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**Problem 1 (3 points)** Given

$$\tilde{C}(z)=\frac{1}{z+\alpha+\tilde{M}(z)},$$

and

$$\tilde{M}(z)=-\frac{\tilde{M}^{\mathrm{irr}}(z)}{1+\alpha^{-1}\tilde{M}^{\mathrm{irr}}(z)}.$$

(Ref. Pitts2000b,JCP. 113, 8671) Prove that $C(t)$ satisfies the following equation.

$$\frac{dC(t)}{dt}+\alpha C(t)+\frac{1}{\alpha}\int_0^t d\tau M^{\mathrm{irr}}(t-\tau)\frac{dC(\tau)}{d\tau}=0.$$

**Proof**:
One can obtain the relation between $\tilde{M}^{\mathrm{irr}}(z)$ and $\tilde{C}(z)$:

$$\frac{\tilde{M}^{\mathrm{irr}}(z)}{1+\alpha^{-1}\tilde{M}^{\mathrm{irr}}(z)}=\frac{(z+\alpha)\tilde{C}(z)-1}{\tilde{C}(z)}.$$

i.e.,

$$\tilde{C}(z)+\frac{1}{\alpha}(z\tilde{C}(z)-C(t=0))+\frac{1}{\alpha^2}M^{\mathrm{irr}}(z)(z\tilde{C}(z)-C(t=0))=0.$$

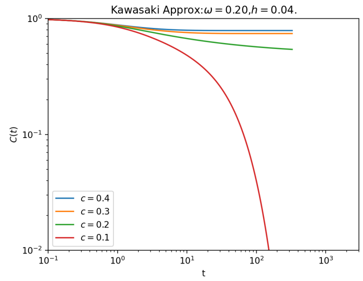
After performing inverse Laplace transform, one obtain the dynamical equation (The initial condition $C(t=0)=1$ is used). Proved.

Note: we have used the following properties of the Laplace transform:

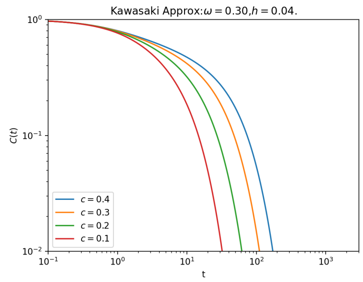
$$\begin{aligned}
&\alpha f(t)\rightarrow\alpha\tilde{F}(z);\\
&\frac{df(t)}{dt}\rightarrow z\tilde{F}(z)-f(0);\\
&\int_0^t f(t-\tau)g(\tau)d\tau\rightarrow\tilde{F}(z)\tilde{G}(z).
\end{aligned}$$

**Problem 2 (3 points)** For different $\omega$ ( $\alpha$ is also denoted as $\omega$ ), solve this integral-differential equation.
**Solution:**
(1) Kawasaki Approximation

$\omega = 0.2$ :

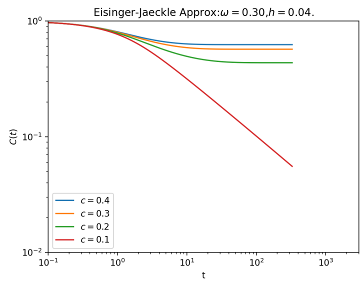



$\omega = 0.3$ :




(2) Eisinger-Jaeckle Approximation

$\omega = 0.3$ :




Discussions：变化趋势与MC模拟的结果不同。原因是什么？最开始我不知道。
但我意识到 $\alpha$ 其实是依赖于 $c$ 或者 $\beta$ 的.
后经金老师提醒, $\alpha$ 与 $c$ 之间可能存在简单的函数关系。

2022年4月28号，我们从文献上找到了这个函数关系. 对一维East模型，有

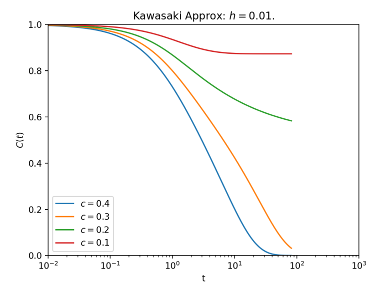
$$\alpha = c.$$

这就找到了上面这些图错误的原因！做这些图时，我错误的认为 $\alpha$ 是不依赖于 $\beta$ 的参数了。

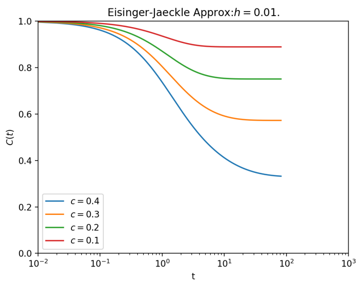
**Solution 2b**

意识到 $\alpha$ 依赖于 $c$ 以后，我重新解方程。得到了于Kawasaki和Eisinger-Jaeckle同样的结果！

(1) Kawasaki Approximation.



(2) EJ approximation.




**Summary**:

a. Now, we have obtained the same solution as Kawasaki and EJ approximation.

b. For the East model, the relation $\alpha = c$ is correct.

**Problem 3 (3 points)**

Does the Laplace transform change the dimension of a dynamical variable $f(t)$ ?

**Solution**:

The Laplace transform

$$\tilde{F}(z) = \int_0^\infty e^{-zt} f(t) dt.$$

We find a general property for the dimensions of $f(t)$ and its Laplace transform $\tilde{F}(z)$ .

**Property 1**

The relation between the dimensions of $\tilde{F}(z)$ and $f(t)$ is:

$$\dim \tilde{F}(z) = (\dim t)(\dim f(t)).$$

For example,

$$\dim C(t) = \dim 1,$$

The dimension of $C(t)$ is pure number. Someone call the quantity like $C(t)$ a "无量纲量". But I prefer to say "its dimension is number", denoted as $\dim 1$ or simply 1. Therefore,

$$\dim \tilde{C}(z) = (\dim t)(\dim 1) = \dim t.$$

(In my opinion, the dimension of a quantity can be described by an abstract mathematical object. It can be represented by a number, a vector or a matrix.)

$$\tilde{C}(z) = \frac{1}{z + \alpha + \tilde{M}(z)},$$

where $\dim \alpha = \dim \tilde{M}(z) = \dim z = \dim t^{-1}.$

From

$$\tilde{M}(z) = -\frac{\tilde{M}^{\mathrm{irr}}(z)}{1 + \alpha^{-1} \tilde{M}^{\mathrm{irr}}(z)}$$

We obtain

$$\dim \tilde{M}^{\text{irr}}(z) = \dim \tilde{M}(z) = \dim z = \dim t^{-1} = T^{-1}.$$

While, from

$$\frac{dC(t)}{dt} + \alpha C(t) + \frac{1}{\alpha}\int_0^t d\tau M^{\text{irr}}(t-\tau)\frac{dC(\tau)}{d\tau} = 0.$$

we get

$$\dim M^{\text{irr}}(t) = \dim \alpha^2 = \dim t^{-2} = T^{-2}.$$

Therefore, the answer is: Yes!

Following **Property 1**, we obtain

$$\dim \tilde{M}^{\text{irr}}(z) = (\dim M^{\text{irr}}(t))(\dim t) = \dim t^{-1} = T^{-1}.$$

**Problem 4 (1 point)**
What is the dimension of the coefficient $M^{\text{irr}}(t)$ ? Is it the same as the dimension of $\tilde{M}^{\text{irr}}(z)$ ?

**Solution:**
$\dim \tilde{M}^{\text{irr}}(z) = \dim t^{-1}$, and $\dim M^{\text{irr}}(t) = \dim t^{-2}$. Their dimensions are different.