Bayesian Model Comparison

Model comparison - Deviance Information Criteria (DIC)

Recall that the AIC is given by $\mathcal{D}(\hat{\boldsymbol{\theta}}) + 2p$, where $\mathcal{D}(\theta) = -2 \log p(\mathbf{y}|\boldsymbol{\theta})$, p is the number of parameters, and $\hat{\boldsymbol{\theta}}$ is the MLE.

The **Deviance Information Criterion (DIC)** is defined as

$$\mathcal{D}(\mathbb{E}(\boldsymbol{\theta}|\mathbf{Y} = \mathbf{y})) + 2p_D$$

where p_D is the "effective number of parameters", defined as

$$p_D = \mathbb{E}(\mathcal{D}(\boldsymbol{\theta})|\mathbf{Y} = \mathbf{y}) - \mathcal{D}(\mathbb{E}(\boldsymbol{\theta}|\mathbf{Y} = \mathbf{y}))$$

It can be shown that for vague priors $p_D \approx p$ and the DIC and AIC will be close.

Like the AIC, the DIC can be used to compare different models for the same observations.

The term $\mathcal{D}(\mathbb{E}(\boldsymbol{\theta}|\mathbf{Y} = \mathbf{y}))$ measures goodness of fit, and the term $2p_D$ measures model complexity.

Smaller values of the DIC are better, but there is no absolute threshold that represents a good model. A difference in the DIC of 5 or more, from one model to another, is considered substantial.

Note that p_D is not completely satisfactory. In some cases it can be negative.

Bayes factors

Bayes factors are a more traditional approach for comparing Bayesian models.

Given two models H_A and H_B, we suppose that we have some *prior* idea of their relative plausibility

Prior odds: $\frac{p(H_A)}{p(H_B)}$

Often we take the prior odds to be 1.

Formally then, we have

Posterior odds: $\frac{p(H_A|\mathbf{y})}{p(H_B|\mathbf{y})} = \frac{p(H_A,\mathbf{y})/p(\mathbf{y})}{p(H_B,\mathbf{y})/p(\mathbf{y})} = \frac{p(\mathbf{y}|H_A)}{p(\mathbf{y}|H_B)}\frac{p(H_A)}{p(H_B)}$

where $p(\mathbf{y}|H_A)/p(\mathbf{y}|H_B)$ is called the *Bayes factor*.

$$\frac{p(\mathbf{y}|H_A)}{p(\mathbf{y}|H_B)} = \frac{\int p(\mathbf{y}|\boldsymbol{\theta}_A, H_A)p(\boldsymbol{\theta}_A|H_A)d\boldsymbol{\theta}_A}{\int p(\mathbf{y}|\boldsymbol{\theta}_B, H_B)p(\boldsymbol{\theta}_B|H_B)d\boldsymbol{\theta}_B}.$$

Bayes factor is a ratio of the marginal likelihood under one model to the marginal likelihood under the second model.

It quantifies the information in the data regarding the relative plausibility of one model and the second one.

$$\text{Posterior Odds} = \text{Bayes Factor} \times \text{Prior Odds} \tag{1}$$

This equation really emphasizes the way that the prior information (prior odds) are combined with the information in the data (Bayes Factor) to give the posterior.

Bayes factor	Evidence for A and against B
>100	decisive
32–100	very strong
10–32	strong
3.2–10	substantial
1–3.2	barely mentionable

Example: identify the origin of a sample based on its DNA

There are two subspecies of African Elephant: savannah and forest elephants, which differ slightly in their genes. Consider measuring them at a single marker (point in their genome) where there are two alleles (types), A and a.

We will assume that allele A occurs at frequency f_S in savannah elephants and at frequency f_F in forest elephants (and that the allele a occurs at frequencies $1 - f_S$ and $1 - f_F$). We impose Beta priors on f_S and f_F: $f_S \sim \text{Beta}(\alpha_S, \beta_S)$ and $f_F \sim \text{Beta}(\alpha_F, \beta_F)$.

Now assume that Interpol have seized an illegally-smuggled tusk, and they measure this marker in DNA from the tusk and find that it carried the A allele.

The question before us is: Which subspecies of elephant did the tusk come from, and how confident should we be in this conclusion?

This is simplified version of a real problem: Interpol and other authorities want to know the origin of poached tusks to help focus efforts on curbing this illegal activity; in practice they are interested in much finer-level discrimination, and measure many genetic markers to get more information.

We consider two models, H_S: the tusk come from savannah elephant and H_F: the tusk come from forest elephant.

$$
\begin{aligned}
\frac{p(H_s|x)}{p(H_f|x)} &= \frac{p(x|H_s)}{p(x|H_f)}\frac{p(H_s)}{p(H_f)} \\
&= \frac{\int p(x|f_S, H_s)p(f_S|H_s)\,df_S}{\int p(x|f_F, H_f)p(f_F|H_f)\,df_F}\frac{p(H_s)}{p(H_f)} \\
&= \frac{\int f_S\, p(f_S|H_s)\,df_S}{\int f_F\, p(f_F|H_f)\,df_F}\frac{p(H_s)}{p(H_f)}
\end{aligned}
$$