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Decision Support

Generalized ordered weighted utility averaging-hyperbolic absolute risk aversion operators and their applications to group decision-making

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ABSTRACT

This paper develops a new class of aggregation operator based on utility function, which introduces the risk attitude of decision makers (DMs) in the aggregation process. First, under the general framework of utility function, we provide a new operator called the generalized ordered weighted utility averaging (GOWUA) operator, and study its properties which are suitable for any utility function. Then, under the hyperbolic absolute risk aversion (HARA) utility function, we propose another new operator named as the generalized ordered weighted utility averaging-hyperbolic absolute risk aversion (GOWUA-HARA) operator, and further investigate its families including a wide range of aggregation operators. To determine the GOWUA-HARA operator weights, we put forward an orness measure of the GOWUA-HARA operator and analyze its properties. Considering that different DMs may have different opinions toward decision-making and their opinions can be characterized by different orness measures, we construct a new optimization model to determine the optimal weights which can aggregate all the individual sets of weights into an overall set of weights. Furthermore, based on the GOWUA-HARA operator, a method for the multiple attribute group decision-making (MAGDM) is developed. Finally, an example is given to illustrate the application of the GOWUA-HARA operator to the MAGDM.

1. Introduction

Multiple attribute group decision-making (MAGDM) considers the problem of evaluating or selecting alternatives that are associated with incommensurate and conflicting criteria by a cooperative group, known as the group decision-making (Vahdani et al., 2013). To choose a desirable alternative, decision makers (DMs) often present their preference information which needs to be aggregated via some proper approaches. There are many methods for aggregating information (Canós & Liern, 2008; Chiclana, Herrera-Viedma, Herrera, & Alonso, 2007; Dong, Xu, Li, & Feng, 2010; Fullér & Majlender, 2003a, 2003b; Liu, 2013; Llamazares, 2004; Maes, Saminger, & De Baets, 2007; Merigó, 2008; Merigó & Casanovas, 2010a, 2010b; Merigó, Casanovas, & Yang, 2014; Merigó, Casanovas, & Zeng, 2014; Merigó & Gil-Lafuente, 2010, 2011, 2013; Merigó & Yager, 2013; Mesiar, 2007; Ribeiro & Pereira, 2003; Xu, 2004, 2006a, 2006b; Xu, Yang, & Wang, 2006; Yager, 1988, 2004, 2010; Yager & Filev, 1994, 1999; Yang, 2001; Yang, Yang, Liu, & Li, 2013; Zeng, Merigó, & Su, 2013; Zhang, 2013; Zhou & Chen, 2014). One of the most popular methods for aggregating decision-making information is the ordered weighted averaging (OWA) operator developed by Yager (1988). It provides a general class of parametric aggregation operators and has shown to be useful for studying many different kinds of aggregation problems. Up to now, the OWA operator has been used in a wide range of applications (Merigó & Gil-Lafuente, 2010, 2011; Yager, 2010).

Motivated by the OWA operator, an extension of the OWA operator is the generalized OWA (GOWA) operator, which combines the OWA operator with the generalized mean operator (Yager, 2004). It generalizes a wide range of aggregation operators such as the OWA operator, the ordered weighted geometric averaging (OWGA) operator (Xu & Da, 2002a), the ordered weighted harmonic averaging (OWHA) operator (Yager, 2004). Based on the optimization theory, Zhou and Chen (2010) presented the generalized ordered weighted logarithm averaging (GOWLA) operator, which is an extension of the OWGA operator. Other extension of the OWA operator can be found in literature (Merigó, 2008; Yager & Filev, 1999). However, the above aggregation operators only focus on using the mean to eliminate the difference, and do not consider the DMs' risk attitude in the aggregation process.

Another important issue of applying the OWA operator for MAGDM is how to determine the associated weights. Many researchers have focused on this issue and developed some useful approaches to obtaining the OWA weights. For example, O'Hagan (1988) suggested a maximum entropy approach to obtaining OWA operator

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weights under a given orness measure. Yager (1993) suggested an interesting way to compute the weights of the OWA operator using linguistic quantifiers. Fullér and Majlender (2003a) proposed an analytic approach for obtaining maximal entropy OWA operator weights under a given orness measure. Wang and Parkan (2005) proposed a minimax disparity approach for obtaining the OWA operator weights under a given orness measure. Majlender (2005) developed a maximal Rényi entropy method for generating a parametric class of the OWA operators and the maximal Rényi entropy OWA weights. Wang, Luo, and Liu (2007) constructed the chi-square (χ^2) model for determining the OWA operator weights under a given orness measure. Other extension approaches to determining the OWA operator weights can be found in literature (Ahn, 2010; Amin & Emrouznejad, 2010; Filev & Yager, 1998; Merigó, 2008; Sang & Liu, 2014; Xu, 2006a, 2006b; Xu & DA, 2002b, 2003; Yager, 1988, 2009a, 2009b). The methods mentioned above assume that any individual weighting vector is equal to the optimal aggregated weighting vector, and correspondingly there is only one orness measure to characterize the DMs' attitude toward decision-making. As a result, there is only one set of the OWA operator weights to be generated. However, this is not consistent with the real situation. In fact, multiple DMs may join in decision-making process to reach a holistic opinion that reflects a collective view of all the participants. In the decision-making process, different DMs may have different orness measures, and therefore the corresponding OWA operator weights may also be different. So it is necessary to introduce a new method to aggregate all the participants' preference in MAGDM.

This paper aims to develop a new class of aggregation operator based on utility function, which incorporates the risk attitude of DMs in the aggregation process. Under the general framework of utility function and based on an optimal deviation model, we firstly provide a new operator called the generalized ordered weighted utility averaging (GOWUA) operator, and then by studying its properties we find that it is commutative, monotonic, bounded and idempotent. These properties are suitable for any utility function. Furthermore, we focus on the hyperbolic absolute risk aversion (HARA) utility function, which is rather rich, e.g., by suitable adjustments of the parameters, power utility function and exponential utility function can be obtained respectively. Under the HARA utility, we propose another new operator called the generalized ordered weighted utility averaging-hyperbolic absolute risk aversion (GOWUA-HARA) operator, and study its families which contain a wide range of aggregation operators such as the OWGA operator, OWA operator, OWHA operator, GOWA operator, maximum operator, minimum operator. The main advantage of the GOWUA-HARA operator is that it cannot only reflect the DMs' risk attitude toward the aggregation information, but also provide a very general formulation including a wide range of aggregation operators.

In order to determine the weights of the GOWUA-HARA operator, we put forward an orness measure of the GOWUA-HARA operator, which is an extension of the orness measure of the GOWA operator presented by Yager (2004). We further investigate some properties associated with this orness measure. Noting that different DMs may have different perspectives toward decision-making, which can be characterized by different orness measures, this situation leads to different sets of the GOWUA-HARA operator weights corresponding to different orness measures. We then construct a new nonlinear optimization model to determine the optimal weighting vector of the GOWUA-HARA operator which can aggregate all the individual sets of weights into an overall set of weights. The main advantage of the nonlinear model is that it cannot only minimize the differences between the orness measures provided by each DM and the comprehensive orness measure corresponding to an optimal weighting vector, but also produce as equally important weights as possible.

Furthermore, based on the GOWUA-HARA operator, a new approach for MAGDM is developed. This approach is also effectively applicable to different group decision-making problems such as engineering management and financial management. In the end, we provide an application of the new approach for MAGDM in an example of the investment selection.

The rest of the paper is organized as follows. Section 2 reviews the OWA, OWGA and GOWA operators and introduces an HARA utility function. Section 3 presents a GOWUA operator and analyzes its properties. Especially, we provide a GOWUA-HARA operator and identify its families. Section 4 proposes an orness measure of the GOWUA-HARA operator and discusses its properties. In particular, we further construct a nonlinear model for determining the optimal weights which can aggregate each DM's opinion. Section 5 develops an approach for MAGDM under the GOWUA-HARA operator. An illustrative example is provided in Section 6 and the conclusions are drawn in Section 7.

2. Preliminaries

This section briefly reviews the OWA, OWGA and GOWA operators and introduces an HARA utility function which later will be used to develop a new aggregation operator in this paper.

2.1. The OWA operator

The ordered weighted averaging (OWA) operator was presented by Yager (1988), which can be defined as follows:

Definition 1. (Yager, 1988). A mapping OWA: $R^n \to R$ is called an ordered weighted aggregation (OWA) operator of dimension n if

$$\mathrm{OWA}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} w_i y_i, \tag{1}$$

where w_i is a weight satisfying $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, y_i is the ith largest of the x_j.

2.2. The OWGA operator

Xu and Da (2002a) provided the ordered weighted geometric averaging (OWGA) operator, which can be defined as follows:

Definition 2. (Xu & Da, 2002a). An ordered weighted geometric averaging (OWGA) operator is a mapping OWGA: $R^{+n} \to R^+$ that has a weighting vector $W = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ with $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, such that

$$\mathrm{OWGA}(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} y_i^{w_i}, \tag{2}$$

where y_i is the ith largest of the x_j.

2.3. The GOWA operator

Yager (2004) developed the generalized ordered weighted averaging (GOWA) operator, which is defined as follows:

Definition 3. (Yager, 2004). A generalized ordered weighted aggregation (GOWA) operator of dimension n is a mapping GOWA: $R^{+n} \to R^+$ that has a weighting vector $W = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ with $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, such that

$$\mathrm{GOWA}\,(x_1, x_2, \ldots, x_n) = \left(\sum_{i=1}^{n} w_i {y_i}^{\lambda}\right)^{1/\lambda}, \tag{3}$$

where y_i is the ith largest of x_j, and λ is a parameter such that $\lambda \in (-\infty, 0) \cup (0, +\infty)$.

By taking different values of the parameter λ in the GOWA operator, a group of particular cases can be derived. For example, the

GOWA operator reduces to the OWA operator when $\lambda = 1$; the GOWA operator degenerates to the OWGA operator when $\lambda \to 0$; the GOWA operator reduces to the OWHA operator as $\lambda = -1$.

In general, the basic feature of all aggregation operators is the non-decreasing monotonicity, expressing the idea that "an increase of any of the input values cannot decrease the output value". The desirable properties of each aggregation operator are commutative, monotonic, bounded and idempotent (Xu & Da, 2002a). Each aggregation operator mentioned above satisfies these properties. However, these operators cannot reflect the risk attitude of the DMs in the aggregation process. Therefore, when aggregating the input arguments, we can introduce the utility function which not only satisfies the basic feature and desirable properties of aggregation operators, but also can reflect the risk attitude of the DMs toward the input argument information.

2.4. HARA utility function

Generally, utility function $u(x)$ is a non-decreasing real valued function, which just captures the idea of aggregation operator that "an increase of any of the input values cannot decrease the output value". We investigate the hyperbolic absolute risk aversion (HARA) utility function (Grasselli, 2003; Jung & Kim, 2012), which is defined as:

$$u(x) = \frac{1-\gamma}{\beta\gamma}\left(\frac{\beta}{1-\gamma}x + \eta\right)^{\gamma}, \tag{4}$$

where $\beta > 0, \eta > 0, \gamma \in (-\infty, 0) \cup (0, 1)$ (for more discussions on the parameters, cf., Cox & Huang, 1989; Merton, 1971). The HARA utility is rather rich, since by suitable adjustments of the parameters one can obtain utility functions with absolute or relative risk aversion. In particular,

(1) if $\beta = 1 - \gamma$ and $\eta \to 0$ in Eq. (4), then we obtain the power utility function:

$$u(x) = \frac{x^{\gamma}}{\gamma}, \quad \gamma \in (-\infty, 0) \cup (0, 1), \tag{5}$$

(2) if $\eta = 1$ and compute the limit as $\gamma \to -\infty$ in Eq. (4), we obtain the exponential utility function:

$$u(x) = -\frac{e^{-\beta x}}{\beta}, \quad \beta > 0. \tag{6}$$

Pratt (1964) and Arrow (1965) suggested the risk aversion coefficient $r(x) = -u''(x)/u'(x)$ which is called the Pratt–Arrow measure of absolute risk aversion. The absolute risk aversion coefficient of HARA utility is $r(x, \gamma, \beta, \eta) = \beta(1-\gamma)/(\beta x + \eta(1-\gamma))$, which implies that with the relative risk aversion coefficient $r(x, \gamma, \beta, \eta)$ increasing, the risk attitude of DM's involved in the evaluation of decision information will become more prudent.

3. GOWUA-HARA operator

In this section, we firstly present a new operator called the generalized ordered weighted utility averaging (GOWUA) operator for the general utility function. Then, under the HARA utility function, we propose another new operator called the generalized ordered weighted utility averaging-hyperbolic absolute risk aversion (GOWUA-HARA) operator and identify its families.

3.1. General framework

Let $x_i\ (i = 1, 2, \ldots, n)$ be a collection of arguments, and $W = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ be a weighting vector such that $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. We assume that the utility aggregation operator of dimension n is a mapping f determined by:

$$u(z) = f(u(x_1), u(x_2), \ldots, u(x_n)). \tag{7}$$

In the aggregation process, we hope that the smaller the deviation between the utility values $u(x_i)\ (i = 1, 2, \ldots, n)$ and the aggregation result $u(z)$ is, the better effect of aggregation method shows. Hence, to minimize the deviation between $u(z)$ and $u(x_i)\ (i = 1, 2, \ldots, n)$, we have

$$\min P = \sum_{i=1}^{n} w_i\left(u^{\lambda}(z) - u^{\lambda}(x_i)\right)^2, \tag{8}$$

where λ is a parameter satisfying $\lambda \in (-\infty, 0) \cup (0, +\infty)$. By taking the first partial derivative w.r.t. z in the model (8), we obtain

$$\frac{\partial P}{\partial z} = 2\sum_{i=1}^{n} w_i(u^{\lambda}(z) - u^{\lambda}(x_i)) \times \lambda u^{\lambda-1}(z) \times u'(z)$$

According to the necessary condition of extreme value, let $\partial P/\partial z = 0$, we obtain

$$z = u^{-1}\left[\left(\sum_{i=1}^{n} w_i u^{\lambda}(x_i)\right)^{1/\lambda}\right]. \tag{9}$$

Based on Eq. (9), we can define a generalized weighted utility averaging (GWUA) operator shown as follows:

Definition 4. A GWUA operator of dimension n is a mapping GWUA: $R^{+n} \to R^{+}$ such that

$$\mathrm{GWUA}(x_1, x_2, \ldots, x_n) = u^{-1}\left[\left(\sum_{i=1}^{n} w_i u^{\lambda}(x_i)\right)^{1/\lambda}\right], \tag{10}$$

where the weighting vector $W = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ satisfies $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, and λ is a parameter such that $\lambda \in (-\infty, 0) \cup (0, +\infty)$.

If reordering the arguments in the GWUA operator in descending order, we can obtain a generalized ordered weighted utility averaging (GOWUA) operator.

Definition 5. A GOWUA operator of dimension n is a mapping GOWUA: $R^{+n} \to R^{+}$ such that

$$\mathrm{GOWUA}(x_1, x_2, \ldots, x_n) = u^{-1}\left[\left(\sum_{i=1}^{n} w_i u^{\lambda}(y_i)\right)^{1/\lambda}\right], \tag{11}$$

where λ is a parameter such that $\lambda \in (-\infty, 0) \cup (0, +\infty)$, and y_i is the ith largest of x_j and the weighting vector $W = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ such that $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$.

The following properties show that the GOWUA operator is idempotent, monotonic, bounded and commutative, which are suitable for any utility function.

Property 1. (*Idempotency*). Let f be the GOWUA operator. If $x_i = x$ for $i = 1, 2, \ldots, n$, then

$$f(x_1, x_2, \ldots, x_n) = x. \tag{12}$$

Proof. If $x_i = x$ for $i = 1, 2, \ldots, n$, then according to Eq. (11), we obtain that

$$f(x_1, x_2, \ldots, x_n) = u^{-1}\left[\left(\sum_{i=1}^{n} w_i u^{\lambda}(x)\right)^{1/\lambda}\right] = x.$$

The property is proved. □

Property 2. (*Monotonicity*). Let f be the GOWUA operator. If $x_i \geq t_i$ for $i = 1, 2, \ldots, n$, then

$$f(x_1, x_2, \ldots, x_n) \geq f(t_1, t_2, \ldots, t_n). \tag{13}$$

Proof. Let

$$f(x_1, x_2, \ldots, x_n) = u^{-1}\left[\left(\sum_{i=1}^{n} w_i u^{\lambda}(y_i)\right)^{1/\lambda}\right].$$

According to the above equation, taking the first partial derivative of f w.r.t. y_i, we have that

$$\frac{\partial f}{\partial y_i} = \left[\left(\sum_{i=1}^{n} w_i u^{\lambda}(y_i)\right)^{1/\lambda-1} \Big/ u'\left(u^{-1}\left[\left(\sum_{i=1}^{n} w_i u^{\lambda}(y_i)\right)^{1/\lambda}\right]\right)\right] \times \sum_{i=1}^{n} w_i u^{\lambda-1}(y_i) \times u'(y_i).$$

Since $u(y_i) > 0$ and $u'(y_i) > 0$, we obtain that $\partial f/\partial y_i > 0$, which implies that f monotonically increases w.r.t. y_i. Note that $x_i \geq t_i$ for $i = 1, 2, \ldots, n$, therefore, we get

$$f(x_1, x_2, \ldots, x_n) \geq f(t_1, t_2, \ldots, t_n).$$

The property is proved. □

Property 3. (*Boundedness*). Let f be the GOWUA operator. If $y_1 = \max_{1\leq i\leq n}\{x_i\}$ and $y_n = \min_{1\leq i\leq n}\{x_i\}$, then

$$y_n \leq f(x_1, x_2, \ldots, x_n) \leq y_1. \tag{14}$$

Proof. If $y_1 = \max_{1\leq i\leq n}\{x_i\}$, then by Property 2, we have that

$$f(x_1, x_2, \ldots, x_n) \leq f(y_1, y_1, \ldots, y_1) = u^{-1}\left[\left(\sum_{i=1}^{n} w_i (u(y_1))^{\lambda}\right)^{1/\lambda}\right] = y_1.$$

By the same token, we can conclude that $f(x_1, x_2, \ldots, x_n) \geq y_n$. Thus,

$$y_n \leq f(x_1, x_2, \ldots, x_n) \leq y_1.$$

The property is proved. □

Property 4. (*Commutativity*). Let f be the GOWUA operator. If v_i is any permutation of the arguments x_i for $i = 1, 2, \ldots, n$, then

$$f(x_1, x_2, \ldots, x_n) = f(v_1, v_2, \ldots, v_n). \tag{15}$$

Proof. Let $f(x_1, x_2, \ldots, x_n) = u^{-1}[(\sum_{i=1}^{n} w_i (u(y_i))^{\lambda})^{1/\lambda}]$ and $f(v_1, v_2, \ldots, v_n) = u^{-1}[(\sum_{i=1}^{n} w_i (u(t_i))^{\lambda})^{1/\lambda}]$.

Since $(v_1, v_2, \ldots, v_n)$ is any permutation of the arguments $(x_1, x_2, \ldots, x_n)$, we can get $y_i = t_i$ for all i. So, we obtain that

$$f(x_1, x_2, \ldots, x_n) = f(v_1, v_2, \ldots, v_n).$$

The property is proved. □

Property 5. (*Monotonicity with respect to parameter* λ). Let F be the GOWUA operator. If $\lambda_1 \geq \lambda_2$, we have

$$F(\lambda_1) \geq F(\lambda_2). \tag{16}$$

Proof. Let

$$F(\lambda) = u^{-1}\left[\left(\sum_{i=1}^{n} w_i u^{\lambda}(y_i)\right)^{1/\lambda}\right].$$

According to the above equation, let $f(\lambda) = (\sum_{i=1}^{n} w_i u^{\lambda}(y_i))^{1/\lambda}$. Taking the first derivative of f w.r.t. λ, we have that

$$f'(\lambda) = \frac{1}{\lambda^2}\left(\sum_{i=1}^{n} w_i u^{\lambda}(y_i)\right)^{1/\lambda-1} \times \left[\sum_{i=1}^{n} w_i u^{\lambda}(y_i)\log u^{\lambda}(y_i) - \sum_{i=1}^{n} w_i u^{\lambda}(y_i)\log\sum_{i=1}^{n} w_i u^{\lambda}(y_i)\right].$$

Let $t_i = u^{\lambda}(y_i)$, $t_0 = \sum_{i=1}^{n} w_i u^{\lambda}(y_i)$ and $h(t) = t\log t$, we have $f'(\lambda) = \lambda^{-2}(f(\lambda))^{1-\lambda}[\sum_{i=1}^{n} w_i h(t_i) - h(t_0)]$. For $t > 0$, we have $h''(t) = 1/t > 0$, which means that the function $h(t)$ is strictly convex such that $h(t_i) > h(t_0) + (t_i - t_0)h'(t_0)$ for all $t_0 > 0$, $t_i \geq 0$ and $t_i \neq t_0$. Hence, we have

$$\sum_{i=1}^{n} w_i h(t_i) \geq \sum_{i=1}^{n} w_i h(t_0) + \sum_{i=1}^{n} w_i (t_i - t_0)h'(t_0) = h(t_0),$$

where the equality holds only if $t_i = t_0$ for all i with $w_i > 0$. Excluding this case, the first derivative $f'(\lambda) > 0$ implies that f monotonically increases w. r. t. λ. Note that $\mathrm{d}F/\mathrm{d}\lambda = f'(\lambda)/u'[u^{-1}(f(\lambda))]$, we have $\mathrm{d}F/\mathrm{d}\lambda \geq 0$, meaning that F monotonically increases w. r. t. λ. That is,

$$F(\lambda_1) \geq F(\lambda_2).$$

The property is proved. □

3.2. GOWUA-HARA operator

According to Eqs. (4), (10) and the reverse function of HARA utility

$$u^{-1}(x) = \frac{1-\gamma}{\beta}\left[\left(\frac{\beta\gamma}{1-\gamma}x\right)^{1/\gamma} - \eta\right], \tag{17}$$

we have

$$\begin{aligned}
\mathrm{GWUA-HARA}(x_1, x_2, \ldots, x_n) &= \frac{1-\gamma}{\beta} \times \left[\left(\frac{\beta\gamma}{1-\gamma}\left(\sum_{i=1}^{n} w_i\left(\frac{1-\gamma}{\beta\gamma}\left(\frac{\beta}{1-\gamma}x_i + \eta\right)^{\gamma}\right)^{\lambda}\right)^{1/\lambda}\right)^{1/\gamma} - \eta\right] \\
&= \frac{1-\gamma}{\beta}\left[\left(\left(\sum_{i=1}^{n} w_i\left(\left(\frac{\beta}{1-\gamma}x_i + \eta\right)^{\gamma}\right)^{\lambda}\right)^{1/\lambda}\right)^{1/\gamma} - \eta\right] \\
&= \frac{1-\gamma}{\beta}\left[\left(\sum_{i=1}^{n} w_i\left(\frac{\beta}{1-\gamma}x_i + \eta\right)^{\lambda\gamma}\right)^{1/\lambda\gamma} - \eta\right].
\end{aligned}$$

Thus, we can obtain

$$\mathrm{GWUA-HARA}(x_1, x_2, \ldots, x_n) = \frac{1-\gamma}{\beta}\left[\left(\sum_{i=1}^{n} w_i\left(\frac{\beta}{1-\gamma}x_i + \eta\right)^{\lambda\gamma}\right)^{1/\lambda\gamma} - \eta\right], \tag{18}$$

which is called the GWUA-HARA operator.

If reordering the arguments of the GWUA-HARA operator in descending order, we can define a generalized ordered weighted utility averaging-hyperbolic absolute risk aversion (GOWUA-HARA) operator.

Definition 6. A generalized ordered weighted utility averaging-hyperbolic absolute risk aversion (GOWUA-HARA) operator of dimension n is a mapping GOWUA-HARA: $R^{+n} \to R^{+}$ such that

$$\mathrm{GOWUA-HARA}(x_1, x_2, \ldots, x_n) = \frac{1-\gamma}{\beta}\left[\left(\sum_{i=1}^{n} w_i\left(\frac{\beta}{1-\gamma}y_i + \eta\right)^{\lambda\gamma}\right)^{1/\lambda\gamma} - \eta\right], \tag{19}$$

where y_i is the ith largest of x_j and the weighting vector $W = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ satisfies $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$.

From Eq. (19), we find that the GOWUA-HARA operator may focus on its structure and importance of arguments rather than the weighting pattern, which leads to the fact that the GOWUA-HARA operator with more effectively theoretic basis is superior to other aggregation operators including the OWA operator, the OWGA operator, the OWHA operator, etc.

Remark 1. Note that HARA is a utility function, then according to Property 1 to Property 5, we can conclude that the GOWUA-HARA operator is commutative, monotonic, bounded and idempotent.

The following theorems and corollaries show that by taking different values of $\lambda, \beta, \eta, \gamma$ in the GOWUA-HARA operator, we can obtain different types of aggregation operators including the OWGA operator, OWA operator, OWHA operator, GOWA operator, maximum operator, minimum operator, OWUGA-HARA operator, OWUA-HARA operator, OWUHA-HARA operator, etc.

Theorem 1. *Let f be the GOWUA-HARA operator. If* $\beta = 1-\gamma$, $\eta \to 0$, $\gamma \to 1$ *and* $\lambda \in (-\infty, 0) \cup (0, +\infty)$*, then*

$$\lim_{\gamma\to 1}\left(\lim_{\eta\to 0} f(x_1, x_2, \ldots, x_n)\right) = \left(\sum_{i=1}^{n} w_i y_i^{\lambda}\right)^{1/\lambda}. \tag{20}$$

Proof. If $\beta = 1-\gamma, \eta \to 0, \gamma \to 1$ and $\lambda \in (-\infty, 0) \cup (0, +\infty)$, we get that

$$\begin{aligned}
&\lim_{\gamma\to 1}\left(\lim_{\eta\to 0} f(x_1, x_2, \ldots, x_n)\right)\\
&= \lim_{\gamma\to 1}\left[\lim_{\eta\to 0}\left(\left(\sum_{i=1}^{n} w_i (y_i+\eta)^{\lambda\gamma}\right)^{1/\lambda\gamma} - \eta\right)\right]\\
&= \lim_{\gamma\to 1}\left(\sum_{i=1}^{n} w_i (y_i)^{\lambda\gamma}\right)^{1/\lambda\gamma} = \left(\sum_{i=1}^{n} w_i y_i^{\lambda}\right)^{1/\lambda}.
\end{aligned}$$

The theorem is proved. □

Remark 2. Theorem 1 is just the GOWA operator (Yager, 2004), meaning that GOWUA-HARA is an extension of the GOWA operator.

Theorem 2. *Let f be the GOWUA-HARA operator. If* $\lambda \to 0$*, then*

$$\lim_{\lambda\to 0} f(x_1, x_2, \ldots, x_n) = \frac{1-\gamma}{\beta}\left(\prod_{i=1}^{n}\left(\frac{\beta}{1-\gamma}y_i+\eta\right)^{w_i} - \eta\right), \tag{21}$$

which is called as the ordered weighted utility geometric averaging-hyperbolic absolute risk aversion (OWUGA-HARA) operator.

Proof. By the L'Hôpital's rule, we have that

$$\begin{aligned}
&\lim_{\lambda\to 0} f(x_1, x_2, \ldots, x_n)\\
&= \lim_{\lambda\to 0}\frac{1-\gamma}{\beta}\left[\left(\sum_{i=1}^{n} w_i\left(\frac{\beta}{1-\gamma}y_i+\eta\right)^{\lambda\gamma}\right)^{1/\lambda\gamma} - \eta\right].
\end{aligned}$$

Since

$$\begin{aligned}
&\lim_{\lambda\to 0}\left(\sum_{i=1}^{n} w_i\left(\frac{\beta}{1-\gamma}y_i+\eta\right)^{\lambda\gamma}\right)^{1/\lambda\gamma}\\
&= \lim_{\lambda\to 0}\exp\left(\frac{1}{\lambda\gamma}\log\left(\sum_{i=1}^{n} w_i\left(\frac{\beta}{1-\gamma}y_i+\eta\right)^{\lambda\gamma}\right)\right)\\
&= \exp\left(\sum_{i=1}^{n} w_i \log\left(\frac{\beta}{1-\gamma}y_i+\eta\right)\right) = \prod_{i=1}^{n}\left(\frac{\beta}{1-\gamma}y_i+\eta\right)^{w_i}.
\end{aligned}$$

So,

$$\lim_{\lambda\to 0} f(x_1, x_2, \ldots, x_n) = \frac{1-\gamma}{\beta}\left(\prod_{i=1}^{n}\left(\frac{\beta}{1-\gamma}y_i+\eta\right)^{w_i} - \eta\right).$$

The theorem is proved. □

According to Eq. (21), it is easy to derive the following conclusions.

Corollary 1. *Let f be the GOWUA-HARA operator. If* $\lambda \to 0$, $\beta = 1-\gamma$, $\eta \to 0$ and $\gamma \in (-\infty, 0) \cup (0, 1)$*, then the GOWUA-HARA operator is an extension of the OWGA operator* (Xu & Da, 2002a).

$$\lim_{\eta\to 0}\left(\lim_{\lambda\to 0} f(x_1, x_2, \ldots, x_n)\right) = \prod_{i=1}^{n} y_i^{w_i}, \tag{22}$$

Theorem 3. *Let f be the GOWUA-HARA operator. If* $\lambda = 1$*, then*

$$f(x_1, x_2, \ldots, x_n) = \frac{1-\gamma}{\beta}\left(\left(\sum_{i=1}^{n} w_i\left(\frac{\beta}{1-\gamma}y_i+\eta\right)^{\gamma}\right)^{1/\gamma} - \eta\right), \tag{23}$$

which is *named* as the ordered weighted utility averaging-hyperbolic absolute risk aversion (OWUA-HARA) operator.

Proof. According to Eq. (19), it is easy to derive the conclusion. □

Corollary 2. *When* $\lambda = 1$*, by choosing different values of* β, γ, η*, we can obtain the following different aggregation operators*:

(1) *Let f be the GOWUA-HARA operator. If* $\beta = 1-\gamma$, $\eta \to 0$ *and* $\gamma \to 1$*, then the GOWUA-HARA will degenerate to the ordered weighted averaging (OWA) operator* (Yager, 1988):

$$\lim_{\gamma\to 1}\left(\lim_{\eta\to 0} f(x_1, x_2, \ldots, x_n)\right) = \sum_{i=1}^{n} w_i y_i. \tag{24}$$

(2) *Let f be the GOWUA-HARA operator. If* $\gamma \to 0$, $\beta > 0$ *and* $\eta > 0$*, then*

$$\lim_{\gamma\to 0} f(x_1, x_2, \ldots, x_n) = \frac{1}{\beta}\left(\prod_{i=1}^{n}(\beta y_i+\eta)^{w_i} - \eta\right), \tag{25}$$

which *is* called the constant coefficient *OWGA* (CC-OWGA) operator.

(3) *Let f be the GOWUA-HARA operator. If* $\beta = 1, \eta \to 0$ and $\gamma \to 0$*, then the GOWUA-HARA will reduce to the OWGA operator* (Xu & Da, 2002a):

$$\lim_{\eta\to 0}\left(\lim_{\gamma\to 0} f(x_1, x_2, \ldots, x_n)\right) = \prod_{i=1}^{n}(y_i)^{w_i}. \tag{26}$$

Proof. (1) If $\beta = 1-\gamma, \eta \to 0$ and $\gamma \to 1$, according to Eq. (23), it is easy to get the conclusion.

(2) If $\gamma \to 0$, $\beta > 0$ and $\eta > 0$, according to Eq. (23), we have

$$\begin{aligned}
&\lim_{\gamma\to 0} f(x_1, x_2, \ldots, x_n)\\
&= \lim_{\gamma\to 0}\frac{1-\gamma}{\beta}\left(\left(\sum_{i=1}^{n} w_i\left(\frac{\beta}{1-\gamma}y_i+\eta\right)^{\gamma}\right)^{1/\gamma} - \eta\right).
\end{aligned}$$

Since,

$$\begin{aligned}
&\lim_{\gamma\to 0}\left(\sum_{i=1}^{n} w_i\left(\frac{\beta}{1-\gamma}y_i+\eta\right)^{\gamma}\right)^{1/\gamma}\\
&= \lim_{\gamma\to 0}\exp\left(\frac{1}{\gamma}\log\left(\sum_{i=1}^{n} w_i\left(\frac{\beta}{1-\gamma}y_i+\eta\right)^{\gamma}\right)\right)\\
&= \exp\left(\sum_{i=1}^{n} w_i \log(\beta y_i+\eta)\right) = \prod_{i=1}^{n}(\beta y_i+\eta)^{w_i}.
\end{aligned}$$

Then,

$$\lim_{\gamma\to 0} f(x_1, x_2, \ldots, x_n) = \frac{1}{\beta}\left(\prod_{i=1}^{n}(\beta y_i+\eta)^{w_i} - \eta\right).$$

(3) According to Eq. (23) and Eq. (25), it is easy to derive the conclusion.

This completes the proof. □

Theorem 4. *Let f be the GOWUA-HARA operator. If $\lambda = -1$, then*

$$f(x_1, x_2, \ldots, x_n) = \frac{1-\gamma}{\beta}\left(1\Bigg/\left(\sum_{i=1}^{n}\frac{w_i}{\left(\frac{\beta}{1-\gamma}y_i+\eta\right)^{\gamma}}\right)^{1/\gamma} - \eta\right), \tag{27}$$

which is named as the ordered weighted utility harmonic averaging-hyperbolic absolute risk aversion (OWUHA-HARA) operator.

Proof. According to Eq. (19), it is easy to get the conclusion. □

Corollary 3. *Let f be the GOWUA-HARA operator. If $\lambda = -1$, $\beta = 1-\gamma$, $\eta \to 0$ and $\gamma \to 1$, then*

$$\lim_{\gamma\to 1}\left(\lim_{\eta\to 0} f(x_1, x_2, \ldots, x_n)\right) = 1\Bigg/\left(\sum_{i=1}^{n}\frac{w_i}{y_i}\right), \tag{28}$$

which is just the ordered weighted harmonic averaging (OWHA) operator (Yager, 2004).

Theorem 5. *Let f be the GOWUA-HARA operator. When $\lambda \to +\infty$, by choosing different values of γ, we have the following statements.*

(1) *If $0 < \gamma < 1$, then $f(x_1, x_2, \ldots, x_n) = y_1$ (i.e., maximum operator).*
(2) *If $\gamma < 0$, then $f(x_1, x_2, \ldots, x_n) = y_n$ (i.e., minimum operator).*

Proof. According to Property 1 and Property 2, we derive that

$$f(x_1, x_2, \ldots, x_n) \le f(y_1, y_1, \ldots, y_1) = y_1,$$

where $y_1 = \max_{1\le i\le n}\{x_i\}$. When $\lambda \to +\infty$, by choosing $0 < \gamma < 1$ and $\gamma < 0$, we will obtain the opposite results in the following two cases:

(1) If $0 < \gamma < 1$ and $\lambda \to +\infty$, then

$$\left(\sum_{i=1}^{n} w_i\left(\frac{\beta}{1-\gamma}y_i+\eta\right)^{\lambda\gamma}\right)^{1/\lambda\gamma} \ge \left(w_1\left(\frac{\beta}{1-\gamma}y_1+\eta\right)^{\lambda\gamma}\right)^{1/\lambda\gamma} = w_1^{1/\lambda\gamma}\left(\frac{\beta}{1-\gamma}y_1+\eta\right).$$

That is,

$$\frac{1-\gamma}{\beta}\left(\left(\sum_{i=1}^{n} w_i\left(\frac{\beta}{1-\gamma}y_i+\eta\right)^{\lambda\gamma}\right)^{1/\lambda\gamma} - \eta\right) \ge \frac{1-\gamma}{\beta}\left(w_1^{1/\lambda\gamma}\left(\frac{\beta}{1-\gamma}y_1+\eta\right)-\eta\right). \tag{29}$$

Taking the limitation on both sides in inequality (29), we get

$$\lim_{\lambda\to+\infty}\frac{1-\gamma}{\beta}\left(\left(\sum_{i=1}^{n} w_i\left(\frac{\beta}{1-\gamma}y_i+\eta\right)^{\lambda\gamma}\right)^{1/\lambda\gamma} - \eta\right) \ge \lim_{\lambda\to+\infty}\frac{1-\gamma}{\beta}\left(w_1^{1/\lambda\gamma}\left(\frac{\beta}{1-\gamma}y_1+\eta\right)-\eta\right) = y_1.$$

Therefore,

$$\lim_{\lambda\to+\infty} f(x_1, x_2, \ldots, x_n) = y_1,$$

which is just the maximum operator.

(2) If $\gamma < 0$ and $\lambda \to +\infty$, we have that $\lambda\gamma \to -\infty$. Just as the above proof, we can get the minimum operator.

The theorem is proved. □

Theorem 6. *Let f be the GOWUA-HARA operator. When $\lambda \to -\infty$, by choosing different values of γ, we have the following statements.*

(1) *If $0 < \gamma < 1$, then $f(x_1, x_2, \ldots, x_n) = y_n$ (i.e., minimum operator).*
(2) *If $\gamma < 0$, then $f(x_1, x_2, \ldots, x_n) = y_1$ (i.e., maximum operator).*

Proof. Similar to the proof of Theorem 5, the conclusion is easily obtained. □

Remark 3. If we consider the possible values of the weighting vector $W = (w_1, w_2, \ldots, w_n)^T$ in the GOWUA-HARA operator, we can obtain a group of particular cases shown as follows:

- The maximum operator is found if $w_1 = 1$ and $w_i = 0\ (i \ne 1)$.
- The minimum operator is derived if $w_n = 1$ and $w_i = 0\ (i \ne n)$.
- Let $y_1 = \max_{1\le i\le n}\{.x_i\}$ and $y_n = \min_{1\le i\le n}\{x_i\}$, if $w_1 = \alpha$, $w_n = 1-\alpha$ and $w_i = 0\ (i \ne 1, n)$, then

$$\mathrm{GOWUA-HARA}(x_1, x_2, \ldots, x_n) = \frac{1-\gamma}{\beta}\left[\left(\alpha\left(\frac{\beta}{1-\gamma}y_1+\eta\right)^{\lambda\gamma} + (1-\alpha)\left(\frac{\beta}{1-\gamma}y_n+\eta\right)^{\lambda\gamma}\right)^{1/\lambda\gamma} - \eta\right],$$

which includes the maximum and minimum aggregation operators.

- An operator named as Window-GOWUA-HARA operator is obtained if $w_i = 1/p (k \le i \le k+p-1)$ and $w_i = 0 (i < k$ and $i \ge k+p)$, where

$$\mathrm{Window-GOWUA-HARA}(x_1, x_2, \ldots, x_n) = \frac{1-\gamma}{\beta}\left[\left(\frac{1}{p}\sum_{i=k}^{k+p-1}\left(\frac{\beta}{1-\gamma}y_i+\eta\right)^{\lambda\gamma}\right)^{1/\lambda\gamma} - \eta\right].$$

- An operator named as Olympic-GOWUA-HARA operator is derived if $w_i = 1/(n-2)\ (i = 2, 3, \ldots, n-1)$ and $w_i = 0\ (i = 1, n)$, where

$$\mathrm{Olympic-GOWUA-HARA}(x_1, x_2, \ldots, x_n) = \frac{1-\gamma}{\beta}\left[\left(\frac{1}{n-2}\sum_{i=2}^{n-1}\left(\frac{\beta}{1-\gamma}y_i+\eta\right)^{\lambda\gamma}\right)^{1/\lambda\gamma} - \eta\right].$$

- If dimension n in GOWUA-HARA operator is an even number, let $w_{n/2} = w_{n/2+1} = 1/2$ and $w_i = 0\ (i \ne n/2, n/2+1)$, then

$$\mathrm{GOWUA-HARA}(x_1, x_2, \ldots, x_n) = \frac{1-\gamma}{\beta} \times \left[\left(\frac{1}{2}\left(\frac{\beta}{1-\gamma}y_{n/2}+\eta\right)^{\lambda\gamma} + \frac{1}{2}\left(\frac{\beta}{1-\gamma}y_{n/2+1}+\eta\right)^{\lambda\gamma}\right)^{1/\lambda\gamma} - \eta\right],$$

where $y_{n/2}$ is the $n/2$th largest of $x_j (j = 1, 2, \ldots, n)$, and $y_{n/2+1}$ is the $(n/2+1)$th largest of $x_j (j = 1, 2, \ldots, n)$.
- If dimension n in GOWUA-HARA operator is an odd number, let $w_{(n+1)/2} = 1$ and $w_i = 0 (i \ne (n+1)/2)$, then $\mathrm{GOWUA-HARA}(x_1, x_2, \ldots, x_n) = y_{(n+1)/2}$, where $y_{(n+1)/2}$ is the $(n+1)/2$th largest of $x_j (j = 1, 2, \ldots, n)$.

4. A model for determining the GOWUA-HARA weights

In order to determine the weights of the GOWUA-HARA operator, we propose an orness measure of the GOWUA-HARA operator and analyze its properties. We further construct a new optimization model which can aggregate DMs' opinion and obtain an optimal weighting vector for the GOWUA-HARA operator.

4.1. An orness measure for the GOWUA-HARA operator

This subsection introduces an orness measure of the GOWUA-HARA operator. The orness measure was presented by Yager (1988).

Definition 7. (Yager, 1988). Assume that *orness* is an OWA aggregation operator with weighting vector $W = (w_1, w_2, \ldots, w_n)$, the degree of "orness" associated with this operator is defined as:

$$\text{orness}(W) = \frac{1}{n-1}\sum_{i=1}^{n}[(n-i)w_i]. \tag{30}$$

It can be shown that when $W = (1, 0, \ldots, 0)$, $\text{orness}(W) = 1$; when $W = (0, 0, \ldots, 1)$, $\text{orness}(W) = 0$; when $W = (1n, 1/n, \ldots, 1/n)$, $\text{orness}(W) = 1/2$.

The orness measure is also called the attitudinal character of the aggregation, which can be regarded as the OWA aggregation of the arguments $x_i = (n-i)/(n-1)$ for $i = 1, 2, \ldots, n$. By using this method, Yager (2004) presented the orness measure of the GOWA operator:

$$\text{orness}(W) = \left[\sum_{i=1}^{n} w_i((n-i)/(n-1))^{\lambda}\right]^{1/\lambda}, \tag{31}$$

where λ is a parameter such that $\lambda \in (-\infty, 0) \cup (0, +\infty)$. If $\lambda = 1$, then the orness measure of the GOWA operator will degenerate to the orness measure of the OWA operator.

Following Yager (2004), we can define an orness measure of the GOWUA-HARA operator.

Definition 8. The orness measure associated with the GOWUA-HARA operator is defined as:

$$\text{orness}(W) = \frac{1-\gamma}{\beta}\left(\left[\sum_{i=1}^{n} w_i\left(\frac{\beta}{1-\gamma}\times\frac{n-i}{n-1}+\eta\right)^{\lambda\gamma}\right]^{1/\lambda\gamma} - \eta\right), \tag{32}$$

where λ is a parameter such that $\lambda \in (-\infty, 0) \cup (0, +\infty)$.

Remark 4. It can be shown that for the GOWUA-HARA operator: When $W = (1, 0, \ldots, 0)$, $\text{orness}(W) = 1$; when $W = (0, 0, \ldots, 1)$, $\text{orness}(W) = 0$. In particular, if $\beta = 1-\gamma$, $\eta \to 0$ and $\gamma \to 1$, then the orness measure of the GOWUA-HARA operator will degenerate to the orness measure of the GOWA operator (Yager, 2004).

Theorem 7. *The orness measure of the GOWUA-HARA operator satisfies*

$$0 \le \text{orness}(W) \le 1. \tag{33}$$

Proof. Noting that $\text{orness}(W)$ is the orness measure of the GOWUA-HARA operator, we have that

$$\text{orness}(W) = \frac{1-\gamma}{\beta}\left(\left[\sum_{i=1}^{n} w_i\left(\frac{\beta}{1-\gamma}\times\frac{n-i}{n-1}+\eta\right)^{\lambda\gamma}\right]^{1/\lambda\gamma} - \eta\right).$$

According to Property 2 and Property 3, we obtain that

$$\frac{1-\gamma}{\beta}\left(\left[\sum_{i=1}^{n} w_i\left(\frac{\beta}{1-\gamma}\times\frac{n-n}{n-1}+\eta\right)^{\lambda\gamma}\right]^{1/\lambda\gamma} - \eta\right) \le \text{orness}(W)$$
$$\le \frac{1-\gamma}{\beta}\left(\left[\sum_{i=1}^{n} w_i\left(\frac{\beta}{1-\gamma}\times\frac{n-1}{n-1}+\eta\right)^{\lambda\gamma}\right]^{1/\lambda\gamma} - \eta\right).$$

That is,
$0 \le \text{orness}(W) \le 1$.
This completes the proof. □

Remark 5. Based on Theorem 5, it is easy to obtain that $\lim_{\lambda\to+\infty} \text{orness}(W) = 1$ and $\lim_{\lambda\to-\infty}\text{orness}(W) = 0$. In addition, if $\lambda_1 \ge \lambda_2$, from Property 5, we can derive

$$\text{orness}_{\lambda_1}(W) \ge \text{orness}_{\lambda_2}(W). \tag{34}$$

4.2. An optimization model for determining the GOWUA-HARA weights under the orness measure

Based on the orness measure and the dispersion measure, O'Hagan (1988) developed a maximum entropy method to determine the OWA operator weights, which requires the solution to the following constrained nonlinear optimization problem under a given orness measure:

$$\begin{aligned} &\text{Minimize } \text{disp}(W) = -\sum_{i=1}^{n} w_i \ln w_i \\ &\text{s.t.} \quad \text{orness}(W) = \alpha = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i, \quad 0 \le \alpha \le 1, \\ &\qquad \sum_{i=1}^{n} w_i = 1, \\ &\qquad 0 \le w_i \le 1, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{35}$$

Fullér and Majlender (2003a) provided a minimum variance method, which demands the solution of the following quadratic programming problem for minimizing the variance of the OWA operator weights under a given orness measure:

$$\begin{aligned} &\text{Minimize } D^2(W) = \frac{1}{n}\sum_{i=1}^{n}\left(w_i - \frac{1}{n}\right)^2 \\ &\text{s.t.} \quad \text{orness}(W) = \alpha = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i, \quad 0 \le \alpha \le 1, \\ &\qquad \sum_{i=1}^{n} w_i = 1, \\ &\qquad 0 \le w_i \le 1, i = 1, 2, \ldots, n. \end{aligned} \tag{36}$$

Wang and Parkan (2005) proposed the following model for minimizing the maximum disparity between two adjacent weights under a given orness measure:

$$\begin{aligned} &\text{Minimize } \{\max_{i\in\{1,2,\ldots,n-1\}} |w_i - w_{i+1}|\} \\ &\text{s.t.} \quad \text{orness}(W) = \alpha = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i, \quad 0 \le \alpha \le 1, \\ &\qquad \sum_{i=1}^{n} w_i = 1, \\ &\qquad 0 \le w_i \le 1, i = 1, 2, \ldots, n. \end{aligned} \tag{37}$$

As pointed out in Wang and Parkan (2007), the above three approaches all imply the use of more information from all attributes, and there are no significant differences among the three alternative approaches except for computational simplicity or complexity. Considering the importance of the OWA weights, Wang et al. (2007) constructed the chi-square (χ^2) model for determining the OWA operator weights under a given orness measure:

$$\begin{aligned} &\text{Minimize } J = \sum_{i=1}^{n-1}(w_i/w_{i+1} + w_{i+1}/w_i - 2) \\ &\text{s.t.} \quad \text{orness}(W) = \alpha = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i, \quad 0 \le \alpha \le 1, \\ &\qquad \sum_{i=1}^{n} w_i = 1, \\ &\qquad 0 \le w_i \le 1, \quad i = 1, \ldots, n. \end{aligned} \tag{38}$$

However, these approaches mentioned above suppose that any individual weighting vector is equal to the optimal weighting vector, and correspondingly there is only one orness measure to characterize the DMs' attitude toward decision-making, and only one set of the OWA operator weights to be generated. This situation is not consistent with the reality. In fact, different DMs may have different perspectives, which can be characterized by different orness measures. As a result, there exist different sets of the OWA operator weights corresponding to different orness measures. Hence, it is necessary to develop a new method to aggregate all the individual sets of weights into an overall set of weights.

In order to determine such an optimal aggregated weighting vector for the GOWUA-HARA operator in MAGDM, we propose a new method for determining the weights.

Let α_k be the orness measure provided by the kth DM ($k = 1, \ldots, l$), and $W^* = (w_1^*, w_2^*, \ldots, w_n^*)^{\mathrm{T}}$ be an optimal aggregated weighting vector for the GOWUA-HARA operator. The orness measure corresponding to such an optimal aggregated weighting vector $W^* = (w_1^*, w_2^*, \ldots, w_n^*)^{\mathrm{T}}$ is generally not equal to the $\alpha_k (k = 1, \ldots, l)$ provided by the DMs. That is,

$$\mathrm{orness}(W^*) = \frac{1-\gamma}{\beta}\left(\left[\sum_{i=1}^{n} w_i^*\left(\frac{\beta}{1-\gamma}\times\frac{n-i}{n-1}+\eta\right)^{\lambda\gamma}\right]^{1/\lambda\gamma} - \eta\right) \neq a_k, \quad (k = 1, \ldots, l).$$

To measure the differences between orness(W^*) and each $\alpha_k (k = 1, \ldots, l)$, the deviation variable ε_k is introduced:

$$\varepsilon_k = \left|\mathrm{orness}(W^*) - \alpha_k\right|, \quad (k = 1, \ldots, l).$$

It is hoped that each deviation variable $\varepsilon_k (k = 1, \ldots, l)$ tends to be zero as much as possible, and meanwhile, following Wang and Parka (2007), the aggregation operator weights should be equally important and all the arguments can be equally aggregated. Therefore, by taking into account the orness measure constraint, the model should make all the weights as close to each other as possible.

Based on the above analysis, the following nonlinear optimization model is constructed to determine the GOWUA-HARA operator weights.

$$\begin{aligned} \text{Minimize } & \varphi\sum_{k=1}^{l}\theta_k\varepsilon_k + (1-\varphi)\sum_{i=1}^{n-1}(w_i/w_{i+1} + w_{i+1}/w_i - 2) \\ \text{s.t.} \quad & \left|\frac{1-\gamma}{\beta}\left(\left[\sum_{i=1}^{n} w_i\left(\frac{\beta}{1-\gamma}\times\frac{n-i}{n-1}+\eta\right)^{\lambda\gamma}\right]^{1/\lambda\gamma} - \eta\right) - \alpha_k\right| \\ & = \varepsilon_k, 0 \le \alpha_k \le 1, \quad k = 1, \ldots, l, \\ & \sum_{i=1}^{n} w_i = 1, \\ & w_i, \quad \varepsilon_k \ge 0, \quad i = 1, \ldots, n, \quad k = 1, \ldots, l, \end{aligned} \tag{39}$$

where φ stands for the relatively important degree of total deviation $\sum_{k=1}^{l}\theta_k\varepsilon_k$ such that $0 < \varphi < 1$, and θ_k denotes the relatively important weight of the kth DM ($k = 1, \ldots, l$).

The model (39) is nonlinear and the optimal aggregated weighting vector $W^* = (w_1^*, w_2^*, \ldots, w_n^*)^T$ can be obtained by using MATLAB or Lingo software package.

If there is only one DM participating in the decision-making process or any individual weighting vector is equal to the optimal aggregated weighting vector, then ε_k will be zero. In these cases, the objective function of model (39) will degenerate to the case of model (38). The main advantage of nonlinear model (39) is that it cannot only minimize the differences between the orness measure provided by each DM and the orness measure corresponding to an optimal weighting vector, but also produce as equally important weights as possible. In addition, note that the model (39) for determining the GOWUA-HARA operator weights does not follow a regular distribution, which is also the merit of the model.

5. An approach to MAGDM based on the GOWUA-HARA operator

This section develops a new approach for MAGDM based on the GOWUA-HARA operator.

For a MAGDM problem, let $X = \{x_1, x_2, \ldots, x_m\}$ be a finite set of alternatives, $C = \{c_1, c_2, \ldots, c_n\}$ a finite set of attributes, $V = (v_1, v_2, \ldots, v_n)^{\mathrm{T}}$ the weighting vector of attributes such that $v_i \ge 0$ and $\sum_{i=1}^{n} v_i = 1$, $D = \{d_1, d_2, \ldots, d_l\}$ a finite set of DMs, and $W = (w_1, w_2, \ldots, w_l)^{\mathrm{T}}$ the weighting vector of DMs satisfying $w_i \ge 0$ and $\sum_{i=1}^{l} w_i = 1$. We assume that α_k is the orness measure provided by the kth DM ($k = 1, \ldots, l$), and θ_k is the relatively important weight of the kth DM ($k = 1, \ldots, l$). In addition, we suppose that each decision maker provides his/her own decision matrix $B^{(k)} = (b_{ij}^{(k)})_{m\times n}$, where $b_{ij}^{(k)}$ is given by the decision maker $d_k \in D$ for the alternative $x_i \in X$ w.r.t. the attribute $c_j \in C$.

In general, MAGDM problems follow a common resolution scheme (Herrera & Martínez, 2000; Roubens, 1997), composed by the following three phases:

(1) *Information processing phase*: Different attributes may have different measurement scales in MAGDM problem, then it is necessary to standardize the attributes so as to avoid the variance among different attributes.
(2) *Aggregation phase*: It combines the individual preferences to obtain a collective preference value for each alternative.
(3) *Exploitation phase*: It orders the collective preference values to obtain the best alternative(s).

In the following we shall utilize the GOWUA-HARA operator and the weighting model (39) to propose an approach to MAGDM problem.

Step 1. Standardize the decision matrixes.

We suppose that there are two types of attributes, i.e., profit type attribute and cost type attribute. Let I_1 be a set of benefit attributes and I_2 a set of cost attributes. The decision matrixes $B^{(k)} = (b_{ij}^{(k)})_{m\times n}$ $(k = 1, 2, \ldots, l)$ can be transformed into the corresponding standardization decision matrixes $R^{(k)} (k = 1, 2, \ldots, l)$:

$$R^{(k)} = \left(r_{ij}^{(k)}\right)_{m\times n} = \begin{bmatrix} r_{11}^{(k)} & r_{12}^{(k)} & \cdots & r_{1n}^{(k)} \\ r_{21}^{(k)} & r_{22}^{(k)} & \cdots & r_{2n}^{(k)} \\ \vdots & \vdots & \vdots & \vdots \\ r_{m1}^{(k)} & r_{m2}^{(k)} & \cdots & r_{mn}^{(k)} \end{bmatrix}, \quad k = 1, 2, \ldots, l,$$

where the real numbers $r_{ij}^{(k)}$ are calculated via the following formulas (Hwang & Yoon, 1981):

$$r_{ij}^{(k)} = \frac{a_{ij}^{(k)}}{\max_i a_{ij}^{(k)}}, \quad j \in I_1, \quad i = 1, 2, \ldots, m, \tag{40}$$

$$r_{ij}^{(k)} = \frac{\min_i a_{ij}^{(k)}}{a_{ij}^{(k)}}, \quad j \in I_2, \quad i = 1, 2, \ldots, m. \tag{41}$$

Step 2. Calculate the weights of the attributes.

Considering that each DM provides the corresponding decision-making matrix and different DMs may have different preferences for the same attribute, it is necessary to calculate the weight of attributes under each DM. In this case, we only need to consider the orness measure of this DM, in other words, ε_k in model (39) will be zero. Then we utilize the following model

$$\text{Minimize} \sum_{i=1}^{n-1} (v_i^{(k)}/v_{i+1}^{(k)} + v_{i+1}^{(k)}/v_i^{(k)} - 2)$$

$$\text{s.t.} \quad \frac{1-\gamma}{\beta}\left(\left[\sum_{i=1}^{n} v_i^{(k)}\left(\frac{\beta}{1-\gamma}\times\frac{n-i}{n-1}+\eta\right)^{\lambda\gamma}\right]^{1/\lambda\gamma} - \eta\right) = \alpha_k, \quad 0 \le \alpha_k \le 1, \tag{42}$$

$$\sum_{i=1}^{n} v_i^{(k)} = 1,$$

$$v_i^{(k)} \ge 0, \quad i = 1, \ldots, n; \quad k = 1, 2, \ldots, l,$$

to calculate the weighting vector $V^{(k)}$ of attributes of the kth decision matrix $(k = 1, 2, \ldots, l)$:

$$V^{(k)} = (v_1^{(k)}, v_2^{(k)}, \ldots, v_n^{(k)})^{\mathrm{T}}, \quad (k = 1, 2, \ldots, l).$$

Step 3. Aggregate the decision matrixes into a collective decision matrix.
Utilize the weighting vector $V^{(k)}(k = 1, 2, \ldots, l)$ of attributes and the GOWUA-HARA operator GOWUA-HARA $(r_{i1}^{(k)}, r_{i2}^{(k)}, \ldots, r_{in}^{(k)})$ to aggregate the whole individual decision matrixes $R^{(k)} = (r_{ij}^{(k)})_{m\times n}(k = 1, 2, \ldots, l)$ into a collective decision matrix $\tilde{R} = (\tilde{r}_{ik})_{m\times l}$, where

$$\tilde{r}_{ik} = \text{GOWUA-HARA}\,(r_{i1}^{(k)}, r_{i2}^{(k)}, \ldots, r_{in}^{(k)}) = \frac{1-\gamma}{\beta}\left[\left(\sum_{j=1}^{n} v_j^{(k)}\left(\frac{\beta}{1-\gamma} r_{ij}^{(k)} + \eta\right)^{\lambda\gamma}\right)^{1/\lambda\gamma} - \eta\right],$$
$$(i = 1, 2, \ldots, m, \; k = 1, 2, \ldots, l).$$

Step 4. Determine the weights of DMs.
For ith alternative $(i = 1, 2, \ldots, m)$ in the collective decision matrix $\tilde{R} = (\tilde{r}_{ik})_{m\times l}$, when aggregating $\tilde{r}_{ik}$ $(k = 1, 2, \ldots, l)$, we need to determine the weights of DMs for the importance of each DM's evaluation may be different. Considering that different DMs may have different orness measures, we utilize model (39) to obtain the optimal weighting vector W of DMs:

$$W = (w_1, w_2, \ldots, w_l)^{\mathrm{T}}, \quad \text{where}$$
$$\sum_{k=1}^{l} w_k = 1 \quad \text{and} \quad w_k \in [0, 1], \; (k = 1, 2, \ldots, l).$$

Step 5. Aggregate the collective overall preference value.
Based on the collective decision matrix $\tilde{R} = (\tilde{r}_{ik})_{m\times l}$ obtained by *Step 3* and the optimal weighting vector W provided by *Step 4*, we utilize the GOWUA-HARA operator GOWUA-HARA $(\tilde{r}_{i1}, \tilde{r}_{i2}, \ldots, \tilde{r}_{il})$ to aggregate the overall preference value t_i of the alternative x_i, where

$$t_i = \text{GOWUA-HARA}(\tilde{r}_{i1}, \tilde{r}_{i2}, \ldots, \tilde{r}_{il}) = \frac{1-\gamma}{\beta}\left[\left(\sum_{k=1}^{l} w_k\left(\frac{\beta}{1-\gamma}\tilde{r}_{ik} + \eta\right)^{\lambda\gamma}\right)^{1/\lambda\gamma} - \eta\right],$$
$$(i = 1, 2, \ldots, m).$$

Step 6. Rank the collective overall preference value $t_i (i = 1, 2, \ldots, m)$ in descending order.

Step 7. Select the best alternative.

Based on the fact that the greater the value t_i is, the better the alternative x_i is, we can rank the alternatives $x_i (i = 1, 2, \ldots, m)$ and select the best one(s).

6. Illustrative example

This section provides a numerical example to examine the validity of the approach developed in Section 5. First, following Zhou and Chen (2014), we study an investment selection problem where the investor is looking for an optimal investment, and examine the effective of our approach. Second, to illustrate the influence of DMs' risk attitude on the decision-making result, we provide a sensitive analysis of the parameters in HARA utility with respect to the best alternative.

6.1. An investment selection problem

Following Zhou and Chen (2014), we suppose that an investment company wants to invest money in the best possible option, and there is a panel of five possible alternatives to invest: (1) x_1 is a computer company; (2) x_2 is a car company; (3) x_3 is a furniture company; (4) x_4 is a food company; and (5) x_5 is a chemical company. The investment company must consider the following six attributes while making the choice: (1) c_1 stands for expected benefit; (2) c_2 denotes technical ability; (3) c_3 indicates competitive power on market; (4) c_4 represents ability to bear risk; (5) c_5 expresses management capability; and (6) c_6 refers to organizational culture. Suppose that there are four decision experts d_1, d_2, d_3 and d_4.

Based on the assumption of Zhou and Chen (2014), we suppose that the decision experts' assessments for alternatives are shown as in Tables A.1–A.4 (see Appendix A). Throughout the numerical analysis, to simplify the calculation, we assume that $\lambda = 2$ in the GOWUA-HARA operator, and that the relatively important weight of DMs and the relatively important degree of total deviation in model (39) are respectively $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 1/4$ and $\varphi = 0.5$. Noting that different DMs may have different perspectives toward decision-making which can be characterized by different orness measures, we then assume that there are four DMs whose orness measures $\alpha_k (k = 1, 2, 3, 4)$ are 0.60, 0.50, 0.70, 0.40, respectively. In addition, the parameters' values of HARA utility are respectively given by: $\beta = 4$, $\eta = 2$ and $\gamma = 0.3$, which are drawn from Cox and Huang (1989) who discussed the problem of portfolio selection for an investor with HARA utility.

According to the approach developed in Section 5 and the given parameters, we can rank the order of enterprises by applying MATLAB or Lingo software package. The concrete steps are shown as follows.

Step 1. Standardize each decision matrixes $B^{(k)}$ into the matrices $R^{(k)}(k = 1, \ldots, 4)$ via Eq. (40), which are shown as in Tables A.5–A.8 (see Appendix A). Here we use only Eq. (40) due to the attribute $u_j \in I_1$ for all j.

Step 2. Utilize the model (42) to calculate the weighting vector of attributes w.r.t. the DM d_k :

$$V^{(1)} = (0.2689, 0.2238, 0.1723, 0.1327, 0.1074, 0.0949)^{\mathrm{T}},$$
$$V^{(2)} = (0.1870, 0.1807, 0.1712, 0.1611, 0.1526, 0.1474)^{\mathrm{T}},$$
$$V^{(3)} = (0.3789, 0.2496, 0.1505, 0.0959, 0.0684, 0.0567)^{\mathrm{T}},$$
$$V^{(4)} = (0.1249, 0.1335, 0.1496, 0.1726, 0.1991, 0.2203)^{\mathrm{T}}.$$

Step 3. Aggregate the matrices $R^{(k)}(k = 1, 2, 3, 4)$ into a collective decision matrix $\tilde{R} = (\tilde{r}_{ik})_{5\times 4}$ by using the GOWUA-HARA operator $\tilde{r}_{ik} =$ GOWUPA-HARA $(r_{i1}^{(k)}, r_{i2}^{(k)}, \ldots, r_{i6}^{(k)})$, $(i = 1, \ldots, 5; k = 1, \ldots, 4)$:

$$\tilde{R} = \begin{bmatrix} 0.9094 & 0.8198 & 0.9928 & 0.7723 \\ 0.8989 & 0.7983 & 0.9473 & 0.8339 \\ 0.7807 & 0.8764 & 0.8855 & 0.7131 \\ 0.8630 & 0.9226 & 0.9443 & 0.7641 \\ 0.9143 & 0.8373 & 0.9195 & 0.7696 \end{bmatrix}.$$

Step 4. Compute the weighting vector of four experts based on the model (39) to derive:

$$W = (0.2633,\ 0.2553,\ 0.2447,\ 0.2367)^T.$$

Step 5. Aggregate the collective overall preference value t_i of the alternative x_i based on the GOWUA-HARA operator $t_i =$ GOWUA-HARA $(\tilde{r}_{i1}, \tilde{r}_{i2}, \tilde{r}_{i3}, \tilde{r}_{i4}),\quad (i = 1, \ldots, 5)$:

$$t_1 = 0.8758,\quad t_2 = 0.8714,\quad t_3 = 0.8159,$$
$$t_4 = 0.8754,\quad t_5 = 0.8619.$$

Step 6. Rank the collective overall preference value $t_i (i = 1, 2, 3, 4, 5)$ in descending order:

$$t_1 > t_4 > t_2 > t_5 > t_3.$$

Step 7. Select the best alternative.

Rank the alternatives $x_i (i = 1, 2, \ldots, m)$ in accordance with the collective overall preference values $t_i (i = 1, 2, 3, 4, 5)$:

$$x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_3.$$

Thus, the best investment alterative is the computer company x_1. This result coincides with the case of Zhou and Chen (2014), which implies that our method is effective.

6.2. Sensitive analysis of the parameters of HARA utility

Note that introducing the utility function influences not only the DMs' evaluation about the input argument information (See formula (18)), but also the weights of attributes and the DMs (see formula (39)). From the formulas of (18) and (39), however, it is difficult to find that what role the parameters of HARA utility function play in the selection of the best alternative.

Based on the parameters given by Subsection 6.1, the following Table A.9 (see Appendix A) shows the impact of changing the parameters of HARA utility function on the best investment alternative, including the changing of single parameter (no. 1 to no. 15), two parameters (no. 16 to no. 30), three parameters (no. 31 to no. 35), respectively.

From the single parameter change in Table A.9, we find that the selection of the best alternative will change with the single parameter γ(or η) decreasing or with the parameter β increasing to a certain point (see, no. 1 to no. 15). In other words, the changing of single parameter has an influence on the selection of the best alternative.

For example, the best alternative will change from x_1 to x_4 as the parameter γ(resp. η) decreases from 0.9 to 0.01(resp. from 5 to 0.6) (see, no. 1 to no. 10), while this change will happen as the parameter β increases from 0.1 to 15 (see, no. 11 to no. 15). Moreover, the absolute risk aversion coefficient $r(x, \gamma, \beta, \eta)$ will decrease with the rise of γ or η, while it will increase with the rise of β (see, no. 1 to no. 15). This can be explained by the first partial derivative of the absolute risk aversion coefficient. Note that the first partial derivative of $r(x, \gamma, \beta, \eta)$ for HARA utility is $r'_\gamma(x, \gamma, \beta, \eta) = -\beta^2 x/(\beta x + \eta(1-\gamma))^2 \le 0$, $r'_\eta(x, \gamma, \beta, \eta) = -\beta(1-\gamma)^2/(\beta x + \eta(1-\gamma))^2 < 0$ and $r'_\beta(x, \gamma, \beta, \eta) = \eta(1-\gamma)^2/(\beta x + \eta(1-\gamma))^2 > 0$, respectively. Then, we can conclude that $r(x, \gamma, \beta, \eta)$ will decrease with the increasing of γ or η, while it will increase with the increasing of β.

In addition, we see that from Table A.9 the changing of two parameters (or three parameters) will also affect the selection of the best alternative. In particular, the best alternative will change with the absolute risk aversion coefficient $r(x, \gamma, \beta, \eta)$ increasing to a certain point, and this changing tendency is also suitable for the single parameter case.

For example, under the condition of single parameter γ, the best alternative will change from x_1 to x_4 when the absolute risk aversion coefficient $r(x, \gamma, \beta, \eta)$ increases from 0.18 to 0.95 (see, no. 1 to no. 5). For the case of two parameters γ and η, the best alternative will change from x_1 to x_4 when $r(x, \gamma, \beta, \eta)$ increases from 0.37 to 1.14 (see, no. 16 to no. 20). Under the changing of three parameters, the best alternative will change from x_1 to x_4 when $r(x, \gamma, \beta, \eta)$ increases from 0.54 to 1.33 (see, no. 31 to no. 35).

This situation can be understood by the meaning of $r(x, \gamma, \beta, \eta)$. Notice that with the absolute risk aversion coefficient increasing, the risk attitude of DM's involved in the evaluation of decision-making information will become more prudent. In other words, the higher the risk aversion is, the more conservative the DM is (Arrow, 1965; Pratt, 1964). From Tables A1 to A4, we find that the alternative x_4 is better than the alternative x_1 with respect to the attributes c_1, c_3, c_4 and c_5. Recall that c_1, c_3, c_4 and c_5 respectively denote the expected benefit, competitive power on market, ability to bear risk and management capability. These attributes, for the security of the investment in reality, draw more attention than those of c_2 (technical ability) and c_6 (organizational culture). Therefore, from the viewpoint of investment safety, the alternative x_4 is superior to the alternative x_1 with the increasing of $r(x, \gamma, \beta, \eta)$.

In addition, Table A.9 shows that no matter what the changing of parameter (single parameter, two or three parameters) is, the best alternative is still x_1 if the value of $r(x, \gamma, \beta, \eta)$ is less than 0.95, while it will become x_4 when the value of $r(x, \gamma, \beta, \eta)$ is larger or equal to 0.95. Here, we should note that the point of $r(x, \gamma, \beta, \eta)$ equal to 0.95 is only suitable for this example because the calculation of the best investment alternative involves other factors such as the values of attributes, weights of attributes and experts, orness measure, the relatively important weight of DMs, the relatively important degree of total deviation.

Compared with the existing aggregation approaches, the main characters of our developed approach can be concluded as follows.

(1) By introducing the utility function in aggregating process, the GOWUA-HARA operator can reflect the DMs' risk attitude toward input argument information which is characterized by the three parameters (γ, η and β) of HARA utility function, while the existing aggregation operators do not take into account this point. For instance, the aggregation operator derived by Zhou and Chen (2014) mainly focuses on its structure and importance of arguments, and does not consider the DMs' risk attitude toward the input argument information.

(2) A new weighting model is constructed to determine the weights of DMs, which aggregates all the individual sets of weights into an overall set of weights. That is, the different perspectives of different DMs can be characterized by a comprehensive orness measure, which overcomes the case of the existing weighting methods only using one orness measure to characterize DMs' perspectives. For example, Zhou and Chen (2014) put forward the generalized least squares method (GLSM) to calculate the weights of DMs, and they supposed that each DM's weights were equal to the optimal weights of DMs. In this case, this method neglects the fact that different DMs may have different perspectives characterized by different orness measures.

(3) Compared with the existing aggregation approaches, in our approach, the DMs' risk attitude characterized by the three parameters (γ, η and β) may influence the selection of the best alternative. Indeed, in practice, the DMs' subjective attitude

toward the evaluation of arguments has an impact on the choice of the best alternative.

7. Conclusion

In this paper, we have developed a new operator called the GOWUA operator and investigated its properties. We have found that it is commutative, monotonic, bounded and idempotent, which are suitable for any utility function. Furthermore, under the HARA utility function, we have proposed a GOWUA-HARA operator and discussed its families including a wide range of aggregation operators. In order to determine the GOWUA-HARA operator weights, we have addressed an orness measure of the GOWUA-HARA operator. Based on this orness measure, we have constructed a new model to determine the optimal weighting vector of the GOWUA-HARA operator. Furthermore, based on the GOWUA-HARA operator, a new approach for MAGDM has been developed. The main advantages of the approach can be concluded from two aspects. First, in the aggregation process, the GOWUA-HARA operator can reflect the DMs' risk attitude toward aggregation information by introducing the HARA utility function, which overcomes the shortcomings of traditional aggregation operators. Second, the model for determining the weights of DMs can minimize the differences between the orness measures provided by each DM and the comprehensive orness measure corresponding to an optimal weighting vector. This improves the existing weighting methods which only use one orness measure to characterize DMs' perspectives.

This approach can also be applied effectively to different group decision-making problems such as engineering management and financial management. While when using the GOWUA-HARA operator to MAGDM, it involves in a tedious calculation for the HARA utility function contains three parameters. In addition, we do not consider the relative gain/loss of attribute values when introducing the utility function, then it will be very interesting to introduce prospect theory in the GOWUA-HARA operator. Nevertheless, we leave that point for future research since our methodology cannot be applied to that extended framework, which will result in more sophisticated calculation and which we cannot tackle here.

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Appendix A

Table A.1
Decision matrix $B^{(1)} - d_1$.

	c_1	c_2	c_3	c_4	c_5	c_6
x_1	70	80	60	70	60	90
x_2	80	60	90	70	60	70
x_3	50	40	80	30	80	80
x_4	60	70	60	70	80	60
x_5	90	80	40	70	70	80

Table A.2
Decision matrix $B^{(2)} - d_2$.

	c_1	c_2	c_3	c_4	c_5	c_6
x_1	80	30	70	70	60	70
x_2	60	80	50	60	40	80
x_3	70	60	80	60	70	70
x_4	70	60	80	70	80	70
x_5	60	70	50	60	80	70

Table A.3
Decision matrix $B^{(3)} - d_3$.

	c_1	c_2	c_3	c_4	c_5	c_6
x_1	70	80	70	70	60	80
x_2	60	40	80	70	60	70
x_3	70	60	60	60	40	70
x_4	70	60	70	60	60	70
x_5	60	50	80	50	50	80

Table A.4
Decision matrix $B^{(4)} - d_4$.

	c_1	c_2	c_3	c_4	c_5	c_6
x_1	60	70	70	50	80	60
x_2	70	80	60	70	60	80
x_3	40	50	90	70	60	60
x_4	70	60	40	80	70	70
x_5	80	70	60	60	70	50

Table A.5
The standardized decision matrix $R^{(1)}$.

	c_1	c_2	c_3	c_4	c_5	c_6
x_1	0.7778	1.0000	0.6667	1.0000	0.7500	1.0000
x_2	0.8889	0.7500	1.0000	1.0000	0.7500	0.7778
x_3	0.5556	0.5000	0.8889	0.4286	1.0000	0.8889
x_4	0.6667	0.8750	0.6667	1.0000	1.0000	0.6667
x_5	1.0000	1.0000	0.4444	1.0000	0.8750	0.8889

Table A.6
The standardized decision matrix $R^{(2)}$.

	c_1	c_2	c_3	c_4	c_5	c_6
x_1	1.0000	0.3750	0.8750	1.0000	0.7500	0.8750
x_2	0.7500	1.0000	0.6250	0.8571	0.5000	1.0000
x_3	0.8750	0.7500	1.0000	0.8571	0.8750	0.8750
x_4	0.8750	0.7500	1.0000	1.0000	1.0000	0.8750
x_5	0.7500	0.8750	0.6250	0.8571	1.0000	0.8750

Table A.7
The standardized decision matrix $R^{(3)}$.

	c_1	c_2	c_3	c_4	c_5	c_6
x_1	1.0000	1.0000	0.8750	1.0000	1.0000	1.0000
x_2	0.8571	0.5000	1.0000	1.0000	1.0000	0.8750
x_3	1.0000	0.7500	0.7500	0.8571	0.6667	0.8750
x_4	1.0000	0.7500	0.8750	0.8571	1.0000	0.8750
x_5	0.8571	0.6250	1.0000	0.7143	0.8333	1.0000

Table A.8
The standardized decision matrix $R^{(4)}$.

	c_1	c_2	c_3	c_4	c_5	c_6
x_1	0.7500	0.8750	0.7778	0.6250	1.0000	0.7500
x_2	0.8750	1.0000	0.6667	0.8750	0.7500	1.0000
x_3	0.5000	0.6250	1.0000	0.8750	0.7500	0.7500
x_4	0.8750	0.7500	0.4444	1.0000	0.8750	0.8750
x_5	1.0000	0.8750	0.6667	0.7500	0.8750	0.6250

Table A.9
The best alternative under the changing of parameters γ, η and β.

No.	γ	η	β	$r(x)$	The overall preference value t_i $(i = 1, 2, 3, 4, 5)$	Ordering
1	0.9	2	4	0.18	$t_1 = 0.8540, t_2 = 0.8418, t_3 = 0.7891, t_4 = 0.8490, t_5 = 0.8350.$	$x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_3.$
2	0.5	2	4	0.67	$t_1 = 0.8681, t_2 = 0.8626, t_3 = 0.8073, t_4 = 0.8672, t_5 = 0.8537.$	$x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_3.$
3	0.1	2	4	0.95	$t_1 = 0.8789, t_2 = 0.8762, t_3 = 0.8202, t_4 = 0.8795, t_5 = 0.8663.$	$x_4 \succ x_1 \succ x_2 \succ x_5 \succ x_3.$
4	−0.1	2	4	1.05	$t_1 = 0.8813, t_2 = 0.8799, t_3 = 0.8234, t_4 = 0.8826, t_5 = 0.8696.$	$x_4 \succ x_1 \succ x_2 \succ x_5 \succ x_3.$
5	−1	2	4	1.33	$t_1 = 0.8895, t_2 = 0.8912, t_3 = 0.8340, t_4 = 0.8924, t_5 = 0.8801.$	$x_4 \succ x_1 \succ x_2 \succ x_5 \succ x_3.$
6	0.3	5	4	0.51	$t_1 = 0.8736, t_2 = 0.8684, t_3 = 0.8132, t_4 = 0.8728, t_5 = 0.8592.$	$x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_3.$
7	0.3	3	4	0.68	$t_1 = 0.8747, t_2 = 0.8700, t_3 = 0.8146, t_4 = 0.8742, t_5 = 0.8606.$	$x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_3.$
8	0.3	1.5	4	0.92	$t_1 = 0.8767, t_2 = 0.8725, t_3 = 0.8169, t_4 = 0.8764, t_5 = 0.8629.$	$x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_3.$
9	0.3	0.6	4	1.16	$t_1 = 0.8794, t_2 = 0.8759, t_3 = 0.8201, t_4 = 0.8795, t_5 = 0.8660.$	$x_4 \succ x_1 \succ x_2 \succ x_5 \succ x_3.$
10	0.3	0.2	4	1.31	$t_1 = 0.8823, t_2 = 0.8792, t_3 = 0.8233, t_4 = 0.8826, t_5 = 0.8691.$	$x_4 \succ x_1 \succ x_2 \succ x_5 \succ x_3.$
11	0.3	2	0.1	0.05	$t_1 = 0.8787, t_2 = 0.8632, t_3 = 0.8079, t_4 = 0.8677, t_5 = 0.8543.$	$x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_3.$
12	0.3	2	1	0.37	$t_1 = 0.8729, t_2 = 0.8673, t_3 = 0.8122, t_4 = 0.8718, t_5 = 0.8582.$	$x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_3.$
13	0.3	2	5	0.90	$t_1 = 0.8765, t_2 = 0.8722, t_3 = 0.8167, t_4 = 0.8762, t_5 = 0.8627.$	$x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_3.$
14	0.3	2	15	1.18	$t_1 = 0.8798, t_2 = 0.8763, t_3 = 0.8205, t_4 = 0.8799, t_5 = 0.8664.$	$x_4 \succ x_1 \succ x_2 \succ x_5 \succ x_3.$
15	0.3	2	20	1.23	$t_1 = 0.8806, t_2 = 0.8773, t_3 = 0.8214, t_4 = 0.8808, t_5 = 0.8673.$	$x_4 \succ x_1 \succ x_2 \succ x_5 \succ x_3.$
16	0.8	0.9	4	0.37	$t_1 = 0.8567, t_2 = 0.8461, t_3 = 0.7928, t_4 = 0.8527, t_5 = 0.8389.$	$x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_3.$
17	0.6	1.7	4	0.60	$t_1 = 0.8651, t_2 = 0.8582, t_3 = 0.8035, t_4 = 0.8634, t_5 = 0.8498.$	$x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_3.$
18	0.4	3	4	0.63	$t_1 = 0.8733, t_2 = 0.8677, t_3 = 0.8126, t_4 = 0.8722, t_5 = 0.8586.$	$x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_3.$
19	0.2	1	4	1.14	$t_1 = 0.8807, t_2 = 0.8780, t_3 = 0.8220, t_4 = 0.8812, t_5 = 0.8679.$	$x_4 \succ x_1 \succ x_2 \succ x_5 \succ x_3.$
20	−0.1	0.3	4	1.89	$t_1 = 0.9040, t_2 = 0.9057, t_3 = 0.8491, t_4 = 0.9065, t_5 = 0.8935.$	$x_4 \succ x_2 \succ x_5 \succ x_1 \succ x_3.$
21	0.1	2	1	0.39	$t_1 = 0.8737, t_2 = 0.8688, t_3 = 0.8134, t_4 = 0.8730, t_5 = 0.8595.$	$x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_3.$
22	0.5	2	7	0.78	$t_1 = 0.8681, t_2 = 0.8626, t_3 = 0.8073, t_4 = 0.8672, t_5 = 0.8537.$	$x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_3.$
23	0.4	2	6	0.86	$t_1 = 0.8744, t_2 = 0.8692, t_3 = 0.8139, t_4 = 0.8735, t_5 = 0.8599.$	$x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_3.$
24	0.2	2	5	0.98	$t_1 = 0.8785, t_2 = 0.8753, t_3 = 0.8193, t_4 = 0.8787, t_5 = 0.8654.$	$x_4 \succ x_1 \succ x_2 \succ x_5 \succ x_3.$
25	−0.2	2	3.5	1.01	$t_1 = 0.8810, t_2 = 0.8799, t_3 = 0.8233, t_4 = 0.8825, t_5 = 0.8696.$	$x_4 \succ x_1 \succ x_2 \succ x_5 \succ x_3.$
26	0.3	4.5	0.5	0.10	$t_1 = 0.8693, t_2 = 0.8638, t_3 = 0.8086, t_4 = 0.8684, t_5 = 0.8549.$	$x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_3.$
27	0.3	1.5	1	0.45	$t_1 = 0.8733, t_2 = 0.8679, t_3 = 0.8128, t_4 = 0.8724, t_5 = 0.8588.$	$x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_3.$
28	0.3	3	5	0.76	$t_1 = 0.8753, t_2 = 0.8707, t_3 = 0.8153, t_4 = 0.8748, t_5 = 0.8613.$	$x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_3.$
29	0.3	1	7	1.17	$t_1 = 0.8796, t_2 = 0.8761, t_3 = 0.8203, t_4 = 0.8797, t_5 = 0.8662.$	$x_4 \succ x_1 \succ x_2 \succ x_5 \succ x_3.$
30	0.3	0.2	3	1.28	$t_1 = 0.8816, t_2 = 0.8785, t_3 = 0.8226, t_4 = 0.8819, t_5 = 0.8684.$	$x_4 \succ x_1 \succ x_2 \succ x_5 \succ x_3.$
31	0.7	1.2	6	0.54	$t_1 = 0.8604, t_2 = 0.8516, t_3 = 0.7976, t_4 = 0.8576, t_5 = 0.8438.$	$x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_3.$
32	0.1	3	2.5	0.57	$t_1 = 0.8750, t_2 = 0.8709, t_3 = 0.8153, t_4 = 0.8748, t_5 = 0.8614.$	$x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_3.$
33	0.4	2.5	4.5	0.72	$t_1 = 0.8737, t_2 = 0.8628, t_3 = 0.8130, t_4 = 0.8727, t_5 = 0.8591.$	$x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_3.$
34	−1	1	2	1.33	$t_1 = 0.8895, t_2 = 0.8912, t_3 = 0.8340, t_4 = 0.8924, t_5 = 0.8801.$	$x_4 \succ x_2 \succ x_1 \succ x_5 \succ x_3.$
35	−0.5	0.8	3	1.67	$t_1 = 0.8960 t_2 = 0.8984, t_3 = 0.8411, t_4 = 0.8991, t_5 = 0.8866.$	$x_4 \succ x_2 \succ x_1 \succ x_5 \succ x_3.$

References

Ahn, B. S. (2010). Parameterized OWA operator weights: An extreme point approach. *International Journal of Approximate Reasoning*, *51*, 820–831.

Amin, G. R., & Emrouznejad, A. (2010). Finding relevant search engines results a minimax linear programming approach. *Journal of the Operational Research Society*, *61*, 1144–1150.

Arrow, K. (1965). Yrjo Hahnsson Foundation. *Aspects of the theory of risk-bearing*, Helsinki.

Canós, L., & Liern, V. (2008). Soft computing-based aggregation methods for human resource management. *European Journal of Operational Research*, *189*, 669–681.

Chiclana, F., Herrera-Viedma, E., Herrera, F., & Alonso, S. (2007). Some induced ordered weighted averaging operators and their use for solving group decision-making problems based on fuzzy preference relations. *European Journal of Operational Research*, *182*, 383–399.

Cox, J. C., & Huang, C. F. (1989). Optimal consumption and portfolio policies when asset prices follow a diffusion process. *Journal of Economic Theory*, *49*(1), 33–83.

Dong, Y. C., Xu, Y. F., Li, H. Y., & Feng, B. (2010). The OWA-based consensus operator under linguistic representation models using position indexes. *European Journal of Operational Research*, *203*, 455–463.

Filev, D., & Yager, R. R. (1998). On the issue of obtaining OWA operator weights. *Fuzzy Sets and Systems*, *94*, 157–169.

Fullér, R., & Majlender, P. (2003a). An analytic approach for obtaining maximal entropy OWA operator weights. *Fuzzy Sets and Systems*, *124*, 53–57.

Fullér, R., & Majlender, P. (2003b). On obtaining minimal variability OWA operator weights. *Fuzzy Sets and Systems*, *136*, 203–215.

Grasselli, M. (2003). A stability result for the HARA class with stochastic interest rates. *Insurance: Mathematics and Economic*, *33*, 611–627.

Herrera, F., & Martínez, L. (2000). A 2-tuple fuzzy linguistic representation model for computing with words. *IEEE Transactions on fuzzy systems*, *8*, 746–752.

Hwang, C. L., & Yoon, K. (1981). Multiple attribute decision making: Methods and applications. Berlin, Germany: Springer-Verlag.

Jung, E. J., & Kim, J. H. (2012). Optimal investment strategies for the HARA utility under the constant elasticity of variance model. *Insurance: Mathematics and Economics*, *51*, 667–673.

Liu, P. D. (2013). Some Hamacher aggregation operators based on the interval-valued intuitionistic fuzzy numbers and their application to group decision making. *IEEE Transactions on Fuzzy Systems*, *22*(1), 83–97.

Llamazares, B. (2004). Simple and absolute special majorities generated by OWA operators. *European Journal of Operational Research*, *158*, 707–720.

Maes, K. C., Saminger, S., & De Baets, B. (2007). Representation and construction of self-dual aggregation operators. *European Journal of Operational Research*, *177*, 472–487.

Majlender, P. (2005). OWA operators with maximal Rényi entropy. *Fuzzy Sets and Systems*, *155*, 340–360.

Merigó, J. M. (2008). New extensions to the OWA operator and its application in business decision making Department of Business Administration, University of Barcelona.

Merigó, J. M., & Casanovas, M. (2010). Decision-making with distance measures and induced aggregation operators. *Computers & Industrial Engineering*, *60*, 66–76.

Merigó, J. M., & Casanovas, M. (2010). Fuzzy generalized hybrid aggregation operators and its application in decision making. *International Journal of Fuzzy Systems*, *12*, 15–24.

Merigó, J. M., Casanovas, M., & Yang, J. B. (2014). Group decision making with expertons and uncertain generalized probabilistic weighted aggregation operators. *European Journal of Operational Research*, *235*, 215–224.

Merigó, J. M., Casanovas, M., & Zeng, S. Z. (2014). Distance measures with heavy aggregation operators. *Applied Mathematical Modelling*, *38*, 3142–3153.

Merigó, J. M., & Gil-Lafuente, A. M. (2010). New decision making techniques and their application in the selection of financial products. *Information Sciences*, *180*, 2085–2094.

Merigó, J. M., & Gil-Lafuente, A. M. (2011). Fuzzy induced generalized aggregation operators and its application in multi-person decision making. *Expert Systems with Applications*, *38*, 9761–9772.

Merigó, J. M., & Gil-Lafuente, A. M. (2013). Induced 2-tuple linguistic generalized aggregation operators and their application in decision-making. *Information Sciences*, *236*, 1–16.

Merigó, J. M., & Yager, R. R. (2013). Generalized moving averages, distance measures and OWA operators. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, *21*, 533–559.

Merton, R. C. (1971). Optimum consumption and portfolio rules in a continuous-time case. *Journal of Economic Theory*, *3*, 373–413.

Mesiar, R. (2007). Fuzzy set approach to the utility, preference relations, and aggregation operators. *European Journal of Operational Research*, *176*, 414–422.

O'Hagan, M. (1988). Aggregating template or rule antecedents in real-time expert systems with fuzzy set logic. Proceedings of the 22nd annual IEEE asilomar conference on signals, *systems and computers* 681–689. CA: Pacific Grove.

Pratt, J. (1964). Risk aversion in the small and in the large. *Econometrica*, *32*, 122–136.

Ribeiro, R. A., & Pereira, R. A. M. (2003). Generalized mixture operators using weighting functions: A comparative study with WA and OWA. *European Journal of Operational Research*, *145*, 329–342.

Roubens, M. (1997). Fuzzy sets and decision analysis. *Fuzzy Sets and Systems*, *90*, 199–206.

Sang, X. Z., & Liu, X. W. (2014). An analytic approach to obtain the least square deviation OWA operator weights. *Fuzzy Sets and Systems*, *240*, 103–116.

Vahdani, B., Mousavi, S. M., & Tavakkoli-Moghaddam, R. (2013). A new design of the elimination and choice translating reality method for multi-criteria group decision-making in an intuitionistic fuzzy environment. *Applied Mathematical Modelling*, *37*, 1781–1799.

Wang, Y. M., Luo, Y., & Liu, X. W. (2007). Two new models for determining OWA operator weights. *Computers & Industrial Engineering*, *52*, 203–209.

Wang, Y. M., & Parkan, C. (2005). A minimax disparity approach for obtaining OWA operator weights. *Information Sciences*, *175*, 20–29.

Wang, Y. M., & Parkan, C. (2007). A preemptive goal programming method for aggregating OWA operator weights in group decision making. *Information Sciences*, *177*, 1867–1877.

Xu, D. L., Yang, J. B., & Wang, Y. M. (2006). The evidential reasoning approach for multi- attribute decision analysis under interval uncertainties. *European Journal of Operational Research*, *174*, 1914–1943.

Xu, Z. S. (2004). A method based on linguistic aggregation operators for group decision making with linguistic preference relations. *Information Sciences*, *166*, 19–30.

Xu, Z. S. (2006a). Dependent OWA operator. *Modeling decision for artificial intelligence* 172–178. Berlin: Springer.

Xu, Z. S. (2006b). Induced uncertain linguistic OWA operators applied to group decision making. *Information Fusion*, *7*, 231–238.

Xu, Z. S., & Da, Q. L. (2002a). The ordered weighted geometric averaging operator. *International Journal of Intelligent Systems*, *17*, 709–716.

Xu, Z. S., & Da, Q. L. (2002b). The uncertain OWA operator. *International Journal of Intelligent Systems*, *17*, 569–575.

Xu, Z. S., & Da, Q. L. (2003). An overview of operators for aggregating information. *International Journal of Intelligent Systems*, *18*, 953–969.

Yager, R. R. (1988). On ordered weighted averaging aggregation operators in multi-criteria decision making. *IEEE Transactions on Systems, Man and Cybernetics B*, *18*, 183–190.

Yager, R. R. (1993). Families of OWA operators. *Fuzzy Sets and Systems*, *59*, 125–148.

Yager, R. R. (2004). Generalized OWA aggregation operators. *Fuzzy Optimization and Decision Making*, *3*, 93–107.

Yager, R. R. (2009a). Prioritized OWA aggregation. *Fuzzy Optimization and Decision Making*, *8*, 245–262.

Yager, R. R. (2009b). On the dispersion measure of OWA operator. *Information Sciences*, *179*, 3908–3919.

Yager, R. R. (2010). Generalized Bonferroni mean operators in multi-criteria aggregation. *Fuzzy Sets and Systems*, *161*, 2227–2242.

Yager, R. R., & Filev, D. P. (1994). Parameterized “andlike” and “orlike” OWA operators. *International Journal of General Systems*, *22*, 297–316.

Yager, R. R., & Filev, D. P. (1999). Induced ordered weighted averaging operators. *IEEE Transactions on Systems, Man and Cybernetics B*, *29*, 141–150.

Yang, J. B. (2001). Rule and utility based evidential reasoning approach for multiple attribute decision analysis under uncertainty. *European Journal of Operational Research*, *131*, 31–61.

Yang, G. L., Yang, J. B., Liu, W. B., & Li, X. X. (2013). Cross-efficiency aggregation in DEA models using the evidential-reasoning approach. *European Journal of Operational Research*, *231*, 393–404.

Zeng, S. Z., Merigó, J. M., & Su, W. H. (2013). The uncertain probabilistic OWA distance operator and its application in group decision making. *Applied Mathematical Modelling*, *37*, 6266–6275.

Zhang, Z. M. (2013). Hesitant fuzzy power aggregation operators and their application to multiple attribute group decision making. *Information Sciences*, *234*, 150–181.

Zhou, L. G., & Chen, H. Y. (2010). Generalized ordered weighted logarithm aggregation operators and their applications to group decision making. *International Journal of Intelligent Systems*, *25*, 683–707.

Zhou, L. G., & Chen, H. Y. (2014). Generalized ordered weighted proportional averaging operator and its application to group decision making. *Informatica*, *25*, 327–360.