Finding the Eigenvalues of the Jacobi Matrix: Solution to the Boundary Value Problem

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Problem Statement

We aim to solve the following boundary value problem for a second-order differential equation:

$$\frac{d^2u}{dy^2} = -P,$$

with boundary conditions:

$$u(0) = 0 \quad \text{and} \quad u(1) = 1.$$

Additionally, we need to:

1. Express the first derivative $\frac{du}{dy}$ as a constant multiple of $u(y)$, and
2. Relate this constant multiple to the concept of eigenvalues in the context of a Jacobi matrix.

Step 1: General Solution to the Differential Equation

The general solution of

$$\frac{d^2u}{dy^2} = -P$$

is obtained by integrating twice:

1. First integration:
$$\frac{du}{dy} = -Py + C_1$$
2. Second integration:
$$u(y) = -\frac{P}{2}y^2 + C_1 y + C_2$$

Thus, the general solution is:

$$u(y) = -\frac{P}{2}y^2 + C_1 y + C_2.$$

Step 2: Apply Boundary Conditions

Using the boundary conditions $u(0) = 0$ and $u(1) = 1$:

1. At $y = 0$:
$$0 = -\frac{P}{2}(0)^2 + C_1(0) + C_2 \implies C_2 = 0.$$

2. At $y = 1$:
$$1 = -\frac{P}{2}(1)^2 + C_1(1) \implies 1 = -\frac{P}{2} + C_1 \implies C_1 = 1 + \frac{P}{2}.$$

Thus, the solution becomes:
$$u(y) = -\frac{P}{2}y^2 + \left(1 + \frac{P}{2}\right) y.$$

Step 3: First Derivative $u_1(y)$

Differentiating $u(y)$:
$$u_1(y) = \frac{du}{dy} = -Py + \left(1 + \frac{P}{2}\right).$$

Thus, the first derivative $u_1(y)$ is:
$$u_1(y) = -Py + \left(1 + \frac{P}{2}\right).$$

Step 4: Express $u_1(y)$ as a Multiple of $u(y)$

In this step, we want to express $u_1(y)$ as a constant multiple of $u(y)$. To do this, we assume that there exists some function $f(u)$ such that:
$$u_1(y) = f(u) \cdot u(y),$$
where $f(u)$ is the proportionality factor (which will be related to the eigenvalue in this case). We aim to determine this function $f(u)$ and express it purely in terms of u rather than y.

Substitute the expressions for $u_1(y)$ and $u(y)$:

We substitute the expressions for $u_1(y)$ and $u(y)$ into the equation $u_1(y) = f(u) \cdot u(y)$:
$$-Py + \left(1 + \frac{P}{2}\right) = f(u) \cdot \left(-\frac{P}{2}y^2 + \left(1 + \frac{P}{2}\right) y\right).$$

At this point, we have an equation where the left-hand side is linear in y, and the right-hand side is a quadratic function of y. To solve this, we need to isolate the terms involving y and then compare the coefficients of y^2 and y on both sides.

Rearrange $u(y)$ into a quadratic form:

Let's first express $u(y)$ as a quadratic equation in y:

$$u(y) = -\frac{P}{2}y^2 + \left(1 + \frac{P}{2}\right) y.$$

This is a quadratic equation in y, and we need to solve for y in terms of $u(y)$. We rearrange it as:

$$\frac{P}{2}y^2 - \left(1 + \frac{P}{2}\right) y + u = 0.$$

This is now a quadratic equation in y, which we can solve using the quadratic formula. The quadratic formula for $ay^2 + by + c = 0$ is:

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where $a = \frac{P}{2}$, $b = -\left(1 + \frac{P}{2}\right)$, and $c = u$. Substituting these values into the quadratic formula, we get:

$$y = \frac{\left(1 + \frac{P}{2}\right) \pm \sqrt{\left(1 + \frac{P}{2}\right)^2 - 2Pu}}{P}.$$

We select the positive root for y, because it corresponds to the physical behavior of the solution (choosing the positive root gives us the correct sign for y).

Substitute y into $u_1(y)$:

Now that we have $y(u)$, we can substitute it back into the expression for $u_1(y)$:

$$u_1(y) = -Py + \left(1 + \frac{P}{2}\right).$$

Substitute the expression for y:

$$u_1(y) = -P \cdot \frac{\left(1 + \frac{P}{2}\right) + \sqrt{\left(1 + \frac{P}{2}\right)^2 - 2Pu}}{P} + \left(1 + \frac{P}{2}\right).$$

Simplifying, we get:

$$u_1(y) = -\left(1 + \frac{P}{2}\right) - \sqrt{\left(1 + \frac{P}{2}\right)^2 - 2Pu} + \left(1 + \frac{P}{2}\right).$$

The terms $\left(1 + \frac{P}{2}\right)$ cancel out, and we are left with:

$$u_1(y) = -\sqrt{\left(1 + \frac{P}{2}\right)^2 - 2Pu}.$$

Express $u_1(y)$ as a multiple of $u(y)$:

Now, we express $u_1(y)$ as a multiple of $u(y)$ by dividing both sides of the equation by $u(y)$. Recall the expression for $u(y)$:

$$u(y) = -\frac{P}{2}y^2 + \left(1 + \frac{P}{2}\right)y.$$

Thus, the final function $f(u)$ is:

$$f(u) = -\frac{\sqrt{\left(1 + \frac{P}{2}\right)^2 - 2Pu}}{u}.$$

Conclusion

In this boundary value problem, the eigenvalue cannot be computed directly from a Jacobi matrix because the problem is defined in a continuous domain, making the traditional discrete Jacobi matrix inapplicable.

Instead, the relationship between $u_1(y)$ and $u(y)$ provides an equivalent measure of the eigenvalue. The eigenvalue function is given by:

$$f(u) = -\frac{\sqrt{\left(1 + \frac{P}{2}\right)^2 - 2Pu}}{u}.$$

This function, derived by relating the first derivative to the solution, encapsulates the system's dynamics in the absence of a conventional Jacobi matrix.