MAST30027: Modern Applied Statistics

Week 12 Lab Sheet

1. The Poisson distribution has the probability density function (pdf)

$$f(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \text{ for } x = 0, 1, \ldots$$

   (a) Show that the Poisson distribution is an exponential family, identifying the parameters θ and ϕ as well as the functions $b(\theta)$ and $a(\phi)$.

   (b) Obtain the canonical link. Show your work.

   (c) Obtain the variance function. Show your work.

2. Suppose that $Y \sim NB(r, p)$ for some known non-negative integer valued r and unknown p. That is

$$\begin{aligned} P(Y = y) &= \binom{y + r - 1}{r - 1} (1 - p)^r p^y \\ E(Y) &= \frac{pr}{1 - p}. \end{aligned}$$

   If p is the probability of success, then Y is the number of successful trials until the r-th failure. What are the log-likelihood, observed information, and Fisher information for this example?

3. Suppose that $y \sim N(\mu, 1)$. What is the Jeffreys prior for μ?

4. Consider a random sample $X_1, \cdots, X_n$ satisfying $X_i \overset{d}{=} \text{pois}(\theta)$, i.e., $f(x_i|\theta) = \frac{\theta^{x_i} e^{-\theta}}{x_i!}$. To assess an estimator $\hat{\theta} = t(X_1, \cdots, X_n)$ of θ we use the loss function $L(\hat{\theta}; \theta) = \dfrac{(\hat{\theta} - \theta)^2}{\theta}$. We assume a gamma prior pdf $\theta \overset{d}{=} \text{gamma}(\beta, \kappa)$ with known β and κ, i.e. $p(\theta) = \dfrac{1}{\beta^\kappa \Gamma(\kappa)} \theta^{\kappa-1} e^{-\theta/\beta}$; $\theta > 0$.

   (a) Show that the Bayes estimator under the given loss function is $\hat{\theta} = (E[\theta^{-1}|\mathbf{x}])^{-1}$.

   (b) What is the posterior distribution of θ. Show your work.

   (c) Find a closed form for $\hat{\theta}$ in (a) by using the result of (b). Show your work. [Hint: $\Gamma(n) = (n - 1)!$].

5. An expert tells you that the growth rate of a population is between 0.2 and 0.5 with 90% probability. Choose a log-normal distribution to match these prior beliefs.

6. **Metropolis-Hastings** [**We already solved problems (a), (b), and (c) in the week 10. This week, solve the problem (d), (e), and (f).**]

   Recall that $\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, with $\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$ and $\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$, iff $\mathbf{X}$ has joint density

$$f_{\boldsymbol{\mu},\boldsymbol{\Sigma}}(\mathbf{x}) = \frac{1}{2\pi|\boldsymbol{\Sigma}|^{1/2}} \exp\left(-\tfrac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})\right),$$

   where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

   (a) Write an R function that evaluates the density of a bivariate normal distribution. The function should take as input the point $\mathbf{x}$, the mean $\boldsymbol{\mu}$ and the covariance matrix $\boldsymbol{\Sigma}$.

   You will find the functions `solve` and `det` useful.

(b) Write a program in R that uses the Metropolis-Hastings algorithm to generate a sample of size $n = 1000$ from the $N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}\right)$ distribution. Use the symmetric random walk proposal distribution $N\left(\mathbf{x}, \sigma^2 I\right)$ with $\sigma = 2.5$.

Use $\mathbf{X}(0) = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$ as your initial state. Report the proportion of accepted values.

(c) Let $\mathbf{X}(n)$ be the n-th sample point. Plot $X_i(n)$ and the cumulative averages $\overline{X}_i(n) = n^{-1}\sum_{j=1}^{n} X_i(j)$, for $i = 1, 2$. The cumulative averages should give a rough idea of how quickly the $\mathbf{X}(n)$ converge in distribution.

(d) You can use the R functions `cor` or `acf` to estimate the lag 1 autocorrelation ρ. Calculate the so called *effective sample size*: $n(1-\rho)/(1+\rho)$ for each variable.

(e) Change the standard deviation of the proposal chain to $\sigma = 0.1$ and then $\sigma = 10$. How do the proportion of accepted values, the cumulative averages, and the effective sample size change?

(f) Now change $\mathbf{\Sigma}$ to $\begin{pmatrix} 4 & 2.8 \\ 2.8 & 4 \end{pmatrix}$ and repeat the above analysis. How do things change?