`pyprop`—Implementation notes

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This document records various theoretical details that underpin the implementation of `pyprop`.

1 The P-SV system

We have (OW18 eqs.55–57)

$$\mathbf{A}^{\text{P-SV}} = \mathbf{Z}\mathbf{A}'\mathbf{Z}^{-1} \tag{1}$$

with

$$\mathbf{Z} = \begin{pmatrix} \frac{1}{\sqrt{\rho}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{\rho}} \\ 0 & 0 & \sqrt{\rho} & -\frac{2k\mu}{\sqrt{\rho}} \\ \frac{2k\mu}{\sqrt{\rho}} & \sqrt{\rho} & 0 & 0 \end{pmatrix} \tag{2}$$

$$\mathbf{A}' = \begin{pmatrix} 0 & 0 & \frac{\rho}{\sigma} & -k \\ 0 & 0 & -k & \omega^2 \\ -\omega^2 & -k & 0 & 0 \\ -k & -\frac{\rho}{\mu} & 0 & 0 \end{pmatrix} \tag{3}$$

We use the notation $\mathcal{P}_6$ to denote the 6×6 minor propagator that carries a minor vector $\mathbf{m}_a$ at z_a to $z_b = z_a + h$. It can be shown that

$$\boldsymbol{\mathcal{P}_6} = \boldsymbol{\mathcal{Z}_6}\mathbf{R_1}\mathbf{U}\boldsymbol{\Lambda}\mathbf{V}\mathbf{R_2}\boldsymbol{\mathcal{Z}_6^{-1}} \tag{4}$$

where $\boldsymbol{\mathcal{Z}_6}$ is the 6×6 minor matrix associated with $\mathbf{Z}$, and where

$$\mathbf{R_1} = \operatorname{diag}\begin{pmatrix} 1 & \frac{k}{\zeta_\sigma} & \frac{k^2}{\omega^2\zeta_\sigma} & \frac{\omega^2}{\zeta_\sigma} & \frac{k}{\zeta_\sigma} & \frac{\zeta_\mu}{\zeta_\sigma} \end{pmatrix} \tag{5}$$

$$\mathbf{U} = \begin{pmatrix} -1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 1 \\ \xi+1 & \xi+1 & \xi-1 & \xi-1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 1 & -1 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 \end{pmatrix} \tag{6}$$

$$\boldsymbol{\Lambda} = \begin{pmatrix} \exp h\left(\zeta_\sigma - \zeta_\mu\right) & \exp h\left(\zeta_\mu - \zeta_\sigma\right) & \exp -h\left(\zeta_\mu + \zeta_\sigma\right) & \exp h\left(\zeta_\mu + \zeta_\sigma\right) & \frac{\zeta_\mu\zeta_\sigma}{k^2} & \frac{\zeta_\mu\zeta_\sigma}{k^2} \end{pmatrix} \tag{7}$$

$$\mathbf{V} = \frac{1}{4}\begin{pmatrix} -1 & -1 & 1 & -\xi-1 & 1 & -1 \\ 1 & -1 & 1 & -\xi-1 & 1 & 1 \\ -1 & -1 & 1 & \xi-1 & 1 & 1 \\ 1 & -1 & 1 & \xi-1 & 1 & -1 \\ 0 & 0 & 0 & 4 & -4 & 0 \\ 0 & 4 & 0 & 4 & 0 & 0 \end{pmatrix} \tag{8}$$

$$\mathbf{R_2} = \operatorname{diag}\begin{pmatrix} 1 & \frac{k}{\zeta_\mu} & \frac{\omega^2}{\zeta_\mu} & \frac{k^2}{\omega^2\zeta_\mu} & \frac{k}{\zeta_\mu} & \frac{\zeta_\sigma}{\zeta_\mu} \end{pmatrix} \tag{9}$$

with

$$\zeta_\sigma = \sqrt{k^2 - \frac{\rho\omega^2}{\sigma}}$$

$$\zeta_\mu = \sqrt{k^2 - \frac{\rho\omega^2}{\mu}}$$

$$\xi = \frac{\zeta_\mu\zeta_\sigma}{k^2}$$

$$\partial_{\zeta_\sigma}\mathbf{R_1} = \mathrm{diag}\begin{pmatrix} 0 & -\frac{k}{\zeta_\sigma^2} & -\frac{k^2}{\omega^2\zeta_\sigma^2} & -\frac{\omega^2}{\zeta_\sigma^2} & -\frac{k}{\zeta_\sigma^2} & -\frac{\zeta_\mu}{\zeta_\sigma^2} \end{pmatrix} \tag{10}$$

$$\partial_{\zeta_\mu}\mathbf{R_1} = \mathrm{diag}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{\zeta_\sigma} \end{pmatrix} \tag{11}$$

$$\partial_{\zeta_\sigma}\mathbf{U} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta_\mu}{k^2} & \frac{\zeta_\mu}{k^2} & \frac{\zeta_\mu}{k^2} & \frac{\zeta_\mu}{k^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{12}$$

$$\partial_{\zeta_\mu}\mathbf{U} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta_\sigma}{k^2} & \frac{\zeta_\sigma}{k^2} & \frac{\zeta_\sigma}{k^2} & \frac{\zeta_\sigma}{k^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{13}$$

$$\partial_h\mathbf{\Lambda} = \mathrm{diag}\begin{pmatrix} \zeta_\sigma-\zeta_\mu & \zeta_\mu-\zeta_\sigma & -\zeta_\mu-\zeta_\sigma & \zeta_\mu+\zeta_\sigma & 0 & 0 \end{pmatrix}\mathbf{\Lambda} \tag{14}$$

$$\partial_{\zeta_\sigma}\mathbf{\Lambda} = \mathrm{diag}\begin{pmatrix} h & -h & -h & h & \frac{1}{\zeta_\sigma} & \frac{1}{\zeta_\sigma} \end{pmatrix}\mathbf{\Lambda} \tag{15}$$

$$\partial_{\zeta_\mu}\mathbf{\Lambda} = \mathrm{diag}\begin{pmatrix} -h & h & -h & h & \frac{1}{\zeta_\mu} & \frac{1}{\zeta_\mu} \end{pmatrix}\mathbf{\Lambda} \tag{16}$$

$$\partial_{\zeta_\sigma}\mathbf{V} = \frac{\zeta_\mu}{4k^2}\begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{17}$$

$$\partial_{\zeta_\mu}\mathbf{V} = \frac{\zeta_\sigma}{4k^2}\begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{18}$$

$$\partial_{\zeta_\mu}\mathbf{R_2} = \mathrm{diag}\begin{pmatrix} 0 & -\frac{k}{\zeta_\mu^2} & -\frac{\omega^2}{\zeta_\mu^2} & -\frac{k^2}{\omega^2\zeta_\mu^2} & -\frac{k}{\zeta_\mu^2} & -\frac{\zeta_\sigma}{\zeta_\mu^2} \end{pmatrix} \tag{19}$$

$$\partial_{\zeta_\sigma}\mathbf{R_2} = \mathrm{diag}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{\zeta_\mu} \end{pmatrix} \tag{20}$$

1.1 Density derivatives

Differentiating the propagator with respect to density, we get

$$\begin{aligned}\frac{\partial\boldsymbol{\mathcal{P}}}{\partial\rho} = \frac{\partial\boldsymbol{\mathcal{Z}_6}}{\partial\rho}\mathbf{R_1}(\mathbf{M_1}+\mathbf{M_2})\mathbf{R_2}\boldsymbol{\mathcal{Z}_6^{-1}} + \boldsymbol{\mathcal{Z}_6}\frac{\partial\mathbf{R_1}}{\partial\rho}(\mathbf{M_1}+\mathbf{M_2})\mathbf{R_2}\boldsymbol{\mathcal{Z}_6^{-1}} + \boldsymbol{\mathcal{Z}_6}\mathbf{R_1}(\frac{\partial\mathbf{M_1}}{\partial\rho}+\frac{\partial\mathbf{M_2}}{\partial\rho})\mathbf{R_2}\boldsymbol{\mathcal{Z}_6^{-1}} \\ + \boldsymbol{\mathcal{Z}_6}\mathbf{R_1}(\mathbf{M_1}+\mathbf{M_2})\frac{\partial\mathbf{R_2}}{\partial\rho}\boldsymbol{\mathcal{Z}_6^{-1}} + \boldsymbol{\mathcal{Z}_6}\mathbf{R_1}(\mathbf{M_1}+\mathbf{M_2})\mathbf{R_2}\frac{\partial\boldsymbol{\mathcal{Z}_6^{-1}}}{\partial\rho}\end{aligned} \tag{21}$$

It turns out that

$$\frac{\partial\boldsymbol{\mathcal{Z}_6}}{\partial\rho} = \boldsymbol{\mathcal{Z}_6}\mathbf{K} \tag{22}$$

where

$$\mathbf{K} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{23}$$

Similarly,

$$\frac{\partial\boldsymbol{\mathcal{Z}_6^{-1}}}{\partial\rho} = -\mathbf{K}\boldsymbol{\mathcal{Z}_6^{-1}} \tag{24}$$

and

$$\frac{\partial \mathbf{R_i}}{\partial \rho} = \mathbf{J_i R_i} \tag{25}$$

where

$$\mathbf{J_1} = \frac{\omega^2}{2\sigma\zeta_\sigma^2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-\frac{\sigma\zeta_\sigma^2}{\mu\zeta_\mu^2} \end{pmatrix} \tag{26}$$

$$\mathbf{J_2} = \frac{\omega^2}{2\mu\zeta_\mu^2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-\frac{\mu\zeta_\mu^2}{\sigma\zeta_\sigma^2} \end{pmatrix} \tag{27}$$

Hence,

$$\begin{split}\frac{\partial \boldsymbol{\mathcal{P}}}{\partial \rho} = \boldsymbol{\mathcal{Z}_6}(\mathbf{J_1}+\mathbf{K})\mathbf{R_1}(\mathbf{M_1}+\mathbf{M_2})\mathbf{R_2}\boldsymbol{\mathcal{Z}_6^{-1}} + \boldsymbol{\mathcal{Z}_6}\mathbf{R_1}(\frac{\partial \mathbf{M_1}}{\partial \rho} + \frac{\partial \mathbf{M_2}}{\partial \rho})\mathbf{R_2}\boldsymbol{\mathcal{Z}_6^{-1}} \\ + \boldsymbol{\mathcal{Z}_6}\mathbf{R_1}(\mathbf{M_1}+\mathbf{M_2})\mathbf{R_2}(\mathbf{J_2}-\mathbf{K})\boldsymbol{\mathcal{Z}_6^{-1}}\end{split} \tag{28}$$

Now, we note that $\mathbf{M_1}$ can be expressed as a power series in ξ

$$\mathbf{M_1} = \mathbf{M_{10}} + \xi\mathbf{M_{11}} + \xi^2\mathbf{M_{12}} \tag{29}$$

and hence

$$\frac{\partial \mathbf{M_1}}{\partial \rho} = \frac{\partial \mathbf{M_{10}}}{\partial \rho} + \frac{\partial \xi}{\partial \rho}\mathbf{M_{11}} + \xi\frac{\partial \mathbf{M_{11}}}{\partial \rho} + 2\xi\frac{\partial \xi}{\partial \rho}\mathbf{M_{12}} + \xi^2\frac{\partial \mathbf{M_{12}}}{\partial \rho} \tag{30}$$

$$\frac{\partial \mathbf{M_{10}}}{\partial \rho} = \frac{h\omega^2}{2}\begin{pmatrix} -Y_{\mathrm{X}\sigma} & Y_{\mathrm{P}\mu} & -Y_{\mathrm{P}\mu} & Y_{\mathrm{P}\mu} & -Y_{\mathrm{P}\mu} & Y_{\mathrm{X}\mu} \\ Y_{\mathrm{P}\sigma} & -Y_{\mathrm{X}\mu} & Y_{\mathrm{X}\mu} & -Y_{\mathrm{X}\mu} & Y_{\mathrm{X}\mu} & -Y_{\mathrm{P}\mu} \\ Y_{\mathrm{P}\sigma} & -Y_{\mathrm{X}\mu} & Y_{\mathrm{X}\mu} & -Y_{\mathrm{X}\mu} & Y_{\mathrm{X}\mu} & -Y_{\mathrm{P}\mu} \\ -Y_{\mathrm{P}\sigma} & Y_{\mathrm{X}\mu} & -Y_{\mathrm{X}\mu} & Y_{\mathrm{X}\mu} & -Y_{\mathrm{X}\mu} & Y_{\mathrm{P}\mu} \\ -Y_{\mathrm{P}\sigma} & Y_{\mathrm{X}\mu} & -Y_{\mathrm{X}\mu} & Y_{\mathrm{X}\mu} & -Y_{\mathrm{X}\mu} & Y_{\mathrm{P}\mu} \\ Y_{\mathrm{X}\mu} & -Y_{\mathrm{P}\sigma} & Y_{\mathrm{P}\sigma} & -Y_{\mathrm{P}\sigma} & Y_{\mathrm{P}\sigma} & -Y_{\mathrm{X}\sigma} \end{pmatrix} \tag{31}$$

$$\frac{\partial \mathbf{M_{11}}}{\partial \rho} = \frac{h\omega^2}{2}\begin{pmatrix} 0 & 0 & 0 & -Y_{\mathrm{P}\sigma} & 0 & 0 \\ 0 & 0 & 0 & Y_{\mathrm{X}\sigma} & 0 & 0 \\ -Y_{\mathrm{P}\mu} & Y_{\mathrm{X}\sigma} & -Y_{\mathrm{X}\sigma} & 2Y_{\mathrm{X}\sigma} & -Y_{\mathrm{X}\sigma} & Y_{\mathrm{P}\sigma} \\ 0 & 0 & 0 & -Y_{\mathrm{X}\sigma} & 0 & 0 \\ 0 & 0 & 0 & -Y_{\mathrm{X}\sigma} & 0 & 0 \\ 0 & 0 & 0 & Y_{\mathrm{P}\mu} & 0 & 0 \end{pmatrix} \tag{32}$$

$$\frac{\partial \mathbf{M_{12}}}{\partial \rho} = \frac{h\omega^2}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -Y_{\mathrm{X}\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{33}$$

$$\frac{\partial \xi}{\partial \rho}\cdot(\mathbf{M_{11}} + 2\xi\mathbf{M_{12}}) = \frac{\omega^2\left(\mu\zeta_\mu^2 + \sigma\zeta_\sigma^2\right)}{2k^2\mu\sigma\zeta_\mu\zeta_\sigma}\begin{pmatrix} 0 & 0 & 0 & -X_1 & 0 & 0 \\ 0 & 0 & 0 & P_C & 0 & 0 \\ -X_2 & P_C & -P_C & 2\left(P_C - \xi P_s\right) & -P_C & X_1 \\ 0 & 0 & 0 & -P_C & 0 & 0 \\ 0 & 0 & 0 & -P_C & 0 & 0 \\ 0 & 0 & 0 & X_2 & 0 & 0 \end{pmatrix} \tag{34}$$

1.2 P-wave modulus derivatives

$$\frac{\partial \boldsymbol{\mathcal{Z}}_\mathbf{6}}{\partial \sigma} = 0 \tag{35}$$

$$\frac{\partial \boldsymbol{\mathcal{Z}}_\mathbf{6}^{-\mathbf{1}}}{\partial \sigma} = 0 \tag{36}$$

$$\frac{\partial \mathbf{R_1}}{\partial \sigma} = -\frac{\rho\omega^2}{2\sigma^2\zeta_\sigma^2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\mathbf{R_1} \tag{37}$$

1.3 S-wave modulus derivatives

$$\frac{\partial \boldsymbol{\mathcal{Z}}_\mathbf{6}}{\partial \mu} = \boldsymbol{\mathcal{Z}}_\mathbf{6}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2k}{\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2k}{\rho} & 0 & 0 & -\frac{2k}{\rho} & 0 \\ 0 & 0 & \frac{2k}{\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{38}$$

$$\frac{\partial \boldsymbol{\mathcal{Z}}_\mathbf{6}^{-\mathbf{1}}}{\partial \mu} = -\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2k}{\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2k}{\rho} & 0 & 0 & -\frac{2k}{\rho} & 0 \\ 0 & 0 & \frac{2k}{\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\boldsymbol{\mathcal{Z}}_\mathbf{6}^{-\mathbf{1}} \tag{39}$$