③ Big-Theta (or) Θ-notation

A function $t(n)$ is said to be in $\Theta(g(n))$, if $t(n)$ is bounded both above and below by some positive constant multiples of $g(n)$ for all large 'n' i.e.; there exists some positive constants 'c_1' and 'c_2' and some non-negative integer 'n_0' such that

$$c_2 g(n) \leq t(n) \leq c_1 g(n) \quad \text{for all } n \geq n_0$$

$$\Rightarrow \boxed{t(n) \in \Theta(g(n))}$$

eg

$$t(n) = \frac{1}{2} n(n-1)$$

$$g(n) = n^2$$

$$\frac{1}{2} n(n-1) = \frac{1}{2} n^2 - \frac{1}{2} n \leq \frac{1}{2} n^2 \quad , n \geq 0$$

$$\boxed{t(n) \leq c_1 g(n)}$$

$$\Rightarrow \frac{1}{2} n(n-1) \in O(n^2) \text{ where}$$

$c_1 = \frac{1}{2}$

$n_0 = 0$

$$\frac{1}{2}n(n-1) = \frac{1}{2}n^2 - \frac{1}{2}n \geq \frac{1}{2}n^2 - \frac{1}{2}n \cdot \frac{1}{2}n$$

$n \geq 2$

$$\frac{1}{2}n^2 - \frac{1}{2}n \geq \frac{1}{4}n^2$$

$$\boxed{t(n) \geq c_2\, g(n)}$$

$$\frac{1}{2}n(n-1) \in \Omega(n^2)$$

where

$c_2 = \frac{1}{4}$

$n_0 = 2$

$$\Rightarrow \boxed{t(n) \in \Theta(g(n))}$$

where $c_1 = \frac{1}{2}$

$c_2 = \frac{1}{4}$

$n \geq 2$

Hence Proved by definition

Using Limits to compare Orders of growth :-

$$\boxed{\lim_{n\to\infty} \frac{t(n)}{g(n)} = \lim_{n\to\infty} \frac{t'(n)}{g'(n)}} \Rightarrow \text{L'Hôpital's rule}$$

3 principal cases

$$\lim_{n\to\infty} \frac{t(n)}{g(n)} = \begin{cases} 0 & \Rightarrow t(n) \text{ has a smaller order of growth than } g(n) \\ c & \Rightarrow t(n) \text{ has the same order of growth as } g(n) \\ \infty & \Rightarrow t(n) \text{ has a larger order of growth than } g(n) \end{cases}$$

'c' → some constant

eg 1

$t(n) = \frac{1}{2} n(n-1)$

$g(n) = n^2$

$$\lim_{n\to\infty} \frac{t(n)}{g(n)} = \lim_{n\to\infty} \frac{\frac{1}{2} n(n-1)}{n^2}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{2}(n-1)}{n}$$

$$= \frac{1}{2} \lim_{n\to\infty} \left(\frac{n-1}{n}\right)$$

$$= \frac{1}{2} \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)$$

$$= \frac{1}{2}$$

$\boxed{\frac{1}{2} n(n-1) \in \Theta(n^2)}$ Hence Proved using limits eg2 $t(n) = \log_2 n$

$g(n) = \sqrt{n}$

$$\lim_{n\to\infty} \frac{t(n)}{g(n)} = \lim_{n\to\infty} \frac{\log_2 n}{\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{\log_2 e \ \log_e n}{\sqrt{n}}$$

$\left[\because \log_a n = \log_a b \log_b n\right]$

$$= \log_2 e \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}}$$

Applying L'Hôpital's rule,

$$= \log_2 e \lim_{n\to\infty} \frac{(\ln n)'}{(\sqrt{n})'}$$

$$= \log_2 e \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}}$$

$$= \log_2 e \lim_{n\to\infty} \left(\frac{2}{\sqrt{n}}\right)$$

$$= 2\log_2 e \lim_{n\to\infty} \left(\frac{1}{\sqrt{n}}\right)$$

$$= 0$$

$$\boxed{\log_2 n \in O(\sqrt{n})}$$

Hence Proved using limits

eg3

$t(n) = n!$

$g(n) = 2^n$

$$\lim_{n\to\infty} \frac{t(n)}{g(n)} = \lim_{n\to\infty} \frac{n!}{2^n}$$

$$= \lim_{n\to\infty} \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{2^n} \quad \text{(Using Stirling's formula)}$$

$$= \lim_{n\to\infty} \sqrt{2\pi n}\left(\frac{n}{2e}\right)^n = \infty$$

$\boxed{n! \in \Omega(2^n)}$ Hence Proved using limits

Stirling's Formula

$$\boxed{n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n}$$

Properties of Orders of Growth :

1. If $f(n) \in O(g(n))$ then $g(n) \in \Omega(f(n))$
2. If $f(n) \in O(g(n))$ and $g(n) \in O(h(n))$ then
   $f(n) \in O(h(n))$
3. If $f_1(n) \in O(g_1(n))$ and $f_2(n) \in O(g_2(n))$ then
   $f_1(n) + f_2(n) \in O(\max\{g_1(n), g_2(n)\})$

Important Formulas

① $$\sum_{i=l}^{u} 1 = u - l + 1 \quad \text{where } l \leq u$$

② $$\sum_{i=0}^{n} i = \sum_{i=1}^{n} i$$

③ $$\sum_{i=l}^{u} c a_i = c \sum_{i=l}^{u} a_i$$

④ $$\sum_{i=l}^{u} (a_i \pm b_i) = \sum_{i=l}^{u} a_i \pm \sum_{i=l}^{u} b_i$$

MATHEMATICAL ANALYSIS OF NON-RECURSIVE ALGORITHMS:

```
ALGORITHM  MaxElement (A[0 . . . n-1])

    max ← A[0]

    for i ← 1 to n-1 do

            if A[i] > max

                    max ← A[i]

    return max
```

Input size → n
Basic Operation → Key comparison

$$\sum_{i=1}^{n-1} 1 = n-1 \cancel{-1} \cancel{+1}$$

$$= n-1 \in \Theta(n)$$