# Assignment 4

Due: Tuesday, December 5 at 4:00pm

1. Let $\mathsf{X}$ and $\mathsf{Y}$ be registers and let $\rho \in \mathrm{D}(\mathcal{X} \otimes \mathcal{Y})$ be a state of the pair $(\mathsf{X}, \mathsf{Y})$. With respect to $\rho$, one defines the *entanglement of formation* between $\mathsf{X}$ and $\mathsf{Y}$ as

$$\mathrm{E_F}(\mathsf{X} : \mathsf{Y}) = \inf \left\{ \sum_{a \in \Sigma} p(a)\, \mathrm{H}\big(\mathrm{Tr}_{\mathcal{Y}}(u_a u_a^*)\big) \;:\; \sum_{a \in \Sigma} p(a)\, u_a u_a^* = \rho \right\},$$

where the infimum is over all choices of an alphabet $\Sigma$, a probability vector $p \in \mathcal{P}(\Sigma)$, and a collection of unit vectors $\{u_a : a \in \Sigma\} \subset \mathcal{X} \otimes \mathcal{Y}$ for which it holds that

$$\sum_{a \in \Sigma} p(a)\, u_a u_a^* = \rho.$$

Now suppose that $\mathsf{Z}$ and $\mathsf{W}$ are registers and $\Phi \in \mathrm{C}(\mathcal{X} \otimes \mathcal{Y}, \mathcal{Z} \otimes \mathcal{W})$ is a channel that can be expressed as

$$\Phi(X) = \sum_{b \in \Gamma} (A_b \otimes B_b) X (A_b \otimes B_b)^*$$

for all $X \in \mathrm{L}(\mathcal{X} \otimes \mathcal{Y})$, for some collection of isometries $\{A_b : b \in \Gamma\} \subset \mathrm{U}(\mathcal{X}, \mathcal{Z})$ and a collection of operators $\{B_b : b \in \Gamma\} \subset \mathrm{L}(\mathcal{Y}, \mathcal{W})$ satisfying

$$\sum_{b \in \Gamma} B_b^* B_b = \mathbb{1}_{\mathcal{Y}}.$$

(Thus, $\Phi \in \mathrm{LOCC}(\mathcal{X}, \mathcal{Z} : \mathcal{Y}, \mathcal{W})$ is a one-way left LOCC channel.) Prove that

$$\mathrm{E_F}(\mathsf{Z} : \mathsf{W})_{\Phi(\rho)} \leq \mathrm{E_F}(\mathsf{X} : \mathsf{Y})_{\rho}$$

where $\mathrm{E_F}(\mathsf{X} : \mathsf{Y})_{\rho}$ and $\mathrm{E_F}(\mathsf{Z} : \mathsf{W})_{\Phi(\rho)}$ denote the entanglement of formation of the pairs $(\mathsf{X}, \mathsf{Y})$ and $(\mathsf{Z}, \mathsf{W})$ with respect to the states $\rho$ and $\Phi(\rho)$, respectively.

Once the above inequality has been established, it is not difficult to conclude that it holds not only for channels $\Phi$ of the form described above, but for all LOCC channels $\Phi$. You do not need to prove this as a part of your solution.

(Challenge problem.) Prove the stronger claim that the inequality

$$\mathrm{E_F}(\mathsf{Z} : \mathsf{W})_{\Phi(\rho)} \leq \mathrm{E_F}(\mathsf{X} : \mathsf{Y})_{\rho}$$

holds for all separable channels $\Phi \in \mathrm{SepC}(\mathcal{X}, \mathcal{Z} : \mathcal{Y}, \mathcal{W})$.

2. Let $\Sigma$ be an alphabet, let $n = |\Sigma|$, and assume $n \geq 2$. Also let $\mathcal{X} = \mathbb{C}^\Sigma$ and $\mathcal{Y} = \mathbb{C}^\Sigma$, and recall that the *swap operator* $W \in \mathrm{L}(\mathcal{X} \otimes \mathcal{Y})$, which satisfies $W(x \otimes y) = y \otimes x$ for all $x, y \in \mathbb{C}^\Sigma$, may alternatively be defined as

$$W = \sum_{a,b \in \Sigma} E_{a,b} \otimes E_{b,a}.$$

Define projections $\Pi_0, \Pi_1 \in \mathrm{Proj}(\mathcal{X} \otimes \mathcal{Y})$ and states $\sigma_0, \sigma_1 \in \mathrm{D}(\mathcal{X} \otimes \mathcal{Y})$ as follows:

$$\Pi_0 = \frac{1}{2}\mathbb{1} \otimes \mathbb{1} + \frac{1}{2}W, \qquad \Pi_1 = \frac{1}{2}\mathbb{1} \otimes \mathbb{1} - \frac{1}{2}W, \qquad \sigma_0 = \frac{1}{\binom{n+1}{2}}\Pi_0, \qquad \sigma_1 = \frac{1}{\binom{n}{2}}\Pi_1.$$

Prove that if $\mu : \{0,1\} \to \mathrm{Pos}(\mathcal{X} \otimes \mathcal{Y})$ is a PPT measurement, meaning that $\mu$ is a measurement and $\mu(0), \mu(1) \in \mathrm{PPT}(\mathcal{X} : \mathcal{Y})$, then

$$\frac{1}{2}\langle \mu(0), \sigma_0 \rangle + \frac{1}{2}\langle \mu(1), \sigma_1 \rangle \leq \frac{1}{2} + \frac{1}{n+1}.$$

(Thus, PPT measurements are not very good at discriminating between $\sigma_0$ and $\sigma_1$, even though they are orthogonal.)

3. Let $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ be a map, for complex Euclidean spaces $\mathcal{X}$ and $\mathcal{Y}$. Prove that

$$|||\Phi|||_1 = \max_{\rho_0, \rho_1 \in \mathrm{D}(\mathcal{X})} \big\| \big(\mathbb{1}_{\mathcal{Y}} \otimes \sqrt{\rho_0}\big) J(\Phi) \big(\mathbb{1}_{\mathcal{Y}} \otimes \sqrt{\rho_1}\big) \big\|_1.$$

4. Let $\mathcal{X}$ be a complex Euclidean space, let $n = \dim(\mathcal{X})$, and let $\mu$ denote the uniform spherical measure on $\mathcal{S}(\mathcal{X})$.

   (a) Define a mapping $\Phi \in \mathrm{CP}(\mathcal{X})$ as

   $$\Phi(X) = n \int \langle uu^*, X \rangle uu^* \,\mathrm{d}\mu(u)$$

   for all $X \in \mathrm{L}(\mathcal{X})$. Give a simpler expression for $\Phi$. Your expression should describe $\Phi$ as a convex combination of channels that we have already encountered many times in this course.

   (b) Define a channel $\Xi \in \mathrm{C}(\mathcal{X}, \mathcal{X} \otimes \mathcal{X})$ as

   $$\Xi(X) = n \int \langle uu^*, X \rangle uu^* \otimes uu^* \,\mathrm{d}\mu(u)$$

   for all $X \in \mathrm{L}(\mathcal{X})$. This channel might seem like it is good for cloning pure states. Calculate the value

   $$\inf_{v \in \mathcal{S}(\mathcal{X})} \big\langle vv^* \otimes vv^*, \Xi(vv^*) \big\rangle,$$

   which quantifies how good $\Xi$ is as a pure state cloner.
   (It so happens that $\Xi$ is a sub-optimal cloning channel, in the sense of Theorem 7.28, aside from the trivial case in which $\dim(\mathcal{X}) = 1$.)