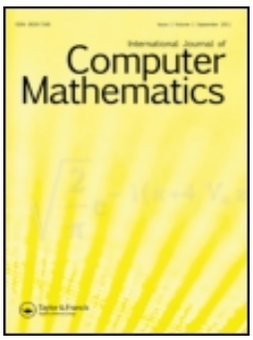

International Journal of Computer Mathematics

ISSN: 0020-7160 (Print) 1029-0265 (Online) Journal homepage: http://www.tandfonline.com/loi/gcom20

Taylor polynomial solution of non-linear Volterra–Fredholm integral equation

Y. Mahmoudi

To cite this article: Y. Mahmoudi (2005) Taylor polynomial solution of non-linear Volterra–Fredholm integral equation, International Journal of Computer Mathematics, 82:7, 881-887, DOI: 10.1080/00207160512331331110

To link to this article: http://dx.doi.org/10.1080/00207160512331331110

International Journal of Computer Mathematics
Vol. 82, No. 7, July 2005, 881–887

Taylor polynomial solution of non-linear Volterra–Fredholm integral equation

Y. MAHMOUDI*

Azad University of Tabriz, Tabriz, Iran

(Received 14 August 2004)

A Taylor polynomial method is developed for the approximation of a non-linear Volterra–Fredholm integral equation of the second kind.

Keywords: Taylor polynomials and series; Non-linear Volterra and Fredholm integral equations

C.R. Category: G.1.9

1. Introduction

Sezer and colleagues have presented a Taylor expansion approach for solving integral equations [1], differential equations [2], non-linear Volterra–Fredholm integral equations [3] and linear Volterra–Fredholm integro-differential equations [4]. We have used this method to solve non-linear high-order Volterra–Fredholm integro-differential equations of the second kind [5].

In this paper we solve non-linear Volterra–Fredholm integral equations of the second kind for the general case

$$y(x) = f(x) + \lambda_1 \int_a^x k_1(x,t)\Phi(y(t))\,\mathrm{d}t + \lambda_2 \int_a^b k_2(x,t)\Psi(y(t))\,\mathrm{d}t \tag{1}$$

where $f(x)$, $k_1(x,t)$, $k_2(x,t)$, $\Phi(x)$, $\Psi(x)$ are functions that have suitable derivatives in the interval $x, t \in [a,b]$, a, b, λ_1 and λ_2 are constants and $y(x)$ is the unknown function.

We consider the solution of equation (1) expressed in terms of Taylor polynomials as

$$y(x) = \sum_{n=0}^{N} \frac{1}{n!} y^{(n)}(c)(x-c)^n \qquad a \le c \le b \tag{2}$$

which is a Taylor polynomial of degree N at $x = c$. The coefficients $y^{(n)}(c)$, $n = 0, 1, \ldots, N$ are to be determined.

*Email: mahmoudi@iaut.ac.ir

International Journal of Computer Mathematics
ISSN 0020-7160 print/ISSN 1029-0265 online
http://www.tandf.co.uk/journals
DOI: 10.1080/00207160512331331110

We also consider $\Phi(y(x))$ and $\Psi(y(x))$ in equation (1) expressed in terms of Taylor polynomials as

$$\Phi(y(x)) = \sum_{n=0}^{N} \frac{1}{n!}\phi_n(x-c)^n \qquad a \le c \le b \tag{3}$$

$$\Psi(y(x)) = \sum_{n=0}^{N} \frac{1}{n!}\psi_n(x-c)^n \qquad a \le c \le b \tag{4}$$

which are Taylor polynomials of degree N at $x = c$. The coefficients $\phi_n, \psi_n, n = 0, 1, \ldots, N$ are non-linear combinations of $y^{(0)}(c), y^{(1)}(c), \ldots, y^{(N)}(c)$ as follows:

$$\begin{aligned}
\phi_0 &= \Phi(y(c)) \\
\phi_1 &= y'(c)\Phi'(y(c)) \\
\phi_2 &= y''(c)\Phi'(y(c)) + [y'(c)]^2\Phi''(y(c)) \\
\phi_3 &= y^{(3)}(c)\Phi'(y(c)) + 3y'(c)y''(c)\Phi''(y(c)) + [y'(c)]^3\Phi^{(3)}(y(c)) \\
&\vdots
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
\psi_0 &= \Psi(y(c)) \\
\psi_1 &= y'(c)\Psi'(y(c)) \\
\psi_2 &= y''(c)\Psi'(y(c)) + [y'(c)]^2\Psi''(y(c)) \\
\psi_3 &= y^{(3)}(c)\Psi'(y(c)) + 3y'(c)y''(c)\Psi''(y(c)) + [y'(c)]^3\Psi^{(3)}(y(c)) \\
&\vdots
\end{aligned} \tag{6}$$

2. Matrix representation of the components

To obtain the solution of equation (1) in the form (2) we first differentiate it n times with respect to x to obtain

$$\begin{aligned}
y^{(n)}(x) = f^{(n)}(x) &+ \lambda_1 \frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_a^x k_1(x,t)\Phi(y(t))\,\mathrm{d}t \\
&+ \lambda_2 \int_a^b \frac{\partial^n}{\partial x^n}k_2(x,t)\Psi(y(t))\,\mathrm{d}t \qquad n = 0, 1, 2, \ldots, N
\end{aligned} \tag{7}$$

and then analyse it as matrix representation.

According to [5], equation (7) can be written as

$$y^{(n)}(x) = f^{(n)}(x) + \lambda_1 V^{(n)}(x) + \lambda_2 F^{(n)}(x) \tag{8}$$

where

$$V^{(n)}(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_a^x k_1(x,t)\Phi(y(t))\,\mathrm{d}t$$

and

$$F^{(n)}(x) = \int_a^b \frac{\partial^n k_2(x,t)}{\partial x^n} \Psi(y(t))\,\mathrm{d}t.$$

Then

$$V^{(n)}(x) = \sum_{i=0}^{n-1} [h_i(x)\Phi(y(x))]^{(n-i-1)} + \int_a^x \frac{\partial^n k_1(x,t)}{\partial x^n} \Phi(y(t))\,\mathrm{d}t \tag{9}$$

where

$$h_i(x) = \left.\frac{\partial^i}{\partial x^i} k_1(x,t)\right|_{t=x}.$$

Using Liebnitz's rule we obtain

$$V^{(n)}(x) = \sum_{m=0}^{n-1} \sum_{i=0}^{n-m-1} \binom{n-i-1}{m} h_i^{(n-m-i-1)}(x)[\Phi(y(x))]^{(m)} + \int_a^x \frac{\partial^n k_1(x,t)}{\partial x^n} \Phi(y(t))\,\mathrm{d}t. \tag{10}$$

Note that, from [1],

$$\sum_{m=0}^{n-1} \sum_{i=0}^{n-m-1} (\ldots) = \sum_{i=0}^{n-1} \sum_{m=0}^{n-i-1} (\ldots)$$

in equation (10).

If we approximate $y(x)$, $\Phi(y(x))$ and $\Psi(y(x))$ as in equations (2), (3) and (4), we obtain

$$V^{(n)}(c) = \sum_{m=0}^{n-1} \sum_{i=0}^{n-m-1} \binom{n-i-1}{m} h_i^{(n-m-i-1)}(c)\phi_m + \sum_{m=0}^{N} \frac{1}{m!}\phi_m \int_a^c \left.\frac{\partial^n k_1(x,t)}{\partial x^n}\right|_{x=c} (t-c)^m\,\mathrm{d}t \tag{11}$$

and

$$F^{(n)}(c) = \sum_{m=0}^{N} \frac{1}{m!}\psi_m \int_a^b \left.\frac{\partial^n k_2(x,t)}{\partial x^n}\right|_{x=c} (t-c)^m\,\mathrm{d}t. \tag{12}$$

Substituting equations (11) and (12) in equation (8) gives

$$y^{(n)}(c) = f^{(n)}(c) + \lambda_1 \sum_{m=0}^{n-1} \sum_{i=0}^{n-m-1} \binom{n-i-1}{m} h_i^{(n-m-i-1)}(c)\phi_m + \lambda_1 \sum_{m=0}^{N} \frac{1}{m!}\phi_m \int_a^c \left.\frac{\partial^n k_1(x,t)}{\partial x^n}\right|_{x=c} (t-c)^m\,\mathrm{d}t + \lambda_2 \sum_{m=0}^{N} \frac{1}{m!}\psi_m \int_a^b \left.\frac{\partial^n k_2(x,t)}{\partial x^n}\right|_{x=c} (t-c)^m\,\mathrm{d}t \tag{13}$$

or, briefly,

$$y^{(n)}(c) = f^{(n)}(c) + \lambda_1 \sum_{m=0}^{N} H_{nm}\phi_m + \lambda_2 \sum_{m=0}^{N} K_{nm}\psi_m \tag{14}$$

where

$$H_{nm} = \begin{cases} \displaystyle\sum_{i=0}^{n-m-1} \binom{n-i-1}{m} h_i^{(n-m-i-1)}(c) + \frac{1}{m!}\int_a^c \left.\frac{\partial^n k_1(x,t)}{\partial x^n}\right|_{x=c} (t-c)^m \,\mathrm{d}t & n > m \\ \displaystyle\frac{1}{m!}\int_a^c \left.\frac{\partial^n k_1(x,t)}{\partial x^n}\right|_{x=c} (t-c)^m \,\mathrm{d}t & n \le m \end{cases} \tag{15}$$

and

$$K_{nm} = \frac{1}{m!}\int_a^b \left.\frac{\partial^n k_2(x,t)}{\partial x^n}\right|_{x=c} (t-c)^m \,\mathrm{d}t. \tag{16}$$

If we take $n = 0, 1, 2, \ldots, N$, relation (14) reduces to a system of $N+1$ non-linear equations for $N+1$ unknown coefficients $y^{(0)}(c), y^{(1)}(c), \ldots, y^{(N)}(c)$, as follows:

$$\mathbf{Y} = \mathbf{F} + \lambda_1 \mathbf{H}\Phi + \lambda_2 \mathbf{K}\Psi \tag{17}$$

where

$$\mathbf{Y} = \begin{bmatrix} y^{(0)}(c) \\ y^{(1)}(c) \\ y^{(2)}(c) \\ \vdots \\ y^{(N)}(c) \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} f^{(0)}(c) \\ f^{(1)}(c) \\ f^{(2)}(c) \\ \vdots \\ f^{(N)}(c) \end{bmatrix}, \quad \Phi = \begin{bmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{bmatrix}, \quad \Psi = \begin{bmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{bmatrix}$$

and $\mathbf{H}$ and $\mathbf{K}$ are $(N+1) \times (N+1)$ matrices defined in (15) and (16). The non-linear system of equations obtained in (17) can be solved using standard mathematics toolboxes such as Maple.

We study the accuracy of the solution as follows. Since the truncated Taylor series in (2) is the approximate solution of the equation (1), it must satisfy the equation approximately. Then, for each $x_0 \in [a, b]$,

$$R(x_0) = \left| \sum_{n=0}^{N} \frac{1}{m!} y^{(n)}(c)(x - x_0)^n - f(x_0) - \lambda_1 V(x_0) - \lambda_2 F(x_0) \right| \approx 0 \tag{18}$$

where

$$V(x_0) = \int_a^{x_0} k_1(x_0, t)\Phi(y(t)) \,\mathrm{d}t$$

$$F(x_0) = \int_a^b k_2(x_0, t)\Psi(y(t)) \,\mathrm{d}t.$$

If $\max R(x_0) \le 10^{-k}$ (k is any positive integer and $x_0 \in [a, b]$) is chosen, then the truncation limit N is increased until the difference $R(x_0)$ at each $x_0 \in [a, b]$ becomes smaller than 10^{-k} [3, 4].

3. Numerical illustrations

We show the efficiency of the method described using the following examples.

Example 1: Let us first consider the non-linear Volterra integral equation

$$y(x) = x^2 - 2 - \frac{1}{2}\, x\mathrm{e}^{x^2-2} + \frac{1}{2}\mathrm{e}^{-2}x + \int_0^x xt\mathrm{e}^{y(t)}\,\mathrm{d}t. \tag{19}$$

When $N = 5$ the required vector and matrices are as follows:

$$\mathbf{H} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 15 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{K} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} -2 & 0 & 2 & -3\mathrm{e}^{-2} & 0 & -30\mathrm{e}^{-2} \end{bmatrix}^{\mathrm{T}}.$$

The solution is

$$\mathbf{Y} = \begin{bmatrix} -2 & 0 & 2 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}.$$

which gives the exact solution $y(x) = x^2 - 2$.

Example 2: We consider the non-linear Volterra integral equation

$$y(x) = 2x - x\,\cos(x) - x^2\sin(x) + \int_0^x xt\,\cos[y(t)]\,\mathrm{d}t. \tag{20}$$

When $N = 6$ the required vector and matrices are as follows:

$$\mathbf{H} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 15 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 24 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{K} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 0 & 1 & 0 & -3 & 0 & 15 & 0 \end{bmatrix}^{\mathrm{T}}.$$

Table 1. Numerical results for example 3.

x_r	Exact solution	Taylor solution $y(x_r)$			
		$N = 3$	$N = 4$	$N = 5$	$N = 6$
0	1	1.000000	1.000000	1.000000	1.000000
0.2	1.24	1.235955	1.239658	1.243272	1.238432
0.4	1.56	1.543819	1.558632	1.573089	1.553726
0.6	1.96	1.923592	1.956921	1.989451	1.945884
0.8	2.44	2.375276	2.434526	2.492358	2.414905
1	3	2.898868	2.991447	3.081809	2.960788

Table 2. Numerical results for example 4.

x_r	Exact solution	Taylor solution $y(x_r)$ $N=3$	$N=4$	$N=5$	$N=6$
0	−1	−1.000000	−1.000000	−1.000000	−1.000000
0.2	−0.8	−0.800826	−0.800141	−0.800020	−0.800003
0.4	−0.6	−0.601663	−0.600284	−0.600041	−0.600005
0.6	−0.4	−0.402525	−0.400433	−0.400062	−0.400008
0.8	−0.2	−0.203424	−0.200593	−0.200085	−0.200011
1	−0	−0.004371	−0.000770	−0.000111	−0.000014

The solution is

$$\mathbf{Y} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$$

which gives the exact solution $y(x) = x$.

Example 3: We consider Fredholm integral equation

$$y(x) = f(x) + \int_0^1 2x^2 t \ \ln[y(t)]\,\mathrm{d}t \tag{21}$$

where

$$f(x) = 1 + x + \left[1 - \frac{3}{2}\ \ln(3) + \frac{\sqrt{3}}{6}\pi\right] x^2.$$

The exact solution is $y(x) = 1 + x + x^2$. Table 1 shows the numerical results for example 3.

Example 4: Finally, we consider the Volterra–Fredholm integral equation

$$y(x) = x - 1 - \mathrm{e}^{x-1} + \mathrm{e}^{-1} + \int_0^x (x - t)\mathrm{e}^{y(t)}\,\mathrm{d}t + \int_0^1 xt\mathrm{e}^{y(t)}\,\mathrm{d}t. \tag{22}$$

The exact solution is $y(x) = x - 1$. Table 2 gives the numerical results for example 4.

4. Conclusion

Non-linear integral equations are usually difficult to solve analytically and so it is necessary to obtain the approximate solutions. The present method is effective for cases where the known functions have sufficient derivatives in the given interval. One of the advantages of this method is that the solution is expressed as a Taylor series truncated at $x = c$. Therefore $y(x)$ can easily be evaluated for arbitrary values of x with a low computation at effort.

This method will not work for cases where the given functions do not have enough derivatives. An interesting feature of this method is that we obtain analytical solution in many cases, as shown in the examples. To obtain the best approximate solution of the given equation, the degree N of the approximate solution must be chosen sufficiently large. If N is too small the solution will not be sufficiently accurate, whereas if N is too large the required non-linear system will be too expensive to solve.

This method can be developed and applied to linear and non-linear systems of integral and integro-differential equations.

References

[1] Sezer, M., 1994, Taylor polynomial solution of Volterra integral equations. *International Journal of Mathematical Education in Science and Technology*, **25**, 625–633.
[2] Sezer, M., 1996, A method for the approximate solution of the second-order linear differential equations in terms of Taylor polynomials. *International Journal of Mathematical Education in Science and Technology*, **27**, 821–834.
[3] Yalcinbas, S., 2002, Taylor polynomial solutions of non-linear Volterra–Fredholm integral equations. *Applied Mathematics and Computation*, **127**, 195–206.
[4] Yalcinbas, S. and Sezer, M., 2000, The approximate solution of high-order linear Volterra–Fredholm integro-differential equations in terms of Taylor polynomials. *Applied Mathematics and Computation*, **112**, 291–308.
[5] Maleknejad, K. and Mahmoudi, Y., 2003, Taylor polynomial solution of high-order non-linear Volterra–Fredholm integro-differential equations. *Applied Mathematics and Computation*, **145**, 641–653.