatives operator with 5th_order polynomial reconstruction



**(b)**

**Plot the root mean-square(RMS) of the error against the inverse of the grid spacing for two discretization schemes(third-order and fifth-order polynomial scheme). The results are liste below.**





relationship between RMS error and dx_inverse(5th poly-fitting scheme)
RMS error
dx_inverse
$10^{0}$
$10^{-2}$
$10^{-4}$
$10^{-6}$
$10^{-8}$
$10^{-10}$
$10^{-12}$
$10^{-14}$
$10^{-16}$
$10^{-18}$
$10^{-20}$
$10^{-22}$
$10^{-24}$
$10^{-26}$
$10^{1}$
$10^{2}$
$10^{3}$
$10^{4}$
$10^{5}$
RMS error
1st-order
2nd-order
3rd-order
4th-order
5th-order
6th-order

**Problem 3**

**(a)**

Taylor expansion

$$
\begin{cases}
\alpha_1 \left\{ u_{i+1} = u_i + u_i' \Delta x + u_i'' \dfrac{(\Delta x)^2}{2!} + u_i''' \dfrac{(\Delta x)^3}{3!} + u_i'''' \dfrac{(\Delta x)^4}{4!} + \cdots \right\} \\
\alpha_2 \left\{ u_i = u_i \right\} \\
\alpha_3 \left\{ u_{i-1} = u_i - u_i' \Delta x + u_i'' \dfrac{(\Delta x)^2}{2!} - u_i''' \dfrac{(\Delta x)^3}{3!} + u_i'''' \dfrac{(\Delta x)^4}{4!} - u_i''''' \dfrac{(\Delta x)^5}{5!} + \cdots \right\} \\
\alpha_4 \left\{ u_{i+2} = u_i + u_i' (2\Delta x) + u_i'' \dfrac{(2\Delta x)^2}{2!} - u_i''' \dfrac{(2\Delta x)^3}{3!} + u_i'''' \dfrac{(2\Delta x)^4}{4!} + \cdots \right\} \\
\alpha_5 \left\{ u_{i-2} = u_i - u_i' (2\Delta x) + u_i'' \dfrac{(2\Delta x)^2}{2!} - u_i''' \dfrac{(2\Delta x)^3}{3!} + u_i'''' \dfrac{(2\Delta x)^4}{4!} + \cdots \right\} \\
\beta_1 \left\{ (\Delta x)^3 u_{i+1}''' = (\Delta x)^3 \cdot \left( u_i''' + u_i'''' \Delta x + u_i''''' \dfrac{(\Delta x)^2}{2!} + u_i'''''' \dfrac{(\Delta x)^3}{3!} + \cdots \right) \right\} \\
\beta_3 \left\{ \Delta x^3 u_{i-1}''' = \Delta x^3 \left( u_i''' - u_i'''' \Delta x + u_i''''' \dfrac{\Delta x^2}{2!} - u_i'''''' \dfrac{(\Delta x)^3}{3!} + \cdots \right) \right\}
\end{cases}
$$

from equation above:

$$\alpha_1 u_{i+1} + \alpha_2 u_i + \alpha_3 u_{i-1} + \alpha_4 u_{i+2} + \alpha_5 u_{i-2} + \beta_1 (\Delta x)^3 u_{i+1}''' + \beta_3 (\Delta x)^3 u_{i-1}'''$$

$$=$$

$$A u_i + B u_i' \Delta x + C u_i'' \Delta x^2 + D u_i''' \Delta x^3 + E u_i'''' \Delta x^4 + F u_i''''' \Delta x^5 + G u_i'''''' \Delta x^6$$

A, B, C, D, E, F are linear combination of $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \beta_1, \beta_3\}$

Because we want to obtain 3rd-derivative

So $\begin{cases} A=0 & E=0 \\ B=0 & F=0 \\ C=0 & G=0 \\ D=1 & \end{cases}$, Solve this linear system $\Longrightarrow \begin{cases} \alpha_1 = -2 & \beta_1 = -\frac{1}{2} \\ \alpha_2 = 0 & \beta_2 = -\frac{1}{2} \\ \alpha_3 = 2 & \\ \alpha_4 = 1 & \\ \alpha_5 = -1 & \end{cases}$

So, the scheme will be

$$u_i''' + \frac{1}{2}u_{i-1}''' + \frac{1}{2}u_{i+1}''' = \frac{-u_{i-2} + 2u_{i-1} - 2u_{i+1} + u_{i+2}}{\Delta x^3}$$