FY2045 Quantum Mechanics I

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Week 3

General formulation of QM

Dirac's $\langle \mathrm{bra} | \mathrm{ket} \rangle$ notation

State vector

A quantum mechanical state of a system is described by a state vector

$$|\psi\rangle$$

in a complex, linear vector space $\mathcal{H}$ — Hilbert space.

Dual vector

For each vector $|a\rangle$ we define the dual vector $\langle a|$ in the dual space $\mathcal{H}^*$,

$$|a\rangle \overset{\text{dual}}{\longleftrightarrow} \langle a|,$$

so that we can define the scalar (inner) product of vectors $|a\rangle$ and $|b\rangle$

$$\langle a||b\rangle \equiv \langle a|b\rangle \in \mathbb{C},$$

with $\langle a|b\rangle = \langle b|a\rangle^*$.

Interpretation

Example

Probability amplitude for particle arriving at point x:

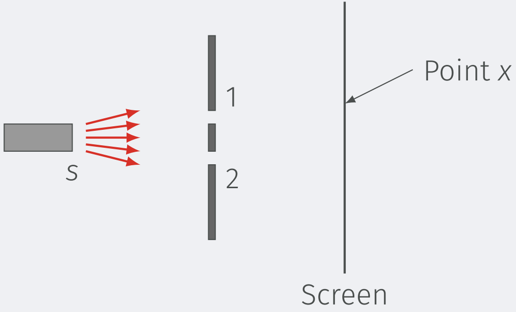
$$\langle \text{Particle arrives at } x | \text{particle leaves } s \rangle$$

or simply

$$\langle x|s \rangle.$$

Can go through either slit 1 or 2:

$$\langle x|s \rangle = \langle x|1 \rangle \langle 1|s \rangle + \langle x|2 \rangle \langle 2|s \rangle.$$



Based on Ch. 3 of Vol. III in the Feynman Lectures.

Interpretation

The wavefunction is probability amplitude of finding state $|\psi\rangle$ at point x:

$$\psi(x) = \langle x|\psi\rangle.$$

The momentum wavefunction is probability amplitude of finding state $|\psi\rangle$ with momentum p:

$$\phi(p) = \langle p|\psi\rangle.$$

The probability amplitude of finding state $|\psi\rangle$ with energy E_n:

$$\langle\psi_n|\psi\rangle.$$

Completeness

n linearly independent vectors $|1\rangle, |2\rangle, |3\rangle, \ldots$ span $\mathcal{H}$ if $\forall\ |\psi\rangle \in \mathcal{H}$ we have

$$|\psi\rangle = \sum_{k=1}^{n} c_n |k\rangle.$$

Assuming orthonormality $\langle m|k\rangle = \delta_{mk}$,

$$\Rightarrow\ |\psi\rangle = \sum_k \langle k|\psi\rangle |k\rangle = \sum_k |k\rangle\langle k||\psi\rangle,$$

meaning we have the completeness relation

$$\sum_k |k\rangle\langle k| = 1.$$

Operators

An operator $\hat{A}$ applied to a vector $|a\rangle \in \mathcal{H}$ results in a new vector $|c\rangle \in \mathcal{H}$,

$$\hat{A}|a\rangle = |c\rangle.$$

Adjoint or Hermitian conjugate of operator:

$$\langle a|\hat{A}^\dagger|b\rangle = \langle b|\hat{A}|a\rangle^* \quad \forall\ |a\rangle, |b\rangle \in \mathcal{H}.$$

meaning that we have the dual vector

$$\hat{A}|a\rangle \overset{\text{dual}}{\longleftrightarrow} \langle a|\hat{A}^\dagger.$$

Properties

$$(\hat{A}^\dagger)^\dagger = \hat{A},$$
$$(\alpha\hat{A})^\dagger = \alpha^*\hat{A}^\dagger,$$
$$(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger.$$

For a Hermitian (self-adjoint) matrix

$$\hat{A}^\dagger = \hat{A}.$$

Eigenvectors and eigenvalues

The eigenvectors $|\alpha\rangle$ and eigenvalues λ_α of an operator $\hat{A}$ are defined by

$$\hat{A}|\alpha\rangle = \lambda_\alpha|\alpha\rangle.$$

The set of eigenvectors $\{|\alpha\rangle\}$ corresponding to physical quantities is assumed to be complete — they form a basis set that span $\mathcal{H}$.

The eigenvectors of a **Hermitian** operator are **real**.

Examples

Energy:

$$\hat{H}|n\rangle = E_n|n\rangle.$$

Position:

$$\hat{x}|x'\rangle = x'|x'\rangle.$$

Momentum:

$$\hat{p}|p'\rangle = p'|p'\rangle.$$

Postulates in general formulation

Postulate A

Each observable quantity F corresponds to a linear, Hermitian operator $\hat{F}$ in Hilbert space. The operators for a generalized coordinate q_n and generalized momentum p_n fulfill

$$[\hat{q}_n, \hat{p}_n] = \hat{q}_n\hat{p}_n - \hat{p}_n\hat{q}_n = i\hbar.$$

Postulate B

Each state of a physical system is represented by a state vector $|\Psi, t\rangle$ in Hilbert space with length 1, $\langle\Psi, t|\Psi t\rangle = 1$. The vector fulfills the time-dependent SE

$$i\hbar\frac{\partial}{\partial t}|\Psi, t\rangle = \hat{H}|\Psi, t\rangle.$$

Postulate C

The expectation value of an observable F, given the state $|\Psi\rangle$, is

$$\langle F\rangle = \langle\Psi|\hat{F}|\Psi\rangle.$$

Postulate D

The measurement of an observable F yields one of the eigenvalues f_n of the corresponding operator $\hat{F}$.

Position representation

Eigenvectors $|x'\rangle$ of operator $\hat{x}$:

$$\hat{x}|x'\rangle = x'|x'\rangle,$$

with eigenvalue x' taking continuous values.
$|x'\rangle$ is δ-function normalized

$$\langle x''|x'\rangle = \delta(x'' - x'),$$

with completeness relation

$$\int dx'|x'\rangle\langle x'| = \mathbb{1}.$$

Position space wavefunction

Given a state vector $|\psi\rangle$, the position space wavefunction is given by

$$\psi(x') = \langle x'|\psi\rangle,$$

the projection of $|\psi\rangle$ on the position basis vector $|x'\rangle$.

Operators

For an operator $\hat{F}$, which is a function of $\hat{x}$ and $\hat{p}$, we have

$$\langle x''|F(\hat{p},\hat{x})|x'\rangle = F\left(\frac{\hbar}{i}\frac{\partial}{\partial x''}, x''\right)\delta(x'' - x').$$

Momentum formulation

Eigenvectors $|p\rangle$ of operator $\hat{p}$:

$$\hat{p}|p\rangle = p|p\rangle,$$

with eigenvalue p taking continuous values.
$|p\rangle$ is δ-function normalized

$$\langle p'|p\rangle = \delta(p' - p),$$

with completeness relation

$$\int dp|p\rangle\langle p| = \mathbb{1}.$$

Momentum space wavefunction

Given a state vector $|\psi\rangle$, the momentum space wavefunction is given by

$$\phi(p) = \langle p|\psi\rangle,$$

the projection of $|\psi\rangle$ on the momentum basis vector $|p\rangle$.

Relation between $|p\rangle$ and $|x\rangle$

Projecting $|p\rangle$ on the position basis $|x\rangle$, we get the momentum eigenfunctions in the position formulation,

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}.$$

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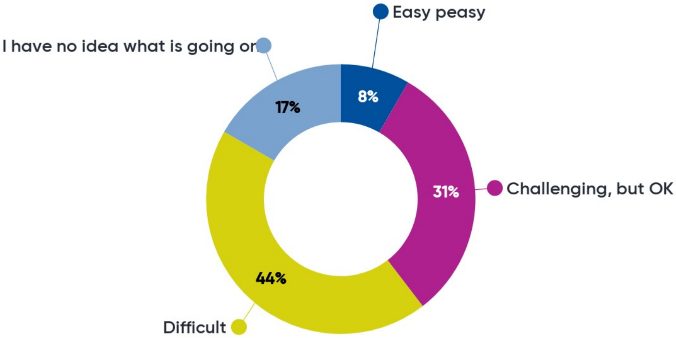
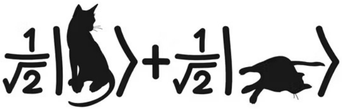
How do you find the general formulation?





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What concepts do you find challenging?

48 Responses

visualising what this mea
what hermitian means
the math and notation
hilbert space
uncountable set of basis
relations
visualizing
adding in completenessrel
basket
new notation
the formalism
unknown rules
vectors
no physical links
what they represent
knowing what to do when
dual vectors
very abstract
bra vs ket
the notation
overview
why we do things
braket
to
using it
to abstract
all is too abstract
dual hibert-space
the math
physical interpretation
how
rules for calculation
use
expectation values
dual
complex numbers
calculations with notatio
the rules for brackets
order of stuff in brakets
momentum space
what is the bras and kets
tricks with formulas

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