ENGR 16100

Previously Derived Equations

A Useful Set of Equations Discussed in Lecture

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Note: You will <u>not</u> have access to these derivations during any exam.

General Derivations

Newton's Second Law

$\sum \vec{F} = m\vec{a}$

Kinematics

Cartesian | Polar

$\vec{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{k}$ $\vec{r} = r\hat{\mathbf{e}}_{\mathbf{r}}$

$\vec{v} = \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{k}$ $\vec{v} = \dot{r}\hat{\mathbf{e}}_{\mathbf{r}} + r\dot{\theta}\hat{\mathbf{e}}_{\theta}$

$\vec{a} = \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{k}$ $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{e}}_{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\mathbf{e}}_{\theta}$

$\vec{p} = m\vec{v}$

$\Delta\vec{p} = \int \vec{F}dt$

Universal Accounting Equation

(Final value − Initial value) = (Input − Output) + (Generated − Consumed)

Energies

Gravitational Potential: $E_g = mgh$

Elastic Potential: $E_{sp} = \frac{1}{2}k(\Delta x)^2$

Kinetic: $E_k = \frac{1}{2}mv^2$

Linear Impulse-Momentum Equations

Single Particle:

$$\Delta \vec{p} = \int_{t_{state1}}^{t_{state2}} \vec{F}_{net} dt$$

$$= m\vec{v}_2 - m\vec{v}_1$$

System of Particles:

$$\Delta \vec{p}_{total} = \int_{t_{state1}}^{t_{state2}} \vec{F}_{net}^{ext}$$

$$= \sum_i \left(m_i \vec{v}_{i,state2} - m_i \vec{v}_{i,state1} \right)$$

Conservation of Momentum:

$$0 = \sum_i \left(m_i \vec{v}_{i,state2} - m_i \vec{v}_{i,state1} \right)$$

$$= \sum_i m_i \vec{v}_{i,state2} - \sum_i m_i \vec{v}_{i,state1}$$

$$\sum_i m_i \vec{v}_{i,state1} = \sum_i m_i \vec{v}_{i,state2}$$

Direct and Oblique Impact

Motion in a Normal Direction

Motion in a normal direction looks exactly the same as a direct impact. Thus, the motion can be described using the Conservation of Momentum and the Coefficient of Restitution equations:

$$m_A \left(v_{A,n}\right)_1 + m_B \left(v_{B,n}\right)_1 = m_A \left(v_{A,n}\right)_2 + m_B \left(v_{B,n}\right)_2$$

$$e = -\frac{\left(v_{A,n}\right)_2 - \left(v_{B,n}\right)_2}{\left(v_{A,n}\right)_1 - \left(v_{B,n}\right)_1}$$

Motion in a Tangent Direction

Motion in a tangent direction does not result in any changes in velocity:

$$\left(v_{A,t}\right)_1 = \left(v_{A,t}\right)_2$$
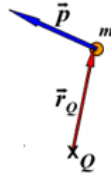$$\left(v_{B,t}\right)_1 = \left(v_{B,t}\right)_2$$

Note: For direct impact problems, you should only use the "motion in a normal direction" equations.

Angular Momentum

The formal definition of the angular momentum $\vec{L}_Q$ of a particle about a point Q is:

$$\vec{L}_Q = \vec{r}_Q \times (m\vec{v})$$
$$= \vec{r}_Q \times \vec{p}$$

where $\vec{r}_Q$ is the position vector of the particle measured from point Q.
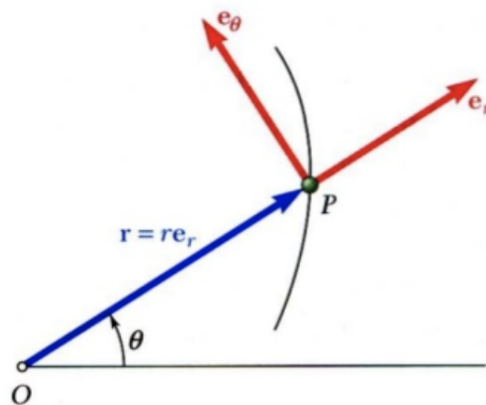
Kinetic Energy

You all know Newton's Second Law:

$$\Sigma \vec{F} = m\vec{a}$$

We can use this equation to find power using kinetic energy T:

$$T = \frac{1}{2}m\vec{v} \cdot \vec{v}$$
$$\frac{dT}{dt} = \frac{1}{2}m\vec{a} \cdot \vec{v} + \frac{1}{2}m\vec{v} \cdot \vec{a}$$
$$= \frac{1}{2}m\vec{a} \cdot \vec{v} + \frac{1}{2}m\vec{a} \cdot \vec{v}$$
$$= m\vec{a} \cdot \vec{v}$$
$$= \left(\Sigma\vec{F}\right) \cdot \vec{v}$$
$$= \text{Power}$$

That's a nice result, but what if we integrate both sides?

$$\int_{t_1}^{t_2} \left(\frac{dT}{dt}\right) dt = \int_{t_1}^{t_2} \left(\Sigma\vec{F} \cdot \vec{v}\right) dt$$
$$\int_{T_1}^{T_2} dT = \int_{t_1}^{t_2} \left(\Sigma\vec{F} \cdot \frac{d\vec{r}}{dt}\right) dt$$
$$T_2 - T_1 = \int_{\vec{r}_1}^{\vec{r}_2} \Sigma\vec{F} \cdot d\vec{r} \ ^{[1]}$$
$$T_2 - T_1 = \int_{S_1}^{S_2} \left(\Sigma\vec{F}\right) \cdot \hat{e}_t ds \ ^{[2]}$$
$$\Delta T = \frac{1}{2}m\left(\vec{v}_2 \cdot \vec{v}_2\right) - \frac{1}{2}m\left(\vec{v}_1 \cdot \vec{v}_1\right)$$

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[1] Since this is a dot product, we can replace $d\vec{r}$ with the tangent—the arc length along the particle's path.
[2] This is equal to the work done on the particle.

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Cartesian and Polar Coordinates

Start with the position vector:

$$\vec{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$

Using the Chain Rule, the derivative of $\vec{r}$ is

$$\vec{v} = \frac{dx}{dt}\hat{\mathbf{i}} + x\frac{d\hat{\mathbf{i}}}{dt} + \frac{dy}{dt}\hat{\mathbf{j}} + y\frac{d\hat{\mathbf{j}}}{dt}$$

Because unit vectors in Cartesian coordinates are fixed and constant, $\frac{d\hat{\mathbf{i}}}{dt} = \frac{d\hat{\mathbf{j}}}{dt} = \vec{0}$, so we can rewrite velocity as

$$\vec{v} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}}$$
$$= \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} \quad ^3$$
$$\vec{a} = \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}}$$

We can also describe the motion of the particle using polar coordinates:

$$\vec{r} = r\hat{\mathbf{e}}_{\mathbf{r}}$$
$$\vec{v} = \dot{r}\hat{\mathbf{e}}_{\mathbf{r}} + r\frac{d\hat{\mathbf{e}}_{\mathbf{r}}}{dt}$$

Before continuing, we need to work out the derivatives of the following equations:

$$\hat{\mathbf{e}}_{\mathbf{r}} = \hat{\mathbf{i}}\cos\theta + \hat{\mathbf{j}}\sin\theta$$
$$\hat{\mathbf{e}}_{\theta} = -\hat{\mathbf{i}}\sin\theta + \hat{\mathbf{j}}\cos\theta$$

The trick is to use the Chain Rule to find their derivatives:

$$\frac{d\hat{\mathbf{e}}_{\mathbf{r}}}{dt} = \frac{d\hat{\mathbf{e}}_{\mathbf{r}}}{d\theta} \cdot \frac{d\theta}{dt}$$
$$= \frac{d\hat{\mathbf{e}}_{\mathbf{r}}}{d\theta}\dot{\theta}$$
$$= (-\hat{\mathbf{i}}\sin\theta + \hat{\mathbf{j}}\cos\theta)\,\dot{\theta}$$
$$= \dot{\theta}\hat{\mathbf{e}}_{\theta}$$
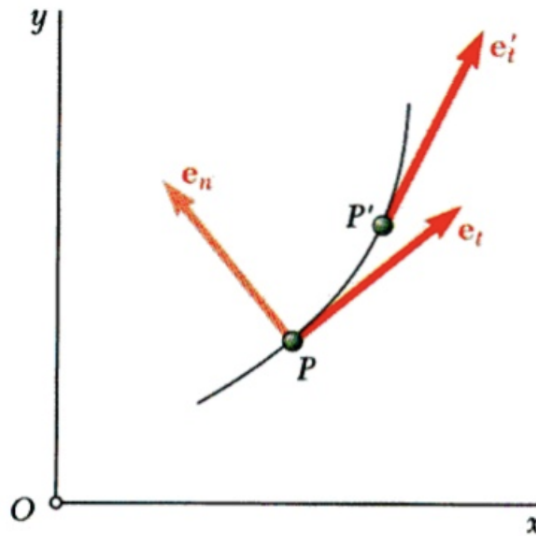
$$\frac{d\hat{\mathbf{e}}_{\theta}}{dt} = \frac{d\hat{\mathbf{e}}_{\theta}}{d\theta} \cdot \frac{d\theta}{dt}$$
$$= \dot{\theta}\left[-\hat{\mathbf{i}}\cos\theta - \hat{\mathbf{j}}\sin\theta\right]$$
$$= \dot{\theta}\left(-\hat{\mathbf{e}}_{\mathbf{r}}\right)$$
$$= -\dot{\theta}\hat{\mathbf{e}}_{\mathbf{r}}$$

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[4]Note that $\dot{x} = \frac{dx}{dt}$

Combining the above derivatives into our equation for velocity, we get

$$\vec{v} = \dot{r}\hat{\mathbf{e}}_{\mathbf{r}} + r\left(\dot{\theta}\hat{\mathbf{e}}_{\theta}\right)$$

$$= \dot{r}\hat{\mathbf{e}}_{\mathbf{r}} + r\dot{\theta}\hat{\mathbf{e}}_{\theta}$$

$$\vec{a} = \ddot{r}\hat{\mathbf{e}}_{\mathbf{r}} + \dot{r}\frac{d\hat{\mathbf{e}}_{\mathbf{r}}}{dt} + \dot{r}\left(\dot{\theta}\hat{\mathbf{e}}_{\theta}\right) + r\ddot{\theta}\hat{\mathbf{e}}_{\theta} + r\dot{\theta}\frac{d\hat{\mathbf{e}}_{\theta}}{dt}$$

$$= \ddot{r}\hat{\mathbf{e}}_{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\mathbf{e}}_{\theta} + \dot{r}\dot{\theta}\hat{\mathbf{e}}_{\theta} + r\ddot{\theta}\hat{\mathbf{e}}_{\mathbf{r}} - r\dot{\theta}^2\hat{\mathbf{e}}_{\mathbf{r}}$$

$$= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{e}}_{\mathbf{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\mathbf{e}}_{\theta}$$

Revision History

Revision	Date	Author(s)	Description
1.0	11/26/17	NSM	Creation
1.1	12/2/17	NSM	Fixed typo in *Motion in a Tangent Direction*