Aggregation of OWA Operators

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***Abstract*—Inspired by the real needs of group decision problems, aggregation of ordered weighted averaging (OWA) operators is studied and discussed. Our results can be applied for data acting on any real interval, such as the standard scales $[0,1]$ and $[0,\infty[$, bipolar scales $[-1,1]$ and $\mathbb{R} =]-\infty,\infty[$, etc. A direct aggregation is shown to be rather restrictive, allowing the convex combinations to be considered only, except the case of dimension $n = 2$. More general is the approach based on the aggregation of related cumulative weighting vectors. The piecewise linearity of OWA operators allows us to consider bilinear forms of aggregation of related weighting vectors. Several interesting examples yielding the link between the aggregation of OWA operators and the related ANDness and ORness measures are also included. Some possible applications and generalizations of our results are also discussed.**

***Index Terms*—Aggregation function, ANDness, multicriteria decision support, ORness, ordered weighted averaging (OWA) operator.**

I. Introduction and Some Preliminaries

Ordered weighted averaging (OWA) operators were introduced by Yager [16] and soon they became an object of research in many papers and were applied in many domains. For an overview of results and papers devoted to OWA operators, we recommend a recent overview paper [5]. Recall that, for a fixed $n \in \{2,3,\ldots\} = \mathbb{N} \setminus \{1\}$, and for a fixed normed weighting vector $\mathbf{w} = (w_1,\ldots,w_n) \in [0,1]^n$, $\sum_{i=1}^n w_i = 1$, the related OWA operator $\mathrm{OWA}_{\mathbf{w}} : [0,1]^n \to [0,1]$ is given by

$$\mathrm{OWA}_{\mathbf{w}}(x_1,\ldots,x_n) = \sum_{i=1}^{n} w_i x_{(i)} \tag{1}$$

where $(\bullet) : \{1,\ldots,n\} \to \{1,\ldots,n\}$ is a permutation satisfying $x_{(1)} \geq,\ldots,\geq x_{(n)}$. Though this permutation need not be unique (in the case of some ties), the formula (1) is well defined and it gives the same result for any acceptable permutation.

Observe that OWA operators can be defined to act on any real interval I, including the distinguished cases $I = [0,\infty[$ or bipolar scales $I = [-1,1]$ or $I = \mathbb{R} =]-\infty,\infty[$, simply applying formula (1) on the considered scale I.

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For any fixed nonempty set L equipped with a (partial) ordering $\leq_L$ and possessing top element 1_L and bottom element 0_L, and for any fixed integer $n \in \mathbb{N}$, the mapping $A : L^n \to L$ is called an aggregation function on L whenever it is increasing in each coordinate, i.e., for any elements $x_1,\ldots, x_n, y_1,\ldots,y_n \in L$ such that $x_1 \leq_L y_1,\ldots,x_n \leq_L y_n$ it holds $A(x_1,\ldots,x_n) \leq_L A(y_1,\ldots,y_n)$, and A satisfies two boundary conditions $A(0_L,\ldots,0_L) = 0_L$ and $A(1_L,\ldots,1_L) = 1_L$. Typical aggregation functions, including OWA operators, are defined on the scale $L = [0,1]$ equipped with the standard linear ordering of reals. As we will show later, also the set of all n-ary OWA operators can be seen as a scale equipped with a partial ordering (which is linear if $n = 2$), and thus aggregation of OWA operators is meaningful.

Observe that the OWA operators can also be characterized axiomatically as symmetric comonotone additive aggregation functions on $[0,1]$, see [6] and [14], i.e., monotone functions $A : [0,1]^n \to [0,1]$ satisfying the boundary conditions $A(0,\ldots,0) = 0$, $A(1,\ldots,1) = 1$, and the following:

1) symmetry, $A(x_1\ldots,x_n) = A(x_{\sigma(1)},\ldots,x_{\sigma(n)})$ for any permutation
$\sigma : \{1,\ldots,n\} \to \{1,\ldots,n\}$ and any $\mathbf{x} \in [0,1]^n$; and
2) comonotone additivity, $A(\mathbf{x}+\mathbf{y}) = A(\mathbf{x}) + A(\mathbf{y})$ for any $\mathbf{x},\mathbf{y},\mathbf{x}+\mathbf{y} \in [0,1]^n$ such that $(x_i - x_j)(y_i - y_j) \geq 0$ for any $i,j \in \{1,\ldots,n\}$.

Recall that OWA operators are also averaging, or, equivalently, idempotent aggregation functions [7], i.e.,

1) $A(x,\ldots,x) = x$ for any $x \in [0,1]$.

Observe that, for any real interval I, a function $A : I^n \to I$ which is symmetric, comonotone additive, and idempotent is necessarily an OWA operator given by formula (1), $A = \mathrm{OWA}_{\mathbf{w}}$. For any fixed reals $a,b \in I$, $a < b$, the related weighting vector $\mathbf{w}$ generating such a function A via (1) is then given by

$$\begin{aligned}
w_1 &= \frac{1}{b-a}(A(b,a,\ldots,a) - A(a,\ldots,a)) \\
&= \frac{1}{b-a}(A(b,a,\ldots,a) - a) \\
w_2 &= \frac{1}{b-a}(A(b,b,a,\ldots,a) - A(b,a,\ldots,a)) \\
&\vdots \\
w_n &= \frac{1}{b-a}(A(b,\ldots,b) - A(b,\ldots,b,a)) = \\
&= \frac{1}{b-a}(b - A(b,\ldots,b,a)).
\end{aligned}$$

In the remaining parts of this paper, we will deal with OWA operators acting on $I = [0,1]$ only, though the majority of results can be directly reformulated for any convex scale $I \subseteq [-\infty, \infty]$.

For a normed weighting vector $\mathbf{w}$, let $\mathbf{W}$ be the related cumulative weighting vector, $\mathbf{W} = (W_1, \ldots, W_n), W_i = w_1 + \cdots + w_i, i = 1, \ldots, n$. Obviously, $\mathrm{OWA}_{\mathbf{w}}(\mathbf{E}_i) = W_i$, where $\mathbf{E}_i = \mathbf{e}_1 + \cdots + \mathbf{e}_i = (1, \ldots, 1, 0, \ldots, 0)$, $\mathbf{e}_1 = (1, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0)$, etc.

Thus, any symmetric comonotone additive aggregation function $A : [0,1]^n \to [0,1]$ is just an OWA operator, $A = \mathrm{OWA}_{\mathbf{w}}$, where $\mathbf{w} = (w_1, \ldots, w_n), w_1 = A(\mathbf{E}_1)$, and for $i = 2, \ldots, n, w_i = A(\mathbf{E}_i) - A(\mathbf{E}_{i-1})$.

We denote with Agg_n the class of all n-ary aggregation functions on $[0,1]$, Owa_n the class of all n-ary OWA operators on $[0,1]$, and $\mathscr{S}_n$ the class of all n-ary normed weighting vectors. Clearly, there is an isomorphism $\varphi : \mathscr{S}_n \to \mathrm{Owa}_n, \varphi(\mathbf{w}) = \mathrm{OWA}_{\mathbf{w}}$. The standard partial ordering of aggregation functions from Agg_n induces the partial ordering on Owa_n, and thus also a partial ordering $\preceq$ on $\mathscr{S}_n$. It is not difficult to check that, for $\mathbf{u}, \mathbf{v} \in \mathscr{S}_n$, $\mathrm{OWA}_{\mathbf{u}}(\mathbf{x}) \le \mathrm{OWA}_{\mathbf{v}}(\mathbf{x})$ for each $\mathbf{x} \in [0,1]^n$ if and only if $u_1 = \mathbf{U}_1 \le v_1 = \mathbf{V}_1$, $u_1 + u_2 = \mathbf{U}_2 \le v_1 + v_2 = \mathbf{V}_2, \ldots, \mathbf{U}_i \le \mathbf{V}_i, \ldots, \mathbf{U}_n = \mathbf{V}_n = 1$, thus $\mathbf{u} \preceq \mathbf{v}$ if and only if $\mathbf{U} \le \mathbf{V}$, where $\mathbf{U}$ and $\mathbf{V}$ are the cumulative weighting vectors related to $\mathbf{u}$ and $\mathbf{v}$, respectively. Hence, $\mathbf{e}_n = (0, \ldots, 0, 1)$ is the bottom element of $\mathscr{S}_n$, and its related OWA operator $\mathrm{OWA}_{\mathbf{e}_n} = \mathrm{Min}$ is the weakest OWA operator. Similarly, $\mathbf{e}_1 = (1, 0, \ldots, 0)$ is the top element of $\mathscr{S}_n$, and it is related to the strongest OWA operator $\mathrm{OWA}_{\mathbf{e}_1} = \mathrm{Max}$. Note that, considering the above mentioned orderings, Agg_n, Owa_n, and $\mathscr{S}_n$ form a bounded lattice, respectively. Moreover, Owa_2 and $\mathscr{S}_2$ are bounded chains.

An important parametric characterization, ORness, was proposed by Yager [16] as follows:

$$\mathrm{ORness}(\mathrm{OWA}_{\mathbf{w}}) = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i = \frac{1}{n-1}\sum_{i=1}^{n-1} W_i. \tag{2}$$

Obviously, ORness is a homomorphism from Owa_n onto $[0,1]$, $\mathrm{ORness}(\mathrm{OWA}_{\mathbf{e}_1}) = 1$, and

$\mathrm{ORness}(\mathrm{OWA}_{\mathbf{e}_n}) = 0$. For any OWA operator A, its ORness can be introduced also directly as a normed expected value of A [4]

$$\begin{aligned}&\mathrm{ORness}(A) = \\ &\left(\int_{[0,1]^n} A(x_1, \ldots, x_n)dx_1 \ldots dx_n \right. \\ &\left. - \int_{[0,1]^n} \mathrm{Min}(x_1, \ldots, x_n)dx_1 \ldots dx_n\right) / \\ &\left(\int_{[0,1]^n} \mathrm{Max}(x_1, \ldots, x_n)dx_1 \ldots dx_n \right. \\ &\left. - \int_{[0,1]^n} \mathrm{Min}(x_1, \ldots, x_n)dx_1 \ldots dx_n\right).\end{aligned} \tag{3}$$

Similarly, dual parameter ANDness : $\mathrm{Owa}_n \to [0,1]$, ANDness $(\mathrm{OWA}_{\mathbf{w}}) = 1 - \mathrm{ORness}(\mathrm{OWA}_{\mathbf{w}})$, is given by

$$\mathrm{ANDness}(\mathrm{OWA}_{\mathbf{w}}) = \frac{1}{n-1}\sum_{i=1}^{n}(i-1)w_i \tag{4}$$

and it can be either seen as a normed expected value satisfying $\mathrm{ANDness}(\mathrm{OWA}_{\mathbf{e}_1}) = 0$ and $\mathrm{ANDness}(\mathrm{OWA}_{\mathbf{e}_n}) = 1$, or it can be derived by the duality. Recall that, for $A \in \mathrm{Agg}_n$, its dual $A^d : [0,1]^n \to [0,1]$ is given by $A^d(x_1, \ldots, x_n) = 1 - A(1 - x_1, \ldots, 1 - x_n)$. Then, it is not difficult to check that $(\mathrm{OWA}_{\mathbf{w}})^d = \mathrm{OWA}_{\overline{\mathbf{w}}}$, where $\overline{\mathbf{w}}$ is the reversed normed weighting vector given by $\overline{\mathbf{w}} = (w_n, \ldots, w_1)$. The duality of ORness and ANDness means that $\mathrm{ORness}(\mathrm{OWA}_{\mathbf{w}}) = \mathrm{ANDness}(\mathrm{OWA}_{\overline{\mathbf{w}}})$, i.e., $\mathrm{ANDness}(\mathrm{OWA}_{\mathbf{w}}) = \mathrm{ORness}$ $(\mathrm{OWA}_{\overline{\mathbf{w}}})$. Note that if a general interval $I \subseteq [-\infty, \infty]$ is considered as the background scale, dual aggregation functions are not considered, in general. However, in the case of OWA operators, one can also in these cases deal with the above-introduced duality $(\mathrm{OWA}_{\mathbf{w}})^d = \mathrm{OWA}_{\overline{\mathbf{w}}}$.

OWA operators are applied in many domains, such as geographic information systems (GIS) [11] or in social sciences [13]. In multicriteria decision methods, OWA's model the utility functions, which are unanimous (i.e., idempotent), anonymous (i.e., symmetric), and comonotone additive. When facing with the problem of a group consensus utility function, when each member of the jury has its own preferences expressed by an OWA operator, the need of aggregating of some given OWA operators $\mathrm{OWA}_{\mathbf{w}^{(1)}}, \ldots, \mathrm{OWA}_{\mathbf{w}^{(k)}}$ into a single OWA operator $\mathrm{OWA}_{\mathbf{w}}$ is a challenging task. This fact was an inspiration leading to the preparation of this paper. It's aim is to take a deeper look on aggregation of OWA operators, i.e., we look for mappings aggregating several n-ary OWA operators into a single one. We will focus mostly on aggregation of two OWA operators.

Based on our approach to the aggregation of two OWA's, aggregation of more OWA's can be easily implemented. Namely, either a direct replacement of binary aggregations from Agg_2 by those from Agg_n can be applied (if applicable), or we can use a sequential aggregation generalizing the extension of binary associative functions to n-ary functions (for more details see [2] or [1]).

In the next section, we will discuss aggregation of OWA's based on some $A \in \mathrm{Agg}_2$. In Section III, A-based aggregation of cumulative weighting vectors is discussed, thus including aggregation of OWA's. In Section IV, we discuss bilinear forms of aggregation of normed weighting vectors including the study of OWA's aggregation with particular properties. Finally, some concluding remarks are added.

II. Direct Aggregation of OWA Operators

For any $A \in \mathrm{Agg}_2$ and $B, C \in \mathrm{Agg}_n$, it is evident that the composite function $A(B,C) : [0,1]^n \to [0,1]$ given by $A(B, C)(\mathbf{x}) = A(B(\mathbf{x}), C(\mathbf{x}))$ is an n-ary aggregation function, see, e.g. [7]. However, it is not difficult to check that if $B, C \in \mathrm{Owa}_n$, then $A(B,C)$ need not be an OWA operator. So, for example, considering $A = \Pi$ the product, and $B = \mathrm{OWA}_{\mathbf{e}_1} = \mathrm{Max}$, $C = \mathrm{OWA}_{\mathbf{e}_n} = \mathrm{Min}$, then $A(B, C) = \mathrm{Max} \cdot \mathrm{Min}$ is not an OWA operator. It is not difficult to see that if A is a weighted mean, $A(x, y) = \lambda x + (1-\lambda)y$ for some $\lambda \in [0, 1]$, then $A(\mathrm{OWA}_{\mathbf{u}}, \mathrm{OWA}_{\mathbf{v}}) = \mathrm{OWA}_{\lambda\mathbf{u}+(1-\lambda)\mathbf{v}}$. We will show that convex combinations of OWA operators are the only direct aggregations on OWA's if $n \geq 3$, but this is not true if $n = 2$. Note that for $n = 2$, Owa_2 and $\mathscr{S}_2$ are chains.

Theorem 1: Let $A : [0, 1]^2 \to [0, 1]$ be an aggregation function. Define $*_A : (\mathrm{Agg}_2)^2 \to \mathrm{Agg}_2$ by

$$(B *_A C)(x, y) = A(B(x, y), C(x, y)). \tag{5}$$

Then, $B *_A C \in \mathrm{Owa}_2$ for any $B, C \in \mathrm{Owa}_2$ if and only if A is the Choquet integral, i.e., there are $a, b \in [0, 1]$ such that

$$A(x, y) = \begin{cases} ax + (1-a)y & \text{if } x \geq y \\ (1-b)x + by & \text{if } x < y. \end{cases} \tag{6}$$

Proof: Suppose that $B *_A C \in \mathrm{Owa}_2$ for any $B, C \in \mathrm{Owa}_2$. Then, for any $(t, s) \in [0, 1]^2$ and $c \in [0, 1]$, due to the homogeneity of OWA operators

$$\begin{aligned} c \cdot (B *_A C)(t, s) &= (B *_A C)(c \cdot (t, s)) \\ &= A(B(c \cdot (t, s)), C(c \cdot (t, s))) \\ &= A(c \cdot B(t, s), c \cdot C(t, s)). \end{aligned} \tag{7}$$

Put $B = \mathrm{OWA}_{\mathbf{e}_1} = \mathrm{Max}$, $C = \mathrm{OWA}_{\mathbf{e}_2} = \mathrm{Min}$, and consider $t \geq s$. Then, $(B(t, s), C(t, s)) = (t, s)$. Similarly, if $t < s$, one can put $B = \mathrm{OWA}_{\mathbf{e}_2}$ and $C = \mathrm{OWA}_{\mathbf{e}_1}$ to obtain $(B(t, s), C(t, s)) = (t, s)$. Thus, due to (7), A is necessarily homogenous. Similarly, due to the shift invariantness of OWA operators, also the shift invariantness of A can be shown. Denote $A(1, 0) = a$ and $A(0, 1) = b$. Then, for any $x \geq y$, $A(x, y) = A(y + x - y, y + 0) = y + A(x - y, 0) = y + (x - y)A(1, 0) = ax + (1-a)y$. Similarly, if $x < y$, we obtain $A(x, y) = (1-b)x + by$. Summarizing the above facts, we see that A is necessarily the Choquet integral given by (6).

To see the sufficiency, suppose A is given by (6). Take $B, C \in \mathrm{Owa}_2$. Then, either $B \geq C$ or $B < C$ (recall that Owa_2 is a chain). Suppose $B \geq C$. Thus, $B *_A C = aB + (1-a)C$ is an OWA operator, and if $B = \mathrm{OWA}_{\mathbf{u}}$ and $C = \mathrm{OWA}_{\mathbf{v}}$, then $B *_A C = \mathrm{OWA}_{\mathbf{w}}$, where $\mathbf{w} = a\mathbf{u} + (1-a)\mathbf{v} \in \mathscr{S}_2$. Similarly, if $B < C$, then $B *_A C = \mathrm{OWA}_{\mathbf{w}}$, where $\mathbf{w} = (1-b)\mathbf{u} + b\mathbf{v} \in \mathscr{S}_2$. ■

Theorem 1 can be directly generalized for aggregating k OWA operators from Owa_2 and then $A \in \mathrm{Agg}_k$ should be the k-ary Choquet integral.

Moreover, for any Choquet integral-based aggregation function $A \in \mathrm{Agg}_2$ ($A \in \mathrm{Agg}_k$) and any OWA operators $B, C \in \mathrm{Owa}_2$ ($B_1, \ldots, B_k \in \mathrm{Owa}_2$) it holds $\mathrm{ORness}(B *_A C) = A(\mathrm{ORness}(B), \mathrm{ORness}(C))$ and $\mathrm{ANDness}(B *_A C) = A^d(\mathrm{ANDness}(B), \mathrm{ANDness}(C))$, $\mathrm{ORness}({*_A}_{i=1}^k B_i) = A(\mathrm{ORness}(B_1), \ldots, \mathrm{ORness}(B_k))$ and $\mathrm{ANDness}({*_A}_{i=1}^k B_i) = A^d(\mathrm{ANDness}(B_1), \ldots, \mathrm{ANDness}(B_k))$.

Example II.1 indicates that Theorem 1 is not valid for OWA operators from Owa_n whenever $n \geq 3$.

Example II.1: Observe that $\mathrm{Min} \in \mathrm{Agg}_2$ can be seen as the Choquet integral related to $a = b = 0$. Consider ternary OWA operators $\mathrm{OWA}_{(0,1,0)}, \mathrm{OWA}_{(\frac{1}{2},0,\frac{1}{2})} \in \mathrm{Owa}_3$. Then, $\mathrm{OWA}_{(0,1,0)} = Med$ is the ternary median and $\mathrm{OWA}_{(\frac{1}{2},0,\frac{1}{2})} = \frac{\mathrm{Max}+\mathrm{Min}}{2}$, also called midrange, and hence $B(x, y, z) = \mathrm{Min}(\mathrm{OWA}_{(0,1,0)}, \mathrm{OWA}_{(\frac{1}{2},0,\frac{1}{2})})(x, y, z) =$

$$\begin{cases} \mathrm{Med}(x, y, z) & \text{if } 2\mathrm{Med}(x, y, z) \leq \\ & \leq \mathrm{Max}(x, y, z) + \mathrm{Min}(x, y, z) \\ \frac{\mathrm{Max}(x,y,z)+\mathrm{Min}(x,y,z)}{2} & \text{otherwise.} \end{cases}$$

Then, $B(1, 0, 0) = 0, B(1, 1, 0) = \frac{1}{2}, B(1, 1, 1) = 1$, but $B(1, 0.4, 0) = 0.4$, showing that $B \notin \mathrm{Owa}_3$. Indeed, if $B \in \mathrm{Owa}_3$, due to the comonotone additivity of OWA operators it would hold $B(1, 0.4, 0) = B(0.4, 0.4, 0) + B(0.6, 0, 0) = 0.2$, a contradiction.

Theorem 2: Let $A : [0, 1]^2 \to [0, 1]$ be an aggregation function. For $n \geq 3$ define $*_A : (\mathrm{Agg}_n)^2 \to \mathrm{Agg}_n$ by

$$B *_A C(\mathbf{x}) = A(B(\mathbf{x}), C(\mathbf{x})), \mathbf{x} \in [0, 1]^n \tag{8}$$

(we will use the notation $*_A$ for any arity $n \geq 2$). Then, $B *_A C \in \mathrm{Owa}_n$ for any $B, C \in \mathrm{Owa}_n$ if and only if A is a weighted arithmetic mean, i.e., there is $\lambda \in [0, 1]$ such that $A(x, y) = \lambda x + (1-\lambda)y$.

Proof: The sufficiency is obvious. To see the necessity, using similar arguments as in the proof of Theorem 1 we can prove the homogeneity and shift invariantness of A, i.e., A is necessarily the Choquet integral linked to constants $a, b \in [0, 1,]$, compare (6).

Consider $\mathbf{u} = (0, 1, 0, \ldots, 0), \mathbf{v} = (\frac{1}{2}, 0, \frac{1}{2}, 0, \ldots, 0) \in \mathscr{S}_n$, and $\mathrm{OWA}_{\mathbf{u}} *_A \mathrm{OWA}_{\mathbf{v}} = \mathrm{OWA}_{\mathbf{w}}$. Then, necessarily $w_1 = A(u_1, v_1) = A(0, \frac{1}{2}) = \frac{b}{2}$ $w_2 = A(u_1 + u_2, v_1 + v_2) - A(u_1, v_1) = A(1, \frac{1}{2}) - A(0, \frac{1}{2}) = a + \frac{1-a}{2} - \frac{b}{2} = \frac{1+a-b}{2}$, $w_3 = A(u_1 + u_2 + u_3, v_1 + v_2 + v_3) - A(u_1 + u_2, v_1 + v_2) = = A(1, 1) - A(1, \frac{1}{2}) = 1 - (a + \frac{1-a}{2}) = \frac{1-a}{2}$.

As $w_1 + w_2 + w_3 = 1$, clearly $w_4 =, \ldots, = w_n = 0$. For $\mathbf{x} = (0.6, 0.5, 0, \ldots, 0) \in [0, 1]^n$ it holds: $\mathrm{OWA}_{\mathbf{u}}(\mathbf{x}) = 0.5$, $\mathrm{OWA}_{\mathbf{v}}(\mathbf{x}) = 0.3$, $\mathrm{OWA}_{\mathbf{w}}(\mathbf{x}) = 0.3b + \frac{1+a-b}{4} = \frac{1}{4} + \frac{a}{4} + \frac{b}{20}$, and $(\mathrm{OWA}_{\mathbf{u}} *_A \mathrm{OWA}_{\mathbf{v}})(\mathbf{x}) = A(0.5, 0.3) = \frac{a}{2} + \frac{3}{10}(1-a) = \frac{3}{10} + \frac{a}{5}$. Therefore, $\frac{3}{10} + \frac{a}{5} = \frac{1}{4} + \frac{a}{4} + \frac{b}{20}$, i.e., $a + b = 1$. However, putting $a = \lambda$, we see that $A(x, y) = \lambda x + (1-\lambda)y$, i.e., A is a weighted arithmetic mean. ■

Similarly, one can show that for $A \in \mathrm{Agg}_k$ the A-aggregation of OWA operators $B_1, \ldots, B_k \in \mathrm{Owa}_n$ is an OWA operator from Owa_n, $n \geq 3$, if and only if $A = W_{\mathbf{w}}$ is a weighted arithmetic mean, where $\mathbf{w} \in \mathscr{S}_k$, i.e., $A(x_1, \ldots, x_k) = \sum_{i=1}^k w_i x_i$. Clearly, then $*_{i=1}^k \mathrm{OWA}_{\mathbf{u}_i} = \mathrm{OWA}_{\sum_{i=1}^k w_i \mathbf{u}_i}$. Moreover, $\mathrm{ORness}({*_A}_{i=1}^k)\mathrm{OWA}_{\mathbf{u_i}} = \sum_{i=1}^k w_i \mathrm{ORness}(\mathrm{OWA}_{\mathbf{u}_i})$ and $\mathrm{ANDness}({*_A}_{i=1}^k)\mathrm{OWA}_{\mathbf{u_i}} = \sum_{i=1}^k w_i \mathrm{ANDness}(\mathrm{OWA}_{\mathbf{u}_i})$.

III. Aggregation of OWA Operators Based on Aggregation of Cumulative Weighting Vectors

As already mentioned, bounded lattices Owa_n and $\mathscr{S}_n$ are isomorphic. Another lattice isomorphic to them is formed by all cumulative weighting vectors $\mathbf{W} = (W_1, \ldots, W_n) \in [0,1]^n$, $W_1 \leq, \ldots, \leq W_n = 1$, equipped with the standard partial ordering of vectors. We denote this lattice by $\mathcal{W}_n$. Any aggregation on $\mathcal{W}_n$ can be mapped into an aggregation on Owa_n. Though there are several kinds of aggregation functions on $\mathcal{W}_n$, see [3] for example, we will focus only on representable aggregation functions.

Definition 1: Let $A : [0,1]^k \to [0,1]$, $A \in \mathrm{Agg}_k$, $k \geq 2$ be an aggregation function. Define a mapping $\mathbb{A} : {\mathcal{W}_n}^k \to \mathcal{W}_n$ by $\mathbb{A}(\mathbf{W}^{(1)}, \ldots, \mathbf{W}^{(k)}) =$

$$(A(W_1^{(1)}, \ldots, W_1^{(k)}), \ldots, A(W_n^{(1)}, \ldots, W_n^{(k)})).$$

Then, $\mathbb{A}$ is called a (k-dimensional) representable aggregation function on $\mathcal{W}_n$.

Note that $A(W_n^{(1)}, \ldots, W_n^{(k)}) = A(1, \ldots, 1) = 1$, and due to the monotonicity of A, also $(W_i^{(1)}, \ldots, W_i^{(k)}) \leq (W_j^{(1)}, \ldots, W_j^{(k)})$ whenever $1 \leq i \leq j \leq n$, i.e., $\mathbb{A}$ introduced in Definition 1 is well defined.

Each aggregation function $\mathbb{A} : {\mathcal{W}_n}^k \to \mathcal{W}_n$ induces an aggregation function $\circledast_{\mathbb{A}} : {\mathrm{Owa}_n}^k \to \mathrm{Owa}_n$ given by ${\circledast_{\mathbb{A}}}_{i=1}^k \mathrm{OWA}_{\mathbf{w}^{(i)}} = \mathrm{OWA}_{\mathbf{w}}$, where $\mathbf{w}, \mathbf{w}^{(1)}, \ldots, \mathbf{w}^{(k)} \in \mathscr{S}_n$ are linked to $\mathbf{W}, \mathbf{W}^{(1)}, \ldots, \mathbf{W}^{(k)} \in \mathcal{W}_n$, and $\mathbf{W} = \mathbb{A}(\mathbf{W}^{(1)}, \ldots, \mathbf{W}^{(k)})$.

It is not difficult to check that the algebraic properties of $\mathbb{A}$ are inherited by $\circledast_{\mathbb{A}}$. We exemplify the case of binary representable aggregation functions only.

Theorem 3: Let $A : [0,1]^2 \to [0,1]$ be an aggregation function on $[0,1]$ and $\circledast_{\mathbb{A}} : Owa_n^2 \to Owa_n$ its related aggregation on Owa_n. Then, the following conditions are satisfied:

1) A is symmetric if and only if $\circledast_{\mathbb{A}}$ is symmetric.
2) A is associative if and only if $\circledast_{\mathbb{A}}$ is associative.
3) A is bisymmetric, i.e., $A(A(x,y), A(u,v)) = A(A(x,u), A(y,v))$ for any $x, y, u, v \in [0,1]$, if and only if $\circledast_{\mathbb{A}}$ is bisymmetric.
4) A has a neutral element $e \in [0,1]$ if and only if $\circledast_{\mathbb{A}}$ has neutral element $\mathrm{OWA}_{(e,0,\ldots,0,1-e)}$.
5) A has an annihilator $a \in [0,1]$ if and only if $\circledast_{\mathbb{A}}$ has an annihilator $\mathrm{OWA}_{(a,0,\ldots,0,1-a)}$.

Proof: We will prove the item 4) only, the proof of the other items being similar. ■

Suppose that $e \in [0,1]$ is a neutral element of A. For the vector $\mathbf{E} = (e, \ldots, e, 1) \in \mathcal{W}_n$ and any $\mathbf{V} \in \mathcal{W}_n$, it holds $\mathbb{A}(\mathbf{E}, \mathbf{V}) = (A(e, V_1), \ldots, A(e, V_{n-1}), A(1,1)) = (V_1, \ldots, V_{n-1}, 1) = \mathbf{V}$ (recall that $V_n = 1$). Similarly, $\mathbb{A}(\mathbf{V}, \mathbf{E}) = \mathbf{V}$. Then, $\mathrm{OWA}_{\mathbf{v}} \circledast_{\mathbb{A}} \mathrm{OWA}_{\mathbf{e}} = \mathrm{OWA}_{\mathbf{e}} \circledast_{\mathbb{A}} \mathrm{OWA}_{\mathbf{v}} = \mathrm{OWA}_{\mathbf{v}}$, where $\mathbf{e} = (e, 0, \ldots, 0, 1-e) \in \mathscr{S}_n$ is linked to $E \in \mathcal{W}_n$, and $\mathbf{v}$ is linked to $\mathbf{V}$. Hence, $\mathrm{OWA}_{\mathbf{e}}$ is a neutral element of $\circledast_{\mathbb{A}}$.

On the other hand, if $\mathrm{OWA}_{(e,0,\ldots,0,1-e)}$ is a neutral element of $\circledast_{\mathbb{A}}$, for any $\mathrm{OWA}_{\mathbf{v}} \in \mathrm{Owa}_n$, it holds $\mathrm{OWA}_{\mathbf{e}} \circledast_{\mathbb{A}} \mathrm{OWA}_{\mathbf{v}} = \mathrm{OWA}_{\mathbf{v}} \circledast_{\mathbb{A}} \mathrm{OWA}_{\mathbf{e}} = \mathrm{OWA}_v$, and hence $\mathbb{A}(E, V) = (A(e, V_1), \ldots, A(e, V_{n-1}), A(1,1)) = (V_1, \ldots, V_{n-1}, 1)$. Then, $A(e, V_1) = V_1$ for any $V_1 \in [0,1]$.

Similarly, $A(V_1, e) = A(e, V_1) = V_1$, and thus e is neutral element of A. ■

Obviously, if $A = T$ is a t-norm on $[0,1]$, then $\circledast_{\mathbb{A}}$ is a t-norm on Owa_n. As a typical example consider the product $\Pi, \Pi(x,y) = xy$. Then, for any $\mathbf{U}, \mathbf{V} \in \mathcal{W}_n$, $\mathbf{\Pi}(\mathbf{U}, \mathbf{V}) = (U_1V_1, \ldots, U_{n-1}V_{n-1}, U_nV_n)$. Denote, $\mathbf{W} = \mathbf{\Pi}(\mathbf{U}, \mathbf{V})$. The corresponding normed weighting vector $\mathbf{w} \in \mathscr{S}_n$ is then given by $\mathbf{w} = (U_1V_1, U_2V_2 - U_1V_1, \ldots, U_nV_n - U_{n-1}V_{n-1})$. Hence, for any $\mathrm{OWA}_{\mathbf{u}}, \mathrm{OWA}_{\mathbf{v}} \in \mathrm{Owa}_n$, it holds $\mathrm{OWA}_{\mathbf{u}} *_{\mathbf{\Pi}} \mathrm{OWA}_{\mathbf{v}} = \mathrm{OWA}_{\mathbf{w}}$, where $\mathbf{w}_1 = u_1v_1, \mathbf{w}_2 = U_2V_2 - U_1V_1 = (u_1+u_2)(v_1+v_2) - u_1v_1 = u_2v_1 + u_1v_2 + u_2v_2, \ldots, \mathbf{w}_i = \sum_{\mathrm{Max}(j,k)=i} u_jv_k, \ldots$. $*_{\mathbf{\Pi}} : \mathrm{Owa}_n^2 \to \mathrm{Owa}_n$ is a t-norm on Owa_n, and its k-ary extension $*_{\mathbf{\Pi}} : \mathrm{Owa}_n^k \to \mathrm{Owa}_n$ is given by ${*_{\mathbf{\Pi}}}_{j=1}^k \mathrm{OWA}_{\mathbf{w}^{(j)}} = \mathrm{OWA}_{\mathbf{w}}$, where $w_i = \sum_{\mathrm{Max}(j_1,\ldots,j_k)=i} \Pi_{r=1}^k w_{j_r}^{(r)}$, $i = 1, .., n$.

Note that it can be shown that

$$\mathrm{ORness}((*_{\Pi})_{j=1}^k \mathrm{OWA}_{\mathbf{w}^{(j)}}) \geq \Pi_{j=1}^k \mathrm{ORness}(\mathrm{OWA}_{\mathbf{w}^{(j)}}).$$

A similar result holds for any t-norm T increasingly dominated by the arithmetic mean AM, i.e., $T(\mathrm{AM}(x_1^{(1)}, \ldots, x_k^{(1)}), \ldots, \mathrm{AM}(x_1^{(n)}, \ldots, x_k^{(n)})) \leq \mathrm{AM}(T(x_1^{(1)}, \ldots, x_1^{(n)}) \ldots, T(x_k^{(1)}, \ldots, x_k^{(n)}))$ for any increasing sequences $(x_1^{(1)}, \ldots, x_k^{(1)}), \ldots, (x_1^{(n)}, \ldots, x_k^{(n)}) \in [0,1]^k$. This is, for example, the case of the Łukasiewicz t-norm $T_{\mathbf{Ł}}$ given by $T_{\mathbf{Ł}}\ (x_1, \ldots, x_k) = \max(0, 1 + \sum_{i=1}^k (x_i - 1))$, but not the case of the strongest t-norm Min.

Note that due to the associativity of t-norms, it is enough to consider only their binary form, and then each ultramodular t-norm T [10] is increasingly dominated by AM.

Note also that any aggregation $*_A$ discussed in Theorems 1 and 2 can be obtained also in the form $\circledast_{\mathbb{A}}$, and $*_A = \circledast_{\mathbb{A}}$.

Definition 1 can be further generalized, considering an ordered $(n-1)$-tuple $(A_1, \ldots, A_{n-1}) \in \mathrm{Agg}_k^{n-1}$, $A_1 \leq A_2 \leq, \ldots, \leq A_{n-1}$. It is not difficult to check that the mapping $K : \mathcal{W}_n^k \to \mathcal{W}_n$ given by $K(\mathbf{W}^{(1)}, \ldots, \mathbf{W}^{(k)}) = (A_1(W_1^{(1)}, \ldots, W_1^{(k)}), \ldots, A_{n-1}(W_{n-1}^{(1)}, \ldots, W_{n-1}^{(k)}), 1)$ is a well-defined k-ary aggregation function on $\mathcal{W}_n$.

IV. Bilinear Aggregation of OWA Operators

Each OWA operator $\mathrm{OWA}_{\mathbf{w}} \in \mathrm{Owa}_n$ can be seen as a convex combination of order statistics, $\mathrm{OWA}_{\mathbf{w}} = \sum_{j=1}^n w_i \mathrm{OS}_i$. Recall that for any $\mathbf{x} \in [0,1]^n$ and any permutation $(\bullet) : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $x_{(1)} \geq x_{(2)} \geq, \ldots, \geq x_{(n)}$, $\mathrm{OS}_i(\mathbf{x}) = x_{(i)}, i = 1, \ldots, n$.

Definition 2: An aggregation function $\otimes : (\mathrm{Owa}_n)^2 \to \mathrm{Owa}_n$ is called bilinear whenever for any OWA operators $A_1, \ldots, A_k, B_1, \ldots, B_r \in \mathrm{Owa}_n$ and nonnegative constants $a_1, \ldots, a_k, b_1, \ldots, b_r$ such that $\sum_{i=1}^k a_i = \sum_{j=1}^r b_j = 1$ it holds

$$\left(\sum_{i=1}^k a_i A_i\right) \otimes \left(\sum_{j=1}^r b_j B_j\right) = \sum_{i=1}^k \sum_{j=1}^r a_i b_j (A_i \otimes B_j). \quad (9)$$

We have the next characterization of bilinear aggregation function on Owa_n, stressing the role of order statistics.

Theorem 4: A mapping $\otimes : (\mathrm{Owa}_n)^2 \to \mathrm{Owa}_n$ is a bilinear aggregation function if and only if for any $\mathrm{OWA}_{\mathbf{u}}, \mathrm{OWA}_{\mathbf{v}} \in \mathrm{Owa}_n$ it holds

$$\mathrm{OWA}_{\mathbf{u}} \otimes \mathrm{OWA}_{\mathbf{v}} = \sum_{i=1}^{n}\sum_{j=1}^{n} u_i v_j (\mathrm{OS}_i \otimes \mathrm{OS}_j) \tag{10}$$

$\mathrm{OS}_1 \otimes \mathrm{OS}_1 = \mathrm{OS}_1$, $\mathrm{OS}_n \otimes \mathrm{OS}_n = \mathrm{OS}_n$ and $\mathrm{OS}_i \otimes \mathrm{OS}_j \le \mathrm{OS}_{i'} \otimes \mathrm{OS}_{j'}$ whenever $i \ge i'$ and $j \ge j'$.

Proof: The necessity follows from the bilinearity of $\otimes$ [namely, (10)], boundary conditions, and monotonicity of aggregation functions. Similarly, when discussing the sufficiency, the boundary conditions and monotonicity of $\otimes$ are obvious. Consider $A_1, \ldots, A_k, B_1, \ldots, B_r \in \mathrm{Owa}_n$, nonnegative constants $a_1, \ldots, a_k, b_1, \ldots, b_r$ such that $\sum_{i=1}^k a_i = \sum_{j=1}^r b_j = 1$, and let $\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)}, \mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(r)} \in \mathscr{S}_n$ be the normed weighting vectors corresponding to $A_1, \ldots, A_k, B_1, \ldots, B_r$, respectively. Then, $(\sum_{i=1}^k a_i A_i) \otimes (\sum_{j=1}^r b_j B_j) = (\sum_{i=1}^k a_i (\sum_{t=1}^n u_t^{(i)} \mathrm{OS}_t)) \otimes (\sum_{j=1}^r b_j (\sum_{t=1}^n v_t^{(j)} \mathrm{OS}_t)) = (\sum_{t=1}^n (\sum_{i=1}^k a_i u_t^{(i)}) \mathrm{OS}_t) \otimes \sum_{t=1}^n (\sum_{j=1}^r b_j v_t^{(j)}) \mathrm{OS}_t) = \sum_{t=1}^n \sum_{s=1}^n (\sum_{i=1}^k a_i u_t^{(i)})(\sum_{j=1}^r b_j v_s^{(j)})(\mathrm{OS}_t \otimes \mathrm{OS}_s) = \sum_{i=1}^k \sum_{j=1}^r (a_i b_j (\sum_{t=1}^n \sum_{s=1}^n u_t^{(i)} v_s^{(j)} (\mathrm{OS}_t \otimes \mathrm{OS}_s))) = \sum_{i=1}^k \sum_{j=1}^r a_i b_j (A_i \otimes B_j)$, proving the bilinearity of $\otimes$. ∎

Due to Theorem 4, any bilinear aggregation function $\otimes$ on Owa_n is determined by its action on order statistics, i.e., one should determine $\mathrm{OS}_i \otimes \mathrm{OS}_j, i, j \in \{1, \ldots, n\}$ fitting the constraints of Theorem 4. The simplest case is when $\mathrm{OS}_i \otimes \mathrm{OS}_j$ is again an order statistic for each i, j. This situation is described in the next theorem.

Theorem 5: Let $H : \{1, \ldots, n\}^2 \to \{1, \ldots, n\}$ be an aggregation function on finite chain $\{1, \ldots, n\}$. Then, the formula

$$\mathrm{OS}_i \otimes_H \mathrm{OS}_j = \mathrm{OS}_{H(i,j)} \tag{11}$$

satisfies the constraints of Theorem 1, and hence $\otimes_H : (\mathrm{Owa}_n)^2 \to \mathrm{Owa}_n$ given by

$$\mathrm{OWA}_{\mathbf{u}} \otimes_H \mathrm{OWA}_{\mathbf{v}} = \sum_{i=1}^{n}\sum_{j=1}^{n} u_i v_j \mathrm{OS}_{H(i,j)} \tag{12}$$

is a bilinear aggregation function on Owa_n.

The proof is obvious and therefore omitted. Note that denoting $\mathrm{OWA}_{\mathbf{u}} \otimes_H \mathrm{OWA}_{\mathbf{v}} = \mathrm{OWA}_{\mathbf{w}}$, one can define (with an abuse of notation) a bilinear aggregation function $\otimes_H$ on $\mathscr{S}_n$, $\mathbf{u} \otimes_H \mathbf{v} = \mathbf{w}$, where, for $k = 1, \ldots, n$, $w_k = \sum_{H(i,j)=k} u_i v_j$.

Obviously, $\otimes_H$ is symmetric if and only if H is symmetric. Similarly, $\otimes_H$ is associative if and only if H is associative. If i is a neutral element (annihilator) of H then OS_i is a neutral element (annihilator) of $\otimes_H$, and vice versa. Thus, the next corollary is obvious.

Corollary 1: Let $H : \{1, \ldots, n\}^n \to \{1, \ldots, n\}$ be an aggregation function. Then, the bilinear aggregation function $\otimes_H$ is a t-norm (t-conorm) on Owa_n if and only if H is a t-conorm (t-norm) on $\{1, \ldots, n\}$.

Example IV.1:

1) Consider $H = \mathrm{MAX}$, which is a t-conorm on $\{1, \ldots, n\}$. Then, $\otimes_{\mathrm{MAX}}$ is a t-norm on Owa_n. Observe that if $\mathrm{OWA}_{\mathbf{u}} \otimes_H \mathrm{OWA}_{\mathbf{v}} = \mathrm{OWA}_{\mathbf{w}}$, then $w_k = \sum_{\mathrm{MAX}(i,j)=k} u_i v_j$, and thus we recognize the aggregation function $*_{\mathbf{\Pi}}$ based on the product (on $[0, 1]$) and discussed in the previous section, i.e., $\otimes_{\mathrm{MAX}} \equiv *_{\mathbf{\Pi}}$.
2) Put $L(i, j) = \max(1, i + j - n)$. Then, L is a t-norm on $\{1, \ldots, n\}$ and the t-conorm $\otimes_L$ on Owa_n satisfies $\mathrm{OWA}_{\mathbf{u}} \otimes_L \mathrm{OWA}_{\mathbf{v}} = \mathrm{OWA}_{\mathbf{w}}$, where $w_n = u_n v_n, w_{n-1} = u_n v_{n-1} + u_{n-1} v_n, \ldots, w_2 = \sum_{j=2}^n u_{n+2-j} v_j$, but $w_1 = 1 - \sum_{j=2}^n w_j = \sum_{i+j \le n+1} u_i v_j$.

Obviously, not each bilinear aggregation function $\otimes$ on Owa_n is constructed by means of Theorem 5. For example, the standard convex sum of OWA operators discussed in Section II is also bilinear, and it is characterised by $\mathrm{OS}_i \otimes \mathrm{OS}_j = \lambda \mathrm{OS}_i + (1 - \lambda)\mathrm{OS}_j$. Recall that these convex sums commute with ORness (ANDness) operators, see Section II. We expect that if, for $n > 2$, the ORness of $\mathrm{OWA}_{\mathbf{u}} \otimes \mathrm{OWA}_{\mathbf{v}}$ can be derived from ORness ($\mathrm{OWA}_{\mathbf{u}}$) and ORness ($\mathrm{OWA}_{\mathbf{v}}$), then necessarily $\mathrm{OS}_i \otimes \mathrm{OS}_j$ is not an order statistic for some i, j.

Example IV.2: For $n = 3$, let $H = \mathrm{Max}$, $u = (0, 1, 0)$ and $v = (\frac{1}{2}, 0, \frac{1}{2})$. Then, $u \otimes_H u = (0, 1, 0)$, $u \otimes_H v = (0, \frac{1}{2}, \frac{1}{2})$. Note the $\mathrm{ORness}(\mathrm{OWA}_u) = \mathrm{ORness}(\mathrm{OWA}_v) = \frac{1}{2}$ and ORness $(\mathrm{OWA}_u \otimes_H \mathrm{OWA}_v) = \frac{1}{4}$, showing that though (ORness (OWA_u), $\mathrm{ORness}(\mathrm{OWA}_v)$) $= (\mathrm{ORness}(OWA_u)$, ORness $(\mathrm{OWA}_u)) = (\frac{1}{2}, \frac{1}{2})$, $\mathrm{ORness}(\mathrm{OWA}_u \otimes_H \mathrm{OWA}_v) \ne$ ORness $(\mathrm{OWA}_u \otimes_H \mathrm{OWA}_u)$.

Note that an example of a bilinear aggregation function $\otimes$ on Owa_n which is a t-norm on Owa_n and satisfies

$$\begin{aligned}&\mathrm{ANDness}(\mathrm{OWA}_u \otimes \mathrm{OWA}_v)\\ &= \mathrm{ANDness}(\mathrm{OWA}_u) \cdot \mathrm{ANDness}(\mathrm{OWA}_v)\end{aligned}$$

was recently introduced in [9]. The method of introducing this bilinear aggregation functions is based on transformation of normed weighting vectors into a particular $n \times n$ matrix on the product of matrices, and backward transformation of a particular matrix into a normed weighting vector. By duality, one can introduce a dual bilinear aggregation function $\otimes^d$, $u \otimes^d v = (u^d \otimes v^d)^d$, which is a t-conorm on $\mathscr{S}_n$, and by isomorphism on Owa_n, which is related to ORness by

$$\begin{aligned}&\mathrm{ORness}(\mathrm{OWA}_\mathrm{u} \otimes^\mathrm{d} \mathrm{OWA}_\mathrm{v})\\ &= \mathrm{ORness}(\mathrm{OWA}_\mathrm{u}) \cdot \mathrm{ORness}(\mathrm{OWA}_\mathrm{v}).\end{aligned}$$

We introduce a more transparent example with similar properties. For the sake of transparency, we will work with normed weighting vectors from $\mathscr{S}_n$.

Theorem 6: Define a mapping $\otimes : (\mathscr{S}_n)^2 \to \mathscr{S}_n$ by $\mathbf{u} \otimes \mathbf{v} = \mathbf{w}$, where

$$w_k = \frac{1}{(n-1)(n-k)} \sum_{Max(i,j)=k} (n-i)(n-j) u_i v_j \tag{13}$$

for $k = 1, \ldots, n-1$, and $w_n = 1 - \sum_{k=1}^{n-1} w_i$.

Then, $\otimes$ is a t-norm on $\mathscr{S}_n$, the related operation $\otimes$ on Owa_n is a bilinear t-norm on Owa_n, and $\mathrm{ORness}(\mathrm{OWA}_{\mathbf{u}} \otimes \mathrm{OWA}_{\mathbf{v}}) = \mathrm{ORness}(\mathrm{OWA}_{\mathbf{u} \otimes \mathbf{v}}) = \mathrm{ORness}(\mathbf{u}) \cdot \mathrm{ORness}(\mathbf{v})$.

Proof: Obviously, $w_k \ge 0$ for $k = 1, \ldots, n-1$ and $\sum_{k=1}^{n-1} w_k \le (\sum_{i=1}^{n-1} u_i)(\sum_{j=1}^{n-1} v_j) \le 1$, thus also $w_n \ge 0$.

Hence, $\mathbf{w}$ is a weighting vector from $\mathscr{S}_n$. It is not difficult to see that $\otimes$ is symmetric, $\mathbf{u} \otimes \mathbf{v} = \mathbf{v} \otimes \mathbf{u}$, and that $\mathbf{e}_1$ is its neutral element and $\mathbf{e}_n$ is its annihilator (thus, on Owa_n, $\mathrm{OS}_1 = \mathrm{Max}$ is neutral element of $\otimes$, and $\mathrm{OS}_n = \mathrm{Min}$ is its annihilator).

To see the monotonicity of $\otimes$, note that $\mathbf{u} \preceq \mathbf{u}'$ if and only if $\mathbf{U} \preceq \mathbf{U}'$. Thus, $U_1' = U_1 + \varepsilon_1$, $U_2' = U_2 + \varepsilon_2, \ldots, U_{n-1}' = U_{n-1} + \varepsilon_{n-1}$, and $U_n' = U_n = 1$, where $\varepsilon_i \geq 0$. For $\mathbf{u}, \mathbf{v} \in \mathscr{S}_n$, denoting $\mathbf{w} = \mathbf{u} \otimes \mathbf{v}$, we see that for the corresponding cumulative weighting vector $\mathbf{W} \in \mathcal{W}_n$, it holds $W_n = 1$, and for $k \in \{1, \ldots, n-1\}$

$$W_k = w_1 + \cdots + w_k$$

$$= \sum_{r=1}^{k} \frac{1}{(n-1)(n-k)} \left(\sum_{Max(i,j)=r} (n-i)(n-j) u_i v_j \right).$$

After a short processing it can be shown that

$$W_k = U_k V_k - \frac{1}{n-1} \sum_{r=1}^{k-1} (U_k - U_r)(V_k - V_r).$$

Denoting $\mathbf{w}' = \mathbf{u}' \otimes \mathbf{v}$, for $\mathbf{W}'$ it holds, for $k \in \{1, \ldots, n-1\}$

$$W_k' = (U_k + \varepsilon_k) V_k - \frac{1}{n-1} \sum_{r=1}^{k-1} (U_k - U_r + \varepsilon_k - \varepsilon_r)(V_k - V_r)$$

and thus

$$W_k' - W_k = \varepsilon_k V_k - \frac{1}{n-1} \sum_{r=1}^{k-1} (\varepsilon_k - \varepsilon_r)(V_k - V_r)$$

$$= \varepsilon_k V_k \left(1 - \frac{k-1}{n-1} \right) + \frac{1}{n-1} \sum_{r=1}^{k-1} \varepsilon_r (V_k - V_r) \geq 0$$

i.e. $W_k' \geq W_k$. Consequently, $\mathbf{u} \otimes \mathbf{v} \preceq \mathbf{u}' \otimes \mathbf{v}$, proving the increasingness of $\otimes$.

Finally, the associativity of $\otimes$ follows from the fact that, for all $\mathbf{u}, \mathbf{v}, \mathbf{p} \in \mathscr{S}_n$, $(\mathbf{u} \otimes \mathbf{v}) \otimes \mathbf{p} = \mathbf{u} \otimes (\mathbf{v} \otimes \mathbf{p}) = \mathbf{w} \in \mathscr{S}_n$, where, for $k \in \{1, \ldots, n-1\}$, $\mathbf{w}_k = \frac{1}{n-1} \sum_{\mathrm{Max}(i,j,r)=k} (n - \mathrm{Min}(i,j,r)) u_i v_j p_r$ and $\mathbf{w}_n = 1 - \sum_{k=1}^{n-1} \mathbf{w}_k$.

Moreover, $\otimes$ is also related to the ORness operator. Indeed, denote $\mathbf{w} = \mathbf{u} \otimes \mathbf{v}$. Then

$$\mathrm{ORness}(\mathrm{OWA}_{\mathbf{w}})$$

$$= \frac{1}{n-1} \left(\sum_{k=1}^{n-1} \frac{n-k}{(n-1)(n-k)} \sum_{\mathrm{Max}(i.j)=k} (n-1)(n-j) u_i v_j \right)$$

$$= \left(\frac{1}{n-1} \right)^2 \sum_{i,j \in \{1, \ldots, n-1\}} (n-i)(n-j) u_i v_j$$

$$= \left(\sum_{i=1}^{n-1} \frac{n-i}{n-1} u_i \right) \cdot \left(\sum_{j=1}^{n-1} \frac{n-j}{n-1} v_j \right)$$

$$= \mathrm{ORness}(\mathrm{OWA}_{\mathbf{u}}) \cdot \mathrm{ORness}(\mathrm{OWA}_{\mathbf{v}}).$$

■

Remark 1:

1) By duality, we can introduce a t-conorm $\otimes^d$ on Owa_n (*on* $\mathscr{S}_n$), $\mathrm{OWA}_{\mathbf{u}} \otimes^d \mathrm{OWA}_{\mathbf{v}} = (\mathrm{OWA}_{\mathbf{u}}^d \otimes \mathrm{OWA}_{\mathbf{v}}^d)^d$. Note that, denoting $\mathbf{u} \otimes^d \mathbf{v} = \mathbf{w}$, it holds $W_k = \frac{1}{(n-1)(k-1)} \sum_{Min(i,j)=k} (i-1)(j-1) u_i v_j$ for any $k = 2, \ldots, n$, and $w_1 = 1 - \sum_{k=2}^{n} w_k$. Obviously, then $\mathrm{ANDness}(\mathrm{OWA}_{\mathbf{u}} \otimes^d \mathrm{OWA}_{\mathbf{v}}) = \mathrm{ANDness}(\mathrm{OWA}_{\mathbf{u}}) \cdot \mathrm{ANDness}(\mathrm{OWA}_{\mathbf{v}})$.
2) For any $\mathrm{OWA}_{\mathbf{u}} \in \mathrm{Owa}_n \backslash \{\mathrm{OS}_1\}$, we have $\mathrm{ORness}(\mathrm{OWA}_{\mathbf{u}}) \in [0, 1[$, and thus $\lim_{m \to \infty} \mathrm{ORness}(\mathrm{OWA}_{\mathbf{u}}^{[m]}) = 0$, where $\mathrm{OWA}_{\mathbf{u}}^{[1]} = \mathrm{OWA}_{\mathbf{u}}$, and for $m > 1$, $\mathrm{OWA}_{\mathbf{u}}^{[m]} = \mathrm{OWA}_{\mathbf{u}} \otimes \mathrm{OWA}_{\mathbf{u}}^{[m-1]}$.

However, $\otimes$ is not an Archimedean t-norm. Indeed, if $u_1 > 0$ then $u_1^{[m]} = (u_1)^m > 0$, and thus $\mathrm{OWA}_{\mathbf{u}}^{[m]} \preceq \mathrm{OWA}_{\mathbf{v}}$ cannot hold whenever $v_1 = 0$, independently of the integer m.

V. Concluding Remarks

We have introduced and discussed aggregation of OWA operators (normed weighting vectors and cumulative weighting vectors). We illustrate some of our results, based on the next data.

For $n = 3$, let $\mathbf{u} = (0.5, 0.2, 0.3)$ and $\mathbf{v} = (0.2, 0.7, 0.1)$. Clearly $\mathbf{u}, \mathbf{v} \in \mathscr{S}_3$ and related $\mathbf{U}, \mathbf{V} \in \mathscr{S}_3$ are given by $\mathbf{U} = (0.5, 0.7, 1)$ and $\mathbf{V} = (0.2, 0.9, 1)$. The corresponding OWA operators $\mathrm{OWA}_{\mathbf{u}}, \mathrm{OWA}_{\mathbf{v}} \in \mathrm{Owa}_3$ are given by $\mathrm{OWA}_{\mathbf{u}}(x, y, z) = 0.5\mathrm{Max}(x, y, z) + 0.2\mathrm{Med}(x, y, z) + 0.3\mathrm{Min}(x, y, z)$, where Med is the median, and $\mathrm{OWA}_{\mathbf{v}}(x, y, z) = 0.2\mathrm{Max}(x, y, z) + 0.7\mathrm{Med}(x, y, z) + 0.1\mathrm{Min}(x, y, z)$. Here, $\mathrm{Max} = \mathrm{OS}_1$, $\mathrm{Med} = \mathrm{OS}_2$ and $\mathrm{Min} = \mathrm{OS}_3$.

Note that $\mathrm{ORness}(\mathrm{OWA}_{\mathbf{u}}) = \frac{1}{2}(2u_1 + u_2) = 0.6$ and $\mathrm{ORness}(\mathrm{OWA}_{\mathbf{v}}) = \frac{1}{2}(2v_1 + v_2) = 0.55$, $\mathrm{ANDness}(\mathrm{OWA}_{\mathbf{u}}) = 0.4$, $\mathrm{ANDness}(\mathrm{OWA}_{\mathbf{v}}) = 0.45$.

1) Let $A(x, y) = 0.4x + 0.6y$. Then, $\mathrm{OWA}_{\mathbf{u}} *_A \mathrm{OWA}_{\mathbf{v}} = \mathrm{OWA}_{\mathbf{w}^{(1)}}$, where $\mathbf{w}^{(1)} = 0.4\mathbf{u} + 0.6\mathbf{v} = (0.32, 0.5, 0.18)$. Note that $\mathrm{ORness}(\mathrm{OWA}_{\mathbf{w}^{(1)}}) = \frac{1}{2}(0.64 + 0.5) = 0.57 = 0.4 \times 0.6 + 0.6 \times 0.55 = A(\mathrm{ORness}(\mathrm{OWA}_{\mathbf{u}}), \mathrm{ORness}(\mathrm{OWA}_{\mathbf{v}}))$. Similarly, $\mathrm{ANDness}(\mathrm{OWA}_{\mathbf{w}^{(1)}}) = 0.43 = A(\mathrm{ANDness}(\mathrm{OWA}_{\mathbf{u}}), \mathrm{ANDness}(\mathrm{OWA}_{\mathbf{v}}))$.
2) Let $B(x, y) = \mathrm{Min}(x, y)$. Then, $\mathrm{OWA}_{\mathbf{u}} *_{\mathbb{B}} \mathrm{OWA}_{\mathbf{v}} = \mathrm{OWA}_{\mathbf{w}^{(2)}}$, where $W^{(2)} = \mathrm{Min}(U, V) = (0.2, 0.7, 1)$, i.e., $\mathbf{w}^{(2)} = (0.2, 0.5, 0.3)$.
3) Let $C(x, y) = xy$. Then, $\mathrm{OWA}_{\mathbf{u}} *_{\mathbb{C}} \mathrm{OWA}_{\mathbf{v}} = \mathrm{OWA}_{\mathbf{w}^{(3)}}$, where $W^{(3)} = (0.1, 0.63, 1)$, i.e., $\mathbf{w}^{(3)} = (0.1, 0.53, 0.37)$. Recall then $\mathrm{OWA}_{\mathbf{w}^{(3)}} = \mathrm{OWA}_{\mathbf{u}} \otimes_{Min} \mathrm{OWA}_{\mathbf{v}}$, too.
4) Consider the t-norm $\otimes$ introduced in Theorem 6, and its dual t-conorm $\otimes^d$. Then, $\mathrm{OWA}_{\mathbf{u}} \otimes \mathrm{OWA}_{\mathbf{v}} = \mathrm{OWA}_{\mathbf{w}^{(4)}}$, where $w_1^{(4)} = u_1 v_1 = 0.1$, $w_2^{(4)} = \frac{1}{2}(2u_1 v_2 + 2u_2 v_1 + u_2 v_2) = 0.46$, $w_3^{(4)} = 1 - (w_1^{(4)} + w_2^{(4)}) = 0.44$, i.e., $\mathbf{w}^{(4)} = (0.1, 0.46, 0.44)$. Observe that $\mathrm{ORness}(\mathrm{OWA}_{\mathbf{w}^{(4)}}) = \frac{1}{2}(2w_1^{(4)} + w_2^{(4)}) = 0.33 = 0.6 \times 0.55 = \mathrm{ORness}(\mathrm{OWA}_{\mathbf{u}}) \cdot \mathrm{ORness}(\mathrm{OWA}_{\mathbf{v}})$. On the other hand, $\mathrm{OWA}_{\mathbf{u}} \otimes^d \mathrm{OWA}_{\mathbf{v}} = \mathrm{OWA}_{\mathbf{w}^{(5)}}$, where $w_3^{(4)} = u_3 v_3 = 0.03$, $w_2^{(4)} = \frac{1}{2}(2u_3 v_2 + 2u_2 v_3 + u_2 v_2) = 0.3$, $w_1^{(4)} = 1 - (w_2^4 + w_3^4) = 0.67$, i.e., $\mathbf{w}^4 = (0.67, 0.3, 0.03)$. Then, $\mathrm{ANDness}(\mathrm{OWA}_{\mathbf{w}^{(4)}}) = \frac{1}{2}(w_2^{(4)} + 2w_3^{(4)}) = 0.18 = 0.4 \times 0.45 = \mathrm{ANDness}(\mathrm{OWA}_{\mathbf{u}}) \cdot \mathrm{ANDness}(\mathrm{OWA}_{\mathbf{v}})$.

Observe that the concept of entropy of OWA operators [16] is not compatible with the aggregation of OWA operators. This is caused by the fact that for any permutation

$\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ and any normed weighting vector $\mathbf{w} \in \mathscr{S}_n$, putting $\mathbf{w}_\sigma = (w_{\sigma_{(1)}}, \ldots, w_{\sigma_{(n)}})$, the entropy of $\mathrm{OWA}_{\mathbf{w}_\sigma}$ coincides with the entropy of $\mathrm{OWA}_{\mathbf{w}}$.

Our results can be applied in several domains dealing with weighting vectors $\mathbf{w} \in [0,1]^n$, $\sum_{i=1}^n w_i = 1$, but also in domains where such vectors are themselves objects of study. To illustrate these last claims, consider, for example, ordered weighted geometric averages operators (OWGA), see [8]. Then, for $n \geq 3$, the only direct aggregation of n-ary OWGA's is a weighted geometric mean, compare Theorem 2 for OWA's.

On the other hand, each weighting vector $\mathbf{w} \in [0,1]^n$ can be seen as a fuzzy set on the universe $X = \{1, \ldots, n\}$ with cardinality 1 [15]. Similarly such a vector $\mathbf{w}$ can be seen as a discrete probability distribution on X. Considering the ordering based on the related distribution functions (assigned to the random variable $\Phi : X \to \mathbb{R}, \Phi(i) = i$), one can apply all our results and introduce aggregation functions (including t-norms, t-conorms, etc.) acting on the space of all discrete probability distributions on X.

Note that the aggregation of OWA operators cannot be obtained as a serial application of special OWA operators, in general. On the other hand, a challenging topic for the further research could be the study of aggregation of OWA operators from Owa_n in the framework of n-ary aggregation functions from Agg_n. In particular, quasi-arithmetic means of OWA operators could be of interest, for example, geometric means of OWA's. Clearly, the resulting aggregation will be out of the OWA framework, in general, though some properties will be still preserved, such as the symmetry and idempotency.

As another promising direction for the further research, the aggregation of several generalizations of OWA operators, see, e.g., [12], can be considered.

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REFERENCES

[1] G. Beliakov and J. Dujmović, "Extension of bivariate means to weighted means of several arguments by using binary trees," *Inf. Sci.*, vol. 331, pp. 137–147, 2016.

[2] V. Cutello and J. Montero, "Recursive connective rules," *Int. J. Intell. Syst.*, vol. 14, no. 1, pp. 3–20, 1999.

[3] G. Deschrijver and C. Cornelis, "Representability in interval-valued fuzzy set theory," *Int. J. Uncertainty, Fuzziness Knowl.-Based Syst.*, vol. 15, no. 3, pp. 345–361, 2007.

[4] J. J. Dujmović, "Weighted conjunctive and disjunctive means and their application in system evaluation," *Publikacije Elektrotehnikog fakulteta. Serija Matematika i fizika*, no. 461/497, pp. 147–158, 1974.

[5] A. Emrouznejad and M. Marra, "Ordered weighted averaging operators 1988–2014: A citation-based literature survey," *Int. J. Intell. Syst.*, vol. 29, pp. 994–1014, 2014.

[6] M. Grabisch, "Fuzzy integral in multicriteria decision making," *Fuzzy Sets Syst.*, vol. 69, no. 3, pp. 279–298, 1995.

[7] M. Grabisch, J. L. Marichal, R. Mesiar, and E. Pap, *Aggregation Functions (Encyclopedia of Mathematics and its Applications)*. Cambridge, U.K.: Cambridge Univ. Press, 2009.

[8] F. Herrera, E. Herrera-Viedma , and F. Chiclana, "A study of the origin and uses of the ordered weighted geometric operator in multicriteria decision making," *Int. J. Intell. Syst.*, vol. 18, pp. 689–707, 2003.

[9] L. Jin and G. Qian, "On obtaining piled OWA operators," *Int. J. Intell. Syst.*, vol. 30, no. 5, pp. 520–536, 2015.

[10] E. P. Klement, M. Manzi, and R. Mesiar, "Ultramodular aggregation functions," *Inf. Sci.*, vol. 181, pp. 4101–4111, 2011.

[11] X. Liu, "GIS-based local ordered weighted averaging: A case study in London," Ph.D. dissertation, Univ. Western Ontario, London, ON, Canada, 2013.

[12] R. Mesiar, A. Stupňanová, and R. R. Yager, "Generalizations of OWA operators," *IEEE Trans. Fuzzy Syst.*, vol. 23, no. 6, pp. 2154–2162, Dec. 2015.

[13] H. Mianabadi, M. Sheikhmohammady, E. Mostert, and N. Van de Giesen, "Application of the ordered weighted averaging (OWA) method to the Caspian Sea conflict," *Stoch. Environ. Res. Risk Assessment*, vol. 28, no. 6, pp. 1359–1372, 2014.

[14] D. Schmeidler, "Subjective probability and expected utility without additivity," *Econometrica*, vol. 57, pp. 571–587, 1989.

[15] M. Wygralak, "Questions of cardinality of finite fuzzy sets," *Fuzzy Sets Syst.*, vol. 102, no. 2, pp. 185–210, 1999.

[16] R. R. Yager, "On ordered weighted averaging aggregation operators in multicriteria decisionmaking," *IEEE Trans. Syst., Man Cybern.*, vol. 18, no. 1, pp. 183–190, Jan./Feb. 1988.

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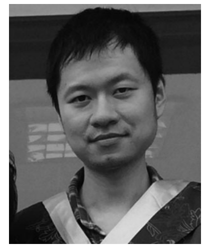

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