FY2045 Quantum Mechanics I

Fall 2023

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Week 9

Addition of angular momenta

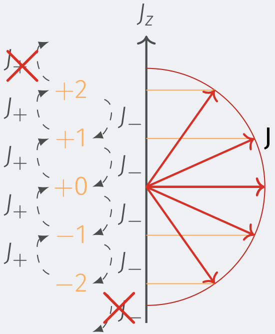
Angular momentum — recap

Hermitian angular momentum operators J_x, J_y, J_z, and $\mathbf{J}^2 = J_x^2 + J_y^2 + J_z^2$ with commutation relations

$$[J_x, J_y] = i\hbar J_z, \ \text{etc.},$$
$$[\mathbf{J}^2, J_i] = 0, \ i = x, y, z.$$

Simultaneous eigenvectors

$$\mathbf{J}^2 |j, m\rangle = \hbar^2 j(j+1) |j, m\rangle, \quad j = 0, \frac{1}{2}, 1, \ldots,$$
$$J_z |j, m\rangle = \hbar m |j, m\rangle, \quad m = -j, -j+1, \ldots, j.$$



Based on figure by Izaak Neutelings

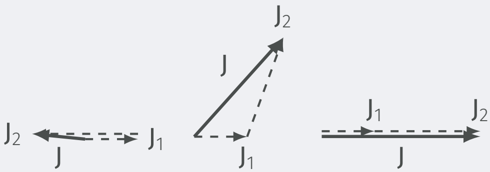
Ladder operators $J_\pm = J_x \pm iJ_y$.

Addition of two angular momenta

How do we treat addition of angular momenta, $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$?

Classical case

$|\mathbf{J}|$ can vary continuously between $||\mathbf{J}_1| - |\mathbf{J}_2||$ and $|\mathbf{J}_1| + |\mathbf{J}_2|$



Quantum mechanical case

More complicated! Two questions arise:

i) What quantum numbers j can we have for $\mathbf{J}$, when $\mathbf{J}_1$ and $\mathbf{J}_2$ have quantum numbers j_1 and j_2.

ii) How do we express the eigenstates of $\mathbf{J}$ in terms of eigenstates of $\mathbf{J}_1$ and $\mathbf{J}_2$?

Two particles with spin ½

Two particles with quantum numbers $s_1 = s_2 = \frac{1}{2}$, and m_1, m_2. Four possible states:

$$
\begin{aligned}
&|\uparrow\uparrow\rangle \,,\; m = 1, \\
&|\uparrow\downarrow\rangle \,,\; m = 0, \\
&|\downarrow\uparrow\rangle \,,\; m = 0, \\
&|\downarrow\downarrow\rangle \,,\; m = -1,
\end{aligned}
$$

where $m = m_1 + m_2$. Since $m = 0, \pm 1$, we would expect $s = 1$, but we have one state too many!

Solution: Two different combinations for the total spin states $|sm\rangle$:

$$
\left.
\begin{aligned}
|1,1\rangle &= |\uparrow\uparrow\rangle \\
|1,0\rangle &= \tfrac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle] \\
|1,\text{-}1\rangle &= |\downarrow\downarrow\rangle
\end{aligned}
\right\} \; s = 1 \text{ (triplet)}
$$

$$
|0,0\rangle = \tfrac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle] \;\Big\}\; s = 0 \text{ (singlet)}
$$

The combination of two particles with spin ½ can carry total spin of **1 or 0**.

General addition of angular momenta

Adding two angular momenta described by

$$\begin{aligned} |j_1, m_1\rangle\,, &\quad m_1 = -j_1, -j_1+1, \ldots, j_1, \\ |j_2, m_2\rangle\,, &\quad m_2 = -j_2, -j_2+1, \ldots, j_2, \end{aligned}$$

the total quantum number j can take the values

$$j = j_1 + j_2,\; j_1 + j_2 - 1,\; \ldots,\; |j_1 - j_2|.$$

The “new” or total spin states are

$$|j, m\rangle = \sum_{m_1+m_2=m} C^{j_1 j_2 j}_{m_1 m_2 m}\, |j_1\, j_2, m_1\, m_2\rangle\,,$$

where $C^{j_1 j_2 j}_{m_1 m_2 m}$ are **Clebsch-Gordan coefficients**.

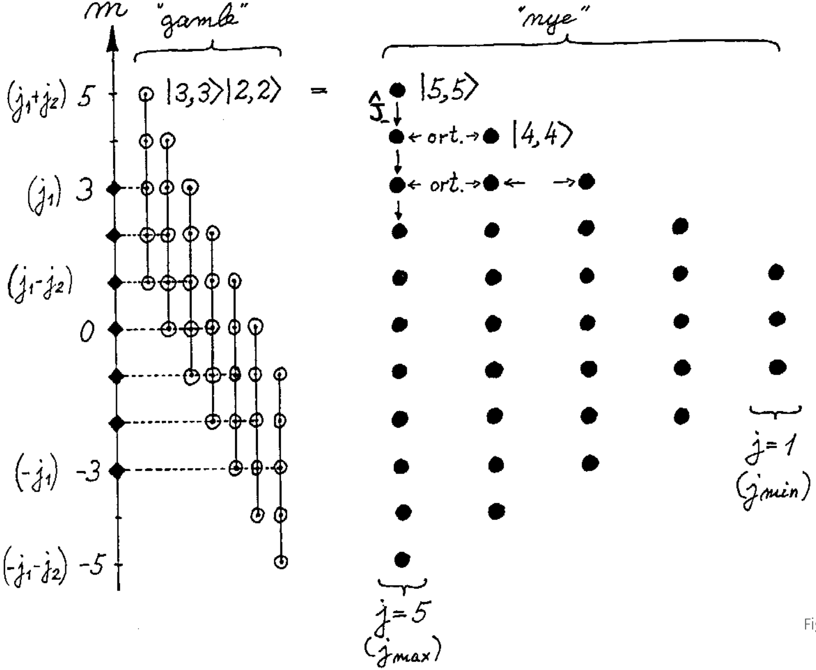


Figure by I. Øverbø.

General addition of angular momenta

Adding two angular momenta described by

$$|j_1, m_1\rangle\,, \quad m_1 = -j_1, -j_1+1, \ldots, j_1,$$
$$|j_2, m_2\rangle\,, \quad m_2 = -j_2, -j_2+1, \ldots, j_2,$$

the total quantum number j can take the values

$$j = j_1 + j_2,\, j_1 + j_2 - 1,\, \ldots,\, |j_1 - j_2|.$$

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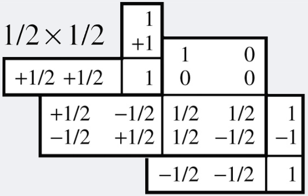
Two spin ½

$C^{j_1 j_2 j}_{m_1 m_2 m}$ given in table



General addition of angular momenta

Adding two angular momenta described by

$$|j_1, m_1\rangle\,, \quad m_1 = -j_1, -j_1+1, \dots, j_1,$$
$$|j_2, m_2\rangle\,, \quad m_2 = -j_2, -j_2+1, \dots, j_2,$$

the total quantum number j can take the values

$$j = j_1 + j_2,\ j_1 + j_2 - 1,\ \dots,\ |j_1 - j_2|.$$

The “new” or total spin states are

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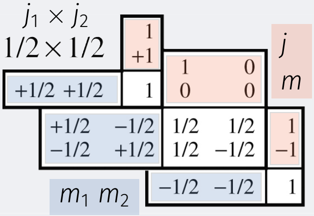


General addition of angular momenta

Adding two angular momenta described by

$$|j_1, m_1\rangle\,, \quad m_1 = -j_1, -j_1+1, \ldots, j_1,$$
$$|j_2, m_2\rangle\,, \quad m_2 = -j_2, -j_2+1, \ldots, j_2,$$

the total quantum number j can take the values

$$j = j_1 + j_2,\, j_1 + j_2 - 1,\, \ldots,\, |j_1 - j_2|.$$

The “new” or total spin states are

$$|j, m\rangle = \sum_{m_1+m_2=m} C^{j_1 j_2 j}_{m_1 m_2 m}\, |j_1\, j_2, m_1\, m_2\rangle\,,$$

where $C^{j_1 j_2 j}_{m_1 m_2 m}$ are **Clebsch-Gordan coefficients**.

Two spin ½

$C^{j_1 j_2 j}_{m_1 m_2 m}$ given in table

$j_1 \times j_2$

$1/2 \times 1/2$

$m_1\ m_2$ \ j, m	1, +1	1, 0	0, 0	1, −1
+1/2 +1/2	1			
+1/2 −1/2		1/2	1/2	
−1/2 +1/2		1/2	−1/2	
−1/2 −1/2				1

Examples:

$$|1,1\rangle = C^{\frac12\frac12 1}_{\frac12\frac12 1}\,|\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle$$
$$|0,0\rangle = C^{\frac12\frac12 0}_{\frac12 -\frac12 0}\,|\uparrow\downarrow\rangle + C^{\frac12\frac12 0}_{-\frac12 \frac12 0}\,|\downarrow\uparrow\rangle$$
$$= \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle].$$

1/2 × 1/2

	1 +1
+1/2 +1/2	1

	1 0	0 0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

	1 −1
−1/2 −1/2	1

1 × 1/2

	3/2 +3/2
+1 +1/2	1

	3/2 +1/2	1/2 +1/2
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3

	3/2 −1/2	1/2 −1/2
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3

	3/2 −3/2
−1 −1/2	1

2 × 1

	3 +3
+2 +1	1

	3 +2	2 +2
+2 0	1/3	2/3
+1 +1	2/3	−1/3

	3 +1	2 +1	1 +1
+2 −1	1/15	1/3	3/5
+1 0	8/15	1/6	−3/10
0 +1	6/15	−1/2	1/10

	3 0	2 0	1 0
+1 −1	1/5	1/2	3/10
0 0	3/5	0	−2/5
−1 +1	1/5	−1/2	3/10

	3 −1	2 −1	1 −1
0 −1	6/15	1/2	1/10
−1 0	8/15	−1/6	−3/10
−2 +1	1/15	−1/3	3/5

	3 −2	2 −2
−1 −1	2/3	1/3
−2 0	1/3	−2/3

	3 −3
−2 −1	1

1 × 1

	2 +2
+1 +1	1

	2 +1	1 +1
+1 0	1/2	1/2
0 +1	1/2	−1/2

	2 0	1 0	0 0
+1 −1	1/6	1/2	1/3
0 0	2/3	0	−1/3
−1 +1	1/6	−1/2	1/3

	2 −1	1 −1
0 −1	1/2	1/2
−1 0	1/2	−1/2

	2 −2
−1 −1	1

3/2 × 1

	5/2 +5/2
+3/2 +1	1

	5/2 +3/2	3/2 +3/2
+3/2 0	2/5	3/5
+1/2 +1	3/5	−2/5

	5/2 +1/2	3/2 +1/2	1/2 +1/2
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6

	5/2 −1/2	3/2 −1/2	1/2 −1/2
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2

	5/2 −3/2	3/2 −3/2
−1/2 −1	3/5	2/5
−3/2 0	2/5	−3/5

	5/2 −5/2
−3/2 −1	1

2 × 1/2

	5/2 +5/2
+2 1/2	1

	5/2 3/2	3/2 +3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

	5/2 +1/2	3/2 +1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

	5/2 −1/2	3/2 −1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

	5/2 −3/2	3/2 −3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

	5/2 −5/2
−2 −1/2	1

3/2 × 1/2

	2 +2
+3/2 +1/2	1

	2 +1	1 +1
+3/2 −1/2	1/4	3/4
+1/2 +1/2	3/4	−1/4

	2 0	1 0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

	2 −1	1 −1
−1/2 −1/2	3/4	1/4
−3/2 +1/2	1/4	−3/4

	2 −2
−3/2 −1/2	1

All Clebsch-Gordan coefficient tables from Griffiths and Schroeter, ©Cambridge University Press 2018. Background changed and clipped.

Example — Electron with orbital angular momentum and spin

The electron in a hydrogen atom can have both orbital and spin angular momentum, $\mathbf{J} = \mathbf{L} + \mathbf{S}$;

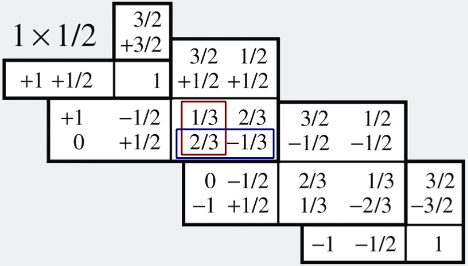
$$j = l - \frac{1}{2}, \text{ or } j = l + \frac{1}{2}, \text{ (assuming } l > 0).$$

For $l = 1$, we can construct e.g. the "new" state

$$\left|\frac{3}{2}, \frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}} \left|1\,\frac{1}{2}, 1 - \frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}} \left|1\,\frac{1}{2}, 0\,\frac{1}{2}\right\rangle,$$

or the "old" state

$$\left|1\,\frac{1}{2}, 0\,\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}} \left|\frac{3}{2}, \frac{1}{2}\right\rangle - \frac{1}{\sqrt{3}} \left|\frac{1}{2}, \frac{1}{2}\right\rangle.$$



Addition of several angular momenta

When adding several angular momenta

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3 + \dots,$$

we do the additions in a step-wise fashion:

1) The combination $\mathbf{J}_{12} = \mathbf{J}_1 + \mathbf{J}_2$ has possible quantum numbers $|j_1 - j_2| \leq j_{12} \leq j_1 + j_2$,
2) The combination $\mathbf{J}_{13} = \mathbf{J}_{12} + \mathbf{J}_3$ has possible quantum numbers $|j_{12} - j_3| \leq j_{12} \leq j_{12} + j_3$,
3) ...

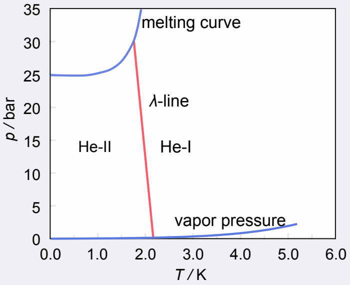
Example — Helium-4 vs. Helium-3

Helium-4

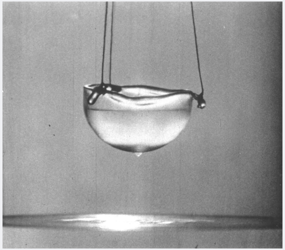
${}^4_2$He has two protons, **two** neutrons and two electrons.

Total spin $S = 0$ — this is a **boson**!

Can form Bose-Einstein condensate*, becomes a superfluid below $\sim 2\,\mathrm{K}$.



Wikipedia, CC BY-SA 3.0. Background added.



By Alfred Leitner - Own work, Public Domain.

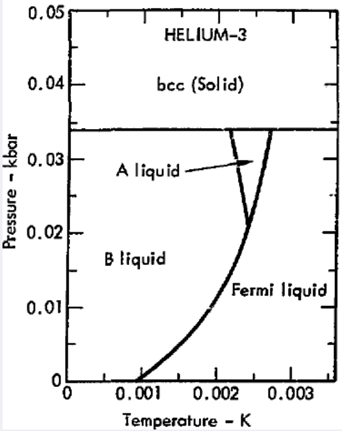
Example — Helium-4 vs. Helium-3

Helium-3

${}^{3}_{2}$He has two protons, **one** neutron and two electrons.

Total spin $S = \frac{1}{2}$ — this is a **fermion**!

A superfluid below $\sim 1\,\text{mK}$ — how?



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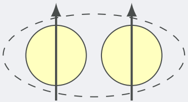
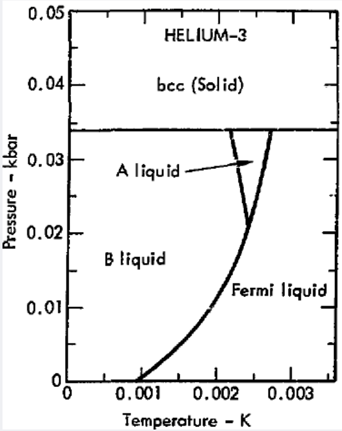
A superfluid below $\sim 1\,\mathrm{mK}$ — how?

Helium-3 atoms can form pairs — with integer spin!



Helium-3 pair



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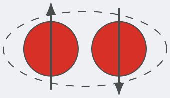
For more information see e.g. Nobel Focus: Helium Impersonates a Superconductor.

Electrons can also form pairs!



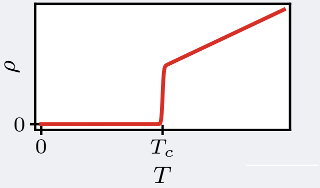
Electron pair

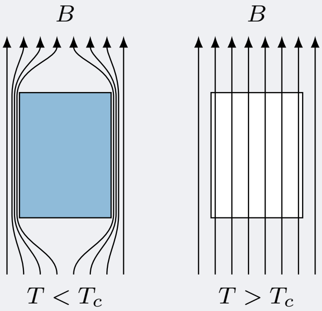
(spin singlet)

If they condense we get a charged superfluid — a **superconductor**.

Below a certain temperature T_c

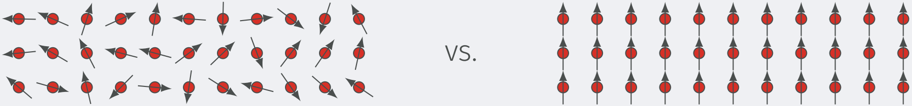
- currents flow without resistance ($\rho = 0$),
- magnetic fields are expelled.



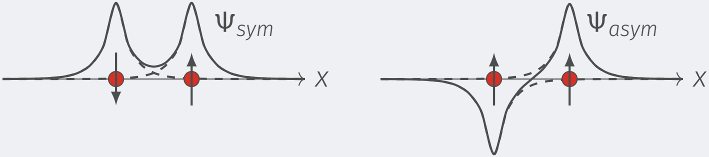


Since electrons have magnetic moments due to spin, they behave like tiny magnets.

If many spins align spontaneously in the same direction we have a **ferromagnet**!



Why do they sometimes align in this way? Due to **exchange interactions**:



The antisymmetric configuration reduces the **electrostatic energy** of the electrons — hence parallel spins are favored.

Tangent #3 — Quantum information

In traditional computers the building blocks are **bits** that can be either 1 or 0.

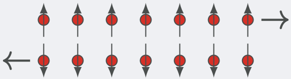
In quantum computers one wants to use quantum bits (**qubits**) that can be in states $|1\rangle$ or $|0\rangle$ — or a **superposition** of them.

Spin qubit

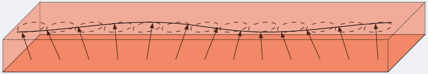
Use spins as bits?

$$|\uparrow\rangle = |1\rangle\,,$$
$$|\downarrow\rangle = |0\rangle\,.$$

Spin currents

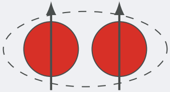


or



Spin supercurrents

Generate triplet electron pairs using e.g. magnets



Electron pair