Applied Soft Computing 124 (2022) 108985

Contents lists available at ScienceDirect

Applied Soft Computing

journal homepage: www.elsevier.com/locate/asoc

The OWA operator in multiple linear regression

Martha Flores-Sosa [a,*], Ezequiel Avilés-Ochoa [a], José M. Merigó [b,c], Janusz Kacprzyk [d]

[a] University Autonomous of Occidente, Av. Lola Beltran 80020, Culiacán, Mexico
[b] School of Information, Systems & Modelling, Faculty of Engineering and Information Technology, University of Technology Sydney, 81 Broadway, Ultimo, 2007 NSW, Australia
[c] Department of Management Control and Information Systems, School of Economics and Business, University of Chile, Av. Diagonal 257, 8330015 Santiago, Chile
[d] Systems Research Institute, Polish Academy of Sciences, Newelska 6, 01-447 Warsaw, Poland

ARTICLE INFO

Article history:
Received 29 April 2020
Received in revised form 21 March 2022
Accepted 4 May 2022
Available online 14 May 2022

Keywords:
OWA operator
Multiple linear regression
GOWA operator
OLS

ABSTRACT

Multiple linear regression (MLR) is one of the most widely used statistical procedures for scholarly and research. The main limitation of MLR is that when being estimated with linear methodologies as ordinary least squares (OLS) becomes not functional with complex data. The ordered weighted average (OWA) is an aggregation operator that provides means that collect complex information. This work presents a new application that uses MLR and OWA operators in the same formulation. We developed two applications called MLR-OWA operator and MLR-GOWA operator. The main advantage of the MLR with OWA operators is that we can consider the degree of optimism and pessimism of the environment. We study some of its main properties and particular cases. Finally, an application is tested for a volatility exchange rate estimation problem.

1. Introduction

Multiple linear regression (MLR) is the generalization of the simple linear regression model, which provides an approach to understanding the mutual effects of the dependent variable on the independent variables. MLR analysis is broadly applicable to hypotheses generated by researchers in economics [1,2], finance [3,4], behavioral sciences [5,6] and business administration [7,8].

The most popular method to estimate the parameters in MLR is the ordinary least squares (OLS) [1,9]. The OLS method has shown promising results because of reasonable accuracy and relatively simple implementation compared to other methods [10,11]. However, the diversity for some types of information has yielded an array of large and complex data with particular characteristics that the OLS method cannot consider [12,13].

In practice, many special situations can affect the behavior of the analyzed data. For example, the convergence of multiple factors and causal sequences can cause particular statistical characteristics. The non-linearity [14,15], nonnormality [16,17] and the imprecision of subjective decision variables [18,19] are common. This causes inefficiencies in traditional models such as MLR that assume linearity and normality in the data.

Therefore, the search for new alternatives in estimation and modeling has improved and adapted existing models using alternative procedures. It is common in modeling, where the data is complex, the use of the model fits atypical cases. Kott [20] proposes a weighted ordinary least squares estimator incorporating weights into estimated regression coefficients. Within these scoring methodologies, Zanutto [21] includes weights from complex data and compares results in linear regressions. In this sense, robust regression methods are proposed by assigning convenient weights to the observations that deviate from the usual behavior. Filzmoser and Nordhausen [22] present a summary about it in low-dimensional and high-dimensional situations.

Additionally, when human estimation is influential on the data, we must deal with a fuzzy structure [23,24]. A fuzzy linear regression model might be more useful in areas of decision making where there is a great deal of uncertainty and vague phenomena. Zadeh [25] described the difference between real situations and applied mathematics, where classical mathematics was not too precise to describe systems where a human element was involved. In this sense, there is some research around linear regression and the use of fuzzy sets for processing complex data. Bargiela et al. [26] propose the use of fuzzy data to create fuzzy variables in combination with non-fuzzy variables. With a systematic method, Chen and Chen [27] use a granular computing approach to construct a fuzzy time series model. In regression procedures, Yager and Beliakov [28] propose a function of the residuals based on ordered weighted aggregation operators (OWA) in order to replace least squares, least absolute deviation, and maximum likelihood criteria. This framework considers OWA operators to minimize the error function used in OLS.

* Corresponding author.
E-mail addresses: martha.flores@udo.mx (M. Flores-Sosa), ezequiel.aviles@udo.com (E. Avilés-Ochoa), jmerigo@fen.uchile.cl (J.M. Merigó).

https://doi.org/10.1016/j.asoc.2022.108985
1568-4946/

The Ordered Weight Average (OWA) operator [29] is a technique for aggregating the information to consider the importance of the weight in the aggregation process, providing the maximum, the minimum and the average. An interesting extension in OWA operators is the Generalized Ordered Weight Average (GOWA) operator [30,31] that uses generalized means with a parameter regulating the intensity in the argument values. Since its introduction, OWA operators have been used in a wide variety of areas [32], such as fuzzy information [33,34], moving averages [35,36], Bonferroni means [37,38] and support vector machines [39].

On linear regression problems, the OWA operator has been used in the OLS method [40] and the estimation parameters with variances and covariances OWA [41,42].

This paper presents a new approach that combines the OWA and GOWA operators with multiple linear regression and the OLS estimator. We call them multiple linear regression with OWA operators (MLR-OWA) and multiple linear regression with GOWA operators (MLR-GOWA). The main advantage of this approach is to consider situations where we give more or less importance in the parameter estimation by using a balanced weight vector towards some of the extremes. This is, using OWA operators, we can insert the degree of optimism or pessimism, which can yield better estimates for complex data. Then if a particular behavior is expected in the data, the weight vector will be oriented towards data with those characteristics. We also analyze the applicability of the new approach in some of the OWA and GOWA operators and their particular cases. We analyze special cases in MLR-OWA. We use the volatility in the exchange rate as an application.

The work is organized as follows. In Section 2, we show a review of the methodologies used; OWA, GOWA, and MLR with OLS estimation. Section 3 presents the proposed approach, MLR-OWA and MLR-GOWA, with the models and the estimation. We apply the new framework in Section 4, where we used an autoregressive model for estimating the volatility exchange rate for USD and MXN. Finally, Section 5 presents the main conclusions of the paper and future research.

2. Preliminaries

In this section, we summarize the OWA operator, GOWA operator, multiple linear regression, variance, and covariance OWA, in order to make a review of its main characteristics.

2.1. OWA operator

The Ordered Weighted Aggregation operator was introduced by Yager [29] and provides a parameterized family of aggregation operators to consider several arguments as the arithmetic mean, the maximum and minimum. The main idea is to put weights on components of vectors after a preliminary ranking of the individual weights. It is defined as follows:

Definition 1. An OWA operator with dimension n is a model $OWA : R^n \rightarrow R$ such that it has a weights vector $W = [w_1, w_2, .., w_n]^T$ thus $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, and then:

$$OWA(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j, \tag{1}$$

where b_j is the jth largest in $a_1, a_2, \ldots, a_n$.

The OWA operator satisfies some conditions like monotonicity, this is, let an additional ordered argument vector $w = [b_1, b_2, \ldots, b_n]$ such that for each j, if $a_j \geq b_j$ then: $OWA(a_1, a_2, \ldots, a_n) \geq OWA(b_1, b_2, \ldots, b_3)$. Symmetry (commutativity), if we have two ordered argument vectors A and A', then A = A', hence: $OWA(a_1, a_2, \ldots, a_n) = OWA(a'_1, a'_2, \ldots, a'_n)$. Idempotent, this is, that if $a_j = a$, for all $j = 1, \ldots, n$, then: $OWA(a_1, a_2, \ldots, a_n) = a$.

Other interesting properties and particular cases of the OWA operators are studied in Yager [39].

2.2. GOWA operator

Important especial OWA cases occur when the arguments are drawn from the unit interval. In this particular case, Yager [31] proposes the Generalized Ordered Weighted Aggregation operator (GOWA), which provides a generalization of the OWA operator by combining it with the generalized mean operator. It can be defined as follows:

Definition 2. A GOWA operator with mapping $GOWA : R^n \rightarrow R$ such that it has a weights vector $W = [w_1, w_2, .., w_n]^T$ thus $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, and we have:

$$GOWA(a_1, a_2, \ldots, a_n) = \left(\sum_{j=1}^{n} w_j {b_j}^{\lambda}\right)^{1/\lambda}, \tag{2}$$

where λ is a parameter such that $\lambda \in [-\infty, \infty]$; b_j is the jth largest of the a_i. As we already noted previously, the GOWA operator is monotonic, symmetric and bounded.

We have some special GOWA cases when λ has different parameters, when $\lambda = 1$, the GOWA operators become the OWA operator [29], When $\lambda = 0$, the GOWA operator is the OWG operator [43]. When $\lambda = -1$, the aggregation becomes the OWHA operator [44], and when $\lambda = 2$, we form the OWQA operator [45].

2.3. Multiple regression

Multiple regression is an approach for generating multiple equations to investigate the relationship between one or more independent parameters and one dependent parameter [46,47]. The multiple linear regression is the generalization of the simple linear regression model, then we have:

Definition 3. Consider a multiple regression problem, given a set of variables (x_k, y_k, z_k), given the $k = 1, \ldots, K : x_k \in U^n, y_k \in U, z_k \in U^n$, in this way, we have a model $f_\theta : R^n \rightarrow R$, parameterized by a parameter vector $\theta \in \Omega \subseteq R^p$, which is composed $\theta = \alpha, \beta_1, \beta_2$. The estimated multiple linear regression model is developed as $y_j = \alpha + \beta_1 x_j + \beta_2 z_j$.

In ordinary least squares (OLS) [48], the optimum parameters vector is found by minimizing the sum of the squared error between the predicted and actual observations $\sum \hat{u}_i^2$ as $Min \sum \hat{u}_i^2 = \sum (y_i - \alpha - \beta_1 x_i - \beta_2 z_i)^2$. To find the values θ that minimizes the previous expression, the OLS estimators are used as follows:

$$\alpha = \bar{y} - \beta_1 \bar{x} - \beta_2 \bar{z} \tag{3}$$

$$\beta_1 = \frac{\left(\sum y_i x_i\right)\left(\sum z_i^2\right) - \left(\sum y_i z_i\right)\left(\sum x_i z_i\right)}{\left(\sum x_i^2\right)\left(\sum z_i^2\right) - \left(\sum x_i z_i\right)^2} = \frac{[Cov(y,x)][var(z)] - [Cov(y,z)][Cov(x,z)]}{[var(x)][var(z)] - [Cov(x,z)]^2} \tag{4}$$

$$\beta_2 = \frac{\left(\sum y_i z_i\right)\left(\sum x_i^2\right) - \left(\sum y_i x_i\right)\left(\sum x_i z_i\right)}{\left(\sum x_i^2\right)\left(\sum z_i^2\right) - \left(\sum x_i z_i\right)^2} = \frac{[Cov(y,z)][var(x)] - [Cov(y,x)][Cov(x,z)]}{[var(x)][var(z)] - [Cov(x,z)]^2}, \tag{5}$$

where cov is the covariance [(z,y)(x,y)(x,z)], then $cov = \sum_{k=1}^{k} (a_k - \bar{a})\left(b_k - \bar{b}\right)$, and var is the variance $(x;z)$, where $var = \sum_{k=1}^{k} (a_k - \bar{a})^2$. The variances and standard errors of the OLS estimators are studied in Gujarati and Porter [49].

2.4. Variance and covariance OWA

Using OWA combined with other methodologies is a common procedure. Yager [50] proposes the use of OWA operators in variance as a form to adapt it to a vector of parameterized weights. It can be defined as follows:

Definition 4. It is a variance OWA with mapping $OWA : R^n \to R$ such that it has an associated weights vector W thus $0 \leq w_i \leq 1$ and $\sum_{i=1}^{n} w_i = 1$, then we have a D_j component associated with a weight w_j in the following way:

$$Var - OWA\left(a_1, \ldots, a_n\right) = \sum_{j=1}^{n} w_j D_j, \tag{6}$$

where D_j is the largest $(a_i - \mu)^2$, a_i is the argument variable, μ is the average (the OWA operator).

The Var-OWA has similar properties than other OWA operators, including commutativity, idempotency, monotonicity, and boundary condition. Some extensions and applications of OWA in the variance are analyzed in Laengle et al. [51] and Blanco-Mesa et al. [52]. In a generalized form, the variance is as follows:

$$Var - GOWA\left(a_1, \ldots, a_n\right) = \sum_{j=1}^{n} \left(w_j D_j^{\lambda}\right)^{1/\lambda}, \tag{7}$$

where D_j is the largest of the $(a_i - \mu)^2$, a_i is the argument variable, μ is the GOWA mean, $w_i \in [0, 1]$, $\sum_{i=1}^{n} w_i = 1$ and $\lambda \in [-\infty, \infty]$.

The covariance can also be formulated using a similar procedure. Merigó [41] suggested the use of an OWA operator in the covariance. It can be formulated as follows:

Definition 5. It is a covariance OWA if we have a mapping $OWA : R^n \to R$ with a weights vector W thus $0 \leq w_i \leq 1$ and $w_i + \cdots + w_n = 1$, then a D_j component is associated with a weight w_j we have:

$$Cov - OWA(X, Y) = \sum_{j=1}^{n} w_j K_j, \tag{8}$$

where K_j is the jth largest of the $(x_i - \mu)(y_i - \nu)$, x_i is the argument variable of the set of elements $X = \{x_1, \ldots, x_n\}$, y_i is the argument variable of the set $Y = \{y_1, \ldots y_n\}$, and μ and ν are the OWA averages of X and Y respectively, $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. The Cov-OWA also can be generalized as the following equation:

$$Cov - GOWA(X, Y) = \sum_{j=1}^{n} \left(w_j K_j^{\lambda}\right)^{1/\lambda}, \tag{9}$$

where K_j is the jth largest of the $(x_i - \mu)(y_i - \nu)$, x_i and y_i are the argument variable of the set of elements in X and Y, respectively. μ and ν are the GOWA averages of X and Y and $\lambda \in [-\infty, \infty]$.

3. OWA operator in multiple linear regression

OWA multiple linear regression (MLR-OWA) proposes using OWA operators to estimate means by ordering the arguments. We can see that depending on the use of the OWA means in the formulation, and we can obtain OWA variances and covariances (var_{OWA}; cov_{OWA}). It can be defined as follows:

Definition 6. An OWA multiple linear regression with two independent parameters of dimension n is a model $OWA : R^n \to R$ given the parameters $x_k \in U^n$, $y_k \in U$ and $z_k \in U^n$ such have weights vector $W = [w_1, w_2, .., w_n]^T$ with $w_i = \ \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, So we have:

$$y_{OWA} = \alpha_{OWA} + \beta_{1OWA} x_j + \beta_{2OWA} z_j, \tag{10}$$

where α_{OWA}, β_{1OWA} and β_{2OWA} are estimated through OLS method with OWA variances and covariances which are given in Box I, where x_j, z_j, and y_j are the jth most extensive arguments in the variables$x$, z and y severally, and μ, ν and υ are means calculated with the OWA operator. In this way, we obtain different estimating forms depending on where we use the OWA means on the variance or covariance. A representative example of combinations is shown in Appendix A.

We can see that many more combinations with variances and covariances can be made. Additionally, we can consider the combinations formed when the OWA means are used in only one position of the covariances.

In the case of β_2 estimation the formula is developed as given in Box II, where x_j, z_j, and y_j are the jth most extensive arguments in the variables x, z and y severally, and μ, ν and υ are OWA means. One can see that the combinations with variances and covariances are the same as in the β_1 estimation.

In this sense, we can estimate α using β_1 and β_2 as follows:

$$\alpha_{OWA} = \upsilon - \beta_{1OWA}\mu - \beta_{2OWA}\nu, \tag{13}$$

where μ, ν and υ are OWA means, β_{1OWA} and β_{2OWA} are calculated with variances and covariances OWA. We can obtain combinations as Appendix B shows.

We shall further see that we can obtain more combinations by uniting α, β_1 and β_2.

Another interesting issue is the special cases of MLR-OWA. Special cases can appear depending on the ordering weights in variance and covariance. In this sense, we analyze the different variance and covariance cases:

First, we obtain the MAX- Var_{OWA} if we have:

$$MAX - Var_{OWA} = \sum_{j=1}^{n} w_j D_j, \tag{14}$$

where D_j is the largest $(a_i - \mu)^2$, using μ with OWA min, and $w_j = 1$. Then we follow a similar way MAX- Cov_{OWA}:

$$MAX - Cov_{OWA} = \sum_{j=1}^{n} w_j K_j, \tag{15}$$

where K_j is the jth largest of the $(x_i - \mu)(y_i - \nu)$, and μ and ν are OWA min of X and Y respectively and $w_j = 1$.

Second, we develop the MIN- Var_{OWA} if we have:

$$MIN - Var_{OWA} = \sum_{j=1}^{n} w_j D_j, \tag{16}$$

where D_j is the smallest $(a_i - \mu)^2$, using μ with OWA max, and $w = (0, \ldots, 0, 1)$. Note that the use *of* $\mu = OWA\ min$ means that there is a component where $a_i = \mu$, then the smallest D_j is equal to zero and:

$$MIN - Var_{OWA} = \sum_{j=1}^{n} w_j D_j = 0, \tag{17}$$

In this sense, MIN- Cov_{OWA}:

$$MIN - Cov_{OWA} = \sum_{j=1}^{n} w_j K_j = 0, \tag{18}$$

where K_j is the jth smaller of the $(x_i - \mu)(y_i - \nu)$, and μ and ν use OWA max of X and Y respectively and $w = (0, \ldots, 0, 1)$.

$$\beta_{1OWA} = \frac{[Cov_{OWA}(y,x)]\,[var_{OWA}(z)] - [Cov_{OWA}(y,z)]\,[Cov_{OWA}(x,z)]}{[var_{OWA}(x)]\,[var_{OWA}(z)] - [Cov_{OWA}(x,z)]^2}$$

$$\beta_{1OWA} = \frac{\left[\sum_{k=1}^{k} w_j\left(y_j-\upsilon\right)\left(x_j-\mu\right)\right]\left[\sum_{k=1}^{k} w_j\left(z_j-\nu\right)^2\right] - \left[\sum_{k=1}^{k} w_j\left(y_j-\upsilon\right)\left(z_j-\nu\right)\right]\left[\sum_{k=1}^{k} w_j\left(x_j-\mu\right)\left(z_j-\nu\right)\right]}{\left[\sum_{k=1}^{k} w_j\left(x_j-\mu\right)^2\right]\left[\sum_{k=1}^{k} w_j\left(z_j-\nu\right)^2\right] - \left[\sum_{k=1}^{k} w_j\left(x_j-\mu\right)\left(z_j-\nu\right)\right]^2}, \quad (11)$$

Box I.

$$\beta_{2OWA} = \frac{[Cov_{OWA}(y,z)]\,[var_{OWA}(x)] - [Cov_{OWA}(y,x)]\,[Cov_{OWA}(x,z)]}{[var_{OWA}(x)]\,[var_{OWA}(z)] - [Cov_{OWA}(x,z)]^2}$$

$$\beta_{2OWA} = \frac{\left[\sum_{k=1}^{k} w_j\left(y_j-\upsilon\right)\left(z_j-\nu\right)\right]\left[\sum_{k=1}^{k} w_j\left(x_j-\mu\right)^2\right] - \left[\sum_{k=1}^{k} w_j\left(y_j-\upsilon\right)\left(x_j-\mu\right)\right]\left[\sum_{k=1}^{k} w_j\left(x_j-\mu\right)\left(z_j-\nu\right)\right]}{\left[\sum_{k=1}^{k} w_j\left(x_j-\mu\right)^2\right]\left[\sum_{k=1}^{k} w_j\left(z_j-\nu\right)^2\right] - \left[\sum_{k=1}^{k} w_j\left(x_j-\mu\right)\left(z_j-\nu\right)\right]^2}, \quad (12)$$

Box II.

Then, we obtain different MLR-OWA special cases as follows:

1. β_1 and β_2 maximum OWA are obtained when we use the maximum OWA in the numerator and denominator, as follows:

$$\beta_{1MAX} = \frac{[MAXCov_{OWA}(y,x)\;\; MAXvar_{OWA}(z)] - [MAXCov_{OWA}\,(y,z)\,MAX\quad Cov_{OWA}(x,z)]}{[MAXvar_{OWA}(x)\;\; var_{OWA}(z)] - [MAXCov_{OWA}(x,z)]^2}. \quad (19)$$

In the same way, the maximum β_2 is:

$$\beta_{2MAX} = \frac{[MAXCov_{OWA}(y,z)\;\; MAXvar_{OWA}(x)] - [MAXCov_{OWA}\,(y,x)\,MAX\quad Cov_{OWA}(x,z)]}{[MAXvar_{OWA}(x)\;\; MAXvar_{OWA}(z)] - [MAXCov_{OWA}(x,z)]^2}. \quad (20)$$

2. β_1 and β_2 with minimum OWA presents an interesting case as shown:

$$\beta_{1MIN} = \frac{[MINCov_{OWA}(y,x)\;\; MINvar_{OWA}(z)] - [MINCov_{OWA}(y,z)MIN\;\; Cov_{OWA}(x,z)]}{[MINvar_{OWA}(x)MIN\;\; var_{OWA}(z)] - MIN\,[Cov_{OWA}(x,z)]^2}$$

$$\beta_{1MIN} = \frac{0}{0} = IND. \quad (21)$$

Recall that both minimum variances and minimum covariances are zero, then the calculation for β_{1MIN} is indeterminate. However, in the case of β_{1MIN} estimation, the result is equal.

3. The α with maximum OWA is calculated as follows:

$$\alpha_{OWA-MAX} = \upsilon - \beta_{1MAX}\mu - \beta_{2MAX}\nu. \quad (22)$$

4. We have the maximum MLR-OWA when we use all parameters (α, β_1 and β_2) with the maximum OWA, this is:

$$MLR - OWA_{MAX} = \alpha_{OWA-MAX} + \beta_{2MAX}x_i + \beta_{2MAX}z_i. \quad (23)$$

The MLR-OWA application shares the properties of OWA operators as follows:

The arguments values are monotonic, then, let a further ordered argument vector $w = [b_1, b_2, \ldots, b_n]$ such that for each j, if $a_j \geq b_j$ then we have $F(y_{owa}\,(a_1, a_2, \ldots, a_n)) \geq F(y_{OWA}(b_1, b_2, \ldots, b_3))$.

Symmetry, this is, if A and A' are the ordered argument vectors ($A = a_1, a_2, \ldots, a_n$; $A' = a'_1 a'_2, \ldots, a'_n$) then $A = A'$, consequently $F\,(y_{owa}\,(a_1, a_2, \ldots, a_n)) = F(y_{owa})(a'_1, a'_2, \ldots, a'_n)$.

Idempotent. if $a_j = a$, for all $j = 1, \ldots, n$, then $F(y_{owa}(a_1, a_2, \ldots, a_n)) = a$.

Additionally, the weights vector in the OWA operator (w) can be analyzed using different measures such as the degree of orness and the entropy of dispersion [29].

If the first weight $w_1 = 1$, we have a pure "or" operator. Then, the closer all the total weights to being in w_1 is the degree of orness in an OWA operator, so it is obtained by:

$$\alpha\,(W) = \sum_{j=1}^{n} w_j^* \left(\frac{n-j}{n-1}\right), \quad (24)$$

where w_j^* is the w_j weight with the jth largest a_i.

In order to capture the variability and the use of the inputs by the OWA weights is introduced the entropy of dispersion, we get:

$$H\,(W) = -\sum_{j=1}^{n} w_j \ln(w_j). \quad (25)$$

Other interesting measures of the OWA weights vector are the balance operator [50] and divergence [53]. To measure the degree of balance between favoring the higher valued elements or lower-valued elements in an OWA operator, we use:

$$BAL\,(W) = \sum_{j-1}^{n} \left(\frac{n+1-2j}{n-1}\right) w_j. \quad (26)$$

$$\beta_{1GOWA} = \frac{[Cov_{GOWA}(y,x)]\,[var_{GOWA}(z)] - [Cov_{GOWA}(y,z)]\,[Cov_{GOWA}(x,z)]}{[var_{GOWA}(x)]\,[var_{GOWA}(z)] - [Cov_{GOWA}(x,z)]^2}$$

$$\beta_{1GOWA} = \frac{\left[\sum_{k=1}^{k} w_j\left(y_j-\upsilon\right)\left(x_j-\mu\right)\right]\left[\sum_{k=1}^{k} w_j\left(z_j-\nu\right)^2\right]-\left[\sum_{k=1}^{k} w_j\left(y_j-\upsilon\right)\left(z_j-\nu\right)\right]\left[\sum_{k=1}^{k} w_j\left(x_j-\mu\right)\left(z_j-\nu\right)\right]}{\left[\sum_{k=1}^{k} w_j\left(x_j-\mu\right)^2\right]\left[\sum_{k=1}^{k} w_j\left(z_j-\nu\right)^2\right]-\left[\sum_{k=1}^{k} w_j\left(x_j-\mu\right)\left(z_j-\nu\right)\right]^2}, \quad (29)$$

$$\beta_{2GOWA} = \frac{[Cov_{GOWA}(y,z)]\,[var_{GOWA}(x)] - [Cov_{GOWA}(y,x)]\,[Cov_{GOWA}(x,z)]}{[var_{GOWA}(x)]\,[var_{GOWA}(z)] - [Cov_{GOWA}(x,z)]^2}$$

$$\beta_{2GOWA} = \frac{\left[\sum_{k=1}^{k} w_j\left(y_j-\upsilon\right)\left(z_j-\nu\right)\right]\left[\sum_{k=1}^{k} w_j\left(x_j-\mu\right)^2\right]-\left[\sum_{k=1}^{k} w_j\left(y_j-\upsilon\right)\left(x_j-\mu\right)\right]\left[\sum_{k=1}^{k} w_j\left(x_j-\mu\right)\left(z_j-\nu\right)\right]}{\left[\sum_{k=1}^{k} w_j\left(x_j-\mu\right)^2\right]\left[\sum_{k=1}^{k} w_j\left(z_j-\nu\right)^2\right]-\left[\sum_{k=1}^{k} w_j\left(x_j-\mu\right)\left(z_j-\nu\right)\right]^2}, \quad (30)$$

$$\alpha_{GOWA} = \upsilon - \beta_{1GOWA}\mu - \beta_{2GOWA}\nu, \quad (31)$$

Box III.

In order to distinguish between two OWA weights vectors, we use the divergence of W, we get:

$$DIV\ (W) = \sum_{j=1}^{n} w_j\left(\frac{n-j}{n-1} - \alpha(W)\right)^2. \quad (27)$$

We can use this measurement to calculate the OWA weights vector of α, β_1 and β_2 in multiple linear regression.

4. Generalized operators in multiple linear regression

Yager [31] proposed a generalization in OWA operators (GOWA). It provides a combination with the generalized mean operator introduced by Dyckhoff and Pedrycz [45] and the OWA operator. The GOWA operator adds a parameter that controls the intensity to which the argument values are elevated. It can be defined as follows:

Definition 7. A GOWA multiple linear regression that has two independent variables of dimension n is a model $GOWA: R^n \rightarrow R$ given the parameters $x_k \in U^n$, $y_k \in U$ and $z_k \in U^n$ with a weights vector $W = [w_1, w_2, .., w_n]^T$ then $w_i = \ \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, So we have:

$$y_{GOWA} = \alpha_{GOWA} + \beta_{1GOWA}x_j + \beta_{2GOWA}z_j, \quad (28)$$

where α_{GOWA}, β_{1GOWA}, and β_{2GOWA} are estimators of the variable dependent y behavior. We estimated α, β_1, and β_2 through the OLS method with GOWA variances and covariances which are given in Box III, where GOWA replaces the means of x, z and y with a parameter $\lambda \in (-\infty. + \infty)$. We can see that the calculation of the parameters α, β_1 and β_2 is similar to the previous section. Then, parameters have many forms of analysis, as we saw. Another essential combination that we can get is with the linear regression model, as Appendix C shows.

Some special cases in GOWA operators are applied in MLR-GOWA when λ takes different values as follows:

1. If $\lambda = 1$, the MLR-GOWA application becomes the MLR-OWA, then the means are calculated with $GOWA(a_1 \ldots..a_n) = \sum_{j=1}^{n} w_j b_j$.
2. When $\lambda = 0$, the MLR-GOWA application is the MLR-OWG. This is, the means are represented with the ordered weighted geometric $GOWA\,(a_1 \ldots.a_n) = \prod_{j=1}^{n} b_j^{w_j}$.
3. If $\lambda = -1$ the aggregation in MLR-GOWA is with OWHA operator, we have means with $GOWA\,(a_1 \ldots\ldots a_n) = \frac{1}{\sum_{j=1}^{n}\frac{w_j}{b_j}}$.
4. When $\lambda = 2$, we get an MLR-OWQA. This is, we calculated means with the ordered weighted quadratic averaging operator, thus $GOWA(a_1 \ldots.a_n) = \left(\sum_{j=1}^{n} w_j b_j^2\right)^{1/2}$.

We note this application is also symmetric, monotonic and idempotency like GOWA operators.

5. Estimating volatility in USD/MXN exchange rate using multiple regression with OWA operators

5.1. Theoretical background

Volatility is considered a measure of the vulnerability of financial markets and the economy [54–56]. It refers to the risk measurement of the intensity in fluctuations expected return of prices. On the one hand, volatility can be caused by occasional structural breaks because of various factors such as financial crisis, significant changes in markets, generation of speculative bubbles, and regime changes in monetary and debt management policies [57,58]. On the other hand, volatility can be generated by spillovers. The importance of such volatilities in different market participants or the volatilities around the world impacts each other [59].

Volatility in the exchange rate affects the corporate, leverage decisions and macro-economic variables [60,61]. In addition, there has been vital interest in predicting the exchange rate movements because of growing volatility in the forex market [62] and its influence on the exchange rate currencies around the world [63].

The fluctuation of exchange rates might not be understood entirely due to quite complicated patterns (trends, abrupt changes, and volatility clustering). In this sense, the exchange rates are complex to predict using economic models [64,65]. In particular, a simple theoretical model such as the random walk is frequently found to generate better exchange rate forecasts [66–68].

However, many volatility time series models have been developed for financial data with changing variance over time [69,70], considering past information for the estimation. The autoregressive (AR) models are a simple proposal for estimating volatility, applied in different contexts [71–74].

Methods of fuzzy modeling are promising techniques for describing complex dynamic systems [75,76]. The volatility estimation using fuzzy systems contributes to capturing the complicated patterns in the financial time series [77,78]. We propose the use of OWA operators in multiple linear regression in order to obtain an estimate adaptable to scenarios of optimism and pessimism. For example, optimism arises if we expect a future of low volatility. Therefore, the weight vector is balanced towards the first elements. Otherwise, a pessimistic scenario implies that the weights are oriented towards the last elements.

Because volatility is an unobservable variable [79,80], we must measure it with a variability framework. The standard deviation is a measure usually used in statistics for financial volatility [81]. We have:

$$h_t = \frac{\sqrt{\sum_i^n (x_i - \bar{x})^2}}{n}, \tag{32}$$

were h_t is the volatility studied, which is calculated with the 2018 FIX exchange rate USD/MXN. An autoregressive AR (2) model is used to estimate volatility as follows:

$$h_t = \alpha + \beta_1 h_{t-1} + \beta_2 h_{t-2}, \tag{33}$$

where h_{t-1} is the volatility FIX exchange rate with one lag and h_{t-2} is the volatility FIX exchange rate with two lags. Case 1 in Table 3 is applied for each of the operators previously seen. The models are as follows:

$$y_{OWA} = \alpha_{OWA} + \beta_{1OWA} x_j + \beta_{2OWA} z_j \tag{34}$$

$$y_{OWA_{MAX}} = \alpha_{OWA_{MAX}} + \beta_{1OWA_{MAX}} x_j + \beta_{2OWA_{MAX}} z_j \tag{35}$$

$$y_{OWA_{MIN}} = \alpha_{OWA_{MIN}} + \beta_{1OWA_{MIN}} x_j + \beta_{2OWA_{MIN}} z_j \tag{36}$$

$$y_{GOWA} = \alpha_{GOWA_{\lambda=-1}} + \beta_{1GOWA_{\lambda=-1}} x_j + \beta_{2GOWA_{\lambda=-1}} z_j \tag{37}$$

$$y_{GOWA} = \alpha_{GOWA_{\lambda=2}} + \beta_{1GOWA_{\lambda=2}} x_j + \beta_{2GOWA_{\lambda=2}} z_j \tag{38}$$

$$y_{GOWA} = \alpha_{GOWA_{\lambda=0}} + \beta_{1GOWA_{\lambda=0}} x_j + \beta_{2GOWA_{\lambda=0}} z_j, \tag{39}$$

where the $y_{OWA_{MAX}}$ and the $y_{OWA_{MIN}}$ are estimated with OWA_{MAX} mean and OWA_{MIN} mean in all variances and covariances.

5.2. The MLR-OWA parameters

The example considers the FIX exchange rate for the Mexican peso-US dollar. The information was extracted directly from the official website of the Mexican central bank (https://www.banxico.org.mx/). The data is daily from January 1, 2018, to December 31, 2018. The information was gathered monthly to calculate volatility. Fig. 1 shows the behavior.

Note that the volatility in the indicator is highly variable. This instability makes estimating and forecasting challenges.

To start the estimation, process the parameters α, β_1 and β_2 are calculated using OWA and GOWA means as follows: α_{OWA} in (13), β_{1OWA} in (11) and β_{2OWA} in (12); α_{GOWA} in (31), β_{1GOWA} in (29) and β_{2GOWA} in (30). Then, we perform the following steps:

Step 1: Define the number of elements in the attribute vector B and its components; we use a sequence $n = 12$ because the decision-maker believes that these are the periods that influence the determination of the estimated variable (volatility). We have three different vectors; vector h_t with the current exchange rate volatility, vector h_{t-1} with the exchange rate volatility with a delay period and vector h_{t-2}, which is the exchange rate volatility with two delays periods.

Step 2: we continue with the construction of weights vector w for the vectors B in h_t, h_{t-1} and h_{t-2}. The weights vector can be the same according to the criteria considered. The larger weights are assigned to the closest months considering that they impact current volatility the most. Vector B and W are shown in Table 1.

The resulting means are shown in Table 2. One can see that here we use the simple OWA corresponding in each case. The results consider the special cases in both OWA and GOWA operators.

Step 3: we calculate the variance and covariance OWA using the OWA means and a vector of weights. Table 3 shows the data. Note that the results are similar in $GOWA_{\lambda=2}$ and $GOWA_{\lambda=0}$, while with four decimals, the same data is displayed.

Step 4: Next, the parameters α, β_1 and β_2 are estimated. The results in OWA_{MIN} cannot be calculated. As we saw earlier,

Table 1
OWA vectors data.

h_t		h_{t-1}		h_{t-2}	
B_1	W_1	B_2	W_2	B_3	W_3
0.3409	0.15	0.3732	0.15	0.2623	0.15
0.1296	0.1	0.3409	0.1	0.3732	0.1
0.1819	0.05	0.1296	0.05	0.3409	0.05
0.3296	0.05	0.1819	0.05	0.1296	0.05
0.2525	0.05	0.3296	0.05	0.1819	0.05
0.3186	0.05	0.2525	0.05	0.3296	0.05
0.1915	0.05	0.3186	0.05	0.2525	0.05
0.2421	0.05	0.1915	0.05	0.3186	0.05
0.2378	0.05	0.2421	0.05	0.1915	0.05
0.4284	0.1	0.2378	0.1	0.2421	0.1
0.2080	0.15	0.4284	0.15	0.2378	0.15
0.2893	0.15	0.2080	0.15	0.4284	0.15

Table 2
Parameters results.

	υ	μ	ν
OWA	0.2692	0.2916	0.2880
OWA_{MAX}	0.4284	0.4284	0.4284
OWA_{MIN}	0.1296	0.1296	0.1296
$GOWA_{\lambda=-1}$	0.2413	0.2616	0.2625
$GOWA_{\lambda=2}$	0.2819	0.3052	0.2999
$GOWA_{\lambda=0}$	0.2556	0.2769	0.2755

Table 3
Analysis of variances and covariances OWA.

	$var(x)$	$var(z)$	$Cov\ (y, x)$	$Cov\ (y, z)$	$Cov\ (x, z)$
OWA	0.0081	0.0070	−0.0018	−0.0016	−0.0022
OWA_{MAX}	0.0561	0.0561	0.0441	0.0417	0.0260
OWA_{MIN}	0	0	0	0	0
$GOWA_{\lambda=-1}$	0.0090	0.0076	−0.0010	−0.0009	−0.0015
$GOWA_{\lambda=2}$	0.0083	0.0071	−0.0016	−0.0014	−0.0021
$GOWA_{\lambda=0}$	0.0083	0.0071	−0.0016	−0.0014	−0.0021

Table 4
β_1 results.

	OWA	OWA_{MAX}	OWA_{MIN}	$GOWA_{\lambda=-1}$	$GOWA_{\lambda=2}$	$GOWA_{\lambda=0}$
β_1	−0.3151	0.5631	Ind	−0.1321	−0.2689	−0.2620

Table 5
β_2 results.

	OWA	OWA_{MAX}	OWA_{MIN}	$GOWA_{\lambda=-1}$	$GOWA_{\lambda=2}$	$GOWA_{\lambda=0}$
β_2	−0.3276	0.4816	Ind	−0.1396	−0.2794	−0.2731

Table 6
Results.

	OWA	OWA_{MAX}	OWA_{MIN}	$GOWA_{\lambda=-1}$	$GOWA_{\lambda=2}$	$GOWA_{\lambda=0}$
α	0.4372	−0.0212	Ind	0.3364	0.4116	0.4080

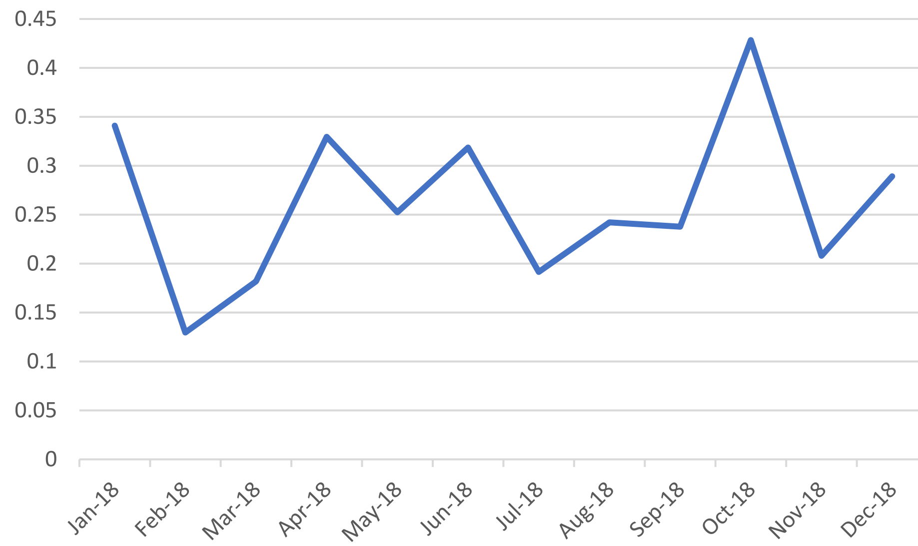

Fig. 1. Exchange rate volatility of the Mexican peso-US dollar.

Table 7
Forecast results.

	h_{t1}	Real	MAD	MSE	MAPE
Multiple linear regression	0.2295 0.2909 0.2713	0.1941 0.1005 0.1850	0.1040	0.0150	0.8477
Regression OWA	0.2313 0.2962 0.2681		0.1054	0.0155	0.8628
Regression OWA_{MAX}	0.3022 0.2491 0.2646		0.1121	0.0134	0.8220
Regression $GOWA_{\lambda=-1}$	0.2491 0.2744 0.2654		0.1047	0.0132	0.8162
Regression $GOWA_{\lambda=2}$	0.2359 0.2900 0.2677		0.1031	0.0148	0.8494
Regression $GOWA_{\lambda=0}$	0.2365 0.2892 0.2676		0.1046	0.0147	0.8476

variance and covariance minimum is zero then $\beta_1, \beta_2 = 0/0 =$ *indeterminated*.

The estimate of β_1 applies case 1 in Appendix A. The results are shown in Table 4. One can see that $GOWA_{\lambda=2}$ and $GOWA_{\lambda=0}$ data are similar. Thus, the OWA_{MAX} result is the one that makes the difference.

For β_2, we were using case 1 in Table 1. The results are shown in Table 5.

For α we using case 1 in Table 2, the results are in Table 6. Thus, the data in the estimation OWA simple and GOWA is similar.

Step 5: we estimate the volatility with the OWA multiple linear regression models. The OWA volatility models are the following:

$$h_{OWA} = 0.4372 - 0.3151 h_{t-1} - 0.3276\, h_{t-2} \tag{40}$$

$$h_{OWA_{max}} = -0.0212 + 0.5631 h_{t-1} + 0.4816 h_{t-2} \tag{41}$$

$$h_{OWA_{min}} = Indeterminated \tag{42}$$

$$h_{GOWA_{\lambda=-1}} = 0.3364 - 0.1321 h_{t-1} - 0.1396 h_{t-2} \tag{43}$$

$$h_{GOWA_{\lambda=2}} = 0.4116 - 0.2689 h_{t-1} - 0.2794 h_{t-2} \tag{44}$$

$$h_{GOWA_{\lambda=0}} = 0.4080 - 0.2620 h_{t-1} - 0.2731 h_{t-2} \tag{45}$$

The h_{t-1} and h_{t-2} are replaced with the last data values in each case. Finally, the results are compared with simple ordinary least squares in multiple linear regression. Table 7 summarizes the results.

The results show that the estimates with the most coincidences are between the linear regression with OLS and the regression $GOWA_{\lambda=0}$. Even the error measures are very similar. The estimated values for the examples are between 0.2295 and 0.2962, resulting in common OLS and maximum OWA.

The linear regression that minimizes MAD, MSE and MAPE errors is regression $GOWA_{\lambda=-1}$. The maximum error is in the regression OWA. One can analyze these results for future estimates because the data reflects a level of estimation of the parameters that can be useful in different scenarios.

6. Conclusions

The traditional application of multiple linear regression with the ordinary least squares is generally suitable for forecasting and estimation when there are phenomena originated by multiple variables. However, the linear solution that this methodology offers makes it ineffective in dealing with complex data. On the other hand, the OWA operators are a methodology that uses

Table A.1
Analysis of β parameter cases.

Case	Formula	Description
1	$\beta_{1OWA} = \frac{[Cov_{owa}(x,y)][var_{OWA}(z)] - [Cov_{owa}(z,y)][Cov_{owa}(x,z)]}{[var_{OWA}(x)][var_{OWA}(z)] - [Cov_{owa}(x,z)]^2}$	All the arithmetic means in the variances and covariances are replaced by the OWA means.
2	$\beta_{1OWA} = \frac{[Cov_{owa}(x,y)][var(z)] - [Cov_{owa}(z,y)][Cov_{owa}(x,z)]}{[var(x)][var(z)] - [Cov_{owa}(x,z)]^2}$	In the covariances OWA means are used, while the calculation in the variances use arithmetic means.
3	$\beta_{1OWA} = \frac{[Cov(x,y)][var_{OWA}(z)] - [Cov(z,y)][Cov(x,z)]}{[var_{OWA}(x)][var_{OWA}(z)] - [Cov(x,z)]^2}$	In the covariances arithmetic means are used, while the calculation of the variances use OWA means.
4	$\beta_{1OWA} = \frac{[Cov_{owa}(x,y)][var(z)] - [Cov(z,y)][Cov(x,z)]}{[var(x)][var(z)] - [Cov(x,z)]^2}$	Only the covariance (x,y) use OWA means, the leftover covariances and variances are calculated with arithmetic means.
5	$\beta_{1OWA} = \frac{[Cov(x,y)][var(z)] - [Cov_{owa}(z,y)][Cov(x,z)]}{[var(x)][var(z)] - [Cov(x,z)]^2}$	The covariance (z,y) use OWA means, the leftover covariances and variances are calculated with arithmetic means.
6	$\beta_{1OWA} = \frac{[Cov(x,y)][var(z)] - [Cov(z,y)][Cov_{owa}(x,z)]}{[var_{OWA}(x)][var_{OWA}(z)] - [Cov_{owa}(x,z)]^2}$	The covariance (z,y) use OWA means, the leftover covariances and variances are calculated with arithmetic means.
7	$\beta_{1OWA} = \frac{[Cov(x,y)][var_{OWA}(z)] - [Cov(z,y)][Cov(x,z)]}{[var(x)][var_{OWA}(z)] - [Cov(x,z)]^2}$	The variance (z) is calculated with OWA means, while the variance (x) and covariances use arithmetic means.
8	$\beta_{1OWA} = \frac{[Cov(x,y)][var(z)] - [Cov(z,y)][Cov(x,z)]}{[var_{OWA}(x)][var(z)] - [Cov(x,z)]^2}$	The variance (x) is calculated with OWA means, while the variance (z) and covariances use arithmetic means.
9	$\beta_{1OWA} = \frac{[Cov_{owa}(x,y)][var_{OWA}(z)] - [Cov_{owa}(z,y)][Cov_{owa}(x,z)]}{[var(x)][var(z)] - [Cov(x,z)]^2}$	OWA means are used in the variances and covariances in the fraction numerator, and the variances and covariances in the denominator use arithmetic mean.
10	$\beta_{1OWA} = \frac{[Cov(x,y)][var(z)] - [Cov(z,y)][Cov(x,z)]}{[var_{OWA}(x)][var_{OWA}(z)] - [Cov_{owa}(x,z)]^2}$	In the variances and covariances in the fraction numerator, arithmetic means are used. The variances and covariances in the denominator used OWA means.

Table B.1
Analysis of α parameter.

Case	Formula	Description
1	$\alpha_{OWA} = \upsilon - \beta_{1OWA}\mu - \beta_{2OWA}\nu$	All parameters use OWA means
2	$\alpha_{OWA} = \upsilon - \beta_1\bar{x} - \beta_2\bar{z}$	y arithmetic mean is replaced by OWA mean υ, the rest of the parameters ($\beta_1, \bar{x}, \beta_{2,}\bar{z}$) are calculated with arithmetic means.
3	$\alpha_{OWA} = \upsilon - \beta_1\mu - \beta_2\bar{z}$	y and x arithmetic means are replaced by OWA means (υ, μ), the rest of the parameters ($\beta_1, \beta_{2,}\bar{z}$) are calculated with arithmetic means.
4	$\alpha_{OWA} = \upsilon - \beta_{1OWA}\bar{x} - \beta_2\bar{z}$	OWA means are used in averages of υ and β_1. The rest of the parameters ($\bar{x}, \beta_{2,}\bar{z}$) are calculated with arithmetic means.
5	$\alpha_{OWA} = \upsilon - \beta_1\bar{x} - \beta_{2OWA}\bar{z}$	OWA means are used in averages of υ and β_2. The rest of the parameters ($\bar{x}, \beta_{1,}\bar{z}$) are calculated with arithmetic means.
6	$\alpha_{OWA} = \upsilon - \beta_1\bar{x} - \beta_2\nu$	z and y arithmetic means are replaced by OWA means (υ, ν), the rest of the parameters ($\beta_1, \beta_{2,}\bar{x}$) are calculated with arithmetic means.
7	$\alpha_{OWA} = \upsilon - \beta_{1OWA}\mu - \beta_2\bar{z}$	OWA means are used in averages of υ, μ, and β_1, the parameter β_2 and $\bar{z}$ are calculated with arithmetic means.
8	$\alpha_{OWA} = \upsilon - \beta_1\mu - \beta_2\nu$	OWA means are used in averages of υ, μ, and ν, the parameters β_2 and β_1 are calculated with arithmetic means.
9	$\alpha_{OWA} = \upsilon - \beta_1\mu - \beta_{2OWA}\bar{z}$	OWA means are used in averages of υ, μ, and β_2, the parameter β_1 and z are calculated with arithmetic means.
10	$\alpha_{OWA} = \upsilon - \beta_{1OWA}\bar{x} - \beta_{2OWA}\bar{z}$	OWA means are used in averages of υ, β_1, and β_2. x and z are calculated with arithmetic means.
11	$\alpha_{OWA} = \upsilon - \beta_{1OWA}\bar{x} - \beta_2\nu$	OWA means are used in averages of ν, β_1 and υ, the parameter β_2 and x are calculated with arithmetic means.
12	$\alpha_{OWA} = \upsilon - \beta_1\bar{x} - \beta_{2OWA}\nu$	OWA means are used in averages of υ, β_2 and ν, the parameter β_1 and x are calculated with arithmetic means.
13	$\alpha_{OWA} = \upsilon - \beta_{1OWA}\mu - \beta_{2OWA}\bar{z}$	OWA means are used in averages of υ, β_1, β_2, and μ, the parameter z is calculated with arithmetic means.
14	$\alpha_{OWA} = \upsilon - \beta_{1OWA}\mu - \beta_2\nu$	Only the parameter β_2 uses arithmetic means. Leftover parameters are calculated with OWA means.
15	$\alpha_{OWA} = \upsilon - \beta_1\mu - \beta_{2OWA}\nu$	Only the parameter β_1 uses arithmetic means. Leftover parameters are calculated with OWA means.
16	$\alpha_{OWA} = \upsilon - \beta_{1OWA}\bar{x} - \beta_{2OWA}\nu$	Only the x uses the arithmetic mean. Leftover parameters are calculated with OWA means.

(continued on next page)

Table B.1 (*continued*).

Case	Formula	Description
17	$\alpha_{OWA} = \bar{y} - \beta_1\mu - \beta_2\bar{z}$	Only the μ average uses OWA means, and leftover parameters are calculated with the arithmetic mean.
18	$\alpha_{OWA} = \bar{y} - \beta_{1OWA}\bar{x} - \beta_2\bar{z}$	Only the averages in β_1 use OWA means. Leftover parameters are calculated with arithmetic means.
19	$\alpha_{OWA} = \bar{y} - \beta_1\bar{x} - \beta_{2OWA}\bar{z}$	Only the averages in β_2 use OWA means. Leftover parameters are calculated with arithmetic means.
20	$\alpha_{OWA} = \bar{y} - \beta_1\bar{x} - \beta_2\nu$	Only the ν average uses OWA means. Leftover parameters are calculated with an arithmetic mean.
21	$\alpha_{OWA} = \bar{y} - \beta_{1OWA}\mu - \beta_2\bar{z}$	OWA means are used in averages of μ and β_1, the rest of the parameters $(\bar{z}, \beta_{2,}\bar{y})$ are calculated with arithmetic means.
22	$\alpha_{OWA} = \bar{y} - \beta_1\mu - \beta_2\nu$	OWA means are used in averages of μ and ν, the rest of the parameters $(\bar{y}, \beta_{2,}\beta_1)$ are calculated with arithmetic means.
23	$\alpha_{OWA} = \bar{y} - \beta_1\mu - \beta_{2OWA}\bar{z}$	OWA means are used in averages of μ and β_2. The rest of the parameters $(\bar{z}, \bar{y}, \beta_1)$ are calculated with arithmetic means.
24	$\alpha_{OWA} = \bar{y} - \beta_{1OWA}\bar{x} - \beta_{2OWA}\bar{z}$	OWA means are used in averages of β_1 and β_2. The rest of the parameters $(\bar{z}, \bar{y}, \bar{x})$ are calculated with arithmetic means.
25	$\alpha_{OWA} = \bar{y} - \beta_{1OWA}\bar{x} - \beta_2\nu$	OWA means are used in averages of β_1 and ν. The rest of the parameters $(\bar{y}, \beta_2, \bar{x})$ are calculated with arithmetic means.
26	$\alpha_{OWA} = \bar{y} - \beta_1\bar{x} - \beta_{2OWA}\nu$	OWA means are used in averages of β_2 and ν. The rest of the parameters $(\bar{y}, \beta_1, \bar{x})$ are calculated with arithmetic means.
27	$\alpha_{OWA} = \bar{y} - \beta_{1OWA}\mu - \beta_{2OWA}\bar{z}$	OWA means are used in averages of β_1, β_2, and $\bar{x}$. The rest of the parameters $(\bar{z}, \bar{y})$ are calculated with arithmetic means.
28	$\alpha_{OWA} = \bar{y} - \beta_{1OWA}\mu - \beta_2\nu$	OWA means are used in averages of β_1, ν, and $\bar{x}$. The rest of the parameters $(\bar{z}, \beta_2)$ are calculated with arithmetic means.
29	$\alpha_{OWA} = \bar{y} - \beta_1\mu - \beta_{2OWA}\nu$	OWA means are used in averages of μ, β_2, and ν, the rest of the parameters $(\bar{y}, \beta_1)$ are calculated with arithmetic means.
30	$\alpha_{OWA} = \bar{y} - \beta_{1OWA}\bar{x} - \beta_{2OWA}\nu$	OWA means are used in averages of β_1, β_2, and ν. The rest of the parameters $(\bar{y}, \bar{x})$ are calculated with arithmetic means.

Table C.1
Analysis of multiple linear regression cases.

Case	Formula	Description
1	$y_{Gowa} = \alpha_{GOWA} + \beta_{1GOWA}x_j + \beta_{2GOWA}z_j$	GOWA means are used in averages of β_1, β_2 and α.
2	$y_{Gowa} = \alpha + \beta_{1GOWA}x_j + \beta_{2GOWA}z_j$	GOWA means are used in averages of β_1 and β_2. The parameter α is calculated with arithmetic means.
3	$y_{Gowa} = \alpha_{GOWA} + \beta_1x_j + \beta_{2GOWA}z_j$	GOWA means are used in averages of α and β_2. The parameter β_1 is calculated with arithmetic means.
4	$y_{Gowa} = \alpha_{GOWA} + \beta_{1GOWA}x_j + \beta_2z_j$	GOWA means are used in averages of α and β_1, the parameter β_2 is calculated with arithmetic means.
5	$y_{Gowa} = \alpha_{GOWA} + \beta_1x_j + \beta_2z_j$	Only the averages in use GOWA means, leftover parameters are calculated with arithmetic means.
6	$y_{Gowa} = \alpha + \beta_{1GOWA}x_j + \beta_2z_j$	Only the averages in β_1 use GOWA means, leftover parameters are calculated with arithmetic means.
7	$y_{Gowa} = \alpha + \beta_1x_j + \beta_{2GOWA}z_j$	Only the averages in β_2 use GOWA means. Leftover parameters are calculated with arithmetic means.

weightings and attributes that can be aggregated as a means, which we can overestimate or underestimate, forming a non-linear sequence. In the same way, GOWA operators can give more or less importance in the parameter estimation using generalized means with an additional parameter.

We have developed a new methodology that unifies the multiple linear regression and the OWA-GOWA operators. We have called it the MLR-OWA and the MLR-GOWA. The main characteristic of MLR-OWA is that we can obtain a multiple regression where its parameters are estimated by OWA means. As a result, we can yield regressions according to the optimism or pessimism scenarios considered by the decision-maker.

Finally, we developed an illustrative example in volatility exchange rate estimation between USD/MXN. We analyzed the application in the traditional multiple linear regression with OLS estimation, comparing it with special cases in the OWA and GOWA operators. Additionally, we showed the special cases of the new MLR-OWA approach.

In future research, we expect to develop further extensions to this approach by adding other extensions of OWA operators with moving averages [82], Distances [83] and theory of expertons [84]. We will also consider other fundamental applications in linear regression models such as the ARCH and GARCH models.

CRediT authorship contribution statement

Martha Flores-Sosa: Conceptualization, Validation, Investigation, Resources, Data curation, Writing – original draft, Writing – review & editing, Supervision. **Ezequiel Avilés-Ochoa:** Validation, Formal analysis, Resources, Data curation, Writing – original draft. **José M. Merigó:** Conceptualization, Methodology, Formal analysis, Investigation, Writing – original draft, Writing – review &

editing, Supervision. **Janusz Kacprzyk:** Methodology, Validation, Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A

See Table A.1.

Appendix B

See Table B.1.

Appendix C

See Table C.1.

References

[1] D. Moreira, The microeconomic impact on growth of SMEs when the access to finance widens: evidence from internet & high-tech industry, Procedia - Soc. Behav. Sci. 220 (2016) 278–287.

[2] M. Bun, T. Harrison, OLS and IV estimation of regression models including endogenous interaction terms, Econometric Rev. 38 (7) (2018) 814–827.

[3] D. Bhanot, V. Bapat, Sustainability index of micro finance institutions (MFIs) and contributory factors, Int. J. Soc. Econ. 42 (4) (2015) 383–407.

[4] J. Lqbal, Does gold hedge stock market, inflation and exchange rate risks? An econometric investigation, Int. Rev. Econ. Finance 48 (2017) 1–17.

[5] S. Winer, D. Cervone, J. Bryant, C. Mckinney, R. Liu, M. Nadorff, Distinguishing mediational models and analyses in clinical psychology: atemporal associations do not imply causation, J. Clin. Psychol. 72 (9) (2016) 947–955.

[6] N. Scott, J. Siltanen, Intersectionality and quantitative methods: assessing regression from a feminist perspective, Int. J. Soc. Res. Methodol. 20 (4) (2016) 373–385.

[7] L. Muñoz-Pascual, J. Galende, The impact of knowledge and motivation management on creativity: employees of innovative spanish companies, Empl. Relat. 39 (5) (2017) 732–752.

[8] A. Singla, I. Singh, A. Sethi, Technology push and demand pull practices for achieving sustainable development in manufacturing industries, J. Manuf. Technol. Manag. 29 (2) (2018) 240–272.

[9] J. Wooldridge, Introductory Econometrics: A Modern Approach, Cengage Learning, Mason, OH: South-Western, 2013.

[10] M. Stone, R. Brooks, Continuum regression: cross-validated sequentially constructed prediction embracing ordinary least squares, partial least squares and principal components regression, J. R. Stat. Soc. Ser. B Stat. Methodol. 52 (2) (1990) 237–258.

[11] F. Huang, Multilevel modeling and ordinary least squares regression: how comparable are they? J. Exp. Educ. 86 (2) (2017) 265–281.

[12] R. Oliveira, The lower tail of random quadratic forms with applications to ordinary least squares, Probab. Theory Related Fields 166 (3–4) (2016) 1175–1194.

[13] D. Cook, L. Forzani, Partial least squares prediction in high-dimensional regression, Ann. Statist. 47 (2) (2019) 884–908.

[14] Y. Bar-Yam, S. McKay, W. Christian, Dynamics of complex systems (studies in nonlinearity), Comput. Phys. 12 (1998) 335.

[15] Y. Peng, M. Nagata, An empirical overview of nonlinearity and overfitting in machine learning using COVID-19 data, Chaos Solitons Fractals 139 (2020) 110055.

[16] S. Gao, P. Monkhtarian, R. Johnston, Nonnormality of data in structural equation models, Transp. Res. Rec.: J. Transp. Res. Board 2082 (1) (2008) 116–124.

[17] A. Schmidt, C. Finan, Linear regression and the normality assumption, J. Clin. Epidemiol. 98 (2018) 146–151.

[18] J. Helliwell, How's life? Combining individual and national variables to explain subjective well-being, Econ. Model. 20 (2) (2003) 331–360.

[19] D. Bartram, Cross-sectional model-building for research on subjective well-being: Gaining clarity on control variables, Soc. Indic. Res. 155 (2021) 725–742.

[20] D. Bartram, Cross-sectional model-building for research on subjective well-being: Gaining clarity on control variables, Soc. Indic. Res. 155 (2021) 725–742.

[21] P.S. Kott, A model-based look at linear regression with survey data, Amer. Statist. 45 (2) (1991) 107–112.

[22] E.L. Zanutto, A comparison of propensity score and linear regression analysis of complex survey data, J. Data Sci. 4 (1) (2006) 67–91.

[23] P. Filzmoser, K. Nordhausen, Robust linear regression for high-dimensional data: An overview, Wiley Interdiscip. Rev. Comput. Stat. 13 (4) (2021) e1524.

[24] H. Tanaka, S. Uejima, K. Asai, Linear regression analysis with fuzzy model, IEEE Trans. Syst. Man Cybern. 12 (6) (1982) 903–907.

[25] W. Zeng, Q. Feng, J. Li, Fuzzy least absolute linear regression, Appl. Soft Comput. 52 (2017) 1009–1019.

[26] L. Zadeh, Fuzzy sets, Inf. Control 8 (3) (1965) 338–353.

[27] A. Bargiela, W. Pedrycz, T. Nakashima, Multiple regression with fuzzy data, Fuzzy Sets and Systems 158 (19) (2007) 2169–2188.

[28] M. Chen, B. Chen, A hybrid fuzzy time series model based on granular computing for stock price forecasting, Inform. Sci. 294 (2016) 227–241.

[29] R. Yager, G. Beliakov, OWA operators in regression problems, IEEE Trans. Fuzzy Syst. 18 (1) (2010) 106–113.

[30] R. Yager, On ordered weighted averaging aggregation operators in multi-criteria decision making, IEEE Trans. Syst., Man, Cybern. B 18 (1988) 183–190.

[31] J. Fodor, J. Marichal, M. Roubens, Characterization of the ordered weighted averaging operators, IEEE Trans. Fuzzy Syst. 3 (1995) 236–240.

[32] R. Yager, Generalized OWA aggregation operators, Fuzzy Optim. Decis. Mak. 3 (2004) 93–107.

[33] F. Blanco-Mesa, E. Leon-Castro, J.M. Merigo, A bibliometric analysis of aggregation operators, Appl. Soft Comput. 81 (2019) 105488.

[34] J. Merigó, Gil-Lafuente, Fuzzy induced generalized aggregation operators and its application in multi-person decision making, Expert Syst. Appl. 38 (2011) 9761–9772.

[35] J. Kacprzyk, R.R. Yager, J.M. Merigó, Towards human centric aggregation via the ordered weighted aggregation operators and linguistic data summaries: A new perspective on zadeh's inspirations, IEEE Comput. Intell. Mag. 14 (1) (2019) 16–30.

[36] E. Leon-Castro, E. Avilés-Ochoa, J. Merigó, Induced heavy moving averages, Int. J. Intell. Syst. 33 (9) (2018) 1823–1839.

[37] J. Merigó, R. Yager, Generalized moving averages, distances measures and OWA operator, Int. J. Uncertain. Fuzziness Knowl.-Based Syst. 21 (4) (2013) 533–559.

[38] R. Yager, On generalized Bonferroni mean operators for multi-criteria aggregation, Internat. J. Approx. Reason. 50 (8) (2009) 1279-1268.

[39] F. Blanco-Mesa, E. Leon-Castro, J. Merigó, Z.S. Xu, Bonferroni means with induced OWA operators, Int. J. Intell. Syst. 35 (2019) 3–23.

[40] S. Maldonado, J.M. Merigó, J. Miranda, Redefining support vector machines with the ordered weighted average, Knowl.-Based Syst. 148 (2018) 41–46.

[41] J. Merigó, A unified model between the weighted average and the induced OWA operator, Expert Syst. Appl. 38 (2011) 11560–11572.

[42] J. Merigó, M. Guillen, J. Sarabia, The ordered weighted average in the variance and the covariance, Int. J. Intell. Syst. 30 (2015) 985–1005.

[43] R. Yager, On generalized measures of realization in uncertain environments, Theory and Decision 33 (1992) 41–69.

[44] F. Chiclana, F. Herrera, E. Herrera-Viedma, The ordered weighted geometric operator: properties and application, in: Proc of 8th Int Conf on Information Processing and Management of Uncertainty in Knowledge-Based Systems, Madrid, 2000, pp. 985–991.

[45] H. Chen, C. Liu, Z. Sheng, Induced ordered weighted harmonic averaging (IOWHA) operator and its application to combination forecasting method, Chin. J. Manag. Sci. 12 (2004) 35–44.

[46] H. Dyckhoff, W. Pedrycz, Generalized means as model of compensative connectives, Fuzzy Sets and Systems 14 (1984) 143–154.

[47] J. Cohen, P. Cohen, S. West, L. Aiken, Applied Multiple Regression/Correlation Analysis for the Behavioral Sciences, Lawrence Erlbaum Associates, Publishers, Mahwah, New Jersey, 2003.

[48] L. Plonsky, F. Oswald, Multiple regession as a flexible alternative to ANOVA in L2 research, Stud. Second Lang. Acquis. 39 (3) (2017) 579–592, 2017.

[49] F. Bass, A new product growth for model consumer durables, Manage. Sci. 15 (5) (1969) 215–227.

[50] D. Gujarati, D. Porter, Basic Econometrics (Vol. 5ta Ed), McGraw-Hill Irwin, New York, NY, 2009.

[51] R. Yager, Constrained OWA aggregation, Fuzzy Sets and Systems 81 (1996) 89–101.

[52] S. Laengle, G. Loyola, J.M. Merigó, Mean–variance portfolio selection with the ordered weighted average, IEEE Trans. Fuzzy Syst. 25 (2) (2017) 350–362.

[53] F. Blanco-Mesa, E. Leon-Castro, J.M. Merigo, E. Herrera-Viedma, Variances with Bonferroni means and ordered weighted averages, Int. J. Intell. Syst. 34 (11) (2019) 3020–3045.

[54] R. Yager, Heavy OWA operators, Fuzzy Optim. Decis. Mak. 1 (2002) 379-297.

[55] H. Markowitz, Portfolio selection, J. Finance 7 (1) (1952) 77–91.

[56] W. Tung, C. Quek, Financial volatility trading using a self-organising neural-fuzzy semantic network and option straddle-based approach, Expert Syst. Appl. 38 (5) (2011) 4668–4688.

[57] A. Jiang, X. Liu, Volatility forecasting in the Chinese commodity futures market with intraday data, Rev. Quant. Financ. Account. 48 (4) (2016) 1–51.
[58] M. Pasaran, A. Timmenman, Market timing and return prediction under model instability, J. Empir. Financ. 9 (2002) 495–510.
[59] K. Yang, L. Chen, Realized volatility forecast: structural breaks, long memory, asymmetry, and day-of-the-week effect, Int. Rev. Finance 14 (3) (2014) 345–392.
[60] T. Bollerslev, B. Hood, J. Huss, L. Pedersen, Risk everywhere: modeling and managing volatility, Rev. Financ. Stud. 31 (7) (2018) 2729–2773.
[61] S. Gupta, S. Kashyap, Modelling volatility and forecasting of exchange rate of british pound sterling and Indian rupee, J. Model. Manag. 11 (2) (2016) 389–404.
[62] S. Lahmiri, Modeling and predicting historical volatility in exchange rate markets, Physica A 471 (2017) 387–395.
[63] J. Baruník, L. Vácha, Asymmetric volatility connectedness on the forex market, J. Int. Money Finance 77 (2017) 39–56.
[64] C. Shu, D. He, X. Cheng, One currency, two markets: the renminbi's growing influence in Asia-Pacific, China Econ. Rev. 33 (2015) 163–178.
[65] C. Engel, N. Mark, K. West, Factor model forecasts of exchange rates. NBER Working Paper, working paper (18382), 2012.
[66] B. Rossi, Exchange rate predictability, Am. Econ. Assoc. 51 (4) (2013) 1063–1119.
[67] R. Meese, K. Rogoff, Empirical exchange rate models of the seventies: Do they fit out of sample? J. Int. Econ. 14 (1) (1983) 3–24.
[68] J. Byrne, D. Korobilis, P. Ribeiro, Exchange rate predictability in a changing world, J. Int. Money Finance 62 (2016) 1–24.
[69] M. Zorzi, M. Kolasa, M. Rubaszek, Exchange rate forecasting with DSGE models, J. Int. Econ. 107 (2017) 127–146.
[70] G. Kastner, S. Fruhwirth-Schnatter, Ancillarity-sufficiency interweaving strategy (ASIS) for boosting MCMC estimation of stochastic volatility models, Comput. Statist. Data Anal. 76 (2014) 408–423.
[71] R. Dash, P. Dash, An evolutionary hybrid fuzzy computationally efficient EGARCH model for volatility prediction, Appl. Soft Comput. 45 (2016) 40–60.
[72] G. Chandler, W. Polonik, Discrimination of locally stationary time series based on the excess mass functional, J. Amer. Statist. Assoc. 101 (473) (2006) 240–253.
[73] P. Fryzlewicz, T. Sapatinas, S. Rao, Haar– fisz technique for locally stationary volatility estimation, Biometrika 93 (3) (2006) 687–704.
[74] M. McAleer, M. Medeiros, A multiple regime smooth transition heterogeneous autoregressive model for long memory and asymmetries, J. Econometrics 147 (2008) 104–119.
[75] R. Zhang, S. Ling, Asymptotic inference for AR models with heavy-tailed G-GARC noises, Econom. Theory 31 (4) (2015) 880–890.
[76] J. Hung, A fuzzy GARCH model applied to stock market scenario using a genetic algorithm, Expert Syst. Appl. 36 (9) (2009) 11710–11717.
[77] J. Merigó, M. Peris-Ortiz, A. Navarro-Garcia, C. Rueda-Armengot, Aggregation operators in economic growth analysis and entrepreneurial group decision-making, Appl. Soft Comput. 47 (2016) 141–150.
[78] L. Maciel, R. Ballini, F. Gomide, Evolving possibilistic fuzzy modeling for realized volatility forecasting with jumps, IEEE Trans. Fuzzy Syst. 25 (2) (2017) 302–314.
[79] E. Leon-Castro, L. Espinoza-Audelo, J.M. Merigó, A.M. Gil-Lafuente, J. Kacprzyk, A new measure of volatility using induced heavy moving averages, Technol. Econ. Dev. Econ. 25 (4) (2019) 576–599.
[80] T. Andersen, T. Bollerslev, F. Diebold, P. Labys, The distribution of realized exchange rate volatility, J. Amer. Statist. Assoc. 96 (453) (2001) 42–55.
[81] R. Becker, A. Clements, A. Doolan, Hurn, Selecting volatility forecasting models for portfolio allocation purposes, Int. J. Forecast. 31 (3) (2015) 849–891.
[82] K. Daly, Financial volatility: Issues and measuring techniques, Physica A 387 (2008) 2377–2393.
[83] J.M. Merigó, R.R. Yager, Aggregation operators with moving averages, Soft Comput. 23 (21) (2009) 10601–10615.
[84] V. Alfaro-Garcia, J.M. Merigó, L. Plata-Perez, G.G. Alfaro-Calderon, A.M. Gil-Lafuente, Induced and logarithmic distances with multi-region aggregation operators, Technol. Econ. Dev. Econ. 25 (4) (2019) 664–692.