Lecture notes 7

Written by Ingjald Øverbø

These notes repeat some central points from *Lecture notes 2*, and cover some new stuff as well, namely measurements of degenerate eigenvalues, together with the momentum representation of quantum mechanics.

7. Introduction to FY2045

FY2045 *Quantum mechanics I* is a continuation of — and is built on — FY1006 / TFY4215 *Introduction to quantum physics*. It is therefore very important to master the contents of the latter in order to be able to follow the present course. This means that you will need to repeat on your own the contents of the introductory course. The present notes are confined to a reminder about the fundamental postulates, as formulated in *Lecture notes 2*. We go on to cover some new stuff, namely how to formulate the measurement postulate when measuring a degenerate eigenvalue. We then repeat the (Fourier) expansion of the system wave function in terms of momentum eigenfunctions, showing that the square of the coefficient function (the Fourier transform), $|\Phi(p,t)|^2$, is the probability density in momentum space. This is used as an introduction to the last subsection, in which we derive the momentum representation of quantum mechanics, showing that $\Phi(p,t)$ works as a wave function in momentum space.

7.1 Basic postulates (Hemmer 2.1, B&J, Lecture notes 2)

The basic postulates, as formulated in *Lecture notes 2* were:

A. The operator postulate

> To each physical observable quantity F there corresponds in quantum-mechanical theory a linear operator $\widehat{F}$. (T7.1)

B. The wave-function postulate

> The state of a system is described, as completely as possible, by the wave function $\Psi(q_n,t)$. The time development of the wave function (and hence of the state) is determined by the Schrödinger equation,
>
> $$i\hbar\frac{\partial\Psi}{\partial t} = \widehat{H}\Psi,$$
>
> where $\widehat{H}$ is the Hamiltonian of the system. (T7.2)

Updated August 2022 by H.G. Hugdal.

C. Expectation value postulate

When a large number of measurements of an observable F is made on a system which is prepared in a state $\Psi(q_1, q_2, \cdots, q_n, t)$ (before each measurement), the average $\overline{F}$ of the measured values will approach the theoretical expectation value, which is postulated to be

$$\langle F \rangle_\Psi = \int \Psi^* \widehat{F} \Psi \, d\tau,$$

where $d\tau = dq_1 dq_2 \cdots dq_n$ and where the integration goes over the whole range of each of the variables. (T7.3)

D. Measurement postulate

(i) The only possible result of a precise measurement of an observable F is one of the eigenvalues f_n of the corresponding linear operator $\widehat{F}$.

(ii) Immediately after the measurement of the eigenvalue f_n, the system is in an eigenstate of $\widehat{F}$, namely, the eigenstate ψ_n corresponding to the measured eigenvalue f_n. (T7.4)

7.2 Measurement of a degenerate eigenvalue

If the measured eigenvalue f_n is non-degenerate, that is, if the eigenvalue equation

$$\widehat{F}\psi_n = f_n \psi_n$$

has only one solution ψ_n for the eigenvalue f_n, it follows from (ii) above that the system is left in this state ψ_n (immediately) after the measurement. This state is unique, apart from an undetermined phase factor which has no physical significance.

In the opposite case, when the eigenvalue f_n is degenerate with (degree of) degeneracy g_n, the above eigenvalue equation has g_n solutions ψ_{ni} which we may enumerate with an extra index i:

$$\widehat{F}\psi_{ni} = f_n \psi_{ni} \; ; \quad i = 1, \cdots, g_n.$$

In this case we shall now see that point (ii) in the measurement postulate (stating that the measurement leaves the system in an eigenstate corresponding to the measured eigenvalue) must be formulated more precisely.

Let us suppose that the set f_n of eigenvalues is discrete, and that the set

$$\{\psi_{ni} |\, n = 1, 2, \cdots ; \, i = 1, \cdots, g_n\}$$

of eigenfunctions is orthonormalized. Since these eigenfunctions form a complete set (basis), the state of the system prior to the measurement can be expanded in this set:

$$\Psi = \sum_n \sum_{i=1}^{g_n} c_{ni} \psi_{ni}. \qquad \text{(T7.5)}$$

By projecting Ψ onto ψ_{ni},

$$\langle\, \psi_{ni}, \Psi\,\rangle = \left\langle\, \psi_{ni}\,,\, \sum_k \sum_j c_{kj}\psi_{kj}\,\right\rangle = \sum_k \sum_j c_{kj}\delta_{nk}\delta_{ij} = c_{ni},$$

we find in the usual manner that the expansion coefficient is

$$c_{ni} = \langle\, \psi_{ni}, \Psi\,\rangle \equiv \int \psi_{ni}^* \Psi d\tau.$$

As in section 2.5.d of Lecture notes 2, we consider a series of measurements of the observable F on an ensemble which is prepared in the state Ψ. According to the measurement postulate, the theoretical expectation value of F is

$$\langle\, F\,\rangle = \sum_n P_n\, f_n,$$

where P_n is the probability of measuring the eigenvalue f_n. At the same time it follows from the expectation-value postulate that

$$\begin{aligned}
\langle\, F\,\rangle_\Psi &= \int \Psi^*\, \widehat{F}\, \Psi d\tau = \int (\widehat{F}\Psi)^* \Psi d\tau \\
&= \int \left(\sum_n \sum_i c_{ni}\, \widehat{F}\psi_{ni} \right)^* \Psi\, d\tau \\
&= \sum_n \sum_i c_{ni}^* f_n \underbrace{\int \psi_{ni}^* \Psi d\tau}_{c_{ni}} = \sum_n \left(\sum_{i=1}^{g_n} |c_{ni}|^2 \right) f_n.
\end{aligned}$$

Since the two formulae for the expectation value are valid for an arbitrary Ψ, we can conclude that the probability of measuring the eigenvalue f_n when the system is in the state (T7.5) is

$$P_n = \sum_{i=1}^{g_n} |c_{ni}|^2. \qquad \text{(T7.6)}$$

Note that the corresponding result in the non-degenerate case is given by (T2.80) in Lecture notes 2, namely $P_n = |c_n|^2$.

Before reformulating point (ii) in the measurement postulate (T7.4), we note that Ψ (equation (T7.5)) may be written as

$$\Psi = \sum_n \Psi_n, \qquad \text{with} \quad \Psi_n \equiv \sum_{i=1}^{g_n} c_{ni}\psi_{ni}.$$

Here, we also note that Ψ_n — as the g_n contributions ψ_{ni} — has the eigenvalue f_n. We may therefore call Ψ_n the part of Ψ which is "compatible with the eigenvalue f_n". It turns out that point (ii) in the measurement postulate must be formulated as follows:

> Immediately after the measurement of the eigenvalue f_n the system is left in the (normalized) state
>
> $$\frac{\Psi_n}{||\Psi_n||} = \frac{\sum_{i=1}^{g_n} c_{ni}\psi_{ni}}{||\sum_{i=1}^{g_n} c_{ni}\psi_{ni}||}. \qquad \text{(T7.7)}$$

(In this expression, the denominator is the norm of the numerator.) The moral is that the part of the wave function Ψ before the measurement which is *not* compatible with the measured eigenvalue f_n "is removed" by the measurement. This is often called a "wave-function collapse". Note by the way that the above result for the probability P_n is the squared norm of Ψ_n,

$$P_n = \sum_i |c_{ni}|^2 = ||\Psi_n||^2$$

(so that $\sum_n P_n = \sum_n ||\Psi_n||^2 = 1$).

Example Let us as an example consider a three-dimensional isotropic harmonic oscillator, with the orthonormalized eigenfunction set

$$\psi_{n_x}(x)\psi_{n_y}(y)\psi_{n_z}(z) \equiv \psi_{n_x n_y n_z}$$

and the energy eigenvalues

$$\hbar\omega(n_x + n_y + n_z + 3/2) \equiv \hbar\omega(N + 3/2).$$

Suppose that the oscillator is at time $t = 0$ prepared in the state

$$\Psi = \underbrace{\sqrt{0.5}\,\psi_{000}}_{\Psi_0} + \underbrace{\sqrt{0.1}(\psi_{100} + \psi_{010} + \psi_{001})}_{\Psi_1} + \underbrace{\sqrt{0.1}(\psi_{200} + i\psi_{020})}_{\Psi_2}.$$

Here, we observe that the squared norms of the three contributions to Ψ are

$$||\Psi_0||^2 = 0.5\,, \quad ||\Psi_1||^2 = 3 \cdot 0.1\,, \quad ||\Psi_2||^2 = 2 \cdot 0.1,$$

so that $||\Psi||^2 = 1$. Thus Ψ is normalized. Furthermore, Ψ_0, Ψ_1 and Ψ_2 have respectively $N = 0$, 1 and 2. Therefore, the possible measured values of the energy are $E_0 = \frac{3}{2}\hbar\omega$ ($N = 0$), $E_1 = \frac{5}{2}\hbar\omega$ ($N = 1$) and $E_2 = \frac{7}{2}\hbar\omega$ ($N = 2$). The respective probabilities are

$$P_0 = ||\Psi_0||^2 = 0.5\,, \qquad P_1 = 0.3 \quad \text{and} \quad P_2 = 0.2.$$

The corresponding normalized states immediately after the measurement are

$$\frac{\Psi_0}{||\Psi_0||} = \psi_{000}\,, \qquad \frac{\Psi_1}{||\Psi_1||} = \frac{1}{\sqrt{3}}(\psi_{100} + \psi_{010} + \psi_{001}) \quad \text{and} \quad \frac{\Psi_2}{||\Psi_2||} = \frac{1}{\sqrt{2}}(\psi_{200} + i\psi_{020}).$$

In the next subsection we shall repeat the physical interpretation of the expansion coefficients in the continuous case, by taking as an example the expansion of Ψ in terms of momentum eigenfunctions.

7.3 Physical interpretation in the continuous case

The physical interpretation of the expansion coefficients in the continuous case is described in a very clear and concise way page 33–34 in Hemmer. (Sea also Griffiths page 102–105, B&J page 208 and Lecture notes 2.)

With the continuous spectrum of momentum eigenfunctions

$$\psi_p(x) = (2\pi\hbar)^{-1/2} e^{ipx/\hbar}, \qquad \widehat{p}_x\psi_p(x) = p\psi_p(x), \qquad p \in (-\infty, \infty) \tag{T7.8}$$

as a basis, the expansion of an arbitrary quadratically integrable function becomes a Fourier integral. For the time-dependent system wave function, we have for example

$$\Psi(x,t) = \int_{-\infty}^{\infty} \Phi(p,t)\, \psi_p(x)\, dp. \tag{T7.9}$$

Since this function depends on time, so will also the Fourier transform $\Phi(p,t)$:

$$\Phi(p,t) = \langle\, \psi_p, \Psi(t)\,\rangle \equiv \int_{-\infty}^{\infty} \psi_p^*(x)\, \Psi(x,t)\, dx. \tag{T7.10}$$

In analogy with the previous section, we now assume that a series of measurements of the momentum p_x are made on an ensemble prepared in the state Ψ. Since the measured values are distributed continuously over the entire spectrum, we may then write the expectation value on the form

$$\langle\, p_x\,\rangle = \int_{-\infty}^{\infty} P(p,t)\, p\, dp,$$

where $P(p,t)dp$ is the probability of finding p_x in the interval $[p, p+dp]$ and $P(p,t)$ is the probability density in "p-space", at time t. On the other hand, we have from the expectation-value postulate that

$$\begin{aligned}
\langle\, p_x\,\rangle_\Psi &= \int_{-\infty}^{\infty} dx\, \Psi^*(x,t)\, \widehat{p}_x\, \Psi(x,t) = \int_{-\infty}^{\infty} dx\, (\widehat{p}_x \Psi(x,t))^*\, \Psi(x,t) \\
&= \int_{-\infty}^{\infty} dx \left(\widehat{p}_x \int_{-\infty}^{\infty} dp\, \Phi(p,t)\, \psi_p(x)\right)^* \Psi(x,t).
\end{aligned}$$

Here we replace $\widehat{p}_x \psi_p(x)$ with $p\psi_p(x)$ and change the order of the integrations:

$$\begin{aligned}
\langle\, p_x\,\rangle_\Psi &= \int_{-\infty}^{\infty} dp\, \Phi^*(p,t)\, p \left(\int_{-\infty}^{\infty} dx\, \psi_p^*(x)\, \Psi(x,t)\right) \\
&= \int_{-\infty}^{\infty} \Phi^*(p,t)\, p\, \Phi(p,t)\, dp = \int_{-\infty}^{\infty} p\, |\Phi(p,t)|^2 dp.
\end{aligned}$$

By comparing the two formulae for $\langle\, p_x\,\rangle$ we see that the physical interpretation of the "expansion coefficient" (the Fourier transform $\Phi(p,t)$) may be formulated as follows:

> When the system is in the state $\Psi(x,t)$ before the measurement, the probability of measuring p_x in the interval $(p, p+dp)$ is
>
> $$P(p)dp = |\Phi(p,t)|^2 dp = |\langle\, \psi_p, \Psi\,\rangle|^2\, dp \equiv \left| \int \psi_p^*(x) \Psi(x,t)\, d\tau \right|^2 dp. \tag{T7.11}$$

Thus the probability density in "p space" is the square of the Fourier transform $\Phi(p,t)$. This is analoguous to $|\Psi(x,t)|^2$ being the probability density in x space .

7.4 The momentum-space formulation of quantum mechanics

This "similarity" between the position-space and momentum-space probability densities is not accidental. As explained in section 4.6 in Hemmer and in 3.9 in B&J, it is straightforward to obtain a formulation of the theory in which the Fourier transform $\Phi(p,t)$ of $\Psi(x,t)$ plays the role of a "wave function" in momentum space. This role is analogous to that played by the ordinary wave function $\Psi(x,t)$ in the **position-space formulation of quantum mechanics**, which we are now beginning to get used to, and which is most commonly used on the introductory level.

In the new **momentum-space formulation of quantum mechanics**, we already know how to obtain the expectation values of observables which depend only on p_x, like e.g. $K = p_x^2/2m$. Since the probability density in momentum space is $|\Phi(p,t)|^2$, we have that

$$\langle\, F(p_x)\,\rangle_\Phi = \int_{-\infty}^{\infty} |\Phi(p,t)|^2 F(p)dp = \int_{-\infty}^{\infty} \Phi^*(p,t)\, F(p)\, \Phi(p,t)\, dp, \qquad \text{(T7.12)}$$

which is analogous to

$$\langle\, V(x)\,\rangle_\Psi = \int_{-\infty}^{\infty} \Psi^*(x,t)\, V(x)\, \Psi(x,t)\, dx$$

in the position-space formulation. The "moral" is that in the momentum-space formulation, the observable p_x is represented by an operator which simply is (multiplication by) the *number* p,

$$\widehat{p}_x = p. \qquad \text{(T7.13)}$$

This is analogous to $\widehat{x} = x$ in the position-space formulation.

What about the operators representing x and functions of x (like e.g. $V(x)$) in the new formulation? To find the answer, we shall assume that the potential $V(x)$ can be expanded in a Taylor series,

$$V(x) = \sum_n v_n x^n,$$

where the expansion coefficients are v_n. The expectation values of x and powers of x can now be found starting with the old formulation, where the expectation value of x^n is

$$\begin{aligned}\langle\, x^n\,\rangle &= \int_{-\infty}^{\infty} \Psi^*(x,t)\, x^n\, \Psi(x,t)\, dx = \int_{-\infty}^{\infty} (x^n\Psi)^*\Psi\, dx \\ &= \int_{-\infty}^{\infty} dx \left(x^n \int_{-\infty}^{\infty} dp\, \Phi(p,t)\, \psi_p(x)\right)^* \Psi(x,t).\end{aligned}$$

Here we apply the identity

$$x\, e^{ipx/\hbar} = \left(\frac{\hbar}{i}\frac{\partial}{\partial p}\right) e^{ipx/\hbar}, \qquad \text{(T7.14)}$$

which means that

$$x^n\, \psi_p(x) = \left(\frac{\hbar}{i}\frac{\partial}{\partial p}\right)^n \psi_p(x). \qquad \text{(T7.15)}$$

Inserting this and rearranging we then have

$$\begin{aligned}\langle\, x^n\,\rangle &= \int_{-\infty}^{\infty} dx \left(\int_{-\infty}^{\infty} dp\, \Phi(p,t) \left(\frac{\hbar}{i}\frac{\partial}{\partial p}\right)^n \psi_p(x)\right)^* \Psi(x,t) \\ &= \int_{-\infty}^{\infty} dp\, \Phi^*(p,t) \left(-\frac{\hbar}{i}\frac{\partial}{\partial p}\right)^n \underbrace{\int_{-\infty}^{\infty} dx\, \psi_p^*(x)\Psi(x,t)}_{\Phi(p,t)} \\ &= \int_{-\infty}^{\infty} dp\, \Phi^*(p,t) \left(-\frac{\hbar}{i}\frac{\partial}{\partial p}\right)^n \Phi(p,t). \qquad \text{(T7.16)}\end{aligned}$$

Comparing this expression with the general "sandwich" recipe for expectation values,

$$\langle\, F\,\rangle_\Phi = \int_{-\infty}^{\infty} dp\, \Phi^*(p,t)\, \widehat{F}\, \Phi(p,t), \tag{T7.17}$$

we can conclude that the observable x^n is represented in the momentum-space formulation by the n^{th} power of the operator

$$\widehat{x} = -\frac{\hbar}{i}\frac{\partial}{\partial p}. \tag{T7.18}$$

For a *function* of x like e.g. $V(x) = \sum v_n x^n$ we find that

$$\langle\, V(x)\,\rangle_\Phi = \int_{-\infty}^{\infty} dp\, \Phi^*(p,t) \left[\sum_n v_n \left(-\frac{\hbar}{i}\frac{\partial}{\partial p}\right)^n\right] \Phi(p,t). \tag{T7.19}$$

Thus the potential energy is in the new formulation represented by the operator

$$\sum_n v_n \left(-\frac{\hbar}{i}\frac{\partial}{\partial p}\right)^n \equiv \widehat{V}\left(-\frac{\hbar}{i}\frac{\partial}{\partial p}\right).$$

This is called an **operator function**, and the Taylor expansion on the left shows exactly what we mean by this. As an example, the harmonic-oscillator potential $V(x) = \frac{1}{2}m\omega^2 x^2$ is in this formulation represented by the operator $\widehat{V} = \frac{1}{2}m\omega^2(-\frac{\hbar}{i}\frac{\partial}{\partial p})^2$.

We have now learnt how to calculate expectation values of observables depending on x and p_x in the new formulation, from the "wave function" $\Phi(p,t)$. But can we be sure that this function contains all possible information about the system, as is the case for $\Psi(x,t)$ according to the wave-function postulate on page 1? The answer is yes: If we know the function $\Phi(p,t)$, then we also know $\Psi(x,t)$, via the Fourier integral

$$\Psi(x,t) = \int_{-\infty}^{\infty} \Phi(p,t)\, \psi_p(x)\, dp,$$

and vice versa, via the Fourier transform

$$\Phi(p,t) = \int_{-\infty}^{\infty} \psi_p^*(x)\, \Psi(x,t)\, dx.$$

Thus the two functions contain the same information.

But isn't $\Psi(x,t)$ still more special, since it satisfies a wave equation, the Schrödinger equation? The answer is no: There exists a wave equation also for $\Phi(p,t)$. We can find this equation by taking the derivative $i\hbar(\partial/\partial t)\Phi(p,t)$ as our starting point: Using the last formula above, we find that

$$\begin{aligned}
\underline{i\hbar\frac{\partial}{\partial t}\Phi(p,t)} &= \int_{-\infty}^{\infty} \psi_p^*(x)\, i\hbar\frac{\partial}{\partial t}\,\Psi(x,t)\, dx \qquad \left(i\hbar\frac{\partial}{\partial t}\Psi = \left[\widehat{p}_x^2/2m + V(x)\right]\Psi\right) \\
&= \int_{-\infty}^{\infty} \psi_p^*(x)\, \underbrace{\left[\widehat{p}_x^2/2m + V(x)\right]}_{\text{Hermitian}}\, \Psi(x,t)\, dx \\
&= \int_{-\infty}^{\infty} \left(\left[\widehat{p}_x^2/2m + \sum_n v_n x^n\right]\psi_p(x)\right)^* \Psi(x,t)\, dx \\
&= \int_{-\infty}^{\infty} \left(\left[p^2/2m + \sum_n v_n \left(\frac{\hbar}{i}\frac{\partial}{\partial p}\right)^n\right]\psi_p(x)\right)^* \Psi(x,t)\, dx.
\end{aligned}$$

Here we have applied the identities

$$\widehat{p}_x\psi_p(x) = p\psi_p(x) \qquad \text{and} \qquad x\,\psi_p(x) = \left(\frac{\hbar}{i}\frac{\partial}{\partial p}\right)\psi_p(x).$$

In the last expression we can move the operator $[\ \ ]^*$ to the left of the integral, because it does not depend on x. We then have

$$\begin{aligned}
\cdots &= \left[p^2/2m + \sum_n v_n\left(-\frac{\hbar}{i}\frac{\partial}{\partial p}\right)^n\right]\underbrace{\int_{-\infty}^{\infty}\psi_p^*(x)\Psi(x,t)\,dx}_{\Phi(p,t)} \\
&= \underline{\left[p^2/2m + \widehat{V}\left(-\frac{\hbar}{i}\frac{\partial}{\partial p}\right)\right]\Phi(p,t)} \equiv \underline{\widehat{H}\Phi(p,t)}. \qquad \text{(T7.20)}
\end{aligned}$$

This must be called a success: $\Phi(p,t)$ does satisfy a wave equation,

$$i\hbar\frac{\partial\Phi}{\partial t}(p,t) = \widehat{H}\Phi(p,t),$$

and the form of this equation allows us to call it a Schrödinger equation.

Thus we have two equivalent versions of quantum mechanics, the **position-space formulation** and the **momentum-space formulation**. With the symbols x, y, z (or $x_i,\ i = 1, .., 3$) for the cartesian coordinates, the situation can be summarized by the following table, where we see that both wave functions satisfy the Schrödinger equation, with a Hamiltonian given by the general formula

$$\widehat{H}(\widehat{x}_i, \widehat{p}_i) = \frac{\widehat{p}_x^2 + \widehat{p}_y^2 + \widehat{p}_z^2}{2m} + \widehat{V}(\widehat{x}, \widehat{y}, \widehat{z}).$$

	Position-space formulation	Momentum-space formulation
Wave function	$\Psi(x, y, z, t)$	$\Phi(p_x, p_y, p_z, t)$
Operator $\widehat{x}_i$	x_i	$-\dfrac{\hbar}{i}\dfrac{\partial}{\partial p_i}$
Operator $\widehat{p}_i$	$\dfrac{\hbar}{i}\dfrac{\partial}{\partial x_i}$	p_i
Wave equation	$i\hbar\dfrac{\partial\Psi}{\partial t} = \widehat{H}(\widehat{x}_i, \widehat{p}_i)\Psi$	$i\hbar\dfrac{\partial\Phi}{\partial t} = \widehat{H}(\widehat{x}_i, \widehat{p}_i)\Phi$

In Lecture Notes 10 we shall see that the momentum-space formulation of quantum mechanics, as well as the original position-space formulation, are special cases of a more general formulation.

Example: Free particle

For a free particle ($V = 0$) we see that the Schrödinger equation in the momentum-space formulation looks like this:

$$i\hbar \frac{\partial \Phi(p,t)}{\partial t} = \frac{p^2}{2m} \Phi(p,t) \qquad (p = p_x)\,.$$

Here, $\partial/\partial t$ means differentiation with p kept fixed. Then it is easy to see that the time-dependent wave function in momentum space becomes

$$\Phi(p,t) = \Phi(p,0)\, e^{-i(p^2/2m)t/\hbar}. \tag{T7.21}$$

Here, $\Phi(p,0)$ is the momentum-space wave function at $t = 0$, which we are allowed to prepare arbitrarily, but we assume that it is normalized:

$$\int_{-\infty}^{\infty} |\Phi(p,0)|^2 dp = 1.$$

From the solution (T7.21) we see that the probability density in momentum space becomes *time independent* for the free particle,

$$|\Phi(p,t)|^2 = |\Phi(p,0)|^2,$$

and this should not be surprising. The same should then be the case for all purely p-dependent observables, like e.g.

$$\langle\, p\,\rangle = \int_{-\infty}^{\infty} \Phi^*(p,t)\, p\, \Phi(p,t)\, dp = \int_{-\infty}^{\infty} \Phi^*(p,0)\, p\, \Phi(p,0)\, dp = \langle\, p\,\rangle_{t=0}\,,$$

$\langle\, p^2\,\rangle$, Δp, etc. We can also find out how the expectation value of the position behaves: From (T7.21) we have

$$\begin{aligned}
\langle\, x\,\rangle_t &= \int_{-\infty}^{\infty} \Phi^*(p,t) \left(-\frac{\hbar}{i}\frac{\partial}{\partial p}\right) \Phi(p,t)\, dp \\
&= \int_{-\infty}^{\infty} \Phi^*(p,0) e^{i(p^2/2m)t/\hbar} \left[-\frac{\hbar}{i}\frac{\partial}{\partial p}\Phi(p,0) - \Phi(p,0)\frac{\hbar}{i}\frac{\partial}{\partial p}\left(-\frac{ip^2t}{2m\hbar}\right)\right] e^{-i(p^2/2m)t/\hbar}\, dp \\
&= \int_{-\infty}^{\infty} \Phi^*(p,0) \left(-\frac{\hbar}{i}\frac{\partial}{\partial p}\right) \Phi(p,0)\, dp + \frac{t}{m}\int_{-\infty}^{\infty} \Phi^*(p,0)\, p\, \Phi(p,0)\, dp \\
&= \langle\, x\,\rangle_{t=0} + \frac{\langle\, p\,\rangle}{m}\, t.
\end{aligned}$$

Thus the expectation value $\langle\, x\,\rangle_t$ is moving with constant velocity $\langle\, p\,\rangle/m$, from $\langle\, x\,\rangle_{t=0}$ at $t = 0$. This agrees with Newton's first law.