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In [1]:
```
import pandas as pd
import numpy as np
import seaborn as sns
import matplotlib.pyplot as plt
```

Problem 1

Null

Under the null hypothesis we are given:

$$p_t = p_{t-1} + \epsilon_t$$

where ϵ_t iid following $\mathcal{N}(0, 1)$. The returns are given by:

$$r_t = p_t - p_{t-1}$$

therefore we have $r_t = \epsilon_t$ This means that r iid $\mathcal{N}(0, 1)$.

Alternative

To solve for σ_ν^2 we can decompose the variance of the given return model and find the variance of the transitory component. Then we can use the properties of $u_t = \rho u_{t-1} + \nu_t$ to find σ_u^2 such that 1/3 of the variance is due to the transitory component. This can be done as follows.

First decompose return variance

$$r_t = p_t - p_{t-1}$$

$$r_t = (p_t^* + u_t) - (p_{t-1}^* + u_{t-1})$$

$$r_t = (p_t^* - p_{t-1}^*) + (u_t - u_{t-1})$$

We can substitute ϵ for the increment in p^*.

$$r_t = \epsilon_t + (u_t - u_t)$$

Now for the variance

$$Var(r_t) = Var(\epsilon_t) + Var(u_t - u_{t-1})$$

We can now solve for the $Var(u_t - u_{t-1})$ component as follows

$$\frac{Var(u_t - u_{t-1})}{Var(r_t)} = 0.25$$

Sub in our derivation of $Var(r_t)$

$$\frac{Var(u_t - u_{t-1})}{Var(\epsilon_t) + Var(u_t - u_{t-1})} = 0.25$$

Plugging in the variance of ϵ_t and solving for $Var(u_t - u_{t-1})$ we get

$$Var(u_t - u_{t-1}) = \frac{1}{3}$$

To match this variance to our particular process we need to take the variance of the increment of u_t again, but using AR properties to reduce and find σ_u^2.

$$Var(u_t - u_{t-1}) = Var(u_t) + Var(u_{t-1}) - 2Cov(u_t, u_t - 1)$$

$$Var(u_t - u_{t-1}) = 2\sigma_u^2 - 2\rho\sigma_u^2$$

$$Var(u_t - u_{t-1}) = 2\sigma_u^2(1 - \rho)$$

Rewrite σ_u^2 using our unconditional variance formula.

$$Var(u_t - u_{t-1}) = 2\frac{\sigma_\nu^2}{1 - \rho^2}(1 - \rho)$$

$$Var(u_t - u_{t-1}) = 2\frac{\sigma_\nu^2}{(1 - \rho)(1 + \rho)}(1 - \rho)$$

$$Var(u_t - u_{t-1}) = \frac{2\sigma_\nu^2}{1 + \rho}$$

$$\sigma_\nu^2 = Var(u_t - u_{t-1})\frac{1 + \rho}{2}$$

We can now plug in our results and get the following

$$\sigma_\nu^2 = \frac{1.98}{6} \approx 0.33$$

In [2]:
```
def variance_ratio(data, k):
    one_month_returns = data.var()
    k_month_returns = data.rolling(window=k).sum()
    k_month_returns = k_month_returns.var()
    return k_month_returns / (k * one_month_returns)
```

```
def pt_summers_null(T):
    result = [10]
    for _ in range(T):
        result.append(result[-1] + np.random.normal(loc=0, scale=1, size=1)[0])
    result = np.diff(np.array(result))
    return result


def pt_summers(T):
    rho = 0.98
    transitory_variance = 1 / 3
    sigma_nu = np.sqrt(transitory_variance * (1 + rho) / 2)
    p_star = np.zeros(T + 1)
    ut = np.zeros(T + 1)
    epsilon = np.random.normal(loc=0, scale=1, size=T)
    nu = np.random.normal(loc=0, scale=sigma_nu, size=T)
    for i in range(1, T + 1):
        p_star[i] = p_star[i - 1] + epsilon[i - 1]
        ut[i] = rho * ut[i - 1] + nu[i - 1]
    pt = p_star + ut
    rt = np.diff(pt)
    return rt
```

In [3]:
```
df = pd.read_csv('crsp_vwretd_1926_2021.csv')
df['date'] = pd.to_datetime(df['date'].astype(str), format='%Y%m%d')
df.set_index('date', inplace=True)
k_vals = [12, 24, 36, 48, 60, 90, 120]
actual_var_ratios = pd.DataFrame({k:variance_ratio(df, k) for k in k_vals})
alt_df = pd.DataFrame({f'Sim {i}':pt_summers(len(df)) for i in range(1, 1001)})
null_df = pd.DataFrame({f'Sim {i}':pt_summers_null(len(df)) for i in range(1, 1001)
null_var_ratios = {}
alt_var_ratios = {}

for k in k_vals:
    null = []
    alt = []
    for col in alt_df.columns:
        null.append(variance_ratio(null_df[col], k))
        alt.append(variance_ratio(alt_df[col], k))
    null_var_ratios[k] = null
    alt_var_ratios[k] = alt


null_var_ratios = pd.DataFrame(null_var_ratios)
alt_var_ratios = pd.DataFrame(alt_var_ratios)
```

In [4]:
```
for k in k_vals:
    print(f'For K = {k}')
    null_percent = (null_var_ratios[k] < actual_var_ratios[k].iloc[0]).sum() / len(
    alt_percent = (alt_var_ratios[k] < actual_var_ratios[k].iloc[0]).sum() / len(al
    print(f'{actual_var_ratios[k].values[0]} for the data')
    print(f'{null_percent} of the empirical distribution under the null hypothesis
    print(f'{alt_percent} of the empirical distribution under the alternative hypot
```

```
For K = 12
1.1910031335812559 for the data
0.954 of the empirical distribution under the null hypothesis is smaller than the po
int estimate
0.978 of the empirical distribution under the alternative hypothesis is smaller than
the point estimate

For K = 24
1.1845489681371817 for the data
0.886 of the empirical distribution under the null hypothesis is smaller than the po
int estimate
0.948 of the empirical distribution under the alternative hypothesis is smaller than
the point estimate

For K = 36
1.0922930591790234 for the data
0.743 of the empirical distribution under the null hypothesis is smaller than the po
int estimate
0.842 of the empirical distribution under the alternative hypothesis is smaller than
the point estimate

For K = 48
0.9971692801012134 for the data
0.59 of the empirical distribution under the null hypothesis is smaller than the poi
nt estimate
0.712 of the empirical distribution under the alternative hypothesis is smaller than
the point estimate

For K = 60
0.9139054547907252 for the data
0.478 of the empirical distribution under the null hypothesis is smaller than the po
int estimate
0.619 of the empirical distribution under the alternative hypothesis is smaller than
the point estimate

For K = 90
0.6377771143818571 for the data
0.165 of the empirical distribution under the null hypothesis is smaller than the po
int estimate
0.279 of the empirical distribution under the alternative hypothesis is smaller than
the point estimate

For K = 120
0.6591174613504467 for the data
0.265 of the empirical distribution under the null hypothesis is smaller than the po
int estimate
0.39 of the empirical distribution under the alternative hypothesis is smaller than
the point estimate
```

Explanation

As we can see from the findings, we see a higher percentage of the empirical distribution under the alternative hypothesis be smaller than the variance ratio estimate, than under the

null hypothesis.

In both models we notice that for small K (12,24) the variance ratio is likely $VR > 1$, while for large K (60,90,120) $VR < 1$. Also, for middle values of K (36,48), we see $VR \approx 1$ therefore indicating that mean reversion begins to become noticable for larger horizons.

In the case of the null hypothesis, the variance ratios should all be equal to one. This only occurs for middle values of K, so we can reject all other values.

In the case of the alternative hypothesis, since here we have $\rho = 0.98$, using the half life formula we can see that $h = \frac{log(0.5)}{log(\rho)} = 34.3$, so it takes about 34 months for half the shock to disappear. This means that u_t should decay slowly. Indeed, from the data we see mean reversion, but only for large values of K.

Extra credit

In [5]:
```
df = pd.read_csv('crsp_vwretd_1926_2021.csv')
df['date'] = pd.to_datetime(df['date'].astype(str), format='%Y%m%d')
df.set_index('date', inplace=True)
df = df[df.index > '1952-01-01']
k_vals = [12, 24, 36, 48, 60, 90, 120]
actual_var_ratios = pd.DataFrame({k:variance_ratio(df, k) for k in k_vals})
alt_df = pd.DataFrame({f'Sim {i}':pt_summers(len(df)) for i in range(1, 1001)})
null_df = pd.DataFrame({f'Sim {i}':pt_summers_null(len(df)) for i in range(1, 1001)
null_var_ratios = {}
alt_var_ratios = {}

for k in k_vals:
    null = []
    alt = []
    for col in alt_df.columns:
        null.append(variance_ratio(null_df[col], k))
        alt.append(variance_ratio(alt_df[col], k))
    null_var_ratios[k] = null
    alt_var_ratios[k] = alt


null_var_ratios = pd.DataFrame(null_var_ratios)
alt_var_ratios = pd.DataFrame(alt_var_ratios)
```

In [6]:
```
for k in k_vals:
    print(f'For K = {k}')
    null_percent = (null_var_ratios[k] < actual_var_ratios[k].iloc[0]).sum() / len(
    alt_percent = (alt_var_ratios[k] < actual_var_ratios[k].iloc[0]).sum() / len(al
    print(f'{actual_var_ratios[k].values[0]} for the data')
    print(f'{null_percent} of the empirical distribution under the null hypothesis
    print(f'{alt_percent} of the empirical distribution under the alternative hypot
```

```
For K = 12
1.1166894183704497 for the data
0.853 of the empirical distribution under the null hypothesis is smaller than the po
int estimate
0.869 of the empirical distribution under the alternative hypothesis is smaller than
the point estimate

For K = 24
0.9936035861065797 for the data
0.612 of the empirical distribution under the null hypothesis is smaller than the po
int estimate
0.654 of the empirical distribution under the alternative hypothesis is smaller than
the point estimate

For K = 36
0.8616612986108448 for the data
0.398 of the empirical distribution under the null hypothesis is smaller than the po
int estimate
0.461 of the empirical distribution under the alternative hypothesis is smaller than
the point estimate

For K = 48
0.7825649715151503 for the data
0.313 of the empirical distribution under the null hypothesis is smaller than the po
int estimate
0.404 of the empirical distribution under the alternative hypothesis is smaller than
the point estimate

For K = 60
0.8016665363261704 for the data
0.403 of the empirical distribution under the null hypothesis is smaller than the po
int estimate
0.488 of the empirical distribution under the alternative hypothesis is smaller than
the point estimate

For K = 90
0.7147566466472679 for the data
0.37 of the empirical distribution under the null hypothesis is smaller than the poi
nt estimate
0.482 of the empirical distribution under the alternative hypothesis is smaller than
the point estimate

For K = 120
0.7735390573034906 for the data
0.504 of the empirical distribution under the null hypothesis is smaller than the po
int estimate
0.631 of the empirical distribution under the alternative hypothesis is smaller than
the point estimate
```

Removing the data from 1926-1952 changes the results. As we can see, the variance ratio for the null hypothesis is now closer to the expected value of 1. Meanwhile, the mean reversion of the alternative hypothesis is also more apparent, as we can see that $VR \approx 1$ for K = 24, and $VR < 1$ for larger K values.

The time period between 1926 and 1952 had some of the most extreme financial shock in US history. The biggest events that occured in that period were: i) the roaring 20s, where the economy saw rapid growth and as a result the CRSP returns where strongly positive, ii) the great depression, where almost 1/3 of the US GDP was lost, and the CRSP index saw extremely negative returns peak to trough, and iii) WWII. These events, along with others (New Deal, Depression within the Depression) make the returns between 1926-1952 vary a lot, therefore influencing our results.

Problem 2

$$y_t = \eta + u_t$$

The distribution does not vary throughout time, it stays exactly the same. The random variable η provides a mean for the process, which stays constant for all t, then at each step random noise from u_t is added. Due to η not changing and our random noise u_t being i.i.d., the joint distribution between y_t and y_{t+1} is the same as y_{t+k} and y_{t+k+1}, making the process strictly stationary.

The first and second moments are as follows.

$$\mathbb{E}[y_t] = \mathbb{E}[\eta] + \mathbb{E}[u_t]$$

$$\mathbb{E}[y_t] = 0 + \frac{1}{2}$$

$$\mathbb{E}[y_t] = \frac{1}{2}$$

$$\mathbb{E}[y_t^2] = Var(y_t) + \mathbb{E}[y_t]^2$$

$$Var(y_t) = Var(\eta) + Var(u_t) - 2Cov(\eta, u_t)$$

$$Var(y_t) = 1 + \frac{1}{12}$$

$$Var(y_t) = \frac{13}{12}$$

$$\mathbb{E}[y_t^2] = \frac{13}{12} + \frac{1}{4}$$

$$\mathbb{E}[y_t^2] = \frac{4}{3}$$

The autocovariance is as follows

$$Cov(y_t, y_{t-1}) = \mathbb{E}[y_t * y_{t-1}] - \mu_t * \mu_{t-1}$$

$$Cov(y_t, y_{t-1}) = \mathbb{E}[y_t * y_{t-1}] - \frac{1}{4}$$

$$Cov(y_t, y_{t-1}) = \mathbb{E}[y_t] * \mathbb{E}[y_{t-1}] - \frac{1}{4}$$

$$Cov(y_t, y_{t-1}) = \frac{1}{4} - \frac{1}{4}$$

$$Cov(y_t, y_{t-1}) = 0$$

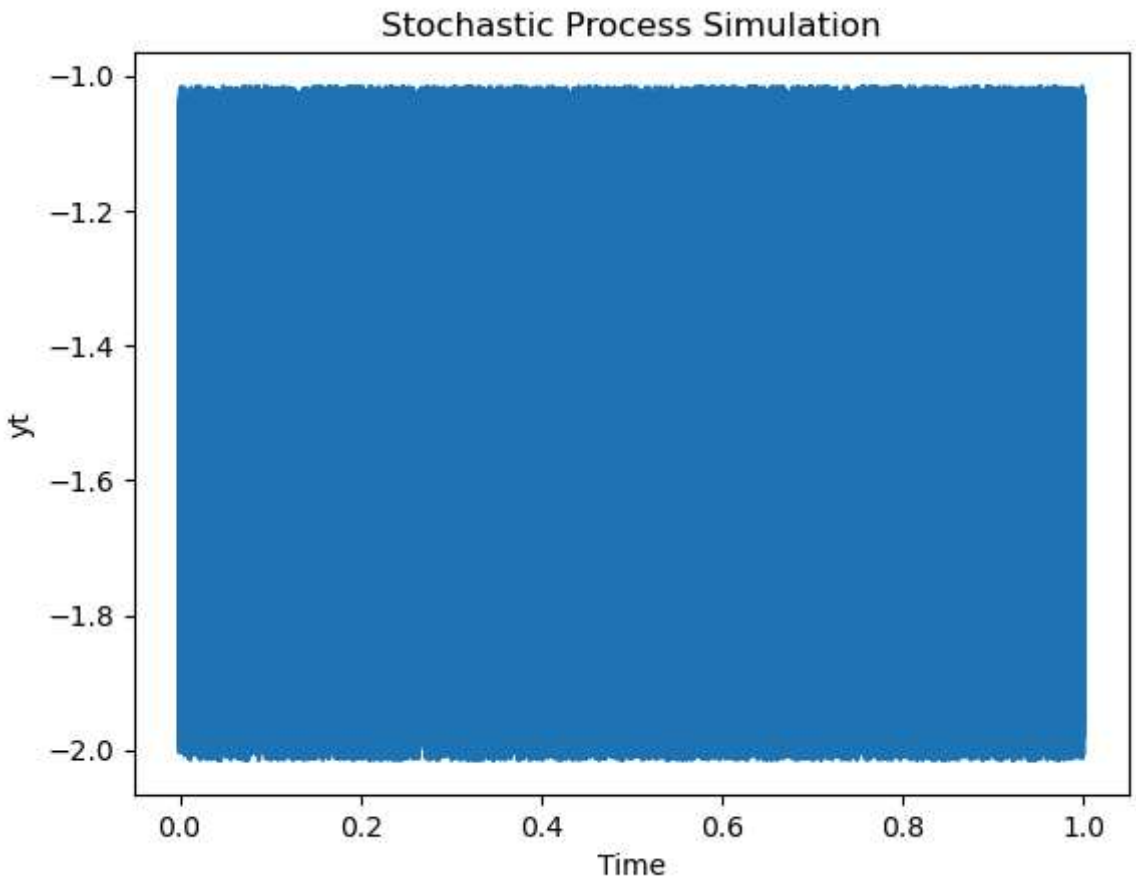
This process is not ergodic as our sample mean depends on our draw of η. Over time the process will not converge to the theoretical mean of $\frac{1}{2}$, it will converge to whatever the draw of η is. Furthermore, the longer term dynamics of the sample don't give more information about the mean of the process, only η. We can run some simulations to confirm our numbers.

In [7]:
```
def stochastic_process(iterations):
    eta = np.random.normal(loc=0, scale=1, size=1)[0]
    u = np.random.uniform(low=0, high=1, size=iterations)
    return (eta + u), eta


def autocovariance(process, lag):
    process = pd.Series(process)
    return process.cov(process.shift(lag))
```

In [8]:
```
sim, draw = stochastic_process(100000)
t = np.linspace(0, 1, num=len(sim))
```

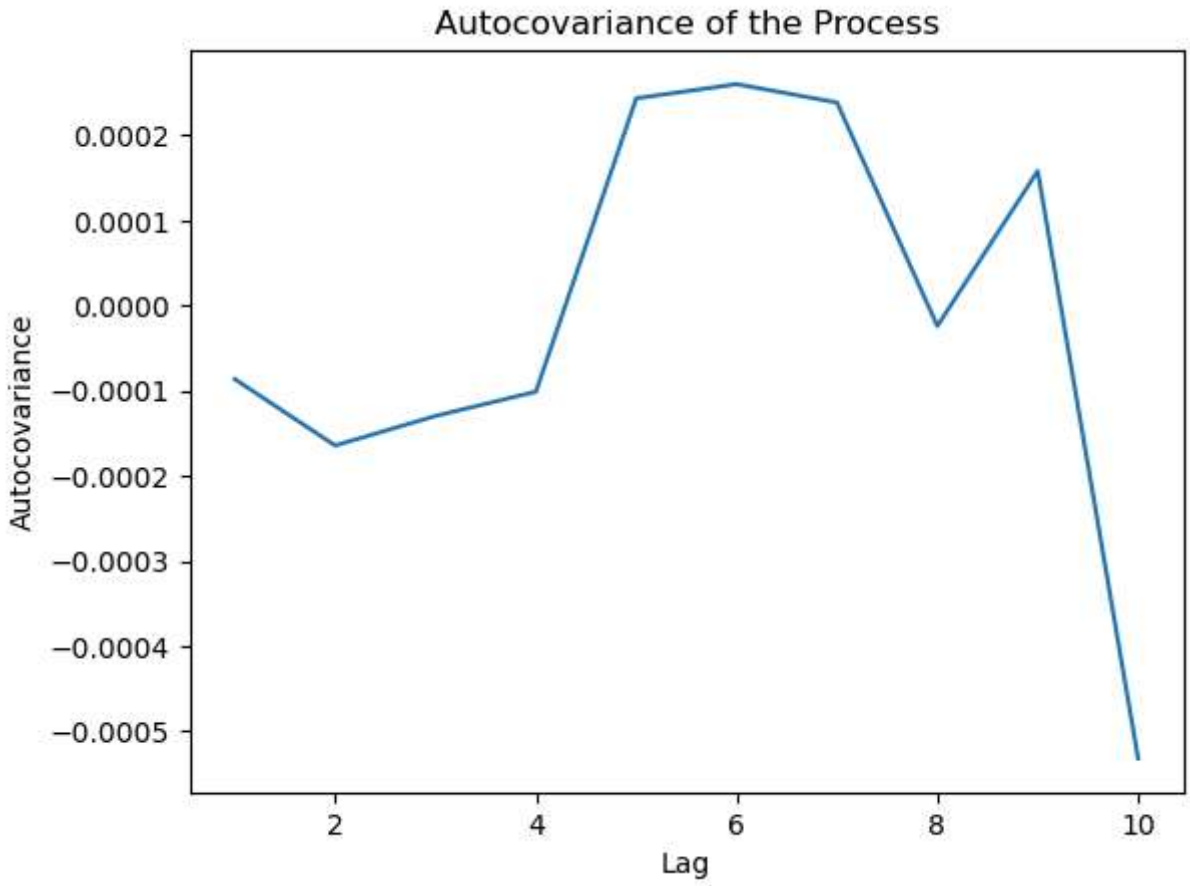
In [9]:
```
sns.lineplot(x=t, y=sim)
plt.xlabel('Time')
plt.ylabel('yt')
plt.title('Stochastic Process Simulation')
plt.show()
```



In [10]:
```
print(f'Eta Draw = {draw}')
print(f'Mu = {np.mean(sim)}')
print(f'Second Moment = {sum(val**2 for val in sim) / len(sim)}')
```

```
Eta Draw = -2.0155016013354756
Mu = -1.5156236736558588
Second Moment = 2.3805439516450515
```

In [11]:
```
auto_cov = {i:autocovariance(sim, i) for i in range(1, 11, 1)}
sns.lineplot(x=auto_cov.keys(), y=auto_cov.values())
plt.title('Autocovariance of the Process')
plt.ylabel('Autocovariance')
plt.xlabel('Lag')
plt.show()
```


Autocovariance of the Process
Autocovariance
0.0002
0.0001
0.0000
−0.0001
−0.0002
−0.0003
−0.0004
−0.0005
2
4
6
8
10
Lag