A Quick Guide to Writing Proofs

Proofs are semi-formal arguments that make a case for the truth of some precise statement. The key word as far as proofs are concerned is *convincing*: a proof is a way of communicating your belief that something is true in a way that will convince another party.

Some proofs are more convincing than others. In mathematics, the statements that we want to give proofs for only have certain shapes — they are built from a few, standard logical operators. Over the years, mathematicians have developed equally standard approaches to dealing with the logical operators in proofs. By following these standard approaches, the benefit is that you will end up with a proof that will be convincing. Proofs of this kind are typically blocks of English text with a scattering of mathematical formulas.

The way that I recommend you approach this guide is to start by very quickly reading through the sections on "Assumptions", "Forward reasoning" and "Backward reasoning". These three sections describe the rules of the game, and we all know that just reading through a rulebook is of limited value if you have never tried to play the game. Next, spend some time trying to understand the section "Playing the game". This gives practical advice on how to read and write proofs based on a few examples. You will want to refer back to the rules whilst trying to understand this section. Try the exercises at the end of this section. If you are happy with them, continue by reading "The excluded middle" and, finally, the section on "Theory reasoning". This guide is a bit experimental and I am keen to hear how you get on with it, so please let me know if something doesn't make sense to you because I would like to improve it.

1 Assumptions

Proofs never happen in a vacuum. There are almost always facts that are available to you to use in a proof. For example, lemmas that you have proven previously, or axioms from the particular system you are working in. I will refer to all these kinds of facts collectively as *assumptions* (they are sometimes also referred to as hypotheses). Part of the process of proving involves temporarily assuming propositions, so, whilst writing a proof, the list of assumptions changes. For this reason, when thinking about a proof it is important to have in mind *two* separate entities: the formula you want to prove and the assumptions that you are allowed to use when proving it.

The most important thing to know about an assumption is that you can conclude it. In particular, if you aim to prove A, and A is already one of your assumptions, then you are done. You can conclude the proof using one of your existing assumptions by writing some English to this effect. Some typical forms of words are:

- If the assumption A was introduced into the proof by writing "assume A", then typically you will write “this is true by assumption.”

- If the assumption is actually part of a definition of some object D, then typically you will write “by definition of D”.
- If the assumption is actually a lemma L that has been proven previously, then you will typically write “by Lemma L”.
- If the assumption is a fact that you have derived by forward reasoning (see later) then often nothing is said, or sometimes, if you labelled this derived assumption by some name (H), then you might write “by (H)”.

I summarise this information, along with the correspondence to the form of natural deduction that you may have seen in Language Engineering, in a table:

Statement	Natural deduction rule	Typical form of words
A	$\dfrac{A}{A}$	“By assumption ...” “By definition ...” “By lemma ...” “”

2 Backward reasoning

Backward reasoning is an umbrella term for those proof steps that involve reducing the problem of proving some formula to a number of simpler subproblems. It is quite typical that a proof starts by performing backward reasoning.

Backward reasoning steps correspond to the use of “introduction rules” to construct a natural deduction proof tree starting from the bottom and building upwards. Each step is associated with a particular kind of formula or statement.

Implication

To prove $A \Rightarrow B$ starting from some assumptions, it suffices to prove B starting from the same assumptions and additionally A.

Statement	Natural deduction rule	Typical form of words
$A \Rightarrow B$ A implies B If A then B Suppose A. Then B.	$\dfrac{\boxed{\begin{array}{c}A\\ \vdots\\ B\end{array}}}{A \Rightarrow B}$	“Assume A. We prove B as follows...” “To show $A \Rightarrow B$, we assume A.”

Conjunction

To prove $A \land B$ from some assumptions, it suffices to give a proof of A from the assumptions and a separate proof of B from the assumptions.

Statement	Natural deduction rule	Typical form of words
$A \land B$ A and B	$\dfrac{\begin{array}{c}A\\B\end{array}}{A \land B}$	“We prove A and B separately. To prove A...” “For A, we argue as follows ... For B, ...”

Disjunction

To prove $A \lor B$ from some assumptions, it suffices to either give a proof of A from the assumptions or to give a proof of B from the assumptions.

Statement	Natural deduction rules	Typical form of words
$A \lor B$ A or B	$\dfrac{A}{A \lor B}$ $\dfrac{B}{A \lor B}$	“To see $A \lor B$, observe that A is true since...” “To see $A \lor B$, observe that B is true since...”

Negation

To prove $\neg A$ starting from some assumptions, it suffices to derive a contradiction (give a proof of false) starting from the same assumptions with A added.

Statement	Natural deduction rule	Typical form of words
$\neg A$ not A A is false	$\dfrac{\boxed{\begin{array}{c}A\\ \vdots\\ \bot\end{array}}}{\neg A}$	“We assume A and try to obtain a contradiction.” “To show $\neg A$, we assume A...”

Universal quantification

To give a proof of $\forall x : X.\, A$ from some assumptions, it suffices to give a proof of A from the same assumptions with $x \in X$ added, as long as you have not made any prior assumptions about the name x (x doesn’t appear in any of your existings assumptions).

Statement	Natural deduction rule	Typical form of words
$\forall x \in X. A$ $\forall x : X. A$ forall x in X, A A holds of all x in X	$\dfrac{\boxed{\begin{array}{c} x \in X \\ \vdots \\ A \end{array}}}{\forall x : X. A}$	“Let $x \in X$. We show A.” “Let x be an arbitrary member of X. We show A.” “Suppose x is in X. ... therefore A.”

Existential Quantification

To prove $\exists x : X. A$ from some assumptions, it suffices to find a witness $t \in X$ and give a proof of A with every occurence of x replaced by t, starting from the same assumptions.

Statement	Natural deduction rule	Typical form of words
$\exists x \in X. A$ $\exists x : X. A$ there exists x in X such that A A holds of some x in X there is some x in X with A true	$\dfrac{\begin{array}{c} t \in X \\ A[t/x] \end{array}}{\exists x : X. A}$	“We show that A holds of t.” “We take t as witness. To see $A[t/x]$...” “We show that t is such an x.”

Example

If your goal is to prove a formula and that formula is built using logical connectives, often the best thing to do is to immediately perform as many backwards proof steps as possible. Doing these backwards proof steps takes apart the formula, breaking your original goal into smaller subgoals, *and* often gives you extra assumptions that you can use in order to prove them. This is an example:

Lemma 1. *For all $n, m, k \in \mathbb{N}$: if $n + m = 0$ then, $n \neq k + 1$*

Proof. Let n, m and k be natural numbers. Assume $n + m = 0$ (A1). To see that $n \neq k + 1$, we assume $n = k+1$ (A2) and try to obtain a contradiction. By associativity and commutativity of addition $(k+1)+m = (k + m) + 1$ (A3). Since m and k are natural numbers, $(k + m) + 1 \neq 0$ (A4). We can therefore conclude the contradictory statement:

$$\begin{array}{rcll} n + m & = & (k+1) + m & \text{by (A2)} \\ & = & (k+m) + 1 & \text{by (A3)} \\ & \neq & 0 & \text{by (A4)} \\ & = & n + m & \text{by (A1)} \end{array}$$

□

The first part of the proof consists of doing backward reasoning in order to reduce the problem down to something simpler, in this case *false* (a contradiction), and extract additional assumptions, $n+m=0$ and $n=k+1$, that help us to prove it. The statement that we wish to prove is of the form $\forall n\ m\ k : \mathbb{N}.\ A$ and, to prove a formula of this shape, it suffices to give a proof of A starting from whatever assumptions we had before with $n, m, k \in \mathbb{N}$ added. I signal that this is how I wish to proceed by using the phrase "Let n, m and k be natural numbers". In this case, to give a proof of A is to give a proof of $n+m=0 \Rightarrow n \neq k+1$ and I know that to give a proof of an implication $B \Rightarrow C$ it suffices to give a proof of C after assuming B. I signal that this is my intention by the phrase "Assume $n+m=0$". I have decided to label this assumption (A1) because I want to refer to it later. Here, C is $\neg(n=k+1)$, so it suffices to derive a contradiction after additionally assuming $n=k+1$. I signal that this is how I want to proceed by the phrase "We assume $n=k+1$ and try to obtain a contradiction". Everything after the word "By" is forward reasoning, in which we try to show that the simpler formula that we now want to prove is a consequence of all that we have assumed, and domain specific reasoning, in which we use reasoning principles that are specific to certain datatypes or predicates[1].

3 Forward reasoning

In forward reasoning, we start from our assumptions and work forwards, deriving their consequences. The steps that are available to you to use in backward reasoning depend only on the shape of the formula that you want to prove. The steps available to you for forward reasoning depend only on the shape of your assumptions. The best way to think about forward reasoning is to think of the formula that you want to prove being fixed and that you are just applying proof steps to generate a larger and larger set of assumptions from which to prove it.

Implication

If you have assumed $A \Rightarrow B$ and you have assumed A, then you can consider B to be assumed.

Statement	Natural deduction rule	Typical form of words
$A \Rightarrow B$	$\dfrac{\begin{array}{c}A\\ A \Rightarrow B\end{array}}{B}$	"From $A \Rightarrow B$ and A we conclude B."

Disjunction

If you have assumed $A \vee B$ and, additionally, you can give the following two proofs:

- a proof of C starting from your assumptions and A
- a proof of C starting from your assumptions and B

Then you can consider C to be assumed.

[1] I include equational reasoning in this category.

Statement	Natural deduction rule	Typical form of words
$A \vee B$	$\dfrac{\begin{array}{c} A \vee B \\ \boxed{\begin{array}{c} A \\ \vdots \\ C \end{array}} \\ \boxed{\begin{array}{c} B \\ \vdots \\ C \end{array}} \end{array}}{C}$	“We proceed by cases on $A \vee B$. Assume A ... Hence C. Assume B ... Hence C.” “We analyse the two cases in order to show C.” “We proceed by case analysis on $A \vee B$”

Conjunction

If you have assumed $A \wedge B$, then you can consider A and B to be assumed.

Statement	Natural deduction rules	Typical form of words
$A \wedge B$	$\dfrac{A \wedge B}{A}$ $\dfrac{A \wedge B}{B}$	“”

Negation

If you have assumed $\neg A$ and you have also assumed A, then you can consider false one of your assumptions.

Statement	Natural deduction rules	Typical form of words
$A \wedge B$	$\dfrac{\begin{array}{c} A \\ \neg A \end{array}}{\bot}$	“From A and $\neg A$ we obtain a contradiction.”

Absurdity

If you have assumed false, then you can assume anything.

Statement	Natural deduction rules	Typical form of words
$\bot$	$\dfrac{\bot}{A}$	“Hence we obtain the desired result.” “*Ex falso quodlibet*” “A follows trivially”

Universal quantification

If you have assumed $\forall x : X.\, A$ and you have assumed $t \in X$, then you can consider A with all occurrences of x replaced by t to be another of your assumptions.

Statement	Natural deduction rules	Typical form of words
$\forall x : X.\, A$	$\dfrac{\begin{array}{l}\forall x : X.\, A \\ t \in X\end{array}}{A[t/x]}$	“It follows that A holds of t”

Existential quantification

If you have assumed $\exists x : X.\, A$ and you can give a proof of C from your assumptions, then you can consider C as an assumption.

Statement	Natural deduction rule	Typical form of words
$\exists x : X.\, A$	$\dfrac{\begin{array}{c}\exists x : X.\, A \\ \boxed{\begin{array}{c}A[y/x] \\ \vdots \\ C\end{array}}\end{array}}{C}$	“Let y be witness to $\exists x : X.\, A$. ... Therefore C.”

Example

In the following example, we derive a consequence of assuming a disjunction. The example uses the following recursive definition of mutliplication of natural numbers:

$$\begin{array}{rcl} 0 * m & = & 0 \\ (n+1) * m & = & n * m + m \end{array}$$

Lemma 2. *for all $n, m \in \mathbb{N}$: if $n = 0$ or $m = 0$ it follows that $n * m = 0$*

Proof. Let $n, m \in \mathbb{N}$ and assume $n = 0 \vee m = 0$. We show that $n * m = 0$ by case analysis.

- If $n = 0$ then $n * m = 0$ is $0 * m = 0$ which is true by the definition of multiplication.
- If $m = 0$ then $n * m = 0$ is $n * 0 = 0$ which, by the commutativity of mutliplication, is $0 * n = 0$ and this also true by definition.

□

The forward proof starts with “We show that...”. At that point I have already assumed $n = 0 \vee m = 0$ and I wish to proceed by deriving $n * m = 0$ from this assumption by forward reasoning, this is signalled by the phrase “We show $n + m = 0$ by case analysis”. The rest of the proof is forward reasoning, but using specific facts to do with arithmetic.

4 Playing the game

The following is an example of pure logic:

Lemma 3. *P implies $\neg\neg P$*

Proof. Assume P. We wish to show $\neg\neg P$, so we assume $\neg P$ and try to obtain a contradiction. From P and $\neg P$ we obtain the desired contradiction. □

It can be very helpful to think of the individual steps of a proof as being operations that manipulate the *proof state*. The proof state is a pair consisting of a set of *assumptions* and the current *goal*. The goal is the formula you are currently trying to prove and the assumptions are the resources that you have available in order to try to prove it. Each step of the proof (i.e. each application of one of the rules of natural deduction, signalled using an appropriate form of words) changes the proof state by either adding to the set of assumptions or changing the goal. For example, if we have assumptions $A_1,\dots,A_k$ and we are aiming to prove $A \Rightarrow B$, then the magic words "assume A", which is a form of words used to signal a use of implication introduction, lead us to a proof state in which the assumptions are $A_1,\dots,A_k,A$ and the goal is B. When you are reading a proof, it is very helpful to picture the proof state in your mind after each step of the proof. Let's take the above proof as an example and let Cxt be the set of assumptions consisting of all the lemmas and theorems that we have proven so far and all the definitions we have made and any axioms we assume. The the following table gives a listing of the proof state at the end of each step in the proof.

Proof step	Assumptions	Goal
"Lemma 3. P implies $\neg\neg P$ \n *Proof.*"	Cxt	$P \Rightarrow \neg\neg P$
"Assume P"	Cxt, P	$\neg\neg P$
"... so we assume $\neg P$ and ... contradiction."	Cxt, P, $\neg P$	$\perp$
"From P and $\neg P$ we obtain..."	Cxt, P, $\neg P$	□

In the following example, we collect quite a few assumptions from our initial backward reasoning. In such cases it can sometimes be helpful to label them as we go.

Lemma 4. *if P implies Q then $\neg Q$ implies $\neg P$*

Proof. Assume $P \Rightarrow Q$ (A1) and assume $\neg Q$ (A2). We aim to prove $\neg P$, so we will assume P (A3) and try to obtain a contradiction. From (A1) and (A3), we obtain Q (A4). From (A4) and (A2) we obtain the desired contradiction. □

This is what is going on in my head when I read this proof back to myself. Let Cxt$'$ be Cxt with Lemma 3 added as an extra assumption:

Proof step	Assumptions	Goal
"Lemma 4. ... *Proof.*"	Cxt$'$	$(P \Rightarrow Q) \Rightarrow \neg Q \Rightarrow P$
"Assume $P \Rightarrow Q$"	Cxt$'$, A1:$P \Rightarrow Q$	$\neg Q \Rightarrow \neg P$
"assume $\neg Q$"	Cxt$'$, A1:$P \Rightarrow Q$, A2:$\neg Q$	$\neg P$
"assume P ... try ... contradiction."	Cxt$'$, A1:$P \Rightarrow Q$, A2:$\neg Q$, A3:P	$\perp$
"From (A1) and (A3), we obtain Q."	Cxt$'$, A1:$P \Rightarrow Q$, A2:$\neg Q$, A3:P, A4:Q	$\perp$
"From ... obtain ... contradiction."	Cxt$'$, A1:$P \Rightarrow Q$, A2:$\neg Q$, A3:P, A4:Q	□

This is a typical shape of proof and it is a shape that will serve you well in this unit. First, the goal is decomposed by backward reasoning until it is something atomic and all the delicious extra assumptions have been extracted. Second, more assumptions are generated from the existing ones using forward reasoning until eventually an assumption matching the goal is generated.

One important reason that it is helpful to have the proof state in your head is that, because the proof steps are just certain phrases in English, sometimes the proof state is needed for disambiguation. For example, it may only be possible to understand whether the word "assume" heralds a use of implication introduction or a use of negation introduction, based on goal at the point at which the word was used. Here's one last example.

Lemma 5. *$P \Rightarrow (Q \Rightarrow R)$ implies $P \wedge Q \Rightarrow R$*

Proof. Assume $P \Rightarrow (Q \Rightarrow R)$ (*). Assume P and Q. From (*) and P, $Q \Rightarrow R$ follows. From this and Q, R follows. □

Proof step	Assumptions	Goal
"Lemma 5. ... *Proof.*"	Cxt$''$	$(P \Rightarrow (Q \Rightarrow R)) \Rightarrow (P \wedge Q) \Rightarrow R$
"Assume $P \Rightarrow (Q \Rightarrow R)$ (*)"	Cxt$''$, *:$P \Rightarrow (Q \Rightarrow R)$	$P \wedge Q \Rightarrow R$
"Assume P and Q"	Cxt$''$, *:$P \Rightarrow (Q \Rightarrow R)$, P, Q	R
"From (*) and P, $Q \Rightarrow R$ follows."	Cxt$''$, *:$P \Rightarrow (Q \Rightarrow R)$, P, Q, $Q \Rightarrow R$	R
"From this and Q, R follows."	Cxt$''$, *:$P \Rightarrow (Q \Rightarrow R)$, P, Q, $Q \Rightarrow R$	□

For practice you might like to attempt proofs of the following formulas, keeping track of the proof state as you go:

(i) $\neg P \Rightarrow P \Rightarrow Q$

(ii) $(P \wedge Q \Rightarrow R) \Rightarrow P \Rightarrow Q \Rightarrow R$

(iii) $\neg(P \wedge \neg P)$

(iv) $(P \Rightarrow Q) \Rightarrow (Q \Rightarrow R) \Rightarrow P \Rightarrow R$

(v) $(\exists x : X.\, P \vee Q) \Rightarrow (\exists x : X.\, P) \vee (\exists x : X.\, Q)$
(Hint: it is quite common to defer backward reasoning on a goal of the form $A \vee B$.)

If you can write proofs of these formulas, then you are in an excellent position to learn types and λ-calculus: all that remains is to give you a few extra tools to work with. If not, my quick guide has failed, and you should come and talk to me in a drop-in session so that I may try to give reddress for its inadequacies.

5 The excluded middle, contradiction and contrapositive

This is the last "rule" in our arsenal as far as pure logic is concerned (if we exclude equality) and it takes us from intuititionistic to classical logic.

Statement	Natural deduction rule	Typical form of words
$A \vee \neg A$	$\dfrac{}{A \vee \neg A}$	"By excluded middle either A or $\neg A$" "$A \vee \neg A$ by LEM"

This rule is needed in order to prove the converse of the two previous lemmas. For example:

Lemma 6. *$\neg\neg P$ implies P*

Proof. Assume $\neg\neg P$. By excluded middle, $P \vee \neg P$. We proceed by case analysis to conclude P. In the first case we assume P and then the result is true by assumption. In the second case we assume $\neg P$ and then from this and our original assumption we obtain a contradiction. Therefore, P follows trivially. □

Here's one last print-out of my brain. Doing a case analysis spawns two processes, each with their own proof state. I have rather clumsily represented this by dedicating two lines to each step of the proof at this point and writing in the upper or lower half depending on which case is being addressed.

Proof step	Assumptions	Goal
"Lemma 5. ... *Proof.*"	Cxt'''	$\neg\neg P \implies P$
"Assume $\neg\neg P$"	$\text{Cxt}''', \neg\neg P$	P
"By excluded middle, $P \vee \neg P$."	$\text{Cxt}''', \neg\neg P, P \vee \neg P$	P
"... case analysis to conclude P."	$\text{Cxt}''', \neg\neg P, P \vee \neg P, P$	P
	$\text{Cxt}''', \neg\neg P, P \vee \neg P, \neg P$	P
"in the first case we assume P"	$\text{Cxt}''', \neg\neg P, P \vee \neg P, P$	P
	$\text{Cxt}''', \neg\neg P, P \vee \neg P\ ,\neg P$	P
"P is true by assumption"	$\text{Cxt}''', \neg\neg P, P \vee \neg P, P$	□
	$\text{Cxt}''', \neg\neg P, P \vee \neg P, \neg P$	P
"In the second case we assume $\neg P$"	$\text{Cxt}''', \neg\neg P, P \vee \neg P, P$	□
	$\text{Cxt}''', \neg\neg P, P \vee \neg P, \neg P$	P
"from this ... we obtain a contradiction"	$\text{Cxt}''', \neg\neg P, P \vee \neg P, P$	□
	$\text{Cxt}''', \neg\neg P, P \vee \neg P, \neg P, \bot$	P
"P follows trivially"	$\text{Cxt}''', \neg\neg P, P \vee \neg P, P$	□
	$\text{Cxt}''', \neg\neg P, P \vee \neg P, \neg P, \bot$	□

In the way that I think about this proof retrospectively, the assumptions P and $\neg P$ are made as soon as the case analysis is announced. The phrases "in the first case we assume P" and "in the second case we assume $\neg P$" seem redundant (they don't change the proof state) but they are very helpful for confirming to the reader that we are doing a case analysis on $P \vee \neg P$ and, simultaneously, signalling which case we are about to attempt and what this entails.

Proof by contradiction

The law of the excluded middle is quite clumsy to use in a proof. However, it is equivalent to the principle of *proof by contradiction* or *reductio ad absurdum*. A proof by contradiction proves the goal A by assuming $\neg A$ and attempting to obtain a contradiction.

Statement	Natural deduction rule	Typical form of words
A	$\dfrac{\boxed{\begin{array}{c}\neg A\\ \vdots\\ \bot\end{array}}}{A}$	"We prove the result by contradiction, so assume $\neg A$" "Assume $\neg A$, ... hence we have obtained a contradiction. We conclude A by *reductio ad absurdum*."

You might think this rule is quite reminiscent of the introduction rule for negation. Indeed it is! The only difference is that the rule of negation introduction concludes with $\neg A$. So we can't use negation introduction to prove A directly (unless A happens to be of shape $\neg B$). We can, however, use negation introduction to prove $\neg\neg A$, and we know from Lemmas 3 and 6 that $\neg\neg A$ is logically equivalent to A (in the sense that they each imply the other).

What is interesting about this is that Lemma 6 is only true by virtue of excluded middle. If we didn't have that rule, then it would be impossible to conclude that A and $\neg\neg A$ are logically equivalent. There is a sect of mathematicians called *intuitionists* that deny the validity of excluded middle and therefore all that follows from it (such as Lemma 6). To intuitionists, proof by contradiction is a not a principle but an abomination, spawned in the firey pits of Satan's insatiable lust for the depraved. I have some sympathy, but I am personally quite relaxed about it. Also, I was once given a completely unfair parking ticket by Oxford City Council which claimed that I was parked on a double yellow line when I was not (actually the situation was a little ambiguous), and I wrote back to them with a proof by contradiction (demonstrating, beyond any shadow of doubt, that if I was parked on double yellows then absurdity followed) and the ticket was rescinded. Anyway, we will talk about intuitionism more in Week 7, because it is very closely related to type theory.

One, somewhat more practical, objection to proof by contradiction is that it is not suggested by any particular logical operator. If you consider all the rules we have looked at until this point, your choice of rule is restricted by the shape of the goal or the shape of the assumptions. Proof by contradiction, however, can be used at any time. This complicates matters a bit when writing proofs because you have more choices. There is no syntactic clue in any of the formulas in the proof state, you just have to decide to do it based on your intuition. It is wise to accept that writing proofs, like writing programs, sometimes requires a bit of trial and error. Here is an example; you might like to write out the evolution of the proof state as practice:

Lemma 7. *$\neg Q \Rightarrow \neg P$ implies $P \Rightarrow Q$*

Proof. Assume $\neg Q \Rightarrow \neg P$ (A) and assume P. We prove Q by contradiction, so assume $\neg Q$. It follows from (A) that, additionally, $\neg P$. Hence, from P and $\neg P$ we have obtained a contradiction. □

This implication that we have now proven gives us another well known proof principle called *proof by contrapositive*: in order to prove $P \Rightarrow Q$, it is enough to prove $\neg Q \Rightarrow \neg P$.

6 Theory specific reasoning

We have talked about the generic, purely logical aspects of proof but, in reality, most proofs argue something in a specific domain (or, in the logical jargon, a specific theory). When you are working with a particular theory, e.g. the theory of arithmetic over the natural numbers, the theory gives you more to work with: for example, the language of formulas is enlarged to include new constants, function symbols and relations like 0, + and < respectively. Furthermore, you are given axioms, like $(x+y)+z = x+(y+z)$, which you are free to use as assumptions in your proofs, and even new proof rules, like induction, that are specific to the datatype introduced by the theory.

Statement	Natural deduction rule	Typical form of words
$\forall x : \mathbb{N}.\, A$	$\dfrac{\begin{array}{l} A[0/x] \\ \boxed{\begin{array}{l} A[k/x] \\ \vdots \\ A[k+1/x] \end{array}} \end{array}}{\forall x : \mathbb{N}.\, A}$	“We prove the result by induction on $x \in \mathbb{N}$. When $x = 0$ ... hence A is true of 0. When $x = k+1$, assume A holds of k ... A is true of $k+1$.” “By induction on $x \in \mathbb{N}$. We analyse the two cases. ...”

So, in the sense above, natural number induction is an alternative introduction rule (backward reasoning) for those formulas of shape $\forall x : X.\, A$ in the specific case that X is $\mathbb{N}$.

If we were just proving statements about logic, the purely formal, tree-structured proofs such as those you used in the second half of Language Engineering would be fine. Actually, they are always fine *in principle*, but in practice they do not scale so well to the sorts of proofs you want to write when particular theories are involved. That is because, typically, people approach forward reasoning in theories with some preconceptions of what is “obviously true”. For example, it would not be unusual to see a sentence like “Since $x^2 - 1 = (x+1)(x-1)$, it follows that...” in a proof, without any justification. It is actually a little bit time consuming to justify the equation $x^2 - 1 = (x+1)(x-1)$ from first principles in the theory of the real field. Not only will it increase the size of your natural deduction proof tree uncomfortably but it will also be quite an annoying distraction.

The trouble is that “obviously true” (and similar phrases such as “clearly” and even “follows from”, which is sometimes used to hide some steps) is a dangerous form of words. Remember that proofs are there to convince another party about the truth of some statement, and stating that a proposition is “obviously true” is only convincing if you are certain that it is already believed by reader. Types and λ-calculus is a great place to learn to write proofs because we build our theory starting from nothing[2]. Consequently, at the start of the unit, nobody has any preconceptions about what is “obviously true”. As the unit progresses, we will have built up sufficient experience that some things will be stated without justification, but it will be a natural evolution and we will all be on the same page ***assuming that we have all been attempting the exercises***.

Extra proof principles, like natual number induction, give you yet more alternative approaches to how to structure the proof. When faced with a goal like $\forall x : \mathbb{N}.\, P \Rightarrow Q$ you have several possible first steps. To

[2] Actually, we starting from a set of strings, so you do need to know something, but we will not be proving anything about the theory of strings.

dispense with the $\forall$ you could attempt a proof by induction or you could simply assume x as an arbitrary element. To prove $P \Rightarrow Q$ you could either assume P and try to show Q or, alternatively, you could try to prove the contrapositive. Maybe you would prefer to assume $\exists x : \mathbb{N}.\, \neg(P \Rightarrow Q)$ instead and derive a contradiction, concluding by *reductio ad absurdum*? Sometimes all of these approaches will work, though some may be shorter or more clear than others. This, a careful use of phrases like "clearly", and a sense of when to split a large proof into several lemmas, are what make writing proofs a kind of art, and you will only get better at choosing the best approach with practice. Since I do not assume you have much experience at writing proofs, throughout the first half of the unit I will always signal to you in a question which of the commonly used alternative strategies (e.g. induction, contradiction or contrapositive) is a good choice.