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FY2045 Solutions Problem set 3 fall 2023

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Problem 1

a) Normalization:

$$\langle 1 | 1 \rangle = \langle 2 | 2 \rangle = \langle 3 | 3 \rangle = 1.$$

Orthogonality:

$$\langle 1 | 2 \rangle = \langle 1 | 3 \rangle = \langle 2 | 3 \rangle = 0.$$

b) Using the completeness relation

$$\sum_{n=1}^{3} |n\rangle \langle n| = |1\rangle \langle 1| + |2\rangle \langle 2| + |3\rangle \langle 3| = \mathbb{1},$$

we find the expansion

$$|\psi\rangle = \mathbb{1} |\psi\rangle = [|1\rangle \langle 1| + |2\rangle \langle 2| + |3\rangle \langle 3|] |\psi\rangle = \langle 1 | \psi \rangle |1\rangle + \langle 2 | \psi \rangle |2\rangle + \langle 3 | \psi \rangle |3\rangle ,$$

where we have moved $\langle n | \psi \rangle \in \mathbb{C}$ past $|n\rangle$.

c) With $P_1 = |1\rangle\langle 1|$ we have that

$$P_1 |\psi\rangle = \langle 1|\, \psi\rangle\, |1\rangle \,,$$

which is the component of $|\psi\rangle$ in the "$|1\rangle$ direction". Furthermore, we find that

$$P_1^2 = |1\rangle\langle 1|\, 1\rangle\langle 1| = |1\rangle\langle 1| = P_1, \quad \text{q.e.d.}$$

In order to show that P_1 is Hermitian, we must show that it is self-adjoint,

$$P_1^\dagger = (|1\rangle\langle 1|)^\dagger = |1\rangle\langle 1| = P_1, \quad \text{q.e.d.}$$

Thus, P_1 is Hermitian. For P_12 we find that

$$P_{12}^2 = (|1\rangle\langle 1| + |2\rangle\langle 2|)(|1\rangle\langle 1| + |2\rangle\langle 2|) = |1\rangle\langle 1| + |2\rangle\langle 2| = P_{12}.$$

It is also Hermitian, meaning that P_{12} is a projection operator.

d) Here we have

$$\begin{aligned}
\langle b| &= (1+i)^* \langle 1| = (1-i)\langle 1| \,, \\
\langle a|\, b\rangle &= \langle 1|\, 1+i\, |1\rangle = (1+i)\langle 1|\, 1\rangle = 1+i, \\
\langle b|\, a\rangle &= \langle a|\, b\rangle^* = 1-i, \\
\langle b|\, b\rangle &= (1-i)(1+i)\langle 1|\, 1\rangle = 2.
\end{aligned}$$

e) For $|\psi_1\rangle$ to be normalized, we must have

$$\begin{aligned}
1 = \langle \psi_1|\, \psi_1\rangle &= \left(3^{-1/2}\langle 1| + c_1^*\langle 2|\right)\left(3^{-1/2}|1\rangle + c_1|2\rangle\right) \\
&= \frac{1}{3}\langle 1|\, 1\rangle + \frac{c_1}{\sqrt{3}}\langle 1|\, 2\rangle + \frac{c_1^*}{\sqrt{3}}\langle 2|\, 1\rangle + |c_1|^2\langle 2|\, 2\rangle = \frac{1}{3} + |c_1|^2,
\end{aligned}$$

meaning $c_1 = \sqrt{2/3}$ when choosing c_1 real and positive.

First we ensure that $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal:

$$\begin{aligned}
0 = \langle \psi_1|\, \psi_2\rangle &= \left(\frac{1}{\sqrt{3}}\langle 1| + \sqrt{\frac{2}{3}}\langle 2|\right)(c_2|1\rangle + c_3|2\rangle) = c_2\left(\frac{1}{\sqrt{3}} + \frac{c_3}{c_2}\sqrt{\frac{2}{3}}\right) \\
\Rightarrow \quad \frac{c_3}{c_2} &= -\sqrt{\frac{1}{2}}.
\end{aligned}$$

We next require normalization of $|\psi_2\rangle$:

$$\begin{aligned}
1 = \langle \psi_2|\, \psi_2\rangle &= |c_2|^2 + |c_3|^2 = |c_2|^2\left(1 + \frac{1}{2}\right) = \frac{3}{2}|c_2|^2 \\
\Rightarrow \quad c_2 &= \sqrt{\frac{2}{3}}, \text{ and in turn } c_3 = -\frac{1}{\sqrt{3}},
\end{aligned}$$

when c_2 is chosen real and positive.

Since neither of the two vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ has a component in the "$|3\rangle$ direction", it follows that

$$|\psi_3\rangle = |3\rangle\,,$$

is orthogonal to both these vectors. We have now constructed $|\psi_1\rangle\,, |\psi_2\rangle$ and $|\psi_3\rangle$ in such a way that they make up an orthonormalized set of vectors. These may be used as a new basis set instead of the original set $|1\rangle$, $|2\rangle$ and $|3\rangle$, if we want to.

f) From the definition of the adjoint we have that

$$\langle a|\,(|c\rangle\,\langle d|)^\dagger\,|b\rangle = \langle b|\,(|c\rangle\,\langle d|)\,|a\rangle^* = \langle b|\,c\rangle^*\,\langle d|\,a\rangle^* = \langle a|\,d\rangle\,\langle c|\,b\rangle = \langle a|\,(|d\rangle\,\langle c|)\,|b\rangle\,.$$

Since this holds for all $|a\rangle$ and $\langle b|$, we can conclude that

$$(|c\rangle\,\langle d|)^\dagger = |d\rangle\,\langle c|\,.$$

Problem 2

a) As mentioned, the scalar products are the same as for the corresponding wavefunctions. Thus we have

$$\left\langle x'\right|\left. p\right\rangle = \int dx\;\psi^*_{x'}(x)\psi_p(x) = \int dx\;\delta(x-x')\psi_p(x) = \psi_p(x').$$

Applying the same method, or using the property $\langle a|\,b\rangle^* = \langle b|\,a\rangle$, we find

$$\left\langle p\right|\left. x'\right\rangle = \psi^*_p(x').$$

b) Since $\langle x'|\,\hat{x} = x'\,\langle x'|$ and $\hat{x}\,|x'\rangle = x'\,|x'\rangle$, we have

$$\left\langle x'\right|\hat{x}\,|p\rangle = x'\left\langle x'\right|\left. p\right\rangle = x'\psi_p(x'),$$

and

$$\langle p|\,\hat{x}\left|x'\right\rangle = x'\left\langle p\right|\left. x'\right\rangle = x'\psi^*_p(x').$$

c) Using the completeness relation, on the form

$$\int dx\;\;|x\rangle\,\langle x| = \mathbb{1},$$

we find

$$|\psi\rangle = \mathbb{1}\,|\psi\rangle = \int dx\;\;|x\rangle\,\langle x|\,\psi\rangle = \int dx\;\psi(x)\,|x\rangle\,,$$

where $\psi(x) = \langle x | \psi \rangle$. An example is,

$$|p\rangle = \int dx \; |x\rangle \langle x| \, p\rangle = \int dx \; \psi_p(x) \, |x\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dx \; e^{ipx/\hbar} \, |x\rangle \, ,$$

where we have inserted the expression for $\psi_p(x) = \langle x | \, p \rangle$ the momentum eigenstates in the position representation.

d) We now get

$$|x\rangle = \mathbb{1} \, |x\rangle = \int dp \; |p\rangle \langle p| \, x\rangle = \int dp \; \langle x| \, p\rangle^* \, |p\rangle = \int dp \; \psi_p^*(x) \, |p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dp \; e^{-ipx/\hbar} \, |p\rangle \, .$$

Problem 3

We consider the vector

$$|h\rangle \equiv |\psi_1\rangle - \frac{\langle \psi_2 | \, \psi_1 \rangle}{\langle \psi_2 | \, \psi_2 \rangle} \; |\psi_2\rangle \, .$$

We then have that

$$\begin{aligned} 0 \leq \; \langle h | \, h \rangle &= \left(\langle \psi_1 | - \frac{\langle \psi_2 | \, \psi_1 \rangle^*}{\langle \psi_2 | \, \psi_2 \rangle} \; \langle \psi_2 | \right) \left(|\psi_1\rangle - \frac{\langle \psi_2 | \, \psi_1 \rangle}{\langle \psi_2 | \, \psi_2 \rangle} \; |\psi_2\rangle \right) \\ &\leq \; \langle \psi_1 | \, \psi_1 \rangle - \frac{| \, \langle \psi_2 | \, \psi_1 \rangle \, |^2}{\langle \psi_2 | \, \psi_2 \rangle}, \end{aligned}$$

that is,

$$| \, \langle \psi_1 | \, \psi_2 \rangle \, |^2 \leq \langle \psi_1 | \, \psi_1 \rangle \, \langle \psi_2 | \, \psi_2 \rangle \, , \; \text{q.e.d.}$$

Hence, if two vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ are in Hilbert space, meaning they have a finite norm/are quadratically integrable, the Schwarz inequality ensures that the scalar product $\langle \psi_1 | \, \psi_2 \rangle$ exists, i.e. it is finite.

Note that Schwarz' inequality is a *mathematical* result, which holds for arbitrary, quadratically integrable functions ψ_1 and ψ_2.