ENGR 16100

Previously Derived Equations

A Useful Set of Equations Discussed in Lecture

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Note: You will not have access to these derivations during any exam.

General Derivations

Newton's Second Law

$\sum \vec{F} = m\vec{a}$

Kinematics

Cartesian	Polar
$\vec{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{k}$	$\vec{r} = r\hat{\mathbf{e}}_\mathbf{r}$
$\vec{v} = \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{k}$	$\vec{v} = \dot{r}\hat{\mathbf{e}}_\mathbf{r} + r\dot{\theta}\hat{\mathbf{e}}_\theta$
$\vec{a} = \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{k}$	$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{e}}_\mathbf{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\mathbf{e}}_\theta$

$\vec{p} = m\vec{v}$

$\Delta\vec{p} = \int \vec{F} dt$

Universal Accounting Equation

(Final value − Initial value) = (Input − Output) + (Generated − Consumed)

Energies

Gravitational Potential: $E_g = mgh$

Elastic Potential: $E_{sp} = \frac{1}{2}k(\Delta x)^2$

Kinetic: $E_k = \frac{1}{2}mv^2$

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Linear Impulse-Momentum Equations

Single Particle:

$$\begin{aligned}\Delta\vec{p} &= \int_{t_{state1}}^{t_{state2}} \vec{F}_{net} dt \\ &= m\vec{v}_2 - m\vec{v}_1\end{aligned}$$

System of Particles:

$$\begin{aligned}\Delta\vec{p}_{total} &= \int_{t_{state1}}^{t_{state2}} \vec{F}_{net}^{ext} \\ &= \sum_i (m_i\vec{v}_{i,state2} - m_i\vec{v}_{i,state1})\end{aligned}$$

Conservation of Momentum:

$$\begin{aligned}0 &= \sum_i (m_i\vec{v}_{i,state2} - m_i\vec{v}_{i,state1}) \\ &= \sum_i m_i\vec{v}_{i,state2} - \sum_i m_i\vec{v}_{i,state1} \\ \sum_i m_i\vec{v}_{i,state1} &= \sum_i m_i\vec{v}_{i,state2}\end{aligned}$$

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Direct and Oblique Impact

Motion in a Normal Direction

Motion in a normal direction looks exactly the same as a direct impact. Thus, the motion can be described using the Conservation of Momentum and the Coefficient of Restitution equations:

$$m_A \left(v_{A,n}\right)_1 + m_B \left(v_{B,n}\right)_1 = m_A \left(v_{A,n}\right)_2 + m_B \left(v_{B,n}\right)_2$$

$$e = -\frac{\left(v_{A,n}\right)_2 - \left(v_{B,n}\right)_2}{\left(v_{A,n}\right)_1 - \left(v_{B,n}\right)_1}$$

Motion in a Tangent Direction

Motion in a tangent direction does not result in any changes in velocity:

$$\left(v_{A,n}\right)_1 = \left(v_{A,n}\right)_1$$
$$\left(v_{B,n}\right)_1 = \left(v_{B,n}\right)_2$$

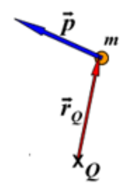
Note: For direct impact problems, you should only use the "motion in a normal direction" equations.

Angular Momentum

The formal definition of the angular momentum $\vec{L}_Q$ of a particle about a point Q is:

$$\begin{aligned}\vec{L}_Q &= \vec{r}_Q \times (m\vec{v}) \\ &= \vec{r}_Q \times \vec{p}\end{aligned}$$

where $\vec{r}_Q$ is the position vector of the particle measured from point Q.



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Kinetic Energy

You all know Newton's Second Law:

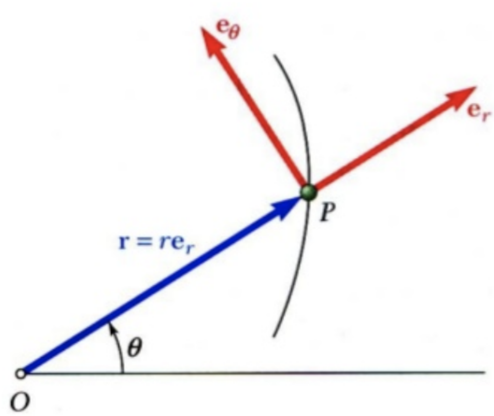
$$\Sigma\vec{F} = m\vec{a}$$

We can use this equation to find power using kinetic energy T:

$$\begin{aligned}
T &= \frac{1}{2}m\vec{v}\cdot\vec{v} \\
\frac{dT}{dt} &= \frac{1}{2}m\vec{a}\cdot\vec{v} + \frac{1}{2}m\vec{v}\cdot\vec{a} \\
&= \frac{1}{2}m\vec{a}\cdot\vec{v} + \frac{1}{2}m\vec{a}\cdot\vec{v} \\
&= m\vec{a}\cdot\vec{v} \\
&= \left(\Sigma\vec{F}\right)\cdot\vec{v} \\
&= \text{Power}
\end{aligned}$$

That's a nice result, but what if we integrate both sides?

$$\begin{aligned}
\int_{t_1}^{t_2}\left(\frac{dT}{dt}\right)dt &= \int_{t_1}^{t_2}\left(\Sigma\vec{F}\cdot\vec{v}\right)dt \\
\int_{T_1}^{T_2} dT &= \int_{t_1}^{t_2}\left(\Sigma\vec{F}\cdot\frac{d\vec{r}}{dt}\right)dt \\
T_2 - T_1 &= \int_{\vec{r}_1}^{\vec{r}_2}\Sigma\vec{F}\cdot d\vec{r}\ ^{1} \\
T_2 - T_1 &= \int_{S_1}^{S_2}\left(\Sigma\vec{F}\right)\cdot\hat{e}_t ds\ ^{2} \\
\Delta T &= \frac{1}{2}m\left(\vec{v}_2\cdot\vec{v}_2\right) - \frac{1}{2}m\left(\vec{v}_1\cdot\vec{v}_1\right)
\end{aligned}$$



[1] Since this is a dot product, we can replace $d\vec{r}$ with the tangent—the arc length along the particle's path.
[2] This is equal to the work done on the particle.

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Cartesian and Polar Coordinates

Start with the position vector:

$$\vec{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$

Using the Chain Rule, the derivative of $\vec{r}$ is

$$\vec{v} = \frac{dx}{dt}\hat{\mathbf{i}} + x\frac{d\hat{\mathbf{i}}}{dt} + \frac{dy}{dt}\hat{\mathbf{j}} + y\frac{d\hat{\mathbf{j}}}{dt}$$

Because unit vectors in Cartesian coordinates are fixed and constant, $\frac{d\hat{\mathbf{i}}}{dt} = \frac{d\hat{\mathbf{j}}}{dt} = \vec{0}$, so we can rewrite velocity as

$$\begin{aligned}
\vec{v} &= \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} \\
&= \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} \quad {}^{3} \\
\vec{a} &= \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}}
\end{aligned}$$

We can also describe the motion of the particle using polar coordinates:

$$\begin{aligned}
\vec{r} &= r\hat{\mathbf{e}}_{\mathbf{r}} \\
\vec{v} &= \dot{r}\hat{\mathbf{e}}_{\mathbf{r}} + r\frac{d\hat{\mathbf{e}}_{\mathbf{r}}}{dt}
\end{aligned}$$

Before continuing, we need to work out the derivatives of the following equations:

$$\begin{aligned}
\hat{\mathbf{e}}_{\mathbf{r}} &= \hat{\mathbf{i}}\cos\theta + \hat{\mathbf{j}}\sin\theta \\
\hat{\mathbf{e}}_{\theta} &= -\hat{\mathbf{i}}\sin\theta + \hat{\mathbf{j}}\cos\theta
\end{aligned}$$

The trick is to use the Chain Rule to find their derivatives:

$$\begin{aligned}
\frac{d\hat{\mathbf{e}}_{\mathbf{r}}}{dt} &= \frac{d\hat{\mathbf{e}}_{\mathbf{r}}}{d\theta} \cdot \frac{d\theta}{dt} \\
&= \frac{d\hat{\mathbf{e}}_{\mathbf{r}}}{d\theta}\dot{\theta} \\
&= \left(-\hat{\mathbf{i}}\sin\theta + \hat{\mathbf{j}}\cos\theta\right)\dot{\theta} \\
&= \dot{\theta}\hat{\mathbf{e}}_{\theta}
\end{aligned}$$

$$\begin{aligned}
\frac{d\hat{\mathbf{e}}_{\theta}}{dt} &= \frac{d\hat{\mathbf{e}}_{\theta}}{d\theta} \cdot \frac{d\theta}{dt} \\
&= \dot{\theta}\left[-\hat{\mathbf{i}}\cos\theta - \hat{\mathbf{j}}\sin\theta\right] \\
&= \dot{\theta}\left(-\hat{\mathbf{e}}_{\mathbf{r}}\right) \\
&= -\dot{\theta}\hat{\mathbf{e}}_{\mathbf{r}}
\end{aligned}$$

[4]Note that $\dot{x} = \frac{dx}{dt}$

Combining the above derivatives into our equation for velocity, we get

$$\begin{aligned}
\vec{v} &= \dot{r}\hat{\mathbf{e}}_{\mathbf{r}} + r\left(\dot{\theta}\hat{\mathbf{e}}_{\theta}\right) \\
&= \dot{r}\hat{\mathbf{e}}_{\mathbf{r}} + r\dot{\theta}\hat{\mathbf{e}}_{\theta}
\end{aligned}$$

$$\begin{aligned}
\vec{a} &= \ddot{r}\hat{\mathbf{e}}_{\mathbf{r}} + \dot{r}\frac{d\hat{\mathbf{e}}_{\mathbf{r}}}{dt} + \dot{r}\left(\dot{\theta}\hat{\mathbf{e}}_{\theta}\right) + r\ddot{\theta}\hat{\mathbf{e}}_{\theta} + r\dot{\theta}\frac{d\hat{\mathbf{e}}_{\theta}}{dt} \\
&= \ddot{r}\hat{\mathbf{e}}_{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\mathbf{e}}_{\theta} + \dot{r}\dot{\theta}\hat{\mathbf{e}}_{\theta} + r\ddot{\theta}\hat{\mathbf{e}}_{\mathbf{r}} - r\dot{\theta}^2\hat{\mathbf{e}}_{\mathbf{r}} \\
&= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{e}}_{\mathbf{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\mathbf{e}}_{\theta}
\end{aligned}$$