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Numerical solution of nonlinear Volterra integral equations of the second kind by using Chebyshev polynomials

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Abstract

Orthogonal Chebyshev polynomials are developed to approximate the solutions of linear and nonlinear Volterra integral equations. Properties of these polynomials and some operational matrices are first presented. These properties are then used to reduce the integral equations to a system of linear or nonlinear algebraic equations. Numerical examples illustrate the pertinent features of the method.

Keywords: Linear and nonlinear Volterra integral equation; Chebyshev polynomials; Operational matrix; Product operation

1. Introduction

Many problems of theoretical physics and other disciplines lead to linear or nonlinear Volterra integral equations of the form

$$y(x) - (\mathscr{V} y)(x) = f(x), \quad a \leqslant x \leqslant b \ \ (b < \infty), \tag{1}$$

where

$$(\mathscr{V} y)(x) := \int_a^x k(x,t)[y(t)]^p \, \mathrm{d}t.$$

The functions $f(x)$ and $k(x,t)$ are known, $y(x)$ is the unknown function to be determined and $p \geqslant 1$ is a positive integer. For $p = 1$, Eq. (1) is a linear and for $p \geqslant 2$ is a nonlinear Hammerstein-type integral equation. We suppose without loss of generality that the interval of integration in (1) is $[-1, 1]$, which is the domain of the Chebyshev polynomials of the first kind, since any finite interval $[a, b]$ can be transformed to interval $[-1, 1]$ by linear maps [2].

For numerical solution of these equations several numerical approaches have been proposed. Many different authors present numerical solutions for (1) by using Galerkin, collocation and other methods [3,6]. In [1],

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Brunner applied a collocation-type method to Volterra integral equations. Delves and Mohamed [2] studied some numerical methods for linear integral equations of the second kind. In general these numerical methods transform the integral equation to a linear or nonlinear system of algebraic equations that can be solved by direct or iterative methods. For big matrices this requires a huge number of arithmetic operations and a large storage capacity. A lot of computing time is saved if we succeed in replacing the fully populated transform matrix with a sparse matrix. One possibility for this gives the wavelet method and orthogonal basis that lead to a sparse matrix representation. An orthogonal basis has the advantage that it guarantees the stability of the matrix equations [2]. In recent years, Maleknejad et al. [5] used Legendre wavelets to numerical solution of linear and nonlinear Volterra integral equations of the form (1), and in [7,8] Taylor polynomials are applied for solving of nonlinear Volterra and Fredholm integral equations.

Consider the well-known Chebyshev polynomials of the first kind of degree n, defined by [4]

$$T_n(x) = \cos(n\cos^{-1}x), \quad n \geqslant 0.$$

Also they are derived from the following recursive formula:

$$\begin{aligned} &T_0(x) = 1,\\ &T_1(x) = x,\\ &T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n = 1,2,3,\ldots \end{aligned}$$

These polynomials are orthogonal on $[-1,1]$ with respect to the weight function $\omega(x) = (1-x^2)^{-\frac{1}{2}}$:

$$\int_{-1}^{1} T_i(x)T_j(x)(1-x^2)^{-\frac{1}{2}}\mathrm{d}x = \begin{cases} \pi, & i=j=0,\\ \frac{\pi}{2}\delta_{ij}, & i,j>0. \end{cases} \tag{2}$$

Chebyshev polynomials are important in approximation theory and numerical analysis and in some quadrature rules based on this polynomials that appear in the theory of numerical integration [2].

In this paper, we apply these polynomials, as a basis on $[-1,1]$, to solve Eq. (1) and reduce it to a set of algebraic equations by expanding the unknown function as Chebyshev polynomials series with unknown coefficients. The properties of these basis functions are then utilized to evaluate the unknown coefficients. The method is tested with the aid of the some numerical examples.

2. Function approximation

A function $f(x)$ defined over $[-1,1]$ may be represented by first kind Chebyshev polynomials series as [2,4]

$$f(x) = \sum_{i=0}^{\infty} c_i T_i(x). \tag{3}$$

If the infinite series in (3) is truncated, then (3) can be written as

$$f(x) \simeq \sum_{i=0}^{N} c_i T_i(x) = \mathbf{C}^{\mathrm{T}} T(x), \tag{4}$$

where $\mathbf{C}$ and T are $(N+1)\times 1$ vectors given by

$$\mathbf{C} = [c_0, c_1, c_2, \ldots, c_N]^{\mathrm{T}}, \tag{5}$$

and

$$T(x) = [T_0(x), T_1(x), T_2(x), \ldots, T_N(x)]^{\mathrm{T}} \tag{6}$$

with coefficients c_i, given by [2]

$$c_i = (f(x), T_i(x)) = \begin{cases} \frac{1}{\pi}\int_{-1}^{1}\omega(x)f(x)\mathrm{d}x, & i=0,\\ \frac{2}{\pi}\int_{-1}^{1}\omega(x)T_i(x)f(x)\mathrm{d}x, & i>0, \end{cases} \tag{7}$$

where $\omega(x)$ is the weight function $(1-x^2)^{-\frac{1}{2}}$.

Similarly a function of two variables, $k(x,t)$, on $[-1,1]$ may be approximated as

$$k(x,t) \simeq T^{\mathrm{T}}(x)\mathbf{K}T(t),$$

where $\mathbf{K}$ is a $(N+1)\times(N+1)$ matrix, with

$$\mathbf{K}_{ij} = (T_i(x),(k(x,t),T_j(t))).$$

Also the positive integer powers of a function may be approximate as

$$[y(x)]^p = [\mathbf{Y}^{\mathrm{T}}T(x)]^p = \mathbf{Y}_p^{*\mathrm{T}}T(x),$$

where $\mathbf{Y}_p^*$ is a column vector, whose elements are nonlinear combinations of the elements of the vector $\mathbf{Y}$. $\mathbf{Y}_p^*$ is called the operational vector of the pth power of the function $y(x)$.

For the Chebyshev polynomials with $N=3$ the second and tired product operation vector of $y(x)$ is computed as follows:

$$\mathbf{Y}_2^* = \frac{1}{2}\begin{bmatrix} 2y_0^2+y_1^2+y_2^2+y_3^2 \\ 4y_0y_1+2y_1y_2+2y_2y_3 \\ y_1^2+4y_0y_2+y_1y_3 \\ 2y_1y_2+4y_0y_3 \end{bmatrix},$$

also

$$\mathbf{Y}_3^* = \frac{1}{4}\begin{bmatrix} 4y_0^3+6y_0y_1^2+3y_1^2y_2+6y_0y_2^2+6y_1y_2y_3+6y_0y_3^2 \\ 12y_0^2y_1+3y_1^3+12y_0y_1y_2+6y_1y_2^2+3y_1^2y_3+12y_0y_2y_3+3y_2^2y_3+6y_1y_3^2 \\ 6y_0y_1^2+12y_0^2y_2+6y_1^2y_2+3y_2^3+12y_0y_1y_3+6y_1y_2y_3+6y_2y_3^2 \\ y_1^3+12y_0y_1y_2+3y_1y_2^2+12y_0^2y_3+6y_1^2y_3+6y_2^2y_3+3y_3^3 \end{bmatrix}.$$

One of the advantages of this method is that, the coefficients of expansion of each function in this base, could be computed directly without estimation.

3. The operational matrices

The integration of the vector $T(x)$ defined in (6) can be obtained as

$$\int_{-1}^{x} T(t)\mathrm{d}t = \mathbf{P}T(x), \tag{8}$$

where $\mathbf{P}$ is the $(N+1)\times(N+1)$ operational matrix for integration as follows:

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{4} & 0 & \frac{1}{4} & 0 & \cdots & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{2} & 0 & \frac{1}{6} & \cdots & 0 & 0 \\ \frac{1}{8} & 0 & -\frac{1}{4} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{(-1)^N}{(N-1)^2-1} & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2N} \\ \frac{(-1)^{N+1}}{N^2-1} & 0 & 0 & 0 & \cdots & -\frac{1}{2N-2} & 0 \end{bmatrix}. \tag{9}$$

Also we obtain the product operation matrix of two Chebyshev polynomial vector functions $\widetilde{\mathbf{C}}$ which satisfies the following relation:

$$T(x)T^{\mathrm{T}}(x)\mathbf{C} \simeq \widetilde{\mathbf{C}}^{\mathrm{T}}T(x), \tag{10}$$

where $\mathbf{C}$ is a vector given in (5) and $\widetilde{\mathbf{C}}$ is a $(N+1)\times(N+1)$ square matrix

$$\widetilde{\mathbf{C}}=\frac{1}{2}\begin{bmatrix} 2c_0 & c_1 & \cdots & c_i & \cdots & c_{N-1} & c_N \\ 2c_1 & 2c_0+c_2 & \cdots & c_{i-1}+c_{i+1} & \cdots & c_{N-2}+c_N & c_{N-1} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\ 2c_i & c_{i-1}+c_{i+1} & \cdots & 2c_0+c_{2i} & \cdots & c_{N-i-1} & c_{N-i} \\ \vdots & \vdots & \cdots & \vdots & \ddots & \vdots & \vdots \\ 2c_{N-1} & c_{N-2}+c_N & \cdots & c_{N-i-1} & \cdots & 2c_0 & c_1 \\ 2c_N & c_{N-1} & \cdots & c_{N-i} & \cdots & c_1 & 2c_0 \end{bmatrix},$$

where $i=\left[\frac{N}{2}\right]$.

4. Volterra integral equations

In this section, we solve Volterra integral equation of the form (1) by using Chebyshev polynomials of the first kind.

If we approximate $y(x)$, $k(x,t)$ and $[y(x)]^p$ by the way mentioned in Section 2 as

$$y(x)=T^{\mathrm{T}}(x)\mathbf{Y},\quad k(x,t)=T^{\mathrm{T}}(x)\mathbf{K}T(t),\quad [y(x)]^p=T^{\mathrm{T}}(x)\mathbf{Y}_p^*, \tag{11}$$

where $T(x)$ is defined in (6), $\mathbf{Y}$ is unknown vector defined similarly to $\mathbf{C}$ in (5) and $\mathbf{Y}_p^*$ is a column vector function of the elements of the vector $\mathbf{Y}$.

Then from (10) and (11) we get

$$\begin{aligned}\int_{-1}^{x}k(x,t)[y(t)]^p\,\mathrm{d}t &\simeq \int_{-1}^{x}T^{\mathrm{T}}(x)\mathbf{K}T(t)T^{\mathrm{T}}(t)\mathbf{Y}_p^*\,\mathrm{d}t=T^{\mathrm{T}}(x)\mathbf{K}\int_{-1}^{x}T(t)T^{\mathrm{T}}(t)\mathbf{Y}_p^*\,\mathrm{d}t\\ &=T^{\mathrm{T}}(x)\mathbf{K}\int_{-1}^{x}\widetilde{\mathbf{Y}}_p^{*\mathrm{T}}T(t)\mathrm{d}t=T^{\mathrm{T}}(x)\mathbf{K}\widetilde{\mathbf{Y}}_p^{*\mathrm{T}}\mathbf{P}T(x),\end{aligned}$$

where $\mathbf{P}$ is a $(N+1)\times(N+1)$ matrix given in (9).

With substituting in (1) we have

$$T^{\mathrm{T}}(x)\mathbf{Y}-T^{\mathrm{T}}(x)\mathbf{K}\widetilde{\mathbf{Y}}_p^{*\mathrm{T}}\mathbf{P}T(x)=f(x). \tag{12}$$

To find the solution $y(x)$ in (11), we first collocate Eq. (12) in $(N+1)$ points $\{x_i\}_{i=1}^{N+1}$ in the interval $[-1,1]$,

$$T^{\mathrm{T}}(x_i)\mathbf{Y}-T^{\mathrm{T}}(x_i)\mathbf{K}\widetilde{\mathbf{Y}}_p^{*\mathrm{T}}\mathbf{P}T(x_i)=f(x_i). \tag{13}$$

The resulting equation (13) generates a system of linear or nonlinear equations which can be solved by direct or iterative methods.

5. Numerical examples

In this section, we applied the method presented in this paper for solving integral equation (1) and solved four examples. In Examples 3 and 4 we used Newton's iterative method for solving introduced nonlinear system. The computations associated with the examples were performed using Mathematica 5.

Example 1. Consider the following linear Volterra integral equation of the second kind [9]:

$$y(x)=\mathrm{e}^{-x^2}-\frac{1}{2}\left(\frac{1}{\mathrm{e}}-\mathrm{e}^{-x^2}\right)x+\int_{-1}^{x}xty(t)\mathrm{d}t.$$

The exact solution of this problem is $y(x)=\mathrm{e}^{-x^2}$. The absolute values of error for $N=8$ and $N=10$ are presented in Table 1.

Table 1
Absolute errors for Examples 1 and 2

x_i	Example 1		Example 2	
	$N = 8$	$N = 10$	$N = 8$	$N = 10$
−1.0	8.423×10^{-9}	3.524×10^{-9}	6.362×10^{-6}	8.387×10^{-7}
−0.75	1.137×10^{-5}	1.144×10^{-7}	9.489×10^{-5}	3.064×10^{-7}
−0.5	5.914×10^{-6}	5.431×10^{-7}	1.855×10^{-4}	3.002×10^{-5}
−0.25	1.331×10^{-6}	2.922×10^{-7}	2.208×10^{-4}	1.417×10^{-5}
0.0	0	0	2.706×10^{-5}	6.618×10^{-5}
0.25	1.346×10^{-6}	3.396×10^{-7}	2.757×10^{-4}	3.636×10^{-5}
0.5	6.267×10^{-6}	2.902×10^{-7}	3.455×10^{-4}	6.407×10^{-5}
0.75	1.424×10^{-5}	1.593×10^{-6}	7.672×10^{-5}	4.091×10^{-5}
1.0	8.690×10^{-6}	7.823×10^{-7}	2.007×10^{-3}	5.104×10^{-5}

Table 2
Absolute errors for Examples 3 and 4

x_i	Example 3		Example 4	
	$N = 5$	$N = 6$	$N = 8$	$N = 10$
−1.0	1.125×10^{-2}	1.493×10^{-16}	2.390×10^{-6}	2.419×10^{-8}
−0.75	1.052×10^{-2}	5.551×10^{-17}	6.634×10^{-5}	6.435×10^{-9}
−0.5	2.005×10^{-3}	1.110×10^{-16}	2.734×10^{-4}	3.037×10^{-6}
−0.25	6.214×10^{-3}	2.220×10^{-16}	3.614×10^{-4}	2.896×10^{-6}
0.0	9.198×10^{-3}	1.110×10^{-16}	3.966×10^{-4}	1.175×10^{-5}
0.25	5.237×10^{-3}	1.110×10^{-16}	1.284×10^{-3}	1.501×10^{-5}
0.5	4.065×10^{-3}	2.220×10^{-16}	1.822×10^{-3}	2.940×10^{-5}
0.75	1.371×10^{-2}	2.220×10^{-16}	2.177×10^{-3}	3.186×10^{-5}
1.0	1.524×10^{-2}	9.422×10^{-17}	1.594×10^{-3}	2.485×10^{-5}

Example 2. As the second example consider the following linear integral equation:

$$y(x) = 2x - (1 - e^{1-x^2}) + \int_{-1}^{x} e^{-x^2+t^2} y(t)\,dt$$

with exact solution $y(x) = 2x$. Table 1 illustrate the numerical results for Example 2.

Example 3. For the following nonlinear second kind Volterra integral equation:

$$y(x) = \frac{1}{15}(2x^6 - 5x^4 + 15x^2 - 8x - 20) + \int_{-1}^{x} (x - 2t)[y(t)]^2\,dt.$$

For $N = 6$, the method gives the exact solution $y(x) = x^2 - 1$. Table 2 shows the absolute errors for $N = 5$ and $N = 6$.

Example 4. As the last example consider the nonlinear second kind Volterra integral equation

$$y(x) = 3x - 1 - \frac{1}{3}e^{x}(13 - 10e^{1+x} + 12x + 9x^2) + \int_{-1}^{x} -\frac{1}{3}e^{2x-t}[y(t)]^2\,dt$$

with exact solution $y(x) = 3x - 1$. The absolute values of error for $N = 8$ and $N = 10$ are shown in Table 2.

6. Conclusion

Nonlinear integral equations are usually difficult to solve analytically. In many cases, it is required to obtain the approximate solutions, for this purpose the presented method can be proposed. In this work we approximate the linear and nonlinear part of the Volterra integral equation by the orthogonal Chebyshev polynomials. A quadrature method is used to compute the coefficients of expansion of any given function.

This method can be extended and applied to the system of linear and nonlinear integral equations, linear and nonlinear integro-differential equations, but some modifications are required.

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