tions and remembering that we picture [1] $\alpha(t)$ as representing spin reserved clusters in the ferromagnet, then we might expect that in the critical region $\alpha(t) \sim N^{-\frac{1}{2}}$ where N is the number of spins in a cluster. If we call the characteristic size of the cluster ξ and parameterize ξ in the critical region as $\xi \sim t^{-\nu'}$, we obtain $\nu' = 2\beta$ since $N \propto \xi^3$ in 3-d. This relationship agrees with the scaling law $3\nu' = \beta(\delta+1)$ with $\delta = 5$, the correct value, where here ν' is the critical exponent for the correlation length describing the range of the critical fluctuations. In addition to being consistent with the scaling laws in three dimensions, this argument further suggests that the correlation length in fluctuation theories of critical phenomena corresponds in our model to the actual characteristic size of a real cluster of spins in a real ferromagnet.

Although more theoretical work on the problem of selfconsistency is required, we believe that there is already strong evidence to support the view that our model is a new non-classical phenomenological model of critical phenomena in three dimensions.

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A NUMERICAL ALGORITHM FOR SOLVING NON-LINEAR SINGULAR INTEGRAL EQUATIONS THAT ARISE IN CONNECTION WITH MANY BODY PROBLEMS*

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A method is given for transforming non-linear singular integral equations that arise in connection with finite-temperature many-body problems into a set of algebraic equations to be solved numerically. The method can be used with either Fermi-Dirac or Bose-Einstein statistics and it can be made arbitrarily accurate.

In formulating many-body problems at finite temperature either by Green function or scattering theory techniques one is often led to non-linear singular integral equations which in general cannot be solved analytically. In this note we give a method for transforming these integral equations into a set of algebraic ones in which the analytical properties and the problems of numerical solution are clearly exhibited. We shall use [1] the language of double time retarded and advanced Green functions, although the same method is applicable in other contexts.

In general one derives a set of non-linear singular integral equations for spectral weights or correlation functions. The integral term is of the form

$$\lim_{\delta \to 0^+} \int_{-\infty}^{\infty} (f(\omega) - \tfrac{1}{2})[G(\omega + i\delta) - G(\omega - i\delta)]d\omega\,, \qquad (1)$$

where the unknown $G(z)$ may be written

$$\sum_k A(z, \xi_k)\,[z - F(\xi_k)]^{-1} \qquad (2)$$

and is assumed to be a sectionally meromorphic function i.e., a function analytic in the (upper/lower) half plane, for z in the (upper/lower) half plane except for a finite number of iso-

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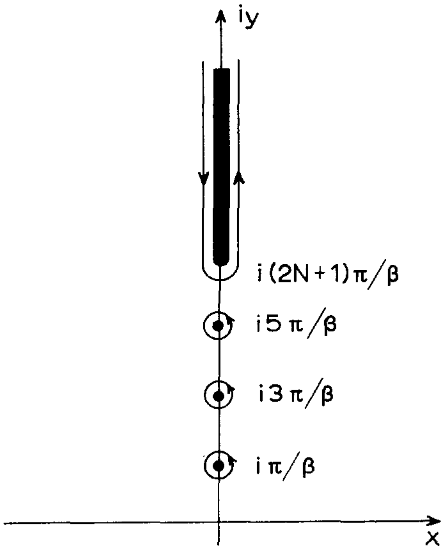

Fig. 1. The represantation of the fm distribution as a series of poles and branch cut and the contour used to evaluate the integrals in the upper half plane.

lated poles; the real axis is to be a branch cut and $f(\omega)$ is either the Fermi-Dirac or Bose-Einstein distribution function. The unknowns are contained in $A(z, \xi_k)$ and perhaps $F(\xi_k)$. If $A(z,\xi_k) \sim 1/z$ as $|z| \to \infty$ or if the asymptotic form of $G(z)$ is subtracted off, giving another unknown, then the integral can be separated into the $G(\omega + \mathrm{i}\delta)$ part and the $G(\omega - \mathrm{i}\delta)$ part, since each separately converges. The contour can be closed in the (upper/lower) half plane as the analytic properties dictate. The finite number of pole terms in $A(z, \xi_k)$, together with the asymptotic term subtracted out, give only a finite number of constants to be solved for self consistently. The problem is that of handling the infinite number of poles in $(f(z) - \frac{1}{2})$. In some cases these poles can be made to cancel [2], but this is not generally the case. We note the following representation for the Fermi-Dirac or Bose-Einstein distribution

$$f_{\mathrm{F.D.}}(z) = -f_{\mathrm{B.E.}}(z - \mathrm{i}\pi/\beta) = \qquad (3)$$
$$= (2\pi\mathrm{i})^{-1}(\psi(\tfrac{1}{2} + z\beta/2\pi\mathrm{i}) - \psi(\tfrac{1}{2} - z\beta/z\pi\mathrm{i}) + \pi\mathrm{i})$$

where $\beta = (kT)^{-1}$ and $\psi(z)$ is the digamma function. Repeated application of the recursion relation for the digamma function [3] yields

$$\psi(W) = -W^{-1} + \psi(W+1) = \sum_{l=1}^{N} (W+l-1)^{-1} + \psi(W+N) \qquad (4)$$

We next use the asymptotic expansion

$$\psi(W+N) = \ln(W+N) - [2(W+N)]^{-1} - [12(W+N)^2]^{-1} + $$
$$+ [120(W+N)^4]^{-1} + \dots \qquad (5)$$

where N or the number of terms in the asymtotic expansion can be chosen so the distribution function is represented as accurately as desired. For Fermi statistics, except perhaps for a finite number of poles elsewhere in the plane, the integral with the contour distorted into the upper half plane is given in terms of the simple poles of the Fermi function at $\mathrm{i}(2l-1)\pi/\beta$ and an integral around a logarithmic branch cut (see fig. 1). The accuracy of the expansion depends only on the choice of N and the number of terms one keeps in the asymptotic expansion and not on the temperature; however, the integral around the branch cut dominates as the temperatures goes to zero so N can be chosen smaller. For Bose-Einstein statistics the poles are shifted to the even multiples of $\mathrm{i}\pi/\beta$. (It should be pointed out in this case the pole at $z = 0$ can be excluded because the general analytical properties of the Green function insure regularity at $z = 0$). The integral around the branch cut can be reduced to

$$\int_{(2N+1)\pi/\beta}^{\infty} G(\mathrm{i}y)\,\mathrm{d}y$$

and this can be evaluated by a variety of numerical procedures. Usually, the interval $((2N+1)\pi/\beta, \infty)$ can be broken up into subintervals and each appropriately mapped onto $(-1, 1)$ and the resultant integrals calculated by Gaussian quadrature. The set of integral equations is thus reduced to a set of nonlinear algebraic or transcendental equations for a finite number of unknowns. A part of the set of unknows consists of the values of the unknown function (and derivatives) at the poles and quadrature points. The accuracy required in the integration along the branch cut is dependent on the location of the other poles of $A(z, \xi_k)$ and the importance of the various regions on the complex plane in representing the desired functions. These considerations depend on the individual problem under study. This method has been successfully applied to the finite magnetic field s-d exchange interaction model [4]; and it should be useful in other applications.

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