School of Computing
National University of Singapore
Biometrics Course
July 2016

Assignment #3

Q1. Let $\mathbf{A} = \begin{bmatrix} 7 & 5 \\ 3 & -7 \end{bmatrix}$. Solve the following by hand:

(a) Factorize $\mathbf{A}$ into $\mathbf{A} = \mathbf{S\Lambda S}^{-1}$, where $\mathbf{S}$ is invertible and $\mathbf{\Lambda}$ is diagonal.

(b) Find $\mathbf{B}$ such that $\mathbf{B}^3 = \mathbf{A}$.

Q2. Let $\mathbf{S} = \left\{ a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \text{for all a,b} \in \mathbb{R} \right\}$.

(a) Find a 3×3 matrix $\mathbf{A}$ such that $Col(\mathbf{A}) = \mathbf{S}$.

(b) Find a 2×3 matrix $\mathbf{B}$ such that $Null(\mathbf{B}) = \mathbf{S}$.

Q3. Let $\mathbf{B}$, $\mathbf{C}$ be two symmetric, positive semi-definite $n \times n$ matrices. Define:

$$J(\mathbf{w}) = \frac{\mathbf{w}^T\mathbf{C}\mathbf{w}}{\mathbf{w}^T\mathbf{B}\mathbf{w}} \tag{1}$$

Note that J is a scalar function of the vector $\mathbf{w}$. The goal is to find $\mathbf{w}$ that maximizes J.

(a) The following is called a *Generalized Eigenvalue Problem*:

$$\mathbf{C}\mathbf{w} = \lambda\mathbf{B}\mathbf{w} \tag{2}$$

where $\mathbf{w}$ is the eigenvector, and λ is the eigenvalue.

Show that the solution to Equation 2 maximizes J. Hint: Use the Quotient Rule, given by:

$$\text{Let} \quad s = \frac{f(x)}{g(x)}. \quad \text{Then} \quad \frac{ds}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

(b) Suppose $\mathbf{B}$ is invertible. Transform Equation 2 into a regular eigenvalue problem. Since there are many eigenvectors, which one maximizes J? what is the maximum value of J?

Q4. (Using SVD for compression.) Stepping gingerly from the spaceship onto Martian soil, you heave a sigh of relief. All these years of training at the School of Cosmology (SOC) has finally paid off: you are on your first mission to Mars. As you survey the barren and rocky landscape, your eyes spot something unusual in the distance. You move closer and discover what looks like a Martian flower. Very excitedly, you whip out your high resolution digital camera and take a few shots of the exotic plant.

Back at the spaceship, you wonder how to transmit the images back to Earth. Each image is large, and because of bandwidth limitations, you can send only a small amount of data at a time. How best to do it? Fortunately, you remember the SVD technique. You decide to use it to transmit an image of the flower progressively: a coarse approximation at first, finer details later.

- In Python, import all the necessary libraries as following:
  ```
  import Image
  import ImageOps
  import numpy
  from numpy import linalg
  import matplotlib
  from matplotlib import pyplot
  ```
- Read the image using `flower = Image.open("flower.jpg")`
- To see the image, use: `flower.show()`
- This is an RGB image. Convert it to grayscale: `flower = ImageOps.grayscale(flower)`
- Convert image type to array using:
  ```
  aflower = numpy.asarray(flower) # aflower is unit8
  aflower = numpy.float32(aflower)
  ```
- Compute the SVD: `U,S,Vt = linalg.svd(aflower);`
- The singular values in S have been sorted in descending order. Plot it with the command:
  ```
  pyplot.plot(S,'b.')
  pyplot.show()
  ```
- Print out the plot and submit it. What do you notice?
- Let $K = 20$. Extract the first K singular values and their corresponding vectors in U and V:
  ```
  K = 20
  Sk = numpy.diag(S[:K])
  Uk = U[:, :K]
  Vtk = Vt[:K, :]
  ```
- Uk, Vk, Sk contain the compressed version of the image. To see this, form the compressed image using:
  ```
  aImk = numpy.dot(Uk, numpy.dot( Sk, Vtk))
  Imk = Image.fromarray(aImk)
  ```
  and display it: `Imk.show()`
- Print out a copy of this and submit it.
- Repeat for $K = 50, 100, 200$. Print and submit the compressed images for the four different values of K. Briefly describe what you notice.

- Thus, instead of transmitting the original image, you can transmit Uk, Vk, Sk, which should be much less data than the original. Is it worth transmitting when $K = 200$?

Note: Please read the file `printing.pdf` for printing tips.

Deadline: 20 July 2:00pm