Solving Boundary Value Problems (BVPs) Using the Finite Difference Method

Abstract

This document outlines a systematic approach to solving boundary value problems (BVPs) for second-order ordinary differential equations (ODEs) using the finite difference method. The focus is on the equation

$$u''(y) = -P$$

subject to boundary conditions $u(a) = u_a$ and $u(b) = u_b$, where P is a constant. The process involves discretizing the interval, applying finite difference approximations, setting up a linear system, and solving for the unknown values.

1 Introduction

Boundary value problems arise in various fields such as physics, engineering, and applied mathematics. They typically involve finding a function that satisfies a differential equation and meets specific conditions at the boundaries of the domain. The finite difference method provides an effective numerical approach for approximating solutions to BVPs.

2 Problem Statement

Consider the boundary value problem defined as follows:

2.1 Differential Equation

$$u''(y) = -P$$

2.2 Boundary Conditions

$$u(a) = u_a \quad \text{and} \quad u(b) = u_b$$

where P is a constant.

3 Discretization of the Interval

To solve the BVP, we first discretize the interval $[a, b]$ into N points using a uniform step size h.

3.1 Step Size

$$h = \frac{b - a}{N - 1}$$

3.2 Discretized Points

The discretized points are given by:

$$y_i = a + (i-1) \cdot h, \quad i = 1, 2, \ldots, N$$

Table 1: Discretized Points and Unknowns

Index i	**Point** y_i	$u(y_i)$ **Value**
1	$y_1 = a$	$u_1 = u_a$ (known)
2	$y_2 = a + h$	u_2 (unknown)
3	$y_3 = a + 2h$	u_3 (unknown)
$\vdots$	$\vdots$	$\vdots$
$N-1$	$y_{N-1} = b - h$	u_{N-1} (unknown)
N	$y_N = b$	$u_N = u_b$ (known)

4 Finite Difference Approximations

To approximate the derivatives in the differential equation, we use finite difference approximations.

4.1 First Derivative Approximation

Using central differences, the first derivative $u'(y)$ is approximated as:

$$u'(y_i) \approx \frac{u_{i+1} - u_{i-1}}{2h}$$

4.2 Second Derivative Approximation

For the second derivative $u''(y)$:

$$u''(y_i) \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$$

Substituting into the differential equation gives:

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = -P$$

Rearranging this, we obtain:

$$u_{i+1} - 2u_i + u_{i-1} = -P \cdot h^2$$

This equation holds for interior points $y_2, y_3, \ldots, y_{N-1}$.

5 Setting Up the System of Equations

For each interior point i, we obtain a linear equation:

5.1 General Form

$$u_{i+1} - 2u_i + u_{i-1} = -P \cdot h^2$$

This leads to a system of $N-2$ equations, as the boundary conditions provide values for u_1 and u_N.

Table 2: System of Equations for Interior Points

Equation Number	Equation
1	$u_3 - 2u_2 + u_1 = -P \cdot h^2$
2	$u_4 - 2u_3 + u_2 = -P \cdot h^2$
$\vdots$	$\vdots$
$N-2$	$u_N - 2u_{N-1} + u_{N-2} = -P \cdot h^2$

5.2 Matrix Formulation

This system can be represented in matrix form as:

$$A \cdot u_{\text{internal}} = \text{rhs}$$

where $u_{\text{internal}} = [u_2, u_3, \ldots, u_{N-1}]^T$.

6 Constructing the Coefficient Matrix A and RHS Vector

6.1 Coefficient Matrix A

The matrix A is a tridiagonal matrix of size $(N-2) \times (N-2)$:

$$A = \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -2 \end{bmatrix}$$

6.2 Right-Hand Side (RHS) Vector

The RHS vector incorporates the boundary conditions and the term $-P \cdot h^2$:

$$\text{rhs} = \begin{bmatrix} -P \cdot h^2 - u_a \\ -P \cdot h^2 \\ \vdots \\ -P \cdot h^2 - u_b \end{bmatrix}$$

7 Solving the System

To solve the system of equations $A \cdot u_{\text{internal}} = \text{rhs}$, we can use the Thomas algorithm, which is efficient for tridiagonal matrices.

7.1 Steps of the Thomas Algorithm

1. **Forward Elimination**: Modify the matrix A and vector rhs to create an upper triangular matrix.

2. **Backward Substitution**: Solve for the unknowns u_{internal} starting from the last equation upwards.

8 Reconstructing the Full Solution

Once u_{internal} is computed, the full solution vector u is constructed as follows:

$$u = \begin{bmatrix} u_a \\ u_{\text{internal}} \\ u_b \end{bmatrix}$$

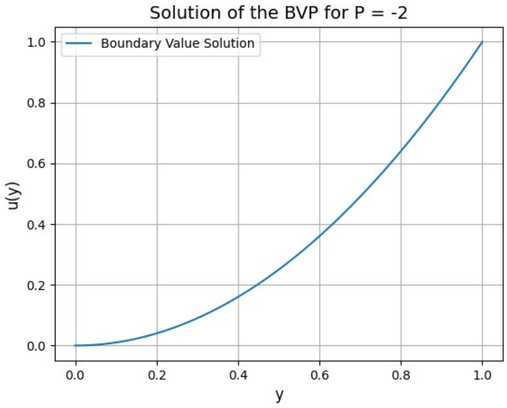
where u_a and u_b are the known boundary values.

9 Conclusion

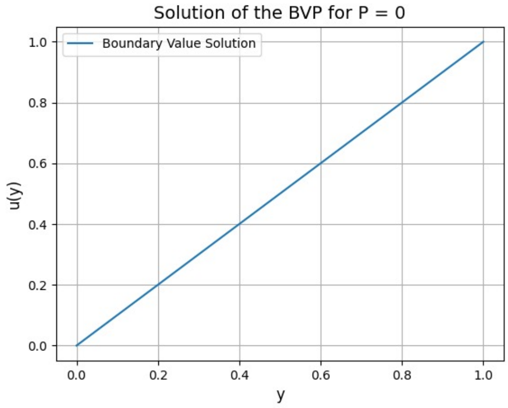
The finite difference method provides a robust approach to numerically solving boundary value problems. By discretizing the interval and utilizing finite difference approximations, we transform the BVP into a solvable linear system. The tridiagonal structure of the coefficient matrix enables efficient computation, making this method suitable for a wide range of applications.

10 Graphs of Solutions for Different Values of P

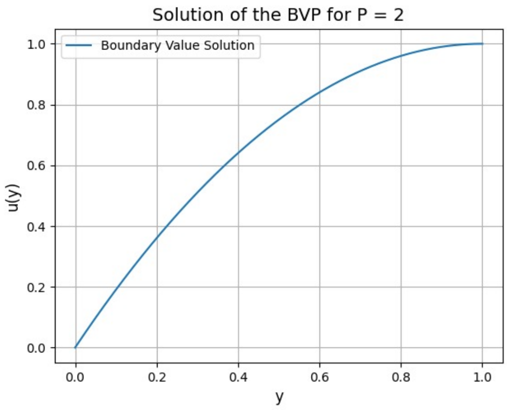
This section presents graphs illustrating the solutions $u(y)$ for various values of the constant P. The graphs demonstrate how the parameter P influences the behavior of the solution.



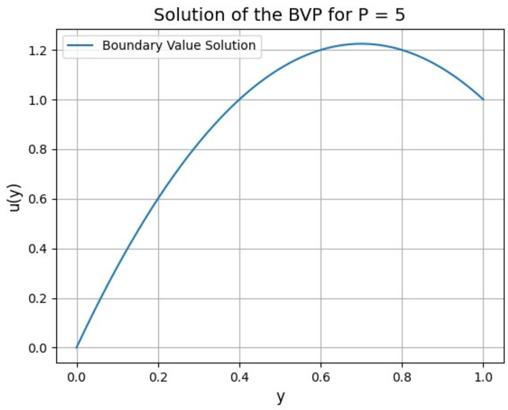
(a) Solution for $P = -2$



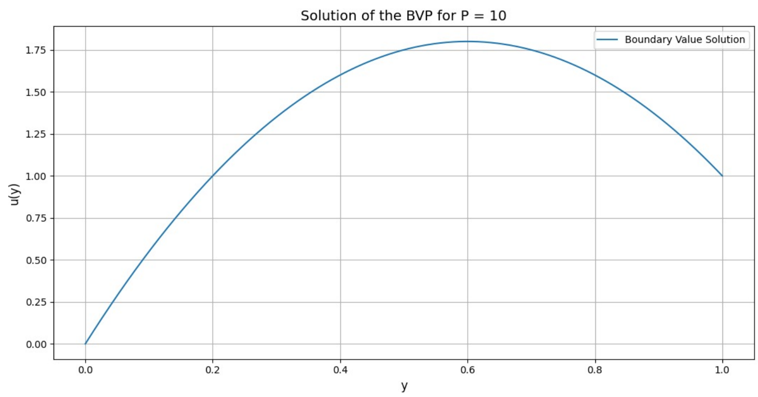
(b) Solution for $P = 0$



(c) Solution for $P = 2$



(d) Solution for $P = 5$



(e) Solution for $P = 10$

Figure 1: Graphs of solutions for different values of P.