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## FY2045 Problem set 9 fall 2023

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### Problem 1

Find an approximate solution of the equation

$$x^3 - 4x + \lambda = 0, \quad (1)$$

where  $\lambda$  is assumed small by expanding

$$x = x_0 + \lambda x_1 + \lambda^2 x_2 + \dots, \quad (2)$$

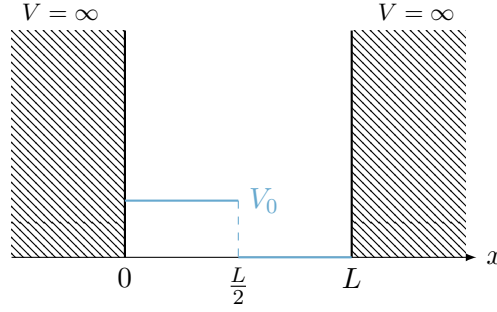
and inserting into the equation. Find the roots to second order in  $\lambda$ . For  $\lambda = 0.1$  you can compare your results to the approximate roots  $x \approx 0.0250$ ,  $x \approx 1.9874$  and  $x \approx -2.0124$ .

### Problem 2

In the lectures we considered a perturbation  $\delta V$  to the infinite square well potential

$$V(x) \rightarrow V(x) + \lambda \delta V \equiv V(x) + \lambda V_0 \Theta\left(\frac{L}{2} - x\right), \quad (3)$$

where  $\Theta(x)$  is the Heaviside step function, as illustrated below.



Without the perturbation ( $\lambda = 0$ ), the eigenenergies and eigenfunctions are

$$E_n^0 = \frac{\hbar^2 \pi^2 n^2}{2mL^2} \equiv E_1^0 n^2, \quad (4)$$

$$\psi_n^0(x) = \sqrt{\frac{2}{L}} \sin \frac{\pi n x}{L}, \quad (5)$$

with  $n = 1, 2, 3, \dots$ . In the lectures, we showed that with the perturbation ( $\lambda > 0$ ) the first order correction to the energies was  $E_n^{(1)} = \frac{V_0}{2}$  for all states.

**a)** The second order corrections to the energies are given by

$$E_n^{(2)} = \sum_{m \neq n} \frac{|\langle m | \delta V | n \rangle|^2}{E_n^0 - E_m^0}. \quad (6)$$

Show that the *matrix elements* can be written

$$\langle m | \delta V | n \rangle = \frac{2V_0}{L} \int_0^{\frac{L}{2}} dx \sin \frac{\pi m x}{L} \cdot \sin \frac{\pi n x}{L} \quad (7)$$

in the position representation.

**b)** Solve the integral, and show that it results in

$$\langle m | \delta V | n \rangle = \frac{V_0}{\pi} \left[ \frac{\sin \frac{\pi(m-n)}{2}}{m-n} - \frac{\sin \frac{\pi(m+n)}{2}}{m+n} \right]. \quad (8)$$

**c)** For what  $m$  and  $n$  are the matrix elements zero? Why do you think this is the case?

**d)** Assume that  $n = 1$  and write down the non-zero matrix elements with  $m \leq 10$ , and compare the contribution to the sum in eq. (6) from the term with  $m = 2$  and  $m = 10$ . Does it seem necessary to include *all* terms in the sum to get a good approximation for  $E_n^{(2)}$ ?

e) It's quite straightforward numerically determine the (approximate) eigenenergies and wavefunctions of the perturbed system. Using [this example code](#) (or [this one](#)) from [www.numfys.net](#), determine the ground state energy as a function of the parameter  $V_0/E_1^0$  (with  $\lambda = 1$ ) and compare it with the result from perturbation theory to second order in  $\lambda$  including terms with  $m \leq 10$  in the sum in eq. (6).

f) Calculate and plot the ground state wavefunction, and compare it to the approximate wavefunction from first-order perturbation theory,

$$\psi_n(x) \approx \psi_n^0(x) + \lambda \sum_{m \neq n} \frac{\langle m | \delta V | n \rangle}{E_n^0 - E_m^0} \psi_m^0(x). \quad (9)$$

Does the change to the ground state wavefunction seem reasonable?

g) What is the ground state energy in the limit  $V_0 \rightarrow \infty$ ?