Short Paper

On WA Expressions of Induced OWA Operators and Inducing Function Based Orness With Application in Evaluation

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***Abstract*—Induced ordered weighted averaging (OWA) operators are important extensions of OWA operators and can suit more different decision environments. However, when practitioners perform induced OWA operators, they frequently do not consider the tied values in inducing information, therein lay the problem of possible non-uniqueness of aggregation results. From a novel perspective that can easily remove this flaw, this study uses a systematical frame to better redefine induced OWA operators, which not only covers all the spirits of the original definition of induced OWA operator but also extends it to some wider understandings and usages. One significant feature related to this frame is that we can express induced OWA operators fully in terms of weighted averaging operators, making the usages and expressions of induced OWA operators much more flexible and stricter. In detail, we propose several new definitions, such as positioning transformation, permutation-based orness, and inducing function based orness. As an example of flexibly formulating complex preferences under the proposed concepts, a novel investigators-consultants-decision maker (ICD) two-stage IOWA evaluation model is clearly proposed and illustrated.**

***Index Terms*—Aggregation operators, evaluation, induced ordered weighted averaging (OWA) operators, information fusion, orness/andness.**

I. Introduction

Information fusion techniques and aggregation operators [1] (also known as aggregation functions) are the basis and underpinning for enormous evaluation problems. Without loss of generality, we confine the discourse to be unit interval [0,1] (which also pertains to the aggregation of fuzzy information granules), and then define any inputs information under aggregation to be a function $x : S(n) \to [0,1]$, where $S(n) = \{1,\dots,n\}$; we denote by $\mathcal{X}^{<n>}$ the set of all such functions x. Then, an aggregation operator (of dimension n) is a bounded and monotonic function $A : \mathcal{X}^{<n>} \to [0,1]$ satisfying the following two conditions [1]: 1) $A((0,\dots,0)) = 0$ and $A((1,\dots,1)) = 1$; 2) for any $x, y \in \mathcal{X}^{<n>}$ with $x \le y$, $A(x) \le A(y)$. We also denote by $\mathcal{A}^{<n>}$ the space of all aggregation operators of dimension n.

Among plenty of diverse aggregation operators, the ordered weighted averaging (OWA) operators are one type of notably powerful and flexible information fusion technique, which were introduced by Yager [2] and have been further studied in many areas from both theoretical and application respects [3]–[10]. The reason why OWA operators are widely accepted and used by numerous practitioners is presumably because it can accurately and readily model the "bipolar" preference (which conforms with the decisional cognitions of most practitioners), such as optimistic/pessimistic preference in decision making and evaluation problems.

By extension, Yager and Filev [11] soon put forward the induced ordered weighted averaging (IOWA) operators, which can accommodate to a wider range of real evaluation problems involving different types of preference rather than optimistic/pessimistic preference only [2]. In OWA aggregation, although a weight function w is predetermined, the actual weights rearrangement is in accordance with the magnitude of inputs function x. For IOWA operators and related aggregation processes, rather than using inputs function x to conduct the weights rearrangement and prioritization, an attached inducing function $t : S(n) \to [0,1]$ to x will in actual play this role, in order to direct how to assign weights to each input.

While IOWA operators have already been successfully applied to numerous applications, the forms and definitions to express the same principle in it are diverse, with some of them not correctly or well formulated. Another shortcoming is the unintentional neglecting of the effect of possibly tied values in the inducing function t, that is, scholars often take for granted that t is always an injective mapping, which is actually not always the case in reality. For instance, when t is a constant function, then any permutation involved in IOWA aggregation can be suitable, which leads to non-uniqueness in results.

This paper will provide a systematical and strict frame to reconstruct IOWA operators from different angles. Moreover, in some applications, where more complex inducing process pertains, we still require a type of more generalized formulations for IOWA operators. Thus, we will later introduce some corresponding new concepts, definitions, and formulations for that purpose, and then some preference involved evaluation model, can be well presented and formulated. In addition, the new concepts and definitions will also strengthen and better standardize the original proposals by Yager from theoretical or mathematical viewpoints.

The remaining of this paper is organized as follows. In Section II, after going through some reviews and definitions, we gradually reformulate IOWA in the expression of weighted averaging (WA) operators via positioning transformation, with some other necessary concepts and discussions proposed. In Section III, pertaining to the earlier proposals, a pertinent decision model in business management and evaluation is presented to illustrate the usage and show the advantage of the new concepts proposed under our frame. Finally, Section VI concludes this article.

II. Induced OWA Operators in Expression of Weighted Averaging Operators and the Positioning Transformation

In this section, we start with paraphrasing the definition of Yager OWA operators.

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Definition 1 [2]: Given any normalized weight function $w : S(n) \to [0, 1]$ (i.e., $\sum_{i=1}^{n} w(i) = 1$), the OWA operator (of dimension *n*) with *w* is defined to be a function $OWA_w \in \mathcal{A}^{<n>}$ such that

$$\mathrm{OWA}_w(x) = \sum_{i=1}^{n} w(i)x(\sigma(i)) \tag{1}$$

where $\sigma : S(n) \to S(n)$ is any suitable permutation on $S(n)$ satisfying $x(\sigma(i)) \geq x(\sigma(j))$ whenever $i < j$.

Example 1: For $n = 4$, suppose $x = (x(i))_{i=1}^{4} = (0.2, 0.5, 1, 0.5)$, then $\sigma = (3, 2, 4, 1)$ is one suitable permutation. Suppose $w = (w(i))_{i=1}^{4} = (0.4, 0.3, 0.2, 0.1)$, then

$$\begin{aligned} OWA_w(x) &= \sum_{i=1}^{4} w(i)x(\sigma(i)) \\ &= w(1)x(\sigma(1)) + w(2)x(\sigma(2)) + w(3)x(\sigma(3)) + w(4)x(\sigma(4)) \\ &= w(1)x(3) + w(2)x(2) + w(3)x(4) + w(4)x(1) \\ &= (0.4)(1) + (0.3)(0.5) + (0.2)(0.5) + (0.1)(0.2) = 0.67. \end{aligned}$$

(Observe that if we choose $\sigma = (3, 4, 2, 1)$, it will lead to the same aggregation result.) □

In this paper, we denote by $\mathcal{W}^{<n>}$, the space of all normalized weight functions. The weight function *w* in Definition 1 (a detailed situational context) is also called OWA weight function, whose orness/andness is defined as follows.

Definition 2 [2]: Given OWA weight function $w : S(n) \to [0, 1]$, its orness is defined to be a function $\mathrm{orness} : \mathcal{W}^{<n>} \to [0, 1]$ such that

$$\mathrm{orness}(w) = \sum_{i=1}^{n} \frac{n-i}{n-1} \cdot w(i). \tag{2}$$

Its andness is defined to be a function $\mathrm{andness} : \mathcal{W}^{<n>} \to [0, 1]$ such that

$$\mathrm{andness}(w) = \sum_{i=1}^{n} \frac{i-1}{n-1} \cdot w(i) = 1 - \mathrm{orness}(w). \tag{3}$$

Remark: Although we called *w* as "OWA weight function," in this study, we will not distinguish it from other normalized weight functions. No matter how we possibly abuse terminologies for $w \in \mathcal{W}^{<n>}$ in different situations, whether or not there is a type of preference involved in *w* depends on the detailed context.

From the perspective of operators, the orness/andness of $\mathrm{OWA}_w \in \mathcal{A}^{<n>}$ is strictly defined as follows.

Definition 3 [1], [2]: The orness of any OWA operator $\mathrm{OWA}_w \in \mathcal{A}^{<n>}$ is defined to be a function $\mathrm{orness} : \mathcal{A}^{<n>} \to [0, 1]$ such that

$$\mathrm{orness}(\mathrm{OWA}_w) = \sum_{i=1}^{n} \frac{n-i}{n-1} \cdot w(i). \tag{4}$$

The andness of OWA operator $\mathrm{OWA}_w \in \mathcal{A}^{<n>}$ is defined to be a function $\mathrm{andness} : \mathcal{A}^{<n>} \to [0, 1]$ such that

$$\mathrm{andness}(\mathrm{OWA}_w) = \sum_{i=1}^{n} \frac{i-1}{n-1} \cdot w(i) = 1 - \mathrm{orness}(\mathrm{OWA}_w). \tag{5}$$

Remark: We have that $\mathrm{orness}(\mathrm{OWA}_w) = \mathrm{orness}(w)$ and $\mathrm{andness}(\mathrm{OWA}_w) = \mathrm{andness}(w)$.

Next, we also briefly rephrase Yager and Filev's IOWA [11] in some convenient and consistent way.

Definition 4 [11]: Given any normalized weight function $w : S(n) \to [0, 1]$ and inducing function $t : S(n) \to [0, 1]$, which is injective, then the IOWA operator (of dimension *n*) with *w* is defined to be a function $\mathrm{IOWA}_w \in \mathcal{A}^{<n>}$ such that

$$\mathrm{IOWA}_w(x) = \sum_{i=1}^{n} w(i)x(\sigma(i)) \tag{6}$$

where $\sigma : S(n) \to S(n)$ is the unique permutation on $S(n)$ satisfying $t(\sigma(i)) > t(\sigma(j))$ whenever $i < j$.

The OWA operators and IOWA operators will be expressed in terms of the well-known WA operators, which are reviewed as follows.

Definition 5 [1]: Given any normalized weight function $w \in \mathcal{W}^{<n>}$, the WA operator (of dimension *n*) with *w* is defined to be a function $WA_w \in \mathcal{A}^{<n>}$ such that

$$WA_w(x) = \sum_{i=1}^{n} w(i)x(i). \tag{7}$$

We denote by $P(n)$ the set of all permutations on $S(n) = \{1, \ldots, n\}$. Next, we define a transformation on $\mathcal{W}^{<n>}$ called positioning transformation (of any $w \in \mathcal{W}^{<n>}$) according to any permutation $\sigma \in P(n)$.

Definition 6: (Positioning transformation according to σ): Given permutation $\sigma \in P(n)$, the positioning transformation on $\mathcal{W}^{<n>}$ with σ, $PT_\sigma : \mathcal{W}^{<n>} \to \mathcal{W}^{<n>}$, is defined such that $\hat{w} = PT_\sigma(w)$ satisfies

$$\hat{w}(i) = w(\sigma^{-1}(i)) \tag{8}$$

or equivalently

$$\hat{w}(\sigma(i)) = w(i). \tag{9}$$

Example 2: For $n = 4$, suppose $\sigma = (\sigma(i))_{i=1}^{4} = (3, 2, 4, 1)$ (which means $\sigma^{-1} = (\sigma^{-1}(i))_{i=1}^{4} = (4, 2, 1, 3)$), and assume $w = (w(i))_{i=1}^{4} = (0.4, 0.3.0.2, 0.1)$. If $\hat{w} = PT_\sigma(w)$, then

$$\begin{aligned} \hat{w} &= (\hat{w}(i))_{i=1}^{4} = (w(\sigma^{-1}(i)))_{i=1}^{4} \\ &= (w(4), w(2), w(1), w(3)) = (0.1, 0.3, 0.4, 0.2). \end{aligned}$$

□

Observe that if an input $x \in \mathcal{X}^{<n>}$ is injective (i.e., for any $i, j \in S(n)$, $x(i) \neq x(j)$), then the permutation $\sigma \in P(n)$ satisfying $x(\sigma(i)) \geq x(\sigma(j))$ whenever $i < j$ is unique. Thus, the OWA operator with *w* can be defined equivalent to the WA operator with $\hat{w}$, due to the following deduction:

$$\begin{aligned} \mathrm{OWA}_w(x) &= \sum_{i=1}^{n} w(i)x(\sigma(i)) = \sum_{i=1}^{n} \hat{w}(\sigma(i))x(\sigma(i)) \\ &= \sum_{i=1}^{n} \hat{w}(i)x(i) = \mathrm{WA}_{\hat{w}}(x). \end{aligned} \tag{10}$$

This motivates us to define OWA operators in a general way of using WA operators since WA operators are very well-known and used in many evaluation problems. For example, recall the expected value of a given random variable is actually with the same essence to a WA operator, and a normalized weight function and probability distribution are only with some terminological difference. Thus, finding a unified way to express different operators using the same frame of WA operator is quite appealing, especially in mathematics (e.g., recall that the well-known Choquet integral with any probability measure can also be expressed as a special WA operator [1]). More advantages of this expression will be found in some practical evaluation problems, as will be discussed later.

Note that $x \in \mathcal{X}^{<n>}$ is not necessarily injective (i.e., there may be "tied" values for *x*). When *x* is not injective, any suitable permutation σ we choose will not affect the OWA aggregation result. That is, if *x* is not injective, then for any two permutations $\sigma_1, \sigma_2 \in P(n)$ satisfying $x(\sigma_1(i)) \geq x(\sigma_1(j))$ and $x(\sigma_2(i)) \geq x(\sigma_2(j))$ whenever $i < j$, we clearly have $x(\sigma_1(i)) = x(\sigma_2(i))$ for all $i \in S(n)$. Then, irrespective of the possibly different permutations chosen, by Definition 4, we have the following new frame for defining OWA operators.

Definition 7: (OWA operators in the expression of WA operators): Given any normalized weight function $w : S(n) \to [0, 1]$, let $\sigma : S(n) \to S(n)$ be any suitable permutation on $S(n)$ satisfying $x(\sigma(i)) \geq x(\sigma(j))$ whenever $i < j$, and let $PT_\sigma : \mathcal{W}^{<n>} \to \mathcal{W}^{<n>}$ be the positioning transformation on $\mathcal{W}^{<n>}$ with σ, then the OWA operator with *w* is defined to be a WA operator $WA_{\hat{w}} \in \mathcal{A}^{<n>}$, where $\hat{w} = PT_\sigma(w)$.

Example 3: For $n = 4$, suppose $x = (x(i))_{i=1}^4 = (0.2, 0.5, 1, 0.5)$, then $\sigma = (3, 2, 4, 1)$ is one suitable permutation. As in Example 2, where $w = (w(i))_{i=1}^4 = (0.4, 0.3 0.2, 0.1)$, if $\hat{w} = PT_\sigma(w)$, then $\hat{w} = (0.1, 0.3, 0.4, 0.2)$. Thus, by Definition 7, the OWA operator to *x* with *w* is computed by

$$\mathrm{WA}_{\hat{w}}(x) = \sum_{i=1}^{4} \hat{w}(i)x(i) = (0.1)(0.2) + (0.3)(0.5) + (0.4)(1) + (0.2)(0.5) = 0.67.$$

One may compare this same result with what has been obtained in Example 1. □

For IOWA operators, especially note that when we consider $t \in \mathcal{X}^{<n>}$, the inducing function, instead of $x \in \mathcal{X}^{<n>}$, the above discussion and results will no longer remain pertinent and suitable. That is, for any two suitable permutations $\sigma_1, \sigma_2 \in P(n)$, we only have $t(\sigma_1(i)) = t(\sigma_2(i))$ rather than $x(\sigma_1(i)) = x(\sigma_2(i))$, which may lead to different results if we similarly use original Definition 2. Thus, for IOWA operators and when *t* is not injective, Definition 2 of IOWA operators should be redefined. We next address this problem by firstly listing all suitable permutations and then taking an average for the corresponding results. With such reasonable handling, we will naturally propose a novel IOWA definition, which is desired and with more generality, by using the language of WA operators as before.

Definition 8: (Positioning transformation according to inducing function *t*): Defines a mapping $\psi^{<n>} : \mathcal{X}^{<n>} \to 2^{P(n)} \backslash \varnothing$ $(n \geq 2)$ such that $\psi^{<n>}(t)$ consists of all the permutation σ satisfying $t(\sigma(i)) \geq t(\sigma(j))$ whenever $i < j$. Given any inducing function $t : S(n) \to [0, 1]$, the positioning transformation on $\mathcal{W}^{<n>}$ according to inducing function t, $PT_t : \mathcal{W}^{<n>} \to \mathcal{W}^{<n>}$, is defined such that

$$PT_t(w) = \frac{1}{|\psi^{<n>}(t)|} \cdot \sum_{\sigma \in \psi^{<n>}(t)} PT_\sigma(w) \tag{11}$$

where $|\psi^{<n>}(t)|$ denotes the cardinality of set $\psi^{<n>}(t)$.

Example 4: Suppose $w = (w(i))_{i=1}^4 = (0.4, 0.3 0.2, 0.1)$, and consider $t = (t(i))_{i=1}^4 = (0.2, 0.5, 1, 0.5)$, then we obtain $\psi^{<4>}(t) = \{(3, 2, 4, 1), (3, 4, 2, 1)\}$. Take $\sigma_1 = (3, 2, 4, 1)$ and $\sigma_2 = (3, 4, 2, 1)$, then from Example 2, we have known that $PT_{\sigma_1}(w) = (0.1, 0.3, 0.4, 0.2)$. Similarly, we obtain $PT_{\sigma_2}(w) = (0.1, 0.2, 0.4, 0.3)$. Consequently

$$\begin{aligned} PT_t(w) &= \frac{1}{|\{\sigma_1, \sigma_2\}|} \cdot \sum_{\sigma \in \{\sigma_1, \sigma_2\}} PT_\sigma(w) \\ &= \frac{1}{2} \cdot ((0.1, 0.3, 0.4, 0.2) + (0.1, 0.2, 0.4, 0.3)) \\ &= (0.1, 0.25, 0.4, 0.25). \end{aligned}$$

□

Definition 9: (IOWA operators in the expression of WA operators): Given any normalized weight function $w \in \mathcal{W}^{<n>}$, and any inducing function $t : S(n) \to [0, 1]$, let $PT_t : \mathcal{W}^{<n>} \to \mathcal{W}^{<n>}$ be the positioning transformation on $\mathcal{W}^{<n>}$ according to *t*, then the IOWA operator with *w* under inducing function *t* is defined to be a WA operator $WA_{\hat{w}} \in \mathcal{A}^{<n>}$, where $\hat{w} = PT_t(w)$.

Example 5: Continued to Example 4, if $x = (x(i))_{i=1}^4 = (0.6, 0.8, 0, 1)$ with other conditions unchanged (i.e., $w = (w(i))_{i=1}^4 = (0.4, 0.3 0.2, 0.1)$, $t = (t(i))_{i=1}^4 = (0.2, 0.5, 1, 0.5)$), then $\hat{w} = (0.1, 0.25, 0.4, 0.25)$. By Definition 9, the IOWA operator to *x* with *w* is computed with □

$$\begin{aligned} \mathrm{WA}_{\hat{w}} &= \sum_{i=1}^{4} \hat{w}(i)x(i) \\ &= (0.1)(0.6) + (0.25)(0.8) + (0.4)(0) + (0.25)(1) = 0.51. \end{aligned}$$

So far, we have used a new frame to neatly and accurately redefine IOWA, by which all related shortcomings or misuses existing in earlier usages are completely eradicated.

Next, we discuss the permutation-based orness and the inducing function based orness, which are useful in some applications, including the one that will be introduced in Section III.

Definition 10: Denote by $\mathcal{A}_{\mathrm{WA}}^{<n>}$, the space of all WA operators of dimension *n*. For any weight function, $w \in \mathcal{W}^{<n>}$, and any permutation on $S(n)$, $\sigma \in P(n)$, the $\sigma - \mathrm{orness}$ of WA operator with *w*, $\mathrm{WA}_w \in \mathcal{A}^{<n>}$, is defined to be a function $\sigma - \mathrm{orness} : \mathcal{A}_{\mathrm{WA}}^{<n>} \to [0, 1]$ such that

$$\begin{aligned} \sigma - \mathrm{orness}(\mathrm{WA}_w) &= \sum_{i=1}^{n} \frac{n - i}{n - 1} \cdot w(\sigma(i)) \\ &= \sum_{i=1}^{n} \frac{n - \sigma^{-1}(i)}{n - 1} \cdot w(i). \end{aligned} \tag{12}$$

Having some similar sense to the relation between Definitions 6 and 8, we directly present the following one.

Definition 11: For any weight function, $w \in \mathcal{W}^{<n>}$, and any Inducing Function $t \in \mathcal{X}^{<n>}$, the $t - \mathrm{orness}$ of WA operator with *w*, $\mathrm{WA}_w \in \mathcal{A}^{<n>}$, is defined to be a function $t - \mathrm{orness} : \mathcal{A}_{\mathrm{WA}}^{<n>} \to [0, 1]$ such that

$$t - \mathrm{orness}(\mathrm{WA}_w) = \frac{1}{|\psi^{<n>}(t)|} \cdot \sum_{\sigma \in \psi^{<n>}(t)} \sigma - \mathrm{orness}(\mathrm{WA}_w) \tag{13}$$

where $\psi^{<n>} : \mathcal{X}^{<n>} \to 2^{P(n)} \backslash \varnothing$ has been defined in Definition 8.

In what follows, from different aspects, we make some discussions and analyses about WA expression of IOWA, positioning transformation, $\sigma - \mathrm{orness}$, and $t - \mathrm{orness}$.

1) Clarity and flexibility

In Definition 11, *t* can be any function satisfying $t \in \mathcal{X}^{<n>}$. It can have different meanings according to different situations. For example, $t(i)$ may represent the certain degree of input $x(i)$, meaning the extent to which $x(i)$ is correctly or accurately provided or reckoned (cf., [12] and [13] for basic unit information). This also means that for a fixed weight function *w*, if we choose some other inducing functions t' with different practical meanings, we can have $t' - orness$ as some other measures to be used in different decisional scenarios. In addition, when $t = x$, i.e., dealing with standard OWA operator, it conveys the information that the related WA operators will be performed depending upon the ordering of magnitudes in inputs, indicating some optimistic/pessimistic preference of decision maker is involved. Hence, the proposed formulations in Definition 11 provide a much more flexible and clearer way in decision making to well model several different preferences expressed by different inducing functions. In Section III, we will present a detailed example in preference involved evaluation (not limited only to optimistic/pessimistic preference). In that example, we will use those formulations to handle the preferences from different sources, and it may otherwise cause some confusion.

2) Possible further extensions on complex structures

The convenience and flexibility of WA expressions of IOWA operators offer possibilities for them to be applied in many types of applications, especially when fuzzy methods will be used [14], [15]. Furthermore, from a theoretical point, it can also have further extensions to be in tune with other developed theories and methods. For example, when the inputs information under aggregation is not a function $x : S(n) \to [0, 1]$ but a function $x : S(n) \to L$ where $(L, \precsim)$ is a vector lattice, then the extended OWA aggregation introduced in [16] requires to redefine a corresponding adapted permutation $\sigma : S(n) \to S(n)$ called permissible permutation satisfying $x(\sigma(j)) \precsim x(\sigma(i))$ whenever $i < j$. Thus, analogue to what has been defined in Definition 8, we also need to redefine $\psi^{<n>} : \tilde{\mathcal{X}}^{<n>} \to 2^{P(n)}\backslash\varnothing$ such that for any $\sigma \in \psi^{<n>}(x)$, it holds that $x(\sigma(j)) \precsim x(\sigma(i))$ whenever $i < j$.

3) Some further variations borrowing the similar ideas and methods.

There can be more variations of the proposed methods in theoretical extensions and more preferences involved decision-making problems. For instance, the distance between two OWA operators OWA_{w_1} and OWA_{w_2} with the consideration of the bipolar structure of OWA weight functions are discussed in [17]. For different forms of those well-defined distances, all are in actual defined to be different functions of multi-variables $(w(i))_{i=1}^n$. When some decisional scenario will change, using a similar idea to $\sigma - \mathrm{orness}$, we can naturally define $\sigma - \mathrm{distance}$ based on certain σ, which embodies some different bipolar preference. Similarly, with some more handlings, we can also define $t - \mathrm{distance}$ with given inducing function *t*.

4) Some discussions concerning one different definition of orness.
   a) Recall that another definition for orness of any idempotent aggregation operator A $(A \in \mathcal{A}^{<n>})$ is expressed by the function $\mathrm{orness}' : \mathcal{A}^{<n>} \to [0, 1]$ such that $\mathrm{orness}'(A) = \frac{1}{n-1}[\int_0^1 \cdots \int_0^1 A(x_1, \ldots, x_{n-1})dx_1 \cdots dx_{n-1} - 1]$; and when $A = \mathrm{OWA}_w$, we have $\mathrm{orness}'(\mathrm{OWA}_w) = \mathrm{orness}(\mathrm{OWA}_w)$ [18], [19]. We recall that for an OWA operator $\mathrm{OWA}_w \in \mathcal{A}^{<n>}$, it may take different orness $\mathrm{orness}(\mathrm{OWA}_w)$ according to different weight functions $w \in \mathcal{W}^{<n>}$ assigned to it, which well indicates a type of bipolar preference extent involved in that operator.
   b) Interestingly, for an IOWA operator $\mathrm{IOWA}_w \in \mathcal{A}^{<n>}$, the orness of it is a constant value, irrespective of the weight function $w \in \mathcal{W}^{<n>}$ assigned to it, i.e., it always has $\mathrm{orness}'(\mathrm{IOWA}_w) = \mathrm{orness}'(\mathrm{WA}_w) = 0.5$ [18], [19].
   c) On the other hand, for a WA operator with *w*, $\mathrm{WA}_w \in \mathcal{A}^{<n>}$, its $\sigma - \mathrm{orness}$, $\sigma - \mathrm{orness}(\mathrm{WA}_w)$, is dependent on both $\sigma \in P(n)$ and $w \in \mathcal{W}^{<n>}$. Moreover, $\sigma - \mathrm{orness}(\mathrm{WA}_w) = 1$ if and only if $w(\sigma(1)) = 1$; $\sigma - \mathrm{orness}(\mathrm{WA}_w) = 0$ if and only if $w(\sigma(n)) = 1$.
   d) Thus, for a WA operator with *w*, $\mathrm{WA}_w \in \mathcal{A}^{<n>}$, its $t - \mathrm{orness}$, $t - \mathrm{orness}(\mathrm{WA}_w)$, is dependent on both $t \in \mathcal{X}^{<n>}$ and $w \in \mathcal{W}^{<n>}$. Moreover, $t - \mathrm{orness}(\mathrm{WA}_w) = 1$ if and only if there is $k \in S(n) = \{1, \ldots, n\}$ so that $w(k) = 1$ and for all $\sigma \in \psi^{<n>}(t)$ we have $\sigma(1) = k$; and in this case $\mathrm{WA}_w(x) = x(\sigma(1))$. Similarly, $t - \mathrm{orness}(\mathrm{WA}_w) = 0$ if and only if there is $k \in S(n) = \{1, \ldots, n\}$ so that $w(k) = 1$ and for all $\sigma \in \psi^{<n>}(t)$ it holds $\sigma(n) = k$; and in this case $\mathrm{WA}_w(x) = x(\sigma(n))$.

III. Application in Business Management and Evaluation

In this section, directly using those concepts, expressions, and definitions discussed in Section II, we will elaborately present an application of them in business management and evaluation.

The prediction of the market share of a certain product in near future is important and helpful in business management and evaluation. To obtain relatively reasonable predicted value for the related market share, objectivity and subjectivity often should be considered simultaneously. This is presumably because objectivity appears to be more a transparent phenomenon related to current economical and market circumstance, and, meanwhile, subjectivity of managements and decision makers is often the very embodiment of their long time experiences and commercial intuitions.

Suppose a decision maker sends out *n* different investigators from her company to manage to obtain their own research works on the market share of that product in future. These *n* investigators are represented by a set $S(n) = \{1, 2, \ldots, n\}$. After their respective market research works have been finished, they all collected necessary information and speculated the possible values for market share. Namely, the predicted values for market share are presented by an inputs function $x : S(n) \to [0, 1]$ such that $x(i)$ is the predicted market share reckoned by investigator *i*. With *x* being suggested to the decision maker, she is then able to use aggregation operators, e.g., a preference involved operator embodying her trust ranking over those *n* investigators, to comprehensively consider all of their opinions and catch a final value, which is the most convincible one to her. However, she is unconfident since, as a management in high position, she very often needs to take into account the opinions of the consultants in her think tank, which is composed of *m* different consultants who are with different preferences and experiences, and represented by $S(m) = \{1, 2, \ldots, m\}$.

Thus, instead of aggregating *x* by using her own chosen operator, she routinely consults with her think tank members and chooses to take into account the opinions suggested by them. That is to say, each of such opinions is an aggregation result $A_j(x)$ obtained from consultant *j* whose selected aggregation operator is $A_j \in \mathcal{A}^{<n>}$. Finally, she is ready to consider those *m* different suggestions $(A_j(x))_{j=1}^m$ and, according to her own "trust" preference over those consultants, uses another aggregation operator $B \in \mathcal{A}^{<m>}$ to obtain $B((A_j(x))_{j=1}^m)$ as the predicted value for market share she mostly believes. As another type of important measurement information, those *m* consultants all have their own different evaluations about the "experiences or abilities" of those *n* investigators; that is, *m* different inducing functions $(t_j)_{j=1}^m$ $(t_j \in \mathcal{X}^{<n>})$ indicates that $t_j(i)$ represents the "ability value" of investigator *i* from the view of consultant *j*.

In the foregoing problem, we assume that their preferences can be well modeled by IOWA operators with different preferences (embodied by weight functions) and inducing information (represented by inducing functions). Next, we elaborately formulate this problem using IOWA operators, as we discussed in Section II.

An investigators-consultants-decision maker (ICD) two-stage IOWA model with WA expressions for predicting market share.

Stage 1 Consultants' aggregation preferences and aggregation results

Step 1: Designate *n* investigators that are represented by set $S(n) = \{1, 2, \ldots, n\}$.

Step 2: Collect the investigated market share values from all of the *n* investigators, $x = (x(i))_{i=1}^n$.

Step 3: Invite *m* consultants that are indicated by a set $S(m) = \{1, 2, \ldots, m\}$.

Step 4: Each consultant *j* $(j \in S(m))$ provides his own belief to the "ability values" of all the *n* investigators; that is, an inducing function $t_j \in \mathcal{X}^{<n>}$ tells that the ability of investigator *i* $(i \in S(n))$ is thought to be $t_j(i)$ from consultant *j*'s belief.

Step 5: Each consultant *j* $(j \in S(m))$ exhibits his preference per se over the ability values of investigators from higher to lower; that is, a normalized weight function $w_j \in \mathcal{W}^{<n>}$, whose orness (see Definition 2) indicates the extent of that preference. In addition, all $w_j \in \mathcal{W}^{<n>}$ need to be monotonic non-decreasing, showing the consistency of those preferences $w_j \in \mathcal{W}^{<n>}$ to their preference over investigators according to t_j.

Step 6: By Definition 9 (and Definition 8), using IOWA operator with w_j under inducing function t_j, $\mathrm{WA}_{\hat{w}_j} \in \mathcal{A}^{<n>}$

($\hat{w}_j = PT_{t_j}(w_j)$), to aggregate x and obtain $\mathrm{WA}_{\hat{w}_j}(x)$ as the predicted market share by consultant j ($j \in S(m)$).

(It is noteworthy that at this stage, m WA operators $\mathrm{WA}_{\hat{w}_j}$ with $\hat{w}_j$ are obtained and will be used in the next stage. However, the processes for obtaining them and the preference information t_j and w_j will not be directly used in the next stage.)

Stage 2 Decision maker's aggregation preference and aggregation result

Step 7: Determine the preference of the decision maker that embodies her trust ranking over those n investigators and represent it by an inducing function $t_D \in \mathcal{X}^{<n>}$. (Note that this "trust" preference can be with very different in concept and practical meaning to those m preferences over the "ability values" of investigators; this fact also underscores the flexibility of the proposed model and Definition 9. If we do not use the proposed definitions, it is hard to clearly formulate the problem with such more concepts and different preferences.)

Step 8: By Definition 11, calculate $t_D - \mathrm{orness}$ of WA operator with $\hat{w}_j$, $t_D - \mathrm{orness}(\mathrm{WA}_{\hat{w}_j})$ for $j \in S(m)$. (Using $t_D - \mathrm{orness}$ of $\mathrm{WA}_{\hat{w}_j}$ is clearly because the involved preference herein is the "trust" ranking of the decision maker who will consult with her think tank members, not other preferences, e.g., over the "ability values" of those investigators determined in Step 5.)

Step 9: The preference extent for $t_D - \mathrm{orness}(\mathrm{WA}_{\hat{w}_j})$ ($j \in S(m)$) is also expressed by a weight function $v \in \mathcal{W}^{<m>}$, where v is monotonic nondecreasing, showing the consistency of this preference v to her "trust" preference over those investigators. (Particularly note the subtle difference between the preference in Step 9 and the preferences in Step 5. Although both types are concerned with preferences over investigators according to their preferred inducing functions t_j and t_D, respectively, the preference of v according to t_D is not directly embodied by immediately over investigators themselves, but via $t_D - \mathrm{orness}(\mathrm{WA}_{\hat{w}_j})$ to indirectly and explicitly realize. Such simple and clear ways for this otherwise complex realization, also embody the very advantage of the proposed expressions.)

Step 10: Define inducing function $t^* \in \mathcal{X}^{<m>}$ such that $t^*(j) = t_D - \mathrm{orness}(\mathrm{WA}_{\hat{w}_j})$; define $y \in \mathcal{X}^{<m>}$ such that $y = (y(j))_{j=1}^m = (\mathrm{WA}_{\hat{w}_j}(x))_{j=1}^m$. Then, use IOWA operator with v under t^*, $\mathrm{WA}_{\hat{v}} \in \mathcal{A}^{<m>}$, to aggregate y to obtain a final predicted value for the market share of that product, where $\hat{v} = PT_{t^*}(v)$ with $PT_{t^*} : \mathcal{W}^{<m>} \to \mathcal{W}^{<m>}$ being the positioning transformation on $\mathcal{W}^{<m>}$ according to t^*.

We make some analyzes and remarks about the above preferences involved aggregation procedures.

There are also some simplistic preferences merging methods in the above management decision problem, without considering the difference between the management roles of that decision maker and her consultants. For example, the decision maker and her m consultants can be at parallel positions, with all their preferences over n investigators being melted directly to generate a single inducing function by $(t_D + \sum_{j=1}^m t_j)/(m+1)$. Alternatively, we can firstly perform $m+1$ times of aggregation and take the average by $(\mathrm{WA}_{\hat{w}}(x) + \sum_{j=1}^m \mathrm{WA}_{\hat{w}_j}(x))/(m+1)$, where $\hat{w}_j = PT_{t_j}(w_j)$ as explained previously, and $\hat{w} = PT_{t_D}(w)$ being assumed to be the preference of that decision maker over n investigators. The above proposed ICD model, however, differentiates the roles of that decision maker and her consultants properly; the preference of that decision maker $w \in \mathcal{W}^{<n>}$ has not been directly put over n investigators, but indirectly over her m consultants with the preference $v \in \mathcal{W}^{<m>}$ instead. Clearly, in management decision making, it is preferable and helpful to devise more suitable preferences involved aggregation methods, including the one we proposed above. This will not only provide more choices in decision making and evaluation methods, but further diversify the aggregation techniques and practices.

Moreover, with the aggregation techniques becoming more complex, it may be difficult to realize well all the intentions of complex preferences involved aggregations. Nevertheless, the proposed formulations in this study can be flexibly applied and thus may provide help to fulfill such aim.

Example 6: As a simple illustrative example of ICD two-stage IOWA model for practitioners to better refer to, take $n = 4$, $m = 3$. Next, we go through the above mentioned two stages with ten steps.

Stage 1 Consultants' aggregation preferences and aggregation results

Step 1: Designate four investigators that are represented by the set $S(4) = \{1, 2, 3, 4\}$.

Step 2: Collect the investigated market share values from all of those four investigators, $x = (x(i))_{i=1}^4 = (0.6, 0.8, 1, 0.5)$.

Step 3: Invite three consultants that are indicated by the set $S(3) = \{1, 2, 3\}$.

Step 4: Each consultant provides his own belief to the "ability values" of all those four investigators with $t_1 = (t_1(i))_{i=1}^4 = (0.8, 1, 0.6, 0.8)$, $t_2 = (t_2(i))_{i=1}^4 = (0.7, 1, 1, 1)$, and $t_3 = (t_3(i))_{i=1}^4 = (1, 0.8, 0.9, 0.6)$, respectively.

Step 5: Each consultant exhibits his preference per se over the ability values of investigators from higher to lower; in detail, we assume $w_1 = (w_1(i))_{i=1}^4 = (0.4, 0.3, 0.2, 0.1)$ whose orness $orness(w_1) = 2/3$, $w_2 = (w_2(i))_{i=1}^4 = (0.7, 0.3, 0, 0)$ whose orness $orness(w_2) = 0.9$, and $w_3 = (w_3(i))_{i=1}^4 = (0.55, 0.3, 0.15, 0)$ whose orness $orness(w_3) = 0.8$.

Step 6: Using Definition 8, calculate

$$\begin{aligned}\hat{w}_1 = PT_{t_1}(w_1) &= \frac{1}{2}((0.3, 0.4, 0.1, 0.2) \\ &\quad + (0.2, 0.4, 0.1, 0.3)) \\ &= (0.25, 0.4, 0.1, 0.25), \\ \hat{w}_2 &= PT_{t_2}(w_2) \\ &= \frac{1}{6}((0, 0.7, 0.3, 0) + (0, 0.7, 0, 0.3) \\ &\quad + (0, 0.3, 0.7, 0) + (0, 0, 0.7, 0.3) \\ &\quad + (0, 0.3, 0, 0.7) + (0, 0, 0.3, 0.7)) \\ &= (0, 1/3, 1/3, 1/3),\end{aligned}$$

and

$$\hat{w}_3 = PT_{t_3}(w_3) = (0.55, 0.15, 0.3, 0).$$

Then, by Definition 9, aggregate using $\mathrm{WA}_{\hat{w}_j}$ ($j \in \{1, 2, 3\}$) to obtain

$$\begin{aligned}\mathrm{WA}_{\hat{w}_1}(x) &= (0.25)(0.6) + (0.4)(0.8) + (0.1)(1) \\ &\quad + (0.25)(0.5) = 0.695, \\ \mathrm{WA}_{\hat{w}_2}(x) &= (0)(0.6) + (1/3)(0.8) + (1/3)(1) \\ &\quad + (1/3)(0.5) = 0.7667,\end{aligned}$$

and

$$\mathrm{WA}_{\hat{w}_3}(x) = (0.55)(0.6) + (0.15)(0.8) + (0.3)(1) + (0)(0.5) = 0.75,$$

as the predicted market share by those 3 consultants, respectively.

Stage 2 Decision maker's aggregation preference and aggregation result

Step 7: Determine the preference of the decision maker that embodies her trust ranking over those four investigators and represent it by an inducing function $t_D = ((t_D(i))_{i=1}^4 = (0.5, 0.6, 0.8, 0.6)$.

Step 8: By Definition 11, calculate $t_D -$ orness of WA operator with $\hat{w}_j$, $t_D - \mathrm{orness}(\mathrm{WA}_{\hat{w}_j})$ for $j \in \{1, 2, 3\}$, respectively. In detail

$$\begin{aligned} &t_D - \mathrm{orness}(\mathrm{WA}_{\hat{w}_1}) \\ &= (1/2)(((3/3)\hat{w}_1(3) + (2/3)\hat{w}_1(2) + (1/3)\hat{w}_1(4) \\ &\quad + (0)\hat{w}_1(1)) + ((3/3)\hat{w}_1(3) + (2/3)\hat{w}_1(4) \\ &\quad + (1/3)\hat{w}_1(2) + (0)\hat{w}_1(1))) \\ &= (1/2)(((3/3)(0.1) + (2/3)(0.4) + (1/3)(0.25) \\ &\quad + (0)(0.25)) + ((3/3)(0.1) + (2/3)(0.25) \\ &\quad + (1/3)(0.4) + (0)(0.25))) \\ &= (1/2)(0.45 + 0.4) = 0.425; \end{aligned}$$

similarly, $t_D - \mathrm{orness}(\mathrm{WA}_{\hat{w}_2}) = 2/3$, and $t_D - \mathrm{orness}(\mathrm{WA}_{\hat{w}_3}) = 0.375$.

Step 9: Assume $v = (v(j))_{j=1}^3 = (0.6, 0.3, 0.1)$ whose orness is $\mathrm{orness}(v) = 2/3$.

Step 10: Define inducing function $t^* = (t^*(j))_{j=1}^3 = (0.425, 2/3, 0.375)$; define $y = (y(j))_{j=1}^3 = (0.695, 0.7667, 0.75)$. Then, use IOWA operator with v under t^*, $\mathrm{WA}_{\hat{v}} \in \mathcal{A}^{<3>}$, to aggregate y to obtain a final predicted value for the market share of that product, where $\hat{v} = PT_{t^*}(v) = (0.3, 0.6, 0.1)$. In detail

$$\begin{aligned} \mathrm{WA}_{\hat{v}}(y) &= (0.3)(0.695) \\ &\quad + (0.6)(0.7667) + (0.1)(0.75) \\ &= 0.74352. \end{aligned}$$

□

IV. Conclusion

Induced OWA has already been widely used for over two decades, but when tied values appear in inducing information, practitioners often might be in a dilemma, and as yet, there is no systematical or stricter way to address this problem. By introducing a new frame to perfectly restructure IOWA, this study firstly went through a series of definitions and finally proposed IOWA operators in the expression of WA operators. The crucial underpinning to support this expression is the positioning transformation based on permutations and inducing functions. With this expression, IOWA can also be regarded as a special type of WA operators.

We defined the permutation-based orness and the inducing function based orness, from which more applications involving induced preference models can be readily and strictly built and formulated. As one application, we proposed an investigators-consultants-decision maker (ICD) two-stage IOWA decision model with a simple numerical realization.

The significance and further potential in more applications is direct because the proposals in this study actually provided a practical, strict, and universal language for preference involved applications, rather than some expedient methods which may easily become improper when facing with tied inducing information, or when inducing variables becoming very complex and tangled as in the decision model proposed in this study. In addition, it is noteworthy that our approach can also be easily modified to accommodate some other developed types of generalizations of OWA operators.

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