Assignment 2

Due: Thursday, October 26 at 4:00pm

1. This first problem consists of two separate questions about the fidelity function.

   (a) This one is inspired by a question asked in the lecture (for which I did not know an answer on the spot). Give an example of two states $\rho, \sigma \in \mathrm{D}(\mathcal{X})$, for your choice of a complex Euclidean space $\mathcal{X}$, such that

   $$1 - \frac{1}{2}\|\rho - \sigma\|_1 = \mathrm{F}(\rho, \sigma) < \sqrt{1 - \frac{1}{4}\|\rho - \sigma\|_1^2}.$$

   Such an example shows that the first of the Fuchs–van de Graaf inequalities is tight, and not just in the trivial case when ρ and σ are either equal or orthogonal (where all three expressions are all equal, either to 1 or to 0, respectively).

   Hint: consider the proof of the first Fuchs–van de Graaf inequality, and think about what is needed to force the inequality to be an equality.

   (b) Let $\mathcal{X}$ be a complex Euclidean space, and define the *fidelity distance* between any two states $\rho, \sigma \in \mathrm{D}(\mathcal{X})$ as

   $$d_F(\rho, \sigma) = \min\bigl\{\|u - v\| \,:\, u, v \in \mathcal{X} \otimes \mathcal{Y},\ \mathrm{Tr}_{\mathcal{Y}}(uu^*) = \rho,\ \mathrm{Tr}_{\mathcal{Y}}(vv^*) = \sigma\bigr\}.$$

   Here, you should assume that $\mathcal{Y}$ is a complex Euclidean space with $\dim(\mathcal{Y}) = \dim(\mathcal{X})$ (although the dimension of $\mathcal{Y}$ does not actually matter, so long as it is large enough to allow for the existence of purifications of ρ and σ). Prove that

   $$d_F(\rho, \sigma) = \sqrt{2 - 2\,\mathrm{F}(\rho, \sigma)}.$$

   Also prove that the fidelity distance obeys the triangle inequality:

   $$d_F(\rho, \sigma) \leq d_F(\rho, \xi) + d_F(\xi, \sigma)$$

   for all $\rho, \sigma, \xi \in \mathrm{D}(\mathcal{X})$.

2. This problem is concerned with an extension of Theorem 3.9 in the text. A formal statement of the problem is as follows.

   Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Euclidean spaces, let $H \in \mathrm{Herm}(\mathcal{Y} \otimes \mathcal{X})$ be a Hermitian operator, and let $\Phi \in \mathrm{C}(\mathcal{X}, \mathcal{Y})$ be a channel. Prove that these two statements are equivalent:

   Statement 1. It holds that

   $$\langle H, J(\Phi)\rangle = \max_{\Psi \in \mathrm{C}(\mathcal{X}, \mathcal{Y})} \langle H, J(\Psi)\rangle.$$

   Statement 2. The operator $\mathrm{Tr}_{\mathcal{Y}}(HJ(\Phi))$ is Hermitian and satisfies

   $$\mathbb{1}_{\mathcal{Y}} \otimes \mathrm{Tr}_{\mathcal{Y}}(HJ(\Phi)) \geq H.$$

The short discussion that follows is not needed to solve the problem, but is only intended to motivate it. Suppose that $\psi : \mathrm{C}(\mathcal{X}, \mathcal{Y}) \to \mathbb{R}$ is an arbitrary linear function from the set of channels $\mathrm{C}(\mathcal{X}, \mathcal{Y})$ to the real numbers. It can be proved, for any such choice of a function ψ, that there must exist a Hermitian operator $H \in \mathrm{Herm}(\mathcal{Y} \otimes \mathcal{X})$ such that $\psi(\Phi) = \langle H, J(\Phi) \rangle$ for all $\Phi \in \mathrm{C}(\mathcal{X}, \mathcal{Y})$. The aim of the problem above is therefore to prove that there is a simple criterion (represented by statement 2) that allows one to easily check, for a given channel Φ, whether or not Φ is an optimal channel for maximizing the function ψ, meaning that

$$\psi(\Phi) = \max_{\Psi \in \mathrm{C}(\mathcal{X}, \mathcal{Y})} \psi(\Psi),$$

which is equivalent to statement 1.

3. Suppose $\mathcal{X}$ and $\mathcal{Y}$ are complex Euclidean spaces, $P, Q \in \mathrm{Pos}(\mathcal{X})$ are positive semidefinite operators, and $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ is a trace-preserving and positive (but not necessarily completely positive) map. Prove that
$$\mathrm{F}(P, Q) \leq \mathrm{F}(\Phi(P), \Phi(Q)).$$

   We already know that the inequality holds for Φ being a channel, but the proof we discussed in lecture relies on Φ being completely positive, so that proof will not work here. However, if you use the right characterization of the fidelity, the required inequality can be proved in a different way that only requires Φ to be positive and trace preserving.

4. Let X, Y, and Z be registers.

   (a) Prove that, for every state $\rho \in \mathrm{D}(\mathcal{X} \otimes \mathcal{Y} \otimes \mathcal{Z})$ of these registers, it holds that
   $$\mathrm{I}(\mathsf{X}, \mathsf{Y} : \mathsf{Z}) \leq \mathrm{I}(\mathsf{Y} : \mathsf{X}, \mathsf{Z}) + 2\,\mathrm{H}(\mathsf{X}).$$

   Hint: you do not need strong subadditivity to prove this inequality.

   (b) Give an example of a state ρ for which the inequality in part (a) is an equality. In order to disqualify trivial examples, be sure that your example is such that $\mathrm{H}(\mathsf{X})$ is nonzero.

   Hint: thinking about dense coding may help you to find a simple example!