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OWA aggregation with an uncertainty over the arguments

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ABSTRACT

We discuss the OWA aggregation operation and the role the OWA weights play in determining the type of aggregation being performed. We introduce the idea of a weight generating function and describe its use in obtaining the OWA weights. We emphasize the importance of the weight generating function in prescribing the type of aggregation to be performed. We consider the problem of performing a prescribed OWA aggregation in the case when we have a probability distribution over the argument values. We show how we use the weight generating function to enable this type of aggregation. Next we consider the situation when we have a more general measure based uncertainty over the argument values. Here again we show how we can use the weight generating function to aid in performing the prescribed OWA aggregation in the face of this more general type of uncertainty. Finally we look at the task of obtaining a weight generating function from a given set of OWA weights.

1. Introduction

The aggregation of a collection of values is a pervasive problem in the modern technologically focused world, it is central to the such fields as data science [1,2], data mining [3–5] and decision making [6,7]. The ordered weighted averaging (OWA) operator is extensively used to perform this aggregation operation [8]. One reason for the popularity of the OWA operator is its ability to implement many different types of aggregation by simply changing the values of the parameters associated with the OWA operator, these parameters are called the OWA weights. In a given application these OWA weights can be provided directly or indirectly via a weight generating function f, which, as the name implies can be used to obtain the weights used in the aggregation. As discussed by Yager [9,10] the weight generating function can, in some situations, be expressed in a linguistic manner which can make it easier for users to specify the type of aggregation they want to be performed. However, the important point we want to emphasize here is that the type of aggregation to be performed is implicit in the mathematical form of the weight generating function. A frequent situation where we want to aggregate a collection of values is when we want to find one representative value of the collection of values. One prominent case where we want to find a representative value of a collection values occurs in decision-making under uncertainty. Here the collection of values are the possible outcomes of a decision alternative and we need a representative value to compare different decision alternatives. Our interest here is in the extension of the OWA aggregation to the situation in which there is some uncertainty over the values being aggregated, the collection of possible outcomes of a course of action. Formally the problem that we focus on is the OWA aggregation, guided by a weight generating function f, of collection of values over which there is an uncertainty. While we initially consider a probabilistic uncertainty as we did in our earlier work [11–13] and our more recent work [14] we then we look at a more general fuzzy measure based formulation of the uncertainty, μ [15].

2. The ordered weighted average (OWA) aggregation

In [16] we introduced the ordered weighted average (OWA) operator, which provides a general class of mean like aggregation operators. These operators are defined as follows.

Definition. An OWA operator of dimension n is a mapping F: $R^n \rightarrow R$ which has an associated collection of n weights w_j for j = 1 to n such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$ with

$$F(a_1, \dots, a_n) = \sum_{j=1}^{n} w_j a_{\rho(j)}$$

where ρ is an index function so that ρ(j) is index of the j largest a_i,

The key feature of this aggregation operation is the ordering of the arguments by value, via the index function ρ. This ordering process introduces nonlinearity into the operation. It can be shown this is a mean, it is commutative, monotonic and bounded $\mathrm{Min}_i[a_i] \leq F(a_1, \dots, a_n) \leq \mathrm{Max}_i[a_i]$. It is also idempotent, if all $a_i = a$ then $F(a_1, \dots, a_n) = a$. At times we shall find it convenient to denote the collection of weights

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as $W = (w_1, \ldots, w_n)$ and denote $F(a_1, \ldots, a_n)$ as $F_W(a_1, \ldots, a_n)$. We refer to W as the weighting vector.

The generality of the OWA operator lies in the fact that by appropriately selecting the weights we can implement different aggregation operations. Consider the situation where the weight are such that $w_1 = 1$ and $w_j = 0$ for $j \neq 1$ which we denote as W^*. In this case we get that $F(a_1, \ldots, a_n) = \text{Max}_j[a_j]$. If the weights are such that $w_n = 1$ and $w_j = 0$ for $j \neq n$, denoted as W_*, we get, $F(a_1, \ldots, a_n) = \text{Min}_j[a_j]$. If the weights are such that $w_j = 1/n$ for all j, denoted as W_{AVE}, in this case we get $F(a_1, \ldots, a_n) = \frac{1}{n}\sum_{j=1}^{n} a_j$

An interesting OWA operation is the Olympic aggregation operator. For the Olympic aggregator $w_1 = w_n = 0$ and $w_j = \frac{1}{n-2}$ for $j \neq 1$ or n. In this case $F(a_1, \ldots, a_n) = \frac{1}{n-2}\sum_{j=2}^{n-1} a_{\rho(j)}$. Here we eliminate the biggest and smallest arguments and take the average of the rest.

If $w_k = 1$ and $w_j = 0$ for $j \neq k$, denoted $W_{(k)}$, we get $F(a_1, \ldots, a_n) = a_{\rho(k)}$, the kth largest a_i. Another notable aggregation, is the so-called Hurwicz aggregation. Here with $0 \le \alpha \le 1$ we let $w_1 = \alpha$ and $w_n = 1 - \alpha$ and we get $F(a_1, \ldots, a_n) = \alpha\,\text{Max}_j[a_j] + (1 - \alpha)\,\text{Min}[a_j]$.

In [16,17] Yager associated with a weighting vector W a characterizing feature called its attitudinal character. $AC(W) = \sum_{j=1}^{n} \frac{n-j}{n-1} w_j$. In [16] it was shown $AC(W) \in [0, 1]$. The closer AC(W) to one the more Max like the aggregation and the closer AC(W) is to zero the more Min like the aggregation. It can be shown that $AC(W^*) = 1$ and $AC(W_*) = 0$ and $AC(W_{AVE}) = 0.5$. For the Hurwicz aggregation we have $AC(W) = \alpha$. For $W_{(k)}$ we have $AC(W) = \frac{n-k}{n-1}$. Typically the larger the attitudinal character the bigger the aggregated value.

In [18] we suggested a way of obtaining the OWA weights with the use of a weight-generating function f. Here f: $[0, 1] \to [0, 1]$ is a function having the properties 1) $f(0) = 0$, 2) $f(1) = 1$ and 3) monotonicity, $f(a) \ge f(b)$ if $a > b$. Using this weight-generating function we get the $w_j = f(\frac{j}{n}) - f(\frac{j-1}{n})$. We see that these generated weights have the required properties of OWA weights. Since f is monotonic the $f(\frac{j}{n}) \ge f(\frac{j-1}{n})$ and hence $w_j > 0$. Since $f(1) = 1$ then $f(\frac{j}{n}) - f(\frac{j-1}{n}) \le 1$ and hence $w_j \in [0, 1]$. Furthermore $\sum_{j=1}^{n} w_j = \sum_{j=1}^{n} (f(\frac{j}{n}) - f(\frac{j-1}{n})) = f(1) - f(0) = 1$. With $w_j = f(\frac{j}{n}) - f(\frac{j-1}{n})$ it is interesting to see that

$$\sum_{k=1}^{j} w_k = \sum_{k=1}^{j} \left(f\left(\frac{k}{n}\right) - f\left(\frac{k-1}{n}\right) \right) = f\left(\frac{j}{n}\right) - f(0) = f\left(\frac{j}{n}\right).$$

If we let $F_f(a_1, \ldots, a_n)$ denote the OWA aggregation of the a_i based on the weight-generating function f we have

$$F_f(a_1, \ldots, a_n) = \sum_{j=1}^{n} \left(f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right) \right) a_{\rho(j)}$$

Consider attitudinal character in the case where $w_j = f(\frac{j}{n}) - f(\frac{j-1}{n})$, using this we see that

$$AC(W) = \sum_{j=1}^{n} \left(\frac{n-j}{n-1}\right) w_j = \sum_{j=1}^{n} \left(\frac{n-j}{n-1}\right) \left(f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right) \right)$$

Here we see that AC(W) depends on n. In [17,18] we suggested a formulation for the attitudinal character in the case of a weight-generating function f independent of n. In particular it can be shown that

$$\lim_{n\to\infty} \sum_{j=1}^{n} \left(\frac{n-j}{n-1}\right) \left(f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right) \right) = \int_0^1 f(x)dx$$

Thus here the $AC(f) = \int_0^1 f(x)dx$ provides a cardinality free formulation of the attitude character of f.

In [10] Yager described a technique for obtaining a weight-generating function based on a quantifier expressing the aggregation imperative of the a_i. Consider a linguistic quantifier Q such is most, at least about α %. Using Zadeh's idea of computing with words [19] we can represent the linguistic qualifier Q as a proportional fuzzy subset Q of the unit interval. If Q is a monotonic quantifier, then the membership function of Q, f_Q, has the form of a weight-generating function.

3. The OWA aggregation with probability distribution over the arguments

In [13] we extended the weight-generating-function based OWA aggregation of the a_i to the case where we have an additional probability distribution over the a_i. Thus here $p_i \in [0, 1]$ is the probability associated with a_i and $\sum_{i=1}^{n} p_i = 1$. Here we desire $F_f((a_1, p_1), (a_2, p_2), \ldots, (a_n, p_n))$. Again letting $\rho(j)$ be the index of j^{th} largest a_i in [13] we indicated that

$$F_f\left((a_i,p_i) \text{ for } i = 1 \text{ to } n\right) = \sum_{j=1}^{n} \left(f(T_j) - f(T_{j-1})\right) a_{\rho(j)}$$

where $T_j = \sum_{k=1}^{j} p_{\rho(k)}$. We see that T_j is the probability of the set of arguments with the j largest values for a_i, thus if $H_j = \{a_{\rho(1)}, \ldots, a_{\rho(j)}\}$ then $T_j = \text{Prob}(H_j)$.

We can now indicate some properties of the $f(T_j)$.

(1) $f(T_j) \in [0, 1]$, since $T_j \in [0, 1]$
(2) $f(T_n) = 1$, since $T_n = 1$
(3) $f(T_0) = 0$, since $T_0 = 0$
(4) $f(T_j) \ge f(T_{j-1})$, since $T_j \ge T_{j-1}$
(5) $f(T_j) - f(T_{j-1}) \in [0, 1]$, follows from the above
(6) $\sum_{j=1}^{n} (f(T_j) - f(T_{j-1})) = 1$

Here we see that the $f(T_j) - f(T_{j-1})$ have the properties of OWA weights. Let us denote $\tilde{w}_j = f(T_j) - f(T_{j-1})$ and we shall refer to these as joint OWA-Probability weights, O-P weights. Thus here we have $F_f((a_i, p_i)) = \sum_{j=1}^{n} \tilde{w}_j a_{\rho(j)}$. Let us look at some special cases.

Assume $p_i = \frac{1}{n}$ for all i. In this case $T_j = \sum_{k=1}^{j} p_{\rho(k)} = \sum_{k=1}^{j} \frac{1}{n} = \frac{j}{n}$. From this we get

$$F_f\left((a_i,p_i)\right) = \sum_{j=1}^{n} \left(f(T_j) - f(T_{j-1})\right) a_{\rho(j)} = \sum_{j=1}^{n} \left(f\left(\frac{1}{n}\right) - f\left(\frac{j-1}{n}\right) \right) a_{\rho(j)}$$
$$= \sum_{j=1}^{n} w_j a_{\rho(j)}$$

It is just the OWA aggregation of the a_i as determined by the weight generating function f.

Consider now the case where f is linear, $f(x) = x$, here

$$F_f\left((a_i,p_i) \text{ for } i = 1, n\right) = \sum_{j=1}^{n} \left(f(T_j) - f(T_{j-1})\right) a_{\rho(j)} = \sum_{j=1}^{n} (T_j - T_{j-1}) a_{\rho(j)}$$

Since $T_j - T_{j-1} = p_{\rho(j)}$ then

$$F_f\left((a_i,p_i) \text{ for } i = 1, n\right) = \sum_{j=1}^{n} p_{\rho(j)} a_{\rho(j)} = \sum_{i=1}^{n} p_i a_i$$

It is simply the expected value of the a_i under the probability distribution p_i. In passing we note that when f is linear then the regular OWA weights are $1/n$.

We see this proposed calculation for $F_f((a_i, p_i)$ for $i = 1, n)$ includes both the formulations for the expected values of a_i under the probability distribution p_i and the ordinary OWA aggregation based on the weight-generating function f.

Let us look at $F_f((a_i, p_i)$ for $i = 1, n)$ for some other notable cases of f and probability distribution P. Assume we have the case where one element, a_r, has probability 1. Here $p_r = 1$ and $p_i = 0$ for us $i \neq r$. In this case $T_j = 1$ if $a_r \in H_j$ and $T_j = 0$ for if $a_r \notin H_j$. Assume a_r is the j^* largest argument. Here then $T_j = 0$ for $j < j^*$ and $T_j = 1$ for $j \ge j^*$. In this case

$f(T_j) = 0$ for $j < j^*$ and $F(T_j) = 1$ for $j \geq j^*$. From this we see

$$F_f((a_i,p_i)) = \sum_{j=1}^{n} (f(T_j) - f(T_{j-1}))a_{\rho(j)}$$

$$F_f((a_i,p_i)) = \sum_{j=1}^{j^*-1} (f(T_j) - f(T_{j-1}))a_{\rho(j)} + (f(T_{j^*}) - f(T_{j^*-1}))a_{\rho(j^*)} + \sum_{j=j^*+1}^{n} (f(T_j) - f(T_{j-1}))a_{\rho(j)}$$

$$F_f((a_i,p_i)) = \sum_{j=1}^{j^*-1} 0\, a_{\rho(j)} + a_{\rho(j^*)} + \sum_{j=j^*+1}^{n} 0\, a_{\rho(j)} = a_{\rho(j^*)} = a_r$$

Thus here independent of f if $p_r = 1$ then the aggregation of $f_f((a_i, p_i)) = a_r$.

Consider now the complementary situation here $p_r = 0$. Again assume a_r is the j^* largest argument. Here we see that $T_{j^*} = \sum_{j=1}^{j^*} p_{\rho(j)} = \sum_{j=1}^{j^*-1} p_{\rho(j)} = T_{j^*-1}$. Thus here $f(T_{j^*}) = f(T_{j^*-1})$ and here $f(T_{j^*}) - f(T_{j^*-1}) = 0$. Since $F_f((a_i, p_i)) = \sum_{j=1}^{n} (f(T_j) - f(T_{j-1}))a_{\rho(j)}$ we easily see that the weight associate with $a_{\rho(j^*)}$is zero and since $a_{\rho(j^*)} = a_r$ and if a_r has $p_r = 0$ then it does not contribute to the aggregation.

We have already shown if f is linear we get the simple expected value. Let us now look at some other notable weight generating functions f. Consider a step like weight generating function, $f(x) = 0$ for $x < b$ and $f(x) = 1$ for $x \geq b$. Here $f(T_j) = 0$ for $T_j < b$ and $f(T_j) = 1$ for $T_j \geq b$. We recall that $F_f((a_i, p_i)) = \sum_{j=1}^{n} (f(T_j) - f(T_{j-1}))a_{\rho(j)}$. Assume j^* is such that $T_{j^*} \geq b$ and $T_{j^*-1} < b$. Here we see $f(T_j) = f(T_{j-1})$ for all $j \neq j^*$ and $f(T_{j^*})$ - $f(T_{j^*-1}) = 1$. Thus we have $F_f((a_i, p_i)) = a_{\rho(j^*)}$. Here the aggregated value is the value of the ordered element that is the position where T_j transitions from being less than b to greater or equal b.

Actually the step like weight generating function at b corresponds to an ordinary OWA aggregation with $w_k = 1$ and $w_j = 0$ for $j \neq k$ where $(k - 1) < bn \leq k$.

Two special cases of step-like weight generating function are the max and min. For the min we have $b = 1$ and for the max we have $b = \varepsilon$, where ε is some arbitrary small value near zero. In the case of min, $b = 1$, we can show that $F_f(a_i, p_i)$ is the smallest argument with a non-zero probability and in the case of max, $b = \varepsilon$, the $F_f(a_i, p_i)$ is the largest argument with a non-zero probability.

Actually the max is obtained from weight generating function f^* where $f^*(0) = 0$ and $f^*(x) = 1$ for $x > 0$. The min is obtained from f_* where $f_*(x) = 0$ for $x < 1$ and $f^*(1) = 1$.

We see that $AC(f^*) = 1$ and $AC(f_*) = 0$. We note that for a step-like weight generating function with step at b we have $AC(f) = \int_0^1 f(x)dx = \int_b^1 dx = 1 - b$. Thus the smaller value of b, the larger the AC. For the linear function $f(x) = x$ we get

$$AC(f) = \int_0^1 x\, dx = \frac{X^2}{2}\Big|_0^1 = \frac{1}{2}$$

Thus we see that the attitudinal character of f locates it on a scale, from 0 to 1 where 0 is the minimum and one is the max. Here the linear function is 0.5.

An interesting class of weight generating functions are $f(x) = x^\alpha$ for $\alpha > 0$. In this case $AC(f) = \int_0^1 x^\alpha dx = \frac{1}{\alpha+1}$. Here we see that the smaller α the more max like aggregation. In particular as $\alpha \to 0$ then $AC \to 1$ while as $\alpha \to \infty$, $AC \to 0$. In this case $f(T_j) = (T_j)^\alpha$ and $f(T_j)$ - $f(T_{j-1}) = (T_j)^\alpha - (T_{j-1})^\alpha = \tilde{w}_j$ using this we see

$$F_f((a_i,p_i)) = \sum_{j=1}^{n} ((T_j)^\alpha - (T_{j-1})^\alpha)a_{\rho(j)}$$

where ρ is an index function so that $\rho(j)$ is the index of the argument with the j^{th} largest a_i and $T_j = \sum_{k=1}^{j} p_{\rho(k)}$, the sum of the probabilities of j largest arguments.

Let us provide an illustrated example of the calculation

$$F_f((a_i,p_i)) = \sum_{j=1}^{n} (f(T_j) - f(T_{j-1}))a_{\rho(j)}$$

for some examples of f.

Example. Assume $n = 4$, with $a_1 = 20$, $a_2 = 10$, $a_3 = 40$, $a_4 = 30$ and $p_1 = 0.4$, $p_2 = 0.2$, $p_3 = 0.3$ and $p_4 = 0.1$. Here since $a_3 > a_4 > a_1 > a_2$ we have $\rho(1) = 3$, $\rho(2) = 4$, $\rho(3) = 1$ and $\rho(4) = 2$. In this example $a_{\rho(1)} = 40$, $a_{\rho(2)} = 30$, $a_{\rho(3)} = 20$ and $a_{\rho(4)} = 10$. We also see that $p_{\rho(1)} = 0.3$, $p_{\rho(2)} = 0.1$, $p_{\rho(3)} = 0.4$, and $p_{\rho(4)} = 0.2$. With $T_j = \sum_{k=1}^{j} p_{\rho(k)}$ we get $T_1 = 0.3$, $T_2 = 0.4$, $T_3 = 0.8$ and $T_3 = 1$.

Here we have $\tilde{w}_j = f(T_j) - f(T_{j-1})$ and

$$F((a_i,p_i)) = \sum_{j=1}^{n} \tilde{w}_j a_{p(j)} = \tilde{w}_1(40) + \tilde{w}_2(30) + \tilde{w}_3(20) + \tilde{w}_4(10)$$

Let us look at this for different weight generating functions

1. $f(x) = x^2$ here $f(T_1) = 0.09$, $f(T_2) = 0.16$, $f(T_3) = 0.64$ and $f(T_4) = 1$. For this weight generating function
   $\tilde{w}_1 = 0.09 - 0 = 0.09$
   $\tilde{w}_2 = 0.16 - 0.09 = 0.07$
   $\tilde{w}_3 = 0.64 - 0.16 = 0.48$
   $\tilde{w}_4 = 1 - 0.64 = 0.36$
   For this example $F_f(a_i, p_i)) = 18.9$
2. $f(x) = x^{1/2}$ here $f(T_1) = 0.55$, $f(T_2) = 0.63$, $f(T_3) = 0.089$ and $f(T_4) = 1$. For this weight generating function

   $\tilde{w}_1 = 0.55$
   $\tilde{w}_2 = 0.08$
   $\tilde{w}_3 = 0.27$
   $\tilde{w}_4 = 0.1$

   For this example $F_f((a_i, p_i)) = 30.7$.
3. $f(x) = x$. Here $f(T_i) = T_i$ and hence
   $\tilde{w}_1 = 0.3$
   $\tilde{w}_2 = 0.1$
   $\tilde{w}_3 = 0.4$
   $\tilde{w}_4 = 0.2$
   For this example $F_f((a_i, p_i)) = 25$.
4. Consider a step-like weight generating function

   $f(x) = 0$ if $x < a$
   $f(x) = 1$ if $x \geq a$

   where $a = 0.5$. Here we get
   $\tilde{w}_1 = 0$
   $\tilde{w}_2 = 0$
   $\tilde{w}_3 = 1$
   $\tilde{w}_4 = 0$
   and hence $F_f((a_i, p_i)) = 20$

Consider $F_f((a_i, p_i), i = 1 \text{ to } n) = \sum_{j=1}^{n} (f(T_j) - f(T_{j-1}))a_{\rho(j)}$ where $\rho(j)$ is the index of jth largest a_i and $T_j = \sum_{k=1}^{j} p_{\rho(k)}$the sum of probabilities of the j largest arguments. Denoting $\tilde{w}_j = f(T_j)–f(T_{j-1})$, the, OWA-probability, O-P, weights we see

$$F_f((a_i,p_i), i = 1 \text{ for } n) = \sum_{j=1}^{n} \tilde{w}_j a_{p(j)}$$

This formulation inspires us to consider a joint OWA-probability attitudinal character AC(f, P).

$$AC(f, P) = \sum_{j=1}^{n} \left(\frac{n-j}{n-1}\right)\tilde{w}_j = \sum_{j=1}^{n} \left(\frac{n-j}{n-1}\right)(f(T_j) - f(T_{j-1}))$$

We note in the case where we have no probability we get $w_j = f(\frac{j}{n}) - f(\frac{j-1}{n})$ and hence $AC(f) = \sum_{j=1}^{n} (\frac{n-j}{n-1})w_j$, the normal value of AC.

In [14] we helped develop a related approach for combining OWA aggregation with probabilistic uncertainty called the Crescent method. In that approach instead of using the weight generating function we described the OWA aggregation directly in terms of the OWA weights.

4. Uncertainty modeling with fuzzy measure

Assume $X = \{x_1, \ldots, x_n\}$ are a set of elements. A fuzzy measure [20-22] on the set X is a set function on X, μ: $2^X \to [0, 1]$ having the properties: (1) $\mu(\emptyset) = 0$, (2) $\mu(X) = 1$ and (3) if $A \subset B$ then $\mu(A) \leq \mu(B)$.

We see that μ associates with each subset of X a value in the unit interval. Assume μ_1 and μ_2 are two fuzzy measures on X such that for any subset A of X, $\mu_1(A) \geq \mu_2(A)$ we say μ_1 is larger than μ_2 and denote it $\mu_1 \geq \mu_2$.

Assume μ is a fuzzy measure of X the set function $\hat{\mu}$ defined such that $\hat{\mu}(A) = 1 - \mu(\bar{A})$ is also a measure on X and is called the dual of μ. We easily see that $\hat{\hat{\mu}} = \mu$.

A measure μ is called self-dual $\hat{\mu}(A) = \mu(A)$. We observe that if μ is self-dual then $\mu(A) = 1 - \mu(\bar{A})$ and hence $\mu(A) + \mu(\bar{A}) = 1$. This is a very special property.

In [23] we discussed how to obtain new fuzzy measures from other fuzzy measures. Let h: $[0, 1] \to [0, 1]$ be a monotonic function on the unit interval, if $b_1 > b_2$ then $h(b_1) \geq h(b_2)$, having the properties $h(0) = 0$ and $h(1) = 1$ Assume μ is a fuzzy measure on X, then the set function μ_h defined such that $\mu_h(A) = h(\mu(A))$ is also a fuzzy measure on X,

(1) $\mu_n(\emptyset) = h(\mu(\emptyset)) = h(0) = 0$
(2) $\mu_h(X) = h(\mu(X)) = h(1) = 1$
(3) If $A \subseteq B$, then $\mu(B) \geq \mu(A)$, consider μ_h here $\mu_h(B) = h(\mu(B)) \geq h(\mu(A)) = \mu_n(A)$.

We observe that if h is linear, $h(x) = x$, then $\mu_h = \mu$. We also observe two notable cases of h, $h_\vee$ and $h_\wedge$. These are defined so that $h_\vee(x) \geq x$ for all x and $h_\wedge(x) \leq x$ for all x. We easily see that $\mu_{h_\vee} \geq \mu_{h_\wedge}$.

Assume V is an uncertain variable that takes its value in the space X, a fuzzy measure μ can provide a very general formulation for modeling the uncertainty associated with V [15]. Under this interpretation of A for any subset $A \subseteq X$, $\mu(A)$ indicates the anticipation that the value of V lies in A.

If μ is such that $\mu(A) = 1$ if $x_k \in A$ and $\mu(A) = 0$ if $x_k \not\subset A$ then this is the fuzzy measure representation of the information that $V = x_k$. If μ is such that for any subset B such that $x_k \notin B$ we have $\mu(B \cup \{x_k\}) = \mu(B)$ then this is a representation of the information that $V \neq x_k$.

μ is a probability measure if μ is such that $\mu(\{x_i\}) = p_i$ and $\mu(A) = \sum_{x_i \in A} \mu(\{x_i\})$. We note that since $\mu(X) = 1$ then $\sum_{\text{all } x_i} p_i = 1$. We can easily show that the probability measure is self-dual and $\mu(\bar{A}) = 1 - \mu(A)$.

If $\mu(x_i) = \tau_i$ with $\mu(A) = \mathrm{Max}_{i_{1}x_i \in A}[\tau_i]$ then μ is a possibility measure. We note here that at least one $\tau_i = 1$. For a possibility measure $\mu(A \cup B) = \mathrm{Max}[\mu(A), \mu(B)]$.

Another class of measures useful for modeling uncertainty are the cardinality-based measures. If α_i for $i = 0$ to n are parameters where $\alpha_0 = 0$, $\alpha_n = 1$ and $\alpha_{i+1} \geq \alpha_i$ then μ defined such that $\mu(A) = \alpha_{|A|}$ is a cardinality-based measure. Here the anticipation that $V \in A$ just depends on the number of elements in A, independent of which elements they are. A special case of cardinality-based measure is one in which $\alpha_i = \frac{i}{n}$, we shall call this the neutral measure and denote this neutral measure $\mu_\$$. We note for the neutral measure $\mu_\$(A) = \frac{|A|}{n}$.

5. OWA aggregation with fuzzy measure based uncertainty over the arguments

Assume $X = \{x_1, \ldots, x_n\}$ where each x_i is a numerical values. Let μ be a fuzzy measure on X. The Choquet integral of the collection X with respect to the measure μ, $\mathrm{Choq}_\mu(X)$, is defined [24] as

$$\mathrm{Choq}_\mu(X) = \sum_{j=1}^{n} \left(\mu(H_j) - \mu(H_{j-1})\right) x_{\rho(j)}$$

where ρ is an index function so that $\rho(j)$ is the index of the jth largest of the x_i and $H_j = \{x_{\rho(k)}$ for $k = 1$ to j$\}$, it is the subset of the j largest x_i. Since each $(\mu(H_j) - \mu(H_{j-1})) \in [0, 1]$ and $\sum_{j=1}^{n} (\mu(H_j) - \mu(H_{j-1})) = 1$ we see that $\mathrm{Choq}_\mu(X)$ is a weighted average of the x_i. The Choquet integral is a mean or averaging function that has the following properties

(1) $\mathrm{Min}_i(x_i) \leq \mathrm{Choq}_\mu(X) \leq \mathrm{Max}_i(x_i)$. It is bounded by the min and max of the arguments.
(2) If $Y = \{y_1, \ldots, y_n\}$ where $x_i \geq y_i$ for all i then $\mathrm{Choq}_\mu(X) \geq \mathrm{Choq}_\mu(Y)$. The Choquet integral is monotonic.
(3) Choquet integral is idempotent, if $x_i = a$ for all i then $\mathrm{Choq}_\mu(X) = a$.

With some algebraic manipulation we can show that

$$\mathrm{Choq}_\mu(x) = \sum_{j=1}^{n} \mu(H_j)\left(x_{p(j)} - x_{p(j+1)}\right)$$

where $x_{p(n+1)} = 0$ by convention. From this it follows that if μ_1 and μ_2 are two measures on X such that $\mu_1 \geq \mu_2$ then $\mathrm{Choq}_{\mu_1}(X) \geq \mathrm{Choq}_{\mu_2}(X)$.

Recalling the neutral measure, $\mu_\$(A) = \frac{|A|}{n}$, consider the $\mathrm{Choq}_{\mu\$;}(X)$. Here

$$\mathrm{Choq}_{\mu\$}(X) = \sum_{j=1}^{n} \left(\mu_\$(H_j) - \mu_\$(H_{j-1})\right) x_{\rho(j)}$$
$$= \sum_{j=1}^{n} \left(\frac{j}{n} - \frac{j-1}{n}\right)(x_{\rho(j)}) = \frac{1}{n} \sum_{j=1}^{n} x_{\rho(j)}$$

It is the simple average of the elements in X.

Observation: $\mathrm{Choq}_{\mu\$}(\{x_1, \ldots, x_n\})$ is the simple average of $\{x_1, \ldots, x_n\}$.

Assume f is a weight-generating function associated with an OWA aggregation. We recall $f(0) = 0$, $f(1) = 1$ and f is monotonic, if $a \geq b$ then $f(a) \geq f(b)$. Consider now the set function $\mu = f\mu_\$$ defined such that $\mu(A) = f(\mu_\$(A))$. In this case μ is also a fuzzy measure on X. Consider now the Choquet integral of X with respect to $\mu = f\mu_\$$, here

$$\mathrm{Choq}_\mu(X) = \sum_{j=1}^{n} \left(f(\mu_\$(H_j)) - f(\mu_\$(H_{j-1})\right) x_{\rho(j)}$$

Since $\mu_\$$ is cardinality based with $\mu_\$;(A) = \frac{|A|}{n}$ and H_j has j elements then $\mu_\$(H_j) = \frac{j}{n}$ and hence

$$\mathrm{Choq}_\mu(X) = \sum_{j=1}^{n} \left(f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right)\right) x_{\rho(j)}$$

Observation: If $\mu = f\mu_\$$ then $\mathrm{Choq}_\mu(X)$ is the OWA aggregation of the arguments $\{x_1, \ldots, x_n\}$ with respect to the weights obtained using the weight-generating function f.

Assume P is a probability distribution on X. Here p_i is the probability of x_i and $P(A) = \sum_{i, x_i \in A} p_i$, it is the probability of the subset A. Consider a measure μ_P such that $\mu_p(\{x_i\}) = p_i$ and $\mu_P(A) = \sum_{x_i \in A} p_i$. Here μ_P is a probability measure.

Consider the Choquet integral of $\{x_1, \ldots, x_n\}$ with respect to this measure μ_P. In this case

$$\mathrm{Choq}_{\mu_P}(\{x_1, \cdot \ldots, \cdot x_n\}) = \sum_{j=1}^{n} \left(\mu_p(H_j) - \mu_P(H_{j-1})\right) x_{\rho(j)}.$$

Here ρ is an index function so that $x_{\rho(j)}$ is the index of jth largest x_i and $H_j = \{x_{\rho(k)}, k = 1$ to j$\}$. Since $H_j = \{x_{\rho(k)}, k = 1$ to j$\}$ then $\mu_P(H_j) = \mathrm{Prob}(H_j) = \sum_{k=1}^{j} p_{\rho(k)}$ and $\mu_P(H_{j-1}) = \mathrm{Prob}(H_{j-1}) = \sum_{k=1}^{j-1} p_{\rho(k)}$ so in this case $\mu_P(H_j)$ - $\mu_P(H_{j-1}) = p_{\rho(j)}$. From this we see that

$$\mathrm{Choq}_{\mu_P}(\{x_1, \cdot \ldots, \cdot x_n\}) = \sum_{j=1}^{n} p_{\rho(j)} x_{p(j)} = \sum_{i=1}^{n} p_i x_i,$$

this is the expected value of $\{x_1, \ldots, x_n\}$ with the probability distribution P.

Observation: The Choquet integral of X with respect to the measure μ_P is the expected value of the elements in X.

Consider now the measure $\mu = f\mu_p$ where f is the weight generating function associated with an OWA aggregation. Let us look at the Choquet integral of X with respect to the measure $\mu = f\mu_p$

$$\mathrm{Choq}_\mu(\{x_1, \cdot \ldots, \cdot x_n\}) = \sum_{j=1}^{n} (\mu(H_j) - \mu(H_{j-1}))x_{\rho(j)}$$
$$= \sum_{j=1}^{n} (f(\mu_P(H_j)) - f(\mu_p(H_{j-1}))x_{\rho(j)}$$

Since $H_j = \{x_{\rho(k)}$ for $k = 1$ to j}, the set of elements with the j largest values for x_i

$$\mathrm{Choq}_\mu(\{x_1, \cdot \ldots, \cdot x_n\}) = \sum_{j=1}^{n} \left(f\left(\sum_{k=1}^{j} p_{\rho(k)} \right) \right) - f\left(\sum_{k=1}^{j-1} p_{\rho(k)} \right) x_{\rho(j)}$$

We see here that $\mathrm{Choq}_\mu(x_1, \ldots, x_n)$ is the OWA aggregation of the x_i with this probability distribution P on X.

Let μ_1 be some arbitrary uncertainty measure of the space X. Let f be the weight generating function associated with an OWA aggregation on X. Consider now the measure $\mu = f\mu_1$, with $\mu(A) = f\mu_1(A)$. Consider now the Choquet integer $\mathrm{Choq}_\mu(\{x_1, \ldots, x_n\})$ this is the OWA aggregation of the collection $X = \{x_1, \ldots, x_n\}$ where there is some measure μ on elements in X. In this case

$$\mathrm{Choq}_\mu(\{x_1, \cdot \ldots, \cdot x_n\}) = \sum_{j=1}^{n} (f(\mu(H_j)) - f(\mu(H_{j-1}))x_{\rho(j)}$$

Here we note that $\mu(H_j)$ is the measure of subset of X consisting of the elements with the j largest values for x_i.

Thus we see we have developed a formulation for the OWA aggregation of a collection of arguments X where there is some uncertainty associated with the elements in X.

Assume f_1 and f_2 are two weight generating functions so that $f_1(a) \geq f_2(a)$ for all a. In this case for any uncertainty measure μ on X, $f_1(\mu(H_j)) \geq f_2(\mu(H_j))$ and we have $\mu_1 = f_1\mu \geq f_2\mu = \mu_2$. In this case we see

$$\mathrm{Choq}_{\mu 1}(X) = \sum_{j=1}^{n} \mu_1(H_j)\ (x_{p(j)} - x_{p(j+1)}) \geq \sum_{j=1}^{n} \mu_2(H_j)\ (x_{p(j)} - xp_{(j+1)})$$
$$= \mathrm{Choq}_{\mu 2}(X)$$

Consider the special case of possibility measure μ over X. Here $\mu(\{x_i\}) = \tau_i$ and $\mu(A) = \mathrm{Max}_{x_i \in A}(\tau_i)$. Since $H_j = \{x_{\rho(k)}$ for $k = 1$ to j} then $\mu(H_j) = \mathrm{Max}_{k=1\,\mathrm{to}\,j}[\tau_{\rho(k)}]$ and $f(H_j) = f(\mathrm{Max}_{k=1\,\mathrm{to}\,j}[\tau_{\rho(k)}]$ thus for the case where we have an possibility measure μ over X and we are interested in OWA aggregation of X guided by the weight generating function f then the aggregated value is

$$\mathrm{Choq}_{f\mu}(x) = \sum_{j=1}^{n} (f(\underset{k=1\,\mathrm{to}\,j}{\mathrm{Max}}[\tau_{\rho(k)}]) - f(\underset{k=1\,\mathrm{to}\,j-1}{\mathrm{Max}}[\tau_{\rho(k)}]))x_{\rho(j)}$$

Thus the above formula provides the OWA aggregation guided by a weight generating function f over a set of element X having a possibilistic uncertainty.

Consider the case where μ is a cardinality-based measure with parameters α_j where $\alpha_j \leq \alpha_{\ j+1}$, $\alpha_{\ 0} = 0$ and $\alpha_{\ n} = 1$. First for any subset A of X, $\mu(A) = \alpha_{\mathrm{Card}(A)}$. In this situation since H_j has j elements then $\mu(H_j) = \alpha_j$. Here if f is the weight generating function associated with the OWA aggregation then

$$\mathrm{Choq}_{f\mu}(X) = \sum_{j=1}^{n} (f(\alpha_j) - f(\alpha_{j-1}))x_{\rho(j)}$$

Example. Assume $x_1 = 40$, $x_2 = 30$, $x_3 = 20$, $x_4 = 10$. Here $p(j) = j$.

Here

$H_0 = \emptyset$
$H_1 = \{x_1\} = \{40\}$
$H_2 = \{x_1, x_2\} = \{40, 30\}$
$H_3 = \{x_1, x_2, x_3\} = \{40, 30, 20\}$
$H_4 = \{x_1, x_2, x_3, x_4\} = \{40, 30, 20, 10\}$

Assume μ is a possibility measure with $\tau_1 = 0.7$, $\tau_2 = 0.4$, $\tau_3 = 1.0$ and $\tau_4 = 0.2$

We see that since $\mu(H_j) = \mathrm{Max}_{x_k \in H_j}[\tau_k]$ we have

$$\mu(H_0) = 0, \mu(H_1) = 0.7, \mu(H_2) = 0.7, \mu(H_3) = 1,\ \mu(H_4) = 1$$

We see that

$$\mathrm{Choq}_\mu(40, 30, 20, 10) = \sum_{j=1}^{4} (f\ (\mu(H_j) - f(\mu\ (H_{j-1}))x_j$$

Since $\mu(H_1) = \mu(H_2)$ and $\mu(H_3) = \mu(H_4)$ we have

$$\mathrm{Choq}_{f\mu}(X) = f(\mu(H_1)x_1 + (f(\mu\ (H_3)) - f(\mu(H_2))x_3 = f(0.7)(40)$$
$$+ (f(1) - f(0.7))20$$
$$\mathrm{Choq}_\mu(X) = f(0.7)(40) + (1 - f(0.7))20$$

Consider the following cases of weight generating function f

(i) $f(y) = y^2$. Here $f(0.7) = 0.49$ and $(1 - 0.49) = 0.51$
$\mathrm{Choq}_{f_\mu}(y) = 29.8$
(ii) $f(y) = y^{1/2}$. Here $f(0.7) = 0.84$ and $(1 - 0.84) = 0.16$
$\mathrm{Choq}_\mu(y) = 30.7$
(iii) $f(y) = y$. Here $f(0.7) = 0.7$ and $(1 - 0.7) = 0.3$
$\mathrm{Choq}_\mu(y) = 34$

6. Obtaining a weight generating from OWA weights

Many of the procedures developed in the preceding took advantage of the availability of a weight generating function. A natural question that arises is that given a set of n OWA weights, w_j for $j = 1$ to n, can we associate with these OWA weights a weight generating function f. Clearly the function f should have the following properties.

(1) f is a mapping from the unit interval into the unit interval, f: [0, 1] → [0, 1]
(2) $f(0) = 0$
(3) $f(1) = 1$
(4)f is monotonic, if $a > b$ then $f(a) \geq f(b)$
(5) The w_j should be obtained from f,

$$w_j = f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right) \text{for } j = 1 \text{ to } n$$

It is clear that there are many functions f that have these properties. Let us look at some further features that we can associate to any f and some notable special cases in f.

We can associate with any f a number of fixed points in addition to $f(0) = 0$ and $f(1) = 1$. If by convention we let $w_0 = 0$ we see that for $j = 0$ to n we have $f(\frac{j}{n}) = \sum_{k=0}^{j} w_k$. From this we easily see that for $j = 1$ to n we have $f(\frac{j}{n})$ - $f\frac{(j-1)}{n} = \sum_{k=0}^{j} w_k - \sum_{k=1}^{j-1} w_k = w_j$. Thus we see that any f is fixed at the $n + 1$ points $f(\frac{j}{n})$ for $j = 0$ to n. We see any freedom we have in choosing f lies in our determination of the value of f between these fixed points. Thus f may be described in a precise manner by expressing its value in the open intervals $(\frac{j-1}{n}, \frac{j}{n})$ for $j = 1$ to n. Because of the required monotonicity of f we see two properties of f in these intervals are

$$f\left(\frac{j-1}{n}\right) \leq f(y) \leq f\left(\frac{j}{n}\right) \qquad \text{for } y \in \left(\frac{j-1}{n}, \frac{j}{n}\right)$$

$$f(y_2) \geq f(y_1) \text{ if } y_2\ \geq y_1 \text{ and } y_1 \text{ and } y_2 \in \left(\frac{j-1}{n}, \frac{j}{n}\right)$$

We note that any function f that satisfies these two conditions will have monotonicity in the whole [0, 1] interval, for a, b ∈ [0, 1] and $a > b$ then $f(a) \geq f(b)$.

Using these bounds we can obtain the smallest, f_+, and largest, f^+, weight-generating functions associated with a set of OWA weights. Here f_+ is defined as

$$f_+(y)=\sum_{k=0}^{j-1} w_k \quad \text{for } y \in \left[\frac{j-1}{n}, \frac{j}{n}\right) \text{ for each } j = 1 \text{ to } n$$

$$f_+(1)= 1$$

Thus f_+ is essentially a step function takes a step w_j at the end of jth interval. At the other extreme is the largest function f^+ defined so that

$$f^+(0)= 0$$

$$f^+(y)= \sum_{k=0}^{j} w_k \text{ for } y \in \left(\frac{j-1}{n}, \frac{j}{n}\right] \text{foreach} j = 1 \text{ton}$$

This is also a step function it takes a step of w_j at the beginning of the j^{th} interval.

We see that any weight generating function f associated the OWA weights will be such that $f_+(y) \le f(y) \le f^+(y)$.

Another notable example of weight generating function associated with a set of OWA weights is the piecewise linear function f_L. In this case for $y \in [\frac{j-1}{n}, \frac{j}{n}]$ we have

$$f_L(y)= \sum_{k=0}^{j-1} w_k + w_j(ny - j + 1)$$

We note here that

$$f_L\left(\frac{j}{n}\right) = \sum_{k=0}^{j-1} w_k + w_j\left(\frac{nj}{n} - j + 1\right) = \sum_{k=0}^{j-1} w_k + w_j = \sum_{k=0}^{j} w_k$$

We see in this case that $f_+(y) \le f_L(y) \le f^+(y)$.

We can consider a family of weight generating functions f_α based on a generalized f_L. Here for $0 < \alpha < \infty$ we have

$$f_\alpha(y)= \sum_{k=0}^{j-1} w_k + w_j(ny - j + 1)^\alpha \text{for } y \in \left[\frac{j-1}{n}, \frac{j}{n}\right]$$

We see here that if $\alpha_1 < \alpha_2$ then $f_{\alpha_1}(y) \ge f_{\alpha_2}(y)$.

Thus we see that we have an ordered family of weight generating functions associated with a set of OWA weights. Furthermore as $\alpha \to 0$ then $f_\alpha \to f^+$, as $\alpha \to \infty$ then $f_\alpha \to f_+$ and for $\alpha = 1$ we get f_L. We see that f_L with $\alpha = 1$ is in the middle. The following discussion leads us to believe f_L is a good choice.

Assume $X = \{x_1, \ldots, x_n\}$ is a bag of data and we want to aggregate this data using an OWA aggregation with weights w_j for $j = 1$ to n. However we have some uncertainty over the collection X as expressed by a measure μ. To perform this aggregation we can select a weight generating function f_α associated with the given OWA weights and calculate

$$\text{Choq}_{f\alpha\mu}(x_1, \ldots, x_n)= \sum_{j=1}^{n} \left(f_\alpha(\mu(H_j)) - f\alpha(\mu(H_{j-1}))\right) x_{\rho(j)}$$

Here we recall $\rho(j)$ is the index of the x_i with the j^{th} largest value and H_j is the subset of X with the j largest values for x_i. We observe that Choq $(x_1, \ldots, x_n)$ will depend on our choice of f_α. In particular we have earlier shown that the bigger the weight generating function the larger the aggregation. Thus if $f_{\alpha_1} > f_{\alpha_2}$ then $\text{Choq}_{f_{\alpha_1}\mu}(X) \ge \text{Choq}_{f_{\alpha_2}\mu}(X)$. Since f_L is in the middle of the ordered weight generating functions the use of f_L is a reasonable choice. In addition its being linear is always a good feature. Thus a good solution to the aggregation problem is

$$\text{Cho}\, q_{f_L\mu}(X) = \sum_{j=1}^{n} \left(f_L(\mu(H_j)) - f_L(\mu(H_j - 1))\right) x_{\rho(j)}$$

Here we provide an illustrative example.

Example. : Let $x_1 = 40$, $x_2 = 30$, $x_3 = 20$, $x_4 = 10$, here $\rho(j) = j$. Assume we have as the given OWA weights $w_1 = 0.3$, $w_2 = 0.1$, $w_3 = 0.4$ and $w_4 = 0.2$. Here we shall assume a probability measure μ on the X with $p_1 = 0.2$, $p_2 = 0.3$, $p_3 = 0.4$ and $p_4 = 0.1$. In this case since

$$H_1 = \{x_1\}, H_2 = \{x_1, x_2\}, H_3 = \{x_1, x_2, x_3\} \text{ and } H_4 = X$$

we have $\mu(H_j) = \sum_{k=1}^{j} p_k$ and hence $\mu(H_1) = 0.2$, $\mu(H_3) = 0.5$, $\mu(H_3) = 0.9$ and $\mu(H_4) = 1$.

In this example we shall use f_L. Here $f_L(j/n) = \sum_{k=0}^{j} w_k$

$f_L(0)= 0$,
$f_L(1/4)= f(0.25)= 0.3$
$f_L(2/4)= f(0.5)= 0.4$
$f_L(3/4)= f(0.75)= 0.8$
$f_L(n/L)= f(1)= 1$

Here we see that

$\mu(H_1)= 0.2 \in \left[\frac{0}{4}, \frac{1}{4}\right)$
$\mu(H_2) = 0.5 \in \left[\frac{1}{2}, \frac{1}{2}\right)$
$\mu(H_3) = 0.9 \in \left[\frac{3}{4}, 1\right)$
$\mu(H_4) = 1 \in [1, 1]$

We recall that for $y \in [\frac{j-1}{n}, \frac{j}{n}]$ we have $f_L(y) = \sum_{k=0}^{j-1} w_k + w_j$ (ny - j + 1). In this example

$f_L(\mu(H_1))= \frac{0.2(0.3)}{0.25} = \frac{4}{5}0.3 = 0.24$
$f_L(\mu(H_2))= f(0.5)= 0.4$
$f_L(\mu(H_3))=0.92$
$f_L(\mu(H_4))1.00$

Since $\tilde{w}_j = f(\mu(H_j))$ - $f(\mu(H_{j-1}))$ we obtain

$\tilde{w}_1= 0.24, \tilde{w}_2= 0.66, \tilde{w}_3= 0.52, \tilde{w}_4= 0.08$

and hence

$$\text{Choq}_\mu(X)= \sum_{k=1}^{4} \tilde{w}_j x_j = (0.24)40 + (0.16)(30) + (0.52)20 + (0.08)10 = 25.6$$

7. The attitudinal character of weight generating functions

We recall that the attitudinal character of a set of n OWA weights W, w_j for $i = 1$ to n is

$AC(W) = \sum_{j=1}^{n} \frac{n-j}{n-1} w_j$. We also recall that the attitudinal character associated with a weight generating function f is $AC(f) = \int_0^1 f(y)dy$.

In the preceding we suggested a procedure for obtaining a family, f_α of weight generating functions associated with the OWA weights w_j for $j = 1$ to n.

Since for any f_α we have $f_+(y) \le f_\alpha(y) \le f^+(y)$ for all y hence for any f_α

$$AC(f_+) \le AC(f_\alpha) \le AC(f^+)$$

Let us look at these bounding cases. We recall $f^+(y) = \sum_{k=0}^{j} w_k = f(\frac{j}{n})$ for $y \in (\frac{j-1}{n}, \frac{j}{n}]$. Thus $\int_0^1 f^+(y)dy = \sum_{j=1}^{n} \int_{(j-1)/n}^{j/n} f(\frac{j}{n})dy = \frac{1}{n}\sum_{j=1}^{n} f(\frac{j}{n}) = \frac{1}{n}(\sum_{j=1}^{n}(\sum_{k=0}^{j} w_k)) = \frac{1}{n}$ $(nw_1 + (n-1)w_2 + \ldots + w_n)$

$$\int_0^1 f^+(y)dy = \sum_{j=1}^{n} \frac{n+1-j}{n} w_j$$

While AC(W) and $\int_0^1 f^+(y)\,dy$ are similar it is clear that $\frac{n+1-j}{n} \ge \frac{n-j}{n-1}$ for all j and hence we have

$$\int_0^1 f^+(y)dy \ge AC(W)$$

More specifically since $\frac{n+1-j}{n} - \frac{n-j}{n-1} = \frac{j-1}{(n)(n-1)}$ then AC(f) - $AC(W) = \sum_{j=1}^{n} \frac{j-1}{n(n-1)} w_j$

Further recalling that $f_+(y) = f(\frac{j-1}{n}) = \sum_{k=0}^{j-1} w_k$ for $y \in [\frac{j-1}{n}, \frac{j}{n})$ we have

$$AC(f_+) = \int_0^{+} f_+(y)dy = \sum_{j=1}^{n} \int_{\frac{j-1}{n}}^{n} f\left(\frac{(j-1)}{n}\right)dy = \frac{1}{n}\sum_{j=1}^{n} f\left(\frac{j-1}{n}\right)$$

$$= \frac{1}{n}\left(\sum_{j=1}^{n}\left(\sum_{k=0}^{j-1} w_k\right)\right) = \sum_{j=1}^{n} \frac{(n-j)}{n} w_j$$

Since $\frac{n-j}{n} \leq \frac{n-j}{n-1}$ for all j then $AC(f_+) = \int_0^{+} f_+(y)dy \leq AC(W)$.

More specifically since $\frac{n-j}{n-1}$ - $\frac{n-j}{n} = \frac{n-j}{(n)(n-1)}$ we have AC(W) - $AC(f_+) = \sum_{j=1}^{n} \frac{n-j}{(n)(n-1)} w_j$

Thus we see for any f_α we have $AC(f_+) \leq AC(f_\alpha) \leq AC(f^+)$ and hence

$$\sum_{j=1}^{n} \frac{(n-j)}{n} w_j \leq AC(f_\alpha) \leq \sum_{j=1}^{n} \left(\frac{n+1-j}{n}\right) w_j$$

While in general the calculation of $AC(f_\alpha)$ is rather complex because of the piecewise linearity of F_L we can obtain a closed form for $AC(f_L)$. In this case we shall let $f(m_j)$ denote the middle portion for the interval $[\frac{j-1}{n}, \frac{j}{n}]$. In this case because of piecewise linearity

$$AC(f_L) = \int_0^1 f_L(y)dy = \sum_{j=1}^{n} \int_{(j-1)/n}^{j/n} f(m_j)dy = \sum_{j=1}^{n} \frac{1}{n} f(m_j)$$

In this case

$$f(m_j) = \frac{1}{2}\left(f\left(\frac{j-1}{n}\right) + f\left(\frac{j}{n}\right)\right) = \frac{1}{2}\left(\sum_{k=0}^{j-1} w_k + \sum_{k=0}^{j} w_k\right)$$

$$= \sum_{k=0}^{j-1} w_k + \frac{1}{2} w_j = f\left(\frac{j-1}{n}\right) + \frac{1}{2} w_j$$

Here then

$$AC(f_L) = \sum_{j=1}^{n} \frac{1}{n}\left(f\left(\frac{j-1}{n}\right) + \frac{1}{2} w_j\right) = AC(f_+) + \sum_{j=1}^{n} \frac{1}{n}\frac{1}{2} w_j = AC(f_+) + \frac{1}{2n}$$

$$= \sum_{j=1}^{n} \frac{(n-j+0.5)}{n} w_j$$

8. Conclusion

We discussed the OWA aggregation operation and the role the OWA weights play in determining the type of aggregation being performed. We introduced the idea of a weight generating function and described its use in obtaining the OWA weights. We emphasized the importance of the weight generating function in prescribing the type of aggregation to be performed. We considered the problem of performing a prescribed OWA aggregation in the case when we have a probability distribution over the argument values. We showed how we could use the weight generating function to enable this type of aggregation. Next we considered the situation when we have a more general measure based uncertainty over the argument values. Here again we showed how we can use the weight generating function to aid in performing the prescribed OWA aggregation in the face of this more general type of uncertainty. Finally we looked at the task of obtaining a weight generating function from a given set of OWA weights.

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