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Crack Problems in the Classical Theory of Elasticity

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Crack Problems in the Classical Theory of Elasticity

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Preface

Interest in crack problems in the mathematical theory of elasticity arises from the theory of brittle fracture, which itself originated nearly fifty years ago in the classical work of Griffith (1921). Since the number of materials that fail under normal conditions in a brittle fashion is relatively small, for many years this theory was regarded as of academic rather than practical interest; more as a source of interesting mixed boundary problems (e.g., see Sneddon, 1966) than as a growing part of solid mechanics. Interest has revived in the theory in recent years as a result of the experimental discovery that at high or low temperatures many structural elements composed of commonly used materials that display plastic properties in standard tensile tests fail by a "quasi-brittle" process. By this it is meant that failure occurs by the propagation of cracks and that although there is a plastic zone it is of limited extent and concentrated at the crack tip.

In this book an account is given of calculations in the mathematical theory of elasticity relating to Griffith cracks and their three-dimensional analogues and has some relevance to the theory of brittle fracture. In Chapter 1 we include a brief discussion of the physical considerations that indicate the need for the calculation of certain features of the stress field. The basic equations used in this theory are discussed in Sections 1.3–1.6 of Chapter 1. Heavy use is made of a modification of the Papkovitch-Boussinesque-Neuber solution of the equations of elastic equilibrium due to Sneddon (1961). Chapter 2 deals with two-dimensional crack problems, beginning with a discussion of Griffith's (1921) calculations and ending with a brief consideration of cohesive forces at the tip of Griffith cracks due to Barenblatt (1961). A brief account is given of solutions of dynamical crack problems. Chapter 3 contains a discussion of three dimensional problems in which the crack is in the form of a flat desk. Except in Section 3.8, in which we consider a disk in the shape of an ellipse, we shall confine our attention to flat cracks in the form of circular disks; these are usually called *penny-shaped cracks*. An extensive bibliography is also included. We have attempted to include in the bibliography those papers relevant to the mathematical theory rather than the physical theory.

This book is based on a set of lectures given in the mathematics department at North Carolina State University by the first named author during the spring of 1961. These lectures have been expanded and brought up to date. We are indebted to the late Dr. J. W. Cell who provided the initial opportunity for us to work on the manuscript. Part of the work was supported by various air-force contracts at both North Carolina State and Duke universities. In addition the second author was supported in part by a Leverhulme fellowship at the University of Glasgow during the 1962–1963 academic year and by a National Science foundation science-faculty fellowship during 1966–1967. We should like to express our gratitude to the many secretaries at Duke University, Wesleyan University, University of Glasgow, and Indiana University who took great care in the typing of various stages of the manuscript.

We are indebted to Dr. R. P. Srivastav, now of the Department of Applied Analysis, SUNY at Stony Brook, for many helpful suggestions while the manuscript was being prepared and for valuable help in the reading of the proof sheets.

Glasgow, Scotland
Bloomington, Indiana
June 1969

Ian N. Sneddon
Morton Lowengrub

Contents

Crack Problems in the Classical Theory of Elasticity

CHAPTER 1

Introduction

1.1 INTRODUCTION

We shall be concerned in this book with the calculation of the distribution of stress in the neighborhood of a crack in an elastic solid. The problems are of two distinct kinds. The first group of problems (treated in Chapter 2) concerns the determination of the stress fields governed by the equations of plane strain or by the equations of plane stress. In the first instance a crack, which in a two-dimensional diagram is represented by a segment of a straight line, is in reality a long flat ribbon-shaped cavity in a solid; it is stressed in such a way that there is no variation in the stress pattern as we pass in a direction parallel to the plane of the crack. In both cases we shall call the crack a *Griffith crack*. For clarity we shall treat the plane strain case since the plane stress case can then be deduced merely by altering the value of Poisson's ratio of the material.

Chapter 3 contains a discussion of the second group of problems, in which the crack is in the form of a flat disk. Except in Section 3.8, where we consider a disk in the shape of an ellipse, we shall confine our attention to flat cracks in the form of circular disks; these are usually called *penny-shaped cracks.**

Only the simplest problems will be considered but in the main we shall derive exact solutions of them. For instance, we shall assume that the infinitesimal theory holds and that the elastic body is both homogeneous and isotropic. Most of the methods of solution we shall develop could easily be extended to the anisotropic case. The calculations are essentially no more

* This term seems to have first appeared in print in Sneddon (1946) but was originated by Sir Neville Mott who suggested the problem in conversation with the author.

difficult; it is just that the arithmetic is more complicated. We also make an assumption which in this field is analogous to what is called the "surface wave approximation" in hydrodynamics; that is, when we are specifying the conditions on a crack surface, we relate these to the *undisturbed* surface. This is consistent with the use of the equations of the classical theory of elasticity.

No account is taken of the fact that plastic strains will develop at the tips of Griffith cracks and along the rims of penny-shaped cracks. Irwin (1958) has adduced physical arguments to suggest that this will not result in significant loss of accuracy in the calculation of relevant quantities in fracture theory.

We shall consider problems in which the crack is situated within a body with definite boundaries (e.g., a Griffith crack in a thick strip is considered in Section 2.11) but, because of the simplicity of the results and their value in the theory of fracture, most attention is devoted to the discussion of stress fields near a crack in an infinite elastic body.

In two dimensions there are three basic problems corresponding to three distinct modes of displacement. In the first mode, shown in Fig. 1, there is a

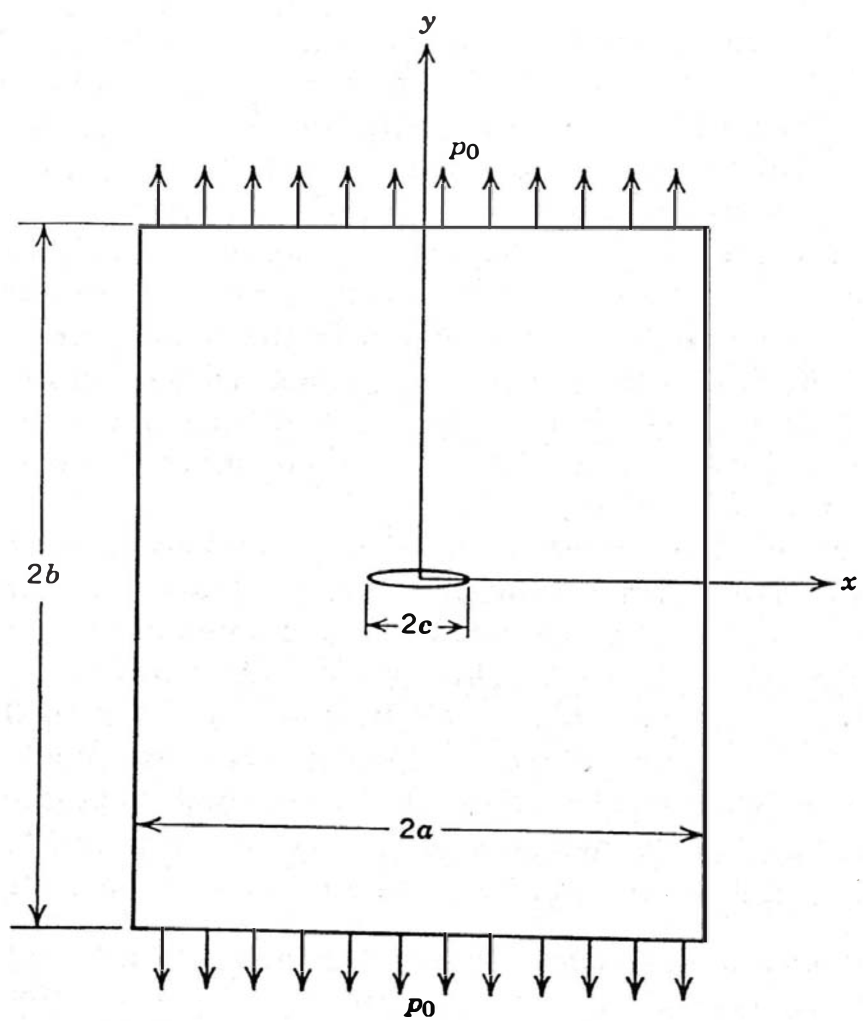


Figure 1 Griffith crack in a plate: Mode I displacement.

Griffith crack (of length $2c$) in a solid under a tension which is in a direction perpendicular to the line of the crack. Here we are interested in calculating the *stress intensity factor* K_1, defined by the equation

$$K_1 = \lim_{x \to c+} \sqrt{2(x - c)}\, \sigma_{yy}(x,0), \tag{1.1.1}$$

where the x-axis is along the crack and the origin is at the crack center. It is also useful to calculate the strain energy W_1 required to form the crack. In two dimensions we define the strain energy release rate of a Griffith crack of length Z by the equation

$$\mathcal{G} = \frac{\partial W}{\partial Z}.$$

Thus in this case, we have the equation

$$\mathcal{G}_1 = \frac{1}{2}\frac{\partial W_1}{\partial c}. \tag{1.1.2}$$

The *Griffith criterion* for failure is that the crack will begin to spread when the applied tension reaches the value given by the equation

$$\frac{\partial}{\partial c}(W - U) = 0, \tag{1.1.3}$$

where U is the surface energy of the crack. Griffith expresses U in terms of a "surface tension" T of the solid by the equation $U = 4cT$. Thus, by using equation (1.1.2) we can write (1.1.3) in the form

$$\mathcal{G} = 2T. \tag{1.1.4}$$

It is interesting to note that recently Burns and Lawn (1968) devised a simulated crack experiment to test the validity of the energy balance criterion of Griffith. Their results indicate that, at least for the materials they tested, Griffith's criteria is a viable assumption.

In the second mode, shown in Fig. 2, the solid is under an applied shear parallel to the crack. In this case we should be interested in calculating the stress intensity factor

$$K_2 = \lim_{x \to c+} \sqrt{2(x - c)}\, \sigma_{xy}(x,0), \tag{1.1.5}$$

the strain energy W_2, and the corresponding strain energy release rate.

In the third mode, shown in Fig. 3, the solid is under an applied shear perpendicular to the crack. Here, we should be interested in calculating

$$K_3 = \lim_{x \to c+} \sqrt{2(x - c)}\, \sigma_{xz}(x,0,z). \tag{1.1.6}$$

The three-dimensional analogues for these modes of displacement are shown for a penny-shaped crack in Fig. 4. To help the reader visualize the physical

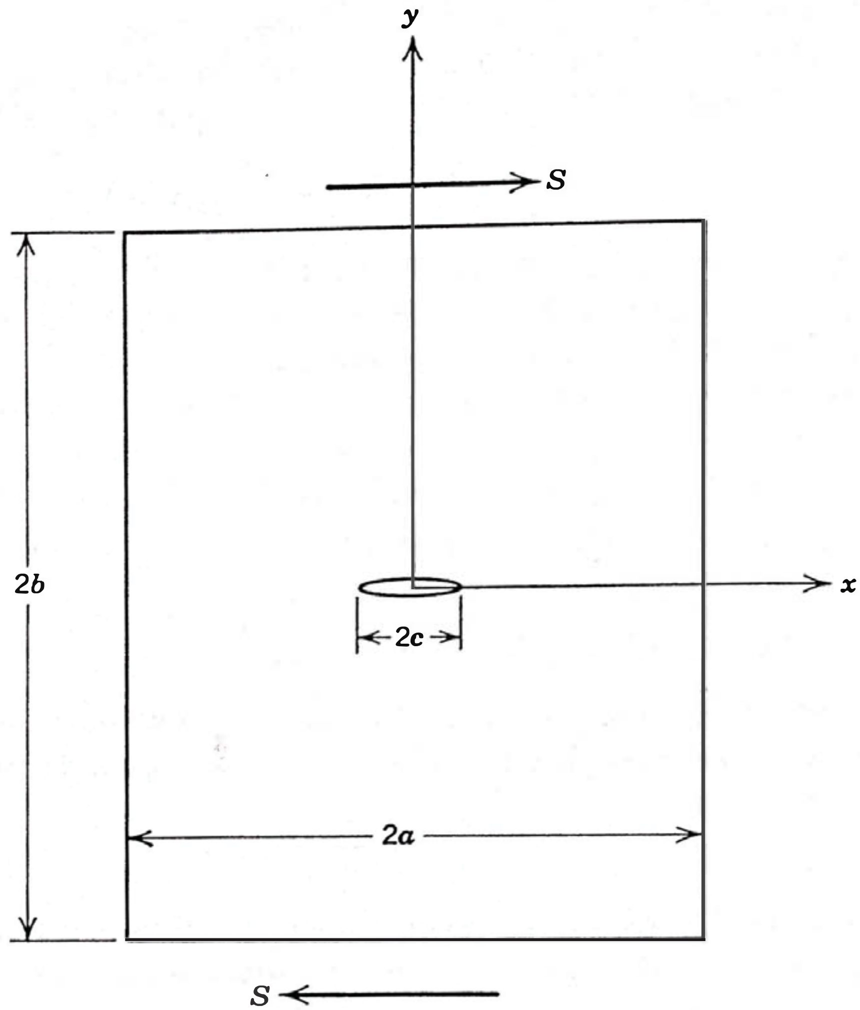


Figure 2 Griffith crack in a plate: Mode II displacement.

situation, we have drawn a penny-shaped crack of radius c in a right circular cylinder of radius a; the displacement fields in which we are interested correspond to the case in which $a \gg c$. In Fig. 4(i), the solid is under tension, in Fig. 4(ii) it is under shear, and in Fig. 4(iii) it is under torsion.

Equations (1.1.1), (1.1.2), and (1.1.4) have their three-dimensional counterparts. If we take a system of cylindrical coordinates (ρ,θ,z) with origin at the center of the crack and z-axis along the axis of symmetry, then the stress intensity factor in which we are interested, in first mode, is defined by

$$K_1 = \lim_{\rho \to c+} \sqrt{2(\rho - c)}\, \sigma_{zz}(\rho,0). \tag{1.1.7}$$

In three dimensions, we define the strain energy release rate of a disk-shaped crack of radius A by

$$\mathscr{G} = \frac{\partial W}{\partial A},$$

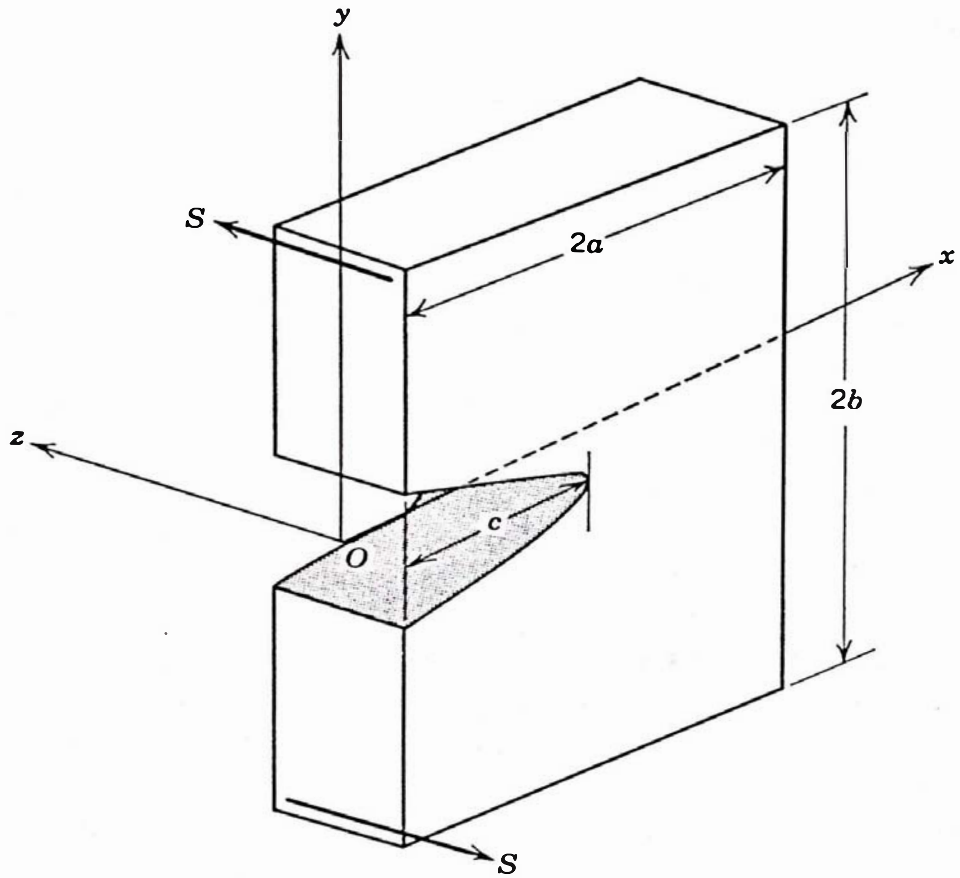


Figure 3 Section of a plate with a Griffith crack: Mode III displacement.

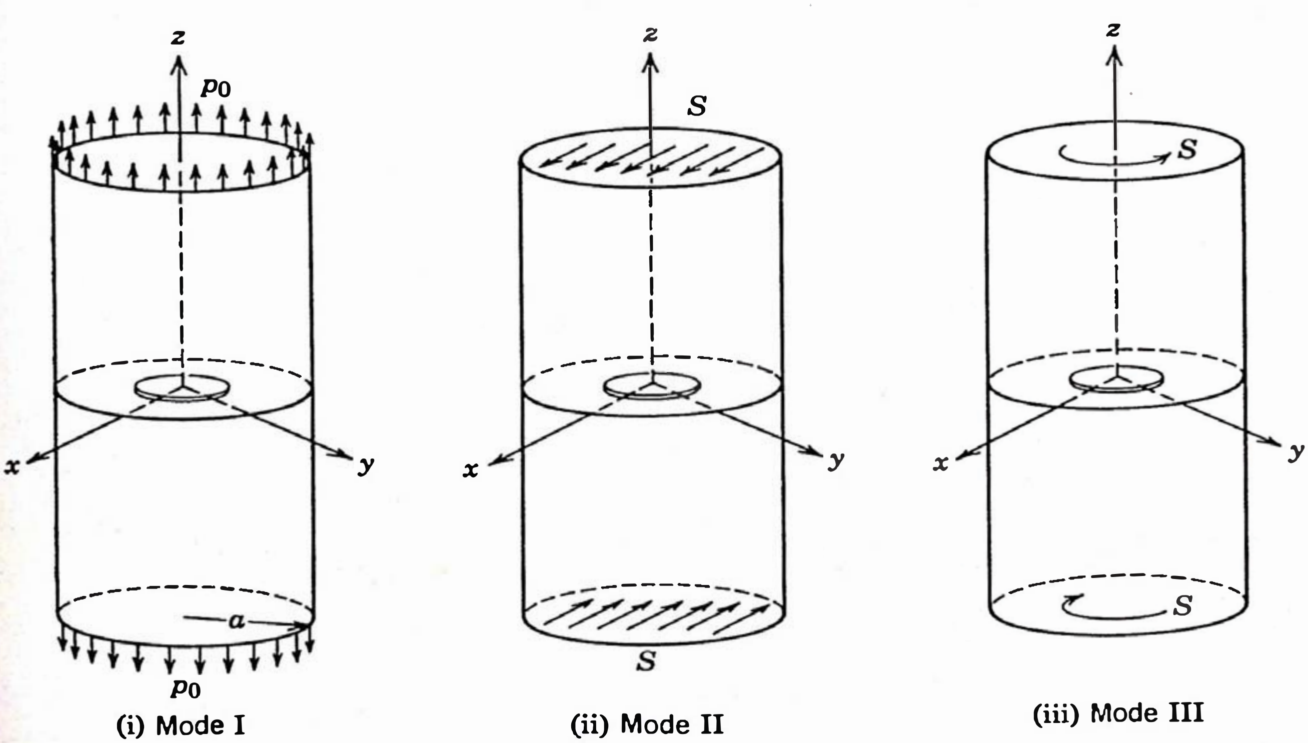


Figure 4 Penny-shaped crack in a circular cylinder.

so that, in this case, we have the equation

$$\mathscr{G}_1 = \frac{1}{2\pi c}\frac{\partial W_1}{\partial c}\,. \tag{1.1.8}$$

The Griffith criterion can again be written in the form (1.1.3), but now we have $U = 2\pi c^2 T$. With this value of U and the value of $\mathscr{G}$ given by (1.1.8), we see that in the case of penny-shaped cracks, the Griffith criterion again takes the form (1.1.4).

The quantities K and $\mathscr{G}$ can be defined analogously for the second and third modes.

It is of interest now to give an account of the physical considerations which lead us to believe that there is some merit in calculating the stress intensity factor on the basis of the classical theory of elasticity. Irwin (1957) has proposed an approach based on the following considerations. Suppose that the shape of a Griffith crack, when its length is $2c$, is shown by the full curve in Fig. 4*a* and that its shape when its length is $2(c + \delta c)$ is shown by the dotted curve. Irwin argues that if the tensile stress $\sigma_{yy}(x,0)$ obtained from the solution in the case of a crack of length $2c$ is exerted on the faces $c \leq |x| \leq c + \delta c$ of the slightly enlarged crack, then it will close up the crack to its original length and the work done must be equivalent to $4T(\delta c)$ for a brittle solid, where T is the surface tension discussed previously. For a partially brittle solid, where plastic zones of limited extent come into existence Irwin (1948) replaces $2T$ by a constant $\mathscr{G}_c$ which he calls *crack extension force*. (See also

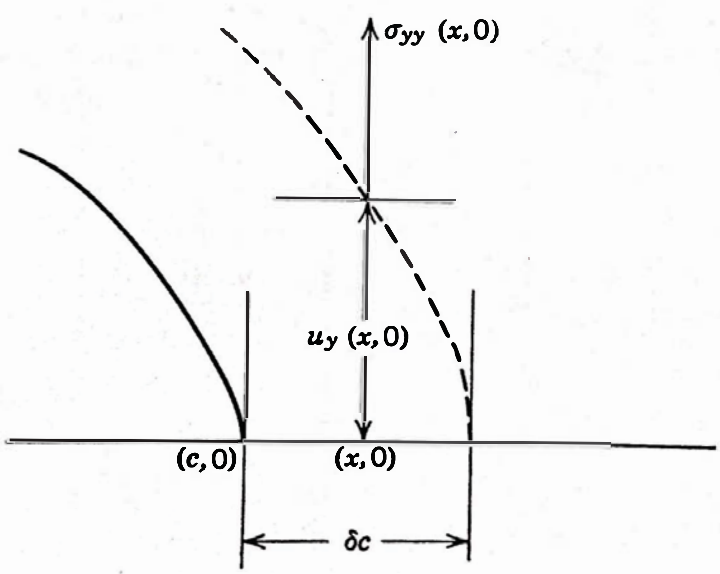


Figure 4a

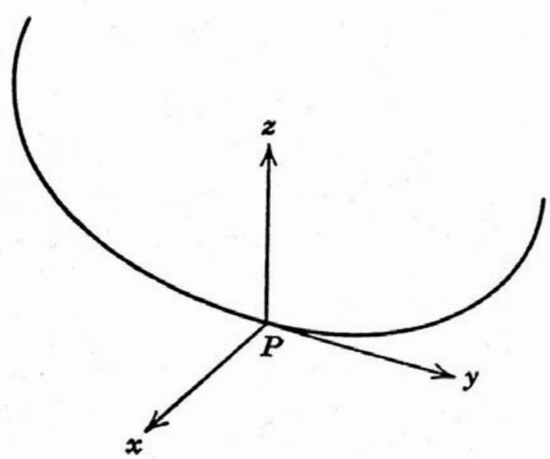


Figure 4b

Orowan, 1948–9.) Irwin's relation is therefore of the form

$$\frac{1}{\delta c}\int_c^{c+\delta c} u_y(x,0)\sigma_{yy}(x,0)\,dx = \mathscr{G}_c \tag{1.1.9}$$

where $u_y(x,0)$ is the surface displacement of the crack. If we define the stress intensity factor K_1 by (1.1.1), then we may take

$$\sigma_{yy}(x,0) = \frac{K_1}{\sqrt{2(x-c)}}$$

in the region of the tip $c \leq |x| \leq c + \delta c$. It turns out that we can approximate to $u_y(x,0)$ to the same degree of accuracy by the formula

$$u_y(x,0) = \frac{4(1-\eta^2)}{E} K_1\sqrt{2(c+\delta c - x)}. \tag{1.1.10}$$

Thus we see that equation (1.1.7) is equivalent to the condition

$$\frac{2\pi(1-\eta^2)}{E} K_1 = \mathscr{G}_c. \tag{1.1.11}$$

There are corresponding expressions in the case of penny-shaped cracks. Irwin's relation (1.1.9), may be written in the form

$$\frac{1}{c\,\delta c}\int_c^{c+\delta c} \rho\sigma_{zz}(\rho,0)u_z(\rho,0)\,d\rho = \mathscr{G}_c, \tag{1.1.12}$$

where $u_z(\rho,0)$ is the surface displacement of the crack. The stress intensity factor is given by (1.1.7), hence for $c \leq \rho \leq c + \delta c$ we may write

$$\sigma_{zz}(\rho,0) = \frac{K_1}{\sqrt{2(\rho - c)}}, \tag{1.1.13}$$

and it turns out that we may take

$$u_z(\rho,0) = \frac{4(1-\eta^2)}{E} K_1(c+\delta c-\rho)^{1/2} \tag{1.1.14}$$

in the range so that equation (1.1.12) can be written in terms of the stress intensity factor in the form (1.1.11), but with K_1 defined by (1.1.7) instead of (1.1.1).

An entirely different approach was formulated by Barenblatt. (For references to earlier work see Barenblatt, 1961.) In a paper on the mechanism of hydraulic fracture of an oil-carry stratum, Zhelton and Khristianovich put forward the hypothesis that the position of the tip of a Griffith crack was determined by the condition that the stress intensity factor K should vanish there. This condition implies the finiteness of the stresses and the smooth closure of opposite faces at the tip of a Griffith crack, or along the edge of a penny-shaped crack. Using the principle of virtual work, Barenblatt (1960) was able to establish this earlier conjecture.

Barenblatt's ideas are more clearly seen if we consider a plane crack of arbitrary profile in an infinite solid body (see Fig. 4*b*). We introduce a local coordinate system at a typical point P on the rim of the crack, the x-axis being normal to the rim and pointing into the solid, the y-axis being tangential to the rim, and the z-axis normal to the plane of the crack. It can be shown that at points of the x-axis near the origin P the z-components of the stress tensor are of the form

$$\sigma_{xz} = \frac{K_2}{\sqrt{x}} + O(1), \qquad \sigma_{yz} = \frac{K_3}{\sqrt{x}} + O(1), \qquad \sigma_{zz} = \frac{K_1}{\sqrt{x}} + O(1) \quad (1.1.15)$$

while the remaining components σ_{xx}, σ_{xy}, σ_{yy} are all $O(1)$. In any particular case, the stress intensity factors K_1, K_2, K_3 will depend on the position of the point P, the shape of the crack profile, the load applied to the crack faces, and the presence of other cracks in the solid, but they will not depend on x. Barenblatt and Cherepanov (1961) have shown that all the components of stress at an arbitrary point on the rim of the crack must be finite, so that we must have

$$K_1 = K_2 = K_3 = 0. \qquad (1.1.16)$$

In most problems (including the simplest one in which the pressure on the crack faces is constant) the values of the stress intensity factors, as calculated on the basis of the classical theory of elasticity, turn out to be non-zero. It is obvious that this discrepancy arises because the physical conditions at the rim of the crack profile cannot be adequately represented within the framework of the classical theory. To deal with this difficulty, Barenblatt (1959) has proposed that the surface of the crack be divided into two regions: an "inner" region in which the opposite faces of the crack are well separated so that cohesive forces may be neglected, and an "end" region in which the opposing crack faces are so close to each other that cohesive forces are brought into play. For instance, in the case of quasi-brittle fracture, the end region of the crack surface can be taken to be the plastic zone surrounding the crack, and the roles of the cohesive forces are played by forces applied by the plastic tip of the crack. In any event, we usually assume that the linear dimensions of the end region are very small in comparison with the greatest diameter of the crack. It is also assumed that the distribution of the cohesive force in the neighborhood of points at which its intensity attains its maximum value is independent of the loading conditions on the crack.*

* Barenblatt calls this the "autonomy" hypothesis.

These assumptions enable us to calculate the stress intensity factors K_{1_α}, K_{2_α}, and K_{3_α} due to the cohesive forces acting on the end region of the crack. If the corresponding expressions calculated by the infinitesimal theory for the forces acting on the inner region are K_{1_0}, K_{2_0}, K_{3_0} respectively, then equations (1.1.16) reduce to

$$K_{1_0} + K_{1_\alpha} = 0, \qquad K_{2_0} + K_{2_\alpha} = 0, \qquad K_{3_0} + K_{3_\alpha} = 0. \quad (1.1.17)$$

For instance, Barenblatt has shown that the stress intensity factor K_{2_α} can be expressed in terms of the cohesion modulus $\mathscr{K}$, a constant of the material, through

$$K_{2_\alpha} = -\frac{\mathscr{K}}{\pi}.$$

These considerations lead to the condition that everywhere on the rim of a crack,

$$K_{2_0} \leq \frac{\mathscr{K}}{\pi} \qquad (1.1.18)$$

At points on the rim at which $K_{2_0} = \mathscr{K}/\pi$ the state is a limiting one so that any change in the load which would have led to an increase in the value of K_{2_0} leads instead to a movement of the crack at these points.

The crack may, of course, be oriented in any way relative to the applied load, so the shape of the end region at the start of crack propagation will not be unique. To determine the conditions for the initiation of crack propagation Barenblatt put forward the hypothesis that *for any body in which failure results from brittle or quasi-brittle fracture there exists a universal function $\Phi(-K_{1_\alpha}, -K_{2_\alpha}, -K_{3_\alpha})$ of the stress intensity factors of the cohesive forces such that*

$$\Phi(-K_{1_\alpha}, -K_{2_\alpha}, -K_{3_\alpha}) \leq 0, \qquad (1.1.19)$$

at all points on the rims of all cracks within the body. At points at which $\Phi = 0$ the state of stress is limiting in that the attainment of this state at some point on the rim makes the crack move at that point and any increase in the load which would have led to $\Phi < 0$ in fact leads to crack propagation.

Because of equations (1.1.17), we see that the limiting condition can be written in the form

$$\Phi(K_{1_0}, K_{2_0}, K_{3_0}) = 0. \qquad (1.1.20)$$

For example, if the limiting condition corresponds to the constant energy of rupture so that the density of energy Γ, expended in forming a new crack surface is constant, equation (1.1.20) takes the form

$$\frac{\pi(1+\eta)}{E}\left[(1-\eta)(K_{2_0}{}^2 + K_{3_0}{}^2) + K_{1_0}{}^2\right] = \Gamma.$$

But Barenblatt has also shown that

$$\Gamma = \frac{(1-\eta^2)\mathscr{K}^2}{\pi E},$$

so that this relation can be written in the form

$$\pi[(1-\eta)(K_{2_0}{}^2 + K_{3_0}{}^2) + K_{1_0}{}^2] = \frac{1-\eta}{\pi}\mathscr{K}^2.$$

Therefore we may take the function Φ to be given the formula

$$\Phi(-K_{1_\alpha}, -K_{2_\alpha}, -K_{3_\alpha}) = K_{2_\alpha}{}^2 + K_{3_\alpha}{}^2 + (1-\eta)^{-1}K_{1_\alpha}{}^2 - \pi^{-2}\mathscr{K}^2. \tag{1.1.21}$$

The condition (1.1.20) defines the conditions for the *initiation* of crack propagation. It should be emphasized that it does not say anything about the *subsequent* motion of the crack.

To the mathematician the great merit of these formulations of the physics of crack propagation lies in the fact that for any given situation we need only know the values of the stress intensity factors (i.e., conditions near the rim of the crack) and need not undertake a complete analysis of the stress and displacement fields. It should also be observed that Kambour (1964), using optical interference techniques, has shown that the craze preceding the tip of a true crack in organic glasses has a shape very similar to that suggested by Barenblatt. This observation has been confirmed subsequently by Cotterell (1968).

A comparison of the fracture criteria of Griffith and Barenblatt has been made by Willis (1967). On the basis of the linear theory, Willis examines quantitatively the relation between these two different approaches if they are to yield the same fracture criterion; he extends the analysis to the case of a uniformly moving crack and shows that Barenblatt's modulus of cohesion depends upon the speed of the crack and is therefore not so fundamental a quantity as the surface energy which appears in the Griffith formulation.

1.2 THE BASIC EQUATIONS

We consider the deformation of an elastic body under the action of body forces, temperature fields, and surface tractions. We shall work entirely within the framework of the classical theory of elasticity in which it is assumed that the displacements and strains are small and that the physical constants of the solid (density, elastic, and thermal constants) are independent of temperature and the state of stress in the body. We shall consider a solid which at temperature T, and in the absence of external forces, is in a state of zero stress and strain. We shall be concerned mainly with equilibrium problems.

Denoting the position of a typical point of the solid by three Cartesian coordinates (x,y,z) referred to a rectangular system of axes, we shall suppose that there is a stationary temperature field $T[1 + \theta(x,y,z)]$ in the solid and that the equilibrium of the solid is further disturbed by the action of surface tractions. We shall follow the notation of Green and Zerna (1954) and denote the components of the displacement vector u by (u_x,u_y,u_z); the state of stress at the point (x,y,z) will be specified completely by the symmetrical tensor $(\sigma_{xx},\sigma_{xy},\sigma_{xz},\sigma_{yy},\sigma_{yz},\sigma_{zz})$.

If, further, we assume the solid to be homogeneous and isotropic, the relation between the stress and strain tensors may be written in the form

$$\sigma_{xx} = \lambda\Delta - \gamma\theta + 2\mu\frac{\partial u_x}{\partial x}, \qquad \sigma_{yz} = \mu\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}, \quad \text{etc.,} \tag{1.2.1}$$

where λ and μ denote Lamé's elastic constants, $\gamma = \alpha T(3\lambda + 2\mu)$ (where α is the coefficient of linear expansion of the solid), and Δ denotes the dilatation

$$\Delta = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}. \tag{1.2.2}$$

The equations (1.2.1) express a physical relationship which is known as the *Duhamel-Neumann Law*.

The equations of equilibrium, in the absence of body forces, assume the simple forms

$$\begin{gathered}\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z} = 0, \qquad \frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{yz}}{\partial z} = 0,\\ \frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z} = 0.\end{gathered} \tag{1.2.3}$$

In addition, the temperature deviation θ must satisfy the equation of the steady-state conduction of heat $\nabla^2\theta = 0$.

If we assume that the equations are made dimensionless by introducing a typical length a as the unit of length, and the rigidity modulus μ as the unit of stress, then the relations (1.2.1) assume the simpler form

$$\sigma_{xx} = (\beta^2 - 2)\Delta + 2\frac{\partial u_x}{\partial x} - b\theta, \qquad \sigma_{yz} = \frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}, \quad \text{etc.,} \tag{1.2.4}$$

where, in terms of Poisson's ratio η,

$$\beta^2 = \frac{\lambda + 2\mu}{\mu} = \frac{2(1-\eta)}{1-2\eta}, \tag{1.2.5}$$

and

$$b = \frac{\alpha T(3\lambda + 2\mu)}{\mu} = (3\beta^2 - 4)\alpha T. \tag{1.2.6}$$

1.3 SOLUTION IN TERMS OF POTENTIAL FUNCTIONS

It is well known that a general solution of the equations of elastic equilibrium can be written down in terms of four potential functions—the Papkovitch-Boussinesq-Neuber solution (cf. Sneddon and Berry, 1958, p. 89)—but here we shall make use of a modification of it (Sneddon, 1962*b*).

If we introduce three harmonic functions χ, ϕ, and ψ, and express the components of the displacement vector in terms of them through the vector equation

$$\mathbf{u} = \mathbf{grad}(\chi + \phi) + (\beta^2 - 1)z\frac{\partial}{\partial z}\,\mathbf{grad}\,\phi + z\,\mathbf{grad}\,\chi - \left\{(\beta^2+1)\frac{\partial \phi}{\partial z} + \psi\right\}\mathbf{k}, \tag{1.3.1}$$

where $\mathbf{k}$ is the unit vector in the direction Oz. If we then express the temperature field by

$$\theta = \frac{2}{b}\frac{\partial \psi}{\partial z}, \tag{1.3.2}$$

it is easily verified that the equations of equilibrium are satisfied identically.

Further, we can show that if, in addition to being harmonic, χ has the property that $\partial\chi/\partial z$ vanishes identically when $z = 0$, then on the plane $z = 0$,

$$\sigma_{xz} = \sigma_{yz} = 0, \tag{1.3.3}$$

$$u_z = -\left[\beta^2\frac{\partial \phi}{\partial z} + \Psi\right]_{z=0}, \tag{1.3.4}$$

$$\sigma_{zz} = 2\left[\frac{\partial^2 \chi}{\partial z^2} - (\beta^2 - 1)\frac{\partial^2 \phi}{\partial z^2} - \frac{\partial \Psi}{\partial z}\right]_{z=0}. \tag{1.3.5}$$

In problems in which there is axial symmetry we may take the axis of symmetry to be the z-axis and employ cylindrical coordinates (ρ, θ, z).

Referred to this coordinate system, the components of the displacement become $(u_\rho, 0, u_z)$, where

$$u_\rho = \frac{\partial \chi}{\partial \rho} + \frac{\partial \phi}{\partial \rho} + (\beta^2 - 1)z\,\frac{\partial^2 \phi}{\partial \rho\, \partial z} + z\,\frac{\partial \Psi}{\partial \rho}, \tag{1.3.6}$$

and u_z is given by the third component of the vector equation (1.3.1). For this solution we find that if χ is a harmonic function which vanishes on $z = 0$, then $\sigma_{\rho z} = 0$ and u_z and σ_{zz} take the boundary values (1.3.4) and (1.3.5).

1.4 TWO-DIMENSIONAL PROBLEMS

The equations of elasticity reduce to two-dimensional form in three special cases which are of some interest:

1. *Plane Strain:* In this case the displacement component u_z is identically equal to zero, and none of the physical quantities depends on z.
2. *Plane Stress:* In a state of plane stress parallel to the xy-plane, the stress components σ_{xz}, σ_{yz}, σ_{zz} all vanish but the components of the displacement vector are not, in general, independent of z.
3. *Generalized Plane Stress:* This is a state of stress in a thin plate $-h \leq z \leq h$ when $\sigma_{zz} = 0$ throughout the plate but $\sigma_{xz} = \sigma_{yz} = 0$ only on the surfaces $z = \pm h$ of the plate.

We begin by considering plane strain. Putting $u_z = 0$, $(\partial/\partial z) = 0$ in equations (1.2.4), we find that

$$\sigma_{xx} = (\beta^2 - 2)\,\frac{\partial u_y}{\partial y} + \beta^2\,\frac{\partial u_x}{\partial x} - b\theta, \tag{1.4.1}$$

$$\sigma_{yy} = \beta^2\,\frac{\partial u_y}{\partial y} + (\beta^2 - 2)\,\frac{\partial u_x}{\partial x} - b\theta, \tag{1.4.2}$$

$$\sigma_{zz} = (\beta^2 - 2)\left(\frac{\partial u_y}{\partial y} + \frac{\partial u_x}{\partial x}\right) - b\theta, \tag{1.4.3}$$

$$\sigma_{xy} = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}, \qquad \sigma_{xz} = \sigma_{yz} = 0. \tag{1.4.4}$$

It is obvious from (1.3.1) and (1.3.2) that the expressions

$$u_x = \frac{\partial \chi}{\partial x} + \frac{\partial \phi}{\partial x} + (\beta^2 - 1)y\,\frac{\partial^2 \phi}{\partial x\, \partial y} + y\,\frac{\partial \Psi}{\partial x}, \tag{1.4.5}$$

$$u_y = \frac{\partial \chi}{\partial y} - \beta^2\,\frac{\partial \phi}{\partial y} + (\beta^2 - 1)y\,\frac{\partial^2 \phi}{\partial y^2} + y\,\frac{\partial \Psi}{\partial y} - \Psi, \tag{1.4.6}$$

$$\theta = \frac{2}{b}\,\frac{\partial \Psi}{\partial y}, \tag{1.4.7}$$

substituted into equations (1.4.1) through (1.4.4), lead to stress components which satisfy the equations of equilibrium, provided that χ, ϕ, and Ψ are plane harmonic functions.

The equations of plane stress can be treated similarly. In fact, it is easily shown (Sneddon and Berry, 1958, p. 41) that in the case of plane stress, σ_{xx}, σ_{yy}, σ_{xy} are given by equations (1.4.1), (1.4.2), and (1.4.4) with β and b replaced respectively by β_1 and b_1, where

$$\beta_1{}^2 = \frac{2}{1-\eta}, \qquad b_1 = \frac{1-2\eta}{1-\eta}\, b \tag{1.4.8}$$

with η denoting Poisson's ratio. In this case $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$.

In the case of generalized plane stress we get the same expressions (as for plane stress) for $\bar{\sigma}_{xx}$, $\bar{\sigma}_{yy}$, $\bar{\sigma}_{xy}$ where

$$\bar{\sigma}_{xx} = \frac{1}{2h}\int_{-h}^{h} \sigma_{xx}\, dz$$

and similarly for $\bar{\sigma}_{yy}$ and $\bar{\sigma}_{xy}$.

1.5 THE KOLOSOV-MUSKHELISHVILI FORMULAS

The major development of the present century in the field of two-dimensional elasticity has been Muskhelishvili's work on the complex form of the two-dimensional equations due to G. B. Kolosov (1909). (See Muskhelishvili, 1953*a*, Chapter 5.)

In the units we have employed it is readily shown that the two-dimensional equations of elasticity have solution

$$\sigma_{xx} + \sigma_{yy} = 2[\phi'(z) + \overline{\phi'(z)}] - \gamma\theta(z,\bar{z}), \tag{1.5.1}$$

$$\sigma_{yy} - \sigma_{xx} + 2i\,\sigma_{xy} = 2[\bar{z}\phi''(z) + \chi'(z)] - \gamma\int \frac{\partial\theta}{\partial z}\, d\bar{z}, \tag{1.5.2}$$

$$2(u_x + iu_y) = K\phi(z) - z\overline{\phi(z)} - \overline{\chi(z)} + \tfrac{1}{2}\gamma\int \theta(z,\bar{z})\, dz, \tag{1.5.3}$$

where $z = x + iy$, $\bar{z} = x - iy$, $\theta(z,\bar{z})$ denotes the temperature, $\phi(z)$ and $\chi(z)$ are arbitrary functions of z, and for plane strain, $K = 3 - 4\eta$, $\gamma = 2(1+\eta)\alpha T/(1-\eta)$, while for plane stress, $K = (3-\eta)/(1+\eta)$, and $Y = 2(1+2\eta)\alpha T$.

In most problems we shall be dealing with the isothermal case ($\theta = 0$) in which the equations reduce to the forms

$$\sigma_{xx} + \sigma_{yy} = 2[\phi'(z) + \overline{\phi'(z)}] = 2[\Phi(z) + \overline{\Phi(z)}], \tag{1.5.4}$$

$$\sigma_{yy} - \sigma_{xx} + 2i\,\sigma_{xy} = 2[\bar{z}\phi''(z) + \chi'(z)] = 2[\bar{z}\Phi'(z) + \Psi(z)], \tag{1.5.5}$$

$$2(u_x + iu_y) = K\phi(z) - z\overline{\phi'(z)} - \overline{\chi(z)}, \tag{1.5.6}$$

where

$$\Phi(z) = \phi'(z), \qquad \Psi(z) = \chi'(z). \tag{1.5.7}$$

If we put $\chi(z) = \phi(z) - z\phi'(z)$, we obtain the special solution

$$2(u_x + iu_y) = K\phi(z) - \overline{\phi(z)} - (z - \bar{z})\overline{\phi'(z)}, \tag{1.5.8a}$$

$$\sigma_{xx} + \sigma_{yy} = 2[\phi'(z) + \overline{\phi'(z)}], \tag{1.5.8b}$$

$$\sigma_{yy} - \sigma_{xx} + 2i\,\sigma_{xy} = 2(\bar{z} - z)\phi''(z). \tag{1.5.8c}$$

For this solution we find that on $y = 0$,

$$\sigma_{xy} = 0, \tag{1.5.9a}$$

$$\sigma_{yy} = \phi'(z) + \overline{\phi'(z)}, \tag{1.5.9b}$$

$$4iu_y = (K + 1)[\phi(z) - \overline{\phi(z)}], \tag{1.5.9c}$$

provided $y\phi(z)$ and $y\overline{\phi''(z)}$ tend to zero as $y \to +0$.

The problem of the uniqueness of the functions Φ, Ψ, ϕ, χ, has also been discussed by Muskhelishvili. When the stresses are given, the arbitrary nature of these functions is removed by imposing the conditions

$$\phi(0) = 0, \qquad \phi'(0) = 0, \qquad \chi(0) = 0, \tag{1.5.10}$$

while if the *displacements* are given we must have

$$\phi(0) = 0. \tag{1.5.11}$$

It is easily shown from these formulas that the resultant force (X, Y) of the tractions exerted on the arc AB (from the positive side) can be derived from the equation

$$X + iY = -i[\phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)}]_A^B, \tag{1.5.12}$$

where $[f]_A^B$ denotes the increase in f as z traverses the arc from A to B. The moment of these tractions is

$$M = \mathrm{Re}[\chi(z) - z\psi(z) + z\bar{z}\phi'(z)]_A^B. \tag{1.5.13}$$

The transformation of the Kolosov-Muskhelishvili formulas, when there is a conformal mapping from the z-plane to the ζ-plane (where $z = x + iy$ and $\zeta = \xi + i\gamma$) is of great use in discussing boundary value problems. If we have

$$z = \omega(\zeta), \tag{1.5.14}$$

and if we denote the functions which were written earlier as $\phi(z)$, $\psi(z)$, $\Phi(z)$, $\Psi(z)$ by $\phi_1(z)$, $\psi_1(z)$, $\Phi_1(z)$, $\Psi_1(z)$, and then write

$$\phi(\zeta) = \phi_1[\omega(\zeta)], \qquad \psi(\zeta) = \psi_1[\omega(\zeta)], \qquad \Phi(\zeta) = \Phi_1[\omega(\zeta)],$$
$$\Psi(\zeta) = \psi_1(\omega(\zeta)], \tag{1.5.15}$$

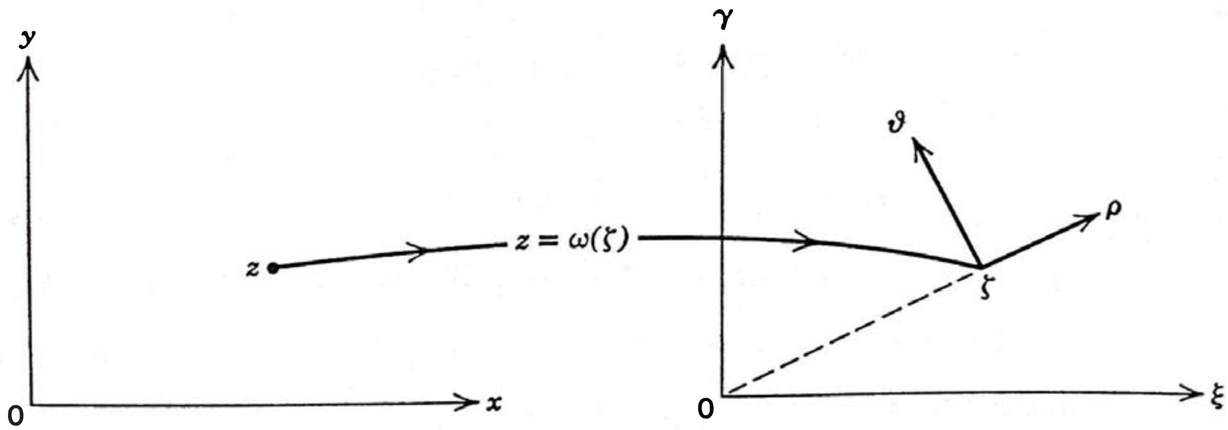


Figure 5 Conformal mapping from the z-plane to the ζ-plane.

we can show (see, for example, Muskhelishvili, (1953a) Chapter 7) that

$$\sigma_{\xi\xi} + \sigma_{\gamma\gamma} = 2[\Phi(\zeta) + \overline{\Phi(\zeta)}], \tag{1.5.16}$$

$$\sigma_{\xi\xi} - \sigma_{\gamma\gamma} + 2i\,\sigma_{\xi\gamma} = -\frac{2}{\omega'(\zeta)}[\omega(\zeta)\bar\Phi'(\bar\zeta) + \bar\omega'(\bar\zeta)\Psi(\zeta)], \tag{1.5.17}$$

$$2(u_\xi + iu_\gamma) = \frac{\bar\omega'(\bar\zeta)}{\omega'(\zeta)}\left[K\phi(\zeta) - \frac{\omega(\zeta)}{\bar\omega'(\bar\zeta)}\,\phi'(\bar\zeta) - \bar\psi(\bar\zeta)\right]. \tag{1.5.18}$$

When mapping into circular regions we put $\zeta = \rho e^{ie}$ to obtain, in polar coordinates, the set of formulas,

$$\sigma_{\rho\rho} + \sigma_{\theta\theta} = 2[\Phi(\zeta) + \overline{\Phi(\zeta)}], \tag{1.5.19}$$

$$\sigma_{\theta\theta} - \sigma_{\rho\rho} + 2i\sigma_{\rho\theta} = \frac{2\zeta^2}{\rho^2\bar\omega'(\bar\zeta)}[\bar\omega(\zeta)\Phi'(\zeta) + \omega'(\zeta)\Psi(\zeta)], \tag{1.5.20}$$

$$2(u_\rho + iu_\theta) = \frac{\zeta\omega'(\zeta)}{\rho\,|\omega'(\zeta)|}\left\{K\phi(\zeta) - \frac{\omega(\zeta)}{\omega'(\zeta)}\,\overline{\phi'(\zeta)} - \bar\Psi(\bar\zeta)\right\}, \tag{1.5.21}$$

for components expressed in terms of polar coordinates (ρ,θ) in the ζ-plane (cf. Fig. 5).

1.6 THE WESTERGAARD FORMULAS

Two special cases of the formulas (1.5.4), (1.5.5) are associated with the name of Westergaard.

If, in (1.5.4) and (1.5.5) we make the substitutions

$$\Phi(z) = \tfrac{1}{2}Z_1(z), \qquad \Psi(z) = -\tfrac{1}{2}zZ_1(z), \tag{1.6.1}$$

we find that they reduce to

$$\sigma_{xx} + \sigma_{yy} = 2 \operatorname{Re} Z_1, \tag{1.6.2}$$

$$\sigma_{yy} - \sigma_{xx} = 2y \operatorname{Im} Z_1', \tag{1.6.3}$$

$$\sigma_{xy} = -y \operatorname{Re} Z_1'. \tag{1.6.4}$$

It should be observed that this solution has the property that, on the line $y = 0$, $\sigma_{xy} = 0$, and $\sigma_{xx} = \sigma_{yy}$.

Similarly, if in (1.5.4) and (1.5.5) we make the substitutions

$$\Phi(z) = -\tfrac{1}{2}iZ_2(z), \qquad \Psi(z) = \tfrac{1}{2}izZ_2' + iZ_2, \tag{1.6.5}$$

we find that the corresponding stress field is specified by the equations

$$\sigma_{xx} + \sigma_{yy} = 2 \operatorname{Im} Z_2, \tag{1.6.6}$$

$$\sigma_{yy} - \sigma_{xx} = -2 \operatorname{Im} Z_2 - 2y \operatorname{Re} Z_2', \tag{1.6.7}$$

$$\sigma_{xy} = \operatorname{Re} Z_2 - y \operatorname{Im} Z_2'. \tag{1.6.8}$$

This solution has the property that, on the line $y = 0$, $\sigma_{yy} = 0$.

1.7 SOLUTION OF PLANE DYNAMICAL EQUATIONS

It is only recently that solutions have been obtained of boundary value problems in the dynamical theory of elasticity. Most of the problems discussed have been two-dimensional and here we restrict our attention to the case of plane strain. We also assume that the temperature is fixed and that we are interested only in problems in which the disturbing influences are moving with a velocity c parallel to the x-axis.

In this case it is convenient to introduce the coordinate $x' = x - ct$.

It can be shown (Sneddon and Berry, 1958, p. 118) that the equations of motion are satisfied by the displacement components

$$u_x = \operatorname{Re}[F_1'(z_1) + \tfrac{1}{2}(1 + \beta_2{}^2)F_2'(z_2)], \tag{1.7.1}$$

$$u_y = \operatorname{Im}\left[\beta_1 F_1'(z_1) + \frac{1 + \beta_2{}^2}{\beta_2} F_2'(z_2)\right], \tag{1.7.2}$$

where $F_1(z_1)$ and $F_2(z_2)$ are analytical functions of the complex variables z_1 and z_2 defined in terms of the wave velocities $c_1 = [(\lambda + 2\mu)/\rho]^{1/2}$, $c_2 = (\mu/\rho)^{1/2}$, by the equations

$$z_1 = x' + i\beta_1 y, \qquad z_2 = x' + i\beta_2 y, \tag{1.7.3}$$

with

$$\beta_1{}^2 = 1 - \frac{c^2}{c_1{}^2}, \qquad \beta_2{}^2 = 1 - \frac{c^2}{c_2{}^2}. \tag{1.7.4}$$

The components of stress can be determined by means of the equations

$$\sigma_{yy} = (1 + \beta_2{}^2)\mathrm{Re}[F_1''(z_1) + F_2''(z_2)], \tag{1.7.5}$$

$$\sigma_{xx} + \sigma_{yy} = -2(\beta_1{}^2 - \beta_2{}^2)\mathrm{Re}[F_1''(z_1)], \tag{1.7.6}$$

$$\sigma_{xy} = 2\,\mathrm{Im}\left[\beta_1 F_1''(z_1) + \frac{(1 + \beta_2{}^2)^2}{4\beta_2} F_2''(z_2)\right]. \tag{1.7.7}$$

The solution in this form is due to Radok (1956) but apart from a simple difference of a notation it is identical to one derived earlier by Sneddon (1952).

CHAPTER 2

Two Dimensional Crack Problems

2.1 THE GRIFFITH CRACK

Although the earliest calculation of the distribution of stress in the neighborhood of an elliptic crack in a thin plate seems to be due to Inglis (1913), interest in such calculations appear to stem from Griffith's paper (1921) in which he put forward an explanation of the fracture strength of glass on the basis that glass contains crack-like flaws. We might remark at this point that cracks can be considered as surfaces of discontinuity of the material (i.e., of the displacement vector). In the Griffith theory it is assumed that cracks exist or are formed in a solid body when it is subjected to tensile forces and that when the tensile forces are increased to a point where the strain-energy release rate with crack extension is greater than the rate at which energy is gained by the creation of new free surface area, rapid crack extension takes place and the solid fractures. To exploit even the simplest physical model of this type, it is necessary to investigate the stress distribution in the neighborhood of a crack and to calculate the increase of strain energy due to the presence of the crack.

Griffith made use of Inglis' calculations which enable us (in principle, if not so easily in practice) to calculate the distribution of stress in an infinite plate containing an elliptic crack with semi-major axis c and semi-minor axis b. Griffith considered the case in which $b = 0$ when the ellipse degenerates to a straight line of length $2c$. For this reason a crack occupying, say, the segment

$$y = 0, \quad -c \leq x \leq c, \tag{2.1.1}$$

is now called a *Griffith crack*.

In most problems, we are interested in the case in which the surfaces of the crack are stress free and there is a prescribed tensile stress system at infinity so that

$$\sigma_{yy} = \sigma_{yx} = 0 \qquad \text{on} \qquad -c \leq x \leq c, \quad y = 0, \tag{2.1.2}$$

and

$$\sigma_{yx} \to 0, \qquad \sigma_{yy} \to p_0, \qquad \sigma_{xx} \to q_0 \qquad \text{as} \qquad \sqrt{x^2 + y^2} \to \infty. \tag{2.1.3}$$

It is a simple matter to use the superposition principle to turn the solution of such a problem into a discussion of the problem

$$\sigma_{yy} = -p_0, \qquad \sigma_{yx} = 0 \qquad \text{on} \qquad -c \leq x \leq c, \quad y = 0, \tag{2.1.4}$$

and

$$\sigma_{yx} \to 0, \qquad \sigma_{yy} \to 0, \qquad \sigma_{xx} \to 0 \qquad \text{as} \qquad \sqrt{x^2 + y^2} \to \infty. \tag{2.1.5}$$

In the above equations p_o, q_o are constants.

The boundary value problem posed by these equations is that of finding the stress distribution in the neighborhood of a Griffith crack when it is opened out by the application of a constant pressure to its free surface. This problem can easily be generalized to the case in which the internal pressure opening the crack varies along the length of the crack. Instead of equations (2.1.4) and (2.1.5) we then have

$$\sigma_{yy} = -p(x), \qquad \sigma_{xy} = 0 \qquad \text{on} \qquad -c \leq x \leq c, \qquad y = 0, \tag{2.1.6}$$

and

$$\sigma_{xy} \to 0, \qquad \sigma_{xx} \to 0, \qquad \sigma_{yy} \to 0 \qquad \text{as} \qquad \sqrt{x^2 + y^2} \to \infty, \tag{2.1.7}$$

where, now, $p(x)$ is a prescribed function of x in the closed interval $[-c, c]$.

One way of treating such a problem is to consider the stress distribution in the plane region bounded by two ellipses E, E' (cf. Fig. 6) when in the case of the boundary value problem described by the set of relations (2.1.2) and (2.1.3), the perimeter of the ellipse E is free from stress and that of E' has certain prescribed tractions. The limiting form of the solution can then

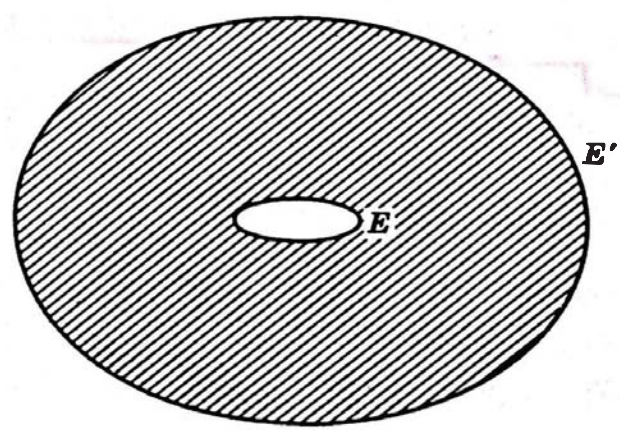


Figure 6 Elliptic hole in an elliptic plate.

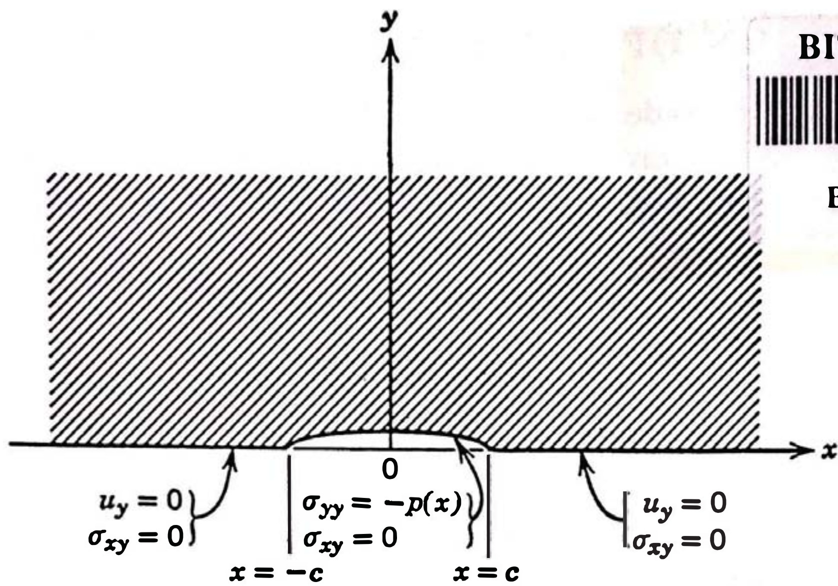


Figure 7 The boundary value problem for the half-plane $y \geq 0$ corresponding to Mode I displacement in a Griffith crack.

be obtained by letting E degenerate into a Griffith crack and the minimum diameter of E' tend to infinity. This method is considered in Section 2.2.

Another approach (adopted in Section 2.3) consists of changing the boundary value problem for an infinite solid into one for a semi-infinite solid. If we assume that the pressure is the same for both faces of the crack, it is obvious that the stress field is symmetrical about the line $y = 0$. As a result of the symmetry (cf. Fig. 7) we see that, outside the crack, the component of the displacement vector normal to the x-axis and the shearing stress σ_{xy} are both zero. Inside the crack the normal component of stress is prescribed to be $-p(x)$, say, and the shearing stress to be zero. In this way we arrive at the boundary conditions

$$\sigma_{xy}(x,0) = 0, \qquad \sigma_{yy}(x,0) = -p(x), \quad |x| \leq c, \tag{2.1.8}$$

$$\sigma_{xy}(x,0) = 0, \qquad u_y(x,0) = 0, \qquad |x| \geq c, \tag{2.1.9}$$

and the conditions at infinity

$$\sigma_{xx} \to 0, \qquad \sigma_{xy} \to 0, \qquad \sigma_{yy} \to 0 \qquad \text{as} \qquad \sqrt{x^2 + y^2} \to \infty \quad y \geq 0, \tag{2.1.10}$$

for the half-plane $y \geq 0$.

If, in addition, $p(x)$ is an *even* function of x, we can replace conditions (2.1.8) by the relations

$$\sigma_{xy}(x,0) = 0, \qquad \sigma_{yy}(x,0) = -p(x), \qquad 0 \leq x \leq c \tag{2.1.11}$$

and equations (2.1.9) by the relations

$$\sigma_{xy}(x,0) = 0, \qquad u_y(x,0) = 0, \quad x \geq c. \tag{2.1.12}$$

2.2 SOLUTION OF THE ELLIPTIC HOLE PROBLEM

The physical considerations of Griffith are based on an analysis by Inglis (1913) of the stress in the neighborhood of an elliptic hole E in an elastic plate whose boundary was in the form of a concentric ellipse E' (cf. Fig. 8).

Inglis uses the curvilinear coordinate system defined by

$$x = c \cosh \xi \cos \gamma, \qquad y = c \sinh \xi \sin \gamma, \tag{2.2.1}$$

so that the curve $\xi = \alpha$ is an ellipse with center the origin, semi-major axis $c \cosh \alpha$ and semi-minor axis $c \sinh \alpha$. In particular, $\xi = 0$ is the Griffith crack $|x| \leq c$, $y = 0$. It is easily shown that the equations of plane strain are satisfied if we take

$$\sigma_{\xi\xi} = \frac{p_0 \sinh 2\xi(\cosh 2\xi - \cosh 2\alpha)}{(\cosh 2\xi - \cos 2\gamma)^2}, \tag{2.2.2}$$

$$\sigma_{\gamma\gamma} = \frac{p_0 \sinh 2\xi(\cosh 2\xi + \cosh 2\alpha - 2\cos 2\gamma)}{(\cosh 2\xi - \cos 2\gamma)^2}, \tag{2.2.3}$$

$$\sigma_{\xi\gamma} = \frac{p_0 \sin 2\gamma(\cosh 2\xi - \cosh 2\alpha)}{(\cosh 2\xi - \cos 2\gamma)^2}, \tag{2.2.4}$$

and that these expressions satisfy the conditions

$$\sigma_{\xi\xi} = \sigma_{\xi\gamma} = 0$$

on the elliptic hole $\xi = \alpha$. Also as $\xi \to \infty$, $\sigma_{\xi\xi} \to p_o$, $\sigma_{\gamma\gamma} \to p_o$, $\sigma_{\xi\gamma} \to 0$ so that the solution corresponds to an elliptic hole in a plate under an

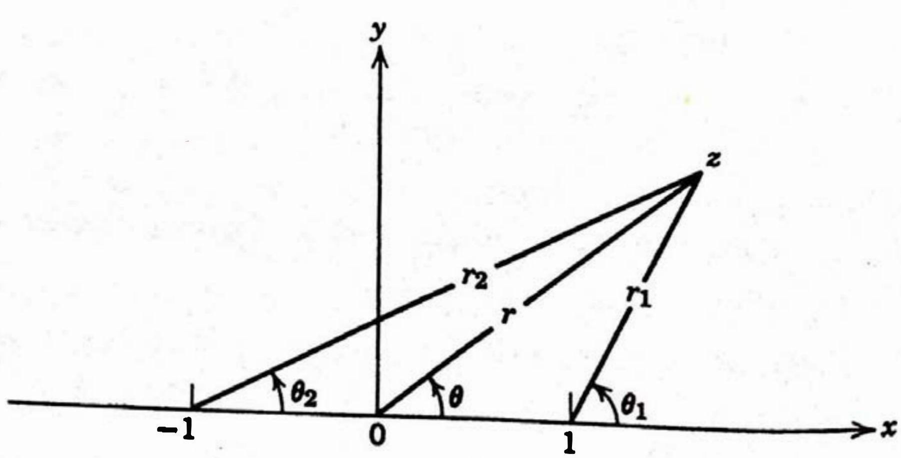


Figure 8 The coordinates (r,θ), (r_1,θ_1), (r_2,θ_2).

"all-round" tension p_0. In the case of plane strain, the corresponding displacement field is given by the pair of equations

$$u_\xi = -\frac{p_0(1+\eta)c}{\sqrt{2E}}(\cosh 2\xi - \cos 2\gamma)^{-\frac{1}{2}}$$
$$\times \{(1-2\eta)\cosh 2\xi + 2\eta \cos 2\gamma - \cosh 2\alpha\}, \quad u_\gamma = 0. \quad (2.2.5)$$

The integral expression for the strain energy per unit thickness of the material within the region bounded by the ellipses $\xi = \xi_1$ and $\xi = \alpha$ is difficult to evaluate as it stands but since we are interested only in the case in which ξ_1 is very large we can show that it is given approximately by the expression

$$-\frac{\pi p_0{}^2 c^2(1+\eta)}{4E}\{(1-2\eta)e^{2\xi_1} - 4(1-\eta)\cosh 2\alpha\} + O(e^{-2\xi_1}).$$

Hence W, the increase of strain energy due to the elliptic cavity $\xi = \alpha$ in an infinite plane is given by

$$W = \frac{\pi p_0{}^2 c^2(1-\eta^2)}{E}\cosh 2\alpha.$$

In the limit $\alpha = 0$ we obtain the expression

$$W = \frac{\pi p_0{}^2 c^2(1-\eta^2)}{E}. \quad (2.2.6)$$

Inglis also showed that if the crack $\alpha = 0$ is subjected to an applied stress q_o parallel to the length of the crack, $W = 0$. In other words, the general form of the crack strain energy W is determined by the applied stress p_o, perpendicular to the crack.

The strain energy release rate corresponding to the energy (2.2.6) is

$$\mathscr{G} = \frac{\pi p_0{}^2 c(1-\eta^2)}{E} \quad (2.2.7)$$

so that the Griffith criterion (1.1.4) leads to the relation

$$\boxed{p_{cr} = \left\{\frac{2ET}{\pi c(1-\eta^2)}\right\}^{\frac{1}{2}},} \quad (2.2.8)$$

for the critical stress producing spread of the crack (in the case of plane strain).

This formula was obtained by Griffith from Inglis' solution. Spencer (1965) has pointed out that, in performing energy calculations of this kind, care must be exercised to formulate *precisely* the boundary conditions at the outer of the two confocal ellipses bounding the solid. He shows that if either tractions or displacements are specified at the boundary of a circle of large radius R to within quantities of an order to ensure that the work done tends to zero as $R \to \infty$, the equation (2.2.6) is correct. Spencer's analysis demonstrates clearly the need to specify surface tractions to order p_0c^2/R^2 or surface displacements to order p_0c^2/ER if a meaningful expression for the crack energy is to be maintained.

A solution of the problem of an elliptic crack in an infinite plate was given by Muskhelishvili (1919) using a simple conformal mapping of the exterior of an ellipse into the exterior of the unit circle. The method used is a particular case of a more general method given by Muskhelishvili; for a discussion of this work, the reader is referred to Chapters 15 and 21 of Muskhelishvili (1953a).

The problem of determining the stresses in the neighborhood of an elliptic crack provides a good illustration of the usefulness of the method indicated in the latter part of Section 1.5. We consider the case of an unstressed elliptic hole in an infinite plate. The elliptic coordinates defined by (2.2.1) can be considered as being defined by the conformal mapping

$$z = \psi(\zeta) = c \cosh \zeta, \tag{2.2.9}$$

where $z = x + iy$, $\zeta = \xi + i\gamma$ and in these coordinates the elliptic boundary is defined, as before, by $\xi = \alpha$. It is assumed that at infinity the plate is in a state of "all-round" tension p_o, so that, from (1.5.16) and (1.5.17) we must have

$$\Phi(\zeta) + \overline{\Phi(\zeta)} \to p_0, \qquad \overline{\omega(\zeta)}\Phi'(\zeta) + \omega'(\zeta)\Psi(\zeta) \to 0 \tag{2.2.10}$$

as $|z| \to \infty$. Stevenson (1943) found that complex potentials of the forms

$$\phi(z) = Ac \sinh \zeta, \qquad \psi(z) = Bc \operatorname{cosech} \zeta, \tag{2.2.11}$$

with A and B real constants, were sufficient to describe the stress distribution. It is easily verified that these functions satisfy the conditions (1.5.10), while from (1.5.12) and (1.5.13) we find that the resulting stresses and moment over any circuit enclosing the hole are zero. Now

$$\Phi(\zeta) = \phi'(z) = A \coth \zeta, \qquad \Psi(\zeta) = \psi'(z) = -B \operatorname{cosech}^2 \zeta \coth \zeta, \tag{2.2.12}$$

so that from (2.2.10) it is easily verified that to satisfy these equations we must take A to be $\frac{1}{2}p_o$. If we now substitute these results in (1.5.16) and (1.5.17), we find, on subtraction, that

$$\sigma_{\xi\xi} - i\sigma_{\xi\gamma} = \coth \zeta (A + A \coth \zeta \coth \bar{\zeta} + B \operatorname{cosech} \zeta \operatorname{cosech} \bar{\zeta}).$$

This expression vanishes on the boundary $\zeta + \bar{\zeta} = 2\alpha$ if

$$A[\sinh \zeta \sinh(2\alpha - \zeta) + \cosh \zeta \cosh(2\alpha - \zeta)] + B = 0;$$

i.e., if $B = -A \cosh 2\alpha = -\frac{1}{2}p_o \cosh 2\alpha$. We therefore find that the required potentials are

$$\phi(z) = \tfrac{1}{2}p_o c \sinh \zeta, \qquad \psi(z) = -\tfrac{1}{2}p_o c \cosh 2\alpha \operatorname{cosech} \zeta, \tag{2.2.13}$$

that is,

$$\phi(z) = \tfrac{1}{2}p_o\sqrt{z^2 - c^2}, \qquad \psi(z) = -\tfrac{1}{2}p_o \frac{c \cosh 2\alpha}{\sqrt{z^2 - c^2}}. \tag{2.2.14}$$

The stress components at any point in the neighborhood of the ellipse can be calculated by inserting the potentials (2.2.14) into equations (1.5.1) through (1.5.3) with, of course, $\theta(z,\bar{z})$ put identically equal to zero. In the case of a Griffith crack $\alpha = 0$ and it follows from (2.2.14) with $\cosh 2\alpha = 1$ and (1.6.1) that the stress field is determined by equations (1.6.2) through (1.6.4) with the Westergaard potential

$$Z_1(z) = \frac{p_o z}{\sqrt{z^2 - c^2}}. \tag{2.2.15}$$

This is the stress function assumed by Westergaard (1939).

2.3 SOLUTION OF THE HALF-PLANE PROBLEM: MODE I DISPLACEMENT

The problem of determining the distribution of stress in the neighborhood of a Griffith crack which is subject to an internal pressure varying along the length of the crack, has been solved by Sneddon and Elliott (1946). They considered the corresponding mixed boundary value problem for the half-plane $y \geq 0$ posed by equations (2.1.11) and (2.1.12) and solved that problem by means of Fourier transform techniques. This problem has also been considered by England and Green (1963) but we shall follow the Sneddon-Elliott solution here.

For convenience we shall choose our unit of length to be half the width of the crack, or $c = 1$.

If, in (1.4.5) and (1.4.6) we put

$$\chi = \psi = 0, \qquad \phi(x,y) = \tfrac{1}{2}(\beta^2 - 1)^{-1}\mathscr{F}_c[\xi^{-1}\psi(\xi)e^{-\xi y}; \xi \to x],$$

where we have used the notation

$$\mathscr{F}_c[f(\xi,y); \xi \to x] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(\xi,y)\cos(\xi x)\, d\xi \tag{2.3.1}$$

to denote the Fourier cosine transform of a function, we obtain the displacement field

$$u_x = -\tfrac{1}{2}\mathscr{F}_s[(1 - 2\eta - \xi y)\psi(\xi)e^{-\xi y}; \xi \to x] \tag{2.3.2}$$

$$u_y = \tfrac{1}{2}\mathscr{F}_c[(2 - 2\eta + \xi y)\psi(\xi)e^{-\xi y}; \xi \to x] \tag{2.3.3}$$

where $\mathscr{F}_s[f(\xi,y);\ \xi \to x]$ denotes the Fourier sine transform of the function $f(\xi,y)$ defined by an equation of the type (2.3.1) but with $\cos(\xi x)$ replaced by $\sin(\xi x)$. From equations (1.4.1) through (1.4.4) we easily deduce that in this case the stress field is given by the equations

$$\sigma_{xx}(x,y) = -\frac{d}{dx}\mathscr{F}_s[(1-\xi y)\psi(\xi)e^{-\xi y};\ \xi \to x], \tag{2.3.4}$$

$$\sigma_{yy}(x,y) = -\frac{d}{dx}\mathscr{F}_s[(1+\xi y)\psi(\xi)e^{-\xi y};\ \xi \to x], \tag{2.3.5}$$

$$\sigma_{xy}(x,y) = y\mathscr{F}_s[\xi^2\psi(\xi)e^{-\xi y};\ \xi \to x]. \tag{2.3.6}$$

It follows immediately from this last equation that $\sigma_{xy}(x,0) = 0$ for all values of x, so that if we are to satisfy the boundary conditions $\sigma_{yy}(x,0) = -p(x)$, $0 \le x \le 1$, $u_y(x,0) = 0$, $x > 1$ we must choose the function $\psi(\xi)$ to satisfy the pair of dual integral equations

$$\frac{d}{dx}\mathscr{F}_s[\psi(\xi);\ x] = p(x), \quad 0 \le x \le 1, \tag{2.3.7}$$

$$\mathscr{F}_c[\psi(\xi);\ x] = 0, \qquad x > 1. \tag{2.3.8}$$

These equations are a special case of a pair of dual integral equations considered by Busbridge (1938); the complete solution is given in Sneddon and Elliott (1946). Here, we shall outline a slightly different method of solution.

If we make the integral representation

$$\psi(\xi) = \sqrt{\frac{\pi}{2}}\int_0^1 f(t)J_0(\xi t)\,dt, \tag{2.3.9}$$

we see that the equation (2.3.8) is satisfied identically whatever the form of $f(t)$, and if we use the fact that

$$\int_0^\infty J_0(\xi t)\sin(\xi x)\,d\xi = \frac{H(x-t)}{\sqrt{x^2-t^2}}$$

we see that (2.3.7) is equivalent to the Abel integral equation

$$\frac{d}{dx}\int_0^x \frac{f(t)\,dt}{\sqrt{x^2-t^2}} = p(x), \quad 0 \le x \le 1,$$

which has the solution

$$f(t) = \frac{2t}{\pi}q(t), \qquad q(t) = \int_0^t \frac{p(u)\,du}{\sqrt{t^2-u^2}}. \tag{2.3.10}$$

The expression for the normal component of stress along the line of the crack is given by (2.3.5) and (2.3.9) as

$$\sigma_{yy}(x,0) = -\frac{d}{dx}\int_0^1 f(t)\,dt\int_0^\infty \sin(\xi x)J_0(\xi t)\,d\xi$$

so that we have the formula

$$\sigma_{yy}(x,0) = -\frac{d}{dx}\int_0^1 \frac{f(t)\,dt}{\sqrt{x^2-t^2}}, \quad (x > 1). \tag{2.3.11}$$

If we substitute from the first of equations (2.3.10) into (2.3.11) and integrate by parts, we find, on carrying out the differentiation, that

$$\sigma_{yy}(x,0) = \frac{2}{\pi}\left[\frac{x}{\sqrt{x^2-1}}\,q(1) + x\int_0^1 \frac{q'(t)\,dt}{\sqrt{x^2-t^2}}\right]$$

from which it follows that

$$\lim_{x\to 1+}\sqrt{(x-1)}\,\sigma_{yy}(x,0) = \frac{\sqrt{2}}{\pi}\int_0^1 \frac{p(u)\,du}{\sqrt{1-u^2}}$$

so that the stress intensity factor is given by the equation

$$K_1 = \frac{2}{\pi}\int_0^1 \frac{p(u)\,du}{\sqrt{1-u^2}}. \tag{2.3.12}$$

From (2.3.3) we see that the shape of the crack is given by

$$u_y(x,0) = (1-\eta)\mathscr{F}_c[\psi(\xi);x].$$

Substituting from (2.3.9) and (2.3.10) we see that

$$\begin{aligned} u_y(x,0) &= \frac{2(1-\eta)}{\pi}\int_0^1 tq(t)\mathscr{F}_c[J_0(\xi t);x]\,dt \\ &= \frac{2(1-\eta)}{\pi}\int_x^1 \frac{tq(t)\,dt}{\sqrt{t^2-x^2}}. \end{aligned} \tag{2.3.13}$$

We can easily show that the energy required to form the crack is given by

$$\begin{aligned} W &= 2\int_0^1 p(x)u_y(x,0)\,dx \\ &= \frac{4(1-\eta)}{\pi}\int_0^1 p(x)\,dx\int_x^1 \frac{tq(t)\,dt}{\sqrt{t^2-x^2}}. \end{aligned}$$

Interchanging the order of the integrations we find that in our system of units

$$W = \frac{4(1-\eta)}{\pi}\int_0^1 t[q(t)]^2\,dt.$$

For the consideration of problems in fracture mechanics it is useful to have these results in conventional units. For a crack of length $2c$ subject to an internal pressure $p_o p_1(x)$, where p_o is a constant with the dimensions of a pressure and $p_1(x)$ is an even, dimensionless function of x defined on $[-1, 1]$, we have

$$u_y(x,0) = b\int_{x/c}^1 \frac{tq_1(t)\,dt}{\sqrt{(t^2 - x^2/c^2)}}, \tag{2.3.14}$$

and

$$W = \frac{2\pi(1-\eta^2)p_0{}^2c^2}{E}\int_0^1 t[q_1(t)]^2\,dt, \tag{2.3.15}$$

where b and $q_1(t)$ are defined respectively by

$$b = \frac{2(1-\eta^2)p_0c}{E}, \qquad q_1(t) = \frac{2}{\pi}\int_0^t \frac{p_1(u)\,du}{\sqrt{t^2-u^2}}. \tag{2.3.16}$$

We can easily express this result in terms of a Westergaard function. Comparing equations (2.3.4) through (2.3.6) with equations (1.6.2) through (1.6.4) we see that the present solution is equivalent to the Westergaard function

$$Z_1(z) = -\sqrt{\frac{2}{\pi}}\int_0^\infty \xi\psi(\xi)e^{i\xi z}\,d\xi.$$

Inserting the form (2.3.9) for $\psi(\xi)$ and interchanging the order of the integrations we find that (in dimensional form)

$$Z_1(z) = -ip_0z\int_0^1 \frac{tq_1(t)\,dt}{(t^2 - z^2)^{3/2}}. \tag{2.3.17}$$

2.4 GRIFFITH CRACK OPENED UNDER CONSTANT PRESSURE

An important special case arises when the normal pressure on the faces of the Griffith crack is constant; e.g., p_o. Taking $p_1(u) = 1$, $0 \le u \le 1$ in equation (2.3.12) we find that, in this instance, the stress concentration factor in conventional units is

$$K_1 = p_0\sqrt{c} \tag{2.4.1}$$

while the strain energy is given by equation (2.3.15) in the form

$$W_1 = \frac{\pi(1-\eta^2)p_0{}^2c^2}{E}. \tag{2.4.2}$$

Similarly, from (2.3.14), we see that

$$u_y(x,0) = b(1 - x^2/c^2)^{1/2},$$

and hence that the crack opens out into an ellipse.

From (2.3.17) we deduce that the appropriate Westergaard function is

$$Z_1(z) = p_0\left[\frac{z}{\sqrt{z^2-1}} - 1\right]. \tag{2.4.3}$$

Apart from the constant term (which is explicable by the different boundary conditions) this is the stress function derived above [cf. (2.2.7)]. Introducing polar coordinates r, θ, r_1, θ_1, r_2, θ_2 through the equations

$$z = re^{i\theta}, \qquad z-1 = r_1e^{i\theta_1}, \qquad z+1 = r_2e^{i\theta_2},$$

(cf. Fig. 8) we see that

$$Z_1(z) = p_0\left[\frac{r}{\sqrt{r_1r_2}}\exp\{i(\theta - \tfrac{1}{2}\theta_1 - \tfrac{1}{2}\theta_2)\} - 1\right],$$

and that

$$Z_1'(z) = -\frac{p_0}{(r_1r_2)^{3/2}}e^{-\frac{3}{2}i(\theta_1+\theta_2)}.$$

Hence from equations (1.6.6) through (1.6.7) we deduce that

$$\tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) = \frac{rp_0}{\sqrt{r_1r_2}}\cos(\theta - \tfrac{1}{2}\theta_1 - \tfrac{1}{2}\theta_2), \tag{2.4.4}$$

and that

$$\tfrac{1}{2}(\sigma_{yy} - \sigma_{xx}) + i\sigma_{xy} = p_0\frac{r\sin\theta}{(r_1r_2)^{3/2}}ie^{-\frac{3}{2}i(\theta_1+\theta_2)}. \tag{2.4.5}$$

If we now revert to conventional units, we deduce immediately from these equations that

$$\sigma_{xx} = \frac{rK_1}{\sqrt{cr_1r_2}}\left[\cos(\theta - \tfrac{1}{2}\theta_1 - \tfrac{1}{2}\theta_2) - \frac{1}{r_1r_2}\sin\theta\sin\tfrac{3}{2}(\theta_1+\theta_2)\right], \tag{2.4.6}$$

$$\sigma_{yy} = \frac{rK_1}{\sqrt{cr_1r_2}}\left[\cos(\theta - \tfrac{1}{2}\theta_1 - \tfrac{1}{2}\theta_2) + \frac{1}{r_1r_2}\sin\theta\sin\tfrac{3}{2}(\theta_1+\theta_2)\right], \tag{2.4.7}$$

$$\sigma_{xy} = \frac{rK_1}{\sqrt{cr_1r_2}}\cos\tfrac{3}{2}(\theta_1+\theta_2), \tag{2.4.8}$$

where now r, r_1, r_2 are distances measured from the center and the tips of a crack of length $2c$, and θ, θ_1, θ_2 are the corresponding angles. K_1 is the stress intensity factor given by (2.4.1).

By means of these equations we can calculate the components of stress at any point in the elastic solid. The maximum shearing stress, τ, across any plane through the point (x,y) can be obtained from the relation

$$\tau = |\tfrac{1}{2}(\sigma_{yy} - \sigma_{xx}) + i\sigma_{xy}|. \tag{2.4.9}$$

Hence we find that

$$\tau(x,y) = \frac{K_1 c^2 r}{(r_1 r_2)^{3/2}} \sin \theta. \tag{2.4.10}$$

To get some idea of the distribution of stress in the vicinity of a Griffith crack, the maximum shearing stress $\tau(x,y)$ was calculated for several values of x and y by means of equation (2.4.10). The results are given in Sneddon (1946, Table 1, p. 235) and the variation of τ with x and y is shown graphically in Fig. 1 of the same paper. In the case of plane strain (which is the case considered here) the condition of constant energy of distortion (Nadai, 1931, p. 184) states that plastic flow is initiated when τ reaches the value k, where k is related to s_o, the yield point in the tension of the material, by the equation $3k^2 = s_o^2$. The maximum shear theory gives precisely the same condition except that now $2k = s_o$. Thus, in both cases, the inception of plastic flow can be predicted from the behavior of the function $\tau(x,y)$. A convenient way of showing the variation of this function and hence of visualizing the distribution of stress in the interior of the elastic solid consists of plotting the level curves of the function $\tau(x,y)$, i.e., constructing the family of curves $\tau(x,y) = \alpha p_o$, where α is a parameter. These curves are the isochromatic lines of photoelasticity. In this case, it follows from (2.4.10) that the isochromatic lines are the family with equation

$$y^8 = \alpha^2 c^2[(x - c)^2 + y^2]^3[(x + c)^2 + y^2]^3$$

where α is a parameter. The family of curves is shown in Fig. 9, which is taken from Sneddon (1946).

The fact that all the isochromatic lines pass through the points $(\pm c,0)$ shows that at both of these points (the tips of the crack) the principal shearing stress is infinite. Thus, even for small internal pressures p_o, plastic flow occurs at the tips of the crack to relieve this infinite stress. There is, in fact, no purely elastic solution of the problem. If, however, the internal pressure is not too large, the region of plastic flow will be small and will not appreciably affect the distribution of stress at points in the solid not too close to the tips of the crack.

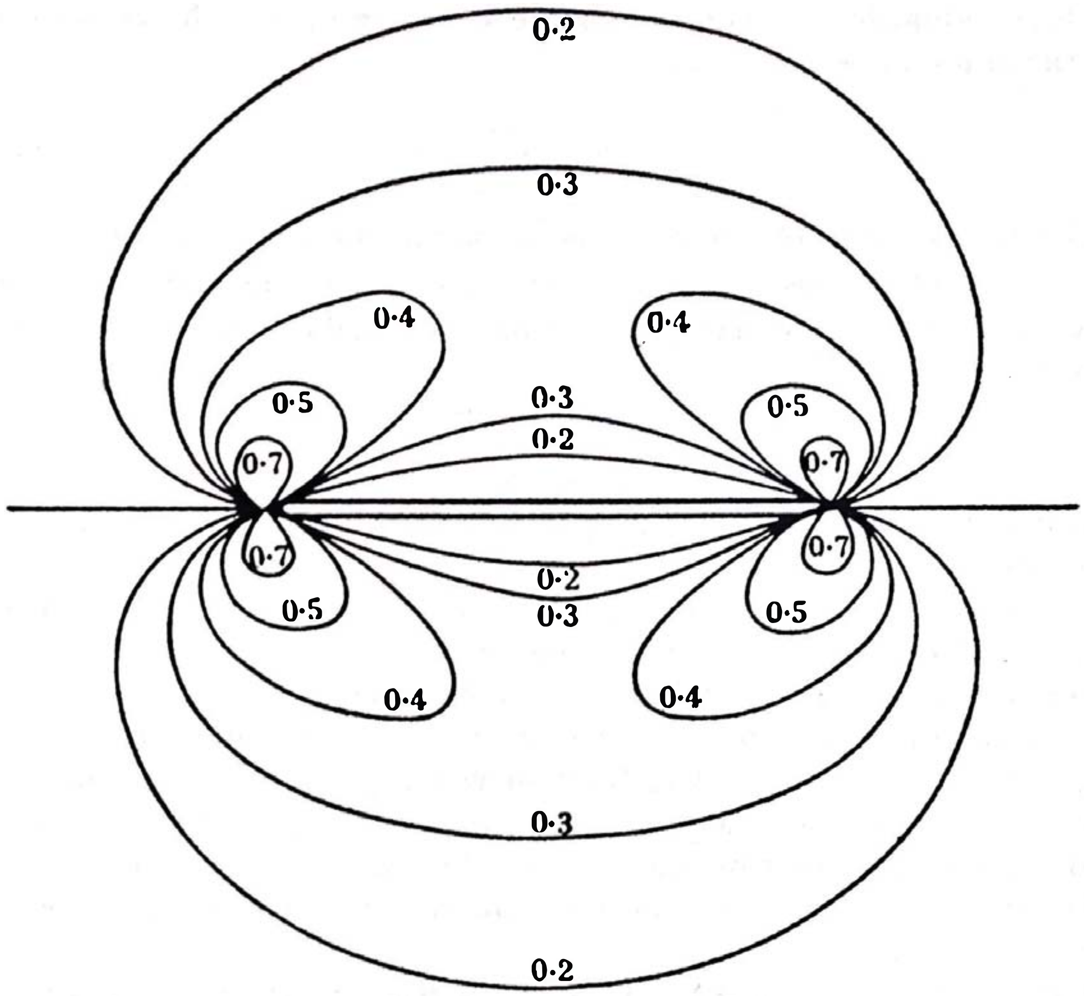


Figure 9 The isochromatic lines in the vicinity of a Griffith crack.

In certain discussions of fracture mechanics it is useful to have available expressions for the components of stress in the vicinity of the tip $x = c$, $y = 0$ of the crack. We then assume that r_1/c is very small so that we can write $r = c + r_1 \cos \theta_1$, $r_2 = 2c + 2r_1 \cos \theta_1$, $\theta = (r_1/c)\sin \theta_1$, and $\theta_2 = \frac{1}{2}(r_1/c)\sin \theta_1$ in equations (2.4.6) through (2.4.8). We then find (in conventional units) that

$$\sigma_{xx} = \frac{K_1}{\sqrt{(2r_1)}} \cos \tfrac{1}{2}\theta_1(1 - \sin \tfrac{1}{2}\theta_1 \sin \tfrac{3}{2}\theta_1), \tag{2.4.11}$$

$$\sigma_{yy} = \frac{K_1}{\sqrt{(2r_1)}} \cos \tfrac{1}{2}\theta_1(1 + \sin \tfrac{1}{2}\theta_1 \sin \tfrac{3}{2}\theta_1), \tag{2.4.12}$$

$$\sigma_{xy} = \frac{K_1}{2\sqrt{(2r_1)}} \sin \theta_1 \cos \tfrac{3}{2}\theta_1. \tag{2.4.13}$$

From these equations we can immediately deduce that in the vicinity of the tip of the crack ($x = c$, $y = 0$),

$$\tau(x,y) = \frac{K_1}{2\sqrt{2r_1}} \sin \theta_1. \tag{2.4.14}$$

The shape of the isochromatic lines in the neighborhood of the tip of the crack can readily be deduced from the last of these equations. In terms of polar coordinates r_1, θ_1 with origin at the point $(c,0)$, the family of isochromatic curves has the equation

$$r_1 = \alpha \sin^2 \theta_1$$

where α is a parameter, showing that near the tip of a Griffith crack the isochromatics are curves of two loops situated symmetrically with respect to the axis of the crack.

The presence of ties and rivets in an elastic plate can be simulated in the mathematical theory by body forces (usually considered to be concentrated at points in the plate). The calculation of the stress field in the vicinity of a Griffith crack in a plate with ties or rivets is therefore reduced to that in the vicinity of a crack in an elastic body in which prescribed body forces are acting. The case in which there is a symmetric distribution of body forces has been discussed in a paper by Sneddon and Tweed (1967*b*). The unsymmetric case is treated in a paper by the same authors to appear shortly in *Proc. Edinburgh Roy. Soc.*

The effect of couple stresses on the stress intensity at the tip of a Griffith crack has been considered by Sternberg and Muki (1967). The method of solution is similar to that used here but the calculations are more complicated; the reader is referred to their paper for details. The related problem of determining the stress singularities in the vicinity of a Griffith crack in a "Cosserat" plate has been discussed by Pagano and Sih (1968).

2.5 SOLUTION OF THE HALF-PLANE PROBLEM: MODE II DISPLACEMENT

The problem of determining the distribution of stress in the case of a Mode II displacement can be solved by using Fourier transforms in a way similar to that employed in Section 2.3. It is easily shown that the displacement field defined by the components

$$u_x(x,y) = \frac{1+\eta}{E}\mathscr{F}_c[(2 - 2\eta - \xi y)\psi(\xi)e^{-\xi y};\ \xi \to x] + \frac{2(1+\eta)}{E} Sy, \tag{2.5.1}$$

$$u_y(x,y) = \frac{1+\eta}{E}\mathscr{F}_s[(1 - 2\eta + \xi y)\psi(\xi)e^{-\xi y};\ \xi \to x], \tag{2.5.2}$$

leads to the stress field given by the components

$$\sigma_{xx} = \frac{d}{dx}\mathscr{F}_c[(2-\xi y)\psi(\xi)e^{-\xi y};\ \xi \to x], \tag{2.5.3}$$

$$\sigma_{yy} = y\frac{d}{dx}\mathscr{F}_c[\xi\psi(\xi)e^{-\xi y};\ \xi \to x], \tag{2.5.4}$$

$$\sigma_{xy} = -\frac{d}{dx}\mathscr{F}_s[(1-\xi y)\psi(\xi)e^{-\xi y};\ \xi \to x] + S. \tag{2.5.5}$$

Hence we see that as $y \to \infty$, $\sigma_{xy} \to S$ and the boundary conditions

$$\sigma_{yy}(x,0) = 0; \qquad \sigma_{xy}(x,0) = 0, \quad |x| < 1; \qquad u_x(x,0) = 0, \quad |x| > 1,$$

will be satisfied provided that $\psi(\xi)$ is a solution of the pair of dual integral equations

$$\frac{d}{dx}\mathscr{F}_s[\psi(\xi);x] = S, \quad 0 < x < 1; \tag{2.5.6}$$

$$\mathscr{F}_c[\psi(\xi);x] = 0, \quad x > 1. \tag{2.5.7}$$

These equations are identical with equations (2.3.7) and (2.3.8) so that their solution is given by equation (2.3.9) with $f(t) = St$, and we have

$$\psi(\xi) = \frac{\pi}{2}S\int_0^1 tJ_0(\xi t)\,dt = \sqrt{\frac{\pi}{2}}\,S\,\frac{J_1(\xi)}{\xi}.$$

If we substitute this result into equations (2.5.3) and (2.5.4) and add, we find that

$$\tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) = -S\,\mathscr{F}_s[J_1(\xi)e^{-\xi y};\ \xi \to x].$$

Comparing this result with equation (1.6.6) we see that the appropriate Westergaard function in this case is

$$Z_2(z) = -S\int_0^\infty J_1(\xi)e^{i\xi z}\,dz = \left[\frac{z}{\sqrt{z^2-1}} - 1\right]S.$$

If we take the length of the crack to be $2c$ and add to this function a constant term S to account for the second term on the right in equation (2.5.5) we find that Z_2 is given by

$$Z_2(z) = \frac{zS}{\sqrt{z^2-c^2}}. \tag{2.5.8}$$

Substituting from this equation into equations (1.6.6) through (1.6.8) and writing

$$K_2 = 2S\sqrt{c}, \tag{2.5.9}$$

we find that in the notation of Section 2.4, the components of stress are given by

$$\sigma_{xx} = \frac{K_2 r}{2\sqrt{c r_1 r_2}}\left[2\sin(\theta - \tfrac{1}{2}\theta_1 - \tfrac{1}{2}\theta_2) - \frac{c^2}{r_1 r_2}\sin\theta\cos\tfrac{3}{2}(\theta_1+\theta_2)\right], \tag{2.5.10}$$

$$\sigma_{yy} = \frac{K_2 r c^2 \sin\theta}{2\sqrt{c r_1^2 r_2^3}}\cos\tfrac{3}{2}(\theta_1+\theta_2), \tag{2.5.11}$$

$$\sigma_{xy} = \frac{K_2 r}{2\sqrt{c r_1 r_2}}\left[\cos(\theta - \tfrac{1}{2}\theta_1 - \tfrac{1}{2}\theta_2) - \frac{c^2}{r_1 r_2}\sin\theta\sin\tfrac{3}{2}(\theta_1+\theta_2)\right]. \tag{2.5.12}$$

Near the tip $x = c$, $y = 0$ of the crack, we have the approximate equations

$$\sigma_{xx} = -\frac{K_2}{\sqrt{(2r_1)}}\sin\tfrac{1}{2}\theta_1[2 + \cos\tfrac{1}{2}\theta_1\cos\tfrac{3}{2}\theta_1], \tag{2.5.13}$$

$$\sigma_{yy} = \frac{K_2}{\sqrt{(2r_1)}}\tfrac{1}{2}\sin\theta_1\cos\tfrac{3}{2}\theta_1, \tag{2.5.14}$$

$$\sigma_{xy} = \frac{K_2}{\sqrt{(2r_1)}}\cos\tfrac{1}{2}\theta_1[1 - \sin\tfrac{1}{2}\theta_1\sin\tfrac{3}{2}\theta_1]. \tag{2.5.15}$$

From the last of these equations we see that K_2, defined by (2.5.9) is the stress intensity factor.

It is easily shown from equation (2.5.1) that

$$u_x(x,0) = \frac{2(1-\eta^2)}{E}S\int_0^\infty \frac{J_1(\xi)}{\xi}\cos(\xi x)\,d\xi,$$

showing that if $0 \le x \le 1$,

$$u_x(x,0) = \frac{2(1-\eta^2)S}{E}\sqrt{1-x^2}.$$

Now it is easily shown that the work done in forming the crack is

$$W_2 = 2\int_0^1 S u_x(x,0)\,dx$$

$$= \frac{4(1-\eta^2)S^2}{E}\int_0^1 \sqrt{(1-x^2)}\,dx$$

$$= \frac{\pi(1-\eta^2)S^2}{E},$$

or, in conventional units, for a crack of length $2c$,

$$W_2 = \frac{\pi(1-\eta^2)S^2}{E}c^2. \tag{2.5.16}$$

The strain-energy release rate with crack extension in this case is

$$\mathscr{G}_2 = \frac{1}{2}\frac{\partial W_2}{\partial c} = \frac{\pi(1-\eta^2)S^2c}{E}. \tag{2.5.17}$$

In terms of K_2 defined by (2.5.9) we can write this equation in the form

$$\mathscr{G}_2 = \frac{\pi(1-\eta^2)K_2^{\,2}}{4E}. \tag{2.5.18}$$

These results are the ones usually accepted by workers on the theory of fracture (see, for instance, Irwin, 1958, and Paris and Sih, 1965) but they would seem to be in error. Westergaard's solution (2.5.8) on which they are based does indeed provide a solution of the half-plane problem posed by the boundary conditions

$$\sigma_{yy}(x,0) = 0, \quad 0 < |x| < \infty, \qquad \sigma_{xy}(x,0) = 0, \quad |x| < 1;$$
$$u_x(x,0) = 0, \quad |x| > 1. \tag{2.5.19}$$

This would lead to a solution of the corresponding crack problem if it also turned out that $u_y(x,0) = 0$, $|x| > 1$. We should expect this condition to be satisfied as a result of the symmetry of the problem but it is easily seen from (2.5.2) that this condition is not satisfied by the solution derived above. On the other hand there does not seem to be a simple solution to the half-plane problem with the boundary conditions:

$$\begin{aligned} \sigma_{yy}(x,0) = \sigma_{xy}(x,0) = 0, &\quad |x| \leq 1, \\ u_x(x,0) = u_y(x,0) = 0, &\quad |x| > 1. \end{aligned} \tag{2.5.20}$$

A possible way of proceeding is the following: If we take

$$\phi(x,y) = (1+\eta)(1-2\eta)$$
$$\times \frac{S}{E}\left[\mathscr{F}_s[\xi^{-1}\{A(\xi) - B(\xi)\}e^{-\xi y}; \xi \to x] + \frac{1}{(1-\eta)}\,xy\right],$$

$$\chi(x,y) = (1+\eta)(1-2\eta)$$
$$\times \frac{S}{E}\mathscr{F}_s[\xi^{-1}\{\beta^2 A(\xi) + B(\xi)\}e^{-\xi y}; \xi \to x] + 2(1+\eta)\frac{S}{E}\,xy,$$

$$\psi(x,y) = 0$$

in equations (1.4.5) and (1.4.6), we obtain the displacement field

$$u_x(x,y) = 2(1+\eta)\frac{S}{E}y + (1+\eta)\frac{S}{E}\mathscr{F}_c[\{(3-4\eta)A - \xi y(A-B)\}e^{-\xi y};\ \xi \to x], \quad (2.5.21)$$

$$u_y(x,y) = -(1+\eta)\frac{S}{E}\mathscr{F}_s[\{(3-4\eta)B - \xi y(A-B)\}e^{-\xi y};\ \xi \to x]. \quad (2.5.22)$$

For this solution we find that

$$u_x(x,0) = (1+\eta)(3-4\eta)\frac{S}{E}\mathscr{F}_c[A(\xi);x], \quad (2.5.23)$$

$$u_y(x,0) = (1+\eta)(3-4\eta)\frac{S}{E}\mathscr{F}_s[B(\xi);x], \quad (2.5.24)$$

$$\sigma_{yy}(x,0) = -S\frac{d}{dx}\mathscr{F}_c[(1-2\eta)A(\xi) + 2(1-\eta)B(\xi);\ x], \quad (2.5.25)$$

$$\sigma_{xy}(x,0) = S\frac{d}{dx}[x - \mathscr{F}_s\{2(1-\eta)A(\xi) + (1-2\eta)B(\xi);\ x\}], \quad (2.5.26)$$

so that the boundary conditions (2.5.20) are satisfied if we can find two functions $A(\xi)$, $B(\xi)$ satisfying the simultaneous set of dual integral equations

$$2(1-\eta)\mathscr{F}_s[A(\xi);x] + (1-2\eta)\mathscr{F}_s[B(\xi);x] = x, \quad 0 \le x \le 1, \quad (2.5.27)$$
$$(1-2\eta)\mathscr{F}_c[A(\xi);x] + 2(1-\eta)\mathscr{F}_c[B(\xi);x] = 0, \quad 0 \le x \le 1, \quad (2.5.28)$$
$$\mathscr{F}_c[A(\xi);x] = 0, \quad x > 1, \quad (2.5.29)$$
$$\mathscr{F}_s[B(\xi);x] = 0, \quad x > 1. \quad (2.5.30)$$

There are two obvious ways to proceed. In the first, we regard equations (2.5.27) and (2.5.30) as defining $\mathscr{F}_s[B(\xi);x]$. Inverting by means of the Fourier sine inversion theorem, we find that $A(\xi)$, $B(\xi)$ are related through the equation

$$(1-2\eta)B(\xi) + \frac{2(1-\eta)}{\pi}\int_0^\infty A(u)\left[\frac{\sin(\xi-u)}{\xi-u} - \frac{\sin(\xi+u)}{\xi+u}\right]du = f(\xi), \quad (2.5.31)$$

where $f(\xi)$ is defined by

$$f(\xi) = \sqrt{\frac{2}{\pi}}\,\frac{\sin\xi - \xi\cos\xi}{\xi^2}. \quad (2.5.32)$$

Similarly, if we regard equations (2.5.28) and (2.5.29) as defining $\mathscr{F}_c[A(\xi);x]$ and use the Fourier cosine inversion theorem, we obtain

$$(1-2\eta)A(u)+\frac{2(1-\eta)}{\pi}\int_0^\infty B(\zeta)\left[\frac{\sin(\zeta-u)}{\zeta-u}+\frac{\sin(\zeta+u)}{\zeta+u}\right]du=0. \tag{2.5.33}$$

The functions $A(\xi)$, $B(\xi)$ are then found from the solution of the pair of simultaneous integral equations (2.5.31) and (2.5.33). Alternatively, we could determine $B(\xi)$ from the integral equation

$$B(\xi)-\int_0^\infty B(\zeta)K(\xi,\zeta)\,d\zeta=(1-2\eta)^{-1}f(\xi), \tag{2.5.34}$$

whose kernel is defined by

$$K(\xi,\zeta)=\frac{4(1-\eta)^2}{\pi^2(1-2\eta)^2}\int_0^\infty\left[\frac{\sin(\xi-u)}{\xi-u}-\frac{\sin(\xi+u)}{\xi+u}\right]$$
$$\times\left[\frac{\sin(\zeta+u)}{\zeta+u}+\frac{\sin(\zeta-u)}{\zeta-u}\right]du. \tag{2.5.35}$$

The function $A(\xi)$ can then be determined by means of equation (2.5.33), via standard methods for solving Fredholm equations of the second kind (see Tricomi, 1957).

2.6 SOLUTION OF THE HALF-PLANE PROBLEM: MODE III DISPLACEMENT

The stress field in the half-plane $y \geq 0$ in the case of Mode III displacement can be derived from the displacement field

$$u_x=0,\qquad u_y=0,\qquad u_z=w(x,y). \tag{2.6.1}$$

The corresponding stress components all vanish with the exception of

$$\sigma_{xz}=\mu\frac{\partial w}{\partial x},\qquad \partial_{yz}=\mu\frac{\partial w}{\partial y},$$

so that the equations of equilibrium will be satisfied if w is a plane harmonic function. If we take

$$w(x,y)=\frac{S}{\mu}[y+\mathscr{F}_c\{\psi(\xi)e^{-\xi y};\ \xi\to x\}],$$

then

$$u_z(x,0,z)=\frac{S}{\mu}\mathscr{F}_c[\psi(\xi);x], \tag{2.6.2}$$

and

$$\sigma_{yz}(x,0,z)=S\left[1-\frac{d}{dx}\mathscr{F}_s\{\psi(\xi);x\}\right].$$

The conditions $\sigma_{yz} \to S$ as $y \to \infty$, $\sigma_{yz}(x,0,z) = 0$, $|x| < 1$, $u_2(x,0,z) = 0$, $|x| > 1$ will therefore be satisfied if $\psi(\xi)$ satisfies the dual integral equations

$$\frac{d}{dx}\mathscr{F}_s[\psi(\xi);x] = 1, \quad 0 \le x < 1,$$

$$\mathscr{F}_c[\psi(\xi);x] = 0, \quad x > 1.$$

These equations are of exactly the same type as (2.5.6) and (2.5.7) so that we have

$$\psi(\xi) = \sqrt{\frac{\pi}{2}} \cdot \frac{J_1(\xi)}{\xi}.$$

Hence we deduce that

$$\sigma_{xz} = -S\int_0^\infty J_1(\xi)e^{-\xi y}\sin(\xi x)\,d\xi, \qquad \sigma_{yz} = S\left[1 - \int_0^\infty J_1(\xi)e^{-\xi y}\cos(\xi x)\,d\xi\right].$$

We can write these results in the form

$$\sigma_{yz} + i\sigma_{xz} = S\left[1 - \int_0^\infty J_1(\xi)e^{i\xi(x+iy)}\,d\xi\right].$$

The integral on the righthand side of this equation is easily evaluated and, in the notation of Section 2.4, we find that

$$\sigma_{yz} = \frac{rS}{\sqrt{(r_1 r_2)}}\cos(\theta - \tfrac{1}{2}\theta_1 - \tfrac{1}{2}\theta_2), \qquad \sigma_{xz} = \frac{rS}{\sqrt{(r_1 r_2)}}\sin(\theta - \tfrac{1}{2}\theta_1 - \tfrac{1}{2}\theta_2). \tag{2.6.3}$$

Near the tip of a crack of length $2c$ (in conventional) units we have the approximate expressions

$$\sigma_{yz} = \frac{K_3}{\sqrt{2r_1}}\cos\tfrac{1}{2}\theta_1, \qquad \sigma_{xz} = -\frac{K_3}{\sqrt{2r_1}}\sin\tfrac{1}{2}\theta_1, \tag{2.6.4}$$

where the stress intensity factor K_3 is defined by

$$K_3 = \sqrt{c}\,S. \tag{2.6.5}$$

Similarly, from equation (2.6.2) we find that

$$u_z(x,0,z) = \frac{S}{\mu}\int_0^\infty \xi^{-1}J_1(\xi)\cos(\xi x)\,d\xi = \frac{S}{\mu}\sqrt{1 - x^2}, \quad 0 \le x < 1. \tag{2.6.6}$$

The energy of crack formation is

$$W_3 = \frac{2S^2c^2}{\mu}\int_0^1 \sqrt{(1 - x^2)}\,dx = \frac{\pi(1 + \eta)S^2c^2}{E}, \tag{2.6.7}$$

so that the corresponding strain energy release rate with crack extension is

$$\mathscr{G}_3 = \frac{\pi(1+\eta)S^2c}{E}, \tag{2.6.8}$$

or, in terms of the stress intensity factor,

$$\mathscr{G}_3 = \frac{\pi(1+\eta)K_3^{\,2}}{E}. \tag{2.6.9}$$

2.7 REDUCTION OF A CRACK PROBLEM TO THE HILBERT PROBLEM

The problem of the distribution of stress in the neighborhood of a Griffith crack (or a series of Griffith cracks) opened out by internal pressure can be reduced to the solution of a Hilbert problem. For a description of the Hilbert problem, see Green and Zerna (1954, pp. 49 and 270).

Suppose that we have a series of colinear cracks occupying the segments $L_1, L_2, \ldots, L_n$ of the x-axis X, and that the cracks are opened by application of prescribed pressures. If we write

$$L = L_1 \cup L_2 \cup \cdots \cup L_n,$$

then the boundary conditions are

(i) on $y = 0$:
$$\begin{aligned} \sigma_{xy} &= 0 && \text{for all } x, \\ \sigma_{yy} &= -p(x), && x \in L, \\ u_y &= 0, && x \in X - L \end{aligned}$$

(ii) the stress components vanish at infinity and there is no resultant force over any crack.

We note that equations (1.5.8) have the property that on $y = 0$, $\sigma_{xy} = 0$, and in addition, condition (ii) above requires that we must have at least

$$\phi(z) = O(z^{-1}) \qquad \text{as} \qquad z \to \infty. \tag{2.7.1}$$

In equations (1.5.8) and (1.5.9) $\phi(z)$ is defined in the half-plane $y > 0$ which we shall denote by S^+. We extend the definition of this function to the half-plane $y < 0$ (which we shall denote by S^-) by the formulas

$$\phi(z) = \Phi(z), \qquad \phi'(z) = \overline{\phi'(z)}, \quad z \in S^- \tag{2.7.2}$$

so that

$$\phi(\bar{z}) = \overline{\phi(z)}, \qquad \phi'(\bar{z}) = \overline{\phi'(z)}, \quad z \in S^+ \tag{2.7.3}$$

It follows from (2.7.1) that for the whole plane,

$$\phi(z) = O(z^{-1}) \qquad \text{as} \qquad z \to \infty. \tag{2.7.4}$$

If we put $y = 0$ in equations (1.5.9) and make use of the conditions (i) we find that these equations are equivalent to the conditions

$$[\phi'(x)]^{+} + [\phi'(x)]^{-} = -p(x), \quad x \in L, \tag{2.7.5}$$

$$\phi^{+}(x) - \phi'(x) = 0, \quad x \in X - L. \tag{2.7.6}$$

It follows that $\phi'(z)$ is holomorphic in $\complement L$, the complement of L, and is $O(z^{-2})$ as $|z| \to \infty$. Its form is then given by the solution of the Hilbert problem (2.7.5).

For simplicity we restrict our attention to the case of one crack $-c \leq x \leq c$ in which case it follows from equation (8.11.5) in Green and Zerna (1954, p. 270), with $K = 1$, $\gamma = 0$ that

$$\phi'(z) = \frac{-1}{2\pi i(c^2 - z^2)^{1/2}} \int_{-c}^{c} \frac{p(t)\sqrt{c^2 - t^2}}{t - z}\, dt.$$

We take $H = 0$ in Green and Zerna's equation since $\phi'(z) = O(z^{-2})$ as $|z| \to \infty$. In particular, if $p(t) = p_0$, a constant, the integral may be evaluated by the methods of the calculus of residues to give

$$\phi'(z) = -\tfrac{1}{2}p_0\left[1 - \frac{z}{\sqrt{z^2 - c^2}}\right],$$

which on integration yields

$$\phi(z) = -\tfrac{1}{2}p_0[z - \sqrt{(z^2 - c^2)}]. \tag{2.7.8}$$

If we substitute from this equation into (1.5.8) we obtain the results (2.4.6) through (2.4.8).

2.8 TWO EQUAL COLLINEAR GRIFFITH CRACKS

Willmore (1949) has considered the distribution of stress in the neighborhood of two equal collinear Griffith cracks in an isotropic material, when a uniform pressure, p_0, acts normally across the surfaces of each crack and there is no shearing stress. Tranter (1961) has used the method of Section 2.4 to discuss the case when the internal pressure p varies along the length of each crack. Recent methods due to England and Green (1963) and Sneddon and Srivastav (1965) may also be applied to this problem. We shall discuss their methods in the Section 2.9. An alternative method of solution has been given recently by Lowengrub and Srivastava (1968*a*). For the present, we restrict our attention to the problems of Willmore and Tranter.

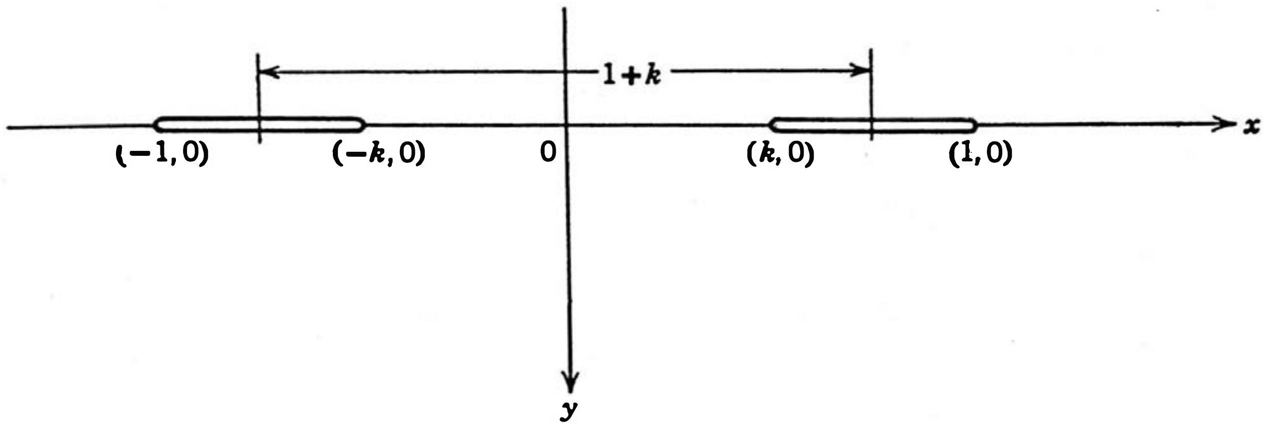


Figure 10 Coordinate system for the problem of two collinear Griffith cracks.

The problem considered by Willmore is illustrated in Fig. 10. The collinear cracks lie on the x-axis from $x = k$ to $x = 1$ and from $x = -k$ to $x = -1$. Willmore points out that the corresponding boundary value problem is equivalent to the hydrodynamical problem of the uniform motion of two equal co-planar flat plates through a fluid at rest at infinity, and deduces from the known solution of that problem (Durand, 1935, vol. II, p. 83) that suitable expressions for the Kolosov-Muskhelishvili potentials are

$$\phi'(z) = \tfrac{1}{2}p_0\left[\frac{z^2 - \lambda^2}{\sqrt{(z^2 - 1)(z^2 - k^2)}} - 1\right], \quad (k < 1) \tag{2.8.1}$$

$$\psi(z) = \phi(z) - z\phi'(z), \tag{2.8.2}$$

where

$$\lambda^2 = \frac{E'}{K'}, \tag{2.8.3}$$

and K', E' are respectively the complete elliptic integrals of the first and second kind associated with

$$k' = \sqrt{1 - k^2}, \tag{2.8.4}$$

the modulus complementary to k. Integrating (2.8.1), we see that a suitable form for $\phi(z)$ is

$$\phi(z) = \tfrac{1}{2}p_0\int_1^z \frac{(t^2 - \lambda^2)\,dt}{\sqrt{(t^2 - 1)(t^2 - k^2)}} - \tfrac{1}{2}p_0 z. \tag{2.8.5}$$

If we make the substitutions $z = dc(\zeta,k)$, $t = dc(u,k)$ we transform (2.8.5) to

$$\phi(z) = \tfrac{1}{2}p_0[sn\,\zeta\,dc\zeta + \zeta - E(\zeta) - \lambda^2\zeta - dc\zeta], \tag{2.8.6}$$

where

$$E(\zeta) = \int_0^\zeta dn^2u\,du, \tag{2.8.7}$$

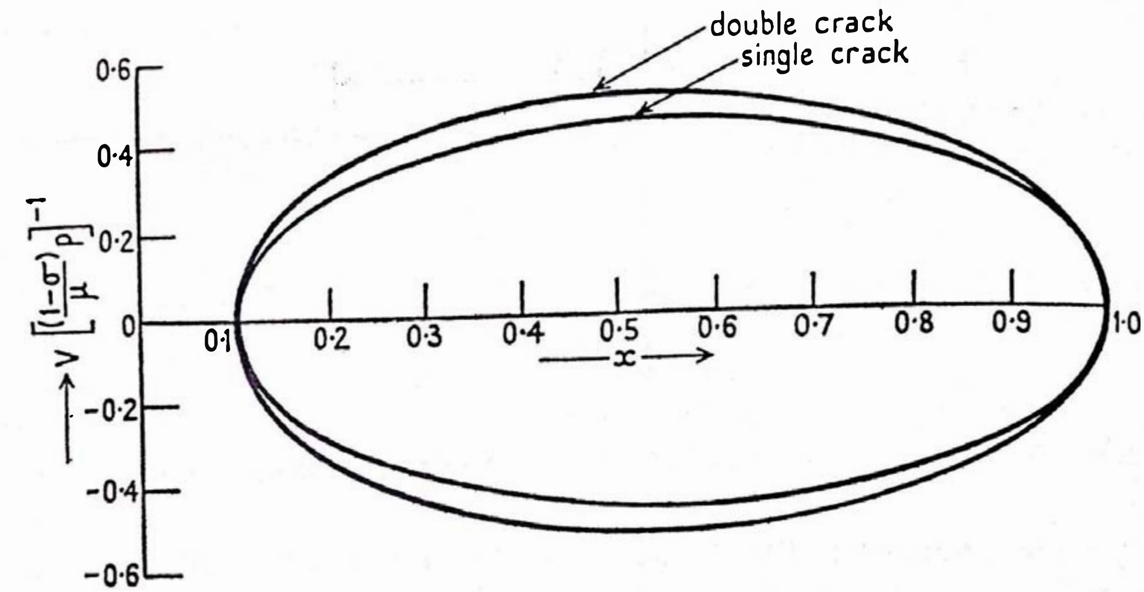


Figure 11 The shape of the crack in the case $k = 0.1$ (i.e., $\gamma = 2/9$).

and the elliptic functions are referred to modulus k. If we make the substitution $\zeta = i\tau$, equation (2.8.6) yields

$$\phi(x) = \tfrac{1}{2} i\, p_0[E(\tau,k') - \lambda^2\tau + ix] \tag{2.8.8}$$

for points x on the crack, with

$$x = dn(\tau,k'). \tag{2.8.9}$$

It follows from substituting (2.8.8) into equation (1.5.3) that the displacement along the crack is given by

$$u_y = (1 - \eta)p_0[E(\tau,k') - \lambda^2\tau] \tag{2.8.10}$$

The elliptic function in 2.8.9 has been tabulated by Milne-Thompson (1931) and the incomplete elliptic integral in (2.8.10) has been tabulated by Legendre (1934), so that the shape of the cracks can easily be obtained. The shape of the crack in the case $k = 0.1$ is shown in Fig. 11 (Fig. 5 of Willmore's paper) which also shows, for comparison, the elliptic shape of a single crack of the same length opened up by an internal pressure of the same magnitude. The presence of the second crack will not produce an appreciable change in shape. The strain energy of the pair of cracks is given by

$$W = 2(1 - \eta)p_0^2\int_0^{k'} [E(\tau,k') - \lambda^2\tau]k'^2 sn(\tau,k')cn(\tau,k')d_\tau$$

which reduces after some calculation to

$$W = \pi(1 - \eta)p_0^2(1 - \lambda^2 - \tfrac{1}{2}k'^2). \tag{2.8.11}$$

If we assume that the crack is opened by the application of an average tension p_0 to the surface of the body and that the surface of the crack is free from stress, then the presence of the two cracks lowers the potential energy by an amount W. But the cracks have a surface energy $U = 4(1 - k)T$ so that

$$W - U = \pi(1 - \nu)p_0^2(1 - \lambda^2 - \tfrac{1}{2}k'^2) - 4(1 - k)T.$$

The condition

$$\frac{\partial}{\partial k}(W - U) = 0$$

leads to the equation

$$p_{cr} = \left(\frac{K'k'k^{1/2}}{E' - k^2K'}\right)\sqrt{\frac{4T\mu}{\pi(1 - \eta)}} \tag{2.8.12}$$

(in conventional units). The corresponding critical value for a single crack of length $1 - k$ is

$$p'_{cr} = \frac{1}{\sqrt{1 - k}}\sqrt{\frac{4T\mu}{\pi(1 - \eta)}},$$

so

$$\frac{p_{cr}}{p'_{cr}} = \frac{K'k'k^{1/2}}{\sqrt{(1 - k)(E' - k^2K')}}. \tag{2.8.13}$$

The variation of this ratio with the ratio

$$\gamma = \frac{\text{distance between cracks}}{\text{length of crack}} = \frac{2k}{1 - k} \tag{2.8.14}$$

is shown in Fig. 12. From this diagram we see that if the cracks are separated by a distance comparable with the length of each crack, then the effect of the second crack is small.

The maximum shearing stress σ can be calculated easily by inserting the functions (2.8.1), (2.8.2) into equation (1.5.2) and taking the modulus of both sides. We find that

$$\frac{\sigma}{p_0} = \frac{(1 + k^2 - 2\lambda^2)y\sqrt{(x^2 + y^2)}\sqrt{\{x^2 + (y + \beta)^2\}}\sqrt{\{x^2 + (y - \beta)^2\}}}{[(x + 1)^2 + y^2]^{3/4}[(x - 1)^2 + y^2]^{3/4}[(x + k)^2 + y^2]^{3/4}[(x - k)^2 + y^2]^{3/4}}$$

where

$$\beta^2 = \frac{\lambda^2(1 + k^2) - 2k^2}{1 + k^2 - 2\lambda^2}. \tag{2.8.15}$$

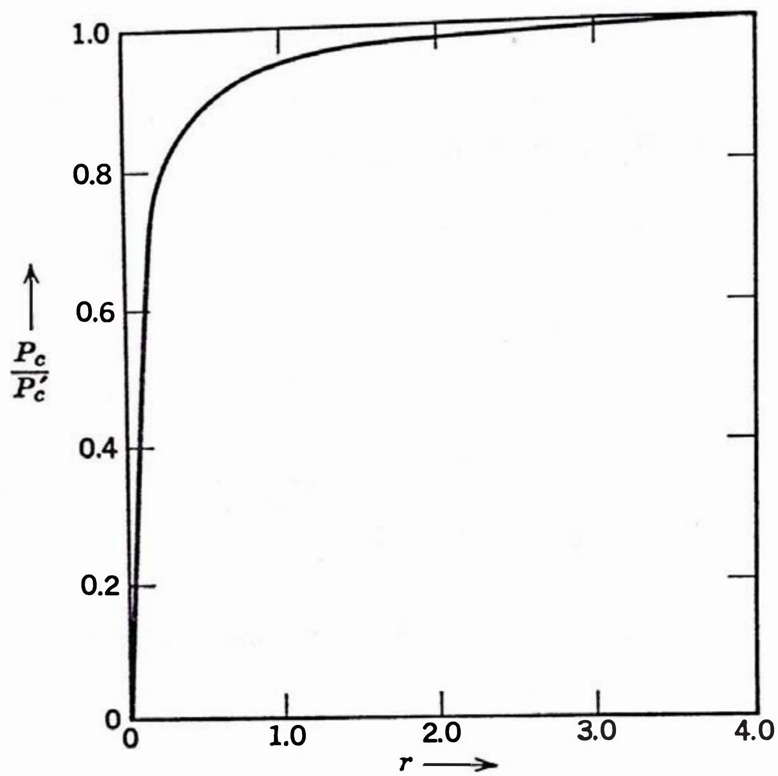


Figure 12 Variation with γ of the critical pressure p_{cr}. The pressure p'_{cr} is the corresponding critical value for a single crack of length $1 - k$.

The variation of σ/p_0 with x and y for the case $k = 0.1$ (i.e., $\eta = 2/9$) was calculated by Willmore. The results of these calculations are given in Table 1 and Fig. 2 of Willmore's paper. These calculations show that the presence of the second crack does markedly increase the stresses near the end of each crack, particularly the edge nearer to the other crack, but that the general character of the stress field is unaltered.

The isochromatic lines $\sigma/p_0 =$ constant for this case are shown in Fig. 13. These should be compared with the isochromatic lines for the single crack (Fig. 9 above).

We now consider Tranter's solution of the more general problem:

on $y = 0$:

$$\sigma_{xy} = 0, \qquad \sigma_{yy} = -p(x), \qquad k < |x| < 1; \tag{2.8.16}$$

$$\sigma_{xy} = u_y = 0, \qquad \begin{cases} 0 < |x| < k, \\ \quad\;\; |x| > 1; \end{cases} \tag{2.8.17}$$

with the additional condition that all three stress components vanish as $\sqrt{x^2 + y^2} \to \infty$. $p(x)$ is assumed to be an even function of x.

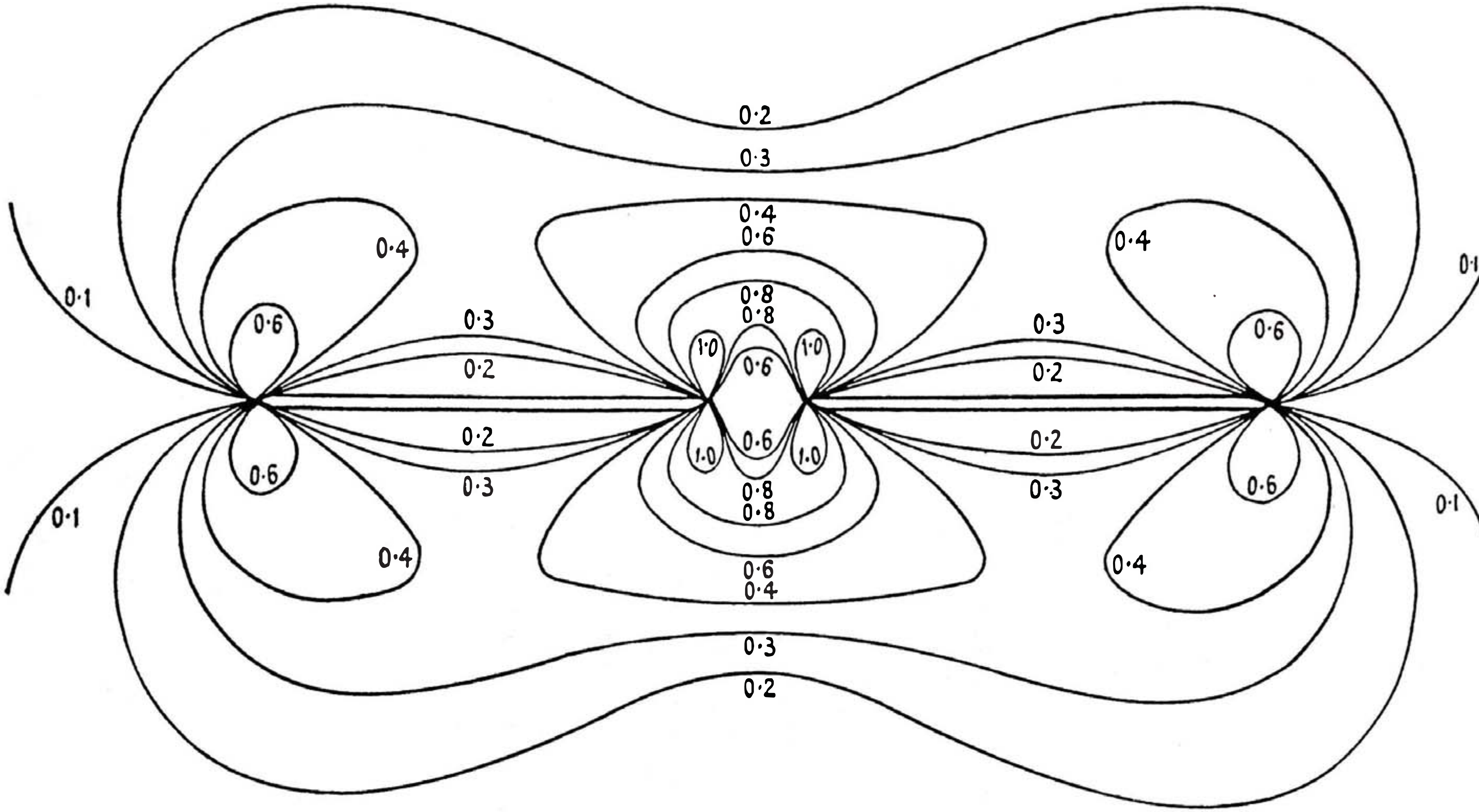


Figure 13 The isochromatic lines in the vicinity of the double crack.

This problem has the solution (2.3.2), (2.3.3) provided that the function $\psi(\xi)$ satisfies the triple integral equations

$$\mathscr{F}_c[\psi(\xi);x] = 0, \qquad 0 < x < k,$$

$$\frac{d}{dx}\mathscr{F}_s[\psi(\xi);x] = p(x), \quad k < x < 1,$$

$$\mathscr{F}_c[\psi(\xi);x] = 0, \qquad x > 1.$$

Making use of a special case of the Weber-Schafheitlin integral (Watson, 1944, p. 405) Tranter shows that the representation

$$\psi(\xi) = \xi^{-1}\sum_{n=1}^{\infty} a_n J_{2n-1}(\xi) \tag{2.8.18}$$

satisfies the third of these equations automatically, and satisfies the first two if the sequence of constants (a_n) is a solution of the dual series relations

$$\sum_{n=1}^{\infty}(2n-1)^{-1}a_n\cos[(n-\tfrac{1}{2})\theta] = 0, \quad 0 < \theta < \beta, \tag{2.8.19}$$

$$\sum_{n=1}^{\infty} a_n\cos[(n-\tfrac{1}{2})\theta] = (\tfrac{1}{2}\pi)^{1/2}\cos(\tfrac{1}{2}\theta)p(\sin\tfrac{1}{2}\theta), \quad \beta < \theta < \pi, \tag{2.8.20}$$

where $k = \sin\frac{1}{2}\beta$. Dual trigonometric series relations of the type (2.8.19), (2.8.20) have been considered by Tranter (1959) and Srivastav (1964*b*). Either method of solution leads to complicated results which will not be reproduced here; for the details when Tranter's method is used, see Tranter (1961).

In the case when $p(x) = p_0$, Tranter's results agree with Willmore's. In addition, Tranter derives the formula

$$\sigma_{yy}(x,0) = p_0\left[\frac{x^2 - \{E(k')/K(k')\}}{\sqrt{\{(x^2-1)(x^2-k^2)\}}} - 1\right], \qquad x > 1, \quad k' = \sqrt{1-k^2}, \tag{2.8.21}$$

so that, in dimensionless units, the stress intensity factor at the "outer" tip $x = 1$, $y = 0$ of the crack is

$$K_0 = \frac{[K(k') - E(k')]p_0}{k'K(k')}.$$

If the cracks are each of length $2c$ and their centers are at a distance $2a$ apart, this formula becomes

$$K_0 = \frac{[K(\kappa) - E(\kappa)]}{\kappa K(\kappa)}p_0\sqrt{a+c}, \qquad \kappa = \frac{2\sqrt{ac}}{a+c}. \tag{2.8.22}$$

Similarly, from Tranter's formula

$$\sigma_{yy}(x,0) = \frac{[E(k')/K(k') - x^2]p_0}{\sqrt{(k^2 - x^2)(1 - x^2)}} - p_0, \quad 0 \leq x \leq k$$

we deduce that the stress concentration factor at the "inner" tip of the crack is

$$K_i = \frac{[E(\kappa) - (1 - \kappa^2)K(\kappa)]}{\kappa(1 - \kappa^2)^{1/4}K(\kappa)} p_0\sqrt{a + c}, \qquad \kappa = \frac{2\sqrt{ac}}{a + c}. \quad (2.8.23)$$

If k is small, then we have that, to the first order in k,

$$\frac{K(\kappa) - E(\kappa)}{\kappa K(\kappa)} = \frac{E(\kappa) - (1 - \kappa^2)K(\kappa)}{\kappa(1 - \kappa^2)^{1/4}K(\kappa)} = \tfrac{1}{2}\kappa$$

so that if we write $K_1 = p_0\sqrt{c}$, and compare equation (2.4.1) above we find that

$$\frac{K_0}{K_1} = \frac{K_i}{K_1} = 1 - \frac{c}{2a}, \quad (c \ll a). \quad (2.8.24)$$

Similarly if a is approximately equal to c we find that

$$\frac{K_0}{K_1} = \left(1 + \frac{a}{c}\right)^{1/2}, \frac{K_i}{K_1} = -\frac{(a - c)^{3/2}}{2c^{3/2}}, \quad (a \simeq c). \quad (2.8.25)$$

The problem of determining the stress field in the vicinity of two collinear Griffith cracks in an infinitely long elastic strip has been considered by Lowengrub and Srivastava (1968*b*).

2.9 INFINITE ROW OF COLLINEAR CRACKS

We now consider the problem of determining the stress field caused by the presence of an infinite row of collinear Griffith cracks of equal length when each is subjected to the same constant pressure. For convenience we shall choose the unit of length in this problem in such a way that the distance between the center of a pair of neighboring cracks is 2π (cf. Fig. 14). We assume that with this unit the length of each crack is $2c$. Assuming that each line crack is opened by equal and opposite normal pressures on each face of the crack, and that this pressure does *not* vary from crack to crack, we see that the boundary conditions take the form

$$\sigma_{yy}(x,0) = -p_0, \quad |x - 2n\pi| \leq c, \quad (2.9.1)$$

$$\sigma_{xy}(x,0) = 0, \quad -\infty < x < \infty, \quad (2.9.2)$$

$$u_y(x,0) = 0, \quad c \leq |x - 2n\pi| \leq 2\pi - c \quad (2.9.3)$$

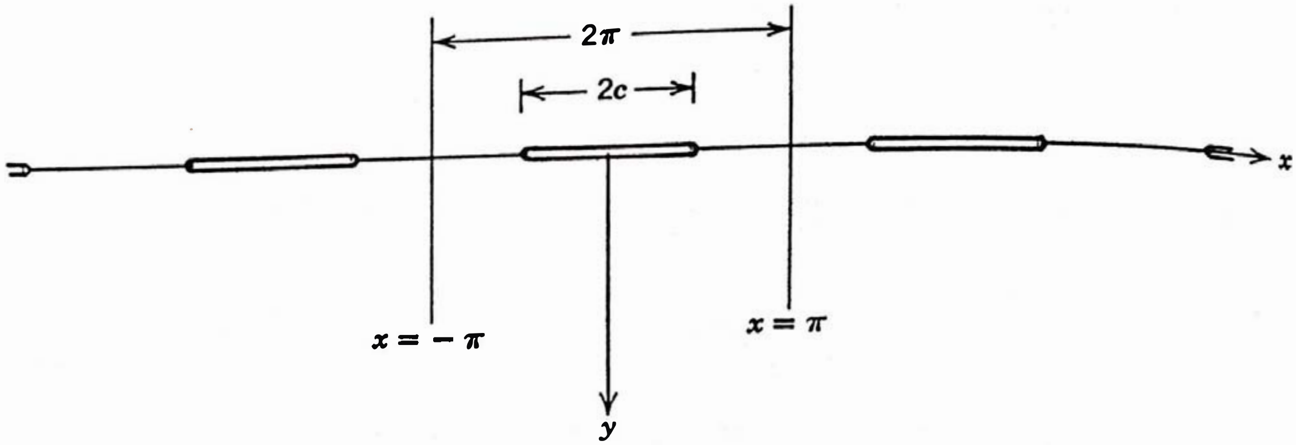


Figure 14 The coordinate system for the problem of a row of collinear cracks.

for $n = 0, \pm 1, \pm 2, \ldots$. It is also assumed that the components of stress all vanish as $|y| \to \infty$.

This problem has been considered by Koiter (1959a) and by England and Green (1963). Westergaard (1939) and Sneddon and Srivastav (1965) consider the same problem in the form of a discussion of the state of stress in a very long elastic strip $|x| \leq \pi$, $|y| < \delta$, $(\delta \gg 1)$ containing a Griffith crack $|x| \leq c$, $y = 0$. In this second form of the problem we have the boundary conditions

$$\sigma_{xy}(x,0) = 0, \qquad |x| \leq \pi \tag{2.9.4}$$

$$\sigma_{yy}(x,0) = -p_0, \quad |x| \leq c, \tag{2.9.5}$$

$$u_y(x,0) = 0, \qquad c < |x| \leq \pi, \tag{2.9.6a}$$

on the line $y = 0$, the conditions

$$\sigma_{xy}(\pm\pi,y) = u_x(\pm\pi,y) = 0, \quad 0 \leq y \leq \delta. \tag{2.9.6b}$$

Finally, on the line $y = \delta$, where $\delta \gg 1$, we may impose the conditions

$$u_y(x,\delta) = \sigma_{xy}(x,\delta) = 0, \tag{2.9.7}$$

and then consider what happens as we pass to the limit $\delta \to \infty$.

We shall outline very briefly the method of solution employed by England and Green to solve the half-plane problem subject to the boundary conditions (2.9.1) through (2.9.3) and then discuss more fully the method of Sneddon and Srivastav to solve the strip problem with the boundary conditions (2.9.4) through (2.9.7).

If, in equations (1.5.8) we take the substitution

$$\phi(z) = 2\cos(\tfrac{1}{2}z)\int_0^c \frac{f(\xi)\tan\frac{1}{2}\xi \, d\xi}{\sqrt{\cos\xi - \cos z}}, \tag{2.9.8}$$

where $f(\xi)$ is a real continuous function of ξ on $[0,c]$, then it is easily shown from equations (1.5.9) that the conditions (2.9.2) and (2.9.3) are automatically satisfied whatever the form of the function $f(\xi)$, but that equation (2.9.1) is satisfied only if $f(\xi)$ is the solution of the integral equation

$$4\frac{d}{dx}\cos(\tfrac{1}{2}x)\int_0^x \frac{f(\xi)\tan(\tfrac{1}{2}\xi)\,d\xi}{\sqrt{\cos\xi-\cos x}} = -p_0, \quad 0 \leq x \leq c. \tag{2.9.9}$$

The solution of this equation is elementary (see Srivastav, 1963, or Burlak, 1965). We find that

$$f(\xi) = +\frac{p_0}{2\pi}\int_0^\xi \frac{\cos(\tfrac{1}{2}u)\,du}{\sqrt{\cos u-\cos\xi}} = \frac{p_0}{2\sqrt{2}}. \tag{2.9.10}$$

Substituting from (2.9.10) into (2.9.8) we find that

$$\phi(z) = \frac{p_0}{2}\cos(\tfrac{1}{2}z)\int_0^c \frac{\tan(\tfrac{1}{2}\xi)\,d\xi}{\sqrt{\cos\xi-\cos z}} = p_0\left[\sin^{-1}\left(\frac{\cos\frac{1}{2}z}{\cos\frac{1}{2}c}\right) - \sin^{-1}(\cos\tfrac{1}{2}z)\right]. \tag{2.9.11}$$

The stress and displacements fields can be obtained by substituting this expression in equations (1.5.8).

If we represent the displacement field by equations of the form

$$u_x(x,y) = -\sum_{n=1}^{\infty} a_n \operatorname{cosech}(n\delta)\{n(\delta-y)\sinh n(\delta-y) + (1-2\eta-n\delta\coth n\delta)\cosh n(\delta-y)\}\sin(nx), \tag{2.9.12}$$

$$u_y(x,y) = \tfrac{1}{2}u_0\left(1-\frac{y}{\delta}\right) - \sum_{n=1}^{\infty} a_n \operatorname{cosech}(n\delta)\{n(\delta-y)\cosh n(\delta-y) - (2-2\eta+n\delta\coth n\delta)\sinh n(\delta-y)\}\cos(nx), \tag{2.9.13}$$

then, taking μ as the unit of stress, we find that the stress field is determined by the equations

$$\sigma_{xx} = -\frac{\eta u_0}{(1-2\eta)\delta} - \sum_{n=1}^{\infty} na_n \operatorname{cosech}(n\delta) \times \{(1-n\delta\coth n\delta)\cosh n(\delta-y) + n(\delta-y)\sinh n(\delta-y)\}\cos(nx)$$

$$\sigma_{yy} = -\frac{(1-\eta)u_0}{(1-2\eta)\delta} - \sum_{n=1}^{\infty} na_n \operatorname{cosech}(n\delta) \times \{(1+n\delta\coth n\delta)\cosh n(\delta-y) - n(\delta-y)\sinh n(\delta-y)\}\cos(nx)$$

$$\sigma_{xy} = 2\sum_{n=1}^{\infty} n^2 a_n \operatorname{cosech}(n\delta) \times \{(\delta-y)\cosh n(\delta-y) - \delta\coth(n\delta)\sinh n(\delta-y)\}\sin(nx).$$

This solution obviously satisfies the conditions (2.9.4), (2.9.6), and (2.9.7). Also on $y = 0$ we have

$$u_y(x,0) = 2(1-\eta)\left[\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)\right]$$

$$\tfrac{1}{2}\sigma_{yy}(x,0) = -\tfrac{1}{2}\alpha a_0 - \sum_{n=1}^{\infty} n a_n[1 + k(n\delta)]\cos(nx)$$

where $a_0 = u_0/[2(1-\eta)]$, $\alpha = 2(1-\eta)^2/[(1-2\eta)\delta]$ and the function $k(x)$ is defined by

$$k(x) = (x + 2e^{-x}\sinh x)\operatorname{cosech}^2 x.$$

Hence we shall have satisfied the remaining conditions (2.9.5) if we can find a sequence of constants a_n satisfying the dual series relations

$$\tfrac{1}{2}\alpha a_0 + \sum_{n=1}^{\infty} n a_n[1 + k(n\delta)]\cos(nx) = \tfrac{1}{2}p_0, \quad 0 \le x \le c$$

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) = 0, \quad 0 < x \le \pi.$$

If we now consider the case of a very long strip so that $\delta \gg \pi$ we find that these equations reduce to

$$\tfrac{1}{2}\alpha a_0 + \sum_{n=1}^{\infty} n a_n \cos(nx) = \tfrac{1}{2}p_0, \quad 0 \le x \le c, \tag{2.9.14}$$

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) = 0, \quad c \le x \le \pi, \tag{2.9.15}$$

and that the corresponding components of displacement and stress are given by the equations

$$u_x = -\sum_{n=1}^{\infty} a_n(1 - 2\eta - ny)e^{-ny}\sin(nx), \tag{2.9.16}$$

$$u_y = \tfrac{1}{2}u_0\left(1 - \frac{y}{\delta}\right) + \sum_{n=1}^{\infty} a_n(2 - 2\eta + ny)e^{-ny}\cos(nx), \tag{2.9.17}$$

$$\tfrac{1}{2}\sigma_{xx} = -\tfrac{1}{2}\bar{\beta}a_0 - \sum_{n=1}^{\infty} n a_n(1 - ny)e^{-ny}\cos(nx), \tag{2.9.18}$$

$$\tfrac{1}{2}\sigma_{yy} = -\tfrac{1}{2}\alpha a_0 - \sum_{n=1}^{\infty} n a_n(1 + ny)e^{-ny}\cos(nx), \tag{2.9.19}$$

$$\tfrac{1}{2}\sigma_{xy} = -y\sum_{n=1}^{\infty} n^2 a_n e^{-ny}\sin(nx), \tag{2.9.20}$$

where the constant $\bar{\beta}$ is defined by

$$\bar{\beta} = \frac{2\eta(1-\eta)}{(1-2\eta)\delta}.$$

In the case of an infinitely long strip we may take $\alpha = \bar{\beta} = 0$.

To solve the pair of dual series relations (2.9.14) and (2.9.15), we make use of a method due to Srivastav (1964c). We assume that when $0 \leq x \leq c$,

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) = \tfrac{1}{2}p_0 \cos(\tfrac{1}{2}x)\int_x^c \frac{g(t)\,dt}{\sqrt{\cos x - \cos t}}, \tag{2.9.21}$$

and reduce the problem to that of determining the function $g(t)$.

We note that the surface displacement of the crack is given by

$$u_y(x,0) = p_0(1-\eta)\cos(\tfrac{1}{2}x)\int_x^c \frac{g(t)\,dt}{\sqrt{\cos x - \cos t}}, \quad 0 \leq x \leq c. \tag{2.9.22}$$

From the formulas for the coefficients in a Fourier half-range cosine series we find that

$$a_0 = \sqrt{2}\int_0^c g(t)\,dt, \qquad a_n = \frac{1}{\sqrt{2}}\int_0^c g(t)[p_n(\cos t) + p_{n-1}(\cos t)]\,dt \tag{2.9.23}$$

where p_n denotes the Legendre polynomial of degree n. Now if $\alpha = 0$ in equation (2.9.14) and we integrate both sides of the equation with respect to x from 0 to $x < c$ we find, on substituting the second of the expressions (2.9.23) into the resulting equation that we obtain an integral equation of Schlömilch type with solution

$$g(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{x\sin(\tfrac{1}{2}x)\,dx}{\sqrt{\cos x - \cos t}} = \sqrt{2}\tan(\tfrac{1}{2}t). \tag{2.9.24}$$

Substituting this expression into equation (2.9.22) we find that

$$u_y(x,0) = \sqrt{2}\,(1-\eta)p_0\cos(\tfrac{1}{2}x)\int_x^c \frac{\tan(\tfrac{1}{2}t)\,dt}{\sqrt{\cos x - \cos t}}.$$

The integration is elementary and it is readily shown that

$$u_y(x,0) = 2(1-\eta)p_0 \log\left[\frac{\cos(\tfrac{1}{2}x) + \sqrt{(\cos^2\tfrac{1}{2}x - \cos^2\tfrac{1}{2}c)}}{\cos\tfrac{1}{2}c}\right], \quad |x| < c. \tag{2.9.25}$$

Written in terms of conventional units with $2a$ as the spacing (instead of 2π) this equation takes the form

$$u_y(x,0) = \frac{2a}{\pi c}\,\epsilon^{(\infty)}\left\{\log\left[\cos\left(\frac{\pi x}{2a}\right) + \sqrt{\cos^2\frac{\pi x}{2a} - \cos^2\frac{\pi c}{2a}}\right] - \log\cos\left(\frac{\pi c}{2a}\right)\right\}, \tag{2.9.26}$$

where $\epsilon^{(\infty)}$ denotes the depth of the crack in the case $a \gg c$ and is given by

$$\epsilon^{(\infty)} = \frac{2(1-\eta^2)p_0 c}{E}. \tag{2.9.27}$$

The depth of the crack in the general case is $\epsilon = u_y(0,0)$ so that

$$\epsilon = \epsilon^{(\infty)} d\left(\frac{c}{a}\right) \tag{2.9.28}$$

where the function $d(\xi)$ is defined by

$$d(\xi) = \frac{2}{\pi\xi}\log[\tan(\tfrac{1}{4}\pi\xi + \tfrac{1}{4}\pi)]. \tag{2.9.29}$$

The variation of the ratio $\epsilon/\epsilon^{(\infty)}$ with c/a for fixed internal pressure p_0 is shown in Fig. 15, and that of the ratio $u_y(x,0)/\epsilon^{(\infty)}$ with x for four values of

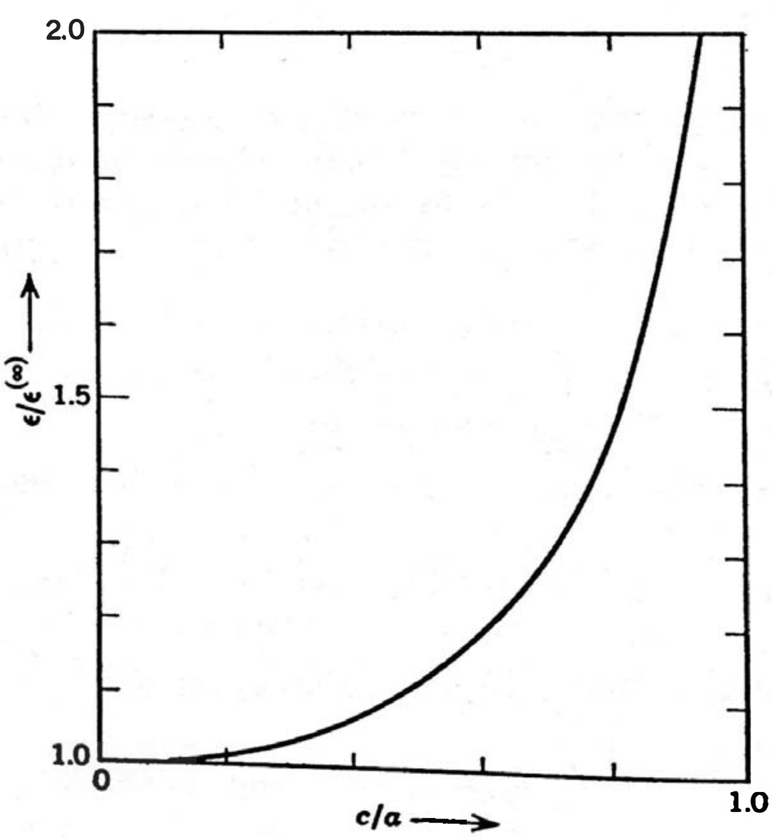


Figure 15 The variation with c/a of the depth, ϵ, of the crack in the case of constant internal pressure p_0; $\epsilon^{(\infty)} = 2(1-\eta^2)p_0c/E$ is depth in the case $a/c = \infty$.

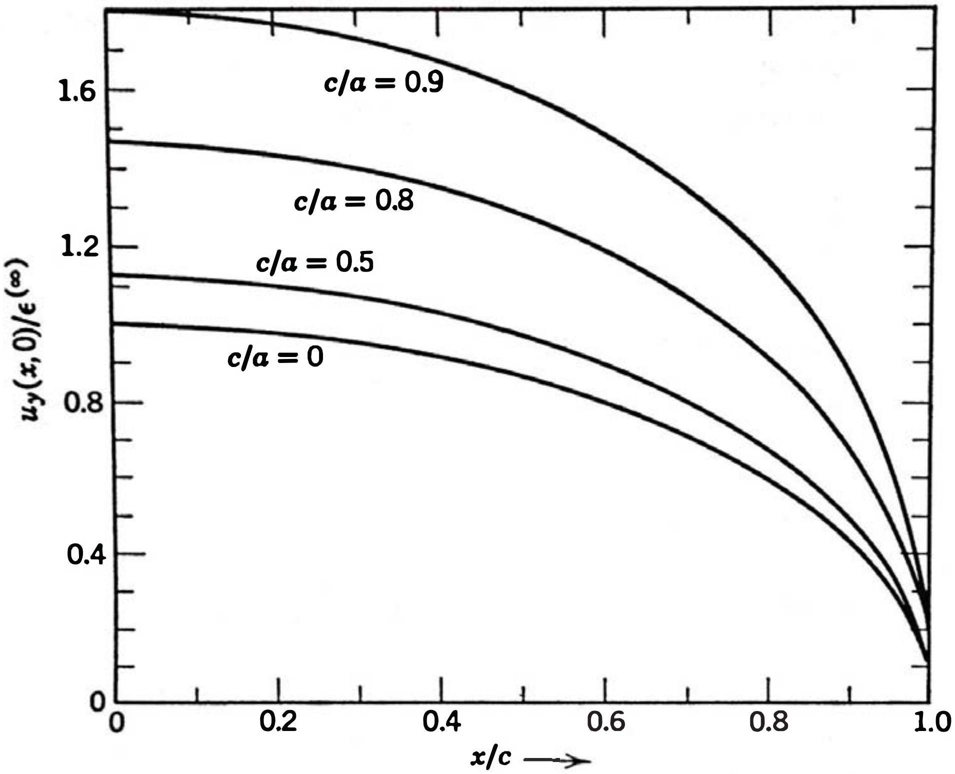


Figure 16 The shape of the crack in the case of constant internal pressure for four values of c/a; $\epsilon^{(\infty)}$ is as defined in Fig. 15.

the ratio c/a is shown in Fig. 16. The curve corresponding to the value $c/a = 0$ is the ellipse

$$u_y(x,0) = \epsilon^{(\infty)}\sqrt{\left(1 - \frac{x^2}{c^2}\right)}$$

and it can be seen from the curves of Fig. 16 that even a substantial increase in the ratio c/a does not appreciably affect the *shape* of the curve although (as we should expect) the "minor" axis increases by nearly 80% as c/a increases from 0 to 0.9.

Similarly, the normal component of stress along the line of the crack is given by equation (2.9.19) in the form

$$\sigma_{yy}(x,0) = -2\frac{d}{dx}\left[\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \sin(nx)\right]$$

and substituting from equations (2.9.23) we find that this is equivalent to

$$\sigma_{yy}(x,0) = -p_0\frac{d}{dx}\left[\cos(\tfrac{1}{2}x)\int_0^c \frac{g(t)\,dt}{\sqrt{\cos t - \cos x}}\right], \quad c < x \leq \pi,$$

$$= -\sqrt{2}\,p_0\frac{d}{dx}\left[\cos(\tfrac{1}{2}x)\int_0^c \frac{\tan(\tfrac{1}{2}t)\,dt}{\sqrt{\cos t - \cos x}}\right].$$

The integration is elementary. Carrying out both it and the subsequent differentiation, and then expressing the result in conventional units, we find that the normal component of stress across the line of the crack is given by

$$\sigma_{yy}(x,0) = p_0\left\{\sin\left(\frac{\pi x}{2a}\right)\left[\cos^2\left(\frac{\pi c}{2a}\right) - \cos^2\left(\frac{\pi x}{2a}\right)\right]^{-1/2} - 1\right\}, \quad c \leq |x| < a. \tag{2.9.30}$$

The form of the variation of $\sigma_{yy}(x,0)$ with x in three cases is shown in Fig. 17 (the same value of p_0 being chosen in each case). As we should expect, we find that the *shape* of the curve is much the same whatever the value of the ratio c/a, but the intensity is greater the higher the value of c/a.

The stress intensity factor in this case is defined by

$$K_a = \lim_{x\to c+} \sqrt{2(x-c)}\,\sigma_{yy}(x,0) = p_0\left(\frac{2a}{\pi}\tan\frac{\pi c}{2a}\right)^{1/2}. \tag{2.9.31}$$

If $c \gg a$, K_a takes the value K_1 defined by (2.4.1) so that we may write the percentage increase of K_a over K_1, $100(K_a - K_1)/K_1$, as $n(c/a)$ where the function $n(\xi)$ is defined by

$$n(\xi) = 100\left\{\left[\frac{2}{\pi\xi}\tan(\tfrac{1}{2}\pi\xi)\right]^{1/2} - 1\right\}. \tag{2.9.32}$$

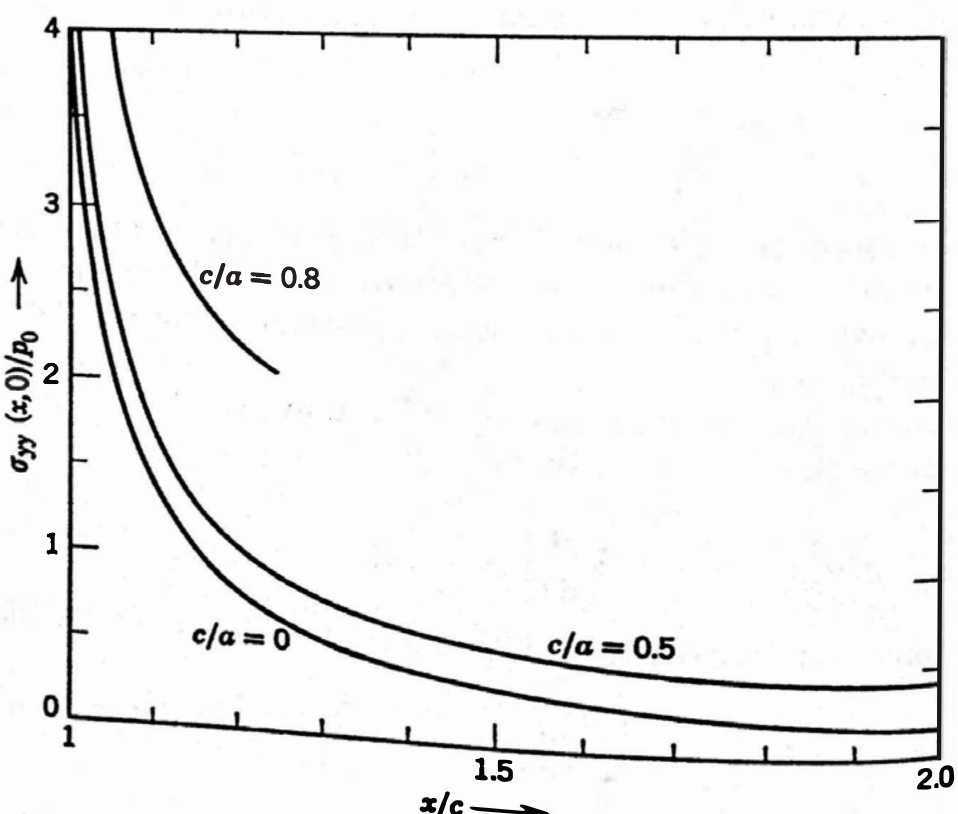


Figure 17 The variation with x of $\sigma_{yy}(x,0)$, the normal component of stress across the line of the crack, in the case of constant internal pressure p_0.

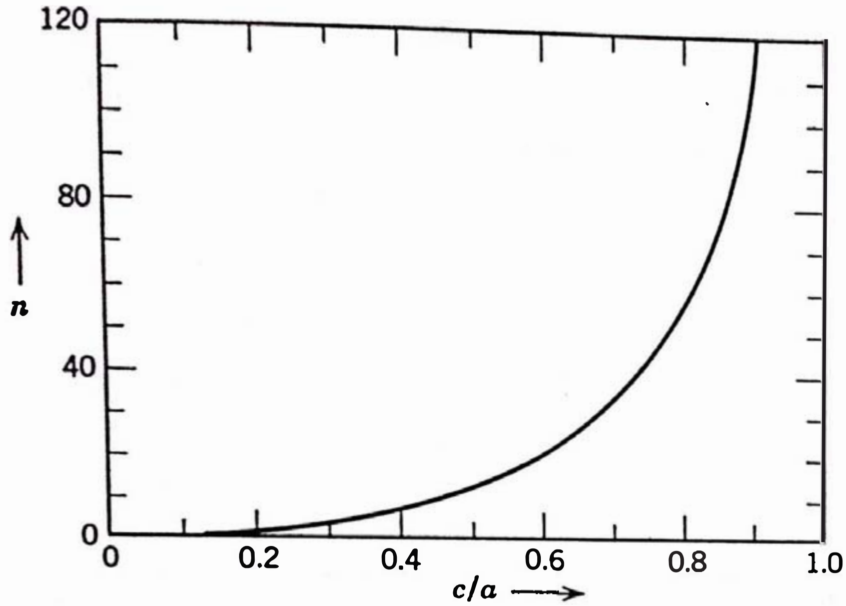


Figure 18 The variation with c/a of n, the percentage increase of the stress intensity factor over its value in the case $a/c = \infty$, for constant internal pressure.

The variation with c/a of n, the percentage increase of the stress intensity factor over its value in the case $c/a = 0$ is shown in Fig. 18.

The strain energy of the crack is given (in dimensionless form) by

$$W_a = 2p_0 \int_0^c u_y(x,0)\, dx,$$

so that inserting the integral form for $u_y(x,0)$ and interchanging the order in which we carry out the integrations, we find that

$$\begin{aligned} W_a &= 2\sqrt{2}\,(1-\eta)p_0^{\,2} \int_0^c \tan(\tfrac{1}{2}t)\, dt \int_0^t \frac{\cos(\tfrac{1}{2}x)\, dx}{\sqrt{\cos x - \cos t}} \\ &= 4(1-\eta)p_0^{\,2}\pi \log(\sec \tfrac{1}{2}c). \end{aligned}$$

Reverting to conventional units we see that

$$W_a = \frac{8(1-\eta^2)p_0^{\,2}c^2}{\pi E} \log\left(\sec \frac{\pi c}{2a}\right).$$

Griffith's criterion

$$\frac{\partial W_a}{\partial c} = 4T,$$

where T is the surface tension of the material, leads to the equation

$$p_{cr} = \left[\frac{ET}{(1-\eta^2)a} \cot \frac{\pi c}{2a}\right]^{1/2}$$

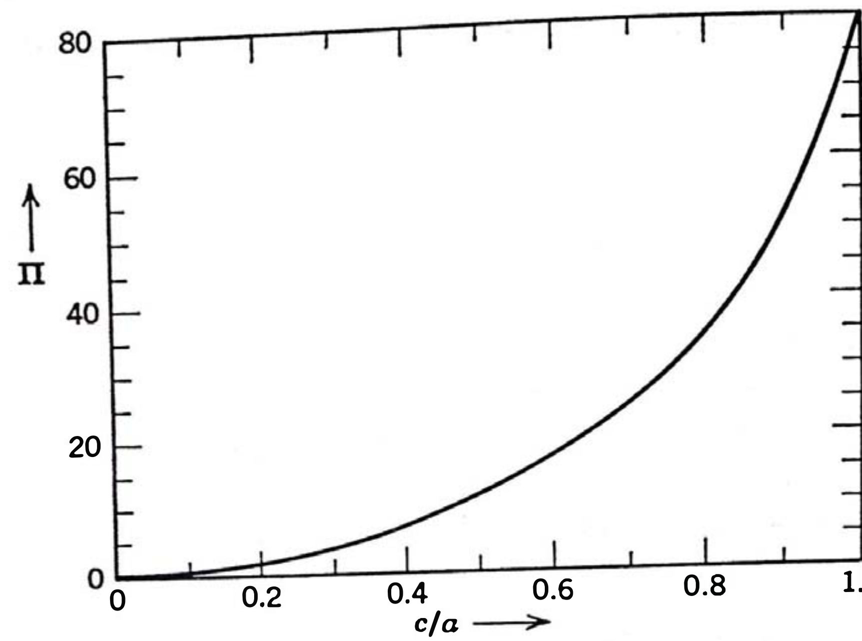


Figure 19 The variation with c/a of Π, the percentage increase in the critical pressure over its value in the case $a/c = \infty$, for constant internal pressure.

for the critical value of p_0 when the crack length is $2c$. In the infinite case ($a/c = \infty$) the corresponding value is $p_{cr}^{(\infty)} = [2ET/\pi(1 - \eta^2)c]^{1/2}$ so that we may write Π, the percentage change of the critical pressure p_{cr} from the value $p_{cr}^{(\infty)}$, in the form

$$\Pi = 100\left[1 - \left(\frac{\pi c}{2a} \cot \frac{\pi c}{2a}\right)^{1/2}\right]. \tag{2.9.33}$$

The variation with c/a of the ratio Π is shown in Fig. 19.

A closely related problem to the one considered here is that of determining the stress field in two semi-infinite planes of dissimilar materials bonded partially along a straight line. The situation is simulated by considering two such half-planes bonded together but containing Griffith cracks at the interface (cf. Williams, 1959). Based on the results of Cherepanov (1962), the discussion of Salganik (1963) revealed that violent oscillations occur in the stresses near the tip of a Griffith crack between two bonded materials. This phenomenon is accompanied by interpenetration of the crack surfaces which is, of course, physically impossible. However, this singular behavior is confined to a small region around the crack tip where there would in any case be plastic flow, so that, for all practical purposes, this oscillatory behavior of the local stresses may be ignored. Similar observations were made by England (1965). The papers by Erdogan (1962, 1963, 1965) and by Rice and Sih (1964, 1965) should also be consulted.

This problem has been extended to cracks lying along a curved interface by Perlman and Sih (1967). The problem in which the cracks are of Griffith type but the bonded materials are anisotropic has been investigated by Gotoh (1967), while the distribution of thermal stresses in bonded materials containing Griffith cracks on the interface has been analyzed by Brown and Erdogan (1968).

2.10 INFINITE ROW OF PARALLEL CRACKS

In this section we discuss the problem of determining the stress field in the xy-plane resulting from the presence of an infinite row of line cracks, each parallel to the x-axis and equally spaced along the y-axis. We shall first of all briefly outline England and Green's (1963) solution of this problem and then discuss that obtained by Lowengrub (1966b).

We take as our unit of length one-half of the crack length and (in this unit) denote the distance between two neighboring cracks by 2δ (see Fig. 20). Let us suppose that each face of each crack is opened up by a constant normal pressure p_0. The lines $y = (2n + 1)\delta$, $n = 0, \pm 1, \pm 2, \ldots$ are lines of symmetry along which $u_y = \sigma_{xy} = 0$, so that we need only consider the distribution of stress in the infinite strip $|y| \leq \delta$ when it is subjected to the boundary conditions

$$\sigma_{yy}(x,0) = -p_0, \qquad |x| \leq 1, \tag{2.10.1}$$

$$u_y(x,0) = 0, \qquad |x| \geq 1, \tag{2.10.2}$$

$$\sigma_{xy}(x,0) = 0, \qquad -\infty < x < \infty, \tag{2.10.3}$$

$$\sigma_{xy}(x, \pm\delta) = u_y(x, \pm\delta) = 0, \quad -\infty < x < \infty. \tag{2.10.4}$$

For the potential functions $\phi(z)$ and $\psi(z)$ occurring in equations (1.5.4) through (1.5.6), England and Green assume the forms

$$\phi(z) = -\tfrac{1}{2}p_0\int_0^1 \frac{f(t)\,dt}{\sqrt{z^2 - t^2}} - \tfrac{1}{2}p_0\int_0^\infty a(\xi)\sin(\xi z)\,d\xi, \tag{2.10.5}$$

$$\psi(z) = \phi(z) - z\phi'(z) - \tfrac{1}{2}p_0\int_0^\infty b(\xi)\sin(\xi z)\,d\xi. \tag{2.10.6}$$

For these potential functions equations (2.10.2) and (2.10.3) are satisfied identically and it can easily be shown that equations (2.10.4) can be satisfied if we take

$$a(\xi) = \operatorname{cosech}(\xi\delta)e^{-\xi\delta}\int_0^1 f(t)J_0(\xi t)\,dt \tag{2.10.7}$$

$$b(\xi) = 2\xi\delta(\operatorname{cosech}\xi\delta + \operatorname{sech}\xi\delta)e^{-\xi\delta}\int_0^1 f(t)J_0(\xi t)\,dt, \tag{2.10.8}$$

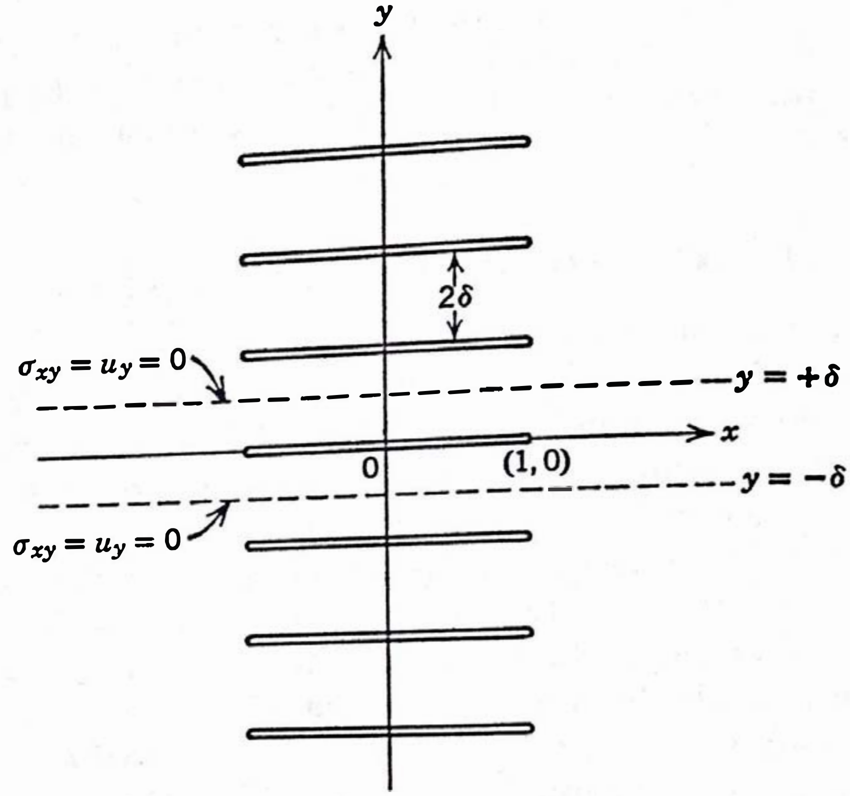


Figure 20 The coordinate system for the problem of a row of parallel cracks.

and that the remaining boundary condition (2.10.1) is equivalent to the integral equation

$$\frac{d}{dx}\left\{\int_0^x \frac{f(t)\,dt}{(\sqrt{(x^2-t^2)})} + \int_0^\infty [a(\xi) + \tfrac{1}{2}b(\xi)]\sin(\xi x)\,d\xi\right\} = 1, \quad 0 \le x < 1.$$

Regarding this as an Abel integral equation with the second term on the left hand side a known function of x, we see that $f(t)$, $a(\xi)$ and $b(\xi)$ are related through the equation

$$f(t) + t\int_0^\infty \xi[a(\xi) + \tfrac{1}{2}b(\xi)]J_0(\xi t)\,dt = t, \quad 0 \le t < 1.$$

Substituting the forms (2.10.7) and (2.10.8) for $a(\xi)$ and $b(\xi)$, respectively, into this last equation, we find that it is equivalent to the Fredholm integral equation of the second kind,

$$f(t) + t\int_0^1 K(s,t)f(s)\,ds = t, \quad 0 \le t < 1, \tag{2.10.9}$$

for the function $f(t)$, the kernel being defined by

$$K(s,t) = \frac{1}{\delta^2}\int_0^\infty \frac{u}{\sinh u}[1 + u(1 + \coth u)]e^{-u}J_0\left(\frac{su}{\delta}\right)J_0\left(\frac{tu}{\delta}\right)du,$$

$$(0 \le s \le 1, 0 \le t \le 1). \tag{2.10.10}$$

We note also from (1.5.6), (2.10.5) and (2.10.6) that the normal displacement over the surfaces of a crack is given by

$$u_y(x,0) = \tfrac{1}{4}p_0(1+\kappa)\int_x^1 \frac{f(t)\,dt}{\sqrt{t^2-x^2}}, \quad 0 \le x \le 1, \tag{2.10.11}$$

where, as in Section 1.5, $\kappa = 3 - 4\eta$ for plane strain and $\kappa = (3-\eta)/(1+\eta)$ for plane stress. From this expression we deduce that the change in internal energy (per crack) because of the opening of the cracks is given by

$$W = \tfrac{1}{4}\pi(1+\kappa)p_0^{\,2}\int_0^1 f(t)\,dt. \tag{2.10.12}$$

For general values of δ, the integral equation (2.10.9) has to be solved numerically and W calculated from the numerical values of $f(t)$. When $\delta \gg 1$, the following approximate method gives useful results.

It is easily seen that equation (2.10.9) has the solution

$$f(t) = t\left[1 + \sum_{r=1}^{\infty}(-1)^r f_r(t)\right], \tag{2.10.13}$$

where the sequence of functions $\{f_r(t)\}$ is determined by the iterative scheme

$$f_0(t) = 1, \qquad f_r(t) = \int_0^1 s f_{r-1}(s)K(s,t)\,ds, \quad (r \ge 1). \tag{2.10.14}$$

If we use the expansion

$$J_0\left(\frac{su}{\delta}\right)J_0\left(\frac{tu}{\delta}\right) = 1 - \frac{s^2+t^2}{4\delta^2}u^2 + \frac{s^4+4s^2t^2+t^4}{64\delta^4} + O(\delta^{-6}), \tag{2.10.15}$$

we see that an approximate form for the kernel $K(s,t)$ when $\delta \gg 1$ is

$$\frac{1}{\delta^2}\left[K_0 - \frac{s^2+t^2}{4\delta^2}K_1 + \frac{s^4+4s^2t^2+t^4}{64\delta^4}\right] + O(\delta^{-8}),$$

where

$$K_r = \int_0^\infty \frac{u^{2r+1}}{\sinh u}\,[1 + u(1+\coth u)]e^{-u}\,du = \frac{(2r+3)(2r+1)!}{2^{2r+1}}\zeta(2r+2),$$

where ζ denotes the Riemann zeta-function. Since

$$K_0 = \frac{\pi^2}{4}, \qquad K_1 = \frac{\pi^2}{24}, \qquad K_2 = \frac{\pi^6}{36},$$

we see that for $\delta \gg 1$,

$$K(s,t) = \frac{\pi^2}{4\delta^2}\left[1 - \frac{\pi^2}{24\delta^2}(s^2+t^2) + \frac{\pi^2}{576\delta^2}(s^4+4s^2t^2+t^4)\right] + O(\delta^{-8}) \tag{2.10.16}$$

and consequently that

$$f_1(t) = \frac{\pi^2}{8\delta^2}\left[1 - \frac{\pi^2}{48\delta^2}(2t^2+1) + \frac{\pi^4}{1728\delta^4}(3t^4 + 6t^2 + 1)\right] + O(\delta^{-8})$$

$$f_2(t) = \frac{\pi^4}{64\delta^2}\left[1 - \frac{\pi^2}{24\delta^2}(2t^2+1)\right] + O(\delta^{-8})$$

$$f_3(t) = \frac{\pi^6}{512\delta^6} + O(\delta^{-8})$$

for which it follows that

$$\int_0^1 f(t)\,dt = \frac{1}{2}\left[1 - \frac{\pi^2}{8\delta^2} + \frac{\pi^4}{48\delta^4}\right] + O(\delta^{-6}).$$

Substituting this value for the integral in (2.10.12) and converting to conventional units we find that for cracks of length $2c$, distance a apart,

$$W = \frac{\pi(1+\kappa)p_0{}^2c^2}{8\mu}\left[1 - \frac{\pi^2}{2}\left(\frac{c}{d}\right)^2 + \frac{\pi^4}{3}\left(\frac{\pi c}{d}\right)^4\right], \quad (d \gg c).$$

Hence for *plane strain* we have the expression

$$W = \frac{\pi(1-\eta^2)p_0{}^2c^2}{E}\left[1 - \frac{1}{2}\left(\frac{\pi c}{d}\right)^2 + \frac{1}{3}\left(\frac{\pi c}{d}\right)^4\right], \quad (d \gg c), \qquad (2.10.17)$$

for the strain energy and

$$\mathscr{G} = \frac{\pi(1-\eta^2)p_0{}^2c}{E}\left[1 - \left(\frac{\pi c}{d}\right)^2 + \left(\frac{\pi c}{d}\right)^4\right], \quad (d \gg c), \qquad (2.10.18)$$

for the strain-energy release rate. The corresponding formulas for *plane stress* are obtained by replacing the constant factor in these expressions by $\pi p_0{}^2c^2/E$ and $\pi p_0{}^2c/E$ respectively.

Lowengrub (1966b) gave a solution of the plane strain boundary value problem expressed by equations (2.10.1) through (2.10.4) by taking

$$\begin{gathered}\psi = 0, \qquad \chi = -\frac{p_0}{2}\delta\mathscr{F}_c\left[\frac{\cosh \xi\delta}{\cosh^2 \xi\delta}f(\xi); \xi \to x\right], \\ \phi = \frac{p_0}{2(\beta^2-1)}\mathscr{F}_c\left[\frac{\cosh \xi(\delta - y)}{\sinh \xi\delta}\xi^{-1}\psi(\xi); \xi \to x\right]\end{gathered} \qquad (2.10.19)$$

in equations (1.4.5) and (1.4.6). The resulting equations then lead to the equations

$$u_y(x,y) = \frac{p_0}{2}\mathscr{F}_c\left[\left\{\frac{2(1-\eta)\sinh \xi(\delta - y)}{\sinh \xi\delta} - \frac{\xi\delta \sinh \xi y}{\sinh^2 \xi\delta} + \frac{y\cosh \xi(\delta - y)}{\sinh \xi\delta}\right\}\psi(\xi);\ \xi \to x\right],$$

$$\sigma_{yy}(x,y) = -p_0\frac{d}{dx}\mathscr{F}_s\left[\left\{\frac{\xi\delta\cosh \xi y}{\sinh^2 \xi\delta} + \frac{\cosh \xi(\delta - y)}{\sinh \xi\delta} + \frac{\xi y\cosh \xi(\delta - y)}{\sinh \xi\delta}\right\}\psi(\xi);\ \xi \to x\right],$$

$$\sigma_{xy}(x,y) = p_0\mathscr{F}_c\left[\left\{\frac{\xi\delta\sinh \xi y}{\sinh^2 \xi\delta} - \frac{\xi y\cosh \xi(\delta - y)}{\sinh \xi\delta}\right\}\xi\psi(\xi);\ \xi \to x\right].$$

It is clear that these expressions automatically satisfy equations (2.10.3) and (2.10.4) and that the remaining conditions (2.10.1) and (2.10.2) will be satisfied if the function $\psi(\xi)$ is chosen to be the solution of the dual integral equations

$$\mathscr{F}_s[\{1 + H(\xi\delta)\}\psi(\xi); x] = x, \quad 0 \le x \le 1, \tag{2.10.20}$$

$$\mathscr{F}_c[\psi(\xi); x] = 0, \quad x > 1, \tag{2.10.21}$$

where the function $H(u)$ is defined by the equation

$$H(u) = \frac{u + e^{-u}\sinh u}{\sinh^2 u}. \tag{2.10.22}$$

By making the substitution

$$\psi(\xi) = \sqrt{(\tfrac{1}{2}\pi)}\int_0^1 f(t)J_0(\xi t)\,dt, \tag{2.10.23}$$

we see that equation (2.10.21) is automatically satisfied and that equation (2.10.20) is also if $f(t)$ satisfies the equation

$$\int_0^t \frac{f(t)\,dt}{\sqrt{t^2 - x^2}} + \int_0^1 f(s)\,ds\int_0^\infty H(\xi\delta)J_0(\xi s)\sin(\xi x)\,d\xi = x, \quad 0 \le x \le 1.$$

Regarding this as an integral equation of Abel type in which the second term on the lefthand side is a known function of x, we see that $f(t)$ must satisfy the integral equation (2.10.9) with the kernel now given by the formula

$$K(s,t) = \frac{2}{\pi t}\int_0^\infty H(\xi\delta)J_0(\xi s)\,d\xi\,\frac{d}{dt}\int_0^t \frac{x\sin(\xi x)}{\sqrt{t^2 - x^2}}\,dx.$$

Using the fact that

$$\frac{2}{\pi t}\frac{d}{dt}\int_0^t \frac{x\sin(\xi x)}{\sqrt{t^2-x^2}}\,dx = \frac{2\xi}{\pi}\int_0^t \frac{\cos(\xi x)\,dx}{\sqrt{t^2-x^2}} = \xi J_0(\xi t),$$

we see that $K(s,t)$ is again given by equation (2.10.10).

Also we note that, if $|x| < 1$, then

$$u_y(x,0) = p_0(1-\eta)\mathscr{F}_c[\psi(\xi);x] = p_0(1-\eta)\int_0^1 f(t)\,dt\int_0^\infty \cos(\xi x)J_0(\xi t)\,d\xi$$
$$= p_0(1-\eta)\int_x^1 \frac{f(t)\,dt}{\sqrt{t^2-x^2}},$$

in agreement with equation (2.10.11). Equation (2.10.12) then follows and the rest of the analysis proceeds as before.

From equations (1.5.4), (1.4.1), (1.4.2), (1.4.5), (1.4.6) we have that the relation between the Muskhelishvili function $\phi(z)$ and the function $\phi(x,y)$ of section 1.4 is given by

$$\phi'(z) + \overline{\phi'(z)} = 2(\beta^2-1)\frac{\partial^2\phi}{\partial x^2},$$

so that in this problem

$$\phi'(z) + \overline{\phi'(z)} = -p_0\mathscr{F}_c\left[\xi\,\frac{\cosh\xi(\delta-y)}{\sinh\xi\delta}\,\psi(\xi);\,\xi\to x\right].$$

Now

$$\frac{\cosh\xi(\delta-y)}{\sinh\xi\delta} = e^{-\xi y} + \frac{e^{-\xi\delta}}{\sinh\xi\delta}\cosh(\xi y),$$

and

$$\mathscr{F}_c[\xi e^{-\xi y}\psi(\xi);x] = \frac{1}{2}\int_0^1 f(t)\,dt\int_0^\infty \xi(e^{i\xi z} + e^{-i\xi\bar{z}})J_0(\xi t)\,d\xi$$

$$\mathscr{F}_c\left[\xi\,\frac{e^{-\xi\delta}}{\sinh(\xi\delta)}\cosh(\xi y)\psi(\xi);\,x\right] = \frac{1}{2}\int_0^\infty \xi\{\cos(\xi z) + \cos(\xi\bar{z})\}a(\xi)\,d\xi,$$

where $a(\xi)$ is defined by equation (2.10.7). We therefore deduce that

$$\phi(z) = -\tfrac{1}{2}ip_0\int_0^1 f(t)\,dt\int_0^\infty J_0(\xi t)e^{i\xi z}\,d\xi - \tfrac{1}{2}p_0\int_0^\infty a(\xi)\sin(\xi z)\,d\xi,$$

which is identical with equation (2.10.5). The second part of the England-Green solution (2.10.6) can similarly be related to Lowengrub's solution.

2.11 CRACKS IN A STRIP WITH STRESS FREE EDGES

In this section we consider the effect of internal cracks on the distribution of stress in thin elastic strips where the edges of the strips are assumed to be

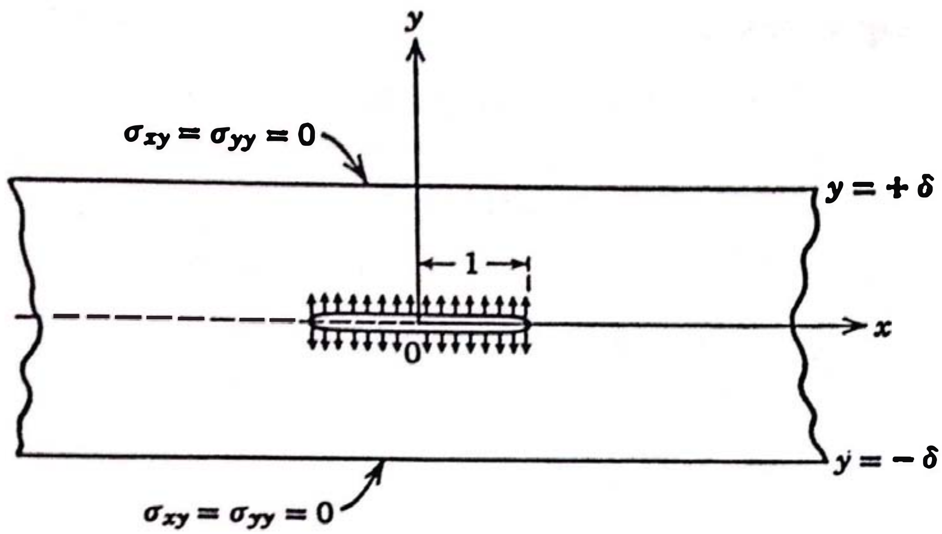


Figure 21 Griffith crack parallel to the boundary of a semi-infinite strip.

free from stress. We shall discuss two separate problems: (a) the case where the crack is assumed to be parallel to the edges of the strip, and (b) the case where the crack is perpendicular to the edges of the strip. Lowengrub (1966b) considers case (a) and Sneddon and Srivastav (1969) consider case (b).

We begin with a discussion of case (a). Let us assume that the crack is opened by a uniform internal pressure p_0 and that the edges are stress free. The crack is taken to lie in the interior of a strip, $-\infty < x < \infty$, $-\delta \le y \le \delta$, on the line $y = 0$, $-1 \le x \le 1$ (see Fig. 21). The boundary conditions take the form

on $y = 0$:

$$\sigma_{yy} = -p_0, \quad 0 \le x \le 1 \tag{2.11.1}$$

$$\sigma_{xy} = 0, \quad x \ge 0 \tag{2.11.2}$$

$$u_y = 0, \quad x > 1 \tag{2.11.3}$$

on $|y| = \delta$:

$$\sigma_{yy} = \sigma_{xy} = 0, \quad x \le 0 \tag{2.11.4}$$

In equations (1.4.6) we take $\psi = 0$ and substitute

$$\phi = \frac{-p_0}{\beta^2 - 1}\mathscr{F}_c\left[\xi^{-1}g(\xi)\left\{\tfrac{1}{2}\sinh \xi y - \frac{\sinh^2 \xi\delta \cosh \xi y}{2\xi\delta + \sinh 2\xi\delta}\right\}; x\right] \tag{2.11.5}$$

$$\chi = +p_0\delta^2\mathscr{F}_c\left[\xi g(\xi)\frac{\cosh \xi y}{2\xi\delta + \sinh 2\xi\delta}; x\right] \tag{2.11.6}$$

to obtain the equations

$$u_y = +p_0\mathscr{F}_c\left[g(\xi)\,\frac{\xi^2\delta^2\sinh(\xi y) - 2(1-\eta)\sinh^2(\xi\delta)\sinh(\xi y) + \xi y\sinh^2(\xi\delta)\cosh(\xi y)}{2\xi\delta + \sinh 2\xi\delta}\,;\ \xi\to x\right]$$

$$+\ p_0\mathscr{F}_c[g(\xi)\{(1-\eta)\cosh(\xi y) - \tfrac{1}{2}\xi y\sinh(\xi y)\};\ \xi\to x], \tag{2.11.7}$$

$$\sigma_{yy} = +2p_0\frac{d}{dx}\mathscr{F}_s\left[g(\xi)\left\{\frac{\xi^2\delta^2\cosh(\xi y) - \sinh^2\xi\delta\cosh(\xi y)}{2\xi\delta + \sinh 2\xi\delta}\right.\right.$$

$$\left.\left. +\ \frac{\xi y\sinh^2(\xi\delta)\sinh(\xi y)}{2\xi\delta + \sinh 2\xi\delta} + \tfrac{1}{2}\sinh(\xi y) - \tfrac{1}{2}\xi y\cosh(\xi y)\right\};\ \xi\to x\right], \tag{2.11.8}$$

$$\sigma_{xy} = -p_0\mathscr{F}_s\left[\xi g(\xi)\left\{\xi^2 y\sinh(\xi y)\right.\right.$$

$$\left.\left. -\ \frac{2\xi^2\delta^2\sinh\xi y + 2\xi^2 y^2\sinh^2\xi\delta\cosh\xi y}{2\xi\delta + \sinh 2\xi\delta}\right\};\ \xi\to x\right]. \tag{2.11.9}$$

It is a simple matter to verify that these expressions satisfy the conditions (2.11.2), (2.11.4) and that the remaining boundary conditions (2.11.1), (2.11.2) will be satisfied if the function $g(\xi)$ satisfies the dual integral equations

$$\begin{aligned} \mathscr{F}_s[g(\xi)\{1 - M(\xi\delta)\};\ x] &= x, \quad 0 \le x < 1 \\ \mathscr{F}_c[g(\xi);x] &= 0, \qquad x > 1 \end{aligned} \tag{2.11.10}$$

where the function M is defined by

$$M(u) = \frac{2u(u+1) + 1 - e^{-2u}}{2u + \sinh 2u}. \tag{2.11.11}$$

The pair of dual integral equations (2.11.10) is exactly the same as the pair (2.10.20), (2.10.21) but with H replaced by $-M$. It follows that the solution of the present pair is given by

$$g(\xi) = \sqrt{\tfrac{1}{2}\pi}\int_0^1 f(t)J_0(\xi t)\,dt \tag{2.11.12}$$

where $f(t)$ is the solution of the Fredholm integral equation

$$f(t) - t\int_0^1 M(s,t)f(s)\,ds = t, \tag{2.11.13}$$

the kernel $M(s,t)$ defined, by analogy with equation (2.10.10), by the relation

$$M(s,t) = \frac{1}{\delta^2}\int_0^\infty M(u)J_0\left(\frac{su}{\delta}\right)J_0\left(\frac{tu}{\delta}\right)du. \tag{2.11.14}$$

If, now, we assume that $\delta \gg 1$ we find, by analogy with equation (2.10.6), that

$$M(s,t) = \frac{1}{\delta^2}\left[M_1 - \frac{t^2 + s^2}{4\delta^2} M_3 + \frac{t^4 + 4s^2t + s^4}{16\delta^4} M_5\right] + O(\delta^{-8}), \quad (2.11.15)$$

where the numbers M_n are the moments of the function $M(u)$, namely,

$$M_n = \int_0^\infty u^n M(u)\, du. \quad (2.11.16)$$

The numbers M_n are readily calculated in terms of Howland's integrals

$$I_n = \frac{2^n}{n!}\int_0^\infty \frac{u^n\, du}{\sinh 2u + 2u}, \qquad \mathrm{II}_n = \frac{2^n}{n!}\int_0^\infty \frac{u^n e^{-2u}\, du}{\sinh 2u + 2u}, \quad (2.11.17)$$

which have been evaluated numerically by Ling (1957). We find that

$$M_1 = 2.34974 \qquad M_3 = 9.96195 \qquad M_5 = 101.39415. \quad (2.11.18)$$

Using the same iterative procedure as in Section 2.10, we see that

$$f(t) = t\left[1 + \frac{M_1}{2\delta^2} + \frac{M_1^2}{4\delta^4} - \frac{(2t^2+1)M_3}{16\delta^4} + \frac{M_1^3}{2\delta^6} - \frac{6t^2+7}{96\delta^6} M_1M_3 + \frac{3t^4+6t+1}{96\delta^6} M_5\right] + O(\delta^{-8}). \quad (2.11.19)$$

From equation (2.11.7) we find that

$$u_y(x,0) = +p_0(1-\eta)\int_x^1 \frac{f(t)\, dt}{\sqrt{t^2 - x^2}}$$

and from equation (2.11.19) we therefore deduce that if $0 \le x \le 1$,

$$u_y(x,0) = p_0(1-\eta)\sqrt{1-x^2}$$
$$\times \left[1 + \frac{M_1}{2\delta^2} + \frac{4M_1^2 - M_3}{16\delta^4} + \frac{48M_1^3 - 7M_1M_3 + M_5}{96\delta^6} - \frac{(1+2x^2)}{24\delta^4}\right.$$
$$\left. \times \left(M_3 + \frac{M_1M_3 - M_5}{2\delta^2}\right) + \frac{3 + 4x^2 - 12x^4}{480\delta^6} M_5\right] + O(\delta^{-8}).$$

The crack energy is given (in dimensionless units) by the expression

$$W = \pi p_0^2(1-\eta)\int_0^1 f(t)\, dt = \tfrac{1}{2}\pi p_0^2(1-\eta)$$
$$\times \left[1 + \frac{M_1}{2\delta^2} + \frac{2M_1^2 - M_3}{8\delta^4} + \frac{48M_1^3 - 10M_1M_3 + 5M_5}{96\delta^6} + O(\delta^{-8})\right].$$

Reverting to conventional units we see that, if the crack is of length $2c$ and the strip is of width $2a$, then the energy of the crack is

$$W = \frac{\pi p_0^2(1-\eta^2)c^2}{E}\left[1 + \mu_1\left(\frac{c}{a}\right)^2 + \mu_2\left(\frac{c}{a}\right)^4 + \mu_3\left(\frac{c}{a}\right)^6 + O\left(\frac{c^8}{a^8}\right)\right], \quad c \gg a,$$

where

$$\mu_1 = \tfrac{1}{2}M_1 = 1.1987, \qquad \mu_2 = \tfrac{1}{4}M_1^{\,2} - \tfrac{1}{8}M_3 = 0.1351,$$

$$\mu_3 = \tfrac{1}{2}M_1^{\,3} - \frac{5M_1M_3}{48} + \frac{5M_5}{96} = 9.3291.$$

The corresponding formula for strain-energy release rate with crack extension is

$$\mathscr{G} = \mathscr{G}_1\left[1 + 2\mu_1\left(\frac{c}{a}\right)^2 + 3\mu_2\left(\frac{c}{a}\right)^4 + 4\mu_3\left(\frac{c}{a}\right)^6 + O\left(\frac{c^8}{a^8}\right)\right], \quad c \gg a, \qquad (2.11.20)$$

where $\mathscr{G}_1$ is the strain-energy release rate for a crack in an infinite plane.

We next consider the solution of the equations of elastic equilibrium for the semi-infinite strip $|x| \leq a$, $y \geq 0$ subject to the boundary conditions

$$\sigma_{yy}(x,0) = -p_0, \qquad |x| \leq 1, \qquad (2.11.21)$$

$$u_y(x,0) = 0, \qquad 1 \leq |x| \leq a, \qquad (2.11.22)$$

$$\sigma_{xy}(x,0) = 0, \qquad |x| \leq a, \qquad (2.11.23)$$

$$\sigma_{xy}(\pm a,y) = \sigma_{xx}(\pm a,y) = 0, \quad y \geq 0, \qquad (2.11.24)$$

which corresponds to the opening under a constant pressure p_0 of a Griffith crack (whose half-length is taken to be the unit of length in the problem) in a strip of width $2a$ (see Fig. 22).

Sneddon and Srivastav (1965) reduce the solution of this problem to that of a Fredholm integral equation of the second kind by a method similar to that employed by Sneddon (1962a) for the solution of a similar boundary value problem in electrostatics. We assume a displacement field of the form

$$u_x(x,y) =$$
$$-\tfrac{1}{2}p_0\mathscr{F}_c[\xi^{-1}\{f(\xi) - (1-2\eta)g(\xi)\}\sinh(\xi x) + xg(\xi)\cosh(\xi x); \xi \to y]$$
$$-\tfrac{1}{2}p_0\mathscr{F}_s[\zeta^{-1}\phi(\zeta)(1 - 2\eta - \zeta y)e^{-\zeta y}; \zeta \to x] \qquad (2.11.25)$$

$$u_y(x,y) =$$
$$-\tfrac{1}{2}p_0\mathscr{F}_s[\xi^{-1}\{f(\xi) + 2(1-\eta)g(\xi)\}\cosh(\xi x) + xg(\xi)\sinh(\xi x); \xi \to y]$$
$$+\tfrac{1}{2}p_0\mathscr{F}_c[\zeta^{-1}\phi(\zeta)(2 - 2\eta + \zeta y)e^{-\zeta y}; \zeta \to x]. \qquad (2.11.26)$$

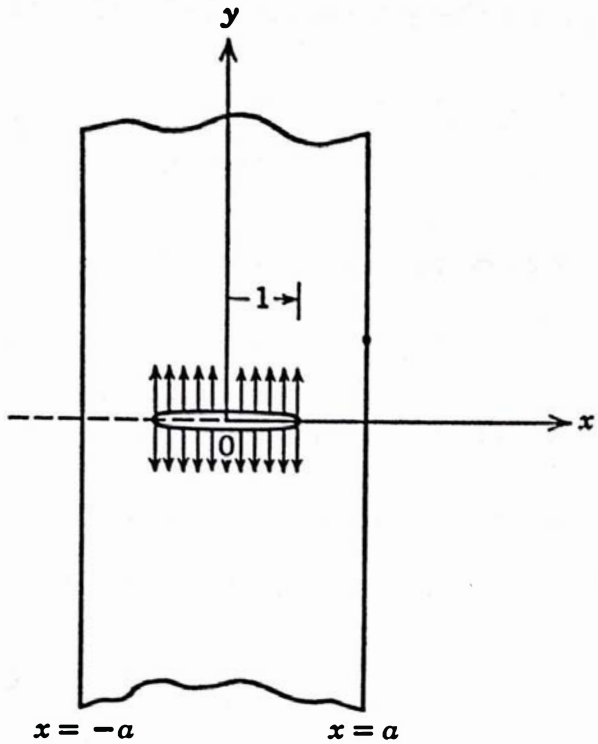


Figure 22 Griffith crack perpendicular to the boundary of a semi-infinite strip.

The corresponding stress field is given by the equations

$$\sigma_{xx}(x,y) = -p_0\mathscr{F}_c[f(\xi)\cosh(\xi x) + \xi x g(\xi)\sinh(\xi x);\ \xi \to y] - p_0\mathscr{F}_c[\phi(\zeta)(1 - \zeta y)e^{-\zeta y};\ \zeta \to x] \quad (2.11.27)$$

$$\sigma_{yy}(x,y) = p_0\mathscr{F}_c[\{f(\xi) + 2g(\xi)\}\cosh(\xi x) + \xi x g(\xi)\sinh(\xi x);\ \xi \to y] - p_0\mathscr{F}_c[\phi(\zeta)(1 + \zeta y)e^{-\zeta y};\ \zeta \to x] \quad (2.11.28)$$

$$\sigma_{xy}(x,y) = p_0\mathscr{F}_s[\{f(\xi) + g(\xi)\}\sinh(\xi x) + \xi x g(\xi)\cosh(\xi x);\ \xi \to y] - p_0 y\mathscr{F}_s[\zeta\phi(\zeta)e^{-\zeta y};\ \zeta \to x]. \quad (2.11.29)$$

This solution therefore has the property that

$$u_y(x,0) = (1 - \eta)p_0\mathscr{F}_c[\zeta^{-1}\phi(\zeta);\ \zeta \to x]$$

$$\sigma_{yy}(x,0) = -p_0\frac{d}{dx}\mathscr{F}_s[\zeta^{-1}\phi(\zeta);\ \zeta \to x] + p_0\sqrt{\frac{2}{\pi}}\int_0^\infty [\{f(\xi) + 2g(\xi)\}\cosh(\xi x) + \xi x g(\xi)\sinh(\xi x)]\,d\xi$$

$$\sigma_{xy}(x,0) = 0.$$

We note that we can write equations (2.11.27), (2.11.29) in the form

$$\mathscr{F}_c[\sigma_{xx}(x,y);\, y \to \xi] = -p_0 f(\xi)\cosh(\xi x) - p_0 \xi x g(\xi)\sinh(\xi x) - \frac{4\xi^2}{\pi} p_0 \int_0^\infty \frac{\zeta\phi(\zeta)\cos(\zeta x)\, d\zeta}{(\xi^2+\zeta^2)^2}$$

$$\mathscr{F}_s[\sigma_{xy}(x,y);\, y \to \xi] = p_0\{f(\xi) + g(\xi)\}\sinh(\xi x) + p_0 \xi x g(\xi)\cosh(\xi x) - \frac{4\xi}{\pi} p_0 \int_0^\infty \frac{\zeta^2\phi(\zeta)\sin(\zeta x)\, d\zeta}{(\xi^2+\zeta^2)^2},$$

and equations (2.11.24) in the forms

$$\mathscr{F}_s[\sigma_{xy}(\pm a,y);\, y \to \xi] = 0, \qquad \mathscr{F}_c[\sigma_{xx}(\pm a,y);\, y \to \xi] = 0,$$

so that the conditions (2.11.24) will be satisfied if we choose $f(\xi)$ and $g(\xi)$ to be such that

$$f(\xi)\cosh(\xi a) + \xi a g(\xi)\sinh(\xi a) = -\frac{4\xi^2}{\pi}\int_0^\infty \frac{\zeta\phi(\zeta)\cos(\zeta a)\, d\zeta}{(\xi^2+\zeta^2)^2} \tag{2.11.30}$$

$$f(\xi)\sinh(\xi a) + g(\xi)[\sinh(\xi a) + \xi a \cosh(\xi a)] = \frac{4\xi}{\pi}\int_0^\infty \frac{\zeta^2\phi(\zeta)\sin(\zeta a)\, d\zeta}{(\xi^2+\zeta^2)^2}. \tag{2.11.31}$$

On the other hand the conditions (2.11.21), (2.11.22) will be satisfied if

$$\frac{d}{dx}\mathscr{F}_s[\zeta^{-1}\phi(\zeta);\, \zeta \to x] - \sqrt{\frac{2}{\pi}}\int_0^\infty [\{f(\xi) + 2g(\xi)\}\cosh(\xi x) + \xi x g(\xi)\sinh(\xi x)]\, d\xi = 1, \quad 0 \le x \le 1 \tag{2.11.32}$$

$$\mathscr{F}_c[\zeta^{-1}\phi(\zeta);\, \zeta \to x] = 0, \quad 1 \le x \le a. \tag{2.11.33}$$

The second of these two dual integral equations will be satisfied identically if we make the representation

$$\phi(\zeta) = \sqrt{\frac{\pi}{2}}\,\zeta\int_0^1 t\psi(t)J_0(\zeta t)\, dt \tag{2.11.34}$$

of the function $\phi(\zeta)$. If we substitute this form in equation (2.11.32) we find that $\psi(t)$ is related to $f(\xi)$ and $g(\xi)$ through the equation

$$\sqrt{\frac{\pi}{2}}\int_0^x \frac{t\psi(t)\, dt}{\sqrt{x^2-t^2}} - \int_0^\infty \{\xi^{-1}[f(\xi) + g(\xi)]\sinh(\xi x) + x g(\xi)\cosh(\xi x)\}\, d\xi = \sqrt{\frac{\pi}{2}}\,x, \quad 0 \le x \le 1$$

and it is easily shown that this is equivalent to the relation

$$\sqrt{\frac{\pi}{2}}\, t\psi(t) = \frac{2}{\pi}\int_0^\infty \xi^{-1}\{f(\xi) + g(\xi)\}\, d\xi \,\frac{d}{dt}\int_0^t \frac{u \sinh(\xi u)\, du}{\sqrt{t^2 - u^2}}$$

$$+ \frac{2}{\pi}\int_0^\infty g(\xi)\, d\xi \,\frac{d}{dt}\int_0^t \frac{u^2 \cosh(\xi u)\, du}{\sqrt{t^2 - u^2}} + \sqrt{\frac{\pi}{2}}\, t, \quad 0 \le t \le 1.$$

Since

$$\frac{d}{dt}\int_0^t \frac{u \sinh(\xi u)\, du}{\sqrt{t^2 - u^2}} = \tfrac{1}{2}\pi\xi t I_0(\xi t)$$

$$\frac{d}{dt}\int_0^t \frac{u^2 \cosh(\xi u)\, du}{\sqrt{t^2 - u^2}} = \tfrac{1}{2}\pi t[I_0(\xi t) + \xi t I_1(\xi t)]$$

we can write this equation in the form

$$\psi(t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \{[f(\xi) + 2g(\xi)]I_0(\xi t) + \xi t g(\xi) I_1(\xi t)\}\, d\xi + 1. \quad (2.11.35)$$

On the other hand, if we substitute from (2.11.34) we find that

$$i_1(\xi) = \frac{2\xi^2}{\pi}\int_0^\infty \frac{\zeta\phi(\zeta)\cos(\zeta a)\, d\zeta}{(\xi^2 + \zeta^2)^2} = \sqrt{\frac{\pi}{2}}\,\xi\int_0^1 u\psi(u) i_3(\xi,u,a)\, du$$

$$i_2(\xi) = \frac{2\xi}{\pi}\int_0^\infty \frac{\zeta^2\phi(\zeta)\sin(\zeta a)\, d\zeta}{(\xi^2 + \zeta^2)^2} = -\sqrt{\frac{\pi}{2}}\int_0^1 u\psi(u)\frac{\partial}{\partial a} i_3(\xi,u,a)\, du,$$

where we have written

$$i_3(\xi,u,a) = \frac{1}{\pi}\frac{\partial}{\partial\xi}\,\xi^2\int_0^\infty \frac{\cos(\zeta a) J_0(\zeta u)\, d\zeta}{\xi^2 + \zeta^2}.$$

The integral on the righthand side of this last equation can be evaluated by the use of formula (14) in Erdelyi (1954, Vol. 1, p. 45), to give

$$i_3(\xi,u,a) = \tfrac{1}{2}e^{-\xi a}[(1 - \xi a)I_0(\xi u) + \xi u I_1(\xi u)]$$

so that

$$i_1(\xi) = \frac{1}{2}\sqrt{\frac{\pi}{2}}\,\xi e^{-a\xi}\int_0^1 u\psi(u)[(1 - a\xi)I_0(\xi u) + \xi u I_1(\xi u)]\, du$$

$$i_2(\xi) = \frac{1}{2}\sqrt{\frac{\pi}{2}}\,\xi e^{-a\xi}\int_0^1 u\psi(u)[(2 - a\xi)I_0(\xi u) + \xi u I_1(\xi u)]\, du.$$

We now solve equations (2.11.30), (2.11.31) to obtain the equations

$$f(\xi) = \frac{-4}{2\xi a + \sinh(2\xi a)} \{[\xi a \cosh(\xi a) + \sinh(\xi a)]i_1(\xi) + \xi a \sinh(\xi a) i_2(\xi)\} \tag{2.11.36}$$

$$g(\xi) = \frac{4}{2\xi a + \sinh(2\xi a)} [\sinh(\xi a) i_1(\xi) + \cosh(\xi a) i_2(\xi)], \tag{2.11.37}$$

and if we substitute from these equations into (2.11.35) we find that $\psi(t)$ is the solution of the integral equation

$$\psi(t) - \int_0^1 \psi(u) K(t,u)\, du = 1, \quad 0 \leq t \leq 1, \tag{2.11.38}$$

where the kernel is defined by

$$\begin{aligned} K(t,u) = u \int_0^\infty & \frac{\xi\, d\xi}{2\xi a + \sinh(2\xi a)} \\ & \times \{(2a^2\xi^2 - 6a\xi + 5 + 3e^{-2a\xi}) I_0(\xi u) I_0(\xi t) + (3 - 2a\xi + e^{-2a\xi}) \\ & \times [\xi u I_0(\xi t) I_1(\xi u) + \xi t I_0(\xi u) I_1(\xi t)] + 2\xi^2 ut I_1(\xi u) I_1(\xi t)\}\, d\xi. \end{aligned} \tag{2.11.39}$$

If we now let

$$\psi(t) = \psi_1\left(\frac{t}{a}\right), \tag{2.11.40}$$

$$\begin{aligned} L(t,v) &= aK(at,au) \\ &= v \int_0^\infty \frac{\zeta\, d\zeta}{2\zeta + \sinh 2\zeta} \\ & \quad \times \{(2\zeta^2 - 6\zeta + 5 + 3e^{-2\zeta}) I_0(\zeta v) I_0(\zeta t) + (3 - 2\zeta + e^{-2\zeta}) \\ & \quad \times [\zeta v I_0(\zeta t) I_1(\zeta v) + \zeta t I_0(\zeta v) I_1(\zeta t)] + 2\zeta^2 tv I_1(\zeta v) I_1(\zeta t)\}\, d\zeta, \end{aligned} \tag{2.11.41}$$

we may write (2.11.39) in the form

$$\psi_1(t) - \int_0^\alpha \psi_1(v) L(t,v)\, dv = 1, \qquad 0 \leq t \leq \alpha \quad \left(\alpha = \frac{1}{a}\right). \tag{2.11.42}$$

An approximate solution of equation (2.11.42) is

$$\psi_1(t) = 1 + \int_0^\alpha L(t,v)\, dv. \tag{2.11.43}$$

Now, from equation (2.11.25), we see that

$$u_y(x,0) = p_0(1-\eta)\mathscr{F}_c[\zeta^{-1}\phi(\zeta);x] = p_0(1-\eta)\int_x^1 \frac{t\psi(t)\, dt}{\sqrt{t^2 - x^2}},$$

so that (in dimensionless units) the energy required to form the crack is

$$W = 2p_0\int_0^1 u_y(x,0)\,dx = \pi p_0^{\,2}(1-\eta)\int_0^1 t\psi(t)\,dt$$

and, by equation (2.11.40), this can be written as

$$W = \pi p_0^{\,2}a^2(1-\eta)\int_0^\alpha t\psi_1(t)\,dt.$$

If we now substitute the approximate expression (2.11.43) into this last equation we find that

$$W = \tfrac{1}{2}\pi p_0^{\,2}a^2(1-\eta)\left[1 + \frac{2}{\alpha^2}\int_0^\alpha\int_0^\alpha tL(t,v)\,dt\,dv\right]. \qquad (2.11.44)$$

Using the results

$$\int_0^\alpha vI_0(\zeta v)\,dv = \frac{\alpha I_1(\zeta\alpha)}{\zeta}, \qquad \int_0^\alpha v^2 I_1(\zeta v)\,dv = \frac{\alpha^2 I_2(\zeta\alpha)}{\zeta}$$

we find that

$$\begin{aligned}\int_0^\alpha L(t,v)\,dv = \alpha\int_0^\infty &\frac{d\zeta}{2\zeta + \sinh 2\zeta}\\ &\times \{(2\zeta^2 - 6\zeta + 5 + 3e^{-2\zeta})I_1(\zeta\alpha)I_0(\zeta t) + (3 - 2\zeta + e^{-2\zeta})\\ &\times [\zeta\alpha I_0(\zeta t)I_2(\zeta\alpha) + \zeta t I_0(\zeta\alpha)I_1(\zeta t) + 2\zeta^2\alpha t I_2(\zeta\alpha)I_1(\zeta t)]\}\,d\xi\end{aligned}$$

and hence that

$$\begin{aligned}&\int_0^\alpha t\,dt\int_0^\alpha L(t,v)\,dv\\ &= \alpha^2\int_0^\infty \frac{d\zeta}{2\zeta + \sinh 2\zeta}\{(2\zeta^2 - 6\zeta + 5 + 3e^{-2\zeta})\zeta^{-1}I_1(\zeta\alpha)I_1(\zeta\alpha)\\ &\quad + (3 - 2\zeta + e^{-2\zeta})[\alpha I_1(\zeta\alpha)I_1(\zeta\alpha) + \alpha I_0(\zeta\alpha)I_2(\zeta\alpha)] + 2\zeta\alpha I_2(\zeta\alpha)I_2(\zeta\alpha)\}\,d\zeta\\ &= \tfrac{1}{4}\alpha^4\int_0^\infty \frac{\zeta(2\zeta^2 - 6\zeta + 5 + 3e^{-2\zeta})}{2\zeta + \sinh 2\zeta}\,d\zeta + O(\alpha^5).\end{aligned}$$

Reverting to conventional units, we see that, if the crack is of length $2c$ and the strip of width $2a$,

$$W = \frac{\pi p_0^{\,2}(1-\eta)a^2}{E}\left[1 + \omega_1\left(\frac{c}{a}\right)^2 + O\left(\frac{c^3}{a^3}\right)\right], \quad a \gg c \qquad (2.11.45)$$

where ω_1 is the number defined by

$$\omega_1 = \frac{1}{2}\int_0^\infty \frac{\zeta(2\zeta^2 - 6\zeta + 5 + 3e^{-2\zeta})}{2\zeta + \sinh 2\zeta}\,d\zeta = 0.59482, \qquad (2.11.46)$$

the numerical value determined from Ling's tables (Ling, 1957).

2.12 THE DISTRIBUTION OF STRESS IN THE NEIGHBORHOOD OF AN EXTERNAL CRACK

In this section we consider the distribution of stress in the vicinity of the external crack $|x| \geq 1$, $y = 0$ in an infinite two-dimensional elastic medium when the crack is opened up by the application of pressure to its faces (see Fig. 23). We assume that the pressure on the upper face of the crack is equal to that on the lower face (i.e., that there is symmetry about the x-axis but not necessarily about the y-axis). We follow here the method of Lowengrub (1966a).

With the assumption we have made of symmetrical loading about the x-axis we can consider the solution of the equations of elastic equilibrium in the half-space $y \geq 0$ when the boundary $y = 0$ is subjected to the conditions

$$\sigma_{xy}(x,0) = 0, \quad -\infty < x < \infty; \qquad \sigma_{yy}(x,0) = -\frac{1}{1-\eta}\, g(x), \quad |x| > 1;$$
$$u_y(x,0) = 0, \quad |x| < 1, \tag{2.12.1}$$

where the function $g(x)$ is prescribed.

If we superpose two solutions of the elastic equations of the type (2.3.2) and (2.3.3), we obtain the displacement field

$$u_x(x,y) = \frac{1}{2(1-\eta)} \int_0^\infty \xi^{-1} e^{-\xi y}(1 - 2\eta - \xi y)[\psi_a(\xi)\cos(\xi x) - \psi_s(\xi)\sin(\xi x)]\, d\xi \tag{2.12.2}$$

$$u_y(x,y) = \frac{1}{2(1-\eta)} \int_0^\infty \xi^{-1} e^{-\xi y}(2 - 2\eta + \xi y)[\psi_s(\xi)\cos(\xi x) + \psi_a(\xi)\sin(\xi x)]\, d\xi \tag{2.12.3}$$

and it is easily shown that this leads to a stress field for which $\sigma_{xy}(x,0) = 0$ and

$$\sigma_{yy}(x,0) = -\frac{1}{1-\eta} \int_0^\infty [\psi_s(\xi)\cos(\xi x) + \psi_a(\xi)\sin(\xi x)]\, d\xi. \tag{2.12.4}$$

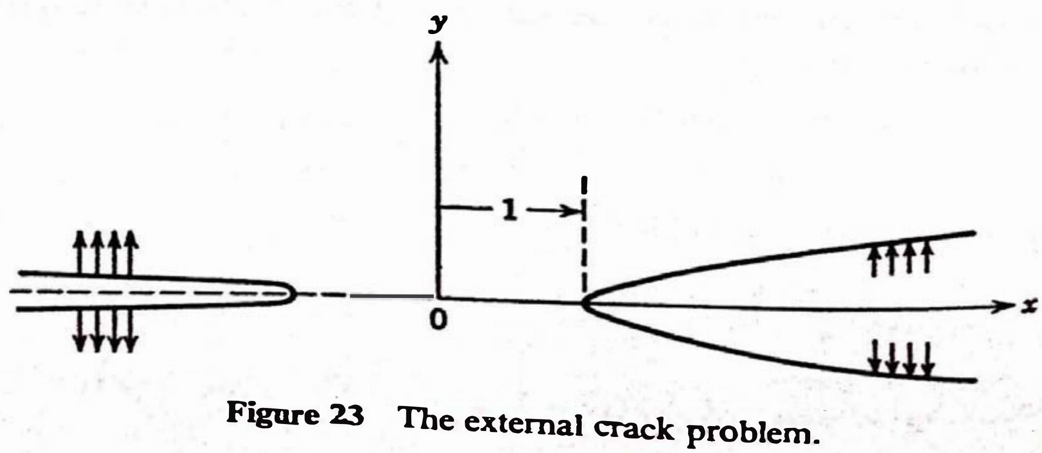


Figure 23 The external crack problem.

Hence if we write $g_s(x) = \frac{1}{2}[g(x) + g(-x)]$, $g_a(x) = \frac{1}{2}[g(x) - g(-x)]$, we see that the boundary conditions (2.12.1) will be satisfied if we can find a function $\psi_s(\xi)$ which satisfies the dual integral equations

$$\begin{aligned} \int_0^\infty \xi^{-1}\psi_s(\xi)\cos(\xi x)\,d\xi &= 0, \qquad 0 \le x \le 1, \\ \int_0^\infty \psi_s(\xi)\cos(\xi x)\,d\xi &= g_s(x), \qquad x > 1, \end{aligned} \tag{2.12.5}$$

and a function $\psi_a(\xi)$ which satisfies the dual integral equations

$$\begin{aligned} \int_0^\infty \xi^{-1}\psi_a(\xi)\sin(\xi x)\,d\xi &= 0, \qquad 0 \le x \le 1, \\ \int_0^\infty \psi_a(\xi)\cos(\xi x)\,d\xi &= g_a(x), \qquad x > 1. \end{aligned} \tag{2.12.6}$$

If we write

$$\psi_s(\xi) = \psi_0(\xi) + \psi_1(\xi) \tag{2.12.7}$$

where

$$\psi_1(\xi) = \frac{2}{\pi}\int_1^\infty g_s(x)\cos(\xi x)\,dx, \tag{2.12.8}$$

we observe that equations (2.12.5) are reduced to the form

$$\begin{aligned} \int_0^\infty \xi^{-1}\psi_0(\xi)\cos(\xi x)\,d\xi &= f(x), \quad 0 \le x \le 1 \\ \int_0^\infty \psi_0(\xi)\cos(\xi x)\,d\xi &= 0, \qquad x > 1 \end{aligned} \tag{2.12.9}$$

where the function $f(x)$ is defined by the relation

$$f(x) = -\int_0^\infty \xi^{-1}\psi_1(\xi)\cos(\xi x)\,d\xi, \tag{2.12.10}$$

and $\psi_1(\xi)$ is defined by equation (2.12.8). Assuming, now, that when $0 \le x \le 1$,

$$\int_0^\infty \psi_0(\xi)\cos(\xi x)\,d\xi = \frac{d}{dx}\left[x\int_x^1 \frac{p(t)\,dt}{\sqrt{t^2 - x^2}}\right], \tag{2.12.11}$$

we find that the unknown function $p(t)$ is determined by the relation

$$\int_0^x \frac{p(t)\,dt}{\sqrt{x^2 - t^2}} = \frac{1}{x}[C - f(x)],$$

where C denotes the constant

$$\int_0^1 p(t)\,dt.$$

Solving this Abel type equation we obtain the expression

$$p(t) = -\frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{f(x)\,dx}{\sqrt{t^2 - x^2}},$$

so that

$$p(t) = \frac{2}{\pi t}\int_1^\infty \frac{x g_s(x)\,dx}{\sqrt{x^2 - t^2}} - \frac{2}{\pi t}\int_1^\infty g_s(x)\,dx \tag{2.12.12}$$

and

$$\psi_0(\xi) = -\frac{2}{\pi}[1 - J_0(\xi)]\int_1^\infty g_s(x)\,dx - \frac{2}{\pi}\xi\int_0^1 J_1(\xi t)\,dt\int_1^\infty \frac{x g_s(x)\,dx}{\sqrt{x^2 - t^2}}. \tag{2.12.13}$$

The function $\psi_s(\xi)$ is therefore given by equations (2.12.7), (2.12.8), and (2.12.13). It is also easily shown that when $0 \le x \le 1$

$$\int_0^\infty \psi_0(\xi)\cos(\xi x)\,d\xi = \frac{2}{\pi\sqrt{1 - x^2}}\int_1^\infty g_s(u)\,du + \frac{2}{\pi}\int_1^\infty \frac{u\sqrt{u^2 - 1}\,g_s(u)\,du}{(u^2 - x^2)\sqrt{1 - x^2}}. \tag{2.12.14}$$

The equations (2.12.6) can be treated in a similar fashion. If we put

$$\psi_a(\xi) = \psi_0^*(\xi) + \psi_1^*(\xi), \tag{2.12.15}$$

where

$$\psi_1^*(\xi) = \frac{2}{\pi}\int_1^\infty g_a(x)\sin(x\xi)\,dx \tag{2.12.16}$$

we find that

$$\psi_0^*(\xi) = \frac{2}{\pi}\int_0^1 t^2 J_1(\xi t)\,dt\int_1^\infty \frac{g_a(u)\,du}{(u^2 - t^2)^{3/2}} \tag{2.12.17}$$

When $0 \le x \le 1$ we now have

$$\int_0^\infty \psi_0^*(\xi)\sin(\xi x)\,d\xi = \frac{2}{\pi}x\sqrt{1 - x^2}\int_1^\infty \frac{g_a(u)\,du}{(u^2 - x^2)\sqrt{u^2 - 1}}. \tag{2.12.18}$$

Combining this result we see that

$$\sigma_{yy}(x,0) = -\frac{2}{\pi(1 - \eta)}\sigma(x), \quad 0 \le x < 1 \tag{2.12.19}$$

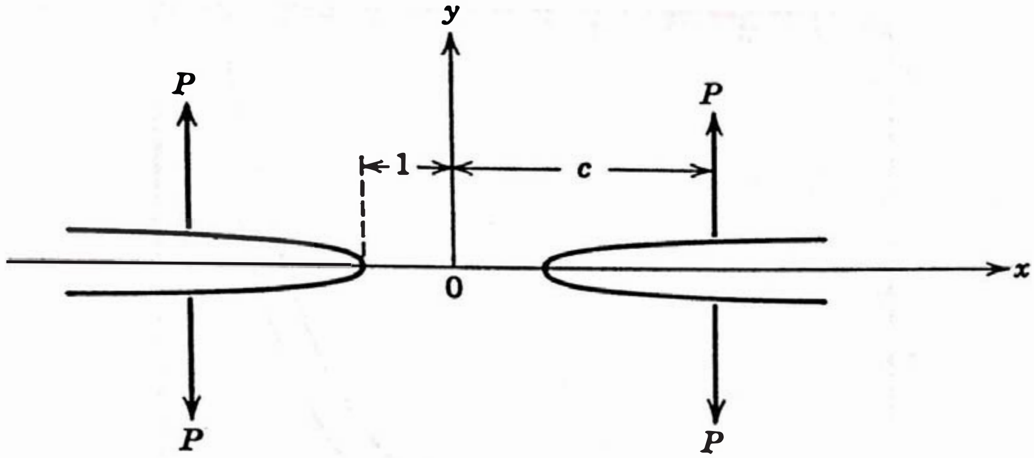


Figure 24 Symmetrical loading of an external crack by point forces.

where the function $\sigma(x)$ is defined by

$$\sigma(x) = \frac{1}{\sqrt{1-x^2}}\left\{\int_1^\infty g_s(u)\,du + \int_1^\infty \frac{u\sqrt{u^2-1}\,g_s(u)\,du}{u^2-x^2}\right\} + x\sqrt{1-x^2}\int_1^\infty \frac{g_a(u)\,du}{(u^2-x^2)\sqrt{u^2-1}}. \quad (2.12.20)$$

We consider two special cases:

Case (a) If we take

$$g_s(x) = (1-\eta)P\delta(c-x), \qquad g_a(x) = 0, \quad (c > 1), \quad (2.12.21)$$

we get the symmetrical loading shown in Fig. 24. In this case we find that

$$\sigma(x) = \frac{P(1-\eta)}{\sqrt{1-x^2}}\left[1 + \frac{c\sqrt{c^2-1}}{c^2-x^2}\right]$$

so that the stress intensity factor

$$K = \lim_{x\to 1-} \sqrt{2(1-x)}\,\sigma_{yy}(x,0)$$

is given by the formula

$$K = -\frac{2P}{\pi}\left[1 + \frac{c}{\sqrt{c^2-1}}\right]. \quad (2.12.22)$$

The variation of $-(1/2)\pi\sigma_{yy}(x,0)/P \equiv \sigma(x)/P(1-\eta)$ with x for the two cases $c = 1.2$, $c = 1.6$ is shown in Fig. 25.

Case (b) If we take

$$g_s(x) = (1-\eta)P\delta(c-x), \qquad g_a(x) = (1-\eta)P\delta(c-x), \quad (c > 1), \quad (2.12.23)$$

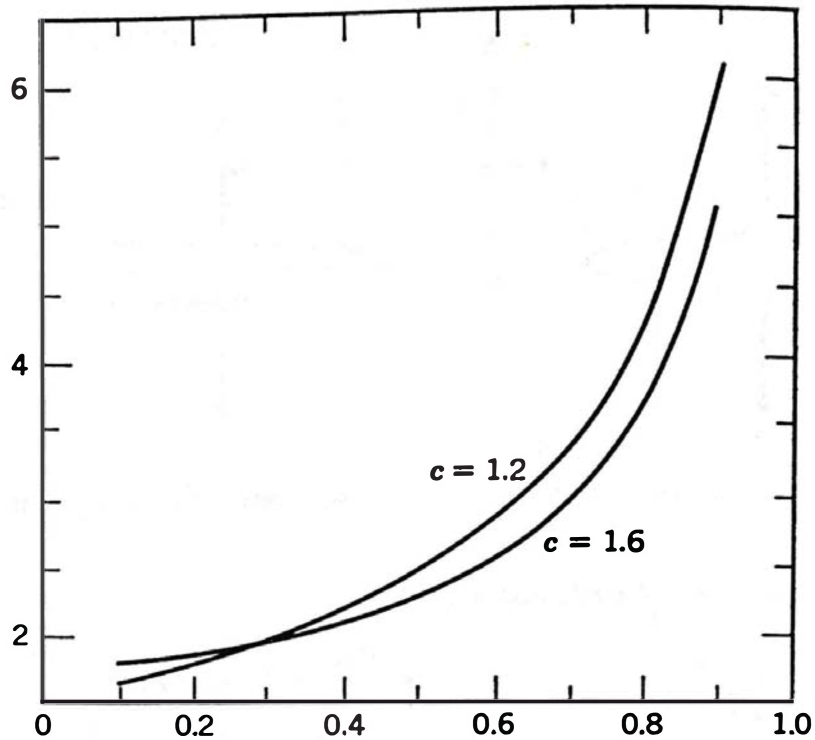


Figure 25 The variation with x of $\sigma_{yy}(x,0)$, $0 \leq x \leq 1$, for the loading shown in Fig. 24.

we get the symmetrical loading shown in Fig. 26. In this case we have

$$\sigma(x) = \tfrac{1}{2}(1-\eta)P\left[(1-x^2)^{-1/2}\left(1+\frac{c\sqrt{c^2-1}}{c^2-x^2}\right)+\frac{x\sqrt{1-x^2}}{(c^2-x^2)\sqrt{c^2-1}}\right]. \tag{2.12.24}$$

The variation of $\sigma(x)$ with x is shown for the two cases, $c = 1.2, 1.6$, in Fig. 27. It is easily seen from this last equation that the stress intensity factor is given by the formula

$$K = -\frac{P}{\pi}\left(1+\frac{c}{\sqrt{c^2-1}}\right), \tag{2.12.25}$$

(i.e., it is exactly half of the value in the symmetrical case).

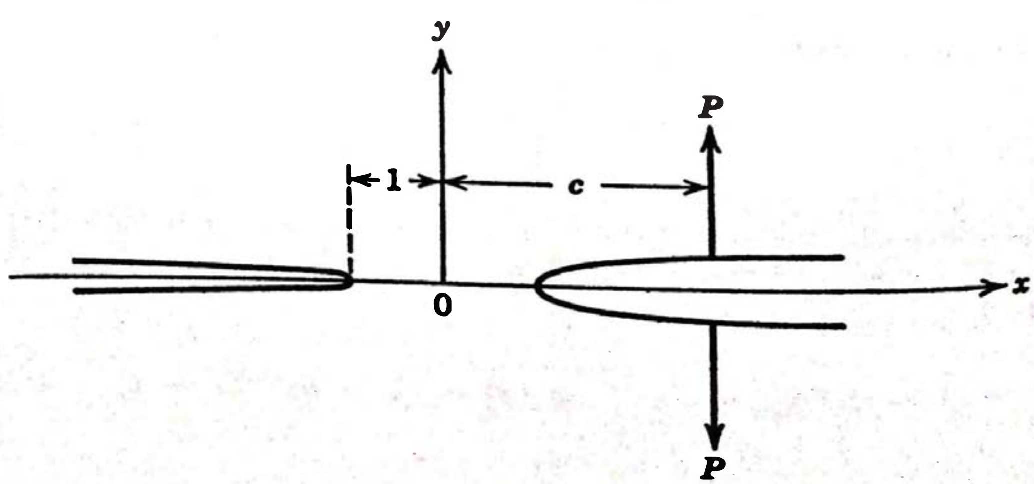


Figure 26 Unsymmetrical loading of an external crack by point forces.

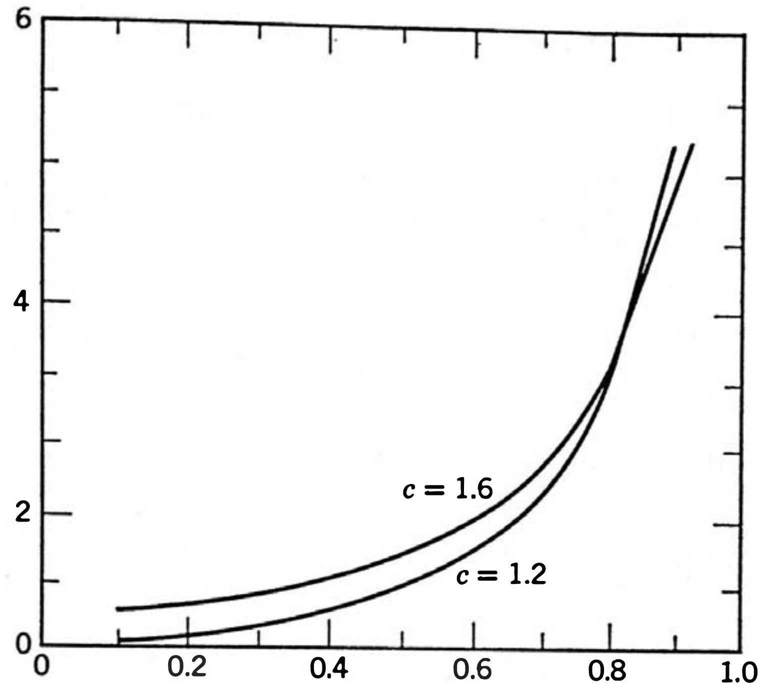


Figure 27 The variation with x of $\sigma_{yy}(x,0)$, $0 \leq x \leq 1$, for the loading shown in Fig. 26.

2.13 THE DISTRIBUTION OF PRESSURE NECESSARY TO PRODUCE A GRIFFITH CRACK OF PRESCRIBED SHAPE

In the discussion of certain physical problems, particularly in Barenblatt's theory of cohesive forces at the tips of cracks, it is important to determine what distribution of pressure is necessary to produce a Griffith crack of prescribed shape. This is equivalent to determining the value of the normal component of the surface stress $\sigma_{yy}(x,0)$ when the half-plane $y \geq 0$ is deformed as a result of the surface conditions

$$u_y(x,0) = W(x)H(1 - |x|), \quad W(1) = 0, \tag{2.13.1}$$

$$\sigma_{xy}(x,0) = 0, \tag{2.13.2}$$

valid for all real values of x. For simplicity we shall consider only the symmetrical case in which $W(x)$ is an *even* function of x (cf. Sneddon, 1969), and choose our unit of length to be half the width of the crack.

If we write $p(x) = -\sigma_{yy}(x,0)$ for the surface pressure, it follows from equations (2.3.3) and (2.3.5) that $W(x)$ and $p(x)$ are related through the equations

$$W(x)H(1 - x) = (1 - \eta)\mathscr{F}_c[\psi(\xi);x],$$

$$p(x) = \mathscr{F}_c[\xi\psi(\xi);x].$$

Using the notation

$$W_s^{(1)}(\xi) = \mathscr{F}_s[W'(x)H(1 - x);\ \xi], \tag{2.13.3}$$

we see that this relationship can be expressed through the equation

$$p(x) = -(1-\eta)^{-1}\mathscr{F}_c[W_s^{(1)}(\xi);x]. \tag{2.13.4}$$

If we substitute this expression into equations (2.3.10) we find that the auxiliary function $f(t)$ introduced through equation (2.3.9) is given by the equations

$$f(t) = -\frac{4t}{\pi^2(1-\eta)}\int_0^t \frac{du}{\sqrt{(t^2-u^2)}}\int_0^\infty \cos(\xi u)\,d\xi \int_0^1 W'(v)\sin(\xi v)\,dv.$$

Reversing the order in which the integrations are performed we find that

$$\begin{aligned} f(t) &= -\frac{2t}{\pi(1-\eta)}\int_0^1 W'(v)\,dv\int_0^\infty \sin(\xi v)\,d\xi\,\frac{2}{\pi}\int_0^t \frac{\cos(\xi u)\,du}{\sqrt{(t^2-u^2)}} \\ &= -\frac{2t}{\pi(1-\eta)}\int_0^1 W'(v)\,dv\int_0^\infty J_0(\xi t)\sin(\xi v)\,d\xi. \end{aligned}$$

That is, that

$$f(t) = -\frac{2t}{\pi(1-\eta)}\int_t^1 \frac{W'(v)\,dv}{\sqrt{v^2-t^2}}. \tag{2.13.5}$$

From equation (2.3.11) we find that

$$p(x) = \frac{d}{dx}\,x\int_0^1 \frac{f(x\sqrt{u})\,du}{\sqrt{1-u}}. \tag{2.13.6}$$

The required relation between the functions $p(x)$ and $W(x)$ is provided by (2.13.5) and (2.13.6).

For example, suppose that

$$W(x) = \epsilon(1-x^2)^{n+1/2}, \quad (n > -\tfrac{1}{2}). \tag{2.13.7}$$

Then by making use of (2.13.5) we find that

$$f(t) = \frac{2\epsilon}{(1-\eta)\pi}\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\cdot\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}(1-t^2)^n,$$

and substituting this expression for $f(t)$ into (2.13.6), we find that the pressure distribution corresponding to the shape (2.13.7) is

$$p(x) = \frac{4\epsilon}{(1-\eta)\sqrt{\pi}}\cdot\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}q_n(x^2), \quad |x| < 1, \tag{2.13.8}$$

where the function $q_n(z)$ is defined by the equation

$$q_n(z) = {}_2F_1(-n,1;\tfrac{1}{2};z), \quad |z| < 1 \tag{2.13.9}$$

It will be observed that

$$\lim_{z\to 1} q_n(z)$$

exists only if $n > \frac{1}{2}$ and that then the use of the Gauss summation theorem for a generalized hypergeometric function of unit argument gives

$$\lim_{x\to 1-} p(x) = -\frac{2(2n+1)\epsilon}{(1-\eta)\sqrt{\pi}} \cdot \frac{\Gamma(n-\frac{1}{2})}{\Gamma(n+1)}. \tag{2.13.10}$$

When n is a positive integer, the functions $q_n(z)$ are polynomials; the low-order polynomials are given by the equations

$$\begin{aligned}
q_0(z) &= 1 \\
q_2(z) &= 1 - 2z \\
q_2(z) &= 1 - 4z + \tfrac{8}{3}z^2 \\
q_3(z) &= 1 - 6z + 8z^2 - \tfrac{16}{5}z^3 \\
q_4(z) &= 1 - 8z + 16z^2 - \tfrac{64}{5}z^3 + \tfrac{128}{35}z^4 \\
q_5(z) &= 1 - 10z + \tfrac{80}{3}z^2 - 32z^3 + \tfrac{128}{7}z^4 - \tfrac{256}{63}z^5.
\end{aligned}$$

If the shape of the crack is prescribed to be

$$W(x) = \epsilon \sum_{n=1}^{\infty} c_n (1 - x^2)^{n+\frac{1}{2}}$$

then the pressure necessary to preserve that shape is

$$p(x) = \frac{2\epsilon}{1-\eta} \sum_{n=1}^{\infty} \frac{(\frac{3}{2})_n}{n!} c_n q_n(x^2).$$

The polynomials $q_n(x)$ have the interesting property that

$$\sum_{r=1}^{n} (-1)^{r-1} \binom{n}{r} q_r(x) = 1 - \frac{2^{2n}(n!)^2}{(2n)!} x^n$$

so that the pressure necessary to keep the crack in the shape

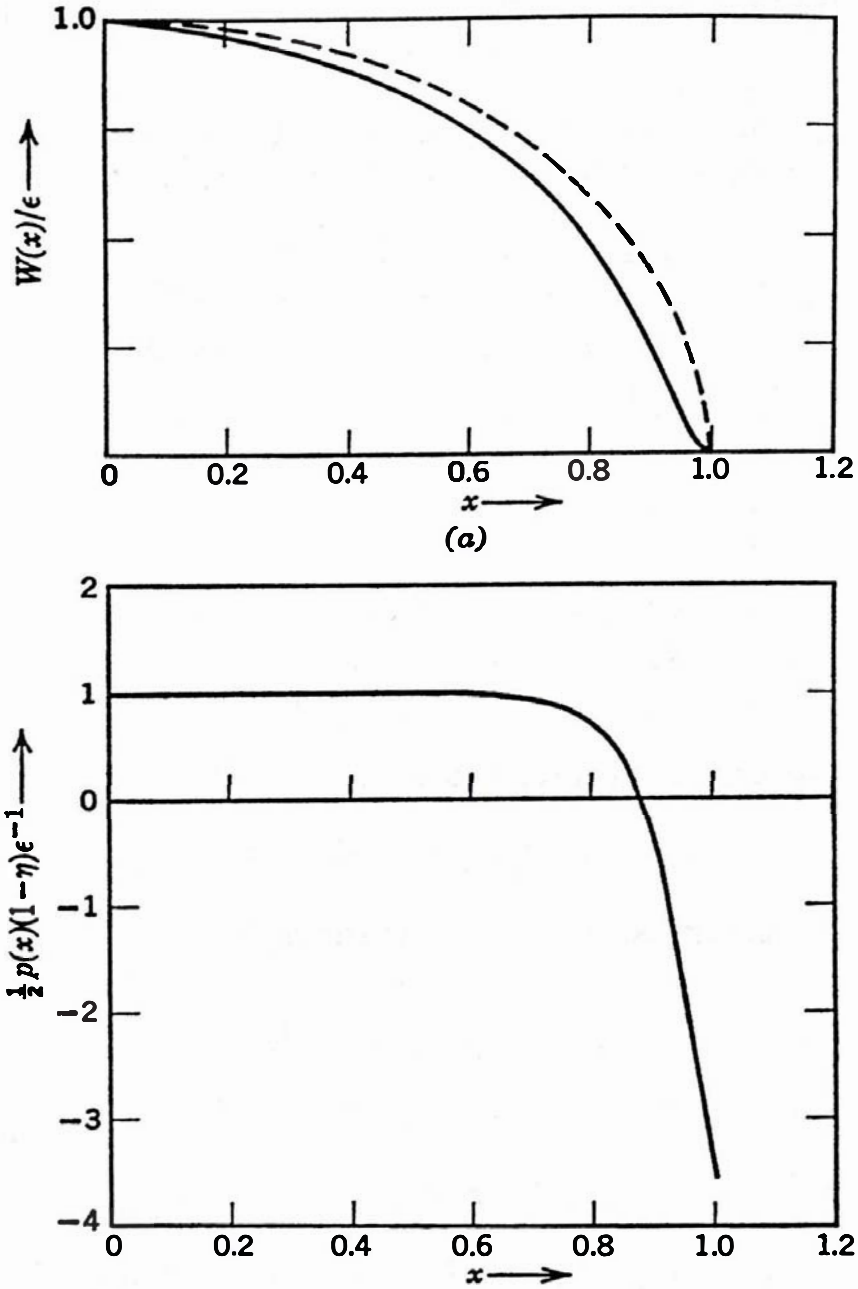
$$W(x) = \frac{2n+1}{2n} \epsilon (1 - x^2)^{\frac{1}{2}} [1 - {}_2F_1(-n, 1; \tfrac{3}{2}; 1 - x^2)]$$

is

$$p(x) = \frac{2\epsilon}{1-\eta}\left[1 - \frac{2^{2n}(n!)^2}{(2n)!} x^{2n}\right].$$

For example if we put $n = 6$ in this result we see that the pressure necessary to keep the crack in the shape

$$\begin{aligned}
W(x) = \frac{\epsilon(1-x^2)^{\frac{3}{2}}}{693} [3003 &- 6006(1-x^2) + 6864(1-x^2)^2 \\
&- 4576(1-x^2)^3 + 1664(1-x^2)^4 - 256(1-x^2)^5]
\end{aligned} \tag{2.13.11}$$



(b)

Figure 28 The variation with x of the functions $w(x)$, $\rho(x)$ defined by equations (2.13.11) and (2.13.12) respectively. The broken line in (a) shows $w(x)$ in the case in which $\rho(x) = 2\epsilon/(1 - \eta)$.

is given by the formula

$$p(x) = \frac{2\epsilon}{1 - \eta}\left[1 - \frac{1024}{231}\, x^{12}\right]. \tag{2.13.12}$$

The variation with x of this pair of functions is shown in Fig. 28. To afford a comparison with the case in which the applied pressure is constant we have included in Fig 28a a broken curve which shows the variation of $W(x)/\epsilon$ in the case in which the applied pressure has the constant value $2\epsilon/(1 - \eta)$.

From the shape of the curve shown in Fig. 28b we see that the force applied to the faces of the crack is *compressive* in the body of the crack but that near the tip of the crack the applied force becomes *tensile*. This is precisely the kind of variation which would be predicted from an examination of equation (2.13.10).

2.14 STRESS DISTRIBUTION IN A NOTCHED PLATE

The problem of determining the distribution of stress in a semi-infinite elastic plate which contains a thin notch perpendicular to its edge, and is in a state of plane strain or generalized plane stress under the action of given loads has been considered by Wigglesworth (1957).* We may take the x- and y-axes in such a way that the origin 0 is the point where the crack cuts the free surface of the elastic body; the x-axis is taken along the free surface and the y-axis is perpendicular to it lying along the crack (cf. Fig. 29). We may take the length of the crack to be our unit of length so that the crack occupies the space $x = \pm 0,\ 0 \leq y \leq 1$.

In the general case, where the loads include body forces, tractions on the boundary $y = 0$ and nonvanishing stresses at infinity, the stress distribution can be expressed as the superposition of the stresses caused by these loads in a plate occupying the complete half-plane, and those in the notched plate only because of certain tractions on the notch. These are the original applied tractions suitably modified by the addition of the appropriate terms from the solution for the complete plate. Since this latter solution is easily derived by standard methods, we need only consider the reduced problem for the

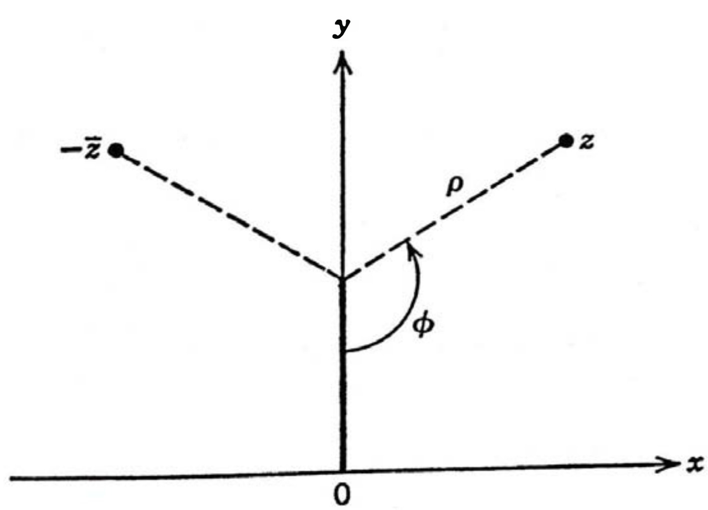


Figure 29 Polar coordinates in the problem of the notched plate.

* An approximate solution of this problem using the 'collocation' method has been given by Gross, Roberts, and Srawley (1968).

notched plate. The vector tractions at the points $(\pm 0,y)$ of the notch are $\mp(\sigma_{xx} + i\,\sigma_{xy})$ so that the boundary conditions on the notch are

$$\sigma_{xx} + i\,\sigma_{xy} = \begin{cases} P_1(y) = f_1(y) - f(y) & (x = +0,\ 0 \leq y \leq 1) \\ P_2(y) = f_2(y) - f(y) & (x = -0,\ 0 \leq y \leq 1) \end{cases} \tag{2.14.1}$$

where $f(y)$ is the value of $\sigma_{xx} + i\,\sigma_{xy}$ at both points $(\pm 0,y)$ in the solution for the complete plate* and $f_1(y), f_2(y)$ are the original loads applied to the notch.

We might expect to have some considerable simplification when the boundary values in (2.14.1) are equal. This occurs in many cases of great practical interest as, for example, when the notch is free from traction so that $f_1(y) = f_2(y) = 0$ and $P_1(y) = P_2(y) = -f(y)$. A complete solution of the problem of the notched plate, based on the Kolosov-Muskhelishvili equations, has been given by Wigglesworth (1957). If we consider the case in which the notched plate is under a uniform tension T at infinity parallel to the x-axes, we have in equation (2.14.1)

$$p_1(y) = p_2(y) = -T.$$

The calculation of the stress components is fairly complicated. Wigglesworth's expressions for the components of stress near the root of the crack are

$$\frac{(\sigma_{xx} + \sigma_{yy})}{T} = -1 - \rho^{-1/2} \sum_{n=0}^{\infty} (n + \tfrac{1}{2}) A_n \rho^n \sin(n - \tfrac{1}{2})\phi$$
$$- \sum_{n=0}^{\infty} (n+1) B_{n+1} \rho^n \cos n\phi$$

$$\frac{(\sigma_{xx} - \sigma_{yy})}{T} = 1 + \rho^{-1/2} \sin\phi \sum_{n=0}^{\infty} (n^2 - \tfrac{1}{4}) A_n \rho^n \cos(n - \tfrac{3}{2})\phi$$
$$- \sin\phi \sum_{n=0}^{\infty} n(n+1) B_{n+1} \rho^n \sin(n-1)\phi$$
$$+ \sum_{n=0}^{\infty} (n+1) B_{n+1} \rho^n \cos n\phi$$

$$\frac{\sigma_{xy}}{T} = -\rho^{-1/2} \sin\phi \sum_{n=0}^{\infty} (n^2 - \tfrac{1}{4}) A_n \rho^n \sin(n - \tfrac{3}{2})\phi$$
$$- \sin\phi \sum_{n=0}^{\infty} n(n+1) B_{n+1} \rho^n \cos(n-1)\phi$$
$$- \sum_{n=0}^{\infty} (n+1) B_{n+1} \rho^n \sin n\phi,$$

* It should be noted that if there are no isolated body-forces, the stresses in the complete plate are continuous.

TABLE 1

VALUES OF COEFFICIENTS IN WIGGLESWORTH'S SOLUTION

n	$(n + \frac{1}{2})A_n$	$(n^2 - \frac{1}{4})A_n$	$(n + 1)B_{n+1}$	$n(n + 1)B_{n+1}$
0	1.58607	−0.79304	−0.4740	0
1	−0.68385	−0.34193	−0.3850	−0.3850
2	0.1583	0.2375	−0.2144	−0.4288
3	0.2183	0.5458	−0.0872	−0.2614
4	0.1692	0.5923	−0.0143	−0.057
5	0.1091	0.4909	0.0190	0.095
6	0.0617	0.3396	0.0304	0.182
7	0.0299	0.1942	0.0304	0.213
8	0.0105	0.0788	0.0260	0.208

where ρ and ϕ are the polar coordinates defined in Fig. 29 and the values of the powers are given in Table 1.

For sufficiently small values of ρ these expressions reduce to the approximate forms

$$\frac{(\sigma_{xx} + \sigma_{yy})}{T} = 1.586\rho^{-\frac{1}{2}} \sin \tfrac{1}{2}\phi \tag{2.14.2}$$

$$\frac{(\sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy})}{T} = -0.793\rho^{-\frac{1}{2}} \sin \phi e^{3i\phi/2}. \tag{2.14.3}$$

It follows from equation (2.14.3) that the maximum shear stress τ is given by the relation $\tau \simeq \tau_0$ where

$$\frac{\tau_0}{T} = 0.793\rho^{-\frac{1}{2}} \sin \phi. \tag{2.14.4}$$

Hence in the neighborhood of the root of the notch, the isochromatic lines are circles touching the notch at its root. On the circle $\rho = a$, a constant, τ_0 takes its maximum value when $\phi = 90°$. When $a = 0.1$, the maximum of τ, as calculated by the series expansion, occurs when $\phi = 100°$. In a similar way, we obtain the values

$$a = 0.1 \qquad \left(\frac{\tau}{T}\right)_{\max} = 1.45 \qquad \left(\frac{\tau_0}{T}\right)_{\max} = 1.26$$

$$a = 0.2 \qquad \left(\frac{\tau}{T}\right)_{\max} = 1.10 \qquad \left(\frac{\tau_0}{T}\right)_{\max} = 0.96$$

which show that the values of the maxima of τ are underestimated by equation (2.14.4).

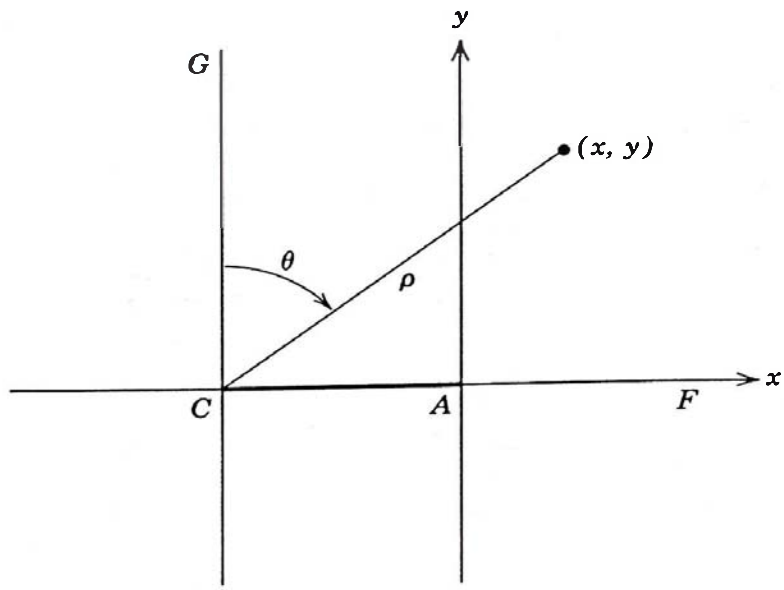


Figure 30 Polar coordinates in Koiter's solution.

A solution of a problem of this type in terms of an Airy stress function expressed in terms of plane polar coordinates has been given by Koiter (1956) who considered the case in which the notch was opened by compressive stresses varying in a linear way along the notch. Koiter's system of co-ordinates is shown in Fig. 30. The boundary conditions he assumes are that the line CG is free from tractions but that the notch CA is loaded by compressive stresses $e_0 - k_0 x$ where e_0 and k_0 are constants. By the symmetry of the problem, the line AF remains straight and no shearing stress is transmitted across it. We thus have the boundary conditions

$$\sigma_{yy} = -E(e_0 - k_0 x), \qquad -1 \leq x \leq 0$$
$$\sigma_{xy} = 0, \qquad x \geq -1$$
$$u_y = 0, \qquad x \geq -1$$

it being assumed that the length of the notch is taken to be the unit of length. To these equations we might add the equation

$$\sigma_{yy} = p_y(x), \quad x \geq 0$$

where the function $p_y(x)$ is unknown.

If we introduce polar coordinates (ρ,θ) as shown, and an Airy stress function $f(\rho,\theta)$ satisfying the biharmonic equation

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}\right)^2 f = 0, \tag{2.14.5}$$

then the stress components assume the forms

$$\sigma_{\rho\rho} = \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho}, \qquad \sigma_{\theta\theta} = \frac{\partial^2 f}{\partial \rho^2}, \qquad \sigma_{\rho\theta} = \frac{1}{\rho^2}\frac{\partial f}{\partial \theta} - \frac{1}{\rho}\frac{\partial^2 f}{\partial \rho\, \partial \theta}. \quad (2.14.6)$$

The boundary conditions then become

$$\theta = 0: \quad \frac{\partial^2 f}{\partial \rho^2} = 0, \qquad \frac{1}{\rho^2}\frac{\partial f}{\partial \theta} - \frac{1}{\rho}\frac{\partial^2 f}{\partial \rho\, \partial \theta} = 0, \quad (2.14.7)$$

$$\theta = \frac{\pi}{2}: \quad \frac{1}{\rho^2}\frac{\partial f}{\partial \theta} - \frac{1}{\rho}\frac{\partial^2 f}{\partial \rho\, \partial \theta} = 0, \quad (2.14.8)$$

$$\theta = \frac{\pi}{2}: \quad \frac{\partial^2 f}{\partial \rho^2} = \begin{cases} -(e_0 + k_0 - k_0\rho), & \rho < 1 \\ P_y(\rho), & \rho > 1. \end{cases} \quad (2.14.9)$$

We also require the following relations:

$$E\left(\frac{1}{\rho}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\rho}{\rho}\right) = \left[\frac{\partial^2 f}{\partial \rho^2} - \nu\left(\frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho}\right)\right], \quad (2.14.10)$$

$$E\left(\frac{\partial u_\rho}{\partial \rho}\right) = \left[\frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} - \nu\frac{\partial^2 f}{\partial \rho^2}\right], \quad (2.14.11)$$

$$E\left(\frac{1}{\rho}\frac{\partial u_\rho}{\partial \theta} + \frac{\partial u_\theta}{\partial \rho} - \frac{u_\theta}{\rho}\right) = 2(1+\nu)\left[\frac{1}{\rho^2}\frac{\partial f}{\partial \theta} - \frac{1}{\rho}\frac{\partial^2 f}{\partial \rho\, \partial \theta}\right], \quad (2.14.12)$$

which follow from the stress-strain relations in generalized plane stress, E denoting Young's modulus, and ν the Poisson's ratio.

We shall now employ Mellin transforms to obtain the desired solution. Let

$$\mathscr{M}[g(\rho);s] = \int_0^\infty \rho^{s-1} g(\rho)\, d\rho$$

denote the Mellin transform of a function $g(\rho)$. If we assume that $\rho^{\mu-2}f(\rho,\theta)$, $\rho^{\mu}p_y(\rho)$ are absolutely integrable with respect to ρ over $(0,\infty)$, and $(1,\infty)$ respectively, for all μ in a range, $0 < \mu < \mu_1$, we see that

$$F(s,\theta) = \mathscr{M}[f; s-1]$$
$$U(s,\theta) = \mathscr{M}[u_\rho;s]$$
$$V(s,\theta) = \mathscr{M}[u_\theta;s]$$

exist and are regular in the strip $0 < \mathrm{Re}(s) < u_1$. We also note that the function

$$P_{-}(s) = \int_1^\infty \rho^s P_y(\rho)\, d\rho \quad (2.14.13)$$

is regular for $\mathrm{Re}(s) < u_1$ and the function

$$V\left(s, \frac{\pi}{2}\right) = V_+(s) = \int_0^1 \rho^{s-1} u_\theta\left(\rho, \frac{\pi}{2}\right) d\rho \tag{2.14.14}$$

is regular in the half plane $\mathrm{Re}(s) > 0$.

Applying Mellin transforms to equation (2.14.5), we see that $F(s,\theta)$ satisfies the fourth order ordinary differential equation

$$[D^2 + (s-1)^2][D^2 + (s+1)^2]F(s,\theta) = 0 \tag{2.14.15}$$

where D denotes the operator $d/d\theta$, subject to the boundary conditions

$$\theta = 0: \quad F = 0, \qquad DF = 0$$

$$\theta = \frac{\pi}{2}: \quad DF = 0$$

$$\theta = \frac{\pi}{2}: \quad s(s-1)F = -\frac{E(e_0 + k_0)}{s+1} + \frac{Ek_0}{s+2} + P_-(s).$$

The solution of (2.14.15) satisfying these conditions is easily seen to be

$$F = \left[-\frac{E(e_0+k_0)}{s+1} + \frac{Ek_0}{s+2} + P_-(s)\right]\frac{s+1}{\cos \pi s + 2s^2 - 1}$$
$$\times \left[-\cos\left(\frac{\pi}{2}s\right)\frac{\sin(s-1)\theta}{s-1} + \cos\left(\frac{\pi}{2}s\right)\frac{\sin(s+1)\theta}{s+1}\right.$$
$$\left. + \frac{1}{s}\sin\left(\frac{\pi}{2}s\right)\{\cos(s-1)\theta - \cos(s+1)\theta\}\right]. \tag{2.14.16}$$

If we now transform the relations (2.14.7) through (2.14.9) by multiplying by ρ^s and integrating, we find that

$$E(DV + U) = [s^2 - (1-\nu)s - \nu]F - \nu D^2F$$
$$-EsU = [-\nu s^2 - (1-\nu)s + 1]F + D^2F$$
$$E[DU - (s+1)V] = 2(1+\nu)s\, DF$$

which may be solved for $V(s,\theta)$ to yield the equation

$$Es(s+1)V = -[\{(2+\nu)s^2 - (1-\nu)s + 1\}DF + D^3F] \tag{2.14.17}$$

Using (2.14.17) and (2.14.16), we note that

$$V_+(s) = \frac{2\sin(\pi s)}{s\cos(\pi s) + 2s^2 - 1}\left[-\frac{e_0+k_0}{s+1} + \frac{k_0}{s+2} + \frac{P_-(s)}{E}\right] \tag{2.14.18}$$

is valid in the strip $0 < \mathrm{Re}(s) < \mu_1$.

To solve this equation we introduce the function

$$H(s) = -\frac{s\sin(\pi s)}{[\cos(\pi s) + 2s^2 - 1]\sqrt{B^2 - s^2}}, \tag{2.14.19}$$

where B is a constant greater than 1. This function is regular in a strip $-\mu_1 < \operatorname{Re}(s) < \mu_1$ where $\mu_1 > 1$. In this strip $H(s) \to 1$ as $s \to \infty$. We now write

$$H(s) = \frac{H_+(s)}{H_-(s)}, \tag{2.14.20}$$

where $H_+(s)$, $H_-(s)$ are defined by

$$\left.\begin{matrix}\log H_+(s)\\ \log H_-(s)\end{matrix}\right\} = -\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} \frac{\log H(z)}{z-s}\,dz \left\{\begin{matrix}\mu < \operatorname{Re}(s)\\ \mu > \operatorname{Re}(s)\end{matrix}\right. \tag{2.14.21}$$

and $-\mu_1 < \mu < \mu_1$. Equation (2.14.18) can now be written in the form

$$V_+(s) = -2\frac{\sqrt{B+s}\sqrt{B-s}\,H_+(s)}{s^2 H_-(s)}\left[-\frac{e_0+k_0}{s+1} + \frac{k_0}{s+2} + \frac{P_-(s)}{E}\right], \tag{2.14.22}$$

where the arguments of the square roots do not exceed $\pi/2$ in absolute value. The solution for $P_-(s)$ and $V_+(s)$ must satisfy the following requirements:

(a) $V_+(s)$ is regular in the half-plane $\operatorname{Re}(s) > 0$.

(b) $P_-(s)$ is regular in the half-plane $\operatorname{Re}(s) < \mu_1$.

(c) $V_+(s)$, $P_-(s)$ tend to zero as $|s| \to \infty$ in their half-planes of regularity.

The homogeneous equation obtained by putting $e_0 = k_0 = 0$ in (2.14.22) has the solution

$$V_+(s) = Q\frac{\sqrt{B+s}\,H_+(s)}{s^2}, \qquad P_-(s) = -\tfrac{1}{2}QE\frac{H_-(s)}{\sqrt{B-s}} \tag{2.14.23}$$

for arbitrary values of the constant Q. The solution of (2.14.22) which satisfies the above conditions is therefore not unique. The general solution of (2.14.22) is now obtained by adding a suitable particular solution of (2.14.22) to the solution (2.14.23). It is a simple matter to verify that

$$V_+(s) = -\frac{2(e_0+k_0)}{s+1}\frac{\sqrt{B+1}}{H_-(-1)}\frac{1}{s}\sqrt{B+s}\,H_+(s) + \frac{k_0}{s+2}\frac{\sqrt{B+2}}{H_-(-2)}\frac{1}{s}\sqrt{B+s}\,H_+(s)$$

$$P_-(s) = E\frac{e_0+k_0}{s+1}\left[1 + \frac{(B+1)}{H_-(-1)}\frac{sH_-(s)}{(B-S)}\right] - \frac{Ek_0}{s+2}\left[1 + \frac{B+2}{2H_-(2)}\frac{sH(s)}{(B-s)}\right] \tag{2.14.24}$$

is just such a particular solution. To find a condition that removes this ambiguity we note that the function $P(s)$ defined by the equation

$$P(s) = -\frac{E(e_0 + k_0)}{s + 1} + \frac{Ek_0}{s + 1} + P_{-}(s) \tag{2.14.25}$$

can be written as the integral

$$-E\int_0^1 (e_0 + k_0 - k_0\rho)\rho^s \, d\rho + \int_1^\infty P_y(\rho)\rho^s \, d\rho$$

so that $P(0)$ is the resulting normal force along the entire boundary $\theta = \pi/2$. If, therefore, we make the additional physical requirement that the resulting normal force over the entire boundary is zero we have the condition

$$P(0) = 0. \tag{2.14.26}$$

The particular solution (2.14.24) satisfies this condition whereas the general solution (2.14.23) of the homogeneous equation does not unless $Q = 0$. Hence (2.14.24) is the required solution of (2.14.22).

The closely related problem of determining the stress field in the vicinity of an edge slot with rounded tip in a semi-infinite sheet under tension has been discussed by Bowie and Neal (1967).

2.15 RADIAL CRACKS ORIGINATING AT THE BOUNDARY OF AN INTERNAL CIRCULAR HOLE

The problem of determining the stress distribution in an infinite plate which contains straight radial cracks originating from the boundary of a circular hole and which is subjected to prescribed loads is one of great importance. For instance, the solution of the problem in the case of a single crack might provide some theoretical justification of the engineering practice of "stress relief," in which circular holes are drilled at the ends of a thin slot with the object of reducing the concentration of stress at the base of the crack; the problem considered here would certainly be relevant when the diameters of the holes are small compared with the length of the crack since (as we saw in Fig. 13) the distribution of stress near one hole will not be greatly influenced by the presence of the other.

The plane strain problem is relevant to the study of the action of internal pressure in the interior of hollow cylinders of large wall thickness and with radial cracks originating from the inner surface.

Approximate solutions of some special problems of this kind have been derived by Morkovin (1944) and Bueckner (1960), but the most complete discussion of the general case would appear to be that due to Bowie (1956); in the case of a single crack the analysis of Wigglesworth (1958) is relevant.

We shall begin by considering Bowie's method. The problem considered is that of the stressing of an infinite thin plate in the z-plane whose internal boundary, which we shall denote by τ, is free from tractions. The boundary τ consists of a circular hole of unit radius centered at the point $z = 0$ and with k radial cracks of equal length, L, lying along the radii $\theta = 0, 2\pi/k, \ldots, 2(k-1)\pi/k$.

Bowie's method consists of mapping τ and the region of the z-plane exterior to τ into the unit circle γ in the ζ-plane and the region of the ζ-plane exterior to it. The required mapping can be expressed in the differential form

$$\frac{dz}{z} = \frac{(1 - \zeta^{-k})\, d\zeta}{\zeta(1 + 2\epsilon\zeta^{-k} + \zeta^{-2k})^{1/2}}, \tag{2.15.1}$$

where ϵ is a real parameter such that $0 \leq |\epsilon| \leq 1$ and the denominator is considered positive at $\zeta = 1$ in order to define the desired branch of the mapping function.

It is desirable to find a series representation of (2.15.1) which is convergent on the boundary of the unit circle γ and in the whole ζ-plane exterior to γ. The form of such a series is

$$z = \omega(\zeta) = C\left[\zeta + \sum_{n=1}^{\infty} A_n \zeta^{1-kn}\right], \tag{2.15.2}$$

where C is a constant, and where the coefficients A_n, which are real, may be obtained numerically from simple recursive formulas found by expanding both sides of equation (2.15.1) in series form and equating coefficients of equal powers of ζ. It can be shown that the series (2.15.2) converges in that part of the ζ-plane which is exterior to γ and at all points of γ except at the roots of the equation

$$\zeta^{2k} + 2\epsilon\zeta^k + 1 = 0. \tag{2.15.3}$$

To avoid difficulties in establishing the convergence of series expansions for stress functions, Bowie represents the mapping function (2.15.2) by polynomial approximations. By using polynomial approximations of the internal boundary, Bowie obtains an accurate description of the stress distribution at the crack roots by introducing cusps in the polynomial mappings to describe the crack roots.

The existence of cusps at points corresponding to the crack roots is ensured in polynomial mapping approximations by assuming that $\omega(\zeta)$ is of the form

$$\omega'(\zeta) = (1 - \zeta^{-k})g(\zeta), \tag{2.15.4}$$

where $g(\zeta)$ is a polynomial whose coefficients are chosen in such a way that all the zeros of $g(\zeta)$ fall within γ. Because of the convergence of (2.15.2) at all but a finite number of points on the unit circle, suitable approximations of the type

$$\omega(\zeta) = C\left[\zeta + \sum_{n=1}^{N} \epsilon_n \zeta^{1-kn}\right] \tag{2.15.5}$$

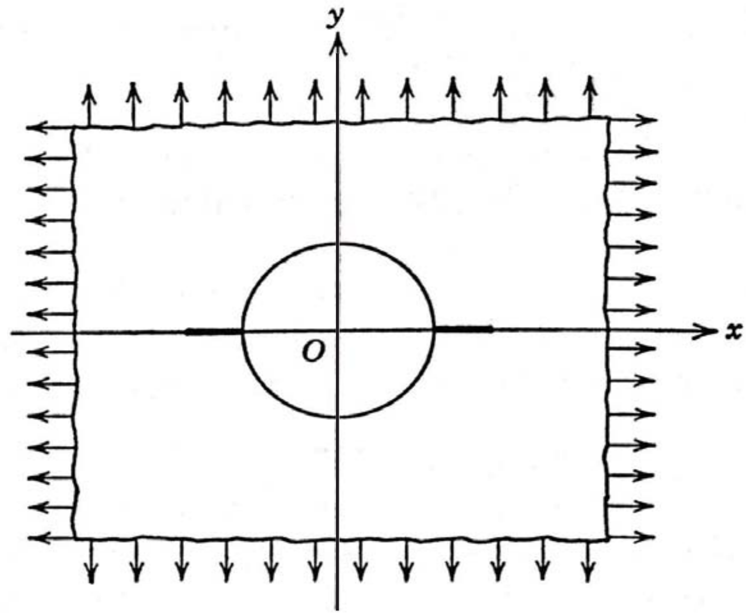


Figure 31 Circular hole with two radial cracks in a plate under "all-round" tension.

can be obtained by setting $\epsilon_n \simeq A_n$ and then modifying the ϵ_n's to ensure that equation (2.15.4) is satisfied.

We now consider the case of a uniform "all-round" tension at infinity illustrated in Fig. 31. It is easily shown that the loading condition

$$\sigma_{xx} = \sigma_{yy} = p_0 \qquad \text{on} \qquad |z| = R,$$

where R is large, is satisfied by choosing $\phi(\zeta)$, $\psi(\zeta)$ in such a way that, for large values of $|\zeta|$, they approach the values $(1/2)Cp_0\zeta$, $Cp_0\gamma_0\zeta$, respectively, where γ_0 is a constant. We therefore assume that $\phi(\zeta)$ is a polynomial of the form

$$\phi(\zeta) = p_0 C\left[\tfrac{1}{2}\zeta + \sum_{n=1}^{N} a_n \zeta^{1-kn}\right], \tag{2.15.6}$$

which would be the form it would take if there were no boundary irregularities implied by the mapping function (2.15.5).

From equation (1.5.12) we see that the condition that the boundary τ should be load free is that

$$\phi(\sigma) + \omega(\sigma)\overline{\frac{\phi'(\sigma)}{\omega'(\sigma)}} + \psi(\sigma) = 0 \tag{2.15.7}$$

where $\zeta = \sigma$ on the circle γ. We can write this condition in the form

$$\omega'(\sigma)\psi(\sigma) = -\omega'(\sigma)\overline{\phi(\sigma)} - \overline{\omega(\sigma)}\phi'(\sigma), \quad \sigma \in \gamma. \tag{2.15.8}$$

The function $\omega'(\sigma)\psi(\sigma)$ is analytic in the region exterior to γ and with the form of $\phi(\zeta)$ assumed in (2.15.6) is a continuous function on γ. Thus, if the coefficients α_n can be chosen so that the coefficients of all positive powers of ζ in the Laurent expansion of

$$\omega'(\zeta)\phi\left(\frac{1}{\zeta}\right) + \omega\left(\frac{1}{\zeta}\right)\phi'(\zeta) \tag{2.15.9}$$

vanish, we can determine $\psi(\zeta)$ explicitly. Multiplying both sides of (2.15.8) by

$$\frac{1}{2\pi i(\sigma - \zeta)}$$

and integrating round γ we obtain, by a well-known theorem (Titchmarsh, 1939, p. 145), that

$$\omega'(\zeta)\psi(\zeta) = -\omega'(\zeta)\phi\left(\frac{1}{\zeta}\right) - \omega\left(\frac{1}{\zeta}\right)\phi'(\zeta). \tag{2.15.10}$$

It should be observed that the cusp roots are reflected by singularities in $\psi(\zeta)$ in the form of simple poles.

We have, of course, still to verify that the coefficients α_n can be determined in this way. If we equate to zero the coefficients of all positive powers of ζ in the Laurent expansion of the function (2.15.9), we obtain the system of linear equations

$$\alpha_i + \sum_{j=1}^{N-i} \alpha_{i+j}\epsilon_j(1 - jk) + \sum_{j=1}^{N-i} \epsilon_{i+j}\alpha_j(1 - jk) + \tfrac{1}{2}\epsilon_i = 0, \tag{2.15.11}$$

($i = 1, 2, \ldots, N$) whose solution determines the N coefficients, $\alpha_1, \alpha_2, \ldots, a_N$.

In order to compare the difference in potential energies in the physical systems with and without cracks, we refer to the exact geometry rather than polynomial approximations. For the case of uniform tension at infinity, the potential energy of the system with cracks, V_c, under the assumption of plane stress, is given by the expression

$$V_c = \lim_{R\to\infty}\left[-\tfrac{1}{2}h\int_0^{2\pi}(\sigma_{rr}u_r + \sigma_{r\theta}u_\theta)R\,d\theta\right], \tag{2.15.12}$$

where h is the thickness of the plate, and (in conventional units)

$$\sigma_{rr} - i\sigma_{r\theta} = \phi'(z) + \overline{\phi'(z)} - e^{2i\theta}[\bar{z}\phi''(z) + \psi'(z)]$$

$$2\mu(u_r + iu_\theta) = e^{-i\theta}[K\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}]$$

In the part of z-plane exterior to τ, $\phi(z)$ and $\psi(z)$ are analytic except at the point at infinity. Hence, we may expand these functions in the form

$$\phi(z) = p_0[\tfrac{1}{2}z + a_1z^{1-k} + \cdots], \qquad \psi(z) = p_0[b_0z^{-1} + b_1z^{-1-k} + \cdots]. \tag{2.15.13}$$

Substituting these expressions in (2.15.12) we find, after some algebra, that

$$V_c = -\frac{\pi p_0^2 h}{E}\lim_{R\to\infty}[(1-\nu)R^2 - 2\nu b_0], \tag{2.15.14}$$

where E denotes Young's modulus.

To obtain the expression for the potential energy of the system without cracks, we consider a ring with inner radius R_1 and outer radius R. It is readily shown that the potential energy of this system is given by

$$V_0 = -\frac{\pi p_0^2 h}{E}\lim_{R\to\infty}[(1-\nu)R^2 + 2R_1^2 + 2(1-\nu)b_0]. \tag{2.15.15}$$

From equations (2.15.14) and (2.15.15) we see that the reduction in potential energy in the case of "all-round" tension at infinity (and plane stress) is given by

$$V = V_c - V_0 = \frac{2\pi p_0^2 h}{E}(R_1^2 + b_0). \tag{2.15.16}$$

Now the stress function $\psi(\zeta)$ given by (2.15.10) for polynomial approximations of the geometry can be expanded in a series of the form

$$\psi(\zeta) = Cp_0[\gamma_0' + \gamma_0\zeta^{-1} + \gamma_1\zeta^{-1-k} + \cdots]. \tag{2.15.17}$$

On the other hand, a series expansion of $\psi(\zeta)$ for the exact geometry can be found by substituting the exact mapping (2.15.1) into the second of the equations (2.15.13). If we assume that the polynomial approximation can be made to converge to the exact solution, we find by the comparison of the coefficients of ζ^{-1} in the expansions of $\psi(\zeta)$ that

$$C^2\gamma_0 = b_0. \tag{2.15.18}$$

The constant C occurring in the mapping function can now be chosen in such a way that the radius of the circular hole in the z-plane has unit length. Thus, if $\sigma = \sigma_1$ is that point on γ which corresponds to the junction of the crack and the circle in the z-plane, then C is chosen so that

$$R_1 = |\omega(\sigma_1)| = 1. \tag{2.15.19}$$

Thus, if a sufficiently accurate polynomial mapping is taken, we can write equation (2.15.16) in the form

$$V = \frac{2\pi p_0^2 h}{E}\left\{\frac{\gamma_0 C^2}{[\omega(\sigma_1)]^2} + 1\right\}. \tag{2.15.20}$$

If we note that, measured in units of the radius of the circular hole, the length, L, of the crack is given by

$$L = \frac{\omega(1)}{\omega(\sigma_1)} - 1, \tag{2.15.21}$$

we may write (2.15.20) in the form

$$V = \frac{(2\pi p_0{}^2 h)}{E} f(L), \tag{2.15.22}$$

where

$$f(L) = \gamma_0 \left[\frac{L+1}{\omega(1)/c}\right]^2 + 1. \tag{2.15.23}$$

It should be noted that, in this equation, both γ_0 and $\omega(1)/C$ are functions of L.

It should be emphasized that the results (2.15.22) and (2.15.23) relate to an "all-round" tension p_0 at infinity in the case of plane stress. From this expression it is readily shown that the critical load, calculated according to the Griffith criterion, is given by

$$p_{cr} = \sqrt{[-kET/\pi f'(L)]} \tag{2.15.24}$$

where, as before, T is the surface tension per unit area of the crack surface.

For the case of plane strain the only change is in the value of the multiplicative constant. We find that

$$p_{cr} = \sqrt{[-kET/\pi(1-\eta^2)f'(L)]}. \tag{2.15.25}$$

The plane strain criterion (2.15.25) is useful in the study of radial cracks in cylinders under internal pressure. Such a load system is obtained simply by superposing a hydrostatic pressure on the load system considered above. Since the superposition of a hydrostatic pressure does not affect the critical stress as calculated above, the critical internal pressure, p_{cr} is also given by equation (2.15.25).

Bowie has carried out the numerical calculations in two cases; $k = 1$ (single crack) and $k = 2$ (two cracks), for nine values of the parameter ϵ.

For example, when $k = 2$, the coefficient γ_0 in equation (2.15.23) is given by

$$\gamma_0 = -\left[1 + 2\sum_{i=1}^{N} \epsilon_i \alpha_i (1-2i)\right],$$

and Bowie found that γ_0 can be obtained to three significant figure accuracy for the crack lengths chosen if about thirty terms of the polynomial approximation of the mapping function are retained. For moderately long cracks, only a few terms are required, but for small cracks the convergence is poor. The results are shown in Table 2. To find $f'(L)$, the data in the final column are differentiated (numerically) with respect to L. The critical stress p_{cr} is then easily calculated from (2.15.24).

TABLE 2

VALUES OF BOWIE'S FUNCTION $f(L)$

ϵ	L	$\omega(1)/c$	γ_0	$f(L)$
−1.000	0.000	1.000	−1.000	0.000
−0.866	0.303	1.259	−1.075	−0.152
−0.707	0.497	1.383	−1.160	−0.359
−0.500	0.732	1.500	−1.267	−0.689
0.000	1.414	1.707	−1.509	−2.018
+0.500	2.732	1.866	−1.752	−6.008
+0.707	4.027	1.924	−1.852	−11.640
+0.866	6.596	1.966	−1.933	−27.860
+1.000	∞	2.000	−2.000	$-\infty$

For large values of L,

$$f(L) = 1 - \tfrac{1}{2}(L + 1)^2$$

so that we get the Griffith criterion for a crack of length $2(1 + L)$.

The variation of the critical load with crack length is shown in Fig. 32. The results for a single crack are included on the same diagram. In this diagram we have used the notation

$$P = \begin{cases} \sqrt{[ET/\pi(1 - \nu^2)]}, & \text{plane strain;} \\ \sqrt{ET/\pi}, & \text{plane stress;} \end{cases}$$

and it will be recalled that, in measuring L, we have taken the radius of the circular hole as our unit of length.

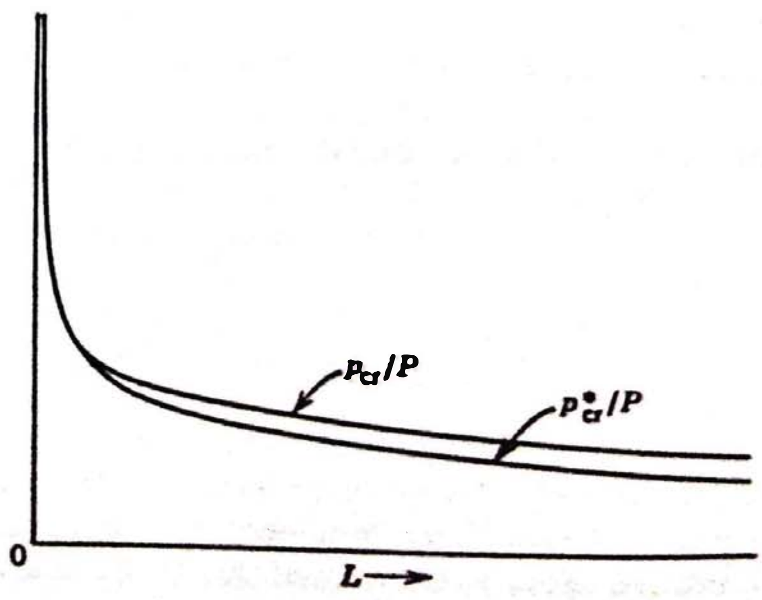


Figure 32 Variation of the critical load with crack length for the case of "all-round" tension at infinity (cf. Fig. 31). p_{cr} is the critical load for a single radial crack; p^*_{cr} is the critical load for two radial cracks.

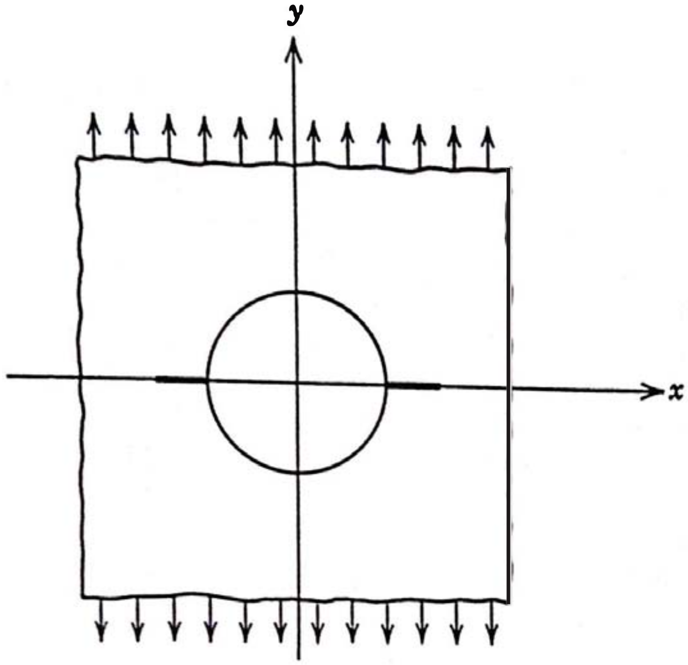


Figure 33 Circular hole with two radial cracks in a plate under simple (uni-directional) tension.

Bowie has carried out, in a similar manner, the analysis for the case of tension in one direction at infinity (cf. Fig. 33). The analysis is more complicated in this case because of the lack of symmetry and Bowie considers only the cases $k = 1, 2$. He finds that in the case of plane stress the value of the critical tension is given by

$$p_{cr} = \sqrt{[-kET/\pi g'(L)]} \tag{2.15.26}$$

where the function $g(L)$ is defined by

$$g(L) = \tfrac{1}{2}(\gamma_0 + 2\alpha_2 - A)\left[\frac{L+1}{\omega(1)/c}\right]^2 + \frac{3}{4} \tag{2.15.27}$$

with $Cp_0\gamma_0$, α_2, and CA being respectively the coefficients of ζ^{-1} in the Laurent expansions of $\psi(\zeta)$, $\phi(\zeta)$, $\omega(\zeta)$.

Bowie's numerical calculations for the case $k = 2$ are summarized in Table 3. Numerical differentiation of the last column gives $g'(L)$ which, when substituted in equation (2.15.26) gives the appropriate value of the critical load. The variation of the critical load with crack length for this case is shown in Fig. 34.

It will be observed from Figs. 32 and 34 that for large cracks ($L > 1$) the presence of the circular hole has no appreciable effect on the value of the critical stress, so that the solution for a single crack of length $2(L + 1)$ (for $k = 2$) or $L + 2$ (for $k = 1$) can be used with a fair degree of accuracy. On the other hand, the critical load for very small cracks appears to be governed essentially by the local stress field of the hole.

TABLE 3

VALUES OF BOWIE'S FUNCTION $g(L)$

ϵ	ϵ_1	α_2	γ_0	$g(L)$
-1.000	0.00	-0.50	-0.50	0.00
-0.866	0.07	-0.61	-0.62	-0.28
-0.707	0.15	-0.69	-0.74	-0.57
-0.500	0.25	-0.75	-0.87	-1.00
0.000	0.50	-0.83	-1.13	-2.53
$+0.500$	0.75	-0.82	-1.33	-6.69
$+0.707$	0.85	-0.80	-1.41	-12.40
$+0.866$	0.93	-0.78	-1.46	-28.80
$+1.000$	1.00	-0.75	-1.50	$-\infty$

The problem considered by Wigglesworth (1958) is shown in Fig. 35. The circular hole is again taken to be the unit circle in the z-plane and the crack is taken to be the line segment $y = 0$, $1 \le x \le a$ so that the length of the crack is $L = a - 1$. The determination of stress distribution due to a general set of loads (body forces, tractions on the hole and nonvanishing stresses at infinity) can be reduced by known methods to that of finding the stress distribution when the only loads are the boundary stresses on the crack; there are no body forces and no tractions on the hole and the components of stress all vanish at infinity. Thus the boundary conditions are given by

$$\sigma_{yy} = -p_0(x), \qquad \sigma_{xy} = -t_0(x), \qquad y = 0, \quad 1 \le x \le a. \tag{2.15.28}$$

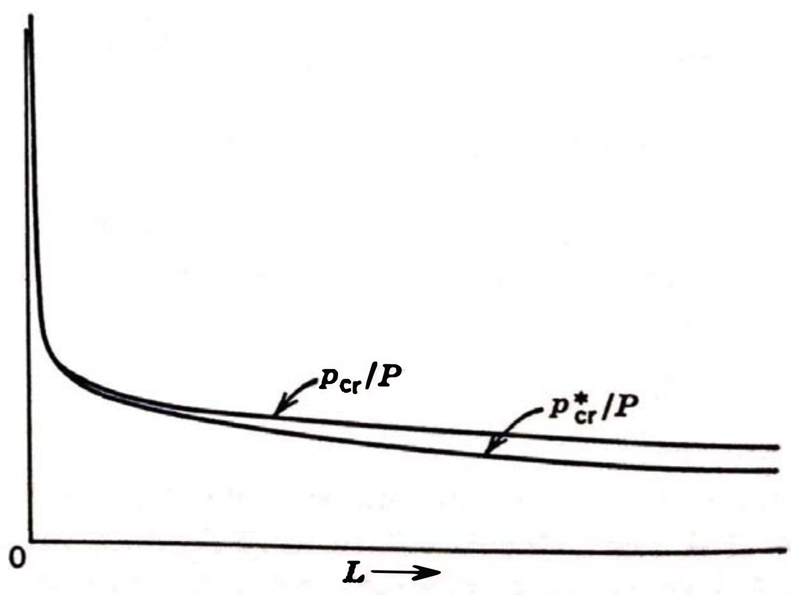
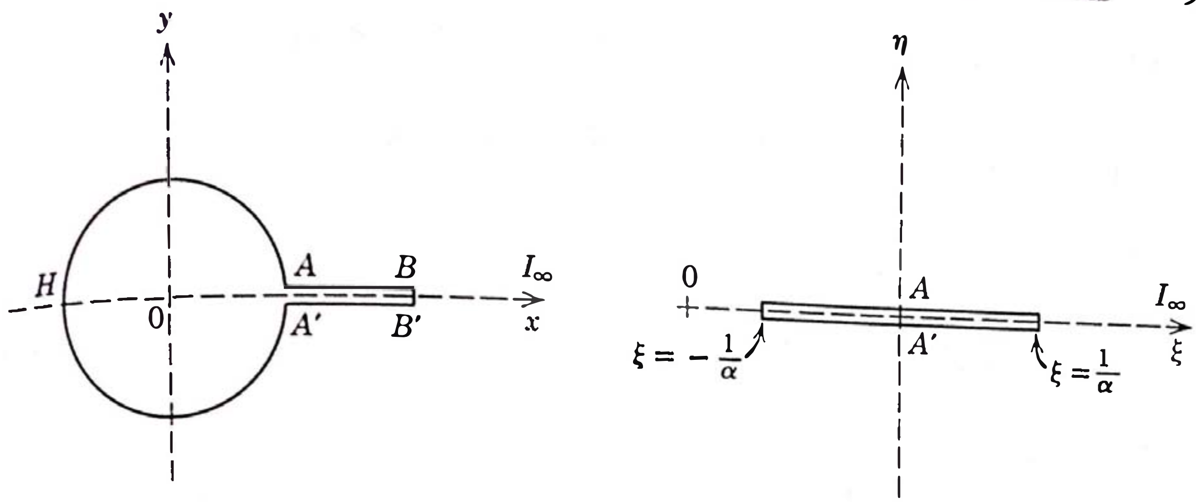


Figure 34 Variation of the critical load with crack length for the case of simple tension at infinity (cf. Fig. 33). p_{cr} is the critical load for a single radial crack; p^*_{cr} is the critical load for two radial cracks.



(a) z-plane (b) ζ-plane

Figure 35 Wigglesworth's problem.

In these circumstances it can be shown (Green and Zerna, 1954, p. 280) that the components of stress and displacement can be expressed in terms of a single analytic function $\Omega(z)$, defined both in the physical region and its geometrical inverse with respect to the boundary of the hole. In the case considered here, this region is the whole z-plane bounded by the segment $y = 0$, $1/a \leq x \leq a$. In the case of plane strain, we then have the equations

$$2(\sigma_{xx} + \sigma_{yy}) = \frac{\partial}{\partial z} F(z,\bar{z}), \tag{2.15.29}$$

$$2(\sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy}) = -\frac{\partial}{\partial \bar{z}} F(z,\bar{z}), \tag{2.15.30}$$

with

$$F(z,\bar{z}) = G(x,y) = \left(z - \frac{1}{\bar{z}}\right)\bar{\Omega}'(\bar{z}) + \Omega(z) - \Omega\left(\frac{1}{\bar{z}}\right), \tag{2.15.31}$$

for the determination of the stress components, and the equation

$$8(u_x + iu_y) = 4(1 - \eta)\Omega(z) - F(z,\bar{z}), \tag{2.15.32}$$

for the determination of the displacement components. In the case of plane stress, η is replaced, as usual, by the modified ratio $\eta/(1 + \eta)$.

If there are no isolated body forces on the edges of the crack, it can easily be shown that $\Omega(z)$ and $f(z,\bar{z})$ must be single-valued there; and also, that the values of $G(x,y)$ at the points $(x,\pm 0)$ are therefore equal, both for points on the crack and outside it. Taking $G(1,0) = 0$ we have, for points on the crack,

$$G(x,0) = -G_0(x), \quad (1 \leq x \leq a), \tag{2.15.33}$$

where

$$G_0(x) = -4i\int^{x} [t_0(x) + ip_0(x)]\, dx, \quad (1 \leq x \leq a). \tag{2.15.34}$$

Putting $z = x \pm i0$ in (2.15.31) we also have the relations

$$\left(x - \frac{1}{x}\right)\bar{\Omega}'(x \mp i0) + \Omega(x \mp i0) - \Omega\left(\frac{1}{x} \mp i0\right) = G(x,0), \quad (2.15.35)$$

for $|x| \geq 1$. We now introduce the function

$$Q(x) = \Omega(x + i0) - \Omega(x - i0), \quad \left(\frac{1}{a} \leq x \leq a\right), \quad (2.15.36)$$

the right-hand side being zero except for $(1/a) \leq x \leq a$. Subtracting the two equations (2.15.35), we find that they are equivalent to the relation

$$Q\left(\frac{1}{x}\right) = Q(x) - \left(x - \frac{1}{x}\right)\bar{Q}'(x), \quad (1 \leq x \leq a), \quad (2.15.37)$$

where $\bar{Q}(x) = \bar{\Omega}(x - i0) - \bar{\Omega}(x + i0)$ is the complex conjugate of $Q(x)$.

Further, it follows from equations (2.15.32), (2.15.36), that the function $Q(x)$ is proportional to the relative displacement of the points $(x, \pm 0)$ on the crack. Continuity of the displacement at $x = a$ leads, therefore, to the condition

$$Q(a) = 0 \quad (2.15.38)$$

Also, the continuity of the displacement requires continuity of $\Omega(z)$, and the vanishing of the stresses at infinity requires $\Omega(z)$ to be regular at $z = 0$ and, at most, $O(1/z)$ at infinity. From these conditions and (2.15.36), and a simple application of Cauchy's theorem, we find that

$$\Omega(z) = \frac{1}{2\pi i}\int_1^a \left[\frac{Q(s)}{s - z} + \frac{Q\left(\frac{1}{s}\right)}{s(1 - sz)}\right] ds. \quad (2.15.39)$$

Substituting from (2.15.37), integrating by parts, and using the boundary condition (2.15.38), we find that

$$\Omega(z) = W(z) - W\left(\frac{1}{z}\right) + (z^3 - z)\frac{d}{dz}\,\overline{W}\left(\frac{1}{z}\right) + cz, \quad (2.15.40)$$

where

$$W(z) = \frac{1}{2\pi i}\int_1^a \frac{Q(s)\,ds}{s - z} \quad (2.15.41)$$

and

$$c = \mathrm{Re}[W'(0)]. \quad (2.15.42)$$

In equation (2.15.40) we have omitted terms of the form $A + iBz$ with B real, since they correspond to a rigid body displacement and do not affect the stresses and, hence, the boundary conditions.

If we write

$$Q(x) = Q_1(x) + Q_2(x), \qquad W(z) = W_1(z) + W_2(z), \tag{2.15.43}$$

where $Q_1(x)$ is real and $Q_2(x)$ is imaginary and W_1, W_2 correspond to Q_1, Q_2 respectively, we can show, after some reduction, that for $|x| \geq 1$

$$W_1(x + i0) + W_1(x - i0) + (4\Theta^2 - 2)W_1\left(\frac{1}{x}\right) = i\,\mathrm{Im}[G(x,0)], \tag{2.15.44}$$

$$W_2(x + i0) + W_2(x - i0) + \left[4\Theta^2 - 2 + 4\left(x - \frac{1}{x}\right)\Theta\right]W_2\left(\frac{1}{x}\right)$$
$$= \mathrm{Re}[G(x,0)] - 2c\left(x - \frac{1}{x}\right), \tag{2.15.45}$$

where Θ denotes the differential operator

$$\Theta = \tfrac{1}{2}(x^2 - 1)\frac{d}{dx}. \tag{2.15.46}$$

The lefthand side of equation (2.15.45) can be put into the simpler form of (2.15.44) by means of a substitution for $W_2(z)$ and an integration with respect to x. We define $P(x)$ by the equation

$$(x^2 - 1)P(x) = \begin{cases} \displaystyle\int_1^x Q_2(s)\,ds, & (1 \leq x \leq a), \\ \displaystyle\int_1^a Q_2(s)\,ds, & (x \geq a) \end{cases} \tag{2.15.47}$$

so that $P(x)$ is continuous at $x = a$. We then define $V(z)$ by the equation

$$V(z) = \frac{1}{2\pi i}\int_1^\infty \frac{P(x')\,dx'}{x' - z}. \tag{2.15.48}$$

From these definitions and equation (2.15.41) we can easily show that

$$W_2(z) = \frac{d}{dz}[(z^2 - 1)V(z)]. \tag{2.15.49}$$

There should be a constant of integration on the righthand side of this equation but it can be taken to be zero since it makes no contribution to the expressions for the components of stress and displacement. Substituting from (2.15.49) into equation (2.15.45) and integrating, we find that V satisfies the equation

$$V(x + i0) + V(x - i0) + (4\Theta^2 - 2)V\left(\frac{1}{x}\right)$$
$$= (x^2 - 1)\int_1^x \mathrm{Re}[G(x',0)]\,dx' - c\left[1 - \frac{2}{x^2 - 1}\log x\right], \tag{2.15.50}$$

for $|x| \geq 1$, and the lefthand side of this equation is of the same form as (2.15.44).

The constant c occurring in this last equation has a simple physical significance. It can be shown that c is the hoop stress σ_{yy} at the point H of Fig. 35. It can also be proved that

$$c = -\lim_{x \to \infty}\left[V(x + i0) + V(x - i0) + (4\Theta^2 - 2)V\left(\frac{1}{x}\right)\right], \tag{2.15.51}$$

a result which is used in the determination of c and the other constant

$$\int_1^a Q_2(s)\, ds$$

which enters into $P(x)$ and $V(z)$.

The problem can now be put into a simpler form by mapping the z-plane into the ζ-plane with the transformation,

$$z = \frac{1 + \alpha\zeta}{1 - \alpha\zeta}, \qquad \left(\alpha = \frac{a - 1}{a + 1}\right), \tag{2.15.52}$$

which maps the exterior (interior) of the unit circle $|z| = 1$ onto the half-plane $\xi > 0 (\xi < 0)$ and the segment $y = 0$, $1/a \leq x \leq a$ onto the segment $-1 \leq \xi \leq 1$. The points $z = 0$, and $z = \infty$ correspond respectively to $\zeta = -1/\alpha$, $\zeta = 1/\alpha$ (cf. Fig. 35).

If we write

$$P(x) = p(\xi), \qquad V(z) = v(\zeta) \tag{2.15.53}$$

then equation (2.15.48) is transformed to

$$v(\xi) = \frac{1}{2\pi i}\int_0^{1/\alpha}\left\{\frac{1}{\xi' - \zeta} + \frac{\alpha}{1 - \alpha\xi'}\right\}p(\xi')\, d\xi' \tag{2.15.54}$$

and the lower of equations (2.15.47) becomes

$$-2ip(\xi) = \frac{c_1(1 - \alpha\xi)^2}{\alpha\xi}, \tag{2.15.55}$$

with

$$c_1 = \frac{1}{2i}\int_1^a Q_2(s)\, ds \tag{2.15.56}$$

while the condition (2.15.38) is equivalent to the condition

$$(1 - \alpha)p'(1) + (1 + \alpha)p(1) = 0. \tag{2.15.57}$$

Further, the operator Θ becomes $\xi \dfrac{d}{d\xi}$.

Suppose now that $l(\xi)$ denotes the lefthand side of equation, (2.15.50), i.e., that

$$l(\xi) = l_0(\xi) - c\left[1 - \frac{(1-\alpha\xi)^2}{2\alpha\xi}\log\frac{1+\alpha\xi}{1-\alpha\xi}\right], \quad (0 \le \xi \le 1), \quad (2.15.58)$$

where

$$l_0(\xi) = -(x^2-1)^{-1}\int_1^x \mathrm{Re}[G_0(x')]\,dx', \quad 0 \le \xi \le 1, \qquad (2.15.59)$$

and $x = (1+\alpha\xi)/(1-\alpha\xi)$. Then from equations (2.15.50) and (2.15.53) we have the equation

$$v(\xi + i0) + v(\xi - i0) + \left[4\left(\xi\frac{d}{d\xi}\right)^2 - 2\right]v(-\xi) = l(\xi), \quad (0 \le \xi < \infty), \qquad (2.15.60)$$

Further, the condition (2.15.51) takes the form

$$c = -l\left(\frac{1}{\alpha}\right), \qquad (2.15.61)$$

an equation which, together with (2.15.57), is used to determine the constants c and c_1. In the form given by equations (2.15.54), (2.15.58), and (2.15.60), the problem is effectively the same as the one considered in Section 2.14 also due to Wigglesworth. For complete details of the analysis, see Wigglesworth (1958).

Wigglesworth also considers the special case in which the original load is a uniform tension, p_0, at infinity perpendicular to the crack. A tension in this direction produces the greatest stresses near the hole and it is practically certain that the hoop stress on the hole will be greatest at the point H (Fig. 35). It is well known that in the absence of the crack, this hoop stress is $3p_0$, and it was previously shown that c is the value of the hoop stress in the reduced problem. The stress concentration factor is therefore

$$s = 3 + \frac{c}{p_0}. \qquad (2.15.62)$$

When the external load consists of a uniform stress p_0 in the y-direction, the formulas (2.15.29) and (2.15.30) apply to the plate with the hole alone if $\Omega(z)$ is taken to be the function $\Omega_0(z)$ defined by the equation

$$\Omega_0(z) = p_0\left(\frac{z-2}{z}\right).$$

From equation (2.15.31) we then find that

$$G_0(x) = 2\left(\frac{x-1}{\ }\right) + 2\left(\frac{x-1}{3}\right).$$

TABLE 4

r	0.05	0.1	0.15	0.3	0.4	0.5
s	10.44	7.64	6.34	4.74	4.20	3.83

The evaluation of the relevant integrals is discussed in Wigglesworth's paper. His results are summarized in Table 4. In this table s is the stress concentration factor defined by equation (2.15.62) and r is the ratio defined by the equation

$$r = 1 - \alpha = \frac{\text{diameter of hole}}{\text{diameter of hole and length of crack}}.$$

The results are shown graphically in Fig. 36.

By the definition of r, we see that $r = 1$ when there is no crack present, in which case $s = 3$. Wigglesworth has shown that at the other end of the scale, when the diameter of the hole is very small in comparison with the length of the crack (i.e., when $r \ll 1$),

$$s \sim 3 + 2.115 r^{-1/2}$$

so that, as we should expect, $s \to \infty$ as the diameter of the hole tends to zero.

2.16 STAR CRACKS UNDER PRESSURE

The problem of determining the distribution of stress in a thin plate in the neighborhood of a star crack formed by the intersection of a number of Griffith cracks of equal length, when the faces of the crack are subjected to a pressure p_0, has been considered by Westmann (1965) and by Srivastav and Narain (1965).

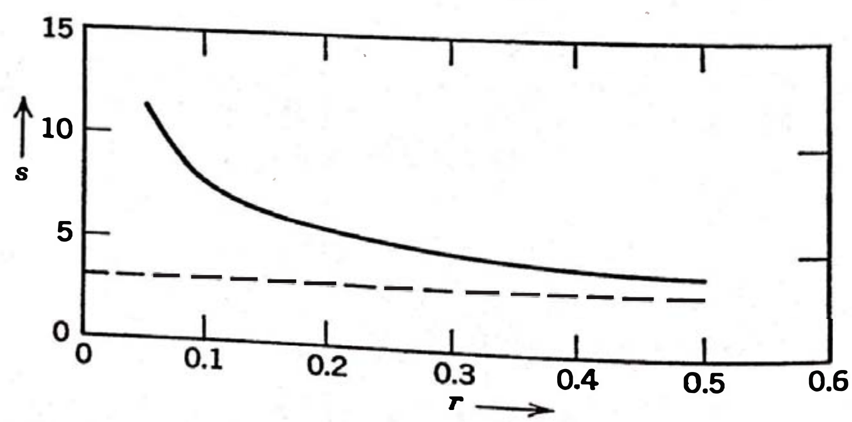


Figure 36 Variation of the stress concentration factor s with the ratio r.

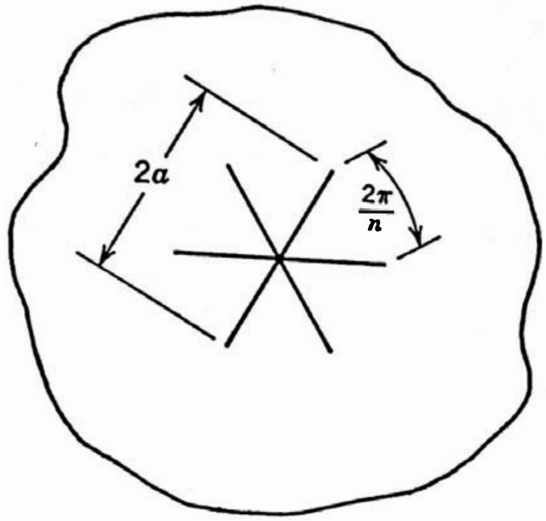


Figure 37 A symmetrical star crack.

Westmann considers the stress in a thin elastic plate in which there are n Griffith cracks each of length c radiating from a common point and having angles $2\pi/n$ between adjacent cracks, n being a positive integer (cf. Fig. 37). The faces of each crack are assumed to be free from shear but subjected to normal pressure p_0. It is also assumed that the stress field vanishes at large distances from the crack. From the symmetry of the problem we may reduce it to one concerning the wedge $0 \leq \rho < \infty$, $|\theta| \leq \pi/n$, ρ and θ being polar coordinates in the plane. If we take the length of a crack to be the unit of length, the boundary conditions appropriate to this problem are then

$$\sigma_{\rho\theta}(\rho, \pm\pi/n) = 0, \qquad \rho \geq 0, \tag{2.16.1}$$

$$\sigma_{\theta\theta}(\rho, \pm\pi/n) = -p_0, \quad 0 \leq \rho < 1 \tag{2.16.2}$$

$$u_\theta(\rho, \pm\pi/n) = 0, \qquad \rho > 1. \tag{2.16.3}$$

It is further assumed that all the components of stress tend to zero as $\rho \to \infty$ and that $u_\theta = O[(1-\rho)^\alpha]$, $0 < \alpha < 1$ as $\rho \to 1-$, $\theta = \pm\pi/n$.

From (2.14.6) we have the relations

$$\mathscr{M}[\sigma_{\rho\rho}; \rho \to s+1] = [D^2 - (s-1)]F(s,\theta), \tag{2.16.4}$$

$$\mathscr{M}[\sigma_{\theta\theta}; \rho \to s+1] = s(s-1)F(s,\theta), \tag{2.16.5}$$

$$\mathscr{M}[\sigma_{\rho\theta}; \rho \to s+1] = sDF(s,\theta), \tag{2.16.6}$$

where D denotes $d/d\theta$; $F(s,\theta)$ is the Mellin transform of order $s-1$ of the Airy stress function $f(\rho,\theta)$, and satisfies the fourth ordinary differential equation (2.14.15). Also, from the equations

$$\sigma_{\rho\rho} - \eta\sigma_{\theta\theta} = E\frac{\partial u_\rho}{\partial \rho}, \qquad 2(1+\eta)\sigma_{\rho\theta} = E\left[\frac{1}{\rho}\frac{\partial u_\rho}{\partial \theta} + \frac{\partial u_\theta}{\partial \rho} - \frac{u_\theta}{\rho}\right]$$

we deduce the relations

$$Es\mathscr{M}[u_\rho;\, \rho \to s] = \mathscr{M}[\eta\sigma_{\theta\theta} - \sigma_{\rho\rho};\, \rho \to s+1]$$

$$ED\mathscr{M}[u_\rho;\, \rho \to s] - E(s+1)\mathscr{M}[u_\theta;\, \rho \to s] = 2(1+\eta)\mathscr{M}[\sigma_{\rho\theta};\, \rho \to s+1]$$

from which it follows that

$$\mathscr{M}[Eu_\rho;\, \rho \to s] = -s^{-1}[D^2 - (s-1)(1+\eta s)]F(s,\theta), \tag{2.16.7}$$

$$\mathscr{M}[Eu_\theta;\, \rho \to s] = -(s+1)^{-1}[D^3 + \{(2+\eta)s^2 - (1-\eta)s + 1\}D]F(s,\theta). \tag{2.16.8}$$

We note that the solution

$$F(s,\theta) = -\frac{p_0}{4}\psi(s)\left\{\frac{(s+1)\cos[(s-1)\theta]}{(s-1)\sin[(s-1)\pi/n]} - \frac{\cos[(s+1)\theta]}{\sin[(s+1)\pi/n]}\right\} \tag{2.16.9}$$

of equation (2.14.15) has the correct symmetry about $\theta = 0$ and has the property that $DF(s,\theta) = 0$ when $\theta = \pm\pi/n$. It is therefore the symmetrical solution appropriate to the boundary condition (2.16.1). If we now insert the function $F(s,\theta)$, defined by equation (2.16.9), into equations (2.16.5) and (2.16.8), we find that the remaining conditions (2.16.2) and (2.16.3) are satisfied if the function $\psi(s)$ satisfies the dual integral equations

$$\mathscr{M}^{-1}[k(s)\psi(s);\, s \to \rho] = \rho, \quad 0 \le \rho \le 1 \tag{2.16.10}$$

$$\mathscr{M}^{-1}[\psi(s);\, s \to \rho] = 0, \qquad \rho > 1 \tag{2.16.11}$$

where the function $k(s)$ is defined by

$$k(s) = \tfrac{1}{4}s\,\frac{\sin(2s\pi/n) + s\sin(2\pi/n)}{\sin[(s-1)\pi/n]\sin[(s+1)\pi/n]}. \tag{2.16.12}$$

Westmann solves these equations by making use of the Wiener-Hopf technique. He finds that

$$\psi(s) = \frac{1}{c(n)sF_+(s)F_+(1)}, \tag{2.16.13}$$

where the function $c(n)$ is defined by the equation

$$c(n) = \frac{2\pi/n + \sin(2\pi/n)}{2\sin^2(\pi/n)} \tag{2.16.14}$$

and the function $F_+(s)$ by the equation

$$F_+(s) = \prod_{m=1}^{\infty} \frac{(1 + s/a_m)(1 + s/\bar{a}_m)}{[1 + s/(mn+1)][1 + s/(mn-1)]}, \tag{2.16.15}$$

the a_m being the roots of the equation

$$\sin\frac{2s\pi}{n}+s\sin\frac{2\pi}{n}=0. \tag{2.16.16}$$

From equation (2.16.5) we can deduce that the stress intensity factor

$$K=\lim_{\rho\to 1+}\sqrt{2(\rho-1)}\,\sigma_{\theta\theta}(\rho,\pi/n)$$

can be calculated in terms of the function $K(n)$ defined by the equation

$$K(n)=\frac{2[\Gamma(\frac{3}{4})]^2}{\sqrt{(\pi n)}\,\Gamma(1+\alpha)\Gamma(1-\alpha)}\prod_{m=1}^{\infty}\frac{m(m-\frac{1}{4})^2(m+2\alpha)}{\alpha^2(m^2-\alpha^2)[1+2\,\mathrm{Re}(a_m)+|a_m|^2]},$$

$$\left(\alpha=\frac{1}{n}\right), \tag{2.16.17}$$

by means of the formula

$$K=\sqrt{2}\,p_0K(n). \tag{2.16.18}$$

Numerical values of $K(n)$ for several values of n were calculated by Hideo Igawa and reported by Westmann. These values are shown by the full circles in Fig. 38. On the same diagram the curve $K(n)=\sqrt{(2/n)}$ is shown by the broken line. It will be noticed that for $n>10$ the value of $\sqrt{(2/n)}$ is indistinguishable from that of $K(n)$ so that we have the approximate formula

$$K=\frac{2}{n}p_0,\quad (n>10). \tag{2.16.19}$$

It should also be noted that the solution of the problem of a stress free star crack in a biaxial tension field can be obtained from Westmann's solution

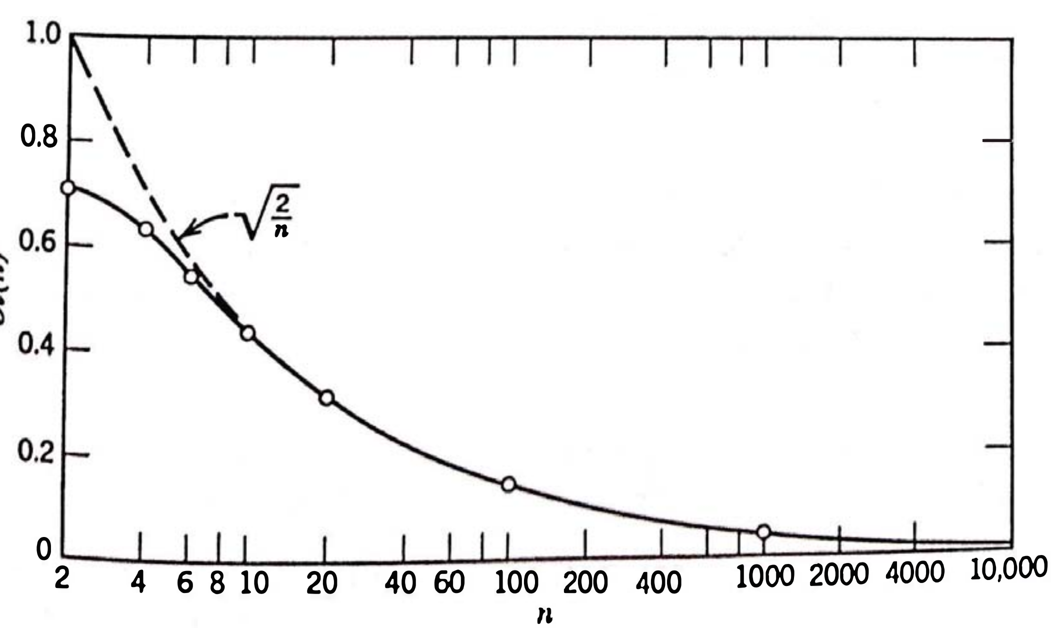


Figure 38 Variation of the function $K(n)$ with n.

by simply superposing the solution corresponding to a uniform biaxial tension field of magnitude p_0.

If we use conventional units, then, for cracks of length c, formulas (2.16.18) and (2.16.19) are replaced by

$$K = \sqrt{2c}\, p_0 K(n) \tag{2.16.20}$$

and

$$K = \frac{2}{n} p_0 \sqrt{c} \tag{2.16.21}$$

respectively.

The method used by Srivastav and Narain consists essentially of observing that the equation (2.16.11) is satisfied identically if we take the representation

$$\psi(s) = \frac{\Gamma(\frac{1}{2}s + \frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}s + 1)} \int_0^1 t^s g(t)\, dt \tag{2.16.22}$$

of the unknown function $\psi(s)$. If we now substitute from this equation into (2.16.10) we obtain a Fredholm integral equation of the second kind for the determination of the function $g(t)$. For the details the reader is referred to the original paper.

The case $n = 4$, when we have the problem of a cruciform crack, has been discussed recently by Stallybrass (1969). His method consists of using the appropriate solution of the half-plane problem and the symmetry of the present problem to reduce the latter to that of solving a Fredholm integral equation of the second kind with a simple kernel. (This equation had been derived earlier by Sneddon; see, e.g., pp. 51–54 of Sneddon (1964).) Stallybrass transforms this equation to an integral equation of Wiener-Hopf type which he solves by the standard procedure. The beauty of Stallybrass' method is that in the calculation of K for a constant pressure on the crack faces the only numerical work involved is the calculation of a simple definite integral, so his results can be made as accurate as we please by computing the value of this integral to a sufficient degree of accuracy. Rooke and Sneddon (1969) show that the integral equation can be reduced to an infinite system of simultaneous linear algebraic equations. By comparing the numerical values with those calculated by Stallybrass, they show that a remarkably accurate approximation is obtained by replacing the infinite system by five or six equations.

2.17 AN ELECTRICAL ANALOGUE SOLUTION OF CRACK PROBLEMS

Experimental methods involving the use of brittle lacquers, strain gauges, or photoelastic materials have been used to determine the distribution of stress in the vicinity of a hole or a crack in a plate under prescribed loads.

In investigations on cracks, such methods suffer from two defects. Firstly, though it is easy to make a hole, it is difficult to manufacture a fine crack; secondly, and a much more serious problem, it is difficult to measure strains over a short length and at sufficient stations in the vicinity of the crack. However, Palmer and Redshaw (1955) have established an electrical analogue which provides a solution of the biharmonic equation when this equation is expressed in finite difference form, and Redshaw and Rushton (1960) have used a very enlarged version of the prototype apparatus to investigate the plane stress problem for a flat rectangular plate under uniaxial tension. This method has the advantage that it permits the representation of an infinitely thin crack. The plate has either a central circular hole, an interior crack, or an exterior crack.

For the Palmer-Redshaw electrical analogue to be used in this way, the problem has to be expressed in terms of an Airy stress function ϕ giving the stress components in the forms

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2}, \qquad \sigma_{yy} = \frac{\partial^2\phi}{\partial x^2}, \qquad \sigma_{xy} = -\frac{\partial^2\phi}{\partial x\,\partial y}. \tag{2.17.1}$$

When the boundary conditions, formulated in terms of ϕ, have been represented on the electrical analogue network, the measurement of the electrical potentials at various nodes gives the values of the Airy stress function at those points. The components of stress can be obtained by double differentiation but, in addition, the sum of the principal stresses $\sigma_{xx} + \sigma_{yy}$ can be measured directly from the network.

For a simply-connected region (exterior crack) the boundary conditions may be expressed directly in terms of the Airy stress function, but for a multiply-connected region (interior crack) an additional boundary condition is required.

If we denote the applied forces on the boundary by X, Y (cf. Fig. 39) and suppose that the normal direction n makes an angle ψ with the x-axis, then

$$\begin{aligned} X &= \sigma_{xx}\cos\psi + \sigma_{xy}\sin\psi, \\ Y &= \sigma_{xy}\sin\psi + \sigma_{yy}\cos\psi. \end{aligned} \tag{2.17.2}$$

Using equations (2.17.1), we can write these equations in terms of the Airy stress function ϕ and then integrate them to give

$$\frac{\partial\phi}{\partial n} = A\cos\psi + B\sin\psi + \alpha\cos\psi + \beta\sin\psi, \tag{2.17.3}$$

where α and β are constants of integration. And

$$A = \int_0^s Y\,ds, \qquad B = \int_0^s X\,ds.$$

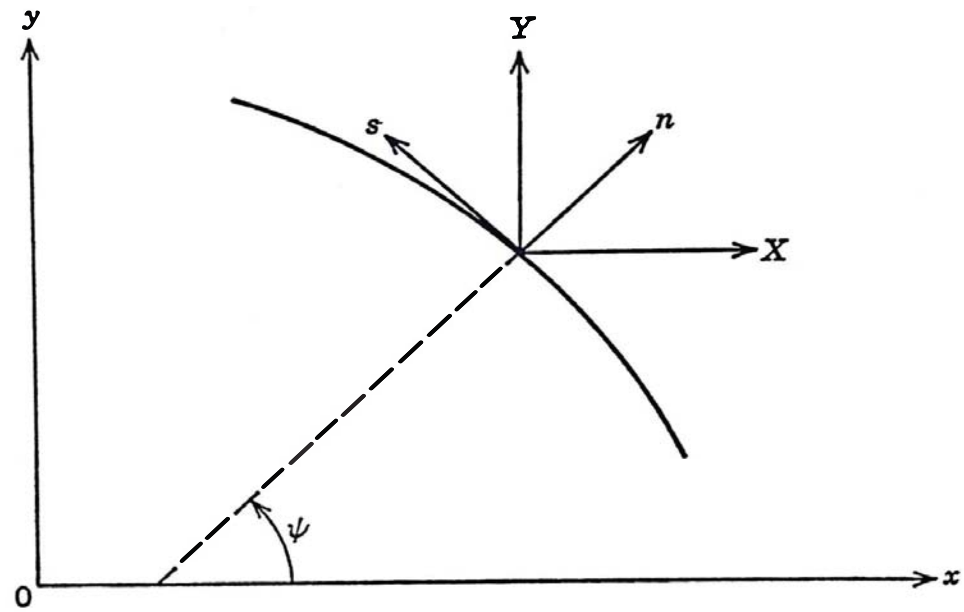


Figure 39

Hence,

$$\phi = \int_0^s (B \cos \psi + A \sin \psi)\, ds + \alpha x + \beta y + \gamma \tag{2.17.4}$$

where γ is a constant of integration. If the region is simply connected a, β, γ can be chosen arbitrarily. If the region is multiply connected, the constants must be chosen to insure that the rotations and displacements are continuous; on the external boundary the constants are arbitrary but on the internal boundaries they are restricted by the continuity condition. Therefore, when the stresses are specified on the boundary, they can be expressed in terms of the Airy stress function and its normal gradient.

The continuity conditions were first expressed in terms of the Airy stress function by Michell (1899), who deduced that, to insure single-valuedness on the boundary, the equations

$$\oint_{\mathrm{I}} \frac{\partial}{\partial n} (\nabla^2 \phi)\, ds = 0, \tag{2.17.5}$$

$$\oint_{\mathrm{I}} \left\{ y \frac{\partial}{\partial n} (\nabla^2 \phi) - x \frac{\partial}{\partial s} (\nabla^2 \phi) \right\} ds = 0, \tag{2.17.6}$$

$$\oint_{\mathrm{I}} \left\{ y \frac{\partial}{\partial s} (\nabla^2 \phi) + x \frac{\partial}{\partial n} (\nabla^2 \phi) \right\} ds = 0, \tag{2.17.7}$$

must be satisfied on each internal boundary. Equation (2.17.5) is the rotation condition and equations (2.17.6) and (2.17.7) the displacement conditions.

In the problems considered by Redshaw and Rushton there was twofold symmetry so that $\alpha = \beta = 0$, and the equations (2.17.6) and (2.17.7) are automatically satisfied. The remaining constant γ can be determined either by using a superposition method due to Prager (1945) or an iterative method due to Southwell (1948). Redshaw and Rushton used the latter method since it is easily applied in the analogue; various values of γ were tried until the correct solution of (2.17.5) was found.

In addition to considering the distribution of stress in the vicinity of a central circular hole, Redshaw and Rushton considered the following problems:

(i) Two external cracks (i.e., a simply connected region),
(ii) An internal crack.

In both cases the experiments were carried out on a rectangular plate with a length/breadth ratio of $\frac{3}{2}$.

(i) *External Crack*

The problem considered was that of a flat rectangular plate of length/breadth ratio equal to $\frac{3}{2}$, having the symmetrically disposed cracks each $\frac{3}{16}$ of the plate width. In the electrical analogue, the half-length of the plate was represented by 48 mesh intervals, so that the half-width of the plate and each crack length were represented by 32 and 12 units, respectively. The stress components were derived from the electrical measurements of ϕ by a numerical procedure; but, as observed above, the analogue gives a direct value of $\sigma_{xx} + \sigma_{yy}$ which serves as a check.

It should be noted that the stress at the end of the crack should be infinite, since an infinitely thin crack has been represented on the electrical analogue, but, owing to the use of a finite mesh, the recorded stress at the end of the crack gives the average value of the stress over the area represented by the discrete interval. Redshaw and Rushton suggest that the network may be considered to be acting like a metallic plate which will experience plastic yielding at the edge of the crack.

(ii) *Internal Crack*

With a plate of the same size as in the last experiment, Redshaw and Rushton found the stress distribution in a rectangular plate under tension when it has an internal crack. Six cracks, varying in length from $\frac{1}{4}$ to $\frac{5}{8}$ of the width of the plate, were considered. During the course of the experiments, Redshaw and Rushton found that a crack whose length exceeded $\frac{5}{8}$ of the width of the plate could not be set up on the electrical analogue because the stress at the end of the crack was so large that it could not be represented by an equivalent electrical potential.

For the purpose of making a comparison between the infinite plate and the rectangular plate considered here, Redshaw and Rushton plotted the lines of equal values of $\sigma_{xx} + \sigma_{yy}$ in both cases. Their results show that reasonable agreement is obtained in the immediate vicinity of the crack but, as we should expect, there are appreciable differences near the external boundary of the plate.

The results of a photoelastic experiment (Cheng et al., 1959) on a rectangular plate with a central crack, but with different proportions to those considered by Redshaw and Rushton, show the same trend in the stress distribution at points remote from the crack. In the immediate vicinity of the crack, the agreement between the two methods is poor, but this may have been because of the fact that in the photoelastic case the crack was formed by making a saw cut whose width was $\frac{1}{24}$ of the crack width, whereas for the purpose of the electrical analogue method, the crack was assumed to be infinitely thin. Also, the crack was formed by first drilling a central hole whose diameter was $\frac{5}{16}$ of the crack width.

2.18 RELATED TWO-DIMENSIONAL PROBLEMS

In this section we shall mention briefly some investigations in the mathematical theory of elasticity which are closely related to the kind of problems we have been considering above.

Of direct interest is a paper by Isida (1958) in which he considers the distribution of stress in a semi-infinite plate under tension when the plate has an elliptical hole, one of whose axes is parallel to the free edge. It is assumed that the deformation is due to the action of a uniform tension at infinity in a direction parallel to the free edge (cf. Fig. 40). The distance from the center of the crack to the free edge of the plate is taken to be the unit of

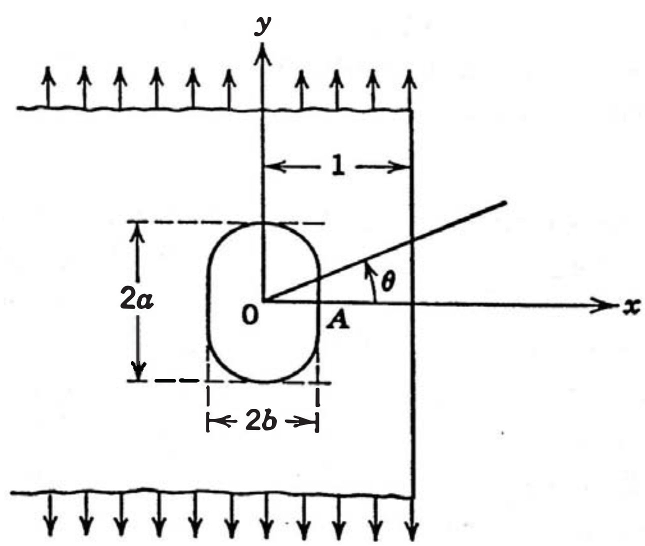


Figure 40 Isida's problem.

length and the axis of the coordinate system is taken to be at the center of the ellipse. The axes of the ellipse are taken to be $2a$ and $2b$, as shown. The tensile stress at infinity is taken to be p_0. Isida's analysis, which is long and complicated, depends on assuming Laurent series for the functions $\varphi(z)$; $\psi(z)$ occurring in the Kolosov-Muskhelishvili equations. The problem of the convergence of the series obtained is not resolved but a direct comparison between Isida's numerical results for the stress concentration factor in the case $b = a$ and those obtained for a circular hole by Udogoti (1949) show the two sets of results to be in remarkable agreement. Isida finds that, on the circumference of the ellipse, the maximum stress occurs at the point A (i.e., at the nearest point from the straight edge). Denoting this maximum value by σ_m, he finds that

$$\frac{\sigma_m}{p_0} = A_0 + A_2\alpha^2 + A_3\alpha^3 + A_4\alpha^4 + A_5\alpha^5 + A_6\alpha^6$$

where in terms of

$$\epsilon = 1 - b/a$$

the coefficients $A_0, A_2, \ldots, A_a$ are given by the equations

$$\begin{aligned}
A_0 &= 1 + 2a/b, \qquad A_2 = 1.5 + 0.5\epsilon^2 a/b,\\
A_3 &= 2 - 1.5\epsilon + 0.75\epsilon^2 + 0.25\epsilon^3 a/b,\\
A_4 &= 1.5 - 0.75\epsilon + 0.375\epsilon^2 + 0.3125\epsilon^3 + 0.2656\epsilon^4 a/b,\\
A_5 &= 3 - 4.125\epsilon^2 + 2.4375\epsilon^3 - 0.25\epsilon^4 + 0.1563\epsilon^5 a/b,\\
A_6 &= -0.375 + 4.5625\epsilon - 6.4688\epsilon^2 + 4.0547\epsilon^3\\
&\quad - 0.6578\epsilon^4 + 0.2226\epsilon^5 + 0.1836\epsilon^6 a/b.
\end{aligned}$$

When $a \gg b$, we may put $\epsilon \simeq 1$, $\epsilon^n(a/b) \simeq n + a/b$, so that

$$\begin{aligned}
\frac{\sigma_m}{p_0} &= (2 + 0.5\alpha^2 + 0.25\alpha^3 + 0.2656\alpha^4 + 0.1563\alpha^5 + 0.1836\alpha^6)\frac{a}{b}\\
&\quad + 1 + 0.5\alpha^2 + 0.5\alpha^3 + 0.375\alpha^4 + 0.2813\alpha^5 + 0.2189\alpha^6.
\end{aligned}$$

The values of σ_m/p_0 for various values of a/b and $0 < \alpha < 0.5$ are shown in Fig. 41.

Similarly, it is found that the tensile stress along the free edge is given by the expression

$$\sigma_{yy}/p_0 = 1 + B_2\alpha^2 + B_4\alpha^4 + B_6\alpha^6,$$

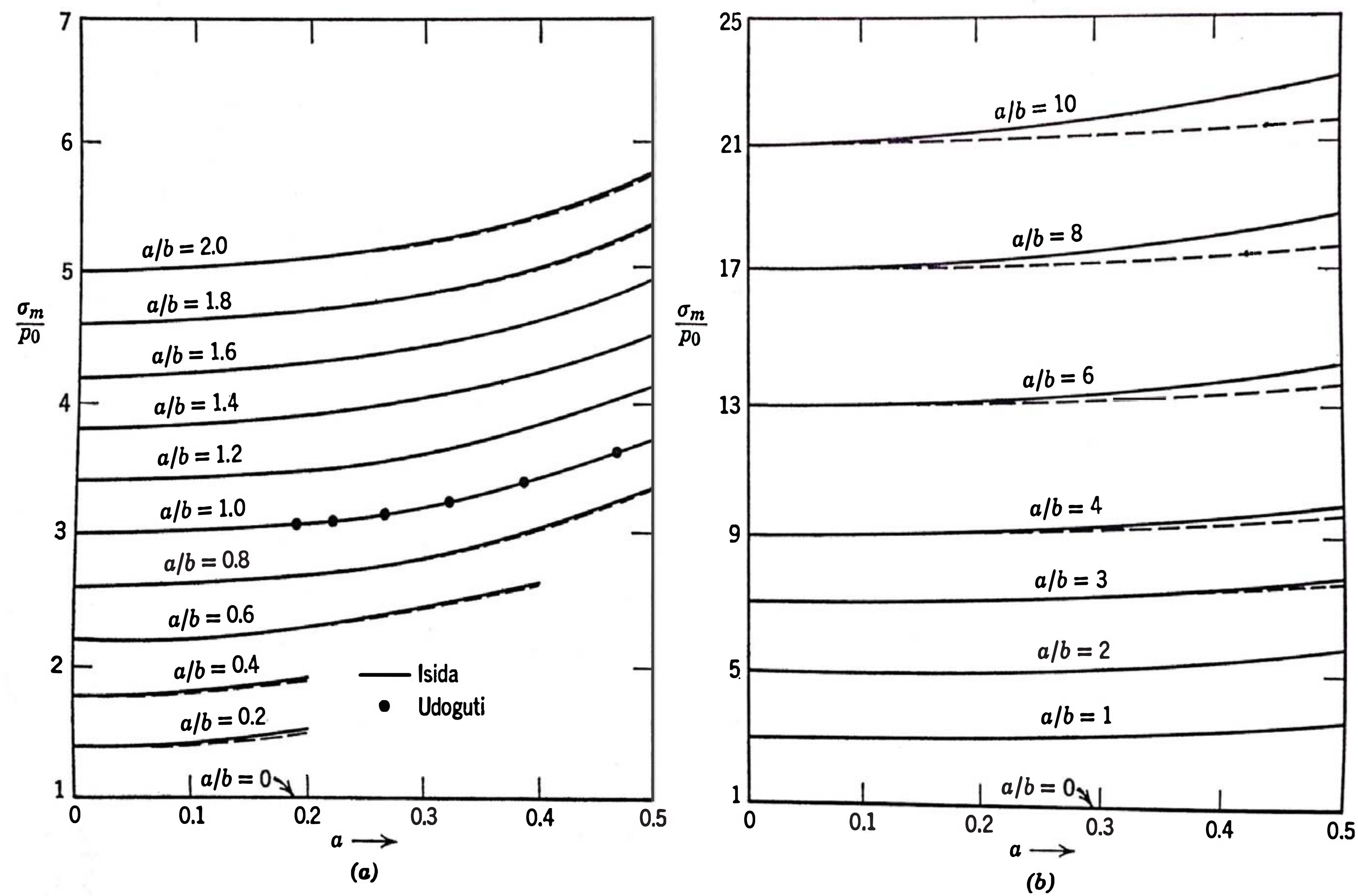


Figure 41 The variation of σ_m/p_0 with a and b.

where

$$B_2 = (6 - 2\epsilon)S_2 + (8 - 4\epsilon)S_3,$$

$$B_4 = (3 - 3\epsilon + \epsilon^2)S_2 - (6 - 10\epsilon + 6.5\epsilon^2 - 1.5\epsilon^3)S_3$$
$$+ (6 + 6\epsilon - 7.5\epsilon^2 + 1.5\epsilon^3)S_4 - 6\epsilon(2 - \epsilon)^2 S_5,$$

$$B_6 = (-0.75 + 5.25\epsilon - 7.3125\epsilon^2 + 4\epsilon^3 - 0.75\epsilon^4)S_2$$
$$+ (8.5 - 28\epsilon + 36.625\epsilon^2 - 24\epsilon^3 - 7.813\epsilon^4 - \epsilon^5)S_3$$
$$- (10.5 - 39\epsilon + 49.125\epsilon^2 - 28.875\epsilon^3 + 7.878\epsilon^4 - 0.75\epsilon^5)S_4$$
$$- (18\epsilon - 39\epsilon^2 + 34.5\epsilon^3 - 14.25\epsilon^4 + 2.25\epsilon^5)S_5$$
$$+ (20\epsilon - 5\epsilon^2 - 15\epsilon^3 + 8.75\epsilon^4 - 1.25\epsilon^5)S_6$$
$$- (60\epsilon^2 - 90\epsilon^3 + 45\epsilon^4 - 7.5\epsilon^5)S_7.$$

The functions S_p are defined by

$$S_p(y) = \mathrm{Re}(1 - iy)^{-p} = (1 + y^2)^{-p}\left[1 - \binom{p}{2}y^2 + \binom{p}{4}y^4 - \cdots\right].$$

Values of this function for $p = 2, 3, \ldots, 7$ and $0 \leq y \leq 1$ are listed in Table 4 of Isida's paper.

Similarly, the resultant force across the minimum section is given by

$$\frac{P_y}{p_0} = 1 - p_2 a^2 - p_4 a^4 - p_6 a^6,$$

where

$$p_2 = \tfrac{1}{4}(2 - \epsilon), \qquad p_4 = \tfrac{1}{34}(4 + 4\epsilon - 8\epsilon^2 + 3\epsilon^3),$$

$$p_6 = \tfrac{1}{128}(8 - 14\epsilon + 59\epsilon^2 - 88\epsilon^3 + 51\epsilon^4 - 10\epsilon^5).$$

The mean stress over the minimum section is $P_y/(1 - a)$ and we define the stress concentration factor s to be the ratio

$$s = \sigma_m \frac{P_y}{(1 - a)}.$$

The values of s can readily be calculated by means of the formulas given above for σ_m and P_y. The results of these calculations are shown in Table 5.

The most comprehensive study of the concentration of stress in the neighborhood of holes of various shapes is Savin's book (Savin, 1951) of which a German translation already exists (Neuber, 1956) and of which an English translation has been published by Pergamon Press (1961). Savin

TABLE 5

VALUES OF THE STRESS CONCENTRATION FACTOR s

a/b	$a = 0$	$a = 0.1$	$a = 0.2$	$a = 0.3$	$a = 0.4$	$a = 0.5$
0	1.0	1.0	1.0	1.0	1.0	1.0
0.2	1.4	1.3186	1.29			
0.4	1.8	1.6570	1.588	1.56		
0.6	2.2	2.0112	1.885	1.813	1.78	
0.8	2.6	2.3694	2.196	2.075	1.999	1.95
1.0	3.0	2.7291	2.514	2.349	2.229	2.14
1.2	3.4	3.0896	2.835	2.629	2.468	2.34
1.4	3.8	3.4505	3.157	2.914	2.714	2.55
1.6	4.2	3.8116	3.481	3.201	2.965	2.76
1.8	4.6	4.1729	3.806	3.490	3.218	2.98
2.0	5.0	4.5343	4.130	3.780	3.473	3.20
3.0	7.0	6.3423	5.760	5.239	4.768	4.33
4.0	9.0	8.1509	7.391	6.705	6.075	5.48
6.0	13.0	11.769	10.658	9.643	8.702	7.81
8.0	17.0	15.387	13.926	12.585	11.335	10.14
10.0	21.0	19.006	17.194	15.528	13.970	12.48

discusses many particular problems; the theory is illustrated by tables, graphs, and charts. The holes are assumed to be with or without reinforcement and the stresss distribution is assumed to be the result of uniform normal stress or uniform shear or simple bending.

The holes considered by Savin are of four types:

(a) rectangle with rounded corners,
(b) triangle with rounded corners,
(c) ellipse,
(d) circle.

The detailed results are given for shapes of types (a) and (b), which can be derived from a circle by a conformal mapping. Savin uses a technique similar to that employed by Bowie (see Section 2.15 above). For instance, the mapping function for a square is

$$z = \omega(\zeta) = c\left(\frac{1}{\zeta} - \tfrac{1}{6}\zeta^4 + \tfrac{1}{56}\zeta^7 - \tfrac{1}{176}\zeta^{11} + \cdots\right), \tag{2.18.1}$$

and if only the first two, three, or four terms are retained, we obtain three shapes which become progressively more like the shape of the square. In retaining a term of a certain order it is not, of course, necessary to retain exactly the same numerical value for the coefficient as given by the infinite expansion.

Problems relating to the distribution of stress in an infinite plate containing a hole of general curvilinear polygonal shape were first discussed by Stevenson

(1943) and Green (1945). The material is admirably reviewed in Green and Zerna (1954, pp. 296–306). Such polygonal holes are given by the transformation

$$z = \omega(\zeta) = a(\zeta + b\zeta^{-n}). \tag{2.18.2}$$

The practical value of one of these calculations will depend on whether they approximate sufficiently closely the actual holes occurring in engineering practice, which usually have straight edges, but with the corners rounded off by circular arcs.

A practical method capable of dealing directly with a hole of prescribed shape has been devised by Kikukawa (1951, 1953, 1954) and this method has been reviewed briefly by Goodier (1958, pp. 8–10). This method proceeds from an initial simple mapping $z = \omega_0(\zeta)$ which is amended to bring the hole sufficiently close to the prescribed shape. Kikukawa has illustrated his method by considering some special cases in detail. One case which is of some interest concerns the hole of rhombic form with circular arc fillets at the corners; Kikukawa's results for this case show that the polygonal hole approximation to the shape [$n = 3$ in equation (2.18.2)] is not a good approximation.

The stress distribution in an infinite thin plate containing an infinite number of holes arranged in a single row has been discussed by Howland (1930, 1935) and Savin (1939). The case when the plate has a double infinite row of circles with arbitrary stagger has been discussed by Howland and Knight (1939); the paper by Howland and McMullan (1936) is also relevant in this case. The general theory developed by Knight (1934) and Green (1940) is also of interest.

Similar problems for an elastic strip of finite width can be discussed on the basis of a paper by Howland (1929). The calculation of the stresses in the neighborhood of a single hole in a strip under tension has been carried out by Howland (1930), Howland and Stevenson (1933), and Knight (1934). The case of one pair or two pairs of circular holes symmetrically situated in the strip has been treated fully by Howland and Knight (1939).

The problem of determining stress distributions in a thin perforated sheet with a doubly-periodic set of holes is of some importance in engineering. Natanson (1935) gave a complete theoretical account of the problem for a plate with circular holes basing his solution on Kolosov-Muskhelishvili complex potentials, but did not give any numerical results. Some approximate calculations have been performed by Horvay (1951, 1952). These are reviewed in a survey article by Dally and Durelli (1956) who also discuss the relatively extensive experimental results obtained by Stiebel and Kopf. More recently, Saito (1957), discussed the case of a thin plate with circular holes whose centers are the nodal points of a square mesh. Saito's analysis does not fully exploit the doubly-periodic nature of the problem and contains certain operations on infinite series which are not justified. However, as Koiter remarks, his numerical results may still be correct since "it often happens in theoretical investigations that unjustified operations on series expansions yield a correct final result."

In the earlier, more rigorous solution, Natanson employs series expansions of both the stress functions $\phi(z)$, $\psi(z)$ in terms of elliptic and associated functions and obtains the values of the coefficients in these expansions from a doubly infinite set of equations. He gives no numerical results but proves that the series converge and satisfy all the conditions of the problem. Like Saito's solution, Natanson's suffers from the defect that both $\phi(z)$ and $\psi(z)$ are represented by series expansions and, hence, that the solution of the doubly infinite set of complex equations determining the coefficients in these series leads to very heavy numerical work.

These defects are overcome in a recent paper by Koiter (1960) in which he develops a more general theory of the problem relating to an infinite sheet with a doubly-periodic set of holes of arbitrary shape, all holes being congruent, in a form which is more convenient for the actual numerical solution in the case of circular holes. In the course of his analysis, Koiter makes use of the properties of a modified Cauchy integral and of a general theorem on the boundary values of a holomorphic quasiperiodic function which leads to the elimination of the stress function $\psi(z)$. The remaining stress function $\phi(z)$ is shown to satisfy a functional equation which is reduced to an ordinary Fredholm integral equation of the second kind. Making the further assumptions that the curvature of the crack profile and the surface tractions on the crack satisfy Hölder conditions, Koiter then proves that a solution of this integral equation exists with a derivative tangential to the crack profile which also satisfies a Hölder condition. Although Koiter's account of the general theory is both rigorous and complete and lays down a possible scheme for numerical calculations, it does not report any numerical results.

Great interest has also been shown in the calculation of stress fields in an elastic body containing external cracks or notches. The classic work in this subject is Neuber's book (Neuber, 1937) of which an English translation is available (Raven, 1946). A second German edition has appeared under a slightly different title (Neuber, 1958), but we shall refer here to the earlier edition because it does not differ markedly from the later one, and because there is, as yet, no English edition of the new German text. Two-dimensional problems are discussed in Chapter IV of Neuber's book, where he considers the distribution of stress in a thin plate containing notches or cracks when the applied stress field takes one of the three forms:

(a) Pure tension,
(b) Pure bending,
(c) Pure shear.

Neuber considers the case where there is an external notch on both sides of the plate; he represents this geometry by taking an infinite plate with a double notch whose boundary is the hyperbola with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{2.18.3}$$

He employs essentially the same set of curvilinear coordinates as we used in Section 2.8. The hyperbola whose equation is (2.18.3) can then be represented by $\eta = \beta$ where $a = c \cos \beta$, $b = c \sin \beta$. In this connection it is of interest to note that, using the method referred to above, Kikukawa has obtained a solution which is in good agreement with Neuber's. In Section 3, Neuber uses the solution of the last problem to find the stress in a plate with a deep external notch on one side. The title of Section 4 ("A circular and an elliptic hole in a very wide bar") is a little misleading in that Neuber, in fact, is considering the plate to be of infinite extent; he justifies the approximation on the grounds that at points in an infinite plate remote from the crack, the stress has practically vanished. If we look at the numerical tables given in Sneddon (1946) we see that this is a reasonable approximation to make if the minimum linear dimension of the plate is at least twice as great as the length of the crack. The solution given by Neuber in case (a) is, of course, equivalent to Inglis' solution considered earlier in Section 2.2. In Section 5, Neuber considers the case of a shallow external notch; i.e., the case when a plate of width b (or $2b$) has one (or two) notches of depth t lying normal to the length of the plate, and the ratio t/b is very small. Again, the solution he derives is approximate in that Neuber represents the actual surface profile by a smooth curve with continuously turning tangent by taking the system of curvilinear coordinates (u,v) where

$$x = u + \frac{u}{u^2 + v^2}, \qquad y = v - \frac{v}{u^2 + v^2}, \tag{2.18.4}$$

and using the smooth curve $u = u_0$ to represent the boundary of the plate with a notch running normal to it.

In Section 6, Neuber uses the limiting values of the stress concentration factors of shallow and deep notches and of elliptic holes in infinite plates to establish general stress concentration factors for external notches and lengthened holes.

Finally, in Section 7, he considers the case of a plate under tension when its free surface, instead of having a notch, has a small projection at one place. Such a geometrical configuration is approximated by considering the boundary to be the curve of the system (2.18.4) corresponding to the value $v = v_0$.

Recent studies to which the reader is referred are the investigation of the effect of a circular inclusion on the stresses in the vicinity of a Griffith crack in a plate under tension (Tamate, 1968) and the calculation of the stress field in the neighborhood of a Griffith crack situated symmetrically at the apex of an elastic wedge (Chakrabarti, 1969). The mixed boundary value problem for an isotropic elastic cylinder containing a strip crack opened by internal pressure is a two-dimensional problem concerning a Griffith crack whose center coincides with the center of a circular elastic region; it has been solved by Srivastav and Narain (1968).

2.19 DYNAMICAL PROBLEMS

We shall now consider the solution of two dynamical problems relating to the spread of cracks. These problems are idealizations of the kind of situation which is of interest in the theory of rupture, but they provide some indication of the dynamical effects coming into play.

The problem considered by Yoffe (1951) is that of finding the stresses near the tip of a Griffith crack which is propagating rapidly in a plate of isotropic elastic material. If a sufficiently large transverse tension p_0 is applied to the plate, the crack may be expected to spread in both directions along its length. In the Yoffe model, it is assumed that the crack propagates to the right only, remaining of constant length $2a$. This is justified by the fact that the stress distribution close to one end of the crack is not influenced by its distance from the other end. This assumption is confirmed later.

By keeping the crack of constant length, the problem becomes that of a plate in uniform tension, across which a disturbance passes at a constant velocity c and without change of form.

The boundary conditions for a crack of length $2a$ moving along the axis with uniform velocity c is that

$$\begin{aligned} \sigma_{yy} &= 0 \quad \text{on} \quad y = 0, \qquad |x'| < a \\ \sigma_{xy} &= 0 \quad \text{on} \quad y = 0, \quad -\infty < x' < \infty \end{aligned} \tag{2.19.1}$$

where $x' = x - ct$ and also that at infinity

$$\sigma_{xx} = 0, \qquad \sigma_{yy} = p_0, \qquad \sigma_{xy} = 0. \tag{2.19.2}$$

Yoffe's solution is obtained from simple wave solutions by means of Fourier integrals. Here we shall make use of the solution of the dynamical equations stated in Section 1.7.

As a generalization of the Westergaard solution (2.4.3) we consider the functions

$$F_1'(z_1) = -A_1 z_1 - B_1(z_1 - \sqrt{z_1^{\,2} - a^2}) \tag{2.19.3}$$

$$F_2'(z_2) = A_2 z_2 + B_2(z_2 - \sqrt{z_2^{\,2} - a^2}) \tag{2.19.4}$$

where A_1, A_2, B_1, B_2 are *real* constants. If we substitute these expressions into equations (1.6.5), (1.6.6), (1.6.7) we obtain the equations

$$\sigma_{yy} = (1 + \beta_2^2)\mathrm{Re}\left[A_2 - A_1 + B_2 - B_1 + \frac{B_1 z_1}{\sqrt{z_1^2 - a^2}} - \frac{B_2 z_2}{\sqrt{z_2^2 - a^2}}\right], \tag{2.19.5}$$

$$\sigma_{xx} + \sigma_{yy} = 2(\beta_1^2 - \beta_2^2)\mathrm{Re}\left[A_1 + B_1\left(1 - \frac{z_1}{\sqrt{z_1^2 - a^2}}\right)\right], \tag{2.19.6}$$

$$\sigma_{xy} = \frac{1}{2\beta_2}\mathrm{Im}\left[(1 + \beta_2^2)^2(A_2 + B_2) - 4\beta_1\beta_2(A_1 + B_1) + \frac{4\beta_1\beta_2 B_1 z_1}{\sqrt{z_1^2 - a^2}} - \frac{(1 + \beta_2^2)^2 B_2 z_2}{\sqrt{z_2^2 - a^2}}\right], \tag{2.19.7}$$

From the boundary conditions (2.19.1) and (2.19.2) and the relations (2.19.5) through (2.19.7), we obtain four equations for the determination of the constants A_1, A_2, B_1, B_2. These can easily be solved to give

$$\left.\begin{aligned} A_1 &= \frac{p_0}{2(\beta_1^2 - \beta_2^2)}, & B_1 &= \frac{(1 + \beta_2^2)p_0}{(1 + \beta_2^2)^2 - 4\beta_1\beta_2} \\ A_2 &= \frac{p_0}{2(\beta_1^2 - \beta_2^2)} + \frac{p_0}{1 + \beta_2^2}, & B_2 &= \frac{4\beta_1\beta_2 p_0}{(1 + \beta_2^2)[(1 + \beta_2^2)^2 - 4\beta_1\beta_2]} \end{aligned}\right\}. \tag{2.19.8}$$

Substituting from equations (2.19.8) into equations (2.19.5), (2.19.6), (2.19.7) we obtain

$$\sigma_{xx} = \frac{p_0}{(1 + \beta_2^2)^2 - 4\beta_1\beta_2}\mathrm{Re}\left\{(2\beta_1^2 - \beta_2^2 + 1)(1 + \beta_2^2)\left[1 - \frac{z_1}{\sqrt{z_1^2 - a^2}}\right] - 4\beta_1\beta_2\left[1 - \frac{z_2}{\sqrt{z_2^2 - a^2}}\right]\right\} \tag{2.19.9}$$

$$\sigma_{yy} = \frac{p_0}{(1 + \beta_2^2)^2 - 4\beta_1\beta_2}\mathrm{Re}\left[(1 + \beta_2^2)\frac{z_1}{\sqrt{z_1^2 - a^2}} - 4\beta_1\beta_2\frac{z_2}{\sqrt{z_2^2 - a^2}}\right] \tag{2.19.10}$$

$$\sigma_{xy} = \frac{2\beta_1(1 + \beta_2^2)p_0}{(1 + \beta_2^2)^2 - 4\beta_1\beta_2}\mathrm{Im}\left[\frac{z_1}{\sqrt{z_1^2 - a^2}} - \frac{z_2}{\sqrt{z_2^2 - a^2}}\right]. \tag{2.19.11}$$

If we substitute from equations (2.19.8) into equations (2.19.3) and (2.19.4), and from these equations into equations (1.7.1) and (1.7.2), we find that the components of the displacement vector are given by

$$u_x = \mathrm{Re}\left\{p_0(1-\beta_2{}^2)\left[\frac{1}{4(\beta_1{}^2-\beta_2{}^2)}+\frac{1}{2(1+\beta_2{}^2)}\right]z_2 + \frac{2\beta_1\beta_2 p_0(z_2-\sqrt{z_2{}^2-a^2})}{(1+\beta_2{}^2)[(1+\beta_2{}^2)^2-4\beta_1\beta_2]} - \frac{p_0 z_1}{2(\beta_1{}^2-\beta_2{}^2)} - \frac{(1+\beta_2{}^2)p_0(z_1-\sqrt{z_1{}^2-a^2})}{[(1+\beta_2{}^2)^2-4\beta_1\beta_2]}\right\} \tag{2.19.12}$$

$$u_y = \mathrm{Im}\left\{\frac{(1+\beta_2{}^2)p_0}{2\beta_2}\left[\frac{1}{2(\beta_1{}^2-\beta_2{}^2)}+\frac{1}{1+\beta_2{}^2}\right]z_2 + \frac{2\beta_1 p_0(z_2-\sqrt{z_2{}^2-a^2})}{[(1+\beta_2{}^2)^2-4\beta_1\beta_2]} - \frac{\beta_1 p_0 z_1}{2(\beta_1{}^2-\beta_2{}^2)} - \frac{\beta_1(1+\beta_2{}^2)p_0(z_1-\sqrt{z_1{}^2-a^2})}{(1+\beta_2{}^2)^2-4\beta_1\beta_2}\right\}. \tag{2.19.13}$$

We can evaluate the stress components from equations (2.19.9), (2.19.10), and (2.19.11) by the same methods as we employed in Section 2.4 above. In fracture theory the significant quantity to calculate is the direct stress acting across a radius from the tip of the crack; i.e., we write

$$x' + iy = a(1+\delta e^{i\theta}) \tag{2.19.14}$$

and consider the stresses for small values of δ. The required stress $\sigma_{\theta\theta}$ is given by

$$\sigma_{\theta\theta} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\sigma_{xy}\sin\theta\cos\theta.$$

Substituting from equations (2.19.9), (2.19.10), and (2.19.11), expanding in powers of δ and retaining only the highest order term, we find that

$$\sigma_{\theta\theta} = \frac{p_0}{\sqrt{\delta}}f(c,\theta) \tag{2.19.15}$$

where

$$f(c,\theta) = \frac{1}{K}\left[(\beta^2-1)(1-\beta_1{}^2)-(1+\beta_1{}^2\cos 2\theta)\cos\frac{\theta_1}{2} - 2\beta_1\sin 2\theta\sin\frac{\theta_1}{2}\right] \times (\cos^2\theta+\beta_1{}^2\sin^2\theta)^{-1/4} + \frac{2\beta_1}{1+\beta_2{}^2}\left[(1+\beta_2{}^2)\sin 2\theta\sin\frac{\theta_2}{2} + 2\beta_2\cos\theta\cos\frac{\theta_2}{2}\right] \times (\cos^2\theta+\beta_2{}^2\sin^2\theta)^{-1/4}, \tag{2.19.16}$$

K denoting the constant defined by

$$K = \sqrt{2}\left[\beta^2 - 2 - \beta^2\beta_1{}^2 + \frac{4\beta\beta_1}{1 + \beta^2}\right] \tag{2.19.17}$$

and θ_1 and θ_2 being defined by the equations

$$\tan\theta_1 = \beta_1 \tan\theta, \qquad \tan\theta_2 = \beta_2 \tan\theta. \tag{2.19.18}$$

It will be observed that, to this order of magnitude, a, the half-length of the crack, does not appear in the expression for $\sigma_{\theta\theta}$ except through δ. It cannot therefore affect the variation of $\sigma_{\theta\theta}$ with θ, but only its absolute magnitude, so that, as mentioned earlier, for the purpose of this particular calculation the actual length of the crack is irrelevant.

In the case when $\lambda = \mu$ (i.e., $\eta = 0.25$) we have $\beta^2 = 3$, and using the relation

$$\beta_1{}^2 = \frac{1 - c^2}{c_1{}^2} = 1 - \frac{(1 - \beta_2{}^2)}{\beta^2}$$

we see that $\beta_1{}^2 = (2 + \beta_2{}^2)/3$. The values of the function $f(c,\theta)$ in this case for a few values of c and θ are shown in Table 6. The results are also shown graphically in Fig. 42. In the limiting case $c = 0$,

$$\sigma_{\theta\theta} = \frac{p_0}{\sqrt{2\delta}} \cos^3\frac{\theta}{2},$$

which has its maximum value at $\theta = 0°$. As c increases, the curve for $\sigma_{\theta\theta}$ as a function of θ first flattens and finally shows a definite maximum at an appreciable angle from the x-axis. From these results it is to be expected that if the material is such that a crack propagates in a direction normal to the maximum tensile stress, there is a critical velocity of about $0.6c_2$ at which the crack tends to become curved. At a lower velocity the crack spreads in a

TABLE 6

VARIATION OF YOFFE'S FUNCTION $f(c,\theta)$ WITH c AND θ IN THE CASE $\nu = 0.25$

c/c_2 \ θ	0°	30°	60°	90°
0	0.707	0.64	0.46	0.25
0.5	0.707	0.70	0.55	0.35
0.8	0.707	0.85	1.04	0.78
0.9	0.707	2.3	5.0	4.6

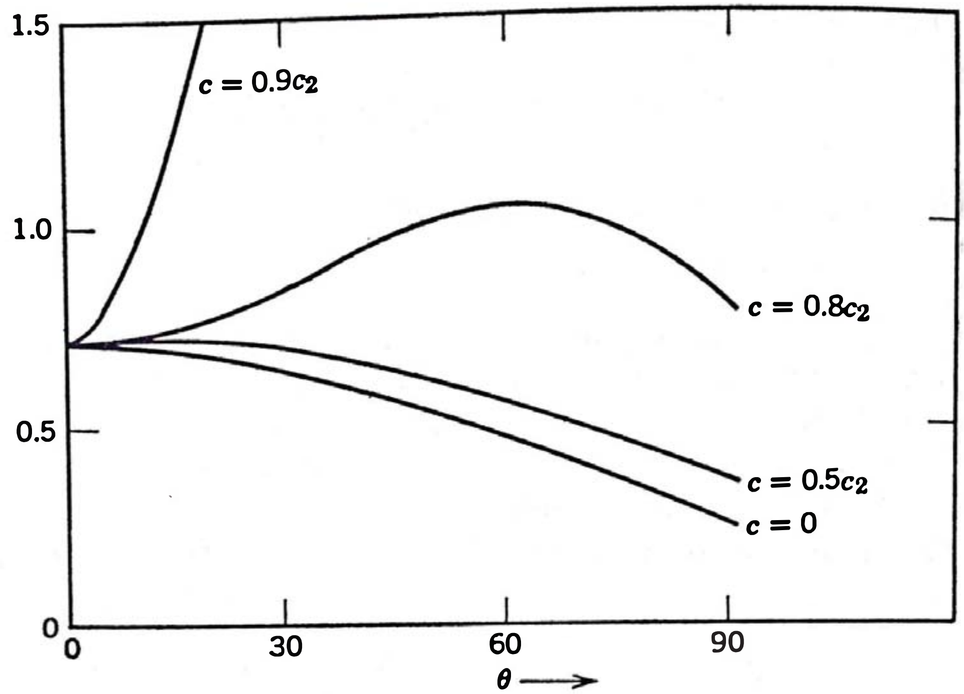


Figure 42 Variation of Yoffe's function with c and θ.

straight line, but as the speed increases the $\sigma_{\theta\theta}$ curve becomes flat and the crack may form branches since nearly equal stresses exist over a wide arc about the tip of the crack. At higher velocities the $\sigma_{\theta\theta}$ curve as a definite maximum again, and each branch tends to curve as is often observed in cracked glass. These effects have been demonstrated experimentally in sheets of cellophane by Orowan.

A similar problem has been considered by Craggs (1960). He consider an infinite elastic body in a state of plane strain in which there is a cut along the line $y = 0$, $x < ct$, and in which the stress distribution in the body is produced by the surface tractions

$$\sigma_{yy} = -f(x'), \qquad \sigma_{xy} = g(x'), \quad x' < 0. \tag{2.19.19}$$

He also assumes that $R\sigma_{xx}$, $R\sigma_{yy}$, $R\sigma_{xy}$ in which

$$R = \sqrt{x'^2 + y^2}$$

all tend to zero as $R \to \infty$ uniformly in y. As above, x' denotes $x - ct$. In particular, Craggs discusses the case

$$\sigma_{yy} = \begin{cases} 0 & x' < -a, \\ -p & -a < x' < 0, \end{cases}$$

and

$$\sigma_{xy} = \begin{cases} 0 & x' < -a, \\ S & -a < x' < 0. \end{cases}$$

In his paper Craggs gives a solution of the general problem (2.19.19), but here we shall consider only the latter special case. The method of solution given by Craggs is complicated. Guided by his analysis we consider the two functions:

$$\left.\begin{aligned} F_1''(z_1) &= (A_1 + iB_1)\left[\log\frac{\sqrt{a} - i\sqrt{z_1}}{\sqrt{a} + i\sqrt{z_1}} + i\pi - \frac{2i\sqrt{a}}{\sqrt{z_1}}\right] \\ F_2''(z_2) &= (A_2 + iB_2)\left[\log\frac{\sqrt{a} - i\sqrt{z_2}}{\sqrt{a} + i\sqrt{z_2}} + i\pi - \frac{2i\sqrt{a}}{\sqrt{z_1}}\right] \end{aligned}\right\} \tag{2.19.20}$$

where A_1, A_2, B_1, B_2 are constants which are, as yet, undetermined. Substituting these expressions into equation (1.7.5) we find that on $y = 0$,

$$\sigma_{yy} = (1 + \beta_2^2)\left[(A_1 + A_2)\left\{\log\frac{\sqrt{a} + \sqrt{-x'}}{\sqrt{a} - \sqrt{-x'}} - \frac{2\sqrt{a}}{\sqrt{-x'}}\right\} - (B_1 + B_2)\pi\right],$$

$$-a < x' < 0$$

and

$$\sigma_{yy} = (1 + \beta_2^2)(A_1 + A_2)\left\{\log\frac{\sqrt{-x'} + \sqrt{a}}{\sqrt{-x'} - \sqrt{a}} - \frac{2\sqrt{a}}{\sqrt{-x'}}\right\}, \qquad x' < -a.$$

Hence to satisfy the crack conditions on σ_{yy} we must take

$$A_1 + A_2 = 0 \tag{2.19.21}$$

$$(B_1 + B_2)\pi(1 + \beta_2^2) = -p. \tag{2.19.22}$$

Similarly, we can show that on $y = 0$,

$$\sigma_{xy} = 2\left[\beta_1 B_1 + \frac{(1 + \beta_2^2)^2}{4\beta_2} B_2\right]$$
$$\times \left[\log\frac{\sqrt{a} + \sqrt{-x'}}{\sqrt{a} - \sqrt{-x'}} - \frac{2\sqrt{a}}{\sqrt{-x'}} + 2\pi\left(\beta_1 A_1 + \frac{(1 + \beta_2^2)^2}{4\beta_2} A_2\right)\right],$$

$$-a < x' < 0,$$

and

$$\sigma_{xy} = 2\left[\beta_1 B_1 + \frac{(1 + \beta_2^2)^2}{4\beta_2} B_2\right]\left[\log\frac{\sqrt{-x'} + \sqrt{a}}{\sqrt{-x'} - \sqrt{a}} - \frac{2\sqrt{a}}{\sqrt{-x'}}\right], \qquad x' < -a,$$

so that to satisfy the conditions on σ_{xy} on the crack surface we must choose our constants in such a way that

$$2\pi\left[\beta_1 A_1 + (1 + \beta_2^2)^2 \frac{A_2}{4\beta_2}\right] = S \tag{2.19.23}$$

$$4\beta_1\beta_2 B_1 + (1 + \beta_2^2)^2 B_2 = 0. \tag{2.19.24}$$

Solving equations (2.19.21) and (2.19.23) we find that

$$A_1 = -A_2 = -\frac{2\beta_2 S}{\pi[(1+\beta_2{}^2)^2 - 4\beta_1\beta_2]}, \tag{2.19.25}$$

and solving equations (2.19.22) and (2.19.24) we find that

$$\frac{B_1}{(1+\beta_2{}^2)^2} = -\frac{B_2}{4\beta_1\beta_2} = \frac{-P}{\pi(1+\beta_2{}^2)[(1+\beta_2{}^2)^2 - 4\beta_1\beta_2]}. \tag{2.19.26}$$

If we substitute from equations (2.19.19) and (2.19.20) with the constants A_1, A_2, B_1, B_2 given by equations (2.19.25), (2.19.26), into equations (1.5.5) through (1.5.7), we obtain the appropriate expressions for the stress components. Also, from equations (1.5.1) and (1.5.2) we find that

$$\frac{\partial u}{\partial t} = -c \operatorname{Re}[F_1''(z_1) + \tfrac{1}{2}(1 - \beta_2{}^2)F_2''(z_2)] \tag{2.19.27}$$

$$\frac{\partial v}{\partial t} = -c \operatorname{Im}\left[\beta_1 F_1''(z_1) + \frac{1+\beta_2{}^2}{2\beta_2} F_2''(z_2)\right] \tag{2.19.28}$$

so that these time derivatives can also be calculated in terms of the functions (2.19.19) and (2.19.10).

It is easily verified from these expressions that the stress components and the velocity components are all $O(R^{-3/2})$ as $R \to \infty$, so that the energy density is $O(R^{-3})$ and there is no transport of energy to infinity. The rate at which the external forces do work is

$$U = 2\int_{-a}^{0} \left[P\left(\frac{\partial v}{\partial t}\right)_{y=0} - S\left(\frac{\partial u}{\partial t}\right)_{y=0}\right] dx'.$$

From equations (2.19.27), (2.19.28) with $F_1''(z_1)$, $F_2''(z_2)$ given by equations (2.19.19), (2.19.20) it can be shown that (in conventional units)

$$U = \frac{4ac^3[(1-\beta_1{}^2)P^2 + (1-\beta_2{}^2)S^2]}{\pi\mu c_2{}^2[4\beta_1\beta_2 - (1+\beta_2{}^2)^2]}.$$

If we assume that the surface tension T is the same whether the surface is formed by direct or tangential stress, the rate of increase of surface energy of the material is $2Tc$. Equating these two expressions we obtain

$$4ac^2[\beta_1 P^2 + \beta_2 S^2] = 2\pi\mu T[4\beta_1\beta_2 - (1+\beta_2{}^2)^2] \tag{2.19.29}$$

which gives the condition that the applied forces P, S are sufficient to maintain the steady extension of the crack. Table 7 shows the variation of P with c for $S = 0$ and of S with c for $P = 0$, calculated from this equation in the case $\eta = 0.25$. The second line gives the direct stress required to maintain

TABLE 7

c/c_2	0.0	0.1	0.2	0.3	0.4	0.5
$a^{1/2}P/(\pi T)^{1/2}$	0.8165	0.8137	0.8002	0.7878	0.7629	0.7284
$a^{1/2}S/(\pi T)^{1/2}$	0.8165	0.8150	0.8098	0.8004	0.7860	0.7675
c/c_2	0.6	0.7	0.8	0.9	c'/c_2	
$a^{1/2}P/(\pi T)^{1/2}$	0.6799	0.6092	0.4957	0.2305	0.0000	
$a^{1/2}S/(\pi T)^{1/2}$	0.7363	0.6893	0.6026	0.3228	0.0000	

$c' = 0.9194c_2$ is the velocity of a Rayleigh wave on a free surface.

the motion of a crack, in the absence of shear, and the third line represents the tangential stress needed in the absence of normal stress.

As c increases the force required to maintain the motion decreases, so that if a stress is applied which causes a crack to spread, the crack may continue to spread even when the applied load is decreased, and catastrophic cracking may occur even under a load which, after reaching a maximum, decreases fairly rapidly. As $c \to c'$, where $c' = 0.9194c_2$ is the velocity of a Rayleigh wave on a free surface, the cracking process would seem to be self-maintaining.* However, it is almost certain that crack-branching will take place before this velocity is reached.

Craggs also calculates the stress σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{r\theta}$ near the advancing tip of the crack. If we take $S = 0$, we get the Yoffe model where it is assumed that the cracking of the material is caused by large direct stress P. As $c \to 0$, $\sigma_{\theta\theta}$ has a maximum for $\theta = 0$, but as c increases through a critical value $\bar{c}$, at which $[d^2 \sigma_{\theta\theta}/d\theta^2] = 0$, the value of $\sigma_{\theta\theta}$ at $\theta = 0$ becomes a minimum and there are maxima on either side of $\theta = 0$. This probably means that $\bar{c}$ is the limiting velocity of propagation of a simple crack, and that branching takes place if the applied stresses are further increased.

If, on the other hand, it is assumed that rupture is associated with strain, the limiting velcity c will then be that for which $[d^2\gamma_\theta/d\theta^2]_{\theta=0} = 0$, where γ_θ is the strain component. Again, if it is assumed that hydrostatic pressure is unimportant, the limiting velocity $\bar{v}$ would be obtained by considering $\sigma_{\theta\theta} - \sigma_{rr}$.

Finally, if it is assumed that rupture results from shearing stress, we take $P = 0$ in these equations and assume that branching will occur at a velocity

* The velocity c' of the Rayleigh wave is a root of the equation $4\beta_1\beta_2 = (1 + \beta_2{}^2)^2$.

TABLE 8

v	$\bar{c}/c_2$	c/c_2	$\bar{v}/c_2$	c_s
$\frac{1}{4}$	0.629	0.611	0.565	0.772
$\frac{1}{3}$	0.667	0.621	0.571	0.810
$\frac{1}{4}$*	0.612	0.598	0.562	0.762
$\frac{1}{3}$*	0.629	0.597	0.565	0.772
$\frac{1}{2}$*	0.667	0.571	0.571	0.810

c_s for which $(d^2\sigma_{r\theta}/d\theta^2)_{\theta=0} = 0$. The values of $\bar{c}$, C, $\bar{v}$, and c_s are given in Table 8 for $v = \frac{1}{4}$, $v = \frac{1}{3}$, the third, fourth, and fifth lines, marked with asterisks, are the corresponding results for plane stress.

Papadopoulos (1963*b*) examines the diffraction of plane waves in an isotropic elastic solid by a semi-infinite plane crack. Such a crack is an obstacle to the propagation of plane pulses of the scalar and vector velocity potential so that both reflected and diffracted fields will be set up. One new feature that arises in his work is that following the arrival of a tensile field which tends to open up the crack, there is necessarily a scattered field which causes the crack to close itself with the velocity of free surface waves. Papadopoulos' analysis is extended to throw light on the possible initiation of brittle fracture and the subsequent propagation of a smooth crack in the plane of the existing one. It is shown how the analysis leads to the result that since the crack is extended without the application of external forces to the crack itself, it can only travel freely with the velocity of Rayleigh waves.

In his analysis, Papadopoulos considers two possible states: when the crack is opened there are two distinct surfaces at which the stresses must vanish, but when it is closed the normal velocity, displacement, and stress components must be continuous, leaving, in the absence of friction, the tangential stresses to vanish. His analysis, depending on results reported in his paper (Papadopoulos, 1963*a*), is quite complicated. The reader is referred to this paper for the details.

Transient problems in which a semi-infinite crack appears instantaneously in a uniformly stressed medium have been investigated by Ang (1960). Baker (1962) extended this work to the case where the crack propagates at constant velocity after it has appeared suddenly in the stretched elastic body. Most work on the dynamics of fracture has been limited to the model of a semi-infinite crack. In a recent paper, Sih (1968) places special emphasis on the configuration of a crack having finite length. By retaining the additional characteristic dimension of the crack length, several features of the dynamical solution are exhibited; these are the peak values of the stress-intensity factor at a certain wave length for cracks opened up under periodic cyclic loadings and at a given time for cracks opened by rapidly applied tractions. In this paper a general solution is given of the problem of cracks traveling at constant speed and the extension of Dugdale's model to such cracks is established.

Experimental values of the limiting velocity of crack propagation in thin plates of different materials are given by Irwin (1958). The measured values lie between $c/c_2 = 0.44$ (for cellulose acetate) and $c/c_2 = 0.59$ (for silica glass). The velocities in Table 8 are by comparison consistently too high, but from the nature of the experiments it would seem that the measured values are more likely to be too low rather than too high. The solution of the dynamical equations for the stress field near the tip of a crack subjected to longitudinal shear, and traveling in an elastic medium, has recently been considered by McClintock and Sukhatme (1960). As in the case of tensile cracks, the applied stress required for constant velocity is lower for higher crack velocity and there is a critical velocity, approximately equal to $0.6c_2$, above which the crack will branch. Similar stress levels are found using two different fracture criteria:

(a) the Griffith energy criterion;

(b) the criterion of local average shear strain Neuber, 1937 and McClintock, 1958), according to which fracture will occur when the average shear strain over a distance related to the structure of the material along the line directly ahead of the tip of the crack reaches some critical value γ_f. In an elastic material this structural distance is taken to be an atomic or molecular spacing.

It should also be noted that several authors (e.g., Mott, 1948, Gilman, 1959, Stoh, 1960, and Berry, 1960) have produced simple models to describe the behavior of propagating cracks. These usually depend, not on exact solutions of the dynamical equations of elasticity, but on simple energy considerations.

2.20 COHESIVE FORCES AT THE TIPS OF A GRIFFITH CRACK

The calculations discussed in the preceding sections have all been made on the basis of the linear theory of elasticity. For certain loadings, this theory predicts infinite stresses at the tips of a crack and these obviously cannot be sustained by a real material. The classical theory takes no account of the fact that cohesive forces between the crack faces are brought into play.

Barenblatt has suggested a method of utilizing the model of an elastic body to incorporate cohesive forces into the theory of cracks. (A review of Barenblatt's work is given in Barenblatt, 1962.) The real difficulty is that we know neither the distribution of the cohesive forces over the crack surfaces nor the variation with the distance between opposite faces of the intensity of these forces. In addition, the distribution of the cohesive forces will depend, in general, on the manner in which the crack is loaded.

To circumvent these difficulties, Barenblatt introduces two simplifying assumptions. The first is that if the crack occupies the slit $|x| \leq c$, $y = 0$, the cohesive forces act only over the region $c - \delta \leq |x| \leq c$, $y = 0$, where δ is small compared with c; this region is called the *edge region* of the crack. The second assumption is that the form of the crack surface in the edge region (and the crack surface) does not depend on the manner in which the crack is loaded and is always the same for a given material under given conditions of temperature, external pressure, etc.

With these two hypotheses, Barenblatt is able to reformulate the results of the qualitative analysis of fracture phenomena; this theory appears in Barenblatt (1959*a*,*b*) and in the review article already cited.

Barenblatt considers the body to be linearly elastic up to fracture, the stress held being represented as the sum of two fields: a field evaluated neglecting cohesive forces and one corresponding to the action of forces of cohesion and no others. If K_0 and K_c are the stress intensity factors corresponding respectively to these fields, then the total stress intensity factor is

$$K = K_0 + K_c. \tag{2.20.1}$$

Denoting the cohesive force by $C(s)$ where s is distance along the crack measured from the tip, we see from (2.3.12) that, in conventional units

$$K_c = -\frac{2\sqrt{c}}{\pi}\int_0^c \frac{C(c-x)\,dx}{\sqrt{c^2 - x^2}}.$$

By the first hypothesis we may replace the lower limit of integration by $c - \delta$, and then, by a simple change of variable, we find that

$$K_c = -\frac{2\sqrt{c}}{\pi}\int_0^\delta \frac{C(s)\,ds}{\sqrt{2cs - s^2}}.$$

If $\delta \gg c$ we may replace $(2cs - s^2)^{-1/2}$ by $(2cs)^{-1/2}$ and so obtain the relation

$$K_c = -\sqrt{\frac{2}{\pi}}\int_0^\delta \frac{C(s)\,ds}{\sqrt{s}}.$$

According to the second hypothesis, the form of the function $C(s)$ and the value of the number δ are both independent of the load on the crack; in other words

$$C = \int_0^\delta \frac{C(s)\,ds}{\sqrt{s}} \tag{2.20.2}$$

is a constant. Barenblatt calls this constant, which characterizes the resistance of the material to an extension of its cracks, *the module of cohesion* of the material.

In terms of this constant,

$$K_c = -\frac{\sqrt{2}}{\pi} C. \tag{2.20.3}$$

If we assume that for any given loading of the crack faces, the cohesive forces are brought into play in such a way that the stresses at the tips of the crack remain finite, we see that $K = 0$ so that equations (2.20.1) and (2.20.3) are equivalent to

$$K_0 = \frac{\sqrt{2}}{\pi} C, \tag{2.20.4}$$

where C is defined by (2.20.2). From the dimensional form of equation (2.3.12) we find that the relation between the applied pressure $p(x)$ and the crack length c is given by

$$\int_0^c \frac{p(x)\,dx}{\sqrt{c^2 - x^2}} = \frac{C}{\sqrt{2c}}, \tag{2.20.5}$$

where C is the modulus of cohesion.

In the case in which $p(x)$ is a constant (e.g., p_0), (2.20.5) reduces to

$$p_0 \pi \sqrt{\tfrac{1}{2} c} = C. \tag{2.20.6}$$

CHAPTER 3

Three-Dimensional Crack Problems

3.1 THE PENNY-SHAPED CRACK

In this chapter, we shall consider some three-dimensional problems in the mathematical theory of elasticity relating to cracks. In the analysis of these problems we shall find that, just as in the two-dimensional case, the method of curvilinear coordinates and the method of integral transforms play central roles. In the three-dimensional case, however, there is no analogue of the powerful methods based on the theory of functions of a complex variable which were at our disposal in the two-dimensional case.

The first few sections of this chapter are concerned with the penny-shaped crack which is, in some ways, the natural analogue in three dimensions of the Griffith crack. Suppose that we have an infinite solid and that we use the cylindrical coordinates (ρ,θ,z) to describe points in it. We assume that we have a penny-shaped crack lying in the xy-plane, with its center at the origin of coordinates. If the radius of the crack is c, then we may take the crack to be the disk

$$\rho \leq c, \quad z = 0. \tag{3.1.1}$$

In most problems, we shall take the radius of the crack to be our unit of length, so that the crack becomes

$$\rho \leq 1, \quad z = 0. \tag{3.1.2}$$

Usually we are interested in problems in which the surfaces of the crack are stress free and there is a prescribed tensile stress at infinity so that

$$\sigma_{zz} = \sigma_{\rho z} = \sigma_{\theta z} = 0, \quad \rho \leq 1, \quad z = 0 \tag{3.1.3}$$

and all the stress components tend to zero as $\sqrt{\rho^2 + z^2} \to \infty$ except σ_{zz} which tends to p_0. However, it is a simple matter to use the superposition principle to turn the solution of such a problem into a discussion of the problem

$$\sigma_{zz} = -p_0, \qquad \sigma_{\rho z} = \sigma_{\theta z} = 0, \quad \rho \leq 1, \quad z = 0 \tag{3.1.4}$$

and all the components of stress tend to zero as $\sqrt{\rho^2 + z^2} \to \infty$.

In equation (3.1.4), p_0 is a constant; i.e., the problem is that of finding the stress distribution in the neighborhood of a penny-shaped crack when it is opened out by the application of a constant pressure to its free surface. This problem can be generalized immediately to the case in which the internal pressure opening the crack varies over the surface of the crack. In place of equations (3.1.4) we then have

$$\sigma_{zz} = -p(\rho,\theta), \qquad \sigma_{z\rho} = \sigma_{z\theta} = 0, \quad \rho \leq 1, \quad z = 0, \tag{3.1.5}$$

it being assumed that the radius of the penny-shaped crack is taken to be the unit of length.

In general, we shall be considering only cases in which there is symmetry about the z-axis so that $\sigma_{z\theta}$ is identically zero and equations (3.1.5) reduce to

$$\sigma_{zz} = -p(\rho), \qquad \sigma_{z\rho} = 0, \quad p \leq 1, \quad z = 0. \tag{3.1.6}$$

If we write $r = \sqrt{\rho^2 + z^2}$ and take R to be a constant which is very much greater than unity, we may take the boundary conditions at points remote from the crack to be such that all the components of stress and displacement should be zero at all points on the surface of the sphere $r = R$.

One method of treating such a problem is to consider the three-dimensional analogue of Inglis' method (Section 2.2 above); i.e., to determine the stress distribution in the region bounded by two ellipsoids E, E' (cf. Fig. 43) when the surface of the inner ellipsoid E is free from stress and that of the outer ellipsoid E' has certain prescribed tractions. The limiting form of the solution can then be obtained by letting E degenerate into a penny-shaped crack and letting E' tend to the sphere $r = R$ with $R \to \infty$. This is the method used by Sack (1946).

If we define a set of oblate spheroidal coordinates ψ, θ, ϕ by the equations

$$x = \cosh\psi \sin\theta \cos\phi, \qquad y = \cosh\psi \sin\theta \sin\phi, \qquad z = \sinh\psi \cos\theta \tag{3.1.7}$$

then the surfaces $\psi = constant$ form a family of oblate spheroids; in particular, $\psi = 0$ corresponds to the disk defined by the relations (3.1.2), and $\psi = \infty$ corresponds to the sphere of infinite radius. Laplace's equation in

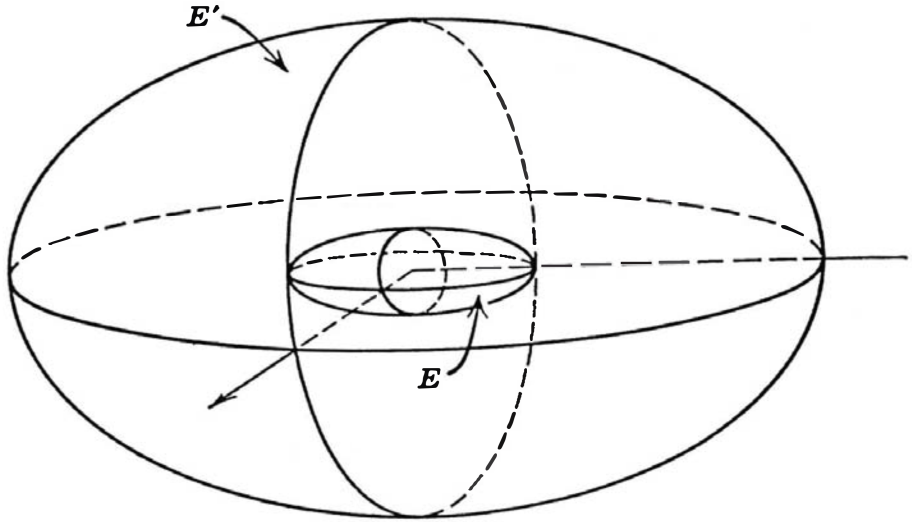


Figure 43 Ellipsoidal cavity in a large ellipsoid.

these coordinates takes the form

$$\frac{1}{\cosh\psi}\frac{\partial}{\partial\psi}\left(\cosh\psi\frac{\partial V}{\partial\psi}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{\cosh^3\psi\sin^3\theta}\frac{\partial^2 V}{\partial\phi^2}=0,$$

with solutions

$$\Pi_n^m(\sinh\psi)P_n^m(\cos\theta)e^{im\phi},\qquad T_n^m(\sinh\psi)P_n^m(\cos\theta)e^{im\phi}$$

where m and n are positive integers (or zero) and the functions P_n^m are the associated Legendre polynomials, while Π_n^m, T_n^m represent the Legendre functions of the first and second kind respectively, with imaginary argument.

If we are considering a penny-shaped crack opened out by constant internal pressure, then the stress components will have symmetry about the z-axis and will fall off in magnitude with increasing distance from the center of the crack. Because of the axial symmetry, we need only consider associated Legendre functions which have $m = 0$, and because of the condition of diminishing as ψ increases, our solution must not contain functions of the type Π_n^o. Only functions of the type T_n^o will occur in our solution.

Sack has shown that the appropriate solution of the equations of elastic equilibrium can be written in the form

$$u_\psi=\frac{1}{h}\left[\frac{\partial u}{\partial x}+(3-4\eta)\cosh\psi\cos\phi V-\sinh\psi\cos\phi\frac{\partial V}{\partial x}\right],\tag{3.1.8}$$

$$u_\theta=\frac{1}{h}\left[\frac{\partial u}{\partial x}-(3-4\eta)\sinh\psi\sin\phi V-\sinh\psi\cos\phi\frac{\partial V}{\partial x}\right],\tag{3.1.9}$$

where $h = \sqrt{\sinh^2 \psi + \cos^2 \theta}$ and

$$U = \frac{(1 - 2\eta)p_0}{3\pi} [T_0^0(\sinh \psi) + \tfrac{1}{4}(3 \cos^2 \theta - 1)T_2^0(\sinh \psi)], \quad (3.1.10)$$

$$V = \frac{p_0}{\pi} T_1^0(\sinh \psi)\cos \theta. \quad (3.1.11)$$

The three functions of type T_n^o involved are given by

$$T_0^0(\sinh \psi) = \cot^{-1}(\sinh \psi), \qquad T_1^0(\sinh \psi) = \sinh \psi \, T_0^0(\sinh \psi) - 1,$$
$$T_2^0(\sinh \psi) = (3 \sinh^2 \psi + 1)T_0^0(\sinh \psi) - 3 \sinh \psi,$$

and it is easily shown that for large values of r

$$T_0^0(\sinh \psi) = \frac{1}{r} - \frac{1}{3r^3} + \cdots, \qquad T_1^0(\sinh \psi) = -\frac{1}{3r^2} + \cdots$$

and also that $T_2^0(\sinh \psi)$ can be neglected since it falls off in a higher power. After a little calculation it can be shown that the displacement components are given by

$$u_r = \frac{p_0}{9r^2} [2(1 + \eta) + (5 - 4\eta)(3 \cos^2 \theta - 1) + \cdots]$$

$$u_\theta = -\frac{2p_0}{3r^2} (1 - 2\eta)\sin \theta \cos \theta + \cdots$$

and that the components of stress required to calculate the energy are given by

$$\sigma_{rr} = -\frac{4p_0}{9r^3} [2(1 + \eta) + (5 - \eta)(3 \cos^2 \theta - 1) + \cdots]$$

$$\sigma_{r\theta} = -\frac{4p_0}{9r^3} (1 + \eta)\sin \theta \cos \theta + \cdots$$

In these equations the row of dots denotes terms of higher powers in r^{-1} which can be neglected.

The energy W_1 of the solid containing the crack is given by the integral

$$W_1 = -\pi \int_0^\pi (u_r\sigma_{rr} + u_\theta\sigma_{r\theta})_{r=R} R^2 \sin \theta \, d\theta,$$

where $R \gg 1$. Using these expressions, Sack has shown that the presence of a penny-shaped crack of unit radius in a solid under a uniform tension p_0 normal to the surface of the crack alters the free energy of the solid by an amount

$$W = -\frac{4(1 + \eta)}{3} p_0^2.$$

In conventional units, if the crack has radius c,

$$W = \frac{8(1-\eta^2)}{3E} p_0^2 c^3. \tag{3.1.12}$$

Sack also showed that, provided that the component of the applied stress in the z-direction is tensile, this result is independent of the values of the components of the applied stress in the x- and y-directions.

The strain energy release rate is therefore given by the equation

$$\mathscr{G} = \frac{4(1-\eta^2)p_0^2 c}{\pi E}. \tag{3.1.13}$$

Applying the Griffith criterion (1.1.3) we find that the critical value of the applied tension is

$$p_{cr} = \left[\frac{ET}{2c(1-\eta^2)}\right]^{1/2}. \tag{3.1.14}$$

The three-dimensional model thus gives a critical tensile stress differing from the plane-strain Griffith value (2.2.8) by a factor $\frac{1}{2}\pi$.

Just as in the two-dimensional case, we can formulate the problem of the penny-shaped crack as a boundary value problem of mixed type for the half-space $z \geq 0$. By considerations of symmetry we know that at points on the plane $z = 0$ lying outside the circle $\rho = 1$, the normal component of displacement, u_z, is zero as is also the shearing stress $\sigma_{\rho z}$. We therefore have the boundary conditions

$$\sigma_{\rho z}(\rho,0) = 0, \qquad \rho \geq 0, \tag{3.1.15}$$

$$\sigma_{zz}(\rho,0) = -p(\rho), \quad 0 \leq \rho \leq 1, \tag{3.1.16}$$

$$u_z(\rho,0) = 0, \qquad \rho > 1. \tag{3.1.17}$$

In addition, we have the condition that all the components of displacement and stress should tend to zero as $r = \sqrt{\rho^2 + z^2} \to \infty$ through positive (or zero) values of z. The solution of this problem has been discussed by Sneddon (1946), Green (1949), Payne (1953), and others; we shall consider these methods in Sections 3.2 through 3.4.

In discussing the half-space problem we are interested in calculating the energy, W, of the crack and the stress intensity factor K at the tip of the crack. They are given respectively by the equations

$$W = 2\pi \int_0^1 \rho p(\rho) u_z(\rho,0)\, d\rho, \tag{3.1.18}$$

$$K = \lim_{\rho \to 1+} \sqrt{2(\rho - 1)}\, \sigma_{zz}(\rho,0). \tag{3.1.19}$$

The problem can, of course, be generalized to flat cracks of arbitrary shape. Suppose that we have a flat crack occupying the region S in the plane $z = 0$. Then we may consider the equivalent problem of determining the distribution of stress in the half-plane $z \geq 0$ when the boundary is subjected to the conditions

$$\sigma_{xz}(x,y,0) = \sigma_{yz}(x,y,0) = 0, \qquad (x,y) \in Z, \tag{3.1.20}$$
$$\sigma_{zz}(x,y,0) = -p(x,y), \qquad (x,y) \in S, \tag{3.1.21}$$
$$u_z(x,y,0) = 0, \qquad (x,y) \in Z - S, \tag{3.1.22}$$

where Z denotes the entire xy-plane. We shall discuss a problem of this kind in Section 3.8 below.

3.2 SOLUTION OF THE HALF-PLANE PROBLEM FOR A PENNY-SHAPED CRACK

In the solution of the half-plane problem for a penny-shaped crack expressed by equations (3.1.15), (3.1.16), and (3.1.17), we find it convenient to use the Hankel transform of a function. We use the notation

$$\mathscr{H}_\nu[G(\rho,z);\ \rho \to \xi]$$

to denote the Hankel transform

$$\int_0^\infty \rho G(\rho,z) J_\nu(\rho\xi)\, d\rho$$

of order ν with respect to the variable ρ of an axisymmetric function $G(\rho,z)$.

If in the third component of equation (1.3.1) and in equation (1.3.6) we take $\chi = \psi = 0$ and

$$\phi = \tfrac{1}{2}(1 - 2\eta)\mathscr{H}_0[\xi^{-2}\psi(\xi)e^{-\xi z};\ \xi \to \rho],$$

we find that a solution appropriate to the half-space $z \geq 0$ of the equations of elastic equilibrium is given by the equations

$$u_\rho(\rho,z) = -\tfrac{1}{2}\mathscr{H}_1[(1 - 2\eta - \xi z)\xi^{-1}\psi(\xi)e^{-\xi z};\ \xi \to \rho], \tag{3.2.1}$$
$$u_z(\rho,z) = \tfrac{1}{2}\mathscr{H}_0[(2 - 2\eta + \xi z)\xi^{-1}\psi(\xi)e^{-\xi z};\ \xi \to \rho]. \tag{3.2.2}$$

In dimensionless units the corresponding expressions for the z-components of the stress tensor are given by the equations

$$\sigma_{\theta z} \equiv 0, \tag{3.2.3}$$
$$\sigma_{\rho z}(\rho,z) = -z\mathscr{H}_1[\xi\psi(\xi)e^{-\xi z};\ \xi \to \rho], \tag{3.2.4}$$
$$\sigma_{zz}(\rho,z) = -\mathscr{H}_0[(1 + \xi z)\psi(\xi)e^{-\xi z};\ \xi \to \rho].$$

Equation (3.2.3) shows that equation (3.2.15) is satisfied whatever form is taken for $\psi(\xi)$, and equations (3.2.2) and (3.2.4) show that the remaining boundary conditions (3.1.16) and (3.1.17) are satisfied if the function $\psi(\xi)$ is chosen to be the solution of the dual integral equations

$$\mathscr{H}_0[\psi(\xi);\rho] = p(\rho), \quad 0 \leq \rho \leq 1 \tag{3.2.5}$$

$$\mathscr{H}_0[\xi^{-1}\psi(\xi);\rho] = 0, \quad \rho > 1 \tag{3.2.6}$$

The solution of these equations is elementary (Sneddon, 1960; see also Section 3.5 of Sneddon, 1966). We find that to satisfy equation (3.2.6) we may represent the function $\psi(\xi)$ by the integral

$$\psi(\xi) = \int_0^1 g(t)\sin(\xi t)\, dt, \quad g(0) = 0, \tag{3.2.7}$$

in which case (3.2.5) reduces to the Abel type integral equation

$$\int_0^x \frac{g'(t)\, dt}{\sqrt{x^2 - t^2}} = p(x), \quad 0 \leq x < 1 \tag{3.2.8}$$

with solution

$$g(t) = \frac{2}{\pi}\int_0^t \frac{sp(s)\, ds}{\sqrt{t^2 - s^2}}. \tag{3.2.9}$$

From equation (3.2.2) we have

$$u_z(\rho,0) = (1 - \eta)\mathscr{H}_0[\xi^{-1}\psi(\xi);\rho],$$

so that if we use the representation (3.2.7) and the well-known result

$$\int_0^\infty J_0(\xi\rho)\sin(\xi t)\, d\xi = \frac{H(t - \rho)}{\sqrt{t^2 - \rho^2}},$$

we find that

$$u_z(\rho,0) = (1 - \eta)\int_\rho^1 \frac{g(t)\, dt}{\sqrt{t^2 - \rho^2}}, \quad (0 \leq \rho < 1). \tag{3.2.10}$$

On the other hand, we find from equation (3.2.4) that the normal component of stress across the plane of the crack is given by

$$\sigma_{zz}(\rho,0) = -\mathscr{H}_0[\psi(\xi);\rho].$$

Using the form

$$\psi(\xi) = -\xi^{-1}\cos\xi g(1) + \xi^{-1}\int_0^1 g'(t)\cos(\xi t)\, dt$$

of equation (3.2.7) and the integral

$$\int_0^\infty J_0(\xi\rho)\cos(\xi t)\, d\xi = \frac{H(\rho - t)}{\sqrt{\rho^2 - t^2}},$$

we find that

$$\sigma_{zz}(\rho,0) = \frac{g(1)}{\sqrt{\rho^2 - 1}} - \int_0^1 \frac{g'(t)\,dt}{\sqrt{\rho^2 - t^2}}, \quad \rho > 1. \tag{3.2.11}$$

Now if $g(t)$ is differentiable in the neighborhood of $t = 1$, we have that

$$\int_0^1 \frac{g'(t)\,dt}{\sqrt{\rho^2 - t^2}} = O(1) \qquad \text{as} \qquad \rho \to 1+$$

so that the stress intensity factor defined by equation (3.1.19) is given by the simple expression

$$K = g(1). \tag{3.2.12}$$

This, of course, is the expression for K in our system of dimensionless units. If we substitute for $g(t)$ from equation (3.2.9) and then convert to conventional units, we find that for a penny-shaped crack of radius c, equation (3.2.12) reduces to

$$K = \frac{2G(c)}{\pi\sqrt{c}} \tag{3.2.13}$$

where the function $G(t)$ is $\frac{1}{2}\pi g(t)$ so that

$$G(t) = \int_0^t \frac{\rho p(\rho)\,d\rho}{\sqrt{t^2 - \rho^2}}. \tag{3.2.14}$$

In dimensionless units the work done in opening the crack is given by the equation

$$W = 2\pi \int_0^1 \rho p(\rho) u_z(\rho,0)\,d\rho.$$

If we substitute from equation (3.2.10) and invert the order of integration in the resulting integral, we find that

$$W = 2\pi(1 - \eta)\int_0^1 g(t)\,dt \int_0^t \frac{\rho p(\rho)\,d\rho}{\sqrt{t^2 - \rho^2}}$$

and from (3.2.9) we see that this relation can be put into the form

$$W = \pi^2(1 - \eta)\int_0^1 [g(t)]^2\,dt.$$

In conventional units this can be written in the form

$$W = \frac{8(1 - \eta^2)}{E}\int_0^c [G(t)]^2\,dt \tag{3.2.15}$$

where $G(t)$ is defined by equation (3.2.14).

3.3 PENNY-SHAPED CRACK OPENED UP BY CONSTANT PRESSURE

In this section we examine a little more closely the case in which $p(\rho) = p_0$, a constant. It follows from equation (3.2.14) that $G(t) = p_0 t$ so that equation (3.2.13) gives the expression

$$K = \frac{2p_0\sqrt{c}}{\pi} \tag{3.3.1}$$

for the stress intensity factor, and that equation (3.2.15) gives the expression

$$W = \frac{8(1-\eta^2)p_0{}^2c^3}{3E} \tag{3.3.2}$$

for the strain energy of the crack. The corresponding expression for the strain energy release rate $\mathscr{G}$ can be calculated by means of equation (1.1.8). We find that

$$\mathscr{G} = \frac{4(1-\eta^2)p_0{}^2c}{\pi E}, \tag{3.3.3}$$

a result which we can also write as

$$\mathscr{G} = \pi(1-\eta^2)\frac{K^2}{E}. \tag{3.3.4}$$

It is also of interest to calculate the stress field in this particular case. If we put $g(t) = 2p_0 t/\pi$ in equation (3.2.7), we find that

$$\psi(\xi) = \frac{2p_0}{\pi}\left(\frac{\sin\xi - \xi\cos\xi}{\xi^2}\right) = -\frac{2p_0}{\pi}\frac{d}{d\xi}\left(\frac{\sin\xi}{\xi}\right). \tag{3.3.5}$$

If we substitute from equation (3.3.5) into equations (3.2.1) and (3.2.2), we obtain the pair of equations

$$u_\rho = \frac{(1-2\eta)p_0}{\pi}\int_0^\infty \xi^{-2}(\sin\xi - \xi\cos\xi)e^{-\xi z}J_1(\xi\rho)\,d\xi - \frac{p_0 z}{\pi}\int_0^\infty \xi^{-1}(\sin\xi - \xi\cos\xi)e^{-\xi z}J_1(\xi\rho)\,d\xi, \tag{3.3.6}$$

$$u_z = \frac{2(1-\eta)p_0}{\pi}\int_0^\infty \xi^{-2}(\sin\xi - \xi\cos\xi)e^{-\xi z}J_0(\xi\rho)\,d\xi + \frac{p_0 z}{\pi}\int_0^\infty \xi^{-1}(\sin\xi - \xi\cos\xi)e^{-\xi z}J_0(\xi\rho)\,d\xi, \tag{3.3.7}$$

Performing an integration by parts on the first integral in equation (3.3.7) and adopting the notation of George (1962), namely,

$$S(m,n) = \int_0^\infty \xi^n \sin \xi J_m(\xi\rho) e^{-\xi z}\, d\xi,$$

$$C(m,n) = \int_0^\infty \xi^{n-1}(1 - \cos \xi) J_m(\xi\rho) e^{-\xi z}\, d\xi \tag{3.3.8}$$

we find that

$$u_z = \epsilon[1 - zS(0,-1) - \rho S(1,-1)] + \frac{\epsilon z}{2(1-\eta)}[zS(0,0) + \rho S(1,0) - S(0,-1)] \tag{3.3.9}$$

where, in dimensionless units

$$\epsilon = \frac{2(1-\eta)p_0}{\pi},$$

is the value of u_z at the origin $\rho = 0$, $z = 0$. We note that in conventional units $\epsilon = 4(1 - \eta^2)p_0 c/\pi E$, c being the radius of the crack. Putting $z = 0$ in equation (3.3.9) and using the fact that, when $z = 0$ and $0 < \rho < 1$,

$$S(1,-1) = \frac{1 - \sqrt{1-\rho^2}}{\rho} \tag{3.3.10}$$

(Watson, 1944, p. 405), we find that

$$u_z(\rho,0) = \epsilon\sqrt{1-\rho^2}. \tag{3.3.11}$$

The components of the stress tensor can be computed from the set of equations

$$\sigma_{zz} = \frac{2p_0}{\pi}[I(0,1) - S(0,-1) + zI(0,2) - zS(0,0)], \tag{3.3.12}$$

$$\sigma_{\rho z} = \frac{2p_0 z}{\pi}[I(1,2) - S(0,1)], \tag{3.3.13}$$

$$\sigma_{\rho\rho} + \sigma_{zz} + \sigma_{\theta\theta} = \frac{4(1-\eta)p_0}{\pi}[I(0,1) - S(0,-1)], \tag{3.3.14}$$

$$\sigma_{\theta\theta} - \sigma_{\rho\rho} = \frac{2p_0}{\pi}\{(1-2\eta)[I(2,1) - S(2,-1)] - z[I(2,2) - S(2,0)]\}, \tag{3.3.15}$$

where $C(m,n)$ and $S(m,n)$ denote the integrals defined by equations (3.3.8) and

$$I(m,n) = Z(m,n) - C(m,n), \tag{3.3.16}$$

where $Z(m,n)$ is defined by

$$Z(m,n) = \int_0^\infty J_m(\xi\rho)\xi^{n-1}e^{-\xi z}\,d\xi. \tag{3.3.17}$$

The values of the integrals $C(m,n)$, $S(m,n)$ can be taken from George's tables, and those of the integral $Z(m,n)$ can easily be calculated from the formula

$$Z(m,n) = \frac{\rho^m(m+n-1)!}{m!\,2^m(\rho^2+z^2)^{\frac{1}{2}m+\frac{1}{2}n-\frac{1}{2}}} \times {}_2F_1\left(\tfrac{1}{2}m+\tfrac{1}{2}n, \tfrac{1}{2}m-\tfrac{1}{2}n+\tfrac{1}{2}; m+1; \frac{\rho^2}{\rho^2+z^2}\right)$$

(Sneddon, 1956, p. 112). In particular, we have

$$Z(0,1) = r^{-1}, \qquad Z(0,2) = zr^{-3}, \qquad Z(1,2) = \rho r^{-3} \tag{3.3.18}$$

and, as a result of the recurrence formula

$$x[J_0(x) + J_2(x)] = 2J_1(x),$$

we have the relation

$$Z(2,n) = \frac{2}{\rho}Z(1, n-1) - Z(0,n). \tag{3.3.19}$$

With the help of George's tables and the auxiliary formulas (3.3.18) and (3.3.19), we can calculate the value of the maximum shearing stress τ at every point of the solid. Curves showing the variation of each of the stress components with ρ and z are given in Sneddon (1946). The curves showing the variation with ρ and z of the maximum shearing stress τ and of the normal displacement of planes parallel to the central plane of an ellipsoidal crack of small depth are shown in Sneddon's paper and also in Sneddon (1951, pp. 498, 499). Probably the most convenient way of visualizing the distribution of stress in an elastic solid is to draw the surfaces of equal principal shearing stress. These surfaces of revolution which have parametric equations $\tau/p_0 = k$, where k is a parameter, are the three-dimensional analogue of the isochromatic lines of photoelasticity. A section of these surfaces by a plane through the z-axis is shown in Fig. 44. It will be observed that except in the vicinity of the edges of the crack (and apart from numerical values) these contours of equal principal shearing stress are very much like the isochromatic lines of the two-dimensional case (Fig. 9). The main difference between the two systems of curves lies in their behavior near the edges of the crack. In the two-dimensional case the points $x + \pm c$ are nodes separating two loops of the curves (cf. the curves corresponding to the values $\tau/p_0 = 0.4, 0.5, 0.7$ in Fig. 9), but in the three-dimensional case, the

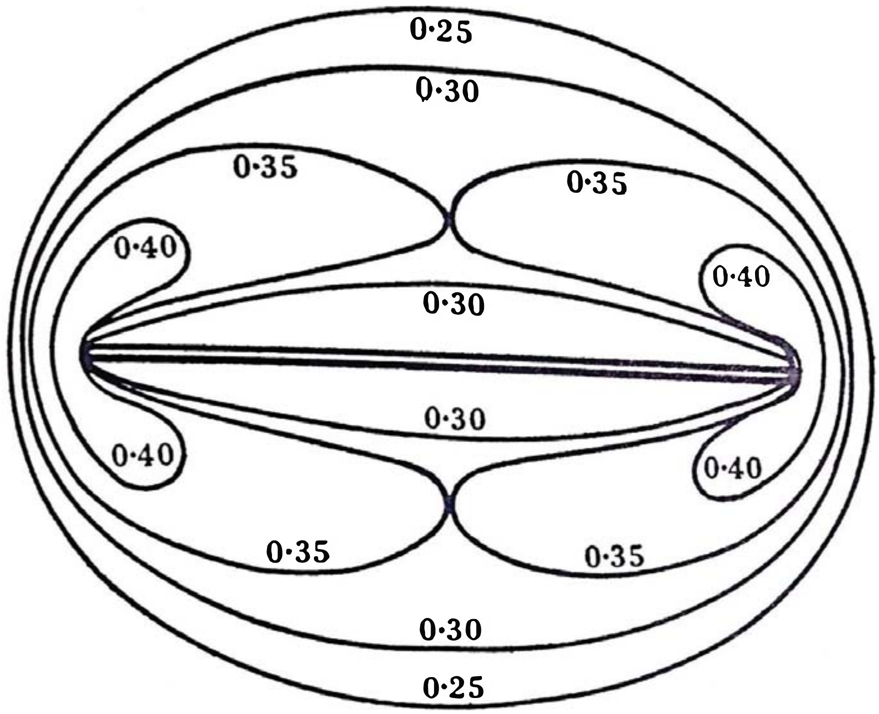


Figure 44 The curves obtained by an intersection with a plane through the z-axis of the surfaces of equal principal shearing stress in the vicinity of a circular crack. The numerical values are the relevant values of τ/p_0.

point $\rho = 1$ is a simple point of the curve, the curve cutting the r-axis again at a point where $\rho > 1$ (cf. curves with r/p_0 in Fig. 44). It should, of course, be remembered that these "contours" are actually surfaces of equal principal shearing stress obtained by rotating Fig. 36 about the z-axis.

As in the two-dimensional case the principal shearing stress becomes infinite at the edges of the crack, indicating that a certain amount of plastic flow will occur and hence that there is in reality no purely elastic problem. Except in the neighborhood of the crack, however, the elastic stresses are dominant and the energy relation (3.3.2) very closely approximates the true result.

In certain applications of these calculations (e.g., in the theory of fracture) it is useful to have expressions for the stress components at points in the immediate vicinity of the periphery of the crack. Taking the radius of the crack to be unity and writing

$$\rho = 1 + \delta \cos \psi, \qquad z = \delta \sin \psi$$

we obtain results which are analogous to those of section 2.4. It is readily shown from the formulas for the integrals involved that

$$S(1,0) = I(0,1) = -I(2,1) = (2\delta)^{-\frac{1}{2}} \cos \tfrac{1}{2}\psi,$$
$$I(1,2) = (2\delta)^{-\frac{3}{2}} \cos \tfrac{3}{2}\psi,$$
$$I(0,2) = -I(2,2) = (2\delta)^{-\frac{3}{2}} \sin \tfrac{3}{2}\psi,$$
$$S(0,0) = -S(2,0) = (2\delta)^{-\frac{1}{2}} \sin \tfrac{1}{2}\psi,$$

the other integrals being negligible to this approximation. With these values for the integrals involved, we obtain from equations (3.3.12) through (3.3.15) the expressions

$$
\begin{aligned}
\sigma_{\rho\rho} &= \frac{2p_0}{\pi}(2\delta)^{-1/2}[\tfrac{3}{4}\cos\tfrac{1}{2}\psi + \tfrac{1}{4}\cos\tfrac{5}{2}\psi],\\
\sigma_{zz} &= \frac{2p_0}{\pi}(2\delta)^{-1/2}[\tfrac{5}{4}\cos\tfrac{1}{2}\psi - \tfrac{1}{4}\cos\tfrac{5}{2}\psi],\\
\sigma_{z\rho} &= \frac{p_0}{\pi}(2\delta)^{-1/2}\sin\psi\cos\tfrac{3}{2}\psi,
\end{aligned}
\tag{3.3.20}
$$

for the stresses $\sigma_{\rho\rho}$, σ_{zz}, $\sigma_{\rho z}$ in the vicinity of the periphery of the crack. It will be observed that these formulas differ from the expressions for the components of stress in the two-dimensional case only by the presence of a factor $2/\pi$. There is no parallel in the two-dimensional analysis to the hoop stress $\sigma_{\theta\theta}$; in the three-dimensional case its value near the edge of the crack is given by

$$
\sigma_{\theta\theta} = \frac{4\eta p_0}{\pi}(2\delta)^{-1/2}\cos\tfrac{1}{2}\psi. \tag{3.3.21}
$$

The distribution of stress in the vicinity of a crack can also be illustrated by calculating the normal component of the displacement vector for points in the elastic body and then drawing the normal displacement of planes which were originally parallel to the plane of the crack. By means of the formula (3.3.9) the variation of the normal displacement along the planes $z = 0.05, 0.2, 0.4$ was calculated and the results incorporated in Fig. 11 of Sneddon (1946). Approximate formulas for u_z in planes close to the crack can be derived from equation (3.3.9). If $\rho \ll 1$ we can write

$$
J_0(\rho\xi) = 1 - \tfrac{1}{4}\rho^2\xi^2, \qquad J_1(\rho\xi) = \rho\xi
$$

in the expression for $S(m,n)$ and obtain approximate expressions for the values of these integrals at points just off the axis of symmetry $\rho = 0$. Substituting these approximate forms into equation (3.3.9) we obtain

$$
u_z = \epsilon[f_1(z) - \rho^2 f_2(z)] + O(\epsilon\rho^4), \quad p \ll 1 \tag{3.3.22}
$$

where the functions $f_1(z)$ and $f_2(z)$ are given by the equations

$$
\begin{aligned}
f_1(z) &= 1 - \frac{(1-2\eta)}{2(1-\eta)}z\left(\frac{\pi}{2} - \tan^{-1} z\right) - \frac{z^2}{2(1-\eta)(1+z^2)},\\
f_2(z) &= \left[\frac{1}{2} - \frac{3-\eta}{2(1-\eta)}z^2\right](1+z^2)^{-3}.
\end{aligned}
\tag{3.3.23}
$$

By means of equation (3.3.22) we may calculate u_z at points just off the axis of symmetry $\rho = 0$.

Similarly, by taking ρ to be large (i.e., $\rho \gg 1$) and $z \ll \rho$ we obtain, from the explicit expressions for $S(m,n)$, the approximate formula

$$u_z = \frac{\epsilon z}{\rho}\left[1 - \frac{z}{2(1-\eta)}\right] + O\left(\frac{\epsilon}{\rho^2}\right). \tag{3.3.24}$$

The value of w in the neighborhood of the periphery of the crack can be determined by the method outlined above. We find that

$$u_z = \epsilon\sqrt{2\delta}\,\sin \tfrac{1}{2}\psi\left[1 - \frac{1+\cos\psi}{4(1-\eta)}\right], \quad \delta \ll 1, \tag{3.3.25}$$

where $z = \delta \sin \psi$ and $\rho = 1 + \delta \cos \psi$.

The relation of these approximate formulas to the exact expression for u_z is shown in Fig. 45 where the component u_z for points on the plane $z = 0.05$ is plotted; the full line gives the value of u_z as calculated from the exact expression in the case $\eta = 0.25$. The broken lines show the values given by the approximate formulas, for the same value of η. It will be observed that the agreement is good in the range specified.

A similar method of solution has been devised by Sneddon and Tweed (1967*a*) for the problem of calculating the stress intensity factor for a penny-shaped crack in an elastic body under the action of symmetrically distributed body forces.

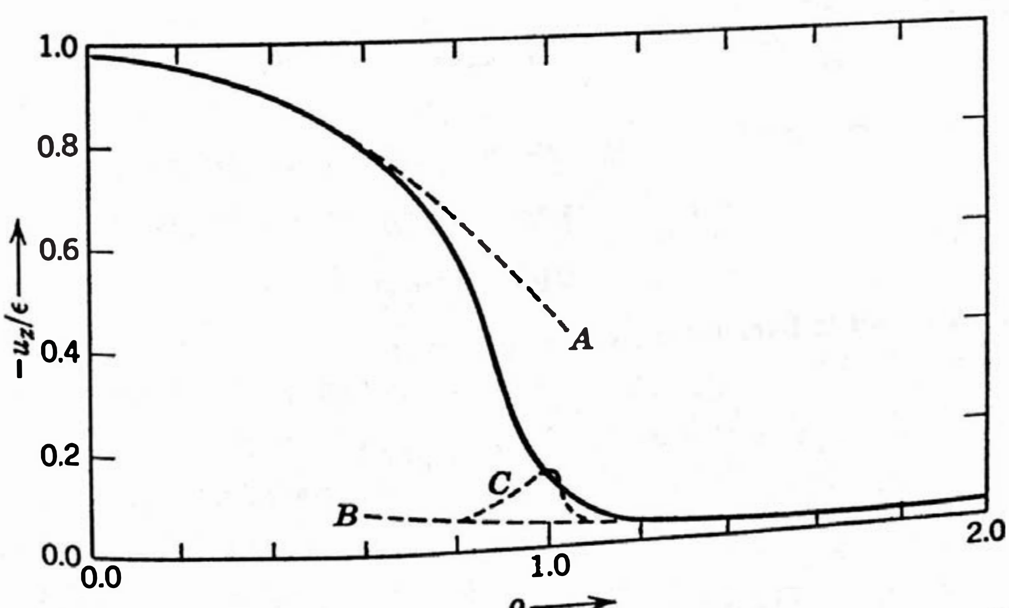


Figure 45 The variation of the normal displacement u_z. The full curve shows the exact value; the broken curves correspond to the approximate formulas: A to (3.3.23), B to (3.3.24), and C to (3.3.25).

There has also been a limited amount of work on dynamical problems concerning penny-shaped cracks. The problem of the expanding circular crack has been considered by Craggs (1966); see also Atkinson (1968). The response of a penny-shaped crack to a loading in the form of a plane harmonic dilatational wave propagating along the axis of the crack has been discussed by Mal (1968*a*); the response to an incident plane harmonic shear wave polarized in a plane normal to the plane of the crack and propagating along the axis of the crack has been investigated by Mal (1968*b*).

3.4 OTHER METHODS OF SOLVING THE PROBLEM OF THE PENNY-SHAPED CRACK

(a) Green's Solution

If we let $\chi = \phi = 0$, $\phi = \frac{1}{2}(\beta^2 - 1)^{-1}f(\rho,z)$ in equations (1.3.1), (1.3.6) we see that the displacement field

$$u_\rho = \frac{1}{2(\beta^2 - 1)} \cdot \frac{\partial f}{\partial \rho} + \tfrac{1}{2}z \frac{\partial^2 f}{\partial \rho\, \partial z}, \qquad u_z = -\frac{\beta^2}{2(\beta^2 - 1)} \cdot \frac{\partial f}{\partial z} + \tfrac{1}{2}z \frac{\partial^2 f}{\partial z^2} \tag{3.4.1}$$

will give a solution of the boundary value problem for the half-space $z \geq 0$ set by equations (3.1.15) through (3.1.17) provided that we can find a harmonic function $f(\rho,z)$ satisfying the boundary conditions

$$\frac{\partial^2 f}{\partial z^2} = p(\rho), \quad 0 \leq \rho \leq 1, \quad z = 0, \tag{3.4.2}$$

$$\frac{\partial f}{\partial z} = 0, \qquad \rho \geq 1, \quad z = 0. \tag{3.4.3}$$

Since $f(\rho,z)$ is a harmonic function, we have the relation

$$\frac{\partial^2 f}{\partial z^2} = -\frac{\partial^2 f}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial f}{\partial \rho},$$

so that if we can find a function $h(\rho)$ such that

$$\frac{d^2 h}{d\rho^2} + \frac{1}{\rho}\frac{dh}{d\rho} = -p(\rho), \tag{3.4.4}$$

we can replace equation (3.4.2) by

$$f(\rho,0) = h(\rho), \quad 0 \geq \rho \geq 1. \tag{3.4.5}$$

This form of the boundary value problem and its solution is due to Green (1949); see also Green and Zerna (1954, pp. 172–178). To solve the boundary value problem posed by equations (3.4.3) and (3.4.5) we consider the real potential function

$$f(\rho,z) = \frac{1}{2}\int_0^1 \frac{g(t)\,dt}{\sqrt{[\rho^2 + (z + it)^2]}} + \frac{1}{2}\int_0^1 \frac{g(t)\,dt}{\sqrt{[\rho^2 + (z - it)^2]}}, \tag{3.4.6}$$

where $g(t)$ is a real continuous function of t in (0,1). It can be shown that, whatever the form of $g(t)$, this function satisfies (3.4.3) and that the appropriate form of $g(t)$ to satisfy (3.4.5) is determined by the integral equation

$$\int_0^\rho \frac{g(t)\,dt}{\sqrt{\rho^2 - t^2}} = h(\rho), \quad 0 \leq \rho \leq 1$$

so that

$$g(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{\rho h(\rho)\,d\rho}{\sqrt{t^2 - \rho^2}}. \tag{3.4.7}$$

If we substitute this value of $g(t)$ into equation (3.4.6) we obtain a function $f(\rho,z)$ which, together with equations (3.4.1), furnish a solution to the problem.

The surface displacement of the crack is given by

$$u_z(\rho,0) = -\frac{\beta^2}{2(\beta^2 - 1)}\left[\frac{\partial f}{\partial z}\right]_{z=0}$$

so that

$$u_z(\rho,0) = -\frac{\beta^2}{2(\beta^2 - 1)\rho}\frac{\partial}{\partial\rho}\int_\rho^1 \frac{tg(t)\,dt}{\sqrt{t^2 - \rho^2}}. \tag{3.4.8}$$

For example, if $p(\rho) = p_o$, the appropriate solution of (3.4.4) is

$$h(\rho) = \tfrac{1}{4}p_0(c_1 - \rho^2) \tag{3.4.9}$$

where c_1 is a constant whose value we shall determine later. If we substitute from equation (3.4.9) into equation (3.4.7) we find that

$$g(t) = \frac{p_0}{2\pi}(c_1 - 2t^2). \tag{3.4.10}$$

If we substitute from equation (3.4.10) into (3.4.8) we find that

$$u_z(\rho,0) = \frac{\beta^2 p_0}{(\beta^2 - 1)\pi}\left[\sqrt{1 - \rho^2} + \frac{c_1 - 2}{4\sqrt{1 - \rho^2}}\right].$$

Now as $p \to 1-$, $u_z(\rho,0)$ must tend to zero so that we must take $c_1 = 2$, in which case

$$g(t) = \frac{p_0}{\pi}(1 - t^2). \tag{3.4.11}$$

The solution of the problem is thus reduced to the straightforward calculation of integrals.

Alternatively, we can show that

$$f(\rho,z) = \frac{ip_0}{4\pi}\left[(\rho^2 - 2z^2 - 2)\log\frac{R_2 + z + i}{R_1 + z - i} + 3z(R_2 - R_1) - i(R_2 + R_1)\right],$$

where R_1 and R_2 are defined by the relations

$$R_1 = \sqrt{\rho^2 + (z - i)^2}, \qquad R_2 = \sqrt{\rho^2 + (z + i)^2}.$$

The relevant physical quantities can then be derived from equations (3.4.1).

(b) Collins' Solution

Closely related to Green's solution is one due to Collins (1961) who considers the more general problem in which the surfaces of a penny-shaped crack are subjected to a variable internal pressure and a variable shear stress so that

$$\sigma_{zz}(\rho,0) = f_1(\rho), \qquad \sigma_{\rho z}(\rho,0) = f_2(\rho), \quad 0 \le \rho \le 1.$$

The problem can be split into two parts:

$$\text{I.}\quad \sigma_{zz}(\rho,0) = f_1(\rho), \qquad \sigma_{\rho z}(\rho,0) = 0, \qquad 0 \le \rho \le 1; \tag{3.4.12}$$

$$\text{II.}\quad \sigma_{zz}(\rho,0) = 0, \qquad \sigma_{\rho z}(\rho,0) = f_2(\rho), \quad 0 \le \rho \le 1. \tag{3.4.13}$$

To obtain the solution of Problem I we put

$$\Psi = \chi + \phi, \qquad \Phi = -(\beta^2 - 1)\frac{\partial\phi}{\partial z}$$

in the solution of Section 1.3 to get the displacement field

$$u_\rho = \frac{\partial\Psi}{\partial\rho} - z\frac{\partial\Phi}{\partial\rho}, \qquad u_z = \frac{\partial\Psi}{\partial z} + (3 - 4\eta)\Phi - z\frac{\partial\Psi}{\partial z}. \tag{3.4.14}$$

If we now choose Φ and Ψ such that

$$(1 - 2\eta)\Phi = -\frac{\partial\Psi}{\partial z}, \tag{3.4.15}$$

we find that the stresses are given by the equations

$$\sigma_{\rho\rho} = 2\left[2\eta\frac{\partial\Phi}{\partial z} - z\frac{\partial^2\Phi}{\partial\rho^2} + \frac{\partial^2\Psi}{\partial\rho^2}\right]$$

$$\sigma_{00} = 2\left[2\eta\frac{\partial\Phi}{\partial z} - \frac{z}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{1}{\rho}\frac{\partial\Psi}{\partial\rho}\right],$$

$$\sigma_{zz} = 2\left[\frac{\partial\Phi}{\partial z} - z\frac{\partial^2\Phi}{\partial z^2}\right],$$

$$\sigma_{\rho z} = -2z\frac{\partial^2\Phi}{\partial\rho\,\partial z}.$$

On the crack we therefore have the condition

$$2\frac{\partial\Phi}{\partial z} = f_1(\rho), \quad 0 \leq \rho \leq 1, \quad z = 0.$$

If we take the representation

$$\Phi(\rho,z) = \frac{1}{2i}\int_{-1}^{1}\frac{g_1(t)\,dt}{\sqrt{\rho^2 + (z+it)^2}}, \tag{3.4.16}$$

where it is assumed that $g_1(t)$ is an *odd* function of t, then

$$2i\int_0^\rho u\frac{\partial\Phi}{\partial z}\,du = \int_{-1}^{1}\frac{(z+it)g_1(t)\,dt}{\sqrt{\rho^2+(z+it)^2}}.$$

Letting $z \to 0$ we find that $g_1(t)$ is determined as the solution of the integral equation

$$\int_0^\rho \frac{tg_1(t)\,dt}{\sqrt{\rho^2 - t^2}} = \frac{1}{2}\int_0^\rho xf_1(x)\,dx,$$

so that it is given by the equation

$$g_1(t) = \frac{1}{\pi}\int_0^t \frac{\rho f_1(\rho)\,d\rho}{\sqrt{t^2-\rho^2}}. \tag{3.4.17}$$

From equation (3.4.15) we find that

$$\Psi(\rho,z) = \tfrac{1}{2}(1-2\eta)i\int_{-1}^{1} g_1(t)\log[z + it + \sqrt{\rho^2 + (z+it)^2}]\,dt. \tag{3.4.18}$$

The solution of Problem I is then given by the set of equations (3.4.14), (3.4.16), (3.4.17), and (3.4.18).

For the solution of Problem II, we again take (3.4.14) but this time we replace the relation (3.4.15) by

$$2(1-\eta)\Phi = -\frac{\partial\Psi}{\partial z} \tag{3.4.19}$$

so that the displacement field is given by the pair of equations

$$u_\rho = \frac{\partial\Psi}{\partial\rho} - z\frac{\partial\Phi}{\partial\rho}, \qquad u_z = (1-2\eta)\Phi - z\frac{\partial\Phi}{\partial z}. \tag{3.4.20}$$

We find also that

$$\sigma_{zz} = -2z\frac{\partial^2\Phi}{\partial z^2}, \qquad \sigma_{\rho z} = -2\left[\frac{\partial\Phi}{\partial\rho} + z\frac{\partial^2\Phi}{\partial\rho\,\partial z}\right]$$

so that we must have

$$\Phi(\rho,0) = C - \frac{1}{2}\int_0^\rho f_2(x)\,dx, \quad 0 \le \rho \le 1.$$

where C is an arbitrary constant.

If we take

$$\Phi(\rho,z) = \frac{1}{2}\int_{-1}^{1} \frac{g_2(t)\,dt}{\sqrt{\rho^2 + (z + it)^2}}, \tag{3.4.21}$$

where $g_2(t)$ is an *even* function of t, then

$$\int_0^\rho \frac{g_2(t)\,dt}{\sqrt{\rho^2 - t^2}} = C - \frac{1}{2}\int_0^\rho f_2(x)\,dx$$

so that

$$g_2(t) = \frac{1}{\pi}\left[2C - \frac{d}{dt}\int_0^t f_2(\rho)\sqrt{(t^2 - \rho^2)}\,d\rho\right]. \tag{3.4.22}$$

Now we can show that, as $z \to 0+$,

$$\frac{\partial \Psi}{\partial \rho} \to -\frac{1-\eta}{\rho}\int_{-1}^{1} g_2(t)\,dt, \quad \rho > 1$$

while, as $z \to 0-$,

$$\frac{\partial \Psi}{\partial \rho} \to +\frac{1-\eta}{\rho}\int_{-1}^{1} g_2(t)\,dt, \quad \rho > 1$$

so that if u_ρ is continuous over $z = 0$, $\rho > 1$ we must have

$$\int_0^1 g_2(t)\,dt = 0. \tag{3.4.23}$$

Substituting the form (3.4.22) for $g_2(t)$ we see that the arbitrary constant C must be assigned the value

$$C = \frac{1}{2}\int_0^1 f(\rho)\sqrt{(1 - \rho^2)}\,d\rho. \tag{3.4.24}$$

The function $\Phi(\rho,z)$ is therefore determined by the equations (3.4.21), (3.4.22), and (3.4.24); the function $\Psi(\rho,z)$ is similarly given by the formula

$$\Psi(\rho,z) = -(1-\eta)\int_{-1}^{1} g_2(t)\log[z + it + \sqrt{\rho^2 + (z + it)^2}]\,dt. \tag{3.4.25}$$

(c) Payne's Solution

If in equations (1.3.1) and (1.3.6) we write

$$\Phi(\rho,z) = 2(\beta^2 - 1)\frac{\partial \phi}{\partial z},$$

where $\Phi(\rho,z)$ is a harmonic function, and introduce the associate function $\Psi(\rho,z)$ defined by the equations

$$\rho\frac{\partial\Phi}{\partial z} = \frac{\partial\Psi}{\partial\rho}, \qquad \rho\frac{\partial\Phi}{\partial\rho} = -\frac{\partial\Psi}{\partial z},$$

we obtain Payne's solution

$$u_\rho = -\frac{1}{2\rho}\left[(1-2\eta)\Psi + z\frac{\partial\Psi}{\partial z}\right] \tag{3.4.26}$$

$$u_z = -(1-\eta)\Phi + \frac{z}{2}\frac{\partial\Phi}{\partial z} \tag{3.4.27}$$

of the equations of elastic equilibrium (Payne, 1953). The corresponding components of the stress tensor can be calculated from the equations

$$\sigma_{\rho\rho} + \sigma_{\theta\theta} = -(1+2\eta)\frac{\partial\Phi}{\partial z} - z\frac{\partial^2\Phi}{\partial z^2} \tag{3.4.28}$$

$$\sigma_{\rho\rho} - \sigma_{\theta\theta} = -(1-2\eta)\left[\frac{1}{\rho}\frac{\partial\Psi}{\partial\rho} - \frac{2}{\rho^2}\Psi\right] + z\frac{\partial^2\Phi}{\partial\rho^2} + \frac{2z}{\rho}\frac{\partial\Phi}{\partial\rho} \tag{3.4.29}$$

$$\sigma_{\rho z} = z\frac{\partial^2\Phi}{\partial\rho\,\partial z} \tag{3.4.30}$$

$$\sigma_{zz} = -\frac{\partial\Phi}{\partial z} + z\frac{\partial\Phi}{\partial z}. \tag{3.4.31}$$

It follows from these equations that we can solve the boundary value problem expressed by equations (3.1.15) through (3.1.17) if we can find a harmonic function $\Phi(\rho,z)$ satisfying the boundary conditions:

On $z = 0$:

$$\frac{\partial\Phi}{\partial z} = p(\rho), \quad 0 < \rho < 1, \tag{3.4.32}$$

$$\Phi = 0, \qquad \rho > 1. \tag{3.4.33}$$

Payne solves this boundary value problem by expressing the function Φ in terms of oblate spheroidal coordinates defined by the equation

$$z + i\rho = \sinh(\xi + i\eta). \tag{3.4.34}$$

The disk $\rho \leq 1$, $z = 0$ is then given by $\xi = 0$ and its exterior in the z-plane is given by $\eta = \pi/2$. In terms of these coordinates Φ can be expressed in the form of an infinite series

$$\Phi = \sum_{n=0}^{\infty} A_{2n+1}P_{2n+1}(\cos\eta)Q_{2n+1}(i\sinh\xi). \tag{3.4.35}$$

Since $P_{2n+1}(0) = 0$ it follows that $\Phi = 0$ when $\eta = \pi/2$ so that the condition (3.4.33) is automatically satisfied. Also when $\xi = 0$,

$$\frac{\partial \xi}{\partial z} = \frac{1}{\cos \eta}, \qquad \frac{\partial \eta}{\partial z} = 0$$

so that on the disk $\xi = 0$ we have

$$\frac{\partial \Phi}{\partial z} = \frac{1}{\cos \eta} \sum_{n=0}^{\infty} A_{2n+1} P_{2n+1}(\cos \eta) Q_{2n+1}(+i0). \tag{3.4.36}$$

This type of representation is obviously suitable for the discussion of problems in which the function $p(\rho)$ can be expanded in a series of the type

$$p(\rho) = \sum_{n=0}^{\infty} a_n (1 - \rho^2)^n. \tag{3.4.37}$$

Since, on the disk itself, $\xi = 0$, $\rho = \sin \eta$ we find that

$$p(\rho) = \sum_{n=0}^{\infty} a_n \cos^{2n} \eta \tag{3.4.38}$$

so that equation (3.4.35) will be the solution of the problem posed by equations (3.4.32) and (3.4.33) if we can find constants A_{2n+1} such that

$$\sum_{n=0}^{\infty} A_{2n+1} Q_{2n+1}(+i0) P_{2n+1}(\cos \eta) = \sum_{m=0}^{\infty} a_m \cos^{2m+1} \eta. \tag{3.4.39}$$

From the theory of Legendre series (Hobson, 1931, p. 42) we therefore have

$$A_{2n+1} Q_{2n+1}(+i0) = (2n + \tfrac{3}{2}) \sum_{m=0}^{\infty} a_m \int_{-1}^{1} x^{2m+1} P_{2n+1}(x)\, dx.$$

The integral occurring on the right side of this equation is a standard form (Sneddon, 1956, p. 57) and we obtain the equation

$$A_{2n+1} Q_{2n+1}(+i0) = (2n + \tfrac{3}{2}) \sum_{m=n}^{\infty} \frac{(2m+1)!\, \Gamma(m - n + \frac{1}{2}) a_m}{2^{2n+1} (2m - 2n)!\, (m + n + \frac{5}{2})}. \tag{3.4.40}$$

by means of which to calculate A_{2n+1}.

The case of constant pressure p_o is, of course, particularly simple. In this instance we must choose the A_{2n+1} in such a way that

$$\sum_{n=0}^{\infty} A_{2n+1} Q_{2n+1}(+i0) P_1(\cos \eta) = p_0 P_1(\cos \eta)$$

so that $A_3, A_5, A_7, \ldots$ are all zero and

$$A_1 = \frac{p_0}{Q_1(+i0)}.$$

Now from Hobson (1931, pp. 52, 73) we have

$$Q_1(\mu \pm i0) = \mu \log \frac{1+\mu}{1-\mu} - 1 \pm i\pi\mu, \quad |\mu| < 1,$$

so that letting $\mu \to 0$ we find that $Q_1(+i0) = -1$ and so

$$A_1 = -p_0, \tag{3.4.41}$$

giving the solution

$$\Phi = -p_0 P_1(\cos \eta) Q_1(i \sinh \xi);$$

that is,

$$\Phi = -p_0 \cos \eta \; Q_1(i \sinh \xi). \tag{3.4.42}$$

Payne also discusses the special case in which

$$p(\rho) = -p_0[\sqrt{(a^2 - \rho^2)} - \sqrt{(a^2 - 1)}], \quad \rho < 1,$$

corresponding to a spherical distribution of pressure. In this case

$$\Phi = \frac{p_0}{2\pi} \sum_{n=0}^{\infty} \frac{\Gamma(n - \frac{1}{2})\Gamma(n + \frac{3}{2})}{(a^2 - 1)^{n-1/2}} \sum_{m=0}^{n} \frac{(-1)^m (4n - 4m + 3)(n - m)!}{m!\, \Gamma(2n - m + \frac{5}{2})\Gamma(n - m + \frac{3}{2})}$$
$$\times P_{2n-2m+1}(\cos \eta) Q_{2n-2m+1}(i \sinh \xi).$$

If we introduce the toroidal transformation

$$z + i\rho = -\cot \tfrac{1}{2}(\xi + i\eta) = \frac{i \sinh \eta - \sin \xi}{\cosh \eta - \cos \xi} \tag{3.4.43}$$

it is possible to express a harmonic function Φ in the form

$$\sqrt{(\cosh \eta - \cos \xi)} K_\alpha(\cosh \eta)[A(\alpha)\sinh(\alpha\xi) + B(\alpha)\cosh(\alpha\xi)] \tag{3.4.44}$$

where $K_\alpha(s)$, $(s = \cosh \eta)$ is a Legendre function of the Mehler type

$$K_\alpha(s) = P_{i\alpha - 1/2}(s). \tag{3.4.45}$$

For the coordinate system (3.4.43) we note that as z tends to zero through positive values, $\xi \to -\pi$ for $\rho < 1$ and $\xi \to 0$ for $\rho > 1$.

To satisfy the condition (3.4.33) we therefore assume an expression for Φ of the type

$$\Phi = \sqrt{s - \cos \xi} \int_0^\infty H(\alpha) \frac{\tanh a\pi}{\cosh a\pi} \sinh \alpha\xi K_\alpha(s)\, d\alpha. \tag{3.4.46}$$

Now when $\xi = -\pi$,

$$\frac{\partial \xi}{\partial z} = (s + 1),$$

so that on the disk

$$\frac{\partial \Phi}{\partial z} = (s + 1)\left(\frac{\partial \Phi}{\partial \xi}\right)_{\xi = -\pi}.$$

Hence from the boundary condition (3.4.32) we have

$$(s+1)^{3/2}\int_0^\infty \alpha H(\alpha)\tanh \alpha\pi K_\alpha(s)\,d\alpha = h(s),$$

where $h(s)$ denotes the value of $p[\rho(-\pi,\eta)] = p\left[\sinh \dfrac{\eta}{(s+1)}\right]$. Using the Mehler-Fock inversion theorem* we find that the unknown function $H(\alpha)$ is given by the equation

$$H(\alpha) = \int_1^\infty \frac{h(s)K_\alpha(s)\,ds}{(s+1)^{3/2}}. \tag{3.4.47}$$

If we substitute from equation (3.4.47) into equation (3.4.46) we obtain the required solution.

Payne has carried out the calculations in the case of an oblate spheroidal distribution of pressure; that is,

$$p(\rho) = p_0\sqrt{1-\rho^2}$$

so that

$$h(s) = \frac{2p_0}{s+1}.$$

The resulting expression for Φ is

$$\Phi = \frac{\pi p_0}{8}\sqrt{(s-\cos\xi)}\int_0^\infty (1+4\alpha^2)\tanh\alpha\pi\cdot\operatorname{sech}^2\alpha\pi\cdot\sinh\alpha\xi K_\alpha(s)\,d\alpha.$$

(d) Kobayashi's Solution

The problem of the penny-shaped crack can be solved by a method developed by Kobayashi (1931) for the solution of the corresponding boundary value problem in hydrodynamics. This solution is of interest even though it is not useful for numerical calculations. Working in cylindrical coordinates (ρ,θ,z) we assume, as before, that the crack is a disk of unit radius lying in the z-plane and that it is opened up by the application of a nonsymmetrical pressure $p(\rho,\theta)$.

If, in the equations of Section 1.3, we take $\chi = \psi = 0$ and write

$$\Phi = 2(\beta^2-1)\frac{\partial\psi}{\partial z},$$

* The Mehler-Fock inversion theorem states that a continuous function $f(s)$ which tends to zero faster than $1/\sqrt{s}$ as $s \to \infty$ admits the integral representation

$$f(s) = \int_0^\infty pC(p)\tanh(\pi p)K_p(s)\,dp$$

where

$$C(p) = \int_1^\infty f(s)K_p(s)\,ds.$$

then it follows from equations (2.14) and (2.15) that we can satisfy the following boundary conditions on $z = 0$:

$$\sigma_{zz} = -p(\rho,\theta), \quad 0 \leq \rho \leq 1,$$
$$u_z = 0, \quad \rho > 1,$$
$$\sigma_{\rho z} = 0, \quad \rho > 0$$

provided we can find a solution of Laplace's equation

$$\frac{\partial^2\Phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{1}{\rho^2}\cdot\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2} = 0, \tag{3.4.48}$$

satisfying the boundary conditions

$$\frac{\partial\Phi}{\partial z} = -p(\rho,\theta) \quad 0 \leq \rho \leq 1, \quad z = 0; \tag{3.4.49}$$

$$\Phi = 0, \quad \rho > 1, \quad z = 0. \tag{3.4.50}$$

We now introduce the function

$$\Phi = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(A_m^n \cos m\theta + B_m^n \sin m\theta)\int_0^\infty J_m(\rho\xi)J_{m+2n+3/2}(\xi)e^{-\xi z}\frac{d\xi}{\sqrt{\xi}} \tag{3.4.51}$$

for which

$$\frac{\partial\Phi}{\partial z} = -\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(A_m^n \cos m\theta + B_m^n \sin \theta)\int_0^\infty \xi^{1/2}J_m(\rho\xi)J_{m+2n+3/2}(\xi)e^{-\xi z}\,d\xi. \tag{3.4.52}$$

From the Weber-Schaftheitlin integral (Watson, 1944, p. 402) we know that if n is an integer,

$$\int_0^\infty \xi^{1-k}J_{\nu+2n+k}(\xi)J_\nu(\rho\xi)\,d\xi = 0, \qquad \text{if } \rho > 1,$$

so that taking $\nu = m$, $k = \frac{3}{2}$ we see that the form (3.4.51) for Φ automatically satisfies the equation (3.4.50). Further, from the Weber-Schaftheilin integral, we know that

$$\int_0^\infty \sqrt{\xi}\,J_m(\rho\xi)J_{m+2n+3/2}(\xi)\,d\xi = \frac{\sqrt{2}\,\rho^m\Gamma(m+n+\frac{3}{2})}{m!\,n!}\mathscr{F}_n(m+\tfrac{3}{2}, m+1, \rho^2), \tag{3.4.53}$$

$0 < \rho < 1$, where $\mathscr{F}_n(a,b,x)$ denotes the Jacobi polynomial ${}_2F_1(-n, a+n; b; x)$. If we substitute from (3.4.52) and (3.4.53) into (3.4.49) we see that the function Φ defined by (3.4.51) will be a solution if the constants A_m^n, B_m^n are chosen so that

$$\rho(\sqrt{x},\theta) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sqrt{2}\,(A_m^n \cos m\theta + B_m^n \sin m\theta)$$
$$\times \frac{\Gamma(m+n+\frac{3}{2})}{m!\,n!}\, x^{\frac{1}{2}m} \mathscr{F}_n(m+\tfrac{3}{2}, m+1, x). \quad (3.4.54)$$

Now the orthogonality relation for the Jacobi polynomials $\mathscr{F}_n(\alpha,\gamma,x)$ is

$$\int_0^1 x^{\gamma-1}(1-x)^{\alpha-\gamma} \mathscr{F}_n \mathscr{F}_{m'}\, dx = \frac{\Gamma(\gamma)\Gamma(\alpha+1-\gamma+n)n!}{(\gamma)_n(\alpha+2n)\Gamma(\alpha+n)}\,\delta_{nm'} \quad (3.4.55)$$

provided $\mathrm{Re}(\gamma) > 0$, $\mathrm{Re}(\alpha-\gamma) > -1$, $\delta_{nm'}$ denotes the Kronecker delta and

$$(\gamma)_n = \gamma(\gamma+1)\cdots(\gamma+n-1) = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)},$$

(Magnus and Oberhettinger, 1949, p. 83) and this may be written in the form

$$\frac{\Gamma(m+n+\frac{3}{2})}{m!\,n!}\int_0^1 x^m \sqrt{(1-x)}\,\mathscr{F}_n(m+\tfrac{3}{2}, m+1, x)\mathscr{F}_{n'}(m+\tfrac{3}{2}, m+1, x)\,dx$$
$$= \frac{m!\,\Gamma(n+\frac{3}{2})}{(m+n)!\,(m+2n+\frac{3}{2})}\,\delta_{nn'}. \quad (3.4.56)$$

If we multiply both sides of equation (3.4.54) by

$$x^{m/2}\sqrt{(1-x)}\cos m\theta\,\mathscr{F}_n(m+\tfrac{3}{2}, m+1, x)$$

and integrate with respect to θ and x we obtain the expression

$$A_m^n = \frac{(m+2n+\frac{3}{2})(m+n)!}{\sqrt{2\pi}\,\Gamma(n+\frac{3}{2})}\int_0^{2\pi} \cos m\theta\, d\theta$$
$$\times \int_0^1 x^{m/2}\sqrt{(1-x)}\; p(\sqrt{x},\theta)\mathscr{F}_n(m+\tfrac{3}{2}, m+1, x)\,dx$$

Similarly, we obtain the expressions

$$B_m^n = \frac{(m+2n+\frac{3}{2})\cdot(m+n)!}{\sqrt{2\pi}\,\Gamma(n+\frac{3}{2})}\int_0^{2\pi} \sin m\theta\, d\theta$$
$$\times \int_0^1 x^{m/2}\sqrt{(1-x)}\; p(\sqrt{x},\theta)\mathscr{F}_n(m+\tfrac{3}{2}, m+1, x)\,dx$$

$$A_0^n = \frac{(2n+\frac{3}{2})n!}{2\sqrt{2\pi}\,\Gamma(n+\frac{3}{2})}\int_0^{2\pi} d\theta \int_0^1 \sqrt{(1-x)}\; p(\sqrt{x},\theta)\mathscr{F}_n(\tfrac{3}{2},1,x)\,dx.$$

In Kobayashi's form, these can be written as

$$\begin{matrix} A_m^n \\ B_m^n \end{matrix} = \frac{(m + 2n + \frac{3}{2})}{\sqrt{2\pi}\,\Gamma(n + \frac{3}{2})} \int_0^{2\pi} \begin{matrix} \cos \\ \sin \end{matrix} m\theta\, d\theta$$

$$\times \int_0^1 x^{-m/2} p(\sqrt{x},\theta) \frac{d^n}{dx^n} [x^{m+n}(1 - x)^{n-\frac{1}{2}}]\, dx, \quad (m \neq 0),$$

$$A_0^n = \frac{2n + \frac{3}{2}}{2\sqrt{2\pi}\,\Gamma(n + \frac{3}{2})} \int_0^{2\pi} d\theta \int_0^1 p(\sqrt{x},\theta) \frac{d^n}{dx^n} [x^2(1 - x)^{n+\frac{1}{2}}]\, dx.$$

In particular, if

$$p(\rho,\theta) = p_0$$

then all these constants are zero except A_0^0 which has the value

$$A_0^0 = \sqrt{\frac{2}{\pi}} \cdot p_0$$

so that we obtain the solution

$$\Phi = \frac{2}{\pi} p_0 \int_0^\infty \frac{J_0(\rho\xi) J_{3/2}(\xi)}{\sqrt{\xi}} e^{-\xi z}\, d\xi.$$

Using the fact that

$$J_{3/2}(\xi) = \sqrt{\frac{2}{\pi\xi}} \left(\frac{\sin \xi - \xi \cos \xi}{\xi} \right),$$

we obtain the solution in the form

$$\Phi(\rho,z) = \frac{2p_0}{\pi} \int_0^\infty \frac{\sin \xi - \xi \cos \xi}{\xi^2} e^{-\xi z} J_0(\xi\rho)\, d\xi$$

$$= \frac{-2p_0}{\pi} \int_0^\infty \frac{d}{d\xi} \left(\frac{\sin \xi}{\xi} \right) e^{-\xi z} J_0(\xi\rho)\, d\xi,$$

which is in agreement with the solution given by (3.3.6) and (3.3.7).

3.5 PENNY-SHAPED CRACK IN A SOLID UNDER SHEAR

The three-dimensional analogue of a Mode II displacement is the problem of determining the stress field in the neighborhood of the penny-shaped crack $0 \leq \rho \leq 1$, $z = 0$ in a solid under a shear such that

$$\sigma_{xz} \sim S \tag{3.5.1}$$

as $r = (\rho^2 + z^2) \to \infty$ [cf. Fig. 4(ii)]. In terms of cylindrical coordinates (ρ,θ,z), this condition is equivalent to the pair

$$\sigma_{\rho z} \sim S \cos \theta, \qquad \sigma_{\theta z} \sim -S \sin \theta, \quad r \to \infty, \tag{3.5.2}$$

and these in turn are equivalent to

$$u_z \sim \frac{S}{\mu} \rho \cos \theta, \quad r \to \infty. \tag{3.5.3}$$

If we assume that the surfaces of the crack are free from stress, we have

$$\sigma_{\rho z}(\rho,\theta,0) = \sigma_{\theta z}(\rho,\theta,0) = \sigma_{zz}(\rho,\theta,0) = 0, \quad \rho \leq 1. \tag{3.5.4}$$

If we assume that the displacement is of "Westergaard type," i.e., is the three-dimensional analogue of the displacement given by the conditions (2.5.19), we have on the plane of the crack the conditions

$$\sigma_{zz}(\rho,\theta,0) = 0, \; 0 < \rho < \infty, \; u_\rho(\rho,\theta,0) = u(\rho,\theta,0) = 0, \quad \rho \geq 1 \tag{3.5.5}$$

whereas for a displacement considered in Section 2.5 we must take

$$u_\rho(\rho,\theta,0) = u_\theta(\rho,\theta,0) = u_z(\rho,\theta,0) = 0, \quad \rho \geq 1. \tag{3.5.6}$$

From Muki's solution (Muki, 1960) of the equations of elastic equilibrium we have the displacement field in the half-space $z \geq 0$,

$$u_\rho(\rho,\theta,z) = \tfrac{1}{2}[U_2(\rho,z) - V_0(\rho,z)]\cos \theta + \frac{S}{\mu} z \cos \theta \tag{3.5.7}$$

$$u_\theta(\rho,\theta,z) = \tfrac{1}{2}[U_2(\rho,z) + V_0(\rho,z)]\sin \theta - \frac{S}{\mu} z \sin \theta \tag{3.5.8}$$

$$u_z(\rho,\theta,z) = \mathscr{H}_1[(1 - 2\eta)G''(z) - 2(1 - \eta)\xi^2 G(z); \, \xi \to \rho]\cos \theta \tag{3.5.9}$$

where

$$U_2(\rho,z) = \mathscr{H}_2[\xi G'(z) + 2H(z); \, \xi \to \rho],$$
$$V_0(\rho,z) = \mathscr{H}_0[\xi G'(z) - 2H(z); \, \xi \to \rho]$$

and we may take

$$G(\xi,z) = 4(1 + \eta)S\xi^{-3}E^{-1}[2(1 - 2\eta)(A + B) + (C - A - B)\xi z]e^{-\xi z}$$
$$H(\xi,z) = 2(1 + \eta)S\xi^{-2}E^{-1}(B - A)e^{-\xi z}$$

where A, B, C are functions of ξ alone. For this solution we have that

$$u_\rho(\rho,\theta,0) = -4(1+\eta)(3-4\eta)\frac{S}{E}\cos\theta\{\mathscr{H}_2[\xi^{-1}B(\xi);\rho] - \mathscr{H}_0[\xi^{-1}A(\xi);\rho]\}$$

$$u_\theta(\rho,\theta,0) = -4(1+\eta)(3-4\eta)\frac{S}{E}\sin\theta\{\mathscr{H}_2[\xi^{-1}B(\xi);\rho] + \mathscr{H}_0[\xi^{-1}A(\xi);\rho]\}$$

$$u_z(\rho,\theta,0) = -4(1+\eta)(3-4\eta)\frac{S}{E}\cos\theta\,\mathscr{H}_1[\xi^{-1}C(\xi);\rho]$$

$$\frac{\sigma_{\rho z}(\rho,\theta,0)}{S} = \mathscr{H}_2[(7-8\eta)A(\xi) + B(\xi) + 2(1-2\eta)C(\xi);\ \rho]\cos\theta$$
$$- \mathscr{H}_0[A(\xi) + (7-8\eta)B(\xi) + 2(1-2\eta)C(\xi);\ \rho]\cos\theta,$$

$$\frac{\sigma_{\theta z}(\rho,\theta,0)}{S} = \mathscr{H}_2[(7-8\eta)A(\xi) + B(\xi) + 2(1-2\eta)C(\xi);\ \rho]\sin\theta$$
$$+ \mathscr{H}_0[A(\xi) + (7-8\eta)B(\xi) + 2(1-2\eta)C(\xi);\ \rho]\sin\theta,$$

$$\frac{\sigma_{zz}(\rho,\theta,0)}{S} = 2\mathscr{H}_1[2(1-2\eta)(A+B) + 4(1-\eta)C;\ \rho]\cos\theta.$$

Hence if $\sigma_{zz}(\rho,\theta,0) = 0$ for all positive values of ρ we must take

$$C(\xi) = -\frac{1-2\eta}{2(1-\eta)}[A(\xi) + B(\xi)], \tag{3.5.10}$$

and it follows that the conditions (3.5.4), (3.5.5) are equivalent to the relations

$$\begin{aligned}
(3-4\eta)\mathscr{H}_0[(2-\eta)B(\xi) - \eta A(\xi);\rho] &= 1-\eta, & 0 \le \rho \le 1\\
(2-\eta)\mathscr{H}_2[A(\xi);\rho] - \eta\mathscr{H}_2[B(\xi);\rho] &= 0, & 0 \le \rho \le 1\\
\mathscr{H}_0[\xi^{-1}A(\xi);\rho] &= 0, & \rho > 1\\
\mathscr{H}_2[\xi^{-1}B(\xi);\rho] &= 0, & \rho > 1.
\end{aligned}$$

On the other hand the boundary conditions (3.5.4), (3.5.6) lead to the even more complicated set

$$\mathscr{H}_0[A(\xi);\rho] + (7-8\eta)\mathscr{H}_0[B(\xi);\rho] + 2(1-\eta)\mathscr{H}_0[C(\xi);\rho] = 1, \quad 0 \le \rho \le 1$$

$$2(1-2\eta)\mathscr{H}_1[A(\xi) + B(\xi);\rho] + 4(1-\eta)\mathscr{H}_1[C(\xi);\rho] = 0, \quad 0 \le \rho \le 1$$

$$(7-8\eta)\mathscr{H}_2[A(\xi);\rho] + \mathscr{H}_2[B(\xi);\rho] + 2(1-2\eta)\mathscr{H}_2[C(\xi);\rho] = 0, \quad 0 \le \rho \le 1.$$

$$\begin{aligned}
\mathscr{H}_0[\xi^{-1}A(\xi);\rho] &= 0, & \rho \ge 1,\\
\mathscr{H}_2[\xi^{-1}B(\xi);\rho] &= 0, & \rho \ge 1,\\
\mathscr{H}_1[\xi^{-1}C(\xi);\rho] &= 0, & \rho \ge 1.
\end{aligned}$$

Solutions of both sets of equations can be obtained by using the method of Lowengrub (1967) or Erdogan and Bahar (1964).

3.6 PENNY-SHAPED CRACK IN A SOLID UNDER TORSION

We now consider the problem of calculating the stress field in the vicinity of the penny-shaped crack $0 \leq \rho \leq 1$, $z = 0$ in a solid under torsion so that, in terms of cylindrical coordinates (ρ,θ,z),

$$\sigma_{\theta z}(\rho,\theta,z) \sim S, \tag{3.6.1}$$

as $r = \sqrt{\rho^2 + z^2} \to \infty$. In such a stress distribution the components u_ρ and u_z of the displacement field will be identically zero and the remaining component u_θ will satisfy the equation

$$\frac{\partial^2 u_\theta}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u_\theta}{\partial \rho} - \frac{u_\theta}{\rho^2} + \frac{\partial^2 u_\theta}{\partial z^2} = 0.$$

If we take the solution of this equation in the form

$$u_\theta(\rho,z) = \frac{S}{\mu} z + \frac{S}{\mu} \mathscr{H}_1[\xi^{-1}A(\xi)e^{-\xi|z|}; \xi \to \rho], \tag{3.6.2}$$

where μ is the rigidity modulus, we find that

$$\sigma_{\theta z}(\rho,z) = S - S\mathscr{H}_1[A(\xi)e^{-\xi|z|}; \xi \to \rho] \tag{3.6.3}$$

so that the condition (3.6.1) will be satisfied if the Hankel transform exists at all and the boundary conditions

$$\sigma_{\theta z}(\rho,0) = 0, \quad 0 \leq \rho \leq 1$$
$$u_\theta(\rho,0) = 0, \quad \rho \geq 1$$

will be satisfied if $A(\xi)$ satisfies the pair of dual integral equations

$$\mathscr{H}_1[A(\xi);\rho] = 1, \quad 0 \leq \rho \leq 1$$
$$\mathscr{H}_1[\xi^{-1}A(\xi);\rho] = 0 \quad \rho > 1$$

The solution of this pair of equations is easily shown to be

$$A(\xi) = \frac{1 - \cos \xi}{\xi^2} - \frac{\sin \xi}{2\xi}. \tag{3.6.4}$$

This can be derived by putting $\alpha = -\frac{1}{2}$, $\nu = 1$ in the general solution (4.2.23) of Sneddon (1966) or merely by inspection from the equations

$$\int_0^\infty \left(\frac{1-\cos\xi}{\xi} - \tfrac{1}{2}\sin\xi\right) J_1(\rho\xi)\,d\xi = \begin{cases} 1, & 0 \le \rho \le 1 \\ \dfrac{1}{\rho}\left(\rho - \sqrt{\rho^2-1}\right) - \dfrac{1}{2\rho\sqrt{\rho^2-1}}, & \rho > 1 \end{cases}$$

$$\int_0^\infty \left(\frac{1-\cos\xi}{\xi^2} - \frac{\sin\xi}{2\xi}\right) J_1(\rho\xi)\,d\xi = \begin{cases} \tfrac{1}{2}\rho \log \dfrac{1+\sqrt{1-\rho^2}}{\rho}, & 0 \le \rho \le 1 \\ 0, & \rho > 1 \end{cases}$$

From these equations we deduce also that

$$u_\theta(\rho,0) = \frac{S}{2\mu}\rho \log \frac{1+\sqrt{1-\rho^2}}{\rho}, \quad 0 \le \rho \le 1 \tag{3.6.5}$$

and that

$$\sigma_{\theta z} = S\left[\frac{\sqrt{\rho^2-1}}{\rho} + \frac{1}{2\rho\sqrt{\rho^2-1}}\right], \quad \rho > 1 \tag{3.6.6}$$

from which it follows that the factor K_t defined by the equation

$$K_t = \lim_{p\to 1+} \sqrt{2(\rho-1)}\,\sigma_{\theta z}(\rho,0) \tag{3.6.7}$$

has the value $\frac{1}{2}S$.

The energy of the crack is given by the formula

$$W_t = 2\pi S \int_0^1 \rho u_\theta(\rho,0)\,d\rho$$

so that

$$W_t = \frac{\pi S^2}{\mu}\int_0^1 \rho^2 \log \frac{1+\sqrt{1-\rho^2}}{\rho}\,d\rho.$$

Evaluating the integral by the rule for integrating by parts we find that

$$W_t = \frac{\pi^2 S^2}{12\mu}.$$

If the crack radius had been taken to be c this would have taken the form

$$W_t = \frac{\pi^2 S^2 c^3}{12\mu}, \tag{3.6.8}$$

so that the corresponding strain energy release rate is

$$\mathscr{G}_t = \frac{\pi S^2 c}{8\mu}, \tag{3.6.9}$$

a result which can be written in the alternative form

$$\mathscr{G}_t = \frac{\pi K_t^2 c}{2\mu}. \tag{3.6.10}$$

If the values of u_θ and $\sigma_{\theta z}$ are required at a general point in the solid, they may be derived from equations (3.6.2) through (3.6.4) by the method of Section 3.3 above.

Using the relation

$$\sigma_{\rho\theta} = \mu\rho \frac{\partial}{\partial\rho}(\rho^{-1}u_\theta)$$

and the recurrence formula

$$\frac{\partial}{\partial\rho}[\rho^{-1}J_1(\xi\rho)] = -\frac{\xi}{p}J_2(\xi\rho),$$

we immediately deduce from (3.6.2) that the remaining stress component is given by

$$\sigma_{\rho\theta} = -S\mathscr{H}_2[A(\xi)e^{-\xi|z|}; \xi \to \rho]. \tag{3.6.11}$$

3.7 THERMAL STRESSES AROUND A PENNY-SHAPED CRACK

In this section we shall discuss the distribution of thermal stress in the vicinity of a penny-shaped crack of unit radius (cf. Olesiak and Sneddon, 1960). In addition to the assumptions made above about crack problems, we shall assume that the thermal conditions on the upper surface of the crack are identical with those on the lower surface. In this way we can reduce the problem to one concerning the distribution of thermal stress in a semi-infinite elastic solid.

If we take $\chi = 0$,

$$\phi = \tfrac{1}{2}\mathscr{H}_0[\xi^{-2}\{(\beta^2-1)^{-1}f(\xi) + g(\xi)\}e^{-\xi z}; \xi \to \rho], \tag{3.7.1}$$

$$\psi = \tfrac{1}{2}\beta^2\mathscr{H}_0[\xi^{-1}g(\xi)e^{-\xi z}; \xi \to \rho], \tag{3.7.2}$$

in the equations of Section 1.3 we obtain a solution of the equations of thermoelasticity which satisfies the conditions:

$$\sigma_{zz}(\rho,0) = \mathscr{H}_0[g(\xi) - f(\xi); \rho], \tag{3.7.3}$$

$$u_z(\rho,0) = \frac{\beta^2}{2(\beta^2-1)}\mathscr{H}_0[\xi^{-1}f(\xi);\rho], \tag{3.7.4}$$

and

$$\left[\frac{\partial\theta}{\partial z}\right]_{z=0} = \frac{\beta^2}{b}\mathscr{H}_0[\xi g(\xi);\rho]. \tag{3.7.5}$$

This solution also gives $\sigma_{\rho z} = 0$ on $z = 0$ for all values of ρ.

If we suppose that there is a flux of heat across the faces of the crack then we have the relation

$$\frac{\partial\theta}{\partial z} = Q(\rho), \quad 0 \le \rho \le 1, \quad z = 0; \tag{3.7.6}$$

and, because of our assumption of symmetry with respect to the z-plane, there is no heat flow across that plane outside the crack region. Hence, we have the equation

$$\frac{\partial\theta}{\partial z} = 0, \quad \rho > 1, \quad z = 0. \tag{3.7.7}$$

If we apply the Hankel inversion theorem to the equations (3.7.5), (3.7.6), and (3.7.7) we find that

$$g(\xi) = b\beta^{-2}\xi^{-1}q(\xi), \tag{3.7.8}$$

where

$$q(\xi) = \int_0^1 \rho Q(\rho) J_0(\xi\rho)\, d\rho. \tag{3.7.9}$$

If, in addition to the thermal flux, there is an internal pressure $\rho(\rho)$ then we have the boundary conditions

$$\sigma_{zz}(\rho,0) = -p(\rho), \quad 0 \le \rho \le 1 \tag{3.7.10}$$

$$u_z(\rho,0) = 0, \qquad \rho > 1 \tag{3.7.11}$$

Hence if we write

$$F(\rho) = p(\rho) + b\beta^{-2}\mathscr{H}_0[\xi^{-1}g(\xi);\rho], \tag{3.7.12}$$

we see from equations (3.7.3), (3.7.4) that the boundary conditions (3.7.10), (3.7.11) are equivalent to the dual integral equations

$$\mathscr{H}_0[f(\xi);\rho] = F(\rho), \quad 0 \le \rho \le 1, \tag{3.7.13}$$

$$\mathscr{H}_0[\xi^{-1}f(\xi);\rho] = 0, \qquad \rho > 1, \tag{3.7.14}$$

with solution

$$f(\xi) = \frac{2}{\pi}\int_0^1 \sin(\xi s)\, ds \int_0^s \frac{xF(x)\, dx}{\sqrt{s^2 - x^2}}. \tag{3.7.15}$$

Interchanging the orders of integration and making use of standard properties of Bessel functions, we find that

$$\int_0^s \frac{x\, dx}{\sqrt{s^2 - x^2}} \int_0^\infty q(\xi) J_0(\xi x)\, d\xi = \int_s^1 \rho Q(\rho) \sin^{-1}\frac{s}{\rho}\, d\rho + \tfrac{1}{2}\pi \int_0^s \rho Q(\rho)\, d\rho.$$

Applying the rule for integration by parts and introducing the function

$$Q_1(\rho) = \int_\rho^1 uQ(u)\,du, \tag{3.7.16}$$

we find that the righthand side of this equation is equal to

$$\tfrac{1}{2}\pi Q_1(0) - s\int_s^1 \frac{Q(u)\,du}{u(u^2 - s^2)}$$

and, after a little manipulation, that

$$f(\xi) = \frac{2}{\pi}\int_0^1 \sin(\xi s)\,ds \int_0^s \frac{xp(x)\,dx}{\sqrt{s^2 - x^2}} + b\beta^{-2} \times \left[Q_1(0)\,\frac{1 - \cos\xi}{\xi} - \int_0^1 Q_1(\rho)J_1(\xi\rho)\,d\rho\right]. \tag{3.7.17}$$

To illustrate the use of these results we consider the stress distribution in the solid when the flux function takes two special forms, it being assumed that $p(\rho) = 0$. The latter restriction is not at all a severe limitation since the solution for non-zero $p(\rho)$ is known (Section 3.2 above).

Case* (i) $Q = Q_0$, *a Constant

If we put $Q(\rho) = Q_0$ in equation (3.7.16) we find that

$$Q_1(\rho) = \tfrac{1}{2}Q_0(1 - \rho^2), \tag{3.7.18}$$

and it follows from equation (3.7.17) that

$$f(\xi) = \tfrac{1}{2}b\beta^{-2}Q_0\xi^{-2}[2J_1(\xi) - \xi\cos\xi] \tag{3.7.19}$$

and from equations (3.7.8) and (3.7.9) that

$$g(\xi) = b\beta^{-2}\xi^{-2}Q_0J_1(\xi). \tag{3.7.20}$$

If we substitute from equations (3.7.19) and (3.7.20) into equations (3.7.1), (3.7.2) and then substitute from these into the equations of Section 1.3, we find the expression

$$u_\rho(\rho,z) = \tfrac{1}{2}b\beta^{-2}Q_0\mathscr{H}_1[\xi^{-3}\{(z\xi - 1 + 2\eta) \times [J_1(\xi) - \tfrac{1}{2}\xi\cos\xi] - (1 + \xi z)J_1(\xi)\}e^{-\xi z};\ \xi \to \rho] \tag{3.7.21}$$

for the radial component of the displacement vector and the expression

$$u_z(\rho,z) = \tfrac{1}{2}b(\beta^2 - 1)^{-1}Q_0 \times \mathscr{H}_0[\xi^{-3}\{2J_1(\xi) - \xi\cos\xi - \tfrac{1}{4}z(1 - \eta)^{-1}\cos\xi\}e^{-\xi z};\ \xi \to \rho] \tag{3.7.22}$$

for the other nonvanishing component of the displacement vector.

It follows from (3.7.22) that the normal displacement of the crack is found by substituting $z = 0$, $0 \le \rho \le 1$ in this integral to obtain the expression

$$u_z(\rho,0) = \tfrac{1}{2}b(\beta^2 - 1)^{-1}Q_0 \times \left[\int_0^\infty \xi^{-1}(1 - \cos \xi)J_0(\xi\rho)\, d\xi + \int_0^\infty \xi^{-2}\{2J_1(\xi) - \xi\}J_0(\xi\rho)\, d\xi\right].$$

Now if $x \ge 0$, $\rho \ge 0$,

$$x\int_0^\infty \xi^{-1}[1 - J_0(\xi x)]J_0(\xi\rho)\, d\xi = x \log \frac{x}{\rho} H(x - \rho)$$

so that integrating both sides with respect to x from 0 to 1, we obtain the result

$$\frac{1}{2}\int_0^\infty \xi^{-2}[\xi - 2J_1(\xi)]J_0(\xi\rho)\, d\xi = -\tfrac{1}{2}\log \rho - \tfrac{1}{4}(1 - \rho^2).$$

Also,

$$\int_0^\infty \xi^{-1}(1 - \cos \xi)J_0(\xi\rho)\, d\xi = \cosh^{-1}\frac{1}{\rho} H(1 - \rho).$$

Using these results we find that, when $0 < \rho < 1$,

$$\int_0^\infty \xi^{-1}(1 - \cos \xi)J_0(\xi\rho)\, d\xi + \int_0^\infty \xi^{-2}[2J_1(\xi) - \xi]J_0(\xi\rho)\, d\xi = \log(1 + \sqrt{1 - \rho^2}) + \tfrac{1}{2}(1 - \rho^2),$$

so that, finally, we have

$$u_z(\rho,0) = \frac{bQ_0}{4(\beta^2 - 1)}[2 \log(1 + \sqrt{1 - \rho^2}) + (1 - \rho^2)], \quad 0 < \rho < 1. \tag{3.7.23}$$

We note that $u_z(1,0) = 0$ and $u_z(0,0) = w_0$ where

$$w_0 = \frac{bQ_0}{4(\beta^2 - 1)}(2 \log 2 + 1). \tag{3.7.24}$$

In a similar way we can readily find the corresponding expressions for the components of the stress tensor. For instance, if we substitute from equations (3.7.1), (3.7.2) [with $f(\xi)$ and $g(\xi)$ given by equations (3.7.19), (3.7.20)] into (1.3.5) we obtain

$$\sigma_{zz}(\rho,z) = -\tfrac{1}{2}b\beta^{-2}Q_0 f(\rho,z), \tag{3.7.25}$$

where the function $f(\rho,z)$ is defined by the integral

$$f(\rho,z) = \int_0^\infty (1 + \xi z)e^{-\xi z} \cos \xi J_0(\rho\xi)\, d\xi. \tag{3.7.26}$$

This integral can be easily evaluated by using either George's tables or the formula

$$f(\rho,z) = R^{-1}\cos\tfrac{1}{2}u + zR^{-3}(z\cos\tfrac{3}{2}u + \sin\tfrac{3}{2}u), \tag{3.7.27}$$

with R and u defined by

$$R^4 = (\rho^2 + z^2 - 1)^2 + 4z^2, \qquad \tan u = \frac{2z}{\rho^2 + z^2 - 1}.$$

The variation of $f(\rho,z)$ with ρ and z for a set of values of these variables is shown in Table 9. This variation is also shown graphically in Fig. 46. From the values of this function we can easily calculate the normal component of stress σ_{zz} by means of equation (3.7.25).

Near the tip of the crack we can write $\rho = 1 + \delta\cos\psi$, $z = \sin\psi$, where δ is small. We then find that $f(\rho,z) = (2\delta)^{-1/2}f(\psi)$ where

$$f(\psi) = \cos\tfrac{1}{2}\psi(1 + \tfrac{1}{2}\sin\psi\sin\tfrac{3}{2}\psi) \tag{3.7.28}$$

so that

$$\sigma_{zz} = -\tfrac{1}{2}b\beta^{-2}Q_0(2\delta)^{-1/2}f(\psi) \tag{3.7.29}$$

and the stress intensity factor

$$K_t = -\lim_{\rho\to 1+}\sqrt{2(\rho-1)}\,\sigma_{zz}(\rho,0)$$

has the value $\frac{1}{2}b\beta^{-2}Q_0$.

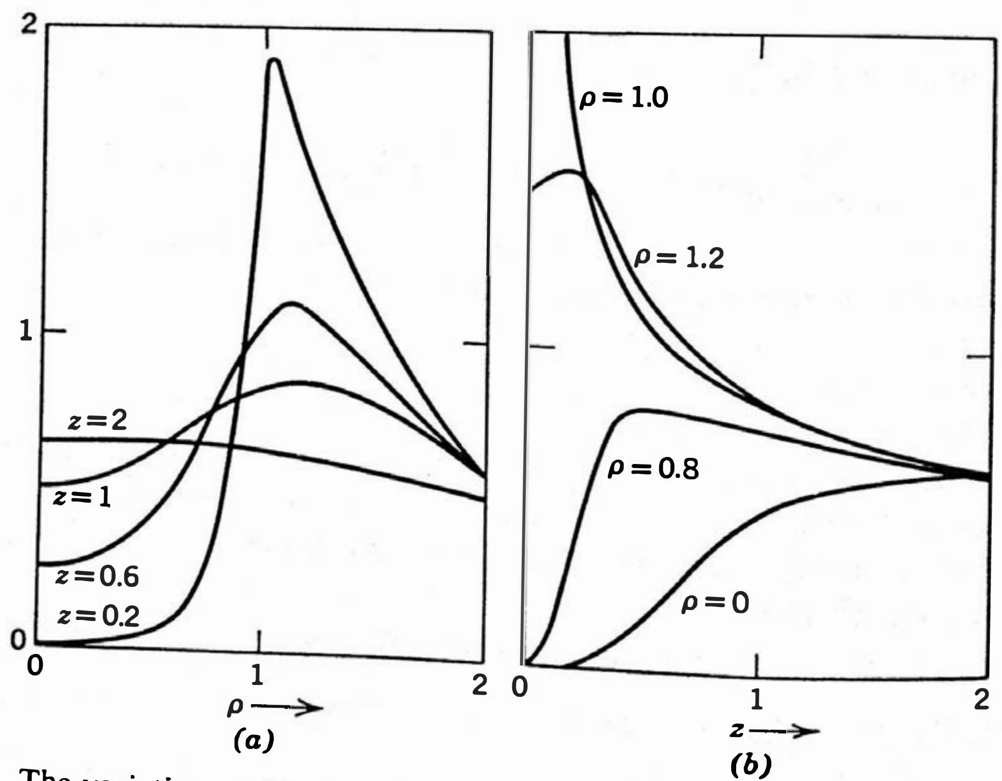


Figure 46 The variation of $f(\rho,z)$ with ρ and z. (*a*) shows the form of $f(\rho,z)$ as ρ varies, z being kept fixed, and (*b*) shows the form of $f(\rho,z)$ as z varies, ρ being kept fixed.

TABLE 9

VALUES OF THE FUNCTION $f(\rho,z)$

ρ \ z	0	0.2	0.4	0.6	0.8	1	2	3	4	5	10
	0	0.0148	0.0951	0.2336	0.3807	0.5000	0.6400	0.5400	0.4429	0.3698	0.1961
	0	0.0171	0.1097	0.2578	0.4057	0.5199	0.6410	0.5391	0.4422	0.3694	0.1960
.2	0	0.0305	0.1666	0.3407	0.4833	0.5782	0.6433	0.5365	0.4404	0.3681	0.1958
.4	0	0.0832	0.3214	0.5132	0.6149	0.6670	0.6453	0.5320	0.4369	0.3659	0.1954
.6	0	0.3763	0.7915	0.8113	0.7799	0.7656	0.6464	0.5255	0.4322	0.3629	0.1949
.8	0	1.7075	1.2185	1.0314	0.9161	0.8399	0.6396	0.5169	0.4263	0.3591	0.1943
	∞	1.5800	1.3166	1.0990	0.9558	0.8614	0.6281	0.5062	0.4193	0.3546	0.1935
.2	1.5076	0.5844	0.5997	0.6120	0.6152	0.6092	0.5175	0.4454	0.3812	0.3301	0.1890
2	0.5774	0.3547	0.3581	0.3625	0.3679	0.3727	0.3769	0.3539	0.3222	0.2906	0.1810
3	0.3536	0.2586	0.2598	0.2616	0.2641	0.2666	0.2774	0.2766	0.2656	0.2495	0.1740
4	0.2582	0.2043	0.2049	0.2058	0.2070	0.2084	0.2179	0.2226	0.2190	0.2124	0.1596
5	0.2041	0.1005	0.1006	0.1007	0.1009	0.1010	0.1024	0.1043	0.1063	0.1079	0.1061
	0.1005										

Case (ii): $Q(\rho)$ *Represented by a Fourier-Bessel Series*

If the flux function is represented by the Fourier-Bessel expansion

$$Q(\rho) = \sum_{m=1}^{\infty} Q_m J_0(j_m \rho), \tag{3.7.30}$$

where $j_1, j_2, \ldots$ are the positive zeros of $J_0(z)$, by equation (3.7.16), we then have

$$Q_1(\rho) = \sum_{m=1}^{\infty} j_m{}^{-1} Q_m [J_1(j_m) - \rho J_1(j_m \rho)], \tag{3.7.31}$$

with

$$Q_1(0) = \sum_{m=1}^{\infty} j_m{}^{-1} Q_m J_1(j_m). \tag{3.7.32}$$

From equations (3.7.8), (3.7.9), (3.7.17) we find that

$$g(\xi) = b\beta^{-2} J_0(\xi) \sum_{m=1}^{\infty} \frac{j_m Q_m J_1(j_m)}{\xi(j_m{}^2 - \xi^2)}, \tag{3.7.33}$$

$$f(\xi) = b\beta^{-2} \sum_{m=1}^{\infty} j_m{}^{-1} Q_m J_1(j_m) \left[\frac{1 - \cos\xi}{\xi} - \frac{1 - J_0(\xi)}{\xi} + \frac{\xi J_0(\xi)}{j_m{}^2 - \xi^2}\right]. \tag{3.7.34}$$

If we make use of equations (3.7.4) and the results

$$\int_0^{\infty} \xi^{-1}[J_0(\xi) - \cos\xi] J_0(\rho\xi)\, d\xi$$
$$= [\cosh^{-1}(\rho^{-1}) - \log(\rho^{-1})] H(1 - \rho), \quad (\rho > 0),$$

$$\int_0^{\infty} \xi(j_m{}^2 - \xi^2)^{-1} J_0(\xi) J_0(\rho\xi)\, d\xi = \frac{J_0(j_m\rho)}{j_m J_1(j_m)},$$

we see that the normal component of the surface displacement is given by

$$u_z(\rho,0) = \frac{b}{\beta^2 - 1}\left\{\log[1 + \sqrt{1-\rho^2}]\sum_{m=1}^{\infty} j_m^{-1} Q_m J_1(j_m) + \sum_{m=1}^{\infty} j_m^{-2} Q_m J_0(j_m \rho)\right\}. \tag{3.7.35}$$

If we write

$$C = \sum_{m=1}^{\infty} j_m^{-1} Q_m J_1(j_m) = \int_0^1 sQ(s)\, ds, \tag{3.7.36}$$

and note that

$$\int_\rho^1 t^{-1}\, dt \int_0^t sQ(s)\, ds = \sum_{m=1}^{\infty} j_m^{-2} Q_m J_0(j_m \rho),$$

we find that we can put equation (3.7.35) into the form

$$u_z(\rho,0) = \frac{b}{(\beta^2 - 1)}\left\{C \log[1 + \sqrt{1-\rho^2}] + \int_\rho^1 t^{-1}\, dt \int_0^t sQ(s)\, ds\right\}, \quad 0 < \rho < 1. \tag{3.7.37}$$

For instance, for flux functions of the type

$$Q(\rho) = \sum_{n=0}^{\infty} C_n \rho^{2n}, \quad 0 < \rho < 1 \tag{3.7.38}$$

we find that

$$u_z(\rho,0) = \frac{b}{\beta^2 - 1}[C \log(1 + \sqrt{1-\rho^2}) + w(\rho)], \quad 0 < \rho < 1, \tag{3.7.39}$$

with

$$C = \frac{1}{2}\sum_{n=0}^{\infty}(n+1)^{-1} C_n, \qquad w(\rho) = \frac{1}{4}\sum_{n=0}^{\infty}(n+1)^{-2} C_n (1 - \rho^{2n+2}), \quad 0 < \rho < 1. \tag{3.7.40}$$

Similarly, if we subtract equations (3.7.33) and (3.7.34) we find that

$$g(\xi) - f(\xi) = bC\beta^{-2}\xi^{-1} \cos \xi, \tag{3.7.41}$$

where C is defined by equations (3.7.36). If we substitute from equation (3.7.41) into equations (3.7.1), (3.7.2) and then into equation (1.3.5) we obtain the formula

$$\sigma_{zz}(\rho,z) = b\beta^{-2} C f(\rho,z), \tag{3.7.42}$$

where $f(\rho,z)$ is defined by equations (3.7.26), (3.7.27). This shows that, apart from the value of the multiplicative constant C, the form of the variation of σ_{zz} with ρ and z is independent of the values of the constants $Q_1, Q_2, \ldots$ occurring in the Fourier-Bessel expansion of the flux-function $Q(\rho)$. We find that the stress intensity factor is given by

$$K_t = \frac{b}{\beta^2}\int_0^1 sQ(s)\, ds. \tag{3.7.43}$$

As special examples Olesiak and Sneddon consider the flux-functions

$$Q(\rho) = Q_0(1 - \rho^{2n})$$

with $n = 1, 2$; they show graphically the shape of the crack in each case.

We can also derive the solution in the case when the crack is opened out by the application of a prescribed temperature to its flat surfaces. In this case the boundary conditions (3.7.6), (3.7.7) are replaced by the pair

$$\theta = -\theta_0 h(\rho), \quad 0 \leq \rho \leq 1, \tag{3.7.44}$$

$$\frac{\partial \theta}{\partial z} = 0, \qquad \rho > 1. \tag{3.7.45}$$

It is easily shown that the temperature $\theta(\rho,z)$ is then given by

$$\theta(\rho,z) = -b\beta^2 \mathscr{H}_0[g(\xi)e^{-\xi z}; \xi \to \rho], \tag{3.7.46}$$

provided that $g(\xi)$ is chosen to be the solution of the dual integral equations

$$\mathscr{H}_0[g(\xi);\rho] = b\theta_0\beta^{-2}h(\rho), \quad 0 \leq \rho \leq 1$$
$$\mathscr{H}_0[\xi g(\xi);\rho] = 0, \qquad \rho > 1.$$

The solution of these equations is elementary (Sneddon, 1960) and we find that

$$g(\xi) = \frac{2b\theta_0 \cos \xi}{\pi\beta^2\xi}\int_0^1 \frac{yh(y)\,dy}{\sqrt{1-y^2}} + \frac{2b\theta_0}{\pi\beta^2}\int_0^1 \frac{y\,dy}{\sqrt{1-y^2}}\int_0^1 ug(yu)\sin(\xi u)\,du. \tag{3.7.47}$$

If we substitute from (3.7.47) into (3.7.5) we can calculate the expression for

$$Q(\rho) = \left[\frac{\partial \theta}{\partial z}\right]_{z=0} = \beta^2 b^{-1}\mathscr{H}_0[\xi g(\xi);\rho], \tag{3.7.48}$$

the heat flux across the surfaces of the crack. Once we have found this function we can proceed using the method developed above. In applying this method the following results are useful:

$$Q_1(\rho) = \beta^2 b^{-1}\int_0^\infty \xi g(\xi)[J_1(\xi) - \rho J_1(\rho\xi)]\,d\xi, \tag{3.7.49}$$

$$C = \beta^2 b^{-1}\int_0^\infty \xi g(\xi)J_1(\xi)\,d\xi, \tag{3.7.50}$$

$$w(\rho) = \beta^2 b^{-1}\int_0^\infty g(\xi)[J_0(\xi\rho) - J_0(\xi)]\,d\xi. \tag{3.7.51}$$

To illustrate the procedure, we consider the simple case given by the equations

$$\theta = -\theta_0, \quad 0 \leq \rho \leq 1,$$

$$\frac{\partial\theta}{\partial z} = 0, \qquad \rho > 1.$$

In this case $h(\rho) = 1$ and we find that

$$g(\xi) = \frac{2b\theta_0}{\pi\beta^2\xi^2}\sin\xi \tag{3.7.52}$$

so that

$$Q(\rho) = \frac{2\theta_0}{\pi}\int_0^\infty \sin\xi J_0(\rho\xi)\,d\xi = \frac{2\theta_0}{\pi\sqrt{1-\rho^2}}, \quad 0 \leq \rho < 1,$$

and

$$Q_1(\rho) = \frac{2\theta_0}{\pi}\sqrt{1-\rho^2}, \quad 0 \leq \rho \leq 1.$$

From equation (3.7.17) we find that

$$f(\xi) = \frac{b\theta_0}{\pi\beta^2\xi}\left(2 - \cos\xi - \frac{\sin\xi}{\xi}\right). \tag{3.7.53}$$

The solution is therefore given by equations (3.7.52) and (3.7.53).

For this problem*

$$\int_0^t sQ(s)\,ds = \frac{2\theta_0}{\pi}\left[1 - \sqrt{1-t^2}\right]$$

so that $C = 2\theta_0/\pi$ and

$$w(\rho) = \frac{2\theta_0}{\pi}\left[\sqrt{1-\rho^2} - \log(1 + \sqrt{1-\rho^2})\right].$$

Substituting this expression into equation (3.7.39) we find that, in this case, the crack surface takes the elliptical shape

$$u_z(\rho,0) = L_0\sqrt{1-\rho^2}, \quad 0 \leq \rho \leq 1.$$

where $L_0 = 2b\theta_0/(\beta^2 - 1)$.

This problem has been discussed by Florence and Goodier (1963) and by Shail (1964) who also considers the associated problem of determining the thermal stress in the vicinity of a crack in a thick plate.

* The corresponding equation in Olesiak and Sneddon (1960, p. 251) is in error.

3.8 THE FLAT ELLIPTICAL CRACK

In the previous sections we have considered only the case of a penny-shaped crack. We shall now consider the distribution of stress near a flat elliptical crack in an infinite body. Sadowsky and Sternberg (1949) have solved the more general problem of the stress concentration around a tri-axial ellipsoidal cavity in an elastic body of infinite extent, the body at infinity being in a uniform state of stress whose principal axes are parallel to the axes of the cavity. A different approach to the crack problem has been made by Green and Sneddon (1950); since it is a simple solution we shall present it here. A more recent paper by Sakadi (1958) should also be noted in this connection.

If we denote by E the crack whose boundary has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad z = 0,$$

we see that the problem of determining the stress distribution in the vicinity of a crack E in an infinite solid opened up by a constant internal pressure p_0 is equivalent to that of a semi-infinite solid $z \geq 0$ whose surface $z = 0$ is deformed as follows:

$$\sigma_{zz} = -p_0, \quad (x,y) \in E, \tag{3.8.1}$$

$$u_z = 0, \quad (x,y) \in Z - E, \tag{3.8.2}$$

$$\sigma_{xz} = \sigma_{yz} = 0, \quad (x,y) \in Z, \tag{3.8.3}$$

where Z denotes the whole xy-plane.

It follows from the equations of Section 1.3 with $\chi = \psi = 0$ that

$$u_x = \frac{\partial \phi}{\partial x} + (\beta^2 - 1)z\frac{\partial^2 \phi}{\partial x\, \partial z}, \qquad u_y = \frac{\partial \phi}{\partial y} + (\beta^2 - 1)z\frac{\partial^2 \phi}{\partial y\, \partial z},$$
$$u_z = -\beta^2\frac{\partial \phi}{\partial z} + (\beta^2 - 1)z\frac{\partial^2 \phi}{\partial z^2} \tag{3.8.4}$$

will be a solution of the problem provided we can find a potential function $\varphi(x,y,z)$ such that on $z = 0$,

$$2(\beta^2 - 1)\frac{\partial^2 \phi}{\partial z^2} = p_0, \quad (x,y) \in E, \tag{3.8.5}$$

$$\frac{\partial \phi}{\partial z} = 0, \quad (x,y) \in Z - E. \tag{3.8.6}$$

Green and Sneddon solved this problem by noting that $\partial\varphi/\partial z$ is equivalent to the velocity potential of a flat elliptical disk moving with uniform velocity $p_o/2(\beta^2 - 1)$ perpendicular to its plane through an infinite incompressible fluid which is at rest at infinity. The solution of this problem is well known (Lamb, 1932, pp. 151–2), and will now be given in terms of ellipsoidal coordinates (λ,μ,ν) defined by the equations

$$\begin{aligned} a^2(a^2 - b^2)x^2 &= (a^2 + \lambda)(a^2 + \mu)(a^2 + \nu), \\ b^2(b^2 - a^2)y^2 &= (b^2 + \lambda)(b^2 + \mu)(b^2 + \nu), \\ a^2b^2z^2 &= \lambda\mu\nu, \end{aligned} \tag{3.8.7}$$

where $-a^2 \leq \nu \leq -b^2 \leq \mu \leq 0 \leq \lambda < \infty$. The crack E is given by $\lambda = 0$ and $Z - E$ by $\mu = 0$.

We need the following partial derivatives

$$\frac{\partial\lambda}{\partial x} = \frac{x}{2h_1^2(a^2 + \lambda)}, \qquad \frac{\partial\lambda}{\partial y} = \frac{y}{2h_1^2(b^2 + \lambda)}, \qquad \frac{\partial\lambda}{\partial z} = \frac{z}{2\lambda h_1^2} \tag{3.8.8}$$

where

$$4h_1^2Q(\lambda) = (\lambda - \mu)(\lambda - \nu), \qquad Q(\lambda) = \lambda(a^2 + \lambda)(b^2 + \lambda). \tag{3.8.9}$$

If necessary, x, y, z can be expressed in terms of elliptic functions (Whittaker and Watson, 1927, Ch. 23).

By the hydrodynamic analogy mentioned above we may write

$$\frac{\partial\phi}{\partial z} = Az\int_\lambda^\infty \frac{ds}{s\sqrt{Q(s)}}, \tag{3.8.10}$$

where A is a constant, and hence

$$\phi = \frac{A}{2}\int_\lambda^\infty \left(\frac{x^2}{a^2 + s} + \frac{y^2}{b^2 + s} + \frac{z^2}{s} - 1\right)\frac{ds}{\sqrt{Q(s)}}. \tag{3.8.11}$$

The function ϕ in (3.8.11) is known to be harmonic, since, apart from a multiplicative constant, it represents the gravitational potential at an external point of a uniform elliptic plate and it can be verified immediately that $\partial\phi/\partial z$ is given by (3.8.10) since

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} + \frac{z^2}{\lambda} - 1 = 0. \tag{3.8.12}$$

Except for the determination of A, the solution is complete, and the components of the displacement vector and the stress tensor can be derived from the equations of Section 1.3. For numerical calculations it is convenient to express some of the integrals in terms of Jacobian elliptic functions by writing

$$\lambda = \frac{a^2 cn^2 u}{sn^2 u} \tag{3.8.13}$$

where $sn\,u$ is the Jacobian elliptic function with real period K corresponding to the modulus

$$k = \sqrt{1 - b^2/a^2} \tag{3.8.14}$$

and complementary modulus

$$k' = \frac{b}{a}. \tag{3.8.15}$$

The integral in (3.8.10) diverges when $\lambda = 0$, so it is convenient to express this integral in the alternative form

$$\begin{aligned} \frac{\partial \phi}{\partial z} &= Az\left[\frac{2}{\sqrt{Q(\lambda)}} - \int_\lambda^\infty \frac{(2s + a^2 - b^2)\,ds}{(a^2 + s)(b^2 + s)\sqrt{Q(s)}}\right] \\ &= \frac{2Az}{ab^2}\left[\frac{sn u\, dn u}{cn u} - E(u)\right] \end{aligned} \tag{3.8.16}$$

where

$$E(u) = \int_0^u dn^2 t\, dt \tag{3.8.17}$$

in order to evaluate $\partial\phi/\partial z$ when $\lambda = 0$. Differentiating $\partial\phi/\partial z$ with respect to z with the help of (3.8.8) gives

$$\begin{aligned} \frac{\partial^2 \phi}{\partial z^2} &= A\left\{\frac{2\sqrt{\lambda[\lambda(a^2b^2 - \mu\nu) - a^2b^2(\mu + \nu) - (a^2 + b^2)\mu\nu]}}{a^2b^2(\lambda - \mu)(\lambda - \nu)\sqrt{a^2 + \lambda}\sqrt{b^2 + \lambda}}\right. \\ &\quad \left. - \int_\lambda^\infty \frac{(2s + a^2 - b^2)\,ds}{(a^2 + s)(b^2 + s)\sqrt{Q(s)}}\right\} \\ &= A\left\{\frac{2\sqrt{\lambda[\lambda(a^2b^2 - \mu\nu) - a^2b^2(\mu + \nu) - (a^2 + b^2)\mu\nu]}}{a^2b^2(\lambda - \mu)(\lambda - \nu)\sqrt{a^2 + \lambda}\sqrt{b^2 + \lambda}}\right. \\ &\quad \left. - \frac{2}{a^2b^2}\left[E(u) - \frac{sn u\, cn u}{dn u}\right]\right\}. \end{aligned} \tag{3.8.18}$$

Putting $\mu = 0$ in (3.8.16) we can verify (3.8.6). Putting $\lambda = 0$ in (3.8.18) and using (3.8.6) we have $u_x = K$ and

$$\frac{\partial^2 \phi}{\partial z^2} = -\frac{2A}{a^2b^2}E(K) = \frac{p_0}{2(\beta^2 - 1)}.$$

Hence the constant A is given by

$$A = -\frac{a^2b^2p_0}{4(\beta^2 - 1)E(K)}. \tag{3.8.19}$$

Also, when $\lambda = 0$

$$u_z(x,y,0) = -\beta^2 \frac{\partial \phi}{\partial z}$$

$$= -\frac{2A\beta^2\sqrt{\mu\nu}}{a^2b^2}$$

$$= -\frac{2A}{ab}\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \tag{3.8.20}$$

which gives the shape of the crack.

A very useful and interesting theorem concerning the distribution of stress in the vicinity of the elliptic crack E in an infinite elastic solid has recently been proved by Kassir and Sih (1967). This states that if the normal component of the surface displacement of the crack is prescribed by the equation

$$u_z(x,y,0) = \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{1/2} w_n(x^2,y^2), \quad (x,y) \in E, \tag{3.8.21}$$

where $w_n(x^2,y^2)$ is a polynomial of degree n in the variable x^2, y^2, then the normal pressure acting over the surface of the elliptic crack is also a polynomial $\phi_n(x^2,y^2)$ of degree n in the variables x^2, y^2. The application of this theorem reduces the problem of determining the stress field in the neighborhood of an elliptic crack under a normal pressure $p_n(x^2,y^2)$ to that of determining the distribution of stress in a half-space $z \geq 0$ when the boundary conditions are

$$u_z(x,y,0) = \begin{cases} \left(1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}\right)^{1/2} w_n(x^2,y^2), & (x,y) \in E; \\ 0, & (x,y) \in Z - E. \end{cases}$$

We take $w_n(x^2,y^2)$ to be of the form

$$w_n(x^2,y^2) = \sum_{r=0}^{n} c_r x^{2n-2r} y^{2r}$$

where the coefficients $c_0, c_1, \ldots, c_n$ have to be determined. The solution of this problem is a straightforward matter (although the algebraic details may be tiresome) and we can find the normal pressure on E, $-\sigma_{zz}(x,y,0)$, as a function of the form

$$\sum_{r=0}^{n} \gamma_r x^{2n-2r} y^{2r}$$

where the constants $\gamma_0, \gamma_1, \ldots, \gamma_n$ are linear combinations of the constants $c_0, c_1, \ldots, c_n$. Equating the coefficients of this polynomial with those of $p_n(x^2,y^2)$ we obtain a set of $n + 1$ linear algebraic equations by means of which we may determine the constants $c_0, c_1, \ldots, c_n$.

The problem of determining the stress field in the vicinity of an elliptical crack in an anisotropic elastic solid has been discussed recently by Willis (1968).

3.9 THE STRESS IN THE VICINITY OF AN EXTERNAL CRACK

Uflyand (1959) and Lowengrub and Sneddon (1965) have given exact solutions of the equations of elastic equilibrium appropriate to an infinite space containing a flat crack covering the outside of a circle whose radius is taken to be the unit of length.

In his solution Uflyand uses toroidal coordinates α, β, ϕ defined by the relations

$$x = \frac{\sinh \alpha \cos \phi}{\cosh \alpha + \cos \beta}, \qquad y = \frac{\sinh \alpha \sin \phi}{\cosh \alpha + \cos \beta}, \qquad z = \frac{\sin \beta}{\cosh \alpha + \cos \beta}, \tag{3.9.1}$$

with

$$0 \le \alpha \le \infty, \qquad -\pi \le \beta \le +\pi, \qquad 0 \le \phi < 2\pi.$$

In these coordinates the upper face of the crack is $\beta = +\pi$, the lower face is $\beta = -\pi$, the inside of the circle $x^2 + y^2 = 1$ is the surface $\beta = 0$ and the circumference of the circle is $\alpha = \infty$. Uflyand also uses the Boussinesq-Papkovich solution of the equations of elastic equilibrium (Sneddon and Berry, 1958, p. 89) in the form

$$2\mathbf{u} = -\mathbf{grad}\, F + 4(1 - \nu)\mathbf{\Phi} \tag{3.9.2}$$

where $\mathbf{\Phi} = (\Phi_1, \Phi_2, \Phi_3)$ is a vector, each of whose components is a harmonic function, Φ_0 is a scalar harmonic function and

$$F = \Phi_0 + (\mathbf{r} \cdot \mathbf{\Phi}).$$

For this solution the z-components of stress are given by the equations

$$\sigma_{zz} = \frac{\partial}{\partial z}[2(1 - \eta)\Phi_3 - \Phi_4] + 2\eta\left(\frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_2}{\partial y}\right) - \mathbf{r} \cdot \frac{\partial^2 \mathbf{\Phi}}{\partial z^2}$$

$$\sigma_{xz} = \frac{\partial \Phi}{\partial x} + 2(1 - \eta)\frac{\partial \Phi_1}{\partial z}, \qquad \sigma_{yz} = \frac{\partial \Phi}{\partial y} + 2(1 - \eta)\frac{\partial \Phi_2}{\partial z} \tag{3.9.3}$$

$$\Phi_4 = \frac{\partial \Phi_0}{\partial z}, \qquad \Phi = (1 - 2\eta)\Phi_3 - \Phi_4 - \mathbf{r} \cdot \frac{\partial \mathbf{\Phi}}{\partial z}.$$

The problem may now be split into two parts: one symmetric (with respect to the plane $z = 0$) and the other antisymmetric. In each case we may lay down certain conditions on the surface $\beta = 0$ and consider the problem for the upper half-space ($z > 0$; i.e., $0 \leq \beta \leq \pi$) only.

Symmetric problem

In the case of a stress distribution which is symmetric with respect to the plane $z = 0$, the equilibrium of the upper half-space can be considered under the following boundary conditions:
On $\beta = 0$:

$$u_z = \sigma_{xz} = \sigma_{yz} = 0. \tag{3.9.4}$$

On $\beta = \pi$:

$$\sigma_{zz} = \sigma(\alpha,\phi), \qquad \sigma_{xz} = \sigma_x(\alpha,\phi), \qquad \sigma_{yz} = \sigma_y(\alpha,\phi). \tag{3.9.5}$$

Since one of the harmonic functions contained in the Papkovich-Neuber solution is arbitrary we may add to (3.9.4), (3.9.5) the additional conditions

$$\Phi = 0 \qquad \text{on} \qquad \beta = 0, \quad \beta = \pi. \tag{3.9.6}$$

Hence the harmonic functions Φ_1, Φ_2 satisfy the boundary conditions

$$\begin{aligned} &\text{on } \beta = 0, \quad \frac{\partial \Phi_1}{\partial z} = \frac{\partial \Phi_2}{\partial z} = 0, \\ &\text{on } \beta = \pi, \quad \frac{\partial \Phi_1}{\partial z} = \frac{\sigma_x(\alpha,\phi)}{2(1-\eta)}, \qquad \frac{\partial \Phi_2}{\partial z} = \frac{\sigma_y(\alpha,\phi)}{2(1-\eta)}, \end{aligned} \tag{3.9.7}$$

showing that they are solutions of the Neumann problem for a half-space.

From the conditions (3.9.6) we find that the function

$$\psi = (1 - 2\eta)\Phi_3 - \Phi_4 \tag{3.9.8}$$

is a solution of the Dirichlet problem

$$\begin{aligned} &\nabla^2 \psi = 0, \qquad \psi = 0 \qquad \text{on} \qquad \beta = 0, \\ &\psi = \frac{1}{2(1-\eta)}(x\sigma_{xz} + y\sigma_{yz})_{\beta=\pi}. \end{aligned} \tag{3.9.9}$$

Finally, the remaining conditions

$$\begin{aligned} u_z &= 0 \qquad \text{on} \qquad \beta = 0, \\ \sigma_{zz} &= \sigma(\alpha,\phi) \qquad \text{on} \qquad \beta = \pi, \end{aligned}$$

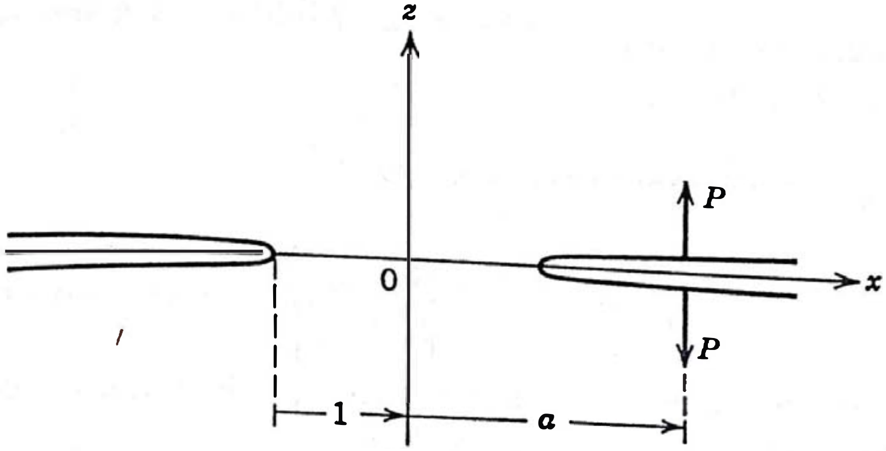


Figure 47 The two-point loading problem.

after the substitution of values of $\Phi_4 = (1 - 2\eta)\Phi - \psi$, lead to the mixed boundary conditions

on $\beta = 0$: $\quad \Phi_3 = 0,$

on $\beta = \pi$: $$\frac{\partial \Phi_3}{\partial z} = \sigma(\alpha,\phi) + \left[\frac{\partial^2}{\partial z^2}(x\Phi_1 + y\Phi_2) - 2\eta\left(\frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_2}{\partial y}\right) - \frac{\partial \psi}{\partial z}\right]_{\beta=\pi},$$

for the determination of the harmonic function Φ_3.

To illustrate the method of solution of a mixed boundary value problem of this type we consider the case when the crack is deformed by two normal point forces P acting in opposite directions and applied at the points $\alpha = \alpha_o$, $\beta = \pm\pi$, $\phi = 0$ (cf. Fig. 47).

Since there are no external shears acting on the crack it follows from equations (3.9.7) that we may take $\Phi_1 = \Phi_2 = \psi = 0$, $\Phi_4 = (1 - 2\eta)\Phi_3$ and the problem is reduced to finding a single harmonic function $\Phi = \Phi_3$ satisfying the boundary conditions

$$\Phi = 0 \quad \text{on} \quad \beta = 0, \qquad \frac{\partial \Phi}{\partial z} = \sigma(\alpha,\phi) \quad \text{on} \quad \beta = \pi. \tag{3.9.10}$$

If we assume that

$$\Phi = \sqrt{\cosh\alpha + \cos\beta}\sum_{m=0}^{\infty}\cos m\phi\int_0^{\infty}A_m(\tau)\frac{\sinh\beta\tau}{\tau\cosh\pi\tau}P^m_{-\frac{1}{2}+i\tau}(\cosh\alpha)\,d\tau, \tag{3.9.11}$$

where $P^m_v(x)$ denotes the associated Legendre function, then the first of the conditions (3.9.10) is satisfied automatically and the second leads to the relation

$$\sigma(\alpha,\phi) = -(\cosh\alpha - 1)^{\frac{3}{2}}\sum_{m=0}^{\infty}\cos m\phi\int_0^{\infty}A_m(\tau)P^m_{-\frac{1}{2}+i\tau}(\cosh\alpha)\,d\tau \tag{3.9.12}$$

for the determination of $A_m(\tau)$. Using the Mehler-Fock inversion formula, we find that, if $m \geq 1$,

$$A_m(\tau) = \frac{2}{\pi}(-1)^{m+1}\tau \tanh(\pi\tau)\int_0^\pi \cos m\phi \, d\phi \times \int_0^\infty \frac{\sigma(\alpha,\phi)}{(\cosh \alpha - 1)^{3/2}} P^{-m}_{-1/2+i\tau}(\cosh \alpha)\sinh \alpha \, d\alpha. \quad (3.9.13)$$

With $m = 0$, the last expression has a factor $\frac{1}{2}$. For the concentrated load under consideration it is easily shown that

$$A_m(\tau) = (-1)^m \frac{P}{\pi b} \frac{\sinh \alpha_0}{\sqrt{\cosh \alpha_0 - 1}} \tau \tanh(\pi\tau) P^{-m}_{-1/2+i\tau}(\cosh \alpha_0). \quad (3.9.14)$$

The function Φ is then given by the expression

$$\Phi = \frac{P}{\pi}[(\cosh \alpha_0 - 1)(\cosh \alpha + \cos \beta)]^{1/2} \sum_{m=0}^{\infty}{}' (-1)^m \cos m\phi \times \int_0^\infty \frac{\tanh \pi\tau}{\cosh \pi\tau} \cdot P^{-m}_{-1/2+i\tau}(\cosh \alpha_0) P^{-m}_{-1/2+i\tau}(\cosh \alpha)\, d\tau, \quad (3.9.15)$$

where the dash denotes that the term corresponding to $m = 0$ should be multiplied by the factor $\frac{1}{2}$.

Using the integral representation

$$(-1)^m P^m_{-1/2+i\tau}(\cosh \alpha) P^{-m}_{-1/2+i\tau}(\cosh \alpha_0)\sqrt{\sinh \alpha \sinh \alpha_0} = \frac{2}{\pi^2}\cosh(\pi\tau)\int_0^\infty Q_{m-1/2}\left(\frac{\cosh s + \cosh \alpha \cosh \alpha_0}{\sinh \alpha \sinh \alpha_0}\right)\cos \tau s \, ds \quad (3.9.16)$$

and the expansion

$$[2(\cosh u - \cos \phi)]^{-1/2} = \frac{2}{\pi} \sum_{m=0}{}' Q_{m-1/2}(\cosh u)\cos m\phi \quad (3.9.17)$$

we obtain the solution in the closed form

$$\Phi = \frac{P}{\pi^2 \rho} \tan^{-1}\left[\frac{1}{\rho}(b^2 - 1)^{1/2}\left(\frac{1 - \cos \beta}{\cosh \alpha + \cos \beta}\right)\right] \quad (3.9.18)$$

where

$$\rho^2 = (x - b)^2 + y^2 + z^2. \quad (3.9.19)$$

Furthermore, in the mid-section $z = 0$, $x^2 + y^2 < 1$, the normal component of stress takes the form

$$[\sigma_{zz}]_{\beta=0} = \frac{P}{\pi^2}(b^2 - 1)^{1/2} \frac{(\cosh \alpha_0 - 1)\cosh^3 \frac{1}{2}\alpha}{\cosh \alpha \cosh \alpha_0 + 1 - \sinh \alpha \sinh \alpha_0 \cos \phi}. \quad (3.9.20)$$

Antisymmetric Problem

In this case we consider the distribution of stress in the upper half-space $0 \leq \beta \leq \pi$ under the boundary conditions

$$\beta = 0: \quad u_x = u_y = 0, \qquad \sigma_{zz} = 0; \tag{3.9.21}$$
$$\beta = \pi: \quad \sigma_{zx} = \sigma_x(\alpha,\phi), \qquad \sigma_{yz} = \sigma_y(\alpha,\phi), \qquad \sigma_{zz} = \sigma(a,\phi). \tag{3.9.22}$$

Through a special choice of two additional boundary conditions, this problem can be reduced to separated boundary problems (Dirichlet, Neumann, or mixed) for each of four functions, harmonic in the upper half-space.

There are some additional difficulties in this case. When the method outlined above is used systematically, one of the boundary conditions is satisfied to within a single plane harmonic function. Consequently, a plane harmonic function containing unknown coefficients, to be determined at a later stage, has to be introduced from the start.

Also, when carrying out the calculations, we find that some of the functions occurring on the righthand side of the boundary conditions do not converge to zero as $\alpha \to \infty$, and therefore cannot be represented by a Mehler-Fock integral. This difficulty can be overcome by introducing "special" solutions of Laplace's equation,

$$f(\alpha,\beta,\phi) = (\cosh \alpha + \cos \beta)^{1/2} e^{\pm 1/2 i\beta} \sum_{m=0}^{\infty} f_m \tanh^m (\tfrac{1}{2}\alpha) e^{im\phi} \tag{3.9.23}$$

discontinuous on $\alpha = \infty$, the line of separation of the boundary conditions.

In conformity with the above, we choose two additional conditions of the following form:

$$\text{on } \beta = 0: \quad F = \mathrm{Re} \sum_{m=0}^{\infty} F_m r^m e^{im\phi}, \tag{3.9.24}$$
$$\text{on } \beta = \pi: \quad \Phi = 0$$

where the constants F_m are, as yet, unspecified.

Then from (3.3.21) we have the boundary conditions

$$\Phi_1 = \tfrac{1}{4}(1-\eta)^{-1} \frac{\partial F}{\partial x} \quad \text{on} \quad \beta = 0; \qquad \frac{\partial \Phi_1}{\partial z} = \tfrac{1}{2}(1-\eta)^{-1}\sigma_x \quad \text{on} \quad \beta = \pi; \tag{3.9.25}$$

$$\Phi_2 = \tfrac{1}{4}(1-\eta)^{-1} \frac{\partial F}{\partial y} \quad \text{on} \quad \beta = 0; \qquad \frac{\partial \Phi_2}{\partial z} = \tfrac{1}{2}(1-\eta)^{-1}\sigma_y \quad \text{on} \quad \beta = \pi, \tag{3.9.26}$$

for the determination of the functions Φ_1, Φ_2.

Once Φ_1 and Φ_2 are known, we can find the harmonic function ω, defined by

$$\omega = 2(1 - \eta)\Phi_3 - \Phi_4, \tag{3.9.27}$$

from the boundary conditions:

$$\text{on } \beta = 0: \quad \frac{\partial \omega}{\partial z} = 0,$$

$$\text{on } \beta = \pi: \quad \frac{\partial \omega}{\partial z} = \sigma(\alpha,\phi) + \left(x \frac{\partial^2 \Phi_1}{\partial z^2} + y \frac{\partial^2 \Phi_2}{\partial z^2}\right)_{\beta=\pi} - 2\eta\left(\frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_2}{\partial y}\right)_{\beta=\pi} \tag{3.9.28}$$

In this way we find that ω is the solution of a Neumann problem for the upper half-space.

Using (3.9.27) we find that the additional condition (3.9.24) leads to the condition

$$\Phi_4 = (1 - 2\eta)(\omega)_{\beta=\pi} - (x\sigma_{zx} + y\sigma_{zy})_{\beta=\pi} \tag{3.9.29}$$

on $\beta = \pi$ for the function Φ_4. The second boundary condition for this function can be obtained by applying the operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

to equation (3.9.24), rewritten as follows:

$$\text{On } \beta = 0: \quad \Phi_0 = \operatorname{Re} \sum_{m=1}^{\infty} F_m \left[1 - \frac{m}{4(1 - \eta)}\right] r^m e^{im\phi} + \text{constant}. \tag{3.9.30}$$

Remembering that

$$\Phi_4 = \frac{\partial \Phi_0}{\partial z},$$

we get

$$\frac{\partial \Phi_4}{\partial z} = 0 \qquad \text{on} \qquad \beta = 0. \tag{3.9.31}$$

Hence the function Φ_4 can be obtained by means of the Mehler-Fock integral transformation, as above.

In the solution of Lowengrub and Sneddon (1965) we use cylindrical coordinates (ρ,ϕ,z). We assume (i) that the crack face $z = 0+$, $\rho \geq 1$ is loaded in exactly the same way as the face $z = 0-$, $\rho \geq 1$, and (ii) that the loading function $p(\rho,\phi)$, is an *even* function of ϕ. The case in which (ii) is not satisfied can be discussed in exactly the same way, the only difference being that in the general case the Fourier series for $p(\rho,\phi)$, $0 < \phi < 2\pi$ contains sine terms as well as cosine terms. We cannot remove the restriction

(i) by such a simple extension; we need to retain this condition if we are to apply the method of reducing the problem of finding the stress field in the infinite solid to that of finding it in a half-space $z \geq 0$ when the surface $z = 0$ is subjected to boundary conditions of mixed type. Because of the symmetry of the loading with respect to the plane $z = 0$, we see that the displacement component $u_z(\rho,\phi,0)$ is zero inside the circle $\rho \leq 1$, $z = 0$. Also, as a result of the symmetry, we can postulate that the shearing stress vanishes on the boundary plane of the half-space.

In this way, we obtain the boundary conditions

$$u_z(\rho,\phi,0) = 0, \qquad 0 \leq \phi \leq 2\pi, \quad 0 \leq \rho \leq 1, \tag{3.9.32}$$

$$\sigma_{zz}(\rho,\phi,0) = -p(\rho,\phi), \quad 0 \leq \phi \leq 2\pi, \qquad \rho > 1, \tag{3.9.33}$$

$$\sigma_{\rho z}(\rho,\phi,0) = 0, \qquad 0 \leq \phi \leq 2\pi, \qquad \rho \geq 0, \tag{3.9.34}$$

and we assume that $p(\rho,\phi)$ is an even function of ϕ.

Following Muki (1960) we may take a solution of the equations of elastic equilibrium in the form

$$u_\rho = \sum_{m=0}^{\infty} \cos m\phi \{\mathscr{H}_{m-1}[\xi^{-2}\psi_m(\xi)(1 - 2\eta - \xi z)e^{-\xi z}; \xi \to \rho]$$
$$- m\rho^{-1}\mathscr{H}_m[\xi^{-3}\psi_m(\xi)(1 - 2\eta - \xi z)e^{-\xi z}; \xi \to \rho]\}, \tag{3.9.35}$$

$$u_\phi = -2\rho^{-1}\sum_{m=0}^{\infty} m \sin m\phi \mathscr{H}_m[\xi^{-3}\psi_m(\xi)(1 - 2\eta - \xi z)e^{-\xi z}; \xi \to \rho], \tag{3.9.36}$$

$$u_z = \sum_{m=0}^{\infty} \cos m\phi \mathscr{H}_m[\xi^{-2}\psi_m(\xi)(2 - 2\eta + \xi z)e^{-\xi z}; \xi \to \rho]. \tag{3.9.37}$$

It is easily verified that on the plane $z = 0$,

$$\sigma_{\rho z} = \sigma_{\phi z} = 0 \tag{3.9.38}$$

$$u_z = 2(1 - \eta)\sum_{m=0}^{\infty} \mathscr{H}_m[\xi^{-2}\psi_m(\xi);\rho]\cos m\phi, \tag{3.9.39}$$

$$\sigma_{zz} = -\frac{E}{1+\eta}\sum_{m=0}^{\infty} \mathscr{H}_m[\xi^{-1}\psi_m(\xi);\rho]\cos m\phi, \tag{3.9.40}$$

If we write

$$p(\rho,\phi) = \frac{E}{1+\eta}\sum_{m=0}^{\infty} g_m(\rho)\cos m\phi, \tag{3.9.41}$$

then we see that the boundary conditions (3.9.32) through (3.9.34) are satisfied if the function $\psi_m(\xi)$ is the solution of the pair of dual integral equations

$$\int_0^\infty \xi^{-1}\psi_m(\xi)J_m(\xi\rho)\, d\xi = 0, \qquad 0 < \rho < 1, \tag{3.9.42}$$

$$\int_0^\infty \psi_m(\xi)J_m(\xi\rho)\, d\psi = g_m(\rho), \qquad \rho > 1. \tag{3.9.43}$$

The dual integral equations (3.9.42), (3.9.43) have been considered by Noble (1958), W. E. Williams (1961), and Lowengrub and Sneddon (1962). Putting $\beta = \frac{1}{2}$, $\nu = m$ in the general solution of the last paper, we see that we may take

$$\psi_m(\xi) = \left(\frac{2\xi}{\eta}\right)^{1/2} \int_1^\infty t^{m+1/2} G_m(t) J_{m-1/2}(\xi t)\, dt, \tag{3.9.44}$$

where

$$G_m(t) = \int_t^\infty \frac{\rho^{1-m} g_m(\rho)\, d\rho}{\sqrt{\rho^2 - t^2}}\,. \tag{3.9.45}$$

The general solution of the problem is obtained by substituting from equations (3.9.44) and (3.9.45) into equations (3.9.35) through (3.9.37). We can, however, derive simple expressions for the normal displacement and stress on the plane $z = 0$.

For instance, from equation (3.9.39) we note that

$$u_z(\rho,\phi,0) = \frac{2\sqrt{2}}{\sqrt{\pi}}(1 - \eta) \sum_{m=0}^\infty \cos m\phi$$

$$\times \int_1^\infty t^{m+1/2} G_m(t)\, dt \int_0^\infty \xi^{1/2 m} J_m(\xi\rho) J_{m-1/2}(\xi t)\, d\xi.$$

Using formula (34) in Erdélyi (1954, Vol. 2, p. 92), we find that

$$\int_0^\infty \xi^{1/2} J_m(\xi\rho) J_{m-1/2}(\xi t)\, d\xi = \sqrt{\left(\frac{2}{\pi}\right)}\, t^{m-1/2} \rho^{-m} (\rho^2 - t^2)^{-1/2} H(\rho - t),$$

from which it follows that

$$u_z(\rho,\phi,0) = \frac{4(1 - \eta)}{\pi} \sum_{m=0}^\infty \rho^{-m} \cos m\phi \int_1^\rho \frac{t^{2m} G_m(t)\, dt}{\sqrt{\rho^2 - t^2}}, \quad \rho > 1. \tag{3.9.46}$$

Similarly, if we make use of the result

$$\int_0^\infty \xi^{3/2} J_m(\xi\rho) J_{m-1/2}(\xi t)\, d\xi = \frac{2\sqrt{2}}{\pi} \rho^m t^{1/2-m} (t^2 - \rho^2)^{-3/2} H(t - \rho),$$

we obtain the equation

$$\sigma_{zz}(\rho,\phi,0) = -\frac{4E}{\pi^2(1 + \eta)} \sum_{m=0}^\infty \rho^m \cos m\phi \int_1^\infty t(t^2 - \rho^2)^{-3/2} G_m(t)\, dt,$$

$$0 < \rho < 1 \tag{3.9.47}$$

for the distribution of stress across the central circular area.

As an example, we consider the loading shown in Fig. 47 in which point forces of magnitude P act at the points $(a,0,0+)$, $(a,0,0-)$, $(a > 1)$, of the crack surface. Expanding the Dirac delta function $\delta(\phi)$ formally in a Fourier series, we find that we can write

$$g_0(\rho) = \frac{P(1+\eta)}{4\pi^2 E\rho}\,\delta(\rho - a), \qquad g_m(\rho) = \frac{P(1+\eta)}{2\pi^2 E\rho}\,\delta(\rho - a)$$

and hence that

$$G_0(t) = \frac{P(1+\eta)}{4\pi^2 E}(a^2 - t^2)^{-1/2}H(a - t),$$

$$G_m(t) = \frac{P(1+\eta)}{2\pi^2 E}\,a^{-m}(a^2 - t^2)^{-1/2}H(a - t).$$

We see that

$$\int_1^\rho \frac{t^{2m}G_m(t)\,dt}{\sqrt{\rho^2 - t^2}} = \frac{P(1+\eta)}{2\pi^2 E}\,a^{-m}\int_1^\rho \frac{t^{2m}H(a-t)\,dt}{\sqrt{(\rho^2 - t^2)(a^2 - t^2)}}.$$

The two cases $1 < \rho < a$ and $\rho > a$ must be treated separately. For $1 < t < \rho < a$ we have the equation

$$\int_1^\infty \frac{t^{2m}\,dt}{\sqrt{(\rho^2 - t^2)(a^2 - t^2)}} = \frac{\rho^{2m}}{a} I_m(k_1,\beta_1)$$

where

$$I_m(k,\beta) = \int_\beta^{\frac{1}{2}\pi} \frac{\sin^{2m}\theta\,d\theta}{\sqrt{1 - k^2\sin^2\theta}} \tag{3.9.48}$$

and

$$k_1 = \frac{\rho}{a}, \qquad \beta_1 = \sin^{-1}\frac{1}{\rho} \tag{3.9.49}$$

so that, in the usual notation for elliptic integrals (Byrd and Friedman, 1954), we have the formula

$$u_z(\rho,\phi,0) = \frac{(1-\eta^2)P}{\pi^3 a^2 E}\left[K(k_1) - F(k_1,\beta_1) + 2a\sum_{m=1}^{\infty}\left(\frac{\rho}{a}\right)^m I_m(k_1,\beta_1)\cos m\phi\right],$$

$$1 < \rho < a. \tag{3.9.50}$$

Similarly, if $\rho > a$, we find that

$$\int_1^\rho \frac{t^{2m}G_m(t)\,dt}{\sqrt{\rho^2 - t^2}} = \frac{P(1+\eta)}{2\pi^2 E}\,a^{-m}\,\frac{a^{2m}}{\rho}\,I_m(k_2,\beta_2)$$

where

$$k_2 = \frac{a}{\rho}, \qquad \beta_2 = \sin^{-1}\left(\frac{1}{\alpha}\right) \tag{3.9.51}$$

so that

$$u_z(\rho,\phi,0)=\frac{(1-\eta^2)P}{\pi^3\rho^2E}\left[K(k_2)-F(k_2,\beta_2)+2\rho\sum_{m=1}^{\infty}\left(\frac{a}{\rho}\right)^m I_m(k_2,\beta_2)\cos m\phi\right]\rho>a. \tag{3.9.52}$$

The integrals (3.9.48) can be evaluated numerically for any particular values of m, k, β.

The solution of the problem when the loading function $p(\rho,\phi)$ does not depend on the angle ϕ assumes a much simpler form.

In problems in which the loading on the crack surface is axisymmetric, equations (3.9.35) through (3.9.37) reduce to the simpler forms

$$u_\rho(\rho,z) = -\mathscr{H}_1[\xi^{-2}\psi(\xi)(1-2\eta-\xi z)e^{-\xi z};\ \xi\to\rho], \tag{3.9.53}$$

$$u_\phi(\rho,z) = 0, \tag{3.9.54}$$

$$u_z(\rho,z) = \mathscr{H}_0[\xi^{-2}\psi(\xi)(2-2\eta+\xi z)e^{-\xi z};\ \xi\to\rho], \tag{3.9.55}$$

where for a symmetric load

$$\sigma_{zz}(\rho,0) = -\frac{E}{1+\eta}\,g(\rho), \qquad \rho>1. \tag{3.9.56}$$

The function $\psi(\xi)$ entering into these equations is the solution of the dual integral equations

$$\int_0^\infty \xi^{-1}\psi(\xi)J_0(\xi\rho)\,d\xi = 0, \qquad 0<\rho<1$$

$$\int_0^\infty \psi(\xi)J_0(\xi\rho)\,d\xi = g(\rho), \qquad \rho>1.$$

The solution of these equations is readily shown to be

$$\psi(\xi) = \xi\int_1^\infty \phi(t)\cos(\xi t)\,dt, \qquad \lim_{t\to\infty}\phi(t)=0, \tag{3.9.57}$$

where

$$\phi(t) = \frac{2}{\pi}\int_t^\infty \frac{\rho g(\rho)\,d\rho}{\sqrt{\rho^2-t^2}}. \tag{3.9.58}$$

(see Lowengrub and Sneddon, 1962). It is easily shown that

$$u_z(\rho,0) = 2(1-\eta)F(\rho),\quad \rho>1;\quad \sigma_{zz}(\rho,0) = \frac{E}{1+\eta}G(\rho),\quad 0\le\rho<1, \tag{3.9.59}$$

where the functions $F(\rho)$ and $G(\rho)$ are defined by the pair of equations

$$F(\rho) = \int_1^\rho \frac{\phi(t)\,dt}{\sqrt{\rho^2 - t^2}}, \qquad \rho > 1; \tag{3.9.60}$$

$$G(\rho) = \int_1^\infty \frac{\phi'(t)\,dt}{\sqrt{t^2 - \rho^2}} + \frac{\phi(1)}{\sqrt{1 - \rho^2}} \qquad 0 \le \rho < 1. \tag{3.9.61}$$

If the function $\phi(t)$ is analytic in a neighborhood of $t = 1$, the first term on the right side of equation (3.9.61) will not have a singularity at $\rho = 1$ so that from equations (3.9.59) we deduce that the stress intensity factor is given by the equation

$$K = \lim_{\rho \to 1-} \sqrt{[2(1 - \rho)]}\sigma_{zz}(\rho,0) = \frac{E\phi(1)}{(1 + \eta)}. \tag{3.9.62}$$

We shall now derive the forms of the solution corresponding to certain simple forms of the loading function $g(\rho)$ of equation (3.9.56).

Case (a)

If

$$g(\rho) = \frac{1 + \eta}{E} p_0 H(a - \rho), \qquad a > 1, \tag{3.9.63}$$

then

$$\phi(t) = \frac{2(1 + \eta)p_0}{\pi E}(a^2 - t^2)^{1/2} H(a - t). \tag{3.9.64}$$

It follows immediately from the definition (3.9.60) that if $1 < \rho < a$,

$$F(\rho) = \frac{2(1 + \eta)p_0 a}{\pi E}\left[E\left(\frac{\rho}{a}\right) - E\left(\frac{\rho}{a}, \alpha_1\right)\right], \tag{3.9.65}$$

in the usual notation for elliptic integrals with $\alpha_1 = \sin^{-1}(1/\rho)$, $0 < \alpha_1 < \frac{1}{2}\pi$, and that if $\rho > a$,

$$F(\rho) = \frac{2(1 + \eta)p_0}{\pi E}\left[\rho\left\{E\left(\frac{a}{\rho}\right) - E\left(\frac{a}{\rho}, \alpha_2\right)\right\} - \frac{\rho^2 - a^2}{\rho}\left\{K\left(\frac{a}{\rho}\right) - F\left(\frac{a}{\rho}, \alpha_2\right)\right\}\right] \tag{3.9.66}$$

with $\alpha_2 = \sin^{-1}\left(\frac{1}{a}\right)$, $0 < \alpha < \frac{1}{2}\pi$.

On the other hand from equations (3.9.61) and (3.9.64) we find that if $0 \le \rho < 1$,

$$G(\rho) = \frac{2(1 + \eta)p_0}{\pi E}\left[\left(\frac{a^2 - 1}{1 - \rho^2}\right)^{1/2} - \sin^{-1}\left(\frac{a^2 - 1}{a^2 - \rho^2}\right)^{1/2}\right]. \tag{3.9.67}$$

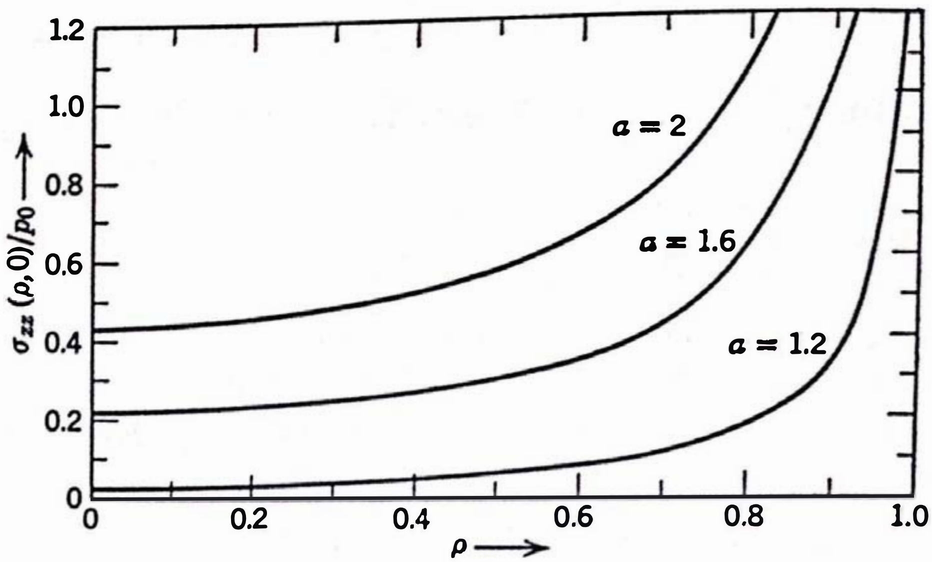


Figure 48 The variation of $\sigma_{zz}(\rho,0)/p_0$ with ρ for the loading defined by (3.9.63).

The variation of $\sigma_{zz}(\rho,0)$ with ρ is shown in Fig. 48 and that of $u_z(\rho,0)$ with ρ is shown in Fig. 49 for three values of a.

From the second equation of the pair (3.9.59) and equation (3.9.67), we see that for this type of loading the stress intensity factor is given by

$$K = 2\left(\frac{p_0}{\pi}\right)\sqrt{a^2 - 1} \tag{3.9.68}$$

Case (b)

If

$$g(\rho) = \frac{1+\eta}{E}\, p_0\rho^{-m}; \qquad m > 1 \tag{3.9.69}$$

we find that

$$\phi(t) = \frac{(1+\eta)p_0\Gamma(\frac{1}{2}m - \frac{1}{2})}{\sqrt{\pi}E\Gamma(\frac{1}{2}m)}\, t^{1-m}. \tag{3.9.70}$$

From equation (3.9.60) we find that

$$F(\rho) = \frac{(1+\eta)p_0}{\sqrt{\pi}E} \cdot \frac{\Gamma(\frac{1}{2}m - \frac{1}{2})}{\Gamma(\frac{1}{2}m)} \int_1^\rho \frac{t^{1-m}\, dt}{\sqrt{\rho^2 - t^2}},$$

a result which in terms of the incomplete beta function

$$B_x(p,q) = \int_0^x u^{p-1}(1-u)^{q-1}\, du$$

can be written in the form

$$F(\rho) = \frac{(1+\eta)p_0}{\sqrt{\pi}E} \cdot \frac{\Gamma(\frac{1}{2}m - \frac{1}{2})}{\Gamma(\frac{1}{2}m)} B_r(\tfrac{1}{2}, 1 - \tfrac{1}{2}m)\rho^{1-m} \qquad (\rho \geq 1) \tag{3.9.71}$$

where $r = 1 - \rho^{-2}$. Numerical values of the incomplete beta function can be obtained from the tables of Pearson (1934).

Similarly, from (3.9.61), we have the relation

$$G(\rho) = \frac{(1+\eta)p_0}{\sqrt{\pi E}} \cdot \frac{\Gamma(\frac{1}{2}m - \frac{1}{2})}{\Gamma(\frac{1}{2}m)} [(1-\rho^2)^{-\frac{1}{2}} - \tfrac{1}{2}(m-1)\rho^{1-m} B_r(\tfrac{1}{2}m, \tfrac{1}{2})], \tag{3.9.72}$$

where in this instance $r = \rho^{\frac{1}{2}}$.

From equations (3.9.59) and (3.9.72) we deduce that in this case

$$\lim_{\rho \to 1-} \sqrt{(1-\rho)}\sigma_{zz}(\rho,0) = \frac{\Gamma(\frac{1}{2}m - \frac{1}{2})}{\sqrt{(2\pi)}\Gamma(\frac{1}{2}m)} p_0. \tag{3.9.73}$$

Case (c)

The results in Case (b) take a simpler form when $m = 2$; that is, when

$$g(\rho) = \frac{(1+\eta)p_0}{E\rho_2}, \qquad \rho > 1, \tag{3.9.74}$$

so that we have

$$\phi(t) = \frac{(1+\eta)p_0}{Et};$$

and from equations (3.9.60) and (3.9.61), we can easily deduce the equations

$$F(\rho) = \frac{(1+\eta)p_0}{E\rho} \log(\rho + \sqrt{\rho^2 - 1}), \qquad \rho > 1, \tag{3.9.75}$$

$$G(\rho) = \frac{(1+\eta)p_0}{E} [(1-\rho^2)^{-\frac{1}{2}} - \rho^{-2}(1 - \sqrt{1-\rho^2})], \quad 0 \le \rho < 1. \tag{3.9.76}$$

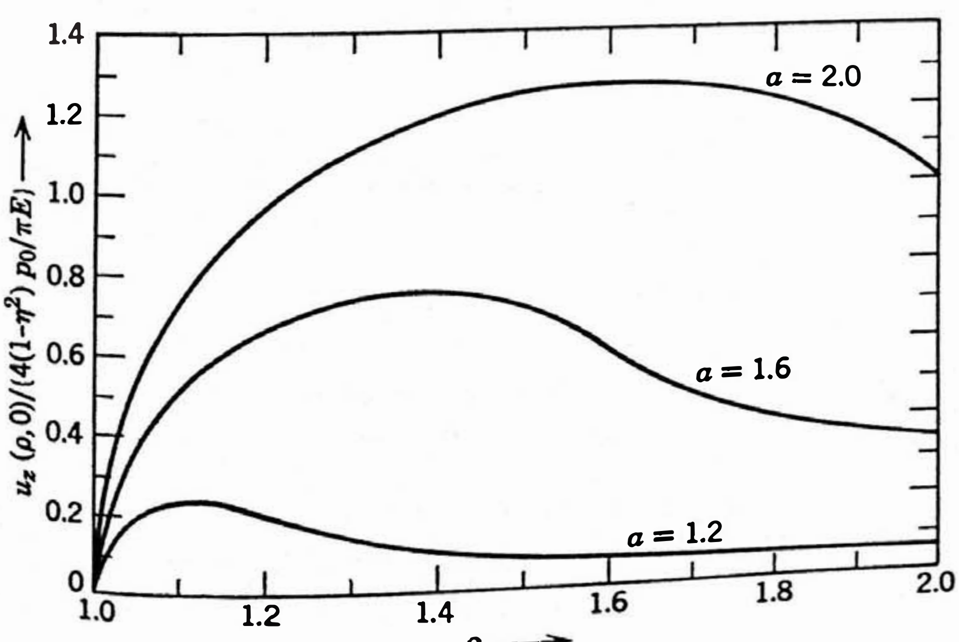


Figure 49 The variation of $u_z(\rho,0)$ with ρ for the loading defined by (3.9.63).

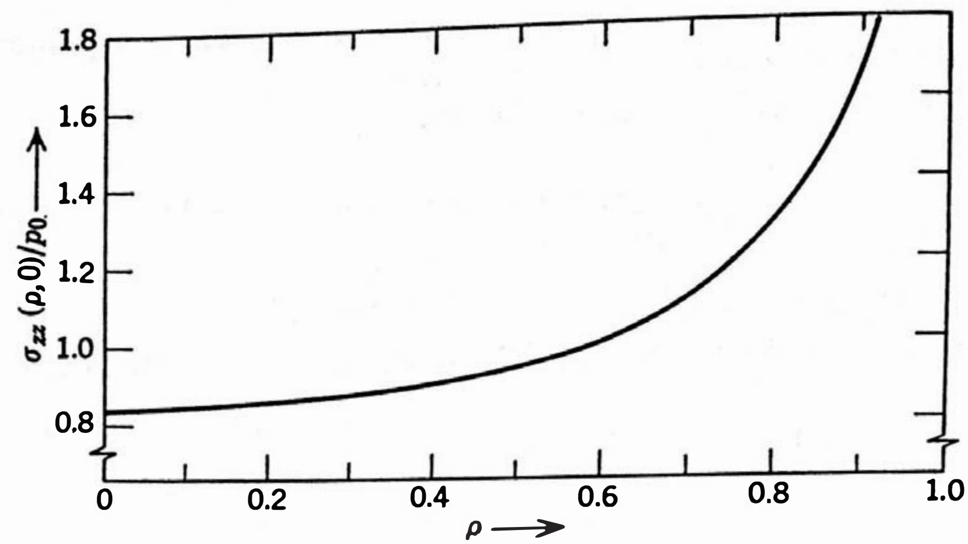


Figure 50 The variation of $\sigma_{zz}(\rho,0)/p_0$ with ρ for the loading $p_0\rho^{-2}(\rho > 1)$.

In this instance,

$$K = p_0. \tag{3.9.77}$$

The variation of $\sigma_{zz}(\rho,0)$ with ρ is shown in Fig. 50 and that of $u_z(\rho,0)$ in Fig. 51.

Case (d)

Finally we consider the case in which

$$g(\rho) = \frac{1+\eta}{E}\cdot\frac{P}{2\pi\rho}\,\delta(\rho - c), \qquad (c > 1), \tag{3.9.78}$$

corresponding to a force of magnitude P distributed uniformly over the

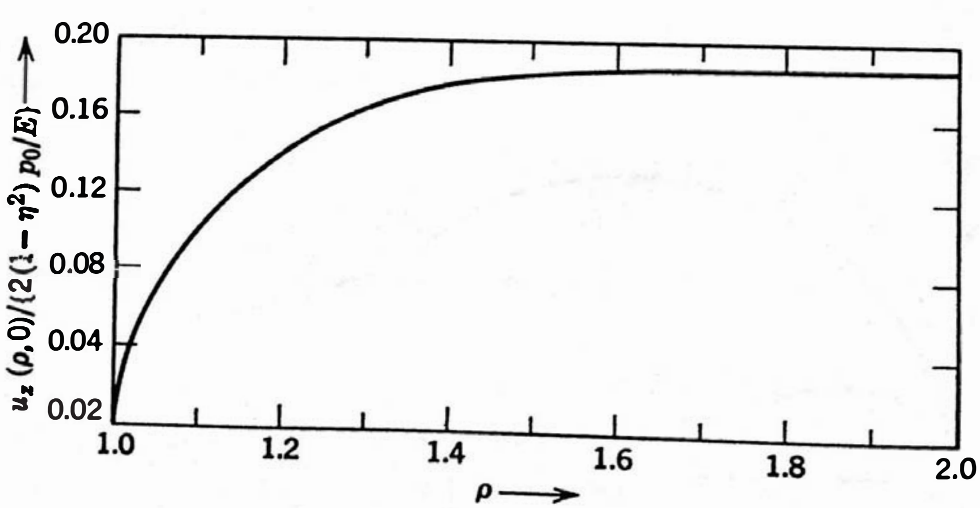


Figure 51 The variation of $u_z(\rho,0)$ with ρ for the loading $p_0\rho^{-2}(\rho > 1)$.

circumference of the circle $\rho = c$. Here we have

$$\phi(t) = \frac{(1+\eta)PH(c-t)}{\pi^2 E\sqrt{c^2 - t^2}}. \tag{3.9.79}$$

Inserting this expression into equation (3.9.60) we find that

$$F(\rho) = \frac{(1+\eta)P}{\pi^2 E}\int_1^{\min(\rho,c)} \frac{dt}{\sqrt{(c^2-t^2)(\rho^2-t^2)}},$$

from which we deduce that

$$F(\rho) = \begin{cases} \dfrac{(1+\eta)P}{\pi^2 E}\left[K\left(\dfrac{\rho}{c}\right) - F\left(\dfrac{\rho}{c}, \alpha_1\right)\right], & 1 < \rho < c, \\[2ex] \dfrac{(1+\eta)P}{\pi^2 E}\left[K\left(\dfrac{c}{\rho}\right) - F\left(\dfrac{c}{\rho}, \alpha_2\right)\right], & \rho > c, \end{cases}$$

with $\alpha_1 = \sin^{-1}(1/\rho)$, $0 < a_1 < \frac{1}{2}\pi$, and $\alpha_2 = \sin^{-1}(1/c)$, $0 < \alpha_2 < \frac{1}{2}\pi$. For a concentrated ring force of this kind

$$K = \frac{P}{\pi^2\sqrt{c^2 - 1}} \tag{3.9.80}$$

The problem of the deformation of an external crack by thermal means has been discussed by Shail (1968).

3.10 PENNY-SHAPED CRACK IN A THICK PLATE

The distribution of stress in the vicinity of a penny-shaped crack in an elastic plate of finite thickness but infinite radius has been discussed by Lowengrub (1961). The crack is taken to lie in the central plane of the plate with its surfaces parallel to those of the plate. We assume that the deformation results from the application of prescribed forces or displacements to the plane boundaries of the plate, these forces being distributed symmetrically about an axis which passes through the center of the crack and is perpendicular to the plane of the crack. If we take the radius of the crack to be our unit of length, the crack therefore occupies the region $0 \le \rho \le 1$, $z = 0$ where (ρ,θ,z) are cylindrical coordinates referred to the center of the crack as origin and having the z-axis along the axis of symmetry of the system. In this system of coordinates, we take the components of the displacement vector to be $(u_\rho, 0, u_z)$. The thickness of the plate will be taken to be 2δ. The problem can best be solved by converting it to a mixed boundary value problem for the elastic layer $0 \le z \le \delta$.

If we assume that the surfaces of the plate are given a uniform displacement ϵ then we have the boundary conditions:

$$\text{on } z = \delta: \quad u_z = \epsilon, \qquad \sigma_{\rho z} = 0, \quad \rho \geq 0. \tag{3.10.1}$$

$$\begin{aligned} \text{on } z = 0: \quad \sigma_{\rho z} &= 0, \quad \rho \geq 0, \\ \sigma_{zz} &= 0, \quad 0 \leq \rho \leq 1, \\ u_z &= 0, \quad \rho > 1. \end{aligned} \tag{3.10.2}$$

The second equation of the set (3.10.2) is valid if there is no pressure applied to the surface of the crack, and the last one is a consequence of the symmetry. If we take the components of the displacement vector to be $(u_\rho, 0, u_z + \epsilon z/\delta)$ we see that u_ρ and u_z must be such as to provide a displacement field satisfying the conditions:

$$\text{on } z = 0: \quad u_z = 0, \qquad \rho \geq 1, \tag{3.10.3}$$

$$\sigma_{\rho z} = 0, \qquad \rho \geq 0, \tag{3.10.4}$$

$$\sigma_{zz} = -\frac{\beta^2\epsilon}{\delta}, \qquad 0 \leq \rho \leq 1. \tag{3.10.5}$$

$$\text{on } z = \delta: \quad w = 0, \qquad \rho \geq 0, \tag{3.10.6}$$

$$\sigma_{\rho z} = 0, \qquad \rho \geq 0. \tag{3.10.7}$$

On the other hand, if we assume that the free faces of the plate are pulled by a uniform tension F, we have the boundary conditions (3.10.2) again but the conditions (3.10.1) are replaced by the pair:

$$\text{on } z = \delta: \quad \sigma_{zz} = -F, \qquad \sigma_{zz} = 0, \quad \rho \geq 0. \tag{3.10.8}$$

In this case, if we take the components of the displacement vector to be $(u_\rho, 0, u_z - \beta^2 Fz)$, we must choose the functions u_ρ, u_z in such a way that they lead to a displacement field satisfying the boundary conditions (3.10.3), (3.10.4), (3.10.7) and the conditions:

$$\text{on } z = 0: \quad \sigma_{zz} = -F, \quad 0 < \rho < 1 \tag{3.10.9}$$

$$\text{on } z = \delta: \quad \sigma_{zz} = 0, \qquad \rho > 0. \tag{3.10.10}$$

If we put

$$\psi = 0,$$

$$\chi = -\frac{\delta}{2}\mathscr{H}_0\left[\xi^{-1}f(\xi)\frac{\cosh \xi z}{\sinh^2 \xi\delta}; \xi \to \rho\right] \tag{3.10.11}$$

$$\phi = \frac{1}{2(\beta^2 - 1)}\mathscr{H}_0\left[\xi^{-2}f(\xi)\frac{\cosh \xi(\delta - z)}{\sinh \xi\delta}; \xi \to \rho\right] \tag{3.10.12}$$

in the equations of Section 1.3 we obtain

$$u_z = -\tfrac{1}{2}\mathscr{H}_0\left[\xi^{-1}f(\xi)\left\{\frac{\xi\delta\sinh\xi z}{\sinh^2\xi\delta} - \frac{\xi z\cosh\xi(\delta - z)}{\sinh\xi\delta} - \frac{\beta^2\sinh\xi(\delta - z)}{(\beta^2 - 1)\sinh\xi\delta}\right\};\ \xi \to \rho\right], \quad (3.10.13)$$

$$\sigma_{\rho z} = \mathscr{H}_1\left[\xi f(\xi)\left\{\frac{\delta\sinh\xi z}{\sinh^2\xi\delta} - \frac{z\cosh\xi(\delta - z)}{\sinh\xi\delta}\right\};\ \xi \to \rho\right], \quad (3.10.14)$$

$$\sigma_{zz} = -\mathscr{H}_0\left[f(\xi)\left\{\frac{\xi\delta\cosh\xi z}{\sinh^2\xi\delta} + \frac{\xi z\sinh\xi(\delta - z)}{\sinh\xi\delta} + \frac{\cosh\xi(\delta - z)}{\sinh\xi\delta}\right\};\ \xi \to \rho\right], \quad (3.10.15)$$

from which it follows immediately that this solution satisfies equations (3.10.4), (3.10.6), and (3.10.7). Putting $z = 0$ in equations (3.10.13) and (3.10.15), we see that this solution will satisfy the remaining boundary conditions (3.10.5) and (3.10.3) if $f(\xi)$ is chosen to be the solution of the dual integral equations

$$\mathscr{H}_0[\{1 + H(\xi\delta)\}f(\xi);\ \rho] = \frac{\beta^2\epsilon}{\delta}, \qquad 0 \le \rho \le 1, \quad (3.10.16)$$

$$\mathscr{H}_0[\xi^{-1}f(\xi);\ \rho] = 0, \qquad \rho \ge 1, \quad (3.10.17)$$

where the function $H(x)$ is defined by

$$H(x) = \frac{x + 2e^{-x}\sinh x}{\sinh^2 x}. \quad (3.10.18)$$

Similarly, if we put

$$\chi = \delta^2\mathscr{H}_0\left[\frac{f(\xi)\cosh(\xi z)}{2\xi\delta + \sinh 2\xi\delta};\ \xi \to \rho\right], \quad (3.10.19)$$

$$\phi = -\frac{1}{\beta^2 - 1}\mathscr{H}_0\left[\xi^{-2}f(\xi)\left\{\tfrac{1}{2}\sinh\xi z - \frac{\sinh^2\xi\delta\cosh\xi z}{2\xi\delta + \sinh 2\xi\delta}\right\};\ \xi \to \rho\right] \quad (3.10.20)$$

in the equations of Section 1.3, we obtain a solution of the equations of elastic equilibrium which satisfies the boundary conditions (3.10.3), (3.10.4), (3.10.7), (3.10.9), and (3.10.10), provided that $f(\xi)$ satisfies the dual integral equations

$$\mathscr{H}_0[M(\xi\delta)f(\xi);\ \xi \to \rho] = F(\rho), \quad 0 < \rho < 1 \quad (3.10.21)$$

$$\mathscr{H}_0[\xi^{-1}f(\xi);\ \xi \to \rho] = 0, \qquad \rho > 1 \quad (3.10.22)$$

where

$$M(u) = \frac{\sinh^2 u - u^2}{2u + \sinh 2u}. \quad (3.10.23)$$

Lowengrub derives the solution of the pair of dual integral equations (3.10.16), (3.10.17) using the method of Lebedev and Ufliand (1958). Equations (3.10.21) and (3.10.22) can be solved by a similar procedure. The essence of the Lebedev-Ufliand method is that it reduces the solution of a pair of dual integral equations to a Fredholm integral equation of the second kind which can then be solved by standard methods.

Since we are interested in values of δ much greater than 1, we write

$$H(u) = 2\sum_{m=1}^{\infty} (1 + 2mu)e^{-2mu}. \tag{3.10.24}$$

Following Lebedev and Ufliand we write

$$f(\xi) = \int_0^1 g(t)\sin(\xi t)\, dt, \qquad g(0) = 0. \tag{3.10.25}$$

It is easily shown that this representation of $f(\xi)$ provides an automatic solution of equation (3.10.17). Substituting from (3.10.25) into (3.10.16) we find that

$$\int_0^\infty J_0(\xi\rho)\, d\xi \int_0^1 g(t)\xi \sin(\xi t)\, dt + \int_0^\infty \xi H(\xi\delta) J_0(\xi\rho)\, d\xi \int_0^1 g(t)\sin(\xi t)\, dt$$
$$= \frac{\beta^2\epsilon}{\delta}, \qquad 0 \le \rho \le 1. \tag{3.10.26}$$

Since $g(0) = 0$, we see on integrating by parts that

$$\int_0^1 g(t)\xi \sin(\xi t)\, dt = -g(1)\cos\xi + \int_0^1 g'(t)\cos(\xi t)\, dt,$$

and using the well-known integral

$$\int_0^\infty J_0(\xi\rho)\cos(\xi t)\, dt = (\rho^2 - t^2)^{-1/2} H(\rho - t), \tag{3.10.27}$$

we find, after a simple change in the order of integrations, that the first term on the left side of equation (3.10.26) is equal to

$$\int_0^\rho \frac{g'(t)\, dt}{\sqrt{\rho^2 - t^2}}.$$

If in the second term on the left side of equation (3.10.26) we replace $J_o(\xi\rho)$ by its integral representation

$$\frac{2}{\pi}\int_0^\rho \frac{\cos(u\xi)\, du}{\sqrt{\rho^2 - u^2}},$$

we find that this term reduces to

$$\frac{1}{\pi}\int_0^\rho \frac{du}{\sqrt{\rho^2 - u^2}}\int_0^1 [H^*(u+t) - H^*(u-t)]\phi(t)\,dt,$$

where

$$H^*(w) = \delta\int_0^\infty \xi H(\xi\delta)\sin(\xi w)\,d\xi. \tag{3.10.28}$$

In this way we obtain the relation

$$\int_0^\rho \frac{g'(t)\,dt}{\sqrt{\rho^2 - t^2}} + \frac{2}{\pi\delta}\int_0^\rho \frac{du}{\sqrt{\rho^2 - u^2}}\int_0^1 g(t)K_1(u,t)\,dt = \frac{\beta^2\epsilon}{\delta}, \tag{3.10.29}$$

for $0 \leq \rho \leq 1$, where

$$K_1(u,t) = \tfrac{1}{2}[H^*(u+t) - H^*(u-t)]. \tag{3.10.30}$$

We now let

$$G(\rho) = \int_0^\rho \frac{g'(t)\,dt}{\sqrt{\rho^2 - t^2}}, \tag{3.10.31}$$

so that

$$g'(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{vG(v)\,dv}{\sqrt{t^2 - v^2}}.$$

Since $g(0) = 0$ we may integrate to obtain

$$g(t) = \frac{2}{\pi}\int_0^t \frac{vG(v)\,dv}{\sqrt{t^2 - v^2}}. \tag{3.10.32}$$

If we now substitute this expression into (3.10.29), change the order of the integrations, and write

$$K(\rho,v) = v\int_v^1 \frac{dt}{\sqrt{t^2 - v^2}}\int_0^\rho \frac{\delta^2 K_1(u,t)\,du}{\sqrt{\rho^2 - u^2}}, \qquad 0 < \rho < 1,$$

we find that (3.10.29) reduces to the Fredholm equation

$$G(\rho) + \frac{4}{\pi^2\delta^3}\int_0^1 G(v)K(\rho,v)\,dv = \frac{\beta\epsilon^2}{\delta}, \qquad 0 < \rho < 1. \tag{3.10.33}$$

By the classical theory of these equations (Tricomi, 1957, p. 49) we can obtain a solution of this equation as a power series in $4\pi^{-2}\delta^{-3}$, provided that δ is sufficiently large for

$$\frac{1}{4\pi^2\delta^3} > M,$$

where M is the bound on the kernel $K(\rho,v)$. In this way we obtain the solution

$$G(\rho) = \beta^2\epsilon\,\delta^{-1}\left[1 - 4\pi^{-2}\,\delta^{-3}\int_0^1 K(\rho,v)\,dv\right.$$
$$\left. + 16\pi^{-4}\,\delta^{-6}\int_0^1\int_0^1 K(p,v)K(v,w)\,dv\,dw + \cdots\right].$$

Since $K(\rho,v)$ is of order zero in δ, this means that

$$G(\rho) = \beta^2\epsilon\,\delta^{-1}\left[1 - 4\pi^{-2}\,\delta^{-3}\int_0^1 K(\rho,v)\,dv\right] + O(\delta^{-7}). \quad (3.10.34)$$

To obtain the solution of equation (3.2.33) to this order we need only calculate

$$\int_0^1 K(\rho,\,v)\,dv = \int_0^\rho \frac{dw}{\sqrt{\rho^2 - w^2}}\int_0^1 t[\delta^2K_1(w,t)]\,dt,$$

where $K_1(w,t)$ is defined by the equations (3.10.30), (3.10.28), and (3.10.24). The integrations involved are elementary and Lowengrub shows that to the order stated

$$\int_0^1 t[\delta^2K_1(w,t)]\,dt = \tfrac{2}{3}\zeta(3) - \frac{3(5w^2+2)}{20\delta^2}\zeta(5) + O(\delta^{-4}), \quad (3.10.35)$$

where $\zeta(3)$, $\zeta(5)$ denote the Riemann zeta-functions

$$\zeta(3) = \sum_{m=1}^{\infty} m^{-3} \simeq 1.202, \qquad \zeta(5) = \sum_{m=1}^{\infty} m^{-5} \simeq 1.037.$$

Substituting from (3.10.35) into (3.10.34), we find that

$$G(\rho) = \frac{\beta^2\epsilon}{\delta}\left[1 - \frac{4\zeta(3)}{3\pi\delta^3} + \frac{3(5\rho^2+4)\zeta(5)}{20\pi\delta^5} + O(\delta^{-7})\right]. \quad (3.10.36)$$

From equations (3.10.32) and (3.10.25) we find that

$$f(\xi) = \frac{2}{\pi}\int_0^1 \sin(\xi t)\,dt\int_0^t \frac{\rho G(\rho)\,d\rho}{\sqrt{t^2-\rho^2}},$$
$$= -\frac{2\beta^2\epsilon}{\pi}\,[f_0D - \tfrac{2}{3}f_1D^3]\frac{\sin\xi}{\xi} + O(\delta^{-7}), \quad (3.10.37)$$

where D denotes the operator $d/d\xi$ and

$$f_0 = \delta^{-1}\left[1 - \frac{4\zeta(3)}{3\pi\delta^3} + \frac{3\zeta(5)}{5\pi\delta^5}\right] \quad (3.10.38)$$
$$f_1 = \frac{3\zeta(5)}{4\pi\delta^5}.$$

If we now put $z = 0$ in equation (3.10.13), we find that the shape of the crack is given by

$$u_z(\rho,0) = \frac{\beta^2}{2(\beta^2 - 1)} \mathscr{H}_0[\xi^{-1}f(\xi); \rho], \qquad 0 \le \rho \le 1.$$

Making use of the representation (3.10.25) and the well-known integral

$$\int_0^\infty J_0(\xi\rho)\sin(\xi t)\, d\xi = (t^2 - \rho^2)^{-1/2} H(t - \rho),$$

we find that

$$u_z(\rho,0) = \frac{\beta^2}{2(\beta^2 - 1)} \int_\rho^1 \frac{g(t)\, dt}{\sqrt{t^2 - \rho^2}},$$

which, because of (3.10.32), can be written in the form

$$u_z(\rho,0) = \frac{\beta^2}{\pi(\beta^2 - 1)} \int_\rho^1 \frac{dt}{\sqrt{t^2 - \rho^2}} \int_0^t \frac{vG(v)\, dv}{\sqrt{t^2 - v^2}}.$$

Substituting for $G(v)$ from (3.10.36) we find that

$$u_z(\rho,0) = \frac{\beta^4\epsilon}{\pi(\beta^2 - 1)} \sqrt{(1 - \rho^2)}k(\rho,\delta), \tag{3.10.39}$$

with

$$k(\rho,\delta) = 1 - \frac{4\zeta(3)}{3\pi\delta^3} - \frac{(23 + 10\rho^2)\zeta(5)}{30\pi\delta^5} + O(\delta^{-7}). \tag{3.10.40}$$

Using the numerical values quoted above for the zeta-functions we find that

$$k(\rho,\delta) = 1 - 0.510\delta^{-3} - 0.011\delta^{-5}(23 + 10\rho^2) + O(\delta^{-7}). \tag{3.10.41}$$

3.11 THE EFFECT OF A PENNY-SHAPED CRACK ON THE DISTRIBUTION OF STRESS IN A LONG CIRCULAR CYLINDER UNDER TENSION

In this section we shall give an analysis (based on Sneddon and Tait, 1963, and Sneddon and Welch, 1963) of the state of stress in an infinitely long cylinder $0 \le \rho \le a$, $-\infty < z < \infty$, containing a penny-shaped crack lying in the plane $z = 0$. For convenience we take the unit of length in our problem to be the radius of the crack, so that we have the relations $0 \le \rho \le 1$, $z = 0$ defining the region occupied by the crack. We shall consider separately two sets of boundary conditions:

Case (i): In the first case we assume that the cylindrical surface is free from shear and is supported in such a way that the radial component of the displacement vector vanishes on the surface. Such a situation would arise physically if the elastic cylinder were resting in a cylindrical hollow (of exactly the same radius) in a rigid body and were then deformed by the application of a known pressure to the surfaces of the crack. The problem of determining the distribution of stress in the vicinity of the crack is equivalent to that of finding the distribution of stress in the semi-infinite cylinder $z \geq 0, 0 \leq \rho \leq a$ when its plane boundary $z = 0$ is subjected to the conditions:

$$\sigma_{\rho z} = 0, \qquad 0 \leq \rho \leq a, \tag{3.11.1}$$

$$\sigma_{zz} = -p(\rho), \quad 0 \leq \rho \leq 1, \tag{3.11.2}$$

$$u_z = 0, \qquad 1 \leq \rho \leq a, \tag{3.11.3}$$

and its curved boundary, $\rho = a$, is subjected to the conditions

$$u_\rho = 0, \qquad \sigma_{\rho z} = 0, \quad z \geq 0 \tag{3.11.4}$$

which may be written in the alternative forms

$$\mathscr{F}_c[u_\rho(a,z); z \to \xi] = 0 \tag{3.11.5}$$

$$\mathscr{F}_s[\sigma_{\rho z}(a,z); z \to \xi] = 0 \tag{3.11.6}$$

We further assume that as $r = \sqrt{\rho^2 + z^2} \to \infty$, all the components of the stress and of the displacement vector tend to zero.

Case (ii)*:* In the second case we assume that the cylindrical surface is stress free. The conditions (3.11.1) through (3.11.3) remain the same, but equations (3.11.4) are replaced by the pair

$$\sigma_{\rho\rho} = 0, \qquad \sigma_{\rho z} = 0, \quad \rho = a, \quad z \geq 0$$

so that equation (3.11.6) remains unaltered but equation (3.11.5) is replaced by

$$\mathscr{F}_c[\sigma_{\rho\rho}(a,z); z \to \xi] = 0. \tag{3.11.7}$$

A particular instance of both cases is of special interest. If the cylinder is in a state of uniform tension p_o in the direction of its axis then the conditions at infinity are replaced by $\sigma_{zz} = p_o$, $\sigma_{\rho\rho} = \sigma_{\rho z} = \sigma_{\theta\theta} = 0$ and equation (3.11.2) is replaced by the equation $\sigma_{zz} = 0, 0 \leq \rho \leq 1, z = 0$. The solutions to the two cases will be obtained if we add to the solution for a cylinder in a state of tension, the solution of the equations of elastic equilibrium satisfying the boundary conditions of either Case (i) or Case (ii). The problem for a cylinder in a state of tension containing a penny-shaped crack whose surfaces are stress free is therefore equivalent to that for a cylinder containing a crack opened out by constant pressure but otherwise free from external forces.

Conditions on the Crack Face

Because of the similarity between the two problems, we divide the solution of each into two parts. In the first part, which is common to both Case (i) and Case (ii), we make use of the fact (cf. Sneddon, 1951, p. 505.) that a solution of the equations of elastic equilibrium in the axially symmetric case is given by

$$u_\rho(\rho,z) = -\frac{1}{2\mu}\frac{\partial^2 V}{\partial\rho\,\partial z}, \qquad u_z(\rho,z) = \frac{1}{2\mu}\left[2(1-\eta)\,\nabla^2 V - \frac{\partial^2 V}{\partial z^2}\right], \tag{3.11.8}$$

where $V(\rho,z)$ is an axisymmetric biharmonic function. The stress components can easily be determined from the stress-strain relations. We have

$$\sigma_{\rho\rho} = \frac{\partial}{\partial z}\left(\eta\,\nabla^2 V - \frac{\partial^2 V}{\partial \rho^2}\right), \qquad \sigma_{\rho z} = \frac{\partial}{\partial \rho}\left[(1-\eta)\,\nabla^2 V - \frac{\partial^2 V}{\partial z^2}\right],$$

$$\sigma_{\theta\theta} = \frac{\partial}{\partial z}\left(\eta\,\nabla^2 V - \frac{1}{\rho}\frac{\partial V}{\partial \rho}\right), \qquad \sigma_{zz} = \frac{\partial}{\partial z}\left[(2-\eta)\,\nabla^2 V - \frac{\partial^2 V}{\partial z^2}\right].$$

A suitable type of biharmonic function for a problem of this type is defined by

$$V(\rho,z) = -\mathscr{F}_s[\xi^{-2}\{a(\xi) + 4(1-\eta)b(\xi)\}I_0(\xi\rho) - \xi^{-1}\rho b(\xi)I_1(\xi\rho);\ \xi \to z]$$
$$-\mathscr{H}_0[\xi^{-3}\psi(\xi)(2\eta + \xi z)e^{-\xi z};\ \xi \to \rho], \tag{3.11.9}$$

where $a(\xi)$, $b(\xi)$, and $\psi(\xi)$ are functions of ξ alone. A solution of this form automatically satisfies equation (3.11.1). The corresponding expressions for σ_{zz} and u_z on the plane $z = 0$ are given respectively by the equations

$$\sigma_{zz} = -\sqrt{\frac{2}{\pi}}\int_0^\infty \xi[\{a(\xi) - 2\eta b(\xi)\}I_0(\xi\rho) - \xi\rho b(\xi)I_1(\xi\rho)]\,d\xi - \mathscr{H}_0[\psi(\xi);\rho], \tag{3.11.10}$$

$$\mu u_z = (1-\eta)\mathscr{H}_0[\xi^{-1}\psi(\xi);\rho]. \tag{3.11.11}$$

From these equations it follows immediately that the boundary conditions (3.11.2) and (3.11.3) are satisfied if $a(\xi)$, $b(\xi)$, and $\psi(\xi)$ satisfy the dual integral equations

$$\mathscr{H}_0[\xi\psi(\xi);\rho] + \sqrt{\frac{2}{\pi}}\int_0^\infty \xi[\{a(\xi) - 2\eta b(\xi)\}I_0(\xi\rho) - \xi\rho b(\xi)I_1(\xi\rho)]\,d\xi = p(\rho),$$
$$0 \le \rho \le 1, \tag{3.11.12}$$

$$\mathscr{H}_0[\xi^{-1}\psi(\xi);\rho] = 0, \qquad \rho \ge 1. \tag{3.11.13}$$

Equation (3.11.13) is identical with equation (3.2.6) so it also has the solution (3.2.7). Substituting this form into equation (3.11.12) we find by comparison with equation (3.2.9) that a relation between $a(\xi)$, $b(\xi)$, and the function $g(t)$ is

$$g(t) = h(t) - \left(\frac{2}{\pi}\right)^{3/2} \int_0^\infty \xi\left[(a - 2\eta b)\int_0^t \frac{\rho I_0(\xi\rho)\,d\rho}{\sqrt{t^2-\rho^2}} - \xi b\int_0^t \frac{\rho^2 I_1(\xi\rho)\,d\rho}{\sqrt{t^2-\rho^2}}\right] d\xi,$$

where $h(t)$ is defined in terms of $p(\rho)$ by

$$h(t) = \frac{2}{\pi}\int_0^t \frac{\rho p(\rho)\,d\rho}{\sqrt{t^2-\rho^2}}. \tag{3.11.14}$$

Making use of the results

$$\int_0^t \frac{\rho I_0(\xi\rho)\,d\rho}{\sqrt{t^2-\rho^2}} = \frac{\sinh(\xi t)}{\xi}, \qquad \int_0^t \frac{\rho^2 I_1(\xi\rho)\,d\rho}{\sqrt{t^2-\rho^2}} = \frac{\xi t\cosh(\xi t) - \sinh(\xi t)}{\xi^2},$$

we find that the relation connecting $g(t)$ with $a(\xi)$ and $b(\xi)$ may be written in the form

$$g(t) = h(t) - \left(\frac{2}{\pi}\right)^{3/2}\int_0^\infty [\{a(\xi) + (1-2\eta)b(\xi)\}\sinh(\xi t) - \xi t b(\xi)\cosh(\xi t)]\,d\xi. \tag{3.11.15}$$

Further, if we substitute from (3.2.7) into (3.11.11) we find that the normal component of the surface displacement is given by

$$\mu u_z(\rho,0) = (1-\eta)\int_\rho^1 \frac{g(t)\,dt}{\sqrt{t^2-\rho^2}}, \qquad 0 \le \rho \le 1. \tag{3.11.16}$$

The energy required to open out the crack is given by

$$W_1 = 2\pi\int_0^1 \rho p(\rho) u_z(\rho,0)\,d\rho.$$

If we substitute the expression (3.11.16) for $u_z(\rho,0)$ into this last equation and interchange the order in which we perform the integrations, we find that

$$W_1 = \pi^2\mu^{-1}(1-\eta)\omega(a^{-1}) \tag{3.11.17}$$

where

$$\omega(a^{-1}) = \int_0^1 h(t)g(t)\,dt. \tag{3.11.18}$$

The expression (3.11.17) for the energy W_1 is in dimensionless form. If the length of the crack were c, we should have in conventional units

$$W_1 = \frac{2\pi^2(1-\eta^2)c^3}{E}\,\omega\left(\frac{c}{a}\right), \tag{3.11.19}$$

it being assumed that $p(\rho)$ is expressed in conventional units.

In particular, if a constant pressure p_o is applied to the crack, $h(t) = 2p_o t/\pi$, and equations (3.11.17) and (3.11.18) reduce to

$$W_1 = \frac{8(1-\eta^2)c^3 p_0^2}{3E}\,\omega_1\left(\frac{c}{a}\right) = W_\infty \omega_1\left(\frac{c}{a}\right),$$

where W_∞ is the value, $8(1-\eta^2)c^3 p_o^2/3E$, of the energy in the case $a \gg c$, and

$$\omega_1\left(\frac{c}{a}\right) = \tfrac{3}{2}\pi \int_0^1 t\phi(t)\,dt,$$

the function $\phi(t) \equiv g(t)/p_o$ is the solution of the equation

$$\phi(t) = \frac{2}{\pi}t - \left(\frac{2}{\pi}\right)^{1/2} p_0^{-1} \int_0^\infty [\{a(\xi) + (1-2\eta)b(\xi)\}\sinh(\xi t) - \xi t b(\xi)\cosh(\xi t)]\,d\xi. \tag{3.11.20}$$

The corresponding expression for the strain-energy release rate is

$$\mathscr{G}_1 = \frac{1}{2\pi c}\frac{\partial W_1}{\partial c} = \frac{4(1-\eta^2)p_0^2 c}{E}\,g_1\left(\frac{c}{a}\right),$$

where

$$g_1(x) = \omega_1(x) + \tfrac{1}{3}x\omega_1'(x).$$

The Griffith criterion therefore leads to the expression

$$p_{cr} = \left[\frac{\pi E T_0}{2(1-\eta^2)c}\right]^{1/2} \Omega_1\left(\frac{c}{a}\right), \qquad \Omega_1(x) = [g_1(x)]^{-1/2},$$

for the critical value of the applied pressure to cause the crack to spread when its radius is c. If $a \gg c$, the critical pressure takes the value

$$p_{cr}^{(\infty)} = \left[\frac{\pi E T_0}{2(1-\eta^2)c}\right]^{1/2},$$

so that we may write this last equation in the form

$$p_{cr} = p_{cr}^{(\infty)}\Omega_1\left(\frac{c}{a}\right).$$

It is convenient to represent the numerical results by calculating the ratio Π_1 defined by the equation

$$\Pi_1 = \frac{p_{cr}^{(\infty)} - p_{cr}}{p_{cr}^{(\infty)}} \times 100,$$

so that Π_1 denotes the percentage decrease in the value of the critical pressure p_{cr} from the value $p_{cr}^{(\infty)}$ corresponding to a crack in an infinite solid.

***Conditions on the Surface of the Cylinder: Case* (i)**

We now complete the solution by satisfying the boundary conditions on the surface of the cylinder. Equation (3.11.15) gives one relation connecting the unknown functions $a(\xi)$, $b(\xi)$, $g(t)$, and the known function $h(t)$. The two remaining relations necessary for the determination of the unknown functions are given by the prescribed conditions on the curved surface $\rho = a$. It is easily shown that the value of $\sigma_{\rho z}$ and u_ρ on the surface $\rho = a$ corresponding to the form (3.11.9) for $V(\rho,z)$ are given by the equations

$$\sigma_{\rho z}(a,z) = -z\mathscr{H}_1[\zeta\psi(\zeta)e^{-\zeta z}; \zeta \to a] - \mathscr{F}_s[\xi\{a(\xi) + 2(1-\eta)b(\xi)\}I_1(\xi a) - \xi a b(\xi) I_0(\xi a); \xi \to z] \tag{3.11.21}$$

$$u_\rho(a,z) = -\tfrac{1}{2}\mathscr{H}_1[\zeta^{-1}\psi(\zeta)(1 - 2\eta - \zeta z)e^{-\zeta z}; \zeta \to a] + \tfrac{1}{2}\mathscr{F}_c[\{a(\xi) + 4(1-\eta)b(\xi)\}I_1(\xi a) - \xi a b(\xi) I_0(\xi a); \xi \to z]. \tag{3.11.22}$$

If we substitute from equation (3.11.21) into equation (3.11.6) we obtain

$$a(\xi)I_1(\xi a) - b(\xi)[\xi a I_0(\xi a) - 2(1-\eta)I_1(\xi a)] = i_2(\xi) - i_1(\xi), \tag{3.11.23}$$

where the functions $i_1(\xi)$, $i_2(\xi)$ are defined by the integrals

$$i_1(\xi) = 2\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\zeta\psi(\zeta)J_1(\zeta a)}{\xi^2+\zeta^2}\,d\zeta, \qquad i_2(\xi) = 2\sqrt{\frac{2}{\pi}}\,\xi^2\int_0^\infty \frac{\zeta\psi(\zeta)J_1(\zeta a)}{(\xi^2+\zeta^2)^2}\,d\zeta. \tag{3.11.24}$$

Similarly, if we substitute from equation (3.11.22) into equation (3.11.5), we obtain

$$a(\xi)I_1(\xi a) - b(\xi)[\xi a I_0(\xi a) - 4(1-\eta)I_1(\xi a)] = i_2 - \eta i_1. \tag{3.11.25}$$

Solving the equations (3.11.23) and (3.11.25) we find that

$$a(\xi) = \frac{i_1(\xi)}{I_1(\xi a)}\left[\frac{\xi a I_0(\xi a)}{2I_1(\xi a)} - (2-\eta)\right] + \frac{i_2(\xi)}{I_1(\xi a)}, \qquad b(\xi) = \frac{i_1(\xi)}{2I_1(\xi a)}. \tag{3.11.26}$$

Substituting from equation (3.2.7) into the first of the equations (3.11.24) and interchanging the order of the integrations we find that the integral $i_1(\xi)$ is related to $g(t)$ through the equation

$$i_1(\xi) = 2\sqrt{\frac{2}{\pi}}\int_0^1 g(t)\,dt\int_0^\infty \frac{\zeta\sin(\zeta t)J_1(\zeta a)}{\xi^2+\zeta^2}\,d\zeta.$$

The value of the inner integral can be derived easily from entry (5) in Erdelyi

(1954, Vol. 2, p. 10), and we find that

$$i_1(\xi) = 2\sqrt{\frac{2}{\pi}}\, K_1(\xi a)\int_0^1 g(u)\sinh(\xi u)\, du. \tag{3.11.27}$$

Similarly, it is easily shown that

$$i_1(\xi) = \sqrt{\frac{2}{\pi}}\int_0^1 g(u)\{\sinh(\xi u)[\xi a K_0(\xi a) + K_1(\xi a)] - \xi u \cosh(\xi u) K_1(\xi a)\}\, du. \tag{3.11.28}$$

If we substitute from equations (3.11.27) and (3.11.28) into equations (3.11.26) and then into equation (3.11.15), we find that this last equation reduces to the integral equation

$$g(t) - \int_0^1 K(t,u) g(u)\, du = h(t), \qquad 0 \le t \le 1, \tag{3.11.29}$$

in which the kernel $K(t,u)$ is defined by the equation

$$K(t,u) = \frac{4}{\pi^2}\int_0^\infty \left[\frac{2I_1(\xi a)K_1(\xi a) - 1}{I_1^{\,2}(\xi a)}\sinh(\xi t)\sinh(\xi u) \right.$$
$$\left. + \frac{\xi K_1(\xi a)}{I_1(\xi a)}\{u\cosh(\xi u)\sinh(\xi t) + t\cosh(\xi t)\sinh(\xi u)\}\right] d\xi.$$

Using standard properties of Bessel functions (see, for instance, Watson, 1944, pp. 79–80), we have

$$\frac{d}{d\xi}\left[\frac{K_1(\xi a)}{I_1(\xi a)}\right] = a\,\frac{K_1'(\xi a)I_1(\xi a) - I_1'(\xi a)K_1(\xi a)}{I_1^{\,2}(\xi a)}$$
$$= -\tfrac{1}{2}a\,\frac{[I_0(\xi a) + I_2(\xi a)]K_1(\xi a) + [K_0(\xi a) + K_2(\xi a)]I_1(\xi a)}{I_1^{\,2}(\xi a)}$$
$$= -\frac{1}{\xi}\cdot\frac{1}{I_1^{\,2}(\xi a)},$$

so that

$$\int_0^\infty \frac{1}{I_1^{\,2}(\xi a)}\sinh(\xi t)\sinh(\xi u)\, d\xi = -\int_0^\infty \frac{d}{d\xi}\left[\frac{K_1(\xi a)}{I_1(\xi a)}\right]\xi\sinh(\xi t)\sinh(\xi u)\, d\xi.$$

Integrating by parts, we can reduce this last term to

$$\int_0^\infty \frac{K_1(\xi a)}{I_1(\xi a)}[\sinh(\xi t)\sinh(\xi u) + \xi t\cosh(\xi t)\sinh(\xi u) + \xi u\sinh(\xi t)\cosh(\xi u)]\, d\xi.$$

We therefore find that the kernel $K(t,u)$ of the integral equation (3.11.29) is given by the equation

$$K(t,u) = H_1\left(\frac{u+t}{a}\right) - H_1\left(\frac{u-t}{a}\right), \tag{3.11.30}$$

where the function H_1 is defined by the equation

$$H_1(s) = \frac{2}{\pi^2 a} \int_0^\infty \frac{K_1(\zeta)}{I_1(\zeta)} [\cosh(s\zeta) - 1] \, d\zeta. \tag{3.11.31}$$

If in the equation

$$K = \lim_{\rho \to 1+} \sqrt{2(\rho - 1)} \sigma_{zz}(\rho,0), \tag{3.11.32}$$

we use the expression (3.11.10) for $\sigma_{zz}(\rho,0)$ with the values (3.11.26) substituted for the functions $a(\xi)$, $b(\xi)$, we find that the first integral on the righthand side of equation (3.11.10) makes no contribution to this limit and the second integral gives the result

$$K = 2g(1).$$

If we denote the stress intensity factor in the case $a \gg 1$ by $K^{(\infty)}$ then, since in that case the solution of the integral equation (3.11.29) is $g(t) = h(t)$, we have that $K^{(\infty)} = 2h(1)$ and hence that the percentage increase in the stress intensity factor due to the effect of the finite radius of the cylinder

$$n_1 = \frac{K - K^{(\infty)}}{K^{(\infty)}} \times 100,$$

is given by

$$n_1 = \frac{100}{h(1)} [g(1) - h(1)]. \tag{3.11.33}$$

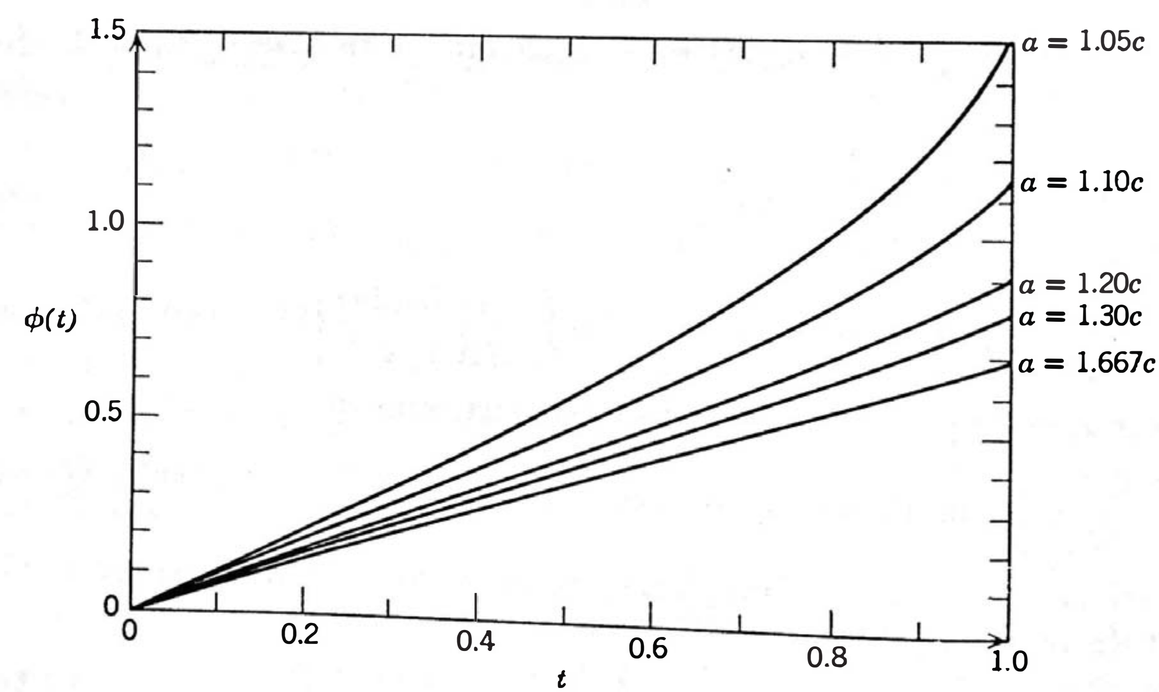


Figure 52 The variation with t and a/c of the function $\phi(t)$.

TABLE 10

a/c	1.05	1.10	1.20	1.30	1.6667	2.5	5.0
W_1/W_∞	1.9704	1.6649	1.3670	1.2725	1.0994	1.0205	1.0027
Π_1	33.3	26.9	16.3	12.5	6.9	2.1	0.3
n_1	140	88	42	27	9.5	2.3	0.3

In the case in which the constant pressure p_o is applied to the crack $g(t) = p_o\phi(t)$ where

$$\phi(t) - \int_0^1 \phi(u)K(t,u)\,du = \frac{2}{\pi}\,t, \qquad 0 \leq t \leq 1. \tag{3.11.34}$$

The solutions of the integral equation (3.11.34) corresponding to seven values of the parameter a were obtained by Sneddon and Tait, using a high-speed computer. The results of the calculations of $\phi(t)$ are shown graphically in Fig. 52. It will be observed that even for moderately small values of a (for instance $a/c > 1.20$) the graph of the function $g(t)$ differs very little from a straight line. Using these values of the function $g(t)$ and the above formulas we can calculate the values of W_1/W_∞, Π, and n_1 for each value of a. The results are shown in Table 10.

To illustrate the variation of the critical pressure p_{cr} necessary to cause the spread of the crack when its radius is c, the ratio Π_1 was calculated. The results are shown graphically in Fig. 53. From this diagram we see that the

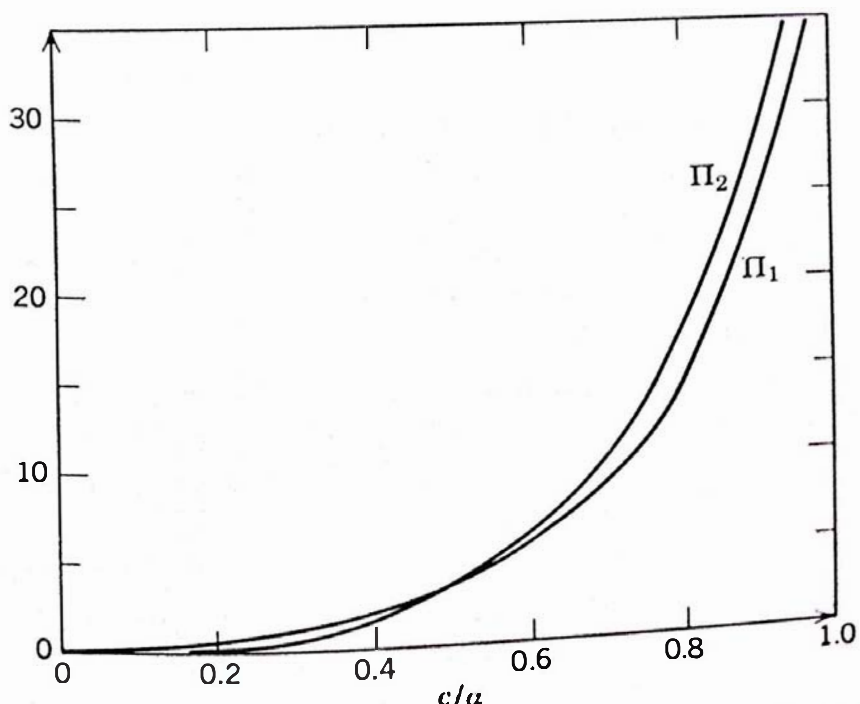


Figure 53 The variation with c/a of the percentage increase in p_{cr} for constant pressure on the crack surfaces. The curve Π_1 corresponds to the case in which $u_\rho = 0$ on the cylindrical surface and is independent of η; the curve Π_2 corresponds to the case in which $\sigma_{\rho\rho} = 0$ on the cylindrical surface and $\eta = 0.25$.

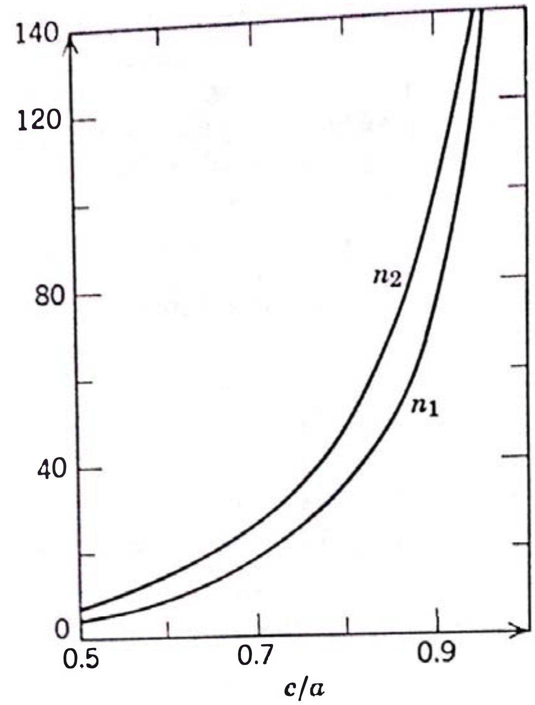


Figure 54 The variation with c/a of the percentage increase in the stress intensity factor for constant pressure on the crack surfaces. The curve n_1 corresponds to the case in which $u_\rho = 0$ on the cylindrical surface and is independent of η; the curve n_2 corresponds to the case in which $\sigma_{\rho\rho} = 0$ on the cylindrical surface and $\eta = 0.25$.

drop in the value of the critical pressure p_{cr} from the value $p_{cr}^{(\infty)}$ corresponding to a crack in an infinite solid is less than 5% if $a > 2c$, and is less than 10% if $a > 1.4c$.

If we look at the variation with c/a of the stress intensity factor K, we get much the same kind of picture. To illustrate the variation of K, the percentage increase, n_1, in the stress intensity factor is plotted as a function of c/a in Fig. 54. The effect of the finite value of the radius a of the cylinder is more pronounced in this case. For instance if $a < 1.2c$, the change in the stress intensity factor is greater than 40%. However, it is less than 5% if $a > 2c$ and is less than 10% if $a > 1.7c$.

It is of interest to compare these results with those obtained in the analogous problem in plane strain (Section 2.9). If we plot the ratio Π_1 as a function of x, the ratio of the area of the crack to the cross-sectional area of the cylinder (i.e., $x = c^2/a^2$) we get the curve shown in Fig. 55. The curve Π in the same diagram shows the value of the percentage change in p_{cr} in the plane strain problem, but here x is taken to be the ratio of the length of the crack to the width of the strip. In Fig. 56 we show a comparison of n_1 the percentage increase in the stress intensity factor in the axisymmetric case with n the corresponding quantity in the plane strain case, x being given the same interpretation as before. Using this basis of comparison we see that as far as an "engineering" approximation is concerned, the size effect in the axisymmetric case can be guaged from that in the plane strain case.

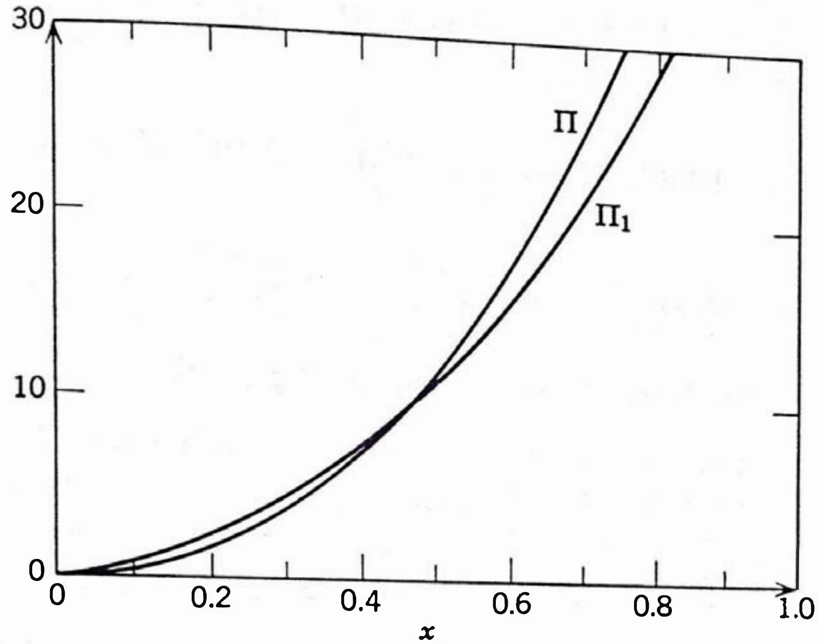


Figure 55 A comparison of the axisymmetric solution and the plane strain solution. Π_1 is the percentage change in the value of the critical pressure in the axisymmetric case with x equal to the ratio of the *area* of the crack to the cross-sectional *area* of the cylinder; Π is the analogous quantity in the plane strain case with x equal to the ratio of the *length* of the crack to the *width* of the strip.

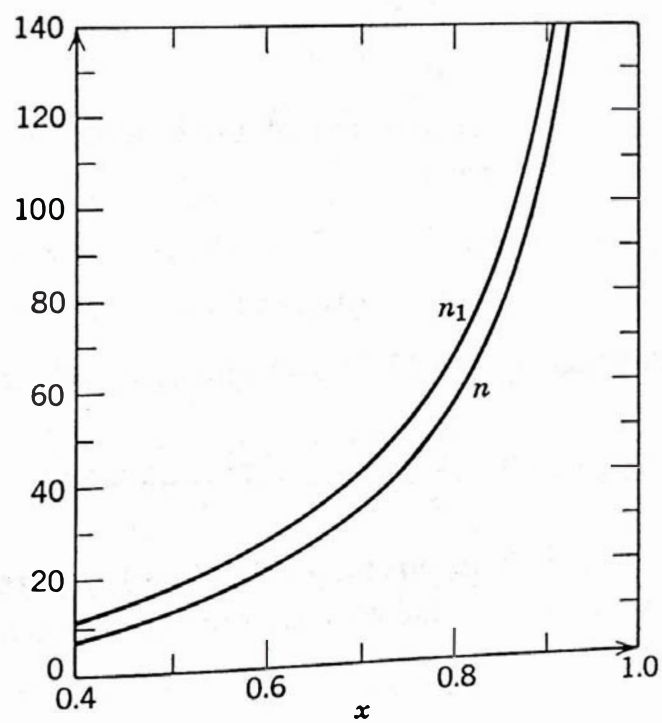


Figure 56 A comparison of the axisymmetric solution and the plane strain solution. n_1 is the percentage increase in the stress intensity factor in the axisymmetric case, n that in the plane strain case. x is defined as in Fig. 55.

Sneddon and Tait (1963) also contains an iterative solution of the integral equation (3.11.34), valid when $a \gg c$. Using this solution it is possible to deduce the approximate formulas

$$\Pi_1 = 33.82\left(\frac{c}{a}\right)^3 + 10.87\left(\frac{c}{a}\right)^5 + 18.56\left(\frac{c}{a}\right)^6 + 2.19\left(\frac{c}{a}\right)^7 + o\left(\frac{c^8}{a^8}\right), \tag{3.11.35}$$

$$n_1 = 33.8\left(\frac{c}{a}\right)^3 + 10.2\left(\frac{c}{a}\right)^5 + 11.4\left(\frac{c}{a}\right)^6 + o\left(\frac{c^7}{a^7}\right). \tag{3.11.36}$$

***Conditions on the Surface of the Cylinder: Case* (ii)**

The solution of the second mixed boundary value problem is given by equations (3.11.9), (3.2.7), (3.11.23), and by a new equation which replaces (3.11.25). It is therefore sufficient to derive this new relation which is a consequence of equation (3.11.7). From (3.11.9), we easily deduce the relation

$$\begin{aligned}\sigma_{\rho\rho}(a,z) = \frac{1}{a}\sqrt{\frac{2}{\pi}}\int_0^\infty \Big\{&a(\xi)[\xi a I_0(\xi a) - I_1(\xi a)] + b(\xi)[(3-2\eta)\xi a I_0(\xi a) \\ &- \xi^2 a^2 I_1(\xi a) - 4(1-\eta)I_1(\xi a)]\Big\}\cos(\xi z)\, d\xi \\ &- \int_0^\infty \zeta\psi(\zeta)e^{-\zeta z}\Big\{J_0(\zeta a) - (1-2\eta)\frac{J_1(\zeta a)}{\zeta a} \\ &- \zeta z\Big[J_0(\zeta a) - \frac{J_1(\zeta a)}{\zeta a}\Big]\Big\}\, d\zeta. \end{aligned} \tag{3.11.37}$$

If we take the Fourier cosine transform of both sides of this equation and make use of (3.11.7), we find that

$$\begin{aligned}a(\xi)[\xi a I_0(\xi a) - I_1(\xi a)] + b(\xi)[(3-2\eta)\xi a I_0(\xi a) - \xi^2 a^2 I_1(\xi a) \\ - 4(1-\eta)I_1(\xi a)] = \xi a i_3 - i_2 + \eta i_1,\end{aligned} \tag{3.11.38}$$

where i_1 and i_2 are defined by the (3.11.24) and i_3 is defined by

$$i_3(\xi) = 2\sqrt{\frac{2}{\pi}}\,\xi\int_0^\infty \frac{\zeta^2\psi(\zeta)J_0(\zeta a)\, d\zeta}{(\xi^2+\zeta^2)^2}. \tag{3.11.39}$$

Substituting (3.2.7) for $\psi(\zeta)$ into equation (3.11.39), interchanging the order of the integrations, and making use of well-known integrals involving Bessel functions, we find that

$$i_3(\xi) = \sqrt{\frac{2}{\pi}}\int_0^1 g(u)\{\sinh(\xi u)[K_0(\xi a) - \xi a K_0(\xi a)] + \xi u\cosh(\xi u)K_0(\xi a)\}\, du. \tag{3.11.40}$$

Solving the equations (3.11.23) and (3.11.38) for $a(\xi)$ and $b(\xi)$, and inserting the resulting expressions into (3.11.15), we find that $g(t)$ satisfies the integral equation

$$g(t) + \frac{2}{\pi}\int_0^1 L(t,u)g(u)\,du = h(t), \qquad 0 \le t \le 1, \tag{3.11.41}$$

whose kernel $L(t,u)$ is defined by

$$\begin{aligned} L(t,u) = \frac{2}{\pi}\int_0^\infty \frac{1}{G(\xi a)} \{&[H(\xi a) - 1][\xi u \cosh(\xi u)\sinh(\xi t) \\ &+ \xi t \cosh(\xi t)\sinh(\xi u)] \\ &+ [2H(\xi a) - 3 + 2\eta - \xi^2 a^2]\sinh(\xi u)\sinh(\xi t) \\ &- \xi^2 ut \cosh(\xi u)\cosh(\xi t)\}\,d\xi, \end{aligned} \tag{3.11.42}$$

where the functions G and H are defined by

$$G(x) = x^2[I_0(x)]^2 - (2 - 2\eta + x^2)[I_1(x)]^2, \tag{3.11.43}$$

$$H(x) = x^2 I_0(x)K_0(x) + (2 - 2\eta + x^2)I_1(x)K_1(x). \tag{3.11.44}$$

This integral equation is identical with that derived otherwise by Collins (1962a).

If the internal pressure of the crack is constant, p_0 say, then $h(t) = (2p_0 t/\pi)$, and through the transformations

$$g(t) = \frac{2p_0 a}{\pi}\,\phi\left(\frac{t}{a}\right), \qquad K(t,u) = \frac{2a}{\pi}\,L(at,au), \qquad \alpha = \frac{c}{a},$$

we reduce the integral equation (3.11.41) to the form

$$\phi(t) + \int_0^\alpha K(t,u)\phi(u)\,du = t, \qquad 0 \le \alpha < 1, \qquad 0 \le t \le 1, \tag{3.11.45}$$

where the kernel is given by

$$\begin{aligned} K(t,u) = \frac{4}{\pi^2}\int_0^\infty &\{[2H(x) - 3 + 2\eta - x^2]\sinh(ux)\sinh(tx) \\ &- utx^2 \cosh(ux)\cosh(tx) + x[H(x) - 1][u\cosh(ux)\sinh(tx) \\ &+ t\cosh(tx)\sinh(ux)]\}\,\frac{dx}{G(x)}, \end{aligned} \tag{3.11.46}$$

with the functions $H(x)$ and $G(x)$ defined by (3.11.43) and (3.11.44). It will be observed that, in contrast with the situation in Case (i), the kernel $K(t,u)$ depends upon the value of Poisson's ratio η.

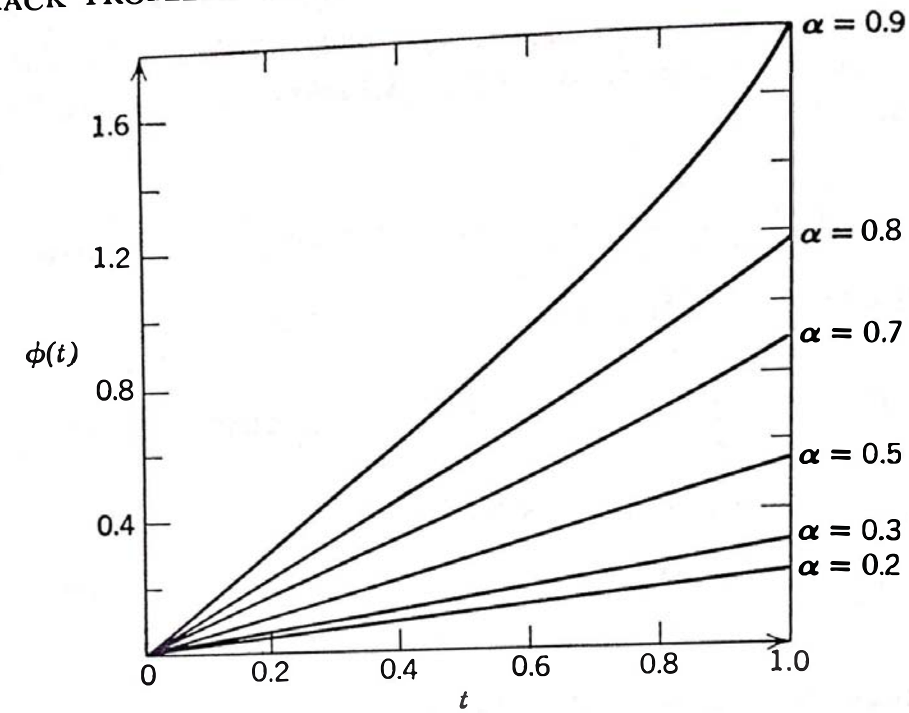


Figure 57 The variation with t and α of the function $\phi(t)$ for constant pressure and $\eta = 0.25$.

Numerical solutions of (3.11.45) appropriate to six values of α were derived by Sneddon and Welch (1963). The results are shown graphically in Fig. 57. The corresponding values values of n_2 and Π_2, the percentage changes in the stress intensity factor, and the critical pressure respectively, in the case $\eta = 0.25$, are listed in Table 11. They are shown in Figs. 53 and 54.

The problem of determining the thermal stresses in a long cylinder containing a penny-shaped crack has been considered by Das (1968).

3.12 COHESIVE FORCES AT THE RIM OF A PENNY-SHAPED CRACK

The discussion of Section 2.20 can be readily extended to the case of circular cracks. In this case the stress-intensity factor, resulting from the cohesive forces, is given by (3.2.13) and (3.2.14) in the form

$$K_c = -\frac{2}{\pi\sqrt{c}} \int_0^c \frac{\rho C(c-\rho)\, d\rho}{(c^2-\rho^2)},$$

TABLE 11

c/a	0.2	0.3	0.5	0.7	0.8	0.9
Ω_2	0.998	0.993	0.965	0.891	0.822	0.707
Π_2	0.2	0.7	3.5	10.9	17.8	29.3
n_2	0.5	1.3	7.2	25.9	47.9	100.2

from which it follows that

$$K_c = -\frac{2}{\pi\sqrt{c}}\int_0^\delta \frac{(c-s)C(s)\,ds}{(2cs-s^2)}.$$

If $\delta \ll c$, the factor $(c-s)(2cs-s^2)^{-1/2}$ in the integrand can be replaced by $c(2cs)^{-1/2}$ so that K_c is again given by equations (2.20.3) and (2.20.2), and the three-dimensional analogue of the relation (2.20.5) is

$$\int_0^c \frac{\rho p(\rho)\,d\rho}{\sqrt{c^2-\rho^2}} = C\sqrt{\tfrac{1}{2}c}, \tag{3.12.1}$$

where C is *the modulus of cohesion* of the material defined by the equation

$$C = \int_0^\delta \frac{C(s)\,ds}{\sqrt{s}}. \tag{3.12.2}$$

In the case in which $p(\rho) = p_0$, a constant, equation (3.12.1) reduces to

$$p_0\sqrt{2c} = C. \tag{3.12.3}$$

Closely related to this problem is that of calculating the "plastic" zone at the rim of a penny-shaped crack using the criterion of Dugdale (1960). The problem for purely mechanical loads has been considered by Olesiak and Wnuk (1965, 1968) and for the stress fields produced by prescribed temperature fields by Olesiak (1968).

Another related problem—that of determining (within the framework of the classical theory of elasticity) the distribution of applied surface stress necessary to produce a crack of the shape postulated by Barenblatt—has been considered recently by Olesiak and Sneddon (1969).

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Index