FY2045 Quantum Mechanics I

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Week 2

Measurement of a degenerate eigenvalue

D: The measurement postulate

(i) The only possible result of a precise measurement of an observable F is **one of the eigenvalues** f_n of the corresponding linear operator $\hat{F}$.

(ii) Immediately after the measurement of the eigenvalue f_n, the system is in an eigenstate of $\hat{F}$, namely, the eigenstate ψ_n corresponding to the measured eigenvalue f_n.

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§7.2

Non-degenerate case

$$\hat{F}\psi_n = f_n\psi_n$$

has only one solution ψ_n for eigenvalue f_n.

If a measurement of observable F gives f_n, the system is in state ψ_n immediately after the measurement.

Measurement of a degenerate eigenvalue

$$\hat{F}\psi_{ni} = f_n\psi_{ni}, \quad i = 1, 2, \dots g_n.$$

Complete set, expand general state as

$$\Psi = \sum_n \Psi_n = \sum_n \sum_{i=1}^{g_n} c_{ni}\psi_{ni},$$

resulting in probability of measuring f_n

$$P_n = \sum_{i=1}^{g_n} |c_{ni}|^2.$$

D (ii) — degenerate case

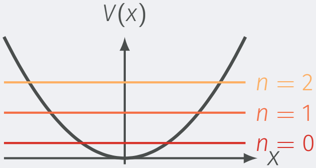
Immediately after a measurement of eigenvalue f_n, the system is in the normalized state

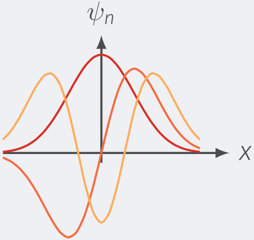
$$\frac{\Psi_n}{||\Psi_n||} = \frac{\sum_{i=1}^{g_n} c_i\psi_{ni}}{||\sum_{i=1}^{g_n} c_i\psi_{ni}||},$$

with $||\Psi_n||$ the norm of Ψ_n.

Example — 3D isotropic harmonic oscillator

1D case





$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2,$$

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \qquad n = 0, 1, 2\ldots$$

$$\psi_n = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} \frac{e^{-m\omega x^2/2\hbar}}{\sqrt{2^n n!}} H_n\left(x\sqrt{\frac{m\omega}{\hbar}}\right),$$

3D case

$$\begin{aligned}
\hat{H} &= -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega^2(x^2 + y^2 + z^2),\\
\psi_{n_x n_y n_z} &= \psi_{n_x}(x)\psi_{n_y}(y)\psi_{n_z}(z),\\
E_{n_x n_y n_z} &= \hbar\omega\left(n_x + n_y + n_z + \frac{3}{2}\right)\\
&= \hbar\omega\left(N + \frac{3}{2}\right) \equiv E_N.
\end{aligned}$$

H3.5 H5.1, G2.3

Eigenfunctions of continuous variables

Momentum eigenfunctions

$$\hat{p}\psi_p(x) = p\psi_p(x) \quad \Rightarrow \quad \psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}.$$

Position eigenfunctions

$$\hat{x}\psi_y(x) = y\psi_y(x) \quad \Rightarrow \quad \psi_y(x) = \delta(x - y).$$

Normalization

For continuous case,

$$\int d\tau \; \Psi_{f'}^* \Psi_f = \delta(f - f'),$$

compared to

$$\int d\tau \; \Psi_{n'}^* \Psi_n = \delta_{nn'},$$

in discrete case.

Physical interpretation of the continuous case

Discrete case

The probability that a measurement of F gives the result f_n, when the system is in the state Ψ, is

$$|c_n|^2 = \left| \int d\tau \ \Psi_n^* \Psi \right|^2 ,$$

where Ψ_n is the eigenstate corresponding to f_n.

Continuous case

The probability that a measurement of F gives a result in the interval $(f, f + df)$ when the system is in the state Ψ, is

$$|c(f)|^2 df = \left| \int d\tau \ \Psi_f^* \Psi \right|^2 df,$$

where Ψ_f is the eigenstate corresponding to the value f.

∅7.3

Momentum-space representation

	Position-space formulation	Momentum-space formulation
Wavefunction	$\Psi(x, y, z, t)$	$\Phi(p_x, p_y, p_z, t)$
Operator $\hat{x}_i$	x_i	$-\frac{\hbar}{i}\frac{\partial}{\partial p_i}$
Operator $\hat{p}_i$	$\frac{\hbar}{i}\frac{\partial}{\partial x_i}$	p_i
Wave equation	$i\hbar\frac{\partial \Psi}{\partial t} = \hat{H}(\hat{x}_i, \hat{p}_i)\Psi$	$i\hbar\frac{\partial \Phi}{\partial t} = \hat{H}(\hat{x}_i, \hat{p}_i)\Phi$

∅7.4

General formulation of QM

Dirac's $\langle \mathrm{bra} | \mathrm{ket} \rangle$ notation

State vector

A quantum mechanical state of a system is described by a state vector

$$|\psi\rangle$$

in a complex, linear vector space $\mathcal{H}$ — Hilbert space.

Dual vector

For each vector $|a\rangle$ we define the dual vector $\langle a|$ in the dual space $\mathcal{H}^*$,

$$|a\rangle \overset{\text{dual}}{\longleftrightarrow} \langle a|,$$

so that we can define the scalar (inner) product of vectors $|a\rangle$ and $|b\rangle$

$$\langle a||b\rangle \equiv \langle a|b\rangle \in \mathbb{C},$$

with $\langle a|b\rangle = \langle b|a\rangle^*$.

Example

Probability amplitude for particle arriving at point x:

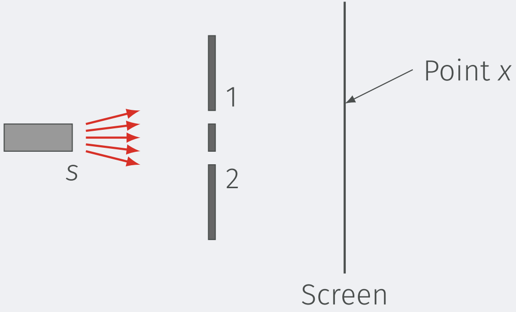
$$\langle \text{Particle arrives at } x | \text{particle leaves } s \rangle$$

or simply

$$\langle x | s \rangle.$$

Can go through either slit 1 or 2:

$$\langle x | s \rangle = \langle x | 1 \rangle \langle 1 | s \rangle + \langle x | 2 \rangle \langle 2 | s \rangle.$$



Based on Ch. 3 of Vol. III in the Feynman Lectures.

Interpretation

The wavefunction is probability amplitude of finding state $|\psi\rangle$ at point x:

$$\psi(x) = \langle x|\psi\rangle.$$

The momentum wavefunction is probability amplitude of finding state $|\psi\rangle$ with momentum p:

$$\phi(p) = \langle p|\psi\rangle.$$

The probability amplitude of finding state $|\psi\rangle$ with energy E_n:

$$\langle\psi_n|\psi\rangle.$$

Completeness

n linearly independent vectors $|1\rangle, |2\rangle, |3\rangle, \ldots$ span $\mathcal{H}$ if $\forall\ |\psi\rangle \in \mathcal{H}$ we have

$$|\psi\rangle = \sum_{k=1}^{n} c_n |k\rangle.$$

Assuming orthonormality $\langle m|k\rangle = \delta_{mk}$,

$$\Rightarrow\ |\psi\rangle = \sum_k \langle k|\psi\rangle |k\rangle = \sum_k |k\rangle\langle k||\psi\rangle,$$

meaning we have the completeness relation

$$\sum_k |k\rangle\langle k| = 1.$$