Assignment 3

Due: Thursday, November 16 at 4:00pm

1. The joint convexity of quantum relative entropy is useful for establishing fundamental and intuitive facts concerning various entropic quantities. The two problems that follow provide examples. (It is not necessary that you directly use the joint convexity of quantum relative entropy to answer the problems—you might, for instance, use a corollary of joint convexity such as strong subadditivity of von Neumann entropy.)

   (a) Prove that the Holevo information (or Holevo χ-quantity) of an ensemble cannot increase under the action of a channel.

   In more precise terms, let $\mathcal{X}$ and $\mathcal{Y}$ be complex Euclidean spaces, let $\Phi \in \mathrm{C}(\mathcal{X}, \mathcal{Y})$ be a channel, let Σ be an alphabet, let $\eta : \Sigma \to \mathrm{Pos}(\mathcal{X})$ be an ensemble, and define an ensemble $\Phi(\eta) : \Sigma \to \mathrm{Pos}(\mathcal{Y})$ as
   $$(\Phi(\eta))(a) = \Phi(\eta(a))$$
   for each $a \in \Sigma$. Prove that $\chi(\Phi(\eta)) \leq \chi(\eta)$.

   (b) Prove that the conditional von Neumann entropy of a register X given a register Y is a concave function of the state of these registers:
   $$\mathrm{H}(\mathsf{X}|\mathsf{Y})_{\lambda\rho_0+(1-\lambda)\rho_1} \geq \lambda\,\mathrm{H}(\mathsf{X}|\mathsf{Y})_{\rho_0} + (1-\lambda)\,\mathrm{H}(\mathsf{X}|\mathsf{Y})_{\rho_1},$$
   or, equivalently,
   $$\begin{aligned}&\mathrm{H}\big(\lambda\rho_0 + (1-\lambda)\rho_1\big) - \mathrm{H}\big(\lambda\rho_0[\mathsf{Y}] + (1-\lambda)\rho_1[\mathsf{Y}]\big)\\ &\quad \geq \lambda\big(\mathrm{H}(\rho_0) - \mathrm{H}(\rho_0[\mathsf{Y}])\big) + (1-\lambda)\big(\mathrm{H}(\rho_1) - \mathrm{H}(\rho_1[\mathsf{Y}])\big),\end{aligned}$$
   for all $\rho_0, \rho_1 \in \mathrm{D}(\mathcal{X} \otimes \mathcal{Y})$ and $\lambda \in [0,1]$.

2. For every positive integer $n \geq 2$, define a unital channel $\Phi_n \in \mathrm{C}(\mathbb{C}^n)$ as
   $$\Phi_n(X) = \frac{\mathrm{Tr}(X)\mathbb{1}_n - X^{\mathsf{T}}}{n-1}$$
   for every $X \in \mathrm{L}(\mathbb{C}^n)$, where $\mathbb{1}_n$ denotes the identity operator on $\mathbb{C}^n$. Prove that Φ_n is not mixed-unitary when n is odd.

   Hint: This is proved in the book in Example 4.3 for the case that $n = 3$, but this proof will not extend to larger odd values of n. Instead, for any fixed choice of $n \geq 2$, think about an arbitrary Kraus representation
   $$\Phi_n(X) = \sum_{a\in\Sigma} A_a X A_a^*$$
   of Φ_n. Try to identify a property that *every* Kraus operator A_a must have, and then prove that no nonzero scalar multiple of a unitary operator can have this property when n is odd.

3. Let $\mathcal{X}$ be a complex Euclidean space, let $n = \dim(\mathcal{X})$, and let $\Phi \in \mathrm{C}(\mathcal{X})$ be a unital channel. Following our usual convention for singular-value decompositions, let $s_1(Y) \geq \cdots \geq s_n(Y)$ denote the singular values of a given operator $Y \in \mathrm{L}(\mathcal{X})$, ordered from largest to smallest, and taking $s_k(Y) = 0$ when $k > \mathrm{rank}(Y)$.

Prove that, for every operator $X \in \mathrm{L}(\mathcal{X})$, it holds that

$$s_1(X) + \cdots + s_m(X) \geq s_1(\Phi(X)) + \cdots + s_m(\Phi(X))$$

for every $m \in \{1, \ldots, n\}$.

Hint: thinking about the block operator

$$\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} = E_{0,1} \otimes X + E_{1,0} \otimes X^*$$

is helpful when solving this problem.

4. This problem asks you to prove two inequalities concerning entropic quantities that hold for separable states, but not necessarily for other states. For both inequalities, let X and Y be registers and let $\rho \in \mathrm{D}(\mathcal{X} \otimes \mathcal{Y})$ be a separable state of these registers, expressed as

$$\rho = \sum_{a \in \Sigma} p(a) \sigma_a \otimes \xi_a,$$

for some choice of an alphabet Σ, a probability vector $p \in \mathcal{P}(\Sigma)$, and two collections of states $\{\sigma_a : a \in \Sigma\} \subseteq \mathrm{D}(\mathcal{X})$ and $\{\xi_a : a \in \Sigma\} \subseteq \mathrm{D}(\mathcal{Y})$.

(a) Prove that, with respect to the state ρ, it holds that $\mathrm{I}(\mathsf{X} : \mathsf{Y}) \leq \mathrm{H}(p)$.

(b) Prove that, with respect to the state ρ, it holds that

$$\mathrm{H}(\mathsf{X}|\mathsf{Y}) \geq \sum_{a \in \Sigma} p(a)\, \mathrm{H}(\sigma_a).$$

(The conditional von Neumann entropy is therefore nonnegative for separable states.)