Auto-correlations of spins in East Model

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East Model is an example of kinetically constrained Ising model.

Our model consists kinetically constrained Ising spins, $n_i = 0$ or 1 on a 1-D lattice, with PBC.
No static interactions between spins. (**Eisinger, Jaeckle 1993; Faggionato, Martinelli, Roberto, and Toninelli 2012**)
A given spin may only flip if the neighboring spin to the right is up, with an acceptance ratio A,

$$A = \begin{cases} e^{-\beta}, & \text{for } 01 \rightarrow 11 \\ 1, & \text{for } 11 \rightarrow 01. \end{cases} \tag{1}$$

East model shows glassy dynamics. (**Pitts and Andersen, 2001**)
Define auto correlation function

$$C(t) = \frac{\langle \delta n_i(t) \delta n_i(0) \rangle}{\langle \delta n_i(0)^2 \rangle}. \tag{2}$$

$\langle \cdots \rangle$: Average for the equilibrium state,
n_i: occupation number of site i.

Relation between β and c

Assumption (A simplifying case):

- The spins are statistically independent at equilibrium;
- All **sites** have the **same distribution function** for the occupation number, ie, $\forall i, t = \infty, P(n_i = 0) = 1 - e^{-\beta}, P(n_i = 1) = e^{-\beta}$.

Relation between β and c

Denote $c = \langle n_i \rangle_{\text{eq}}$ (**Eisinger and Jaeckle, 1991**),

$$c = \frac{e^{-\beta}}{1 + e^{-\beta}} \Rightarrow \beta = \ln \frac{c}{1 - c} \text{ for } c < 1/2. \tag{3}$$

(**Wu Jianlan, 2004, JPC**)

$$\begin{aligned} c &= 0.10 \rightleftarrows \beta = 2.20 \\ c &= 0.20 \rightleftarrows \beta = 1.39 \\ c &= 0.30 \rightleftarrows \beta = 0.84 \\ c &= 0.48 \rightleftarrows \beta = 0.05 \end{aligned} \tag{4}$$

$C(t)$ for 1-dimensional East model

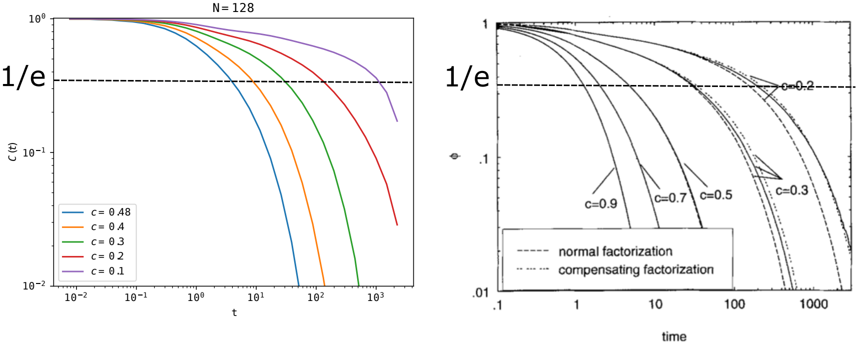

Figure: $C(t)$ from MC simulations and from **Eisinger and Jaeckle (1993)**.

Exact $C(t)$ for 1-dimensional East model

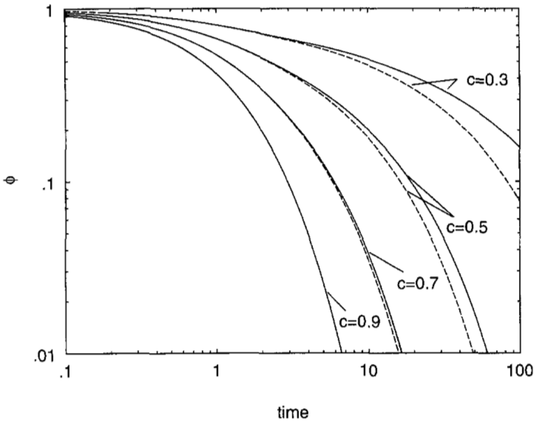

Figure: Exact $C(t)$ (solid lines) from numerical results for finite chains (**Eisinger and Jaeckle, 1991)**.

β-dependence of $C(t)$

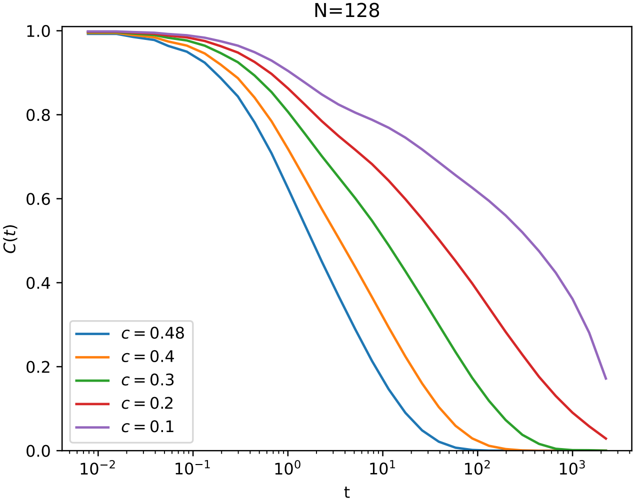

Figure: t-dependence and β-dependence of $C(t)$

β-dependence of relaxation time τ

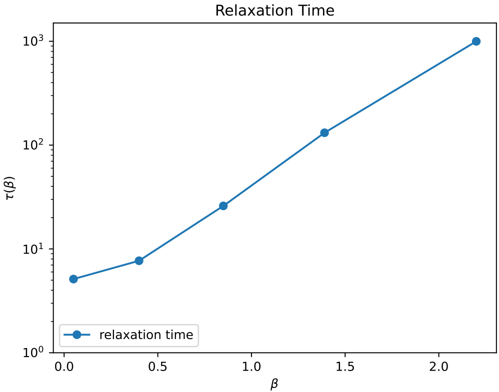

Figure: β-dependence of $C(t)$. $N = 128$.

c-dependence of relaxation time τ

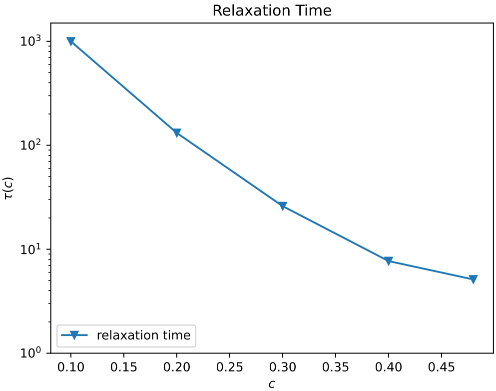

Figure: c-dependence of $C(t)$. $N = 128$.

c-dependence of $C(t)$.

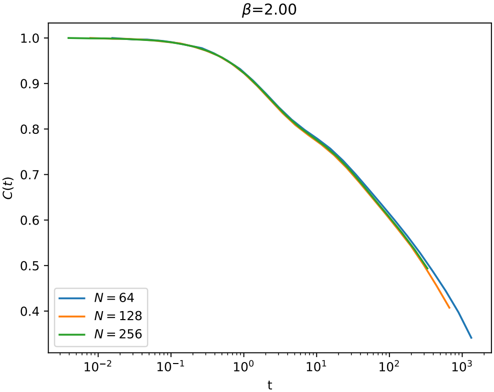

Figure: c-dependence of $C(t)$.

For $C(t)$:

- $\tau(\beta) \sim$

For $C(t)$:

- $\phi(t) = \frac{\langle \delta n_i(t) \delta n_i(0) \rangle}{c(1-c)}$ (**Eisinger and Jaeckle, 1991**, where $\sigma_i = \pm 1$).
- $C(t) = \frac{\langle \delta n_i(t) \delta n_i(0) \rangle}{\langle \delta n_i(0)^2 \rangle}$. What is the difference between $C(t)$ and $\phi(t)$? $n_i = \sigma_i = 0$ or 1.

Theoretical object: to derive an equation of $C(t)$ (kinetic theory of $C(t)$)

In general,

$$\frac{dC(t)}{dt} + \omega C(t) + \omega^{-1} \int_0^t d\tau M^{\text{irr}}(t-\tau) \frac{dC(\tau)}{d\tau} = 0, \qquad (5)$$

where $\omega = k(0)$, $k(t) = -\frac{dC(t)}{dt}$, M^{irr} is irreducible memory function.
In MCT, assume that

$$M^{\text{irr}}(t) = \sum_n a_n(c)[C(t)]^n. \qquad (6)$$

Some approximations:

- $M_{\text{K}}^{\text{irr}}(t) = c(1-c)[C(t)]^2$ (**Kawasaki1995**)
- $M_{\text{EJ}}^{\text{irr}}(t) = c(1-c)C(t)$ (**Eisinger and Jaeckle 1993**)
- diagrammatic representations of the irreducible memory function (**Pitts and Andersen, 2001**).

More about $C(t)$: Detailed balance

Let the state $\alpha = (s_1, s_2, \cdots, s_N)$. In equilibrium, $\frac{\partial P_\alpha(t)}{\partial t} = 0 \Rightarrow P_\alpha(t) W_{\alpha\to\beta} = P_\beta(t) W_{\beta\to\alpha}$, where

- $P_\alpha(t)$ is the probability of the chain being in state α.
- From detailed balance, $W_{a\to b} = e^{-\beta}$; $W_{b\to a} = 1 \Rightarrow$ $P_b(t)/P_a(t) = e^{-\beta}$ (see Figure).

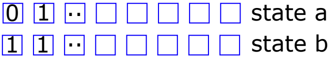

Figure: Two states of the 1-D finite East model with N sites.

More about $C(t)$: Detailed balance condition (In general)

We now restrict attention to systems for which there is a stationary distribution function $\rho_0(\Gamma)$ s.t. the transition probabilities obey the DBC (**Pitts and Andersen, 2001**):

$$W(\Gamma', \Gamma)\rho_0(\Gamma) = W(\Gamma, \Gamma')\rho_0(\Gamma'). \tag{7}$$