Supporting information: "Resilience of networks with community structure behaves as if under an external field"

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Scaling Exponents for Varying Parameters

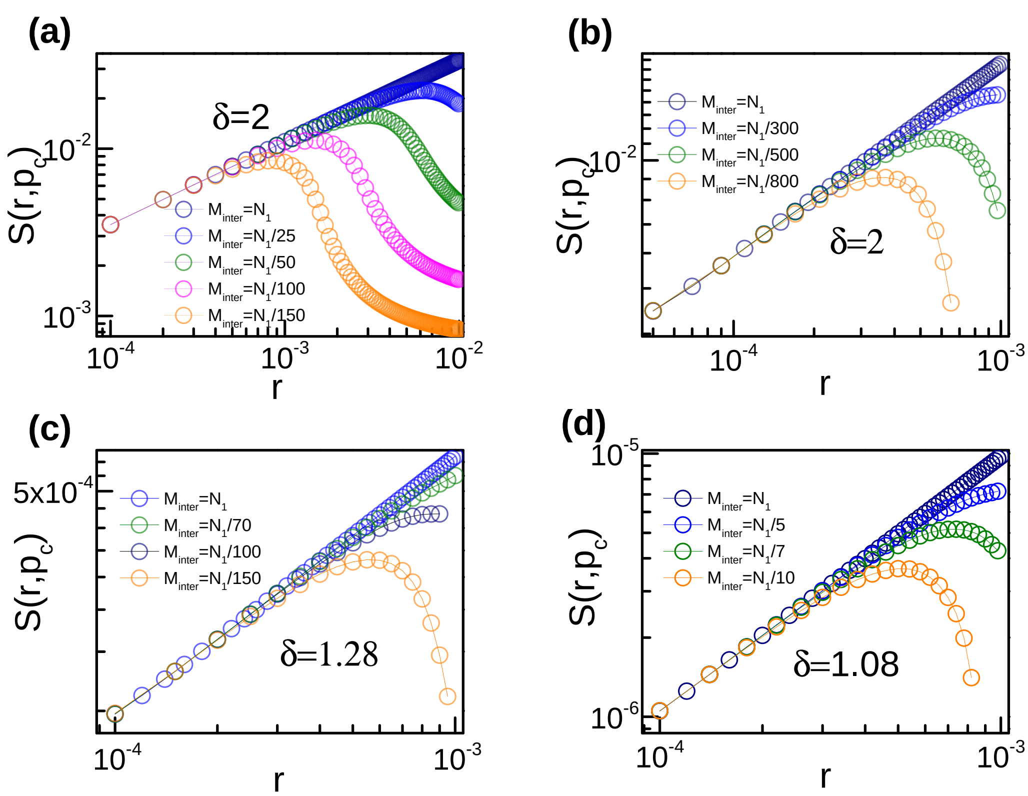

Fig. S1. $S(r, p_c)$ as a function of r with different M_{inter} from Eqs. (7) and (17) for parameters $k_{min} = 2$ and $k_{max} = 10^6$. (a) ER, $k = 4$, (b) $\lambda = 4.5$, (c) $\lambda = 3.35$ and (d) $\lambda = 2.8$. The other parameters are similar to Fig. 3 of the main text.

The dependence of $S(r, p_c)$ on r under different cases of M_{inter} are shown in Fig. S1. In addition, Fig. S2 shows $\frac{\partial S(r,p)}{\partial r}$ as a function of $p_c - p$ for different cases (ER and SF). One observes that the effect of the external field given by decreasing fractions of inter-connected nodes leads the system to show an obvious scaling relationship at criticality.

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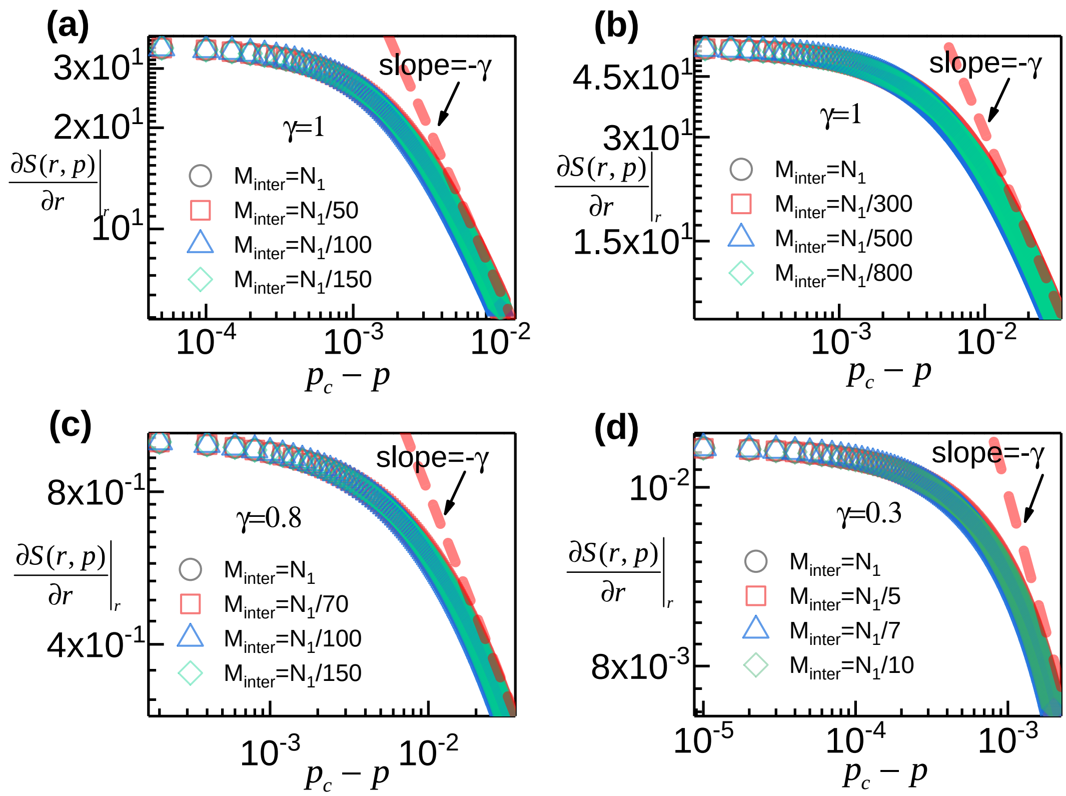

Fig. S2. $\frac{\partial S(r,p)}{\partial r}$ as a function of $p_c - p$ for different M_{inter} (a) The case of ER modules. The parameters are same as Fig.2(c). (b) Scale-free modules with $\lambda = 4.5$ and $r = 0.0001$. (c) Scale-free modules with $\lambda = 3.35$ and $r = 0.0001$. (d) Scale-free modules with $\lambda = 2.8$ and $r = 0.0001$. The other parameters are similar as Fig. 4.

From the above figures, we observe that for small r, γ remains unchanged even for small M_{inter}.

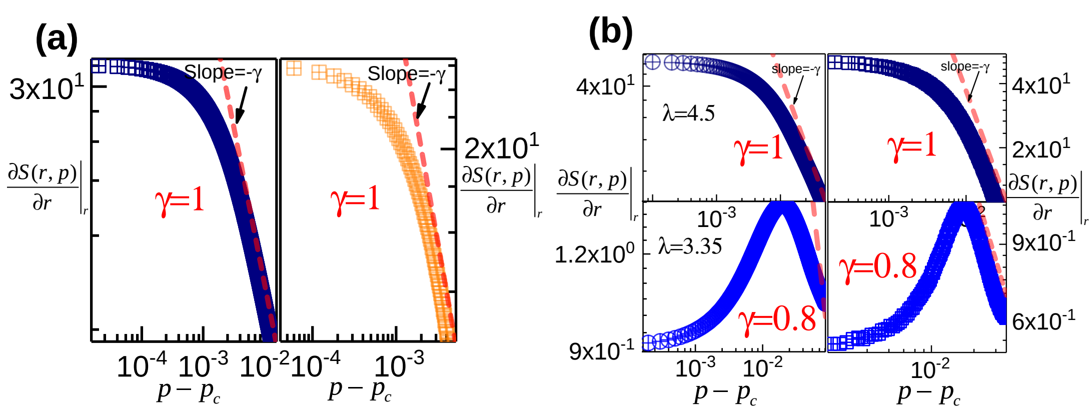

Fig. S3. $\frac{\partial S(r,p)}{\partial r}$ as a function of $p - p_c$ (a) The case of ER modules. The parameters are same as Fig.2(c). (b) Scale-free modules with $\lambda = 4.5$ and $r = 0.0001$. (c) Scale-free modules with $\lambda = 3.35$ and $r = 0.0002$. Analytical and simulations results are shown on the left and right panels respectively. The other parameters are similar to Fig. 4 of the main text.

Generating function formulation

Generating functions are a useful mathematical tool for describing the structure of random network models (1–4). For the case of a single community (ER network) with the likelihood of a node having k links being $p(k)$, the generating function $G_0(x) = \Sigma p(k)x^k$ represents the generating function of degree. After randomly removing a fraction $1-p$ of nodes, the generating function of the remaining network becomes $G_0^p(x) = G_0[1 - p(1 - x)]$ (1, 5–7). For our model of multiple communities, the likelihood of having k_i links to other nodes within module i is given by $P(k_i)$, while the likelihood of a node in module i having k_{ij} links to nodes in module j is given by $P(k_{ij})$. Since only a fraction r_i of nodes in module i have connections to other modules and independent random connection for intra and inter links in this random network model, the generating function characterizing the degree distribution of module i can be expressed as

$$G_i(\mathbf{x}) = (1-r_i)\Sigma P(k_i)x_{ii}^{k_i} + r_i\Sigma P(k_i)x_{ii}^{k_i}\prod_{j\neq i}^{m}\Sigma P(k_{ij})x_{ij}^{k_{ij}} = (1-r_i)G_{ii}(x_{ii}) + r_iG_{ii}(x_{ii})\prod_{j\neq i}^{m}G_{ij}(x_{ij}), i = 1,2,\cdots,m, \qquad [1]$$

where $(1-r_i)\Sigma P(k_i)x_{ii}^{k_i}$ denotes sub-generating function of $P(k_i)$ of module i, and $r_i\Sigma P(k_i)x_{ii}^{k_i}\prod_{j\neq i}^{m}\Sigma P(k_{ij})x_{ij}^{k_{ij}}$ denotes sub-generating function of $P(k_{ij})$ between module i and other $m-1$ modules. From Eq. (1), we can get the following branching generating functions (1, 4, 8),

$$\begin{cases} G_i^{(ii)}(\mathbf{x}) = (1-r_i)\Sigma\frac{P(k_i)k_i}{\Sigma P(k_i)k_i}x_{ii}^{(k_i-1)} + r_i\Sigma\frac{P(k_i)k_i}{\Sigma P(k_i)k_i}x_{ii}^{(k_i-1)}\prod_{j\neq i}^{m}\Sigma P(k_{ij})x_{ij}^{k_{ij}} = \frac{\frac{\partial G_i(\mathbf{x})}{\partial x_{ii}}}{\frac{\partial G_i(\mathbf{x})}{\partial x_{ii}}|_{1,1}}, \\ G_i^{(ij)}(\mathbf{x}) = \Sigma P(k_i)x_{ii}^{k_i}\Sigma\frac{P(k_{ij})k_{ij}}{\Sigma P(k_{ij})k_{ij}}x_{ij}^{(k_{ij}-1)}\prod_{j\neq i}^{m-1}\Sigma P(k_{ij})x_{ij}^{k_{ij}} = \frac{\frac{\partial G_i(\mathbf{x})}{\partial x_{ij}}}{\frac{\partial G_i(\mathbf{x})}{\partial x_{ij}}|_{1,1}}, \end{cases} \qquad [2]$$

where $\frac{P(k_i)k_i}{\Sigma P(k_i)k_i}$ and $\frac{P(k_{ij})k_{ij}}{\Sigma P(k_{ij})k_{ij}}$ represent probabilities of following a randomly chosen intra and inter-link to a node with k_i intra-links within module i, and k_{ij} inter-links between module i and j respectively.

To study the system resilience, we define $1-f_{ii}$ and $1-f_{ij}$ as the probability that a randomly chosen intra and inter-link respectively, belong to the giant component of the network. After the random removal of a fraction $1-p$ of nodes, the probability of an arbitrary node to connect to the giant component through intra-links and inter-links are $p(1-f_{ii})$ and $p(1-f_{ij})$ respectively, and the probability to fail to connect to the giant component is thus $1-p(1-f_{ii})$ and $1-p(1-f_{ij})$ respectively (7–9). This same quantity is also given by the branching generating functions, $G_i^{(ii)}[1-p(1-f_{ii}), 1-p(1-f_{ij})]$ and $G_i^{(ij)}[1-p(1-f_{ii}), 1-p(1-f_{ij})]$, which give the probability that a node with $k_{ii}-1$ intra links and $k_{ij}-1$ inter links is not connected to the giant component. Therefore, the following self-consistent equations, Eqs. (3), can be defined and solved:

$$\begin{cases} f_{ii} = G_i^{(ii)}[1-p(1-f_{ii}), 1-p(1-f_{ij})], \\ f_{ij} = G_i^{(ij)}[1-p(1-f_{ii}), 1-p(1-f_{ij})]. \end{cases} \qquad [3]$$

If we let S_i denote the fraction of nodes within module i belonging to the giant component, we arrive at the following solution

$$S_i = p[1 - G_i(1-p(1-f_{ii}), 1-p(1-f_{ij}))]. \qquad [4]$$

ER modules. For the case of m ER modules with average intra-degree k_{intra}^i and Poisson distribution between any two modules i and j, Eqs. (1) and (2) become

$$G_i(\mathbf{x}) = (1-r_i)e^{k_{\mathrm{intra}}^i(x_{ii}-1)} + r_ie^{k_{\mathrm{intra}}^i(x_{ii}-1)}e^{(m-1)k_{\mathrm{inter}}^{ij}(x_{ij}-1)}, \qquad [5]$$

$$\begin{cases} G_i^{(ii)}(\mathbf{x}) = (1-r_i)e^{k_{\mathrm{intra}}^i(x_{ii}-1)} + r_ie^{k_{\mathrm{intra}}^i(x_{ii}-1)}e^{(m-1)k_{\mathrm{inter}}^{ij}(x_{ij}-1)}, \\ G_i^{(ij)}(\mathbf{x}) = e^{k_{\mathrm{intra}}^i(x_{ii}-1)}e^{(m-1)k_{\mathrm{inter}}^{ij}(x_{ij}-1)}. \end{cases} \qquad [6]$$

For simplicity, we consider $k_{\mathrm{intra}}^i = k$, $k_{\mathrm{inter}}^{ij} = K$ and $r_i = r$, then Eqs. (3) and (4) becomes

$$\begin{cases} S = S_i = p(1-f_{ii}) = p(1 - re^{kp(f_{ii}-1)}e^{(m-1)Kp(f_{ji}-1)} - (1-r)e^{kp(f_{ii}-1)}), \\ f_{ii} = f_i = (1-r)e^{kp(f_i-1)} + re^{kp(f_i-1)+(m-1)Kp(f_j-1)}, \\ f_{ij} = f_{ji} = f_j = e^{kp(f_i-1)}e^{(m-1)Kp(f_j-1)}, \end{cases} \qquad [7]$$

and we can obtain,

$$e^{-Sk}(r-1) + 1 - \frac{S}{p} = re^{(m-1)Kp(\frac{e^{-Sk}(r-1)+1-\frac{S}{p}-r}{r})-Sk}. \qquad [8]$$

As $m=2$, it is equal to Eq. (3) in the main paper. For $r=0$, the above equation becomes $S = p(1-e^{-kS})$, which recovers the standard result for percolation in a single ER module (8, 10).

From Eqs. (7), we get

$$p = \frac{\ln(f_j)}{(m-1)K(f_j-1) + k(f_i-1)}. \qquad [9]$$

Furthermore, by substituting Eq. (9) into Eq. (7), we obtain

$$p = \frac{\ln(\frac{f_i-(1-r)e^{kp(f_i-1)}}{r})}{(m-1)K[(\frac{f_i-(1-r)e^{kp(f_i-1)}}{r})-1] + k(f_i-1)}. \qquad [10]$$

As $f_i \to 1$, and $r \neq 0, 1$, the percolation threshold is,

$$p_c = \frac{[(m-1)K+k] - \sqrt{[(m-1)K-k]^2 + 4(m-1)Kkr}}{2(m-1)Kk(1-r)}, K = \frac{mM_{\mathrm{inter}}}{rN}, \qquad [11]$$

for $r=1$, $p_c = \frac{1}{(m-1)K+k}$, and for $r=0$, $p_c = \frac{1}{k}$. The theoretical results are verified by simulations as shown in Fig. S4 with different r and Fig. S5 with different m.

Modules with Power-law degree distribution. For the case of m modules with power-law exponent λ_i and Poisson distribution between any two modules i and j, Eqs.(1) and (2) become

$$\begin{aligned} G_i(\mathbf{x}) =&(1-r_i)\Sigma[(\frac{k_{\min}}{k})^{\lambda-1}-(\frac{k_{\min}}{k+1})^{\lambda-1}]x_{ii}^k+ \\ & r_i\Sigma[(\frac{k_{\min}}{k})^{\lambda-1}-(\frac{k_{\min}}{k+1})^{\lambda-1}]x_{ii}^k e^{(m-1)k_{\text{inter}}^{ij}(x_{ij}-1)}, \end{aligned} \tag{12}$$

$$\begin{cases} G_i^{(ii)}(\mathbf{x}) = (1-r_i)\frac{\Sigma[(\frac{k_{\min}}{k})^{\lambda-1}-(\frac{k_{\min}}{k+1})^{\lambda-1}]kx_{ii}^{k-1}}{\Sigma[(\frac{k_{\min}}{k})^{\lambda-1}-(\frac{k_{\min}}{k+1})^{\lambda-1}]k} + r_i\frac{\Sigma[(\frac{k_{\min}}{k})^{\lambda-1}-(\frac{k_{\min}}{k+1})^{\lambda-1}]kx_{ii}^{k-1}}{\Sigma[(\frac{k_{\min}}{k})^{\lambda-1}-(\frac{k_{\min}}{k+1})^{\lambda-1}]k}e^{(m-1)k_{\text{inter}}^{ij}(x_{ij}-1)}, \\ G_i^{(ij)}(\mathbf{x}) = \Sigma[(\frac{k_{\min}}{k})^{\lambda-1}-(\frac{k_{\min}}{k+1})^{\lambda-1}]x_{ii}^k e^{(m-1)k_{\text{inter}}^{ij}(x_{ij}-1)}. \end{cases} \tag{13}$$

Further, Eqs. (3) and (4) become,

$$\begin{cases} \begin{aligned} S = p\{1-&(1-r)\Sigma[(\frac{k_{\min}}{k})^{\lambda-1}-(\frac{k_{\min}}{k+1})^{\lambda-1}](1-p(1-f_{ii})^k- \\ & r\Sigma[(\frac{k_{\min}}{k})^{\lambda-1}-(\frac{k_{\min}}{k+1})^{\lambda-1}](1-p(1-f_{ii}))^k e^{(m-1)Kp(f_{ij}-1)}\}, \end{aligned} \\ \begin{aligned} f_{ii} = f_{jj} =&(1-r)\frac{\Sigma[(\frac{k_{\min}}{k})^{\lambda-1}-(\frac{k_{\min}}{k+1})^{\lambda-1}]k(1-p(1-f_{ii}))^{k-1}}{\Sigma[(\frac{k_{\min}}{k})^{\lambda-1}-(\frac{k_{\min}}{k+1})^{\lambda-1}]k}+ \\ & r\frac{\Sigma[(\frac{k_{\min}}{k})^{\lambda-1}-(\frac{k_{\min}}{k+1})^{\lambda-1}]k(1-p(1-f_{ii}))^{k-1}}{\Sigma[(\frac{k_{\min}}{k})^{\lambda-1}-(\frac{k_{\min}}{k+1})^{\lambda-1}]k}e^{(m-1)Kp(f_{ij}-1)}, \end{aligned} \\ f_{ij} = f_{ji} =\Sigma[(\frac{k_{\min}}{k})^{\lambda-1}-(\frac{k_{\min}}{k+1})^{\lambda-1}](1-p(1-f_{ii}))^k e^{(m-1)Kp(f_{ij}-1)}, \end{cases} \tag{14}$$

where $\lambda_i = \lambda$, $k_{\text{inter}}^{ij} = K$ and $r_i = r$.

For the case of m modules with power-law degree distribution and power-law distribution of inter-connections, Eqs. (1) and (2) become

$$\begin{aligned} G_i(\mathbf{x}) =&(1-r_i)\Sigma[(\frac{k_{\min}}{k})^{\lambda_i-1}-(\frac{k_{\min}}{k+1})^{\lambda_i-1}]x_{ii}^k+ \\ & r_i\Sigma[(\frac{k_{\min}}{k})^{\lambda_i-1}-(\frac{k_{\min}}{k+1})^{\lambda_i-1}]x_{ii}^k\{\Sigma[(\frac{k'_{\min}}{k'})^{\lambda_{\text{inter}}^{ij}-1}-(\frac{k'_{\min}}{k'+1})^{\lambda_{\text{inter}}^{ij}-1}]x_{ij}^{k'}\}^{m-1}, \end{aligned} \tag{15}$$

$$\begin{cases} \begin{aligned} G_i^{(ii)}(\mathbf{x}) =&(1-r_i)\frac{\Sigma[(\frac{k_{\min}}{k})^{\lambda_i-1}-(\frac{k_{\min}}{k+1})^{\lambda_i-1}]kx_{ii}^{k-1}}{\Sigma[(\frac{k_{\min}}{k})^{\lambda_i-1}-(\frac{k_{\min}}{k+1})^{\lambda_i-1}]k}+ \\ & r_i\frac{\Sigma[(\frac{k_{\min}}{k})^{\lambda_i-1}-(\frac{k_{\min}}{k+1})^{\lambda_i-1}]kx_{ii}^{k-1}}{\Sigma[(\frac{k_{\min}}{k})^{\lambda_i-1}-(\frac{k_{\min}}{k+1})^{\lambda_i-1}]k}\{\Sigma[(\frac{k'_{\min}}{k'})^{\lambda_{\text{inter}}^{ij}-1}-(\frac{k'_{\min}}{k'+1})^{\lambda_{\text{inter}}^{ij}-1}]x_{ij}^{k'}\}^{m-1}, \end{aligned} \\ \begin{aligned} G_i^{(ij)}(\mathbf{x}) =&\Sigma[(\frac{k_{\min}}{k})^{\lambda_i-1}-(\frac{k_{\min}}{k+1})^{\lambda_i-1}]x_{ii}^k \\ & \{\Sigma[(\frac{k'_{\min}}{k'})^{\lambda_{\text{inter}}^{ij}-1}-(\frac{k'_{\min}}{k'+1})^{\lambda_{\text{inter}}^{ij}-1}]x_{ij}^{k'}\}^{m-2}\frac{\Sigma[(\frac{k'_{\min}}{k'})^{\lambda_{\text{inter}}^{ij}-1}-(\frac{k'_{\min}}{k'+1})^{\lambda_{\text{inter}}^{ij}-1}]k'x_{ij}^{k'-1}}{\Sigma[(\frac{k'_{\min}}{k'})^{\lambda_{\text{inter}}^{ij}-1}-(\frac{k'_{\min}}{k'+1})^{\lambda_{\text{inter}}^{ij}-1}]k'}, \end{aligned} \end{cases} \tag{16}$$

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where λ_i and $\lambda^{ij}_{\text{inter}}$ are the power-law exponents of intra-module and inter- modules respectively. Next, Eqs. (3) and (4) become

$$\begin{cases} S = p\{1-(1-r)\Sigma[(\frac{k_{\min}}{k})^{\lambda-1}-(\frac{k_{\min}}{k+1})^{\lambda-1}](1-p(1-f_{ii}))^{k}- \\ \quad r\Sigma[(\frac{k_{\min}}{k})^{\lambda-1}-(\frac{k_{\min}}{k+1})^{\lambda-1}](1-p(1-f_{ii}))^{k} \\ \quad [\Sigma[(\frac{k'_{\min}}{k'})^{\lambda_{\text{inter}}-1}-(\frac{k'_{\min}}{k'+1})^{\lambda_{\text{inter}}-1}](1-p(1-f_{ij}))^{k'}]^{m-1}\}, \\ f_{ii} = f_{jj} = (1-r)\frac{\Sigma[(\frac{k_{\min}}{k})^{\lambda-1}-(\frac{k_{\min}}{k+1})^{\lambda-1}]k(1-p(1-f_{ii}))^{k-1}}{\Sigma[(\frac{k_{\min}}{k})^{\lambda-1}-(\frac{k_{\min}}{k+1})^{\lambda-1}]k}+ \\ \quad r\frac{\Sigma[(\frac{k_{\min}}{k})^{\lambda-1}-(\frac{k_{\min}}{k+1})^{\lambda-1}]k(1-p(1-f_{ii}))^{k-1}}{\Sigma[(\frac{k_{\min}}{k})^{\lambda-1}-(\frac{k_{\min}}{k+1})^{\lambda-1}]k} \\ \quad \{\Sigma[(\frac{k'_{\min}}{k'})^{\lambda_{\text{inter}}-1}-(\frac{k'_{\min}}{k'+1})^{\lambda_{\text{inter}}-1}](1-p(1-f_{ij}))^{k'}\}^{m-1}, \\ f_{ij} = f_{ji} = \Sigma[(\frac{k_{\min}}{k})^{\lambda-1}-(\frac{k_{\min}}{k+1})^{\lambda-1}](1-p(1-f_{ii}))^{k} \\ \quad \{\Sigma[(\frac{k'_{\min}}{k'})^{\lambda_{\text{inter}}-1}-(\frac{k'_{\min}}{k'+1})^{\lambda_{\text{inter}}-1}](1-p(1-f_{ij}))^{k'}\}^{m-2} \\ \quad \frac{\Sigma[(\frac{k'_{\min}}{k'})^{\lambda_{\text{inter}}-1}-(\frac{k'_{\min}}{k'+1})^{\lambda_{\text{inter}}-1}]k'(1-p(1-f_{ij}))^{k'-1}}{\Sigma[(\frac{k'_{\min}}{k'})^{\lambda_{\text{inter}}-1}-(\frac{k'_{\min}}{k'+1})^{\lambda_{\text{inter}}-1}]k'}, \end{cases} \quad [17]$$

where $\lambda^i = \lambda$, $\lambda^{ij}_{\text{inter}} = \lambda_{\text{inter}}$. Fig. S6 shows that the numerical and simulation results agree well for above formulars.

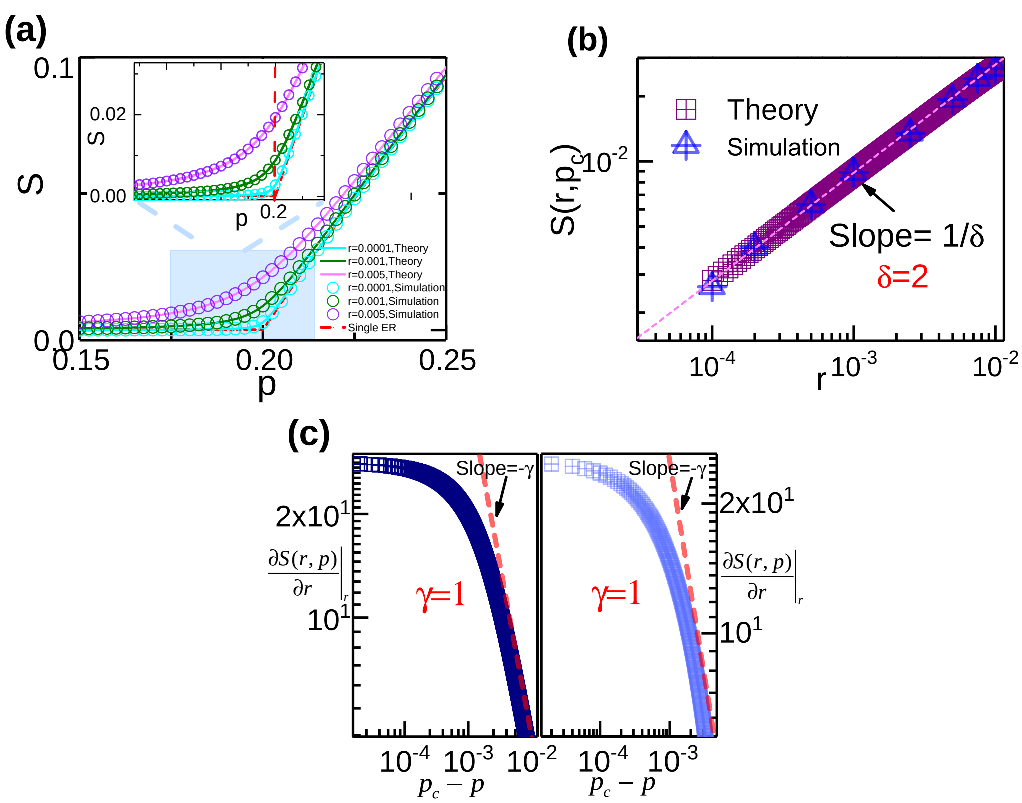

Fig. S4. (a) Comparison of analytical and simulation results for ER networks for the size of the giant component $S(r,p)$ as a function of p. (b) $S(r,p_c)$ as a function of r. (c) $\frac{\partial S(r,p)}{\partial r}$ as a function of $p_c - p$ with $r = 0.0001$. Left and right panels show the numerical and simulation results respectively. The parameters are $m = 2$, $k = 5$, $N_1 = N_2 = 10^8$, $M_{inter} = \frac{N_1}{5}$ and for simulation results we chose same parameters and averaged over 1000 realizations.

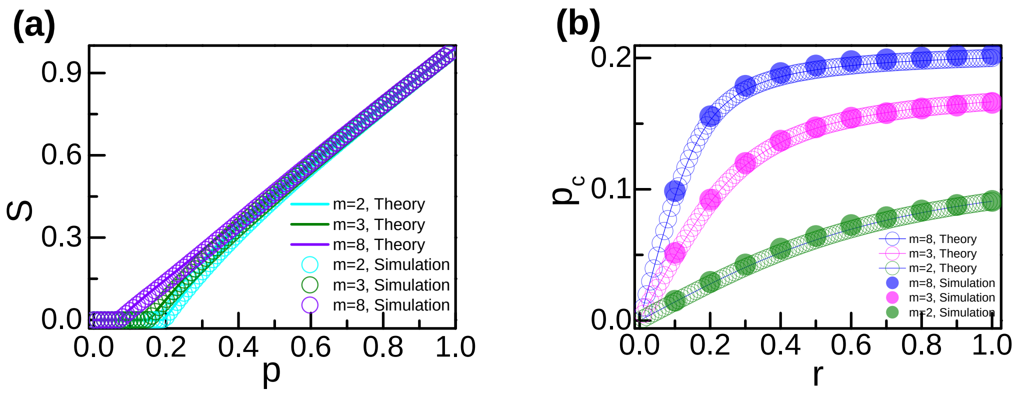

Fig. S5. (a) S as a function of p for different m with parameters $k = 4$, $r = 0.6$, $N = m \times 10^7$, $M_{\mathrm{inter}} = 10^7$ for Eq. (7). (b) p_c as a function of r for different m for Eq. (11) with same parameters as S5(a). Simulation results are averaged over 5000 realizations.

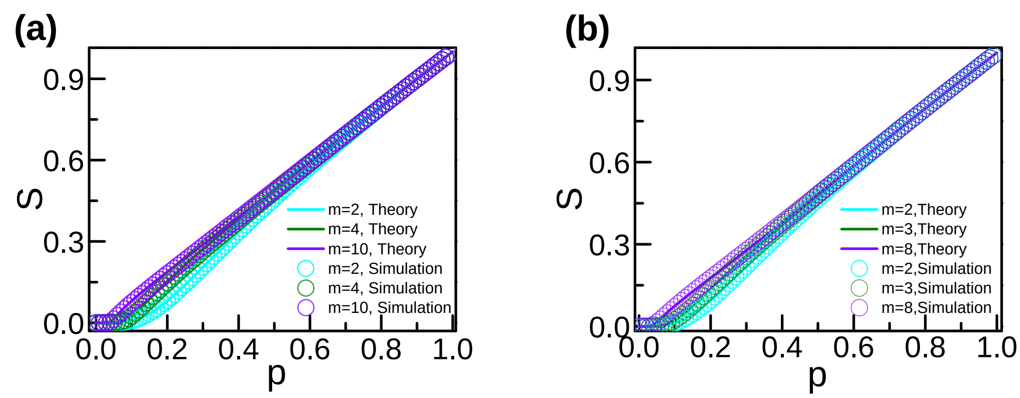

Fig. S6. (a) S as a function of p for different m with parameters $\lambda = 2.8$, $r = 0.9$, $N = m \times 10^7$, $M_{\mathrm{inter}} = 10^7$ for Eq. (14). (b) S as a function of p for different m with parameters $\lambda = 2.7$, $\lambda_{\mathrm{inter}} = 3.5$, $r = 0.8$, $N = m \times 10^7$ for Eq. (17). Simulation results are averaged over 5000 realizations.

In the main paper, two real-world co-authors networks with community structures are studied, (1) the co-author dblp collaboration network with N=317080, M=1049866 (the number of links), k=6.622089 (average degree) and (2) the co-authorship MathSciNet with N=332689, M=820644, k=4.9334. Detailed information about the largest modules of the above two networks are shown in Table S1. To simulate the external field effects in real networks, we choose the largest two modules with the same scaling exponents $\lambda = 2.8$ as our modules *1* and *2* in the co-author dblp network, in which r_1=0.11488, r_2=0.052517, $M_{inter} = 7736$, $p_{c1} = 0.01954$ and $p_{c2} = 0.02049$ for $S_{cutoff} = 0.0001$. Similarly, we choose the largest two modules with the scaling exponent $\lambda = 3.35$ as modules *3* and *4* from the co-authorship MathSciNet network, where r_1=0.12440, r_2=0.07165, $M_{inter} = 11702$, $p_{c1} = 0.01650$ and $p_{c2} = 0.02042$ for $S_{cutoff} = 0.0001$. The degree distributions of these modules (intra-links) are shown in Fig. S7. For demonstrating critical relationships in real modules of finite sizes, critical exponents δ and γ are shown in Fig. S8 and Fig. S9 by choosing different $S_{cutoff} = 0.0002, 0.0005, 0.001$. From Figs. S8 and S9, we can observe that critical exponents are independent of S_{cutoff} and have a universal feature.

i	N_i	M_i	k_i	λ_i
1	46367	133442	5.7559	2.8
2	49926	150443	6.0266	2.8
3	54620	238914	8.7482	2.95
4	9060	22809	5.0351	2.5
5	5479	11662	4.257	2.7
6	11201	25465	4.5469	2.75
7	6464	14707	4.5504	2.6
8	2713	5731	4.2248	2.9
9	54620	238914	8.7482	2.6
10	2975	6342	4.2635	2.9
11	4170	9331	4.4753	3
12	4098	9583	4.6769	2.75
13	11048	24714	4.4739	2.65
14	4556	11862	5.2072	2.9
15	2555	5512	4.3147	2.7
16	15376	52348	6.8091	2.6
17	1183	2714	4.5883	2.7
18	3242	7212	4.4491	2.7
19	6276	13967	4.4509	2.7
20	3146	6248	3.972	2.7
21	3096	7422	4.7946	2.7
22	2025	4221	4.1689	2.7
23	1232	2819	4.5763	2.6
24	2247	4540	4.0409	2.6
25	1172	2016	3.4403	2.7
26	1523	3898	5.1188	2.6

i	N_i	M_i	k_i	λ_i
1	51326	144176	5.618	3.35
2	65805	204983	6.23	3.35
3	52026	110275	4.2392	3.45
4	30445	57917	3.8047	3.4
5	8154	13182	3.2333	3.1
6	10680	17357	3.2504	3.1
7	4688	7326	3.1254	2.6
8	6680	10550	3.1587	2.6
9	4796	6914	2.8832	3.1
10	1923	3248	3.3781	3.1
11	2391	3816	3.192	3.1
12	1169	1772	3.0317	3.1
13	2198	3670	3.3394	2.8
14	1140	1827	3.2053	2.8
15	1607	2370	2.9496	2.8
16	1378	2292	3.3266	2.8
17	3138	5313	3.3862	2.8
18	1133	2127	3.7546	2.9
19	2304	3779	3.2804	2.7
20	1576	2422	3.0736	2.7
21	2286	3724	3.2581	2.7
22	1406	2199	3.128	2.7
23	1043	1589	3.047	2.7
24	1085	1847	3.4046	2.6
25	1573	2095	2.6637	2.7
26	1352	2575	3.8092	3.2

Table S1. Modules with more than 10^3 nodes in the co-author dblp collaboration network and co-authorship MathSciNet network are discribed in the left and right table. N_i, M_i are the numbers of nodes and links respectively, k_i, λ_i are average degree and scaling exponent in module i respectively. Modules *1*, *2*, *3* and *4* are marked by orange background.

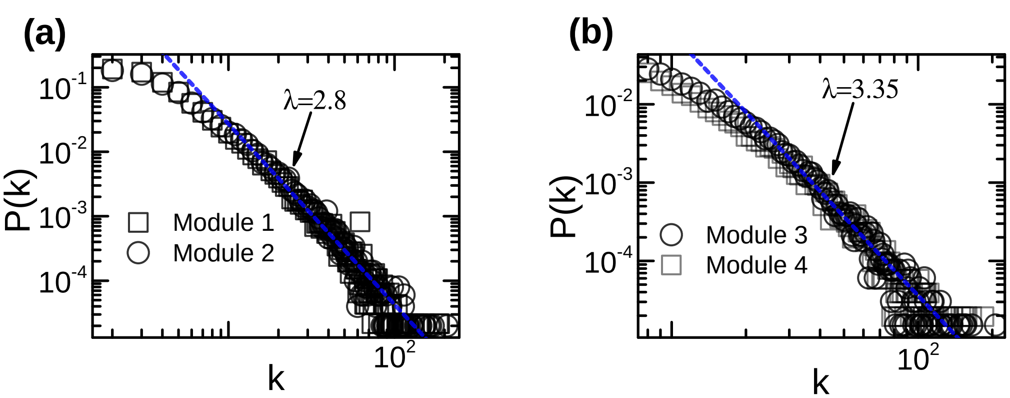

Fig. S7. (a) The degree distributions of modules *1* and *2* within dblp collaboration network. (b) The degree distributions of modules *3* and *4* within MathSciNet collaboration network.

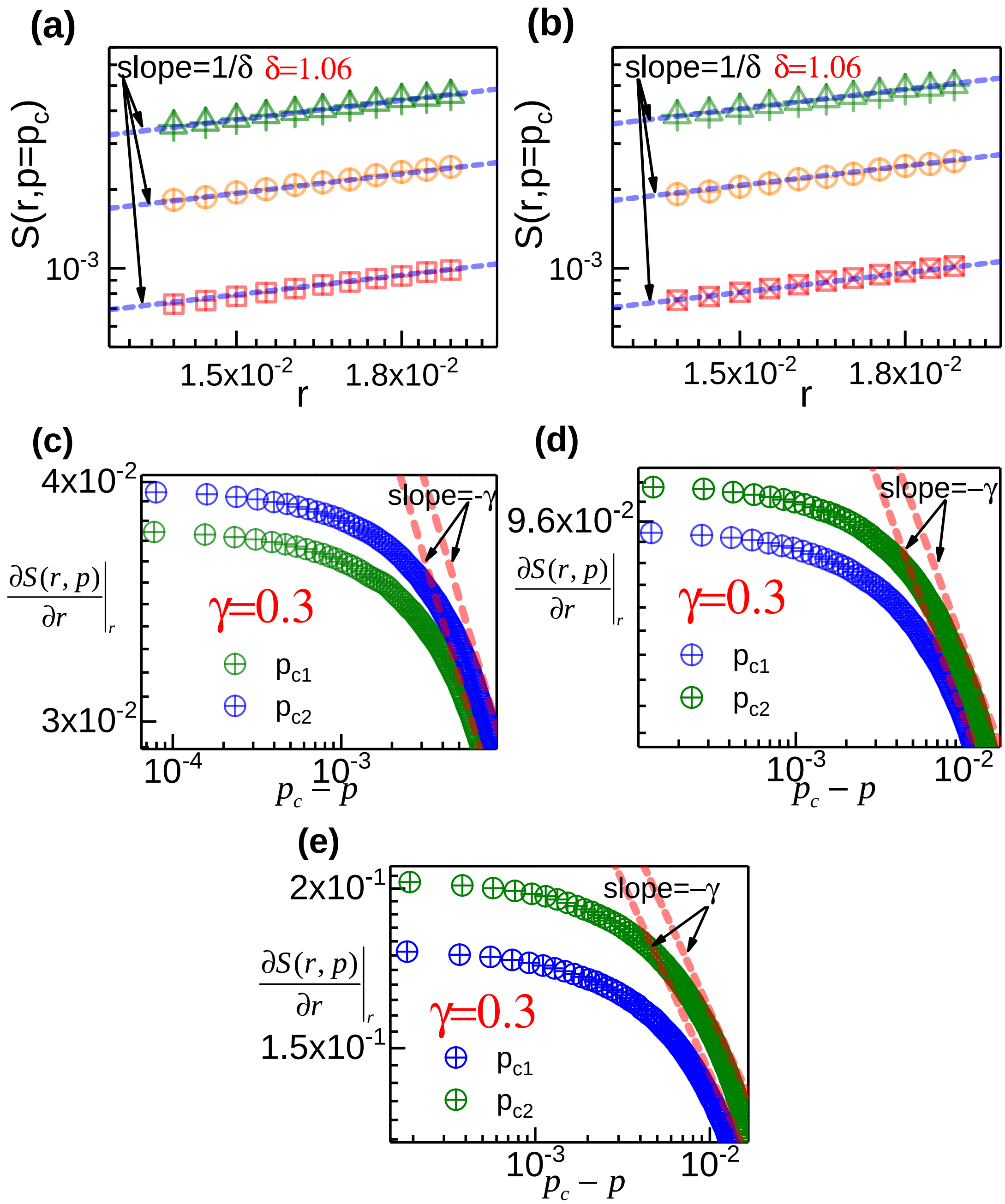

Fig. S8. Critical scaling and exponents for two modules in co-author dblp collaboration network with $\lambda = 2.8$ and $M_{inter} = 4 \times 10^5$. $S(r, p_c)$ as a function of r for the module 1 (a) and module 2 (b) with $S_{cutoff} = 0.0002$ (red squares), $S_{cutoff} = 0.0005$ (orange circles) and $S_{cutoff} = 0.001$ (green triangles). $\frac{\partial S(r,p)}{\partial r}$ as a function of $p_c - p$ for $S_{cutoff} = 0.0002$ (c), $S_{cutoff} = 0.0005$ (d) and $S_{cutoff} = 0.001$ (e). We average over 2000 realizations for each network.

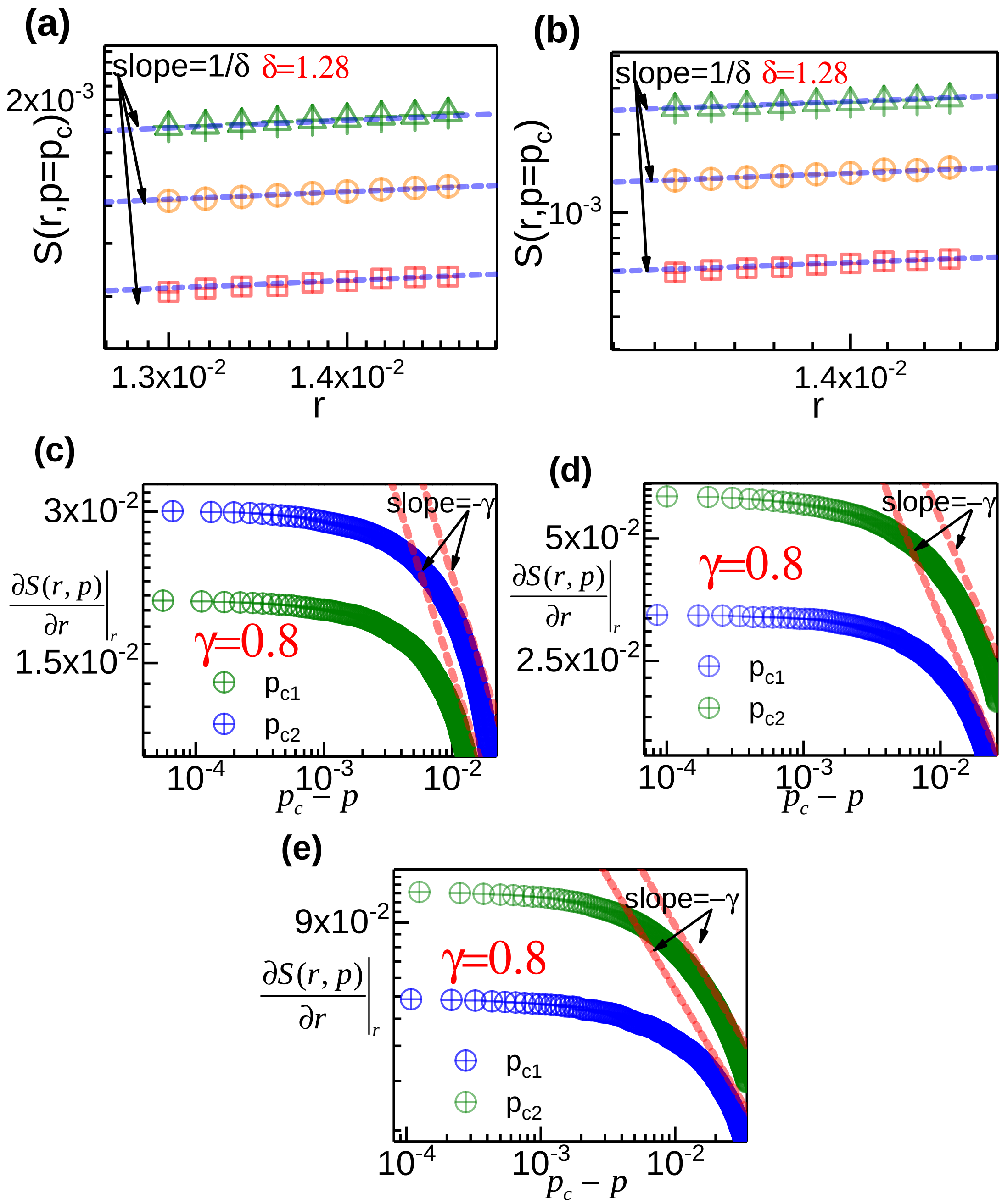

Fig. S9. Critical scaling and exponents for two modules in the co-author MathSciNet collaboration network with $\lambda = 3.35$ and $M_{inter} = 5 \times 10^5$. $S(r, p_c)$ as a function of r for the module 1 (a) and module 2 (b) with $S_{cutoff} = 0.0002$ (red squares), $S_{cutoff} = 0.0005$ (orange circles) and $S_{cutoff} = 0.001$ (green triangles). $\frac{\partial S(r,p)}{\partial r}$ as a function of $p_c - p$ for $S_{cutoff} = 0.0002$ (c), $S_{cutoff} = 0.0005$ (d) and $S_{cutoff} = 0.001$ (e). We average over 2000 realizations for each network.

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