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1 LP and Duality

1.1 Standard Form of LP

Any unbounded variable x can be replaced into a pair of non-negative variables (u, v).

$$\begin{cases} x = u - v \\ u \geq 0 \\ v \geq 0 \end{cases} \tag{1}$$

Any in-equality constraint can be converted into an equality constraint, by introducing an additional assistant non-negative variable x'.

$$\begin{aligned} \mathbf{a}_i^T\mathbf{x} \geq b_i \quad & \Leftrightarrow \quad \begin{cases} \mathbf{a}_i^T\mathbf{x} - x' & = b_i \\ x' & \geq 0 \end{cases} \\ & \Leftrightarrow \quad \begin{cases} \begin{bmatrix} \mathbf{a}_i^T & (-1) \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ x' \end{bmatrix} & = b_i \\ x' & \geq 0 \end{cases} \end{aligned} \tag{2}$$

So we can safely represent any LP problem in its standard form, with only non-negative variables and only equality constraints.

1.2 Definition of Dual

Assume $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^{m\times 1}$, $\mathbf{c} \in \mathbb{R}^{n\times 1}$. The Linear Programming (LP) problem

$$\begin{aligned} \text{Minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & \begin{cases} \mathbf{A}\mathbf{x} & \geq \mathbf{b} & \in \mathbb{R}^{m\times 1} \\ \mathbf{x} & \geq \mathbf{0} & \in \mathbb{R}^{n\times 1} \end{cases} \end{aligned} \tag{3}$$

has a dual problem

$$\begin{aligned} \text{Maximize} \quad & \mathbf{b}^T\lambda \\ \text{s.t.} \quad & \begin{cases} \mathbf{A}^T\lambda & \leq \mathbf{c} & \in \mathbb{R}^{n\times 1} \\ \lambda & \geq \mathbf{0} & \in \mathbb{R}^{m\times 1} \end{cases} \end{aligned} \tag{4}$$

1.3 The Duality Theorem

$$\tag{5}$$