连续介质力学　第 1 次课笔记

庄逸

课程日期：2019 年 2 月 25 日　　最后修改：2019 年 3 月 24 日

- 经典力学：经过伽利略群变换后不变。

- 相对论力学：经过洛伦兹群变换后不变。

固体力学、流体力学、流变学

Kinetic theory: 动理学，运动机理学　　　Dynamics: 动力学

Multidisciplines, multidisciplinary 横断学科：不仅局限于某一领域，而是横向贯穿于众多领域。

作业：看各种空间（书 P11）

1　玻尔兹曼方程

Boltzmann equation(1872)

$$\mathrm{d}N = \int f(\boldsymbol{r}, \boldsymbol{v}, t) \mathrm{d}^3 \boldsymbol{r} \mathrm{d}^3 \boldsymbol{v} \tag{1}$$

f：PDF, probablity density function. 无量纲函数，概率密度分布

六维相空间 $\mathrm{d}^3\boldsymbol{r}\mathrm{d}^3\boldsymbol{v} = \mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}v_x\mathrm{d}v_y\mathrm{d}v_z$

牛顿决定性原理：给定初始位置和初速度，轨迹确定。

$$\mathrm{d}f = f(\boldsymbol{r} + \boldsymbol{v}\mathrm{d}t, \boldsymbol{v} + \frac{\boldsymbol{F}}{m}\mathrm{d}t, t + \mathrm{d}t) - f(\boldsymbol{r}, \boldsymbol{v}, t) \tag{2}$$

Taylor expansion (1st order)

$$\frac{\partial f}{\partial t}\mathrm{d}t + \frac{\partial f}{\partial \boldsymbol{r}} \cdot \boldsymbol{v}\mathrm{d}t + \frac{\partial f}{\partial \boldsymbol{v}} \cdot \frac{\boldsymbol{F}}{m}\mathrm{d}t = \mathrm{d}f \tag{3}$$

其中 $\frac{\partial f}{\partial \boldsymbol{r}}, \frac{\partial f}{\partial \boldsymbol{v}}$ 是向量。

利用 $\boldsymbol{p} = m\boldsymbol{v}$，

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}} f + \boldsymbol{F} \cdot \boldsymbol{\nabla}_{\boldsymbol{p}} f = \left(\frac{\mathrm{d}f}{\mathrm{d}t} \right)_{collision} = \nu \text{ (frequency)} \tag{4}$$

ν : 碰撞频率，具有频率量纲。

    作业：什么是光滑？

- 当 $M \to 0$（? zx）时，compressible $\to$ imcompressible

- 当 $\mu \to 0$ 时，N-S$\to$ Euler

    李雪梅，苏格兰咖啡馆，café，波兰，巴拿赫利沃夫

感谢为本次笔记提供参考的同学：   包雨晨，卢子璇

连续介质力学　第 2 次课笔记

庄逸

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1　复习

1.1　PDF

f: PDF n is the particle number per unit volume

$$n = \int f \mathrm{d}u \mathrm{d}v \mathrm{d}w \tag{1}$$

If m is the mass of one molecule, then $\rho = nm$

$$N = \int n \mathrm{d}x \mathrm{d}y \mathrm{d}z \tag{2}$$

N: total particle numbers.

1.2　玻尔兹曼方程

$$\int_{momenta} \int_{space} f(\boldsymbol{q}, \boldsymbol{p}, t) \mathrm{d}q_x \mathrm{d}q_y \mathrm{d}q_z \mathrm{d}p_x \mathrm{d}p_y \mathrm{d}p_z \tag{3}$$

phase space：相空间　　　$\mathrm{d}q_x$：广义坐标　　$\mathrm{d}p_x$：momenta 广义动量

$$\frac{\partial f}{\partial t} + \frac{\boldsymbol{p}}{m} \cdot \boldsymbol{\nabla}_{\boldsymbol{q}} f + \boldsymbol{F} \cdot \boldsymbol{\nabla}_{\boldsymbol{p}} f = \frac{\mathrm{d}f}{\mathrm{d}t} \tag{4}$$

$\frac{\mathrm{d}f}{\mathrm{d}t}$: external forces + diffusion + collision，具有频率的量纲

1.3　Others

NS 方程

$$\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \otimes \boldsymbol{v} \right) = -\nabla \boldsymbol{p} + \mu \nabla^2 \boldsymbol{v} + \rho \boldsymbol{g} \tag{5}$$

$\mu\nabla^2\boldsymbol{v}$: momentum diffusion，动量扩散

Lorentz force

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \tag{6}$$

conjugate: 共轭，如应力和应变

2 物理定律可逆性与对称性

multidisciplinary, hierarchy: 级联

Kinematics: 运动学　Kinetic theory: 动理学，运动机理学

Dynamics: 动力学

- Newtonian(1687)

$$\boldsymbol{f} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = m\ddot{x} \tag{7}$$

- Lagrangian(1778)

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial q_\alpha} = 0 \tag{8}$$

- Hamiltonian(1834,1835)

$$\dot{p_\alpha} = -\frac{\partial H}{\partial q_\alpha} \qquad \dot{q_\alpha} = \frac{\partial H}{\partial p_\alpha} \tag{9}$$

- Schrödiner equation

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + v\Psi = i\hbar\frac{\partial \Psi}{\partial t} \tag{10}$$

这些都是可逆的。

涨落：fluctuation　　CPT symmetry reversal：CPT 对称

Boltzmann equation 破坏了时间反演不变性

热力学第二定律：微观保守，宏观耗散。

3 BBGKY 级联和量级分类

见课程讲义。

感谢为本次笔记提供参考的同学：　包雨晨，卢子璇

连续介质力学　第 3 次课笔记

庄逸

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希尔伯特公理：完备性，相容性 (与一致性？)，独立性。

- paradiam：范式；paradiam shift：范式转移

- postulate：公设；axiom：公理

- 对流导数：convective (advective) derivative

- aka: also known as

- steady: 定常的；unsteady: 非定常的

1　对流导数

温度场（标量场）$T(\boldsymbol{r}, t)$

$$\frac{\mathrm{D}T}{\mathrm{D}t} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial \boldsymbol{r}} \cdot \frac{\partial \boldsymbol{r}}{\partial t} = \frac{\partial T}{\partial t} + \boldsymbol{v} \cdot \nabla T \tag{1}$$

$\boldsymbol{v} \cdot \nabla T$ 称为对流导数，是非线性项。$\frac{\mathrm{D}T}{\mathrm{D}t}$ 是拉格朗日描述，$\frac{\partial T}{\partial t}$ 是欧拉描述。

作业：证明 NS 方程的非线性性，参见 P16 例 1.7

$\nabla \otimes \boldsymbol{v}$ 二阶张量

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连续介质力学 第 4 次课笔记

庄逸

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1　线性与非线性方程

· Schödinger's equation, **Linear equation**

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + v\right)\Psi(\boldsymbol{r}, t) = i\hbar\frac{\partial\Psi(\boldsymbol{r}, t)}{\partial t} \tag{1}$$

· Principle of superposition 叠加原理

Gross-pitaevskii equation (Bose-Einstein condensation)

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + v(\boldsymbol{r}) + g\underbrace{|\Psi(\boldsymbol{r}, t)|^2}_{\text{nonlinear}}\right)\Psi(\boldsymbol{r}, t) = i\hbar\frac{\partial\Psi(\boldsymbol{r}, t)}{\partial t} \tag{2}$$

· N-S Equation

$$\rho\left(\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\vec{\nabla})\boldsymbol{v}\right) = -\vec{\nabla}p + \mu\nabla^2\boldsymbol{v} + \rho\boldsymbol{g} \tag{3}$$

2　无量纲数和量纲

Of the order of magnitude: 具有数量级

2.1　Capillary number

毛细数，是黏性力与表面张力之比

$$Ca = \frac{\mu v/L}{\gamma/L} = \frac{\mu v}{\gamma} = \frac{\text{danamic viscosity} \times \text{velocity}}{\text{surface tension}} \tag{4}$$

表面张力：形成单位新面积需要做的功。

$$[\gamma] = \frac{F \cdot L}{A} \sim \frac{F \cdot L}{L^2} \sim \frac{F}{L} \tag{5}$$

2.2 Reynolds number

$$Re = \frac{\rho v^2}{\mu v/L} = \frac{\rho v L}{\mu} = \frac{v L}{\nu} \qquad \nu = \frac{\mu}{\rho} \tag{6}$$

2.3 Newtonian fluid

$$应力 \underbrace{\tau}_{\text{Pa}} = -\mu \underbrace{\frac{\partial v}{\partial y}}_{1/t} \tag{7}$$

Dynamic Viscocity 动力黏度 $\rightarrow [\mu] = [\text{Pa} \cdot s]$
Kinematic Viscosity 运动黏度

$$[\nu] = L^2 T^{-1}, \ [\nu] = [D] \tag{8}$$

D: 扩散系数

Diffusion $< r^2 >= 6Dt, \ < x^2 >= 2Dt$ 注: 尖括号表示成正比？

$$\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla \boldsymbol{v}) \right) = -\nabla p + \mu \nabla^2 \boldsymbol{v} + \rho \boldsymbol{g} \tag{9}$$

作业：证明 $[p] = [\rho v^2] = ML^{-3}L^2T^{-2} = \dots$
导数量纲的计算：

$$\left[\frac{\mathrm{d}y}{\mathrm{d}x} \right] = \frac{[y]}{[x]}, \ \left[\frac{\mathrm{d}^n y}{\mathrm{d}x^n} \right] = \frac{[y]^n}{[x]^n} \tag{10}$$

2.4 Deborah number

$$De = \frac{\tau}{T} = \frac{\text{relaxation time}}{\text{observation time}} \tag{11}$$

(Maxwell) relaxation time：施加一个切应力，在介质中能保持的时间。

$De > 1$, Solid; $De < 1$, Liquid; $De \sim 1$, 流变体！

2.5 Water, e.g.

$\tau \sim 10^{-12}$s。分子间力程 10^{-10}m(Å)
已知

$$Ca = \frac{\mu v}{\gamma} \sim 1, \qquad v \sim \frac{\gamma}{\mu} \tag{12}$$

$\gamma = 0.0738\text{N/m} = 7.38 \times 10^{-3}\text{N/m} \quad \mu \sim 1.02 \times 10^{-3}\text{Pa} \cdot \text{s}.$

$\therefore v \sim 72\text{m/s}$ 是水的毛细波速，量级为 10^2m/s. Hence,

$$\tau = \frac{10^{-10}\text{m}(\text{Å})}{10^2\text{m/s}} \sim 10^{-12}\text{s} \tag{13}$$

和已知的结论相符合。

　　作业：雷诺数是如何出来的 (见书 P325-326)；动力黏度和运动黏度有什么关系，和扩散系数又有什么关系？毛细数为什么是粘性力和表面张力的比。

3　无源场和涡量场

　　涡量

$$\boldsymbol{\Omega} = \nabla \times \boldsymbol{v} = \text{curl } \boldsymbol{v} \tag{14}$$

　　由于

$$\nabla \cdot \boldsymbol{\Omega} = \nabla \cdot (\nabla \times \boldsymbol{v}) = 0 \tag{15}$$

因此涡量场散度为零，为无源场。

3.1　无源场的性质

1. 涡管中任一横截面上的涡通量保持同一常数值。

2. 涡管不能在流体中产生或消失。

(《流体力学》吴望一)

感谢为本次笔记提供参考的同学：　包雨晨，卢子璇

连续介质力学　第 5 次课笔记

庄逸

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1　拉格朗日方程

1788 年，Lagrangian Mechanics

HP: Hamilton's principle, first principle (1834,1835)

1.1　作用量与动能势能

action 作用量

$$S = \int_{t_1}^{t_2} L \mathrm{d}t \qquad L = T - V \tag{1}$$

L : Lagrangian，是个标量

Kinetic energy

$$T = T(\dot{q}, t) = \frac{1}{2} m \dot{q}^2 \tag{2}$$

$\dot{q}$: generalized velocity，广义速度

potential energy for conservative $\boldsymbol{f} = -\boldsymbol{\nabla} V$, $V(q), q$: generalized coordinate

1.2　变分

HP: $\delta S = 0$　　stationary value 驻值，一般是最小

$$S = \int_{t_1}^{t_2} L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \mathrm{d}t \tag{3}$$

$$S + \delta S = \int_{t_1}^{t_2} L(\boldsymbol{q} + \delta\boldsymbol{q}, \dot{\boldsymbol{q}} + \delta\dot{\boldsymbol{q}}, t) \mathrm{d}t \tag{4}$$

$$\delta S = \int_{t_1}^{t_2} [L(\boldsymbol{q} + \delta\boldsymbol{q}, \dot{\boldsymbol{q}} + \delta\dot{\boldsymbol{q}}, t) - L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)] \mathrm{d}t \tag{5}$$

variation 变分；isochronous variation 等时变分

derivative $\frac{\mathrm{d}x}{\mathrm{d}t}$, its variation $\delta\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)$

必须要等时变分的原因：

$$\delta\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) = \frac{\delta(\mathrm{d}x)}{\mathrm{d}t} - \frac{\mathrm{d}x}{} \frac{\delta(\mathrm{d}t)}{(\mathrm{d}t)^2} = \frac{\mathrm{d}(\delta x)}{\mathrm{d}t} - \frac{\mathrm{d}x}{} \frac{\mathrm{d}(\delta t)}{(\mathrm{d}t)^2} \tag{6}$$

$$\delta\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) = \frac{\mathrm{d}}{\mathrm{d}t}(\delta x) \Rightarrow \delta t \equiv 0 \tag{7}$$

进行泰勒展开：

$$\delta S = \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial \boldsymbol{q}} \cdot \delta \boldsymbol{q} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \cdot \delta \dot{\boldsymbol{q}}\right)\mathrm{d}t \tag{8}$$

$\delta \dot{\boldsymbol{q}}$：虚速度

对第二项分部积分，integral by parts.

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \cdot \delta \dot{\boldsymbol{q}} \mathrm{d}t = \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \mathrm{d}(\delta \boldsymbol{q}) \tag{9}$$

$$= \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \cdot \delta \boldsymbol{q} \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \cdot \delta \boldsymbol{q} \tag{10}$$

$\delta \dot{\boldsymbol{q}}$: virtual velcity

$\delta \boldsymbol{q}$: virtual displacement

$\frac{\partial L}{\partial \dot{\boldsymbol{q}}}$: generalized momentum

结合变分的重要性质，起点和终点变分为零：

$$\delta \boldsymbol{q}(t_1) = \delta \boldsymbol{q}(t_2) = 0 \tag{11}$$

于是

$$\delta S = \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) \cdot \delta \boldsymbol{q} \mathrm{d}t = 0 \tag{12}$$

1.3 拉格朗日方程

由 $\delta \boldsymbol{q}$ 任意性，可知拉格朗日方程为

$$\frac{\partial L}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} = 0 \tag{13}$$

或

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \tag{14}$$

其中由于

$$\frac{\partial L}{\partial q_i} = -\frac{\partial V}{\partial q_i} = F_i \tag{15}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{q}_i) = m\ddot{q}_i = F_i \tag{16}$$

和牛顿第二定律等价！

广义坐标可以是角度，广义速度可以是角速度。

作业：写单摆和双摆的拉格朗日量和拉格朗日方程。

2 连续性方程

Continuity equation

$$\frac{\partial p}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0 \tag{17}$$

通过散度展开，前两项合并得到等价形式：

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \, \mathrm{div}\boldsymbol{v} = 0 \tag{18}$$

或者写成相对变化率

$$-\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = \mathrm{div}\boldsymbol{v} \tag{19}$$

在不可压缩时，有

$$\mathrm{div}\boldsymbol{v} = 0 \qquad \frac{\mathrm{D}\rho}{\mathrm{D}t} = 0 \tag{20}$$

sonic boom：音爆

作业：证明弦振动方程是双曲型方程。

感谢为本次笔记提供参考的同学： 包雨晨，卢子璇

连续介质力学　第 6 次课笔记

庄逸

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1　场论 Field theory

Vorticity $\boldsymbol{\Omega} = \nabla \times \boldsymbol{u} = \operatorname{curl} \boldsymbol{u}$

$$\nabla \cdot \boldsymbol{\Omega} = \nabla \cdot (\nabla \times \boldsymbol{u}) \equiv 0 \tag{1}$$

$\forall \phi$ as a scalar in soblev space,

$$\nabla \times (\nabla \phi) = \boldsymbol{0} \tag{2}$$

$\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$: base vectors

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y} \right) \boldsymbol{i} + ()\boldsymbol{j} + ()\boldsymbol{k} = \boldsymbol{0} \tag{3}$$

Helmholtz decomposition representation，分解为一个无旋场和无散场

$$\boldsymbol{u} = \nabla \phi + \nabla \times \boldsymbol{A} \qquad \nabla \times (\nabla \phi) = \boldsymbol{0}, \ \nabla \cdot (\nabla \times \boldsymbol{A}) = 0 \tag{4}$$

orthogonal decomposition 正交分解

2　杂项

连续介质假定，适用范围：作用量 $S \gg \hbar$。当 $S \sim \hbar$ 时，假定不适用！
$\hbar = h/2\pi$，h: plank const
近代连续介质力学始于欧拉。哈密顿原理是力学第一性原理。

3 分析力学解振动方程

$[x]$-dimension, tension T, $[T] = MLT^{-2}$; Line density μ, $[\mu] = ML^{-1}$, Then $c \sim \sqrt{T/\mu}$ （标度关系 scality）

trig (trigonometric) function 三角函数

作业：证明弦振动是双曲型偏微分方程。

书 P29-30

Kinetic energy; potential energy

Lagrange equation($L(q, \dot{q}, t) = L(q, q_t, t)$)

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \tag{5}$$

When $L(q, q_t, q_x, t)$

$$\frac{\partial}{\partial t} \frac{\partial L}{\partial u_t} + \frac{\partial}{\partial x} \frac{\partial L}{\partial u_x} - \frac{\partial L}{\partial u} = 0 \tag{6}$$

作业：写鼓的膜振动方程（加一项 y)

双射：bijective; 范式：paradigm; 反对称：antisymmetry

Epsilon ϵ_{ijk}, or Levi-civita（奇维塔）symbol, or permutation symbol.

4 麦克斯韦方程组

$$\nabla \cdot (\nabla \times \boldsymbol{H}) = \nabla \cdot \left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \right) = \nabla \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} \tag{7}$$

作业：证明麦克斯韦方程是线性的。

用麦克斯韦方程推连续性方程。

电磁张量，一个矩阵。P70：反对称、无迹、六个独立分量。

感谢为本次笔记提供参考的同学： 包雨晨，卢子璇

连续介质力学　第 7 次课笔记

庄逸

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1　诺特定理

Neother's theorem (1918), *symmetry*

1.1　齐次函数

Homogeneous function 齐次函数，具有以下性质

$$f(\lambda x_1.\lambda x_2,\ldots,\lambda x_n) = \lambda^m f(x_1,x_2,\ldots,x_n) \tag{1}$$

m: degree of homogeneity. e.g., Kinetic energy $T = 1/2m\dot{q}^2$

1.2　欧拉齐次函数定理

Euler's homogeneous function theorem
Taking derivative about λ:

$$\frac{\partial f}{\partial(\lambda x_1)}x_1 + \frac{\partial f}{\partial(\lambda x_2)}x_2 + \cdots + \frac{\partial f}{\partial(\lambda x_n)}x_n = m\lambda^{m-1}f(x_1\ldots x_n) \tag{2}$$

Let $\lambda = 1$, then

$$\sum_{i=1}^{n}\frac{\partial f}{\partial x_i} = mf \tag{3}$$

1.3　能量守恒

conservation of energy: t-homogeneous

时间平移不变性：拉格朗日量不显含时间 $L(q_\alpha, \dot{q}_\alpha)$. Total differential:

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \sum_\alpha \left(\frac{\partial L}{\partial q_\alpha} \dot{q}_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha} \ddot{q} \right) \tag{4}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_\alpha} \dot{q}_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha} \ddot{q}_\alpha \tag{5}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_\alpha} \dot{q}_\alpha \right) \tag{6}$$

$$\because \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial q_\alpha} = 0 \tag{7}$$

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} \dot{q}_\alpha \right) = \frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_\alpha \frac{\partial T}{\partial \dot{q}_\alpha} \dot{q}_\alpha \right) \tag{8}$$

$(\because L = T - V)$

$$\therefore \frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_\alpha \frac{\partial T}{\partial \dot{q}_\alpha} \dot{q}_\alpha - L \right) \equiv 0 \tag{9}$$

$$\frac{\partial T}{\partial \dot{q}} = m\dot{q} \qquad \frac{\partial T}{\partial \dot{q}} \dot{q} = m\dot{q}^2 = 2T \tag{10}$$

$$\therefore \sum_\alpha \frac{\partial T}{\partial \dot{q}_\alpha} \dot{q}_\alpha - L = const \tag{11}$$

$$2T - (T - V) = const \qquad T + V = const \tag{12}$$

2 爱因斯坦求和约定

convention centre: 会展中心

Einstein summation convention: 约定

$$\sum_{\alpha=1}^{n} \frac{\partial L}{\partial \dot{q}_\alpha} \dot{q}_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} \dot{q}_\alpha \tag{13}$$

Summation is taken over repeated index.

2.1 梯度、散度、旋度算子

$$\nabla = \frac{\partial}{\partial x_i} \boldsymbol{e_i} \tag{14}$$

- gradient

$$\nabla \phi = \frac{\partial \phi}{\partial x_i} \boldsymbol{e_i} = \frac{\partial \phi}{\partial x} \boldsymbol{i} + \frac{\partial \phi}{\partial y} \boldsymbol{j} + \frac{\partial \phi}{\partial z} \boldsymbol{k} \tag{15}$$

- divergence

$$\nabla \cdot \boldsymbol{A} = \boldsymbol{e_i} \frac{\partial}{\partial x_i} \cdot A_j \boldsymbol{e_j} = \frac{\partial A_j}{\partial x_i} \boldsymbol{e_i} \cdot \boldsymbol{e_j} = A_{j,i} \delta_{ij} = A_{i,i} = \frac{\partial A_i}{\partial x_i} \tag{16}$$

kronecker delta

$$\boldsymbol{e_i} \cdot \boldsymbol{e_j} = \delta_{ij} = \begin{cases} 1, & if \ i = j \\ 0, & if \ i \neq j \end{cases} \tag{17}$$

$$A_{k,k} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{18}$$

- *left* curl

$$\nabla \times \boldsymbol{A} = \frac{\partial}{\partial x_i} \boldsymbol{e_i} \times A_j \boldsymbol{e_j} = \frac{\partial A_j}{\partial x_i} \boldsymbol{e_i} \times \boldsymbol{e_j} = \frac{\partial A_j}{\partial x_i} \varepsilon_{ijk} \boldsymbol{e_k} \tag{19}$$

Epsilon symbol, Levi-civita. even/odd permutation 123, 231, 312

- *right* curl

$$\boldsymbol{A} \times \nabla = A_i \boldsymbol{e_i} \times \frac{\partial}{\partial x_j} \boldsymbol{e_j} = \frac{\partial A_i}{\partial x_j} \boldsymbol{e_i} \times \boldsymbol{e_j} = A_{i,j} \varepsilon_{ijk} \boldsymbol{e_k} \tag{20}$$

作业：证明左旋度和右旋度相反。

$$\boldsymbol{a} \cdot \boldsymbol{b} = (a_i \boldsymbol{e_i}) \cdot (b_j \boldsymbol{e_j}) = a_i b_j \delta_{ij} = a_i b_i \tag{21}$$

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \qquad F_i = q(E_i + \varepsilon_{ijk} v_j B_k) \tag{22}$$

方程两边都有的: free index.

Repeated indices- dummy index/ silent index

2.2 矩阵和双缩并

$$\boldsymbol{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \qquad \boldsymbol{A} = A_{ij} \boldsymbol{e_i} \otimes \boldsymbol{e_j}? \tag{23}$$

trace 迹 $\mathrm{tr}(\boldsymbol{A}) = A_{ii}$

$$\mathrm{tr}\boldsymbol{A} = \boldsymbol{A} : \boldsymbol{I} \qquad \boldsymbol{I} = \delta_{ij} \boldsymbol{e_i} \times \boldsymbol{e_j} = \boldsymbol{e_i} \otimes \boldsymbol{e_i} \ \text{identity tensor} \tag{24}$$

colon product

$$A : I = (A_{ij}\overbrace{e_i \otimes e_j) : (e_k}^{} \otimes e_k) = A_{ij}\delta_{ik}\delta_{jk} = A_{ij}\delta_{ij} = A_{ii} \tag{25}$$

(运算方法：相间)

    <u>作业：证明$A : I = I : A$</u>

2.3   projection tensor of rank two

    n: outward normal

$$f_n = (f \cdot n)n = f \cdot (n \otimes n) \tag{26}$$

$$f_\tau = f - f_n = f \cdot (I - n \otimes n) = f \cdot \mathcal{P} \tag{27}$$

其中点积的点号可去。

感谢为本次笔记提供参考的同学：   无

连续介质力学 第 7 次课笔记

庄逸

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1 诺特定理，动量

conservation of momentum $\delta \boldsymbol{q} = \boldsymbol{\varepsilon}$

variation of Lagrangian, isochronous

$$\delta L = \sum_{\alpha=1}^{n} \frac{\partial L}{\partial \boldsymbol{q}_\alpha} \cdot \delta \boldsymbol{q}_\alpha = \boldsymbol{\varepsilon} \cdot \sum_{\alpha=1}^{n} \frac{\partial L}{\partial \boldsymbol{q}_\alpha} \tag{1}$$

Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{\boldsymbol{q}}_\alpha} - \frac{\partial L}{\partial \boldsymbol{q}_\alpha} = 0 \tag{2}$$

Hence

$$\delta L = \boldsymbol{\varepsilon} \sum_{\alpha=1}^{n} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{\boldsymbol{q}}_\alpha} = \boldsymbol{\varepsilon} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \sum_{\alpha} \frac{\partial L}{\partial \dot{\boldsymbol{q}}_\alpha} = \boldsymbol{\varepsilon} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \sum_{\alpha} \boldsymbol{p}_\alpha = 0 \quad (\because L = T - V, \frac{\partial L}{\partial \dot{\boldsymbol{q}}_\alpha} = \boldsymbol{p}_\alpha) \tag{3}$$

$$\therefore \boldsymbol{p}_{total} = \sum_{\alpha} \boldsymbol{p}_\alpha = const \tag{4}$$

2 张量

$$\text{Prove:} \quad (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{\nabla} \otimes \boldsymbol{v} \tag{5}$$

$$\frac{\partial}{\partial x_j} \boldsymbol{e}_j (v_i \boldsymbol{e}_i) = v_{i,j} \boldsymbol{e}_j \otimes \boldsymbol{e}_i \tag{6}$$

$$\boldsymbol{v} \cdot \boldsymbol{\nabla} \otimes \boldsymbol{v} = (v_k \boldsymbol{e}_k)(v_{i,j} \boldsymbol{e}_j \otimes \boldsymbol{e}_i) \tag{7}$$

$$= v_k v_{i,j} \delta_{kj} \boldsymbol{e}_i = v_j v_{i,j} \boldsymbol{e}_i = v_j \frac{\partial v_i}{\partial x_j} \boldsymbol{e}_i \tag{8}$$

$$= \left(v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial v_2} + v_3 \frac{\partial v_1}{\partial v_3} \right) \boldsymbol{e}_1 + \dots \tag{9}$$

free index & dummy index

2.1 拉普拉斯算子

Laplacian (operator)

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \otimes \boldsymbol{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2 \boldsymbol{v} + \boldsymbol{g} \tag{10}$$

$$Re = \frac{\rho v^2}{\mu v/L} = \frac{\rho v L}{\mu} = \frac{vL}{\nu} \quad [\nu] = \mathrm{L}^2\mathrm{T}^{-1} \quad <\boldsymbol{x}^2> = 2pt \tag{11}$$

$$\nabla^2 = \triangle = \nabla \cdot \nabla = \left(\frac{\partial}{\partial x_i}\boldsymbol{e}_i\right) \cdot \left(\frac{\partial}{\partial x_j}\boldsymbol{e}_j\right) = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\delta_{ij} = \frac{\partial^2}{\partial x_i^2} \tag{12}$$

2.2 弹性力学平衡方程

$$\nabla \cdot \boldsymbol{\sigma} = \boldsymbol{0} \tag{13}$$

$$\left(\frac{\partial}{\partial x_k}\boldsymbol{e}_k\right)(\sigma_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j) = \frac{\partial\sigma_{ij}}{\partial x_k}\delta_{ki}\boldsymbol{e}_j = \frac{\partial\sigma_{ij}}{\partial x_i}\boldsymbol{e}_j \tag{14}$$

作业：什么时候 NS 方程中$-1/\rho \cdot \nabla p$项可将系数放入为$-\nabla(p/\rho)$

2.3 双缩并与迹

$$\boldsymbol{T} = T_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \qquad \boldsymbol{S} = S_{kl}\boldsymbol{e}_k \otimes \boldsymbol{e}_l \tag{15}$$

$$\boldsymbol{T} : \boldsymbol{S} = T_{ij}S_{kl}\delta_{ik}\delta_{jl} = T_{ij}S_{ij} \tag{16}$$

$$\boldsymbol{A} = A_{lm}\boldsymbol{e}_l \otimes \boldsymbol{e}_m \tag{17}$$

$$\mathrm{tr}\boldsymbol{A} = A_{lm}\boldsymbol{e}_l \cdot \boldsymbol{e}_m = A_{lm}\delta_{lm} = A_{ll} \tag{18}$$

作业：证明张量 AB 的迹和转置的各种性质。

感谢为本次笔记提供参考的同学： 无

连续介质力学　第 9 次课笔记

庄逸

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1　诺特定理（角动量守恒）

Angular momentum, isotropy, isotropic：各向同性

infinitesimal rotation$\delta\boldsymbol{\phi}$ (pseudo vector)

$$\delta\boldsymbol{r} = \delta\boldsymbol{\phi} \times \boldsymbol{r} \qquad \delta\boldsymbol{v} = \delta\boldsymbol{\phi} \times \boldsymbol{v} \tag{1}$$

variation of Lagrangian　（注意此处 L 表示拉格朗日量）

$$\delta L = \sum_{\alpha=1}^{n} \left(\frac{\partial L}{\partial \boldsymbol{r}_\alpha} \cdot \delta\boldsymbol{r}_\alpha + \frac{\partial L}{\partial \boldsymbol{v}_\alpha} \cdot \delta\boldsymbol{v}_\alpha \right) = 0 \tag{2}$$

$\because L = T - U = m\dot{q}^2/2 - U(q)$, conservative force $\boldsymbol{f} = -\nabla U$

$$\frac{\partial L}{\partial \boldsymbol{v}_\alpha} = \boldsymbol{p}_\alpha \qquad \frac{\partial L}{\partial \boldsymbol{r}_\alpha} = \boldsymbol{f} = \dot{\boldsymbol{p}}_\alpha \tag{3}$$

代入Equation 1,Equation 3

$$\delta L = \sum_{\alpha=1}^{n} [\dot{\boldsymbol{p}}_\alpha \cdot (\delta\boldsymbol{\phi} \times \boldsymbol{r}_\alpha) + \boldsymbol{p}_\alpha \cdot (\delta\boldsymbol{\phi} \times \boldsymbol{v}_\alpha)] \tag{4}$$

$$= \sum_{\alpha=1}^{n} [\delta\boldsymbol{\phi} \cdot (\boldsymbol{r}_\alpha \times \dot{\boldsymbol{p}}_\alpha) + \delta\boldsymbol{\phi} \cdot (\boldsymbol{v}_\alpha \times \boldsymbol{p}_\alpha)] \tag{5}$$

$$= \delta\boldsymbol{\phi} \cdot \sum_{\alpha=1}^{n} (\boldsymbol{r}_\alpha \times \dot{\boldsymbol{p}}_\alpha + \dot{\boldsymbol{r}}_\alpha \times \boldsymbol{p}_\alpha) \tag{6}$$

$$= \delta\boldsymbol{\phi} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \sum_{\alpha=1}^{n} (\boldsymbol{r}_\alpha \times \boldsymbol{p}_\alpha) = 0 \tag{7}$$

$\because \delta\boldsymbol{\phi}$ is arbitrary　（注意下面 $\boldsymbol{L}$ 表示角动量）

$$\therefore \sum_{\alpha=1}^{n} (\boldsymbol{r}_\alpha \times \boldsymbol{p}_\alpha) = \boldsymbol{L} = \text{const} \tag{8}$$

2 转动惯量

recall: moment of inertia, rigid body 注意此处开始斜体 $\boldsymbol{I}$ 表示转动惯量，为二阶张量。

$$\boldsymbol{L} = \boldsymbol{I}\boldsymbol{\omega} \tag{9}$$

$\boldsymbol{\omega}$: angular velocity

contraction, $\mathbf{I}$: identity tensor

continua (continuum mechanics)

$$\boldsymbol{L} = \int \boldsymbol{r} \times (\rho\boldsymbol{v})\mathrm{d}V = \rho \int \boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})\mathrm{d}V \tag{10}$$

$$= \rho \int [\boldsymbol{\omega}(\boldsymbol{r} \cdot \boldsymbol{r}) - \boldsymbol{r}(\boldsymbol{r} \cdot \boldsymbol{\omega})]\mathrm{d}V \tag{11}$$

$$= \rho \int ((\boldsymbol{\omega}r^2) - (\boldsymbol{r} \otimes \boldsymbol{r}) \cdot \boldsymbol{\omega})\mathrm{d}V \tag{12}$$

$$= \left(\rho \int (r^2\mathbf{I} - \boldsymbol{r} \otimes \boldsymbol{r})\mathrm{d}V \right) \cdot \boldsymbol{\omega} \tag{13}$$

$$\therefore \boldsymbol{I} = \rho \int (r^2\mathbf{I} - \boldsymbol{r} \otimes \boldsymbol{r})\mathrm{d}V \tag{14}$$

$$I_{ik} = \rho \int (r^2\delta_{ik} - r_i r_k)\mathrm{d}V \tag{15}$$

$$I_{xx} = \rho \int (x^2 + y^2 + z^2 - x \cdot x)\mathrm{d}V = \rho \int (y^2 + z^2)\mathrm{d}V \tag{16}$$

$$I_{xy} = -\rho \int xy\,\mathrm{d}V \tag{17}$$

In matrix form

$$[I_{ik}] = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \tag{18}$$

作业：把每个分量写出来，并推导质点系的转动惯量。

2.1 二阶投影张量

$$\boldsymbol{f}_n = (\boldsymbol{f} \cdot \boldsymbol{n})\boldsymbol{n} = \boldsymbol{f} \cdot (\boldsymbol{n} \otimes \boldsymbol{n}) \tag{19}$$

$$\boldsymbol{f}_\tau = \boldsymbol{f} - \boldsymbol{f}_n = \boldsymbol{f} \cdot \mathbf{I} - \boldsymbol{f} \cdot (\boldsymbol{n} \otimes \boldsymbol{n}) = \boldsymbol{f} \cdot (\mathbf{I} - \boldsymbol{n} \otimes \boldsymbol{n}) \tag{20}$$

3 双缩并的重要性质

$$A : (BC) = \operatorname{tr}(A^T(BC)) = \operatorname{tr}((A^T B)C) = (B^T A) : C \tag{21}$$

4 凯莱-哈密顿定理

caylay-hamilton, 特征多项式零化矩阵本身

$$X_A(\lambda) = |\lambda I - A| \qquad X_A(A) = 0 \tag{22}$$

adjoint matrix $B = (\lambda I - A)^\wedge = \lambda^2 B_1 + \lambda B_2 + B_3, B^\wedge B = I |B|$

$$B(\lambda I - A) = |\lambda I - A| = \lambda^3 B_1 + \lambda^2 (B_2 - B_1 A) + \lambda(B_3 - B_2 A) - B_3 A \tag{23}$$

$$= I(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) \tag{24}$$

$$B_1 = I \quad B_2 - B_1 A = a_1 I \quad B_3 - B_2 A = a_2 I \quad -B_3 A = a_3 I \tag{25}$$

$$0 = A^3 + a_1 A^2 + a_2 A + a_3 I \tag{26}$$

$$f(A) = a_0 I + a_1 A + \cdots + a_n A^n + \cdots = b_2 A^2 + b_1 A + b_0 I \tag{27}$$

任何一个矩阵函数都能用二次以内的矩阵多项式表示。因为这里考虑的矩阵都是三阶的，因此极小多项式至多为三次。

作业：整理 C-H 定理

parallelepiped 平行六面体 $[u\ v\ w]$

$\forall A \in Lin\ Lin$:All set of tensors

作业：查张量的四种定义方式

$$I_1(A) = \frac{[Au\ v\ w] + [u\ Av\ w] + [u\ v\ Aw]}{[u\ v\ w]} = \operatorname{tr} A \tag{28}$$

$$I_2(A) = \frac{[Au\ Av\ w] + [Au\ v\ Aw] + [u\ Av\ Aw]}{[u\ v\ w]} = \frac{1}{2}(A_{ii}A_{jj} - A_{ij}A_{ji}) \tag{29}$$

$$I_3(A) = \frac{[Au\ Av\ Aw]}{[u\ v\ w]} = |A| \tag{30}$$

作业：为什么可以用 parallelpipied 写成这样的形式？

感谢为本次笔记提供参考的同学：　无

连续介质力学　第 10 次课笔记

庄逸

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1　曲线坐标

或曲纹坐标，curvilinear coordinates. 坐标系 coordinate system.

直角坐标系，rectangular coordinates, Cartesian coordinates orthogonal.

坐标线 line of coordinates 都是直线

曲纹坐标系：至少一个坐标线是曲线。

triple (x^1, x^2, x^3), unit bases $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$, $\boldsymbol{e}_i \cdot \boldsymbol{e}_k = \delta_{ik}$

1637 年，Rene Descartes，《论方法》创立了一套体系，被西方人认为是一套完整的哲学体系。I think, therefore I am.
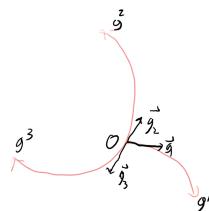
radious vector

$$\boldsymbol{r} = x^1 \boldsymbol{e}_1 + x^2 \boldsymbol{e}_2 + x^3 \boldsymbol{e}_3 = x^i \boldsymbol{e}_i \tag{1}$$

协变基，为矢径对逆变分量求导

$$\boldsymbol{e}_i = \frac{\partial \boldsymbol{r}}{\partial x^i} \tag{2}$$

1852 年, G. Lamé, spherical surface equations of equilibrium. q^1, q^2, q^3, $q^i = q^i(x^1, x^2, x^3)$


Curvilinear cooedinate (according to Equation 2)
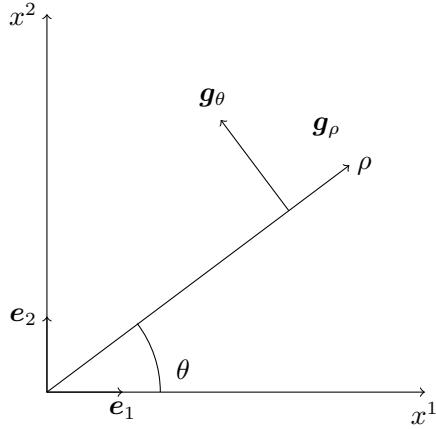
$$\boldsymbol{g}_i = \frac{\partial \boldsymbol{r}}{\partial x^k} \frac{\partial x^k}{\partial q^i} = \frac{\partial x^k}{\partial q^i} \boldsymbol{e}_k \tag{3}$$

归一化，拉梅系数

$$\boldsymbol{b}_i = \frac{\boldsymbol{g}_i}{H_i} \qquad H_i = \sqrt{\left(\frac{\partial x^1}{\partial q^i}\right)^2 + \left(\frac{\partial x^2}{\partial q^i}\right)^2 + \left(\frac{\partial x^3}{\partial q^i}\right)^2} \tag{4}$$

（这里没有求和）

Example polar coordinate.

$$\boldsymbol{r} = x^1 \boldsymbol{e}_1 + x^2 \boldsymbol{e}_2 = \rho \boldsymbol{g}_\rho = \rho\cos\theta \boldsymbol{e}_1 + \rho\sin\theta \boldsymbol{e}_2 \tag{5}$$

$$\begin{cases} \rho = \sqrt{x^2 + y^2} \\ \theta = \arctan\frac{y}{x} = \mathrm{Arg}(x, y) \end{cases} \tag{6}$$

Arg: argument.

$$\boldsymbol{g}_\rho = \frac{\partial x^i}{\partial g^\rho} = \frac{\partial(\rho\cos\theta)}{\partial \rho}\boldsymbol{e}_1 + \frac{\partial(\rho\sin\theta)}{\partial \rho}\boldsymbol{e}_2 = \cos\theta \boldsymbol{e}_1 + \sin\theta \boldsymbol{e}_2 \tag{7}$$

$$\boldsymbol{g}_\theta = \frac{\partial x^i}{\partial g^\theta} = \frac{\partial(\rho\cos\theta)}{\partial \theta}\boldsymbol{e}_1 + \frac{\partial(\rho\sin\theta)}{\partial \theta}\boldsymbol{e}_2 = -\rho\sin\theta \boldsymbol{e}_1 + \rho\cos\theta \boldsymbol{e}_2 \tag{8}$$

Lamé coefficients

$$H_\rho = \sqrt{\cos^2\theta + \sin\theta^2} = 1 \qquad H_\theta = \rho \tag{9}$$

$$\boldsymbol{b}_\rho = \frac{1}{H_\rho}\boldsymbol{g}_\rho = \cos\theta \boldsymbol{e}_1 + \sin_\theta \boldsymbol{e}_2 \tag{10}$$

$$\boldsymbol{b}_\theta = \frac{1}{H_\theta}\boldsymbol{g}_\theta = -\sin\theta \boldsymbol{e}_1 + \cos\theta \boldsymbol{e}_2 \tag{11}$$

$$\left\{ \begin{matrix} \boldsymbol{b}_\rho \\ \boldsymbol{b}_\theta \end{matrix} \right\} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \left\{ \begin{matrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{matrix} \right\} \tag{12}$$

$\boldsymbol{b}_\rho \cdot \boldsymbol{b}_\theta = 1$

velocity in curvilinear coordinates, $q^i = q^i(t)$

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \frac{\partial\boldsymbol{r}}{\partial q^i}\frac{\mathrm{d}q^i}{\mathrm{d}t} = \dot{q}^i\boldsymbol{g}_i = \dot{q}^i H_i \boldsymbol{b}_i \tag{13}$$

作业：在极坐标里写成分量。 $\dot{\rho}\boldsymbol{b}_\rho + \rho\dot{\theta}\boldsymbol{b}_\theta$

projection of acceleration along $\boldsymbol{b}_i$ (注意此处也没有求和)

$$a^i = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \cdot \frac{1}{H_i}\frac{\partial\boldsymbol{r}}{\partial q^i} = \frac{1}{H_i}\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\frac{\partial\boldsymbol{r}}{\partial q^i} \tag{14}$$

$$= \frac{1}{H_i}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{v}\cdot\frac{\partial\boldsymbol{r}}{\partial q^i}\right) - \boldsymbol{v}\cdot\frac{\partial\boldsymbol{v}}{\partial q^i}\right] \tag{15}$$

$$\left(\because \frac{\partial\boldsymbol{r}}{\partial q^i} \equiv \frac{\partial\boldsymbol{v}}{\partial\dot{q}^i}\right) \quad = \frac{1}{H_i}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{v}\cdot\frac{\partial\boldsymbol{v}}{\partial\dot{q}^i}\right) - \boldsymbol{v}\cdot\frac{\partial\boldsymbol{v}}{\partial q^i}\right] \tag{16}$$

Let $T = v^2/2 = \boldsymbol{v}\cdot\boldsymbol{v}/2$

$$a^i = \frac{1}{H_i}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial\dot{q}^i} - \frac{\partial T}{\partial q^i}\right) \tag{17}$$

Example polar coordinate

$$T = \frac{1}{2}\left|\boldsymbol{v}\right|^2 = \frac{\dot{\rho}^2 + \rho^2\dot{\theta}^2}{2} \tag{18}$$

$$a_\rho = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial\dot{\rho}} - \frac{\partial T}{\partial\rho} = \ddot{\rho} - \rho\dot{\theta}^2 \tag{19}$$

$$a_\theta = \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}t}(\rho^2\dot{\theta}) = 2\dot{\rho}\dot{\theta} + \rho\ddot{\theta} \tag{20}$$

作业：用一年级的方法求加速度

2 度规张量

$$g_{ij} = \dot{\boldsymbol{g}}_i \cdot \boldsymbol{g}_j \qquad g_i^k = \boldsymbol{g}_i \cdot \boldsymbol{g}^k \tag{21}$$

注：第二式应该是（δ_i^k？）

$$\boldsymbol{g}_i = \frac{\partial\boldsymbol{r}}{\partial q^i} \qquad \boldsymbol{g}_j = \frac{\partial\boldsymbol{r}}{\partial q^j} \tag{22}$$

$$\mathrm{d}s^2 = \mathrm{d}\boldsymbol{r} \cdot \mathrm{d}\boldsymbol{r} = g_{ij}\mathrm{d}q^i\mathrm{d}q^j \tag{23}$$

3 惯性坐标系的定义

inertial coordinate system

$$\frac{\partial L}{\partial \boldsymbol{r}} = \boldsymbol{0} \tag{24}$$

space is homogeneous & isotropic $L = L(\boldsymbol{v}^2, t)$

time is homogeneous $->$ $L = L(\boldsymbol{v}^2)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^\alpha} - \underbrace{\frac{\partial L}{\partial q^\alpha}}_{=0} = 0 \tag{25}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^\alpha} = 0 \ \Rightarrow \ \frac{\partial L}{\partial \boldsymbol{v}} = \mathrm{const} \tag{26}$$

由于 $\frac{\partial L}{\partial \boldsymbol{v}}$ 是某个只关于 $\boldsymbol{v}$ 的函数，因此

$$\boldsymbol{v} = \mathrm{const} \ vector \tag{27}$$

只要是匀速直线运动的坐标系就是惯性坐标系。

4 伽利略变换群

$$v_x = ..., \ v_y = ..., \ v_z = ... \tag{28}$$

$t \in \mathbb{R}, \boldsymbol{r}, \boldsymbol{v}, \boldsymbol{s} \in \mathbb{R}^3$

$$g_1(t, \boldsymbol{r}) = g_1(t, \boldsymbol{r} + \boldsymbol{v}t) \tag{29}$$

$$g_2(t, \boldsymbol{r}) = g_2(t + \tau, \boldsymbol{r} + \boldsymbol{s}) \tag{30}$$

$$g_3(t, \boldsymbol{r}) = g_3(t, \boldsymbol{r}\boldsymbol{R}) \tag{31}$$

R : 二阶旋转张量

3+1+3+3：十元伽利略群。

感谢为本次笔记提供参考的同学： 无

连续介质力学　第 11 次课笔记

庄逸

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Monge 蒙日

1　证明

注意到 $\boldsymbol{r} = \boldsymbol{r}(q_1, q_2, \ldots, q_n, t)$，按照定义，求导得到速度。

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \frac{\partial \boldsymbol{r}}{\partial t} + \sum \frac{\partial \boldsymbol{r}}{\partial q_i} \frac{\mathrm{d}q_i}{\mathrm{d}t} \tag{1}$$

速度对 $\dot{q}_j$ 求偏导，得到

$$\frac{\partial \boldsymbol{v}}{\partial \dot{q}_j} = \frac{\partial^2 \boldsymbol{r}}{\partial t \partial \dot{q}_j} + \sum \frac{\partial^2 \boldsymbol{r}}{\partial q_i \partial \dot{q}_j} \frac{\mathrm{d}q_i}{\mathrm{d}t} + \sum \frac{\partial \boldsymbol{r}}{\partial q_i} \frac{\partial}{\partial \dot{q}_j} \dot{q}_i \tag{2}$$

因为 $\boldsymbol{r}$ 不显含 $\dot{q}_j$，因此前两项均为零。最后，因为 $\dot{q}_1, \dot{q}_2 \ldots$ 也互不相关，最后一项的求和只有在 $i = j$ 时不为零。

$$\frac{\partial \boldsymbol{v}}{\partial \dot{q}_j} = \sum \frac{\partial \boldsymbol{r}}{\partial q_i} \frac{\partial}{\partial \dot{q}_j} \dot{q}_i \delta_{ij} = \frac{\partial \boldsymbol{r}}{\partial q_j} \frac{\partial \dot{q}_j}{\partial \dot{q}_j} = \frac{\partial \boldsymbol{r}}{\partial q_j} \tag{3}$$

2　第一性原理

First principles

$$\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} \tag{4}$$

Hamilton's principle

$$\delta S = \delta \int_{t_1}^{t_2} L \mathrm{d}t \qquad \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}^\alpha} - \frac{\partial L}{\partial q^\alpha} = 0 \tag{5}$$

Virtual work principle, virtual displacement principle

Real displacement

$$\mathrm{d}\boldsymbol{r}_i(q^1, \ldots, q^n, t) = \frac{\partial \boldsymbol{r}}{\partial t}\mathrm{d}t + \sum_{i=1}^n \frac{\partial \boldsymbol{r}}{\partial q^i}\mathrm{d}q^i \tag{6}$$

$q^1, \ldots, q^n$: generalized coordinates

virtual displacement

$$\delta \boldsymbol{r}_i = \sum \frac{\partial \boldsymbol{r}}{\partial q^i}\delta q^i \tag{7}$$

实质就是等时变分，isochronous variation.

3 虚功原理

$$\sum_i \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i = 0 \tag{8}$$

代入力的分类，作用力 (applied) 和约束力 (constrained).

$$\boldsymbol{F}_i = \boldsymbol{F}_i^{(a)} + \boldsymbol{f}_i^{(c)} \tag{9}$$

得到

$$\sum_i (\boldsymbol{F}_i^{(a)} + \boldsymbol{f}_i^{(c)}) \cdot \frac{\partial \boldsymbol{r}}{\partial q_i}\delta \boldsymbol{q}_i = 0 \tag{10}$$

理想约束：约束反力虚功为零。

$$\sum_i \boldsymbol{f}_i^{(c)} \cdot \frac{\partial \boldsymbol{r}}{\partial q_i}\delta q_i = 0 \qquad Q_i = \boldsymbol{F}_i^{(a)} \cdot \frac{\partial \boldsymbol{r}}{\partial q_i} \qquad \therefore \sum Q_i \delta q_i = 0 \tag{11}$$

把上面的式子代入

$$\boldsymbol{F} - \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = 0 \tag{12}$$

得到，

$$\sum_i \left(\boldsymbol{F}_i^{(a)} - m\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} \right) \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_i}\delta q_i = 0 \tag{13}$$

$$\sum \left(Q_i - m\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_i} \right) \delta q_i = 0 \tag{14}$$

$$m\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\left(m\boldsymbol{v}\frac{\partial \boldsymbol{r}}{\partial q} \right) - m\boldsymbol{v}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \boldsymbol{r}}{\partial q} \tag{15}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left(m\boldsymbol{v}\frac{\partial \boldsymbol{v}}{\partial \dot{q}} \right) - m\boldsymbol{v}\frac{\partial \boldsymbol{v}}{\partial q} \tag{16}$$

$$(\because \; kinetic \; energy \; T = m\boldsymbol{v}^2/2) \quad = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} \tag{17}$$

令 $Q = -\frac{\partial V}{\partial q}, L = T - V$

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial(T-V)}{\partial \dot{q}} - \frac{\partial(T-V)}{\partial q} = 0 \tag{18}$$

D'Almbert's principle.

虚速度原理，虚加速度原理。

4 张量的现代数学观点

modern mathematics point of view

$$f : (V)^r \times (V^*)^s \tag{19}$$

V: vector space $\underbrace{V \times V \times \cdots \times V}_{r}$　　V^*: dual space $\underbrace{V^* \times V^* \times \cdots \times V^*}_{s}$

位移和力组成对偶空间。

Catesian product, direct product 笛卡儿积，直积

Two set X & Y, ordered pair

$$X \times Y = \{(x,y) | x \in X \wedge y \in Y\} \tag{20}$$

f 是 $r+s$ 重线性映射

$$\boldsymbol{T} = T_{ik} \boldsymbol{e}^i \otimes \boldsymbol{e}^k \tag{21}$$

$$\boldsymbol{T} = T_{i_1 i_2 \ldots i_r}^{k_1 k_2 \ldots k_s} \boldsymbol{e}^{i_1} \otimes \boldsymbol{e}^{i_2} \otimes \cdots \otimes \boldsymbol{e}^{i_r} \otimes \boldsymbol{e}_{j_1} \otimes \cdots \otimes \boldsymbol{e}_{j_s} \tag{22}$$

这个张量称为 (r,s) 型，有 $r+s$ 阶，$(0,0)$ 型 scalar, $(1,0)$ 型为 $a_i \boldsymbol{e}^i$, $(0,1)$ 型为 $b^i \boldsymbol{e}_i$

作业：写 (1,1),(2,0),(0,2) 类型的张量

$$\mathrm{tr}\boldsymbol{A} = \boldsymbol{I} : \boldsymbol{A} \tag{23}$$

$$\mathrm{tr}\boldsymbol{A}^2 = \boldsymbol{I} : \boldsymbol{A}^2 = \boldsymbol{A}^T : \boldsymbol{A} \tag{24}$$

作业：P86 三缩并

多重线性是张量的本质, linear map. $\boldsymbol{u} \in V, \boldsymbol{v} \in W, c \in K$　K is a field.

$$f(\boldsymbol{u} + \boldsymbol{v}) = f(\boldsymbol{u}) + f(\boldsymbol{v}) \qquad f(c\boldsymbol{u}) = c(\boldsymbol{u}) \tag{25}$$

homogeneous function of degree 1.

$$f(c_1 \boldsymbol{u}_1 + c_2 \boldsymbol{u}_2 + \cdots + c_n \boldsymbol{u}_n) = c_1 f(\boldsymbol{u_1}) + c_2 f(\boldsymbol{u_2}) + \cdots + c_n f(\boldsymbol{u_n}) \tag{26}$$

感谢为本次笔记提供参考的同学： 无

连续介质力学 第 12 次课笔记

庄逸

课程日期：2019 年 4 月 3 日　　最后修改：2019 年 4 月 14 日

1 黎曼度规

1.1 如何获取坐标——逆变基和协变基

现在我们的坐标系是笛卡尔斜角坐标系，以二维情况为例，假设我们有基底 $\boldsymbol{g}_1, \boldsymbol{g}_2$（画图），则一个向量能够进行分解

$$\boldsymbol{v} = v^1 \boldsymbol{g}_1 + v^2 \boldsymbol{g}_2 \tag{1}$$

为什么写成上标之后再解释。

然后我们想要得到这个向量的分量坐标，也就是 v^1, v^2。

想想在笛卡尔直角坐标系中我们的做法：

$$\boldsymbol{u} = u_x \boldsymbol{i} + u_y \boldsymbol{j} \tag{2}$$

要得到 u_x，我们只需要运算 $\boldsymbol{u} \cdot \boldsymbol{i}$，因为 $\boldsymbol{i} \cdot \boldsymbol{i} = 1, \boldsymbol{i} \cdot \boldsymbol{j} = 0$，剩下的自然只有 u_x。

因此，在斜角坐标系中我们也会想进行运算，结果却得到

$$\boldsymbol{v} \cdot \boldsymbol{g}_1 = v^1 + v^2 \cos \phi \tag{3}$$

其中 ϕ 是两个基底的夹角。这样就不能方便地得到坐标。为了解决这个问题，我们引入另外两个向量 $\boldsymbol{g}^1, \boldsymbol{g}^2$，使得

$$\boldsymbol{g}^1 \cdot \boldsymbol{g}_1 = 1 \qquad \boldsymbol{g}^1 \cdot \boldsymbol{g}_2 = 0 \tag{4}$$

$$\boldsymbol{g}^2 \cdot \boldsymbol{g}_1 = 0 \qquad \boldsymbol{g}^2 \cdot \boldsymbol{g}_2 = 1 \tag{5}$$

或者可以简记为

$$\boldsymbol{g}_i \cdot \boldsymbol{g}^j = \delta_i^j \tag{6}$$

这个是容易实现的。（Figure 1）
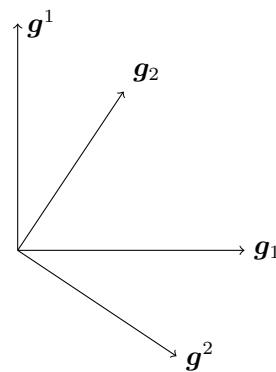
图 1: 逆变基与协变基

我们把 $\boldsymbol{g}_1, \boldsymbol{g}_2$ 称为协变基，$\boldsymbol{g}^1, \boldsymbol{g}^2$ 称为逆变基。v^1, v^2 称为逆变分量。

显然，我们还有协变分量，因为逆变基也是一组基。

$$\boldsymbol{v} = v_1 \boldsymbol{g}^1 + v_2 \boldsymbol{g}^2 \tag{7}$$

因此为了区分协变与逆变分量，并且遵从指标位置规则，就把逆变分量的脚标写成上标。

这些在三维中可以类似推广。之后在三维中讨论，并且采用爱因斯坦求和约定。

1.2　如何得到曲线微元

现在考虑长度微元

$$\mathrm{d}s^2 = \mathrm{d}\boldsymbol{r} \cdot \mathrm{d}\boldsymbol{r} \tag{8}$$

其中 $\mathrm{d}\boldsymbol{r} = r^i \boldsymbol{g}_i$，于是

$$\mathrm{d}s^2 = \mathrm{d}\boldsymbol{r} \cdot \mathrm{d}\boldsymbol{r} = (\mathrm{d}r^i \boldsymbol{g}_i) \cdot (\mathrm{d}r_j \boldsymbol{g}^j) = \mathrm{d}r^i \mathrm{d}r_j \boldsymbol{g}_i \cdot \boldsymbol{g}^j = \mathrm{d}r^i \mathrm{d}r_j \delta_i^j = \mathrm{d}r^i \mathrm{d}r_i \tag{9}$$

然而，这里的 r^i 和 r_i 分别是逆变分量和协变分量，就是说这不在一个坐标系中，不好处理。那么试试在同一个坐标系中求

$$\mathrm{d}s^2 = \mathrm{d}\boldsymbol{r} \cdot \mathrm{d}\boldsymbol{r} = (\mathrm{d}r^i \boldsymbol{g}_i) \cdot (\mathrm{d}r^j \boldsymbol{g}_j) = \mathrm{d}r^i \mathrm{d}r^j \boldsymbol{g}_i \cdot \boldsymbol{g}_j \tag{10}$$

其中出现了 $\boldsymbol{g}_i \cdot \boldsymbol{g}_j$ 因子，我们直接设其值为 g_{ij}，其张成一个矩阵，称为度量张量的协变分量。由定义可知这是一个对称矩阵。于是现在曲线长度微分可表示为

$$\mathrm{d}s^2 = \mathrm{d}r^i \mathrm{d}r^j g_{ij} \tag{11}$$

事实上，这个量的重要之处还在于它联系着逆变基和协变基。尝试用协变基线性表示逆变基，设一个系数

$$\boldsymbol{g}_i = h_{ij}\boldsymbol{g}^j \tag{12}$$

在上式两端点乘 $\boldsymbol{g}_k$，得到

$$\boldsymbol{g}_i \cdot \boldsymbol{g}_k = h_{ij}\boldsymbol{g}^j \cdot \boldsymbol{g}_k = h_{ij}\delta_k^j = h_{ik} \tag{13}$$

可见 h 就是 g。

另有张量被称为"度量张量的逆变分量"，具体的运算性质都是对称类似的。

1.3 几个例子

欧几里得度量（例 15.1） 欧几里得空间中，$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$，如果把三个方向的基向量作为协变基，即 $\boldsymbol{g}_i = \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$，则由于相互垂直，相应的矩阵 g_{ij} 为单位矩阵。

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathrm{d}s^2 = \mathrm{d}x^i\mathrm{d}x^j g_{ij} = (\mathrm{d}x^i)^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{14}$$

坐标变换的度量（基于例 15.5） 在欧几里得空间的基础上，作变换

$$u = x + 2y \qquad v = x - y \qquad w = z \tag{15}$$

相比原坐标系，新坐标系经历了缩放，两轴也不再垂直，故不能直接用 $\mathrm{d}u^2 + \mathrm{d}v^2 + \mathrm{d}w^2$ 直接计算长度，而需要用度规。下面算度规张量的协变分量。

我们已经知道

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{16}$$

因此考虑作逆变换，将 xyz 换为 uvw。简单计算得到

$$x = (u + 2v)/3 \qquad y = (u - v)/3 \qquad z = w \tag{17}$$

于是

$$\mathrm{d}s^2 = \frac{2}{9}\mathrm{d}u^2 + \frac{2}{9}\mathrm{d}u\mathrm{d}v + \frac{5}{9}\mathrm{d}v^2 + \mathrm{d}w^2 \tag{18}$$

利用二次型的表示

$$\mathrm{d}s^2 = (\mathrm{d}u\ \mathrm{d}v\ \mathrm{d}w)g_{ij}(\mathrm{d}u\ \mathrm{d}v\ \mathrm{d}w)^t \tag{19}$$

可以写出度规张量

$$g_{ij} = \begin{pmatrix} \frac{2}{9} & \frac{1}{9} & 0 \\ \frac{1}{9} & \frac{5}{9} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{20}$$

下面我们来计算一个具体的长度。考虑在 xyz 坐标系中，线段 $l : (3, -1, 4) \to (5, 0, 2)$ 显然它的长度是

$$L = \sqrt{(5-3)^2 + (0-(-1))^2 + (2-4)^2} = 3 \tag{21}$$

相对应地，在 uvw 坐标系中，它可以对应地表示成 $l : (1, 4, 4) \to (5, 5, 2)$。我们用积分来算。

首先先写出直线的参数方程

$$u = 1 + 4t \quad v = 4 + t \quad w = 4 - 2t \quad t \in [0, 1] \tag{22}$$

然后积分算长度，注意 $\mathrm{d}s$ 此时带着度量张量

$$L = \int \mathrm{d}s = \int \sqrt{\mathrm{d}s^2} = \int \sqrt{\mathrm{d}x^i \mathrm{d}x^j g_{ij}} \tag{23}$$

$$= \int \sqrt{\frac{2}{9}\mathrm{d}u^2 + \frac{2}{9}\mathrm{d}u\mathrm{d}v + \frac{5}{9}\mathrm{d}v^2 + \mathrm{d}w^2} \tag{24}$$

$$= \int_0^1 \sqrt{\frac{2}{9}\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 + \frac{2}{9}\frac{\mathrm{d}u}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{5}{9}\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}w}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \tag{25}$$

$$= \int_0^1 \sqrt{\frac{2}{9} \times 16 + \frac{2}{9} \times 4 + \frac{5}{9} + 4}\,\mathrm{d}t = \int_0^1 3\mathrm{d}t = 3 \tag{26}$$

算出的长度相同。

1.4 总结

顾名思义，度规就是一把尺子。在简单的欧几里得空间中，因为三个基向量具有很好的例如互相垂直的性质，度规成为单位矩阵，所以计算长度时不需要考虑度规。但是更重要的原因应该是：现在计算的距离就是在欧几里得空间中简单定义的。

而在一般的坐标系中计算"在欧几里得空间中定义的距离"，甚至是由度规本身定义的距离，就需要使用度规，因为此时只有度规才能告诉你在新坐标系中，1 个单位长度等于多少"距离"。考虑上面的例子，如果在新坐标系中直线被扭成了曲线，仍然可以用类似的方法计算。设想把曲线划成无数多个 $\mathrm{d}s$，然后拿着尺子去一段一段量。这个尺子就是度规，能告诉你这一小段应该算多少距离，最后将它们加起来。这就是得出总长度的线积分过程，可以看出度规的名字是很形象的。

在上面的例子中，我们先由坐标变换得出了线微元的展开式，然后得到度规，在计算长度时又将度规代回去得到线微元展开式进行计算。乍看起来求度规有点累赘，但是其实这二者可以说是等价的，而度规不包含装饰性的微元 $\mathrm{d}x$，能集中反映空间和距离的性质，并且作为一个张量，和许多量有联系，因此度规本身更有研究的价值。

1.5 参考

赵老师的《理性力学教程》和黄克智《张量分析》

2 球量和偏量

Cartesian product 笛卡儿积，direct product（直积）

Newtonian mechanics, Newton's principle of determinacy

$$\boldsymbol{x}_0 \in \mathbb{R}^{3n} \qquad \dot{\boldsymbol{x}}_0 \in \mathbb{R}^{3n} \qquad t \in \mathbb{R} \tag{27}$$

$\mathbb{R}^{3n} \times \mathbb{R}^{3n} \times \mathbb{R} \to \mathbb{R}^{3n}$

spherical devitoric tensor

$$\alpha \mathbf{I} = \frac{1}{3}(\text{tr}\boldsymbol{A})\mathbf{I} = \frac{1}{3}(\mathbf{I} : \boldsymbol{A})\mathbf{I} = \frac{1}{3}\mathbf{I}(\mathbf{I} : \boldsymbol{A}) = \frac{1}{3}(\mathbf{I} \otimes \mathbf{I}) : \boldsymbol{A} \tag{28}$$

$$\text{dev}\boldsymbol{A} = \boldsymbol{A} - \alpha\mathbf{I} = \boldsymbol{A} - \frac{1}{3}(\mathbf{I} \otimes \mathbf{I}) : \boldsymbol{A} = (\mathbb{I} - \frac{1}{3}\mathbf{I} \otimes \mathbf{I}) : \boldsymbol{A} = \mathbb{P} : \boldsymbol{A} \tag{29}$$

和二阶投影张量相比。$\boldsymbol{F}_\tau = \boldsymbol{F}(\mathbf{I} - \boldsymbol{n} \otimes \boldsymbol{n})$

四阶单位张量 4th order identity tensor

$$\mathbb{I} = \delta_{ik}\delta_{jl}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \tag{30}$$

口诀：1324（ikjl）。

和单位矩阵的并矢相区分：（指标 1234）

$$\mathbf{I} \otimes \mathbf{I} = \delta_{ij}\delta_{kl}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \tag{31}$$

$$\mathbb{I} : \mathbf{I} = (\delta_{ik}\delta_{jl}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l) : (\delta_{pq}\boldsymbol{e}_p\boldsymbol{e}_q) \tag{32}$$

$$= \delta_{ik}\delta_{jl}\delta_{kp}\delta_{lq}\delta_{pq}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{33}$$

$$= \delta_{ip}\delta_{jq}\delta_{pq}\boldsymbol{e}_i\boldsymbol{e}_j \tag{34}$$

$$= \delta_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{35}$$

作业：算四阶单位张量和二阶张量的双点积 (P123)，算柯西应力张量

stress measure 应力度量, strain measure 应变度量

感谢为本次笔记提供参考的同学： 无

连续介质力学 第 13 次课笔记

庄逸

课程日期：2019 年 4 月 8 日　　最后修改：2019 年 4 月 28 日

1　勒让德变换

Legendre transformation

$$H = \sum_{i=1}^{n} p_i \dot{q}_i - L \tag{1}$$

$$= \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i} \dot{q}_i - L \tag{2}$$

$$= 2T - L = 2T - (T - V) \tag{3}$$

$$= T + V = E \tag{4}$$

Euler homogeneous function

$$f(\alpha \boldsymbol{u}) = \alpha f(\boldsymbol{u}), \qquad T = \frac{1}{2} m \dot{q}^2 \tag{5}$$

2　狄拉克符号

$\langle$ 态 $2|$ 态 $1\rangle$，从右往左读。

1. 拆分：左矢 $\langle$ 态 $2|$ 和右矢 $|$ 态 $1\rangle$。

2. 组分：$\langle 3|2\rangle$, $\langle 2|1\rangle$。

3. 跳转：$\langle 2|\, Q\, |1\rangle$。

4. 前后变换取复数共轭：$\langle 2|1\rangle = \langle 1|2\rangle^*$

证明：

$$\langle A| = |A\rangle^* \qquad |A\rangle = (a_1, \ldots, a_n)^t \qquad \langle A| = (a_1^*, \ldots, a_n^*) \tag{6}$$

$$|A\rangle^* = (a_1, \ldots, a_n)^{t*} = (a_1^*, \ldots, a_n^*) = \langle A| \tag{7}$$

$$\langle B\,|A\rangle = (b_1^*, \ldots, b_n^*)(a_1, \ldots, a_n)^t = \sum_{i=1}^{n} b_i^* a_i \tag{8}$$

$$\langle B\,|A\rangle^* = \left(\sum_{i=1}^{n} b_i^* a_i\right)^* = \sum_{i=1}^{n} a_i^* b_i = \langle A|\,B\rangle \tag{9}$$

bracket:bra-ket

应用：应力集中（裂尖）、奇点

作业：N-S 方程（薛定谔方程?）用狄拉克符号写

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi = i\hbar\frac{\partial\Psi}{\partial t} \tag{10}$$

3 谱定理

3x3 (3 by 3) matrix $\boldsymbol{S}$, eigenvalue and eigenvector, diagonal, diagonized

美国最主流的连续介质力学教材：*Gurtin M E. An Introduction to continuum mechanics*

Spectral theorem

$$\boldsymbol{S}\boldsymbol{e} = w\boldsymbol{e} \;\Rightarrow\; (\boldsymbol{S} - w\mathrm{I})\boldsymbol{e} = \boldsymbol{0} \tag{11}$$

I: identity tensor of rank two

nontrivil solution if and only if(iff)

$$\begin{vmatrix} S_{11} - \omega & S_{12} & S_{13} \\ S_{21} & S_{22} - \omega & S_{23} \\ S_{31} & S_{32} & S_{33} - \omega \end{vmatrix} = 0 \tag{12}$$

characteristic equation $\omega^3 - \boldsymbol{I}_1\omega^2 + \boldsymbol{I}_2\omega - \boldsymbol{I}_3 = 0$, three distinct eigenvalues $\omega_1 \neq \omega_2 \neq \omega_3 \neq 0$; two distinct eigenvalues $\omega_1, \omega_2 = \omega_3$.

Spectral theorem: let $\boldsymbol{S}$ be symmetric. Then thesis on orthonormal basis for V (vector space) consisting entirely of eigenvectors of $\boldsymbol{S}$. Moreover, for any such basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$, the corresponding eigenvalues $\omega_1, \omega_2, \omega_3$, when ordered, form the entire spectrum of $\boldsymbol{S}$，abd(?)

$$\boldsymbol{S} = \sum \omega_i \boldsymbol{e}_i \otimes \boldsymbol{e}_i \tag{13}$$

4　矩阵的不变量

principle invariants

$$\boldsymbol{I}_1 = \text{tr}\boldsymbol{S} = \mathbf{I} : \boldsymbol{S} = S_{ii} = \omega_1 + \omega_2 + \omega_3 \tag{14}$$

$$\boldsymbol{I}_2 = \frac{1}{2}(\text{tr}^2\boldsymbol{S} - \text{tr}\boldsymbol{S}^2) = \omega_1\omega_2 + \omega_2\omega_3 + \omega_3\omega_1 \tag{15}$$

$$\boldsymbol{I}_3 = \frac{1}{6}(\text{tr}^3\boldsymbol{S} - 3\text{tr}\boldsymbol{S}\ \text{tr}\boldsymbol{S}^2 + 2\text{tr}\boldsymbol{S}^3) = \omega_1\omega_2\omega_3 \tag{16}$$

作业：P92 中间，把矩阵写成谱定理的形式

主应力 $\boldsymbol{T}\boldsymbol{n} = \sigma\boldsymbol{n}$

作业：理解 13-10

5　黑森算子

见书 P95-97

作业：由 14-13 推 14-14

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连续介质力学　第 14 次课笔记

庄逸

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主方向和主应力

$$\boldsymbol{S}\boldsymbol{e} = \omega\boldsymbol{e} \tag{1}$$

$\boldsymbol{e}$: unit vector

1　Square root theorem

如果 C 可以进行谱分解，又是 U 的平方

$$\boldsymbol{C} = \boldsymbol{U}^2 = \sum_i \omega_i \boldsymbol{e}_i \otimes \boldsymbol{e}_i \tag{2}$$

则

$$\boldsymbol{U} = \sum_i \sqrt{\omega_i}\boldsymbol{e}_i \otimes \boldsymbol{e}_i \tag{3}$$

2　Polar decomposition theorem

There exist positive definite, symmetrical tensors $\boldsymbol{U}$ and $\boldsymbol{V}$, and a rotation $\boldsymbol{R}$, such that (左极分解和右极分解)

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R} \tag{4}$$

Moreover, each of these decompositions is unique: in fact

$$\boldsymbol{U} = \sqrt{\boldsymbol{F}^T\boldsymbol{F}} \qquad \boldsymbol{V} = \sqrt{\boldsymbol{F}\boldsymbol{F}^T} \tag{5}$$

Proof.(1)

$$\because \boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} \qquad \boldsymbol{F}^T = \boldsymbol{U}^T\boldsymbol{R}^T = \boldsymbol{U}\boldsymbol{R}^{-1} \tag{6}$$

$$\therefore \boldsymbol{F}^T\boldsymbol{F} = \boldsymbol{C} = \boldsymbol{U}\boldsymbol{R}^{-1}\boldsymbol{R}\boldsymbol{U} = \boldsymbol{U}^2 \tag{7}$$

类似可证另一个。

为什么 $\mathbb{I}^T : A = A^T$，并且是 2314

$$(\delta_{jk}\delta_{il}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l) : (A_{pq}\boldsymbol{e}_p \otimes \boldsymbol{e}_q) = \delta_{jk}\delta_{il}\delta_{kp}\delta_{lq}A_{pq}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = A_{ji}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = A^T \tag{8}$$

3 黎曼度规

$$\ddot{x}^i + \Gamma^i_{kl}\dot{x}^k\dot{x}^l = F^i/m \tag{9}$$

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\mathrm{d}}{\mathrm{d}t} + \Gamma v v \tag{10}$$

作业：看 19-20 讲

作业：找到黎曼度规张量和拉梅系数的关系（开方？）

3.1 克里斯托费尔符号

(参考第 16 节 P108)

协变基 $\hat{\boldsymbol{i}}_1, \hat{\boldsymbol{i}}_2, \hat{\boldsymbol{i}}_3$，逆变坐标 $\tilde{x}^1, \tilde{x}^2, \tilde{x}^3$。在位置 $\boldsymbol{r}$ 基矢为 $\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3$，其中

$$\boldsymbol{g}_i = \frac{\partial \boldsymbol{r}}{\partial x^i} = \frac{\partial \boldsymbol{r}}{\partial \tilde{x}^p}\frac{\partial \tilde{x}^p}{\partial x_i} = \hat{\boldsymbol{i}}_p \frac{\partial \tilde{x}^p}{\partial x^i} \qquad x^i = x^i(\tilde{x}^p)$$

2nd kind Christoff symbol.

$$\frac{\partial \boldsymbol{g}_j}{\partial x^i} = \Gamma^k_{ij}\boldsymbol{g}_k \tag{11}$$

对于平直的笛卡儿坐标系而言，为零；但对 Curvilinear 而言不为零。因此克里斯托弗符号不是张量，因为它在坐标变换下会改变。

由于 $\boldsymbol{g}_j = \frac{\partial \boldsymbol{r}}{\partial x^j}$

$$\frac{\partial \boldsymbol{g}_j}{\partial x^i} = \frac{\partial}{\partial x^i}\left(\frac{\partial \boldsymbol{r}}{\partial x^j}\right) = \frac{\partial^2 \boldsymbol{r}}{\partial x^i \partial x^j} = \frac{\partial^2 \boldsymbol{r}}{\partial x^j \partial x^i} \tag{12}$$

于是 $\Gamma^k_{ij} = \Gamma^k_{ji}$，是对称的。

对于非对称三阶张量，共有 $3^3 = 27$ 个自由度。对于对称的而言，只有 18 个

作业：证明三阶 n 维对称张量自由度为

$$\frac{N^2(N+1)}{2} \tag{13}$$

逆变基矢量对坐标导数例 16.3

$$\frac{\partial \boldsymbol{g}^i}{\partial x^j} = -\Gamma^i_{jp}\boldsymbol{g}^p \tag{14}$$

3.2 加速度

$$\boldsymbol{u} = u^i \boldsymbol{g}_i \tag{15}$$

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \frac{\mathrm{d}u^i}{\mathrm{d}t}\boldsymbol{g}_i + u^m \frac{\partial \boldsymbol{g}_m}{\partial x^j} \frac{\mathrm{d}x^j}{\mathrm{d}t} = \frac{\mathrm{d}u^i}{\mathrm{d}t}\boldsymbol{g}_i + u^m v^j \Gamma^i_{jm}\boldsymbol{g}_i = \left(\frac{\mathrm{d}u^i}{\mathrm{d}t} + u^m v^j \Gamma^i_{jm} \right)\boldsymbol{g}_i \tag{16}$$

$$= \frac{\mathrm{D}u^i}{\mathrm{D}t} \tag{17}$$

（注：按照定义，下标应该是 mj，但是由于是对称的所以没关系。

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连续介质力学　第 15 次课笔记

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1　黑洞无毛定理

Black hole no-hair theorem (S. Hawkin, 1972)

hair : complicated informations

almost no-hair, 3 hairs : mass, angular momentum, electric charge

2　张量标量函数的微分

scalar-valued function

strain energy 应变能, extensive quantity $w = w(\boldsymbol{E})$，应变能求导能得到应力。

2.1　Banach Space

（类比社会 = 人群 + 关系）

Topological space » Metric space (distance) » normed vector space » inner product » $\mathbb{R}^n$

范数: norm　　　完备性: completeness

Banach Space ：完备赋范向量空间

Soblev Space

内积：$\langle \boldsymbol{A}, \boldsymbol{B} \rangle = |\boldsymbol{A}|\,|\boldsymbol{B}| \cos\theta$

2.2　一些符号

$f(\boldsymbol{A}), \frac{\partial f(\boldsymbol{A})}{\partial \boldsymbol{A}}$ derivative, 导数。写成偏导的原因是 $\boldsymbol{A}$ 有很多分量。

Tensor of rank two.

1. Lin : the set of all tensors. Multi-**lin**earity, **lin**ear map

$$f(\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_n \boldsymbol{u}_n) = \alpha_1 f(\boldsymbol{u}_1) + \cdots + \alpha_n f(\boldsymbol{u}_n) \tag{1}$$

2. Lin$^+$: the set of all tensors with $\det \boldsymbol{A} > 0$

3. Sym : Symmetrical tensors

4. P_{Sym} : Symmetrical, **p**ositive-definite

5. orth ：orthogonal tensors. $\boldsymbol{Q}^{-1} = \boldsymbol{Q}^T$

6. orth$^+$: rotation $\det(\boldsymbol{Q}\boldsymbol{Q}^T) = \det \boldsymbol{Q}^2 = 1$

2.3 矩阵的不变量

Taylor expansion $\forall \boldsymbol{A}, \boldsymbol{B} \in \text{Lin}$, infinitesimal $\varepsilon \in \mathbb{R}$

$$f(\boldsymbol{A} + \alpha \boldsymbol{B}) = f(\boldsymbol{A}) + \frac{\partial f}{\partial \boldsymbol{A}} : (\alpha \boldsymbol{B}) + o(\alpha^2 \boldsymbol{B}^2) \tag{2}$$

$$\frac{\partial f(\boldsymbol{A})}{\partial \boldsymbol{A}} : \boldsymbol{B} = \lim_{\alpha \to 0} \frac{f(\boldsymbol{A} + \alpha \boldsymbol{B}) - f(\boldsymbol{A})}{\alpha} = \left. \frac{\mathrm{d} f(\boldsymbol{A} + \alpha \boldsymbol{B})}{\mathrm{d} \alpha} \right|_{\alpha = 0} \tag{3}$$

$\forall \boldsymbol{A} \in \text{Lin}$, characteristic equation 特征方程

$$|\boldsymbol{A} - \omega \mathbf{I}| = \begin{vmatrix} A_{11} - \omega & A_{12} & A_{13} \\ A_{21} & A_{22} - \omega & A_{23} \\ A_{31} & A_{32} & A_{33} - \omega \end{vmatrix} = -\omega^3 + I_1 \omega^2 - I_2 \omega + I_3 = 0 \tag{4}$$

$I_1 = \text{tr}\boldsymbol{A} = A_{ii} = \mathbf{I} : \boldsymbol{A} = \omega_1 + \omega_2 + \omega_3$

$$\frac{\partial (\text{tr}\boldsymbol{A})}{\partial \boldsymbol{A}} : \boldsymbol{B} = \lim_{\varepsilon \to 0} \frac{\mathbf{I} : (\boldsymbol{A} + \varepsilon \boldsymbol{B}) - \mathbf{I} : \boldsymbol{A}}{\varepsilon} = \mathbf{I} : \boldsymbol{B} \tag{5}$$

$$\therefore \frac{\partial (\text{tr}\boldsymbol{A})}{\partial \boldsymbol{A}} = \mathbf{I} \tag{6}$$

$I_2 = \frac{1}{2}[\text{tr}^2\boldsymbol{A} - \text{tr}(\boldsymbol{A}^2)] = \omega_1 \omega_2 + \omega_1 \omega_3 + \omega_2 \omega_3$

$$\frac{\partial (\text{tr}\boldsymbol{A})^2}{\partial \boldsymbol{A}} = 2(\text{tr}\boldsymbol{A}) \frac{\partial \text{tr}\boldsymbol{A}}{\partial \boldsymbol{A}} \tag{7}$$

$$\frac{\partial \mathrm{tr}(\boldsymbol{A^2})}{\partial \boldsymbol{A}} : \boldsymbol{B} = \lim_{h \to 0} \frac{\mathbf{I} : (\boldsymbol{A} + h\boldsymbol{B})^2 - \mathbf{I} : \boldsymbol{A}^2}{h} \tag{8}$$

$$= \lim_{h \to 0} \frac{\mathbf{I} : (\boldsymbol{A}^2 + h\boldsymbol{AB} + h\boldsymbol{BA} + h^2\boldsymbol{B}^2) - \mathbf{I} : \boldsymbol{A}^2}{h} \tag{9}$$

$$= \lim_{h \to 0} \frac{\mathbf{I} : (h(\boldsymbol{AB} + \boldsymbol{BA})) + h^2\mathbf{I} : \boldsymbol{B}^2}{h} \tag{10}$$

$$= \mathbf{I} : (\boldsymbol{AB} + \boldsymbol{BA}) = \mathbf{I} : (\boldsymbol{AB}) + \mathbf{I} : (\boldsymbol{BA}) \tag{11}$$

$$= 2\boldsymbol{A}^T : \boldsymbol{B} \tag{12}$$

注意 $(\boldsymbol{A} + h\boldsymbol{B})^2$ 的展开。最后利用了关系式

$$\boldsymbol{A} : (\boldsymbol{BC}) = (\boldsymbol{B}^T\boldsymbol{A}) : \boldsymbol{C} = (\boldsymbol{AC})^T : \boldsymbol{B} \tag{13}$$

作业：计算$\frac{\partial (\mathrm{tr}\boldsymbol{A})^n}{\partial \boldsymbol{A}}$ 和 $\frac{\partial \mathrm{tr}\boldsymbol{A}^3}{\partial \boldsymbol{A}}$

$I_3 = \det\boldsymbol{A} = \omega_1 \omega_2 \omega_3$

parallelpiped volume $[\boldsymbol{u}\ \boldsymbol{v}\ \boldsymbol{w}] = (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$

$\forall \boldsymbol{A} \in \mathrm{Lin}, I_3 = \det\boldsymbol{A}$

考虑性质 $\det(\boldsymbol{AB}) = \det\boldsymbol{A}\det\boldsymbol{B}$，$\det(\varepsilon\boldsymbol{A}) = \varepsilon^3 \det\boldsymbol{A}$

$$\frac{\partial (\det\boldsymbol{A})}{\partial \boldsymbol{A}} : \boldsymbol{B} = \lim_{\varepsilon \to 0} \frac{\det(\boldsymbol{A} + \varepsilon\boldsymbol{B}) - \det\boldsymbol{A}}{\varepsilon} \tag{14}$$

其中

$$\det(\boldsymbol{A} + \varepsilon\boldsymbol{B}) = \det[\varepsilon\boldsymbol{A}(\varepsilon^{-1}\mathbf{I} + \boldsymbol{A}^{-1}\boldsymbol{B})] = \varepsilon^3 \det\boldsymbol{A}\det(\varepsilon^{-1}\mathbf{I} + \boldsymbol{A}^{-1}\boldsymbol{B}) \tag{15}$$

于是

$$\frac{\partial (\det\boldsymbol{A})}{\partial \boldsymbol{A}} : \boldsymbol{B} = \lim_{\varepsilon \to 0} \frac{\varepsilon^3 \det\boldsymbol{A}\det(\varepsilon^{-1}\mathbf{I} + \boldsymbol{A}^{-1}\boldsymbol{B}) - \det\boldsymbol{A}}{\varepsilon} \tag{16}$$

再利用Equation 4

$$\det(\boldsymbol{A}^{-1}\boldsymbol{B} + \varepsilon^{-1}\mathbf{I}) = \varepsilon^{-3} + I_1(\boldsymbol{A}^{-1}\boldsymbol{B})\varepsilon^{-2} + I_2()\varepsilon^{-1} + I_3() \tag{17}$$

$$\frac{\partial (\det\boldsymbol{A})}{\partial \boldsymbol{A}} : \boldsymbol{B} = \lim_{\varepsilon \to 0} \frac{\varepsilon(\det\boldsymbol{A})I_1(\boldsymbol{A}^{-1}\boldsymbol{B}) + \varepsilon^2(\dots) + \varepsilon^3(\dots)}{\varepsilon} \tag{18}$$

$$= (\det\boldsymbol{A})(\mathbf{I} : (\boldsymbol{A}^{-1}\boldsymbol{B})) \tag{19}$$

$$= (\det\boldsymbol{A})(\boldsymbol{A}^{-T} : \boldsymbol{B}) \tag{20}$$

$$\therefore \frac{\partial (\det\boldsymbol{A})}{\partial \boldsymbol{A}} = (\det\boldsymbol{A})\boldsymbol{A}^{-T} \tag{21}$$

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连续介质力学　第 16 次课笔记

庄逸

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1　复习

principal stress

复习转动惯量，$L = m\boldsymbol{v} \times \boldsymbol{r} = m(\boldsymbol{\omega} \times \boldsymbol{r}) \times \boldsymbol{r}$

复习四阶投影张量。$\alpha = \mathrm{tr}/3 = \mathrm{I} : \boldsymbol{A}/3, \mathrm{dev}\boldsymbol{A} = \boldsymbol{A} - \alpha\mathbf{I}$

$$\overline{\dot{\boldsymbol{A^{-1}}}} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{A}^{-1}) \tag{1}$$

$$\overline{\dot{\boldsymbol{A^{-1}A}}} = \dot{\mathbf{I}} = \mathbf{0} \tag{2}$$

$$\therefore \overline{\dot{\boldsymbol{A^{-1}}}}\boldsymbol{A} + \boldsymbol{A}^{-1}\dot{\boldsymbol{A}} = \mathbf{0} \tag{3}$$

$$\therefore \overline{\dot{\boldsymbol{A^{-1}}}} = -\boldsymbol{A}^{-1}\dot{\boldsymbol{A}}\boldsymbol{A}^{-1} \tag{4}$$

应力方向和大小的特性：满足静力平衡条件，即 $\sum F_x = \sum F_y = \sum M_O = 0$

Lin, Lin$^+$ $\rightarrow$ det $\boldsymbol{A} > 0$

Sym, P_{sym} positive definite

Orthogonal $\boldsymbol{Q}^{-1} = \boldsymbol{Q}^T, \det \boldsymbol{Q}\boldsymbol{Q}^T = \det \mathrm{I} = 1$ orth$^+$ $\rightarrow$ det $\boldsymbol{Q} > 0$

复习左极分解和右极分解。

2　克里斯托费符号

期中考题：怎么在曲线坐标系中写运动方程？（16-18）

$$a^i = \frac{\mathrm{D}v^i}{\mathrm{D}t} = \frac{\mathrm{d}v^i}{\mathrm{d}t} + v^m v^k \Gamma^i_{mk} \tag{5}$$

configuration space 位形空间

补充例 16.4

$$L = g_{\mu\nu} \frac{\mathrm{d}q^{\mu}}{\mathrm{d}\lambda} \frac{\mathrm{d}q^{\nu}}{\mathrm{d}\lambda} \tag{6}$$

$$\frac{\partial L}{\partial \dot{q}^{k}} = g_{\mu\nu} \delta^{\mu k} \frac{\mathrm{d}q^{\nu}}{\mathrm{d}\lambda} + g_{\mu\nu} \frac{\mathrm{d}q^{\mu}}{\mathrm{d}\lambda} \delta_{\nu k} = g_{k\nu} \frac{\mathrm{d}g^{\nu}}{\mathrm{d}\lambda} + g_{\mu k} \frac{\mathrm{d}g^{\mu}}{\mathrm{d}\lambda} \tag{7}$$

Curvature 有了曲率，物体就要运动。时空的弯曲告诉物体如何运动，物体告诉时空如何弯曲。

3 两种构型

$t = 0$ undeformed configuration, initial configuration 初始构形, reference configuration 参考构型.

"清华的学生都用'型'，当我年轻的时候我也用'型'，现在我坚定地用'形'。"

$t = t$ current configuration 当前构形，即时构形

$\boldsymbol{x}$: current configuration; $\boldsymbol{X}$: reference

$$\boldsymbol{x} = \chi(\boldsymbol{X}, t) \tag{8}$$

$$\mathrm{d}\boldsymbol{x} = \chi(\boldsymbol{X} + \mathrm{d}\boldsymbol{X}, t) - \chi(\boldsymbol{X}, t) = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} \mathrm{d}\boldsymbol{X} = \boldsymbol{F} \mathrm{d}\boldsymbol{X} \tag{9}$$

3.1 two-point tensor

$\boldsymbol{x}$ is current, $\boldsymbol{X}$ is reference.

$$F_{ik} = \frac{\partial x_i}{\partial X_K} \boldsymbol{e}_i \otimes \boldsymbol{e}_K \tag{10}$$

要区分参考构形和当前构形大小写。

deformation gradient

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连续介质力学　第 17 次课笔记

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1　运动学

Kinematics. 区别于 dynamics, kinetics (复习 Bogliubov 的尺度理论)

固定坐标系 fixed / absolute.

物质流形 material maniform 流形 continuous media, mechanics of material manifold (M^3)

$\mathcal{E}^3 \times \mathbb{R} \to \sum$

configuration ($t = 0$, reference / initial / undeformed configuration, $\mathsf{K}(\boldsymbol{X})$)

($t = \tau$, current / deformed configuration)

bijective 双射，deformation function $\boldsymbol{x} = \chi(\boldsymbol{X}), \boldsymbol{X} = \chi^{-1}\boldsymbol{x}$

$$\mathrm{d}\boldsymbol{x} = \chi(\boldsymbol{X} + \mathrm{d}\boldsymbol{X}, t) - \chi(\boldsymbol{X}, t) \tag{1}$$

$$= \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}\mathrm{d}\boldsymbol{X} = \boldsymbol{F}\mathrm{d}\boldsymbol{X} \tag{2}$$

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{F}\mathrm{d}\boldsymbol{X} = \mathrm{d}\boldsymbol{X}\boldsymbol{F}^T \tag{3}$$

$\boldsymbol{F}$ is a two-point tensor

$$\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \frac{\partial x_i}{\partial X_K}\boldsymbol{e}_i \otimes \boldsymbol{e}_K \tag{4}$$

$$\boldsymbol{F}^T = \frac{\partial x_i}{\partial X_K}\boldsymbol{e}_K \otimes \boldsymbol{e}_i \tag{5}$$

polar decomposition $\boldsymbol{F} \in \mathrm{Lin}, \boldsymbol{U}, \boldsymbol{V} \in P_{sym}, \boldsymbol{R} \in \mathrm{orth}^+$ is a rotation.

$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R}$

P176 切映射场 $\boldsymbol{F}(\boldsymbol{X}) : \mathsf{T} \to \tau(?), \boldsymbol{X} \in \mathsf{K}(\mathsf{B})$, B : material manifold.

$\forall \boldsymbol{A} \in \mathrm{Lin}$ 1st kind, Lagrangian. 2nd kind, Eularian. 3rd king, Mixed (two-point)

（$\boldsymbol{F}^T\boldsymbol{F}$ 消除中间两个基后剩余两个基都在参考构形中，因此称为拉格朗日型，可以通过大小写简单判断。）

$$\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} = \left(\frac{\partial x_j}{\partial X_A}\boldsymbol{e}_A \otimes \boldsymbol{e}_j\right)\left(\frac{\partial x_i}{\partial X_K}\boldsymbol{e}_i \otimes \boldsymbol{e}_K\right) = \frac{\partial x_j}{\partial X_A}\frac{\partial x_i}{\partial X_K}\delta_{ij}\boldsymbol{e}_A \otimes \boldsymbol{e}_K = \frac{\partial x_i}{\partial X_A}\frac{\partial x_i}{\partial X_K}\boldsymbol{e}_A \otimes \boldsymbol{e}_K \tag{6}$$

$$\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T = \left(\frac{\partial x_i}{\partial X_K}\boldsymbol{e}_i \otimes \boldsymbol{e}_K\right)\left(\frac{\partial x_l}{\partial X_M}\boldsymbol{e}_M \otimes \boldsymbol{e}_l\right) = \frac{\partial x_i}{\partial X_K}\frac{\partial x_l}{\partial X_M}\delta_{KM}\boldsymbol{e}_i \otimes \boldsymbol{e}_l = \frac{\partial x_i}{\partial X_K}\frac{\partial x_l}{\partial X_K}\boldsymbol{e}_i \otimes \boldsymbol{e}_l \tag{7}$$

可能是期中考题：证明 $\boldsymbol{c} \in P_{sym}$

作业：例 26.3, 26.4

主伸长 $\lambda_\gamma = \lambda_\Gamma$（书 P180）

2 Four definition of Strain

strain measure, stress measure.

2.1 Green strain

（现长平方-原长平方）/2 原长平方

$$\mathrm{d}\boldsymbol{x}^2 - \mathrm{d}\boldsymbol{X}^2 = \boldsymbol{F}\mathrm{d}\boldsymbol{X} \cdot \boldsymbol{F}\mathrm{d}\boldsymbol{X} - \mathrm{d}\boldsymbol{X} \cdot \mathrm{d}\boldsymbol{X} \tag{8}$$

$$= \mathrm{d}\boldsymbol{X}(\boldsymbol{F}^T\boldsymbol{F})\mathrm{d}\boldsymbol{X} - \mathrm{d}\boldsymbol{x}\mathbf{I}\mathrm{d}\boldsymbol{X} \tag{9}$$

$$= \mathrm{d}\boldsymbol{X}(\boldsymbol{F}^T\boldsymbol{F} - \mathbf{I})\mathrm{d}\boldsymbol{X} \tag{10}$$

$$= \mathrm{d}\boldsymbol{X}(2\boldsymbol{E})\mathrm{d}\boldsymbol{X} \tag{11}$$

$$= \mathrm{d}\boldsymbol{X}^2 2\boldsymbol{E}^T \tag{12}$$

$$\boldsymbol{E}^T = \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \mathbf{I}) = \frac{1}{2}(\boldsymbol{C} - \mathbf{I}) \tag{13}$$

为对称矩阵，因此

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \mathbf{I}) = \frac{1}{2}(\boldsymbol{C} - \mathbf{I}) \tag{14}$$

作业/可能是期末考题，可参考 P252

$$\frac{\partial \omega}{\partial \boldsymbol{E}} = 2\frac{\partial \omega}{\partial \boldsymbol{C}} \tag{15}$$

2.2 Almansi strain

（现长平方-原长平方）/2 现长平方

$$\mathrm{d}\boldsymbol{x}^2 - \mathrm{d}\boldsymbol{X}^2 = \mathrm{d}\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{x} - \boldsymbol{F}^{-1}\mathrm{d}\boldsymbol{x}\boldsymbol{F}^{-1}\mathrm{d}\boldsymbol{x} \tag{16}$$

$$= \mathrm{d}\boldsymbol{x}\boldsymbol{I}\mathrm{d}\boldsymbol{x} - \mathrm{d}\boldsymbol{x}\boldsymbol{F}^{-T}\boldsymbol{F}^{-1}\mathrm{d}\boldsymbol{x} \tag{17}$$

$$= \mathrm{d}\boldsymbol{x}(2\boldsymbol{e})\mathrm{d}\boldsymbol{x} \tag{18}$$

同理,

$$\boldsymbol{e} = \frac{1}{2}(\mathbf{I} - \boldsymbol{F}^{-T}\boldsymbol{F}^{-1}) \tag{19}$$

2.3 Logarithmic strain

infinitesimal

$$\mathrm{d}e = \frac{\mathrm{d}l}{l} \ \rightarrow \ e = \int_L^l \frac{\mathrm{d}l}{l} = \ln\frac{l}{L} = \ln(1+\varepsilon) = \varepsilon - \frac{\varepsilon^2}{2} + \dots \tag{20}$$

2.4 Cauchy strain

$$\varepsilon = \frac{\Delta l}{L} = \frac{l - L}{L} \tag{21}$$

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连续介质力学　第 18 次课笔记

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(53-8)

$$S = \frac{1}{2}(\boldsymbol{v} \otimes \nabla + \nabla \otimes \boldsymbol{v}) = \frac{1}{2}((\nabla \otimes \boldsymbol{v})^T + \nabla \otimes \boldsymbol{v}) \tag{1}$$

作业：用矩阵形式表示 12-24

　　书 P180-182

　　Linear elastic 线弹性：忽略掉二阶项。

　　(27-25)

1　Theory of elastic

elasticity 弹性力学　　　plasticity 塑性力学

作业：例 27.5，27.6

连续介质力学 第 19 次课笔记

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1 复习

1.1 almansi strain

Dirac symbol

$$\frac{(\text{current length})^2 - (\text{original length})^2}{2(\text{current length})^2} \tag{1}$$

$$\langle \mathrm{d}\boldsymbol{x}|\mathrm{d}\boldsymbol{x}\rangle - \langle \mathrm{d}\boldsymbol{X}|\mathrm{d}\boldsymbol{X}\rangle = \langle \mathrm{d}\boldsymbol{x}|\mathrm{d}\boldsymbol{x}\rangle - \langle \boldsymbol{F}^{-1}\mathrm{d}\boldsymbol{x}|\boldsymbol{F}^{-1}\mathrm{d}\boldsymbol{x}\rangle \tag{2}$$

$$= \langle \mathrm{d}\boldsymbol{x}|\mathrm{d}\boldsymbol{x}\rangle - \langle \mathrm{d}\boldsymbol{x}\boldsymbol{F}^{-T}|\boldsymbol{F}^{-1}\mathrm{d}\boldsymbol{x}\rangle \tag{3}$$

$$= \langle \mathrm{d}\boldsymbol{x}|\boldsymbol{I} - \boldsymbol{F}^{-T}\boldsymbol{F}^{-1}|\mathrm{d}\boldsymbol{x}\rangle \tag{4}$$

$$\therefore \boldsymbol{e} = \frac{1}{2}(\boldsymbol{I} - \boldsymbol{F}^{-T}\boldsymbol{F}^{-1}) \tag{5}$$

1.2 测地线方程

$$\frac{\mathrm{d}^2 q^r}{\mathrm{d}\lambda^2} + \Gamma^r_{\mu\nu}\frac{\mathrm{d}q^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}q^\nu}{\mathrm{d}\lambda} = 0 \tag{6}$$

λ: proper time τ, q - generalized coordinate, r - free index, μ, ν - dummy index.

$$m\frac{\mathrm{d}^2 q^r}{\mathrm{d}\tau^2} + \Gamma^r_{\mu\nu}\cdots = ma \tag{7}$$

1.3 Basics

1. Neother's theorem, field equations.

2. superposition $\rightarrow$ Linear $\rightarrow$ Taylor expension (1st)

3. Dimensional analysis, (rough) order-of-magnitude estimate.

2   Dimensional analysis

model, prototype, dimension number

fine structure const, $\alpha = e^2/4\varepsilon\pi\hbar c$, c : speed of light.

作业：有一张有名的图片，杨振宁和李政道的老师 Enrico Fermi 在上课时把精细结构参数写成了 $\hbar^2/ec$

Lederman, Rabi

2.1   foundamental and derived

physical quantitles: foundamental $(M, L, T, I, \Theta, N, J)$ / derived

$$\boldsymbol{F} = -G\frac{m_1 m_2}{\boldsymbol{r}^2}\hat{\boldsymbol{r}} \tag{8}$$

inverse square rule.

Point sorce, area of surface $4\pi r^2$

In $\mathbb{R}^n$, $F \propto r^{-(n-1)}$

2.2   intensive and extensive

intensive $\rho, T, \ldots$

extensive m, S, V

2.3   量纲

universal, primary, dimensional homogeneity.

$$\boldsymbol{F} = -\frac{GMm}{r^2}\hat{\boldsymbol{r}} = m\boldsymbol{a} \tag{9}$$

$$MLT^{-2} = [G]\frac{M^2}{L^2} \qquad [G] = M^{-1}L^3 T^{-2} \tag{10}$$

Kepler's 3rd law, 1618

$$M[G] = L^3 T^{-2} \qquad \frac{a^3}{T^2} = const = \frac{GM}{4\pi^2} \tag{11}$$

$$[MG] = (\frac{L}{T})^2 L \tag{12}$$

L/T: velocity. Second velocity

$$v^2 = \frac{2GM}{R} \qquad v^2 \sim \frac{GM}{R} \tag{13}$$

black hole, $R_S \sim 2GM/c^2$

plank length

$$l_p = \sqrt{\hbar G/c^3} \tag{14}$$

量子引力 quantum gravity

$$[E] = [mc^2] = ML^2T^{-2} \tag{15}$$

$$[\hbar] = ML^2T^{-2}T = ML^2T^{-1} \tag{16}$$

$$[G] = M^{-1}L^3T^{-2} \qquad [G\hbar] = L^5T^{-3} = (\frac{L}{T})^3L^2 = [c^3][l_p^2] \tag{17}$$

$$l_p \sim \sqrt{\hbar G/c^3} \tag{18}$$

$\sim$: order of magnitude

plank time, $t_p = l_p/c \sim \sqrt{\hbar G/c^5} \sim 10^{-44}s$

2.4   无量纲数

Bernoulli eqn

$$p + \frac{1}{2}\rho v^2 + \rho gh = const \tag{19}$$

dimension homogeneity, additive term

$$[\rho v^2] = ML^{-3}L^2T^{-2} = ML^{-1}T^{-2} \tag{20}$$

n terms, $n-1$ dimenionless.

$$\frac{1}{2}\frac{v^2}{gh} + \frac{p}{\rho gh} = \dots \tag{21}$$

Froude number

$$Fr = \frac{V}{\sqrt{gh}} \tag{22}$$

2.5

$$\nabla \cdot \boldsymbol{u} = 0 \tag{23}$$

mass conservation,

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\mathrm{div}\boldsymbol{u} = 0 \tag{24}$$

incompressible $\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0$

$$\rho(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \otimes \boldsymbol{u}) = -\nabla p + \mu\nabla^2\boldsymbol{u} + \rho\boldsymbol{g} \tag{25}$$

1.

$$\frac{\rho V}{T} \Big/ \frac{\rho V^2}{L} \sim \frac{L}{TV} \sim \frac{fL}{v} = St \tag{26}$$

2.

$$\frac{\Delta p}{L} \Big/ \frac{\rho V^2}{L} \sim \frac{\Delta p}{\rho v^2} = Eu \tag{27}$$

3.

$$\frac{\rho V^2}{L} \Big/ \frac{\mu V}{L^2} \sim \frac{\rho V L}{\mu} = Re \tag{28}$$

4. <u>froude number</u>，留作业

$$Fr = sqrt \frac{\rho V^2}{L} \Big/ \rho g \tag{29}$$

Komlogoror, 1941

$$\langle |\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r},t) - \boldsymbol{v}(\boldsymbol{x},t)|^2 \rangle = c\mathcal{E}^{2/3}r^{2/3} \tag{30}$$

$$[\mathcal{E}] = \frac{\text{energy}}{\text{mass x time}} = \frac{ML^2T^{-2}}{MT} = L^2T^{-3} \tag{31}$$

Spectrum $E(k)$, $k \to$ wave number, $[k] = L^{-1}$

$$\frac{1}{2}\langle \boldsymbol{v}(\boldsymbol{x},t) \cdot \boldsymbol{b}(\boldsymbol{x},t) \rangle = \int_0^\infty E(k)\mathrm{d}k \tag{32}$$

self-correlated function

　　<u>作业：计算$[E(k)]$ $c\mathcal{E}^{2/3}k^{-5/3}$</u>

　　Fermi: better be approximately right than precisely wrong.

感谢为本次笔记提供参考的同学：　无

连续介质力学　第 20 次课笔记

庄逸

课程日期：2019 年 5 月 8 日　　最后修改：2019 年 6 月 19 日

1　积分的量纲

$$\left[\int E(k)\mathrm{d}k\right] = [E(k)][k] = [E(k)]L^{-1} = L^3 T^{-2} \tag{1}$$

作业：推积分的量纲表达式

2　plank mass

plank length l_p，plank time l_τ，quantum gravity $\hbar, G, c$

$$[\hbar] = ML^2T^{-1} \qquad [G] = M^{-1}L^3T^{-2} \tag{2}$$

$$\left[\frac{\hbar}{G}\right] = \frac{ML^2T^{-1}}{M^{-1}L^3T^{-2}} = M^2\frac{T}{L} \ \Rightarrow\ l_m = \sqrt{\frac{\hbar c}{G}} \tag{3}$$

3　Buckingham

Edgar Buckingham (1914) Π-theorem.

Variables $x_1, \ldots x_n$, primary dimension k, $n-k$

semi-infinite bar, longituclinal wave c_L, transverse wave c_T

$$\sigma = E\varepsilon,\ [E] = [\sigma] = [\rho v^2] = [p] = [\rho gh] = ML^{-1}T^{-2} \tag{4}$$

$$[\rho] = ML^{-3} \quad \left[\frac{E}{\rho}\right] = \frac{L^2}{T^2} \qquad c \sim \sqrt{\frac{E}{\rho}} \tag{5}$$

$$\sigma(x,t) - \sigma(x+\mathrm{d}x, t) = \rho\mathrm{d}xAu_{tt} = -\sigma_x\mathrm{d}xA \tag{6}$$

$$-\frac{\partial(-E\varepsilon)}{\partial x} = E\frac{\partial\varepsilon}{\partial x} = \rho u_{tt}E \qquad \varepsilon = u_x \tag{7}$$

$$\therefore Eu_{xx} = \rho u_{tt} \qquad c_L = \sqrt{\frac{E}{\rho}} \tag{8}$$

hyperbolic.

$$T_t = x\nabla^2 T \tag{9}$$

轶事 anecdote

作业：P316-317

4   dirac notation

$$2ed\boldsymbol{x}^2 = \langle d\boldsymbol{x}|d\boldsymbol{x}\rangle - \langle d\boldsymbol{X}|d\boldsymbol{X}\rangle \tag{10}$$

$$= \langle d\boldsymbol{x}|d\boldsymbol{x}\rangle - \langle (\mathbf{I} - \boldsymbol{u}\otimes\nabla)\,d\boldsymbol{x}|\,(\mathbf{I} - \boldsymbol{u}\otimes\nabla)\,d\boldsymbol{x}\rangle \tag{11}$$

$$= \langle d\boldsymbol{x}|d\boldsymbol{x}\rangle - \langle d\boldsymbol{x}|\,(\mathbf{I} - \boldsymbol{u}\otimes\nabla)^T\,(\mathbf{I} - \boldsymbol{u}\otimes\nabla)\,|d\boldsymbol{x}\rangle \tag{12}$$

$$= \langle d\boldsymbol{x}|\mathbf{I} - (\mathbf{I} - \boldsymbol{u}\otimes\nabla)^T\,(\mathbf{I} - \boldsymbol{u}\otimes\nabla)\,|d\boldsymbol{x}\rangle \tag{13}$$

5

$$\frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{u} + \boldsymbol{g} \tag{14}$$

ν: diffusivity momentum, kinematic viscosity.

$$Re = \frac{\rho V L}{\mu} = \frac{vL}{\nu} \tag{15}$$

$$\tau = \mu\frac{\partial u}{\partial y} \qquad [\tau] = \left[\frac{F}{A}\frac{T}{T}\right] \tag{16}$$

τ: flux of momentum.

6   Thickness of boundary layer

(laminar).

$$\delta = c\sqrt{\nu\frac{x}{u_0}} \tag{17}$$

$$[u_0] = LT^{-1} \qquad [x/u_0] = T \tag{18}$$

adiabatic shear, non-slip boundary 作业：用雷诺数表示

7 Schrödiner equation

$$-\frac{\hbar^2}{2m}\Psi_{xx} + V\Psi = i\hbar\Psi_t \tag{19}$$

Ψ: wave funcation, non-dimensional.

Let $t = \tau\hat{t}$, τ : characteristic time, $x = L\hat{x}$, L : length. $\Psi(x,t) = \Psi(\hat{x},\hat{t})$

$$-\frac{\hbar^2}{2mL^2}\Psi_{\hat{x}\hat{x}} + V\Psi = \frac{i\hbar}{\tau}\Psi_{\hat{t}} \tag{20}$$

divided by $\frac{1}{2}m(\frac{L}{\tau})^2$

$$-\frac{\hbar^2\tau^2}{4m^2L^4}\Psi_{\hat{x}\hat{x}} + \frac{V}{\frac{1}{2}m\left(\frac{L}{\tau}\right)^2}\Psi = \frac{i\hbar\tau^2}{\tau\frac{1}{2}mL^2}\Psi_{\hat{t}} = \frac{i\hbar\tau}{\frac{1}{2}mL^2}\Psi_{\hat{t}} \tag{21}$$

$$-\frac{\varepsilon^2}{4}\Psi_{\hat{x}\hat{x}} + \hat{V}\Psi = 2i\varepsilon\Psi_{\hat{t}} \tag{22}$$

感谢为本次笔记提供参考的同学： 无

连续介质力学　第 21 次课笔记

庄逸

课程日期：2019 年 5 月 13 日　　最后修改：2019 年 6 月 18 日

1　量纲和普朗克量

力学讲义表 8.1

基本力类型	力的强度
引力	10^{-39}
弱力	10^{-6}
电磁力	$1/137$
强力	1

$$[MG] = L^3 T^{-2} \quad [G] = M^{-1} L^3 T^{-2} \tag{1}$$

$$\frac{Gm_p^2}{\hbar c} = 5.91 \times 10^{-39} \tag{2}$$

Boltzmann constant $S = k_{\mathrm{B}} \ln \Omega$

$$[k_{\mathrm{B}}] = J/K = M L^2 T^{-2} \Theta^{-1} \tag{3}$$

Planck length l_{P}, Planck time t_{P}, planck mass m_P

Planck temperature $T_{\mathrm{P}} = m_p c^2 / k_{\mathrm{B}}$ 作业：查一下普朗克温度

bijection, strain compatibility equation.

$$\varepsilon_{ij,st} + \varepsilon_{st,ij} = \varepsilon_{is,jt} + \varepsilon_{jt,is} \tag{4}$$

$4\pi\varepsilon_0$，高斯制，CGS: centimeter, Gram, second.

$$[e^2] = [E][L] = M L^3 T^{-2} \qquad [\hbar c] = [E \cdot T][V] = [E][L] \tag{5}$$

planck charge

$$q_{\mathrm{P}} = \sqrt{\hbar c} = \sqrt{4\pi\varepsilon_0 \hbar c} \tag{6}$$

作业：如果世界毁灭了，只能有一条信息留下来，那么。。。

类比 analogy, 不完全相似?

2 韦伯数

Weber number, γ, σ. surface tension.

$$[\gamma] = J/A = FLL^{-2} = FL^{-1} = MT^{-2} \tag{7}$$

$$We = \frac{\rho v^2}{\gamma/L} \tag{8}$$

3 邦德数

$$Bo = \frac{\rho g l}{\gamma} 1 \ \Rightarrow \ L = \sqrt{\frac{\gamma}{\rho g}} \tag{9}$$

作业：分析液滴的毛细特征尺寸，什么时候可以忽略重力? 小虫子在水上为什么能站住?

作业：估算水的弛豫时间，毛细特征速度。 让无量纲量等于 1 10^{-12}m/s

Better be approximately right than precisely wrong. 解方程不注意限制条件，会得到错误的结果，还不如粗略估计。

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连续介质力学 第 22 次课笔记

庄逸

课程日期：2019 年 5 月 15 日   最后修改：2019 年 6 月 18 日

1 估算山的最大高度

guesstimation, Enrico Fermi

作业：maximum height of a mountain on earth；思路：熔解热，太高的话底部会发生塑性变形

Mount Everest $\sigma = \rho g h$, E : Young's moculus, $[E] = [p]$

strain $\varepsilon = \sigma/E = \rho g h/E$, plastic strain $\varepsilon_y = 6.977/1000$, Let $\varepsilon = \varepsilon_y$, $h \to h_{max}$

density $\rho = 4 \times 10^3 \text{kg/m}^2$, $E = 64.5\text{GPa}$, $h_{max} = 11.6\text{km}$

$M = 10^6, G = 10^9$, mega-Å

Method II. $\sigma = \rho g h$, $\sigma_Y = 450\text{MPa}$, $h_{max} = \ldots$

Method III.

2 牛顿炒股

quant 宽克（矿工）

royal mint 金本位 gold standard system , 牛顿 1720 stock.

股票走势图见课件（45-46）

先赚了 7000 英镑，又赔了 20000 英镑。

year 1717 金本位 Ounce 盎司 gold = 3 磅 17 先令 10 便士

作业：估算今天牛顿赔的钱的数量级

"I could calculate the orbits of heavily bodied, but could not calculate the madness of people " ——Sir Issac Newton

3   Others

作报告第一句话很重要

$$R = \sum_{\gamma=1,2,3} \sum_{\Gamma=I,II,III} b_\gamma \otimes N_\gamma = n_1 \otimes N_I + n_2 \otimes N_{II} + n_3 \otimes N_{III} \tag{1}$$

R 是联系当前构形和初始构形的两点张量。

strech ratio, 赛斯应变度量见书。

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连续介质力学　第 23 次课笔记

庄逸

课程日期：2019 年 5 月 20 日　　最后修改：2019 年 6 月 18 日

大作业　渗流力学, Darcy's law, (Darcy-) Stokes-Brinkman equation, 应用：石油、天然气
正则: kenadical?

弛豫时间　水 $10^{-12}s$，钢铁则是十的几次方。德博拉数约为 1 的时候是流变体，大于 1 则是固体。

Navier，1822 年方程。1820+ Suspension bridge
对称张量和非对称张量双内积，先把它对称化

1　功共轭

物质流形 material manifold

Refernce configuration, $t = 0$; current configuration, $t = t$

deformation gradient tensor. two-point tensor $\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\boldsymbol{X}}$

$$\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = F_{aB}\boldsymbol{e}_a \otimes \boldsymbol{e}_B \tag{1}$$

determinant of $\boldsymbol{F}$

$$\det \boldsymbol{F} = \begin{vmatrix} \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} & \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{Y}} & \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{Z}} \\ \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{X}} & \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{Y}} & \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{Z}} \\ \frac{\partial \boldsymbol{z}}{\partial \boldsymbol{X}} & \frac{\partial \boldsymbol{z}}{\partial \boldsymbol{Y}} & \frac{\partial \boldsymbol{z}}{\partial \boldsymbol{Z}} \end{vmatrix} = J \tag{2}$$

J : Jacobian, $J = \frac{\mathrm{d}v}{\mathrm{d}V}$; $\mathrm{d}v = \mathrm{d}\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{a} = \mathrm{d}\boldsymbol{x}(\mathrm{d}a)\boldsymbol{n}$

$\mathrm{d}V = \mathrm{d}\boldsymbol{X} \cdot \mathrm{d}\boldsymbol{A}$

stress = d(force) / d(area)

Cauchy stress $\boldsymbol{\sigma} = \mathrm{d}\boldsymbol{f}/\mathrm{d}\boldsymbol{a}$

$$J\mathrm{d}\boldsymbol{X} \cdot \mathrm{d}\boldsymbol{A} = \mathrm{d}\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{a} = \boldsymbol{F}\mathrm{d}\boldsymbol{X} \cdot \mathrm{d}\boldsymbol{a} = \mathrm{d}\boldsymbol{X}\boldsymbol{F}^T\mathrm{d}\boldsymbol{a} \tag{3}$$

$$\mathrm{d}\boldsymbol{X}(J\mathrm{d}\boldsymbol{A} - \boldsymbol{F}^T\mathrm{d}\boldsymbol{a}) = 0 \tag{4}$$

由 $\mathrm{d}\boldsymbol{X}$ 任意性，$J\mathrm{d}\boldsymbol{A} = F^T\mathrm{d}\boldsymbol{a}$

$$\mathrm{d}\boldsymbol{a} = J\boldsymbol{F}^{-T}\mathrm{d}\boldsymbol{A} \tag{5}$$

1.1 PK1

Piola-Kirchhoff stress of the first kind (PK1) $\mathrm{d}\boldsymbol{f} = \boldsymbol{\sigma}\mathrm{d}\boldsymbol{a} = J\boldsymbol{\sigma}\boldsymbol{F}^{-T}\mathrm{d}\boldsymbol{A}$

$$\boldsymbol{P} = \frac{\mathrm{d}\boldsymbol{f}}{\mathrm{d}\boldsymbol{A}} = J\boldsymbol{\sigma}\boldsymbol{F}^{-T} = \boldsymbol{\tau}\boldsymbol{F}^{-T} \tag{6}$$

where $\boldsymbol{\tau} = J\boldsymbol{\sigma}$ in current configuration, τ : kirchhoff stress

$$\boldsymbol{P} = (J\sigma_{kl}\boldsymbol{e}_k \otimes \boldsymbol{e}_l) \cdot (\frac{\partial \boldsymbol{X}_M}{\partial \boldsymbol{x}_n}\boldsymbol{e}_n \otimes \boldsymbol{e}_M) = J\sigma_{kl}\frac{\partial \boldsymbol{X}_M}{\partial \boldsymbol{x}_n}\delta_{ln}\boldsymbol{e}_k \otimes \boldsymbol{e}_m \tag{7}$$

$$\boldsymbol{P} = J\boldsymbol{\sigma}\boldsymbol{F}^{-T} \tag{8}$$

1.2 PK2

$$\boldsymbol{S} = \boldsymbol{S}^T, \boldsymbol{S} = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T}, (J\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T})^T = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T}$$

1.3 velocity gradient tensor

$$\boldsymbol{l} = \boldsymbol{v} \otimes \nabla_{\boldsymbol{x}} = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} = \frac{\partial}{\partial \boldsymbol{x}}\left(\frac{\partial \boldsymbol{x}}{\partial t}\right) = \frac{\partial}{\partial \boldsymbol{X}}\left(\frac{\partial \boldsymbol{x}}{\partial t}\right)\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}} = \frac{\partial}{\partial t}\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}\right)\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1} \tag{9}$$

strain rate

$$\boldsymbol{d} = \frac{\boldsymbol{l} + \boldsymbol{l}^T}{2} \tag{10}$$

symmetrical $\boldsymbol{d} = \boldsymbol{d}^T, \boldsymbol{l} = \boldsymbol{d} + \boldsymbol{\omega}$

spin

$$\boldsymbol{\omega} = \frac{\boldsymbol{l} - \boldsymbol{l}^T}{2} \tag{11}$$

1.4 work conjugate

$$\dot{w} = J\boldsymbol{\sigma} : \boldsymbol{d} = J\boldsymbol{\sigma} : (\boldsymbol{l} - \boldsymbol{\omega}) = J\boldsymbol{\sigma} : \boldsymbol{l} = J\boldsymbol{\sigma} : (\dot{\boldsymbol{F}}\boldsymbol{F}^{-1}) = (J\boldsymbol{\sigma}\boldsymbol{F}^{-T}) : \dot{\boldsymbol{F}} = \boldsymbol{P} : \dot{\boldsymbol{F}} \tag{12}$$

Green strain

$$E = \frac{1}{2}(F^T F - I) \tag{13}$$

$$\dot{E} = \frac{1}{2}(\dot{F}^T F + F^T \dot{F}) \tag{14}$$

$$\dot{w} = (J\sigma F^{-T}) : \dot{F} \tag{15}$$

$$= J F^{-1} F \sigma F^{-T} : \dot{F} \tag{16}$$

$$= (J F^{-1} \sigma F^{-T}) : (F^T \dot{F}) \tag{17}$$

$$= S : \frac{F^T \dot{F} + \dot{F}^T F}{2} = S : \dot{E} \tag{18}$$

Summary

$$\dot{w} = J\sigma : d = J\sigma : l = P : \dot{F} = S : \dot{E} \tag{19}$$

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连续介质力学　第 24 次课笔记

庄逸

课程日期：2019 年 5 月 22 日　　最后修改：2019 年 6 月 30 日

$$[eV] = [k_{\mathrm{B}}T] = [RT/N_A] = [h\nu] = [\hbar\omega] = [hc/\lambda] \tag{1}$$

$[e^2] = [E][L]$, $a_e = e^2/m_e c^2$

1　鸵鸟飞不起来的原因

mg : gravity;　$CA\rho_{air}v^2$: lift;　L : characteristic length;　$m : \rho_{bird}L^3$;　$A : L^2$

$$CA\rho_{air}v^2 \sim mg \qquad v^2 \sim \frac{\rho_{bird}}{C\rho_{air}}gL \tag{2}$$

$v \sim \sqrt{gL}$

2　无量纲数·达西数

solid-like liquid 类固体。达西定律：

$$v = -\frac{\kappa}{\mu}\nabla p \qquad \frac{\mu v}{L} \sim \frac{[\kappa]p}{L^2} \tag{3}$$

$\kappa \sim L^2$

类比，analogy，热传导方程，扩散方程，通量 flux

3　Jacobian

$f(x), x(t)$

$$\int f\,\mathrm{d}x = \int f\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t \tag{4}$$

$$\iint f(x,y)\mathrm{d}x\mathrm{d}y = \iint f(x(u,v), y(u,v))\, |J|\, \mathrm{d}u\mathrm{d}v \tag{5}$$

$$|J_2| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial(x,y)}{\partial(u,v)} \tag{6}$$

$$|J_3| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \tag{7}$$

$$|J_n| = \ldots \tag{8}$$

$$\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \boldsymbol{x} \otimes \nabla_{\boldsymbol{X}} \tag{9}$$

作业：求球坐标的雅可比和球的体积
Professor, Reader, Lecturer
原创

4 速度梯度

deformation gradient $\boldsymbol{F}$, Jacobian $J = \det \boldsymbol{F}$

$$\dot{J} = \frac{\mathrm{d}}{\mathrm{d}t} J = \frac{\mathrm{d}}{\mathrm{d}t} \det(\boldsymbol{F}) = \frac{\partial J}{\partial \boldsymbol{F}} : \dot{\boldsymbol{F}} \tag{10}$$

velocity gradient $\boldsymbol{l}$

$$\boldsymbol{l} = \boldsymbol{v} \otimes \nabla_{\boldsymbol{x}} = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} = \frac{\partial}{\partial \boldsymbol{X}}\left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}\right)\boldsymbol{F}^{-1} \tag{11}$$

$$\boldsymbol{l} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1} \quad \Rightarrow \quad \dot{\boldsymbol{F}} = \boldsymbol{l}\boldsymbol{F} \tag{12}$$

$$\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} \qquad \boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T \qquad \det\boldsymbol{C} = \det(\boldsymbol{F}^T\boldsymbol{F}) = (\det\boldsymbol{F})^2 \tag{13}$$

作业：验证 trC=trB?

5 行列式对矩阵的导数

$$\frac{\partial \det \boldsymbol{F}}{\partial \boldsymbol{F}} : \boldsymbol{A} = \lim_{\varepsilon \to 0} \frac{\det(\boldsymbol{F} + \varepsilon \boldsymbol{A}) - \det \boldsymbol{F}}{\varepsilon} \tag{14}$$

$$\because \det(\boldsymbol{F} + \lambda \mathbf{I}) = \lambda^3 + I_1(\boldsymbol{F})\lambda^2 + I_2(\boldsymbol{F})\lambda + I_3(\boldsymbol{F}) \tag{15}$$

$$\frac{\partial \det \boldsymbol{F}}{\partial \boldsymbol{F}} : \boldsymbol{A} = \lim_{\varepsilon \to 0} \frac{\det[\varepsilon \boldsymbol{F}(\varepsilon^{-1}\mathbf{I} + \boldsymbol{F}^{-1}\boldsymbol{A})]}{\varepsilon} \tag{16}$$

$$= \lim_{\varepsilon \to 0} \frac{\det(\varepsilon \boldsymbol{F}) \det(\varepsilon^{-1}\mathbf{I} + \boldsymbol{F}^{-1}\boldsymbol{A}) - J}{\varepsilon} \tag{17}$$

$$= \lim_{\varepsilon \to 0} \frac{\varepsilon^3 J[\varepsilon^{-3} + I_1(\boldsymbol{F}^{-1}\boldsymbol{A})\varepsilon^{-2} + I_2(\boldsymbol{F}^{-1}\boldsymbol{A})\varepsilon^{-1} + I_3(\boldsymbol{F}^{-1}\boldsymbol{A})]}{\varepsilon} \tag{18}$$

$$= J I_1(\boldsymbol{F}^{-1}\boldsymbol{A}) = J\mathrm{tr}(\boldsymbol{F}^{-1}\boldsymbol{A}) = J\mathbf{I} : (\boldsymbol{F}^{-1}\boldsymbol{A}) = J\boldsymbol{F}^{-T} : \boldsymbol{A} \tag{19}$$

$$\therefore \frac{\partial J}{\partial \boldsymbol{F}} = \frac{\partial \det \boldsymbol{F}}{\partial \boldsymbol{F}} = J\boldsymbol{F}^{-T} \tag{20}$$

Volume strain

$$\varepsilon_V = \frac{(1 + \varepsilon_{11})\mathrm{d}x_1(1 + \varepsilon_{22})\mathrm{d}x_2(1 + \varepsilon_{33})\mathrm{d}x_3 - \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3}{\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3} \approx \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \tag{21}$$

作业：29-45，5.1?

Spectral decomposition

6 常用张量的谱分解

$$x_1 = \lambda_1 X_1 \qquad x_2 = \lambda_2 X_2 \qquad x_3 = \lambda_3 X_3 \tag{22}$$

$$\boldsymbol{F} = \sum_{i=1}^{3} \lambda_i \boldsymbol{n}_i \otimes \boldsymbol{N}_i \qquad \boldsymbol{R} = \sum_{i=1}^{3} \boldsymbol{n}_i \otimes \boldsymbol{N}_i \tag{23}$$

$$\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F} = \sum_i \lambda_i^2 \boldsymbol{N}_i \otimes \boldsymbol{N}_i \qquad \boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T = \sum_i \lambda_i^2 \boldsymbol{n}_i \otimes \boldsymbol{n}_i \tag{24}$$

$$\boldsymbol{U} = \sqrt{\boldsymbol{C}} = \sum \lambda_i \boldsymbol{N}_i \otimes \boldsymbol{N}_i \qquad \boldsymbol{V} = \sqrt{\boldsymbol{B}} = \sum \lambda_i \boldsymbol{n}_i \otimes \boldsymbol{n}_i \tag{25}$$

$$\boldsymbol{E} = \sum \frac{1}{2}(\lambda_i^2 - 1)\boldsymbol{N}_i \otimes \boldsymbol{N}_i \qquad \boldsymbol{e} = \frac{1}{2}(1 - 1/\lambda_i^2)\boldsymbol{n}_i \otimes \boldsymbol{n}_i \tag{26}$$

principal stretch 作业：写比奥应变，物质亨奇应变，空间比奥应变

感谢为本次笔记提供参考的同学： 无

连续介质力学　第 25 次课笔记

庄逸

课程日期：2019 年 5 月 27 日　　最后修改：2019 年 6 月 18 日

1　无量纲数・磁流体力学

lorentz force，magenetic field

$$\rho \left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \right] = -\nabla p + \mu \nabla^2 \boldsymbol{u} + \rho \boldsymbol{g} + \boldsymbol{j} \times \boldsymbol{B} \tag{1}$$

where, $\boldsymbol{j}$ - current density, ohm's law. $\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{B} \times \boldsymbol{u})$

$\boldsymbol{j} \times \boldsymbol{B} \sim \sigma B^2 V \qquad \mu \nabla^2 \boldsymbol{u} \sim \mu V / L^2$

$$\sqrt{\frac{\text{Lorentz force}}{\text{viscous force}}} = \sqrt{\frac{\sigma B^2 V L^2}{\mu V}} = BL \sqrt{\frac{\sigma}{\mu}} \tag{2}$$

Hartman number

作业：用 N-S 方程推各种无量纲数？

2　体积的时间变化率

Time derivative

$$\dot{J} = \frac{\partial J}{\partial \boldsymbol{F}} : \dot{\boldsymbol{F}} = J \boldsymbol{F}^{-T} : \dot{\boldsymbol{F}} \tag{3}$$

$$\boldsymbol{l} = \boldsymbol{v} \otimes \nabla_{\boldsymbol{x}} = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} = \frac{\partial}{\partial \boldsymbol{X}} \left(\frac{\mathrm{d} \boldsymbol{x}}{\mathrm{d} t} \right) \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}} = \dot{\boldsymbol{F}} \boldsymbol{F}^{-1} \tag{4}$$

$\boldsymbol{l} = \boldsymbol{d} + \boldsymbol{w}$

$$\therefore \dot{J} = J \mathrm{tr} \boldsymbol{l} = J \mathrm{tr} \boldsymbol{d} = J \nabla \cdot \boldsymbol{v} \tag{5}$$

$$\dot{\mathrm{d}v} = \dot{J} \mathrm{d}V = J(\nabla \cdot \boldsymbol{v}) \mathrm{d}V = \nabla \cdot \boldsymbol{v} \mathrm{d}v \tag{6}$$

区分速度和体积？

Truesdell, Noll, NLFT: Nonlinear field theory, field equations

杨振宁：宁拙勿巧

推 30-4

3   连续性方程

mass continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \tag{7}$$

charge

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j} = 0 \quad \boldsymbol{j} = \rho \boldsymbol{v}_d \tag{8}$$

$\boldsymbol{j}$：漂移速度

diffusion

$$\frac{\partial c}{\partial t} + \nabla \cdot \boldsymbol{J} = 0 \tag{9}$$

Fick's law, $\boldsymbol{J} = -D\nabla c$

thermal conduction

$$\rho c \frac{\partial \theta}{\partial t} + \nabla \cdot \boldsymbol{q} = 0 \tag{10}$$

$\boldsymbol{q} = -\kappa \nabla \theta$

wave function Ψ

probability density $\rho = |\Psi|^2$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0 \tag{11}$$

probability density flux

$$\boldsymbol{J} = \frac{\hbar}{mi}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) \tag{12}$$

可能是期末考题：用一句话说明为什么克里斯托弗符号不是张量。

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连续介质力学 第 26 次课笔记

庄逸

课程日期：2019 年 5 月 29 日   最后修改：2019 年 6 月 18 日

CPI - Corruption perception index 清廉指数

1 无量纲数

$$\frac{\partial c}{\partial t} = D\nabla^2 c \qquad \frac{c}{\tau} \sim D\frac{c}{L^2} \tag{1}$$

$L \sim \sqrt{D\tau}$

oscillatory flow, pulsatile flow （脉动）, Womersley number 沃姆斯莱数

$$Wo = L\sqrt{\frac{\xi\omega}{\nu}} \tag{2}$$

L : character length; ω : frequency; ν : Kinematic viscosity

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{u} + \boldsymbol{f} \tag{3}$$

$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \rightarrow v/\tau = v\omega$, $\nu\nabla^2\boldsymbol{u} \rightarrow \nu v/L^2$

$$\sqrt{\frac{v\omega}{\nu v/L^2}} = L\sqrt{\frac{\omega}{\nu}} \qquad \sqrt{\frac{\text{inertial force}}{\text{viscous force}}} \tag{4}$$

$$W_0 = \sqrt{\frac{\text{transient force}}{\text{viscous force}}} \tag{5}$$

stokes boundary layer

$$\sqrt{\frac{\nu}{\omega}} - \text{ length} \tag{6}$$

$$Wo = \frac{L}{\delta} = \frac{\text{characteristic length}}{Stokes boundary layer thickness} \tag{7}$$

Roshko number

$$Ro = St \cdot Re = W_o^2 \tag{8}$$

2 Others

二阶投影张量

$$P = I - n \otimes n \tag{9}$$

Cauchy's stress principle (1822)

traction force $t_n = \sigma n$

$n \cdot \sigma \cdot n = \sigma_{ij} n_i n_j$

固体中（?）质量守恒，$\rho \mathrm{d}v = \mathrm{d}m$，求导可以跳过（书 P216）

可能是期末考题：参考构形中的平衡方程怎么写？（32-12)

$$\nabla_X \cdot P = 0 \tag{10}$$

感谢为本次笔记提供参考的同学： 无

连续介质力学　第 27 次课笔记

庄逸

课程日期：2019 年 6 月 3 日　　最后修改：2019 年 6 月 17 日

1　连续性方程

- 质量守恒

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0 \qquad \boldsymbol{J} = \rho \boldsymbol{v} \tag{1}$$

- 电荷守恒

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0 \qquad \boldsymbol{J} = \rho \boldsymbol{v}_d \tag{2}$$

- 量子力学概率密度

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0 \qquad \boldsymbol{J} = \frac{\hbar}{2m\mathrm{i}}(\psi^* \nabla \psi - \psi \nabla \psi^*) \tag{3}$$

- 扩散

$$\frac{\partial c}{\partial t} + \nabla \cdot \boldsymbol{J} = 0 \qquad J = -D\nabla c \tag{4}$$

- 热传导和热扩散

$$\rho c_p \frac{\partial \theta}{\partial t} + \nabla \cdot \boldsymbol{q} = 0 \qquad \boldsymbol{q} = -\kappa \nabla \theta \tag{5}$$

- 动量扩散，momentun diffusion

$$\frac{\partial (\rho \boldsymbol{v})}{\partial t} + \nabla \cdot \boldsymbol{\tau} = \boldsymbol{0} \tag{6}$$

$$\boldsymbol{\tau} = -\mu \left(\nabla \otimes \boldsymbol{v} + \boldsymbol{v} \otimes \nabla - \frac{2}{3}(\nabla \cdot \boldsymbol{v})\mathbf{I} \right) \tag{7}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} = \nu \nabla^2 \boldsymbol{v} \tag{8}$$

不可压缩条件：$\nabla \cdot \boldsymbol{v} = 0$

- 对流热扩散方程

$$\frac{\partial \theta}{\partial t} + \dot{v} \cdot \nabla[\theta] = \alpha \nabla^2 \theta \tag{9}$$

类比，Analogy

Kinematic viscosity

$$[\nu] = [\frac{\mu}{\rho}] = L^2 T^{-1} = [D] \tag{10}$$

2   推导 N-S 方程

conservation equation

$$\frac{\partial(\rho \boldsymbol{v})}{\partial t} + \nabla \cdot \boldsymbol{\tau} = \boldsymbol{0} \tag{11}$$

shear stress

$$\boldsymbol{\tau} = -[\mu(\nabla \otimes \boldsymbol{v} + \boldsymbol{v} \otimes \nabla) + \lambda(\nabla \cdot \boldsymbol{v})\mathbf{I}] \tag{12}$$

$$\sigma = -\rho \mathbf{I} + \lambda(\mathrm{tr}\boldsymbol{d})\mathbf{I} + 2\mu\boldsymbol{d} \tag{13}$$

λ : bulk viscosity; μ : dynamic viscosity

Stokes' hypothesis

$$\lambda + \frac{2}{3}\mu = 0, \ \lambda = -\frac{2}{3}\mu \tag{14}$$

$$\nabla \cdot \boldsymbol{\tau} = -\mu[\nabla^2 \boldsymbol{v} + (\nabla \cdot \boldsymbol{v}) \otimes \nabla - \frac{2}{3}\nabla \cdot (\nabla \cdot \boldsymbol{v})\mathbf{I}] \tag{15}$$

for incompressible fluid, $\nabla \cdot \boldsymbol{v} = 0$

$$\nabla \cdot \boldsymbol{\tau} = -\mu \nabla^2 \boldsymbol{v} \tag{16}$$

$$\therefore \rho \frac{\partial \boldsymbol{v}}{\partial t} = \mu \nabla^2 \boldsymbol{v} \qquad \frac{\partial \boldsymbol{v}}{\partial t} = \nu \nabla^2 \boldsymbol{v} \tag{17}$$

$$[\alpha] = [D] = [\nu] = L^2 T^{-1} \tag{18}$$

diffusion tensor

$$D_{ij} = \frac{\overline{x_i x_j}}{t} \qquad \boldsymbol{D} = \frac{\langle \boldsymbol{x} \otimes \boldsymbol{x} \rangle}{2t} \tag{19}$$

3　无量纲数

普朗特数，施密特数

$$Pr = \frac{\nu}{\alpha} \qquad Sc = \frac{\nu}{D} \tag{20}$$

无量纲化

$$\frac{\partial c}{\partial t} + (\boldsymbol{v} \cdot \nabla)c = D\nabla^2 c \tag{21}$$

$t = \alpha\tau, \mathrm{d}\boldsymbol{x} = \mathrm{d}\boldsymbol{\beta}L, \boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}\alpha}\frac{L}{\tau}$

$$\frac{\partial c}{\partial \alpha} + \left(\frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}\alpha} \cdot \nabla_{\boldsymbol{\beta}}\right)c = \frac{\tau D}{L^2}\nabla^2_{\boldsymbol{\beta}}c \tag{22}$$

作业：推导不可压缩 NS 方程

　　Péclet number, Pe 佩克莱数

$$Pe = Re \cdot Pr \qquad \frac{LV}{\alpha} = \frac{LV}{\nu}\frac{\nu}{\alpha} \tag{23}$$

$[\boldsymbol{v} \otimes \boldsymbol{v}]$

作业：推导 33-8

感谢为本次笔记提供参考的同学：　无

连续介质力学 第 28 次课笔记

庄逸

课程日期：2019 年 6 月 5 日 最后修改：2019 年 6 月 17 日

无量纲化方程

1 无量纲数·傅里叶数

Fourier number （Fo）傅里叶数

$$\rho c_p \frac{\partial \theta}{\partial t} + \nabla \cdot \boldsymbol{q} = 0 \tag{1}$$

Fourier's heat transfer

$$\boldsymbol{q} = -\kappa \nabla T \tag{2}$$

k-thermal diffusivity

$$q = h\Delta T = h(T_2 - T_1) \tag{3}$$

h-heat transfer coefficient

$$L[h] = [\kappa] \tag{4}$$

努塞尔特数，毕奥数。<u>作业：它们的物理意义？除了书上的解释还有没有？</u>

　　q-热通量，单位面积单位时间通过的热量

　　thermal diffusion:

$$Fo = \frac{\alpha t}{L^2} \tag{5}$$

mass diffusion:

$$Fo = \frac{Dt}{L^2} \tag{6}$$

无量纲化不能把算子去了，这和数量级估计不一样。$\frac{\partial \theta}{\partial t} \sim \theta/\tau$，$\alpha \nabla^2 \theta \sim \alpha \theta/L^2$

　　characteristic time for thermal conduction $\tau \sim L^2/\alpha$, so $Fo = t/\tau$, Fo–dimensionless time

2 PDE 的类型

PDE-partial differential equations

- hyperbolic, 双曲, wave motion, 引力波方程?

- parabolic, 抛物, flux

- elliptic, 椭圆

毕奥-萨伐尔定律在流体中的类比。(?)

3 热通量变换公式

q_0: heat flux at reference configuration. (per dA)
q: heat flux at current configuration. (per da)

$$q_0 \cdot \mathrm{d}A = q \cdot \mathrm{d}a \tag{7}$$

南森公式 d$a = J F^{-T} \mathrm{d}A$

$$q_0 = J q F^{-T} = J F^{-1} q \tag{8}$$

重要：当前构形和参考构形的热流变化。

对于各向异性，则要用张量。

isotropic

$$\varepsilon_{ik} = \frac{1}{3}\alpha \Delta T \delta_{ik} \ \Rightarrow \ \varepsilon_{kk} = \alpha \Delta T \tag{9}$$

anisotropic

$$\varepsilon_{ik} = \frac{1}{3}\alpha_{ik}\Delta T \tag{10}$$

α_{ik}- thermal expansion tensor;

$$q = -k\nabla T \qquad q_i = -k_{ik}T_k(?) \tag{11}$$

4 Others

作业：独立分量个数。$\alpha_{ik}, k_{ik} = 1$；高级晶系：立方；$= 2$；中级晶系：四方、六方、三方；$= 3$；低级晶系：单斜、三斜、正交。朗道解释：椭球张量

诺特定理推能量守恒。复习等时变分。

作业：推 34-6

感谢为本次笔记提供参考的同学： 无

连续介质力学 第 29 次课笔记

庄逸

课程日期：2019 年 6 月 10 日　　最后修改：2019 年 6 月 17 日

1　无量纲数·奥内佐格数·水的振动

Ohnesorge number

水的特征速度，Capillary number

$$Ca = \frac{\mu v}{\gamma} \tag{1}$$

Capillary velocity

$$v \sim \frac{\gamma}{\mu} \tag{2}$$

γ : surface tension. Capillary time (viscous)

$$t_{vis} \sim \frac{L}{v} = \frac{L\mu}{\gamma} \tag{3}$$

Lord Rayleigh 写了一本关于振动的好书 *The theory of sound*

$$\tau \sim \sqrt{\frac{\rho R^3}{\gamma}} \sim \sqrt{\frac{m}{\gamma}} \sim \sqrt{\frac{m}{k}} \quad \omega \sim \sqrt{\frac{\gamma}{\rho R^3}} \tag{4}$$

弹簧振子振动方程。谐振子 simple harmonic oscillator

$$\ddot{x} + \omega^2 x = 0 \tag{5}$$

$$[\gamma] = ML^2T^{-2}L^{-2} = MT^{-2} \quad [k] = MT^{-2} \tag{6}$$

analogy between (coefficient of stiffness of spring) & (surface tension). 液滴表面相当于一个小弹簧。可能是期末考题：求液滴的杨氏模量

$$E \sim \frac{\gamma}{r} \tag{7}$$

$$Oh = \frac{t_{vis}}{\tau} = \frac{\sqrt{We}}{Re} \tag{8}$$

2　散度分配律

$$(\rho\boldsymbol{v})\cdot\nabla = \rho\mathrm{div}\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\rho \tag{9}$$

$$(\boldsymbol{v}\boldsymbol{\sigma})\cdot\nabla = \boldsymbol{v}\cdot(\boldsymbol{\sigma}\mathrm{div}) + (\boldsymbol{v}\otimes\nabla):\boldsymbol{\sigma} \tag{10}$$

3　无滑移边界条件

流体的无滑移边界条件：速度沿着边界面切线方向为零，二阶投影张量。$\boldsymbol{uP} = \boldsymbol{0}$

完全滑移边界条件 impermeability condition $\boldsymbol{v}\cdot\boldsymbol{n} = 0$

4　others

作业：推热力学第一第二定律，推 (34-8)(34-12,13)(35-14,15)(35-20)(35-24)(35-26)

固体，free body diagram。流体：contour volume 控制体积。作业：查什么是流体的控制体积?

雷诺输运定理 RTT

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连续介质力学 第 30 次课笔记

庄逸

课程日期：2019 年 6 月 12 日   最后修改：2019 年 6 月 17 日

1 复习

NS 方程及指标形式，q, q_0 之间关系

2 无量纲数

$$\frac{\partial \boldsymbol{u}_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} + f_i \tag{1}$$

$$Oh = \frac{\sqrt{We}}{Re} = \frac{\mu}{\sqrt{\rho \gamma L}} \tag{2}$$

拉普拉斯数 laplace，Suratman number Su

$$La = \frac{\gamma L}{\mu v / L \cdot L^2} = \frac{\rho \gamma L}{\mu \rho v L} = \frac{\rho \gamma L}{\mu^2} = Oh^{-2} \tag{3}$$

3 Invariance

Galilean transformation, classical mechanics, continuum mechanics. 伽利略变换在经典力学和连续介质力学下适用。

$$\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{v}t \qquad t' = t - \tau \tag{4}$$

$\boldsymbol{v}$: uniform motion.

4 inertial frame

惯性坐标系的性质

1. space is isotropic and homogenous.

2. time is homogenous.

3. Simplicity

$L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \ \rightarrow \ L(\dot{\boldsymbol{q}}^2, t) \ \rightarrow \ L(\dot{\boldsymbol{q}}^2)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} - \frac{\partial L}{\partial \boldsymbol{q}} = 0 \quad \rightarrow \quad \frac{\partial L}{\partial \dot{\boldsymbol{q}}} = \mathrm{const} \ \dot{\boldsymbol{q}} = \mathrm{const} \tag{5}$$

(?)

$$\boldsymbol{f} = m\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} \qquad \boldsymbol{f}' = m\frac{\mathrm{d}^2\boldsymbol{r}'}{\mathrm{d}t^2} \tag{6}$$

5 流体力学客观性

Take divergence ∇ onto N-S equations

$$\nabla \cdot \left(-\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{u}\right) = -\frac{1}{\rho}\nabla^2 p + \nu\nabla^2(\nabla \cdot \boldsymbol{u}) \tag{7}$$

$$\nabla \cdot \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla\boldsymbol{u}\right) = \frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{u}) + (\nabla \otimes \boldsymbol{u}) : (\boldsymbol{u} \otimes \nabla) + (\boldsymbol{u} \cdot \nabla)(\nabla \cdot \boldsymbol{u}) \tag{8}$$

$$= \frac{\mathrm{D}}{\mathrm{D}t}(\nabla \cdot \boldsymbol{u}) + (\nabla \otimes \boldsymbol{u}) : (\boldsymbol{u} \otimes \nabla) \tag{9}$$

$$\nabla^2 p = -\rho(\nabla \otimes \boldsymbol{u}) : (\boldsymbol{u} \otimes \nabla) \tag{10}$$

注意

$$(\nabla \otimes \boldsymbol{u}) : (\boldsymbol{u} \otimes \nabla) = \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} \tag{11}$$

原因是后面没东西，所以是右散度。

$$\frac{\partial^2 p}{\partial x_i^2} = -\rho\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} \tag{12}$$

$\mathcal{L}$ $length$, u velocity.

$$\boldsymbol{x}^* = \frac{\boldsymbol{x}}{\mathcal{L}} \quad \boldsymbol{u}^* = \frac{\boldsymbol{u}}{y} \quad t^* = t\frac{u}{\mathcal{L}} \tag{13}$$

1.

$$\nabla \cdot \boldsymbol{u} = 0 \ \Rightarrow \ \frac{\nabla^*}{\mathcal{L}} \cdot (\boldsymbol{u}^*u) = 0 \ \rightarrow \ \nabla^*u^* = 0 \tag{14}$$

2.

$$\frac{\partial(u\boldsymbol{u}^*)}{\partial(t^*u/\mathcal{L})} + (\boldsymbol{u}^*u) \cdot \frac{\nabla^*}{\mathcal{L}}(\boldsymbol{u}^*u) = -\frac{\nabla^*}{\mathcal{L}}(pu^2p^*) \tag{15}$$

$$\frac{\partial\boldsymbol{u}^*}{\partial t^*} + \boldsymbol{u}^* \cdot \nabla^*\boldsymbol{u}^* = -\nabla^*p^* + \frac{1}{Re}\nabla^{*2}\nabla^{*2}\boldsymbol{u}^* \tag{16}$$

3. 作业：把第三个方程无量纲化 (11) 式

1. Re similarity 雷诺相似性. 2. 时空 (space and time) 3. Time reversal（时间反演）

$t^* = -t^*, u^* = -u^*$, Re 项（黏性项）不满足时间反演！

4. 旋转 rotation (fixed angle) & reflection

$$\boldsymbol{e}_i \cdot \hat{\boldsymbol{e}}_j = \cos\theta = a_{ij} \tag{17}$$

$$\hat{\boldsymbol{x}} = a\boldsymbol{x} \qquad \hat{\boldsymbol{u}} = a\boldsymbol{u} \tag{18}$$

感谢为本次笔记提供参考的同学： syj

连续介质力学　第 31 次课笔记

庄逸

课程日期：2019 年 6 月 17 日　　最后修改：2019 年 6 月 17 日

1　复习

麦克斯韦方程组，推电荷密度守恒方程，埃尔曼西应变，格林应变张量，客观性，NS 方程无量纲化，? 的散度。

连续性方程：电荷、浓度、热、动量扩散、概率密度，甚至交通流等。

速度的散度的意义是什么？$\nabla \cdot \boldsymbol{v} = 0$。位移的散度是体积微元变化率 $\dot{\mathrm{d}v} = (\nabla \cdot \boldsymbol{u})\mathrm{d}v$

2　Galiliean invariance

$$\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{v}t \qquad \boldsymbol{v} = \mathrm{const} \qquad t' = t + \alpha \tag{1}$$

惯性参考系中牛顿第二定律也不发生变化，因为求导两次。

2.1　Extended Galiliean invariance

$v = v(t)$ is rectilinear

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} - A_i \tag{2}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial x_i} - A_i = -\frac{1}{\rho}\frac{\partial}{\partial x_i}\left(p + \rho x_j A_j\right) = -\frac{1}{\rho}\frac{\partial}{\partial x_i}\hat{p} \tag{3}$$

$$\hat{p}^* = \frac{p + \rho x_j A_j}{\rho u^2} \tag{4}$$

如果在加速系中做实验，只需要把加速度修正到上面这项即可。

$$\hat{p} = p + \rho \boldsymbol{x} \cdot \boldsymbol{A} \tag{5}$$

2.2 rotation invariance

转动的情况

$$a_{ij} = \boldsymbol{e}_i \cdot \hat{\boldsymbol{e}}_j \tag{6}$$

$$\frac{\mathrm{d}\boldsymbol{e}_i}{\mathrm{d}t} = \Omega_{ij}\boldsymbol{e}_j \tag{7}$$

$\boldsymbol{\Omega}$: spin, $\Omega_{ij} = -\Omega_{ji}$, $\boldsymbol{\Omega} = \boldsymbol{\Omega}^T$ 反对称

frame acceleration

$$A_{ij} = x_j \Omega_{jk} \Omega_{ki} + 2v_j \Omega_{ji} + x_j \frac{\mathrm{d}\omega_{ji}}{\mathrm{d}t} \tag{8}$$

欧拉力

时间反演不满足：粘性项。标架旋转不满足：科氏加速度和角加速度

3 欧几里得客观性

§36

$$\boldsymbol{x}^* - \boldsymbol{x}_0^* = \boldsymbol{Q}(\boldsymbol{x} - \boldsymbol{x}_0) \tag{9}$$

$$\boldsymbol{x}^* = \boldsymbol{x}_0^* + \boldsymbol{Q}(\boldsymbol{x} - \boldsymbol{x}_0)\boldsymbol{x} - \boldsymbol{x}_0 = \boldsymbol{Q}^T(\boldsymbol{x}* - \boldsymbol{x}_0^*) \tag{10}$$

作业：解 (36-4)

4 无量纲数

凑伽利略数：重力/粘性力

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连续介质力学　第 32 次课笔记

庄逸

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1　复习

ε_{ijk}，无量纲数，黏性力是哪个 nian？功共轭，现代张量的数学定义：多重线性映射

2　张量的欧几里得客观性

见书 37 节

首先定义 n 阶张量的客观性。一个 n 阶张量 $\boldsymbol{u}_1 \otimes \boldsymbol{u}_2 \otimes \cdots \otimes \boldsymbol{u}_n$，当更换观察者时，所对应的变换满足下式：

$$(\boldsymbol{u}_1 \otimes \boldsymbol{u}_2 \otimes \cdots \otimes \boldsymbol{u}_n)^* = (\boldsymbol{Q}\boldsymbol{u}_1) \otimes (\boldsymbol{Q}\boldsymbol{u}_2) \otimes \cdots \otimes (\boldsymbol{Q}\boldsymbol{u}_n) \tag{1}$$

对于二阶张量，上式退化为

$$(\boldsymbol{u}_1 \otimes \boldsymbol{u}_2)^* = (\boldsymbol{Q}\boldsymbol{u}_1) \otimes (\boldsymbol{Q}\boldsymbol{u}_2) = (\boldsymbol{Q}\boldsymbol{u}_1) \otimes (\boldsymbol{u}_2\boldsymbol{Q}^T) = \boldsymbol{Q}(\boldsymbol{u}_1 \otimes \boldsymbol{u}_2)\boldsymbol{Q}^T \tag{2}$$

亦即，满足上式的二阶张量就是满足欧几里得客观性。

2.1　变形梯度张量的欧几里得客观性

可从两种途径判断变形梯度的客观性。其一，从其基本定义出发：$\boldsymbol{F} = \partial\boldsymbol{x}/\partial\boldsymbol{X}$，对于另外一个观察者而言，有

$$\boldsymbol{F}^* = \frac{\partial\boldsymbol{x}^*}{\partial\boldsymbol{X}} = \frac{\partial(\boldsymbol{Q}\boldsymbol{x})}{\partial\boldsymbol{x}}\frac{\partial\boldsymbol{x}}{\partial\boldsymbol{X}} = \boldsymbol{Q}\boldsymbol{F} \tag{3}$$

上式反映出，变形梯度的变换就像一个矢量一样，它作为两点张量场，不满足欧几里得客观性的要求。其原因之一，是参考构形中的矢量 $\boldsymbol{X}$ 是不变的，亦即：$\boldsymbol{X}^* = \boldsymbol{X}$，换句话说，$\boldsymbol{X}$ 和观察者无关。

判断变形梯度是否满足客观性要求的途径是由当前构形和参考构形中线段的联系：$\mathrm{d}\boldsymbol{x} = \boldsymbol{F}\mathrm{d}\boldsymbol{X}$，由于：$\mathrm{d}\boldsymbol{x}^* = \boldsymbol{Q}\mathrm{d}\boldsymbol{x}$，$(\mathrm{d}\boldsymbol{x})^* = (\boldsymbol{F}\mathrm{d}\boldsymbol{X})^* = \boldsymbol{F}^*\mathrm{d}\boldsymbol{X}$，因此上式成立。

由于雅可比 $J = \det \boldsymbol{F}$ 作为一个反映体积变化的标量毫无疑问是满足欧几里得客观性的，现利用上式证明如下：

$$J^* = \det \boldsymbol{F}^* = \det(\boldsymbol{Q}\boldsymbol{F}) = (\det \boldsymbol{Q})(\det \boldsymbol{F}) = \det \boldsymbol{F} = J \tag{4}$$

下面我们来看"右柯西-格林"和"左柯西-格林"(Finger) 变形张量的客观性：

$$\begin{cases} \boldsymbol{C}^* = (\boldsymbol{F}^T\boldsymbol{F})^* = (\boldsymbol{Q}\boldsymbol{F})^T\boldsymbol{Q}\boldsymbol{F} = \boldsymbol{F}^T\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{F} = \boldsymbol{F}^T\boldsymbol{F} = \boldsymbol{C} \\ \boldsymbol{B}^* = (\boldsymbol{F}\boldsymbol{F}^T)^* = \boldsymbol{Q}\boldsymbol{F}(\boldsymbol{Q}\boldsymbol{F})^T = \boldsymbol{Q}\boldsymbol{F}\boldsymbol{F}^T\boldsymbol{Q}^T = \boldsymbol{Q}\boldsymbol{B}\boldsymbol{Q}^T \end{cases} \tag{5}$$

2.2 柯西应力的欧几里得客观性

柯西面力 (traction) 矢量 $\boldsymbol{t}$ 是柯西应力 $\boldsymbol{\sigma}$ 和截面外法线 $\boldsymbol{n}$ 的点积：$\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n}$。两个矢量 $\boldsymbol{t}$ 和 $\boldsymbol{n}$ 均满足客观性：$\boldsymbol{t}^* = \boldsymbol{Q}\boldsymbol{t}$，$\boldsymbol{n}^* = \boldsymbol{Q}\boldsymbol{n}$，因此有

$$\boldsymbol{t}^* = \boldsymbol{Q}\boldsymbol{t} = (\boldsymbol{\sigma}\boldsymbol{n})^* = \boldsymbol{\sigma}^*\boldsymbol{n}^* = \boldsymbol{\sigma}^*\boldsymbol{Q}\boldsymbol{n} \ \Rightarrow \ \boldsymbol{Q}\boldsymbol{t} = \boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{n} = \boldsymbol{\sigma}^*\boldsymbol{Q}\boldsymbol{n} \ \Rightarrow \ \boldsymbol{\sigma}^* = \boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^T \tag{6}$$

上式说明，柯西应力作为欧拉型的当前构形中的张量，是满足欧几里得客观性要求的。

基尔霍夫应力由于和可惜盈利之间满足：$\boldsymbol{\tau} = J\boldsymbol{\sigma}$，显然也是客观的应力张量。

2.3 PK1 和 PK2 应力张量的欧几里得客观性

PK1 应力的定义式 $\boldsymbol{P} = J\boldsymbol{\sigma}\boldsymbol{F}^{-T}$，应用 (37-6) 和 (12-13) 两式，故有

$$\boldsymbol{P}^* = J^*\boldsymbol{\sigma}^*(\boldsymbol{F}^*)^{-T} = J\boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^T(\boldsymbol{Q}\boldsymbol{F})^{-T} = J\boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^T\boldsymbol{Q}^{-T}\boldsymbol{F}^{-T} \tag{7}$$

$$= J\boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{F}^{-T} = \boldsymbol{Q}\boldsymbol{P} \tag{8}$$

上式说明，作为两点张量的 PK1 的变换关系和另外一个两点张量——变形梯度张量的变换关系一致，不满足二阶张量的欧几里得客观性要求。

PK2 应力的定义式：$\boldsymbol{T} = \boldsymbol{F}^{-1}\boldsymbol{P}$，变换观察者时，其变换关系为

$$\boldsymbol{T}^* = (\boldsymbol{Q}\boldsymbol{F})^{-1}\boldsymbol{Q}\boldsymbol{P} = \boldsymbol{F}^{-1}\boldsymbol{Q}^{-1}\boldsymbol{Q}\boldsymbol{P} = \boldsymbol{F}^{-1}\boldsymbol{P} = \boldsymbol{T} \tag{9}$$

和右柯西-格林变形张量 $\boldsymbol{C}$ 类似，PK2 作为参考构形中的应力张量，其变换关系和标量一样，不满足张量的欧几里得客观性要求。

3 张量的客观率

3.1 速度梯度、应变率、旋率张量的欧几里得客观性

首先看速度梯度张量的客观性。由于 $l = \dot{F}F^{-1}$，故，速度梯度的变换关系为

$$l^* = \overline{\dot{QF}}(QF)^{-1} = (\dot{Q}F + Q\dot{F})F^{-1}Q^{-1} = \dot{Q}Q^T + QlQ^T = \Omega + QlQ^T \tag{10}$$

在上式中，已经利用了正交张量的基本性质 (20-13) 式，利用 (36-6) 式中的 $\Omega = -\Omega^T$，有

$$\begin{cases} d^* = \dfrac{l^* + l^{*T}}{2} = Q\dfrac{l + l^T}{2}Q^T = QdQ^T \\[2mm] \omega^* = \dfrac{l^* - l^{*T}}{2} = \Omega + Q\omega Q^T \end{cases} \tag{11}$$

上式表明，应变率作为当前构形的欧拉型张量是满足欧几里得客观性要求的；而，旋率的转换关系和速度梯度相同，均不满足张量的客观性要求。

从 (38-2) 式中的第二式：$\omega^* = \dot{Q}Q^T + Q\omega Q^T$ 中容易得到：

$$\begin{cases} \dot{Q} = \omega^* Q - Q\omega \\[2mm] \dot{Q}^T = -Q^T\omega^* + \omega Q^T \end{cases} \tag{12}$$

3.2 客观矢量率的定义

由客观矢量的定义 (36-1) 式，对时间求导，有：$\dot{u}^* = \overline{\dot{Qu}} = \dot{Q}u + Q\dot{u}$，将 (38-3) 式中的第一式代入，得到如下关系：

$$\begin{aligned} \dot{u}^* &= \dot{Q}u + Q\dot{u} = (\omega^* Q - Q\omega)u + Q\dot{u} \\ &= \omega^* u^* + Q\dot{u} - Q\omega u \end{aligned} \tag{13}$$

由上式整理得到：

$$(\dot{u} - \omega u)^* = Q(\dot{u} - \omega u) \tag{14}$$

通过上式可定义满足矢量客观性要求的共旋率 (co-rotational rate)：

$$\bar{u} = \dot{u} - \omega u \tag{15}$$

3.3 客观张量率的定义

对于任意满足客观性要求的二阶张量 A，有 $A^* = QAQ^T$，其时间导数：

$$\dot{A}^* = \dot{Q}AQ^T + Q\dot{A}Q^T + QA\dot{Q}^T \tag{16}$$

将 (38-3) 式代入上式, 整理得到:

$$\left(\dot{A} - \omega A + A\omega\right)^{*} = Q(\dot{A} - \omega A + A\omega)Q^{T} \tag{17}$$

上式便定义了连续介质力学中最为常用的尧曼-扎伦巴 (Jaumann-Zaremba) 客观率:

$$\hat{A} = \dot{A} - \omega A + A\omega \tag{18}$$

特别地, 当 A 取柯西应力 σ, 得到连续介质力学中常用的柯西应力的尧曼-扎伦巴客观率:

$$\hat{\sigma} = \dot{\sigma} - \omega\sigma + \sigma\omega \tag{19}$$

4   Others

连续性方程是客观的, 因为是标量方程?

5   无量纲数·阿基米德数

比较伽利略数和阿基米德数

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连续介质力学 第 33 次课笔记

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1

$$\boldsymbol{B} = \alpha(\operatorname{tr}\boldsymbol{A})\mathbf{I} + \beta\boldsymbol{A} \tag{1}$$

$$\operatorname{tr}\boldsymbol{B} = 3\alpha\operatorname{tr}\boldsymbol{A} + \beta\operatorname{tr}\boldsymbol{A} \quad \Rightarrow \quad \operatorname{tr}\boldsymbol{A} = \frac{1}{3\alpha + \beta}\operatorname{tr}\boldsymbol{B} \tag{2}$$

$$\boldsymbol{B} = \frac{\alpha}{3\alpha + \beta}(\operatorname{tr}\boldsymbol{B})\mathbf{I} + \beta\boldsymbol{A} \tag{3}$$

$$\boldsymbol{A} = \beta^{-1}\left(\boldsymbol{B} - \frac{\alpha}{3\alpha + \beta}(\operatorname{tr}\boldsymbol{B})\mathbf{I}\right) \tag{4}$$

2

Constitutive relation. constitution-宪法. paradigm-范式

理性力学范式：八条公理 axiom

rheology 流变学；theology 神学

注意：NS 方程的不变性，无量纲，雷诺相似性

$$\nabla \cdot \boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{kk} = \operatorname{tr}\boldsymbol{\varepsilon} \tag{5}$$

应变 (?)

3

杆拉伸 tension 压缩 compression. 纵向 longituclinal. 横向 transverse.

拉伸纵向伸长 longituclinal extension

作业：查泊松比的详细定义。

线弹性 linear elastic. 叠加原理 superposition principle

作业：应变是对称的

体模量：施加一个静水压强，水的体积变化率

Stress-strain relation

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda(\mathrm{tr}\boldsymbol{\varepsilon})\mathbf{I} \tag{6}$$

$$\overset{\sigma}{\Delta} = \lambda(\mathrm{tr}\boldsymbol{d})\mathbf{I} + 2\mu\boldsymbol{d} \tag{7}$$

$$\boldsymbol{S} = \boldsymbol{C}^{-1} \tag{8}$$

4 格拉晓夫数

Grashof number

$$\varepsilon_{kk} = \alpha(T - T_0) \qquad \varepsilon_{ik} = \frac{1}{3}\alpha(T - T_0)\delta_{ik} \tag{9}$$

$$Gr = \frac{\text{buoyancy}}{\text{viscous force}} = \frac{gL^3\beta(T_{wall} - T_\infty)}{\nu^2} \tag{10}$$

$\beta = \alpha$ 是热膨胀系数。

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王文标教授；Joseph John Thomoson, four to seven courses per year; Geim

$$\boldsymbol{s} = 2\mu\boldsymbol{\varepsilon} + \lambda(\mathrm{tr}\boldsymbol{\varepsilon})\mathbf{I} \tag{1}$$

力、化耦合的胡克定律，JCM Li，(james) 李振民

$$\varepsilon_{ij} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}] + \frac{C\Omega}{3}\delta_{ij} \tag{2}$$

通过类比法得到。

$$\varepsilon_{ik} = \frac{1}{3}\alpha(T - T_0)\delta_{ik} \tag{3}$$

每当你创立一个新学科的时候首先要考虑公理体系和范式。

作业：求 s8-1 的逆，对比 s8-2,3

1　弹性体的分类

- Cauchy elastic, 胡克定律

- hyperelastic, hyper: to a higher degree 超弹性

- hypoelastic, to a lesser degree 次弹性，低弹性

低弹性本构关系

Jaumann rate $\overset{\triangle}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{\omega}\boldsymbol{\sigma} + \boldsymbol{\sigma}\boldsymbol{\omega}$

$$\overset{\triangle}{\boldsymbol{\sigma}} = 2\mu\boldsymbol{d} + \lambda(\mathrm{tr}\boldsymbol{d})\mathbf{I} \tag{4}$$

work conjugacy

$$\dot{\omega} = J\boldsymbol{\sigma} : \boldsymbol{d} = \boldsymbol{\tau} : \boldsymbol{d} = \boldsymbol{P} : \dot{\boldsymbol{F}} = \boldsymbol{T} : \dot{\boldsymbol{E}} \tag{5}$$

spin

$$\boldsymbol{\omega} = \frac{\boldsymbol{l} - \boldsymbol{l}^{\mathrm{T}}}{2} \quad \boldsymbol{d} = \frac{\boldsymbol{l} + \boldsymbol{l}^{\mathrm{T}}}{2} \tag{6}$$

理论上讲这个是对的：

$$\overset{\triangle}{\boldsymbol{\tau}} = 2\mu\boldsymbol{d} + \lambda(\mathrm{tr}\boldsymbol{d})\mathbf{I} \tag{7}$$

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连续介质力学　第 35 次课笔记

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1　复习

stretch ratio λ_i

超弹性：有势，potential

超弹性是处理非线性弹性的。

材料可压缩–要有 $J-1$

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连续介质力学　第 36 次课笔记

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1　复习

转动惯量的表达、格林应变张量、二维板、轴的一维振动 $\nabla \cdot \boldsymbol{\sigma} = \rho \boldsymbol{a}$

波动方程记得写波速

$\nabla \cdot \boldsymbol{\sigma} = \boldsymbol{0}$ 的物理意义

什么是惯性系？伽利略变换，空间均匀各向同性，时间均匀，简单性（二次方）

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连续介质力学 第 37 次课笔记

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1 复习

质量守恒，连续性方程，雷诺输运定理的推导。

event observer

$$\rho = \rho^* \qquad J = J^* \tag{1}$$

continuity equation, Lagrangian viewpoint, $\rho_0^* = J^* \rho^*$

obvious, mathematical humor, scale 标尺

2 objectivity of continuity equation

$$\frac{\partial \rho^*}{\partial t^*} + \nabla^* \cdot (\rho \boldsymbol{v})^* = 0 \quad \Rightarrow \quad \frac{\partial \rho^*}{\partial t^*} + \frac{\partial}{\partial x_l^*}(\rho^* Q_{lk} v_k) = 0 \tag{2}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_p}(\rho Q_{lk} v_k)\frac{\partial x_p}{\partial x_l^*} = A \tag{3}$$

$$\boldsymbol{x}^* - \boldsymbol{x}_0^* = \boldsymbol{Q}(t)(\boldsymbol{x} - \boldsymbol{x}_0) \tag{4}$$

$$\boldsymbol{Q}\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{x}^*} = \mathbf{I} \tag{5}$$

$$\frac{\partial x_p}{\partial x_l^*} = Q_{lp} \quad \det \boldsymbol{Q} = \pm 1 \tag{6}$$

$$A = \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_p}(\rho v_k)Q_{lk}Q_{lp} \tag{7}$$

$$\because Q_{lk}Q_{lp} = \delta_{kp} \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} + \delta_{kp}\frac{\partial}{\partial x_p}(\rho v_k) = 0 \tag{8}$$

$$\because \delta_{kp} = \frac{\partial x_p}{\partial x_k} \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho v_k) = 0 \tag{9}$$

$[\mathrm{div}(\rho\boldsymbol{v})]^*_k$, $\boldsymbol{v}^* = \boldsymbol{Q}\boldsymbol{v}$

动量方程客观性, equations of equlibrium

$$\nabla \cdot \boldsymbol{\sigma} = \boldsymbol{0} \tag{10}$$

$$\nabla^* \cdot \boldsymbol{\sigma}^* = \frac{\partial}{\partial x_l^*}\boldsymbol{\sigma}^* \tag{11}$$

$$\boldsymbol{\sigma}^* = \boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^T \tag{12}$$

$$(\nabla^* \cdot \boldsymbol{\sigma}^*)_k = \frac{\partial}{\partial x_l^*}(Q_{lr}\sigma_{rs}Q_{ks}) \tag{13}$$

$$= \frac{\partial}{\partial x_q}(Q_{lr}\sigma_{rs}Q_{ks})\frac{\partial x_q}{\partial x_l^*} \tag{14}$$

$$= Q_{lr}Q_{ks}Q_{lq}\frac{\partial \sigma_{rs}}{\partial x_q} = Q_{ks}\delta_{rq}\frac{\partial \sigma_{rs}}{\partial x_q} \tag{15}$$

$$\nabla^* \cdot \boldsymbol{\sigma}^* = Q_{sk}\frac{\partial \sigma_{rs}}{\partial x_r} = \boldsymbol{Q} \cdot (\nabla \cdot \boldsymbol{\sigma}) \tag{16}$$

Duhem 迪昂

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \nabla) \tag{17}$$

作业：证明 s10-1

柱坐标的变形张量？26-6，黎曼度规

$$\boldsymbol{F} \to \tag{18}$$

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1 其他坐标的变形梯度和黎曼度规

$$\boldsymbol{X} = R\boldsymbol{g}^R + \Theta\boldsymbol{g}^\Theta + Z\boldsymbol{g}^Z \tag{1}$$

$$\boldsymbol{x} = r\boldsymbol{g}^r + \theta\boldsymbol{g}^\theta + z\boldsymbol{g}^z \tag{2}$$

$$[\boldsymbol{F}] = \begin{bmatrix} \frac{\partial r}{\partial R} & \frac{1}{R}\frac{\partial r}{\partial \theta} & \frac{\partial r}{\partial z} \\ r\frac{\partial \theta}{\partial R} & \frac{r}{R}\frac{\partial \theta}{\partial \Theta} & r\frac{\partial \theta}{\partial z} \\ \frac{\partial z}{\partial R} & \frac{1}{R}\frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial Z} \end{bmatrix} \tag{3}$$

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = g^{\mu\nu}\mathrm{d}x_\mu\mathrm{d}x_\nu \tag{4}$$

$$[g_{\mu\nu}] = \begin{bmatrix} 1 & & \\ & r^2 & \\ & & 1 \end{bmatrix} \tag{5}$$

2

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\boldsymbol{v}) = 0 \tag{6}$$

满足时间反演 $t \cdot -t$。NS 方程不满足时间反演

$$\boldsymbol{Q} = Q_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \quad \boldsymbol{Q}^\mathrm{T} = Q_{ji}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{7}$$

$$\boldsymbol{Q}\boldsymbol{Q}^\mathrm{T} = (Q_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j) \cdot (Q_{sr}\boldsymbol{e}_r \otimes \boldsymbol{e}_s) = \delta_{jr}Q_{ij}Q_{sr}\boldsymbol{e}_i \otimes \boldsymbol{e}_s = \delta_{is}\boldsymbol{e}_i \otimes \boldsymbol{e}_s = \mathbf{I} \tag{8}$$

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \nabla) \tag{9}$$

$$2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} \tag{10}$$

$$\nabla \times \boldsymbol{\varepsilon} \times \nabla = \boldsymbol{0} \tag{11}$$

$$\frac{\partial}{\partial x_i} \boldsymbol{e}_i \times \boldsymbol{e}_j \varepsilon_{jk} \otimes \frac{\partial}{\partial x_l} \boldsymbol{e}_k \times \boldsymbol{e}_l = \frac{\partial^2 \varepsilon_{jk}}{\partial x_i \partial x_l} \varepsilon_{ijm} \varepsilon_{kln} \boldsymbol{e}_m \otimes \boldsymbol{e}_n \tag{12}$$

$$\varepsilon_{jk,il} \varepsilon_{ijm} \varepsilon_{kln} = 0 \tag{13}$$

有限变形、大变形 finite deformation, compability condition

$$\boldsymbol{F} \times \nabla_{\boldsymbol{X}} = \boldsymbol{0} \tag{14}$$

$$\frac{\partial^2 x_i}{\partial X_K \partial X_L} \varepsilon_{KLM} \boldsymbol{e}_i \otimes \boldsymbol{e}_M = \boldsymbol{0} \tag{15}$$

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$$\boldsymbol{F} \times \nabla_{\boldsymbol{X}} = \boldsymbol{0} \tag{1}$$

测地线方程

非线性经典场论 non-linear classical field theory

公理化和客观性，公理化起码要知道一两个，哥德尔不完备定理

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