4 Influence of the Cohesive Stress on Crack Aperture and Stress

4.1 Introduction

For a sharp crack, classical elasticity predicts a stress singularity at the crack tip, which is often claimed to be unrealistic. Barenblatt [1962] attempted to overcome this problem by including the atomic level cohesive forces across the crack faces, so as to remove the singular stress at the crack tip. The region in the crack plane over which these forces are assumed to act is called the *cohesive zone* in literature.

Although Barenblatt seems to have envisioned the cohesive forces as being atomic in origin, and acting over very small length scales, his model has subsequently been interpreted by fracture mechanicians as being applicable on a continuum scale, for the purposes of numerically modelling crack initiation and growth in ductile materials [Needleman, 1990, Tvergaard and Hutchinson, 1996]. A small non-linear region called the *process zone*, through which energy is dissipated, is assumed to have developed ahead of the crack tip. The non-linearity is assumed to be due to plasticity or microcracking. The cohesive zone helps simplify the complications of the process zone by replacing it with a simple *cohesive law*, that relates the cohesive stress with crack face separation, capable of reproducing the similar effect as that of the process zone. Despite its wide usage, there are certain fundamental issues in both the basic and applied forms of the cohesive stress model that have not been adequately addressed in the literature. Some of them are briefly discussed below.

- Barenblatt [1962] hypothesised that the cohesive force acts between the crack faces for a short length near the crack tip. However, some researchers choose to ignore the cohesive stress acting between the crack faces, and consider it only in the small process zone (cohesive zone) ahead of the crack tip [Carpinteri et al., 2012], while others take it to extend throughout the crack plane [Sinclair et al., 2011]. This confusion has stemmed from a lack of clear understanding of the origin and nature of the cohesive force in a crack, or from a lack of agreement on its definition.

- The cohesive zone theory is widely used to model crack initiation and growth. However, if the objective is to study the direct impact of the cohesive force, then the state of stress in the crack plane and the crack aperture must be analysed. This is a direct test to check the importance of the cohesive forces. Complicated phenomena such as crack initiation and growth can be dealt with only after this aspect is clear.

- The nature of cohesive stress as a function of the crack separation is an issue that has been only weakly addressed in the literature. As there is no direct and accurate way to estimate the cohesive force, some researchers assume a simple law [Miron and Constantinescu, 2011], and then attempt to *prove* that it is reasonable, whereas others appeal to atomic/molecular properties to derive a cohesive force law [Krull and Yuan, 2011].

- The need for the cohesive stress in crack problems is also unclear. It is mostly argued to be a *simplification* of the complicated non-linear process zone. This is not in accordance with Barenblatt's original conception of the cohesive force as something that is always present to counter an increase in aperture.

In the present work, an attempt has been made to address these issues clearly. Because the objectives of this present work are quite basic, certain assumptions will have to be made to simplify the problem to the bare minimum. Though elliptical cracks have been considered, most recently by Sinclair et al. [2011] and Unger [2012], only line cracks (a straight horizontal cut) will be considered in the present work, to allow for simplicity in the analytical treatment. It is possible to derive an expression (as a singular integral equation) for crack aperture subject to a general cohesive law. However, a complete solution to this equation will require selection of a specific law. The difficulty in solving this integral equation leaves little choice but to assume a linear traction-separation law. The effect of the cohesive stress on the crack aperture is thereby studied. It is further shown that stress singularity can be removed by the inclusion of the cohesive force. Linear elasticity is assumed in the bulk of the material, as it has been shown in Chapter 2 that it gives an accurate measure of the stress in the crack-tip region. A brittle material (like glass) is assumed in the present work, with arbitrary shear modulus, as the stresses and displacements can be normalised with respect to the far-field stress and the shear modulus. Inglis [1913] was the first to compute the stresses around a cavity in an elastic solid. As stated by Griffith [1921], the cohesive stress acts if the crack aperture is less than some very small distance called the radius of molecular action. As far as this present study is concerned, which utilises the model of a thin line crack, there will be no place on the crack where the aperture will be non-zero initially. Therefore, the cohesive stress will be acting throughout, and only along, the length of the crack. Although the exact nature and applicability of the cohesive force has been studied by a few researchers (for example, Willis [1967]), these attempts have not been successful in making a solid base for its correct use. As described by Chan et al. [1987], a realistic cohesive interaction is complicated to implement. For the present work, the cohesive force is defined as the force which resists the crack face separation (increase in aperture).

4.2 Origin and Nature of Cohesive Stress

For a general crack, the stress at the crack tip that is predicted by Linear Elastic Fracture Mechanics (LEFM) turns out to be infinite, even for a very small loading, assuming *traction free* crack surfaces. As this does not seem to be realistic, Barenblatt [1962] assumed that other forces besides those of classical elasticity must be at play. The cohesive force is the attractive force between the atoms on the two opposite faces of the crack. The nature of this force maybe ionic, or van der Waals, or any other special kind, but it will be called *cohesive* as long as it is attractive in nature and resists the increase of crack aperture with the application of load. All of the attractive forces are shown in Figure 4.1. The dashed (- -) forces are already accounted for in linear elasticity, therefore only the solid (–) ones are referred to as cohesive stresses.

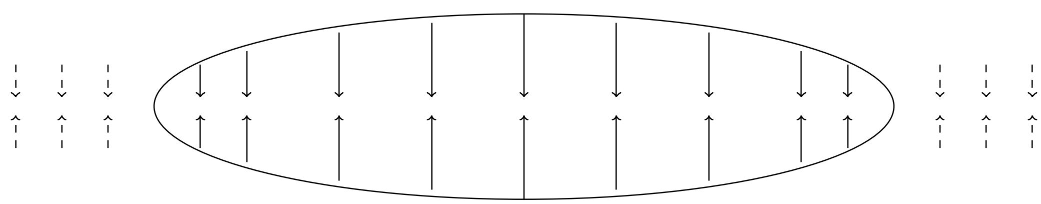


Figure 4.1: Attractive forces along the crack line of an elliptical crack.

The attractive forces in the process zone (if any) ahead of the crack tip are therefore already included in the linear theory of elasticity, and need not be separately considered. This is universally true for a crack in equilibrium that does not intend to grow. For a crack that does intend to grow, the definitions of surface energy and surface tension [Maugis, 2000] will be useful. The surface energy γ of a solid is defined as the work γdA needed to reversibly and isothermally create an elemental area dA of the new surface. This maybe done by overcoming the weak van der Waals forces or the strong ionic forces - both of which depend on the distance between surface atoms on opposite faces of the crack. The asymmetry of interactions at the surface of a solid causes a modification in the ordered arrangement, reflected in non-uniform atomic separation. As a result, a residual stress is set up at the surface, and leads to surface tension. With these revised definitions, it can be safely stated that Griffith [1921] accounted for surface tension in his work, but not surface energy. It is this gap in the mathematical treatment that was filled by Barenblatt [1962]. In the present study, the crack is assumed to be static and not growing, but is free to deform in shape. Barenblatt [1962] asserts that the cohesive forces are distance-dependent forces acting between the crack faces, and are not implicit in the linear theory of elasticity. They are automatically invoked as soon as there is a need to counter an increasing load to oppose the increase in aperture, and are variable in magnitude, just like the force of friction. Keeping this understanding in mind, the next section discusses the analytical derivation of the crack aperture and the stress in crack line under the influence of the cohesive stress.

4.3 Solution for Crack Aperture and Stress

In this section, the crack aperture under the influence of a cohesive stress between the crack faces will be derived, for normal loading. The complete analysis will employ both analytical and numerical methods.

4.3.1 Analytical Formulation

Broberg [1999] derived an equation for the variation of crack aperture with a prescribed stress on the crack faces. Later on in that work, these prescribed stresses were interpreted as the cohesive stresses. Thus, in the present work, the derivation will be reworked with a distinction to the region of cohesive stress (or cohesive zone). It must be pointed out that even though it may appear, in the following analytical derivation, that the cohesive stresses are some sort of a *new* stress; they should in fact be considered as a reaction to the applied load rather than as an active load itself [Broberg, 1999]. For this reason, the cohesive law is valid only in the cohesive zone and not in the bulk, as there is no relationship between the cohesive stress and the stress in the bulk.

Two complex potentials are needed for the treatment and the relationship can be established using the following representation of the displacement field [Papkovich, 1932]

$$2G\mathbf{u} = 4(1-\nu)\psi - \nabla(\phi + \mathbf{r}.\psi) \tag{4.1}$$

where the unit vectors $\hat{\mathbf{x}}$,$\hat{\mathbf{y}}$,$\hat{\mathbf{z}}$ being in the x, y and z directions, $\mathbf{u} = u\hat{\mathbf{x}} + v\hat{\mathbf{y}} + w\hat{\mathbf{z}}$ is the displacement vector, $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ is the position vector while ϕ and $\psi = \psi_x\hat{\mathbf{x}} + \psi_y\hat{\mathbf{y}} + \psi_z\hat{\mathbf{z}}$

are the scalar and vector potentials satisfying

$$\bigtriangledown^2\phi = 0 \qquad\qquad \bigtriangledown^2\psi = 0 \tag{4.2}$$

The four potential components (ϕ,ψ_x,ψ_y,ψ_z) represent only three displacement components, hence one of them can be put equal to zero. In plane strain, where $u = u(x, y), v = v(x, y), w = 0$ holds, it is easy to conclude from Equation 4.1 that $\psi_z = 0$ is the most sensible choice. As ϕ, ψ_x and ψ_y are harmonic functions, they may be represented as

$$\psi_x = \Re f(z) \qquad\qquad \psi_y = \Im f(z) \qquad\qquad \phi = \Re g(z) \tag{4.3}$$

where symbols $\Re$ and $\Im$ represent the real and imaginary parts of functions $f(z)$ and $g(z)$, which are analytic in the entire region. Insertion of Equations 4.3 into Equation 4.1 gives

$$2G(u + iv) = (3 - 4\nu)f(z) - z\overline{f'(z)} - \overline{g'(z)} \tag{4.4}$$

Hooke's law for plane strain gives

$$\sigma_x + \sigma_y = \frac{2G}{1-2\nu}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = \frac{4G}{1-2\nu}\Re[\frac{\partial}{\partial z}(u + iv)] \tag{4.5}$$

$$\sigma_x - \sigma_y = 2G(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}) = 4G\Re[\frac{\partial}{\partial z}(u - iv)] \tag{4.6}$$

$$\tau_{xy} = G(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) = -2G\Im[\frac{\partial}{\partial z}(u - iv)] \tag{4.7}$$

Thus, the stresses are represented by the complex potentials

$$\sigma_x + \sigma_y = 4\Re[f'(z)] \tag{4.8}$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2[\overline{z}f''(z) + g''(z)] \tag{4.9}$$

If the components of the force exerted on the crack surface element $\mathrm{d}s$, where s is the arc length along the crack shown in Figure 4.2, are $\mathrm{X}\,\mathrm{d}s$ and $\mathrm{Y}\,\mathrm{d}s$ in the x and y directions, then the following holds true for equilibrium:

$$\mathrm{X}\,\mathrm{d}s = \sigma_x\,\mathrm{d}y - \tau_{xy}\,\mathrm{d}x \qquad\qquad \mathrm{Y}\,\mathrm{d}s = \tau_{xy}\,\mathrm{d}y - \sigma_y\,\mathrm{d}x \tag{4.10}$$

if the outward normal points to the right when moving in the positive s direction. Furthermore,

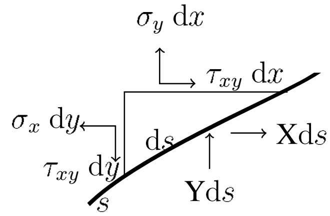


Figure 4.2: Forces X and Y on the crack face, and their relation to the stresses σ_y and τ_{xy}.

the following can be written:

$$(\mathrm{X} + i\mathrm{Y})\,\mathrm{d}s = -\frac{i}{2}[(\sigma_y - \sigma_x - 2i\tau_{xy})\,\mathrm{d}\bar{z} + (\sigma_y + \sigma_x)\,\mathrm{d}z] \tag{4.11}$$

Use of Equations (4.8) - (4.9) in the above equation results in

$$(\mathrm{X} + i\mathrm{Y})\,\mathrm{d}s = -i\,\mathrm{d}[f(z) + z\overline{f'(z)} + \overline{g'(z)}] \tag{4.12}$$

which can be integrated along the arc to give the boundary condition

$$f(z) + z\overline{f'(z)} + \overline{g'(z)} = \int_0^s (i\mathrm{X} - \mathrm{Y})\,\mathrm{d}s \tag{4.13}$$

where the initial value of s can be chosen as zero. The following substitutions [Westergaard, 1939] are now made:

$$f(z) = p(z) - s(z) \tag{4.14}$$

$$g'(z) = p(z) + s(z) - z[p'(z) - s'(z)] \tag{4.15}$$

where $p = (g' + f + zf')/2$ and $s = (g' - f + zf')/2$ are analytic inside the body, as f and g are analytic. Equations (4.8) - (4.9) take the following form:

$$\sigma_x + \sigma_y = 4\Re[p'(z) - s'(z)] \tag{4.16}$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 4s'(z) - 2(z - \overline{z})[p''(z) - s''(z)] \tag{4.17}$$

$$2G(u + iv) = \frac{1 + k^2}{1 - k^2}[p(z) - s(z)] - \overline{p(z)} - \overline{s(z)} - (z - \overline{z})[\overline{p'(z)} - \overline{s'(z)}] \tag{4.18}$$

and hence the boundary condition in Equation (4.13) changes to

$$p(z) - s(z) + \overline{p(z)} + \overline{s(z)} + (z - \overline{z})[\overline{p'(z)} - \overline{s'(z)}] = \int_0^s (iX - Y)\,\mathrm{d}s \tag{4.19}$$

It is assumed that the body and the loading geometry are symmetric with respect to the plane $y = 0$, and pure normal loading prevails. We consider two symmetrically situated points z_0 and $\overline{z_0}$ about the crack axis. If the complex displacement is $u_0 + iv_0$ at z_0, then it is $u_0 - iv_0$ at $\overline{z_0}$. We can write displacements of Equation (4.18) for the two points z_0 and $\overline{z_0}$ as

$$2G(u_0 + iv_0) = \frac{1+k^2}{1-k^2}[p_+(z_0) - s_+(z_0)] - \overline{p_+(z_0)} - \overline{s_+(z_0)} - 2iy_0[\overline{p'_+(z_0)} - \overline{s'_+(z_0)}] \tag{4.20}$$

$$2G(u_0 - iv_0) = \frac{1+k^2}{1-k^2}[p_-(\overline{z_0}) - s_-(\overline{z_0})] - \overline{p_-(\overline{z_0})} - \overline{s_-(\overline{z_0})} + 2iy_0[\overline{p'_-(\overline{z_0})} - \overline{s'_-(\overline{z_0})}] \tag{4.21}$$

with the positive and the negative sign in the subscript of a function representing the upper and the lower half planes separated by the line $y = 0$ (crack axis). Taking the complex conjugate of Equation (4.21) gives

$$2G(u_0 + iv_0) = \frac{1+k^2}{1-k^2}[\overline{p_-(\overline{z_0})} - \overline{s_-(\overline{z_0})}] - p_-(\overline{z_0}) - s_-(\overline{z_0}) - 2iy_0[p'_-(\overline{z_0}) - s'_-(\overline{z_0})] \tag{4.22}$$

By comparison of Equation (4.22) with Equation (4.20), it can be concluded that

$$p_-(\bar{z}) = \overline{p_+(z)} \qquad\qquad s_-(\bar{z}) = \overline{s_+(z)} \tag{4.23}$$

As a general case, the unsymmetrical crack is taken to be $y = 0, b < x < c$, and the condition $(\sigma_y)_+ = (\sigma_y)_- = \sigma_y^0(x)$ and $\tau_{xy}^0 = 0$ holds on it where $\sigma_y^0(x)$ and τ_{xy}^0 are the normal and shear cohesive stress on the crack, respectively. The treatment of non-zero τ_{xy}^0 can be similarly done but is not considered in the present work. Now, the LHS of Equation (4.17) can be written as

$$(\sigma_y - \sigma_x + 2i\tau_{xy})_- = (\sigma_y - \sigma_x + 2i\tau_{xy})_+ \tag{4.24}$$

for $y = 0$, because σ_x and σ_y are symmetrical with respect to $y = 0$, and τ_{xy} vanishes for $y = 0$ throughout the crack. Because of Equation (4.17), the following relation holds true for $y = 0$:

$$s'_-(x) = s'_+(x) \tag{4.25}$$

implying that no branch cut has to be made for $s'(z)$ along the crack line. The points $z = b$ and $z = c$ cannot be poles, as the stress and strain energy here are bounded, directly implying that

$s'(z)$ and $s(z)$ are analytic in the whole region, including the crack line. As the region of study is theoretically infinite, we can invoke Liouville's Theorem to deduce that $s'(z)$ is a constant, which Westergaard [1939] took to be zero. Using Equation (4.23) in Equations (4.16) - (4.19) for $y = 0$, $x \in (b, c)$ gives us

$$(\sigma_y)_+ = 2\Re p'_+(x) = p'(x+i0) + \overline{p'(x+i0)} = p'_+(x) + p'_-(x) = \sigma_y^0(x) \tag{4.26}$$

For $x \notin [b, c]$, the following can be written, using Equation (4.18):

$$2Gi\frac{\partial v_+}{\partial x} = \frac{1}{1-k^2}[p'_+(x) - p'_-(x)] - \frac{k^2}{1-k^2}[s'_+(x) - s'_-(x)] = 0 \tag{4.27}$$

where the last term vanishes because of Equation (4.25) on $y = 0$. Thus,

$$p'_+(x) - p'_-(x) = 0 \qquad \text{for } x \notin [b, c] \tag{4.28}$$

The remote stresses σ_{xx}^∞ and σ_{yy}^∞ are considered, but σ_{xx}^∞ has no influence on crack opening in pure normal loading.

Equations (4.26) and (4.28) constitute Hilbert's 21st problem. This can be solved by bringing both the equations to the same form by the introduction of an auxillary function

$$G(z) = (z-b)^{1/2}(z-c)^{1/2} \tag{4.29}$$

with a branch cut along $y = 0$, $x \in (b, c)$ and with the choice of the branch for which $G(z) \to z$ as $z \to \infty$. Then

$$\frac{G_-(x)}{G_+(x)} = +1 \qquad \text{for } x \notin [b, c] \tag{4.30}$$

$$\frac{G_-(x)}{G_+(x)} = -1 \qquad \text{for } x \in (b, c) \tag{4.31}$$

and Equations (4.26) and (4.28) can be written as one single equation

$$p'_+(x) - \frac{G_-(x)}{G_+(x)}p'_-(x) = 0[U(b-x) + U(x-c)] + \sigma_y^0(x)[U(x-b) - U(c-x)] = s_0(x) \tag{4.32}$$

where U is the unit step function. Multiplication of both sides with $G_+(x)$ results in

$$G_+(x)p'_+(x) - G_-(x)p'_-(x) = G_+(x)s_0(x) \tag{4.33}$$

In the above equation, the LHS contains the difference between the values on the upper and the lower side of the x-axis of a function $G(z)p'(z)$ that is analytic in both the upper and the lower half-planes, though not everywhere on the x-axis. Using Plemelj's Formula, this function is found to be

$$G(z)p'(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{G_+(\xi)s_0(\xi)}{\xi - z}\,\mathrm{d}\xi \tag{4.34}$$

where $s_0(x) = \sigma_y^0(x)$, as can be concluded from Equation (4.32). However, this solution of the Hilbert problem in Equation (4.33) is not complete; a function that is analytic in the whole plane may be added, because the LHS of Equation (4.33) vanishes for such a function. This is the same as adding a general solution to the homogeneous Hilbert problem, obtained by putting the right member of Equation (4.33) equal to zero. Hence, the solution is

$$p'(z) = \frac{1}{2\pi i G(z)}\int_b^c \frac{\sigma_y^0(\xi)G_+(\xi)}{\xi - z}\,\mathrm{d}\xi + \frac{P(z)}{G(z)} \tag{4.35}$$

where $P(z)$ is analytic in the whole plane and thus a polynomial of finite degree (by Liouville's Theorem), the coefficients of which are real because of Equation (4.23). The boundary conditions at infinity give

$$4\Re[p'(z) - s'(z)]_{|z|\to\infty} = \sigma_{yy}^{\infty} + \sigma_{xx}^{\infty} \tag{4.36}$$

$$4\Re[s'(z)]_{|z|\to\infty} = \sigma_{yy}^{\infty} - \sigma_{xx}^{\infty} \tag{4.37}$$

which imply that

$$P(z) = \frac{\sigma_{yy}^{\infty}}{2}z + a_0 \qquad\qquad s'(z) = \frac{\sigma_{yy}^{\infty} - \sigma_{xx}^{\infty}}{4} \tag{4.38}$$

where a_0 is to be determined from the condition that the crack is closed at both ends, meaning

$$\begin{aligned} 2G[\nu_+(c) - \nu_+(b)] &= [\frac{2}{1-k^2}\Im p_+(x) - \frac{2k^2}{1-k^2}\Im s(x)]_b^c && (4.39)\\ &= \frac{2}{1-k^2}\int_b^c \frac{\sigma_y^{\infty}x/2 + a_0}{\sqrt{(x-b)(c-x)}}\,\mathrm{d}x = 0 && (4.40)\end{aligned}$$

which gives

$$a_0 = -\frac{(b+c)}{4}\sigma_{yy}^{\infty} \tag{4.41}$$

This is as far as Broberg [1999] had gone to solve this problem (with his own understanding of the region of cohesive zone). The above treatment gives everything needed to write a formal solution. The normal stress and the displacement gradient on the crack plane are found to be

$$\sigma_y = 2\Re p'(x) = \pm\frac{1}{\sqrt{(x-b)(x-c)}}\Bigg\{\frac{1}{\pi}\int_b^c \frac{\sigma_y^0(\xi)\sqrt{(\xi-b)(c-\xi)}}{\xi - x}\,\mathrm{d}\xi \\ +\sigma_{yy}^{\infty}[x-(b+c)/2]\Bigg\} \tag{4.42}$$

$$\frac{\partial v_+}{\partial x} = -\frac{1}{2(1-k^2)G\sqrt{(x-b)(c-x)}}\Bigg\{\frac{1}{\pi}\int_b^c \frac{\sigma_y^0(\xi)\sqrt{(\xi-b)(c-\xi)}}{\xi - x}\,\mathrm{d}\xi \\ +\sigma_{yy}^{\infty}[x-(b+c)/2]\Bigg\} \tag{4.43}$$

with the upper sign for $x > c$ and lower sign for $x < b$. It is worth noting that the above expressions are valid for a general cohesive stress $\sigma_y^0(\xi)$ and not to a specific type of function, at this stage.

It can be seen in the above Equations (4.42) - (4.43), first term in RHS, that the cohesive zone has been taken to extend across the whole crack, and not just a small region from the crack tip, as in the original work.

4.3.2 Numerical Solution

For a line crack that extends from $b = -a$ to $c = a$, Equation 4.43 can be written as

$$\frac{\partial v_+}{\partial x} = -\frac{1}{2(1-k^2)G\sqrt{(x+a)(a-x)}}\Bigg\{\frac{1}{\pi}\int_{-a}^{+a} \frac{\sigma_y^0(\xi)\sqrt{(\xi+a)(a-\xi)}}{\xi - x}\,\mathrm{d}\xi + \sigma_{yy}^{\infty}x\Bigg\} \tag{4.44}$$

$$\frac{\partial v_+}{\partial x} = -\frac{1}{2(1-k^2)G\sqrt{a^2-x^2}}\Bigg\{\frac{1}{\pi}\int_{-a}^{+a} \frac{\sigma_y^0(\xi)\sqrt{a^2-\xi^2}}{\xi - x}\,\mathrm{d}\xi + \sigma_{yy}^{\infty}x\Bigg\} \tag{4.45}$$

The above equation, upon definite integration from $-a$ to x and a change in the order of integration, gives

$$v_+(x) = -\frac{1}{2(1-k^2)G}\Bigg\{\frac{1}{\pi}\int_{-a}^{x}\Big(\frac{1}{\sqrt{a^2-x^2}}\frac{1}{(\xi-x)}\,\mathrm{d}x\Big)\int_{-a}^{+a}\sigma_y^0(\xi)\sqrt{a^2-\xi^2}\,\mathrm{d}\xi \\ +\int_{-a}^{x}\sigma_{yy}^{\infty}\frac{x}{\sqrt{a^2-x^2}}\,\mathrm{d}x\Bigg\} \tag{4.46}$$

which can be re-written as

$$v_+(x) = -\frac{1}{2(1-k^2)G}\left\{\frac{1}{\pi}\left[(\frac{a\ln(x^2-a^2) - 2a\ln(x-t) - 2t\tanh^{-1}(x/a)}{2(a^3-a\xi^2)})\right]* \right.$$
$$\left. * \int_{-a}^{+a} \sigma_y^0(\xi)\sqrt{a^2-\xi^2}\,\mathrm{d}\xi - \sigma_{yy}^{\infty}\sqrt{a^2-x^2}\right\} \tag{4.47}$$

The linear cohesive law $\sigma^0 = 2\alpha v_+$ is now inserted into the above integral equation, to yield

$$v_+(x) = -\frac{1}{2(1-k^2)G}\left\{\int_{-a}^{+a}\frac{2\alpha}{\pi}\left[\ln(\frac{\sqrt{(a^2-\xi^2)(a^2-x^2)}+a^2-\xi x}{a(\xi-x)})\right]v_+(\xi)\,\mathrm{d}\xi \right.$$
$$\left. -\sigma_{yy}^{\infty}\sqrt{a^2-x^2}\right\} \tag{4.48}$$

The above equation cannot be solved analytically, and hence numerical methods must be used, with kernel K defined as the expression within the square brackets [ ]. Moreover, there is a singularity in this equation at $x = \xi$. The method described by Press et al. [1992] will be used to re-write the kernel K of the above Equation 4.48 as

$$\int_{-a}^{+a}\frac{2\alpha}{\pi}\left[\ln(\frac{\sqrt{(a^2-\xi^2)(a^2-x^2)}+a^2-\xi x}{a(\xi-x)})\right]v_+(\xi)\,\mathrm{d}\xi =$$
$$\int_{-a}^{+a}\frac{2\alpha}{\pi}\left[\ln(\frac{\sqrt{(a^2-\xi^2)(a^2-x^2)}+a^2-\xi x}{a(\xi-x)})\right][v_+(\xi)-v_+(x)]\,\mathrm{d}\xi$$
$$+\int_{-a}^{+a}\frac{2\alpha}{\pi}\left[\ln(\frac{\sqrt{(a^2-\xi^2)(a^2-x^2)}+a^2-\xi x}{a(\xi-x)})\right]v_+(x)\,\mathrm{d}\xi \tag{4.49}$$

where the second term of the RHS can be easily solved numerically. By application of the quadrature rule, Equation 4.48 can be written in discrete form as

$$(v_+)_i = \lambda\sum_{j=1 j\neq i}^{n+1} w_j K_{ij}[(v_+)_j - (v_+)_i] + \lambda r_i (v_+)_i + g_i \tag{4.50}$$

where n is the number of quadrature points, and $\lambda = -1/2(1-k^2)G$. This equation can be re-written after expansion as

$$(v_+)_i = \lambda\sum_{j=1 j\neq i}^{n+1} w_j K_{ij}(v_+)_j - (v_+)_i\lambda\sum_{j=0 j\neq i}^{N-1} w_j K_{ij} + \lambda r_i (v_+)_i + g_i \tag{4.51}$$

where r_i can be numerically found. This is a set of $n+1$ linear equations in $n+1$ unknowns, which can be easily solved. This method has been implemented in a code which is listed in the Appendix. When v_+ is known, the law $\sigma_y^0(\xi) = 2\alpha v_+(\xi)$ can be used to solve Equation 4.42 for σ_y in the entire crack plane. The resulting crack aperture variation needs to be validated. This will be done by comparing it to a perturbation solution, which should be valid for sufficiently small value of the cohesive force constant α.

4.4 Comparison with Perturbation Solution

An exact formulation is known for the crack aperture in the presence of a known spatially-variable normal hydrostatic pressure acting along the crack faces [Sneddon and Lowengrub, 1969]. The present problem is different, in the sense that the cohesive stress is not known $a\ priori$. A regular perturbation technique [Hinch, 1991] will be utilised to address this problem. The classical solution for the case of no cohesive force will be taken as the zeroth-order solution. The aperture that is obtained in the zeroth-order solution is then used to calculate the cohesive force, according to the law $\sigma^0 = 2\alpha v_+$. This force is then applied to the surfaces of the crack, and Sneddon's formulation is used to calculate the additional crack aperture and stress field due to this surface force. This new incremental aperture will give rise to an additional cohesive force, and so this process would need to be iterated until convergence is obtained. Convergence of this perturbation process could be defined, for example, as the stage at which the incremental aperture computed at step N is less than some prescribed fraction of the zeroth-order aperture. For simplicity, only one step of this iterative process will be considered, which is expected to be sufficient for validating the accuracy of the numerical solution procedure.

For the case of a spatially-variable pressure acting on the crack face, the aperture variation for the zeroth-order solution plus the first-order perturbation correction can be written as [Sneddon and Lowengrub, 1969]

$$2v_{sn}(x) = \frac{2p_0(1-\nu^2)a}{E}\left[\sqrt{1-(x^2/a^2)} - \frac{2(1-\nu^2)a}{E}\int_{x/a}^{1}\frac{tq(t)}{\sqrt{t^2-(x^2/a^2)}}\,\mathrm{d}t\right] \tag{4.52}$$

where

$$q(t) = \frac{2}{\pi}\int_0^t \frac{p(u)/p_0}{\sqrt{t^2-u^2}}\,\mathrm{d}u \tag{4.53}$$

with $p(u) = 2\alpha p_0\sqrt{1-(u^2/a^2)}$ being the initial zeroth-order pressure with p_0 being of unit value and dimensions of pressure. The crack-half length is a, Young's modulus E, and Poisson's ratio ν for this case. A built-in function *int* in Matlab has been used to numerically solve for $v_{sn}(x)$. This has been implemented in a code which can be found in the Appendix.

4.5 Results

4.5.1 Crack Aperture

A line crack that extends from $b = -a$ to $c = a$, with $a = 1$ (for concrete numerical purposes), is considered. The loading is biaxial unit normal loading, which is the same as uniaxial normal loading for a horizontal crack, as σ_{xx}^{∞} does not influence the crack opening. The aperture profile $2v_+$ normalised against $E/a\sigma_{yy}^{\infty}$ is plotted as a solution of Equation 4.51. Validation of the numerical results can be done by comparison with the first-order perturbation calculation. The two approaches should agree closely for small values of α, which can be expressed in dimensionless form as $\alpha^* = \alpha a/G$.

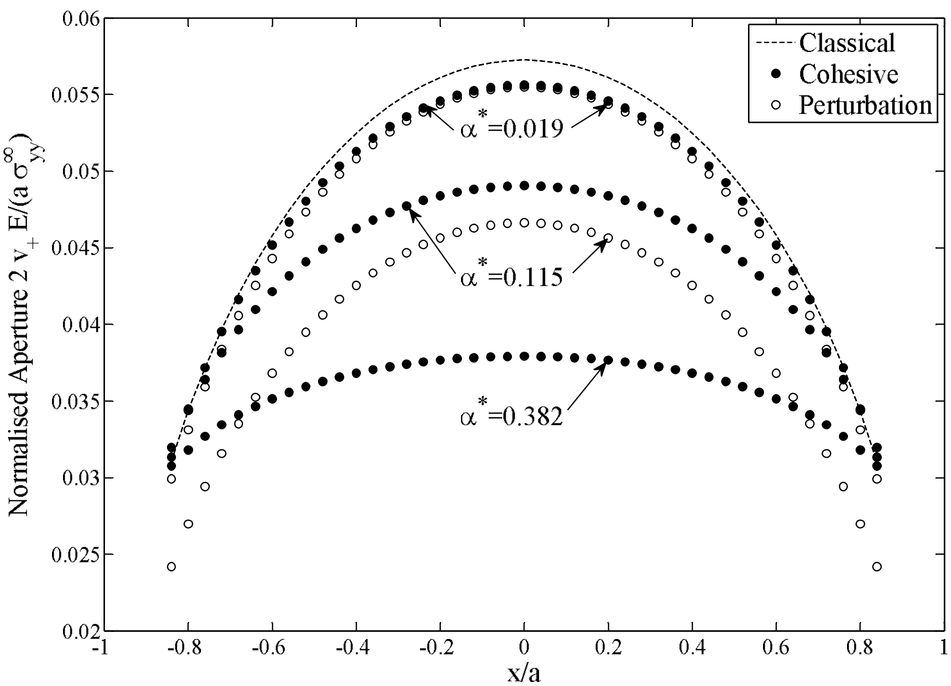


Figure 4.3: Crack face separation in presence of cohesive stress with parameter α^* when applied load is σ_{yy}^{∞}. The dimensionless $\alpha^* = \alpha a/G$.

As can be observed in Figure 4.3, for small values of α^*, the perturbation solution agrees closely with the full cohesive stress solution. As α^* is increased, the perturbation solution becomes less accurate, so that for $\alpha^* = 0.382$ and beyond, the comparisons are no longer meaningful.

A further test to validate the results is to track the error in the calculation of the correction aperture in the cohesive solution against that of the perturbation solution at the mid-point of the crack ($x = 0$). This error is defined as

$$\text{Error}_{x=0} = \frac{(\text{Aperture}_{\text{Cohesive}} - \text{Aperture}_{\text{Perturbation}})_{x=0}}{(\text{Aperture}_{\text{Classical}} - \text{Aperture}_{\text{Perturbation}})_{x=0}} \tag{4.54}$$

As is observed from Figure 4.4, the error reduces as α^* decreases signifying that as α^* becomes small, the cohesive solution comes closer to the perturbation solution. For a very small non-zero α^*, the error converges to 3.86% rather than 0, which maybe due to the numerical integrations which have been performed while calculating cohesive and perturbation solutions.

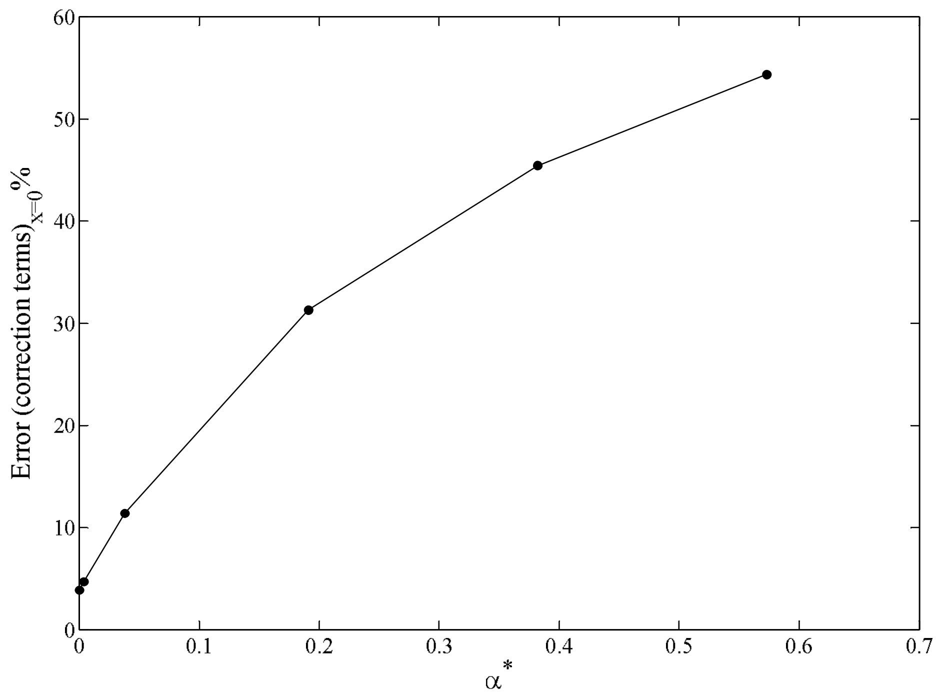


Figure 4.4: Variation of Error at $x = 0$ between the full cohesive solution and the first-order perturbation solution as a function of α^*.

As expected, and seen from Figure 4.3, the crack face separation decreases as the strength of the cohesive stress increases (α^* increases). The extreme case of $\alpha^* \to \infty$ results in a crack that is unable to open at all, as the cohesive force overpowers the applied far-field stress.

4.5.2 Stress Along the Crack Line

After the aperture ($2v_+$) and cohesive stress $\sigma_y^0(\xi) = 2\alpha v_+(\xi)$ have been calculated, Equation 4.42 can be solved for the stress σ_{yy} along the crack line. As can be inferred from Figure 4.5, by reducing the strength of the cohesive stress, α^*, to zero, the stresses reduce to those of the classical solution. It can also be inferred from Figure 4.5 that a large value of the cohesive stress causes a much smaller variation in the stress outside the crack.

It is worth mentioning that even though it would be interesting to compare the normal stress in the crack plane here with those of Chapter 2 as cohesive stress is *in-built* into molecular statics; the current lack of solution for a general cohesive stress in Equation (4.42) restricts this.

For the existence of equilibrium at the crack tip in an infinite system, the *total energy* there must be minimum. Considering the crack tip at $x = a$, it can be stated that the total energy on the RHS of this crack tip (a^+) will be the elastic strain energy, whereas the total energy on the LHS of this crack tip (a^-) will be the cohesive stress energy. It must be recalled that there is no material on the LHS of the crack tip, and hence there is no strain energy contribution from this side of the crack tip. With reference to the LHS of the crack tip (a^-), the cohesive stress opposes the increase in aperture, the work (per unit area) required to increase the aperture from infinitesimal to $2v(x)$ is

$$W_{coh}(x) = \int_0^{2v(x)} \sigma_y^0 \, \mathrm{d}v_+ = \int_0^{v} \alpha 2v_+ \, \mathrm{d}v_+ = 4\alpha v(x)^2 \tag{4.55}$$

which can be stored as energy only where σ_y^0 exists, that is, in the aperture along the crack line, to satisfy the symmetry condition. This understanding of energy storage is true for any conservative field, such as gravitational or electrical. Therefore,

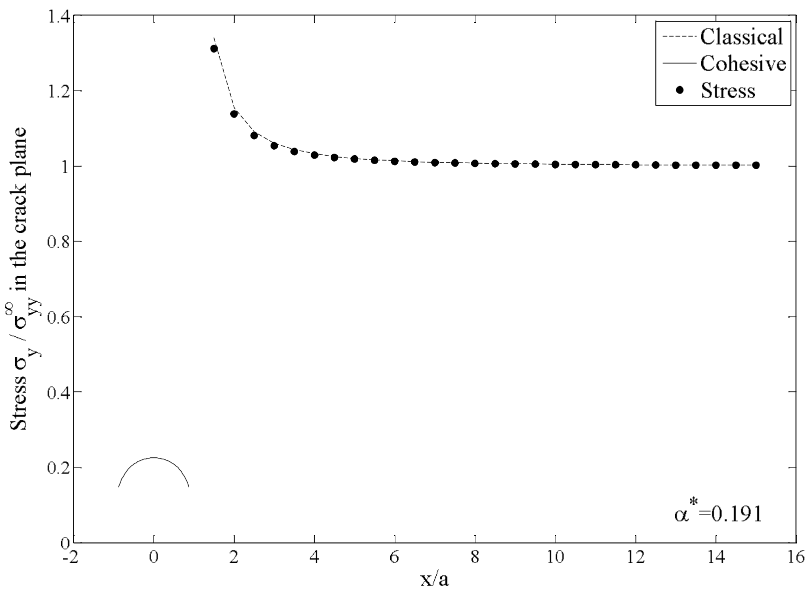
$$E_{coh}(x) = W_{coh}(x) = 4\alpha v(x)^2 \tag{4.56}$$

With reference to the RHS of the crack tip (a^+), the strain energy per unit area is written as

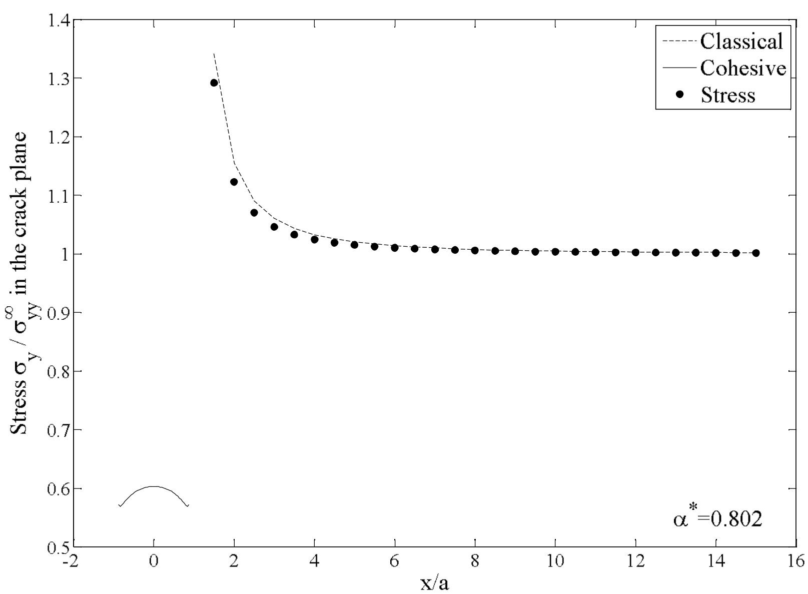
$$E_{str}(x) = \frac{\sigma_y(x)^2}{2E} \tag{4.57}$$

where E is the Young's modulus, and only the σ_y component of the stress is considered, because other components are negligible in magnitude under pure normal loading. The values of total energy when approached from both sides of the crack tip are

$$\lim_{h\to 0} E_{coh}(a-h) = \lim_{h\to 0} 4\alpha v(a-h)^2 = 4\alpha v(a)^2 = 0 \tag{4.58}$$



(a)



(b)

Figure 4.5: Variation of σ_y in Equation (4.42) across the crack line for different values of cohesive stress.

$$\lim_{h\to 0} E_{str}(a+h) = \lim_{h\to 0} \frac{\sigma_y(a+h)^2}{2E} \tag{4.59}$$

For the existence of a local minimum at $x = a$, the following equation can be written:

$$\left(\frac{\mathrm{d}E_{total}}{\mathrm{d}x}\right)_{x=a} = 0 = \lim_{h\to 0} \frac{E_{str}(a+h) - E_{coh}(a-h)}{(a+h)-(a-h)} = \lim_{h\to 0} \frac{E_{str}(a+h) - E_{coh}(a-h)}{2h} \tag{4.60}$$

implying that

$$\lim_{h\to 0} E_{str}(a+h) = \lim_{h\to 0} E_{coh}(a-h) \tag{4.61}$$

into which substituting values from Equations (4.58) - (4.59) gives

$$\lim_{h\to 0} \sigma_y(a+h) = 0 \tag{4.62}$$

Thus, Equation 4.42 can be used to furnish

$$\alpha = \frac{a\pi\sigma_{yy}^{\infty}}{4}\left[\frac{1}{\int_{-a}^{a} v_+(\xi)\sqrt{\frac{a+\xi}{a-\xi}}\,\mathrm{d}\xi}\right] = f(\alpha) \tag{4.63}$$

which holds for equilibrium at the crack tip. It has been numerically concluded (Figure 4.6) that there is only one unique solution to the Equation 4.63, at $\alpha^* \approx 1.5$. In a real material following a linear traction-separation law, a crack at equilibrium would always retain $\alpha^* \approx 1.5$ for the given loading and material properties. If any of the material properties or loading changes, then the equilibrium crack would shift to another value of α^*.

4.6 Conclusion

As can be seen in Figure 4.3, the presence (or rather non-negligence) of the cohesive stress makes the crack opening more difficult. For an infinitely large cohesive stress, the crack would not open at all, whereas in the absence of cohesive stress, the aperture would be the same as that of the classical solution. Observed in Figure 4.5 is the fact that the presence of a cohesive stress causes a reduction in the large stress at the tip to a smaller finite value. A real body in equilibrium has a self-adjusting cohesive stress (similar to the force of friction) so as to maintain equilibrium at the crack-tip to oppose the effect of increasing load. As can be seen from Figure 4.6, this state of equilibrium is achieved at a unique value of α^*. This unique value changes for different conditions. The current state of mathematics is not sufficient for solving an integral equation of a general order, more so for one with a singular kernel. It must be mentioned that it is for the sake of theoretical purposes only that the linear cohesive stress

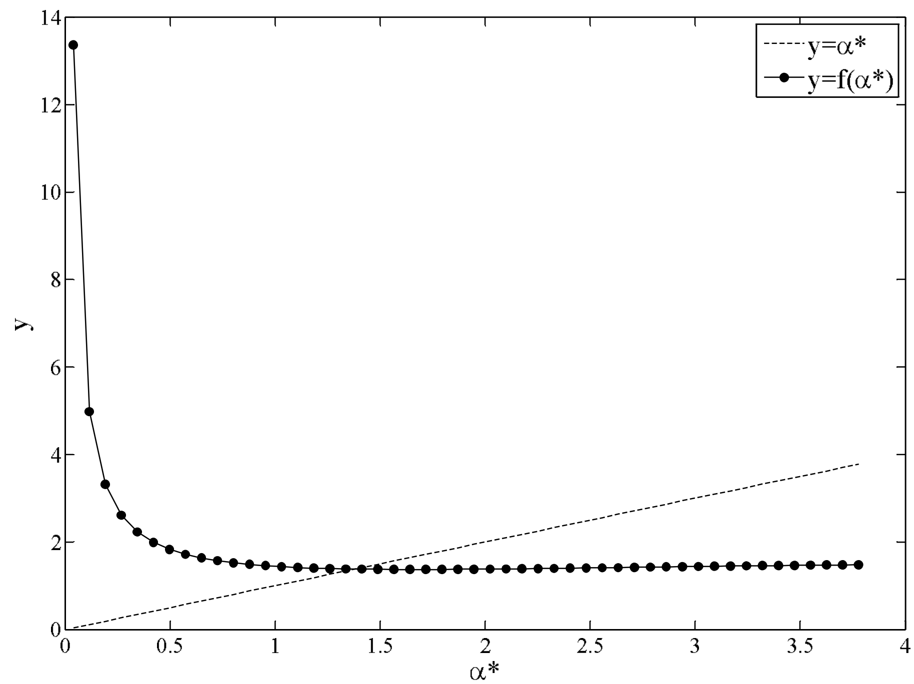


Figure 4.6: Numerical solution of Equation 4.63.

has been assumed; the magnitude of this stress can be controlled by varying α^*. In reality, the value of the cohesive stress cannot be controlled, as it is based on the atomic interactions in the body. However, since equilibrium is theoretically achievable in the crack in the present study - the approximation of a simple linear cohesive law can be justified [Cribb and Tomkins, 1967] on the pretext that any *real* cohesive law will achieve the same.