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Problem 1

In Lecture notes 10 you will find the derivation of the "tool" formulae (T10.62-T10.65)

$$\langle x| \hat{x} = x \langle x| , \tag{1}$$

$$\langle x| \hat{p}_x = \frac{\hbar}{i}\frac{\partial}{\partial x} \langle x| , \tag{2}$$

$$\langle p| \hat{p}_x = p \langle p| , \tag{3}$$

$$\langle p| \hat{x} = -\frac{\hbar}{i}\frac{\partial}{\partial p} \langle p| , \tag{4}$$

where the *ket* vector $|x\rangle$ corresponds to a sharp position x and $|p\rangle$ corresponds to a sharp momentum $p_x = p$.

When the abstract operator $\hat{x}$ acts on a ket vector $|\psi\rangle$, the result is a new vector, which we may call

$$\left|\widetilde{\psi}\right\rangle = \hat{x}\ |\psi\rangle .$$

We can then use tool formula no. (1) above to find the corresponding connection between the *wavefunctions* $\widetilde{\psi}(x) = \langle x|\widetilde{\psi}\rangle$ and $\psi(x) = \langle x|\psi\rangle$. We do this as follows:

$$\underline{\widetilde{\psi}(x)} = \langle x|\widetilde{\psi}\rangle = \langle x|\ \hat{x}\ |\psi\rangle \overset{(1)}{=} x\langle x|\psi\rangle = \underline{x\,\psi(x)}.$$

The moral is: The abstract operator $\hat{x}$ is in the good old position representation of QM represented by multiplication by x:

$$\hat{x}_{\text{p.r.}} = x.$$

a) Use the same method and equation (2) above to find the corresponding representation of the momentum operator ($\hat{p}_{x,\text{p.r.}}$), and notice that the result can be read straight out from equation (2).

b) Consider again

$$\left|\widetilde{\psi}\right\rangle = \hat{p}_x \left|\psi\right\rangle,$$

and use the tool formula (3) above to find the momentum operator in the momentum representation of QM ($\hat{p}_{x,\text{m.r.}}$), and check that the result agrees with section 7.4 in Lecture note 7. *Hint:* The projection of $|\psi\rangle$ onto $|p\rangle$ is the momentum wavefunction: $\langle p|\psi\rangle = \phi(p)$. Therefore it is a good idea to start by multiplying the above equation from the left by $\langle p|$.

Then consider $\left|\widetilde{\psi}\right\rangle = \hat{x}\left|\psi\right\rangle$, and use (4) to find the the position operator ($\hat{x}_{\text{m.r.}}$) in the momentum representation of QM.

Problem 2 — Some general properties of non-stationary oscillator states

We consider a harmonic oscillator (a mass m in the potential $V(q) = \frac{1}{2}m\omega^2 q^2$) prepared in the initial state $\Psi(q,0)$ at time $t=0$. For simplicity, we assume that this initial state is chosen in such a way that $\langle p\rangle_{t=0} \equiv \langle p\rangle_0 = 0$, while $\langle q\rangle_0 = q_0 > 0$. The state $\Psi(q,t)$ then necessarily becomes non-stationary.

a) Show that the time-dependent expectation values of the position and the momentum, $\langle q\rangle_t$ and $\langle p\rangle_t$, then behave classically, in the same way as the coordinate q and the momentum p for a classical oscillation with amplitude q_0. *Hint:* Use Ehrenfest's theorem,

$$\frac{d}{dt}\left\langle q\right\rangle_t = \frac{\langle p\rangle_t}{m}, \qquad \frac{d}{dt}\left\langle p\right\rangle_t = \left\langle -\partial V/\partial q\right\rangle_t,$$

to find a second-order differential equation for $\langle q\rangle_t$, and solve this to find $\langle q\rangle_t$ and $\langle p\rangle_t$.

b) This simple behavior of the expectation values $\langle q\rangle_t$ and $\langle p\rangle_t$ does not tell us much about the behaviour of the wave function $\Psi(q,t) = \langle q|\Psi(t)\rangle$ for such a non-stationary state. This behavior is determined by the Schrödinger equation and the initial state $\Psi(q,0) = \langle q|\Psi(0)\rangle$ and depends of course on the details of the latter. Suppose for example that $\Psi(q,0)$ is a strongly "squeezed" (very narrow) function, centered around the position q_0. We must then expect that the dispersion will soon give a $\Psi(q,t)$ which is spread out all over the place. But strangely enough, we get a revival of the initial form of the wave function at the time

$T = 2\pi/\omega$, corresponding to the classical period of the oscillator. Due to the equidistant energy levels $E_n = \hbar\omega(n + \frac{1}{2})$ of the oscillator, it turns out, namely, that

$$\Psi(q, kT) = (-1)^k \Psi(q, 0), \qquad k = 1, 2, \cdots ,$$

no matter how the initial state is chosen. Show this, using the expansion of the wave function in terms of stationary states:

$$\Psi(q, t) = \sum_{n=0}^{\infty} c_n \, \psi_n(q) \, e^{-iE_n t/\hbar},$$

where $c_n = \langle \psi_n, \Psi(0) \rangle = \langle \psi_n | \Psi(0) \rangle$ is the projection of the initial state $\Psi(q, 0)$ on the energy eigenfunction $\psi_n(q)$ (or equivalently the projection of the initial state vector $|\Psi(0)\rangle$ on the energy eigenvectors $|\psi_n\rangle$. Note that you don't need to know $\Psi(q, 0)$ and hence c_n to carry out the proof. It is sufficient to investigate the exponential

$$e^{-i\omega t(n+1/2)} = e^{-i\omega t/2} (e^{-i\omega t})^n$$

for the times $t = kT = k \cdot 2\pi/\omega$.

Problem 3

In the lectures and Lecture note 11 we introduced the dimensionless (and non-Hermitian) ladder operator

$$a = \sqrt{\frac{m\omega}{2\hbar}} \, \hat{q} + \frac{i\hat{p}}{\sqrt{2m\hbar\omega}} \tag{5}$$

and its adjoint, $a^\dagger$. These operators have the properties

$$a \, |n\rangle = \sqrt{n} \, |n - 1\rangle \, ; \qquad a^\dagger \, |n\rangle = \sqrt{n + 1} \, |n + 1\rangle \, , \tag{6}$$

where $|n\rangle$ is the time-independent Hilbert vector describing the energy eigenstate for the n-th excited state of the harmonic oscillator, such that $\langle q|n\rangle = \psi_n(q)$.

a) According to the formulae eqs. (1) to (4), the *position representation* of the abstract operators $\hat{q}$ and $\hat{p}$ can be read out from the formulae

$$\langle q| \, \hat{q} = q \, \langle q| \, , \qquad \langle q| \, \hat{p} = \frac{\hbar}{i} \frac{\partial}{\partial q} \, \langle q| \ \ .$$

Use these to find the position representation $a_{\text{p.r.}}$ of the lowering operator a, defined by

$$\langle q| \ a = a_{\text{p.r.}} \ \langle q| \, .$$

b) Show that $a_{\text{p.r.}}$ applied to the ground state $\psi_0(q)$ gives zero, in accordance with the "abstract" formula $a\,|0\rangle = 0$; cf. eq. (6).

c) Show that $a^\dagger_{\text{p.r.}}$ applied to the ground state $\psi_0(q)$ gives the first excited state $\psi_1(q)$, in accordance with the "abstract" formula $a^\dagger\,|0\rangle = |1\rangle$; cf. eq. (6).

Given:

$$\psi_0(q) = C_0\, e^{-m\omega q^2/2\hbar}; \qquad \psi_1(q) = C_0\sqrt{\frac{2m\omega}{\hbar}}\, q\, e^{-m\omega q^2/2\hbar}; \qquad C_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}.$$