Notes on Isospin Relations

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Abstract

Notes in the isospin relations among the τ-decay branching fractions. Suggestion on how to reduce the error on B_{SL} is also shown.

I. G-PARITY CONSIDERATIONS

The Kaons are bound states of the following quark-antiquark pairs:

$$\begin{aligned} K^+ &= (u\bar{s}) \, , \\ K^0 &= (d\bar{s}) \, , \\ \overline{K}^0 &= (\bar{d}s) \, , \\ K^- &= (\bar{u}s) \, . \end{aligned} \tag{1}$$

G-parity is a combination of charge conjugation and a rotation around I_2 axis:

$$G = Ce^{i\pi I_2} \, . \tag{2}$$

Note that $G^2 = 1$. For the quarks, which are isospin doublets, $I_2 = \frac{1}{2}\sigma_2$ and

$$e^{i\pi I_2} = e^{i\pi\sigma_2/2} = \cos\frac{\pi}{2} + i\sigma_2 \sin\frac{\pi}{2} = i\sigma_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \, . \tag{3}$$

Therefore,

$$\begin{aligned} \begin{bmatrix} u \\ d \end{bmatrix} &\xrightarrow{e^{i\pi I_2}} \begin{bmatrix} d \\ -u \end{bmatrix} \xrightarrow{C} \begin{bmatrix} \bar{d} \\ -\bar{u} \end{bmatrix} \, , \\ \begin{bmatrix} \bar{d} \\ -\bar{u} \end{bmatrix} &\xrightarrow{e^{i\pi I_2}} \begin{bmatrix} -\bar{u} \\ -\bar{d} \end{bmatrix} \xrightarrow{C} \begin{bmatrix} -u \\ -d \end{bmatrix} \, . \end{aligned} \tag{4}$$

For the Kaons,

$$\begin{aligned} \begin{bmatrix} K^+ \\ K^0 \end{bmatrix} &= \begin{bmatrix} u\bar{s} \\ d\bar{s} \end{bmatrix} \xrightarrow{e^{i\pi I_2}} \begin{bmatrix} d\bar{s} \\ -u\bar{s} \end{bmatrix} \xrightarrow{C} \begin{bmatrix} \bar{d}s \\ -\bar{u}s \end{bmatrix} = \begin{bmatrix} \overline{K}^0 \\ -K^- \end{bmatrix} \, , \\ \begin{bmatrix} \overline{K}^0 \\ -K^- \end{bmatrix} &= \begin{bmatrix} \bar{d}s \\ -\bar{u}s \end{bmatrix} \xrightarrow{e^{i\pi I_2}} \begin{bmatrix} -\bar{u}s \\ -\bar{d}s \end{bmatrix} \xrightarrow{C} \begin{bmatrix} -u\bar{s} \\ -d\bar{s} \end{bmatrix} = \begin{bmatrix} -K^+ \\ -K^0 \end{bmatrix} \, . \end{aligned} \tag{5}$$

Two-Kaon systems can have either $I = 0$ or $I = 1$. The $I = 0$ state is

$$\frac{1}{\sqrt{2}}\left(\overline{K}^0 K^0 + K^- K^+\right) \, , \tag{6}$$

while the $I = 1$ states are

$$\begin{bmatrix} \overline{K}^0 K^+ \\ \frac{1}{\sqrt{2}}\left(\overline{K}^0 K^0 - K^- K^+\right) \\ -K^- K^0 \end{bmatrix} \, . \tag{7}$$

Under G-parity, the $I = 0$ state transform as

$$\frac{1}{\sqrt{2}}\left(\overline{K}^0 K^0 + K^- K^+\right) \xrightarrow{G} \frac{1}{\sqrt{2}}\left(K^+ K^- + K^0 \overline{K}^0\right) = \frac{(-1)^\ell}{\sqrt{2}}\left(K^- K^+ + \overline{K}^0 K^0\right) ,$$

while the $I = 1$ states transform as

$$\begin{bmatrix} \overline{K}^0 K^+ \\ \frac{1}{\sqrt{2}}\left(\overline{K}^0 K^0 - K^- K^+\right) \\ -K^- K^0 \end{bmatrix} \xrightarrow{G} \begin{bmatrix} -K^+ \overline{K}^0 \\ \frac{1}{\sqrt{2}}\left(K^- K^+ - K^0 \overline{K}^0\right) \\ K^0 K^- \end{bmatrix} = -(-1)^\ell \begin{bmatrix} \overline{K}^0 K^+ \\ \frac{1}{\sqrt{2}}\left(\overline{K}^0 K^0 - K^- K^+\right) \\ -K^- K^0 \end{bmatrix} , \tag{8}$$

where ℓ is the orbital angular momentum of the two-Kaon system. Combining the above results we obatin the following relation

$$G_{K\overline{K}} = (-1)^{I_{KK}}(-1)^{\ell_{KK}} . \tag{9}$$

Consider the process $\tau^- \to (\pi K K)^- \nu_\tau$. The amplitude of this decay is proportional to

$$\left\langle \nu_\tau \middle| \bar{\nu}_\tau \gamma_\mu (1-\gamma_5) \tau^- \middle| \tau^- \right\rangle \left\langle \pi K K \middle| \bar{d} \gamma^\mu (1-\gamma_5) u \middle| 0 \right\rangle . \tag{10}$$

The quantum numbers of the $\pi K K$ state is determined by those of the current $\bar{d}\gamma^\mu(1-\gamma_5)u$. It is clear that the isospin of this operator is $I = 1$, $I_3 = +1$, so the final $\pi K K$ state must have isospin $I = 1$, $I_3 = -1$. (The current annihilates $I_3 = +1$ and creates $I_3 = -1$.) The G-parity of the vector and axial-vector parts of the current are different:

$$\begin{array}{lclcll} (\bar{d}\gamma^\mu u) & \xrightarrow{e^{i\pi I_2}} & -(\bar{u}\gamma^\mu d) & \xrightarrow{C} & -(\overline{u^c}\gamma^\mu d^c) & = +(\bar{d}\gamma^\mu u) , \\ (\bar{d}\gamma^\mu \gamma_5 u) & \xrightarrow{e^{i\pi I_2}} & -(\bar{u}\gamma^\mu \gamma_5 d) & \xrightarrow{C} & -(\overline{u^c}\gamma^\mu \gamma_5 d^c) & = -(\bar{d}\gamma^\mu \gamma_5 u) . \end{array} \tag{11}$$

So for decays through the vector current, the total G-parity of the final state is $+1$. Since the pion has G-parity -1, The G-parity of the KK system has to equal to -1. So there are two possible combinations of angular momentum and isospin: $\ell_{KK} = 1$ when $I_{KK} = 0$, and $\ell_{KK} = 0$ when $I_{KK} = 1$. For decays through the axial vector current, the total G-parity of the final state is -1. So the G-parity of the KK system has to be $+1$. Therefore, the two possible combinations in this case are: $\ell_{KK} = 0$ when $I_{KK} = 0$, and $\ell_{KK} = 1$ when $I_{KK} = 1$. Since these four modes have different quantum numbers there are no interference terms, and the total decay width can be written as:

$$\Gamma_{KK\pi} = \Gamma_V^0 + \Gamma_V^1 + \Gamma_A^0 + \Gamma_A^1 , \tag{12}$$

where subscripts V and A indicate vector current and axial-vector current, respectively, and the superscripts 0 and 1 indicate the isospin of the KK system. These decay channels are, in the absence of models that relate them, independent of each other. Now let's investigate each of these four cases.

A. Vector Current

1. For $I_{KK} = 0$, the $KK\pi$ state in the final state of $\tau \to \pi KK\nu$ is

$$\frac{1}{\sqrt{2}}\left(\overline{K}^0 K^0 + K^- K^+\right)\pi^- \ , \tag{13}$$

   so the KK in the final state is $\overline{K}^0 K^0$ half of the time, and K^+K^- half of the time. Furthermore, since $\ell_{KK} = 1$ in this case, the wave-function of the KK system must be anti-symmetric under the interchange of the location of the two Kaons. imposing this anti-symmetry on the $\overline{K}^0 K^0$ wave-function, and rewriting it in terms of K_L and K_S, we find:

$$\begin{aligned}
\overline{K}^0 K^0 &= \frac{1}{\sqrt{2}}\Big[\overline{K}^0(x)K^0(y) - K^0(x)\overline{K}^0(y)\Big] \\
&= \frac{1}{2\sqrt{2}}\Big[(K_S - K_L)_x(K_S + K_L)_y) - (K_S + K_L)_x(K_S - K_L)_y\Big] \\
&= \frac{1}{\sqrt{2}}\Big[K_S(x)K_L(y) - K_L(x)K_S(y)\Big] \ .
\end{aligned} \tag{14}$$

   So the decay widths satisfy the relation:

$$\Gamma_{K^+K^-\pi^-} \ = \ \Gamma_{K_LK_S\pi^-} \ = \ \frac{1}{2}\Gamma_V^0 \ . \tag{15}$$

2. For $I_{KK} = 1$, the $KK\pi$ state is

$$\frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(\overline{K}^0 K^0 - K^- K^+)\pi^- + K^- K^0 \pi^0\right] . \tag{16}$$

   Again, let's look at $K^0\overline{K}^0$. In this case $\ell_{KK} = 0$ so the KK wave-function must be symmetric under the interchange of the two Kaons. By symmetrizing the wave-function, and rewriting $K\overline{K}$ in terms of K_L and K_S, we find:

$$\begin{aligned}
\overline{K}^0 K^0 &= \frac{1}{\sqrt{2}}\Big[\overline{K}^0(x)K^0(y) + K^0(x)\overline{K}^0(y)\Big] \\
&= \frac{1}{2\sqrt{2}}\Big[(K_S - K_L)_x(K_S + K_L)_y) + (K_S + K_L)_x(K_S - K_L)_y\Big]
\end{aligned}$$

$$= \frac{1}{\sqrt{2}}\Big[K_S(x)K_S(y) - K_L(x)K_L(y)\Big] . \quad (17)$$

So the individual decay widths are

$$\begin{aligned}
\Gamma_{K^+K^-\pi^-} &= \frac{1}{4}\Gamma_V^1 , \\
\Gamma_{K^-K^0\pi^0} &= \frac{1}{2}\Gamma_V^1 , \\
\Gamma_{K_SK_S\pi^-} &= \frac{1}{2}\Gamma_{K^0\overline{K}^0\pi^-} = \frac{1}{8}\Gamma_V^1 \\
\Gamma_{K_LK_L\pi^-} &= \frac{1}{2}\Gamma_{K^0\overline{K}^0\pi^-} = \frac{1}{8}\Gamma_V^1 . \quad (18)
\end{aligned}$$

B. Axial Vector Current

1. For $I_{KK} = 0$, same as vector current case, the final state is:

$$\frac{1}{\sqrt{2}}\left(\overline{K}^0K^0 + K^-K^+\right)\pi^- , \quad (19)$$

In this case, $\ell_{KK} = 0$. Therefore, similar to the $I_{KK} = 1$ case for the vector current, we can symmetrize the $\overline{K}^0K^0$ wave-function to get:

$$\overline{K}^0K^0 = \frac{1}{\sqrt{2}}\Big[(K_S(x)K_S(y) - K_L(x)K_L(y))\Big] . \quad (20)$$

So the individual decay widths, in term of Γ_A^0, are:

$$\begin{aligned}
\Gamma_{K^+K^-\pi^-} &= \frac{1}{2}\Gamma_A^0 , \\
\Gamma_{K_SK_S\pi^-} &= \frac{1}{4}\Gamma_A^0 , \\
\Gamma_{K_LK_L\pi^-} &= \frac{1}{4}\Gamma_A^0 . \quad (21)
\end{aligned}$$

2. For $I_{KK} = 1$, the final state is:

$$\frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(\overline{K}^0K^0 - K^-K^+)\pi^- + K^-K^0\pi^0\right] . \quad (22)$$

In this case, $\ell_{KK} = 0$. Similar to the $I_{KK} = 0$ case for the vector current, we can antisymmetrize the $\overline{K}^0K^0$ wave-function to get:

$$\overline{K}^0K^0 = \frac{1}{\sqrt{2}}\Big[K_S(x)K_L(y) - K_L(x)K_S(y)\Big] . \quad (23)$$

So the individual decay widths are

$$\begin{aligned}
\Gamma_{K^+K^-\pi^-} &= \frac{1}{4}\Gamma_A^1 , \\
\Gamma_{K^-K^0\pi^0} &= \frac{1}{2}\Gamma_A^1 , \\
\Gamma_{K_LK_S\pi^-} &= \frac{1}{4}\Gamma_A^1 . \quad (24)
\end{aligned}$$

II. RESULTS

I'm going to use the following shorthands:

$$\begin{aligned}
B_{LL} &= B(\tau^- \to \pi^- K_L^0 K_L^0 \nu_\tau) \ , \\
B_{SS} &= B(\tau^- \to \pi^- K_S^0 K_S^0 \nu_\tau) \ , \\
B_{SL} &= B(\tau^- \to \pi^- K_S^0 K_L^0 \nu_\tau) \ , \\
B_{00} &= B(\tau^- \to \pi^- K^0 \bar{K}^0 \nu_\tau) \ , \\
B_{+-} &= B(\tau^- \to \pi^- K^+ K^- \nu_\tau) \ .
\end{aligned} \tag{25}$$

From equations (15), (18), (21) and (24), we can see that

$$\begin{aligned}
\Gamma_{K_L K_L \pi^-} = \Gamma_{K_S K_S \pi^-} &= \frac{1}{8}\Gamma_V^1 + \frac{1}{4}\Gamma_A^0 \ , \\
\Gamma_{K_S K_L \pi^-} &= \frac{1}{2}\Gamma_V^0 + \frac{1}{4}\Gamma_A^1 \ .
\end{aligned} \tag{26}$$

so the branching fractions B_{LL} and B_{SS} satisfy:

$$B_{LL} = B_{SS} \ , \tag{27}$$

but there is no obvious relation between $B_{SS} = B_{LL}$ and B_{SL}. The total width to $K^0\bar{K}^0\pi^-$ is the following sum:

$$\begin{aligned}
\Gamma_{K^0\bar{K}^0\pi^-} &= \Gamma_{K_S K_S \pi^-} + \Gamma_{K_L K_L \pi^-} + \Gamma_{K_S K_L \pi^-} \\
&= 2\Gamma_{K_S K_S \pi^-} + \Gamma_{K_S K_L \pi^-} \ ,
\end{aligned} \tag{28}$$

from which we deduce

$$B_{00} = 2B_{SS} + B_{SL} \ . \tag{29}$$

On the other hand, from equations (15), (18), (21), and (24), we also find

$$\begin{aligned}
\Gamma_{K^0\bar{K}^0\pi^-} &= \Gamma_{K_S K_S \pi^-} + \Gamma_{K_L K_L \pi^-} + \Gamma_{K_S K_L \pi^-} \\
&= \left(\frac{1}{8}\Gamma_V^1 + \frac{1}{4}\Gamma_A^0\right) + \left(\frac{1}{8}\Gamma_V^1 + \frac{1}{4}\Gamma_A^0\right) + \left(\frac{1}{2}\Gamma_V^0 + \frac{1}{4}\Gamma_A^1\right) \\
&= \frac{1}{2}\Gamma_V^0 + \frac{1}{4}\Gamma_V^1 + \frac{1}{2}\Gamma_A^0 + \frac{1}{4}\Gamma_A^1 \\
&= \Gamma_{K^+K^-\pi^-},
\end{aligned}$$

which gives us another relation:

$$B_{00} = B_{+-} \ . \tag{30}$$

The PDG 2008 averages are:

$$\begin{aligned} B_{SS} &= (2.4 \pm 0.5) \times 10^{-4} \\ B_{SL} &= (1.01 \pm 0.26) \times 10^{-3} \\ B_{+-} &= (1.37 \pm 0.06) \times 10^{-3} \end{aligned} \tag{31}$$

while the fit numbers are:

$$\begin{aligned} B_{SS} &= (2.4 \pm 0.5) \times 10^{-4} \\ B_{SL} &= (1.2 \pm 0.4) \times 10^{-3} \\ B_{+-} &= (1.40 \pm 0.05) \times 10^{-3}. \end{aligned} \tag{32}$$

Using these numbers we find:

$$\begin{aligned} B_{00} &= 2B_{SS} + B_{SL} = (1.49 \pm 0.28) \times 10^{-3} \qquad \text{(average)}\ , \\ B_{00} &= 2B_{SS} + B_{SL} = (1.7 \pm 0.4) \times 10^{-3} \qquad \text{(fit)}\ . \end{aligned} \tag{33}$$

Both agree with B_{+-} within one sigma, which is a good sign. The ratio $2B_{SS}/B_{SL}$ is

$$\begin{aligned} \frac{2B_{SS}}{B_{SL}} &= 0.48 \pm 0.16 \qquad \text{(average)}\ , \\ \frac{2B_{SS}}{B_{SL}} &= 0.40 \pm 0.16 \qquad \text{(fit)}. \end{aligned} \tag{34}$$

So

$$\begin{aligned} \frac{B_{SL}}{B_{00}} &= \frac{1}{1 + 2B_{SS}/B_{SL}} = 0.68 \pm 0.07 \qquad \text{(average)}\ , \\ \frac{B_{SL}}{B_{00}} &= \frac{1}{1 + 2B_{SS}/B_{SL}} = 0.71 \pm 0.08 \qquad \text{(fit)}\ . \end{aligned} \tag{35}$$

Using $B_{00} = B_{+-}$, we can claim

$$\begin{aligned} B_{SL} &= \frac{B_{SL}}{B_{00}} B_{+-} = 0.093 \pm 0.011\% \qquad \text{(average)}\ , \\ B_{SL} &= \frac{B_{SL}}{B_{00}} B_{+-} = 0.100 \pm 0.012\% \qquad \text{(fit)}. \end{aligned} \tag{36}$$

This is probably the best you can do to reduce the error on B_{SL}.