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ENGR 16100

PREVIOUSLY DERIVED EQUATIONS

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A USEFUL SET OF EQUATIONS DISCUSSED IN LECTURE

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Note: You will not have access to these derivations during any exam.

## General Derivations

### Newton's Second Law

$$\sum \vec{F} = m\vec{a}$$

### Kinematics

Cartesian

Polar

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r} = r\hat{e}_r$$

$$\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$$

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$\vec{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

$$\vec{p} = m\vec{v}$$

$$\Delta\vec{p} = \int \vec{F}dt$$

### Universal Accounting Equation

$$(\text{Final value} - \text{Initial value}) = (\text{Input} - \text{Output}) + (\text{Generated} - \text{Consumed})$$

### Energies

Gravitational Potential:  $E_g = mgh$

Elastic Potential:  $E_{sp} = \frac{1}{2}k(\Delta x)^2$

Kinetic:  $E_k = \frac{1}{2}mv^2$

## Impulse-Momentum Equation

Single Particle:

$$\begin{aligned}\Delta \vec{p} &= \int_{t_{initial}}^{t_{final}} \vec{F}_{net} dt \\ &= m\vec{v}_2 - m\vec{v}_1\end{aligned}$$

System of Particles:

$$\begin{aligned}\Delta \vec{p}_{total} &= \int_{t_{initial}}^{t_{final}} \vec{F}_{net}^{ext} dt \\ &= \sum_i (m_i \vec{v}_{i,final} - m_i \vec{v}_{i,initial})\end{aligned}$$

Conservation of Momentum:

$$\begin{aligned}0 &= \sum_i (m_i \vec{v}_{i,final} - m_i \vec{v}_{i,initial}) \\ &= \sum_i \left( m_i \vec{v}_{i,final} - \sum_i m_i \vec{v}_{i,initial} \right) \\ \sum_i m_i \vec{v}_{i,initial} &= \sum_i m_i \vec{v}_{i,final}\end{aligned}$$

## Direct and Oblique Impact

### Motion in a Normal Direction

Motion in a normal direction looks exactly the same as a direct impact. Thus, the motion can be described using the Conservation of Momentum and the Coefficient of Restitution equations:

$$m_A (v_{A,n})_1 + m_B (v_{B,n})_1 = m_A (v_{A,n})_2 + m_B (v_{B,n})_2$$

$$e = - \frac{m_A (v_{A,n})_2 + m_B (v_{B,n})_2}{m_A (v_{A,n})_1 + m_B (v_{B,n})_1}$$

### Motion in a Tangent Direction

Motion in a tangent direction does not result in any changes in velocity:

$$(v_{A,n})_1 = (v_{A,n})_2$$

$$(v_{B,n})_1 = (v_{B,n})_2$$

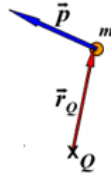
**Note:** For direct impact problems, you should only use the "motion in a normal direction" equations.

## Angular Momentum

The formal definition of the angular momentum  $\vec{L}_Q$  of a particle about a point  $Q$  is:

$$\begin{aligned}\vec{L}_Q &= \vec{r}_Q \times (m\vec{v}) \\ &= \vec{p}\end{aligned}$$

where  $\vec{r}_Q$  is the position vector of the particle measured from point  $Q$ .



## Kinetic Energy

You all know Newton's Second Law:

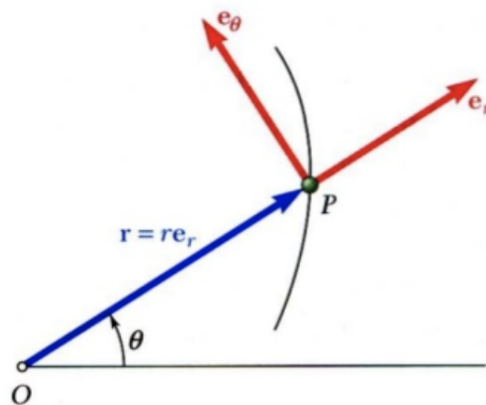
$$\Sigma \vec{F} = m\vec{a}$$

We can use this equation to find power using kinetic energy  $T$ :

$$\begin{aligned} T &= \frac{1}{2} m \vec{v} \cdot \vec{v} \\ \frac{dT}{dt} &= \frac{1}{2} m \vec{a} \cdot \vec{v} + \frac{1}{2} m \vec{v} \cdot \vec{a} \\ &= \frac{1}{2} m \vec{a} \cdot \vec{v} + \frac{1}{2} m \vec{a} \cdot \vec{v} \\ &= m \vec{a} \cdot \vec{v} \\ &= (\Sigma \vec{F}) \cdot \vec{v} \\ &= \text{Power} \end{aligned}$$

That's a nice result, but what if we integrate both sides?

$$\begin{aligned} \int_{t_1}^{t_2} \left( \frac{dT}{dt} \right) dt &= \int_{t_1}^{t_2} (\Sigma \vec{F} \cdot \vec{v}) dt \\ \int_{T_1}^{T_2} dT &= \int_{t_1}^{t_2} \left( \Sigma \vec{F} \cdot \frac{d\vec{r}}{dt} \right) dt \\ T_2 - T_1 &= \int_{\vec{r}_1}^{\vec{r}_2} \Sigma \vec{F} \cdot d\vec{r} \quad ^1 \\ T_2 - T_1 &= \int_{S_1}^{S_2} (\Sigma \vec{F}) \cdot \hat{e}_t ds \quad ^2 \\ \Delta T &= \frac{1}{2} m (\vec{v}_2 \cdot \vec{v}_2) - \frac{1}{2} m (\vec{v}_1 \cdot \vec{v}_1) \end{aligned}$$



<sup>1</sup>Since this is a dot product, we can replace  $d\vec{r}$  with the tangent—the arc length along the particle's path.

<sup>2</sup>This is equal to the work done on the particle.

## Cartesian and Polar Coordinates

Start with the position vector:

$$\vec{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$

Using the Chain Rule, the derivative of  $\vec{r}$  is

$$\vec{v} = \frac{dx}{dt}\hat{\mathbf{i}} + x\frac{d\hat{\mathbf{i}}}{dt} + \frac{dy}{dt}\hat{\mathbf{j}} + y\frac{d\hat{\mathbf{j}}}{dt}$$

Because unit vectors in Cartesian coordinates are fixed and constant,  $\frac{d\hat{\mathbf{i}}}{dt} = \frac{d\hat{\mathbf{j}}}{dt} = \vec{0}$ , so we can rewrite velocity as

$$\begin{aligned}\vec{v} &= \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} \\ &= \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}}^3 \\ \vec{a} &= \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}}\end{aligned}$$

We can also describe the motion of the particle using polar coordinates:

$$\begin{aligned}\vec{r} &= r\hat{\mathbf{e}}_{\mathbf{r}} \\ \vec{v} &= \dot{r}\hat{\mathbf{e}}_{\mathbf{r}} + r\frac{d\hat{\mathbf{e}}_{\mathbf{r}}}{dt}\end{aligned}$$

Before continuing, we need to work out the derivatives of the following equations:

$$\begin{aligned}\hat{\mathbf{e}}_{\mathbf{r}} &= \hat{\mathbf{i}}\cos\theta + \hat{\mathbf{j}}\sin\theta \\ \hat{\mathbf{e}}_{\theta} &= -\hat{\mathbf{i}}\sin\theta + \hat{\mathbf{j}}\cos\theta\end{aligned}$$

The trick is to use the Chain Rule to find their derivatives:

$$\begin{aligned}\frac{d\hat{\mathbf{e}}_{\mathbf{r}}}{dt} &= \frac{d\hat{\mathbf{e}}_{\mathbf{r}}}{d\theta} \cdot \frac{d\theta}{dt} \\ &= \frac{d\hat{\mathbf{e}}_{\mathbf{r}}}{d\theta} \dot{\theta} \\ &= (-\hat{\mathbf{i}}\sin\theta + \hat{\mathbf{j}}\cos\theta) \dot{\theta} \\ &= \dot{\theta}\hat{\mathbf{e}}_{\theta}\end{aligned}$$

$$\begin{aligned}\frac{d\hat{\mathbf{e}}_{\theta}}{dt} &= \frac{d\hat{\mathbf{e}}_{\theta}}{d\theta} \cdot \frac{d\theta}{dt} \\ &= \dot{\theta}[-\hat{\mathbf{i}}\cos\theta - \hat{\mathbf{j}}\sin\theta] \\ &= \dot{\theta}(-\hat{\mathbf{e}}_{\mathbf{r}}) \\ &= -\dot{\theta}\hat{\mathbf{e}}_{\mathbf{r}}\end{aligned}$$

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<sup>4</sup>Note that  $\dot{x} = \frac{dx}{dt}$

Combining the above derivatives into our equation for velocity, we get

$$\begin{aligned}\vec{v} &= \dot{r}\hat{\mathbf{e}}_r + r\left(\dot{\theta}\hat{\mathbf{e}}_\theta\right) \\ &= \dot{r}\hat{\mathbf{e}}_r + r\dot{\theta}\hat{\mathbf{e}}_\theta\end{aligned}$$

$$\begin{aligned}\vec{a} &= \ddot{r}\hat{\mathbf{e}}_r + \dot{r}\frac{d\hat{\mathbf{e}}_r}{dt} + \dot{r}\left(\dot{\theta}\hat{\mathbf{e}}_\theta\right) + r\ddot{\theta}\hat{\mathbf{e}}_\theta + r\dot{\theta}\frac{d\hat{\mathbf{e}}_\theta}{dt} \\ &= \ddot{r}\hat{\mathbf{e}}_r + \dot{r}\dot{\theta}\hat{\mathbf{e}}_\theta + \dot{r}\dot{\theta}\hat{\mathbf{e}}_\theta + r\ddot{\theta}\hat{\mathbf{e}}_\theta - r\dot{\theta}^2\hat{\mathbf{e}}_r \\ &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{e}}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\mathbf{e}}_\theta\end{aligned}$$

