MAST30027: Modern Applied Statistics

Week 8 Lab

1. Let $X_1, \cdots, X_n$ be a random sample from a $N(\theta, \sigma^2)$ population, and suppose that the prior distribution on θ is $N(\mu, \tau^2)$. Here we assume that σ^2, μ and τ^2 are all known.

   Last week we showed that the posterior pdf of θ, denoted as $p(\theta|\mathbf{x}, \sigma^2, \mu, \tau^2)$, is normal with mean and variance given by $E(\theta|\mathbf{x}) = \frac{n\tau^2}{n\tau^2+\sigma^2}\bar{x} + \frac{\sigma^2}{n\tau^2+\sigma^2}\mu$ and $\text{Var}(\theta|\mathbf{x}) = \frac{\sigma^2\tau^2}{n\tau^2+\sigma^2}$.

   Considering the squared error loss, find the Bayes estimators of θ and $\theta(\theta-1)$.

2. Consider a random sample X from a Bernoulli distribution with pdf $f(x|\theta) = \theta^x(1-\theta)^{1-x}$. We consider Beta$(1,2)$ as a prior distribution for θ, where Beta$(a,b) \equiv \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}$ and Beta(a,b) distribution has mean $= a/(a+b)$ and variance $= ab/[(a+b)^2(a+b+1)]$.

   Under the squared error loss function,

   (a) Find the Bayes estimator of θ.

   (b) Find the risk of the Bayes estimator of θ.

   (c) Find the Bayes risk of the Bayes estimator of θ.

3. Let $X_1, \cdots, X_n$ be a random sample from an exponential distribution with pdf $f(x|\theta) = \theta e^{-\theta x}$, $x > 0$. (So the mean of X_i is $\frac{1}{\theta}$ instead of θ.) Assume the prior pdf of θ is $p(\theta) = \beta e^{-\beta\theta}$ where β is known.

   (a) Show that the posterior pdf of θ given $\mathbf{x} = (x_1, \cdots, x_n)$ is gamma$[(\beta + \sum_{i=1}^n x_i)^{-1}, n+1]$.

   (b) Using squared error loss, find the Bayes estimator of θ.

   (c) Using squared error loss, find the Bayes estimator of $\mu = \frac{1}{\theta}$. Then find the risk function of this estimator. Further attempt the associated Bayes risk to see whether it is finite or not. If not finite, what are the possible reason(s) behind this? (*Hint*: no unique answers; think about the appropriateness of the prior.)

   (d) Using squared error loss, find the Bayes estimator of $\xi = 2^{-\theta}$.

4. Show that using the loss function

$$L(t;\theta) = \begin{cases} 0, & |t-\theta| < \epsilon \\ c, & o/w \end{cases}$$

   where ϵ is very small and c is large, results in a Bayes estimator approximately equal to the mode of the posterior distribution.

   Moreover, show that if the prior is uniform over an interval containing the MLE, then the posterior mode *is* the MLE.