technische universität dortmund
Fakultät für Mathematik
Prof. Dr. H. M. Möller

Dortmund, den 28. 5. 2010

**Orthogonale Polynome**

Zusammenfassung der Woche vom 24.5. - 28.5.

## §3.3 The Tschebyscheff polynomials (second kind)

We start with the trigonometric identity

$$\sin((n+1)t) = \sin(nt)\cos(t) + \cos(nt)\sin(t).$$

We divide by $\sin(t)$ and let $x = \cos(t)$. This gives

$$\frac{\sin((n+1)t)}{\sin(t)} = x\frac{\sin(nt)}{\sin(t)} + T_n(x).$$

Because of $\sin(t) = \sqrt{1-x^2}$ if $x = \cos(t)$ and $t \in [0,\pi]$, we obtain therefore in analogy to the Tschebyscheff polynomials of the first kind the *Tschebyscheff polynomials of the second kind*

$$U_n(x) := \frac{\sin((n+1)\ \arccos(x))}{\sqrt{1-x^2}} \Big( = \frac{\sin((n+1)t)}{\sin(t)} \text{ if } x = \cos(t) \Big)$$

which satisfy

$$U_n(x) = xU_{n-1}(x) + T_n(x) \qquad \text{for all } n \in \mathbb{N}.$$

First of all, we have to show, that $U_n$ is a polynomial (of degree $n$). For $n = 0$ we have by definition

$$U_0(x) = \frac{\sin(t)}{\sin(t)} = 1 \quad (\ \cos(t) = x\ ).$$

Then successively

$$\begin{aligned}
U_1(x) &= xU_0(x) + T_1(x) = 2x,\\
U_2(x) &= xU_1(x) + T_2(x) = 4x^2 - 1,\\
U_3(x) &= xU_2(x) + T_3(x) = 8x^3 - 4x,\\
U_4(x) &= xU_3(x) + T_4(x) = 16x^4 - 12x^2 + 1\\
U_5(x) &= xU_4(x) + T_5(x) = 32x^5 - 32x^3 + 6x\\
U_6(x) &= xU_5(x) + T_6(x) = 64x^6 - 80x^4 + 24x^2 - 1\\
U_7(x) &= xU_6(x) + T_7(x) = 128x^7 - 192x^5 + 80x^3 - 8x\\
U_8(x) &= xU_7(x) + T_8(x) = 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1\\
U_9(x) &= xU_8(x) + T_9(x) = 512x^9 - 1024x^7 + 672x^5 - 160x^3 + 10x\\
& \quad \textit{etc.}
\end{aligned}$$

By induction, one can show easily that $U_n$ is a polynomial of degree $n$ with $lc(U_n) = 2^n$ for all $n \in \mathbb{N}_0$ and, since $T_n(-x) = (-1)^n T_n(x)$ the same holds for $U_n$. In other words, $U_n$ is an even polynomial if $n$ is even, and an odd one if $n$ is odd.

**Lemma.** $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$ for all $n \in \mathbb{N}$.

**Proof.** We sum up the following two trigonometric equalities

$$\begin{aligned}\sin((n+1)t) &= \sin(nt)\cos(t) + \cos(nt)\sin(t)\\ \sin((n-1)t) &= \sin(nt)\cos(t) - \cos(nt)\sin(t)\end{aligned}$$

and divide by $\sin(t)$. The substitution $x = \cos(t)$ gives

$$U_n(x) + U_{n-2}(x) = 2xU_{n-1}(x).$$

Subtracting $U_{n-2}(x)$ on both sides and renumbering gives the assertion. $\square$

The lemma shows that $\{T_n(x)\}_{n=0}^{\infty}$ and $\{U_n(x)\}_{n=0}^{\infty}$ satisfy the same three term recursion formula. The polynomial systems differ by the initial conditions

$$\begin{aligned}T_0(x) &= 1, \quad U_0(x) = 1,\\ T_1(x) &= x, \quad U_1(x) = 2x.\end{aligned}$$

This tiny difference only in the definition of the linear polynomials causes the difference of the two systems!

There are many connections between first and second kind polynomials. For instance, if we dismantle $xU_n(x)$ in the three term recursion and make an index transformation,

$$xU_{n-1}(x) = \frac{1}{2}U_n(x) + \frac{1}{2}U_{n-2}(x),$$

then inserting into $U_n(x) = xU_{n-1}(x) + T_n(x)$ and dismantling for $T_n(x)$ gives

$$T_n = \frac{1}{2}U_n - \frac{1}{2}U_{n-2}.$$

An other nice relation between the two systems is obtained by differentiating $T_n$,

$$T_n'(x) = nU_{n-1}(x).$$

Here we used $x = \cos(t)$ and

$$\frac{d\ \cos(nt)}{dt}\frac{dt}{dx} = -n\sin(nt)\frac{1}{\frac{d\ \cos(t)}{dt}} = n\frac{\sin(nt)}{\sin(t)}.$$

We can estimate easily $U_n$ and obtain by $T_n' = nU_{n-1}$ an estimate for $T_n'$. If $x = \cos(t)$, then

$$\begin{aligned} U_n(x) &= \frac{\sin((n+1)t)}{\sin(t)} = \frac{e^{i(n+1)t} - e^{-i(n+1)t}}{e^{it} - e^{-it}} \\ &= \sum_{k=0}^{n} e^{i(n-k)t} e^{-ikt} = \sum_{k=0}^{n} e^{i(n-2k)t}. \end{aligned}$$

Hence

$$|U_n(x)| \le \sum_{k=0}^{n} |e^{i(n-2k)t}| = n+1 \quad \text{for all } \ x \in [-1,1].$$

and for $t \to 0$, i.e., $x \to 1$,

$$U_n(1) = \sum_{k=0}^{n} e^{i(n-2k)0} = n+1.$$

We combine $T_n' = nU_{n-1}$ and $|U_n(x)| \le n+1$

$$|T_n'(x)| = n|U_{n-1}(x)| \le |T_n'(1)| = nU_{n-1}(1) = n^2 \quad \forall\, x \in [-1,1].$$

*Remark.* A result of *V. A. Markov* is

$$\max_{-1\le x\le 1} |p(x)| \le 1 \ \Rightarrow \ \max_{-1\le x\le 1} |p'(x)| \le n^2$$

for all polynomials $p$ of degree at most $n$. Since $\max_{-1\le x\le 1} |T_n(x)| = 1$ we proved that $n^2$, the maximal value for $\max_{-1\le x\le 1} |p'(x)|$, is attained by $p = T_n$.

For obtaining the generating function for $\{U_n(x)\}_{n=0}^{\infty}$, we need to reorder the series $\sum_{n=0}^{\infty} U_n(x) z^n$, i.e., we need the absolute convergence of it for instance for $|x| \le 1$ and $|z| \le a < 1$. This holds because

$$\sum_{n=0}^{\infty} |U_n(x)||z^n| \le \sum_{n=0}^{\infty} (n+1)a^n = \frac{1}{(1-a)^2} < \infty.$$

By similar arguments, we may reorder

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty} (U_n(x) - 2xU_{n-1}(x) + U_{n-2}) z^n \\ &= \sum_{n=2}^{\infty} U_n(x) z^n - 2xz \sum_{n=2}^{\infty} U_{n-1}(x) z^{n-1} + z^2 \sum_{n=2}^{\infty} U_{n-2}(x) z^{n-2} \\ &= \sum_{n=0}^{\infty} U_n(x) z^n \Big(1 - 2xz + z^2\Big) - U_0(x) z^0 - U_1(x) z^1 + 2xzU_0(x) z^0 \end{aligned}$$

Hence

$$\frac{1}{1-2xz+z^2} = \sum_{n=0}^{\infty} U_n(x) z^n.$$

We will now show the orthogonality of the Tschebyscheff polynomials (second kind).

**Theorem 3.6**. *If the inner product for $\mathcal{P}$ is defined by*

$$\langle f, g\rangle := \int_{-1}^{1} f(x)g(x)\sqrt{1-x^2}dx,$$

*then $\{\sqrt{\frac{2}{\pi}}U_n\}_{n=0}^{\infty}$ is an ONS, where $U_n$ denotes the $n$-th Tschebyscheff polynomial of the second kind.*

**Proof.** We have to show

$$\int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2}dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{\pi}{2} & \text{if } m = n \end{cases}$$

The substitution $x = \cos(t)$ ( $\Rightarrow$ $\sin(t) = \sqrt{1-x^2}$) and a trigonometric identity gives

$$\int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2}dx = \int_0^{\pi} \frac{\sin((n+1)t)}{sin(t)} \frac{\sin((m+1)t)}{\sin(t)} \sin(t)^2 dt$$
$$= \frac{1}{2}\int_0^{\pi} \cos((n-m)t)\ dt - \frac{1}{2}\int_0^{\pi} \cos((n+m+2)t)\ dt.$$

Because of $\int_0^{\pi}\cos(kt)dt = \pi$ if $k=0$ and $= \frac{1}{k}\sin(k\pi) - \frac{1}{k}\sin(0\pi) = 0$ if $k \in \mathbb{Z}\setminus\{0\}$, we obtain

$$\int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2}dx = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{\pi}{2} & \text{if } m = n. \end{cases}$$

Hence $\{U_n\}_{n=0}^{\infty}$ is an orthogonal system and $\{\sqrt{\frac{2}{\pi}}U_n\}_{n=0}^{\infty}$ an ONS. □

The zeros of $U_n$ are the inner extremal point of $T_{n+1}$ in $[-1,1]$ because of $T'_{n+1} = (n+1)U_n$, i.e.,

$$U_n(\cos(\frac{k\pi}{n+1})) = 0, \quad k = 1, \dots, n,$$

and since $|\sin((n+1)\ \arccos(x)| \leq 1$ we have for $-1 < x < 1$ in addition to $|U_n(x)| \leq n+1$ the estimate for $-1 < x < 1$

$$|U_n(x)| \leq \frac{1}{\sqrt{1-x^2}}. \tag{1}$$

*Remark.* In analogy to the best approximation property of the polynomial $T_n$ with respect to the norm $\|f\|_\infty := max_{-1\le x\le 1}|f(x)|$ the Tschebyscheff polynomial $U_n$ has a best approximation property with respect to the norm

$$\|f\|_1 := \int_{-1}^{1} |f(x)|\ dx.$$

As shown for instance in the above mentioned book of Th. J. Rivlin, 0 is the best approximation to $U_n$ in $\mathcal{P}_{n-1}$ with respect to the norm $\|f\|_1$, i.e.,

$$\|U_n\|_1 \le \|U_n - p\|_1 \quad \text{for all } p \in \mathcal{P}_{n-1}.$$

If we consider the Tschebyscheff polynomial $\hat{u}_n = 2^{-n}U_n$, then this means

$$\|\hat{u}_n\|_1 \le \|x^n - \sum_{k=0}^{n-1} a_k x^k\|_1 \quad \text{for all } a_0, \ldots, a_{n-1} \in \mathbb{R}.$$

With a little computation, one obtains

$$\|\hat{u}_n\|_1 = 2^{1-n}.$$

## §3.4 The Jacobi polynomials

We consider now the more general inner product

$$\langle f, g\rangle := \int_{-1}^{1} f(x)g(x)\ (1-x)^\alpha(1+x)^\beta dx$$

Here one requires $\alpha > -1$ and $\beta > -1$, because otherwise the inner product is not defined for all $f, g \in \mathcal{P}$.

In the subsections before we have already studied the cases $\alpha = \beta = 0$, (the Legendre polynomials), the case $\alpha = \beta = -\frac{1}{2}$, (Chebyshev polynomials of the first kind), and $\alpha = \beta = \frac{1}{2}$, (Chebyshev polynomials of the second kind). We will not consider separately the case $\alpha = \beta$ (so called *Gegenbauer polynomials* or *hyperspheric polynomials*), but consider the general case of Jacobi polynomials for arbitrary $\alpha > -1, \beta > -1$.

First we need here the binomial coefficients $\binom{a}{k}$ for noninteger $a \in \mathbb{R}$.

**Definition 3.3** (Generalized binomial coefficients)
For $a \in \mathbb{R}$ and $k \in \mathbb{N}$ we define

$$\binom{a}{k} := \frac{a(a-1)\cdots(a-k+1)}{k!}.$$

This definition coincides obviously with the known one for $a \in \mathbb{N}$, $a \geq k$.

An other way of expressing a product $a(a-1)\cdots(a-k+1)$ is possible by using the Gamma function

$$\Gamma(x) := \int_0^\infty e^{-t} t^{x-1}\, dt \qquad \text{for } x > 0.$$

The known functional equation $\Gamma(x+1) = x \cdot \Gamma(x)$ gives

$$a(a-1)\cdots(a-k+1) = \frac{\Gamma(a+1)}{\Gamma(a-k+1)}.$$

Since $\Gamma(1) = 1$ (implying $\Gamma(k+1) = k!$) one can also write

$$\binom{a}{k} = \frac{\Gamma(a+1)}{\Gamma(k+1)\Gamma(a-k+1)}$$

for $a > -1$ and $k \in \mathbb{N}$. Theoretically, but we don't need it, this identity can also be used for defining the binomial coefficient $\binom{a}{k}$ for arbitrary real $k$ with $-1 < k < a+1$.

Starting point is the Rodrigues formula

$$J_n^{(\alpha,\beta)}(x) = K_n \cdot (1-x)^{-\alpha}(1+x)^{-\beta} \frac{d^n}{dx^n}\Big((1-x)^{\alpha+n}(1+x)^{\beta+n}\Big).$$

We show first that $J_n^{(\alpha,\beta)}$ is a polynomial of degree $n$. By direct calculation

$$\begin{aligned}
&\frac{d^n}{dx^n}\Big((1-x)^{\alpha+n}(1+x)^{\beta+n}\Big) \\
&= \textstyle\sum_{k=0}^n \binom{n}{k} \binom{\alpha+n}{k} k!(-1)^k(1-x)^{\alpha+n-k} \binom{\beta+n}{n-k} (n-k)!(1+x)^{\beta+k} \\
&= (1-x)^\alpha(1+x)^\beta \textstyle\sum_{k=0}^n n! \binom{\alpha+n}{k} \binom{\beta+n}{n-k} (-1)^k(1-x)^{n-k}(1+x)^k
\end{aligned}$$

we see that

$$J_n^{(\alpha,\beta)}(x) = n!K_n(-1)^n \sum_{k=0}^n \binom{\alpha+n}{k}\binom{\beta+n}{n-k}(-1)^{n-k}(1-x)^{n-k}(1+x)^k$$

and its leading coefficient is

$$lc(J_n^{(\alpha,\beta)}) = n!K_n(-1)^n \sum_{k=0}^n \binom{\alpha+n}{k}\binom{\beta+n}{n-k}. \qquad (*)$$

In the Rodrigues formula, we needed the $n$-th derivative of $\varphi$,

$$\varphi(x) := (1-x)^{\alpha+n}(1+x)^{\beta+n}.$$

Its lower order derivatives are (with $0 \le m < n$)

$$\varphi^{(m)}(x) = \; m! \sum_{k=0}^{m} \binom{\alpha+n}{k} \binom{\beta+n}{m-k} (-1)^k (1-x)^{\alpha+n-k}(1+x)^{\beta+n-m+k}.$$

Each of the $m+1$ summands is zero in 1 and in $-1$ because of

$$\begin{aligned} \alpha+n-k \;&\ge\; \alpha+n-m \;\ge\; \alpha+1 \;>\; 0, \\ \beta+n-m+k \;&\ge\; \beta+n-m \;\ge\; \beta+1 \;>\; 0. \end{aligned}$$

Hence $\varphi^{(m)}$ is zero in $\pm 1$. Therefore the inner product

$$\langle q, J_n^{(\alpha,\beta)} \rangle = \int_{-1}^{1} q(x) J_n^{(\alpha,\beta)} (1-x)^\alpha (1+x)^\beta dx$$

can be transformed by partial integration into

$$\begin{aligned} \langle q, J_n^{(\alpha,\beta)} \rangle \;&=\; K_n \int_{-1}^{1} q(x) \varphi^{(n)}(x) \; dx \\ &=\; -K_n \int_{-1}^{1} q'(x) \varphi^{(n-1)}(x) \; dx \\ &=\; \ldots \\ &=\; (-1)^n K_n \int_{-1}^{1} q^{(n)}(x) \varphi(x) \; dx \end{aligned}$$

Especially if $q \in \mathcal{P}_{n-1}$, then

$$\langle q, J_n^{(\alpha,\beta)} \rangle = 0.$$

**Theorem 3.8.** *Every orthogonal polynomial of degree n for the inner product*

$$\langle f, g \rangle := \int_{-1}^{1} f(x) g(x) \; (1-x)^\alpha (1+x)^\beta dx$$

*is of type*

$$J_n^{(\alpha,\beta)}(x) = n! K_n (-1)^n \sum_{k=0}^{n} \binom{\alpha+n}{k} \binom{\beta+n}{n-k} (-1)^{n-k} (1-x)^{n-k} (1+x)^k.$$

*If* $K_n = \frac{(-1)^n}{n!} \binom{\alpha+\beta+2n}{n}^{-1}$ *then* $lc(J_n^{(\alpha,\beta)}) = 1$. *If* $K_n := \frac{(-1)^n}{n!} 2^{-n}$ *then* $J_n^{(\alpha,\beta)}(1) = \binom{\alpha+n}{n}$ . *If*

$$K_n = \sqrt{\frac{\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)} (\alpha+\beta+2n+1) \frac{2^{-\alpha-\beta-2n-1}}{n!}}$$

*then* $J_n^{(\alpha,\beta)}$ *is orthonormal.*

**Proof.** A representation for $lc(J_n^{(\alpha,\beta)})$ is already shown in (*),

$$lc(J_n^{(\alpha,\beta)}) = K_n n!(-1)^n \sum_{k=0}^{n} \binom{\alpha+n}{k}\binom{\beta+n}{n-k}.$$

In calculus, one shows for real $a, b$, and $|x| < 1$

$$(1+x)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k \quad \text{and} \quad (1+x)^b = \sum_{n=0}^{\infty} \binom{b}{n} x^n.$$

Therefore

$$(1+x)^{\alpha+\beta} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{\alpha}{k}\binom{\beta}{n-k} x^n = \sum_{n=0}^{\infty} \binom{\alpha+\beta}{n} x^n.$$

Comparing the coefficients of the power series gives

$$\sum_{k=0}^{n} \binom{\alpha}{k}\binom{\beta}{n-k} = \binom{\alpha+\beta}{n}.$$

Hence

$$lc(J_n^{(\alpha,\beta)}) = K_n n!(-1)^n \binom{\alpha+\beta}{n}.$$

This gives the representation for $lc(J_n^{(\alpha,\beta)}) = 1$.

Direct computation shows

$$J_n^{(\alpha,\beta)}(1) = K_n n!(-1)^n \binom{a+n}{n} 2^n.$$

Let the polynomial $J_n^{(\alpha,\beta)}$ with

$$J_n^{(\alpha,\beta)}(x) = c_n x^n + \text{ lower degree terms}$$

be orthogonal. Then

$$\langle J_n^{(\alpha,\beta)}, J_n^{(\alpha,\beta)}\rangle = c_n \langle x^n, J_n^{(\alpha,\beta)}\rangle.$$

Therefore

$$\begin{aligned}
\langle J_n^{(\alpha,\beta)}, J_n^{(\alpha,\beta)}\rangle &= c_n \int_{-1}^{1} x^n J_n^{(\alpha,\beta)}(x)(1-x)^\alpha (1+x)^\beta \, dx \\
&= c_n K_n \int_{-1}^{1} x^n \frac{d^n}{dx^n}\Big((1-x)^{\alpha+n}(1+x)^{\beta+n}\Big) \, dx
\end{aligned}$$

and by partial integration and by using a standard integral formula (Beta function)

$$\begin{aligned}\langle J_n^{(\alpha,\beta)}, J_n^{(\alpha,\beta)}\rangle &= c_n K_n (-1)^n n! \int_{-1}^{1} (1-x)^{\alpha+n}(1+x)^{\beta+n}\, dx \\ &= c_n K_n (-1)^n n! \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{\Gamma(\alpha+\beta+2n+2)} 2^{\alpha+\beta+2n+2}.\end{aligned}$$

We substitute now the leading coefficient

$$c_n = lc(J_n^{(\alpha,\beta)}) = K_n n!(-1)^n \binom{\alpha+\beta+2n}{n} = K_n(-1)^n \frac{\Gamma(\alpha+\beta+2n+1)}{\Gamma(\alpha+\beta+n+1)}$$

and obtain finally

$$\langle J_n^{(\alpha,\beta)}, J_n^{(\alpha,\beta)}\rangle = K_n^2 n! \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{\Gamma(\alpha+\beta+n+1)} \frac{1}{\alpha+\beta+2n+1} 2^{\alpha+\beta+2n+2}.$$

Hence $K_n$ chosen as in the assertion gives the orthonormal polynomial. $\square$

For completeness, we quote without proof the three-term-recursion for the Jacobi polynomials $J_n^{(\alpha,\beta)}$ normed by $J_n^{(\alpha,\beta)}(1) = \binom{\alpha+n}{n}$, i.e., $K_n = \frac{(-1)^n}{n!} 2^{-n}$,

$$\begin{aligned}& 2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta) J_{n+1}^{(\alpha,\beta)}(x) \\ = \; & (2n+\alpha+\beta+1)\Big(\alpha^2-\beta^2+(2n+\alpha+\beta+2)(2n+\alpha+\beta)x\Big) J_n^{(\alpha,\beta)}(x) \\ & -2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2) J_{n-1}^{(\alpha,\beta)}(x)\end{aligned}$$

and $J_n^{(\alpha,\beta)}$ solves the differential equation

$$(1-x^2)y'' + \Big(\beta-\alpha-(\alpha+\beta+2)x\Big)y' + n(n+\alpha+\beta+1)y = 0.$$

## §3.5 The Laguerre polynomials

In this subsection, we consider the inner product

$$\langle f, g\rangle := \int_0^\infty f(x)g(x)\, e^{-x} dx.$$

We begin with the function $\varphi : [0,\infty) \to [0,\infty)$,

$$\varphi(x) = e^{-x} x^n.$$

The first $n$ derivatives of $\varphi$ are

$$\varphi^{(m)}(x) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} e^{-x} \frac{n!}{(n-k)!} x^{n-k}, \quad 1 \le m \le n.$$

Thus,

$$p_n(x) := e^x \varphi^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{n!}{(n-k)!} x^{n-k}$$

is a polynomial of degree $n$ with leading coefficient $(-1)^n$. $\varphi$ together with its first $n-1$ derivatives vanish in $x = 0$ since $x = 0$ is an $n$-fold zero of $\varphi(x) = e^{-x}x^n$. In the other endpoint of the interval $[0, \infty)$, $\varphi$ and its first $n-1$ derivatives vanish, too,

$$\lim_{x\to\infty} \varphi^{(m)}(x) = 0 \quad \text{for } 0 \le m < n.$$

Therefore, by partial integration,

$$\begin{aligned} \langle q, p_n \rangle &= \int_0^\infty q(x) p_n(x) e^{-x} dx = \int_0^\infty q(x) \varphi^{(n)}(x) dx \\ &= -\int_0^\infty q'(x) \varphi^{(n-1)}(x) dx = \ldots = (-1)^n \int_0^\infty q^{(n)}(x) \varphi(x) dx. \end{aligned}$$

Hence

$$\langle q, p_n \rangle = 0 \quad \text{if } \; q \in \mathcal{P}_{n-1},$$

i.e., $p_n$ is orthogonal of degree $n$, and

$$\langle p_n, p_n \rangle = (-1)^n \int_0^\infty n!(-1)^n x^n e^{-x} dx = n!\Gamma(n+1) = (n!)^2.$$

**Theorem 3.9** (The Laguerre polynomials)
*Let $\hat{p}_n$ ( or $\tilde{p}_n$ or $p_n^*$) be the $n$-th degree orthogonal polynomial with respect to the inner product*

$$\langle f, g \rangle := \int_0^\infty f(x) g(x) e^{-x} dx$$

*with $lc(\hat{p}_n) = 1$ (or $\tilde{p}_n(0) = 1$ or $\langle p_n^*, p_n^* \rangle = 1$ resp.). Then*

$$\begin{aligned} \hat{p}_n(x) &:= (-1)^n e^x \tfrac{d^n}{dx^n}(x^n e^{-x}) \\ \tilde{p}_n(x) &:= \tfrac{1}{n!} e^x \tfrac{d^n}{dx^n}(x^n e^{-x}) \qquad \textit{(Rodrigues formula)} \\ p_n^*(x) &:= \tfrac{(-1)^n}{n!} e^x \tfrac{d^n}{dx^n}(x^n e^{-x}) \end{aligned}$$

*The Laguerre polynomials satisfy the three term recursion formulas*

$$\begin{aligned} \hat{p}_{n+1}(x) &= \Big(x - (2n+1)\Big)\hat{p}_n(x) - n^2 \hat{p}_{n-1}(x) \\ (n+1)\tilde{p}_{n+1}(x) &= \Big(2n+1-x\Big)\tilde{p}_n(x) - n\tilde{p}_{n-1}(x) \\ (n+1)p^*_{n+1}(x) &= \Big(x - (2n+1)\Big)p_n^*(x) - np^*_{n-1}(x) \end{aligned}$$