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FY2045 Problem set 8 fall 2023

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Problem 1

Consider an electron in a hydrogen atom. The orbital angular momentum operator is denoted by $\hat{\mathbf{L}}$ and the spin operator by $\hat{\mathbf{S}}$. Since $\hat{\mathbf{L}}^2$, $\hat{\mathbf{S}}^2$, $\hat{L}_z$ and $\hat{S}_z$ commute among themselves and commute with the Hamiltonian $\hat{H}$ of the hydrogen atom, l, m_l , s, and m_s are good quantum numbers. We denote the energy eigenstates of hydrogen by $|nlm_lm_s\rangle_{LS}$, where n is the principal quantum number and the subscript LS denotes that these are eigenstates of $\hat{\mathbf{L}}^2$ and $\hat{\mathbf{S}}^2$. Notice that we suppress the quantum number s since it is always ½.

a) Consider the case with $l = 1$. How many states are there?

b) The total angular momentum is given by $\hat{J} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$. Explain why $\hat{J}^2$ and $\hat{J}_z$ commute with $\hat{\mathbf{L}}^2$, $\hat{\mathbf{S}}^2$, and $\hat{H}$.

c) Instead of using the quantum numbers n, l, m_l, and m_s, we can make a change of basis and label the energy eigenstates by n, l, j, and m_j. These are denoted by $|nljm_j\rangle_J$, where the subscript J indicates that this is an eigenstate of $\hat{\mathbf{J}}^2$. For $l = 1$, what are the possible values for j? Count the number of states with $l = 1$ for one specific value of n, and compare with the result in a).

d) For $l = 1$, express all the states $|nljm_j\rangle_J$ in terms of $|nlm_lm_s\rangle_{LS}$. *Hint:* Start with the state with $j = j_{max}$ and $m_j = j_{max}$, and operate with the total lowering operator $J_- = L_- + S_-$ repeatedly until you reach the state with $m_j = -j_{max}$. Then construct a state with $m_j = j_{max} - 1$ which is orthogonal to the $m_j = j_{max} - 1$ state you already have, and operate with J_- repeatedly to find all states with $j = j_{max} - 1$. Repeat this procedure until you have found all the states. See e.g. Ø13.3 for more details regarding this procedure. Remember that J_- does not affect the quantum number n.

Problem 2

The total spin of two spin ½ particles can be either 1 or 0, where the states $|s, m\rangle$ for the two cases are

$$|1, 1\rangle = |\uparrow\uparrow\rangle\,, \tag{1}$$

$$|1, 0\rangle = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle], \tag{2}$$

$$|1, \text{-}1\rangle = |\downarrow\downarrow\rangle\,, \tag{3}$$

for $s = 1$, and

$$|0, 0\rangle = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle], \tag{4}$$

for $s = 0$, where $|\uparrow\uparrow\rangle = |\uparrow\rangle\,|\uparrow\rangle$, etc., with $|\uparrow (\downarrow)\rangle$ denoting spin up (down) for one particle along the z direction.

a) Using the eigenspinors along x, y and z for spin ½, show that we can write

$$|\uparrow\rangle \equiv |\uparrow_z\rangle = \frac{|\uparrow_x\rangle + |\downarrow_x\rangle}{\sqrt{2}} = \frac{|\uparrow_y\rangle + |\downarrow_y\rangle}{\sqrt{2}}, \tag{5}$$

$$|\downarrow\rangle \equiv |\downarrow_z\rangle = \frac{|\uparrow_x\rangle - |\downarrow_x\rangle}{\sqrt{2}} = \frac{|\uparrow_y\rangle - |\downarrow_y\rangle}{\sqrt{2}i}, \tag{6}$$

where $|\uparrow_x\rangle$ is the spin up state along x, etc.

b) Find expressions for the states $|10\rangle$ and $|00\rangle$ when the total spin is measured along the x and y direction by using the above relations. Any comments?