POSTULATES OF QM [Ø2.2, Ø7.1, H2.1]

We will now formulate the basic set of postulates upon which quantum mechanics is built.

A: THE OPERATOR POSTULATE

To each physical observable quantity F there corresponds in quantum-mechanical theory a linear operator $\hat{F}$.

For instance, in the position-space formulation

$$x \rightarrow \hat{x} = x$$

$$p \rightarrow \hat{p} = -i\hbar \frac{\partial}{\partial x}$$

Observables must be real quantities
$\Rightarrow$ $\hat{F}$ must be hermitian/self-adjoint.

B: THE WAVEFUNCTION POSTULATE

The state of a system is described, as completely as possible, by the wavefunction $\Psi(q_n, t)$. The time development of the wavefunction (and hence of the state) is determined by the Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi,$$

where $\hat{H}$ is the Hamiltonian of the system. Usually the Hamiltonian is the total energy of the system.

This is an equation of motion, which uniquely determines $\Psi(q_n,t)$ given some initial condition $\Psi(q_n,t_0)$.

The concept of a Hamiltonian will become more familiar for those taking classical mechanics. See also Appendix A in Hemmer.

The postulate implies that it is <u>not</u> possible to obtain more information about a system than what is contained in $\Psi(q_n,t)$.

C: THE EXPECTATION VALUE POSTULATE

When a large number of measurements of an observable F is made on a system which is prepared in a state $\Psi(q_1,q_2 \ldots q_n,t)$ (before each measurement), the average $\bar{F}$ of the measured values will approach the theoretical expectation value, which is postulated to be

$$\langle F \rangle = \int \Psi^* \hat{F} \Psi \, d\tau$$

where $d\tau = dq_1 dq_2 \ldots dq_n$ and where the integration goes over the whole range of each of the variables.

q_n: generalized coordinate.
Example x, y, z for free particle in 3D.

We are supposing that the wavefunction is normalized,

$$\int d\tau \, |\Psi|^2 = 1.$$

D: THE MEASUREMENT POSTULATE

(i) The only possible result of a precise measurement of an observable F is one of the eigenvalues f_n of the corresponding linear operator $\hat{F}$.

(ii) Immediately after the measurement of the eigenvalue f_n, the system is in an eigenstate of $\hat{F}$, namely the eigenstate ψ_n corresponding to the measured eigenvalue f_n.

Which of the eigenvalues is measured, and the probability for each, depends on the state before the measurement.

EXAMPLE – INFINITE SQUARE WELL

We will use this example as a chance to recap from Intro to QM, and see how some of the postulates come into play.

The Hamiltonian is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}),$$

$$= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x),$$

with potential energy function

$$V(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq L \\ \infty, & \text{otherwise.} \end{cases}$$

This describes a particle in an infinite, square well. According to postulate B, the system is best described by a wavefunction $\Psi(x,t)$ which is a solution of

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi, \qquad (1)$$

so this is our starting point. $\hat{H}$ does not depend explicitly on t, so we will solve the Schrödinger equation (SE) using separation of variables. We assume

$$\Psi(x,t) = \psi(x)\,\phi(t).$$

Inserted into the SE, we get

$$\left[i\hbar \frac{d\phi}{dt} \right] \psi(x) = \left[-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V\psi \right] \phi$$

or

$$\underbrace{i\hbar \frac{1}{\phi} \frac{d\phi}{dt}}_{\text{depends only on } t} = \underbrace{-\frac{\hbar^2}{2m} \frac{1}{\psi} \frac{d^2\psi}{dx^2} + V}_{\text{depends only on } x}.$$

For this to always hold, both sides must be constant.

We therefore write:

$$i\hbar \frac{1}{\phi} \frac{d\phi}{dt} = E \Rightarrow \phi(t) = \phi_0 e^{-iEt/\hbar}.$$

ϕ_0 can be absorbed into ψ.

and

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V\psi = E\psi$$

This is often referred to as the time-independent Schrödinger equation (TISE).

So far we have

$$\Psi(x,t) = \psi(x) e^{-iEt/\hbar}$$

We now need to find $\psi(x)$ and E.

For $x \leq 0$ and $x \geq L$ we must have $\psi = 0$: a particle cannot be in an area with infinite potential, i.e. inside infinately "hard walls". We therefore use the boundary conditions (BCs)

$$\psi(0) = \psi(L) = 0,$$

when solving the TISE

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi.$$

A general solution is

$$\psi(x) = C_1 \sin\sqrt{\frac{2mE}{\hbar^2}}\,x + C_2 \cos\sqrt{\frac{2mE}{\hbar^2}}\,x.$$

For simplicity we define

$$k = \frac{\sqrt{2mE}}{\hbar}.$$

From $\psi(0) = 0$ we get $C_2 = 0$. From $\psi(L) = 0$ we get the condition

$$\psi(L) = C_1 \sin kL = 0$$

$$\rightarrow kL = \pi \cdot n,$$

$$k = \frac{\pi \cdot n}{L} \qquad n = 1, 2, 3\ldots$$

$n = 0$ just the trivial solution. Negative solutions $\rightarrow C_1$

Hence, we have determined E:

$$\boxed{E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 \pi^2}{2mL^2} \cdot n^2, \qquad n = 1, 2, 3\ldots}$$

But what is C_1?

I said earlier that $|\Psi|^2$ is interpreted as a probability density. From postulate C we also connect Ψ to expectation values of operators. What if we use $\hat{F} = \hat{1}$? ...

$$\Rightarrow \langle 1 \rangle = \int d\tau \Psi^* \cdot 1 \cdot \Psi$$

$$= \int d\tau \, |\Psi|^2 = 1.$$

In other words, the total probability of finding a particle anywhere at all, must be 1. In our case:

$$1 = \int_0^L dx \, |\Psi|^2 = \int_0^L dx \left| \psi(x) e^{-iEt/\hbar} \right|^2$$

$$= \int_0^L dx \, |\psi(x)|^2 = \int_0^L dx \, |C_1|^2 \sin^2 kx$$

$$= |C_1|^2 \frac{L}{2} \Rightarrow |C_1| = \sqrt{\frac{2}{L}}.$$

Choosing C_1 to be real, we therefore have

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} \cdot x\right), \qquad n = 1, 2, 3 \ldots$$

Note:

The wavefunctions are orthonormal

$$\int_0^L dx \, \psi_n^*(x) \, \psi_m(x) = \delta_{nm}.$$

The set of wavefunctions is complete: Any other function can be expressed as a linear combination of them.

$$f(x) = \sum_{n=1}^{\infty} \underset{\llcorner \text{ complex number}}{c_n} \psi_n(x).$$

The general solution to the SE is

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \Psi_n(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_nt/\hbar}$$

Proof:

$$i\hbar \frac{\partial}{\partial t} \Psi(x,t) = \sum_{n=1}^{\infty} c_n E_n \psi_n(x) e^{-iE_nt/\hbar}$$

$$\overset{!}{=} \hat{H} \Psi(x,t)$$

$$= \sum_{n=1}^{\infty} \underbrace{\hat{H} c_n \psi_n(x)}_{= c_n E_n \psi_n(x)} e^{-iE_nt/\hbar}$$

$$= \sum_{n=1}^{\infty} c_n E_n \psi_n(x) e^{-iE_nt/\hbar} \qquad \text{OK.}$$

Since the SE is linear, superposition of solutions works. That makes life easy!

Normalization:

$$\int_0^L dx\, \Psi^*(x,t)\, \Psi(x,t) = \sum_{n,m=1}^{\infty} c_n^* c_m \left[\int_0^L dx\, \psi_n^*(x)\psi_m(x)\right] \times e^{i(E_n-E_m)t/\hbar}$$

$$= \sum_{n,m=1}^{\infty} c_n^* c_m\, e^{i(E_n-E_m)t/\hbar} \delta_{nm} = \sum_n |c_n|^2 \overset{!}{=} 1.$$

Expectation values

We now use postulate C to calculate some expectation values. We choose

$$\Psi(x,t) = \psi_n(x)\, e^{-iE_n t/\hbar}$$

for simplicity, and calculate:

$$\langle E \rangle = \int_0^L dx\, \psi_n^*(x)\, \hat{H}\, \psi_n(x)\, e^{-i(E_n - E_n)t/\hbar}$$

$$= E_n \underbrace{\int_0^L dx\, |\psi_n|^2}_{=1} = E_n.$$

$$\langle x \rangle = \int_0^L dx\, \psi_n^*(x)\, \hat{x}\, \psi_n(x) = \int_0^L dx\, x\, |\psi_n(x)|^2$$

$$= \frac{1}{2}$$

$$\langle p \rangle = \frac{2}{L}\int_0^L dx\, \sin\frac{n\pi x}{L}\left(-i\hbar\frac{d}{dx}\right)\sin\frac{n\pi x}{L}$$

$$= \frac{1}{L}\int_0^L dx\left(-i\hbar\frac{d}{dx}\right)\sin^2\frac{n\pi x}{L}$$

$$= -\frac{i\hbar}{L}\, \sin\frac{n\pi x}{L}\Big|_0^L = 0.$$

No time-dependence, since $|\Psi|^2 = |\psi_n|^2$ is time-independent. However, if we use a superposition of energy eigenstates

$$\Psi(x,t) = c_1\psi_1(x)e^{-iE_1t/\hbar} + c_2\psi_2(x)e^{-iE_2t/\hbar}$$

we get

$$|\Psi(x,t)|^2 = |c_1|^2|\psi_1|^2 + |c_2|^2|\psi_2|^2$$
$$+ c_1\psi_1 c_2^*\psi_2^*\, e^{-i(E_1-E_2)t/\hbar}$$
$$+ c_1^*\psi_1^* c_2\psi_2\, e^{+i(E_1-E_2)t/\hbar}$$

Assuming $c_1, c_2, \psi_1, \psi_2 \in \mathbb{R}$, we get

$$|\Psi(x,t)|^2 = |c_1|^2|\psi_1(x)|^2 + |c_2|^2|\psi_2(x)|^2 + 2c_1c_2\,\psi_1(x)\psi_2(x)\cos\left(\frac{E_1-E_2}{\hbar}t\right)$$

We have an oscillating probability density due to "interference" between the wavefunctions. This also leads to (potentially) time-dependent expectation values $\langle x\rangle(t)$ and $\langle p\rangle(t)$.

[See animation].

Measurements

If we measure the energy of the system described by the state

$$\Psi = c_1\psi_1(x)e^{-iE_1t/\hbar} + c_2\psi_2(x)e^{-iE_2t/\hbar}$$

what values can be got? We can only get E_1, with a probability $|c_1|^2$, or E_2, with probability $|c_2|^2$, according to postulate D. Furthermore, right after measurement, giving e.g. $E=E_1$, the system will be in the state

$$\Psi = \psi_1 e^{-iE_1t/\hbar}.$$

This is sometimes referred to as "wavefunction collapse".

if we instead measure the position, and find the particle at position x', the state immediatly after the measurement is a position eigenstate:

$$\psi_{x'}(x) = \delta(x - x')$$

This is clearly not an energy eigenstate. What energies could you get if you now measured the energy of the system?

TIME-DEPENDENCE OF EXPECTATION VALUE [H4.3]

From the animation, we saw that the expectation values can have a time-dependence. In fact, we can find a general formula for the time-dependence of expectation values. Starting from

$$\langle F \rangle = \int \Psi^* \hat{F} \Psi \, d\tau,$$

we take the time-derivative of both sides:

$$\frac{d\langle F \rangle}{dt} = \int d\tau \left[\frac{\partial \Psi^*}{\partial t} \hat{F} \Psi + \Psi^* \frac{\partial \hat{F}}{\partial t} \Psi + \Psi^* \hat{F} \frac{\partial \Psi}{\partial t} \right].$$

To proceed, we use the Schrödinger eq.

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi \rightarrow \frac{\partial \Psi}{\partial t} = -\frac{i \hat{H} \Psi}{\hbar},$$

Hence, we get

$$\frac{d\langle F \rangle}{dt} = \frac{i}{\hbar} \int d\tau \left[(\hat{H} \Psi)^* \hat{F} \Psi + \Psi^* \hat{F} \hat{H} \Psi \right] + \int d\tau \, \Psi^* \frac{\partial \hat{F}}{\partial t} \Psi$$

We now use the fact that $\hat{H}$ is hermitian,

$$\int d\tau \, (\hat{H} \Psi)^* \Phi = \int d\tau \, \Psi^* \hat{H} \Phi,$$

with $\Phi = \hat{F} \Psi$.

We then get

$$\frac{d\langle F\rangle}{dt} = \frac{i}{\hbar}\int d\tau\, \Psi^*(\hat{H}\hat{F} - \hat{F}\hat{H})\Psi + \int d\tau\, \Psi^* \frac{\partial \hat{F}}{\partial t}\Psi$$

$$= \frac{i}{\hbar}\langle[\hat{H},\hat{F}]\rangle + \left\langle \frac{\partial \hat{F}}{\partial t}\right\rangle$$

We'll call this Ehrenfest's generalized theorem.

If we assume a system described by

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$$

and use $\hat{F} = \hat{x}$ and $\hat{p}$, we get

$$\boxed{\begin{aligned} \frac{d\langle x\rangle}{dt} &= \frac{\langle p\rangle}{m} \\ \frac{d\langle p\rangle}{dt} &= -\langle V'(x)\rangle \end{aligned}}$$

This is known as Ehrenfest's theorem.
This is analogous to the classical equations

$$\frac{dx}{dt} = \frac{p}{m}$$

$$\frac{dp}{dt} = -V'(x) = F.$$

The equations of motion for quantum mechanical expectation values of position, momentum and force coincide with the classical equations of motion. However, this does not generally mean that the mean position of a wavepacket follows the classical trajectory!

Measurement of a degenerate eigenvalue

See also Compendium 7.2 by Øverbø.

If we measure an observable F, and the measured eigenvalue is non-degenerate, i.e.

$$\hat{F}\psi_n = f_n\psi_n,$$

has only one solution ψ_n for the eigenvalue f_n, then from postulate D(ii), the system is in the state ψ_n immediately after the measurement.

However, if the eigenvalue f_n is degenerate, with degeneracy g_n, the eigenvalue equation has g_n solutions ψ_{ni},

$$\hat{F}\psi_{ni} = f_n\psi_{ni}, \quad i = 1, 2, \ldots, g_n.$$

In this case we must formulate D(ii) more precisely, as we shall now see.

Assume a set of discrete eigenvalues f_n , and an orthonormalized set of eigenfunctions

$$\{\psi_{ni}\, | n = 1, 2, \ldots;\ i = 1, 2, \ldots, g_n\}.$$

The eigenfunctions form a complete set, meaning that the state prior to the measurement can be expanded

$$\Psi = \sum_n \sum_{i=1}^{g_n} c_{ni}\ \psi_{ni}. \tag{1}$$

We find the expansion coefficients by projecting Ψ onto ψ_{ni}:

$$\begin{aligned}\langle\psi_{ni}, \Psi\rangle &= \left\langle \psi_{ni}, \sum_k \sum_{j=1}^{g_k} c_{kj}\ \psi_{kj} \right\rangle = \sum_k \sum_{j=1}^{g_k} c_{kj}\langle\psi_{ni}, \psi_{kj}\rangle \\ &= \sum_k \sum_{j=1}^{g_k} \delta_{nk}\delta_{ij}c_{kj} = c_{ni},\end{aligned}$$

where we have used the orthonormality of ψ_{ni}. Hence,

$$c_{ni} = \langle\psi_{ni}, \Psi\rangle = \int d\tau\ \psi_{ni}^*\Psi.$$

Considering a series of experiments of F on the state Ψ, the expectation value is

$$\langle F\rangle = \sum_n P_n f_n,$$

with P_n the probability of measuring f_n. From the expectation value theorem, we have

Definition of a Hermitian operator,

$$\int d\tau\ \Psi^*\hat{F}\Phi = \int d\tau\ (\hat{F}\Psi)^*\Psi.$$

$$\begin{aligned}\langle F\rangle_\Psi &= \int d\tau\ \Psi^*\hat{F}\Psi = \int d\tau\ (\hat{F}\Psi)^*\Psi = \int d\tau \left(\sum_n \sum_{i=1}^{g_n} c_{ni}\hat{F}\psi_{ni}\right)^* \Psi \\ &= \sum_n \sum_{i=1}^{g_n} c_{ni}^* f_n \underbrace{\int d\tau\ \psi_{ni}^*\Psi}_{=c_{ni}} = \sum_n \left(\sum_{i=1}^{g_n} |c_{ni}|^2\right) f_n.\end{aligned}$$

Both formulas are valid for an arbitrary state Ψ. Hence, the probability of measuring eigenvalue f_n when the system is in the state Ψ in eq. (1) is

$$P_n = \sum_{i=1}^{g_n} |c_{ni}|^2. \tag{2}$$

In the non-degenerate case the corresponding relation is simply $P_n = |c_n|^2$.

Notice that we may define

$$\Psi_n = \sum_{i=1}^{g_n} c_{ni}\psi_{ni},$$

which has eigenvalue f_n, allowing us to write

$$\Psi = \sum_n \Psi_n.$$

Ψ_n is therefore the part of Ψ compatible with the eigenvalue f_n. Hence, D(ii) must be formulated

> **Postulate D(ii):** Immediately after a measurement of eigenvalue f_n, the system is in the normalized state
>
> $$\frac{\Psi_n}{||\Psi_n||} = \frac{\sum_{i=1}^{g_n} c_i\psi_{ni}}{||\sum_{i=1}^{g_n} c_i\psi_{ni}||},$$
>
> with $||\Psi_n||$ the norm of Ψ_n.

The part of Ψ not compatible with f_n is "removed" by the measurement.

Example — 3D isotropic harmonic oscillator

Recap: In one dimension (1D) energies of the harmonic oscillator is

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots,$$

with eigenfunctions $\psi_n(x)$.

$$\psi_n = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} \frac{e^{-m\omega x^2/2\hbar}}{\sqrt{2^n n!}} H_n\left(x\sqrt{\frac{m\omega}{\hbar}}\right),$$

with Hermite polynomials H_n.

In three dimensions (3D) the Hamiltonian for the isotropic case

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega^2(x^2 + y^2 + z^2),$$

is separable into the the 1D parts corresponding to each direction. Hence, the solutions can be written in terms of the 1D solutions, giving the orthonormal eigenfunction set

$$\psi_{n_x n_y n_z}(x, y, z) = \psi_{n_x}(x)\psi_{n_y}(y)\psi_{n_z}(z),$$

and energy eigenvalues

$$\begin{aligned} E_{n_x n_y n_z} &= \hbar\omega\left(n_x + n_y + n_z + \frac{3}{2}\right) \\ &= \hbar\omega\left(N + \frac{3}{2}\right) \equiv E_N, \end{aligned}$$

with $N = n_x + n_y + n_z$.

At time $t = 0$ we assume the system is in the state

$$\Psi = \underbrace{\sqrt{0.4}\psi_{000}}_{=\Psi_0} + \underbrace{\sqrt{0.1}(\psi_{100} + \psi_{010} + \psi_{001})}_{=\Psi_1} + \underbrace{\sqrt{0.1}(\psi_{200} + i\psi_{020} + \psi_{011})}_{=\Psi_2}.$$

Ψ_0, Ψ_1 and Ψ_2 have $N = 0, 1$ and 2, respectively. We see that the state is normalized,

$$||\Psi||^2 = ||\Psi_0||^2 + ||\Psi_1||^2 + ||\Psi_2||^2 = 0.4 + 3 \cdot 0.1 + 3 \cdot 0.1 = 1.$$

The possible measured values for the energy and associated probabilities are

$$\begin{aligned} E_0 &= \frac{3}{2}\hbar\omega, & P_0 &= 0.4, \\ E_1 &= \frac{5}{2}\hbar\omega, & P_1 &= 0.3, \\ E_2 &= \frac{7}{2}\hbar\omega, & P_2 &= 0.3. \end{aligned}$$

The corresponding normalized states after measuring E_0, E_1 or E_2 are

$$\begin{aligned} \frac{\Psi_0}{||\Psi_0||} &= \psi_{000}, \\ \frac{\Psi_1}{||\Psi_1||} &= \frac{1}{\sqrt{3}}(\psi_{100} + \psi_{010} + \psi_{001}), \\ \frac{\Psi_2}{||\Psi_2||} &= \frac{1}{\sqrt{3}}(\psi_{200} + i\psi_{020} + \psi_{011}), \end{aligned}$$

respectively.

Eigenfunctions of continuous eigenvalues — Momentum eigenfunctions

The momentum eigenfunctions are solutions of the equation

$$\hat{p}_x\psi = p\psi. \tag{3}$$

Inserting $\hat{p}_x = -i\hbar\frac{d}{dx}$, we get

$$-i\hbar\frac{d\psi}{dx} = p\psi,$$

which has solution

$$\psi_p(x) = Ae^{ipx/\hbar}.$$

This describes a plane wave. p must be real for ψ_p not to diverge when $x \to \pm\infty$. However, $|\psi_p|^2 = |A|^2$ is a constant, so how do we normalize it? We cannot use the same normalization as for discrete eigenvalues, since

$$\int_{-\infty}^{\infty} dx\ |\psi_p|^2 \neq 1.$$

Instead, we use the *δ-function normalization*,[1]

$$\int_{-\infty}^{\infty} dx\ \psi_{p'}^* \psi_p = \delta(p - p'), \tag{4}$$

where $\delta(p)$ is the Dirac δ-function. This is analogous to the relation

$$\int_{-\infty}^{\infty} dx\ \psi_{n'}^* \psi_n = \delta_{nn'},$$

where $\delta_{nn'}$ is the Kronecker delta, seen earlier for discrete spectra.

The δ-function has the integral representation

$$\delta(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx\ e^{iyx}.$$

We compare this with our δ-function normalization:

$$\int_{-\infty}^{\infty} dx\ \psi_{p'}^* \psi_p = \int_{-\infty}^{\infty} dx\ |A|^2 e^{i(p-p')x/\hbar}.$$

Defining $x' = x/\hbar$ and $dx' = dx'/\hbar$, we get

$$\int_{-\infty}^{\infty} dx\ \psi_{p'}^* \psi_p = \int_{-\infty}^{\infty} dx'\ \hbar |A|^2 e^{i(p-p')x'} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx' e^{i(p-p')x'} = \delta(p - p'),$$

when

$$|A| = \frac{1}{\sqrt{2\pi\hbar}}.$$

Choosing A real and positive, we therefore get the plane-wave solutions

$$\psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}. \tag{5}$$

[1] The same normalization must be used also for the position eigenfunctions

$$\psi_{x'}(x) = \delta(x - x')$$

for $x \in (-\infty, \infty)$:

$$\begin{aligned} &\int_{-\infty}^{\infty} dx\ \psi_{x'}^*(x) \psi_{x''}(x) \\ &= \int_{-\infty}^{\infty} dx\ \delta(x - x')\delta(x - x'') \\ &= \delta(x' - x''). \end{aligned}$$

Physical interpretation in the continuous case

See also Compendium 7.3 by Øverbø, Hemmer Ch. 2.5.2, and Griffiths and Schroeter Ch. 3.4.

The momentum eigenfunctions ψ_p in eq. (5) are solutions to the eigenvalue equation in eq. (3) with eigenvalues $p \in (-\infty, \infty)$. Using these as a basis, the expansion of an arbitrary square integrable function[2] becomes a Fourier integral. For instance, for a time-dependent wavefunction, we get

[2] We will discuss this more later.

$$\Psi(x, t) = \int_{-\infty}^{\infty} dp\ \Phi(p, t) \psi_p(x). \tag{6}$$

To find an expression for $\Psi(p, t)$, we project Ψ onto ψ_p:

$$\begin{aligned} \langle \psi_p,\ \Psi \rangle &= \int_{-\infty}^{\infty} dx\ \psi_p^*(x) \Psi(x, t) \\ &= \int_{-\infty}^{\infty} dx\ \psi_p^*(x) \left(\int_{-\infty}^{\infty} dp'\ \Phi(p', t) \psi_{p'}(x) \right) \\ &= \int_{-\infty}^{\infty} dp'\ \Phi(p', t) \underbrace{\int_{-\infty}^{\infty} dx\ \psi_p^*(x) \psi_{p'}(x)}_{\delta(p'-p),\ \text{see eq. (4)}} = \Phi(p, t). \end{aligned}$$

Hence, we find

$$\Rightarrow \qquad \Phi(p, t) = \langle \psi_p,\ \Psi \rangle = \int_{-\infty}^{\infty} dx\ \psi_p^*(x) \Psi(x, t), \tag{7}$$

which also depends on time.

We now assume a series of measurements of the momentum p_x made on an ensemble prepared in the state Ψ. The expectation value is written on the form

$$\langle p_x \rangle = \int_{-\infty}^{\infty} dp\ P(p,t)p, \tag{8}$$

with $P(p,t)dp$ the probability of finding p_x in $[p, p+dp]$, and $P(p,t)$ is the probability in "p-space" at time t. From the expectation-value postulate, we have

$$\begin{aligned}
\langle p_x \rangle_\Psi &= \int_{-\infty}^{\infty} dx\ \Psi^*(x,t)\hat{p}_x\Psi(x,t) = \int_{-\infty}^{\infty} dx\ (\hat{p}_x\Psi)^*\Psi \\
&= \int_{-\infty}^{\infty} dx\ \underbrace{\left[\hat{p}_x \int_{-\infty}^{\infty} dp\ \Phi(p,t)\psi_p(x)\right]^*}_{\text{Use } \hat{p}_x\psi_p = p\psi_p} \Psi(x,t) \\
&= \int_{-\infty}^{\infty} dp\ \Phi^*(p,t)p \underbrace{\left[\int_{-\infty}^{\infty} dx\ \psi_p(x)^*\Psi(x,t)\right]}_{=\Phi(p,t),\text{ see eq. (7)}} \\
&= \int_{-\infty}^{\infty} dp\ |\Phi(p,t)|^2 p.
\end{aligned} \tag{9}$$

Comparing the formulas for $\langle p_x \rangle$ in eqs. (8) and (9), we see that

$$P(p,t) = |\Phi(p,t)|^2,$$

and we get the following physical interpretation of the expansion coefficient:

> When the system is in the state $\Psi(x,t)$ before the measurement, the probability of measuring p_x in the interval $(p, p+dp)$ is
>
> $$\begin{aligned} P(p,t)dp = |\Phi(p,t)|^2\ dp &= |\langle \psi_p,\ \Psi \rangle|^2\ dp \\ &= \left| \int_{-\infty}^{\infty} dx\ \psi_p^*(x)\Psi(x,t) \right|^2 dp. \end{aligned}$$

References

D. J. Griffiths and D. F. Schroeter. *Introduction to Quantum Mechanics*. 3rd edition (Cambridge University Press, Cambridge, 2018).

P. C. Hemmer. *Kvantemekanikk* (Tapir Akademisk forl., Trondheim, 2005).

I. Øverbø. *TFY4215/FY1006 — Lecture Notes 7* (2011).