# MAST30027: Modern Applied Statistics

## Week 7 Lab

1. Suppose that $\theta \sim \text{beta}(a, b)$ and $X \sim \text{bin}(n, \theta)$. That is, $\theta$ has pdf

$$f_\theta(x) = \beta(a, b)^{-1} x^{a-1}(1-x)^{b-1}$$

   and $X|\theta = \theta$ has the conditional pmf

$$p_{X|\theta}(x) = \binom{n}{x} \theta^x (1-\theta)^{n-x}.$$

   (a) Find the joint pdf/pmf of $\theta$ and $X$, and hence show that the condtional pdf of $\theta|X = x$ is beta($u$,$v$), for suitable $u$ and $v$ (depending on $x$).

   (b) Suppose that $a = 2$ and $b = 3$. On the same graph plot the prior distribution of $\theta$, and posterior distributions corresponding to $n = 4$ and $x = 2$, $n = 10$ and $x = 5$, $n = 20$ and $x = 10$, and $n = 50$ and $x = 5$.

   To plot beta densities in R you can use `curve`:

```
curve(dbeta(x, 2, 3), 0, 1, ylim=c(0,10))
curve(dbeta(x, 4, 5), 0, 1, col="red", add=TRUE)
```

   The beta distribution is called the *conjugate* prior for the binomial, because the posterior is from the same family.

2. Suppose that $\theta$ and $X$ are as above. The marginal distribution of $X$ is given by

$$p_X(x) = \int_0^1 p_{X\theta}(x) f_\theta(\theta) d\theta.$$

   $X$ is said to have a beta-binomial distribution.

   It is possible, but not easy, to work out $p_X$ for a beta-binomial. However, it is easy to estimate it using simulation.

   Generate a sample of size 1000,000 from a beta-binomial with $n = 10$, $a = 2$ and $b = 3$. Use it to estimate the pmf of $X$. The following code will help.

```
> rbeta(5, 2, 3)

[1] 0.2445990 0.1878921 0.3057116 0.2492195 0.3008305

> rbinom(5, 10, c(.1, .2, .1, .3, .3, .1))

[1] 1 3 1 3 3

> table(c(1, 4, 3, 1, 3))

1 3 4
2 2 1
```

3. Let $X_1, \cdots, X_n$ be a random sample from a $N(\theta, \sigma^2)$ population, and suppose that the prior distribution on $\theta$ is $N(\mu, \tau^2)$. Here we assume that $\sigma^2$, $\mu$ and $\tau^2$ are all known.

   (a) Find the joint pdf of $\bar{X}$ and $\theta$.

   (b) Show that the posterior pdf of $\theta$, denoted as $p(\theta|\mathbf{x}, \sigma^2, \mu, \tau^2)$, is normal with mean and variance given by $E(\theta|\mathbf{x}) = \frac{n\tau^2}{n\tau^2+\sigma^2}\bar{x} + \frac{\sigma^2}{n\tau^2+\sigma^2}\mu$ and $\text{Var}(\theta|\mathbf{x}) = \frac{\sigma^2\tau^2}{n\tau^2+\sigma^2}$.

   (c) Show that the marginal pdf of $\bar{X}$ is $N(\mu, \frac{\sigma^2}{n} + \tau^2)$.