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On the numerical treatment of the singular integral equation of the second kind

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Abstract

Here, a numerical treatment for solving the integral equation of the second kind with Cauchy kernel is presented. The singular term has been removed and the solution in the Legendre polynomial form has been used to obtain a system of linear algebraic equation. This system is solved numerically.

Keywords: Singular integral equation; Cauchy kernel; Legendre polynomial; A system of linear algebraic equation

1. Introduction

The integral equations appear in many problems of physics and engineering. The singular integral equation considered to be of more interest than the other. Integral equation containing singular kernel appears in studies involving airfoil [18], fracture mechanics [9] contact radiation and molecular conduction [8] and others. Over the past thirty years substantial progress has been made in developing innovative approximate analytical and purely numerical solution to a large class of Fredholm integral equation with singular kernel. Since the theory of singular integral equation developed by Muskhelishvili [15] has assumed various technique and has increasing important applications in different areas

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of science. For this aim, many different methods are established by Tricomi [4], Popov [6], Green [2] and others for obtaining the solution of the integral equations analytically. Since closed form solution to these integral equations are generally not available, great attention has been focused on the numerical treatment. The interested reader should consult the fine exposition by Golberg [14], Linz [17], Atkinson [10], Delves and Mohamed [11]. Since the Fredholm integral equation of the second kind with Cauchy kernel plays an important rule in applied mathematics and physics, so many different numerical solutions are obtained. For example in [1] Gerasolis used a piecewise quadratic polynomials in the solution of the singular integral equation. As the same way, Gerasoulis, Miller and Keer [5] obtained the solution of the integral equation with Cauchy kernel. In [3] Venturino used Galerkin method to obtain the singular integral equation of the second kind with Cauchy kernel. In [8] Frankel used a Galerkin approach for solving the integro-differential equation with Cauchy kernel.

In this paper, a numerical method is used to obtain the potential function of a Fredholm integral equation of the second kind with Cauchy kernel. Firstly, we remove the singularity, secondly the solution is expanded in terms of orthogonal polynomial (we consider the Legendre's polynomial as an example), The solution of the problem reduces to the solution of a linear system equations. At the end, we give a numerical application to test our method.

2. Cauchy kernel

Consider the Fredholm integral equation of the second kind with Cauchy kernel:

$$\phi(x) + v\int_{-1}^{1} \frac{\phi(y)}{x-y}\,\mathrm{d}y = f(x) \quad (v \text{ is a constant}) \tag{2.1}$$

under the static conditions

$$\phi(\pm 1) = 0. \tag{2.2}$$

Here $f(x)$ is continuous function with its derivatives and belongs to the class $C(-1,1)$. Also the sign $\int$ denotes integration with Cauchy principal value sense, and $\phi(x) \in [-1,1]$. The unknown function $\phi(x)$ must satisfy the normality condition $\{\int_{-1}^{1} \phi^2(y)\,\mathrm{d}y\}^{1/2} \leqslant A\|\phi\|_2$ and Hölder condition

$$|\ \phi(x_1) - \phi(x_2)\ | \ \leqslant D\ |\ x_1 - x_2\ |^{\alpha} \quad (0 < \alpha \leqslant 1),$$

where $\|\cdot\|_2$ denotes the L_2 norm, while A and D are constants. Here, we suppose that $\phi(x)$, and $\phi'(x)$ are continuous in the interval $x \in [-1,1]$. Then $\int_{-1}^{1} \phi(y)/(x-y)\,\mathrm{d}y$ exists in the principal value sense.

3. Solution of the problem

In this section, we will solve the Eq. (2.1) under the condition (2.2). For this aim the singularity of the integral term of Eq. (2.1) will be weakened as follows:

$$\int_{-1}^{1} \frac{\phi(y)}{y-x}\,\mathrm{d}y = \int_{-1}^{1} \frac{\phi(y)-\phi(x)}{y-x}\,\mathrm{d}y + \phi(x)\int_{-1}^{1} \frac{1}{y-x}\,\mathrm{d}y.$$

The first of the two right integrals is regular and it will be evaluated later while the second integral is evaluated in [7], it is equal to $-\log((1+x)/(1-x))$. Therefore Eq. (2.1) becomes

$$\phi(x) + v\int_{-1}^{1} \frac{\phi(x)-\phi(y)}{y-x}\,\mathrm{d}y - v\phi(x)\log\frac{1+x}{1-x} = f(x), \quad -1 < x < 1. \tag{3.1}$$

Assume the unknown function, $\phi(x)$ can be expanded in terms of Legendre polynomial form, one may use a Chebyshov form, i.e.

$$\phi(x) = \sum_{j=0}^{\infty} a_j P_j(x), \tag{3.2}$$

where a_j are the constants and $P_j(x)$ are the Legendre polynomials, which satisfy the following orthogonal relation (see [7]),

$$\int_{-1}^{1} P_n(x)P_m(x)\,\mathrm{d}x = \begin{cases} \dfrac{2}{2n+1}, & n = m, \\ 0, & n \neq m. \end{cases} \tag{3.3}$$

Using the orthogonal relation (3.3) in (3.2), when $n = m = 0$, we obtain $a_0 = P$, where P is a constant equivalent to the value $P = 1/2\int_{-1}^{1}\phi(y)\,\mathrm{d}y$. The rest of the coefficients a_j, $j = 1, 2, \ldots$ will to be determined. For this aim, we use the Rodrigues formula of the Legendre polynomial $P_j(x)$ of degree j, where (see [7])

$$P_j(x)\sum_{k=0}^{[j/2]} \alpha_k x^{j-2k},$$

where

$$\alpha_k = \frac{(-1)^k(2j-2k)}{2^k k!(j-k)!(j-2k)!}. \tag{3.4}$$

Then we have:

$$\int_{-1}^{1} \frac{P_j(y)-P_j(x)}{y-x}\,\mathrm{d}y = \sum_{k=0}^{[(j-1)/2]} \alpha_k \sum_{l=0}^{j-2k-1} x^l \int_{-1}^{1} y^{j-2k-1-l}\,\mathrm{d}y,$$

and therefore

$$\int_{-1}^{1} \frac{P_j y - P_j(x)}{y - x}\,\mathrm{d}y = \sum_{k=0}^{[(j-1)/2]} \alpha_k \sum_{l=0}^{j-2k-1} \gamma_{j,k,l\cdot x^l}, \tag{3.5}$$

where

$$\gamma_{j,k,l} = \frac{\alpha_k[l - (-1)^{j-1}]}{j - 2k - l}. \tag{3.6}$$

Using Eqs. (3.2) and (3.5), Eq. (3.1) becomes:

$$\left[1 - v\log\frac{1+x}{1-x}\right]\sum_{j=0}^{\infty} a_j P_j(x) + v\sum_{j=1}^{\infty} a_j \sum_{k=0}^{[(j-1)/2]} \phi_k \sum_{l=0}^{j-2k-1} \gamma_{j,k,l\cdot x^l} = f(x) \quad (|x| \leqslant 1).$$

Multiply both sides of the last equation by x^{i-1} for $i = 1, 2, \ldots, N-1, N$, then integrating the resultant over the interval $[-1, 1]$, we get:

$$\sum_{j=0}^{\infty} a_j \int_{-1}^{1} \left[1 - v\log\frac{1+x}{1-x}\right] x^{i-1} P_j(x)\,\mathrm{d}x + v\sum_{j=1}^{\infty} a_j \sum_{k=0}^{[(j-1)/2]} \phi_k \times \sum_{l=0}^{j-2k-1} \gamma_{j,k,l} \int_{-1}^{1} x^{l+i-1}\,\mathrm{d}x = \int_{-1}^{1} x^{i-1} f(x)\,\mathrm{d}x, \tag{3.7}$$

the term-by-term integration is justified by the uniform convergence of each of previous three series of the left side of the previous equation in the interval $[-1, 1]$ and

$$|\, x^i P_j \,| \leqslant |\, x^i \,| \leqslant 1, \quad |x| \leqslant 1.$$

For $|x| \leqslant 1$, we can assume

$$\log\frac{1+x}{1-x} \simeq -\sum_{\beta=1}^{M} \frac{x^{2\beta}}{\beta}.$$

So the formula (3.7) will take the form

$$\sum_{j=1}^{\infty} a_k \left\{ \int_{-1}^{1} \left(1 + v\sum_{\beta=1}^{M} \frac{x^{2\beta}}{\beta}\right) x^{i-1} P_j(x)\,\mathrm{d}x + v\sum_{k=0}^{[(j-1)/2]} \alpha(k) \sum_{l=0}^{j-2k-1} \frac{2[1 - (-1)(j-l)]}{(j-2k-l)(l+i)} \right\} = \int_{-1}^{1} x^{i-1} f(x)\,\mathrm{d}x - 2p\left[\frac{\delta_{i-1}}{i} - \frac{2v\delta_i}{i+1}\right] \quad (i = 1, 2, \ldots, N-1, N), \tag{3.8}$$

where

$$\delta_c = \begin{cases} 1, & c \text{ even}, \\ 0, & c \text{ odd}. \end{cases}$$

In order to evaluate the integral of the left side of (3.8), we use the famous Rodrigues formula (see [7])

$$\int_{-1}^{1} x^i p_i(x)\,\mathrm{d}x = \sum_{k=0}^{[j/2]} 2\alpha_k \frac{\delta_{i,j}}{j+i-2k+1}, \tag{3.9}$$

where

$$\delta_{c,d} = \begin{cases} 1, & c+d \quad \text{even and } c \geqslant d, \\ 0, & \text{otherwise}. \end{cases}$$

Using Eq. (3.9) in (3.8), we have

$$\begin{aligned} &\sum_{j=1}^{\infty} a_j \Bigg\{ \sum_{k=0}^{[j/2]} \left[\frac{2\alpha_k \delta_{i-1}, j}{j+1-2k} + \sum_{\beta=1}^{M} \frac{2v\alpha_k \delta_{2\beta+i-1,j}}{\beta(j+i-2k+2\beta)} \right] \\ &\quad + \sum_{k=0}^{[(j-1)/2]} \sum_{l=0}^{j-2k-1} \frac{2v\alpha_k[1-(-1)^{j-1}]}{(j-2k-l)(l+i)} \delta_{l+i-1} \Bigg\} \\ &= \int_{-1}^{1} x^{i-1} f(x)\,\mathrm{d}x - 2p\left[\frac{\delta_{i-1}}{i} - \frac{2v\delta_i}{i+1}\right] \quad (i = 1, 2, \ldots, N-1, N). \end{aligned} \tag{3.10}$$

If we truncate the infinite series of the left hand side of Eq. (3.10) to the first N terms. Thus we have

$$\sum_{j=1}^{N} c_{ij} a_j = b_j (i = 1, 2, \ldots, N-1, N), \tag{3.11}$$

where

$$\begin{aligned} C_{ij} = &\sum_{k=0}^{[j/2]} \left\{ \frac{2\alpha_k \delta_{i-1,j}}{j+i-2k} + \sum_{\beta=1}^{M} \frac{2v\alpha_k \delta_{i+(2\beta-1),j}}{\beta(j+i-2k+2\beta)} \right\} \\ &+ \sum_{k=0}^{[(j-1)/2]} \sum_{l=0}^{j-2k-1} \frac{2v\alpha_k[1-(-1)^{j-l}]}{(j-2k-l)(l+i)} \delta_{l+i-1} \end{aligned}$$

and

$$b_i = \int_{-1}^{1} x^{i-1} f(x)\,\mathrm{d}x - 2p\left[\frac{\delta_{i-1}}{i} - \frac{2v\delta_i}{i+1}\right]. \tag{3.12}$$

4. Numerical result

The solution of the integral Eq. (2.1) depends on the Cauchy kernel, the value of v and the given function $f(x)$. Here, we assume the following $f(x) = x$, $p = 0.8$, $v = 0.25$ and $N = 20$ and we use the general assumption of the solution

$$\phi(x) = \sum_{j=0}^{20} \alpha_j p_j(x),$$

where $p_j(x)$ is Legendre's polynomial of degree, and the coefficients a_j are the solution of the linear system (3.1), we used Maple V to solve such system. These coefficients are tabulated in Table 1.

5. Physical meaning

The solution of the integral equation (3.1) under the static condition (3.2) is equivalent to the solution of the following contact problem:

Consider the semi-symmetric contact problem of stamp, of equation $f_1(x)$, is impressed on the strip of equation $f_2(x)$, consider the tangent force $t(x)$ is related with the normal pressure $p(x)$ in the contact region, $\Omega = \{(x, y) \in \Omega : -1 \leqslant x, y \leqslant 1\}$, of the two surfaces by the relation [16]

$$t(x) = kp(x). \tag{5.1}$$

Also, let the tangent stress σ and the normal stress t_{xy} satisfy the relation

$$t_{xy} = k\sigma_y, \tag{5.2}$$

where k is the fraction coefficient. Consider $v_i^*(i = 1, 2)$ the displacement components in the y direction, that satisfy the following [6]

$$\frac{\mathrm{d}v_1^*}{\mathrm{d}x} = \frac{t(x)}{G_1}, \quad \frac{\mathrm{d}V_2^*}{\mathrm{d}x} = \frac{t(x)}{G_2}, \tag{5.3}$$

where G_1 and G_2 are the displacement compressible materials of two surfaces $f_1(x)$ and $f_2(x)$ respectively. It is known that [18] such problem reduces to the following integral equation

Table 1
The coefficients a_j, $j = 1, 2, \ldots, 20$ (ordered in rows)

−0.796513983	1.282960283	−1.39797130	1.456975757	−1.549072869
1.479876614	−1.567379746	1.417647537	−1.559090422	1.178337053
−1.392267336	2.0371206564	−1.596255523	−2.924756126	−5.248330647
8.38415880	6.33866610	1.704701286	−0.2528251584	−1.196720603

$$k_1 \frac{G_1+G_2}{G_1G_2}\int_0^x \phi(t)\,\mathrm{d}t + (v_1+v_2)\int_{-1}^{1} G\Big(\frac{x-y}{\lambda}\Big)\phi(t)\,\mathrm{d}t$$
$$= \delta - f_1(x) - f_2(x), \quad \lambda \in [0,\infty], \tag{5.4}$$
$$G(t) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\tanh u}{u} \mathrm{e}^{\mathrm{i}ut}\,\mathrm{d}u$$

under the condition

$$\int_{-1}^{1} \phi(y)\,\mathrm{d}y = p < \infty \quad (p \text{ is constant}), \tag{5.5}$$

where $\phi(t)$ is the unknown potential function which is continuous through the interval of integration $[-1,1]$, the contact domain between the two surfaces $f_i(x)$ $(i=1,2)$, δ is the rigid displacement under the action of a force P, k_1 is a physical constant, $G(t)$ is the discontinuous kernel of the problem with singularity at the point $x=y$, and $v_i=(1-\mu_i^2)/\pi E_i$ $(i=1,2)$ where μ_i are the Poisson's coefficients and E_i are the coefficients of Young. As in [6], the kernel can be written in the following form

$$G(t) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\tanh u}{u} \mathrm{e}^{\mathrm{i}ut}\,\mathrm{d}u = -\ln\left|\tanh\frac{\pi t}{4}\right|.$$

If $\lambda \to \infty$ and the term $(x-y)/\lambda$ is very small, so that it satisfies the conditions $\tanh z \simeq z$, then we have

$$\ln\left|\tanh\frac{\pi t}{4}\right| = \ln t - d \quad \left(d = \ln\frac{4\lambda}{\pi}\right). \tag{5.6}$$

Hence, Eq. (5.4) with the aide of Eq. (5.6) can be adapted in the form

$$\int_0^x \phi(t)\,\mathrm{d}t + v\int_{-1}^{1}[-\ln|y-x| + d]\phi(y)\,\mathrm{d}y = f^*(x), \tag{5.7}$$

where

$$v = \frac{(v_1+v_2)G_1G_2}{k_1(G_1+G_2)}, \quad f^* = \frac{[\delta - f_1(x) - f_2(x)]G_1G_2}{k_1[G_1+G_2]}.$$

Differentiating Eq. (5.7) with respect to x, we have

$$\phi(x) + v\int_{-1}^{1}\frac{\phi(y)}{y-x}\,\mathrm{d}y = f(x) \quad \left(f(x) = \frac{\mathrm{d}f^*(x)}{\mathrm{d}x}\right). \tag{5.8}$$

Eq. (5.8) represents a Fredholm integral equation of the second kind with Cauchy kernel, which we solved. As an important special case, if $G_1+G_2=0$, $f_2(x)=0$, we have a Fredholm integral equation of the first kind with logarithmic kernel. Abdou and Hassan [12] used potential theory method to obtain the eigenvalue and eigenfunction of such the problem. Also Abdou and Ezz-Eldin [13] used Krein's method to solve the same problem.

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