

Lecture notes FY2045 Quantum Mechanics I

— 3 General formulation of Quantum Mechanics

Henning G. Hugdal

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1 Motivation

So far we have worked in the position representation, where a system is described by the wavefunction $\Psi(\mathbf{r}, t)$. We also briefly looked at the function $\Phi(\mathbf{p}, t)$ which appeared when expanding Ψ in terms of momentum eigenfunctions. Analogous to $|\Psi|^2$, the interpretation of $|\Phi|^2$ is as a probability density in *momentum space*. Moreover, it is possible to formulate quantum mechanics in momentum space, where Φ is the *momentum space wavefunction*.

In fact, there are multiple ways of formulating a quantum theory. This lead Dirac to suggest and introduce a notation which could help with the development of the theory, and provide a unified way of treating different quantities.¹ We will now turn to this more general description, known as Dirac's bra-ket notation, or simply Dirac notation.

2 Dirac's bra-ket notation

In Dirac notation, a quantum mechanical state of a system is described by a state vector $|\Psi\rangle$ in a complex, linear vector space \mathcal{H} called Hilbert space. The dimension of the vector space can be finite or infinite. $|\Psi\rangle$ is a function of time — $|\Psi\rangle = |\Psi(t)\rangle$.

For each vector $|a\rangle$, we define the dual vector $\langle a|$ in the dual space \mathcal{H}^* ,

$$|a\rangle \xleftrightarrow{\text{dual}} \langle a|,$$

so that we can define the *scalar product* (or inner product) of vectors $|a\rangle$ and $|b\rangle$ ²

$$\langle a||b\rangle \equiv \langle a|b\rangle. \quad (1)$$

This is a complex number,³ and has the property

$$\langle a|b\rangle = \langle b|a\rangle^*, \quad (2)$$

where $*$ denotes complex conjugation.⁴ Moreover,

$$c|a\rangle \xleftrightarrow{\text{dual}} c^* \langle a|, \quad c \in \mathbb{C}. \quad (3)$$

We call $|\cdot\rangle$ a *ket*, and $\langle\cdot|$ a *bra*, as in $\langle\text{bra}|\text{ket}\rangle$ — from *bracket*.

3 Interpretation

How should we think of these abstract vectors? Let's consider a double slit experiment with electrons.

See also Compendium 10.1 by Øverbø or Ch. 6.1 in Hemmer.

¹ “In mathematical theories the question of notation, while not of primary importance, is yet worthy of careful consideration, since a good notation can be of great value in helping the development of a theory, by making it easy to write down those quantities or combinations of quantities that are important, and difficult or impossible to write down those that are unimportant.” — Dirac (1939).

² For abstract vector spaces it is common to use the notation $\langle\cdot,\cdot\rangle$ for the inner product rather than the dot operator \cdot . We can therefore write

$$\langle|a\rangle,|b\rangle\rangle = \langle a|b\rangle,$$

which illustrates both why Dirac named the vectors as he did, and why the Dirac notation is so convenient.

³ See Renou *et al.* (2023, 2021) for a discussion of whether quantum mechanics has to be formulated using complex numbers and how to test this experimentally, and Avella (2022) for a synopsis of recent experimental results refuting so-called *real quantum mechanics*.

⁴ This reminds us of

$$\int dx \psi_1^*(x) \psi_2(x) = \left[\int dx \psi_2^*(x) \psi_1(x) \right]^*.$$

3.1 Example — Electron double slit experiment

We want to describe the probability amplitude⁵ of an electron leaving the source at s , and arriving at the detector at position x , see fig. 1. With our new notation, we can write this as

$$\underbrace{\langle \text{Particle arrives at } x |}_{\text{Final condition}} \underbrace{|\text{Particle leaves } s \rangle}_{\text{Initial condition}} \equiv \langle x | s \rangle.$$

However, this can happen in two ways:

$$\langle x | s \rangle = \langle x | s \rangle_{\text{through 1}} + \langle x | s \rangle_{\text{through 2}}.$$

The amplitude $\langle x | s \rangle_{\text{through 1}}$ is the product of amplitude from s to 1, $\langle 1 | s \rangle$, and from 1 to x , $\langle x | 1 \rangle$:

$$\langle x | s \rangle_{\text{through 1}} = \langle x | 1 \rangle \langle 1 | s \rangle,$$

and similarly for from s to x through 2. Hence, we can write

$$\begin{aligned} \langle x | s \rangle &= \langle x | 1 \rangle \langle 1 | s \rangle + \langle x | 2 \rangle \langle 2 | s \rangle = \sum_{\text{slit}=1,2} \langle x | \text{slit} \rangle \langle \text{slit} | s \rangle \\ &= \langle x | \underbrace{\left[\sum_{\text{slit}=1,2} |\text{slit} \rangle \langle \text{slit}| \right]}_{\equiv 1} | s \rangle. \end{aligned}$$

We have found a *completeness relation*,

$$\sum_{\text{slit}=1,2} |\text{slit} \rangle \langle \text{slit}| = 1.$$

3.2 Relation to wavefunction

How can we then understand a state $|\psi\rangle$ and relate it to the wavefunction we are used to? The wavefunction gives the probability amplitude of finding a particle at position x . With the new notation

$$\langle \text{Particle is at } x | \text{Particle is in state } \psi \rangle$$

or simply

$$\langle x | \psi \rangle = \psi(x). \quad (4)$$

This is exactly our wavefunction! For instance, the wavefunction for an energy eigenstate with quantum number n is

$$\langle x | \psi_n \rangle = \psi_n(x).$$

If $|x\rangle$ is the state were a particle is at position x , and $|\psi_n\rangle$ is an energy eigenstate with energy E_n , $\langle x | \psi_n \rangle$ is the probability amplitude of finding the particle in eigenstate $|\psi_n\rangle$ at position x . Hence:

This section is based on Ch. 3 of Vol. III in the Feynman Lectures, Feynman *et al.* (1963)

⁵ The probability is the absolute square of the probability amplitude.

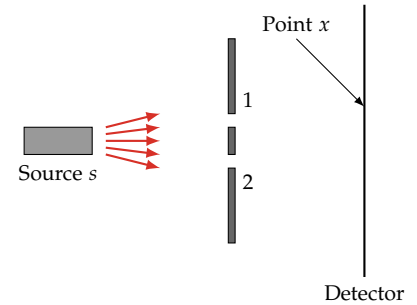


Figure 1: Electrons are sent from the source at s towards the two slits (1 and 2), and are detected by the detector at some position x .

The wavefunction

$$\psi(x) = \langle x | \psi \rangle$$

is the “component of the Hilbert space vector $|\psi\rangle$ in the $|x\rangle$ -direction”, or the projection of $|\psi\rangle$ on the basis vector $|x\rangle$.

There is, therefore, a one-to-one correspondence between wavefunction and state vector,

$$\begin{aligned}\psi(x) &\Leftrightarrow |\psi\rangle, \\ \psi_n(x) &\Leftrightarrow |\psi_n\rangle = |n\rangle = |E_n\rangle, \\ \psi_{x'}(x) &\Leftrightarrow |x'\rangle,\end{aligned}$$

etc. However, they are *different objects*, belonging to different vector spaces.

Other examples are

- $\langle p | \psi \rangle$ — projection on momentum basis vector,
 - $\langle \psi_n | \psi \rangle$ — projection on energy basis vector,
- etc. This requires that the sets of vectors are complete!

4 Completeness

Before looking at completeness for the state vectors in Hilbert space, let's take a look at the analogous case of ordinary vectors in 3D space.

4.1 Analogy — vectors in \mathbb{R}^3

An ordinary vector is often written down in terms of components in a Cartesian coordinate system,

$$\mathbf{A} = A_x \hat{\mathbf{e}}_x + A_y \hat{\mathbf{e}}_y + A_z \hat{\mathbf{e}}_z = \sum_i A_i \hat{\mathbf{e}}_i. \quad (5)$$

The components, A_i , depends on our choice of orthonormal basis vectors $\hat{\mathbf{e}}_i$. However, the vector \mathbf{A} itself is *independent of basis*! This means that we could write it in terms of other basis vectors, for instance in spherical coordinates,

$$\mathbf{A} = A_r \hat{\mathbf{e}}_r + A_\theta \hat{\mathbf{e}}_\theta + A_\phi \hat{\mathbf{e}}_\phi = \sum_i A'_i \hat{\mathbf{e}}'_i.$$

Even though, generally, the components $\hat{A}_i \neq \hat{A}'_i$, the vector \mathbf{A} is still the same. We can relate the basis vectors to the components using the scalar products:

$$\hat{\mathbf{e}}_i \cdot \mathbf{A} = \hat{\mathbf{e}}_i \cdot \sum_j A_j \hat{\mathbf{e}}_j = \sum_j A_j \underbrace{\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j}_{\delta_{ij}} = A_i.$$

Inserting this into the decomposition of \mathbf{A} [eq. (5)], we get

$$\mathbf{A} = \sum_i \hat{\mathbf{e}}_i A_i = \sum_i \hat{\mathbf{e}}_i (\hat{\mathbf{e}}_i \cdot \mathbf{A}) = \underbrace{\left[\sum_i (\hat{\mathbf{e}}_i \hat{\mathbf{e}}_i \cdot) \right]}_{=1} \mathbf{A}. \quad (6)$$

Here,⁶

$$1 = \sum_i (\hat{\mathbf{e}}_i) \hat{\mathbf{e}}_i.$$

is a *completeness relation*,⁷ which can be used to express a vector in terms of different sets of complete basis vectors.

4.2 Completeness relations for Hilbert space vectors

In an n -dimensional vector space, we can choose n linearly independent basis vectors $|1\rangle, |2\rangle, |3\rangle, \dots$, and write

$$|\psi\rangle = \sum_{k=1}^n c_k |k\rangle, \quad c_k \in \mathbb{C}. \quad (7)$$

We assume the basis vectors are orthogonal and with length 1;

$$\langle k|m\rangle = \delta_{km}, \quad (8)$$

such that

$$\langle m|\psi\rangle = \sum_{k=1}^n c_k \langle m|k\rangle = c_m. \quad (9)$$

Inserting this into the expansion of $|\psi\rangle$ in eq. (7), we get

$$|\psi\rangle = \sum_{k=1}^n \underbrace{\langle k|\psi\rangle}_{\in \mathbb{C}} |k\rangle = \sum_{k=1}^n |k\rangle \underbrace{\langle k|\psi\rangle}_{=1} = \left[\sum_{k=1}^n |k\rangle \langle k| \right] |\psi\rangle.$$

Hence, we arrive at the *completeness relation*

$$\sum_{k=1}^n |k\rangle \langle k| = 1 \quad (\text{discrete case}). \quad (10)$$

Notice that the order of the bra and ket matters:

$\langle a b\rangle$	— inner product, a complex number,
$ a\rangle\langle b $	— outer product, an <i>operator</i> .

If basis vectors form an uncountable set, instead of $\langle k'|k\rangle = \delta_{k'k}$, we use⁸

$$\langle k'|k\rangle = \delta(k' - k) \quad (11)$$

and get the completeness relation

$$\int dk |k\rangle \langle k| = 1 \quad (\text{continuous case}). \quad (12)$$

An arbitrary vector $|\Psi\rangle$ may therefore be expanded in terms of different basis sets, for instance

$$\begin{aligned} |\Psi\rangle &= \sum_n c_n |\psi_n\rangle = \sum_n \langle \psi_n | \Psi \rangle |\psi_n\rangle, \\ |\Psi\rangle &= \int dp \Phi(p) |p\rangle = \int dp \langle p | \Psi \rangle |p\rangle, \\ |\Psi\rangle &= \int dx' c(x') |x'\rangle = \int dx' \langle x' | \Psi \rangle |x'\rangle. \end{aligned}$$

⁶ It's more correct to write this as

$$1 = \sum_i \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_i,$$

where \otimes denotes the *outer product*.

⁷ In Cartesian coordinates it reads

$$1 = (\hat{\mathbf{e}}_x) \hat{\mathbf{e}}_x + (\hat{\mathbf{e}}_y) \hat{\mathbf{e}}_y + (\hat{\mathbf{e}}_z) \hat{\mathbf{e}}_z,$$

such that

$$\begin{aligned} \mathbf{A} &= 1 \mathbf{A} \\ &= [(\hat{\mathbf{e}}_x) \hat{\mathbf{e}}_x + (\hat{\mathbf{e}}_y) \hat{\mathbf{e}}_y + (\hat{\mathbf{e}}_z) \hat{\mathbf{e}}_z] \mathbf{A} \\ &= A_x \hat{\mathbf{e}}_x + A_y \hat{\mathbf{e}}_y + A_z \hat{\mathbf{e}}_z, \end{aligned}$$

as expected.

⁸ Notice that this is similar to the δ -function normalization introduced previously for functions with continuous eigenvalues.

5 What is Hilbert space?

In quantum mechanics we refer to the Hilbert \mathcal{H} space as the set of *square-integrable functions* on a specified interval

See Griffiths pp. 92-93.

$$f(x) \text{ such that } \int_a^b dx |f(x)|^2 < \infty, \quad (13)$$

with an inner product of two functions in Hilbert space

$$\langle f|g \rangle = \int_a^b dx f^*(x)g(x). \quad (14)$$

Mathematicians call this space $L^2(a, b)$.

If $f, g \in \mathcal{H}$, the existence of the inner product is guaranteed by the Schwartz inequality

$$|\langle f|g \rangle| \leq \sqrt{\langle f|f \rangle} \sqrt{\langle g|g \rangle},$$

with $\langle f|f \rangle \geq 0$ and real. f is normalized if $\langle f|f \rangle = 1$. A set of functions $\{f_n\}$ is orthonormal if

$$\langle f_n|f_m \rangle = \delta_{nm}. \quad (15)$$

Completeness: For all $f \in \mathcal{H}$, we can write

$$f(x) = \sum_n c_n f_n(x) \longleftrightarrow |f\rangle = \sum_n c_n |f_n\rangle,$$

where

$$c_n = \langle f_n|f \rangle = \int dx f_n^*(x)f(x).$$

Eigenfunctions of continuous eigenvalues The position and momentum space vectors $|x\rangle$ and $|p\rangle$ **do not** satisfy the orthonormality condition in eq. (15), but are rather δ -function normalized

$$\langle x'|x \rangle = \delta(x' - x).$$

This means that the eigenfunctions of \hat{x} and \hat{p} (with real eigenvalues) **do not** live in Hilbert space, but in the nearby “suburbs”⁹, called a *rigged Hilbert space*. These states do not represent possible physical states, but “are still very useful”.⁹

⁹ See Griffiths and Schroeter (2018) p. 100.

6 Operators

An operator \hat{A} is defined as a map

$$\hat{A} : \mathcal{H} \rightarrow \mathcal{H}, \quad \hat{A}|a\rangle = |c\rangle,$$

that is,

$$\text{operator} \cdot \text{vector} = \text{new vector}.$$

The *adjoint* or *Hermitian conjugate* operator \hat{A}^\dagger of \hat{A} is defined by

$$\underbrace{\langle a|\hat{A}^\dagger|b \rangle}_{\in \mathbb{C}} = \langle b|\hat{A}|a \rangle^* \quad \forall |a\rangle, |b\rangle \in \mathcal{H}. \quad (16)$$

The dual vector of $\hat{A}|a\rangle$ is $\langle a|\hat{A}^\dagger$.

The operators have the following properties

See Compendium 10.2 by Øverbø, Hemmer Ch. 6.3, and Griffiths Ch. 3.3.

1. $(\hat{A}^\dagger)^\dagger = \hat{A}$
2. $(\alpha\hat{A})^\dagger = \alpha^* \hat{A}$
3. $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger \hat{A}^\dagger$
4. Hermitian or self-adjoint operator: $\hat{A}^\dagger = \hat{A} \Rightarrow \langle a|\hat{A}|a\rangle \in \mathbb{R}$.

6.1 Eigenvectors and eigenvalues

The vector $|a\rangle$ is an eigenvector of \hat{A} if

$$\hat{A}|a\rangle = a|a\rangle, \quad a \in \mathbb{C}, \quad (17)$$

with eigenvalue a . The collection of eigenvalues of \hat{A} is called the *spectrum* of the operator. The eigenvalues of a Hermitian operator are *real*.

The set of eigenvectors $\{|a\rangle\}$ for an operator corresponding to a physical quantity is assumed to be complete, and can be used as basis vectors.

6.2 Spectral representation

If we have an eigenvector $|a\rangle$ of an operator \hat{A} with eigenvalues a , see eq. (17), we can use this to find a well-defined expression for the operator \hat{A} for any vector $|\psi\rangle$:

$$\hat{A}|\psi\rangle = \int da \hat{A}|a\rangle\langle a|\psi\rangle = \int da a|a\rangle\langle a|\psi\rangle,$$

where we have inserted a completeness relation after the first equality and used the notation for an uncountable set. Since $|\psi\rangle$ is *any* vector, we find the spectral decomposition of the operator

$$\begin{aligned} \hat{A} &= \int da |a\rangle a \langle a|, & (\text{continuous eigenvalues}), \\ \hat{A} &= \sum_n |a_n\rangle a_n \langle a_n|, & (\text{discrete eigenvalues}). \end{aligned}$$

7 Postulates in general formulation

We can reformulate the postulates given earlier to apply to the general formulation of quantum mechanics:

Postulate A: To each physical observable quantity F there corresponds in quantum-mechanical theory a linear, Hermitian operator \hat{F} . Operators for a generalized coordinate q_n and generalized momentum p_n fulfill

$$[\hat{q}_n, \hat{p}_n] = \hat{q}_n \hat{p}_n - \hat{p}_n \hat{q}_n = i\hbar.$$

Instead of postulating the operators themselves, as was the case for the position representation, the postulate now contains the commutation relations of the operators. We will add another commutation relation to the postulate when discussing angular momentum.

Postulate B: Each state of a physical system is represented by a state vector $|\Psi, t\rangle$ in Hilbert space with length 1,

$$\langle \Psi, t | \Psi, t \rangle = 1.$$

The vector is a solution of the time-dependent Schrödinger equation (SE)

$$i\hbar \frac{\partial}{\partial t} |\Psi, t\rangle = \hat{H} |\Psi, t\rangle. \quad (18)$$

Postulate C: The expectation value of an observable F , given the state $|\Psi\rangle$, is

$$\langle F \rangle = \langle \Psi | \hat{F} | \Psi \rangle.$$

Postulate D: The measurement of an observable F yields one of the eigenvalues f_n of the corresponding operator \hat{F} .

8 Different representations

We will now use the general framework defined above to look at the position (x) and momentum (p) representations in one dimension, and show that we can arrive at the familiar expressions and equations using the new framework.

See also Compendium Ch. 10.3 by Øverbø (2011) and Ch. 6.5 in Hemmer (2005).

8.1 Position representation

Eigenvectors $|x'\rangle$ of position operator \hat{x} :

$$\hat{x}|x'\rangle = x'|x'\rangle, \quad (19)$$

with eigenvalue $x' \in (-\infty, \infty)$ taking continuous values. Since the positions do not constitute a countable set, the vectors must be δ -function normalized,

$$\langle x'' | x' \rangle = \delta(x'' - x').$$

Since the position is an observable, we assume the set $\{|x'\rangle\}$ to be complete,

$$\int dx' |x'\rangle \langle x'| = 1.$$

Therefore, for any vector $|\psi\rangle$, we can write

$$|\psi\rangle = \underbrace{\int dx' |x'\rangle \langle x'|}_{=1} |\psi\rangle = \int dx' \underbrace{\langle x' | \psi \rangle}_{\in \mathbb{C}} |x'\rangle.$$

$\langle x' | \psi \rangle$ contributes to $|\psi\rangle$ at position x' . This is our usual position space wavefunction,

$$\psi(x') = \langle x' | \psi \rangle, \quad (20)$$

where time-dependence is not included explicitly in the notation.

We have used some words, like scalar product, orthogonality, normalization and completeness both for the position representation (with wavefunctions) earlier, and now in the general formulation. Let's show that they mean the same thing:

Scalar product

$$\begin{aligned} \langle \psi_1 | \psi_2 \rangle &= \int dx \langle \psi_1 | x \rangle \langle x | \psi_2 \rangle = \int dx \langle x | \psi_1 \rangle^* \langle x | \psi_2 \rangle \\ &= \int dx \psi_1^*(x) \psi_2(x). \quad \blacksquare \end{aligned}$$

Normalization Since the above holds, we have

$$\langle \psi | \psi \rangle = \int dx |\psi(x)|^2 = 1.$$

Completeness

$$\langle x'' | x' \rangle = \sum_n \langle x'' | n \rangle \langle n | x' \rangle = \sum_n \psi_n^*(x'') \psi_n(x') = \delta(x'' - x'). \quad \blacksquare$$

Operators We have for the expectation value

$$\langle \psi | \hat{F} | \psi \rangle = \int dx'' \int dx' \underbrace{\langle \psi | x'' \rangle}_{\psi^*(x'')} \underbrace{\langle x'' | \hat{F} | x' \rangle}_{?} \underbrace{\langle x' | \psi \rangle}_{\psi(x')}, \quad (21)$$

but how do we treat the quantity $\langle x'' | \hat{F} | x' \rangle$?

We start with $\hat{F} = \hat{x}$:

$$\langle x'' | \hat{x} | x' \rangle = x' \langle x'' | x' \rangle = x' \delta(x'' - x').$$

If $\hat{F}(\hat{x})$ can be expanded in powers of \hat{x} , we get

$$\langle x'' | \hat{F}(\hat{x}) | x' \rangle = F(x') \delta(x'' - x').$$

For $\hat{F} = \hat{p}$, we need to use the commutation relation $[\hat{x}, \hat{p}] = i\hbar$:

$$\begin{aligned} \langle x'' | \hat{x} \hat{p} - \hat{p} \hat{x} | x' \rangle &= (x'' - x') \langle x'' | \hat{p} | x' \rangle \\ &= \langle x'' | i\hbar | x' \rangle = i\hbar \delta(x'' - x'). \end{aligned}$$

Using the relation $x\delta'(x) = -\delta(x)$ ¹⁰, we find

$$\langle x'' | \hat{p} | x' \rangle = \frac{\hbar}{i} \frac{\partial}{\partial x''} \delta(x'' - x'). \quad (22)$$

This can be generalized to operator functions of \hat{x} and \hat{p} :

$$\langle x'' | \hat{F}(\hat{p}, \hat{x}) | x' \rangle = F\left(\frac{\hbar}{i} \frac{\partial}{\partial x''}, x'\right) \delta(x'' - x'). \quad (23)$$

Here we use

$$\langle x'' | \hat{p} \hat{x} | x' \rangle = x' \langle x'' | \hat{p} | x' \rangle$$

and

$$\langle x'' | \hat{x} \hat{p} | x' \rangle = \langle x' | \hat{p} \hat{x} | x'' \rangle^* = x'' \langle x'' | \hat{p} | x' \rangle.$$

¹⁰ This relation can be found using integration by parts:

$$\int_a^b dx x \delta'(x) = - \int_a^b dx \delta(x) + \underbrace{x \delta(x)}_{=0} \Big|_{x=a}^b$$

$$= - \int_a^b dx \delta(x),$$

implying that $x\delta'(x) = -\delta(x)$.

Using the above result in the expression for the expectation value, eq. (21), we get

$$\begin{aligned}
 \langle \psi | \hat{F} | \psi \rangle &= \int dx'' \int dx' \psi^*(x'') \langle x'' | \hat{F}(\hat{p}, \hat{x}) | x' \rangle \psi(x') \\
 &= \int dx'' \int dx' \psi^*(x'') F\left(\frac{\hbar}{i} \frac{\partial}{\partial x''}, x''\right) \delta(x'' - x') \psi(x') \\
 &= \int dx' \psi^*(x') F\left(\frac{\hbar}{i} \frac{\partial}{\partial x'}, x'\right) \psi(x') \\
 &= \int dx \psi^*(x) F\left(\frac{\hbar}{i} \frac{\partial}{\partial x}, x\right) \psi(x), \tag{24}
 \end{aligned}$$

where we renamed $x' \rightarrow x$ in the last equality. This is exactly what we had in the position representation. Good!

Schrödinger equation Finally, in our new postulate B, we had the SE

$$i\hbar \frac{\partial}{\partial t} |\Psi\rangle = \hat{H} |\Psi\rangle.$$

Is this equivalent to the one we had before in the position representation? We multiply with $\langle x |$ from the left:

$$\begin{aligned}
 i\hbar \frac{\partial}{\partial t} \langle x | \Psi, t \rangle &= i\hbar \frac{\partial}{\partial t} \Psi(x) \\
 &= \langle x | \hat{H} | \Psi \rangle = \int dx' \underbrace{\langle x | \hat{H} | x' \rangle}_{H\left(\frac{\hbar}{i} \frac{\partial}{\partial x}, x\right) \delta(x-x')} \underbrace{\langle x' | \Psi \rangle}_{\Psi(x')} \\
 &= H\left(\frac{\hbar}{i} \frac{\partial}{\partial x}, x\right) \Psi(x). \quad \blacksquare \tag{25}
 \end{aligned}$$

In conclusion:

When using the eigenvectors $|x\rangle$ of the position operators as basis vectors, the general formulation is reduced to wave-mechanics in position space.

8.2 Momentum representation

The eigenvectors $|p\rangle$ of the momentum operator \hat{p} satisfy

$$\hat{p} |p\rangle = p |p\rangle, \tag{26}$$

with eigenvalue p . The wavefunction in the momentum representation is

$$\Phi(p) = \langle p | \Psi \rangle. \tag{27}$$

Let us show that we have the same relation between $\Phi(p)$ and $\Psi(x)$ as we found before:

$$\begin{aligned}
 \langle x | \Psi \rangle &= \Psi(x) = \int dp \langle x | p \rangle \langle p | \Psi \rangle \\
 &= \int dp \langle x | p \rangle \Phi(p).
 \end{aligned}$$

We here insert the completeness relation

$$\int dp |p\rangle \langle p| = 1.$$

But what is $\langle x|p\rangle$? Multiply eq. (26) by $\langle x|$ from the left:

$$\begin{aligned}\langle x|\hat{p}|p\rangle &= p\langle x|p\rangle \\ &= \int dx' \langle x|\hat{p}|x'\rangle \langle x'|p\rangle = \int dx' \frac{\hbar}{i} \frac{\partial}{\partial x'} \delta(x-x') \langle x'|p\rangle \\ &= \frac{\hbar}{i} \frac{\partial}{\partial x} \langle x|p\rangle.\end{aligned}$$

Hence,

$$p\langle x|p\rangle = \frac{\hbar}{i} \frac{\partial}{\partial x} \langle x|p\rangle \quad \Rightarrow \quad \langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar},$$

with constant chosen such that

$$\begin{aligned}\langle p|p'\rangle &= \int dx \langle p'|x\rangle \langle x|p\rangle = \frac{1}{2\pi\hbar} \int dx e^{i(p-p')x/\hbar} \\ &= \delta(p-p').\end{aligned}$$

We therefore find that the momentum eigenvectors projected on position basis vectors are the plane wave solutions,

$$\langle x|p\rangle = \psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar},$$

and therefore

$$\Psi(x) = \int dp \Phi(p) \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}},$$

as before.

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We here insert the completeness relation

$$\int dx' |x'\rangle \langle x'| = 1,$$

and use the position representation of the operator \hat{p} .