Solution to the Differential Equation $y'' + y' + y + t = 0$

Problem

Solve the differential equation:

$$y'' + y' + y + t = 0 \tag{1}$$

Solution

First, we solve the homogeneous part of the differential equation:

$$y'' + y' + y = 0 \tag{2}$$

The characteristic equation is:

$$r^2 + r + 1 = 0 \tag{3}$$

Solving this characteristic equation, we get the roots:

$$r = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}i}{2} \tag{4}$$

Thus, the general solution to the homogeneous equation is:

$$y_h(t) = e^{-\frac{t}{2}} \left(C_1 \sin\left(\frac{\sqrt{3}t}{2}\right) + C_2 \cos\left(\frac{\sqrt{3}t}{2}\right)\right) \tag{5}$$

Now, we find a particular solution to the non-homogeneous equation. We try a particular solution of the form:

$$y_p(t) = At + B \tag{6}$$

Substituting $y_p(t)$ into the non-homogeneous differential equation:

$$y_p'' + y_p' + y_p + t = 0 \tag{7}$$

$$0 + A + (At + B) + t = 0 \tag{8}$$

$$(1 + A)t + (A + B) = 0 \tag{9}$$

Equating coefficients, we get:

$$1 + A = 0 \tag{10}$$
$$A + B = 0 \tag{11}$$

Solving these equations, we find:

$$A = -1 \tag{12}$$
$$B = 1 \tag{13}$$

Thus, the particular solution is:

$$y_p(t) = -t + 1 \tag{14}$$

Combining the homogeneous and particular solutions, we get the general solution:

$$y(t) = y_h(t) + y_p(t) = e^{-\frac{t}{2}} \left(C_1 \sin\left(\frac{\sqrt{3}t}{2}\right) + C_2 \cos\left(\frac{\sqrt{3}t}{2}\right) \right) - t + 1 \tag{15}$$

Therefore, the solution to the differential equation is:

$$y(t) = -t + \left(C_1 \sin\left(\frac{\sqrt{3}t}{2}\right) + C_2 \cos\left(\frac{\sqrt{3}t}{2}\right) \right) e^{-\frac{t}{2}} + 1 \tag{16}$$