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1 Preliminaries

1.1 Standard Form of LP

Any unbounded variable x can be replaced by a pair of non-negative variables (u, v).

$$\begin{cases} x = u - v \\ u \geq 0 \\ v \geq 0 \end{cases} \tag{1}$$

Any in-equality constraint can be converted into an equality constraint, by introducing an additional assistant non-negative variable x'.

$$\begin{aligned} \mathbf{a}_i^T \mathbf{x} \geq b_i \quad & \Leftrightarrow \quad \begin{cases} \mathbf{a}_i^T \mathbf{x} - x' & = b_i \\ x' & \geq 0 \end{cases} \\ & \Leftrightarrow \quad \begin{cases} \begin{bmatrix} \mathbf{a}_i^T & (-1) \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ x' \end{bmatrix} & = b_i \\ x' & \geq 0 \end{cases} \end{aligned} \tag{2}$$

So we can safely represent any Linear Programming (LP) problem in its standard form, with only non-negative variables and only equality constraints.

1.1.1 Example

The LP with equalities and in-equalities

$$\begin{aligned} \text{Minimize} \quad & \mathbf{c}^T \color{red}{\mathbf{x}} \\ \text{s.t.} \quad & \begin{cases} \mathbf{A}^{(e)} \color{red}{\mathbf{x}} & = \mathbf{b}^{(e)} & \color{gray}{\in \mathbb{R}^{m^{(e)} \times 1}} \\ \mathbf{A}^{(i)} \color{red}{\mathbf{x}} & \geq \mathbf{b}^{(i)} & \color{gray}{\in \mathbb{R}^{m^{(i)} \times 1}} \\ \color{red}{\mathbf{x}} & \geq \mathbf{0} & \color{gray}{\in \mathbb{R}^{n \times 1}} \end{cases} \end{aligned} \tag{3}$$

can be transformed to a standard form without in-equalities

$$\begin{aligned} \text{Minimize} \quad & \mathbf{c}^T \color{red}{\mathbf{x}} \\ \text{s.t.} \quad & \begin{cases} \mathbf{A}^{(e)} \color{red}{\mathbf{x}} & = \mathbf{b}^{(e)} & \color{gray}{\in \mathbb{R}^{m^{(e)} \times 1}} \\ \mathbf{A}^{(i)} \color{red}{\mathbf{x}} - \mathbf{y} & = \mathbf{b}^{(i)} & \color{gray}{\in \mathbb{R}^{m^{(i)} \times 1}} \\ \color{red}{\mathbf{x}} & \geq \mathbf{0} & \color{gray}{\in \mathbb{R}^{n \times 1}} \\ \color{red}{\mathbf{y}} & \geq \mathbf{0} & \color{gray}{\in \mathbb{R}^{m^{(i)} \times 1}} \end{cases} \end{aligned} \tag{4}$$

1.2 In-equality Form of LP

Conversely, any equality constraint can also be converted into in-equality constraints, by using a pair of "opposite" constraints.

$$\begin{aligned} \mathbf{a}_i^T \mathbf{x} = b_i \quad & \Leftrightarrow \quad \begin{cases} \mathbf{a}_i^T \mathbf{x} \geq b_i \\ \mathbf{a}_i^T \mathbf{x} \leq b_i \end{cases} \\ & \Leftrightarrow \quad \begin{cases} +\mathbf{a}_i^T \mathbf{x} \geq +b_i \\ -\mathbf{a}_i^T \mathbf{x} \geq -b_i \end{cases} \\ & \Leftrightarrow \quad \begin{cases} \begin{bmatrix} \mathbf{a}_i^T \\ -\mathbf{a}_i^T \end{bmatrix} \mathbf{x} \geq \begin{bmatrix} +b_i \\ -b_i \end{bmatrix} \end{cases} \end{aligned} \tag{5}$$

1.2.1 Example

The LP with equalities and in-equalities Eq.(3) can also be transformed to an in-equality form without equalities

$$\begin{aligned} \text{Minimize} \quad & \mathbf{c}^T \color{red}{\mathbf{x}} \\ \text{s.t.} \quad & \begin{cases} +\mathbf{A}^{(e)} \color{red}{\mathbf{x}} & \geq +\mathbf{b}^{(e)} & \color{gray}{\in \mathbb{R}^{m^{(e)} \times 1}} \\ -\mathbf{A}^{(e)} \color{red}{\mathbf{x}} & \geq -\mathbf{b}^{(e)} & \color{gray}{\in \mathbb{R}^{m^{(e)} \times 1}} \\ \mathbf{A}^{(i)} \color{red}{\mathbf{x}} & \geq \mathbf{b}^{(i)} & \color{gray}{\in \mathbb{R}^{m^{(i)} \times 1}} \\ \color{red}{\mathbf{x}} & \geq \mathbf{0} & \color{gray}{\in \mathbb{R}^{n \times 1}} \end{cases} \end{aligned} \tag{6}$$

2 LP and Duality

2.1 Definition of Dual

Assume $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^{m\times 1}$, $\mathbf{c} \in \mathbb{R}^{n\times 1}$. The LP problem **(in in-equality form)**

$$\begin{aligned} &\text{Minimize} \quad \mathbf{c}^T\mathbf{x} \\ &\text{s.t.} \quad \begin{cases} \mathbf{A}\mathbf{x} & \geq \mathbf{b} \quad \in \mathbb{R}^{m\times 1} \\ \mathbf{x} & \geq \mathbf{0} \quad \in \mathbb{R}^{n\times 1} \end{cases} \end{aligned} \tag{7}$$

has a dual problem defined as

$$\begin{aligned} &\text{Maximize} \quad \mathbf{b}^T\lambda \\ &\text{s.t.} \quad \begin{cases} \mathbf{A}^T\lambda & \leq \mathbf{c} \quad \in \mathbb{R}^{n\times 1} \\ \lambda & \geq \mathbf{0} \quad \in \mathbb{R}^{m\times 1} \end{cases} \end{aligned} \tag{8}$$

and we can conclude the dual derivation process is simple:

- ($\mathbf{A} \Leftrightarrow \mathbf{A}^T$) coefficient matrix is transposed
- ($\mathbf{b} \Leftrightarrow \mathbf{c}$) cost and constraint vectors are interchanged
- ($\leq \Leftrightarrow \geq$) constraint in-equalities are reversed
- (maximizing $\Leftrightarrow$ minimizing) optimization direction is reversed

and also note that the sizes/lengths of matrices/vectors are changed.

Following this definition, the LP problem **(in standard form)**

$$\begin{aligned} &\text{Minimize} \quad \mathbf{c}^T\mathbf{x} \\ &\text{s.t.} \quad \begin{cases} \mathbf{A}\mathbf{x} & = \mathbf{b} \quad \in \mathbb{R}^{m\times 1} \\ \mathbf{x} & \geq \mathbf{0} \quad \in \mathbb{R}^{n\times 1} \end{cases} \end{aligned} \tag{9}$$

which can be converted into in-equality form as

$$\begin{aligned} &\text{Minimize} \quad \mathbf{c}^T\mathbf{x} \\ &\text{s.t.} \quad \begin{cases} \begin{bmatrix} +\mathbf{A} \\ -\mathbf{A} \end{bmatrix}\mathbf{x} & \geq \begin{bmatrix} +\mathbf{b} \\ -\mathbf{b} \end{bmatrix} \quad \in \mathbb{R}^{(2m)\times 1} \\ \mathbf{x} & \geq \mathbf{0} \quad \in \mathbb{R}^{n\times 1} \end{cases} \end{aligned} \tag{10}$$

should have a dual problem as

$$\begin{aligned} &\text{Maximize} \quad \begin{bmatrix} +\mathbf{b}^T & -\mathbf{b}^T \end{bmatrix}\lambda \\ &\text{s.t.} \quad \begin{cases} \begin{bmatrix} +\mathbf{A}^T & -\mathbf{A}^T \end{bmatrix}\lambda & \leq \mathbf{c} \quad \in \mathbb{R}^{n\times 1} \\ \lambda & \geq \mathbf{0} \quad \in \mathbb{R}^{(2m)\times 1} \end{cases} \end{aligned} \tag{11}$$

Partitioning $\lambda \in \mathbb{R}^{(2m)\times 1}$ as $\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$ and we have

$$\begin{aligned} &\text{Maximize} \quad \mathbf{b}^T(\lambda_1 - \lambda_2) \\ &\text{s.t.} \quad \begin{cases} \mathbf{A}^T(\lambda_1 - \lambda_2) & \leq \mathbf{c} \quad \in \mathbb{R}^{n\times 1} \\ \lambda_1 & \geq \mathbf{0} \quad \in \mathbb{R}^{(m)\times 1} \\ \lambda_2 & \geq \mathbf{0} \quad \in \mathbb{R}^{(m)\times 1} \end{cases} \end{aligned} \tag{12}$$

and we can safely combine the free non-negative variables pairs $\lambda' = (\lambda_1 - \lambda_2)$, resulting in

$$\begin{aligned} &\text{Maximize} \quad \mathbf{b}^T\lambda' \\ &\text{s.t.} \quad \begin{cases} \mathbf{A}^T\lambda' & \leq \mathbf{c} \quad \in \mathbb{R}^{n\times 1} \\ \lambda' & \geq -\infty \quad \in \mathbb{R}^{m\times 1} \\ & \text{(variables not necessarily non-negative)} \end{cases} \end{aligned} \tag{13}$$

2.2 The Duality Theorem

So what is the relationship between Eq.(7) and its dual Eq.(8) defined in section 2.1 ?

We will explore , based on the dual of standard form, Eq.(9) and Eq.(13).

2.2.1 Weak Duality Lemma

If Eq.(9) and Eq.(13) are both feasible, then any feasible solution $\mathbf{x}$ for Eq.(9) as well as any feasible solution λ' for Eq.(13) satisfies:

$$\begin{aligned} &\mathbf{b}^T\lambda' \\ &= (\mathbf{A}\mathbf{x})^T\lambda' \quad \text{by} \quad \mathbf{A}\mathbf{x} = \mathbf{b} \quad \text{in Eq.(9)} \\ &= \mathbf{x}^T(\mathbf{A}^T\lambda') \\ &\leq \mathbf{x}^T\mathbf{c} \quad \text{by} \quad \begin{cases} \mathbf{A}^T\lambda' \leq \mathbf{c} & \text{in Eq.(13)} \\ \mathbf{x} \geq \mathbf{0} & \text{in Eq.(9)} \end{cases} \\ &= \mathbf{c}^T\mathbf{x} \end{aligned} \tag{14}$$

meaning that,

- the objective of **any feasible** solution of a standard form LP as Eq.(9) should always be larger than or equal to the objective of **any feasible** solution of its dual as Eq.(13).

and therefore,

- If either problem has an unbounded optimal objective, the other problem will be in-feasible.

- If both Eq.(9) and Eq.(13) are feasible, the objective of **the optimal** solution of a standard form LP as Eq.(9) should always be larger than or equal to the objective of **the optimal** solution of its dual as Eq.(13).
  - If we come across a pair of feasible solutions $(\mathbf{x}, \lambda')$ where the equality holds, then they should be (one of) the corresponding optimal solutions of each problem.

(Note that Eq.(9) seeks for minimum, and Eq.(13) seeks for maximum.)

2.2.2 Strong Duality Theorem

However, the above comparison is weak, and these questions currently remain:

- If either problem is in-feasible, should the other problem have an unbounded optimal objective?
- If either problem has a finite optimal objective, should the other problem also have a finite optimal objective?
- If both Eq.(9) and Eq.(13) are feasible, in which case the two problems have the equal optimal objective?

TODO

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$$ \tag{16} $$