Problems: Week 2

** 1. Prove by induction on $M \in \Lambda$ that $N \in \mathsf{sub}(M)$ implies $N \sqsubseteq M$.
Hint: $N \in S_1 \cup S_2 \cup S_2$ can be understood as a *disjunction* of three possibilities for sets that contain N.

** 2. Prove, by induction on $M \in \Lambda$, that if $x \neq y$ and $x \notin \mathsf{FV}(Q)$ then:

$$M[P/x][Q/y] = M[Q/y][P[Q/y]/x]$$

Hint: in the variable case, there are three subcases to consider.

* 3. Perform the following substitutions:
   a) $(\lambda x.(\lambda y.xz)z)[(\lambda z.z)/z]$
   b) $(\lambda x.yx)[yz/x]$
   c) $(\lambda y.xy)[yx/x]$

* 4. Perform one step of reduction for each of the following terms:
   a) $(\lambda xy.x)(\lambda x.x)(\lambda z.z)$
   b) $(\lambda xyz.xz(yz))(\lambda xy.x)$
   c) $(\lambda x.xx)(\lambda x.xx)$
   d) $(\lambda xy.x)((\lambda z.zz)(\lambda x.x))$

* 5. Draw the reduction graph of the term $(\lambda xy.yy)((\lambda z.zz)(\lambda z.zz))(\lambda x.x)$.

** 6. Give an example of a *closed* term M for each of the following properties:
   a) M is in β-normal form.
   b) M has exactly one reduct.
   c) M has 2 or more reducts.
   d) M contains fewer redexes than one of its reducts.
   e) A reduct of M contains a redex that did not occur anywhere in M.

** 7.
   a) State the induction principle associated with the inductively defined set $\rightarrow_\beta$, which is a set of pairs of λ-terms.
   b) Prove, by induction on $M \rightarrow_\beta N$ that:
      if $M \rightarrow_\beta N$ then $\mathsf{FV}(N) \subseteq \mathsf{FV}(M)$.
      You may omit one of the cases (AppL) or (AppR) because it is similar to the other.
   c) Give an example of a step $M \rightarrow_\beta N$ where $\mathsf{FV}(M) \neq \mathsf{FV}(N)$.