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FY2045 Solutions Problem set 4 fall 2023

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Problem 1

a) Following the given text we let the abstract momentum operator act on the ket vector $|\psi\rangle$. This results in the vector

$$\left|\widetilde{\psi}\right\rangle = \hat{p}_x \left|\psi\right\rangle .$$

We can then use tool formula number (2) in the text to find the connection between the corresponding *wavefunctions* $\widetilde{\psi}(x) = \langle x|\widetilde{\psi}\rangle$ and $\psi(x) = \langle x|\psi\rangle$. We do this as follows:

$$\underline{\widetilde{\psi}(x)} = \langle x|\widetilde{\psi}\rangle = \langle x|\ \hat{p}_x \left|\psi\right\rangle \stackrel{(2)}{=} \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|\psi\rangle = \underline{\frac{\hbar}{i}\frac{\partial}{\partial x}\,\psi(x)}.$$

The moral is: The abstract operator $\hat{p}_x$ corresponds in the “good old” position-space representation of quantum mechanics to the operator

$$\hat{p}_{x,\text{p.r.}} = \frac{\hbar}{i}\frac{\partial}{\partial x},$$

as expected.

b) We project the equation $\left|\widetilde{\psi}\right\rangle = \hat{p}_x \left|\psi\right\rangle$ onto the ket-vector $|p\rangle$, that is multiply from the left by $\langle p|$, and get

$$\langle p|\widetilde{\psi}\rangle = \underline{\widetilde{\phi}(p)} = \langle p|\,\hat{p}_x \left|\psi\right\rangle \stackrel{(3)}{=} p\langle p|\psi\rangle = \underline{p\phi(p)}.$$

The moral is: The abstract operator $\hat{p}_x$ corresponds in the momentum representation of QM to multiplication by the number p (as we found in Lecture note 7):

$$\hat{p}_{x,\text{m.r.}} = p.$$

Then, let us set $\left|\widetilde{\psi}\right\rangle = \hat{x}\,|\psi\rangle$, and again multiply from the left by $\langle p|$:

$$\langle p|\widetilde{\psi}\rangle = \underline{\widetilde{\phi}(p)} = \langle p|\,\hat{x}\,|\psi\rangle \stackrel{(4)}{=} -\frac{\hbar}{i}\frac{\partial}{\partial p}\langle p|\psi\rangle = \underline{-\frac{\hbar}{i}\frac{\partial}{\partial p}\phi(p).}$$

The moral is: The abstract operator $\hat{x}$ corresponds in the momentum representation of QM to the operator

$$\hat{x}_{\text{m.r.}} = -\frac{\hbar}{i}\frac{\partial}{\partial p}$$

(as we found in Lecture note 7).

We notice that all these operators can be read straight out from the right-hand sides of the given tool formulae.

Problem 2 — Some general properties of non-stationary oscillator states

a) With $\frac{d}{dt}\langle p\rangle_t = \langle -\partial V/\partial q\rangle_t = -m\omega^2\langle q\rangle_t$, we find that

$$\frac{d^2}{dt^2}\langle q\rangle_t = \frac{1}{m}\frac{d}{dt}\langle p\rangle_t = -\omega^2\langle q\rangle_t\,,$$

with the general solution

$$\langle q\rangle_t = A\cos\omega t + B\sin\omega t$$

and

$$\langle p\rangle_t = m\frac{d}{dt}\langle q\rangle_t = -m\omega A\sin\omega t + m\omega B\cos\omega t.$$

Using the values given for $t = 0$, we then have:

$$\langle q\rangle_0 = A = q_0 \qquad \text{and} \qquad \langle p\rangle_0 = m\omega B = 0,$$

that is,

$$\langle q\rangle_t = q_0\cos\omega t \qquad \text{and} \qquad \langle p\rangle_t = -m\omega q_0\sin\omega t.$$

Thus, the expectation values oscillate classically, with amplitudes $\langle q\rangle_{\max} = q_0$ and $\langle p\rangle_{\max} = m\omega q_0$.

b) After k periods, that is, at the times $t = kT$, which correspond to $\omega t = k \cdot 2\pi$, the exponential factor

$$e^{-iE_n t/\hbar} = e^{-i\omega(n+1/2)t} = e^{-i\omega t/2}(e^{-i\omega t})^n$$

takes the form

$$e^{-iE_n kT/\hbar} = e^{-i\pi k}(e^{-2i\pi k})^n = (-1)^k \cdot 1^{kn} = (-1)^k.$$

Inserted into the expansion formula for the wave function this gives

$$\Psi(q, kT) = \sum_{n=0}^{\infty} c_n \psi_n(q) e^{-iE_n kT/\hbar} = (-1)^k \sum_{n=0}^{\infty} c_n \psi_n(q) = (-1)^k \Psi(q, 0). \qquad \square$$

The "narrow" initial state thus is recreated periodically, no matter how much the wave function is "smeared out" *between* these points in time.

Problem 3

a) From the formulae (cf. problem set and Lecture note 10)

$$\langle q| \, \hat{q} = q \, \langle q| \,, \qquad \langle q| \, \hat{p} = \frac{\hbar}{i} \frac{\partial}{\partial q} \, \langle q|$$

it follows that

$$\begin{aligned} \langle q| \; a = \; & \langle q| \left(\sqrt{\frac{m\omega}{2\hbar}} \, \hat{q} + \frac{i\widehat{p}}{\sqrt{2m\hbar\omega}} \right) \\ = \; & \left(\sqrt{\frac{m\omega}{2\hbar}} \, q + \sqrt{\frac{\hbar}{2m\omega}} \, \frac{\partial}{\partial q} \right) \langle q| \equiv a_{\text{p.r.}} \; \langle q| \,. \end{aligned}$$

b) We now apply this operator to the ground state:

$$a_{\text{p.r.}} \, C_0 e^{-m\omega q^2/2\hbar} = C_0 e^{-m\omega q^2/2\hbar} \left(\sqrt{\frac{m\omega}{2\hbar}} \, q + \sqrt{\frac{\hbar}{2m\omega}} \, (-m\omega q/\hbar) \right) = 0, \quad \square$$

Note that this formula is analogous to the relation $a \, |0\rangle = 0$, and that is follows by multiplying the latter from the left by $\langle q|$:

$$\langle q| \, a \, |0\rangle = a_{\text{p.r.}} \langle q|0\rangle = a_{\text{p.r.}} \psi_0(q) = 0.$$

c) The adjoint of $a_{\text{p.r.}}$ is obtained by changing the sign in the last term in the expression for $a_{\text{p.r.}}$. We then have

$$
\begin{aligned}
a^\dagger_{\text{p.r.}}\, \psi_0(q) &= \left(\sqrt{\frac{m\omega}{2\hbar}}\, q - \sqrt{\frac{\hbar}{2m\omega}}\, \frac{\partial}{\partial q} \right) C_0 e^{-m\omega q^2/2\hbar} \\
&= C_0 e^{-m\omega q^2/2\hbar} \left(\sqrt{\frac{m\omega}{2\hbar}}\, q - \sqrt{\frac{\hbar}{2m\omega}}\, (-m\omega q/\hbar) \right) \\
&= \psi_0(q) \cdot q \sqrt{\frac{2m\omega}{\hbar}} = \psi_1(q), \quad \square
\end{aligned}
$$

This result also follows by multiplying the relation $a^\dagger\,|0\rangle = \sqrt{1}\,|1\rangle$ from the left by $\langle q|$.