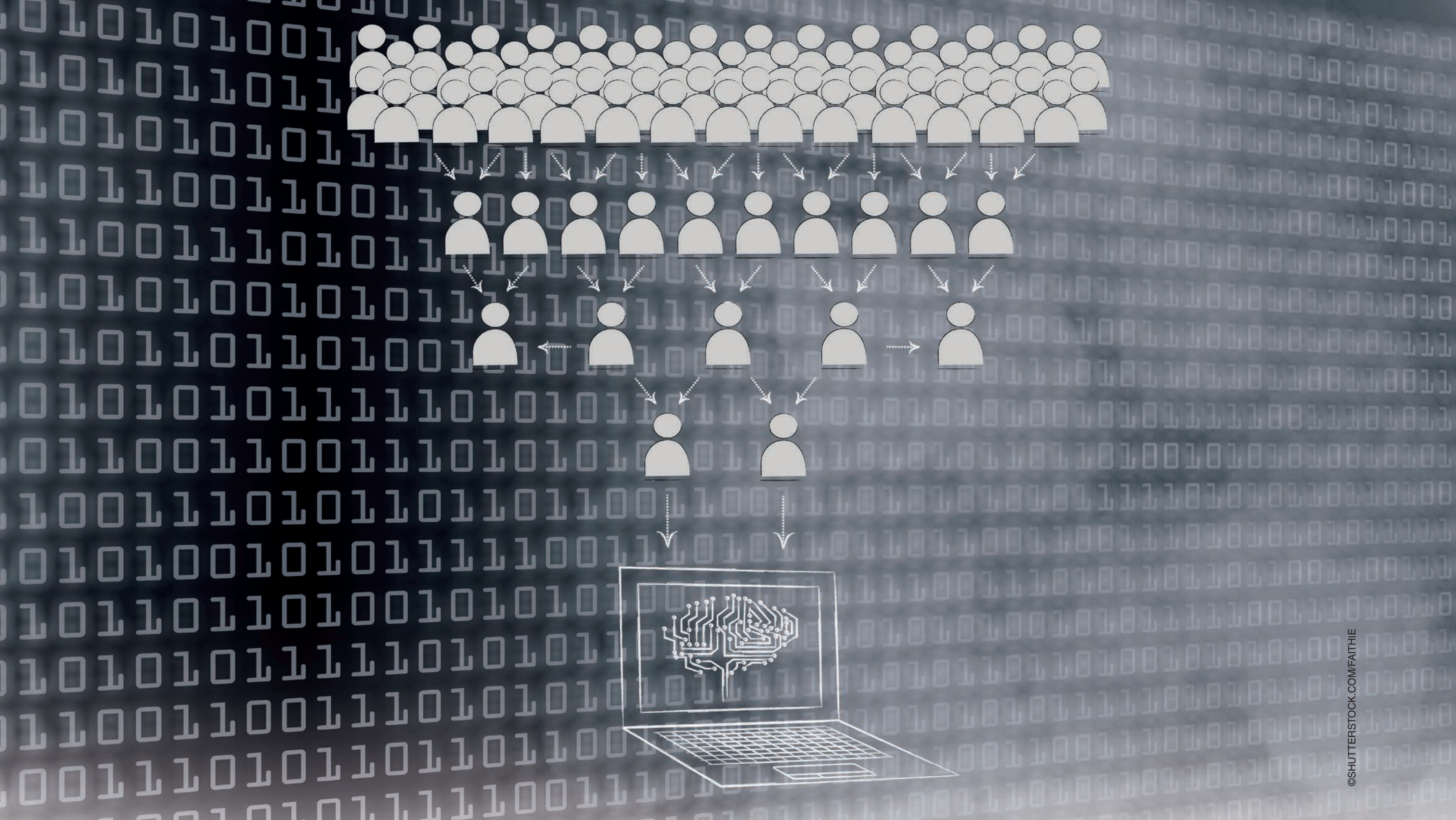

Ordered Weighted Averaging Operators

A Short Review

by Orsolya Csiszár

Aggregation is the process of combining several numerical values into a single representative one, a procedure called an *aggregation function*. Despite the simplicity of this definition, the size of the field of its applications is incredibly huge. Making decisions (in also artificial intelligence) often leads to aggregating preferences or scores on a given set of alternatives. The concept of the ordered weighted averaging (OWA) operator, a symmetric aggregation function that allocates weights according to the input value and unifies in one operator the conjunctive and disjunctive behavior, was introduced by Yager in 1988. Since then, these functions have been axiomatized and extended in various ways. OWA operators provide a parameterized family of aggregation functions, including many of the well-known operators. This function has attracted the interest of several researchers, and therefore, a considerable number of articles in which its properties are studied and its applications are investigated have been published. The development of an appropriate methodology for obtaining the weights is still an issue of great interest. This work provides a short review of OWA operators and gives an overview of some of the most significant results.

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Data Fusion

The problem of data fusion, a synthesis of information or aggregating criteria used to form overall decisions, is of considerable importance in many fields of human knowledge. Because data are obtained more easily, this field is of increasing interest. Some of the fields that use aggregation functions are mathematics, economics, biology, education, artificial intelligence, robotics, vision, fuzzy logic controllers, and knowledge acquisition.

> **The OWA operator is different from the classical weighted average because its weights are not directly associated with a particular attribute but rather with an ordered position.**

One of the most prominent groups of applications of aggregation functions comes from decision theory. Making decisions often leads to aggregating preferences or scores on a given set of alternatives, the preferences being obtained from several decision makers, experts, voters, or representing different points of view, criteria, or objectives. This concerns decisions under multiple criteria or multiple attributes, multiperson decision making, and multi-objective optimization.

Another group comes from artificial intelligence: fuzzy logic. Pattern recognition and classification as well as image analysis are typical examples. Aggregation functions are inevitably used as a generalization of logical connectives in rule-based systems. In artificial intelligence, these techniques are used mainly when a system has to make a decision. The system can have one single criterion for each alternative, or several ones. This case corresponds to a multicriteria decision-making problem. Furthermore, if a system needs a good representation of an environment, it needs the knowledge supplied by information sources to be reliable. However, the information supplied by a single information source (by a single expert or sensor) is often not reliable enough. That is why the information provided from several sensors (or experts) should be combined to improve data reliability and accuracy and also to include some features that are impossible to perceive with individual sensors.

The main factor in determining the structure of the needed aggregation function is the relationship among the criteria. At one extreme is the case in which we desire all the criteria to be satisfied. At the other extreme is the situation in which we want the satisfaction of any of the criteria. These two extreme cases lead to the use of "and" and "or" operators to combine the criteria functions.

In 1988, Yager [26] defined an alternative aggregation function, the so-called OWA operator, to provide the means for aggregating the scores associated with the satisfaction of multiple criteria, which unifies in one operator the conjunctive and disjunctive behavior. This function has attracted the interest of several researchers. A considerable number of articles in which their properties are studied and their applications are investigated have been published. Since then, the family of these functions has been axiomatized and developed in various ways [10], [11], [27].

The goal of this article is to provide a short review of OWA operators, to identify trends in the literature, and to synthesize some of the most significant results.

OWA

Both the weighted mean and the OWA operator are used to combine values with respect to a set of weights. The main difference is that the weights have different meanings in each function. On one hand, the weighted mean allows the system to compute an aggregate value from the ones corresponding to several sources, taking into account the reliability of each information source. Each source has an attached weight that measures its reliability. On the other hand, the OWA operator permits weighting the values in relation to their ordering. In this way, a system can give more importance to a subset of the input values than to another subset. For instance, the influence of extreme values to the result can be diminished, increasing the influence of central values. The function, however, is symmetric, i.e., any permutation of the arguments gives the same result. Therefore, while in the weighted mean the weights measure the importance of an information source with the independence of the value that the source has captured, in the OWA, the weights measure the importance of a value (in relation to other values) independently from the information source where it originated. The need for symmetry is rather frequent in practical applications, which gives preference to OWA operators against weighted arithmetic means.

In other words, the OWA operator is different from the classical weighted average because its weights are not directly associated with a particular attribute but rather with an ordered position. The structure of these operators is in the spirit of combining the criteria under the guidance of a quantifier. This means that the requirement described by a linguistic variable as, for instance, "most" of the criteria is to be satisfied, corresponds to one of these OWA operators [30]. Fodor et. al. showed that it is also possible to describe the class of OWA operators axiomatically; i.e., by means of their properties [11]. It was proved that the class of the OWA aggregators is the operators that satisfy the properties of neutrality, monotonicity, and stability for the same positive linear transformations and ordered linkage.

Definition 1

An OWA operator of dimension n is the mapping $F: I_n \to I$ if it has an associated weighting vector $\mathbf{w} = (w_1, w_2, \ldots, w_n)$,

$w_i \in [0,1], 1 \le i \le n$ with $\Sigma_{i=1}^{n} w_i = 1$ and $F(x_1, x_2, ..., x_n) = OWA_w(\mathbf{x}) = w_1 x_{(1)} + w_2 x_{(2)} + \cdots + w_n x_{(n)}$, where $x_{(j)}$ is the jth largest element of the bag $< x_1, ..., x_n >$.

It is important to emphasize that the fundamental aspect of this operator is the reordering action, i.e., the weights w_i are associated with a particular ordered position rather than with a particular element. It is obvious that different OWA operators are distinguished by their weighting functions. Particularly,

- Max: $\mathbf{w}_* = (1, 0..., 0)$ and $F_{\max}(x_1, ..., x_n) = \max(x_1, ..., x_n)$
- Min: $\mathbf{w}^* = (0, 0..., 1)$ and $F_{\min}(x_1, ..., x_n) = \min(x_1, ..., x_n)$
- Arithmetic mean: $\mathbf{w}_A = (1/n, 1/n..., 1/n)$ and $F_A = ((x_1 + \cdots + x_n)/n)$.

Naturally, OWA operators have the basic properties of an averaging operator: They are always commutative, monotonic, and idempotent. The dual of an OWA function is the so-called reverse OWA, with the vector of weights $\mathbf{w}_d = (w_n, ..., w_1)$. OWA functions are continuous, symmetric, homogeneous, and shift invariant. They do not have neutral or absorbing elements, except for the special cases of min and max. The OWA functions are special cases of the Choquet integral with respect to symmetric fuzzy measures. We can see that the OWA operators provide a parameterized family of aggregation operators, which include many of the well-known operators, such as maximum, minimum, k-order statistics, median, and arithmetic mean.

Obviously, the pure "and," with its lack of compensation (*anding* the criteria means no compensation at all), or the pure "or," with its total submission to any good satisfaction and also with its indifference to the individual criteria (*oring* the criteria means full compensation), are not the desired aggregation functions in most of the cases. A more descriptive name for OWA operators, suggested by Yager [26], would be "orand" operators because they are, in a sense, acting as a combination of the "anding" and "oring" operators.

OWA operators allow for a positive compensation among ratings, which means that a higher degree of satisfying a criterion can compensate, to a certain extent, for a lower degree of satisfying another one. The level of compensation can be chosen between the logical "and" and "or." Given a decision problem, we can find an appropriate OWA aggregation operator from some of the rules and samples, as determined by the decision makers.

Characterizing Measurements

Measurement of Orness

Yager introduced a measurement associated with an OWA operator called the *measurement of orness* [26]. Also called the *degree of orness* or *attitudinal character*, it is an important numerical characteristic of averaging aggregation functions. It was first defined in 1974 by Dujmovic [7] and rediscovered several times. It is applicable to any averaging function and even some other aggregation functions, like STOWA. (OWA functions have recently been mixed with t-norms and t-conorms to provide new mixed aggregation functions, known as OWA operators based on a t-norm (T-OWAs), OWA operators based on an s-conorm (S-OWAs), and OWA operators based on a t-norm or an s-conorm (ST-OWAs); these functions have proved to be useful, in particular, in the context of multicriteria decision making.) Basically, the measurement of orness measures how far a given averaging function is from the max function (the weakest disjunctive function). The definition follows in the next section.

Definition 2

Let F be an OWA operator with a weighting function $\mathbf{w}$.

$$orness(\mathbf{w}) = \left(\frac{1}{n-1}\right) \cdot \sum_{i=1}^{n} ((n-i) w_i).$$

We can readily see that, for any $\mathbf{w}$, $orness(\mathbf{w})$ always lies in the unit interval. Moreover, the nearer $\mathbf{w}$ is to "or," the closer its measurement is to one, while the nearer it is to "and," the closer it is to zero.

- for $\mathbf{w} = [1, 0, 0, ...]$ ("or"), we get $orness(\mathbf{w}) = 1$
- for $\mathbf{w} = [0, 0, 0, ...1]$ ("and"), we get $orness(\mathbf{w}) = 0$
- for $\mathbf{w} = [1/n, 1/n, ...1/n]$ (arithmetic mean), we get $orness(\mathbf{w}) = 0.5$.

Furthermore, the orness is one for only the max function ("or") and zero for only the min function ("and"). However, orness can be 0.5 in cases different from the arithmetic mean as well.

The sum of the orness of an OWA operator and the orness of its dual is always one, i.e., an OWA function is self-dual if and only if $orness(\mathbf{w}) = 0.5$.

If the weighting vector is nondecreasing, i.e., $w_i \le w_{i+1}$ for $i = 1, ..., n-1$, then $orness(\mathbf{w}) \in [0.5, 1]$. Similarly, if the weighting vector is nonincreasing, then $orness(\mathbf{w}) \in [0, 0.5]$.

If two OWA functions have weighing vectors $\mathbf{w}_1, \mathbf{w}_2$ with orness values o_1, o_2, and if

$$\mathbf{w}_3 = a \cdot \mathbf{w}_1 + (1-a) \cdot \mathbf{w}_2,$$

where $a \in [0,1]$, then, for the OWA function with weighting vector $\mathbf{w}_3$, the orness value is

$$orness(\mathbf{w}_3) = a \cdot o_1 + (1-a) \cdot o_2.$$

Obviously, to determine a weighting vector with a desired orness value, we can use different combinations of $\mathbf{w}_1, \mathbf{w}_2$, which all result in different $\mathbf{w}_3$ with the same orness value. For some special weighting vectors, the measurement of orness has been precalculated as well [1].

The measurement of andness can be defined as the *complement of orness*,

$$andness(\mathbf{w}) = 1 - orness(\mathbf{w}).$$

Generally, an OWA operator with much of its nonzero weights near the top will be an orlike operator, with $orness(\mathbf{w}) \geq 0.5$, and when much of its weights are nonzero near the bottom, the OWA operator will be and like, with $andness(\mathbf{w}) \geq 0.5$.

We can also see that, as we move the weight up the vector we increase the orness, while moving the weight down causes us to decrease it [27]; that is, if we have two weighting vectors, $\mathbf{w}_1$ and $\mathbf{w}_2$, such that $\mathbf{w}_1 = a_1, a_2, ..., a_n$ and $\mathbf{w}_2 = a_1, ..., a_j + \epsilon, ..., a_k - \epsilon, ..., a_n$, where $\epsilon > 0, j < k$, then $orness(\mathbf{w}_1) > orness(\mathbf{w}_2)$.

The measurement of the attitudinal character can be directly associated with the weight-generating function f (see the "Quantifier-Guided Aggregation" section). In particular,

$$orness(f) = \int_0^1 f(x)dx.$$

An important class of weighting vectors that generate $orness(\mathbf{w}) = 0.5$ are the symmetric weighting vectors. This measurement of 0.5, of course, does not mean that preference is given to central scores. For example, take $\mathbf{w} = [0.5, 0...0, 0.5]$. If we want to prefer the argument values lying in the middle, we can use the centered OWA (COWA) operators defined in the "COWA Operators" section [35].

Considering the family of regular increasing monotone (RIM) quantifiers (see the "Quantifier-Guided Aggregation" section)

$$Q_\alpha(r) = r^\alpha, \alpha \geq 0,$$

we obtain

$$orness(Q_\alpha) = \frac{1}{1+\alpha},$$

which means that $orness(Q_\alpha) < 0.5$ for $\alpha > 1, orness(Q_\alpha) = 0.5$ for $\alpha = 1$, and $orness(Q_\alpha) > 0.5$ for $\alpha < 1$.

Entropy

Another important measurement is dispersion (a measurement of entropy) [26], which reflects how uniformly the weights are distributed. It has as its extremes the case when all the weights are the same and the case when all the weights are zero except one. It is defined in the following section.

Definition 3

Let $\mathbf{w}$ be a weighting vector $[w_i]$. The measurement of the dispersion of $\mathbf{w}$ is

$$disp(\mathbf{w}) = -\sum_i w_i \ln(w_i).$$

A useful normalization of this measurement is

$$D(W) = -\sum_{i=1}^n w_i \frac{\ln(w_i)}{\ln(n)}.$$

Here, $D(W) \epsilon [0,1]$.

It is easy to see that the entropy measures the degree to which we use all of the aggregates equally.

Special Types of OWA Operators

COWA Operators

Definition 4

An OWA operator is said to be centered if its associated weighting vector $\mathbf{w}$ satisfies the following conditions:

- symmetric: $w_i = w_i + n - 1$
- strongly decaying: If $i < j \leq (n+1)/2$, then $w_i < w_j$, and, if $i > j \geq (n+1)/2$, then $w_i < w_j$
- inclusive: $w_i > 0$.

One can see that these types of aggregation operators give the most weight to the central scores in the argument tuples and less weight to the extreme values. Among other applications, this type of operator can be useful for some kinds of smoothing (e.g., for group preference aggregation). The median, for instance, is a prototypical example of this class. In this case, we want to aggregate so that we can eliminate the extreme values. In the case of COWA operators, with the help of dispersion, a measurement called *strength of centering* can be provided as defined in the next section.

Definition 5

The strength of centering of a centered weighting vector $\mathbf{w}$ is defined as

$$Cent(w) = 1 - \frac{disp(w)}{\ln(n)}.$$

A useful way to specify basic unit interval monotonic (BUM) functions (see the "Quantifier-Guided Aggregation" section) for COWA vectors in terms of their derivative is discussed in [35]. The concept of a centering function is introduced as defined in the following.

Definition 6

A function $g: [0,1] \to R$ is called a *centering function* if

- $g(x) > 0$
- g is symmetric, roughly 0.5, i.e., $g(0.5+z) = g(0.5-z)$ for $z \epsilon [0, 0.5]$
- g is unimodal:
  - $g(x) < g(y)$ for $x < y \leq 0.5$
  - $g(x) < g(y)$ for $x > y \geq 0.5$.

Especially, g can be a piecewise-linear, compressed piecewise-linear, step-like, extreme step-like, or parabolic-type centering function.

A deeply examined class of COWA operators is based on using Gaussian-type weights [35] or the Olympic type introduced by Yager in 1993 [27]. The window type of OWA operators also form an important class. An operator is of window type if it takes the average of the m arguments about the center, omitting the best and worst scores. This means that we have

$$w_i = \begin{cases} 0 & \text{if } i < k \\ 1/m & \text{if } k \le i < k+m \\ 0 & \text{if } k \ge k+m. \end{cases}$$

Weighted OWA

There exists a new combination function, the weighted OWA (WOWA), which combines the advantages of the OWA operator and the ones of the weighted mean [18], [23], [24]. The new function allows the user to weigh the reliability of the information source, as the weighted mean does, and the values in relation to their relative position, as the OWA operator. This aggregation function has two weighting vectors, one of which stands for the role of the weighting vector in weighted means, and the other is for the role of the weighting vector in the OWA functions.

Let us consider the following situation. A robot has to aggregate information coming from n different sensors, meaning distances to the obstacles. On one hand, the reliability of each sensor is known (weights **p**). On the other hand, independent of their reliability, the distances from the nearest obstacles are more relevant; therefore, independent from the reliability of the sensors, the inputs also have to be weighted according to their numerical value (weighting vector **w**). Thus, both the numerical value of the inputs as well as their reliability are to be taken into account. It is exactly WOWA that provides this type of aggregation.

Generalized OWA

OWA functions have been generalized to generalized OWA operators (GOWA, also known as *ordered weighted quasi-arithmetic mean*) with the help of generating functions $g := [0,1] \to (-\infty, \infty)$, as defined in the next section.

Definition 7

Let $g := [0,1] \to (-\infty, \infty)$ be a continuous, strictly monotone function; and let **w** be a weighting vector. Let us define the function $GOWA(x)$ as

$$GOWA(x) = g^{-1}\left(\sum_{i}^{n} w_i g(x_{(i)})\right).$$

As for OWA, $x(i)$ denotes the ith largest value of x.

Special cases, such as the ordered weighted geometric function, the power-based generalized OWA, or the ordered weighted harmonic that function together with trigonometric, quadratic, and exponential OWA operators, were studied in [33].

TOWA

In [34], Yager showed how the evaluation of an alternative involves the determination of the degree to which subsets of criteria are satisfied by the alternative, a calculation based upon an anding of the satisfactions of the individual criteria. He examined the possibility of using other t-norms different from min for the anding operation. Applying this generalization, he introduced a further extension of OWA operators, called *TOWA*, involving the mixing of the t-norm and the OWA operator. He produced various aggregation functions resulting from this mixing. The concept of the "power of a t-norm" to provide an ordering over the t-norm operators was also defined.

Compensative connectives are characterized by a higher degree of satisfaction of one of the criteria, which can compensate for a lower degree of satisfaction of another criterion.

Choosing the Weights for OWA Operators

As we have seen, one important issue in the theory of OWA operators is the determination of the associated weights to obtain the appropriate OWA operators in practical usage [9]. Various methods have been introduced, proving to be useful for obtaining the weights associated with OWA operators. Generally, to obtain the value of the weights, we can use the following two ways.

The first method is the use of a learning mechanism. In this approach, we have some sample data and associated aggregated values. The process involves the use of some kind of a regression model.

A second way is to give some meaning to the weights on the base, of which we can have the decision maker directly provide the values for the weights. Let

$$S_k = \sum_{i=0}^{k} w_i.$$

It is easy to see that $S_n = 1$ and $S_0 = 0$. If we have an input vector of criteria satisfaction B such that $b_j = 1$ for j K, it means that K of the criteria are completely satisfied and the rest are completely unsatisfied.

Thus,

$$F(B) = \sum_{i=1}^{k} w_i = S_k.$$

In other words, S_k expresses the degree of satisfaction if K/N portion of the criteria is satisfied. Moreover, due to the fact that $S_k = S_{k-1} + W_k$, we can interpret W as the degree of additional (marginal) satisfaction we get when we turn from the satisfaction of $K-1$ of the criteria to the satisfaction of K. Note that in this interpretation, the case of $w_i = 1/n$ corresponds to a linear increase for each argument. From a pragmatic point of view, it is more natural for a decision maker to provide the degree to which he/she is happy with k criteria being satisfied (the S_k function). From this, we can easily obtain the $w_j s$:

$$w_k = S_k - S_{k-1}, \text{ where } S_0 = 0.$$

Quantifier-Guided Aggregation

As mentioned in the previous section, an important application of OWA operators is in the area of quantifier-guided aggregations [30]. Compensative connectives are characterized by a higher degree of satisfaction of one of the criteria, which can compensate for a lower degree of satisfaction of another criterion. Although in classic binary logic we have two quantifiers, "there exists" ($\exists$) and "for all" ($\forall$), in natural language, we use a huge amount of additional quantifiers (e.g., "few," "many," "or," and "almost all"). This is how the theory of approximate reasoning extends binary logic. According to Zadeh [40], quantifiers can be represented as fuzzy subsets of the unit interval (or the real line). Zadeh suggested the use of two kinds of quantifiers: those saying something about the number of elements (absolute quantities) and those saying something about the proportion of elements (relative quantities).

Definition 8

Let Q be a function $Q:[0,1] \to [0,1]$ such that $Q(0) = 0$, $Q(1) = 1$ and $Q(x) \geq Q(y)$ for $x > y$. In this case, Q is called a *BUM function* or a *RIM quantifier*. With the help of this RIM quantifier, we get the associated OWA weights the following way:

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right).$$

Let us face a decision problem with n criteria, $A_1, \dots A_n$, where $A_i(x) = a_i$ stands for the degree to which alternative x satisfies criteria A_i. If the decision maker desires that Q of the criteria be satisfied, then Q is an absolute quantity definable on $L = [0,n]$. For $x \epsilon L$, $Q(x)$ indicates the degree to which the decision maker is satisfied with x criteria being solved. We can easily see the following:

- $Q(0) = 0$, i.e., the decision maker gets absolutely no satisfaction if he/she gets no criteria satisfied
- $Q(n) = 1$, i.e., he/she is completely satisfied if he/she gets all the criteria satisfied
- If $r_1 > r_2$, then $Q(r_1) \geq Q(r_2)$, which means that if he/she gets more criteria satisfied he/she will not become less satisfied

The overall valuation of x is $F_Q(a_1, \dots, a_n)$, where F_Q is an OWA operator. We can see the weighting vector as a manifestation of the quantifier underlying the aggregation process. If the decision maker wants Q of the objectives satisfied, then we obtain the following weighting vector:

$$w_k = Q(k) - Q(k-1), \text{ where } k = 1, \dots n \text{ and } Q(0) = 0.$$

Furthermore, if weights are obtained, we have

$$Q(k) = \sum_{i=1}^{k} w_i.$$

For instance, for "and" we get $w_n = 1, w_i = 0$, if $i \neq n$, $Q(k) = 0$, if $k \neq n, Q(n) = 1$. For "or," we obtain $w_1 = 1, w_i = 0$, if $i \neq 1, Q(k) = 1$, if $k \geq 1$. For the pure averaging quantifier, $w_i = (1/n)$, $Q(k) = (k/n)$, which means it is a linear quantifier.

If Q is a relative quantity, then it can be represented as a fuzzy subset of I such that for each $r \epsilon I$, $Q(r)$ indicates the degree to which r portion of the objects satisfies concept Q. For example, the quantifier "for all" can be represented by a fuzzy subset of I such that $Q(1) = 1$ and $Q(r) = 0$, if $r \neq 1$.

Other mentioned quantifiers can be expressed in the following way:

$$Q_{a,b}(t) = \begin{cases} 0, & \text{if } t \leq a, \\ \dfrac{t-a}{b-a}, & \text{if } a < t < b, \\ 1, & \text{if } t \geq b. \end{cases}$$

For instance, for "most," we can choose pairs $(a,b) = (0.3, 0.8)$; for "at least half," $(0, 0.5)$; and for "as many as possible," $(0.5, 1)$. This means that, e.g., for "most," with $(a,b) = (0.3, 0.8)$ and $n = 5$, the weighting vector is $\mathbf{w} = (0, 0.2, 0.4, 0.4, 0)$.

Data-Based Methods

Data-based methods share the common feature of eliminating nonlinearity due to a reordering of the components of $\mathbf{a}$ by restricting the domain to the simplex $S \subset [0,1]^n$ defined by inequalities $a_1 \leq a_2 \leq \dots \leq a_n$. Thus, in that domain, the OWA operator is a linear function (it coincides with the arithmetic mean). By finding the coefficients of this function, the OWA operator can be computed on the whole $[0,1]^n$ by using its symmetry.

One can use least squares or least absolute deviation criterion, employing either quadratic or linear programming techniques. Filev and Yager [9] suggested a nonlinear change in variables to obtain an unrestricted minimization problem; however, the resulting nonlinear optimization problem was rather difficult because of the large number of local minimizers. For example, an approach that relied on quadratic programming was used in [2] and was shown to be numerically efficient and stable. Additionally, a desired value of the measurement of orness is often imposed. This requirement can also be incorporated into a quadratic programming or linear programming problem as an additional linear-equality constraint.

Measurement-Based Methods

Another approach for obtaining weights based on a simple specification of the measurement of orness was suggested by O'Hagan [21]. In this approach, a decision maker specifies one parameter, α, the attitudinal character of the aggregation procedure. O'Hagan developed a way to generate OWA weights that have a predefined degree of orness and that maximize the entropy, referring to them as *maximal entropy OWA operators*. The suggested approach was algorithmically based on the solution of a constrained-optimization problem. In [12], the authors chose a vector of weights that maximizes the dispersion $disp(\mathbf{w})$ for a given n using the method of Lagrange multipliers. In [13], Fullér and Majlender examined a minimum-variance method to obtain the minimal variability OWA operator weights.

The OWAD operator can be further extended by using other types of distances, such as Euclidean, Minkowski, and quasi-arithmetic.

In 2009, Yager [36] also considered the possibility of using minimization of dispersion. He discussed the concerns he had with both the maximization and minimization of dispersion and investigated the possibility of finding an optimal solution intermediate to these extremes. He introduced a fundamental requirement for a measurement of dispersion called the *preference for equal division*.

Methods Based on Weight-Generating Functions

Weight-generating functions make it possible to obtain weighting vectors of OWA operators and weighted means for any number of arguments, which means the acquisition of extended-aggregation functions. This is very important when the number of arguments is not given. It turned out to be possible to learn weight-generating functions from empirical data, similar to determining weighting vectors of the aggregation functions of a fixed dimension [3]. The method relies on representing a weight-generating function with a spline or polynomial and fitting its coefficients by solving a least-squares or least absolute deviation problem, subject to several linear constraints.

Further Applications

A wide range of further applications of OWA operators has been introduced in the literature. Without the desire for completeness, a few examples are provided in this section.

There exists an application of fuzzy connectives in statistical regression. In it, the standard least squares, least absolute deviation, and maximum-likelihood criteria can all be replaced with an OWA function of the residuals. Yager and Beliakov [37] presented various approaches to the numerical solution of regression problems. OWA-based regression is particularly useful in the presence of outliers.

In [28], Yager focused on the problem of maximizing the OWA aggregation of a group of variables that are constrained by a collection of linear inequalities. This procedure is extremely useful to provide a solution to fuzzy linear programming problems in which some linguistically proscribed number of goals must be satisfied.

An extension of the OWA operator to the case in which our argument is a continuous-valued interval rather than a finite set of values is also possible [32]. Moreover, Yager considered the extension of the continuous interval argument OWA operator to the more general case in which the argument values have importance weights. Using this approach, he introduced the idea of an attitudinal-based expected value associated with a continuous random variable.

OWA operators are also applied in decision making under ignorance, an important class of uncertain decision-making problems. In this case, there are alternatives from which one must be selected, and there are different states of nature. The payoff received depends on the state of nature and the alternative selected at the same time. *Uncertainty* means that the decision maker is unaware of the state of nature at the time of choosing the alternative.

An extended class of OWA operators, one based on the relaxation of requirements on the OWA operators, is introduced in [31]. This relaxation allows us to consider a new branch of OWA operators, NOMOWA operators, which have negative weights and exhibit nonmonotonicity. Some special cases of these operators are discussed, and then we investigate the role of these nonmonotonic operators in the formulation of multicriteria decision functions.

The OWA operator can be also used to provide norms. Several different classes of OWA norms are considered in [38]. It is shown that the functional generation of the weights of an OWA norm requires that the weight-generating function have a nonpositive second derivative. The use of OWA operators to induce similarity measurements is also discussed.

In business and economics, the use of different similarity measurements such as the Hamming distance [15] is needed. These measurements also use aggregation operators. The advantage of the Hamming distance in decision making is that it permits comparisons of available values with some ideal ones. Depending on the particular problem we have, different decision makers may have varying opinions or interests; therefore, the "best" results are not always the same for each decision maker. An extreme example of this would be the concept of *dumping*, which means that the seller is selling the product at a price that is lower than its production cost. Thus, in the decision process of fixing this price, the seller is looking for an ideal that it is not the best one. The use of the OWA operator in the Hamming distance, i.e., the OWA distance (OWAD) operator, has also been analyzed [19], [20]. Its main advantage is that it provides a parameterized family of distance aggregation operators between the maximum and minimum distances. The OWAD operator can be further extended by using other types of distances, such as Euclidean, Minkowski, and quasi-arithmetic [19].

An extension of OWA operators for n-dimensional fuzzy sets, denoted by MOWA, has recently been proposed by De

Table 1. A literature overview.

Operator	Main Attributes
OWA [26]	Weights are associated directly with an ordered position
COWA [27], [35]	Gives preference to argument values that lie in the middle; the most weight is given to the central scores
WOWA [18], [23], [24]	Allows for weighing the reliability of the information source and the values in relation to their relative position; it has two weighting vectors
GOWA [3], [33]	Ordered weighted quasi-arithmetic mean; the special cases are, e.g., the ordered weighted geometric function, the power-based generalized OWA, and the ordered weighted harmonic function
TOWA [34]	Using other t-norms different from min for the anding operation; mixing of the t-norm and the OWA operator
OWA-based regression [37]	The standard least squares, least absolute deviation, and maximum-likelihood criteria are replaced by an OWA operator; particularly useful in the presence of outliers
OWA in fuzzy linear programming [28]	Interrelated and constrained variables maximizing OWA aggregation from a collection of linear inequalities
Continuous interval argument [32]	An extension of the OWA operator to a continuous-valued interval as an argument
Nonmonotonic OWA [31]	Negative weights, nonmonotonicity
Inducing norms [38]	A weight-generating function with a nonpositive second derivative
OWAD [15], [19], [20]	A parameterized family of distance aggregation operators between the maximum and the minimum distance
MOWA [6], [22]	An extension of OWA operators for n-dimensional fuzzy sets
OWA operators that are robust against outliers [4]	A penalty-based method comprising both outlier detection and the reallocation of weights of the OWA

Miguel et al. [6] by making use of a linear extension of the product order. Unfortunately, the consideration of a linear extension of the product order extends OWA operators to the multidimensional setting at the cost of losing continuity and robustness in the presence of outliers. In [22], it is shown that this focus ultimately results in a forfeiture of the properties of continuity and robustness in the presence of outliers. A further interesting question is the robustness of the aggregation functions in terms of the ranking of the alternatives they produce. In [4], the authors propose a version of OWA operators that are, in contrast to standard OWA operators, robust against inputs with outliers.

Summary and Outlook

The aim of this work was to provide a short review of OWA operators, to synthesize some of the most significant developments, and to identify trends in the literature. As we have seen, the concept of the OWA operator, a symmetric aggregation function that allocates weights according to the input value and unifies in one operator the conjunctive and disjunctive behavior, was introduced by Yager in 1988. Since then, the family of these functions has been axiomatized and extended in various ways. For an overview of the operators and problems considered in this article, see Table 1. In addition, OWA operators provide a parameterized family of aggregation functions, including many of the well-known operators. The development of an appropriate methodology for obtaining the weights is still an issue of considerable interest. For more details about recent developments in OWA operators, the reader is referred to [39].

Future studies should concentrate on a systematic comparison of outlier-detection methods, that is, to the best of the author's knowledge, still missing in the literature. With respect to multidimensional settings, an interesting open problem is how the use of geometric quantiles, instead of linear extensions of the product order in the construction of OWA operators, could contribute to solving the problem of robustness.

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