Numerical Solution of Polynomial Systems by Homotopy Continuation Methods

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1. Introduction

Let $P(x) = 0$ be a system of n polynomial equations in n unknowns. Denoting $P = (p_1, \ldots, p_n)$, we want to find all isolated solutions of

$$\begin{aligned} p_1(x_1, \ldots, x_n) &= 0, \\ &\vdots \\ p_n(x_1, \ldots, x_n) &= 0 \end{aligned} \tag{1.1}$$

for $x = (x_1, \ldots, x_n)$. This problem is very common in many fields of science and engineering, such as formula construction, geometric intersection problems, inverse kinematics, power flow problems with PQ-specified bases, computation of equilibrium states, etc. Many of those applications has been well documented in TRAVERSO [1997]. Elimination theory based methods, most notably the Buchberger algorithm (BUCHBERGER [1985]) for constructing Gröbner bases, are the classical approach to solving (1.1), but their reliance on symbolic manipulation makes those methods seem somewhat limited to relatively small problems.

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Solving polynomial systems is an area where numerical computations arise almost naturally. Given the complexity of the problem, we must use standard machine arithmetic to obtain efficient programs. Moreover, by Galois theory explicit formulas for the solutions are unlikely to exist. We are concerned with the robustness of our methods and want to be sure that *all* isolated solutions are obtained, i.e. we want exhaustive methods. These criteria are met by homotopy continuation methods. GARCIA and ZANGWILL [1979] and DREXLER [1977] independently presented theorems suggesting that homotopy continuation could be used to find numerically the full set of isolated solutions of (1.1). During the last two decades, this method has been developed and proved to be a reliable and efficient numerical algorithm for approximating all isolated zeros of polynomial systems. Note that continuation methods are the method of choice to deal with numerical solutions of nonlinear systems of equations to achieve global convergence as illustrated by the extensive bibliography listed in ALLGOWER and GEORG [1990].

In the early stage, the homotopy continuation method for solving (1.1) is to define a trivial system $Q(x) = (q_1(x), \ldots, q_n(x)) = 0$ and then follow the curves in the real variable t which make up the solution set of

$$0 = H(x,t) = (1-t)Q(x) + tP(x). \tag{1.2}$$

More precisely, if $Q(x) = 0$ is chosen correctly, the following three properties hold:

PROPERTY 0 (*Triviality*). The solutions of $Q(x) = 0$ are known.

PROPERTY 1 (*Smoothness*). The solution set of $H(x,t) = 0$ for $0 \leqslant t < 1$ consists of a finite number of smooth paths, each parametrized by t in $[0, 1)$.

PROPERTY 2 (*Accessibility*). Every isolated solution of $H(x,1) = P(x) = 0$ can be reached by some path originating at $t = 0$. It follows that this path starts at a solution of $H(x,0) = Q(x) = 0$.

When the three properties hold, the solution paths can be followed from the initial points (known because of property 0) at $t = 0$ to all solutions of the original problem $P(x) = 0$ at $t = 1$ using standard numerical techniques, see ALLGOWER and GEORG [1990], ALLGOWER and GEORG [1993] and ALLGOWER and GEORG [1997].

Several authors have suggested choices of Q that satisfy the three properties, see CHOW, MALLET-PARET and YORKE [1979], LI [1983], MORGAN [1986], WRIGHT [1985], LI and SAUER [1987], and ZULENER [1988] for a partial list. A typical suggestion is

$$\begin{aligned} q_1(x) &= a_1 x_1^{d_1} - b_1, \\ &\vdots \\ q_n(x) &= a_n x_n^{d_n} - b_n, \end{aligned} \tag{1.3}$$

where $d_1, \ldots, d_n$ are the degrees of $p_1(x), \ldots, p_n(x)$, respectively, and a_j, b_j are random complex numbers (and therefore nonzero, with probability one). So in one sense, the original problem we posed is solved. All solutions of $P(x) = 0$ are found at the end of the $d_1 \cdots d_n$ paths that make up the solution set of $H(x,t) = 0$, $0 \leqslant t < 1$.

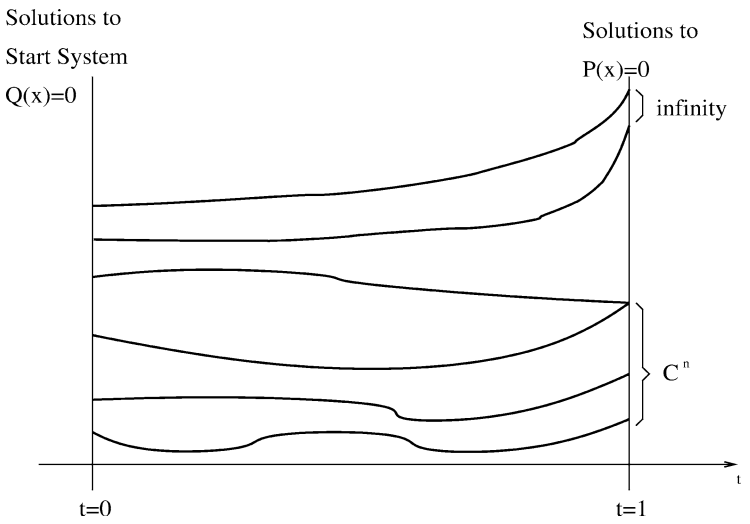


FIG. 1.1.

The book of MORGAN [1987] detailed many aspects of the above approach. A major part of this article will focus on the development afterwards that makes this method more convenient to apply.

The reason the problem is not satisfactorily solved by the above considerations is the existence of *extraneous paths*. Although the above method produces $d = d_1 \cdots d_n$ paths, the system $P(x) = 0$ may have fewer than d solutions. We call such a system *deficient*. In this case, some of the paths produced by the above method will be extraneous paths.

More precisely, even though Properties 0–2 imply that each solution of $P(x) = 0$ will lie at the end of a solution path, it is also consistent with these properties that some of the paths may diverge to infinity as the parameter t approaches 1 (the smoothness property rules this out for $t \to t_0 < 1$). In other words, it is quite possible for $Q(x) = 0$ to have more solutions than $P(x) = 0$. In this case, some of the paths leading from roots of $Q(x) = 0$ are extraneous, and diverge to infinity when $t \to 1$ (see Fig. 1.1).

Empirically, we find that most systems arising in applications are deficient. A great majority of the systems have fewer than, and in some cases only a small fraction of, the "expected number" of solutions. For a typical example of this sort, let's look at the following Cassou–Nogués system (TRAVERSO [1997])

$$\begin{aligned}
p_1 &= 15b^4cd^2 + 6b^4c^3 + 21b^4c^2d - 144b^2c - 8b^2c^2e \\
&\quad - 28b^2cde - 648b^2d + 36b^2d^2e + 9b^4d^3 - 120, \\
p_2 &= 30b^4c^3d - 32cde^2 - 720b^2cd - 24b^2c^3e - 432b^2c^2 + 576ce - 576de \\
&\quad + 16b^2cd^2e + 16d^2e^2 + 16c^2e^2 + 9b^4c^4 + 39b^4c^2d^2 + 18b^4cd^3 \\
&\quad - 432b^2d^2 + 24b^2d^3e - 16b^2c^2de - 240c + 5184, \\
p_3 &= 216b^2cd - 162b^2d^2 - 81b^2c^2 + 1008ce - 1008de + 15b^2c^2de \\
&\quad - 15b^2c^3e - 80cde^2 + 40d^2e^2 + 40c^2e^2 + 5184, \\
p_4 &= 4b^2cd - 3b^2d^2 - 4b^2c^2 + 22ce - 22de + 261.
\end{aligned} \tag{1.4}$$

Since $d_1 = 7, d_2 = 8, d_3 = 6$ and $d_4 = 4$ for this system, the system $Q(x)$ in (1.3) will produce $d_1 \times d_2 \times d_3 \times d_4 = 7 \times 8 \times 6 \times 4 = 1344$ paths for the homotopy in (1.2). However, the system (1.4) has only 16 isolated zeros. Consequently, a major fraction of the paths are extraneous. Sending out 1344 paths in search of 16 solutions appears highly wasteful.

The choice of $Q(x)$ in (1.3) to solve the system $P(x) = 0$ requires an amount of computational effort proportional to $d_1 \times \cdots \times d_n$ and roughly, proportional to the size of the system. The main goal of this article is to derive methods for solving deficient systems for which the computational effort is instead proportional to the actual number of solutions.

This article is written for the readership of the numerical analysis community. Technical terms in algebraic geometry will therefore be confined to a minimum and highly technical proofs of major theorems will be omitted.

2. Linear homotopies

For deficient systems, there are some partial results that use algebraic geometry to reduce the number of extraneous paths with various degrees of success.

2.1. Random product homotopy

For a specific example that is quite simple, consider the system

$$\begin{aligned} p_1(x) &= x_1(a_{11}x_1 + \cdots + a_{1n}x_n) + b_{11}x_1 + \cdots + b_{1n}x_n + c_1 = 0, \\ &\vdots \\ p_n(x) &= x_1(a_{n1}x_1 + \cdots + a_{nn}x_n) + b_{n1}x_1 + \cdots + b_{nn}x_n + c_n = 0. \end{aligned} \tag{2.1}$$

This system has total degree $d = d_1 \cdots d_n = 2^n$. Thus the "expected number of solutions" is 2^n, and the classical homotopy continuation method using the start system $Q(x) = 0$ in (1.3) sends out 2^n paths from 2^n trivial starting points. However, the system $P(x) = 0$ has only $n + 1$ isolated solutions (even fewer for special choices of coefficients). This is a deficient system, at least $2^n - n - 1$ paths will be extraneous. It is never known from the start which of the paths will end up to be extraneous, so they must all be followed to the end, representing wasted computation.

The random product homotopy was developed in LI, SAUER and YORKE [1987a], LI, SAUER and YORKE [1987b] to alleviate this problem. According to that technique, a more efficient choice for the trivial system $Q(x) = 0$ is

$$\begin{aligned} q_1(x) &= (x_1 + e_{11})(x_1 + x_2 + \cdots + x_n + e_{12}), \\ q_2(x) &= (x_1 + e_{21})(x_2 + e_{22}), \\ &\vdots \\ q_n(x) &= (x_1 + e_{n1})(x_n + e_{n2}). \end{aligned} \tag{2.2}$$

Set

$$H(x, t) = (1 - t)cQ(x) + tP(x), \quad c \in \mathbb{C}.$$

It is clear by inspection that for a generic choice of the complex number e_{ij}, $Q(x) = 0$ has exactly $n + 1$ roots. Thus there are only $n + 1$ paths starting from $n + 1$ starting points for this choice of homotopy. It is proved in LI, SAUER and YORKE [1987b] that Properties 0–2 hold for this choice of $H(x, t)$, for almost all complex numbers e_{ij} and c. Thus all solutions of $P(x) = 0$ are found at the end of the $n + 1$ paths. The result of LI, SAUER and YORKE [1987b] is then a mathematical result (that there can be at most $n + 1$ solutions to (2.1)) and the basis of a numerical procedure for approximating the solutions.

The reason this works is quite simple. The solution paths of (1.2) which do not proceed to a solution of $P(x) = 0$ in $\mathbb{C}^n$ diverge to infinity. If the system (1.2) is viewed in projective space

$$\mathbb{P}^n = \{(x_0, \ldots, x_n) \in \mathbb{C}^{n+1} \backslash (0, \ldots, 0)\}/\sim,$$

where the equivalent relation "$\sim$" is given by $x \sim y$ if $x = cy$ for some nonzero $c \in \mathbb{C}$, the diverging paths simply proceed to a "point at infinity" in $\mathbb{P}^n$.

For a polynomial $f(x_1, \ldots, x_n)$ of degree d, denote the associated homogeneous polynomial by

$$\tilde{f}(x_0, x_1, \ldots, x_n) = x_0^d f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right).$$

The solutions of $f(x) = 0$ at infinity are those zeros of $\tilde{f}$ in $\mathbb{P}^n$ with $x_0 = 0$ and the remaining zeros of $\tilde{f}$ with $x_0 \neq 0$ are the solutions of $f(x) = 0$ in $\mathbb{C}^n$ when x_0 is set to be 1.

Viewed in projective space $\mathbb{P}^n$ the system $P(x) = 0$ in (2.1) has some roots at infinity. The roots at infinity make up a nonsingular variety, specifically the linear space $\mathbb{P}^{n-2}$ defined by $x_0 = x_1 = 0$. A Chern class formula from intersection theory (e.g., FULTON [1984], 9.1.1 and 9.1.2) shows that the contribution of a linear variety of solutions of dimension e to the "total degree" $(d_1 \times \cdots \times d_n)$, or the total expected number of solutions, of the system is at least s, where s is the coefficient of t^e in the Maclaurin series expansion of

$$(1 + t)^{e-n} \prod_{j=1}^{n} (1 + d_j t).$$

In our case, $d_1 = \cdots = d_n = 2$, and $e = n - 2$, hence,

$$\frac{(1 + 2t)^n}{(1 + t)^2} = \frac{(1 + t + t)^n}{(1 + t)^2} = \frac{\sum_{j=0}^{n} (1 + t)^{n-j} t^j \binom{n}{j}}{(1 + t)^2} = \sum_{j=0}^{n} (1 + t)^{n-j-2} t^j \binom{n}{j}$$

and $s = \sum_{j=0}^{n-2} \binom{n}{j}$, meaning there are at least $\sum_{j=0}^{n-2} \binom{n}{j}$ solutions of $P(x) = 0$ at infinity. Thus there are at most

$$2^n - s = (1 + 1)^n - \sum_{j=0}^{n-2} \binom{n}{j} = n + 1$$

solutions of $P(x) = 0$ in $\mathbb{C}^n$. The system $Q(x) = 0$ is chosen to have the same non-singular variety at infinity, and this variety stays at infinity as the homotopy progresses from $t = 0$ to $t = 1$. As a result, the infinity solutions stay infinite, the finite solution paths stay finite, and no extraneous paths exist.

This turns out to be a fairly typical situation. Even though the system $P(x) = 0$ to be solved has isolated solutions, when viewed in projective space there may be large number of roots at infinity and quite often high-dimensional manifolds of roots at infinity. Extraneous paths are those that are drawn to the manifolds lying at infinity. If $Q(x) = 0$ can be chosen correctly, extraneous paths can be eliminated.

As another example, consider the algebraic eigenvalue problem,

$$Ax = \lambda x,$$

where

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

is an $n \times n$ matrix. This problem is actually an n polynomial equations in the $n + 1$ variables $\lambda, x_1, \ldots, x_n$:

$$\begin{aligned} \lambda x_1 - (a_{11}x_1 + \cdots + a_{1n}x_n) &= 0, \\ &\vdots \\ \lambda x_n - (a_{n1}x_1 + \cdots + a_{nn}x_n) &= 0. \end{aligned}$$

Augmenting the system with a linear equation

$$c_1x_1 + \cdots + c_nx_n + c_{n+1} = 0,$$

where $c_1, \ldots, c_{n+1}$ are chosen at random, we have a polynomial system of $n + 1$ equations in $n + 1$ variables. This system has total degree 2^n. However, it can have at most n isolated solutions. So, the system is deficient. But the system $Q(x)$ in random product form:

$$\begin{aligned} q_1 &= (\lambda + e_{11})(x_1 + e_{12}), \\ q_2 &= (\lambda + e_{21})(x_2 + e_{22}), \\ &\vdots \\ q_n &= (\lambda + e_{n1})(x_n + e_{n2}), \\ q_{n+1} &= c_1x_1 + \cdots + c_nx_n + c_{n+1} \end{aligned}$$

has n isolated zeros for randomly chosen e_{ij}'s. This $Q(x)$ will produce n curves for the homotopy in (1.3) that proceed to all solutions of the eigenvalue problem. Implicit in this is the fact that the algebraic eigenvalue problem has at most n solutions. Moreover, the generic eigenvalue problem has exactly n solutions.

To be more precise, we state the main random product homotopy result, Theorem 2.2 of LI, SAUER and YORKE [1987b]. Let $V_\infty(Q)$ and $V_\infty(P)$ denote the variety of roots at infinity of $Q(x) = 0$ and $P(x) = 0$, respectively.

THEOREM 2.1. *If $V_\infty(Q)$ is nonsingular and contained in $V_\infty(P)$, then Properties* 1 *and* 2 *hold.*

Of course, Properties 1 and 2 are not enough. Without starting points, the path-following method cannot get started. Thus $Q(x) = 0$ should also be chosen to be of random product forms, as in (2.2), which are trivial to solve because of their form.

This result was superseded by the result in LI and SAUER [1989]. While the complex numbers e_{ij} are chosen at random in LI, SAUER and YORKE [1987b] to ensure Properties 1 and 2, it was proved in LI and SAUER [1989] that e_{ij} can be any fixed numbers Properties 1 and 2 still hold as long as the complex number c is chosen at random. In fact, the result in LI and SAUER [1989] implies that the start system $Q(x) = 0$ in Theorem 2.2 need not be in product form. It can be any chosen polynomial system as long as its zeros in $\mathbb{C}^n$ are known or easy to obtain and its variety of roots at infinity $V_\infty(Q)$ is nonsingular and contained in $V_\infty(P)$.

Theorem 2.1 in LI and WANG [1991] goes one step further. Even when the set $V_\infty(Q)$ of roots at infinity of $Q(x) = 0$ has singularities, if the set is contained in $V_\infty(P)$ counting multiplicities, containment in the sense of *scheme* theory of algebraic geometry, then Properties 1 and 2 still hold. More precisely, let $I = \langle \tilde{q}_1, \ldots, \tilde{q}_n \rangle$ and $J = \langle \tilde{p}_1, \ldots, \tilde{p}_n \rangle$ be the homogeneous ideals spanned by homogenizations of q_i's and p_i's, respectively. For a point p at infinity, if the *local rings* I_p and J_p satisfy

$$I_p \subset J_p$$

then Properties 1 and 2 hold. However, this hypothesis can be much more difficult to verify than whether the set is nonsingular. This limits the usefulness of this approach for practical examples.

2.2. m-homogeneous structure

In MORGAN and SOMMESE [1987b], another interesting approach to reduce the number of extraneous paths is developed, using the concept of m-homogeneous structure.

The complex n-space $\mathbb{C}^n$ can be naturally embedded in the projective space $\mathbb{P}^n$. Similarly, the space $\mathbb{C}^{k_1} \times \cdots \times \mathbb{C}^{k_m}$ can be naturally embedded in $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_m}$. A point $(y_1, \ldots, y_m)$ in $\mathbb{C}^{k_1} \times \cdots \times \mathbb{C}^{k_m}$ with $y_j = (y_1^{(j)}, \ldots, y_{k_j}^{(j)})$, $j = 1, \ldots, m$, corresponds to a point $(z_1, \ldots, z_m)$ in $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_m}$ with $z_j = (z_0^{(j)}, \ldots, z_{k_j}^{(j)})$ and $z_0^{(j)} = 1$, $j = 1, \ldots, m$. The set of such points in $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_m}$ is usually called the *affine space* in this setting. The points in $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_m}$ with at least one $z_0^{(j)} = 0$ are called the *points at infinity*.

Let f be a polynomial in the n variables $x_1, \ldots, x_n$. If we partition the variables into m groups $y_1 = (x_1^{(1)}, \ldots, x_{k_1}^{(1)})$, $y_2 = (x_1^{(2)}, \ldots, x_{k_2}^{(2)}), \ldots, y_m = (x_1^{(m)}, \ldots, x_{k_m}^{(m)})$ with $k_1 + \cdots + k_m = n$ and let d_i be the degree of f with respect to y_i (more precisely, to the variables in y_i), then we can define its m-homogenization as

$$\tilde{f}(z_1, \ldots, z_m) = \left(z_0^{(1)}\right)^{d_1} \times \cdots \times \left(z_0^{(m)}\right)^{d_m} f\left(y_1/z_0^{(1)}, \ldots, y_m/z_0^{(m)}\right).$$

This polynomial is homogeneous with respect to each $z_j = (z_0^{(j)}, \ldots, z_{k_j}^{(j)})$, $j = 1, \ldots, m$. Here $z_i^{(j)} = x_i^{(j)}$, for $i \neq 0$. Such a polynomial is said to be *m-homogeneous*, and $(d_1, \ldots, d_m)$ is its m-homogeneous degree. To illustrate this definition, let us consider the polynomial $p_j(x)$ in (2.1), for $j = 1, \ldots, n$,

$$\begin{aligned} p_j(x) &= x_1(a_{j1}x_1 + \cdots + a_{jn}x_n) + b_{j1}x_1 + \cdots + b_{jn}x_n + c_j \\ &= a_{j1}x_1^2 + x_1(a_{j2}x_2 + \cdots + a_{jn}x_n + b_{j1}) + b_{j2}x_2 + \cdots + b_{jn}x_n + c_j. \end{aligned}$$

If we set $y_1 = (x_1)$, $y_2 = (x_2, \ldots, x_n)$ and $z_1 = (x_0^{(1)}, x_1)$, $z_2 = (x_0^{(2)}, x_2, \ldots, x_n)$, then the degree of $p_j(x)$ is two with respect to y_1 and is one with respect to y_2. Hence, its 2-homogenization with respect to z_1 and z_2 is

$$\begin{aligned} \tilde{p}_j(z_1, z_2) = a_{j1}x_1^2x_0^{(2)} &+ x_1x_0^{(1)}\big(a_{j2}x_2 + \cdots + a_{jn}x_n + b_{j1}x_0^{(2)}\big) \\ &+ \big(x_0^{(1)}\big)^2\big(b_{j2}x_2 + \cdots + b_{jn}x_n + c_jx_0^{(2)}\big). \end{aligned}$$

When the system (2.1) is viewed in $\mathbb{P}^n$ with the homogenization

$$\begin{aligned} \tilde{p}_1(x_0, x_1, \ldots, x_n) &= x_1(a_{11}x_1 + \cdots + a_{1n}x_n) \\ &\quad + (b_{11}x_1 + \cdots + b_{1n}x_n)x_0 + c_1x_0^2 = 0, \\ &\vdots \\ \tilde{p}_n(x_0, x_1, \ldots, x_n) &= x_1(a_{n1}x_1 + \cdots + a_{nn}x_n) \\ &\quad + (b_{n1}x_1 + \cdots + b_{nn}x_n)x_0 + c_nx_0^2 = 0, \end{aligned}$$

its total degree, or the Bézout number, is $d = d_1 \times \cdots \times d_n = 2^n$. However, when the system is viewed in $\mathbb{P}^1 \times \mathbb{P}^{n-1} = \{(z_1, z_2) = ((x_0^{(1)}, x_1), (x_0^{(2)}, x_2, \ldots, x_n))$ where $z_1 = (x_0^{(1)}, x_1) \in \mathbb{P}^1$ and $z_2 = (x_0^{(2)}, x_2, \ldots, x_n) \in \mathbb{P}^{n-1}\}$ with 2-homogenization

$$\begin{aligned} \tilde{p}_1(z_1, z_2) = a_{11}x_1^2x_0^{(2)} &+ x_1x_0^{(1)}\big(a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}x_0^{(2)}\big) \\ &+ \big(x_0^{(1)}\big)^2\big(b_{12}x_2 + \cdots + b_{1n}x_n + c_1x_0^{(2)}\big), \\ &\vdots \\ \tilde{p}_n(z_1, z_2) = a_{n1}x_1^2x_0^{(2)} &+ x_1x_0^{(1)}\big(a_{n2}x_2 + \cdots + a_{nn}x_n + b_{n1}x_0^{(2)}\big) \\ &+ \big(x_0^{(1)}\big)^2\big(b_{n2}x_2 + \cdots + b_{nn}x_n + c_nx_0^{(2)}\big), \end{aligned} \tag{2.3}$$

the Bézout number will be different. It is defined to be the coefficient of $\alpha_1^1\alpha_2^{n-1}$ in the product $(2\alpha_1 + \alpha_2)^n$, which is equal to $2n$.

In general, for an m-homogeneous system

$$\begin{aligned} &\tilde{p}_1(z_1, \ldots, z_m) = 0, \\ &\qquad\vdots \\ &\tilde{p}_n(z_1, \ldots, z_m) = 0, \end{aligned} \tag{2.4}$$

in $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_m}$ with $\tilde{p}_j$ having m-homogeneous degree $(d_1^{(j)}, \ldots, d_m^{(j)})$, $j = 1, \ldots, n$, with respect to $(z_1, \ldots, z_m)$, then the m-homogeneous Bézout number d of the system with respect to $(z_1, \ldots, z_m)$ is the coefficient of $\alpha_1^{k_1} \times \cdots \times \alpha_m^{k_m}$ in the product

$$\left(d_1^{(1)}\alpha_1 + \cdots + d_m^{(1)}\alpha_m\right)\left(d_1^{(2)}\alpha_1 + \cdots + d_m^{(2)}\alpha_m\right) \cdots \left(d_1^{(n)}\alpha_1 + \cdots + d_m^{(n)}\alpha_m\right), \tag{2.5}$$

see SHAFAREVICH [1977]. The classical Bézout theorem says the system (2.4) has no more than d isolated solutions, counting multiplicities, in $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_m}$. Applying this to our example in (2.3), the upper bound on the number of isolated solutions, in affine space and at infinity, is $2n$. When solving the original system in (2.1), we may choose the start system $Q(x)$ in the homotopy

$$H(x, t) = (1 - t)cQ(x) + tP(x) = 0$$

in random product form to respect the 2-homogeneous structure of $P(x)$. For instance, we choose $Q(x) = (q_1(x), \ldots, q_n(x))$ to be

$$\begin{aligned} q_1(x) &= (x_1 + e_{11})(x_1 + e_{12})(x_2 + \cdots + x_n + e_{13}), \\ q_2(x) &= (x_1 + e_{21})(x_1 + e_{22})(x_2 + e_{23}), \\ &\vdots \\ q_n(x) &= (x_1 + e_{n1})(x_1 + e_{n2})(x_n + e_{n3}), \end{aligned} \tag{2.6}$$

which has the same 2-homogeneous structure as $P(x)$ with respect to the partition $y_1 = (x_1)$ and $y_2 = (x_2, \ldots, x_n)$. Namely, each $q_j(x)$ has degree two with respect to y_1 and degree one with respect to y_2. It is easy to see that for randomly chosen complex numbers e_{ij}, $Q(x) = 0$ has $2n$ solutions in $\mathbb{C}^n$ $(= \mathbb{C}^1 \times \mathbb{C}^{n-1})$ (thus, no solutions at infinity when viewed in $\mathbb{P}^1 \times \mathbb{P}^{n-1}$). Hence there are $2n$ paths emanating from $2n$ solutions of $Q(x) = 0$ for this choice of the homotopy. It was shown in MORGAN and SOMMESE [1987b] that Properties 1 and 2 hold for all complex number c except those lying on a finite number of rays starting at the origin. Thus, all solutions of $P(x) = 0$ are found at the end of $n + 1$ paths. The number of extraneous paths, $2n - (n + 1) = n - 1$, is far less than the number of extraneous paths, $2^n - n - 1$, by using the classical homotopy with $Q(x) = 0$ in (1.3).

More precisely, we state the main theorem in MORGAN and SOMMESE [1987b].

THEOREM 2.2. *Let $Q(x)$ be a system of polynomials chosen to have the same m-homogeneous form as $P(x)$ with respect to certain partition of the variables $(x_1, \ldots, x_n)$. Assume $Q(x) = 0$ has exactly the Bézout number of nonsingular solutions with respect to this partition, and let*

$$H(x, t) = (1 - t)cQ(x) + tP(x) = 0,$$

where $t \in [0, 1]$ and $c \in \mathbb{C}^*$. If $c = re^{i\theta}$ for some positive $r \in \mathbb{R}$, then for all but finitely many θ, Properties 1 and 2 hold.

Notice that when the number of isolated zeros of $Q(x)$, having the same m-homogeneous structure of $P(x)$ with respect to a given partition of variables $(x_1, \ldots, x_n)$,

reaches the corresponding Bézout number, then no other solutions of $Q(x) = 0$ exist at infinity.

In general, if $x = (x_1, \ldots, x_n)$ is partitioned into $x = (y_1, \ldots, y_m)$ where

$$y_1 = \big(x_1^{(1)}, \ldots, x_{k_1}^{(1)}\big), \quad y_2 = \big(x_1^{(2)}, \ldots, x_{k_2}^{(2)}\big), \quad \ldots, \quad y_m = \big(x_1^{(m)}, \ldots, x_{k_m}^{(m)}\big)$$

with $k_1 + \cdots + k_m = n$, and for polynomial system $P(x) = (p_1(x), \ldots, p_n(x))$, $p_j(x)$ has degree $(d_1^{(j)}, \ldots, d_m^{(j)})$ with respect to $(y_1, \ldots, y_m)$ for $j = 1, \ldots, n$, we may choose the start system $Q(x) = (q_1(x), \ldots, q_n(x))$ where

$$q_j(x) = \prod_{i=1}^{m} \prod_{l=1}^{d_i^{(j)}} \big(c_{l1}^{(i)} x_1^{(i)} + \cdots + c_{lk_i}^{(i)} x_{k_i}^{(i)} + c_{l0}^{(i)}\big), \quad j = 1, \ldots, n. \tag{2.7}$$

Clearly, $q_j(x)$ has degree $(d_1^{(j)}, \ldots, d_m^{(j)})$ with respect to $(y_1, \ldots, y_m)$, the same degree structure of $p_j(x)$. Furthermore, it is not hard to see that for generic coefficients $Q(x)$ has exactly m-homogeneous Bézout number, with respect to this particular partition $x = (y_1, \ldots, y_m)$, of nonsingular isolated zeros in $\mathbb{C}^n$. They are easy to obtain. In fact, the system $Q(x)$ in (2.6) is constructed according to this principle. In WAMPLER [1994], the product in (2.7) is modified to be more efficient to evaluate.

As mentioned earlier, solving system in (2.3) with start system $Q(x)$ in (2.6), there are still $n - 1$ extraneous paths for the homotopy. This is because, even when viewed in $\mathbb{P}^1 \times \mathbb{P}^{n-1}$, $P(x)$ has zeros at infinity. One can check in (2.3) that

$$S = \big\{\big((x_0^{(1)}, x_1), (x_0^{(2)}, x_2, \ldots, x_n)\big) \in \mathbb{P}^1 \times \mathbb{P}^{n-1} \mid x_0^{(1)} = 0, x_0^{(2)} = 0\big\}$$

is a set of zeros of $P(x)$ at infinity. So, to lower the number of those extraneous paths further, we may choose the start system $Q(x)$ to have the same nonsingular variety of zeros at infinity S as $P(x)$ does, in addition to sharing the same 2-homogeneous structure of $P(x)$. For instance, the system $Q(x) = (q_1(x), \ldots, q_n(x))$ where

$$\begin{aligned}
q_1(x) &= (x_1 + e_{11})(x_1 + x_2 + \cdots + x_n + e_{12}),\\
q_2(x) &= (x_1 + e_{21})(x_1 + x_2 + e_{22}),\\
&\vdots\\
q_n(x) &= (x_1 + e_{n1})(x_1 + x_n + e_{n2})
\end{aligned}$$

shares the same 2-homogeneous structure of $P(x)$ with respect to the partition $y_1 = (x_1)$ and $y_2 = (x_2, \ldots, x_n)$. Furthermore, when viewed in $(z_1, z_2) \in \mathbb{P}^1 \times \mathbb{P}^{n-1}$ with $z_1 = (x_0^{(1)}, x_1)$ and $z_2 = (x_0^{(2)}, x_2, \ldots, x_n)$, this system has the same set of zeros at infinity S as $P(x)$ does. The system $Q(x) = 0$ also has $n + 1$ solutions in $\mathbb{C}^n$ for generic e_{ji}'s, and there are no extraneous paths. The results in LI and WANG [1991] and MORGAN and SOMMESE [1987a] show that if $Q(x)$ in

$$H(x, t) = (1 - t)cQ(x) + tP(x) = 0$$

is chosen to have the same m-homogeneous structure as $P(x)$ and the set of zeros at infinity $V_\infty(Q)$ of $Q(x)$ is nonsingular and contained in the set of zeros at infinity

$V_\infty(P)$ of $P(x)$, then for $c = re^{i\theta}$ for positive $r \in \mathbb{R}$ and for all but finitely many θ Properties 1 and 2 hold.

Most often the zeros at infinity of an m-homogeneous polynomial system $\widetilde{P}(z_1, \ldots, z_m)$ in $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_m}$ is hard to identify. Nevertheless, the choice of $Q(x) = 0$ in Theorem 2.2, assuming no zeros at infinity regardless of the structure of the zeros at infinity of $P(x)$, can still reduce the number of extraneous paths dramatically by simply sharing the same m-homogeneous structure of $P(x)$.

Let's consider the system

$$\begin{aligned} p_1(x) &= x_1(a_{11}x_1 + \cdots + a_{1n}x_n) + b_{11}x_1 + \cdots + b_{1n}x_n + c_1 = 0, \\ &\vdots \\ p_n(x) &= x_1(a_{n1}x_1 + \cdots + a_{nn}x_n) + b_{n1}x_1 + \cdots + b_{nn}x_n + c_n = 0, \end{aligned}$$

in (2.1) once again. This time we partition the variables $x_1, \ldots, x_n$ into $y_1 = (x_1, x_2)$ and $y_2 = (x_3, \ldots, x_n)$. For this partition, the 2-homogeneous degree structure of $p_j(x)$ stays the same, namely, the degree of $p_j(x)$ is two with respect to y_1 and is one with respect to y_2. However, the Bézout number with respect to this partition becomes the coefficient of $\alpha_1^2\alpha_2^{n-2}$ in the product $(2\alpha_1 + \alpha_2)^n$ according to (2.5). This number is

$$\binom{n}{2} \times 2^2 = 2n(n-1),$$

which is greater than the original Bézout number $2n$ with respect to the partition $y_1 = (x_1)$ and $y_2 = (x_2, \ldots, x_n)$ when $n > 2$. If the start system $Q(x)$ is chosen to have the same m-homogeneous structure with respect to this partition, then, assuming $Q(x)$ has no zeros at infinity, we need to follow $2n(n-1)$ paths to find all $n+1$ isolated zeros of $P(x)$. This represents a much bigger amount of extraneous paths.

The m-homogeneous Bézout number is apparently highly sensitive to the chosen partition: different ways of partitioning the variables produce different Bézout numbers. By using Theorem 2.2, we usually follow the Bézout number (with respect to the chosen partition of variables) of paths for finding all the isolated zeros of $P(x)$. In order to minimize the number of paths need to be followed and hence avoid more extraneous paths, it's critically important to find a partition which provides the lowest Bézout number possible. In WAMPLER [1992], an algorithm for this purpose was given. By using this algorithm one can determine, for example, the partition $\mathcal{P} = \{(b), (c, d, e)\}$ which gives the lowest possible Bézout number 368 for the Cassou–Nogués system in (1.4). Consequently, a random product start system $Q(x)$, as in (2.7) for instance, can be constructed to respect the degree structure of the system with respect to this partition. The start system $Q(x)$ will have 368 isolated zeros in $\mathbb{C}^n$. Therefore only 368 homotopy paths need to be followed to find all 16 isolated zeros of the system, in contrast to following 1344 paths if we choose the start system $Q(x)$ as in (1.3).

We shall elaborate below the algorithm given in WAMPLER [1992] for the search of the partition of variables which provides the lowest corresponding m-homogeneous Bézout number of a polynomial system.

First of all, we need a systematic listing of all the possible partitionings of the variables $\{x_1, \ldots, x_n\}$. This can be obtained by considering the reformulated problem:

how many different ways are there to partition n distinct items into m identical boxes for $m = 1, \ldots, n$? Denote those numbers by $g(n, m)$, $m = 1, \ldots, n$. Clearly, we have $g(n, n) = 1$, $g(n, 1) = 1$. Moreover, the recursive relation

$$g(n, m) = m \times g(n-1, m) + g(n-1, m-1)$$

holds, because for each of the $g(n-1, m)$ partitionings of $n-1$ items, we may add the nth item to any one of m boxes, plus for each of the $g(n-1, m-1)$ partitionings of $n-1$ items into $m-1$ boxes the nth item can only be in the mth box by itself. The numbers $g(n, m)$ are known as the *Stirling numbers of the second kind*, see, e.g., SELBY [1971]. Fig. 2.1 illustrates the process for $n = 1, \ldots, 4$. The partitionings can be listed by traversing the tree structure implied by Fig. 2.1 (WAMPLER [1992]).

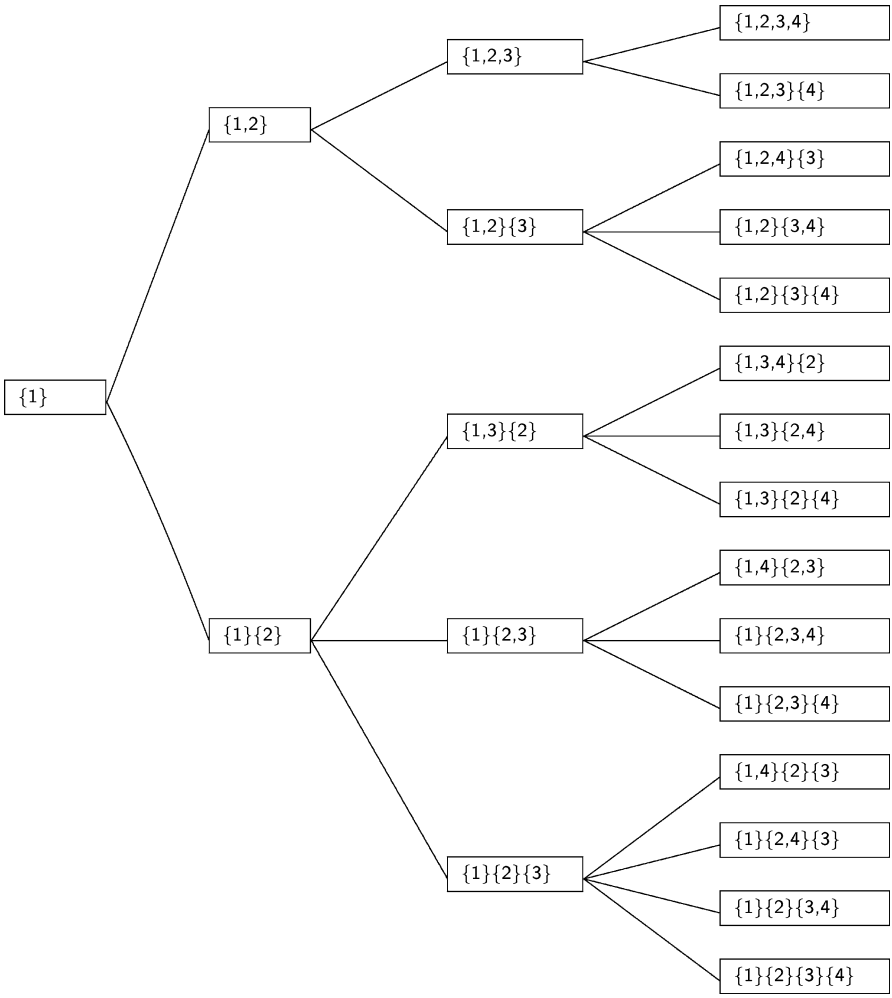


FIG. 2.1. Tree generating all groupings of 4 items.

For a given partition $y_1 = (x_1^{(1)}, \ldots, x_{k_1}^{(1)}), \ldots, y_m = (x_1^{(m)}, \ldots, x_{k_m}^{(m)})$ of the variables $\{x_1, \ldots, x_n\}$ of a polynomial system $P(x) = (p_1(x), \ldots, p_n(x))$ with $k_1 + \cdots + k_m = n$ and

$$d_{ij} = \text{degree of } p_i \text{ with respect to } y_j,$$

a straightforward computation of the Bézout number by expanding the product in (2.5) and finding the appropriate coefficient will not lead to an efficient algorithm except in the simplest cases. A simpler approach is given below.

It's easy to see that the Bézout number given in (2.5) equals the sum of all products of the form

$$d_{1\ell_1} \times d_{2\ell_2} \times \cdots \times d_{n\ell_n},$$

where among $\ell_1, \ldots, \ell_n$ each integer $j = 1, \ldots, m$ appears exactly k_j times. Equivalently, it is the sum of degree products over all possible ways to choose each row once while choosing k_j entries from each column j in the *degree matrix*

$$D = \begin{bmatrix} d_{11} & \cdots & d_{1m} \\ \vdots & \ddots & \vdots \\ d_{n1} & \cdots & d_{nm} \end{bmatrix}. \tag{2.8}$$

Thus, to calculate the Bézout number we may enumerate the permissible combinations in the degree matrix, form the corresponding degree products, and add them up. Since many of the degree products contain common factors, a method resembling the evaluation of a determinant via expansion by minors can be used to reduce the number of multiples, either down the column or across the rows of the degree matrix. The row expansion is generally more efficient, and we shall present only this alternative.

For *partition vector* $K = [k_1, \ldots, k_m]$, we form the degree products in degree matrix D in (2.8) as follows. First, in row 1 of D, suppose element d_{1j} is chosen. Then to complete the degree product we must choose one element from each of the remaining rows while only $k_j - 1$ elements from the jth column are included. So, a *minor* corresponding to d_{1j} is derived by deleting row 1 of D and decreasing k_j by 1. This *minor* has the corresponding Bézout number in its own right, with respect to the partition vector $K' = [k_1, \ldots, k_{j-1}, k_j - 1, k_{j+1}, \ldots, k_m]$. The *row expansion algorithm* for the Bézout number of degree matrix D with respect to the partition vector $K = [k_1, \ldots, k_m]$ is to compute the sum along the first row of each d_{1j} (where $k_j > 0$) times the Bézout number of the corresponding minor. The Bézout number of each minor is then computed recursively by the same row expansion procedure.

More precisely, let $b(D, \overline{K}, i)$ be the Bézout number of the degree matrix

$$D_i = \begin{bmatrix} d_{i1} & \cdots & d_{im} \\ \vdots & \ddots & \vdots \\ d_{n1} & \cdots & d_{nm} \end{bmatrix}$$

consisted of the last $n - i + 1$ rows of D in (2.8), with respect to the partition vector $\overline{K} = [\bar{k}_1, \ldots, \bar{k}_m]$. Here, of course, $\bar{k}_1 + \cdots + \bar{k}_m = n - i + 1$. Let $M(\overline{K}, j)$ be the

partition vector derived by reducing $\bar{k}_j$ in $\overline{K}$ by 1, namely,

$$M(\overline{K}, j) = \big[\bar{k}_1, \ldots, \bar{k}_{j-1}, \bar{k}_j - 1, \bar{k}_{j+1}, \ldots, \bar{k}_m\big].$$

With the convention $b(D, \overline{K}, n+1) := 1$ the row expansion algorithm may be written as

$$b(D, \overline{K}, i) = \sum_{\substack{j=1 \\ k_j \neq 0}}^{m} d_{ij}\, b\big(D, M(\overline{K}, j), i+1\big)$$

and the Bézout number of the original degree matrix D with respect to the partition vector $K = [k_1, \ldots, k_m]$ is simply $B = b(D, K, 1)$.

Note that if the degree matrix D is sparse, we may skip over computations where $d_{ij} = 0$ and avoid expanding the recursion below that branch.

EXAMPLE 2.1 (WAMPLER [1992]). For polynomial system $P(x) = (p_1(x), \ldots, p_4(x))$, $x = (x_1, x_2, x_3, x_4)$, let $y_1 = (x_1, x_2)$ and $y_2 = (x_3, x_4)$. So, the partition vector is $K = [2, 2]$. Let

$$D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \\ d_{31} & d_{32} \\ d_{41} & d_{42} \end{bmatrix}$$

be the degree matrix. Then, by the row expansion algorithm, the Bézout number B of D with respect to K is,

$$\begin{aligned}
B &= d_{11} b\big(D, [1,2], 2\big) + d_{12} b\big(D, [2,1], 2\big) \\
&= d_{11}\big[d_{21} \cdot b\big(D, [0,2], 3\big) + d_{22} \cdot b\big(D, [1,1], 3\big)\big] \\
&\quad + d_{12}\big[d_{21} \cdot b\big(D, [1,1], 3\big) + d_{22} \cdot b\big(D, [2,0], 3\big)\big] \\
&= d_{11}\big[d_{21} d_{32} \cdot b\big(D, [0,1], 4\big) + d_{22}\big(d_{31} \cdot b\big(D, [0,1], 4\big) + d_{32} \cdot b\big(D, [1,0], 4\big)\big)\big] \\
&\quad + d_{12}\big[d_{21}\big(d_{31} \cdot b\big(D, [0,1], 4\big) + d_{32} \cdot b\big(D, [1,0], 4\big)\big) \\
&\qquad\quad + d_{22} \cdot \big(d_{31} \cdot b\big(D, [1,0], 4\big)\big)\big] \\
&= d_{11}\big(d_{21} d_{32} d_{42} + d_{22}(d_{31} d_{42} + d_{32} d_{41})\big) \\
&\quad + d_{12}\big(d_{21}(d_{31} d_{42} + d_{32} d_{41}) + d_{22} d_{31} d_{41}\big).
\end{aligned}$$

EXAMPLE 2.2 (WAMPLER [1992]). Consider the system

$$\begin{aligned}
&x_1^2 + x_2 + 1 = 0, \\
&x_1 x_3 + x_2 + 2 = 0, \\
&x_2 x_3 + x_3 + 3 = 0.
\end{aligned}$$

There are five ways to partition the variables $\{x_1, x_2, x_3\}$. We list the degree matrices and Bézout numbers calculated by the row expansion algorithm for all five partitionings as follows:

(1) $\{x_1, x_2, x_3\}$

$$K = [3], \qquad D = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

Bézout number = 8.

(2) $\{x_1, x_2\}\{x_3\}$

$$K = [2, 1], \qquad D = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Bézout number = 4.

(3) $\{x_1, x_3\}\{x_2\}$

$$K = [2, 1], \qquad D = \begin{bmatrix} 2 & 1 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Bézout number = 8.

(4) $\{x_1\}\{x_2, x_3\}$

$$K = [1, 2], \qquad D = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}.$$

Bézout number = 6.

(5) $\{x_1\}, \{x_2\}, \{x_3\}$

$$K = [1, 1, 1], \qquad D = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Bézout number = 5.

Thus, the grouping $(\{x_1, x_2\}, \{x_3\})$ has the lowest Bézout number (= 4) and will lead to the homotopy continuation with least extraneous paths.

When exhaustively searching for the partitioning of the variables which yields the minimal Bézout number of the system, there are several ways to speed up the process. For instance, as we sequentially test partitionings in search for minimal Bézout numbers, we can use the smallest one found so far to cut short unfavorable partitionings. Since the degrees are all nonnegative, the Bézout number is a sum of nonnegative degree products. If at any time the running subtotal exceeds the current minimal Bézout number, the calculation can be aborted and testing of the next partitioning can proceed. This can save a substantial amount of computation during an exhaustive search. See WAMPLER [1992] for more details.

While the number of partitionings to be tested grows rapidly with the number of variables, the exhaustive search can be easily parallelized by subdividing the tree of partitionings and distributing these branches to multiple processors for examination. Thus, continuing advances in both raw computer speed and in parallel machines will make progressively larger problems feasible.

In VERSCHELDE and HAEGEMANS [1993], a *generalized Bézout number* GB is developed in the GBQ-*algorithm*, in which the partition of variables is permitted to vary among the $p_j(x)$'s. Even lower Bézout number may be achieved when a proper partition structure of the variables for each individual polynomial $p_j(x)$, $j = 1, \ldots, n$, is chosen. This strategy can take a great advantage on certain sparse systems where an appropriate partition of variables is evident. For instance, for the polynomial system in LIU, LI and BAI [Preprint],

$$
\begin{aligned}
p_1(x) &= x_1x_2 + x_1^2x_2^2,\\
p_2(x) &= x_2x_3^2x_{15},\\
p_3(x) &= x_3x_4^2x_{10},\\
p_4(x) &= x_4^2x_5^2,\\
p_5(x) &= x_5^2x_6x_{12}^2,\\
p_6(x) &= x_6^2x_7^2x_{12} + x_6x_7x_{12} + x_6x_7^2x_{12}^2,\\
p_7(x) &= x_7x_8^2 = x_7^2x_8,\\
p_8(x) &= x_8x_9,\\
p_9(x) &= x_9^2x_{10} + x_9^2x_{10}^2,\\
p_{10}(x) &= x_{10}^2x_{11},\\
p_{11}(x) &= x_{11}x_{12},\\
p_{12}(x) &= x_{12}x_{13}x_{21}^2 + x_{12}x_{13}^2x_{21},\\
p_{13}(x) &= x_{13}x_{14},\\
p_{14}(x) &= x_{14}^2x_{15}x_{19}^2,\\
p_{15}(x) &= x_{15}x_{16}^2 + x_{15}^2x_{16},\\
p_{16}(x) &= x_{16}x_{17},\\
p_{17}(x) &= x_8x_{17}^2x_{18}^2 + x_8x_{17}x_{18}^2,\\
p_{18}(x) &= x_{18}^2x_{19} + x_{18}^2x_{19}^2,\\
p_{19}(x) &= x_{19}^2x_{20} + x_{19}x_{20},\\
p_{20}(x) &= x_{20}^2x_{21}^2 + x_{20}x_{21}^2,\\
p_{21}(x) &= x_6x_{21}^2,
\end{aligned}
\tag{2.9}
$$

an obvious partition of variables in each polynomial consists of the set of every variable in the polynomial, such as $(\{x_1\}, \{x_2\})$ for $p_1(x)$, $(\{x_2\}, \{x_3\}, \{x_{15}\})$ for $p_2(x), \ldots$, etc. Accordingly, the generalized Bézout number is 63,488, while the total degree is 79,626,240,000. An efficient algorithm to evaluate the generalized Bézout number for a given partition is proposed in LIU, LI and BAI [Preprint].

2.3. Cheater's homotopy

To organize our discussion in this section, we will at times use a notation that makes the coefficients and variables in the polynomial system $P(x) = 0$ explicit. Thus when

the dependence on coefficients is important, we will consider the system $P(c, x) = 0$ of n polynomial equations in n unknowns, where $c = (c_1, \ldots, c_M)$ are coefficients and $x = (x_1, \ldots, x_n)$ are unknowns.

A method called the *cheater's homotopy* has been developed in LI, SAUER and YORKE [1988], LI, SAUER and YORKE [1989] (a similar procedure can be found in MORGAN and SOMMESE [1989]) to deal with the problem when the system $P(c, x) = 0$ is asked to be solved for several different values of the coefficients c.

The idea of the method is to theoretically establish Properties 1 and 2 by deforming a sufficiently generic system (in a precise sense to be given later) and then to "cheat" on Property 0 by using a preprocessing step. The amount of computation of preprocessing step may be large, but is amortized among the several solving characteristics of the problem.

We begin with an example. Let $P(x)$ be the system

$$\begin{aligned} p_1(x) &= x_1^3 x_2^2 + c_1 x_1^3 x_2 + x_2^2 + c_2 x_1 + c_3 = 0, \\ p_2(x) &= c_4 x_1^4 x_2^2 - x_1^2 x_2 + x_2 + c_5 = 0. \end{aligned} \tag{2.10}$$

This is a system of two polynomial equations in two unknowns x_1 and x_2. We want to solve this system of equations several times for various specific choices of $c = (c_1, \ldots, c_5)$.

It turns out that for any choice of coefficients c, system (2.10) has no more than 10 isolated solutions. More precisely, there is an open dense subset S of $\mathbb{C}^5$ such that for c belonging to S, there are 10 solutions of (2.10). Moreover, 10 is an upper bound for the number of isolated solutions for all c in $\mathbb{C}^5$. The total degree of the system is $6 \times 5 = 30$, meaning that if we had taken a generic system of two polynomials in two variables of degree 5 and 6, there would be 30 solutions. Thus (2.10), with any choice of c, is a deficient system.

The classical homotopy using the start system $Q(x) = 0$ in (1.3) produces $d = 30$ paths, beginning at 30 trivial starting points. Thus there are (at least) 20 extraneous paths.

The cheater's homotopy continuation approach begins by solving (2.10) with *randomly-chosen* complex coefficients $\bar{c} = (\bar{c}_1, \ldots, \bar{c}_5)$; let X^* be the set of 10 solutions. No work is saved there, since 30 paths need to be followed, and 20 paths are wasted. However, the 10 elements of the set X^* are the seeds for the remainder of the process. In the future, for each choice of coefficients $c = (c_1, \ldots, c_5)$ for which the system (2.10) needs to be solved, we use the homotopy continuation method to follow a straight-line homotopy from the system with coefficient c^* to the system with coefficient c. We follow the 10 paths beginning at the 10 elements of X^*. Thus Property 0, that of having trivial-available starting points, is satisfied. The fact that Properties 1 and 2 are also satisfied is the content of Theorem 2.3 below. Thus for each fixed c, all 10 (or fewer) isolated solutions of (2.10) lie at the end of 10 smooth homotopy paths beginning at the seeds in X^*. After the foundational step of finding the seeds, the complexity of all further solving of (2.10) is proportional to the number of solutions 10, rather than the total degree 30.

Furthermore, this method requires no a priori analysis of the system. The first preprocessing step of finding the seeds establishes a sharp theoretical upper bound on the

number of isolated solutions as a by-product of the computation; further solving of the system uses the optimal number of paths to be followed.

We earlier characterized a successful homotopy continuation method as having three properties: triviality, smoothness, and accessibility. Given an arbitrary system of polynomial equations, such as (2.10), it is not too hard (through generic perturbations) to find a family of systems with the last two properties. The problem is that one member of the family must be trivial to solve, or the path-following cannot get started. The idea of the cheater's homotopy is simply to "cheat" on this part of the problem, and run a preprocessing step (the computation of the seeds X^*) which gives us the triviality property in a roundabout way. Thus the name, the "cheater's homotopy".

A statement of the theoretical result we need follows. Let

$$\begin{aligned} &p_1(c_1,\ldots,c_M,x_1,\ldots,x_n)=0,\\ &\qquad\qquad\vdots\\ &p_n(c_1,\ldots,c_M,x_1,\ldots,x_n)=0, \end{aligned} \tag{2.11}$$

be a system of polynomial equations in the variables $c_1,\ldots,c_M,x_1,\ldots,x_n$. Write $P(c,x)=(p_1(c,x),\ldots,p_n(c,x))$. For each choice of $c=(c_1,\ldots,c_M)$ in $\mathbb{C}^M$, this is a system of polynomial equations in the variables $x_1,\ldots,x_n$. Let d be the total degree of the system for a generic choice of c.

THEOREM 2.3. *Let c belong to $\mathbb{C}^M$. There exists an open dense full-measure subset U of $\mathbb{C}^{n+M}$ such that for $(b_1^*,\ldots,b_n^*,c_1^*,\ldots,c_M^*)\in U$, the following holds*:

(a) *The set X^* of solutions $x=(x_1,\ldots,x_n)$ of*

$$\begin{aligned} &q_1(x_1,\ldots,x_n)=p_1\big(c_1^*,\ldots,c_M^*,x_1,\ldots,x_n\big)+b_1^*=0,\\ &\qquad\qquad\vdots\\ &q_n(x_1,\ldots,x_n)=p_n\big(c_1^*,\ldots,c_M^*,x_1,\ldots,x_n\big)+b_n^*=0 \end{aligned} \tag{2.12}$$

consists of d_0 isolated points, for some $d_0\leqslant d$.

(b) *The smoothness and accessibility properties hold for the homotopy*

$$\begin{aligned} H(x,t)=P\big(&n(1-t)c_1^*+tc_1,\ldots,(1-t)c_M^*+tc_M,x_1,\ldots,x_n\big)\\ &+(1-t)b^*, \end{aligned} \tag{2.13}$$

where $b^*=(b_1^*,\ldots,b_n^*)$. It follows that every solution of $P(c,x)=0$ is reached by a path beginning at a point of X^*.

A proof of this theorem can be found in LI, SAUER and YORKE [1989]. The theorem is used as part of the following procedure. Let $P(c,x)=0$ as in (2.11) denote the system to be solved for various values of the coefficients c.

CHEATER'S HOMOTOPY PROCEDURE.

(1) Choose complex number $(b_1^*, \ldots, b_n^*, c_1^*, \ldots, c_M^*)$ at random, and use the classical homotopy continuation method to solve $Q(x) = 0$ in (2.12). Let d_0 denote the number of solutions found (this number is bounded above by the total degree d). Let X^* denote the set of d_0 solutions.

(2) For each new choice of coefficients $c = (c_1, \ldots, c_M)$, follow the d_0 paths defined by $H(x, t) = 0$ in (2.13), beginning at the points in X^*, to find all solutions of $P(c, x) = 0$.

In step (1) above, for random complex numbers $(c_1^*, \ldots, c_M^*)$, using classical homotopy continuation methods to solve $Q(x) = 0$ in (2.12) may itself be computationally expensive. It is desirable that those numbers do not have to be random. For illustration, we regard the linear system

$$\begin{aligned} c_{11}x_1 + \cdots + c_{1n}x_n &= b_1, \\ &\vdots \\ c_{n1}x_1 + \cdots + c_{nn}x_n &= b_n, \end{aligned} \tag{2.14}$$

as a system of polynomial equations with degree one of each. For randomly chosen c_{ij}'s, (2.14) has a unique solution which is not available right away. However, if we choose $c_{ij} = \delta_{ij}$ (the Kronecker delta: $= 1$ if $i = j$, $= 0$ if $i \neq j$), the solution is obvious.

For this purpose, an alternative is suggested in LI and WANG [1992]. When a system $P(c, x) = 0$ with a particular parameter c^0 is solved, this c^0 may be assigned specifically instead of being chosen randomly, then for any parameter $c \in \mathbb{C}^M$ consider the nonlinear homotopy

$$H(a, x, t) = P\big(\big(1 - \big[t - t(1-t)a\big]\big)c^0 + \big(t - t(1-t)a\big)c, x\big) = 0. \tag{2.15}$$

It was shown in LI and WANG [1992] that for randomly chosen complex number a the solution paths of $H(a, x, t) = 0$ in (2.15), emanating from the solutions of $P(c^0, x) = 0$ will reach the isolated solutions of $P(c, x) = 0$ under the natural assumption that for generic c, $P(c, x)$ has the same number of isolated zeros in $\mathbb{C}^n$.

The most important advantage of the homotopy in (2.15) is that the parameter c^0 of the start system $P(c^0, x) = 0$ need not be chosen at random. We only require the system $P(c^0, x) = 0$ to have the same number of solutions as $P(c, x) = 0$ for generic c. Therefore, in some situations, when the solutions of $P(c, x) = 0$ are easily available for a particular parameter c^0, the system $P(c^0, x) = 0$ may be used as the start system in (2.15) and the extra effort of solving $P(c, x) = 0$ for a randomly chosen c would not be necessary.

To finish, we give a non-trivial example of the use of the procedure discussed above. Consider the indirect position problem for revolute-joint kinematic manipulators. Each joint represents a one-dimensional choice of parameters, namely the angular position of the joint. If all angular positions are known, then of course the position and orientation of the end of the manipulator (the hand) are determined. The indirect position problem is the inverse problem: given the desired position and orientation of the hand, find a

set of angular parameters for the (controllable) joints which will place the hand in the desired state.

The indirect position problem for six joints is reduced to a system of eight nonlinear equations in eight unknowns in TSAI and MORGAN [1985]. The coefficients of the equations depend on the desired position and orientation, and a solution of the system (an eight-vector) represents the sines and cosines of the angular parameters. Whenever the manipulator's position is changed, the system needs to be resolved with new coefficients. The equations are too long to repeat here, see the appendix of TSAI and MORGAN [1985]; suffice to say that it is a system of eight degree-two polynomial equations in eight unknowns which is quite deficient. The total degree of the system is $2^8 = 256$, but there are at most 32 isolated solutions.

The nonlinear homotopy (2.15) requires only 32 paths to solve the system with different set of parameters, see LI and WANG [1990], LI and WANG [1992]. The system contains 26 coefficients, and a specific set of coefficients is chosen for which the system has 32 solutions. For subsequent solving of the system, for any choice of the coefficients $c_1, \ldots, c_{26}$, all solutions can be found at the end of exactly 32 paths by using nonlinear homotopy in (2.15) with randomly chosen complex number a.

3. Combinatorial root count

3.1. Bernshteín's theorem

In the middle of 90's, a major computational breakthrough has emerged in solving polynomial systems by the homotopy continuation method. The new method takes a great advantage of the Bernshteín's theorem which provides a much tighter bound in general for the number of isolated zeros of a polynomial system in the algebraic tori $(\mathbb{C}^*)^n$ where $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. Based on this root count, the *polyhedral homotopy* was introduced in HUBER and STURMFELS [1995] to find all isolated zeros of polynomial systems.

We take the following example (HUBER and STURMFELS [1995]) as our point of departure. With $x = (x_1, x_2)$, consider the system $P(x) = (p_1(x), p_2(x))$ where

$$\begin{aligned} p_1(x_1, x_2) &= c_{11}x_1x_2 + c_{12}x_1 + c_{13}x_2 + c_{14} = 0, \\ p_2(x_1, x_2) &= c_{21}x_1x_2^2 + c_{22}x_1^2x_2 + c_{23} = 0. \end{aligned} \tag{3.1}$$

Here, $c_{ij} \in \mathbb{C}^*$. The monomials $\{x_1x_2, x_1, x_2, 1\}$ in p_1 can be written with their explicit exponents as $x_1x_2 = x_1^1x_2^1$, $x_1 = x_1^1x_2^0$, $x_2 = x_1^0x_2^1$ and $1 = x_1^0x_2^0$. The set of their exponents

$$S_1 = \{a = (0,0),\ b = (1,0),\ c = (1,1),\ d = (0,1)\}$$

is called the *support* of p_1, and its convex hull $Q_1 = \text{conv}(S_1)$ is called the *Newton polytope* of p_1. Similarly, p_2 has support $S_2 = \{e = (0,0),\ f = (2,1),\ g = (1,2)\}$ and Newton polytope $Q_2 = \text{conv}(S_2)$. With additional notations $x^q = x_1^{q_1}x_2^{q_2}$ where $q = (q_1, q_2)$, the system in (3.1) becomes

$$p_1(x) = \sum_{q \in S_1} c_{1,q}x^q, \qquad p_2(x) = \sum_{q \in S_2} c_{2,q}x^q.$$

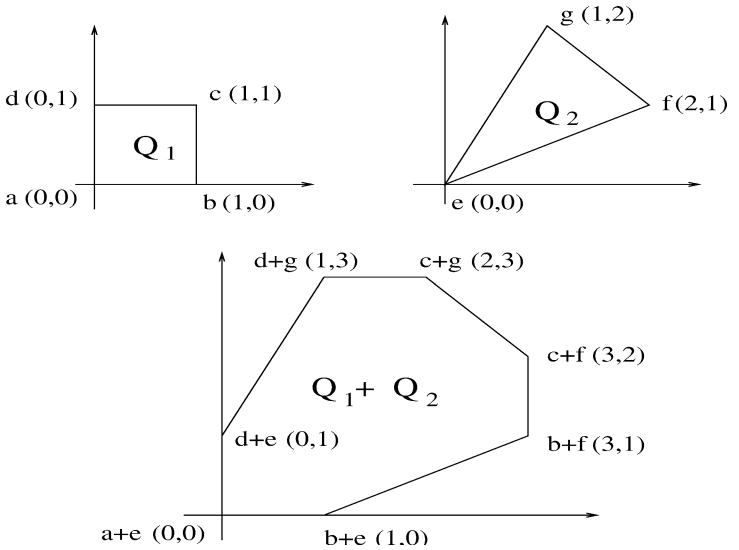


FIG. 3.1.

For polytopes $R_1, \ldots, R_k$ in $\mathbb{R}^n$, their *Minkowski sum* $R_1 + \cdots + R_k$ is defined by

$$R_1 + \cdots + R_n = \{r_1 + \cdots + r_k \mid r_j \in R_j,\ j = 1, \ldots, k\}.$$

(Polytopes Q_1, Q_2 and $Q_1 + Q_2$ for the system in (3.1) are shown in Fig. 3.1.) Let's consider the area of the convex polytope $\lambda_1 Q_1 + \lambda_2 Q_2$ with non-negative variables λ_1 and λ_2. First of all, the area of a triangle on the plane with vertices u, v and w is known to be

$$\frac{1}{2}\left|\det\begin{pmatrix} u - v \\ w - v \end{pmatrix}\right|. \tag{3.2}$$

To compute the area $f(\lambda_1, \lambda_2)$ of $\lambda_1 Q_1 + \lambda_2 Q_2$, we may partition the polytope $\lambda_1 Q_1 + \lambda_2 Q_2$ into a collection of mutually disjoint triangles, $A_1, A_2, \ldots, A_k$. If we choose those triangles in which none of their vertices is an interior point of the polytope $\lambda_1 Q_1 + \lambda_2 Q_2$, then all their vertices take the form $\lambda_1 r_1 + \lambda_2 r_2$ for certain $r_1 \in Q_1$ and $r_2 \in Q_2$. From (3.2), $f(\lambda_1, \lambda_2)$ is a second degree homogeneous polynomial in λ_1 and λ_2. Writing

$$f(\lambda_1, \lambda_2) = a_1\lambda_1^2 + a_2\lambda_2^2 + a_{12}\lambda_1\lambda_2,$$

the coefficient a_{12} of $\lambda_1\lambda_2$ in f is called the *mixed volume* of the polytopes Q_1 and Q_2, and is denoted by $\mathcal{M}(Q_1, Q_2)$.

Clearly,

$$\begin{aligned} a_{12} &= f(1,1) - f(1,0) - f(0,1) \\ &= \text{area of } (Q_1 + Q_2) - \text{ area of } (Q_1) - \text{ area of } (Q_2). \end{aligned}$$

The areas of $Q_1 + Q_2$, Q_1 and Q_2, as displayed in Fig. 3.1, are 6.5, 1 and 1.5, respectively, therefore the mixed volume $\mathcal{M}(Q_1, Q_2)$ of the polytopes Q_1 and Q_2 is

$$\mathcal{M}(Q_1, Q_2) = a_{12} = 6.5 - 1 - 1.5 = 4.$$

On the other hand, system (3.1) has two zeros $(x_0, x_1, x_2) = (0, 0, 1)$ and $(0, 1, 0)$ at infinity when viewed in $\mathbb{P}^2$; hence, it can have at most 4 isolated zeros in $(\mathbb{C}^*)^2$. This is the content of the Bernshteín theory: The number of isolated zeros of (3.1) in $(\mathbb{C}^*)^2$, counting multiplicities, is bounded above by the mixed volume of its Newton polytopes. Furthermore, when c_{ij}'s in (3.1) are chosen generically, these two numbers are exactly the same.

To state the Bernshteín theorem in general form, let the given polynomial system be $P(x) = (p_1(x), \ldots, p_n(x)) \in \mathbb{C}[x]$ where $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$. With $x^a = x_1^{a_1} \cdots x_n^{a_n}$ and $a = (a_1, \ldots, a_n)$, write

$$\begin{aligned} p_1(x) &= \sum_{a \in S_1} c^*_{1,a} x^a, \\ &\vdots \\ p_n(x) &= \sum_{a \in S_n} c^*_{n,a} x^a, \end{aligned} \tag{3.3}$$

where $S_1, \ldots, S_n$ are fixed subsets of $\mathbb{N}^n$ with cardinals $k_j = \#S_j$, and $c^*_{j,a} \in \mathbb{C}^*$ for $a \in S_j$, $j = 1, \ldots, n$. As before, S_j is the *support* of $p_j(x)$, its convex hull $Q_j = \text{conv}(S_j)$ in $\mathbb{R}^n$ is the *Newton polytope* of p_j, and $S = (S_1, \ldots, S_n)$ is the *support* of $P(x)$. For nonnegative variables $\lambda_1, \ldots, \lambda_n$, let $\lambda_1 Q_1 + \cdots + \lambda_n Q_n$ denote the *Minkowski sum* of $\lambda_1 Q_1, \ldots, \lambda_n Q_n$, that is

$$\lambda_1 Q_1 + \cdots + \lambda_n Q_n = \{\lambda_1 r_1 + \cdots + \lambda_n r_n \mid r_j \in Q_j,\ j = 1, 2, \ldots, n\}.$$

Following similar reasonings for calculating the area of $\lambda_1 Q_1 + \lambda_2 Q_2$ of the system in (3.1), it can be shown that the n-dimensional volume, denoted by Vol_n, of the polytope $\lambda_1 Q_1 + \cdots + \lambda_n Q_n$ is a homogeneous polynomial of degree n in $\lambda_1, \ldots, \lambda_n$. The coefficient of the monomial $\lambda_1 \cdots \lambda_n$ in this homogeneous polynomial is called the *mixed volume* of the polytopes $Q_1, \ldots, Q_n$, denoted by $\mathcal{M}(Q_1, \ldots, Q_n)$, or the mixed volume of the supports $S_1, \ldots, S_n$ denoted by $\mathcal{M}(S_1, \ldots, S_n)$. Sometimes, when no ambiguities exist, it is also called the mixed volume of $P(x)$.

We now embed the system $P(x)$ in (3.3) in the systems $P(c, x) = (p_1(c, x), \ldots, p_n(c, x))$ where

$$\begin{aligned} p_1(c, x) &= \sum_{a \in S_1} c_{1,a} x^a, \\ &\vdots \\ p_n(c, x) &= \sum_{a \in S_n} c_{n,a} x^a, \end{aligned} \tag{3.4}$$

and the coefficients $c = (c_{j,a})$ with $a \in S_j$ for $j = 1, \ldots, n$ are taken to be a set of $m := k_1 + \cdots + k_n$ variables. That is, we regard $P(x) = P(c^*, x)$ for a set of specified values of coefficients $c^* = (c^*_{j,a})$ in (3.4).

In what follows, the total number of isolated zeros, counting multiplicities, of a polynomial system will be referred to as the *root count* of the system.

LEMMA 3.1 (HUBER [1996]). *For polynomial systems $P(c, x)$ in* (3.4)*, there exists a polynomial system $G(c) = (g_1(c), \ldots, g_l(c))$ in the variables $c = (c_{j,a})$ for $a \in S_j$ and $j = 1, \ldots, n$ such that for those coefficients $c^* = (c^*_{j,a})$ for which $G(c^*) \neq 0$, the root count in $(\mathbb{C}^*)^n$ of the corresponding polynomial systems in* (3.4) *is a fixed number. And the root count in $(\mathbb{C}^*)^n$ of any other polynomial systems in* (3.4) *is bounded above by this number.*

A simple example that illustrates the assertion of the above lemma is the following 2×2 linear systems:

$$\begin{aligned} c_{11}x_1 + c_{12}x_2 &= b_1, \\ c_{21}x_1 + c_{22}x_2 &= b_2. \end{aligned} \tag{3.5}$$

Here, $c = (c_{11}, c_{12}, c_{21}, c_{22}, -b_1, -b_2)$. Let

$$G(c) = \det\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \times \det\begin{pmatrix} c_{11} & b_1 \\ c_{21} & b_2 \end{pmatrix} \times \det\begin{pmatrix} b_1 & c_{12} \\ b_2 & c_{22} \end{pmatrix}.$$

When the coefficient $c^* = (c^*_{11}, c^*_{12}, c^*_{21}, c^*_{22}, -b^*_1, -b^*_2)$ satisfies $G(c^*) = 0$, its corresponding system in (3.5) has no isolated solution in $(\mathbb{C}^*)^2$; otherwise, the system has a unique solution in $(\mathbb{C}^*)^2$.

Since the zeros of the polynomial $G(c)$ in Lemma 3.1 form an algebraic set with dimension smaller than m, its complement where $G(c) \neq 0$ is open and dense with full measure in $\mathbb{C}^m$. Therefore, polynomial systems $P(c, x)$ in (3.4) with $G(c) \neq 0$ are said to be *in general position*. For $P(x) = (p_1(x), \ldots, p_n(x))$ in general position with support $(S_1, \ldots, S_n)$, let $L(S_1, \ldots, S_n)$ be the fixed number of its isolated zeros in $(\mathbb{C}^*)^n$. This number satisfies the following properties:

(1) (*Symmetry*) $L(S_1, \ldots, S_n)$ remains invariant when S_i and S_j for $i \neq j$ exchange their positions.

(2) (*Shift invariant*) $L(S_1, \ldots, a + S_j, \ldots, S_n) = L(S_1, \ldots, S_n)$ for $a \in \mathbb{N}^n$.

Replacing $p_j(x)$ in $P(x) = (p_1(x), \ldots, p_n(x))$ by $x^a p_j(c, x)$ results in a new system with support $(S_1, \ldots, a + S_j, \ldots, S_n)$. Obviously, the number of its isolated zeros in $(\mathbb{C}^*)^n$ stays the same.

(3) (*Multilinear*) $L(S_1, \ldots, S_j + \overline{S}_j, \ldots, S_n) = L(S_1, \ldots, S_j, \ldots, S_n) + L(S_1, \ldots, \overline{S}_j, \ldots, S_n)$ for $\overline{S}_j \subset \mathbb{N}^n$.

Let $\overline{P}(x) = (p_1(x), \ldots, \bar{p}_j(x), \ldots, p_n(x))$ be a system in general position with support $(S_1, \ldots, \overline{S}_j, \ldots, S_n)$. Then replacing $p_j(x)$ in $P(x)$ by $p_j(x) \cdot \bar{p}_j(x)$ yields a system with support $(S_1, \ldots, S_j + \overline{S}_j, \ldots, S_n)$. It is clear that the number of isolated zeros of the resulting system in $(\mathbb{C}^*)^n$ is the sum of the number of those isolated zeros of $P(x)$ and $\overline{P}(x)$ in $(\mathbb{C}^*)^n$.

(4) (*Automorphism invariant*) $L(S_1, \ldots, S_n) = L(US_1, \ldots, US_n)$ where U is an integer entry $n \times n$ matrix with $\det U = \pm 1$ and $US_j = \{Ua \mid a \in S_j\}$ for $j = 1, \ldots, n$.

Note that in writing $x^a = x_1^{a_1} \cdots x_n^{a_n}$, we regard $a = (a_1, \ldots, a_n)$ as a column vector in convention. Let U_j be the jth column of $U = (u_{ij})$ and $x = y^U := (y^{U_1}, \ldots, y^{U_n})$, i.e.

$$x_j = y^{U_j} = y_1^{u_{1j}} \cdots y_n^{u_{nj}}, \quad j = 1, \ldots, n.$$

This coordinate transformation yields

$$\begin{aligned} x^a &= x_1^{a_1} \cdots x_n^{a_n} \\ &= \left(y^{U_1}\right)^{a_1} \cdots \left(y^{U_n}\right)^{a_n} \\ &= y_1^{u_{11}a_1 + \cdots + u_{1n}a_n} \cdots y_n^{u_{n1}a_1 + \cdots + u_{nn}a_n} \\ &= y^{Ua}, \end{aligned} \tag{3.6}$$

and transforms the system $P(x)$ with support $(S_1, \ldots, S_n)$ to $Q(y) = P(y^U)$ with support $(US_1, \ldots, US_n)$. For a given isolated zeros y_0 of $Q(y)$ in $(\mathbb{C}^*)^n$, $x_0 = y_0^U$ is clearly an isolated zero of $P(x)$ in $(\mathbb{C}^*)^n$. On the other hand, since $\det U = \pm 1$, $V := U^{-1}$ is also an integer-entry matrix, and

$$x^V = \left(y^U\right)^V = y^{(UV)} = y.$$

Therefore, for an isolated zero x_0 of $P(x)$ in $(\mathbb{C}^*)^n$, $y_0 = x_0^V$ is an isolated zero of $Q(y)$ in $(\mathbb{C}^*)^n$. This one-to-one correspondence between isolated zeros of $Q(y)$ and $P(x)$ in $(\mathbb{C}^*)^n$ yields $L(S_1, \ldots, S_n) = L(US_1, \ldots, US_n)$.

Functions that take n finite subsets $S_1, \ldots, S_n$ of $\mathbb{N}^n$ and return with a real number satisfying all the above four properties are rarely available. The mixed volume $\mathcal{M}(S_1, \ldots, S_n)$, emerged in the early 20th century, happens to be one of them:

(1) (*Symmetry*) This property is obvious for $\mathcal{M}(S_1, \ldots, S_n)$ by its definition.

(2) (*Shift invariant*) For $a \in \mathbb{N}^n$ and $Q_k = \text{conv}(S_k)$ for $k = 1, \ldots, n$,

$$\begin{aligned} &\text{Vol}_n\left(\lambda_1 Q_1 + \cdots + \lambda_j(a + Q_j) + \cdots + \lambda_n Q_n\right) \\ &\quad = \text{Vol}_n(\lambda_j a + \lambda_1 Q_1 + \cdots + \lambda_j Q_j + \cdots + \lambda_n Q_n) \\ &\quad = \text{Vol}_n(\lambda_1 Q_1 + \cdots + \lambda_n Q_n). \end{aligned}$$

Hence, $\mathcal{M}(S_1, \ldots, a + S_j, \ldots S_n) = \mathcal{M}(S_1, \ldots, S_n)$.

(3) (*Multilinear*) We shall prove this property for $\overline{S}_1 \subset \mathbb{N}^n$, i.e.

$$\mathcal{M}(S_1 + \overline{S}_1, S_2, \ldots, S_n) = \mathcal{M}(S_1, \ldots, S_n) + \mathcal{M}(\overline{S}_1, \ldots S_n).$$

For positive $\alpha, \beta, \lambda_1, \ldots, \lambda_n$ and $\overline{Q}_1 = \text{conv}(\overline{S}_1)$,

$$\begin{aligned} &\text{Vol}_n\left(\lambda_1(\alpha Q_1 + \beta \overline{Q}_1) + \lambda_2 Q_2 + \cdots + \lambda_n Q_n\right) \\ &\quad = \sum_{j_1 + \cdots + j_n = n} a(\alpha, \beta, j_1, \ldots, j_n) \lambda_1^{j_1} \cdots \lambda_n^{j_n} \end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&\mathrm{Vol}_n(\lambda_1\alpha Q_1+\lambda_1\beta\overline{Q}_1+\cdots+\lambda_n Q_n)\\&\quad=\sum_{j_1+j_1'+\cdots+j_n=n} b(j_1,j_1',\ldots,j_n)(\lambda,\alpha)^{j_1}(\lambda_1\beta)^{j_1'}\cdots\lambda_n^{j_n}.\end{aligned}\tag{3.8}$$

Comparing the coefficients of $\lambda_1\cdots\lambda_n$ in (3.7) and (3.8) gives

$$a(\alpha,\beta,1,\ldots,1)=\alpha b(1,0,1,\ldots,1)+\beta b(0,1,\ldots,1).\tag{3.9}$$

Letting (1) $\alpha=\beta=1$, (2) $\alpha=1,\beta=0$, and (3) $\alpha=0,\beta=1$ in (3.9) respectively, yields

$$\begin{aligned}\mathcal{M}(S_1+\overline{S}_1,\ldots,S_n)&=a(1,\ldots,1)=b(1,0,1,\ldots,1)+b(0,1,\ldots,1)\\&=a(1,0,1,\ldots,1)+a(0,1,\ldots,1)\\&=\mathcal{M}(S_1,\ldots,S_n)+\mathcal{M}(\overline{S}_1,\ldots,S_n).\end{aligned}$$

(4) (*Automorphism invariant*) For linear transformation U,

$$\mathrm{Vol}_n\big(U(\lambda_1Q_1+\cdots+\lambda_nQ_n)\big)=|\det U|\,\mathrm{Vol}_n(\lambda_1Q_1+\cdots+\lambda_nQ_n).$$

Therefore, when $\det U=\pm1$,

$$\begin{aligned}\mathrm{Vol}_n\big(\lambda_1(UQ_1)+\cdots+\lambda_n(UQ_n)\big)&=\mathrm{Vol}_n\big(U(\lambda_1Q_1+\cdots+\lambda_nQ_n)\big)\\&=\mathrm{Vol}_n(\lambda_1Q_1+\cdots+\lambda_nQ_n),\end{aligned}$$

and consequently,

$$\mathcal{M}(US_1,\ldots,US_n)=\mathcal{M}(S_1,\ldots,S_n).$$

The above connection between $L(S_1,\ldots,S_n)$ and $\mathcal{M}(S_1,\ldots,S_n)$ suggested the establishment of the following Bernshteín theorem:

THEOREM 3.1 (BERNSHTEÍN [1975], Theorem A). *The number of isolated zeros, counting multiplicities, in $(\mathbb{C}^*)^n$ of a polynomial system $P(x)=(p_1(x),\ldots,p_n(x))$ with support $S=(S_1,\ldots,S_n)$ is bound above by the mixed volume $\mathcal{M}(S_1,\ldots,S_n)$. When $P(x)$ is in general position, it has exactly $\mathcal{M}(S_1,\ldots,S_n)$ isolated zeros in $(\mathbb{C}^*)^n$.*

In CANNY and ROJAS [1991], the root count in the above theorem was nicknamed the *BKK bound* after the works of BERNSHTEÍN [1975], KUSHNIRENKO [1976] and KHOVANSKIÍ [1978]. In general, it provides a much tighter bound compared to variant Bézout bounds such as those given in the last section. An apparent limitation of this theorem, important in practice, is that it only counts the number of isolated zeros of a polynomial system in $(\mathbb{C}^*)^n$ rather than all isolated zeros in affine space $\mathbb{C}^n$. This problem was first attempted in ROJAS [1994], a bound for the root count in $\mathbb{C}^n$ was obtained via the notion of the *shadowed* sets. Later, a significantly much tighter bound was given in the following

THEOREM 3.2 (LI and WANG [1997]). *The root count in $\mathbb{C}^n$ of a polynomial system $P(x) = (p_1(x), \ldots, p_n(x))$ with supports $S = (S_1, \ldots, S_n)$ is bounded above by the mixed volume $\mathcal{M}(S_1 \cup \{0\}, \ldots, S_n \cup \{0\})$.*

In other words, the root count of a polynomial system $P(x) = (p_1(x), \ldots, p_n(x))$ in $\mathbb{C}^n$ is bounded above by the root count in $(\mathbb{C}^*)^n$ of the polynomial system $\overline{P}(x)$ in general position obtained by augmenting constant terms to those p_j's in $P(x)$ in which the constant terms are absent. As a corollary, when $0 \in S_j$ for all $j = 1, \ldots, n$, namely, all $p_j(x)$'s in $P(x)$ have constant terms, then the mixed volume $\mathcal{M}(S_1, \ldots, S_n)$ of $P(x)$ is a bound for its root count in $\mathbb{C}^n$, more than just a root count in $(\mathbb{C}^*)^n$. This theorem was further extended in several different ways, see HUBER and STURMFELS [1997] and ROJAS and WANG [1996].

3.2. Mixed volume and mixed subdivisions

Let us consider the system in (3.1) with Newton polytopes Q_1 and Q_2, as shown in Fig. 3.1, once again. Recall that

$$\mathcal{M}(Q_1, Q_2) = \text{ area of } (Q_1 + Q_2) - \text{ area of } (Q_1) - \text{ area of } (Q_2).$$

To calculate the area of $Q_1 + Q_2$, we may subdivide the polytope $Q_1 + Q_2$ into convenient pieces, such as squares or triangles, in whichever way as we prefer. Among all the possible subdivisions of $Q_1 + Q_2$, the subdivision I, as shown in Fig. 3.2, exhibits some special properties.

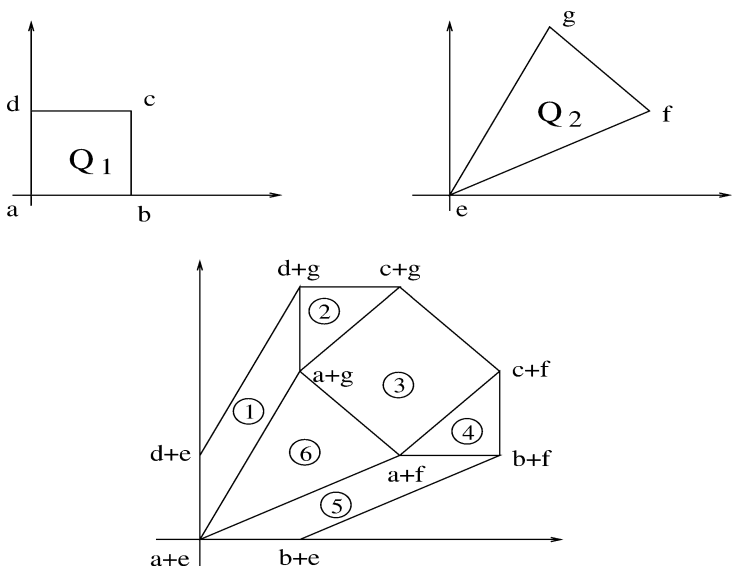


FIG. 3.2. Subdivision I for $Q_1 + Q_2$.

It is easy to verify that all the members, called the *cells*, in this subdivision satisfy the following properties:

(a) Each cell is a Minkowski sum of the convex hulls of $C_1 \subset S_1$ and $C_2 \subset S_2$.
For instance, cell ① $=$ conv$(\{a, d\}) +$ conv$(\{e, g\})$.

(b) When $C_1 \subset S_1$ and $C_2 \subset S_2$ form a cell in the subdivision, conv(C_i) is a simplex of dimension $\#(C_i) - 1$ for $i = 1, 2$. Here, $\#(C_i)$ is the number of points in C_i.
For instance, convex hulls of both $C_1 = \{a, d\}$ and $C_2 = \{e, g\}$ of cell ① are one-dimensional simplices.

(c) Simplices conv(C_1) and conv(C_2) are complementary to each other in the sense: $\dim(\text{conv}(C_1)) + \dim(\text{conv}(C_2)) = \dim(\text{conv}(C_1) + \text{conv}(C_2))$.
That is, $v_1 = a - d$ and $v_2 = e - g$ are linearly independent.

In light of properties (a) and (b), each cell $C = \text{conv}(C_1) + \text{conv}(C_2)$ in subdivision I can be characterized as a cell of type (l_1, l_2) where $l_1 = \dim(\text{conv}(C_1))$ and $l_2 = \dim(\text{conv}(C_2))$.

For $j = 1, \ldots, n$, let A_j be a simplex of dimension $k_j \geqslant 0$ in $\mathbb{R}^n$ with vertices $\{q_0^{(j)}, \ldots, q_{k_j}^{(j)}\}$ where $k_1 + \cdots + k_n = n$. Let V be the $n \times n$ matrix whose rows are $q_i^{(j)} - q_0^{(j)}$ for $1 \leqslant j \leqslant n$ and $1 \leqslant i \leqslant k_j$. Notice that any 0-dimensional simplex consists of only one point, and therefore contributes no rows to V. It can be shown that the n-dimensional volume of the Minkowski sum of $A_1, \ldots, A_n$ is equal to

$$\text{Vol}_n(A_1 + \cdots + A_n) = \frac{1}{k_1! \cdots k_n!} |\det V|. \tag{3.10}$$

Applying (3.10) to cell ① (= conv$(\{a, d\}) +$ conv$(\{e, q\})$) in subdivision I,

$$\text{Vol}_2\big(\text{cell } ①\big) = \left|\det\begin{pmatrix} d-a \\ g-e \end{pmatrix}\right|.$$

When Q_1 and Q_2 are scaled by λ_1 and λ_2 respectively, cell ① becomes a cell ①′ $=$ conv$(\{\lambda_1 a, \lambda_2 d\}) +$ conv$(\{\lambda_2 e, \lambda_2 g\})$ in the subdivision I′ of $\lambda_1 Q_1 + \lambda_2 Q_2$ as shown in Fig. 3.3 and its volume becomes

$$\begin{aligned}\left|\det\begin{pmatrix} \lambda_1 d - \lambda_1 a \\ \lambda_2 g - \lambda_2 e \end{pmatrix}\right| &= \left|\det\begin{pmatrix} d-a \\ g-e \end{pmatrix}\right| \times \lambda_1\lambda_2 \\ &= \big(\text{volume of cell } ①\big) \times \lambda_1\lambda_2.\end{aligned}$$

So, volume of the original cell ① constitutes part of the mixed volume $\mathcal{M}(Q_1, Q_2)$ which, by definition, is the coefficient of $\lambda_1\lambda_2$ in the homogeneous polynomial $\text{Vol}_2(\lambda_1 Q_1 + \lambda_2 Q_2)$. On the other hand, after scaling, cell ② in subdivision I becomes cell ②′ $=$ conv$(\{\lambda_1 a, \lambda_1 c, \lambda_1 d\}) + \{\lambda_2 g\}$ in subdivision I′ of $\lambda_1 Q_1 + \lambda_2 Q_2$, and its volume becomes

$$\begin{aligned}\frac{1}{2}\left|\det\begin{pmatrix} \lambda_1 c - \lambda_1 a \\ \lambda_1 d - \lambda_1 a \end{pmatrix}\right| &= \frac{1}{2}\left|\det\begin{pmatrix} c-a \\ d-a \end{pmatrix}\right| \times \lambda_1^2 \\ &= \big(\text{volume of cell } ②\big) \times \lambda_1^2.\end{aligned}$$

Thus, volume of the original cell ② plays no part in the mixed volume $\mathcal{M}(Q_1, Q_2)$.

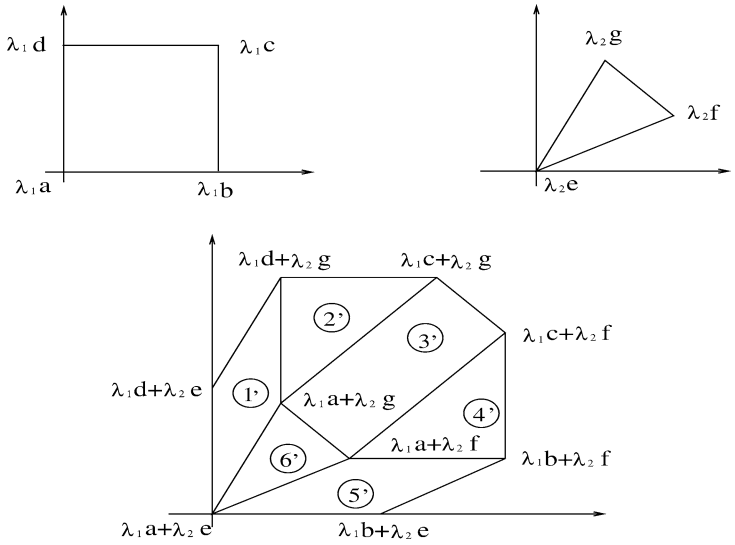


FIG. 3.3. Subdivision I′ for $\lambda_1 Q_1 + \lambda_2 Q_2$.

In general, cells of subdivision I of $Q_1 + Q_2$ become cells of subdivision I′ of $\lambda_1 Q_1 + \lambda_2 Q_2$ with corresponding scalings. Because of properties (a), (b) and (c), volumes of those cells after scaling can all be calculated by (3.10), and only volumes of cells of type $(1, 1)$ in $\lambda_1 Q_1 + \lambda_2 Q_2$ can have the factor $\lambda_1\lambda_2$ with volumes of the corresponding original cells before scaling as their coefficients. Thus,

$$\begin{aligned}\mathcal{M}(Q_1, Q_2) &= \text{the sum of the volumes of the original cells}\\ &\qquad \text{of type } (1,1) \text{ in subdivision I before scaling}\\ &= \text{volume of cell } ① + \text{volume of cell } ③ + \text{volume of cell } ⑤\\ &= 1 + 2 + 1 = 4.\end{aligned}$$

Assembling the mixed volume $\mathcal{M}(Q_1, Q_2)$ in this manner is independent of the scaling factors λ_1 and λ_2, and is valid for any such subdivisions, called the *fine mixed subdivisions*, of $Q_1 + Q_2$ that share the special properties (a)–(c).

To state a formal definition for such subdivisions with less notations, we shall omit those "+" and "conv" in most of the occasions. For instance, rather than formulating the subdivision for $Q_1 + \cdots + Q_n$ $(= \text{conv}(S_1) + \cdots + \text{conv}(S_n))$ we shall deal with the n-tuple $(S_1, \ldots, S_n)$ for short.

Let $S = (S_1, \ldots, S_n)$ be a set of finite subsets of $\mathbb{N}^n$, whose union affinely spans $\mathbb{R}^n$. By a *cell* of $S = (S_1, \ldots, S_n)$ we mean a tuple $C = (C_1, \ldots, C_n)$ where $C_j \subset S_j$ for $j = 1, \ldots, n$. Define

$$\begin{aligned}\text{type}(C) &:= \big(\dim\big(\text{conv}(C_1)\big), \ldots, \dim\big(\text{conv}(C_n)\big)\big),\\ \text{conv}(C) &:= \text{conv}(C_1) + \cdots + \text{conv}(C_n),\end{aligned}$$

and $\mathrm{Vol}(C) := \mathrm{Vol}(\mathrm{conv}(C))$. A *face* of C is a subcell $F = (F_1, \ldots, F_n)$ of C where $F_j \subset C_j$ and some linear functional $\alpha \in (\mathbb{R}^n)^\vee$ attains its minimum over C_j at F_j for $j = 1, \ldots, n$. We call such an α an *inner normal* of F. Clearly, if F is a face of C, then $\mathrm{conv}(F_j)$ is a face of the polytope $\mathrm{conv}(C_j)$ for each $j = 1, \ldots, n$. We call F a *facet* of C if $\mathrm{conv}(F)$ is a co-dimension one face of $\mathrm{conv}(C)$.

DEFINITION 3.1. A mixed subdivision of $S = (S_1, \ldots, S_n)$ is a set of cells $\{C^{(1)}, \ldots, C^{(m)}\}$ such that

(a) For all $i = 1, \ldots, m$, $\dim(\mathrm{conv}(C^{(i)})) = n$,
(b) $\mathrm{conv}(C^{(l)}) \cap \mathrm{conv}(C^{(k)})$ is a proper common face of $\mathrm{conv}(C^{(l)})$ and $\mathrm{conv}(C^{(k)})$ when it is nonempty for $l \neq k$,
(c) $\bigcup_{i=1}^m \mathrm{conv}(C^{(i)}) = \mathrm{conv}(S)$,
(d) For $i = 1, \ldots, m$, write $C^{(i)} = (C_1^{(i)}, \ldots, C_n^{(i)})$, then

$$\dim\big(\mathrm{conv}(C_1^{(i)})\big) + \cdots + \dim\big(\mathrm{conv}(C_n^{(i)})\big) = n.$$

This subdivision is called a fine mixed subdivision if in addition we have

(e) Each $\mathrm{conv}(C_j^{(i)})$ is a simplex of dimension $\#C_j^{(i)} - 1$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

Similar to what we have discussed for the system in (3.1), the following important property holds for a polynomial system $P(x) = (p_1(x), \ldots, p_n(x))$ with support $S = (S_1, \ldots, S_n)$, see HUBER and STURMFELS [1995]:

PROPOSITION 3.1. *The mixed volume $\mathcal{M}(S_1, \ldots, S_n)$ of $S = (S_1, \ldots, S_n)$ equals to the sum of the volumes of cells of type $(1, \ldots, 1)$ in a fine mixed subdivision of $S = (S_1, \ldots, S_n)$.*

So, to calculate mixed volume $\mathcal{M}(S_1, \ldots, S_n)$ following the above result, one must find a fine mixed subdivision of $S = (S_1, \ldots, S_n)$ in the first place. This can be accomplished by the standard process: Choose real-valued functions $\omega_j : S_j \to \mathbb{R}$ for each $j = 1, \ldots, n$. We call the n-tuple $\omega = (\omega_1, \ldots, \omega_n)$ a *lifting function* on S, and ω *lifts* S_j to its graph $\widehat{S}_j = \{(a, \omega_j(a))\colon\ a \in S_j\} \subset \mathbb{R}^{n+1}$. This notation is extended in the obvious way: $\widehat{S} = (\widehat{S}_1, \ldots, \widehat{S}_n)$, $\widehat{Q}_j = \mathrm{conv}(\widehat{S}_j)$, $\widehat{Q} = \widehat{Q}_1 + \cdots + \widehat{Q}_n$, etc. The lifting function $\omega = (\omega_1, \ldots, \omega_n)$ is known as a *generic lifting* if each ω_j for $j = 1, \ldots, n$ is generic in the sense that its images are chosen at random. Let S_ω be the set of cells $\{C\}$ of S satisfying

(a) $\dim(\mathrm{conv}(\widehat{C})) = n$,
(b) $\widehat{C}$ is a facet of $\widehat{S}$ whose inner normal $\alpha \in (\mathbb{R}^{n+1})^\vee$ has positive last coordinate. (In other words, $\mathrm{conv}(\widehat{C})$ is a facet in the *lower hull* of $\widehat{Q}$.)

It can be shown that (see GEL'FAND, KAPRANOV and ZELEVINSKIÍ [1994], HUBER and STURMFELS [1995] and LEE [1991]):

PROPOSITION 3.2. *When $\omega = (\omega_1, \ldots, \omega_n)$ is a generic lifting, then S_ω provides a fine mixed subdivision of $S = (S_1, \ldots, S_n)$.*

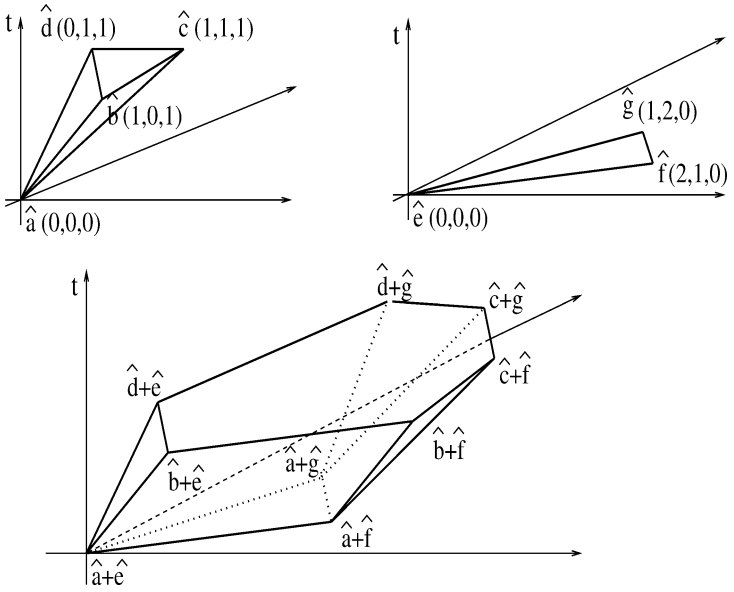


FIG. 3.4.

The subdivision I in Fig. 3.2 for system (3.1) is, in fact, induced by the lifting $\omega = ((0,1,1,1),(0,0,0))$, that is

$$\widehat{S} = \big(\{(a,0),(b,1),(c,1),(d,1)\},\{(e,0),(f,0),(g,0)\}\big)$$

(see Fig. 3.4). While this lifting does not appear to be as generic, it is sufficient to induce a fine mixed subdivision.

4. Polyhedral homotopy method

In light of Theorem 3.2, to find all isolated zeros of a given polynomial system $P(x) = (p_1(x),\ldots,p_n(x))$ in $\mathbb{C}^n$ with support $S = (S_1,\ldots,S_n)$, we first augment the monomial x^0 $(=1)$ to those p_j's in $P(x)$ in which constant terms are absent. Followed by choosing coefficients of all the monomials generically, a new system $Q(x) = (q_1(x),\ldots,q_n(x))$ with support $S' = (S'_1,\ldots,S'_n)$ where $S'_j = S_j \cup \{0\}$ for $j = 1,\ldots,n$ is produced. We will solve this system first, and consider the linear homotopy

$$H(x,t) = (1-t)cQ(x) + tP(x) = 0 \quad \text{for generic } c \in \mathbb{C}^* \tag{4.1}$$

afterwards. Recall that, by Theorem 2.3, Properties 1 and 2 (Smoothness and Accessibility) hold for this homotopy, and because all the isolated solutions of $Q(x) = 0$ are known, Property 0 also holds. Therefore, every isolated zero of $P(x)$ lies at the end of a homotopy path of $H(x,t) = 0$, emanating from an isolated solution of $Q(x) = 0$.

4.1. The polyhedral homotopy

To solve $Q(x) = 0$, write

$$Q(x) = \begin{cases} q_1(x) = \sum_{a \in S_1'} \bar{c}_{1,a} x^a, \\ \quad \vdots \\ q_n(x) = \sum_{a \in S_n'} \bar{c}_{n,a} x^a. \end{cases} \tag{4.2}$$

Since all those coefficients $\bar{c}_{j,a}$ for $a \in S_j'$ and $j = 1, \ldots, n$ are chosen generically, this system can be considered *in general position*. Namely, there exists a polynomial system

$$G(c) = \big(g_1(c), \ldots, g_l(c)\big) \tag{4.3}$$

in the variables $c = (c_{j,a})$ of the coefficients in (4.2) for $a \in S_j'$ and $j = 1, \ldots, n$ where $G(\bar{c}) \neq 0$ for the coefficients $\bar{c} = (\bar{c}_{j,a})$ of $Q(x)$, and all such systems reach the maximum root count $\mathcal{M}(S_1', \ldots, S_n')$ in $\mathbb{C}^n$.

Let t be a new complex variable and consider the polynomial system $\widehat{Q}(x,t) = (\hat{q}_1(x,t), \ldots, \hat{q}_n(x,t))$ in the $n+1$ variables (x,t) given by

$$\widehat{Q}(x,t) = \begin{cases} \hat{q}_1(x,t) = \sum_{a \in S_1'} \bar{c}_{1,a} x^a t^{w_1(a)}, \\ \quad \vdots \\ \hat{q}_n(x,t) = \sum_{a \in S_n'} \bar{c}_{n,a} x^a t^{w_n(a)}, \end{cases} \tag{4.4}$$

where each $w_j : S_j' \to \mathbb{R}$ for $j = 1, \ldots, n$ is a generic function whose images are chosen at random. For a fixed t_0, we rewrite the system in (4.4) as

$$\widehat{Q}(x,t_0) = \begin{cases} \hat{q}_1(x,t_0) = \sum_{a \in S_1'} \big(\bar{c}_{1,a} t_0^{w_1(a)}\big) x^a, \\ \quad \vdots \\ \hat{q}_n(x,t_0) = \sum_{a \in S_n'} \big(\bar{c}_{n,a} t_0^{w_n(a)}\big) x^a. \end{cases}$$

This system is in general position if for $G(c)$ in (4.3),

$$T(t_0) \equiv G\big(\bar{c}_{j,a} t_0^{w_j(a)}\big) \neq 0 \quad \text{for } a \in S_j' \text{ and } j = 1, \ldots, n.$$

The equation $T(t) = 0$ can have at most finitely many solutions, since $T(t)$ is not identically 0 because $T(1) = G(\bar{c}_{j,a}) \neq 0$. Let

$$t_1 = r_1 \mathrm{e}^{\mathrm{i}\theta_1}, \quad \ldots, \quad t_k = r_k \mathrm{e}^{\mathrm{i}\theta_k}$$

be the solutions of $T(t) = 0$. Then, for any $\theta \neq \theta_l$ for $l = 1, \ldots, k$, the systems $\overline{Q}(x,t) = (\bar{q}_1(x,t), \ldots, \bar{q}_n(x,t))$ given by

$$\overline{Q}(x,t) = \begin{cases} \bar{q}_1(x,t) = \sum_{a \in S_1'} \left(\bar{c}_{1,a} \mathrm{e}^{\mathrm{i} w_1(a)\theta}\right) x^a t^{w_1(a)}, \\ \qquad \vdots \\ \bar{q}_n(x,t) = \sum_{a \in S_n'} \left(\bar{c}_{n,a} \mathrm{e}^{\mathrm{i} w_n(a)\theta}\right) x^a t^{w_n(a)}, \end{cases}$$

are in general position for all $t > 0$ because

$$\bar{c}_{j,a} \mathrm{e}^{\mathrm{i} w_j(a)\theta} t^{w_j(a)} = \bar{c}_{j,a} \left(t \mathrm{e}^{\mathrm{i}\theta}\right)^{w_j(a)}$$

and

$$G\left(\bar{c}_{j,a} \left(t \mathrm{e}^{\mathrm{i}\theta}\right)^{w_j(a)}\right) = T\left(t \mathrm{e}^{\mathrm{i}\theta}\right) \neq 0.$$

Therefore, without loss of generality (choose an angle θ at random and change the coefficients $\bar{c}_{j,a}$ to $\bar{c}_{j,a} \mathrm{e}^{\mathrm{i} w_j(a)\theta}$ if necessary) we may suppose the systems $\widehat{Q}(x,t)$ in (4.4) are in general position for all $t > 0$. By Lemma 3.1, they have the same number of isolated zeros in $(\mathbb{C}^*)^n$ for all $t > 0$, the mixed volume $\mathcal{M}(S_1', \ldots, S_n') := k$.

We now regard $\widehat{Q}(x,t) = 0$ as a homotopy, known as the *polyhedral homotopy*, defined on $(\mathbb{C}^*)^n \times [0,1]$ with target system $\widehat{Q}(x,1) = Q(x)$. The zero set of this homotopy is made up of k homotopy paths $x^{(1)}(t), \ldots, x^{(k)}(t)$. Since each $\hat{q}_j(x,t)$ has nonzero constant term for all $j = 1, \ldots, n$, by a standard application of generalized Sard's theorem (ABRAHAM and ROBBIN [1967]), all those homotopy paths are smooth with no bifurcations. Therefore, both Property 1 (Smoothness) and Property 2 (Accessibility) hold for this homotopy. However, at $t = 0$, $\widehat{Q}(x,0) \equiv 0$, so those homotopy paths cannot get started because their starting points $x^{(1)}(0), \ldots, x^{(k)}(0)$ cannot be identified (see Fig. 4.1). This problem can be resolved by the following design.

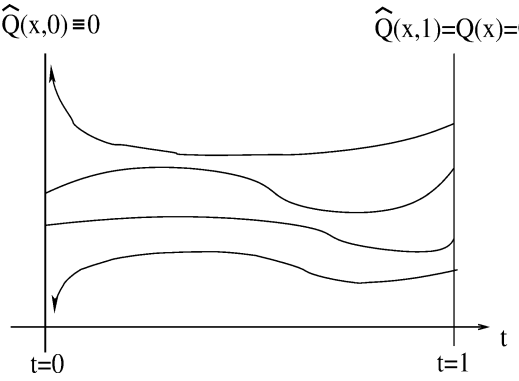


FIG. 4.1.

The function $\omega = (\omega_1, \ldots, \omega_n)$ with $\omega_j : S'_j \to \mathbb{R}$, $j = 1, \ldots, n$, may be considered as a *generic lifting* on the support $S' = (S'_1, \ldots, S'_n)$ of $Q(x)$ which lifts S'_j to its graph

$$\widehat{S}'_j = \{\hat{a} = (a, w_j(a)) \mid a \in S'_j\}, \quad j = 1, \ldots, n.$$

Let $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$ satisfy the following condition:

$$\begin{aligned} &\text{There exists a collection of pairs } \{a_1, a'_1\} \subset S'_1, \ldots, \{a_n, a'_n\} \subset S'_n, \\ &\text{where } \{a_1 - a'_1, \ldots, a_n - a'_n\} \text{ is linearly independent, and} \\ &\qquad \langle \hat{a}_j, \hat{\alpha} \rangle = \langle \hat{a}'_j, \hat{\alpha} \rangle, \\ &\qquad \langle \hat{a}, \hat{\alpha} \rangle > \langle \hat{a}_j, \hat{\alpha} \rangle \quad \text{for } a \in S'_j \backslash \{a_j, a'_j\}. \end{aligned} \tag{A}$$

Here, $\langle \cdot, \cdot \rangle$ stands for the usual inner product in the Euclidean space. For such $\hat{\alpha} = (\alpha, 1)$ where $\alpha = (\alpha_1, \ldots, \alpha_n)$, let

$$\begin{aligned} y_1 &= t^{-\alpha_1} x_1, \\ &\vdots \\ y_n &= t^{-\alpha_n} x_n. \end{aligned} \tag{4.5}$$

In short, we write $y = t^{-\alpha} x$ with $y = (y_1, \ldots, y_n)$ and $y t^{\alpha} = x$. With this transformation and $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$,

$$\begin{aligned} x^a &= x_1^{a_1} \cdots x_n^{a_n} \\ &= \left(y_1 t^{\alpha_1}\right)^{a_1} \cdots \left(y_n t^{\alpha_n}\right)^{a_n} \\ &= y_1^{a_1} \cdots y_n^{a_n} \, t^{\alpha_1 a_n + \cdots + \alpha_n a_n} \\ &= y^a t^{\langle a, \alpha \rangle}, \end{aligned} \tag{4.6}$$

and $\hat{q}_j(x, t)$ of $\widehat{Q}(x, t)$ in (4.4) becomes

$$\begin{aligned} \hat{q}_j(y t^{\alpha}, t) &= \sum_{a \in S'_j} \bar{c}_{j,a} y^a t^{\langle a, \alpha \rangle} t^{w_j(a)} \\ &= \sum_{a \in S'_j} \bar{c}_{j,a} y^a t^{\langle (a, w_j(a)), (\alpha, 1) \rangle} \\ &= \sum_{a \in S'_j} \bar{c}_{j,a} y^a t^{\langle \hat{a}, \hat{\alpha} \rangle}, \quad j = 1, \ldots, n. \end{aligned} \tag{4.7}$$

Let

$$\beta_j = \min_{a \in S'_j} \langle \hat{a}, \hat{\alpha} \rangle, \quad j = 1, \ldots, n, \tag{4.8}$$

and consider the homotopy

$$H^{\alpha}(y, t) = \left(h_1^{\alpha}(y, t), \ldots, h_n^{\alpha}(y, t)\right) = 0 \tag{4.9}$$

on $(\mathbb{C}^*)^n \times [0,1]$ where for $j = 1, \ldots, n$

$$h_j^\alpha(y,t) = t^{-\beta_j}\hat{q}_j(yt^\alpha, t) = \sum_{a\in S_j'} \bar{c}_{j,a} y^a t^{\langle \hat{a},\hat{\alpha}\rangle - \beta_j}$$

$$= \sum_{\substack{a\in S_j' \\ \langle \hat{a},\hat{\alpha}\rangle = \beta_j}} \bar{c}_{j,a} y^a + \sum_{\substack{a\in S_j' \\ \langle \hat{a},\hat{\alpha}\rangle > \beta_j}} \bar{c}_{j,a} y^a t^{\langle \hat{a},\hat{\alpha}\rangle - \beta_j}. \tag{4.10}$$

This homotopy retains most of the properties of the homotopy $\widehat{Q}(x,t) = 0$; in particular, both Properties 1 (Smoothness) and 2 (Accessibility) remain valid and

$$H^\alpha(y,1) = \widehat{Q}(y,1) = Q(y). \tag{4.11}$$

From condition (A), for each $j = 1, \ldots, n$, $\langle \hat{a}_j, \hat{\alpha}\rangle = \langle \hat{a}_j', \hat{\alpha}\rangle = \beta_j$ and $\langle \hat{a}, \hat{\alpha}\rangle > \beta_j$ for $a \in S_j' \backslash \{a_j, a_j'\}$, hence,

$$H^\alpha(y,0) = \begin{cases} h_1^\alpha(y,0) = \displaystyle\sum_{\substack{a\in S_1' \\ \langle \hat{a},\hat{\alpha}\rangle = \beta_1}} \bar{c}_{1,a} y^a = \bar{c}_{1,a_1} y^{a_1} + c_{1,a_1'} y^{a_1'} = 0, \\ \qquad\qquad \vdots \\ h_n^\alpha(y,0) = \displaystyle\sum_{\substack{a\in S_n' \\ \langle \hat{a},\hat{\alpha}\rangle = \beta_n}} \bar{c}_{n,a} y^a = \bar{c}_{n,a_n} y^{a_n} + c_{n,a_n'} y^{a_n'} = 0. \end{cases} \tag{4.12}$$

Such system is known as the *binomial system*, and its isolated solutions in $(\mathbb{C}^*)^n$ are constructively available as shown in the next section.

4.2. Solutions of binomial systems in $(\mathbb{C}^*)^n$

PROPOSITION 4.1. *The binomial system*

$$\begin{aligned} &\bar{c}_{1,a_1} y^{a_1} + \bar{c}_{1,a_1'} y^{a_1'} = 0, \\ &\qquad\vdots \\ &\bar{c}_{n,a_n} y^{a_n} + \bar{c}_{n,a_n'} y^{a_n'} = 0, \end{aligned} \tag{4.13}$$

has

$$k_\alpha := \left| \det \begin{pmatrix} a_1 - a_1' \\ \vdots \\ a_n - a_n' \end{pmatrix} \right| \tag{4.14}$$

nonsingular isolated solutions in $(\mathbb{C}^*)^n$.

PROOF. For $j = 1, \ldots, n$, let $v_j = a_j - a'_j$. To look for solutions of the system (4.13) in $(\mathbb{C}^*)^n$, we rewrite the system as

$$\begin{aligned} y^{v_1} &= b_1, \\ &\vdots \\ y^{v_n} &= b_n, \end{aligned} \tag{4.15}$$

where $b_j = \bar{c}_{j,a'_j}/\bar{c}_{j,a_j}$ for $j = 1, \ldots, n$. Let

$$V = \left[v_1 \mid v_2 \mid \cdots \mid v_n \right] \tag{4.16}$$

and $\mathbf{b} = (b_1, \ldots, b_n)$. Then, (4.15) becomes

$$y^V = \mathbf{b}. \tag{4.17}$$

When matrix V is upper triangular, i.e.

$$V = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ 0 & v_{22} & \cdots & v_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & v_{nn} \end{bmatrix},$$

then, equations in (4.17) become

$$\begin{aligned} y_1^{v_{11}} &= b_1, \\ y_1^{v_{12}} y_2^{v_{22}} &= b_2, \\ &\vdots \\ y_1^{v_{1n}} y_2^{v_{2n}} \cdots y_n^{v_{nn}} &= b_n. \end{aligned} \tag{4.18}$$

By forward substitutions, all the solutions of the system (4.18) in $(\mathbb{C}^*)^n$ can be found, and the total number of solutions is $|v_{11}| \times \cdots \times |v_{nn}| = |\det V|$.

When V is a general matrix, we may upper triangularize it by the following process. Recall that the greatest common divisor d of two nonzero integers a and b, denoted by $\gcd(a, b)$, can be written as

$$d = \gcd(a, b) = ra + lb$$

for certain nonzero integers r and l. Let

$$M = \begin{bmatrix} r & l \\ -\frac{b}{d} & \frac{a}{d} \end{bmatrix}.$$

Clearly, $\det(M) = 1$, and

$$M \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r & l \\ -\frac{b}{d} & \frac{a}{d} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} d \\ 0 \end{bmatrix}.$$

Similar to using Givens rotation to produce zeros in a matrix for its QR factorization, the matrix M may be used to upper triangularize V as follows. For $v \in \mathbb{Z}^n$, let a and b

be its ith and jth (nonzero) components with $i < j$, that is

$$v = \begin{bmatrix} \vdots \\ a \\ \vdots \\ b \\ \vdots \end{bmatrix} \begin{matrix} \\ \to i\text{th} \\ \\ \to j\text{th.} \\ \\ \end{matrix}$$

With $d = \gcd(a, b)$, let

$$U(i,j) := \begin{bmatrix} & & & & i\text{th} & & & & j\text{th} & & & \\ 1 & & & & & & & & & & & \\ & \ddots & & & & & & & & & & \\ & & 1 & & & & & & & & & \\ & & & r & & & & l & & & & i\text{th} \\ & & & & 1 & & & & & & & \\ & & & & & \ddots & & & & & & \\ & & & & & & 1 & & & & & \\ & & & -\frac{b}{d} & & & & \frac{a}{d} & & & & j\text{th} \\ & & & & & & & & 1 & & & \\ & & & & & & & & & \ddots & & \\ & & & & & & & & & & 1 & \end{bmatrix} \tag{4.19}$$

in which all blank entries are zero. Clearly, $U(i, j)$ is an integer matrix with $\det(U(i, j)) = 1$ and

$$U(i,j)v = \begin{bmatrix} \vdots \\ d \\ \vdots \\ 0 \\ \vdots \end{bmatrix} \begin{matrix} \\ i\text{th} \\ \\ j\text{th} \\ \\ \end{matrix}$$

Thus multiplication on the left by a series of matrices in the form of $U(i, j)$ in (4.19) can successively produce zeros in the lower triangular part of the matrix V, resulting in an upper triangular matrix. Let U be the product of all those $U(i, j)$'s. Then $\det U = 1$, and since UV is upper triangular, we may solve the system

$$\left(z^U\right)^V = z^{UV} = \mathbf{b} \tag{4.20}$$

in $(\mathbb{C}^*)^n$ by forward substitutions. And the total number of solutions in $(\mathbb{C}^*)^n$ is

$$\left|\det(UV)\right| = \left|\det(U)\right| \cdot \left|\det(V)\right| = \left|\det(V)\right|.$$

By letting $y = z^U$ for each solution z of (4.20) in $(\mathbb{C}^*)^n$, we obtain $|\det(V)|$ number of solutions of the system (4.13) in $(\mathbb{C}^*)^n$. □

4.3. Polyhedral homotopy procedure

By (4.11), following paths $y(t)$ of the homotopy $H^\alpha(y,t)=0$ in (4.9) that emanate from k_α, as in (4.14), isolated zeros in $(\mathbb{C}^*)^n$ of the binomial start system $H^\alpha(y,0)=0$ in (4.12), yields k_α isolated zeros of the system $Q(x)$ in (4.3) when $t=1$. Moreover, different $\hat{\alpha}=(\alpha,1)\in\mathbb{R}^{n+1}$ that satisfy condition (A) will induce different homotopies $H^\alpha(y,t)=0$ in (4.10) and following corresponding paths of those different homotopies will reach different sets of isolated zeros of $Q(x)$. Those different sets of isolated zeros of $Q(x)$ are actually disjoint from each other, and they therefore provide $\sum_\alpha k_\alpha$ isolated zeros of $Q(x)$ in total. To see they are disjoint, let paths $y^{\alpha^{(1)}}(t)$ of $H^{\alpha^{(1)}}(y,t)=0$ and $y^{\alpha^{(2)}}(t)$ of $H^{\alpha^{(2)}}(y,t)=0$ for $\alpha^{(1)}=(\alpha_1^{(1)},\ldots,\alpha_n^{(1)})$ and $\alpha^{(2)}=(\alpha_1^{(2)},\ldots,\alpha_n^{(2)})\in\mathbb{R}^n$ reach the same point at $t=1$, then their corresponding homotopy paths $x(t)=y(t)t^\alpha$ of $\widehat{Q}(x,t)=0$ are the same since zeros of the system $Q(x)=\widehat{Q}(x,1)$ are isolated and nonsingular. Thus, $x(t)=y^{\alpha^{(1)}}(t)t^{\alpha^{(1)}}=y^{\alpha^{(2)}}(t)t^{\alpha^{(2)}}$ implies

$$1=\lim_{t\to 0}\frac{y_j^{\alpha_j^{(1)}}(t)}{y_j^{\alpha_j^{(2)}}(t)}t^{\alpha_j^{(1)}-\alpha_j^{(2)}},\quad \text{for each } j=1,\ldots,n. \tag{4.21}$$

Hence, $\alpha_j^{(1)}=\alpha_j^{(2)}$ for all $j=1,\ldots,n$.

On the other hand, when $\omega=(\omega_1,\ldots,\omega)$ is a generic lifting, it induces, by Proposition 3.2, a fine mixed subdivision S'_ω of $S'=(S'_1,\ldots,S'_n)$. It's easy to see that when the collection of pairs $C^\alpha=(\{a_1,a'_1\},\ldots,\{a_n,a'_n\})$ with $\{a_j,a'_j\}\subset S'_j$, $j=1,\ldots,n$, satisfies condition (A) with $\hat{\alpha}=(\alpha,1)\in\mathbb{R}^{n+1}$, it is a cell of type $(1,\ldots,1)$ in S'_ω with $\hat{\alpha}$ being the inner normal of $(\{\hat{a}_1,\hat{a}'_1\},\ldots,\{\hat{a}_n,\hat{a}'_n\})$ in $(\widehat{S}'_1,\ldots,\widehat{S}'_n)$ and, by (3.10),

$$\mathrm{Vol}_n(C^\alpha)=\left|\det\begin{pmatrix}a_1-a'_1\\ \vdots\\ a_n-a'_n\end{pmatrix}\right|=k_\alpha. \tag{4.22}$$

By Proposition 3.1, the mixed volume $\mathcal{M}(S'_1,\ldots,S'_n)$, the root count of $Q(x)$ in $(\mathbb{C}^*)^n$, is the sum of all the volumes of C^α. That is,

$$\mathcal{M}\big(S'_1,\ldots,S'_n\big)=\sum_\alpha k_\alpha.$$

In other words, each isolated zero of $Q(x)$ lies at the end of certain homotopy path of the homotopy $H^\alpha(y,t)=0$ induced by certain $\hat{\alpha}=(\alpha,1)\in\mathbb{R}^{n+1}$ that satisfies condition (A).

A key step in the procedure of solving system $Q(x)$ by the polyhedral homotopy method described above is the search for all those vectors $\hat{\alpha}=(\alpha,1)\in\mathbb{R}^{n+1}$ as well as their associated cells $C^\alpha=(\{a_1,a'_1\},\ldots,\{a_n,a'_n\})$ that satisfy condition (A). This step is actually the main bottleneck of the method in practice. We shall address this important issue in the next section.

In conclusion, we list the polyhedral homotopy procedure.

POLYHEDRAL HOMOTOPY PROCEDURE.

Given polynomial system $P(x) = (p_1(x), \ldots, p_n(x))$ with support $S = (S_1, \ldots, S_n)$, let $S' = (S'_1, \ldots, S'_n)$ with $S'_j = S_j \cup \{0\}$ for $j = 1, \ldots, n$.

Step 0: Initialization.

Choose polynomial system $Q(x) = (q_1(x), \ldots, q_n(x))$ with support $S' = (S'_1, \ldots, S'_n)$ and generically chosen coefficients. Write

$$q_j(x) = \sum_{a \in S'_j} c_{j,a} x^a, \quad j = 1, \ldots, n.$$

Step 1: Solve $Q(x) = 0$.

Step 1.1. Choose a set of real valued functions $w_j : S'_j \to \mathbb{R}$, $j = 1, \ldots, n$, their images are generic numbers.

Step 1.2. Find all the cells $C^\alpha = (\{a_1, a'_1\}, \ldots, \{a_n, a'_n\})$ of type $(1, \ldots, 1)$ with $\{a_j, a'_j\} \subset S'_j$, $j = 1, \ldots, n$, in the fine mixed subdivision S'_ω of $S' = (S'_1, \ldots, S'_n)$ induced by $\omega = (\omega_1, \ldots, \omega_n)$ with $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$ being the inner normal of $(\{\hat{a}_1, \hat{a}'_1\}, \ldots, \{\hat{a}_n, \hat{a}'_n\})$ in $(\widehat{S}'_1, \ldots, \widehat{S}'_n)$. (The algorithm of this part will be given in the next section.)

Step 1.3. For each $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$ and its associated cell C^α obtained in Step 1.2.

Step 1.3.1. Solve the binomial system

$$c_{j,a_j} y^{a_j} + c_{j,a'_j} y^{a'_j} = 0, \quad j = 1, \ldots, n,$$

in $(C^*)^n$. Let the solution set be X^*_α.

Step 1.3.2. Follow homotopy paths $y(t)$ of the homotopy $H^\alpha(y, t) = (h^\alpha_1(y, t), \ldots, h^\alpha_n(y, t)) = 0$ with

$$h^\alpha_j(y, t) = \sum_{a \in S'_j} c_{j,a} y^a t^{\langle \hat{a}, \hat{\alpha} \rangle - \beta_j}, \quad j = 1, \ldots, n,$$

where $\beta_j = \langle \hat{a}_j, \hat{\alpha} \rangle$, starting from the solutions in X^*_α. Collect all the points of $y(1)$ as a subset of isolated zeros of $Q(x)$.

Step 2: Solve $P(x) = 0$.

Follow homotopy paths of the homotopy

$$H(x, t) = (1 - t)c\,Q(x) + t\,P(x) = 0 \quad \text{for generic } c \in \mathbb{C}^*$$

starting from the solutions of $Q(x) = 0$ obtained in Step 1 to reach all isolated solutions of $P(x) = 0$ at $t = 1$.

REMARK 4.1. As we can see in the above procedure, in order to find all isolated zeros of $P(x)$ in $\mathbb{C}^n$, there are $k = \mathcal{M}(S'_1, \ldots, S'_n)$ homotopy paths need to be followed in both Step 1.3 and Step 2, hence $2k$ in total. This work may be reduced in half by the following strategy:

For

$$p_j(x) = \sum_{a \in S'_j} \bar{c}_{j,a} x^a, \quad j = 1, \ldots, n,$$

we select the coefficients $c_{j,a}$'s of $q_j(x)$, $j = 1, \ldots, n$, at Step 0 to be $\bar{c}_{j,a} + \varepsilon_{j,a}$, where $\varepsilon_{j,a}$'s are generically chosen small numbers to ensure each $q_j(x)$ is in general position. And at Step 1.3.2, we follow homotopy paths of the homotopy $\overline{H}^\alpha(y,t) = (\bar{h}^\alpha_j(y,t), \ldots, \bar{h}^\alpha_n(y,t)) = 0$, where

$$\bar{h}^\alpha_j(y,t) = \sum_{a \in S'_j} \left[\bar{c}_{j,a} + (1-t)\varepsilon_{j,a}\right] y^a t^{\langle \hat{a}, \hat{\alpha} \rangle - \beta_j}, \quad j = 1, \ldots, n.$$

It can be shown that the starting system $\overline{H}^\alpha(y,0) = 0$ of this homotopy retain the same binomial system as before which was solved at Step 1.3.1 (with different coefficients of course). Most importantly, since $\overline{H}^\alpha(y,1) = \overline{H}^\alpha(x,1) = P(x)$, Step 2 in the above procedure is no longer necessary and we only need to follow k paths.

5. Mixed volume computation

It was mentioned in the last section, a key step in the polyhedral homotopy method for solving polynomial system $P(x) = (p_1(x), \ldots, p_n(x))$ in $\mathbb{C}^n$ with support $S = (S_1, \ldots, S_n)$ is the identification of all the vectors $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$ and their associate pairs $(\{a_1, a'_1\}, \ldots, \{a_n, a'_n\})$ that satisfy condition (A). We repeat the condition here with the assumption that p_j's in $P(x)$ all have constant terms, namely $S_j = S_j \cup \{0\}$ for $j = 1, \ldots, n$:

For a generic lifting $\omega = (\omega_1, \ldots, \omega_n)$ on $S = (S_1, \ldots, S_n)$ with $w_j : S_j \to \mathbb{R}$ for $j = 1, \ldots, n$, write

$$\widehat{S}_j = \left\{\hat{a} = \big(a, \omega_j(a)\big) \mid a \in S_j\right\}.$$

Then $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$ satisfies condition (A) if

$$\begin{aligned} &\text{There exists a collection of pairs } \{a_1, a'_1\} \subset S_1, \ldots, \{a_n, a'_n\} \subset S_n \\ &\text{where } \{a_1 - a'_1, \ldots, a_n - a'_n\} \text{ is linearly independent and} \\ &\quad \langle \hat{a}_j, \hat{\alpha} \rangle = \langle \hat{a}'_j, \hat{\alpha} \rangle, \\ &\quad \langle \hat{a}, \hat{\alpha} \rangle > \langle \hat{a}_j, \hat{\alpha} \rangle \qquad \text{for } a \in S_j \backslash \{a_j, a'_j\}. \end{aligned} \tag{A}$$

The geometric meaning of this problem, as shown in Fig. 5.1, is that with generic *lifting* ω_j on lattice points $S_j \subset \mathbb{N}^n$ for each $j = 1, \ldots, n$, we look for hyperplanes in the form $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$ where each hyperplane supports the convex hull of $\widehat{S}_j$ at exactly two points $\{\hat{a}_j, \hat{a}'_j\}$ of $\widehat{S}_j$ for each $j = 1, \ldots, n$.

The collection of pairs $\{a_1, a'_1\}, \ldots, \{a_n, a'_n\}$ in condition (A), denote it by $C^\alpha = (C_1, \ldots, C_n)$ with $C_j = \{a_j, a'_j\} \subset S_j$ for $j = 1, \ldots, n$, is, as we mentioned before, a cell of type $(1, \ldots, 1)$ in the subdivision $\mathcal{S}_w$ of $S = (S_1, \ldots, S_n)$ induced by the lifting

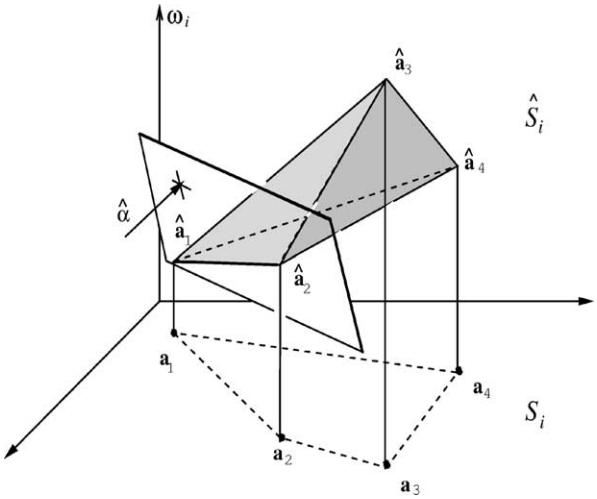


FIG. 5.1. A lifting on lattice points.

$w = (w_1, \ldots, w_n)$. We shall call such C^α a *mixed cell* of S_w without specifying its type. By (3.10), the volume of C^α is

$$\mathrm{Vol}_n(C^\alpha) = \left| \det \begin{pmatrix} a_1' - a_1 \\ \vdots \\ a_n' - a_n \end{pmatrix} \right|.$$

On the other hand, cells of type $(1, \ldots, 1)$ in S_w in the form $(\{a_1, a_1'\}, \ldots, \{a_n, a_n'\})$ where $\{a_j, a_j'\} \subset S_j$ for $j = 1, \ldots, n$ with $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$ being the inner normal of $(\{\hat{a}_1, \hat{a}_1'\}, \ldots, \{\hat{a}_n, \hat{a}_n'\})$ in $\widehat{S} = (\widehat{S}_1, \ldots, \widehat{S}_n)$ automatically satisfies condition (A). Hence, by Proposition 3.1, the mixed volume $\mathcal{M}(S_1, \ldots, S_n)$ is the sum of the volumes of all the mixed cells C^α of S_w. Therefore, when all those mixed cells are identified, the mixed volume $\mathcal{M}(S_1, \ldots, S_n)$ can be assembled with little extra computational effort.

In this section, we shall present an algorithm given in LI and LI [2001] for finding all those mixed cells and their associated vectors $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$ following the route described below.

For $1 \leqslant j \leqslant n$, $\hat{\mathbf{e}} = \{\hat{a}, \hat{a}'\} \subset \widehat{S}_j$ is called a *lower edge* of $\widehat{S}_j$ if there is a vector $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$ for which

$$\langle \hat{a}, \hat{\alpha} \rangle = \langle \hat{a}', \hat{\alpha} \rangle,$$
$$\langle \hat{a}, \hat{\alpha} \rangle \leqslant \langle \hat{b}, \hat{\alpha} \rangle, \quad \forall \hat{b} \in \widehat{S}_j \backslash \{\hat{a}, \hat{a}'\}.$$

For $1 \leqslant k \leqslant n$, $\widehat{E}_k = (\hat{\mathbf{e}}_1, \ldots, \hat{\mathbf{e}}_k)$ with $\hat{\mathbf{e}}_j = \{\hat{a}_j, \hat{a}_j'\} \subset \widehat{S}_j$ for $j = 1, \ldots, k$ is called a *level-k subface* of $\widehat{S} = (\widehat{S}_1, \ldots, \widehat{S}_n)$ if there is a vector $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$ such that for all $j = 1, \ldots, k$,

$$\langle \hat{a}_j, \hat{\alpha} \rangle = \langle \hat{a}_j', \hat{\alpha} \rangle,$$
$$\langle \hat{a}_j, \hat{\alpha} \rangle \leqslant \langle \hat{a}, \hat{\alpha} \rangle, \quad \forall \hat{a} \in \widehat{S}_j \backslash \{\hat{a}_j, \hat{a}_j'\}.$$

Obviously, a level-1 subface of $\widehat{S}$ is just a lower edge of $\widehat{S}_1$ and a level-n subface of $\widehat{S}$ induces a mixed cell in the subdivision $\mathcal{S}_w$ of $S = (S_1, \ldots, S_n)$. Thus, to find the mixed cells of S_w, we may proceed by first finding all the lower edges of $\widehat{S}_j$ for $j = 1, \ldots, n$, followed by extending the level-k subfaces of $\widehat{S}$ from $k = 1$ to $k = n$.

5.1. *A basic linear programming algorithm*

In computing mixed cells, we will repeatedly encounter the LP (linear programming) problems of the following type:

$$\begin{aligned} &\text{Minimize } \langle \mathbf{f}, \mathbf{z} \rangle \\ &\langle \mathbf{c}_j, \mathbf{z} \rangle \leqslant b_j, \quad j = 1, \ldots, m \end{aligned} \tag{5.1}$$

where $\{\mathbf{f}, \mathbf{c}_j, \mathbf{z}\} \subset \mathbb{R}^n$ and $m > n$. To solve those problems, we will employ the classical simplex algorithm instead of using the faster *interior point method* (YE [1997]) because in the course the main algorithm for finding mixed cells takes a great advantage of the rich information generated by the pivoting process in the simplex method.

When the simplex method is used to solve the LP problem in (5.1), it is customary, for historical as well as practical reasons, to convert the form in (5.1) into the following *standard* form:

$$\begin{aligned} &\text{Minimize } \langle \mathbf{f}, \mathbf{y} \rangle \\ &A\mathbf{y} = \mathbf{d}, \\ &\mathbf{y} \geqslant 0, \end{aligned}$$

where $\{\mathbf{f}, \mathbf{y}\} \subset \mathbb{R}^n$, $\mathbf{d} \in \mathbb{R}^r$ and A is a $r \times n$ matrix. In fact, many linear programming software packages apply an automatic internal conversion procedure to create such standard-form problems. However, for our needs, it is critically important to solve the problem in the form given in (5.1) directly. The algorithm for this purpose is briefly outlined below, and the details can be found in, e.g., BEST and RITTER [1985].

The feasible region of (5.1), denoted by R, defines a polyhedral set. A *nondegenerate* vertex of R is a feasible point of R with exactly n active constraints. From a feasible point of the problem, or a point in R, one may attain a nondegenerate vertex of R. Let $\mathbf{z}^0$ be a nondegenerate vertex of R and $J = \{j_1, \ldots, j_n\}$ be the set of indices of active constraints at $\mathbf{z}^0$, that is

$$\begin{aligned} &\left\langle \mathbf{c}_j, \mathbf{z}^0 \right\rangle = b_j, \quad \text{if } j \in J, \\ &\left\langle \mathbf{c}_j, \mathbf{z}^0 \right\rangle < b_j, \quad \text{if } j \notin J. \end{aligned}$$

Since $\mathbf{z}^0$ is nondegenerate, $D^{\mathrm{T}} = [\mathbf{c}_{j_1}, \ldots, \mathbf{c}_{j_n}]$ must be nonsingular. Let $D^{-1} = [\mathbf{u}_1, \ldots, \mathbf{u}_n]$. (When $\mathbf{z}^0$ is degenerate where more than n constraints are active, D^{T} is formulated in accordance with the Blend's rule.) The n edges of the feasible region R emanating from $\mathbf{z}^0$ can be represented in the form

$$\mathbf{z}^0 - \sigma \mathbf{u}_k, \quad \sigma > 0, \ k = 1, \ldots, n.$$

Along all those edges, the objective function $\langle \mathbf{f}, \mathbf{z}^0 - \sigma \mathbf{u}_k \rangle$ decreases as a function of $\sigma > 0$ when $\langle \mathbf{f}, \mathbf{u}_k \rangle > 0$. For such a direction $\mathbf{u}_k$, the largest possible $\sigma > 0$ for $\mathbf{z}^0 - \sigma \mathbf{u}_k$

to stay feasible is

$$\sigma_0 = \min\left\{ \frac{\langle \mathbf{c}_j, \mathbf{z}^0\rangle - b_j}{\langle \mathbf{c}_j, \mathbf{u}_k\rangle} \;\Big|\; j \notin J \text{ with } \langle \mathbf{c}_j, \mathbf{u}_k\rangle < 0 \right\}$$

and the point $\mathbf{z}^1 = \mathbf{z}^0 - \sigma_0 \mathbf{u}_k$ yields an adjacent vertex of R with reduced objective function value. When $\mathbf{z}^0$ is degenerate, it may occur that $\sigma_0 = 0$. In such cases, one may either choose an alternative $\mathbf{u}_k$ or, if no more $\mathbf{u}_k$ is available, reformulate D^{T} with different set of constraints.

Clearly, $\mathbf{z}^0$ is an optimal solution of (5.1) if $\langle \mathbf{f}, \mathbf{u}_k\rangle \leqslant 0$ for all $k = 1, \ldots, n$. To solve the LP problem in (5.1) directly we may move from one vertex of R to an adjacent one in a direction $\mathbf{u}_k$ where $\langle \mathbf{f}, \mathbf{u}_k\rangle > 0$, forcing the objective function to decrease, until a vertex with $\langle \mathbf{f}, \mathbf{u}_k\rangle \leqslant 0$ for all $k = 1, \ldots, n$ is reached. On the other hand, if for all k where $\langle \mathbf{f}, \mathbf{u}_k\rangle > 0$, $\langle \mathbf{c}_j, \mathbf{u}_k\rangle$ are nonnegative for all j, then the problem is unbounded and the solution does not exist.

Most frequently, the LP problems arise in our algorithm are of the following type:

$$\begin{aligned} &\text{Minimize } \langle \mathbf{f}, \mathbf{z}\rangle \\ &\langle \mathbf{a}_i, \mathbf{z}\rangle = b_i, \quad i \in I_1 = \{1, \ldots, q\}, \\ &\langle \mathbf{c}_j, \mathbf{z}\rangle \leqslant b_j, \quad j \in I_2 = \{q+1, \ldots, m\}, \end{aligned} \tag{5.2}$$

where $\{\mathbf{f}, \mathbf{a}_i, \mathbf{c}_j\} \subset \mathbb{R}^n$, $(b_1, \ldots, b_m) \in \mathbb{R}^m$ and $q < n < m$. Actually, problems of this type can be converted to the LP problem in (5.1) by eliminating the equality constraints by reducing an equal number of variables in $\mathbf{z} = (z_1, \ldots, z_n)$. For instance,

$$\begin{aligned} &\langle \mathbf{a}_1, \mathbf{z}\rangle = b_1, \\ &\qquad \vdots \\ &\langle \mathbf{a}_q, \mathbf{z}\rangle = b_q \end{aligned}$$

implies, without loss of generality,

$$\begin{aligned} z_1 \qquad &+ a'_{1,q+1} z_{q+1} + \cdots + a'_{1,n} z_n = b_1, \\ &\ddots \qquad\qquad\qquad\qquad\qquad\quad \vdots \\ &\quad z_q + a'_{q,q+1} z_{q+1} + \cdots + a'_{q,n} z_n = b_q. \end{aligned}$$

Solving $(z_1, \ldots, z_q)$ in terms of $(z_{q+1}, \ldots, z_n)$ in the above and substituting them into the inequality constraints in (5.2), the LP problem in (5.2) becomes

$$\begin{aligned} &\text{Minimize } \langle \mathbf{f}', \mathbf{y}\rangle \\ &\langle \mathbf{c}'_j, \mathbf{y}\rangle \leqslant b'_j, \quad j \in I_2 \end{aligned} \tag{5.3}$$

in the variables $\mathbf{y} = (z_{q+1}, \ldots, z_n)$, where $\{\mathbf{f}', \mathbf{c}'_j\} \subset \mathbb{R}^{n-q}$, and $\mathbf{b} = (b'_q, \ldots, b'_m)^{\mathrm{T}} \in \mathbb{R}^{m-q}$.

5.2. Finding all lower edges of a lifted point set

PROPOSITION 5.1. *For $S = (S_1, \ldots, S_n)$ with $S_j \subset \mathbb{N}^n$ and $Q_j = \mathrm{conv}(S_j)$ for $j = 1, \ldots, n$, the mixed volume $\mathcal{M}(S_1, \ldots, S_n)$ of S equals to*

$$\begin{aligned}\mathcal{M}(S_1, \ldots, S_n) = {} & \mathrm{Vol}_n(Q_1 + \cdots + Q_n) + \cdots \\ & + (-1)^{n-2} \sum_{i<j} \mathrm{Vol}_n(Q_j + Q_j) \\ & + (-1)^{n-1} \sum_{j=1}^{n} \mathrm{Vol}_n(Q_j).\end{aligned}$$

PROOF. For $\lambda = (\lambda_1, \ldots, \lambda_n)$, let $f(\lambda) = \mathrm{Vol}_n(\lambda_1 Q_1 + \cdots + \lambda_n Q_n)$. Let A be the set of all the monomials $\lambda^{\mathbf{k}}$ in $f(\lambda)$ where $\mathbf{k} = (k_1, \ldots, k_n)$ with $k_1 + \cdots + k_n = n$, and for $j = 1, \ldots, n$, let A_j be the monomials in $f(\lambda)$ in which the variable λ_j is absent. For each subset $I \subset \{1, \ldots, n\}$, let

$$A_I = \bigcap_{j \in I} A_j$$

with $A_\emptyset = A$, and for each subset $A_s \subset A$, let

$$g(A_s) := \sum_{\lambda^{\mathbf{k}} \in A_s} \text{the coefficient of } \lambda^{\mathbf{k}}.$$

It is easy to see that

$$\begin{aligned} g(A_I) = f(\lambda_I) \quad \text{where } (\lambda_I)_j = 0 \quad & \text{if } j \in I, \\ (\lambda_I)_j = 1 \quad & \text{if } j \notin I. \end{aligned}$$

Let $\bar{A}_s = A \backslash A_s$ and $|I| :=$ the number of elements in I. By the Principle of Inclusion–Exclusion in combinatorics (see, e.g., STANLEY [1997]), $g(\bar{A}_1 \cap \cdots \cap \bar{A}_n)$, the coefficient of $\lambda_1 \cdots \lambda_n$ in $f(\lambda)$, equals

$$\begin{aligned}\sum_{I \subset \{1,\ldots,n\}} (-1)^{|I|} g(A_I) = {} & f(1, \ldots, 1) - \sum_{j=1}^{n} f(1, \ldots, 1, \overset{j\text{th}}{0}, 1, \ldots, 1) \\ & + \sum_{i<j} f(1, \ldots, 1, \overset{i\text{th}}{0}, 1, \ldots, 1, \overset{j\text{th}}{0}, 1, \ldots, 1) \\ & - \cdots \\ & + (-1)^{n-2} \sum_{i<j} f(0, \ldots, 0, \overset{i\text{th}}{1}, 0, \ldots, 0, \overset{j\text{th}}{1}, 0, \ldots, 0) \\ & + (-1)^{n-1} \sum_{j=1}^{n-1} f(0, \ldots, 0, \overset{j\text{th}}{1}, 0, \ldots, 0)\end{aligned}$$

$$= \mathrm{Vol}_n(Q_1 + \cdots + Q_n) + \cdots$$
$$+ (-1)^{n-2} \sum_{i<j} \mathrm{Vol}_n(Q_j + Q_j)$$
$$+ (-1)^{n-1} \sum_{j=1}^{n} \mathrm{Vol}_n(Q_j). \qquad \square$$

As a consequence of the above, points in $S_j = \{a_{j1}, \ldots, a_{jm_j}\}$ which are not vertices of $Q_j = \mathrm{conv}(S_j)$, called *non-extreme points* of S_j, play no role in the mixed volume $\mathcal{M}(S_1, \ldots, S_n)$. So, when we compute the mixed volume $\mathcal{M}(S_1, \ldots, S_n)$ all those non-extreme points should be eliminated in the first place and we will assume in this section S_j admits only extreme points for each $j = 1, \ldots, n$.

A non-extreme point of S_j is a convex combination of other points of S_j. Namely, if a_{jk} is a non-extreme point of S_j, the following system of equations

$$\lambda_1 a_{j1} + \cdots + \lambda_{k-1} a_{jk-1} + \lambda_{k+1} a_{jk+1} + \cdots + \lambda_{m_j} a_{im_j} = a_{jk},$$
$$\lambda_1 + \cdots + \lambda_{k-1} + \lambda_{k+1} + \cdots + \lambda_{m_j} = 1,$$
$$\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_{m_j} \geqslant 0$$

must have a solution. Testing the existence of solutions of this system by actually finding one is a standard Phase I problem in linear programming, and algorithms for this problem can be found in many standard linear programming books, e.g., PAPADIMITRIOU and STEIGLITZ [1982]. In essence, the existence of solutions of the above system is equivalent to zero optimal value of the optimization problem:

Minimize λ_k
$$\lambda_1 a_{j1} + \cdots + \lambda_{m_j} a_{jm_j} = a_{jk},$$
$$\lambda_1 + \cdots + \lambda_{m_j} = 1,$$
$$\lambda_i \geqslant 0, \quad i = 1, \ldots, m_j.$$

An obvious feasible point of this LP problem is $\lambda_k = 1$ and $\lambda_i = 0$ for $i \neq k$.

For $w = (w_1, \ldots, w_n)$ with generically chosen $w_j : S_j \to \mathbb{R}$ for $j = 1, \ldots, n$, and

$$\widehat{S}_j = \{\hat{a} = (a, w_j(a)) \mid a \in S_j\},$$

denote the set of all lower edges of $\widehat{S}_j$ by $\mathcal{L}(\widehat{S}_j)$. To elaborate the algorithm for finding $\mathcal{L}(\widehat{S}_j)$, we let $\mathcal{B} = \{a_1, \ldots, a_m\} \subset \mathbb{N}^n$ represent general S_j's, and $w : \mathcal{B} \to \mathbb{R}$ be a generic lifting on $\mathcal{B}$. For $\widehat{\mathcal{B}} = \{\hat{a} = (a, w(a)) \mid a \in \mathcal{B}\}$, consider the inequality system

$$\langle \hat{a}_j, \hat{\alpha} \rangle \geqslant \alpha_0, \quad j = 1, \ldots, m \tag{5.4}$$

in the variables $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$ and $\alpha_0 \in \mathbb{R}$. Obviously, when this inequality system has a solution $(\alpha_0, \hat{\alpha})$ for which $\langle \hat{a}_i, \hat{\alpha} \rangle = \langle \hat{a}_j, \hat{\alpha} \rangle = \alpha_0$ for $1 \leqslant i, j \leqslant m$, then $\{\hat{a}_i, \hat{a}_j\}$ is a lower edge of $\widehat{\mathcal{B}}$.

With $a_j = (a_{j,1}, \ldots, a_{j,n})$ for $j = 1, \ldots, m$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$, we write system (5.4) explicitly as

$$\begin{pmatrix} 1 & -a_{1,1} & \cdots & -a_{1,n} \\ 1 & -a_{2,1} & \cdots & -a_{2,n} \\ \vdots & \vdots & & \vdots \\ 1 & -a_{m,1} & \cdots & -a_{m,n} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \leqslant \begin{pmatrix} w(a_1) \\ w(a_2) \\ \vdots \\ w(a_m) \end{pmatrix}. \tag{5.5}$$

Letting $\upsilon \leqslant n$ be the rank of the coefficient matrix on the left-hand side of (5.5), we assume without loss that the first υ rows are linearly independent. By Gaussian elimination, an upper triangular nonsingular matrix U exists by which

$$\begin{pmatrix} 1 & -a_{1,1} & \cdots & -a_{1,n} \\ 1 & -a_{2,1} & \cdots & -a_{2,n} \\ \vdots & \vdots & & \vdots \\ 1 & -a_{\upsilon,1} & \cdots & a_{\upsilon,n} \\ \vdots & \vdots & & \vdots \\ 1 & -a_{m,1} & \cdots & -a_{m,n} \end{pmatrix} \cdot U = \begin{pmatrix} c_{1,1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ c_{2,1} & c_{2,2} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ c_{\upsilon,1} & c_{\upsilon,2} & \cdots & c_{\upsilon,\upsilon} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{m,1} & c_{m,2} & \cdots & c_{m,\upsilon} & 0 & \cdots & 0 \end{pmatrix},$$

where $c_{i,i} \neq 0$ for $i = 1, \ldots, \upsilon$. Replacing $U^{-1}(\alpha_0, \alpha_1, \ldots, \alpha_n)^{\mathrm{T}}$ by $(y_1, \ldots, y_{n+1})^{\mathrm{T}}$ in the following system

$$\begin{pmatrix} 1 & -a_{1,1} & \cdots & -a_{1,n} \\ 1 & -a_{2,1} & \cdots & -a_{2,n} \\ \vdots & \vdots & & \vdots \\ 1 & -a_{\upsilon,1} & \cdots & -a_{\upsilon,n} \\ \vdots & \vdots & & \vdots \\ 1 & -a_{m,1} & \cdots & -a_{m,n} \end{pmatrix} U \cdot U^{-1} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \leqslant \begin{pmatrix} w(a_1) \\ w(a_2) \\ \vdots \\ w(a_\upsilon) \\ \vdots \\ w(a_m) \end{pmatrix} \tag{5.6}$$

yields

$$\begin{pmatrix} c_{1,1} & 0 & \cdots & 0 \\ c_{2,1} & c_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_{\upsilon,1} & c_{\upsilon,2} & \cdots & c_{\upsilon,\upsilon} \\ \vdots & \vdots & & \vdots \\ c_{m,1} & c_{m,2} & \cdots & c_{m,\upsilon} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_\upsilon \end{pmatrix} \leqslant \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_\upsilon \\ \vdots \\ b_m \end{pmatrix}, \tag{5.7}$$

where $b_j = w(a_j)$, $j = 1, \ldots, m$.

The existence of a solution $(\alpha_0, \hat{\alpha})$ of the system in (5.4) satisfying

$$\langle \hat{a}_i, \hat{\alpha} \rangle = \langle \hat{a}_j, \hat{\alpha} \rangle = \alpha_0$$

is now equivalent to the existence of a solution $(y_1, \ldots, y_\upsilon)$ of the system in (5.7) satisfying

$$\begin{aligned} &c_{i,1}y_1 + c_{i,2}y_2 + \cdots + c_{i,\upsilon}y_\upsilon = b_i, \\ &c_{j,1}y_1 + c_{j,2}y_2 + \cdots + c_{j,\upsilon}y_\upsilon = b_j \quad \text{for } 1 \leqslant i, j \leqslant m. \end{aligned}$$

The inequality system (5.7) defines a polyhedron R in $\mathbb{R}^{\upsilon}$, and for any vertex $\mathbf{y}_0$ of R, there are at least υ active constraints, or υ equalities in (5.7). Let $J = \{i_1, \ldots, i_u\}$ with $u \geqslant \upsilon$ be the set of indices of the active constraints at $\mathbf{y}_0$. Clearly, $\{\hat{a}_{i_k}, \hat{a}_{i_l}\}$ is a lower edge of $\widehat{\mathcal{B}}$ for any $i_k, i_l \in J$. On the other hand, if $\{\hat{a}_i, \hat{a}_j\}$ is a lower edge of $\widehat{\mathcal{B}}$, there is a lower facet of $\mathrm{conv}(\widehat{\mathcal{B}})$ with inner normal $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$ that contains the line segment of $\{\hat{a}_i, \hat{a}_j\}$. Let $\alpha_0 = \langle \hat{a}_i, \hat{\alpha} \rangle = \langle \hat{a}_j, \hat{\alpha} \rangle$ and $\mathbf{y}_0 = (y_1, \ldots, y_{\upsilon})$ be the first υ components of $U^{-1}(\alpha_0, \alpha_1, \ldots, \alpha_n)^{\mathrm{T}}$ in (5.6). Then $\mathbf{y}_0$ is a vertex of R.

Therefore, finding all lower edges of $\widehat{\mathcal{B}}$ is equivalent to finding all the vertices of the polyhedron R defined by the inequalities in (5.7). To find all the vertices of R, our strategy is: Find an initial vertex of R in the first place and generate all other vertices of R from this vertex thereafter.

To find an initial vertex of R, we first solve the triangular system

$$\begin{aligned} c_{1,1}y_1 &= b_1, \\ c_{2,1}y_1 + c_{2,2}y_2 &= b_2, \\ &\vdots \\ c_{\upsilon,1}y_1 + c_{\upsilon,2}y_2 + \cdots + c_{\upsilon,\upsilon}y_{\upsilon} &= b_{\upsilon} \end{aligned}$$

in (5.7) and let the solution be $\mathbf{y}_0 = (y_{01}, y_{02}, \ldots, y_{0\upsilon})$. Let

$$\begin{aligned} &d_j = b_j - (c_{j,1}y_{01} + c_{j,2}y_{02} + \cdots + c_{j,\upsilon}y_{0\upsilon}), \\ &\quad j = \upsilon + 1, \ldots, m. \end{aligned}$$

If $d_l := \min_{\upsilon+1 \leqslant j \leqslant m} d_j \geqslant 0$, then $\mathbf{y}_0$ is already a vertex of R. Otherwise we solve the following LP problem in the form given in (5.1):

$$\begin{aligned} &\text{Minimize } \varepsilon \\ &\begin{aligned} c_{1,1}y_1 &\leqslant b_1, \\ c_{2,1}y_1 + c_{2,2}y_2 &\leqslant b_2, \\ &\vdots \\ c_{\upsilon,1}y_1 + c_{\upsilon,2}y_2 + \cdots + c_{\upsilon,\upsilon}y_{\upsilon} &\leqslant b_{\upsilon}, \\ c_{\upsilon+1,1}y_1 + c_{\upsilon+1,2}y_2 + \cdots + c_{\upsilon+1,\upsilon}y_{\upsilon} - c_{\upsilon+1,\upsilon+1}\varepsilon &\leqslant b_{\upsilon+1}, \\ &\vdots \\ c_{m,1}y_1 + c_{m,2}y_2 + \cdots + c_{m,\upsilon}y_{\upsilon} - c_{m,\upsilon+1}\varepsilon &\leqslant b_m, \\ -\varepsilon &\leqslant 0, \end{aligned} \\ &\text{where } c_{j,\upsilon+1} = \begin{cases} 0, & \text{if } d_j \geqslant 0, \\ 1, & \text{if } d_j < 0, \end{cases} \quad \text{for } \upsilon + 1 \leqslant j \leqslant m, \end{aligned} \tag{5.8}$$

in the variables $(\mathbf{y}, \varepsilon) = (y_1, \ldots, y_{\upsilon}, \varepsilon)$ with feasible point $(\mathbf{y}, \varepsilon) = (\mathbf{y}_0, -d_l)$ and initial indices of constraints $J = \{1, 2, \ldots, \upsilon, l\}$. Obviously, ε in the optimal solution $(\bar{\mathbf{y}}, \varepsilon)$ of this LP problem must be zero and in such case $\bar{\mathbf{y}}$ becomes a vertex of R.

To generate all other vertices, we first introduce the following LP problem:

TWO-POINT TEST PROBLEM.

$$\begin{aligned}
&\text{Minimize } -(c_{i_0,1}+c_{j_0,1})y_1-(c_{i_0,2}+c_{j_0,2})y_2-\cdots-(c_{i_0,\upsilon}+c_{j_0,\upsilon})y_\upsilon\\
&c_{1,1}y_1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \leqslant b_1,\\
&c_{2,1}y_1+c_{2,2}y_2 \qquad\qquad\qquad\qquad\qquad\qquad \leqslant b_2,\\
&\qquad\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots\\
&c_{\upsilon,1}y_1+c_{\upsilon,2}y_2+\cdots+c_{\upsilon,\upsilon}y_\upsilon \leqslant b_\upsilon,\\
&\qquad\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots\\
&c_{m,1}y_1+c_{m,2}y_2+\cdots+c_{m,\upsilon}y_\upsilon \leqslant b_m,
\end{aligned} \tag{5.9}$$

where $1 \leqslant i_0, j_0 \leqslant m$.

If the optimal value of this problem is $-b_{i_0}-b_{j_0}$ and attained at $(y_1,\ldots,y_\upsilon)$, then

$$\begin{aligned}
&-(c_{i_0,1}+c_{j_0,1})y_1-(c_{i_0,2}+c_{j_0,2})y_2-\cdots-(c_{i_0,\upsilon}+c_{j_0,\upsilon})y_\upsilon\\
&\quad=(-c_{i_0,1}y_1-c_{i_0,2}y_2-\cdots-c_{i_0,\upsilon}y_\upsilon)+(-c_{j_0,1}y_1-c_{j_0,2}y_2-\cdots-c_{j_0,\upsilon}y_\upsilon)\\
&\quad=-b_{i_0}-b_{j_0}.
\end{aligned}$$

But $(y_1,\ldots,y_\upsilon)$ also satisfies constraints

$$\begin{aligned}
&c_{i_0,1}y_1+c_{i_0,2}y_2+\cdots+c_{i_0,\upsilon}y_\upsilon \leqslant b_{i_0},\\
&c_{j_0,1}y_1+c_{j_0,2}y_2+\cdots+c_{j_0,\upsilon}y_\upsilon \leqslant b_{j_0}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&c_{i_0,1}y_1+c_{i_0,2}y_2+\cdots+c_{i_0,\upsilon}y_\upsilon = b_{i_0},\\
&c_{j_0,1}y_1+c_{j_0,2}y_2+\cdots+c_{j_0,\upsilon}y_\upsilon = b_{j_0}.
\end{aligned}$$

Accordingly, $\{\hat{a}_{i_0},\hat{a}_{j_0}\}$ is a lower edge of $\widehat{\mathcal{B}}$.

The constraints in (5.9) is the same inequality system in (5.7) which defines polyhedron R. Therefore an initial vertex $\bar{\mathbf{y}}$ of R provides a feasible point of the LP problem in (5.9), and we may solve this problem to determine if $\{\hat{a}_{i_0},\hat{a}_{j_0}\}$ for given $1 \leqslant i_0, j_0 \leqslant m$ is a lower edge of $\widehat{\mathcal{B}}$. When the simplex method, as outlined in the last section, is used for solving this problem, we pivot from one vertex of R to another vertex in the direction where the objective function decreases. Every time a newly obtained vertex of R in the process will carry a new set of equalities in (5.7), and therefore provides a new collection of lower edges of $\widehat{\mathcal{B}}$. This important feature makes the exhaustive testings on all the possible pairs in $\widehat{\mathcal{B}}$ unnecessary.

The details of the algorithm for finding all lower edges of $\widehat{\mathcal{B}}$ is given in the following

ALGORITHM 5.1. Given $\widehat{\mathcal{B}}=\{\hat{a}_0,\hat{a}_1,\ldots,\hat{a}_m\}$, construct $\mathcal{L}(\widehat{\mathcal{B}})$.

Step 0: Initialization.

Set up inequality system (5.7). Let $\mathcal{P}=\{\{\hat{a}_i,\hat{a}_j\} \mid 1 \leqslant i, j \leqslant m\}$ be all the possible pairs of $\widehat{\mathcal{B}}$. If $\upsilon = m$, set $\mathcal{L}(\widehat{\mathcal{B}}) := \mathcal{P}$ and stop. Otherwise, solve the optimization problem (5.8), find an initial vertex $\mathbf{y}_0$ of system (5.7) with the set of indices of active constraints $J=\{i_1,\ldots,i_\upsilon\}$ and $D^{-1}=[\mathbf{u}_1,\ldots,\mathbf{u}_\upsilon]$, where $D^{\mathrm{T}}=[\mathbf{c}_{i_1},\ldots,\mathbf{c}_{i_\upsilon}]$ and

$\mathbf{c}_{i_j} = (c_{i_j,1}, \ldots, c_{i_j,\upsilon})$ for $j = 1, \ldots, \upsilon$. Set $\mathcal{L}(\widehat{\mathcal{B}}) := \{\{\hat{a}_k, \hat{a}_l\} \mid k, l \in J\}$ and $\mathcal{P} := \mathcal{P}\backslash\{\{\hat{a}_k, \hat{a}_l\} \mid k, l \in J\}$, go to Step 1.

Step 1: Setting up objective function for the Two-Point Test.

If $\mathcal{P} = \emptyset$, stop. Otherwise select $\{\hat{a}_{i_0}, \hat{a}_{j_0}\} \in \mathcal{P}$, set $\mathbf{f} := (-c_{i_0,1} - c_{j_0,1}, \ldots, -c_{i_0,\upsilon} - c_{j_0,\upsilon})$, and $\mathcal{P} := \mathcal{P}\backslash\{\{\hat{a}_{i_0}, \hat{a}_{j_0}\}\}$, go to Step 2.

Step 2: Solving the LP problem.

Step 2.1. Determine the smallest index k such that

$$\langle \mathbf{f}, \mathbf{u}_k \rangle = \max\big\{\langle \mathbf{f}, \mathbf{u}_j \rangle \mid j = 1, \ldots, \upsilon\big\}.$$

If $\langle \mathbf{f}, \mathbf{u}_k \rangle \leqslant 0$, go to Step 1. Otherwise, set $\mathbf{s} = \mathbf{u}_k$, go to Step 2.2.

Step 2.2. Compute the smallest index l and σ such that

$$\sigma = \frac{\langle \mathbf{c}_l, \mathbf{y}_0 \rangle - b_l}{\langle \mathbf{c}_l, \mathbf{s} \rangle} = \min\left\{ \frac{\langle \mathbf{c}_j, \mathbf{y}_0 \rangle - b_j}{\langle \mathbf{c}_j, \mathbf{s} \rangle} \;\middle|\; j \notin J \text{ with } \langle \mathbf{c}_j, \mathbf{s} \rangle < 0 \right\}.$$

Go to Step 2.3.

Step 2.3. Set $\mathbf{y}_0 := \mathbf{y}_0 - \sigma \mathbf{s}$ and update $J = \{i_1, \ldots, i_\upsilon\}$ and D^{-1}. Set $\mathcal{L}(\widehat{\mathcal{B}}) := \mathcal{L}(\widehat{\mathcal{B}}) \cup (\mathcal{P} \cap \{\{\hat{a}_k, \hat{a}_l\} \mid k, l \in J\})$, and $\mathcal{P} := \mathcal{P}\backslash\{\{\hat{a}_k, \hat{a}_l\} \mid k, l \in J\}$.

Go to Step 2.1.

5.3. Extending level-k subfaces

For a level-k subface $\widehat{E}_k = (\hat{\mathbf{e}}_1, \ldots, \hat{\mathbf{e}}_k)$ of $\widehat{S} = (\widehat{S}_1, \ldots, \widehat{S}_n)$ with $1 \leqslant k < n$ where $\hat{\mathbf{e}}_j = \{\hat{a}_j, \hat{a}'_j\} \in \mathcal{L}(\widehat{S}_j)$ for $j = 1, \ldots, k$, we say $\hat{\mathbf{e}}_{k+1} = \{\hat{a}_{k+1}, \hat{a}'_{k+1}\} \in \mathcal{L}(\widehat{S}_{k+1})$ *extends* $\widehat{E}_k$ if $\widehat{E}_{k+1} = (\hat{\mathbf{e}}_1, \ldots, \hat{\mathbf{e}}_{k+1})$ is a level-$(k+1)$ subface of $\widehat{S}$. Let

$$\mathcal{E}(\widehat{E}_k) = \big\{\{\hat{a}_{k+1}, \hat{a}'_{k+1}\} \in \mathcal{L}(\widehat{S}_{k+1}) \;\big|\; \{\hat{a}_{k+1}, \hat{a}'_{k+1}\} \text{ extends } \widehat{E}_k\big\}.$$

$\widehat{E}_k$ is called *extendible* if $\mathcal{E}(\widehat{E}_k) \neq \emptyset$, it is *nonextendible* otherwise. Obviously, an extendible $\widehat{E}_{n-1}$ yields mixed cells of S_w induced by elements in $\mathcal{E}(\widehat{E}_{n-1})$ (possibly several). So, to find all mixed cells, we may start from $k = 1$ and extend $\widehat{E}_k$ step by step. If $\widehat{E}_k$ is nonextendible, then no mixed cells of S_w whose liftings contain subface $(\{\hat{a}_1, \hat{a}'_1\}, \ldots, \{\hat{a}_k, \hat{a}'_k\})$. Hence, the extension attempt on $\widehat{E}_k$ will only continue when it is extendible.

For a given level-k subface $\widehat{E}_k = (\hat{\mathbf{e}}_1, \ldots, \hat{\mathbf{e}}_k)$ where $\hat{\mathbf{e}}_j = \{\hat{a}_j, \hat{a}'_j\} \subset \widehat{S}_j$ for $j = 1, \ldots, k$, consider the system

$$\begin{aligned} &\langle \hat{a}, \hat{\alpha} \rangle \geqslant \alpha_0, && \hat{a} \in \widehat{S}_{k+1}, \\ &\langle \hat{a}_j, \hat{\alpha} \rangle \leqslant \langle \hat{a}, \hat{\alpha} \rangle, && \text{for } \hat{a} \in \widehat{S}_j \text{ and } j = 1, \ldots, k, \\ &\langle \hat{a}_j, \hat{\alpha} \rangle = \langle \hat{a}'_j, \hat{\alpha} \rangle, && j = 1, \ldots, k, \end{aligned} \tag{5.10}$$

in the variables $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$ and $\alpha_0 \in \mathbb{R}$. Clearly, if this system has a solution $(\alpha_0, \hat{\alpha})$ satisfying

$$\langle \hat{a}_l, \hat{\alpha} \rangle = \langle \hat{a}_i, \hat{\alpha} \rangle = \alpha_0 \quad \text{for } \hat{a}_l, \hat{a}_i \in \widehat{S}_{k+1}$$

then $\{\hat{a}_l, \hat{a}_i\}$ extends $\widehat{E}_k$.

More explicitly, with $S_j = \{a_{j,1}, \ldots, a_{j,m_j}\}$ for $j = 1, \ldots, n$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$, the above system becomes

$$\begin{aligned}
&\langle a_{k+1,i}, \alpha\rangle - \alpha_0 \leqslant -w_{k+1}(a_{k+1,i}), && i = 1, \ldots, m_{k+1},\\
&\langle a_j - a_{j,i}, \alpha\rangle \leqslant w_j(a_{j,i}) - w_j(a_j), && i = 1, \ldots, m_j,\ a_{j,i} \in S_j \backslash \{a_j, a_j'\},\\
& && j = 1, \ldots, k,\\
&\langle a_j - a_j', \alpha\rangle = w_j(a_j') - w_j(a_j), && j = 1, \ldots, k.
\end{aligned}$$

The last k equality constraints can be used to eliminate k variables in $\alpha = (\alpha_1, \ldots, \alpha_n)$, resulting in the following inequality system:

$$\begin{aligned}
&c'_{1,j_1}\alpha_{j_1} + c'_{1,j_2}\alpha_{j_2} + \cdots + c'_{1,j_{\eta'}}\alpha_{j_{\eta'}} \leqslant b_1,\\
&c'_{2,j_1}\alpha_{j_1} + c'_{2,j_2}\alpha_{j_2} + \cdots + c'_{2,j_{\eta'}}\alpha_{j_{\eta'}} \leqslant b_2,\\
&\qquad\vdots \qquad\qquad\qquad\qquad\qquad \vdots\\
&c'_{\mu,j_1}\alpha_{j_1} + c'_{\mu,j_2}\alpha_{j_2} + \cdots + c'_{\mu,j_{\eta'}}\alpha_{j_{\eta'}} \leqslant b_\mu,
\end{aligned} \tag{5.11}$$

where $\mu = \sum_{j=1}^{k+1} m_j - 2k$ and $\eta' = n - k + 1$. As before, by a coordinate transformation $(y_1, \ldots, y_{\eta'}) = (\alpha_{j_1}, \ldots, \alpha_{j_{\eta'}})U$ where $U \in \mathbb{R}^{\eta' \times \eta'}$ is a nonsingular matrix, the system can further be reduced to

$$\begin{aligned}
&c_{1,1}y_1 && \leqslant b_1,\\
&c_{2,1}y_1 + c_{2,2}y_2 && \leqslant b_2,\\
&\qquad\vdots \qquad\qquad \ddots && \qquad\vdots\\
&c_{\eta,1}y_1 + c_{\eta,2}y_2 + \cdots + c_{\eta,\eta}y_\eta && \leqslant b_\eta,\\
&\qquad\vdots && \qquad\vdots\\
&c_{\mu,1}y_1 + c_{\mu,2}y_2 + \cdots + c_{\mu,\eta}y_\eta && \leqslant b_\mu.
\end{aligned} \tag{5.12}$$

Consequently, system (5.10) has a solution $(\alpha_0, \hat{\alpha})$ satisfying

$$\langle \hat{a}_{k+1.i}, \hat{\alpha}\rangle = \langle \hat{a}_{k+1,l}, \hat{\alpha}\rangle = \alpha_0 \quad \text{for } 1 \leqslant i, l \leqslant m_{k+1}$$

if and only if system (5.12) has a solution $(y_1, \ldots, y_\eta)$ satisfying

$$\begin{aligned}
&c_{i,1}y_1 + c_{i,2}y_2 + \cdots + c_{i,\eta}y_\eta = b_i,\\
&c_{l,1}y_1 + c_{l,2}y_2 + \cdots + c_{l,\eta}y_\eta = b_l.
\end{aligned}$$

Inequalities in (5.12) defines a polyhedron $\overline{R}$ in $\mathbb{R}^\eta$, and for any vertex of $\overline{R}$, there are at least η active constraints. If $\mathbf{z}_0$ is a vertex of $\overline{R}$ and $J = \{i_1, \ldots, i_t\}$ with $t \geqslant \eta$ is the indices of active constraints at $\mathbf{z}_0$, then $\{\hat{a}_{k+1,i_p}, \hat{a}_{k+1,i_q}\}$ extends $\widehat{E}_k$ for any pair $\{i_p, i_q\} \subset J \cap \{1, \ldots, m_{k+1}\}$.

Similar to the discussion given before, for $\{\hat{a}_{k+1,i}, a_{k+1,l}\} \in \mathcal{E}(\widehat{E}_k)$, there exists a corresponding vertex $\mathbf{z}_0$ of $\overline{R}$ whose indices of active constraints includes $\{i, l\} \subset I \equiv \{1, \ldots, m_{k+1}\}$. Hence, to construct $\mathcal{E}(\widehat{E}_k) \subset \mathcal{L}(\widehat{S}_{k+1})$, we may look for all those vertices of $\overline{R}$ whose indices of active constraints contain at least a pair of $\{i, l\}$ in I. To this end, we may apply the Two-Point Test introduced in the last section again and confine the

indices of the "two points" to be tested to I. However, most of the $\hat{a}_{k+1,i}$'s that appear in the pairs in $\mathcal{L}(\widehat{S}_{k+1})$ do not exist in any of the pairs in $\mathcal{E}(\widehat{E}_k)$. This never occurs when we compute the set of lower edges $\mathcal{L}(\widehat{\mathcal{B}})$ of $\widehat{\mathcal{B}}$ in the last section since all the points in $\mathcal{B}$ are assumed to be extreme points, and consequently every point of $\widehat{\mathcal{B}}$ appears in certain pairs of $\mathcal{L}(\widehat{\mathcal{B}})$. This important observation stimulates the following One-Point Test to be used in addition to the Two-Point Test in the algorithm.

ONE-POINT TEST PROBLEM.

$$\begin{aligned}
&\text{Minimize } -c_{i_0,1}y_1 - c_{i_0,2}y_2 - \cdots - c_{i_0,\eta}y_\eta \\
&c_{1,1}y_1 \qquad\qquad\qquad\qquad\qquad\qquad \leqslant b_1, \\
&c_{2,1}y_1 + c_{2,2}y_2 \qquad\qquad\qquad\qquad \leqslant b_2, \\
&\quad\vdots \qquad\qquad\quad \ddots \qquad\qquad\quad \vdots \\
&c_{\eta,1}y_1 + c_{\eta,2}y_2 + \cdots + c_{\eta,\eta}y_\eta \leqslant b_\eta, \\
&\quad\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\
&c_{\mu,1}y_1 + c_{\mu,2}y_2 + \cdots + c_{\mu,\eta}y_\eta \leqslant b_\mu
\end{aligned} \tag{5.13}$$

where $1 \leqslant i_0 < m_{k+1}$.

If the optimal value of this LP problem is greater than $-b_{i_0}$, then $\{\hat{a}_{k+1,i_0}, \hat{a}_{k+1,i}\}$ does not extend $\widehat{E}_k$ for all $i \in \{1, \ldots, m_{k+1}\}\backslash\{i_0\}$. For if there exists $1 \leqslant j_0 \leqslant m_{k+1}$ for which $\{\hat{a}_{k+1,i_0}, \hat{a}_{k+1,j_0}\}$ extends $\widehat{E}_k$, then system (5.12) has a solution $(y_1, \ldots, y_\eta)$ satisfying

$$\begin{aligned}
&c_{i_0,1}y_1 + c_{i_0,2}y_2 + \cdots + c_{i_0,\eta}y_\eta = b_{i_0}, \\
&c_{j_0,1}y_1 + c_{j_0,2}y_2 + \cdots + c_{j_0,\eta}y_\eta = b_{j_0},
\end{aligned}$$

and the objective function value at $(y_1, \ldots, y_\eta)$ is then

$$-c_{i_0,1}y_1 - c_{i_0,2}y_2 - \cdots - c_{i_0,\eta}y_\eta = -b_{i_0}.$$

Thus, for the construction of $\mathcal{E}(\widehat{E}_k)$, points which have appeared in the pairs in $\mathcal{L}(\widehat{S}_{k+1})$ will be tested systematically by using the One-Point Test to check for their possible appearance in the pairs in $\mathcal{E}(\widehat{E}_k)$. When the optimal value is not as desired for a particular point $\hat{a}_{k+1,i_0}$, then all the pairs associated with $\hat{a}_{k+1,i_0}$ in $\mathcal{L}(\widehat{S}_{k+1})$ would be deleted from the list of further testing. The constraints in the LP problem in (5.13) is the same inequality system in (5.12) which defines the polyhedron $\overline{R}$ in $\mathbb{R}^\eta$. To find an initial vertex of $\overline{R}$ to start solving the problem, one may employ the same strategy by augmenting a new variable $\varepsilon \geqslant 0$ as in calculating $\mathcal{L}(\widehat{\mathcal{B}})$ of $\widehat{\mathcal{B}}$ in the last section.

In the process of achieving the optimal value, a newly obtained vertex of $\overline{R}$ provides a collection of new pairs of $\mathcal{E}(\widehat{E}_k)$ as long as its active constraints contain a pair of $\{i, l\}$ in $I = \{1, \ldots, m_{k+1}\}$. We will delete those points $\hat{a}_{k+1,i}$ in the pairs in $\mathcal{L}(\widehat{S}_{k+1})$ from the list of testing once their index i have appeared in any of the indices of the active constraints of the vertices of $\overline{R}$ being obtained. After the One-Point Test has exhausted all testings on possible candidates the Two-Point Test will then be used for the remaining pairs in $\mathcal{L}(\widehat{S}_{k+1})$. Empirically, when the One-Point Test is finished, most of the pairs in $\mathcal{E}(\widehat{E}_k)$ have been found and the Two-Point Test only plays a minor role in the process.

REMARK 5.1. Recall that the set of constraints in (5.13) is a modified version of the system in (5.10): the equality constraints in (5.13)

$$\langle \hat{a}_j, \hat{\alpha} \rangle = \langle \hat{a}'_j, \hat{\alpha} \rangle, \quad j = 1, \ldots, k,$$

are used to eliminate equal number of variables in $\alpha = (\alpha_1, \ldots, \alpha_n)$. When we solve the LP problem in (5.13), the inequality constraints

$$\langle \hat{a}_j, \hat{\alpha} \rangle \leqslant \langle \hat{a}, \hat{\alpha} \rangle, \quad \hat{a} \in \widehat{S}_j, \ j = 1, \ldots, k,$$

in (5.10) where either $\{\hat{a}, \hat{a}_j\}$ or $\{\hat{a}, \hat{a}'_j\}$ does not belong to $\mathcal{L}(\widehat{S}_j)$ will obviously never become active in the process. Those constraints should be removed before solving the LP problem in (5.13). In practice, the successive omission of such extraneous constraints in all those LP problems greatly reduces the amount of computation cumulatively.

Combining One-Point Test and Two-Point Test, we list the following algorithm for constructing $\mathcal{E}(\widehat{E}_k)$.

ALGORITHM 5.2. Given $\widehat{E}_k$, construct $\mathcal{E}(\widehat{E}_k)$.

Step 0: Initialization.

Set up inequality system (5.12) after removing extraneous constraints. Start from a vertex $\mathbf{z}_0$ with the set of indices of active constraints $J = \{i_1, \ldots, i_\eta\}$ and $D^{-1} = [\mathbf{u}_1, \ldots, \mathbf{u}_\eta]$ where $D^{\mathrm{T}} = [\mathbf{c}_{i_1}, \ldots, \mathbf{c}_{i_\eta}]$ with $\mathbf{c}_{i_j} = (c_{i_j,1}, \ldots, c_{i_j,\eta})$ and set $\widehat{F}_{k+1} := \mathcal{L}(\widehat{S}_{k+1})$.

Step 1: One-Point Test Problem.

Step 1.0. Set $i_0 := 0$, go to Step 1.1.

Step 1.1. Set up objective function.

Find

$$\tau = \min\big\{ j \mid j > i_0, \{\hat{a}_{k+1,j}, \hat{a}_{k+1,j'}\} \subset \widehat{F}_{k+1} \text{ for some } j' \big\}.$$

If no such τ exists, go to Step 2. Otherwise set $i_0 := \tau$ and $\mathbf{f} = (-c_{i_0,1}, \ldots, -c_{i_0,\eta})$, go to Step 1.2.

Step 1.2. Determine the smallest index k such that

$$\langle \mathbf{f}, \mathbf{u}_k \rangle = \max\big\{ \langle \mathbf{f}, \mathbf{u}_j \rangle \mid j = 1, \ldots, \eta \big\}.$$

If $\langle \mathbf{f}, \mathbf{u}_k \rangle \leqslant 0$, go to Step 1.5. Otherwise, set $\mathbf{s} = \mathbf{u}_k$ and go to Step 1.3.

Step 1.3. Compute the smallest index l and σ such that

$$\sigma = \frac{\langle \mathbf{c}_l, \mathbf{z}_0 \rangle - b_l}{\langle \mathbf{c}_l, \mathbf{s} \rangle} = \min\left\{ \frac{\langle \mathbf{c}_j, \mathbf{z}_0 \rangle - b_j}{\langle \mathbf{c}_j, \mathbf{s} \rangle} \;\middle|\; j \notin J, \ \langle \mathbf{c}_j, \mathbf{s} \rangle < 0 \right\}.$$

Go to Step 1.4.

Step 1.4. Set $\mathbf{z}_0 := \mathbf{z}_0 - \sigma \mathbf{s}$ and update $J = \{i_1, \ldots, i_\eta\}$ and D^{-1}. If $l < m_{k+1}$, check if any lower edge $\{\hat{a}_{k+1,l}, \hat{a}_{k+1,j}\}$ in $\widehat{F}_{k+1}$ extends $\widehat{F}_{k+1}$. Collect these lower edges, if they exist, and delete them from $\widehat{F}_{k+1}$.

Go to Step 1.2.

Step 1.5. If the current value of objective function is different from $-b_{i_0}$, delete all lower edges that contain $\hat{a}_{k+1,i_0}$ from $\widehat{F}_{k+1}$.
Go to Step 1.1.

Step 2: Two-Point Test Problems.

Step 2.1. Set up objective function.
If $\widehat{F}_{k+1} = \emptyset$, stop. Otherwise select a lower edge $\{\hat{a}_{k+1,i_0}, \hat{a}_{k+1,j_0}\} \in \widehat{F}_{k+1}$. Set $\mathbf{f} := (-c_{i_0,1} - c_{j_0,1}, \ldots, -c_{i_0,\eta} - c_{j_0,\eta})$, and $\widehat{F}_{k+1} := \widehat{F}_{k+1} \backslash \{\hat{a}_{k+1,i_0}, \hat{a}_{k+1,j_0}\}$, go to Step 2.2.

Step 2.2. Determine the smallest index k such that

$$\langle \mathbf{f}, \mathbf{u}_k \rangle = \max\big\{ \langle \mathbf{f}, \mathbf{u}_j \rangle \mid j = 1, \ldots, \eta \big\}.$$

If $\langle \mathbf{f}, \mathbf{u}_k \rangle \leqslant 0$, go to Step 2.1. Otherwise, set $\mathbf{s} = \mathbf{u}_k$, go to Step 2.3.

Step 2.3. Compute the smallest index l and σ such that

$$\sigma = \frac{\langle \mathbf{c}_l, \mathbf{z}_0 \rangle - b_l}{\langle \mathbf{c}_l, \mathbf{s} \rangle} = \min\left\{ \frac{\langle \mathbf{c}_j, \mathbf{z}_0 \rangle - b_j}{\langle \mathbf{c}_j, \mathbf{s} \rangle} \;\middle|\; j \notin J,\ \langle \mathbf{c}_j, \mathbf{s} \rangle < 0 \right\}.$$

Go to Step 2.4.

Step 2.4. Set $\mathbf{z}_0 := \mathbf{z}_0 - \sigma \mathbf{s}$ and update $J = \{i_1, \ldots, i_\eta\}$ as well as D^{-1}. If $l < m_{k+1}$, check if any lower edge $\{\hat{a}_{k+1,l}, \hat{a}_{k+1,j}\}$ in $\widehat{F}_{k+1}$ extends $\widehat{F}_{k+1}$. Collect those lower edges, if they exist, and delete them from $\widehat{F}_{k+1}$.
Go to Step 2.2.

REMARK 5.2. Setting up inequality system (5.12) can be very time consuming. To be more efficient, one may save all inequality systems at previous levels to help the establishment of the inequality system in the current level.

5.4. Finding all mixed cells

We now insert the algorithms for finding lower edges of $\widehat{S}_j$ and extending subfaces of $\widehat{S}$ in the following procedure for finding all the mixed cells in S_ω induced by a generic lifting $\omega = (\omega_1, \ldots, \omega_n)$ on $S = (S_1, \ldots, S_n)$.

PROCEDURE FOR FINDING ALL MIXED CELLS.

Find all mixed cells in S_ω induced by a generic lifting $\omega = (\omega_1, \ldots, \omega_n)$ on $S = (S_1, \ldots, S_n)$.

Step 0: Initialization.
Find $\mathcal{L}(\widehat{S}_j)$, for all $j = 1, \ldots, n$ by Algorithm 5.1.
Set $\widehat{\mathcal{F}}_1 := \mathcal{L}(\widehat{S}_1)$, $k := 1$.

Step 1: Backtracking.
If $k = 0$ Stop.
If $\widehat{\mathcal{F}}_k = \emptyset$, set $k := k - 1$ and go to Step 1.
Otherwise go to Step 2.

Step 2: Selecting next level-k subface to extend.
Select $\hat{\mathbf{e}}_k \in \widehat{\mathcal{F}}_k$, and set $\widehat{\mathcal{F}}_k := \widehat{\mathcal{F}}_k \backslash \{\hat{\mathbf{e}}_k\}$.
Let $\widehat{E}_k = (\hat{\mathbf{e}}_1, \ldots, \hat{\mathbf{e}}_k)$ and go to Step 3.

Step 3: Extending the level-k subface.
Find $\mathcal{E}(\widehat{E}_k)$ by Algorithm 5.2.
If $\mathcal{E}(\widehat{E}_k) = \emptyset$, go to Step 1, otherwise set $\widehat{\mathcal{F}}_{k+1} = \mathcal{E}(\widehat{E}_k)$, $k := k+1$ then go to Step 4.

Step 4: Collecting mixed cells.
If $k = n$, all $C = (\hat{\mathbf{e}}_1, \ldots, \hat{\mathbf{e}}_{n-1}, \hat{\mathbf{e}})$ with $\hat{\mathbf{e}} \in \widehat{\mathcal{F}}_n$ induce mixed cells of S_ω, pick up all these mixed cells, then set $k := k-1$, go to Step 1.
Otherwise go to Step 2.

As we suggested earlier, when all the mixed cells are found, the mixed volume $\mathcal{M}(S_1, \ldots, S_n)$ can be computed as the sum of the volumes of all those mixed cells. The algorithm described in this section for calculating the mixed volume by locating all mixed cells in a fine mixed subdivision of the support induced by a generic lifting on the support has been successfully implemented and tested on the supports of a large variety of polynomial systems (LI and LI [2001]). Currently, this algorithm represents the state of the art in computing mixed volumes this way. It leads other existing algorithms (EMIRIS and CANNY [1995], VERSCHELDE [1999], GAO and LI [2000]) in speed as well as storage requirements by a considerable margin. See also TAKEDA, KOJIMA and FUJISAWA [2002].

6. Mixed volume of semi-mixed support

6.1. Semi-mixed polynomial systems

A polynomial system $P(x) = (p_1(x), \ldots, p_n(x))$ with support $S = (S_1, \ldots, S_n)$ is called *semi-mixed* of type $(k_1, \ldots, k_r)$ when the supports S_j's are not all distinct, but they are equal within r blocks of sizes $k_1, \ldots, k_r$. To be more precise, there are r finite subsets $S^{(1)}, \ldots, S^{(r)}$ of $\mathbb{N}^n$ such that

$$S^{(i)} = S_{i1} = \cdots = S_{ik_i},$$

where

$$S_{il} \in \{S_1, \ldots, S_n\} \quad \text{for } 1 \leqslant i \leqslant r,\ 1 \leqslant l \leqslant k_i,$$

and $k_1 + \cdots + k_r = n$. The system $P(x)$ is called *unmixed* if $r = 1$ and is *fully mixed* when $r = n$.

For $1 \leqslant i \leqslant r$, let $P^{(i)}(x)$ be the subset of polynomials in $P(x) = (p_1(x), \ldots, p_n(x))$ having support $S^{(i)}$. Thus each polynomial of $P^{(i)}(x)$ can be written as

$$p_{il}(x) = \sum_{a \in S^{(i)}} c_{ila} x^a \quad \text{for } 1 \leqslant i \leqslant r,\ 1 \leqslant l \leqslant k_i. \tag{6.1}$$

We abbreviate $S = (S^{(1)}, \ldots, S^{(r)})$ and $P(x) = (P^{(1)}(x), \ldots, P^{(r)}(x))$.

We may, of course, solve a semi-mixed polynomial system $P(x) = (p_1(x), \ldots, p_n(x))$ in $\mathbb{C}^n$ by following the standard polyhedral homotopy procedure described in Section 4 without paying special attention to the semi-mixed structure of its supports.

However, when this special structure is taken into account, a revised polyhedral homotopy procedure can be developed with a great reduction in the amount of computation, especially when $P(x)$ is unmixed, such as the 9-point problem in mechanism design (WAMPLER, MORGAN and SOMMESE [1992]).

For $P(x) = (p_1(x), \ldots, p_n(x))$ with support $S = (S_1, \ldots, S_n)$ in general position, we assume as before all the p_j's have constant terms, namely, $S_j = S_j' = S_j \cup \{0\}$ for $j = 1, \ldots, n$. Recall that at the beginning of the polyhedral homotopy procedure, we first assign a generic lifting $\omega = (\omega_1, \ldots, \omega_n)$ on $S = (S_1, \ldots, S_n)$. Now for semi-mixed system $P(x) = (P^{(1)}(x), \ldots, P^{(r)}(x))$ of type $(k_1, \ldots, k_r)$ with support $S = (S^{(1)}, \ldots, S^{(r)})$ and generic coefficients $c_{ila} \in \mathbb{C}^*$ as given in (6.1), we choose generic lifting $\omega = (\omega_1, \ldots, \omega_r)$ on $S = (S^{(1)}, \ldots, S^{(r)})$ where $\omega_i : S^{(i)} \to \mathbb{R}$ for $i = 1, \ldots, r$ and consider the homotopy $Q(x, t) = (Q^{(1)}(x, t), \ldots, Q^{(r)}(x, t)) = 0$ where equations in $Q^{(i)}(x, t) = 0$ for $1 \leqslant i \leqslant r$ are

$$q_{il}(x, t) = \sum_{a \in S^{(i)}} c_{ila} x^a t^{\omega_i(a)} = 0, \quad 1 \leqslant l \leqslant k_i. \tag{6.2}$$

Clearly, $Q(x, 1) = P(x)$. With $\hat{a} = (a, \omega_i(a))$ for $a \in S^{(i)}$, let $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$ satisfy the following condition:

(A′)
There exists $C = (C_1, \ldots, C_r)$ where $C_i = \{a_{i0}, \ldots, a_{ik_i}\} \subset S^{(i)}$ and $\operatorname{conv}(C_i)$ is a simplex of dimension k_i for each $i = 1, \ldots, r$ satisfying $\dim(\operatorname{conv}(C_1) + \cdots + \operatorname{conv}(C_r)) = n$, and for $1 \leqslant i \leqslant r$

$$\langle \hat{a}_{il}, \hat{\alpha} \rangle = \langle \hat{a}_{il'}, \hat{\alpha} \rangle \quad \text{for } 0 \leqslant l, l' \leqslant k_i,$$
$$\langle \hat{a}, \hat{\alpha} \rangle > \langle \hat{a}_{il}, \hat{\alpha} \rangle \quad \text{for } 0 \leqslant l \leqslant k_i,\ a \in S^{(i)} \backslash C_i.$$

For such $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$, the coordinate transformation $y = t^{-\alpha} x$ where $y_j = t^{-\alpha_j} x_j$ for $j = 1, \ldots, n$ transforms equations in (4.3) to

$$q_{il}(yt^{\alpha}, t) = \sum_{a \in S^{(i)}} c_{ila} y^a t^{\langle \hat{a}, \hat{\alpha} \rangle}, \quad 1 \leqslant i \leqslant r,\ 1 \leqslant l \leqslant k_i.$$

Let

$$\beta_i = \min_{a \in S^{(i)}} \langle \hat{a}, \hat{\alpha} \rangle, \quad i = 1, \ldots, r,$$

and consider the homotopy

$$H^{\alpha}(y, t) = \left(H_1^{\alpha}(y, t), \ldots, H_r^{\alpha}(y, t)\right) = 0 \tag{6.3}$$

on $(C^*)^n \times [0, 1]$ where for $1 \leqslant i \leqslant r$ equations in $H_i^{\alpha}(y, t) = 0$ are

$$\begin{aligned} h_{il}^{\alpha}(y, t) &= t^{-\beta_i} q_{il}(yt^{\alpha}, t) \\ &= \sum_{a \in S^{(i)}} c_{ila} y^a t^{\langle \hat{a}, \hat{\alpha} \rangle - \beta_i} \\ &= \sum_{\substack{a \in S^{(i)} \\ \langle \hat{a}, \hat{\alpha} \rangle = \beta_i}} c_{ila} y^a + \sum_{\substack{a \in S^{(i)} \\ \langle \hat{a}, \hat{\alpha} \rangle > \beta_i}} c_{ila} y^a t^{\langle \hat{a}, \hat{\alpha} \rangle - \beta_i} = 0, \quad 1 \leqslant l \leqslant k_i. \end{aligned}$$

When $t = 0$, equations in $H_i^\alpha(y, 0) = 0$ become, by condition (A′),

$$h_{il}^\alpha(y, 0) = \sum_{a \in C_i = \{a_{i0}, \ldots, a_{ik_i}\}} c_{ila} y^a = 0, \quad 1 \leqslant l \leqslant k_i. \tag{6.4}$$

For each $1 \leqslant i \leqslant r$, the above system consists of k_i equations, each one has the same $k_i + 1$ monomials $\{y^{a_{i0}}, \ldots, y^{a_{ik_i}}\}$. By applying Gaussian elimination to its $k_i \times (k_i + 1)$-coefficient matrix (c_{ila}), we can replace $H_i^\alpha(y, 0) = 0$ by an equivalent binomial system

$$\begin{aligned} c'_{i11} y^{a_{i1}} + c'_{i10} y^{a_{i0}} &= 0, \\ &\vdots \\ c'_{ik_i1} y^{a_{ik_i}} + c'_{ik_i0} y^{a_{i0}} &= 0. \end{aligned}$$

If we repeat this process for each $H_i^\alpha(y, 0) = 0$, $i = 1, \ldots, r$, and collect all the resulting binomial equations, a system of $k_1 + \cdots + k_n = n$ binomial equations in n variables is attained. This binomial system is equivalent to the start system $H^\alpha(y, 0) = 0$ of the homotopy $H^\alpha(y, t) = 0$ in (6.3), which, as shown in Proposition 4.1, admits $|\det(V^{(\alpha)})|$ nonsingular isolated zeros in $(\mathbb{C}^*)^n$ where

$$V^{(\alpha)} = \begin{pmatrix} a_{11} - a_{10} \\ \vdots \\ a_{1k_1} - a_{10} \\ \vdots \\ a_{r1} - a_{r0} \\ \vdots \\ a_{rk_r} - a_{r0} \end{pmatrix}.$$

Following homotopy paths of $H^\alpha(y, t) = 0$ emanating from those isolated zeros, we will reach $|\det(V^{(\alpha)})|$ isolated zeros of $P(x)$.

It was shown in HUBER and STURMFELS [1995], the root count of $P(x)$ in $(\mathbb{C}^*)^n$, or its mixed volume, is equal to

$$\mathcal{M}(\overbrace{S^{(1)}, \ldots, S^{(1)}}^{k_1}, \ldots, \overbrace{S^{(r)}, \ldots, S^{(r)}}^{k_r}) = \sum_\alpha \left|\det(V^{(\alpha)})\right|,$$

where the summation is taken over those α where $(\alpha, 1) \in \mathbb{R}^{n+1}$ satisfy condition (A′). Therefore, to find all isolated zeros of $P(x)$ we may repeat this process for all such $(\alpha, 1) \in \mathbb{R}^{n+1}$ along with their associate cells $C^\alpha = (C_1^\alpha, \ldots, C_r^\alpha)$ where $C_i^\alpha = \{a_{i0}, \ldots, a_{ik_i}\} \subset S^{(i)}$ for $i = 1, \ldots, r$.

To identify all those α's and their associate cells $C^\alpha = (C_1^\alpha, \ldots, C_r^\alpha)$, one may follow the same route for finding all $(\alpha, 1) \in \mathbb{R}^{n+1}$ that satisfy condition (A) in the last section with certain modifications. First of all, the definition of *fine mixed subdivision* of $S = (S^{(1)}, \ldots, S^{(r)})$ for $S^{(i)} \subset \mathbb{N}^n$, $i = 1, \ldots, r$, can be repeated word for word in Definition 3.1, but replacing $S = (S_1, \ldots, S_n)$ by $S = (S^{(1)}, \ldots, S^{(r)})$ and cell $C^{(i)} = (C_1^{(i)}, \ldots, C_n^{(i)})$ by $C^{(i)} = (C_1^{(i)}, \ldots, C_r^{(i)})$. Similarly, generic lifting $\omega = (\omega_1, \ldots, \omega_r)$

on $S = (S^{(1)}, \ldots, S^{(r)})$ will induce a fine mixed subdivision S_ω of $S = (S^{(1)}, \ldots, S^{(r)})$. With $\hat{a} = (a, \omega_i(a))$ for $a \in S^{(i)}$ and $\widehat{C}_i = \{\hat{a} \mid a \in C_i\}$ for $C_i \subset S^{(i)}$, it is easy to see that for $\alpha \in \mathbb{R}^n$ along with its associated cell $C^\alpha = (C_1^\alpha, \ldots, C_r^\alpha)$ satisfies condition (A′) if and only if $C^\alpha = (C_1^\alpha, \ldots, C_r^\alpha)$ is a cell of type $(k_1, \ldots, k_r)$ in the fine mixed subdivision S_ω of $S = (S^{(1)}, \ldots, S^{(r)})$ induced by $\omega = (\omega_1, \ldots, \omega_r)$ where $\hat{\alpha} = (\alpha, 1)$ is the inner normal of $\widehat{C}^\alpha = (\widehat{C}_1^\alpha, \ldots, \widehat{C}_r^\alpha)$ in $\widehat{S} = (\widehat{S}^{(1)}, \ldots, \widehat{S}^{(r)})$. The procedure we described in the last section to identify mixed cells, cells of type $(1, \ldots, 1)$, in S_ω is no longer effective for identifying cells of type $(k_1, \ldots, k_r)$ in S_ω, because a straightforward generalization of the Two-Point test in the algorithm to k_i-Point test would enormously increase the number of candidates that need to be tested. In the next few subsections, we shall present a revised procedure given in GAO and LI [2003] for finding cells of type $(k_1, \ldots, k_r)$ in which the Two-Point test is replaced by successive One-Point tests. It is therefore applicable in general to test k_i points consecutively.

We again assume that $S^{(i)}$ admits only extreme points for each $i = 1, \ldots, r$. Meanwhile, when LP problems arise, for the sake of simplicity, we will not deal with the details of the possible degeneracy of the constraints as in the last section. We therefore assume the matrix that represents the set of constraints of any LP problem is always of full rank.

6.2. The relation table

For generic $\omega_i : S^{(i)} \to \mathbb{R}$ and $\widehat{S}^{(i)} = \{\hat{\mathbf{a}} = (\mathbf{a}, \omega_i(\mathbf{a})) \mid \mathbf{a} \in S^{(i)}\}$ for each $i = 1, \ldots, r$, an important primary step of our algorithm for finding cells of type $(k_1, \ldots, k_r)$ in S_ω is to complete the *relation table* consisting of pairwise relation subtables T(i, j) between $\widehat{S}^{(i)}$ and $\widehat{S}^{(j)}$ for all $1 \leqslant i \leqslant j \leqslant r$ as shown in Table 6.1. For $\widehat{S}^{(i)} = \{\hat{\mathbf{a}}_1^{(i)}, \ldots, \hat{\mathbf{a}}_{s_i}^{(i)}\}$, $i = 1, \ldots, r$, Table T(i, j) displays the relationships between elements of $\widehat{S}^{(i)}$ and $\widehat{S}^{(j)}$ in the following sense:

> Given elements $\hat{\mathbf{a}}_l^{(i)} \in \widehat{S}^{(i)}$ and $\hat{\mathbf{a}}_m^{(j)} \in \widehat{S}^{(j)}$ where $l \neq m$ when $i = j$, does there exist an $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$ such that
>
> $$\begin{aligned} &\langle \hat{\mathbf{a}}_l^{(i)}, \hat{\alpha} \rangle \leqslant \langle \hat{\mathbf{a}}_k^{(i)}, \hat{\alpha} \rangle, \quad \forall \hat{\mathbf{a}}_k^{(i)} \in \widehat{S}^{(i)}, \\ &\langle \hat{\mathbf{a}}_m^{(j)}, \hat{\alpha} \rangle \leqslant \langle \hat{\mathbf{a}}_k^{(j)}, \hat{\alpha} \rangle, \quad \forall \hat{\mathbf{a}}_k^{(j)} \in \widehat{S}^{(j)}? \end{aligned} \tag{6.5}$$

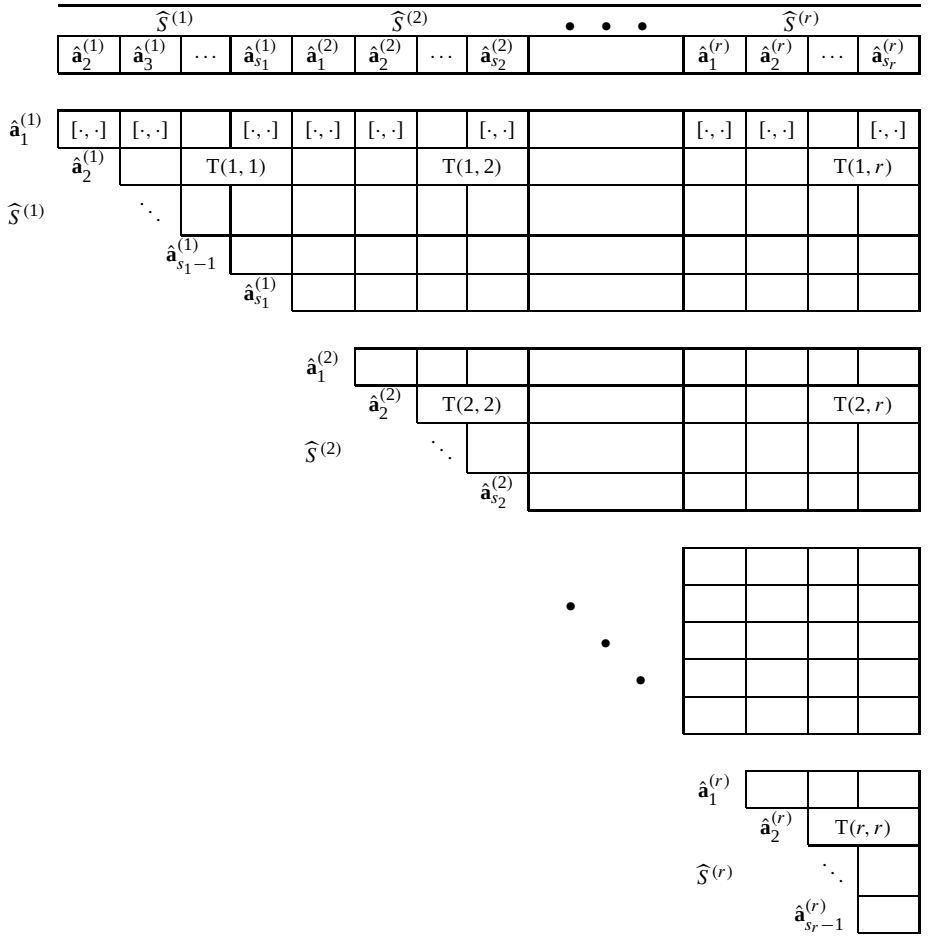
TABLE T(i, i)

$\widehat{S}^{(i)}$	$\hat{\mathbf{a}}_2^{(i)}$	$\hat{\mathbf{a}}_3^{(i)}$	$\cdots$	$\hat{\mathbf{a}}_{s_i-1}^{(i)}$	$\hat{\mathbf{a}}_{s_i}^{(i)}$
$\hat{\mathbf{a}}_1^{(i)}$	$[\hat{\mathbf{a}}_1^{(i)}, \hat{\mathbf{a}}_2^{(i)}]$	$[\hat{\mathbf{a}}_1^{(i)}, \hat{\mathbf{a}}_3^{(i)}]$	$\cdots$	$[\hat{\mathbf{a}}_1^{(i)}, \hat{\mathbf{a}}_{s_i-1}^{(i)}]$	$[\hat{\mathbf{a}}_1^{(i)}, \hat{\mathbf{a}}_{s_i}^{(i)}]$
$\hat{\mathbf{a}}_2^{(i)}$		$[\hat{\mathbf{a}}_2^{(i)}, \hat{\mathbf{a}}_3^{(i)}]$	$\cdots$	$[\hat{\mathbf{a}}_2^{(i)}, \hat{\mathbf{a}}_{s_i-1}^{(i)}]$	$[\hat{\mathbf{a}}_2^{(i)}, \hat{\mathbf{a}}_{s_i}^{(i)}]$
$\widehat{S}^{(i)}$			$\ddots$	$\vdots$	$\vdots$
				$\hat{\mathbf{a}}_{s_i-1}^{(i)}$	$[\hat{\mathbf{a}}_{s_i-1}^{(i)}, \hat{\mathbf{a}}_{s_i}^{(i)}]$
					$\hat{\mathbf{a}}_{s_i}^{(i)}$

TABLE T(i, j)

	$\widehat{S}^{(j)}$				
	$\hat{\mathbf{a}}_1^{(j)}$	$\hat{\mathbf{a}}_2^{(j)}$	$\hat{\mathbf{a}}_3^{(j)}$	$\cdots$	$\hat{\mathbf{a}}_{s_j}^{(j)}$
$\hat{\mathbf{a}}_1^{(i)}$	$[\hat{\mathbf{a}}_1^{(i)}, \hat{\mathbf{a}}_1^{(j)}]$	$[\hat{\mathbf{a}}_1^{(i)}, \hat{\mathbf{a}}_2^{(j)}]$	$[\hat{\mathbf{a}}_1^{(i)}, \hat{\mathbf{a}}_3^{(j)}]$	$\cdots$	$[\hat{\mathbf{a}}_1^{(i)}, \hat{\mathbf{a}}_{s_j}^{(j)}]$
$\widehat{S}^{(i)}$ $\hat{\mathbf{a}}_2^{(i)}$	$[\hat{\mathbf{a}}_2^{(i)}, \hat{\mathbf{a}}_1^{(j)}]$	$[\hat{\mathbf{a}}_2^{(i)}, \hat{\mathbf{a}}_2^{(j)}]$	$[\hat{\mathbf{a}}_2^{(i)}, \hat{\mathbf{a}}_3^{(j)}]$	$\cdots$	$[\hat{\mathbf{a}}_2^{(i)}, \hat{\mathbf{a}}_{s_j}^{(j)}]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\hat{\mathbf{a}}_{s_i}^{(i)}$	$[\hat{\mathbf{a}}_{s_i}^{(i)}, \hat{\mathbf{a}}_1^{(j)}]$	$[\hat{\mathbf{a}}_{s_i}^{(i)}, \hat{\mathbf{a}}_2^{(j)}]$	$[\hat{\mathbf{a}}_{s_i}^{(i)}, \hat{\mathbf{a}}_3^{(j)}]$	$\cdots$	$[\hat{\mathbf{a}}_{s_i}^{(i)}, \hat{\mathbf{a}}_{s_j}^{(j)}]$

TABLE 6.1
The Relation Table.



Denote the entry on Table T(i, j) located at the intersection of the row containing $\hat{\mathbf{a}}_l^{(i)}$ and the column containing $\hat{\mathbf{a}}_m^{(j)}$ by $[\hat{\mathbf{a}}_l^{(i)}, \hat{\mathbf{a}}_m^{(j)}]$. We set $[\hat{\mathbf{a}}_l^{(i)}, \hat{\mathbf{a}}_m^{(j)}] = 1$ when the answer

of problem (6.5) for $\hat{\mathbf{a}}_l^{(i)}$ and $\hat{\mathbf{a}}_m^{(j)}$ is positive and $[\hat{\mathbf{a}}_l^{(i)}, \hat{\mathbf{a}}_m^{(j)}] = 0$ otherwise. When $i = j$, $[\hat{\mathbf{a}}_l^{(i)}, \hat{\mathbf{a}}_m^{(i)}] = [\hat{\mathbf{a}}_m^{(i)}, \hat{\mathbf{a}}_l^{(i)}]$ for $l \neq m$, therefore we always assume $l < m$ in such cases.

To fill out the relation table, Table 6.1, we first *fix* $\hat{\mathbf{a}}_1^{(1)}$ on the first row

$$\hat{\mathbf{a}}_1^{(1)}\colon \overbrace{[\hat{\mathbf{a}}_1^{(1)}, \hat{\mathbf{a}}_2^{(1)}], \ldots, [\hat{\mathbf{a}}_1^{(1)}, \hat{\mathbf{a}}_{s_1}^{(1)}]}^{\mathrm{T}(1,1)}, \ldots, \overbrace{[\hat{\mathbf{a}}_1^{(1)}, \hat{\mathbf{a}}_1^{(r)}], \ldots, [\hat{\mathbf{a}}_1^{(1)}, \hat{\mathbf{a}}_{s_r}^{(r)}]}^{\mathrm{T}(1,r)}. \tag{6.6}$$

To determine $[\hat{\mathbf{a}}_1^{(1)}, \hat{\mathbf{a}}_2^{(1)}]$, we consider the LP problem,

$$\begin{aligned} &\text{Minimize } \langle \hat{\mathbf{a}}_2^{(1)}, \hat{\alpha} \rangle - \alpha_0 \\ &\alpha_0 = \langle \hat{\mathbf{a}}_1^{(1)}, \hat{\alpha} \rangle, \\ &\alpha_0 \leqslant \langle \hat{\mathbf{a}}_k^{(1)}, \hat{\alpha} \rangle, \quad \forall k \in \{2, \ldots, s_1\}, \end{aligned} \tag{6.7}$$

in the variables $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$ and $\alpha_0 \in \mathbb{R}$. More explicitly, we may write this problem in the form (5.1) in Section 5.1:

$$\begin{aligned} &\text{Minimize } \langle \mathbf{a}_2^{(1)} - \mathbf{a}_1^{(1)}, \alpha \rangle + \omega_1(\mathbf{a}_2^{(1)}) - \omega_1(\mathbf{a}_1^{(1)}) \\ &\langle \mathbf{a}_1^{(1)} - \mathbf{a}_k^{(1)}, \alpha \rangle \leqslant \omega_1(\mathbf{a}_k^{(1)}) - \omega_1(\mathbf{a}_1^{(1)}), \quad \forall k \in \{2, \ldots, s_1\}. \end{aligned} \tag{6.8}$$

Since $\mathbf{a}_1^{(1)}$ is a vertex point of $\mathcal{Q}_1 = \mathrm{conv}(S^{(1)})$, $\hat{\mathbf{a}}_1^{(1)}$ must be in the lower hull of $\widehat{\mathcal{Q}}_1 = \mathrm{conv}(\widehat{S}^{(1)})$, and any hyperplane in the form $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$ that supports $\hat{\mathbf{a}}_1^{(1)}$ in $\widehat{\mathcal{Q}}_1$ decides a feasible point of the constraints in (6.7). Such feasible point can be obtained by solving a standard Phase I problem for the constraints in (6.8):

$$\begin{aligned} &\text{Minimize } \varepsilon \\ &\langle \mathbf{a}_1^{(1)} - \mathbf{a}_k^{(1)}, \alpha \rangle - \varepsilon \leqslant \omega_1(\mathbf{a}_k^{(1)}) - \omega_1(\mathbf{a}_1^{(1)}), \quad \forall k \in \{2, \ldots, s_1\}, \\ &-\varepsilon \leqslant 0 \end{aligned}$$

in the variables $\alpha \in \mathbb{R}^n$ and $\varepsilon \geqslant 0$ with feasible point $\alpha = 0$ along with large enough $\varepsilon > 0$.

If the optimal value of the LP problem (6.7) is zero, then at the optimal solution $(\hat{\alpha}, \alpha_0)$ we have

$$\langle \hat{\mathbf{a}}_1^{(1)}, \hat{\alpha} \rangle = \langle \hat{\mathbf{a}}_2^{(1)}, \hat{\alpha} \rangle \leqslant \langle \hat{\mathbf{a}}_k^{(1)}, \hat{\alpha} \rangle, \quad \forall k \in \{3, \ldots, s_1\}. \tag{6.9}$$

This makes $[\hat{\mathbf{a}}_1^{(1)}, \hat{\mathbf{a}}_2^{(1)}] = 1$. Otherwise, $[\hat{\mathbf{a}}_1^{(1)}, \hat{\mathbf{a}}_2^{(1)}]$ must be zero, for if there exists $\hat{\alpha}' = (\alpha, 1) \in \mathbb{R}^{n+1}$ for which the inequalities in (6.9) hold, then this $\hat{\alpha}'$ together with $\alpha_0' = \langle \hat{\mathbf{a}}_1^{(1)}, \hat{\alpha}' \rangle$ yields a feasible point of (6.7) at which the objective function value is zero.

This process essentially uses the *One-Point Test* introduced in the last section to test if $\{\hat{\mathbf{a}}_1^{(1)}, \hat{\mathbf{a}}_2^{(1)}\}$ is a lower edge of $\widehat{S}^{(1)}$. When the simplex method is used, the pivoting process in the algorithm generates rich information on other entries of the relation table. Since the images of $\omega_1 : S^{(1)} \to \mathbb{R}$ is generically chosen, we assume without loss that

there are exactly $n+1$ active constraints at any stage of the pivoting process, say,

$$\begin{aligned}
&\alpha_0 = \langle \hat{\mathbf{a}}_1^{(1)}, \hat{\alpha} \rangle, \\
&\alpha_0 = \langle \hat{\mathbf{a}}_{l_1}^{(1)}, \hat{\alpha} \rangle, \\
&\qquad \vdots \\
&\alpha_0 = \langle \hat{\mathbf{a}}_{l_n}^{(1)}, \hat{\alpha} \rangle, \\
&\alpha_0 < \langle \hat{\mathbf{a}}_k^{(1)}, \hat{\alpha} \rangle, \quad \forall k \in \{2, 3, \ldots, s_1\} \backslash \{l_1, \ldots, l_n\},
\end{aligned}$$

then $[\hat{\mathbf{a}}_{j_1}^{(1)}, \hat{\mathbf{a}}_{j_2}^{(1)}] = 1$ for all $j_1, j_2 \in \{1, l_1, \ldots, l_n\}$ with $j_1 < j_2$. This important feature considerably reduces the number of One-Point tests needed for completely determining the entries of the relation table.

To determine the rest of the unknown entries in the first row of the table in (6.6) from left to right: for $[\hat{\mathbf{a}}_1^{(1)}, \hat{\mathbf{a}}_j^{(1)}]$ for $j > 2$, we apply the One-Point test on $\hat{\mathbf{a}}_j^{(1)}$, or solve the LP problem,

$$\begin{aligned}
&\text{Minimize } \langle \hat{\mathbf{a}}_j^{(1)}, \hat{\alpha} \rangle - \alpha_0 \\
&\alpha_0 = \langle \hat{\mathbf{a}}_1^{(1)}, \hat{\alpha} \rangle, \\
&\alpha_0 \leqslant \langle \hat{\mathbf{a}}_l^{(1)}, \hat{\alpha} \rangle, \quad \forall l \in \{2, \ldots, s_1\},
\end{aligned} \tag{6.10}$$

and for $[\hat{\mathbf{a}}_1^{(1)}, \hat{\mathbf{a}}_j^{(i)}]$ for $i > 1$, $j \in \{1, \ldots, s_i\}$, solve the LP problem

$$\begin{aligned}
&\text{Minimize } \langle \hat{\mathbf{a}}_j^{(i)}, \hat{\alpha} \rangle - \alpha_0 \\
&\langle \hat{\mathbf{a}}_1^{(1)}, \hat{\alpha} \rangle \leqslant \langle \hat{\mathbf{a}}_l^{(1)}, \hat{\alpha} \rangle, \quad \forall l \in \{2, \ldots, s_1\} \\
&\alpha_0 \leqslant \langle \hat{\mathbf{a}}_m^{(i)}, \hat{\alpha} \rangle, \qquad \forall m \in \{1, 2, \ldots, s_i\}.
\end{aligned} \tag{6.11}$$

If the corresponding optimal values are zero, then $[\hat{\mathbf{a}}_1^{(1)}, \hat{\mathbf{a}}_j^{(1)}] = 1$, or $[\hat{\mathbf{a}}_1^{(1)}, \hat{\mathbf{a}}_j^{(i)}] = 1$. They are zero otherwise.

Feasible points of the above LP problems are always available, there is no need to solve the sometimes costly Phase I problem here. Because when we determine $[\hat{\mathbf{a}}_1^{(1)}, \hat{\mathbf{a}}_2^{(1)}]$, there exists $\hat{\alpha} = (\alpha, 1)$ for which

$$\langle \hat{\mathbf{a}}_1^{(1)}, \hat{\alpha} \rangle \leqslant \langle \hat{\mathbf{a}}_l^{(1)}, \hat{\alpha} \rangle, \quad \forall l \in \{2, \ldots, s_1\}.$$

This $\hat{\alpha}$ together with $\alpha_0 = \langle \hat{\mathbf{a}}_1^{(1)}, \hat{\alpha} \rangle$ for (6.10) or

$$\alpha_0 = \min\{ \langle \hat{\mathbf{a}}_m^{(1)}, \hat{\alpha} \rangle \mid m = 1, \ldots, s_i \}$$

for (6.11) provides feasible points for the constraints of the respective LP problems.

An important remark here is, a substantial number of constraints in both (6.10) and (6.11) can be removed. For instance, if we have known $[\hat{\mathbf{a}}_1^{(1)}, \hat{\mathbf{a}}_\mu^{(i)}] = 0$ for certain $\mu \in \{1, \ldots, s_i\}$, $i \geqslant 1$, before solving the LP problems in (6.10) or (6.11), then its corresponding constraint

$$\alpha_0 \leqslant \langle \hat{\mathbf{a}}_\mu^{(i)}, \hat{\alpha} \rangle$$

should be removed, because this constraint will never become active (otherwise, $[\hat{\mathbf{a}}_1^{(1)}, \hat{\mathbf{a}}_\mu^{(i)}] = 1$) during the process of solving the LP problems. In fact, numerous extraneous constraints of this sort appear in all the LP problems below. The successive omission of those extraneous constraints yields a considerable reduction in the amount of computation and plays a crucially important role in the efficiency of the algorithm. We will not elaborate the details of the omission here, they can be found in GAO and LI [2003].

Similarly, when we determine the entries of the row

$$\hat{\mathbf{a}}_\mu^{(\nu)}\colon \overbrace{\left[\hat{\mathbf{a}}_\mu^{(\nu)}, \hat{\mathbf{a}}_{\mu+1}^{(\nu)}\right], \ldots, \left[\hat{\mathbf{a}}_\mu^{(\nu)}, \hat{\mathbf{a}}_{s_\nu}^{(\nu)}\right]}^{\mathrm{T}(\nu,\nu)}, \ldots, \overbrace{\left[\hat{\mathbf{a}}_\mu^{(\nu)}, \hat{\mathbf{a}}_1^{(r)}\right], \ldots, \left[\hat{\mathbf{a}}_\mu^{(\nu)}, \hat{\mathbf{a}}_{s_r}^{(r)}\right]}^{\mathrm{T}(\nu,r)} \tag{6.12}$$

on the relation table assuming all the entries in the previous rows have all been determined, for the unknown entries $[\hat{\mathbf{a}}_\mu^{(\nu)}, \hat{\mathbf{a}}_j^{(\nu)}]$ for $j > \mu$, we solve the LP problem,

$$\begin{aligned} &\text{Minimize } \left\langle \hat{\mathbf{a}}_j^{(\nu)}, \hat{\alpha} \right\rangle - \alpha_0 \\ &\alpha_0 = \left\langle \hat{\mathbf{a}}_\mu^{(\nu)}, \hat{\alpha} \right\rangle, \\ &\alpha_0 \leqslant \left\langle \hat{\mathbf{a}}_l^{(\nu)}, \hat{\alpha} \right\rangle, \quad \forall l \in \{1, \ldots, s_\nu\} \backslash \{\mu\}, \end{aligned} \tag{6.13}$$

and for $[\hat{\mathbf{a}}_\mu^{(\nu)}, \hat{\mathbf{a}}_j^{(i)}]$ for $j \in \{1, \ldots, s_i\}$ and $\nu < i \leqslant r$, solve the LP problem

$$\begin{aligned} &\text{Minimize } \left\langle \hat{\mathbf{a}}_j^{(i)}, \hat{\alpha} \right\rangle - \alpha_0 \\ &\left\langle \hat{\mathbf{a}}_\mu^{(\nu)}, \hat{\alpha} \right\rangle \leqslant \left\langle \hat{\mathbf{a}}_l^{(\nu)}, \hat{\alpha} \right\rangle, \quad \forall l \in \{1, \ldots, s_1\} \backslash \{\mu\}, \\ &\alpha_0 \leqslant \left\langle \hat{\mathbf{a}}_m^{(i)}, \hat{\alpha} \right\rangle, \qquad \forall m \in \{1, \ldots, s_i\}. \end{aligned} \tag{6.14}$$

When the LP problem in (6.14) is solved by the simplex method, information on other unknown entries of the table provided by the pivoting process becomes particularly fruitful. We assume without loss that there are exactly $n+1$ active constraints at any stage of the pivoting process, say

$$\begin{aligned} \left\langle \hat{\mathbf{a}}_\mu^{(\nu)}, \hat{\alpha} \right\rangle &= \left\langle \hat{\mathbf{a}}_{l_1}^{(\nu)}, \hat{\alpha} \right\rangle, \\ &\vdots \\ \left\langle \hat{\mathbf{a}}_\mu^{(\nu)}, \hat{\alpha} \right\rangle &= \left\langle \hat{\mathbf{a}}_{l_s}^{(\nu)}, \hat{\alpha} \right\rangle, \quad \text{and} \\ \alpha_0 &= \left\langle \hat{\mathbf{a}}_{m_1}^{(i)}, \hat{\alpha} \right\rangle, \\ &\vdots \\ \alpha_0 &= \left\langle \hat{\mathbf{a}}_{m_t}^{(i)}, \hat{\alpha} \right\rangle, \end{aligned}$$

where $s + t = n + 1$. Then for $l', l'' \in \{l_1, \ldots, l_s\}$ with $l' < l''$ and $m', m'' \in \{m_1, \ldots, m_t\}$ with $m' < m''$, we have

$$\left[\hat{\mathbf{a}}_{l'}^{(\nu)}, \hat{\mathbf{a}}_{l''}^{(\nu)}\right] = 1, \qquad \left[\hat{\mathbf{a}}_{m'}^{(i)}, \hat{\mathbf{a}}_{m''}^{(i)}\right] = 1.$$

And, for $l_0 \in \{l_1, \ldots, l_s\}$ and $m_0 \in \{m_1, \ldots, m_t\}$, $[\hat{\mathbf{a}}_{l_0}^{(\nu)}, \hat{\mathbf{a}}_{m_0}^{(i)}] = 1$.

6.3. Level-ξ subfaces and their extensions

For $1 \leqslant \xi \leqslant r$ and $\widehat{F}_i \subset \widehat{S}^{(i)}$ with $\dim(\widehat{F}_i) = d_i$ for $i = 1, \ldots, \xi$, $(\widehat{F}_1, \ldots, \widehat{F}_\xi)$ is called a *level-ξ subface* of $\widehat{S} = (\widehat{S}^{(1)}, \ldots, \widehat{S}^{(r)})$ of type $(d_1, \ldots, d_\xi)$ if there exists $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$ such that for each $i = 1, \ldots, \xi$,

$$\begin{aligned} &\langle \hat{\mathbf{a}}^{(i)}, \hat{\alpha} \rangle = \langle \hat{\mathbf{a}}^{(i)'}, \hat{\alpha} \rangle \quad \forall \hat{\mathbf{a}}^{(i)}, \hat{\mathbf{a}}^{(i)'} \in \widehat{F}_i, \\ &\langle \hat{\mathbf{a}}^{(i)}, \hat{\alpha} \rangle \leqslant \langle \hat{\mathbf{a}}^{(i)''}, \hat{\alpha} \rangle \quad \forall \hat{\mathbf{a}}^{(i)} \in \widehat{F}_i \text{ and } \hat{\mathbf{a}}^{(i)''} \in \widehat{S}^{(i)} \backslash \widehat{F}_i. \end{aligned}$$

Equivalently, $\widehat{F}_i$ is a lower face of $\widehat{S}^{(i)}$ of dimension d_i for each $i = 1, \ldots, \xi$. A level-ξ subface $(\widehat{F}_1, \ldots, \widehat{F}_\xi)$ is said to be *extendible* if there is a lower face $\widehat{F}_{\xi+1}$ of $\widehat{S}^{(\xi+1)}$ which makes $(\widehat{F}_1, \ldots, \widehat{F}_\xi, \widehat{F}_{\xi+1})$ a level-$(\xi+1)$ subface. It is *nonextendible* otherwise.

A level-r subface $(\widehat{F}_1, \ldots, \widehat{F}_r)$ of $\widehat{S} = (\widehat{S}^{(1)}, \ldots, \widehat{S}^{(r)})$ of type $(k_1, \ldots, k_r)$ is a lower facet of $\widehat{S}$ of type $(k_1, \ldots, k_r)$ when

$$\begin{aligned} \dim(\widehat{F}_1 + \cdots + \widehat{F}_r) &= \dim(\widehat{F}_1) + \cdots + \dim(\widehat{F}_r) \\ &= k_1 + \cdots + k_r = n. \end{aligned}$$

In such case, $(F_1, \ldots, F_r)$ becomes a cell of type $(k_1, \ldots, k_r)$ in S_ω. To find all such lower facets of $\widehat{S}$ of type $(k_1, \ldots, k_r)$ for the purpose of finding all cells of type $(k_1, \ldots, k_r)$ in S_ω, we first find all level-1 subfaces of $\widehat{S} = (\widehat{S}^{(1)}, \ldots, \widehat{S}^{(r)})$ of type (k_1), followed by extending each such subface step by step from $\xi = 1$ to $\xi = r$ to reach a level-r subface of $\widehat{S}$ of type $(k_1, \ldots, k_r)$.

6.3.1. Level-1 subfaces of $\widehat{S} = (\widehat{S}^{(1)}, \ldots, \widehat{S}^{(r)})$

Clearly, level-1 subfaces of $\widehat{S} = (\widehat{S}^{(1)}, \ldots, \widehat{S}^{(r)})$ of type (k_1) are faces of dimension k_1 in the lower hull of $\widehat{S}^{(1)} = \{\hat{\mathbf{a}}_1^{(1)}, \ldots, \hat{\mathbf{a}}_{s_1}^{(1)}\}$, they are faces of dimension k_1 of $\widehat{S}^{(1)}$ having inner normal of type $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$. When $k_1 = 1$, such subfaces are the pairs of points $\{\mathbf{a}_{l_0}^{(1)}, \mathbf{a}_{l_1}^{(1)}\}$ on the Relation Table T(1, 1) with $[\mathbf{a}_{l_0}^{(1)}, \mathbf{a}_{l_1}^{(1)}] = 1$, $1 \leqslant l_0 < l_1 \leqslant s_1$. So only the case $k_1 > 1$ will be discussed here, and we will find all those faces by extending each lower face of $\widehat{S}^{(1)}$ of dimension one, a lower *edge* of $\widehat{S}^{(1)}$, step by step. More precisely, for lower edge $\{\hat{\mathbf{a}}_{l_0}^{(1)}, \hat{\mathbf{a}}_{l_1}^{(1)}\}$ of $\widehat{S}^{(1)}$ with $l_0 < l_1$, we look for all possible points $\hat{\mathbf{a}}_l^{(1)}$ in $\widehat{S}^{(1)}$ with $l > l_1$ for which $\{\hat{\mathbf{a}}_{l_0}^{(1)}, \hat{\mathbf{a}}_{l_1}^{(1)}, \hat{\mathbf{a}}_l^{(1)}\}$ is a lower face of $\widehat{S}^{(1)}$ of dimension two. And inductively, for known face $\{\hat{\mathbf{a}}_{l_0}^{(1)}, \hat{\mathbf{a}}_{l_1}^{(1)}, \ldots, \hat{\mathbf{a}}_{l_j}^{(1)}\}$ of $\widehat{S}^{(1)}$ of dimension j with $j < k_1$ and $l_0 < l_1 < \cdots < l_j$, we look for all possible points $\hat{\mathbf{a}}_l^{(1)}$ in $\widehat{S}^{(1)}$ with $l > l_j$ for which $\{\hat{\mathbf{a}}_{l_0}^{(1)}, \hat{\mathbf{a}}_{l_1}^{(1)}, \ldots, \hat{\mathbf{a}}_{l_j}^{(1)}, \hat{\mathbf{a}}_l^{(1)}\}$ is a lower face of $\widehat{S}^{(1)}$ of dimension $j+1$. Lower face $\{\hat{\mathbf{a}}_{l_0}^{(1)}, \hat{\mathbf{a}}_{l_1}^{(1)}, \ldots, \hat{\mathbf{a}}_{l_j}^{(1)}\}$ is called *extendible* if such point exists. This task of extension can be carried out systematically by employing the One-Point test successively.

We will extend lower edges of $\widehat{S}^{(1)}$ one by one in the order from left to right and top to bottom of their corresponding entries on the Relation Table T(1, 1).

For $[\hat{\mathbf{a}}_{l_0}^{(1)}, \hat{\mathbf{a}}_{l_1}^{(1)}] = 1$ where $1 \leqslant l_0 < l_1 < s_1$, we first identify on Table T(1, 1) the set

$$\mathcal{C}^{(1)} = \left\{ 1 \leqslant l \leqslant s_1 \mid \hat{\mathbf{a}}_l^{(1)} \text{ has positive relations with both } \hat{\mathbf{a}}_{l_0}^{(1)} \text{ and } \hat{\mathbf{a}}_{l_1}^{(1)} \right\},$$

TABLE T(1, 1)

		$\widehat{S}^{(1)}$			
	$\hat{\mathbf{a}}_2^{(1)}$	$\hat{\mathbf{a}}_3^{(1)}$	$\cdots$	$\hat{\mathbf{a}}_{s_1-1}^{(1)}$	$\hat{\mathbf{a}}_{s_1}^{(1)}$
$\hat{\mathbf{a}}_1^{(1)}$	$[\hat{\mathbf{a}}_1^{(1)}, \hat{\mathbf{a}}_2^{(1)}]$	$[\hat{\mathbf{a}}_1^{(1)}, \hat{\mathbf{a}}_3^{(1)}]$	$\cdots$	$[\hat{\mathbf{a}}_1^{(1)}, \hat{\mathbf{a}}_{s_1-1}^{(1)}]$	$[\hat{\mathbf{a}}_1^{(1)}, \hat{\mathbf{a}}_{s_1}^{(1)}]$
$\hat{\mathbf{a}}_2^{(1)}$		$[\hat{\mathbf{a}}_2^{(1)}, \hat{\mathbf{a}}_3^{(1)}]$	$\cdots$	$[\hat{\mathbf{a}}_2^{(1)}, \hat{\mathbf{a}}_{s_1-1}^{(1)}]$	$[\hat{\mathbf{a}}_2^{(1)}, \hat{\mathbf{a}}_{s_1}^{(1)}]$
$\widehat{S}^{(1)}$		$\hat{\mathbf{a}}_3^{(1)}$	$\cdots$	$[\hat{\mathbf{a}}_3^{(1)}, \hat{\mathbf{a}}_{s_1-1}^{(1)}]$	$[\hat{\mathbf{a}}_3^{(1)}, \hat{\mathbf{a}}_{s_1}^{(1)}]$
			$\ddots$	$\vdots$	$\vdots$
				$\hat{\mathbf{a}}_{s_1-1}^{(1)}$	$[\hat{\mathbf{a}}_{s_1-1}^{(1)}, \hat{\mathbf{a}}_{s_1}^{(1)}]$

and let $\mathcal{T}^{(1)}$ be the elements in $\mathcal{C}^{(1)}$ which are bigger than l_1, i.e.

$$\mathcal{T}^{(1)} = \left\{ l \in \mathcal{C}^{(1)},\ l > l_1 \mid \left[\hat{\mathbf{a}}_{l_0}^{(1)}, \hat{\mathbf{a}}_l^{(1)}\right] = \left[\hat{\mathbf{a}}_{l_1}^{(1)}, \hat{\mathbf{a}}_l^{(1)}\right] = 1 \right\}.$$

Clearly, the set

$$\mathcal{P}^{(1)} = \left\{ \hat{\mathbf{a}}_l^{(1)} \mid l \in \mathcal{T}^{(1)} \right\}$$

contains all the possible points which may subsequently extend $\{\hat{\mathbf{a}}_{l_0}^{(1)}, \hat{\mathbf{a}}_{l_1}^{(1)}\}$ to a k_1-dimensional lower face $\{\hat{\mathbf{a}}_{l_0}^{(1)}, \ldots, \hat{\mathbf{a}}_{l_{k_1}}^{(1)}\}$ of $\widehat{S}^{(1)}$ with $l_0 < l_1 < \cdots < l_{k_1}$. Hence all subsequential extension attempts on $\{\hat{\mathbf{a}}_{l_0}^{(1)}, \hat{\mathbf{a}}_{l_1}^{(1)}\}$ will be restricted to this set. To extend $\{\hat{\mathbf{a}}_{l_0}^{(1)}, \hat{\mathbf{a}}_{l_1}^{(1)}\}$ we apply the One-Point test on points $\hat{\mathbf{a}}_\tau^{(1)}$ in $\mathcal{P}^{(1)}$ by solving the LP problem

$$\begin{aligned} &\text{Minimize } \left\langle \hat{\mathbf{a}}_\tau^{(1)}, \hat{\alpha} \right\rangle - \alpha_0 \\ &\alpha_0 = \left\langle \hat{\mathbf{a}}_{l_0}^{(1)}, \hat{\alpha} \right\rangle = \left\langle \hat{\mathbf{a}}_{l_1}^{(1)}, \hat{\alpha} \right\rangle, \\ &\alpha_0 \leqslant \left\langle \hat{\mathbf{a}}_k^{(1)}, \hat{\alpha} \right\rangle, \quad \forall k \in \mathcal{C}^{(1)}, \end{aligned} \tag{6.15}$$

in the variables $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$ and $\alpha_0 \in \mathbb{R}$.

Since $[\hat{\mathbf{a}}_{l_0}^{(1)}, \hat{\mathbf{a}}_{l_1}^{(1)}] = 1$ implies the existence of $\hat{\alpha} \in \mathbb{R}^{n+1}$ for which

$$\begin{aligned} &\left\langle \hat{\mathbf{a}}_{l_0}^{(1)}, \hat{\alpha} \right\rangle = \left\langle \hat{\mathbf{a}}_{l_1}^{(1)}, \hat{\alpha} \right\rangle, \\ &\left\langle \hat{\mathbf{a}}_{l_0}^{(1)}, \hat{\alpha} \right\rangle \leqslant \left\langle \hat{\mathbf{a}}_k^{(1)}, \hat{\alpha} \right\rangle, \quad \forall k \in \{1, \ldots, s_1\} \backslash \{l_0, l_1\}, \end{aligned}$$

this $\hat{\alpha}$ along with

$$\alpha_0 = \min_{k \in \mathcal{C}^{(1)}} \left\langle \hat{\mathbf{a}}_k^{(1)}, \hat{\alpha} \right\rangle$$

yields a feasible point of the constraints in (6.15). Clearly, zero optimal value of this LP problem makes $\{\hat{\mathbf{a}}_{l_0}^{(1)}, \hat{\mathbf{a}}_{l_1}^{(1)}, \hat{\mathbf{a}}_\tau^{(1)}\}$ a lower face of $\widehat{S}^{(1)}$ of dimension two, and the point $\hat{\mathbf{a}}_\tau^{(1)}$ will be retained for further extension considerations. It will be deleted otherwise.

Again, the pivoting process in solving the LP problem in (6.15) by the simplex method provides abundant information on the extendibility of $\{\hat{\mathbf{a}}_{l_0}^{(1)}, \hat{\mathbf{a}}_{l_1}^{(1)}\}$ by other points

in $\mathcal{P}^{(1)}$. For instance, at any stage of the pivoting process, when the set of active constraints contains

$$\alpha_0 = \langle \hat{\mathbf{a}}_l^{(1)}, \hat{\alpha} \rangle$$

for any $l \in \mathcal{T}^{(1)} \backslash \{\tau\}$, then $\hat{\mathbf{a}}_l^{(1)}$ extends $\{\hat{\mathbf{a}}_{l_0}^{(1)}, \hat{\mathbf{a}}_{l_1}^{(1)}\}$ and can be omitted from the list of further testings.

When the testing on points of $\mathcal{P}^{(1)}$ is completed, we have extended $\{\hat{\mathbf{a}}_{l_0}^{(1)}, \hat{\mathbf{a}}_{l_1}^{(1)}\}$ to all possible 2-dimensional lower faces. This process may be repeated along the same line as we extend a j-dimensional lower face $\{\hat{\mathbf{a}}_{l_0}^{(1)}, \ldots, \hat{\mathbf{a}}_{l_j}^{(1)}\}$ for $j < k_1$ to $(j+1)$-dimensional lower faces. In the end, all k_1-dimensional lower faces $\{\hat{\mathbf{a}}_{l_0}^{(1)}, \hat{\mathbf{a}}_{l_1}^{(1)}, \ldots, \hat{\mathbf{a}}_{l_{k_1}}^{(1)}\}$ of $\widehat{S}^{(1)}$ with $l_0 < l_1 < \cdots < l_{k_1}$ can be obtained if they exist.

6.3.2. The extension of level-ξ subfaces

Let $\widehat{E}_\xi = (\widehat{F}_1, \ldots, \widehat{F}_\xi)$ be a level-ξ subface of $\widehat{S} = (\widehat{S}^{(1)}, \ldots, \widehat{S}^{(r)})$ of type $(k_1, \ldots, k_\xi)$ with $\xi < r$ where $\widehat{F}_i \subset \widehat{S}^{(i)} = \{\hat{\mathbf{a}}_1^{(i)}, \ldots, \hat{\mathbf{a}}_{s_i}^{(i)}\}$ for $i = 1, \ldots, \xi$. To continue the extension of $\widehat{E}_\xi$, we look for lower faces $\{\widehat{F}_{\xi+1}\}$ of $\widehat{S}^{(\xi+1)} = \{\hat{\mathbf{a}}_1^{(\xi+1)}, \ldots, \hat{\mathbf{a}}_{s_{\xi+1}}^{(\xi+1)}\}$ of dimension $k_{\xi+1}$ so that $\widehat{E}_{\xi+1} = (\widehat{F}_1, \ldots, \widehat{F}_{\xi+1})$ is a level-$(\xi+1)$ subface of $\widehat{S}$ of type $(k_1, \ldots, k_{\xi+1})$. To find all such lower faces, we first find all the vertices $\hat{a}_l^{(\xi+1)}$ in the lower hull of $\widehat{S}^{(\xi+1)}$ for which $(\widehat{F}_1, \ldots, \widehat{F}_\xi, \{\hat{a}_l^{(\xi+1)}\})$ is a level-$(\xi+1)$ subface of $\widehat{S}$ of type $(k_1, \ldots, k_\xi, 0)$, followed by extending each such vertex of $\widehat{S}^{(\xi+1)}$ to lower faces $\widehat{F}_{\xi+1}^j$ of $\widehat{S}^{(\xi+1)}$ of dimension j for $j = 1, \ldots, k_{\xi+1}$ consecutively, where for each j, $\widehat{F}_{\xi+1}^j \subset \widehat{F}_{\xi+1}^{j+1}$ and $(\widehat{F}_1, \ldots, \widehat{F}_\xi \widehat{F}_{\xi+1}^j)$ is a level-$(\xi+1)$ subface of $\widehat{S}$ of type $(k_1, \ldots, k_\xi, j)$.

For each $i = 1, \ldots, \xi$, since $\dim(\widehat{F}_i) = k_i$, let

$$\widehat{F}_i = \{\hat{\mathbf{a}}_{l_0}^{(i)}, \ldots, \hat{\mathbf{a}}_{l_{k_i}}^{(i)}\}.$$

To extend $\widehat{E}_\xi$, we begin by collecting on Table $\mathrm{T}(i, \xi+1)$ for $i = 1, \ldots, \xi$ all the points $\hat{\mathbf{a}}_l^{(\xi+1)}$ in $\widehat{S}^{(\xi+1)}$ where $[\hat{\mathbf{a}}_{l_j}^{(i)}, \hat{\mathbf{a}}_l^{(\xi+1)}] = 1$ for all $j = 0, \ldots, k_i$ and $i = 1, \ldots, \xi$, and denote this set by $\mathcal{P}^{(\xi+1)}$. This set clearly contains all the vertex points of any lower face of $\widehat{S}^{(\xi+1)}$ of dimension $k_{\xi+1}$ that extends $\widehat{E}_\xi$. To examine points $\hat{\mathbf{a}}_\tau^{(\xi+1)}$ in $\mathcal{P}^{(\xi+1)}$ for its possibility to extend $\widehat{E}_\xi$, we apply the One-Point test on $\hat{\mathbf{a}}_\tau^{(\xi+1)}$:

$$\begin{aligned}
&\text{Minimize } \langle \hat{\mathbf{a}}_\tau^{(\xi+1)}, \hat{\alpha} \rangle - \alpha_0 \\
&\left.\begin{aligned}
&\langle \hat{\mathbf{a}}_{l_0}^{(i)}, \hat{\alpha} \rangle = \cdots = \langle \hat{\mathbf{a}}_{l_{k_i}}^{(i)}, \hat{\alpha} \rangle, \\
&\langle \hat{\mathbf{a}}_{l_0}^{(i)}, \hat{\alpha} \rangle \leqslant \langle \hat{\mathbf{a}}_l^{(i)}, \hat{\alpha} \rangle, \quad \forall l \in \mathcal{C}^{(i)}
\end{aligned}\right\} \quad i = 1, \ldots, \xi \\
&\alpha_0 \leqslant \langle \hat{\mathbf{a}}_k^{(\xi+1)}, \hat{\alpha} \rangle, \qquad \forall k \in \mathcal{C}^{(\xi+1)},
\end{aligned} \tag{6.16}$$

in the variables $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$ and $\alpha_0 \in \mathbb{R}$, where for $i = 1, \ldots, \xi$, $\mathcal{C}^{(i)}$ is the set of indices of points $\hat{\mathbf{a}}_l^{(i)}$ in $\widehat{S}^{(i)}$ with $[\hat{\mathbf{a}}_{l_j}^{(i)}, \hat{\mathbf{a}}_l^{(i)}] = 1$ for all $j = 0, \ldots, k_i$, and $\mathcal{C}^{(\xi+1)}$

TABLE T$(i, \xi+1)$

		$\widehat{S}^{(\xi+1)}$				
		$\hat{\mathbf{a}}_1^{(\xi+1)}$	$\hat{\mathbf{a}}_2^{(\xi+1)}$	$\hat{\mathbf{a}}_3^{(\xi+1)}$	$\cdots$	$\hat{\mathbf{a}}_{s_{\xi+1}}^{(\xi+1)}$
$\widehat{S}^{(i)}$	$\hat{\mathbf{a}}_1^{(i)}$	$[\hat{\mathbf{a}}_1^{(i)}, \hat{\mathbf{a}}_1^{(\xi+1)}]$	$[\hat{\mathbf{a}}_1^{(i)}, \hat{\mathbf{a}}_2^{(\xi+1)}]$	$[\hat{\mathbf{a}}_1^{(i)}, \hat{\mathbf{a}}_3^{(\xi+1)}]$	$\cdots$	$[\hat{\mathbf{a}}_1^{(i)}, \hat{\mathbf{a}}_{s_{\xi+1}}^{(\xi+1)}]$
	$\hat{\mathbf{a}}_2^{(i)}$	$[\hat{\mathbf{a}}_2^{(i)}, \hat{\mathbf{a}}_1^{(\xi+1)}]$	$[\hat{\mathbf{a}}_2^{(i)}, \hat{\mathbf{a}}_2^{(\xi+1)}]$	$[\hat{\mathbf{a}}_2^{(i)}, \hat{\mathbf{a}}_3^{(\xi+1)}]$	$\cdots$	$[\hat{\mathbf{a}}_2^{(i)}, \hat{\mathbf{a}}_{s_{\xi+1}}^{(\xi+1)}]$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
	$\hat{\mathbf{a}}_{s_i}^{(i)}$	$[\hat{\mathbf{a}}_{s_i}^{(i)}, \hat{\mathbf{a}}_1^{(\xi+1)}]$	$[\hat{\mathbf{a}}_{s_i}^{(i)}, \hat{\mathbf{a}}_2^{(\xi+1)}]$	$[\hat{\mathbf{a}}_{s_i}^{(i)}, \hat{\mathbf{a}}_3^{(\xi+1)}]$	$\cdots$	$[\hat{\mathbf{a}}_{s_i}^{(i)}, \hat{\mathbf{a}}_{s_{\xi+1}}^{(\xi+1)}]$

contains the indices of the points in $\mathcal{P}^{(\xi+1)}$. When the optimal value is zero, point $\hat{\mathbf{a}}_\tau^{(\xi+1)}$ will be retained for further extension considerations, otherwise it would be deleted. As before, beneficial information provided by the pivoting in the simplex method averts the necessity to apply One-Point test on all the points in $\mathcal{P}^{(\xi+1)}$.

When the examination on the points in $\mathcal{P}^{(\xi+1)}$ for the extension of $\widehat{E}_\xi$ is completed, let $\mathcal{E}^{(\xi+1)}$ be the set of points in $\mathcal{P}^{(\xi+1)}$ which are capable of extending $\widehat{E}_\xi$; namely, for each such point $\hat{\mathbf{a}}_l^{(\xi+1)}$,

$$\big(\widehat{F}_1, \ldots, \widehat{F}_\xi, \{\hat{\mathbf{a}}_l^{(\xi+1)}\}\big)$$

is a level-$(\xi+1)$ subface of $\widehat{S}$ of type $(k_1, \ldots, k_\xi, 0)$. Let the indices of its points be

$$\tau_1 < \tau_2 < \cdots < \tau_t$$

and, in this order, we continue our attempt to extend

$$\big(\widehat{F}_1, \ldots, \widehat{F}_\xi, \{\hat{\mathbf{a}}_{\tau_j}^{(\xi+1)}\}\big)$$

for $j = 1, \ldots, t$ by examining points in $\{\hat{\mathbf{a}}_{\tau_l}^{(\xi+1)}\}_{l>j} \subset \mathcal{E}^{(\xi+1)}$.

This procedure may be continued along the same line as we extend the lower faces of $\widehat{S}_1$ until subface

$$\widehat{F}_{\xi+1} = \{\hat{\mathbf{a}}_{l_0}^{(\xi+1)}, \ldots, \hat{\mathbf{a}}_{l_{k_{\xi+1}}}^{(\xi+1)}\}$$

of $\widehat{S}^{(\xi+1)}$ of dimension $k_{\xi+1}$ for which

$$\widehat{E}_{\xi+1} := \big(\widehat{F}_1, \ldots, \widehat{F}_\xi, \widehat{F}_{\xi+1}\big)$$

is a level-$(\xi+1)$ subface of $\widehat{S}$ of type $(k_1, \ldots, k_\xi, k_{\xi+1})$ are all obtained.

6.4. Finding all cells of type $(k_1, \ldots, k_r)$

In summary, we list the procedure for finding all cells of type $(k_1, \ldots, k_r)$ in S_ω induced by a generic lifting $\omega = (\omega_1, \ldots, \omega_r)$ on $S = (S_1^{(1)}, \ldots, S^{(r)})$:

PROCEDURE FOR FINDING ALL CELLS OF TYPE $(k_1, \ldots, k_r)$.

(1) With $\widehat{S}^{(i)} = \{\hat{\mathbf{a}} = (\mathbf{a}, \omega_i(\mathbf{a})) \mid \mathbf{a} \in S^{(i)}\}$ for $i = 1, \ldots, r$, fill out the relation table, Table 6.1, consisting of Tables T(i, j) between $\widehat{S}^{(i)}$ and $\widehat{S}^{(j)}$ for all $1 \leqslant i, j \leqslant r$.
(2) Find all k_1 dimensional faces $\widehat{F}_1$ in the lower hull of $\widehat{S}^{(1)}$.
(3) For $1 \leqslant \xi < r$, extend each level-ξ surface $(\widehat{F}_1, \ldots, \widehat{F}_\xi)$ of $\widehat{S} = (\widehat{S}^{(1)}, \ldots, \widehat{S}^{(r)})$ of type $(k_1, \ldots, k_\xi)$ to level-$(\xi+1)$ subfaces of $\widehat{S}$ of type $(k_1, \ldots, k_{\xi+1})$.
(4) When $\xi + 1 = r$, $(F_1, \ldots, F_r)$ is a cell of type $(k_1, \ldots, k_r)$ in S_ω.

The above procedure has been successfully implemented in GAO and LI [2003] to calculate the mixed volume for semi-mixed polynomial systems. The algorithm achieves a dramatical speed-up, especially when the systems are unmixed, such as the 9-point problem in WAMPLER, MORGAN and SOMMESE [1992].

7. Finding isolated zeros in $\mathbb{C}^n$ via stable cells

7.1. Stable mixed volume

As remarked in the end of Section 3.1, in order to find all isolated zeros of a polynomial system $P(x) = (p_1(x), \ldots, p_n(x))$ with support $S = (S_1, \ldots, S_n)$ in $\mathbb{C}^n$, we need to follow $\mathcal{M}(S_1', \ldots, S_n')$ homotopy paths, where $S_j' = S_j \cup \{0\}$, $j = 1, \ldots, n$. By Theorem 3.2, $\mathcal{M}(S_1', \ldots, S_n')$ provides an upper bound for the root count of the system $P(x)$ in $\mathbb{C}^n$. However, as the following example shows, this bound may not be exact:

EXAMPLE 7.1 (HUBER and STURMFELS [1997]). Using linear homotopy with start system such as (1.3), one finds six isolated zeros in $\mathbb{C}^2$ of the system

$$P(x_1, x_2) = \begin{cases} p_1(x_1, x_2) = ax_2 + bx_2^2 + cx_1x_2^3, \\ p_2(x_1, x_2) = dx_1 + ex_1^2 + fx_1^3x_2 \end{cases}$$

for generic coefficients $\{a, b, c, d, e, f\}$. But for its augmented system

$$Q(x_1, x_2) = \begin{cases} q_1(x_1, x_2) = \varepsilon_1 + ax_2 + bx_2^2 + cx_1x_2^3, \\ q_2(x_1, x_2) = \varepsilon_2 + dx_1 + ex_1^2 + fx_1^3x_2, \end{cases}$$

the mixed volume $\mathcal{M}(S_1 \cup \{0\}, S_2 \cup \{0\})$, easily calculable by hand, is eight. In this case, eight homotopy paths need to be followed to obtain all six isolated zeros of $P(x_1, x_2)$ in $\mathbb{C}^2$, hence two of them are extraneous.

In HUBER and STURMFELS [1997], a tighter upper bound for the root count in $\mathbb{C}^n$ of the system $P(x) = (p_1(x), \ldots, p_n(x))$ was given. Based on this root count, one may employ alternative algorithms, which we will describe in this section, to approximate all isolated zeros of $P(x)$ in $\mathbb{C}^n$ by following fewer homotopy paths.

As before, for a given lifting $\omega = (\omega_1, \ldots, \omega_n)$ on $S' = (S_1', \ldots, S_n')$, we write $\hat{a} = (a, \omega_j(a))$ for $a \in S_j'$ and $\widehat{C}_j = \{\hat{a} \mid a \in C_j\}$ for $C_j \subset S_j'$. Cell $\widehat{C} = (\widehat{C}_1, \ldots, \widehat{C}_n)$ where $C_j \subset S_j'$ for $j = 1, \ldots, n$ is a *lower facet* of $\widehat{S}' = (\widehat{S}_1', \ldots, \widehat{S}_n')$ if $\dim(\mathrm{conv}(\widehat{C})) = n$ and

there exists $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$ satisfying, for $j = 1, \ldots, n$,

$$\langle \hat{a}, \hat{\alpha} \rangle = \langle \hat{b}, \hat{\alpha} \rangle \quad \text{for all } a, b \in C_j,$$
$$\langle \hat{a}, \hat{\alpha} \rangle < \langle \hat{d}, \hat{\alpha} \rangle \quad \text{for } a \in C_j \text{ and } d \in S'_j \backslash C_j.$$

We will refer to the vector $\alpha \in \mathbb{R}^n$ as the *inner normal* of $C = (C_1, \ldots, C_n)$ with respect to the lifting ω, and denote such $C = (C_1, \ldots, C_n)$ by C^α. When α is nonnegative, i.e. $\alpha_j \geqslant 0$ for all $j = 1, \ldots, n$, we call C^α a *stable cell* of $S = (S_1, \ldots, S_n)$ *with respect to the lifting* ω. The term *stable cell* alone, without specification of its corresponding lifting, will be reserved for stable cells with respect to the particular lifting $\omega_0^1 = (\omega_1^{01}, \ldots, \omega_n^{01})$ where $\omega_j^{01} : S'_j \to \mathbb{R}$ for $j = 1, \ldots, n$ is defined as:

$$\omega_j^{01}(0) = 1 \quad \text{if } 0 \notin S_j,$$
$$\omega_j^{01}(a) = 0 \quad \text{for } a \in S_j.$$

Obviously, $S = (S_1, \ldots, S_n)$ itself is a stable cell with inner normal $\alpha = (0, \ldots, 0)$ with respect to this particular lifting.

DEFINITION 7.1. The stable mixed volume of $S = (S_1, \ldots, S_n)$, denoted by $\mathcal{SM}(S_1, \ldots, S_n)$, is the sum of mixed volumes of all stable cells of $S = (S_1, \ldots, S_n)$.

With this definition, a tighter bound for the root count of $P(x)$ in $\mathbb{C}^n$ is given in the following

THEOREM 7.1 (HUBER and STURMFELS [1997]). *For polynomial system* $P(x) = (p_1(x), \ldots, p_n(x))$ *with support* $S = (S_1, \ldots, S_n)$, *the stable mixed volume* $\mathcal{SM}(S_1, \ldots, S_n)$ *satisfies*:

$$\mathcal{M}(S_1, \ldots, S_n) \leqslant \mathcal{SM}(S_1, \ldots, S_n) \leqslant \mathcal{M}\big(S_1 \cup \{0\}, \ldots, S_n \cup \{0\}\big). \tag{7.1}$$

Moreover, it provides an upper bound for the root count of $P(x)$ *in* $\mathbb{C}^n$.

For the system $P(x_1, x_2)$ in Example 7.1,

$$S_1 = \big\{(0,1), (0,2), (1,1)\big\}, \qquad S_2 = \big\{(1,0), (2,0), (3,1)\big\}$$

and

$$S'_1 = S_1 \cup \big\{(0,0)\big\}, \qquad S'_2 = S_2 \cup \big\{(0,0)\big\},$$

as shown in Fig. 7.1. With lifting ω_0^1, there are eight lower facets of $\widehat{S}' = (\widehat{S}'_1, \widehat{S}'_2)$. Among them, four stable cells of the system are induced:

(1) $C^{\alpha^{(1)}} = (\{(0,0), (0,1)\}, \{(1,0), (2,0)\})$ with $\alpha^{(1)} = (0,1)$,
(2) $C^{\alpha^{(2)}} = (\{(0,1), (0,2)\}, \{(0,0), (1,0)\})$ with $\alpha^{(2)} = (1,0)$,
(3) $C^{\alpha^{(3)}} = (\{(0,0), (0,1)\}, \{(0,0), (1,0)\})$ with $\alpha^{(3)} = (1,1)$,
(4) $C^{\alpha^{(4)}} = (S_1, S_2)$ with $\alpha^{(4)} = (0,0)$.

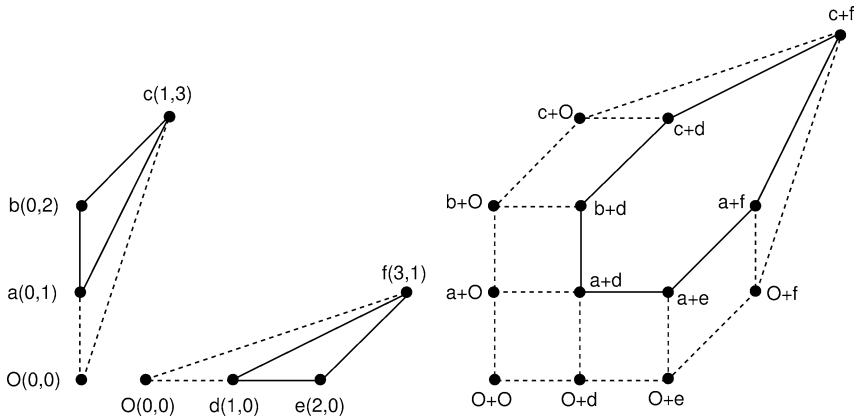


FIG. 7.1.

Clearly, the mixed volumes of $C^{\alpha^{(1)}}$, $C^{\alpha^{(2)}}$, and $C^{\alpha^{(3)}}$ are just their volumes, they all equal to 1. The mixed volume of $C^{\alpha^{(4)}} = (S_1, S_2)$ is

$$\mathcal{M}(S_1, S_2) = \mathrm{Vol}_2(S_1, S_2) - \mathrm{Vol}_2(S_1) - \mathrm{Vol}_2(S_2) = 4 - 0.5 - 0.5 = 3.$$

Therefore, the stable mixed volume $\mathcal{SM}(S_1, S_2) = 6$ for the system, while the mixed volume of its augment system $Q(x_1, x_2)$ with support $(S_1 \cup \{0\}, S_2 \cup \{0\})$ is

$$\begin{aligned}&\mathcal{M}\big(S_1 \cup \{0\}, S_2 \cup \{0\}\big)\\ &\quad = \mathrm{Vol}_2\big(S_1 \cup \{0\}, S_2 \cup \{0\}\big) - \mathrm{Vol}_2\big(S_1 \cup \{0\}\big) - \mathrm{Vol}_2\big(S_2 \cup \{0\}\big)\\ &\quad = 10 - 1 - 1 = 8.\end{aligned}$$

Hence,

$$\mathcal{M}(S_1, S_2) < \mathcal{SM}(S_1, S_2) < \mathcal{M}\big(S_1 \cup \{0\}, S_2 \cup \{0\}\big)$$

for the system and the inequalities in (7.1) are strict in this case.

7.2. An alternative algorithm

Based on the derivation of Theorem 7.1, it was suggested in HUBER and STURMFELS [1997] that one may find all isolated zeros of polynomial system $P(x) = (p_1(x), \ldots, p_n(x))$ in $\mathbb{C}^n$ with support $S = (S_1, \ldots, S_n)$ where

$$p_j(x) = \sum_{a \in S_j} c_{j,a} x^a, \quad j = 1, \ldots, n,$$

by the following procedure:

Step 1: Identify all stable cells $C^\alpha = (C_1, \ldots, C_n)$ of $S = (S_1, \ldots, S_n)$.
Step 2: For each stable cell $C^\alpha = (C_1, \ldots, C_n)$ with inner normal $\alpha = (\alpha_1, \ldots, \alpha_n) \geqslant 0$, find all isolated zeros in $(\mathbb{C}^*)^n$ of the *support* system $P^\alpha(x) = (p_1^\alpha(x), \ldots, p_n^\alpha(x))$

where for $j = 1, \ldots, n$,

$$p_j^{\alpha}(x) = \sum_{a \in C_j \cap S_j} c_{j,a} x^a + \varepsilon_j \tag{7.2}$$

with $\varepsilon_j = 0$ if $0 \in S_j$, an arbitrary nonzero number otherwise.

Step 3: For each isolated zero $z = (z_1, \ldots, z_n)$ of $P^{\alpha}(x)$ in $(\mathbb{C}^*)^n$, let

$$\bar{z}_j = z_j \quad \text{if } \alpha_j = 0,$$
$$\bar{z}_j = 0 \quad \text{if } \alpha_j \neq 0.$$

Then $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_n)$ is an isolated zero of $P(x) = (p_1(x), \ldots, p_n(x))$.

Inevitably, zeros $z = (z_1, \ldots, z_n)$ of $P^{\alpha}(x)$ will depend on $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$. However, it can be shown that the transition from z to $\bar{z}$ given above actually eliminates this dependency as the following example shows:

EXAMPLE 7.2. For the system $P(x_1, x_2) = (p_1(x_1, x_2), p_2(x_1, x_2))$ where

$$p_1(x_1, x_2) = x_1 x_2,$$
$$p_2(x_1, x_2) = x_1 + x_2 - 1,$$

$S_1 = \{(1,1)\}$, $S_2 = \{(1,0), (0,1), (0,0)\}$ and $S_1' = \{(1,1), (0,0)\}$, $S_2' = S_2$.

With lifting ω_0^1, there are two stable cells with nonzero mixed volumes:

(1) $C^{\alpha^{(1)}} = (\{(1,1), (0,0)\}, \{(1,0), (0,0)\})$ with $\alpha^{(1)} = (0,1)$,

(2) $C^{\alpha^{(2)}} = (\{(1,1), (0,0)\}, \{(0,1), (0,0)\})$ with $\alpha^{(2)} = (1,0)$.

The support system of $C^{\alpha^{(1)}}$ is

$$p_1^{\alpha^{(1)}}(x_1, x_2) = x_1 x_2 + \varepsilon = 0, \quad \varepsilon \neq 0,$$
$$p_2^{\alpha^{(1)}}(x_1, x_2) = x_1 - 1 = 0,$$

with isolated zero $(1, -\varepsilon)$ in $(\mathbb{C}^*)^2$. This leads to an isolated zero $(1,0)$ of $P(x_1, x_2)$ in $\mathbb{C}^2$ since the second component of $\alpha^{(1)}$ is positive. Similarly, isolated zero $(-\varepsilon, 1)$ of the support system of $C^{\alpha^{(2)}}$,

$$p_1^{\alpha^{(2)}}(x_1, x_2) = x_1 x_2 + \varepsilon = 0, \quad \varepsilon \neq 0,$$
$$p_2^{\alpha^{(2)}}(x_1, x_2) = x_2 - 1 = 0,$$

in $(\mathbb{C}^*)^2$ gives an isolated zero $(0,1)$ of $P(x_1, x_2)$ since the first component of $\alpha^{(2)}$ is positive.

In Step 2 above, when polyhedral homotopy is used to find all isolated zeros in $(\mathbb{C}^*)^n$ of the support system $P^{\alpha}(x)$ corresponding to the stable cell $C^{\alpha} = (C_1, \ldots, C_n)$, one follows $\mathcal{M}(C_1, \ldots, C_n)$ homotopy paths. Accordingly, the total number of homotopy paths one needs to follow to reach all isolated zeros of $P(x)$ in $\mathbb{C}^n$ equals to the stable mixed volume $\mathcal{SM}(S_1, \ldots, S_n)$, which is strictly fewer than $\mathcal{M}(S_1 \cup \{0\}, \ldots, S_n \cup \{0\})$ in general, therefore admitting less extraneous paths.

However, there are difficulties to implement this procedure efficiently. First of all, types of those stable cells are undetermined in general. They may not be mixed cells, cells of type $(1, \ldots, 1)$, with respect to the lifting $\omega_0^1 = (\omega_1^{01}, \ldots, \omega_n^{01})$, which invalidates the algorithm we developed in Section 5 for finding mixed cells. This makes the identification of all the stable cells in Step 1 rather difficult. Secondly, when polyhedral homotopy is used in Step 2 to solve $P^\alpha(x)$ in $(\mathbb{C}^*)^n$, one must find all mixed cells of a subdivision of $C^\alpha = (C_1, \ldots, C_n)$ induced by a further generic lifting on C^α in the first place. This accumulated work for all the stable cells can be very costly, which may not be more favorable compared to solving $P(x)$ in $\mathbb{C}^n$ by simply following the polyhedral homotopy procedure given in Section 4 directly with a generic lifting on $S' = (S'_1, \ldots, S'_n)$ permitting some of the homotopy paths to be extraneous.

7.3. A revision

We will elaborate in this section a revision given in GAO, LI and WANG [1999] for the procedure suggested by HUBER and STURMFELS [1997] in the last section. To begin, for $k \geqslant 0$, let $\omega_0^k = (\omega_1^{0k}, \ldots, \omega_n^{0k})$ be the lifting on $S' = (S'_1, \ldots, S'_n)$ where for $j = 1, \ldots, n$,

$$\begin{aligned} &\omega_j^{0k}(0) = k \quad \text{if } 0 \notin S_j, \\ &\omega_j^{0k}(a) = 0 \quad \text{for } a \in S_j. \end{aligned} \tag{7.3}$$

Clearly, the set of stable cells with respect to ω_0^k remains invariant for different k's. For instance, if $C = (C_1, \ldots, C_n)$ is a stable cell with respect to the lifting $\omega_0^{k_1}$ with inner normal $\alpha \geqslant 0$, then $C = (C_1, \ldots, C_n)$ is also a stable cell with respect to the lifting $\omega_0^{k_2}$ with inner normal $k_2/k_1 \cdot \alpha \geqslant 0$. Denote this set of stable cells by $\mathcal{T}$. Let $\omega = (\omega_1, \ldots, \omega_n)$ be a generic lifting on $S' = (S'_1, \ldots, S'_n)$ where for $j = 1, \ldots, n$,

$$\begin{aligned} &\omega_j(0) = k && \text{for } 0 \notin S_j, \\ &\omega_j(a) = \text{a generic number in } (0, 1) && \text{for } a \in S_j. \end{aligned} \tag{7.4}$$

For a cell $C = (C_1, \ldots, C_n)$ in the subdivision of $S' = (S'_1, \ldots, S'_n)$ induced by the lifting $\omega_0^k = (\omega_1^{0k}, \ldots, \omega_n^{0k})$, let ω^C be the restriction of ω on C, which can, of course, be regarded as a generic lifting on C. It was shown in GAO, LI and WANG [1999] that if k is sufficiently large, mixed cell $D = (D_1, \ldots, D_n)$ of subdivision S_ω of $S' = (S'_1, \ldots, S'_n)$ induced by the lifting ω is also a mixed cell of subdivision S_{ω^C} induced by the lifting ω^C of certain cell $C = (C_1, \ldots, C_n)$ of S' with respect to the lifting ω_0^k. Accordingly, stable cells $C = (C_1, \ldots, C_n)$ in $\mathcal{T}$ can be assembled by grouping a collection of proper cells in S_ω, and consequently, mixed cells in this collection provides all the mixed cells of subdivision S_{ω^C} of $C = (C_1, \ldots, C_n)$. More precisely, when $k \geqslant n(n+1)d^n$ (see GAO, LI and WANG [1999]) where $d = \max_{1 \leqslant j \leqslant n} \deg p_j(x)$, any mixed cell $D = (D_1, \ldots, D_n)$ in the subdivision S_ω induced by the lifting $\omega = (\omega_1, \ldots, \omega_n)$ on $S' = (S'_1, \ldots, S'_n)$ given in (7.4) is a mixed cell of subdivision S_{ω^C} induced by the lifting ω^C of certain cell $C = (C_1, \ldots, C_n)$ in the subdivision $S_{\omega_0^k}$ induced by the lifting $\omega_0^k = (\omega_1^{0k}, \ldots, \omega_n^{0k})$ on $S' = (S'_1, \ldots, S'_n)$ given in (7.3).

Let $D^* = (D_1, \ldots, D_n)$ be any cell in the subdivision S_w which may or may not be of type $(1, \ldots, 1)$. Let

$$D_j = \{a_{j0}, \ldots, a_{jk_j}\}, \quad j = 1, \ldots, n,$$

where $k_1 + \cdots + k_n = n$. For $j = 1, \ldots, n$ and $a \in S'_j$, write $\hat{a}(k) = (a, \omega_j^{0k}(a))$ and $\widehat{D}_j(k) = \{\hat{a}(k) \mid a \in D_j\}$. Let $\widehat{D}^*(k) = (\widehat{D}_1(k), \ldots, \widehat{D}_n(k))$. Clearly, the $n \times (n+1)$ matrix

$$V\big(\widehat{D}^*(k)\big) = \begin{pmatrix} \hat{a}_{11}(k) - \hat{a}_{10}(k) \\ \vdots \\ \hat{a}_{1k_1}(k) - \hat{a}_{10}(k) \\ \vdots \\ \hat{a}_{n1}(k) - \hat{a}_{n0}(k) \\ \vdots \\ \hat{a}_{nk_n}(k) - \hat{a}_{n0}(k) \end{pmatrix}$$

is of rank n. Let $\alpha \in \mathbb{R}^n$ be the unique vector where $\hat{\alpha} = (\alpha, 1)$ is in the kernel of $V(\widehat{D}^*(k))$. This α is the inner normal of D^* with respect to ω_0^k. Let $\mathcal{T}(\alpha)$ be the collection of all mixed cells in S_ω with the same nonnegative inner normal α with respect to ω_0^k and let $D = (\{a_{10}, a_{11}\}, \ldots, \{a_{n0}, a_{n1}\})$ where $\{a_{j0}, a_{j1}\} \subset S'_j$ for $j = 1, \ldots, n$ be any mixed cell in $\mathcal{T}(\alpha)$. Let $C = (C_1, \ldots, C_n)$ where

$$C_j = \big\{a \in S'_j \mid \langle \hat{a}(k), \hat{\alpha}\rangle = \langle \hat{a}_{j0}(k), \hat{\alpha}\rangle\big\}, \quad j = 1, \ldots, n.$$

This cell satisfies, for $j = 1, \ldots, n$,

$$\big\langle \hat{a}(k), \hat{\alpha}\big\rangle = \big\langle \hat{b}(k), \hat{\alpha}\big\rangle \quad \text{for } a, b \in C_j,$$
$$\big\langle \hat{a}(k), \hat{\alpha}\big\rangle < \big\langle \hat{d}(k), \hat{\alpha}\big\rangle \quad \text{for } a \in C_j,\ d \in S'_j \backslash C_j.$$

It is therefore a stable cell with respect to ω_0^k with inner normal α, which, as mentioned above, is also a stable cell with respect to ω_0^1 with inner normal $\frac{1}{k}\alpha$. In the meantime, the cells in the collection $\mathcal{T}(\alpha)$ gives *all* the mixed cells in the subdivision S_{ω^C} of $C = (C_1, \ldots, C_n)$ induced by the lifting ω^C.

From what we have discussed above, the previously listed procedure for solving system $P(x) = (p_1(x), \ldots, p_n(x))$ with $S = (S_1, \ldots, S_n)$ in $\mathbb{C}^n$ suggested in HUBER and STURMFELS [1997] may now be revised as follows:

FINDING ISOLATED ZEROS IN $\mathbb{C}^n$ VIA STABLE CELLS.

Step 0: Let $d = \max_{1 \leqslant i \leqslant n} \deg p_i(x)$. Choose a real number $k > n(n+1)d^n$ at random.

Step 1: Lift the support $S' = (S'_1, \ldots, S'_n)$ by a random lifting $\omega = (\omega_1, \ldots, \omega_n)$ as defined in (7.4) where for $j = 1, \ldots, n$,

$$\omega_j(0) = k \qquad \text{if } 0 \notin S_j,$$
$$\omega_j(a) = \text{a randomly chosen number in } (0, 1) \quad \text{if } a \in S_j.$$

Find cells of type $(1, \ldots, 1)$ in the induced fine mixed subdivision S_ω of $S' = (S'_1, \ldots, S'_n)$.

Step 2: For cell $D = (\{a_{10}, a_{11}\}, \ldots, \{a_{n0}, a_{n1}\})$ of type $(1, \ldots, 1)$ in S_ω, let $\hat{a}_{ji}(k) = (a_{ji}, l)$ where for $j = 1, \ldots, n$, and $i = 0, 1$,

$$l = k \quad \text{if } a_{ji} = 0 \notin S_j,$$
$$l = 0 \quad \text{if } a_{ji} \in S_j.$$

Form the $n \times (n+1)$ matrix

$$V = \begin{pmatrix} \hat{a}_{11}(k) - \hat{a}_{10}(k) \\ \vdots \\ \hat{a}_{n1}(k) - \hat{a}_{n0}(k) \end{pmatrix},$$

and find the unique vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ where $\hat{\alpha} = (\alpha, 1)$ is in the kernel of V. This α is the inner normal of D with respect to ω_0^k. Let $\mathcal{T}(\alpha)$ be the collection of all cells of type $(1, \ldots, 1)$ in S_ω with the same nonnegative inner normal $\alpha = (\alpha_1, \ldots, \alpha_n)$ with respect to ω_0^k.

Step 3: (a) Choose any mixed cell $D = (\{a_{10}, a_{11}\}, \ldots, \{a_{n0}, a_{n1}\})$ from $\mathcal{T}(\alpha)$, let

$$C_j = \left\{ a \in S_j' \mid \langle \hat{a}(k), \hat{\alpha} \rangle = \langle \hat{a}_{j0}(k), \hat{\alpha} \rangle \right\}, \quad j = 1, \ldots, n,$$

where $\hat{a}(k) = (a, l)$ with

$$l = k \quad \text{if } a = 0 \notin S_j,$$
$$l = 0 \quad \text{if } a \in S_j.$$

Then $C = (C_1, \ldots, C_n)$ is a stable cell of $S = (S_1, \ldots, S_n)$ with respect to the inner normal α in $S_{\omega_0^k}$. Notice that

$$S_{\omega^C} = \left\{ (D_1, \ldots, D_n) \in S_\omega \mid D_j \subseteq C_j \text{ for all } 1 \leqslant j \leqslant n \right\}$$

is the fine mixed subdivision of C induced by ω^C, the restriction of ω on C, and $\mathcal{T}(\alpha)$ gives all the mixed cells, cells of type $(1, \ldots, 1)$, in S_{ω^C}.

(b) Find all the isolated zeros of the system

$$P^\alpha(x) = \left(p_1^\alpha(x), \ldots, p_n^\alpha(x) \right) \tag{7.5}$$

where

$$p_j^\alpha(x) = \sum_{\alpha \in C_j \cap S_j} c_{j,a} x^a + \varepsilon_j, \quad j = 1, \ldots, n,$$

and

$$\varepsilon_j = 0 \quad \text{if } 0 \in S_j,$$
$$\varepsilon_j = 1 \quad \text{if } 0 \notin S_j$$

in $(\mathbb{C}^*)^n$ by employing the polyhedral homotopy procedure developed in Section 4 with lifting ω^C.

(c) For zeros $\mathbf{e} = (e_1, \ldots, e_n)$ of $P^\alpha(x)$ found in (b), let

$$\bar{e}_j = e_j \quad \text{if } \alpha_j = 0,$$
$$\bar{e}_j = 0 \quad \text{if } \alpha_j \neq 0.$$

Then $\bar{\mathbf{e}} = (\bar{e}_1, \ldots, \bar{e}_n)$ is a zero of $P(x)$ in $\mathbb{C}^n$.
Step 4: Repeat Step 3 for all $\mathcal{T}(\alpha)$ with $\alpha \geqslant 0$.

REMARK 7.1. For $d_j = \deg p_j(x)$, $j = 1, \ldots, n$, we assume without loss $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$. It was mentioned in GAO, LI and WANG [1999], in Step 0 of the above procedure, d may be replaced by $d_2 \times \cdots \times d_n \times d_n$ which usually gives a much smaller number.

REMARK 7.2. It is commonly known that when the polyhedral homotopy method is used to solve polynomial systems, large differences between the powers of parameter t in the polyhedral homotopies may cause computational instability when homotopy curves are followed. In the above algorithm, the point 0 often receives very large lifting value k, compared to the rest of the lifting values in $(0, 1)$. It was shown in GAO, LI and WANG [1999] that the stability of the algorithm is independent of the large lifting value k when polyhedral homotopies are used in Step 3(b).

The revised procedure listed above has been successfully implemented in GAO, LI and WANG [1999] with remarkable numerical results.

8. Solutions of positive dimension

8.1. Solutions of positive dimension

For polynomial system $P(x) = (p_1(x), \ldots, p_n(x))$, positive dimensional components of the solution set of $P(x) = 0$ are a common occurrence. Sometimes they are an unpleasant side show (SOMMESE and WAMPLER [1996]) that happens with a system generated using a model for which only the isolated nonsingular solutions are of interest; and sometimes, the positive dimensional solution components are of primary interest. In either case, dealing with positive dimensional solution components, is usually computationally difficult.

Putting aside formal definition with technical terms, by a *generic point* of an irreducible component X of the solution set of $P(x) = 0$, it usually means a point of X which has no special properties not possessed by the whole component X. Numerically, it is modeled by a point in X with random coordinates. In SOMMESE and WAMPLER [1996], a procedure consisted in slicing X with linear subspaces in general position to obtain generic point of X as the isolated solutions of an auxiliary system was presented. By Noether's normalization theorem combined with Bertini's theorem, it can be shown that if X is of k-dimensional, then a general affine linear subspace $\mathbb{C}^{n-k}$ meets X at isolated points. Those are generic points of X. A generic linear subspace $\mathbb{C}^{n-k}$ can be given by

$$\begin{aligned} \lambda_{11}x_1 + \cdots + \lambda_{1n}x_n &= \lambda_1, \\ &\vdots \\ \lambda_{k1}x_1 + \cdots + \lambda_{kn}x_n &= \lambda_k \end{aligned}$$

with all the λ's being random numbers. Thus, the existence of isolated solutions of the system

$$\begin{aligned} p_1(x_1,\ldots,x_n) &= 0,\\ &\vdots\\ p_n(x_1,\ldots,x_n) &= 0,\\ \lambda_{11}x_1+\cdots+\lambda_{1n}x_n &= \lambda_1,\\ &\vdots\\ \lambda_{k1}x_1+\cdots+\lambda_{kn}x_n &= \lambda_k \end{aligned} \tag{8.1}$$

warrants the existence of k-dimensional components of the solution set of the original system $P(x)=(p_1(x),\ldots,p_n(x))=0$. Furthermore, the set of isolated solutions of (8.1) contains at least one generic point of each irreducible component of dimension k of the solution set of $P(x)=0$.

System (8.1) is overdetermined and the procedure suggested in SOMMESE and WAMPLER [1996] for solving its isolated solutions is quite computationally expensive. A more efficient method which we shall describe below is given in SOMMESE and VERSCHELDE [2000], SOMMESE, VERSCHELDE and WAMPLER [2001]. This method can determine the existence of components of various dimensions, including isolated solutions, of the solution set of $P(x)=0$.

With extra parameters $z_1,\ldots,z_n$, consider, for $j=1,\ldots,n$, the embeddings:

$$\mathcal{E}_j(x,z_1,\ldots,z_j)=\begin{cases} p_1(x)+\lambda_{11}z_1+\cdots+\lambda_{1j}z_j,\\ \qquad\vdots\\ p_n(x)+\lambda_{n1}z_1+\cdots+\lambda_{nj}z_j,\\ a_1+a_{11}x_1+\cdots+a_{1n}x_n+z_1,\\ \qquad\vdots\\ a_j+a_{j1}x_1+\cdots+a_{jn}x_n+z_j. \end{cases}$$

For $j=0$, we let $\mathcal{E}_0(x)=P(x)$. Let Y denote the space $\mathbb{C}^{n\times(n+1)}\times\mathbb{C}^{n\times n}$ of parameters

$$\begin{bmatrix} a_1 & a_{11} & \cdots & a_{1n} & \lambda_{11} & \cdots & \lambda_{n1}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ a_n & a_{n1} & \cdots & a_{nn} & \lambda_{1n} & \cdots & \lambda_{nn} \end{bmatrix} \in \mathbb{C}^{n\times(n+1)}\times\mathbb{C}^{n\times n}.$$

LEMMA 8.1 (SOMMESE and VERSCHELDE [2000]). *There is an open dense set U of full measure of points*

$$\begin{bmatrix} a_1 & a_{11} & \cdots & a_{1n} & \lambda_{11} & \cdots & \lambda_{n1}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ a_n & a_{n1} & \cdots & a_{nn} & \lambda_{1n} & \cdots & \lambda_{nn} \end{bmatrix} \in \mathbb{C}^{n\times(n+1)}\times\mathbb{C}^{n\times n} \tag{8.2}$$

such that for each $j=1,\ldots,n$,

(1) *The solution set of* $\mathcal{E}_j(x, z_1, \ldots, z_j) = 0$ *with* $(z_1, \ldots, z_j) \neq 0$ *are isolated and nonsingular*;
(2) *Given any irreducible component* X *of the solution set of* $P(x) = 0$ *of dimension* j, *the set of isolated solutions of* $\mathcal{E}_j(x, z_1, \ldots, z_j) = 0$ *with* $(z_1, \ldots, z_j) = 0$ *contains as many generic points as the degree of* X;
(3) *The solutions of* $\mathcal{E}_j(x, z_1, \ldots, z_j) = 0$ *with* $(z_1, \ldots, z_j) \neq 0$ *are the same as the solutions of* $\mathcal{E}_j(x, z_1, \ldots, z_j) = 0$ *with* $z_j \neq 0$.

By the third assertion of the above lemma, the solutions of $\mathcal{E}_j(x, z_1, \ldots, z_j) = 0$ are either $z_j \neq 0$ or $(z_1, \ldots, z_j) = 0$, namely, $z_j = 0$ implies $z_1 = z_2 = \cdots = z_{j-1} = 0$ for any solutions. Moreover, when all those parameters a's and λ's in (8.2) are chosen generically, the existence of the solutions of $\mathcal{E}_j(x, z_1, \ldots, z_j) = 0$ with $(z_1, \ldots, z_j) = 0$ reveals the possibilities of the existence of components of dimension j of the solution set of $P(x) = 0$. While those solutions might lie in components of dimension higher then j, it is clear that if j is the largest integer for which the solution set of $\mathcal{E}_j(x, z_1, \ldots, z_j) = 0$ is nonempty then the dimension of the solution set V of $P(x) = 0$, defined to be the largest dimension of all the irreducible components of V, must be j.

For j from n to 1 and $t \in [0, 1]$, consider homotopies H_j defined by

$$H_j(x, z_1, \ldots, z_j, t) = (1-t)\mathcal{E}_j(x, z_1, \ldots, z_j) + t \begin{pmatrix} \mathcal{E}_{j-1}(x, z_1, \ldots, z_{j-1}) \\ z_j \end{pmatrix}$$

$$= \begin{cases} p_1(x) + \sum_{i=1}^{j-1} \lambda_{1,i} z_i + (1-t)\lambda_{1,j} z_j, \\ \qquad \vdots \\ p_n(x) + \sum_{i=1}^{j-1} \lambda_{n,i} z_i + (1-t)\lambda_{n,j} z_j, \\ \\ a_1 + a_{11}x_1 + \cdots + a_{1n}x_n + z_1, \\ \qquad \vdots \\ a_{j-1} + a_{j-1,1}x_1 + \cdots + a_{j-1,n}x_n + z_{j-1}, \\ \\ (1-t)(a_j + a_{j,1}x_1 + \cdots + a_{j,n}x_n) + z_j. \end{cases}$$

Let Z_j denote the solutions of $\mathcal{E}_j(x, z_1, \ldots, z_j) = 0$ with $z_j \neq 0$. Following the paths of the homotopy $H_j(x, z_1, \ldots, z_j, t) = 0$ starting at points of Z_j will produce a set of solutions of $\mathcal{E}_{j-1}(x, z_1, \ldots, z_{j-1}) = 0$ at $t = 1$. Among them, let W_{j-1} denote the set of points with $z_{j-1} = 0$. By convention, when $j = 1$ the set W_0 is empty. As discussed above, we have

THEOREM 8.1 (SOMMESE and VERSCHELDE [2000]). *If* j *is the largest integer for which* $W_j \neq \emptyset$, *then the dimension of the solution set* V *of* $P(x) = 0$ *is* j. *Moreover, given any irreducible component* X *of* V *of dimension* $k \leqslant j$, *then the finite set* W_k *contains generic points of* X.

To find the sets W_k for $k = 1, \ldots, n-1$, we may follow the following procedure:

CASCADE OF HOMOTOPIES BETWEEN EMBEDDED SYSTEMS.
Step 0: Initialize the embedding sequences

$$\mathcal{E}_n(x, z_1, \ldots, z_n), \quad \mathcal{E}_{n-1}(x, z_1, \ldots, z_{n-1}), \quad \ldots, \quad \mathcal{E}_1(x, z_1).$$

Step 1: Solve $Z_n :=$ all isolated zeros of $\mathcal{E}_n(x, z_1, \ldots, z_n) = 0$ with $z_n \neq 0$.
Step 2: For $j = n - 1$ down to 0, follow paths of the homotopy

$$H_{j+1} = (1-t)\mathcal{E}_{j+1} + t \begin{pmatrix} \mathcal{E}_j \\ z_{j+1} \end{pmatrix} = 0$$

from $t = 0 \to 1$ with starting points

$$Z_{j+1} := \text{solutions of } \mathcal{E}_{j+1}(x, z_1, \ldots, z_{j+1}) = 0 \quad \text{with } z_{j+1} \neq 0.$$

Let $W_j :=$ limits of solutions of $H_{j+1} = 0$ as $t \to 1$ with $z_j = 0$.

In the above procedure, by default, Step 1 begins by solving Z_n, the set of all isolated zeros of $\mathcal{E}_n(x, z_1, \ldots, z_n) = 0$ with $z_n \neq 0$. But for efficiency reasons, if it can be determined by some other means that there are no components of the solution set V having dimension greater than k, one may start Step 1 with $k < n$.

When we proceed the cascade and identify the largest integer j_0 with $W_{j_0} \neq \emptyset$, then the dimension of the solution set V of $P(x) = 0$ is j_0. We may continue the procedure by following the paths of $H_{j_0}(x, z_1, \ldots, z_{j_0}, t) = 0$ with starting points in Z_{j_0} to identify the lower dimensional components of V. However, as we mentioned earlier, the existence of nonempty W_j for $j < j_0$ may no longer warrant the existence of irreducible components of V of dimension j, because points in W_j may all land in components of V with higher dimensions. In SOMMESE, VERSCHELDE and WAMPLER [2001], an algorithm, called *WitnessClassify*, is developed to clarify this problem:

Let the decomposition of the entire solution set V of $P(x) = 0$ be the nested union:

$$V := \bigcup_{j=0}^{j_0} V_j := \bigcup_{j=0}^{j_0} \bigcup_{i \in I_j} V_{ji},$$

where V_j is the union of all j-dimensional components, the V_{ji} are the irreducible components of dimension j, and the index sets I_j are finite and possibly empty. Let

$$W = \bigcup_{j=0}^{j_0} W_j.$$

The *WitnessClassify* algorithm proceeds to classify the points in each of the nonempty W_j for $j < j_0$ by first separating out the points on higher-dimensional components. This can be accomplished by the construction of the *filtering polynomials*, p_{ji}, each of which vanishes on the entire irreducible component V_{ji} but which is nonzero, with probability 1, for any point in W that is not on V_{ji}. The geometric concept of the filtering polynomials is as follows. For an irreducible component V_{ji}, which by definition has dimension j, pick a generic linear subspace of directions having dimension $n - j - 1$ and define a new set constructed by replacing each point in V_{ji} with its expansion along

the chosen subspace directions. The result is an $(n-1)$-dimensional hypersurface. The filtering polynomial is the unique polynomial that vanishes on this hypersurface.

When filtering polynomials for all components of dimension greater than j are available, those points J_j in W_j that lie on the higher-dimensional components can be identified. Let $\widehat{W}_j = W_j \setminus J_j$. The next task is to sort the points in $\widehat{W}_j$ by their membership in the irreducible component V_{ji}. For some arbitrary point $w \in \widehat{W}_j$, we move the slicing planes that pick w out of the underlying irreducible components V_{ji}, using continuation. In this manner, an arbitrary number of new points can be generated on V_{ji}. After picking a generic linear subspace of directions to expand V_{ji} into a hypersurface, one can find the lowest-degree polynomial that interpolates the samples, always taking extra samples as necessary to ensure the true hypersurface has been correctly determined. This is the filtering polynomial p_{ji}, which can then be used to find any additional points in $\widehat{W}_j$ that lies on V_{ji}. We then choose a new point from those in $\widehat{W}_j$ that are not sorted, and repeat the process until all points are sorted. With all the filtering polynomials p_{ji} in hand, we proceed to dimension $j-1$ and apply the same method.

Instead of giving the details, we shall illustrate the above decomposition procedure by presenting the following example given in SOMMESE, VERSCHELDE and WAMPLER [2001]:

EXAMPLE 8.1. Consider the polynomial system $P(x_1, x_2, x_3) = (p_1(x_1, x_2, x_3), p_2(x_1, x_2, x_3), p_3(x_1, x_2, x_3))$ where

$$\begin{aligned} p_1(x_1, x_2, x_3) &= \big(x_2 - x_1^2\big)\big(x_1^2 + x_2^2 + x_3^2 - 1\big)(x_1 - 0.5), \\ p_2(x_1, x_2, x_3) &= \big(x_3 - x_1^3\big)\big(x_1^3 + x_2^2 + x_3^2 - 1\big)(x_2 - 0.5), \\ p_3(x_1, x_2, x_3) &= \big(x_2 - x_1^2\big)\big(x_3 - x_1^3\big)\big(x_1^2 + x_2^2 + x_3^2 - 1\big)(x_3 - 0.5). \end{aligned}$$

The decomposition of the solution set V of $P(x_1, x_2, x_3) = 0$ into its irreducible components is obvious:

$$V = V_2 \cup V_1 \cup V_0 = \{V_{21}\} \cup \{V_{11}\} \cup \{V_{12}\} \cup \{V_{13}\} \cup \{V_{14}\} \cup \{V_{01}\},$$

where

(1) V_{21} is the sphere $x_1^2 + x_2^2 + x_3^2 = 1$,
(2) V_{11} is the line $(x_1 = 0.5, x_3 = (0.5)^3)$,
(3) V_{12} is the line $(x_1 = \sqrt{0.5}, x_2 = 0.5)$,
(4) V_{13} is the line $(x_1 = -\sqrt{0.5}, x_2 = 0.5)$,
(5) V_{14} is the twisted cubic $(x_2 = x_1^2, x_3 = x_1^3)$,
(6) V_{01} is the point $(x_1 = 0.5, x_2 = 0.5, x_3 = 0.5)$.

Solving $\mathcal{E}_3(x_1, x_2, x_3, z_1, z_2, z_3) = 0$ in Step 1 of the cascade procedure by the polyhedral homotopy method given in Section 4, yields 139 isolated zeros which constitute Z_3. By following 139 paths of

$$H_3 = (1-t)\mathcal{E}_3 + t \begin{pmatrix} \mathcal{E}_2 \\ z_3 \end{pmatrix} = 0$$

starting from points in Z_3 obtained in Step 2, we reach 2 solutions in W_2 consisting of solutions with $z_1 = z_2 = z_3 = 0$, and 38 solutions in Z_2 consisting of solutions with

$z_1 z_2 \neq 0$. The rest of the paths, 99 of them, all went to infinity. Samples by continuation from the first point in W_2 were found to be interpolated by a quadratic surface (the sphere) and the second point was found to also fall on the sphere. Thus, component V_{21} is determined to be a second-degree variety. The sphere equation is appended to the filter, and the algorithm proceeds to the next level.

H_3	Z_3	Z_2	W_2
	139	38	2

W_2	Sphere
2	2

By following 38 paths of the homotopy

$$H_2 = (1-t)\mathcal{E}_2 + t\begin{pmatrix}\mathcal{E}_1\\ z_2\end{pmatrix} = 0$$

starting from points in Z_2 we obtain 14 solutions in W_1 and 20 solutions in Z_1. Among 14 solutions in W_1, 8 of them are found to lie on the sphere and are discarded as J_1. Using the sample and interpolate procedures, the remaining 6 are classified as 3 falling on 3 lines, one on each, and 3 on a cubic. A filtering polynomial for each of these is appended to the filter and the algorithm proceeds to the last level.

H_2	Z_2	Z_1	W_1
	38	20	14

W_1	J_1	Line 1	Line 2	Line 3	Cubic
14	8	1	1	1	3

By following 20 paths of the homotopy

$$H_1 = (1-t)\mathcal{E}_1 + t\begin{pmatrix}\mathcal{E}_0 = P(x_1, x_2, x_3)\\ z_1\end{pmatrix} = 0$$

starting from points in Z_1 yields 19 solutions in W_0. Among them, 13 lie on the sphere, 2 on line 1, 2 on line 2 and 1 on line 3, leaving a single isolated point as W_{01}.

H_1	Z_1	Z_0	W_0
	20	0	19

W_0	J_0				W_{10}
	Sphere	Line 1	Line 2	Line 3	
19	13	2	2	1	1

9. Numerical considerations

9.1. Fundamental procedure for following paths

There are well established numerical techniques to track homotopy paths of a homotopy $H(x,t) = 0$, see ALLGOWER and GEORG [1990], ALLGOWER and GEORG [1993], ALLGOWER and GEORG [1997], in which homotopy paths are usually parametrized by the arc length. However, in the content of solving polynomial systems in $\mathbb{C}^n$ where homotopies $H(x,t)$ are defined on $\mathbb{C}^n \times [0,1]$, we will show below that for any point on the smooth homotopy path $(x(s), t(s))$ of $H(x,t) = 0$, parametrized by the arc length s, dt/ds is always nonzero, and therefore $dt/ds > 0$. Meaning: those paths do not "turn back in t". In other words, they extend across the interval $0 \leqslant t < 1$ and can always be

parametrized by t. Accordingly, standard procedures in tracing general homotopy paths need to be properly adjusted to capitalize this special feature as we will elaborate in this section.

LEMMA 9.1 (CHOW, MALLET-PARET and YORKE [1979]). *Regard the $n \times n$ complex matrix M as a linear transformation of complex variables $(x_1, \ldots, x_n)$ in $\mathbb{C}^n$ into itself. If this transformation is regarded as one on the space $\mathbb{R}^{2n}$ of real variables $(u_1, v_1, \ldots, u_n, v_n)$ where $x_j = u_j + \mathrm{i}v_j$, $j = 1, \ldots, n$, and is represented by the $2n \times 2n$ real matrix N then*

$$\det N = \left|\det M\right|^2 \geqslant 0$$

and

$$\dim_R(\text{kernel } N) = 2 \times \dim_C(\text{kernel } M).$$

Here, $\dim_R$ *and* $\dim_{\mathbb{C}}$ *refer to real and complex dimension.*

PROOF. The relation between M and N is the following: if the (j,k)-entry of M is the complex number $m_{jk} = \xi_{jk} + \mathrm{i}\eta_{jk}$, and N is written in block form as an $n \times n$ array of 2×2 blocks, then the (j,k)-block of N is the real matrix

$$\begin{pmatrix} \xi_{jk} & -\eta_{jk} \\ \eta_{jk} & \xi_{jk} \end{pmatrix}.$$

Denoting this relation by $\alpha(M) = N$, we have $\alpha(AB) = \alpha(A)\alpha(B)$ for complex matrices A and B, and $\alpha(A^{-1}) = \alpha(A)^{-1}$.

Now when M is upper triangular, the assertion is clear. For general M, there exists complex nonsingular matrix A for which $A^{-1}MA$ is upper triangular. Because

$$\alpha(A^{-1}MA) = \alpha(A^{-1})\alpha(M)\alpha(A) = \alpha(A)^{-1}N\alpha(A),$$

it follows that

$$\det\big(\alpha(A^{-1}MA)\big) = \det\big(\alpha(A)\big)^{-1} \times \det N \times \det\big(\alpha(A)\big) = \det N.$$

The assertion holds, since

$$\det\big(\alpha(A^{-1}MA)\big) = \left|\det(A^{-1}MA)\right|^2 = \left|\det M\right|^2. \qquad \square$$

PROPOSITION 9.1. *If (x_0, t_0) is a point on any smooth homotopy paths $(x(s), t(s))$ of the homotopy $H(x,t) = 0$ defined on $\mathbb{C}^n \times [0,1]$ with $t_0 \in [0,1)$, then $H_x(x_0, t_0)$ is nonsingular. Hence, $dt/ds \neq 0$ at (x_0, t_0).*

PROOF. Regard H as a map from $\mathbb{R}^{2n} \times \mathbb{R}$ to $\mathbb{R}^{2n}$. Since the $2n \times (2n+1)$ Jacobian matrix $DH = [H_x, H_t]$ must be of full rank at (x_0, t_0) (otherwise it would be a bifurcation point (ALLGOWER [1984])), its kernel is at most one-dimensional. By the above lemma, the matrix H_x must have zero kernel, so it is nonsingular. Hence, $dt/ds \neq 0$ at

(x_0, t_0), because

$$H_x \frac{dx}{ds} + H_t \frac{dt}{ds} = 0. \qquad \square$$

Algorithms for following homotopy paths vary but are typically of the *predictor–corrector* form, in which the next point on the path is "predicted" by some easy but relatively inaccurate means, and then a series of Newton-like "corrector" iterations is employed to return approximately to the path.

Since homotopy paths of the homotopy $H(x,t) = 0$ in $\mathbb{C}^n \times [0,1]$ can always be parametrized by t, let $x(t)$ be a path in $\mathbb{C}^n$ satisfying the homotopy equation $H(x,t) = 0$, that is,

$$H\big(x(t), t\big) = 0, \quad 0 \leqslant t \leqslant 1. \tag{9.1}$$

From here on we shall denote dx/dt by $x'(t)$. Now, differentiating (9.1) with respect to t, yields

$$H_x x'(t) + H_t = 0, \quad 0 \leqslant t \leqslant 1,$$

or

$$x'(t) = -H_x^{-1} H_t, \quad 0 \leqslant t \leqslant 1. \tag{9.2}$$

For a fixed $0 \leqslant t_0 \leqslant 1$, to proceed from the point $x(t_0)$ already attained on $x(t)$, one takes the following fundamental steps:

Step 1: Euler Prediction:

For an adaptive step size $\delta > 0$, let $t_1 = t_0 + \delta < 1$ and

$$\tilde{x}(t_1) = x(t_0) + \delta x'(t_0). \tag{9.3}$$

Step 2: Newton's Correction:

For fixed t_1, $H(x, t_1) = 0$ becomes a system of n equations in n unknowns. So, Newton's iteration can be employed to solve the solution of $H(x, t_1) = 0$ with starting point $\tilde{x}(t_1)$, i.e.

$$x^{(m+1)} = x^{(m)} - \big[H_x(x^{(m)}, t_1)\big]^{-1} H\big(x^{(m)}, t_1\big), \quad m = 0, 1, \ldots, \tag{9.4}$$

with $x^{(0)} = \tilde{x}(t_1)$. When the iteration fails to converge, Step 1 will be repeat with $\delta \leftarrow \frac{\delta}{2}$. Eventually, an approximate value of $x(t_1)$ can be obtained.

For the efficiency of the algorithm, one seldom stays with a predetermined and fixed step size in practice. Based on the smoothness of the path $x(t)$ around t_0, there are different strategies of choosing step size $\delta > 0$ in Step 1, and of course the smoother the path is, the larger the step size can be adapted. An effective tool to measure the smoothness of $x(t)$ at t_0 is the angle between two consecutive tangent vectors $x'(t_{-1})$ and $x'(t_0)$, where $x(t_{-1})$ is the previous point on the path. For the prediction in Step 1, there are several alternatives for the Euler prediction in (9.3), which are more efficient empirically:

- *The cubic Hermite interpolation*
  For $x(t) = (x_1(t), \ldots, x_n(t))$ and $j = 1, \ldots, n$, let $\tilde{x}_j(t)$ be the cubic polynomial which interpolates $x_j(t)$ and $x_j'(t)$ at $t = t_{-1}$ and $t = t_0$. Namely,

  $$\tilde{x}_j(t_{-1}) = x_j(t_{-1}), \quad \tilde{x}_j(t_0) = x_j(t_0)$$

  and

  $$\tilde{x}_j'(t_{-1}) = x_j'(t_{-1}), \quad \tilde{x}_j'(t_0) = x_j'(t_0).$$

  Writing $\tilde{x}(t) = (\tilde{x}_1(t), \ldots, \tilde{x}_n(t))$, the value $\tilde{x}(t_1)$ will be taken as the prediction of $x(t)$ at $t = t_1$.

  This method usually provides more accurate prediction for $x(t_1)$ with no extra computational cost.
- *The cubic interpolation*
  Let $x(t_{-3}), x(t_{-2}), x(t_{-1})$ be three consecutive points immediately previous to $x(t_0)$ on $x(t)$. For $j = 1, \ldots, n$, let $\tilde{x}_j(t)$ be the cubic polynomial which interpolates $x_j(t)$ at $t = t_{-3}, t_{-2}, t_{-1}$ and t_0. With $\tilde{x}(t) = (\tilde{x}_1(t), \ldots, \tilde{x}_n(t))$, naturally we let $\tilde{x}(t_1)$ be the predicted value of $x(t)$ at $t = t_1$.

  To start this method of prediction, one may use some other means to find the beginning four points on $x(t)$, starting at $t = 0$, such as the Euler method in (9.3). The most important advantage of this interpolation is the absence of the derivative computation in the prediction steps all along the path following (except, perhaps, at the first few points), which may sometimes be very costly.

9.2. Projective coordinates and the projective Newton method

Solution paths of $H(x, t) = 0$ that do not proceed to a solution of the target polynomial equation $P(x) = 0$ in $\mathbb{C}^n$ diverge to infinity: a very poor state of affairs for numerical methods. However, there is a simple idea from classical mathematics which improves the situation. If the system $H(x, t)$ is viewed in the projective space $\mathbb{P}^n$, the diverging paths are simply proceeding to a "point at infinity". Since projective space is compact, we can force all paths, including the extraneous ones, to have finite length by using projective coordinates.

For $P(x) = (p_1(x), \ldots, p_n(x)) = 0$ and $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, we first homogenize the $p_j(x)$'s, that is, for $j = 1, \ldots, n$,

$$\tilde{p}_j(x_0, \ldots, x_n) = x_0^{d_j} p_j\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right), \quad d_j = \text{degree of } p_j(x).$$

Then consider the system of $n + 1$ equations in $n + 1$ unknowns

$$\tilde{P}: \begin{cases} \tilde{p}_1(x_0, \ldots, x_n) & = 0, \\ & \vdots \\ \tilde{p}_n(x_0, \ldots, x_n) & = 0, \\ a_0 x_0 + \cdots + a_n x_n - 1 & = 0, \end{cases}$$

where $a_0, \ldots, a_n$ are generically chosen complex numbers. In short, we augment the homogenization of $P(x)$ with a generic hyperplane

$$a_0x_0 + \cdots + a_nx_n - 1 = 0.$$

When a start system

$$Q(x) = \big(q_1(x), \ldots, q_n(x)\big) = 0$$

is chosen, we augment its homogenization with the same hyperplane,

$$\widetilde{Q}\colon \begin{cases} \tilde{q}_1(x_0, \ldots, x_n) & = 0, \\ & \vdots \\ \tilde{q}_n(x_0, \ldots, x_n) & = 0, \\ a_0x_0 + \cdots + a_nx_n - 1 & = 0. \end{cases}$$

When the classical linear homotopy continuation procedure is used to follow all the solution paths of the homotopy

$$\widetilde{H}(x_0, x, t) = (1-t)c\widetilde{Q}(x_0, x) + t\widetilde{P}(x_0, x) = 0, \quad c \in \mathbb{C}^* \text{ is generic,}$$

the paths stay in $\mathbb{C}^{n+1}$ for almost all choices of $a_0, \ldots, a_n$. It only remains to ignore solutions with $x_0 = 0$ when t reaches 1. For the remaining solutions with $x_0 \neq 0$, $x = (x_1/x_0, \ldots, x_n/x_0)$ are the corresponding solutions of $P(x) = 0$ in $\mathbb{C}^n$.

A similar technique, called *projective transformation* is described in MORGAN and SOMMESE [1987a]. It differs from the above in the following way. Instead of increasing the size of the problem from $n \times n$ to $(n+1) \times (n+1)$, they implicitly consider solving the last equation for x_0 and substituting it in the other equations, essentially retaining n equations in n unknowns. Then the chain rule is used for the Jacobian calculations needed for path following.

A more advanced technique, called the *projective Newton method*, was suggested in BLUM, CUCKER, SHUB and SMALE [1998] and SHUB and SMALE [1993]. For fixed $0 < t_0 < 1$, the homogenization of the homotopy equation $H(x, t) = 0$ becomes $\widetilde{H}(\tilde{x}, t_0) = 0$ with $\tilde{x} = (x_0, x_1, \ldots, x_n)$. It is a system of n equations in $n+1$ variables. Instead of following solution paths of $\widetilde{H}(\tilde{x}, t) = 0$ in $\mathbb{C}^{n+1}$ with an additional *fixed* hyperplane $a_0x_0 + \cdots + a_nx_n - 1 = 0$ as described above, the new strategy follows the paths in $\mathbb{C}^{n+1}$ with variant hyperplanes. Without hyperplane $a_0x_0 + \cdots + a_nx_n - 1 = 0$, Newton's iteration is unsuitable for approximating the solution of an $n \times (n+1)$ system $\widetilde{H}(\tilde{x}, t_0) = 0$ in the fundamental correction step after the prediction. However, for any nonzero constant $c \in \mathbb{C}$, $\tilde{x}$ and $c\tilde{x}$ in $\mathbb{C}^{n+1}$ are considered to be equal in projective space $\mathbb{P}^n$, whence the magnitude of $\tilde{x}$ in $\mathbb{C}^{n+1}$ is no longer significant in $\mathbb{P}^n$. Therefore it is reasonable to *project* every step of Newton's iteration onto the hyperplane that is perpendicular to the current point in $\mathbb{C}^{n+1}$. More precisely, to approximate the solutions of $\widetilde{H}(\tilde{x}, t_0) = 0$ for fixed $0 < t_0 < 1$, at a point $\tilde{x}^{(m)} \in \mathbb{C}^{n+1}$ during the correction process, we augment the equation $\widetilde{H}(\tilde{x}, t_0) = 0$ with the hyperplane $(\tilde{x} - \tilde{x}^{(m)}) \cdot \tilde{x}^{(m)} = 0$ to form an $(n+1) \times (n+1)$ square system

$$\overline{H}(\tilde{x}, t_0) = \begin{cases} \widetilde{H}(\tilde{x}, t_0) = 0, \\ (\tilde{x} - \tilde{x}^{(m)}) \cdot \tilde{x}^{(m)} = 0. \end{cases} \tag{9.5}$$

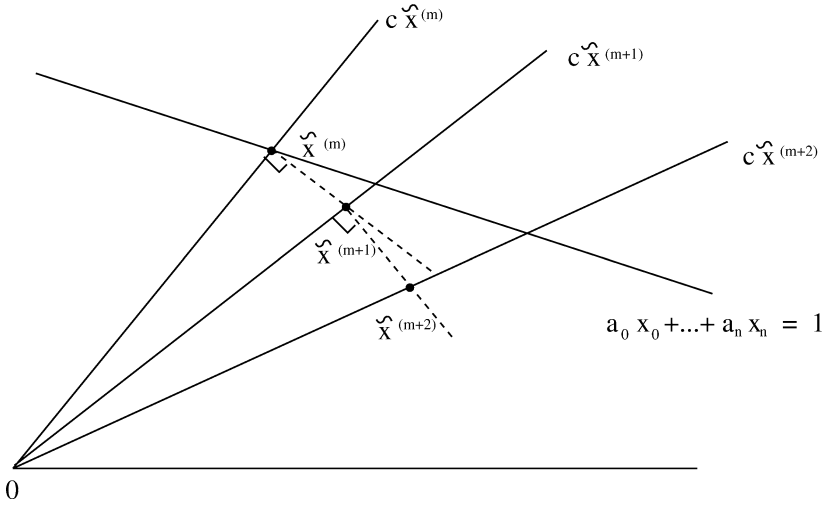


FIG. 9.1.

One step of Newton's iteration applied to this system at $\tilde{x}^{(m)}$ yields

$$\tilde{x}^{(m+1)} = \tilde{x}^{(m)} - \left[\overline{H}_{\tilde{x}}(\tilde{x}^{(m)}, t_0)\right]^{-1} \overline{H}\left(\tilde{x}^{(m)}, t_0\right). \tag{9.6}$$

The iteration is continued by replacing $\tilde{x}^{(m)}$ in (9.5) with $\tilde{x}^{(m+1)}$ obtained in (9.6) until $\widetilde{H}(\tilde{x}^{(m)}, t_0)$ becomes sufficiently small.

The efficiency of this strategy when applied to following the homotopy paths in $\mathbb{P}^n$, is intuitively clear. See Fig. 9.1. It frequently allows a bigger step size at the prediction step.

9.3. Balancing the lifting values in polyhedral homotopies

Polyhedral homotopies are nonlinear in the continuation parameter t. Powers of the continuation parameter too close to zero can be scaled away from zero by a suitable scalar multiplication. After scaling, if very high powers exist in a polyhedral homotopy, small step sizes must be taken in order to successfully trace a solution path of the polyhedral homotopy. Although the end of the solution path can be reached as long as the required step size is not smaller than the available machine precision, the efficiency of the path-tracing is greatly reduced. A more serious problem occurs when the continuation parameter is not yet close enough to 1, some terms of the polyhedral homotopy with high powers of t have values smaller than the machine precision and some solution curves may come close to "valleys" where the values of the homotopy are numerically zero, but no solution paths exist inside the "valleys". This situation can easily cause the path-tracings to be trapped in those "valleys" with no chance of reaching the ends of solution paths unless the paths are retraced with smaller step sizes.

For instance, consider the polynomial system $P(x) = (p_1(x), p_2(x)) = 0$ where $x = (x_1, x_2)$, and

$$p_1(x) = c_1x_1^2 + c_2x_2^2 + c_3 = 0,$$
$$p_2(x) = c_4x_1^3x_2^3 + c_5x_1 + c_6x_2 = 0$$

with support $S_1 = \{(2,0),(0,2),(0,0)\}$ and $S_2 = \{(3,3),(1,0),(0,1)\}$. By a simple calculation, the mixed volume $\mathcal{M}(S_1, S_2)$ of this system is 12. We choose a generic lifting $\omega = (\omega_1, \omega_2)$ where $\omega_1 : S_1 \to \mathbb{R}$ and $\omega_2 : S_2 \to \mathbb{R}$ with

$$\omega_1(2,0) = 0.655416, \quad \omega_1(0,2) = 0.200995, \quad \omega_1(0,0) = 0.893622$$

and

$$\omega_2(3,3) = 0.281886, \quad \omega_2(1,0) = 0.525000, \quad \omega_2(0,1) = 0.314127.$$

Then the mixed-cell configuration $\mathcal{M}_\omega$ in the fine mixed subdivision S_ω of S induced by ω consists of two mixed cells:

$$C^{(1)} = \big(\{(2,0),(0,2)\},\{(3,3),(1,0)\}\big)$$

and

$$C^{(2)} = \big(\{(0,2),(0,0)\},\{(1,0),(0,1)\}\big).$$

To construct a system $G(x) = (g_1(x), g_2(x)) = 0$ in general position with the same support $S = (S_1, S_2)$, we choose the following set of randomly generated coefficients,

$$c_1' = -0.434847 - 0.169859\mathrm{i}, \qquad c_2' = 0.505911 + 0.405431\mathrm{i},$$
$$c_3' = 0.0738596 + 0.177798\mathrm{i}, \qquad c_4' = -0.0906755 + 0.208825\mathrm{i},$$
$$c_5' = 0.175313 - 0.163549\mathrm{i}, \qquad c_6' = 0.527922 - 0.364841\mathrm{i}.$$

The polyhedral homotopy induced by the cell $C^{(1)}$, following the procedure given in Section 4, is $\overline{G}(y,t) = (\bar{g}_1(y,t), \bar{g}_2(y,t)) = 0$, where

$$\bar{g}_1(y,t) = c_1'y_1^2 + c_2'y_2^2 + c_3't^{50.63523},$$
$$\bar{g}_2(y,t) = c_4'y_1^3y_2^3 + c_5'y_1 + c_6'y_2t^2.$$

It is easy to see that $\mathrm{Vol}_2(C^{(1)}) = 10$. Therefore, there are ten solution paths of $\overline{G}(y,t) = 0$ emanating from the ten solutions of $\overline{G}(y,0) = 0$. At $t = 0.65$, five of those ten paths have phase space tangent vectors $(dy_1/dt, dy_2/dt)$ all pointing closely to $y = (0,0)$ at the points on the curves. For the standard prediction–correction method, starting at these points, the prediction step with step size 0.025 yield the predicted points close to $(y,t) = (0,0,0.675)$ for all those five paths. Since $t^{50.63523}$ is about 10^{-9} for $t = 0.675$, the function values of $\overline{G}(y,t)$ in a very small neighborhood of $(0,0,0.675)$, the "valley", are almost zero. Starting from these predicted points at $t = 0.675$, Newton's iterations for the correction step will converge to $(0,0)$, center of the "valley", rather than the points on the paths at $t = 0.675$. But there are no solution paths of $\overline{G}(y,t) = 0$ passing through the "valley". This shows that generic liftings may induce highly nonlinear polyhedral homotopies which may produce numerical instabilities.

Two known geometric approaches to control the numerical stability of polyhedral homotopy continuation methods are recursive liftings as in VERSCHELDE, VERLINDEN and COOLS [1994] and dynamic liftings in VERSCHELDE, GATERMANN and COOLS [1996] and VERSCHELDE [1996]. However, because of using multiple liftings or flattenings, these approaches both require more expensive construction of subdivisions and create more homotopy curves which need to be traced than a random floating-point lifting.

To minimize the height of powers of the continuation parameter t in polyhedral homotopies, we search in the cone of all lifting vectors that induce the same mixed-cell configuration to obtain better-balanced powers of the continuation parameter of polyhedral homotopies. As given in Section 4, the first step of the polyhedral homotopy procedure to find all isolated zeros of a polynomial system $P(x) = (p_1(x), \ldots, p_n(x))$ with $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ and support $S = (S_1, \ldots, S_n)$ where

$$p_j(x) = \sum_{a \in S_j} c_{j,a} x^a, \quad j = 1, \ldots, n,$$

is to solve a generic polynomial system $G(x) = (g_1(x), \ldots, g_n(x)) = 0$ with support $S' = (S'_1, \ldots, S'_n)$ where $S'_j = S_j \cup \{0\}$ for $j = 1, \ldots, n$, but with randomly chosen coefficients $c'_{j,a}$, namely,

$$g_j(x) = \sum_{a \in S'_j} c'_{j,a} x^a, \quad j = 1, \ldots, n.$$

Secondly, the system $G(x) = 0$ is used as the start system in the linear homotopy

$$H(x, t) = (1 - t)cG(x) + tP(x) = 0, \quad c \in \mathbb{C}^* \text{ is generic,}$$

to solve $P(x) = 0$.

To find all isolated zeros of $G(x) = (g_1(x), \ldots, g_n(x))$, we first construct a nonlinear homotopy $\widehat{G}(x, t) = (\hat{g}_1(x, t), \ldots, \hat{g}_n(x, t)) = 0$ where

$$\hat{g}_j(x, t) = \sum_{a \in S'_j} c'_{j,a} x^a t^{\omega_j(a)}, \quad j = 1, \ldots, n, \tag{9.7}$$

and the powers $\omega_j(a)$ of t are determined by the generic *liftings* $\omega_j : S'_j \to \mathbb{R}$ on the support S'_j, $j = 1, \ldots, n$, followed by calculating all mixed cells of the fine mixed subdivision S_ω induced by the lifting $\omega = (\omega_1, \ldots, \omega_n)$. Let $\mathcal{M}_\omega$ denote the set of all those mixed cells and call it the *mixed-cell configuration* of the support $S' = (S'_1, \ldots, S'_n)$. For a mixed cell $(\{a_1, a'_1\}, \ldots, \{a_n, a'_n\}) \in \mathcal{M}_\omega$ with inner normal $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, substituting the coordinate transformation $y = xt^\alpha$ where $y_j = x_j t^{\alpha_j}$, $j = 1, \ldots, n$ into system (9.7), and factoring out the lowest powers of t, we obtain the homotopy $H^\alpha(y, t) = (h_1^\alpha(y, t), \ldots, h_n^\alpha(y, t)) = 0$ defined on $(\mathbb{C}^*)^n \times [0, 1]$ where

$$h_j^\alpha(y, t) = c'_{j,a_j} y^{a_j} + c'_{j,a'_j} y^{a'_j} + \sum_{a \in S'_j \setminus \{a_j, a'_j\}} c'_{j,a} y^a t^{e_a^\omega}, \quad j = 1, \ldots, n,$$

and e_a^ω, the powers of t, are given by

$$e_a^\omega := \langle a, \alpha\rangle - \langle a_j, \alpha\rangle + \omega_j(a) - \omega_j(a_j), \quad \forall a \in S_j'\backslash\{a_j, a_j'\}.$$

To reduce the power of t for numerical stability, the strategy suggested in GAO, LI, VERSCHELDE and WU [2000] is to find a new lifting function $\nu = (\nu_1, \ldots, \nu_n)$ on $S' = (S_1', \ldots, S_n')$ based on the *already computed* mixed-cell configuration $\mathcal{M}_\omega$. The mixed-cell configuration $\mathcal{M}_\nu$ induced by the new lifting ν will be the same as $\mathcal{M}_\omega$, but the highest power of the continuation parameter t in the polyhedral homotopies induced by $\nu = (\nu_1, \ldots, \nu_n)$ is the smallest among all those polyhedral homotopies induced by the liftings having the *same* mixed-cell configuration $\mathcal{M}_\omega$. In this way, very time-consuming procedure for re-identifying the mixed-cell configuration $\mathcal{M}_\nu$ becomes unnecessary.

Let $C = (C_1, \ldots, C_n) \in \mathcal{M}_\omega$ where $C_j = \{a_j, a_j'\} \subset S_j$ for $j = 1, \ldots, n$. To keep $\mathcal{M}_\omega$ invariant, we impose on any new lifting function $\nu = (\nu_1, \ldots, \nu_n)$ the conditions:

$$\begin{aligned}
&\big\langle\big(a_j, \nu_j(a_j)\big), (\gamma, 1)\big\rangle = \big\langle\big(a_j', \nu_j(a_j')\big), (\gamma, 1)\big\rangle,\\
&\big\langle\big(a_j, \nu_j(a_j)\big), (\gamma, 1)\big\rangle < \big\langle\big(a, \nu_j(a)\big), (\gamma, 1)\big\rangle, \quad \forall a \in S_j'\backslash\{a_j, a_j'\},\ j = 1, \ldots, n,
\end{aligned}$$

or,

$$\langle a_j, \gamma\rangle + \nu_j(a_j) = \langle a_j', \gamma\rangle + \nu_j(a_j'), \tag{9.8}$$

$$\langle a_j, \gamma\rangle + \nu_j(a_j) < \langle a, \gamma\rangle + \nu_j(a), \quad \forall a \in S_j\backslash\{a_i, a_j'\},\ j = 1, \ldots, n, \tag{9.9}$$

where γ is the inner normal of C in $\mathcal{M}_\nu$. Since $C = (C_1, \ldots, C_n)$ is a mixed cell in the fine mixed subdivision S_ω, the edges spanned by $\{a_j, a_j'\}$ determine a full-dimensional parallelepiped in $\mathbb{R}^n$, that is, the matrix

$$A = \begin{bmatrix} a_1 - a_1' \\ \vdots \\ a_n - a_n' \end{bmatrix}$$

is nonsingular. So, γ can be expressed uniquely as a linear combination of the lifting values $\nu_j(a_j)$ and $\nu_j(a_j')$ for $j = 1, \ldots, n$. More explicitly, from (9.8), we have

$$\langle a_j - a_j', \gamma\rangle = \nu_j(a_j') - \nu_j(a_j), \quad j = 1, \ldots, n,$$

therefore

$$\gamma^{\mathrm{T}} = A^{-1} \begin{bmatrix} \nu_1(a_1') - \nu_1(a_1) \\ \vdots \\ \nu_n(a_n') - \nu_n(a_n) \end{bmatrix}. \tag{9.10}$$

The polyhedral homotopy induced by the lifting $\nu = (\nu_1, \ldots, \nu_n)$ and mixed cell $C = (\{a_1, a_1'\}, \ldots, \{a_n, a_n'\})$ with inner normal γ is

$$\bar{h}_j^\gamma(y, t) = c_{j,a_j}' y^{a_j} + c_{j,a_j'}' y^{a_j'} + \sum_{a \in S_j'\backslash\{a_j, a_j'\}} c_{j,a}' y^a t^{e_a^\gamma}, \quad j = 1, \ldots, n, \tag{9.11}$$

where e_a^γ, the powers of t, are given by

$$e_a^\gamma := \langle a, \gamma\rangle - \langle a_j, \gamma\rangle + \nu_j(a) - \nu_j(a_j), \quad \forall a \in S_j'\backslash\{a_j, a_j'\}.$$

They are always positive by (9.9). By (9.10), γ may be removed from e_a^γ. We denote the resulting expressions by e_a. Explicitly, $\forall a \in S_j'\backslash\{a_j, a_j'\}$,

$$e_a := (a - a_j)A^{-1}\begin{bmatrix} \nu_1(a_1') - \nu_1(a_1) \\ \vdots \\ \nu_n(a_n') - \nu_n(a_n) \end{bmatrix} + \nu_j(a) - \nu_j(a_j'). \tag{9.12}$$

To avoid the powers e_a being too large or too small, it is natural to consider the following minimization problem:

$$\begin{aligned} &\text{Minimize } M \\ &1 \leqslant e_a \leqslant M, \ e_a \text{ is given in (9.12)} \\ &\forall C = \big(\{a_1, a_1'\}, \ldots, \{a_n, a_n'\}\big) \in \mathcal{M}_\omega, \quad \text{and} \\ &\forall a \in S_j'\backslash\{a_j, a_j'\}, \quad j = 1, \ldots, n. \end{aligned} \tag{9.13}$$

Clearly, the lifting function $\nu = (\nu_1, \ldots, \nu_n)$ having values obtained by the solution of this minimization problem satisfies (9.8) and (9.9) with γ defined in (9.10). Therefore, the mixed-cell configuration $\mathcal{M}_\nu$ induced by ν coincides with $\mathcal{M}_\omega$. Moreover, the powers of the continuation parameter t in the polyhedral homotopies induced by ν are much better balanced.

The minimization problem (9.13) has $1 + \sum_{j=1}^n \#S_j'$ unknowns; they are: M as well as $\nu_j(a)$ for $a \in S_j'$, $j = 1, \ldots, n$. It has $2(\#\mathcal{M}_\omega)\sum_{j=1}^n(\#S_j' - 2)$ inequalities. For practical considerations, we wish to reduce both the number of unknowns and the number of inequalities. We will show below that for a fixed mixed cell $\overline{C} = (\{\bar a_1, \bar a_1'\}, \ldots, \{\bar a_n, \bar a_n'\})$, where $\{\bar a_j, \bar a_i'\} \subset S_j'$, $j = 1, \ldots, n$, in the mixed-cell configuration $\mathcal{M}_\omega$ with inner normal β, we may set $\nu_j(\bar a_j)$ and $\nu_j(\bar a_j)$ to be zero for $j = 1, \ldots, n$ in (9.13), so the number of unknowns is reduced by $2n$. But the lifting function $\nu' = (\nu_1', \ldots, \nu_n')$ defined by the solution of the new minimization problem still induces the same mixed-cell configuration $\mathcal{M}_\omega$.

For a fixed mixed cell $\overline{C} = (\{\bar a_1, \bar a_1'\}, \ldots, \{\bar a_n, \bar a_n'\}) \in \mathcal{M}_\omega$ with inner normal β, we define a lifting function $\omega' = (\omega_1', \ldots, \omega_n')$ as follows: for $j = 1, \ldots, n$ and $a \in S_j'$,

$$\omega_j'(a) := \langle a, \beta\rangle - \langle \bar a_j, \beta\rangle + \omega_j(a) - \omega_j(\bar a_j). \tag{9.14}$$

Then ω_j' vanishes at both $\bar a_j$ and $\bar a_j'$. Let $\mathcal{M}_{\omega'}$ be the mixed-cell configuration in the subdivision $S_{\omega'}$ of $S' = (S_1', \ldots, S_n')$ induced by ω'.

LEMMA 9.2. *$\mathcal{M}_{\omega'} = \mathcal{M}_\omega$. More precisely, $C = (C_1, \ldots, C_n) \in \mathcal{M}_\omega$ with inner normal α with respect to ω if and only if $C = (C_1, \ldots, C_n) \in \mathcal{M}_{\omega'}$ with inner normal $\alpha - \beta$ with respect to ω'.*

PROOF. Let $C_j = \{a_j, a'_j\} \subset S'_j$ for $j = 1, \ldots, n$. Then, $C \in \mathcal{M}_\omega$ with inner normal

$$\alpha \quad \Leftrightarrow \quad \begin{cases} \langle (a_j, \omega_j(a_j)), (\alpha, 1)\rangle = \langle (a'_j, \omega_j(a'_j)), (\alpha, 1)\rangle, \\ \langle (a_j, \omega_j(a_j)), (\alpha, 1)\rangle < \langle (a, \omega_j(a)), (\alpha, 1)\rangle, \\ \quad \forall a \in S'_j \backslash \{a_j, a'_j\}, \ j = 1, \ldots, n. \end{cases}$$

Or,

$$\begin{cases} \langle a_j - a'_j, \alpha\rangle = \omega_j(a'_j) - \omega_j(a_j), \\ \langle a - a_j, \alpha\rangle > \omega_j(a_j) - \omega_j(a), \quad \forall a \in S'_j \backslash \{a_j, a'_j\}, \ j = 1, \ldots, n. \end{cases}$$

On the other hand, $C \in \mathcal{M}_{\omega'}$ with inner normal

$$\alpha - \beta \quad \Leftrightarrow \quad \begin{cases} \langle (a_j, \omega'_j(a_j)), (\alpha - \beta, 1)\rangle = \langle (a'_j, \omega'_j(a'_j)), (\alpha - \beta, 1)\rangle, \\ \langle (a_j, \omega'_j(a_j)), (\alpha - \beta, 1)\rangle < \langle (a, \omega'_j(a)), (\alpha - \beta, 1)\rangle, \\ \quad \forall a \in S_j \backslash \{a_j, a'_j\}, \ j = 1, \ldots, n. \end{cases}$$

By (9.14),

$$\begin{aligned} \langle a_j - a'_j, \alpha - \beta\rangle &= \omega'_j(a'_j) - \omega'_j(a_j) \\ &= \langle a'_j, \beta\rangle - \langle \bar{a}_j, \beta\rangle + \omega_j(a'_j) - \omega_j(\bar{a}_j) \\ &\quad - \big(\langle a_j, \beta\rangle - \langle \bar{a}_j, \beta\rangle + \omega_j(a_j) - \omega_j(\bar{a}_j)\big) \\ &= \langle a_j - a'_j, -\beta\rangle + \omega_j(a'_j) - \omega_j(a_j), \quad \text{i.e.} \\ \langle a_j - a'_j, \alpha\rangle &= \omega_j(a'_j) - \omega_j(a_j), \quad j = 1, \ldots, n. \end{aligned}$$

And, for $a \in S'_j \backslash \{a_j, a'_j\}$,

$$\begin{aligned} \langle a - a_j, \alpha - \beta\rangle &> \omega'_j(a_j) - \omega'_j(a) \\ &= \langle a_j, \beta\rangle - \langle \bar{a}_j, \beta\rangle + \omega_j(a_j) - \omega_j(\bar{a}_j) \\ &\quad - \big(\langle a, \beta\rangle - \langle \bar{a}_j, \beta\rangle + \omega_j(a) - \omega_j(\bar{a}_j)\big) \\ &= \langle a - a_j, -\beta\rangle + \omega_j(a_j) - \omega_j(a), \quad \text{i.e.} \\ \langle a - a_j, \alpha\rangle &> \omega_j(a_j) - \omega_j(a), \quad j = 1, \ldots, n. \end{aligned}$$

The proof is completed. □

Most importantly, a straightforward calculation shows that the polyhedral homotopy, as in (9.11), induced by the cell $C = (C_1, \ldots, C_n)$ in $\mathcal{M}_\omega$ with inner normal α is exactly the same as the one induced by cell $C = (C_1, \ldots, C_n)$ in $\mathcal{M}_{\omega'}$ with inner normal $\alpha - \beta$. So, we may solve the generic polynomial system $G(x) = 0$ by using the polyhedral homotopies induced by mixed cells in $\mathcal{M}_{\omega'}$ along with their corresponding inner normals.

Now, with lifting ω', we consider the minimization problem

$$\begin{aligned}&\text{Minimize } M\\ &1 \leqslant \langle a - a_j, \gamma\rangle + \nu_j(a) - \nu_j(a_j) \leqslant M,\\ &\forall C = (\{a_1, a_1'\}, \ldots, \{a_n, a_n'\}) \in \mathcal{M}_{\omega'},\ \forall a \in S_j \backslash \{a_j, a_j'\},\\ &\nu_j(\bar{a}_j) = \nu_j(\bar{a}_j') = 0, \quad j = 1, \ldots, n.\end{aligned} \tag{9.15}$$

Here, γ can be expressed, as in (9.12), as a linear combination of the values of ν_j's:

$$\gamma^{\mathrm{T}} = \begin{bmatrix} a_1 - a_1' \\ \vdots \\ a_n - a_n' \end{bmatrix}^{-1} \begin{bmatrix} \nu_1(a_1') - \nu_1(a_1) \\ \vdots \\ \nu_n(a_n') - \nu_n(a_n) \end{bmatrix}.$$

This problem has $2n$ fewer unknowns than the original problem in (9.13), and its solution yields a lifting function $\nu' = (\nu_1', \ldots, \nu_n')$, and the mixed-cell configuration $\mathcal{M}_{\nu'}$ induced by which is the same as $\mathcal{M}_{\omega'}$.

For the feasibility of this LP problem, the values of the lifting function $\omega' = (\omega_1', \ldots, \omega_n')$ clearly satisfy

$$0 < \langle \alpha - a_j, \alpha\rangle + \omega_j'(a) - \omega_j'(a_j), \quad j = 1, \ldots, n, \tag{9.16}$$

for all $C = (\{a_1, a_1'\}, \ldots, \{a_n, a_n'\}) \in \mathcal{M}_{\omega'}$ with corresponding inner normal α and $a \in S_j' \backslash \{a_j, a_j'\}$, where

$$\alpha^{\mathrm{T}} = \begin{bmatrix} a_1 - a_1' \\ \vdots \\ a_n - a_n' \end{bmatrix}^{-1} \begin{bmatrix} \omega_1'(a_1') - \omega_1'(a_1) \\ \vdots \\ \omega_n'(a_n') - \omega_n'(a_n) \end{bmatrix}.$$

Therefore the function values of $\nu^{(l)} := l\omega'$ for $l > 0$ also satisfy (9.16) with α replaced by $l\alpha$. We may choose $l_0 > 0$ for which

$$1 \leqslant \langle a - a_j, l_0\alpha\rangle + \nu_j^{(l_0)}(a) - \nu_j^{(l_0)}(a_j), \quad j = 1, \ldots, n, \tag{9.17}$$

for all $C = (\{a_1, a_1'\}, \ldots, \{a_n, a_n'\}) \in \mathcal{M}_{\omega'}$ and $a \in S_j' \backslash \{a_j, a_j'\}$. This makes $(\nu^{(l_0)}, M_0)$, where M_0 is the maximum of the right-hand side of (9.17), a feasible solution of the constraints in (9.15).

The number of variables in each inequality in the constraints of (9.15) is no greater than $2n + 2$ which is usually much smaller than the total number of variables in (9.15). This sparsity in the constraints can be exploited in the algorithm implementation and results in a remarkable speed-up. Furthermore, a substantial amount of the inequalities in (9.15) are exactly the same, and they can easily be detected by comparisons and deleted when the constraints are being generated.

The algorithm in balancing the powers of t by solving the LP problem (9.15) has been successfully implemented (GAO, LI, VERSCHELDE and WU [2000]). The numerical results of applying the algorithm to several well-known polynomial systems are listed in Tables 9.1 and 9.2.

TABLE 9.1
Sizes of the LP problems. Here, n is the number of variables and $\#\mathcal{M}_\omega$ is the number of mixed cells in the mixed-cell configuration $\mathcal{M}_\omega$.

Polynomial system	n	$\#\mathcal{M}_\omega$	Size of the LP in (9.13)		Size of the LP in (9.15)	
			Number of variables	Number of constraints	Number of variables	Number of constraints
Cohn-2	4	17	31	748	23	690
Cassou–Noguès	4	3	28	114	20	106
Planar 4-bar	4	4	33	192	25	168
Cyclic-6	6	25	33	1000	21	692
Cyclic-7	7	126	45	7560	31	4982
Cyclic-8	8	297	59	24948	43	16118

TABLE 9.2
Height of powers and CPU time in seconds.

Polynomial system	Average highest power of t		Average CPU time	
	Before balancing	After balancing	Finding mixed cells	Balancing method
Cohn-2	1391	85	0.21	0.19
Cassou–Noguès	251	11	0.05	0.03
Planar 4-bar	429	8	0.17	0.08
Cyclic-6	425	31	0.46	0.17
Cyclic-7	3152	139	7.1	1.9
Cyclic-8	10281	398	81	16.6

The data in Table 9.1 are generated by the program with one random lifting function ω for each of the polynomial systems Cohn-2 (MOURRAIN [1996], Cassou–Noguès (TRAVERSO [1997]), Planar 4-bar (MORGAN and WAMPLER [1990]), and cyclic-6, -7, -8 problems (BJORCK and FROBERG [1991]). The fourth and fifth columns give the size of the LP problem in (9.13). The last two columns are the size of the LP problem in (9.15) after all repeated constraints are deleted. For cyclic-n polynomial systems, about $1/3$ of the constraints are deleted.

For the data in Table 9.2, the algorithm was run with ten different real random liftings for each polynomial system. The powers of t was first scaled in the polyhedral homotopies before balancing where the lowest power of t in the homotopies is one, and the average of the highest powers of t in the polyhedral homotopies for the ten random liftings are listed in the second column. The third column lists the average of the highest powers of t in the polyhedral homotopies induced by the ten liftings obtained from the optimal solutions of the corresponding LP problems (9.15). The fourth column gives the average time elapsed for finding all mixed cells. The last column is the average time elapsed for finding the optimal lifting functions ν', including the constructing and solving of the LP problems (9.15). From these results, we see that the highest powers of t in the polyhedral homotopies are substantially reduced. The overall reduced powers of

t in the polyhedral homotopies greatly limit the chance of running into "valleys" which may cause the failure of path-tracings.

9.4. The end game

When we approximate all isolated zeros of the polynomial system $P(x)$ in $\mathbb{C}^n$ by following homotopy paths of a homotopy $H(x,t)=0$ on $\mathbb{C}^n \times [0,1]$, every isolated zero of $P(x)$ lies at the end of some path $x(t)$. However, as we mentioned before, there may be many other paths which do not lead to finite solutions. They are divergent in the sense that some coordinates will become arbitrarily large, leaving us with the problem of deciding if a path is indeed diverging or if it is just converging to a solution with large coordinates.

Let $H:\mathbb{C}^n \times [0,1] \to \mathbb{C}^n$ be a homotopy with $H(x,1)=P(x)$ and $H^{-1}(0)$ consisting of finite many smooth paths $x(t)=(x_1(t),\ldots,x_n(t))$. It was shown in MORGAN, SOMMESE and WAMPLER [1992b] that, in the neighborhood of $t=1$, each path can be written in the form

$$x_j(t) = a_j(1-t)^{w_j/m}\left(1+\sum_{i=1}^{\infty} a_{ij}(1-t)^{i/m}\right), \quad j=1,\ldots,n, \tag{9.18}$$

where m, called the *cyclic number*, is a positive integer and $w=(w_1,\ldots,w_n)\in\mathbb{Z}^n$. Evidently, path $x(t)$ diverges to infinity when $w_j<0$ for some j, and $x(t)\in\mathbb{C}^n$ when $t\to 1$ if $w_j \geqslant 0$ for all $j=1,\ldots,n$.

REMARK 9.1. In MORGAN, SOMMESE and WAMPLER [1992b], only expansions in the form (9.18) of those paths which lead to finite solutions in $\mathbb{C}^n$ were discussed. Of course, $w_j \geqslant 0$ for all $j=1,\ldots,n$ in all those expansions. However, the theory established in MORGAN, SOMMESE and WAMPLER [1992b] can easily be extended to cover the case for diverging paths with $w_j<0$ for some j in those expansions.

To decide if $x(t)=(x_1(t),\ldots,x_n(t))$ leads to a solution of $P(x)=0$ at infinity, one must distinguish the signs of w_j/m for all $j=1,\ldots,n$. If none of them are negative, then $x(t)$ will converge to a finite solution of $P(x)=0$. Those (w_j/m)'s can be estimated as follows. Taking the logarithm of the absolute value of the expression in (9.18) yields, for $j=1,\ldots,n$,

$$\log|x_j(t)| = \log|a_j| + \frac{w_j}{m}\log(1-t) + \sum_{i=1}^{\infty} c_{ij}(1-t)^i, \tag{9.19}$$

where $\sum_{i=1}^{\infty} c_{ij}(1-t)^i$ is the Taylor expansion of $\log(1+\sum_{i=1}^{\infty} a_{ij}(1-t)^{i/m})$. During the continuation process, a sequence of points $x(t_k)$, for $k=0,1,\ldots$ with $t_0<t_1<\cdots<1$ were generated. For two consecutive points t_k and t_{k+1} very close to one, computing their differences of (9.19) yields,

$$\frac{\log|x_j(t_k)| - \log|x_j(t_{k+1})|}{\log(1-t_k)-\log(1-t_{k+1})} = \frac{w_j}{m} + 0(1-t_k). \tag{9.20}$$

We may therefore estimate w_j/m by the value on the left hand side of the above equation. While this estimation is only of order 1, this will not cause difficulties in practice. Because, in theory (MORGAN, SOMMESE and WAMPLER [1992b]), m is not a very big number in general, therefore even lower order estimation is capable of distinguishing w_j/m from 0, especially when t_k and t_{k+1} are very close to 1. Nevertheless, higher order approximation of w_j/m can be found in HUBER and VERSCHELDE [1998].

9.5. Softwares

Industrial-Quality software for solving polynomial systems by homotopy continuation methods was first established by MORGAN [1983]. Later, it appeared in HOMPACK by L.T. Watson et al. (WATSON, BILLUPS and MORGAN [1987], MORGAN, SOMMESE and WATSON [1989], WATSON, SOSONKINA, MELVILLE, MORGAN and WALKER [1997]), in which the polyhedral homotopy methods, emerged in the middle of 90's and important in practice, were not implemented. Polyhedral homotopies exist in the package PHC (VERSCHELDE [1999]) written in *Ada* by J. Verschelde, the code HOM4PS written in *Fortran* developed by T. Gao and T.Y. Li (available at: http://www.math.msu.edu/~li/software) and PHoM written in C++ developed by Gunji, Kim, Kojima, Takeda, Fujisawa and Mizutani (available at: http://www.is.titech.ac.jp/~kojima/polynomials/). The excellent performance of these codes on a large collection of polynomial systems coming from a wide variety of application fields provides practical evidence that the homotopy algorithms constitute a powerful general purpose solver for polynomial equations.

Modern scientific computing is marked by the advent of vector and parallel computers and the search for algorithms that are to a large extent parallel in nature. A great advantage of the homotopy continuation algorithm for solving polynomial systems is that it is to a large degree parallel, in the sense that each isolated zero can be computed independently. In this respect, it stands in contrast to the highly serial algebraic elimination methods, which use resultants or Gröbner bases. Excellent speed-ups of parallel algorithms for symmetric eigenvalue problems, considered as polynomial systems, were reported in HUANG and LI [1995], LI and ZOU [1999] and TREFFTZ, MCKINLEY, LI and ZENG [1995]. The performance of the homotopy algorithms for solving general polynomial systems on multi-processor machines with shared or distributed memory is currently under active investigation (ALLISON, CHAKRABORTY and WATSON [1989], HARIMOTO and WATSON [1989]). One may expect a very high level of speed-up on different types of architectures of those algorithms.

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