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FY2045 Solutions Problem set 5 fall 2023

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Problem 1 — Coherent states

a) Requiring normalization, we get

$$\begin{aligned}\langle\alpha|\alpha\rangle &= \left(\sum_{n'=0}^{\infty}\langle n'|\frac{(\alpha^*)^{n'}}{\sqrt{n'!}}c_0^*\right)\left(c_0\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle\right) = |c_0|^2\sum_{nn'}\frac{(\alpha^*)^{n'}\alpha^n}{\sqrt{n'!n!}}\langle n'|n\rangle \\ &= |c_0|^2\sum_{nn'}\frac{(\alpha^*)^{n'}\alpha^n}{\sqrt{n'!n!}}\delta_{nn'} = |c_0|^2\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!} = |c_0|^2 e^{|\alpha|^2},\end{aligned}$$

where we in the last line have used the power series definition of the exponential function. Choosing c_0 real and positive, we get

$$c_0 = e^{-\frac{1}{2}|\alpha|^2}.$$

b) From the definition of the ladder operators a and $a^\dagger$, we get

$$\begin{aligned}\hat{q} &= \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a), \\ \hat{p} &= i\sqrt{\frac{m\hbar\omega}{2}}(a^\dagger - a).\end{aligned}$$

For $\langle q \rangle$ we get,

$$\begin{aligned}\langle q\rangle &= \langle\alpha|\,\hat{q}\,|\alpha\rangle = \sqrt{\frac{\hbar}{2m\omega}}[(\langle\alpha|\,a^\dagger)\,|\alpha\rangle + \langle\alpha|\,(a\,|\alpha\rangle)] = \sqrt{\frac{\hbar}{2m\omega}}[e^{i\omega t}\alpha^* + e^{-i\omega t}\alpha] \\ &= \sqrt{\frac{\hbar}{2m\omega}}|\alpha|\left[e^{i\omega t-i\Theta} + e^{-i\omega t+i\Theta}\right] = \sqrt{\frac{2\hbar}{m\omega}}|\alpha|\cos(\omega t-\Theta),\end{aligned}$$

as given in the lectures. Following the same procedure, we get

$$\langle p\rangle = \langle\alpha|\,\hat{p}\,|\alpha\rangle = i\sqrt{\frac{m\hbar\omega}{2}}[(\langle\alpha|\,a^\dagger)\,|\alpha\rangle - \langle\alpha|\,(a\,|\alpha\rangle)] = -\sqrt{2m\hbar\omega}|\alpha|\sin(\omega t-\Theta).$$

This in agreement with Ehrenfest's theorem:

$$m\frac{d}{dt}\langle q\rangle = -\sqrt{2\hbar\omega m}|\alpha|\sin(\omega t-\Theta) = \langle p\rangle\,.$$

For the other two expectation values, we get

$$\begin{aligned}\left\langle q^2\right\rangle &= \frac{\hbar}{2m\omega}\langle\alpha|\,a^\dagger a^\dagger + a^\dagger a + aa^\dagger + aa\,|\alpha\rangle = \frac{\hbar}{2m\omega}\langle\alpha|\,a^\dagger a^\dagger + 2a^\dagger a + 1 + aa\,|\alpha\rangle \\ &= \frac{\hbar}{2m\omega}|\alpha|^2\left[e^{2i(\omega t-\Theta)} + 2 + e^{-2i(\omega t-\Theta)}\right] + \frac{\hbar}{2m\omega} \\ &= \frac{\hbar}{2m\omega}|\alpha|^2\left[e^{i(\omega t-\Theta} + e^{-i(\omega t-\Theta)}\right]^2 + \frac{\hbar}{2m\omega} \\ &= \frac{2\hbar}{m\omega}\cos^2(\omega t-\Theta) + \frac{\hbar}{2m\omega} = \langle q\rangle^2 + \frac{\hbar}{2m\omega},\end{aligned}$$

and

$$\begin{aligned}\left\langle p^2\right\rangle &= -\frac{m\hbar\omega}{2}\langle\alpha|\,a^\dagger a^\dagger - a^\dagger a - aa^\dagger + aa\,|\alpha\rangle = -\frac{m\hbar\omega}{2}\langle\alpha|\,a^\dagger a^\dagger - 2a^\dagger a - 1 + aa\,|\alpha\rangle \\ &= -\frac{m\hbar\omega}{2}|\alpha|^2\left[e^{2i(\omega t-\Theta)} - 2 + e^{-2i(\omega t-\Theta)}\right] + \frac{m\hbar\omega}{2} \\ &= -\frac{m\hbar\omega}{2}|\alpha|^2\left[e^{i(\omega t-\Theta} - e^{-i(\omega t-\Theta)}\right]^2 + \frac{m\hbar\omega}{2} \\ &= 2m\hbar\omega\sin^2(\omega t-\Theta) + \frac{m\hbar\omega}{2} = \langle p\rangle^2 + \frac{m\hbar\omega}{2},\end{aligned}$$

where in both cases we have used the commutator $[a, a^\dagger] = 1$ to reorder that operators.

c) Using the above results, we get

$$(\Delta q)^2 = \left\langle(q-\langle q\rangle)^2\right\rangle = \left\langle q^2\right\rangle - \langle q\rangle^2 = \frac{\hbar}{2m\omega},$$

and

$$(\Delta p)^2 = \left\langle(p-\langle p\rangle)^2\right\rangle = \left\langle p^2\right\rangle - \langle p\rangle^2 = \frac{m\hbar\omega}{2}.$$

Hence, we get

$$\Delta q \Delta p = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{m\hbar\omega}{2}} = \frac{\hbar}{2},$$

the coherent states are minimal uncertainty states.

d) Inserting the above results into the given expression, we get

$$|\langle q|\alpha\rangle|^2 \propto \exp\left[-\frac{m\omega(q - q_0\cos(\omega t - \Theta))^2}{\hbar}\right],$$

where we have defined $q_0 = |\alpha|\sqrt{\frac{2\hbar}{m\omega}}$. This is in agreement with the expression given in the lectures.

Problem 2 — The Levi-Cevita symbol and Pauli matrices

a) Writing out the vector product, we get

$$\begin{aligned}
\mathbf{A}\times\mathbf{B} &= \hat{e}_x(A_yB_z - A_zBy) - \hat{e}_y(A_xB_z - A_zB_x) + \hat{e}_z(A_xB_y - A_yB_x) \\
&= \hat{e}_x(\epsilon_{xyz}A_yB_z + \epsilon_{xzy}A_zBy) + \hat{e}_y(\epsilon_{yxz}A_xB_z + \epsilon_{yzx}A_zB_x) + \hat{e}_z(\epsilon_{zxy}A_xB_y + \epsilon_{zyx}A_yB_x) \\
&= \hat{e}_x\epsilon_{xjk}A_jB_k + \hat{e}_y\epsilon_{yjk}A_jB_k + \hat{e}_z\epsilon_{zjk}A_jB_k \\
&= \epsilon_{ijk}\hat{e}_iA_jB_k,
\end{aligned}$$

where the last two steps utilize the fact that ϵ_{ijk} is zero if any two indexes are the same to write it as a sum over the indexes i, j, k. Reversing the order of the vectors $\mathbf{A}$ and $\mathbf{B}$ we get

$$\mathbf{B}\times\mathbf{A} = \epsilon_{ijk}\hat{e}_iB_jA_k = -\epsilon_{ikj}\hat{e}_iA_kB_j = -\epsilon_{ijk}\hat{e}_iA_jB_k - \mathbf{A}\times\mathbf{B}.$$

In the penultimate step we have rename the indexes $j \leftrightarrow k$, which we are free to do since we sum over all three directions for all indexes.

b) We calculate the six matrix products between different matrices:

$$\begin{aligned}
\sigma_x\sigma_y &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} = i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = i\sigma_z, \\
\sigma_y\sigma_x &= \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix} = -i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = -i\sigma_z,
\end{aligned}$$

$$\begin{aligned}
\sigma_y\sigma_z &= \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}0 & i\\ i & 0\end{pmatrix} = i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = i\sigma_x, \\
\sigma_z\sigma_y &= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix} = -i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = -i\sigma_x,
\end{aligned}$$

$$\sigma_z\sigma_x = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = i\sigma_y,$$
$$\sigma_x\sigma_y = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = -i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = -i\sigma_y.$$

Combining these into commutators, we get

$$\begin{aligned}
[\sigma_x, \sigma_y] &= i\sigma_z - (-i\sigma_z) = 2i\sigma_z,\\
[\sigma_y, \sigma_z] &= i\sigma_x - (-i\sigma_x) = 2i\sigma_x,\\
[\sigma_z, \sigma_x] &= i\sigma_y - (-i\sigma_y) = 2i\sigma_y,
\end{aligned}$$

and if the matrices in the commutator are the same, we should get zero,

$$[\sigma_i, \sigma_i] = 0, \forall\, i.$$

Using the Levi-Cevita symbol, we can write this as

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k.$$

c) For products between equal matrices, we get

$$\sigma_x\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = I = \sigma_y\sigma_y = \sigma_z\sigma_z.$$

From the matrix products performed between different Pauli matrices above, we notice that $\sigma_i\sigma_j = -\sigma_j\sigma_i$ if $i \neq j$. Hence, for the anticommutator we get

$$\begin{aligned}
\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i &= \begin{cases}\sigma_i\sigma_j - \sigma_i\sigma_j, & \text{if } i \neq j,\\ I + I, & \text{if } i = j,\end{cases}\\
&= 2I\delta_{ij}.
\end{aligned}$$

d) Using the commutation and anticommutation relations, we get

$$\sigma_i\sigma_j = \frac{\sigma_i\sigma_j - \sigma_j\sigma_i + \sigma_i\sigma_j + \sigma_j\sigma_i}{2} = \frac{[\sigma_i, \sigma_j] + \{\sigma_i, \sigma_j\}}{2} = I\delta_{ij} + i\epsilon_{ijk}\sigma_k.$$

The same conclusion could of course be reached by considering the different matrix products directly.

Problem 3 — Spin $\frac{1}{2}$

a) Normalization condition gives

$$1 = \chi^\dagger\chi = |A|^2\begin{pmatrix}1 + 2i & 2\end{pmatrix}\begin{pmatrix}1 - 2i\\ 2\end{pmatrix} = 9|A|^2.$$

Choosing A real and positive, we get

$$\underline{A = \frac{1}{3}}.$$

b) You could get $\hbar/2$ or $-\hbar/2$. The eigenvectors of S_z are

$$\chi_{z+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ and } \chi_{z-} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence, the probabilities are given by

$$P_{z+} = |\chi_{z+}^\dagger \chi|^2 = \left| \frac{1-2i}{3} \right|^2 = \frac{5}{9},$$

and

$$P_{z-} = |\chi_{z-}^\dagger \chi|^2 = \left| \frac{2}{3} \right|^2 = \frac{4}{9},$$

respectively. The expectation value is given by

$$\chi^\dagger S_z \chi = \frac{\hbar}{2}(P_{z+} - P_{z-}) = \frac{\hbar}{18}.$$

c) Again you could get $\hbar/2$ or $-\hbar/2$. The eigenvectors of S_y are

$$\chi_{y+} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \text{ and } \chi_{y-} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix},$$

resulting in the probabilities

$$P_{y+} = |\chi_{y+}^\dagger \chi|^2 = \frac{|1-2i-2i|^2}{18} = \frac{17}{18},$$

and

$$P_{y-} = |\chi_{y-}^\dagger \chi|^2 = \frac{|1-2i+2i|^2}{18} = \frac{1}{18},$$

respectively. The expectation value is given by

$$\chi^\dagger S_y \chi = \frac{\hbar}{2}(P_{y+} - P_{y-}) = \frac{8\hbar}{18}.$$

d) In order to find the spin direction, we need the expectation values of the Pauli matrices. From the two preceding questions we have

$$\langle \sigma_z \rangle = \frac{2}{\hbar}\langle S_z \rangle = \frac{1}{9},$$
$$\langle \sigma_y \rangle = \frac{2}{\hbar}\langle S_y \rangle = \frac{8}{9},$$

meaning we only need to calculate

$$\langle \sigma_x \rangle = \chi^\dagger \sigma_x \chi = \frac{1}{9} \begin{pmatrix} 1+2i & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1-2i \\ 2 \end{pmatrix} = \frac{4}{9}.$$

Hence, the spin direction is

$$\langle \boldsymbol{\sigma} \rangle = \frac{1}{9}[4\hat{x} + 8\hat{y} + \hat{z}].$$

e) The length of $\langle \boldsymbol{\sigma} \rangle$ is equal to 1,

$$\langle \boldsymbol{\sigma} \rangle \cdot \langle \boldsymbol{\sigma} \rangle = \frac{1 + 64 + 16}{81} = 1,$$

and therefore is a unit vector. Thus we have $\mathbf{n} = \langle \boldsymbol{\sigma} \rangle$. Hence, we have

$$\mathbf{n} \cdot \mathbf{S} = \frac{\hbar}{18} \begin{pmatrix} 1 & 4 - 8i \\ 4 + 8i & -1 \end{pmatrix},$$

and when operating on the spinor χ we get

$$\mathbf{n} \cdot \mathbf{S} \chi = \frac{\hbar}{3 \cdot 18} \begin{pmatrix} 1 & 4 - 8i \\ 4 + 8i & -1 \end{pmatrix} \begin{pmatrix} 1 - 2i \\ 2 \end{pmatrix} = \frac{\hbar}{3 \cdot 18} \begin{pmatrix} 9 - 18i \\ 18 \end{pmatrix} = \frac{\hbar}{2} \chi.$$

Hence, χ is an eigenvector of $\mathbf{n} \cdot \mathbf{S}$ with eigenvalue $\hbar/2$.