

FY2045 Quantum Mechanics I

Fall 2023

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Week 6

Quantization of Angular Momentum

Angular momentum operators

Define angular momentum operator \mathbf{J} with components J_x, J_y , and J_z – Hermitian operators with commutation relations

$$[J_x, J_y] = i\hbar J_z,$$

$$[J_y, J_z] = i\hbar J_x,$$

$$[J_z, J_x] = i\hbar J_y.$$

Also define $J^2 = J_x^2 + J_y^2 + J_z^2$, which satisfies $[J^2, J_i] = 0, i = x, y, z.$

Since they commute, we can find simultaneous eigenvectors of J^2 and e.g. J_z .

Eigenvalues

The eigenvectors satisfy the eigenvalue equations

$$\mathbf{J}^2 |j, m\rangle = \hbar^2 j(j+1) |j, m\rangle,$$

$$J_z |j, m\rangle = \hbar m |j, m\rangle,$$

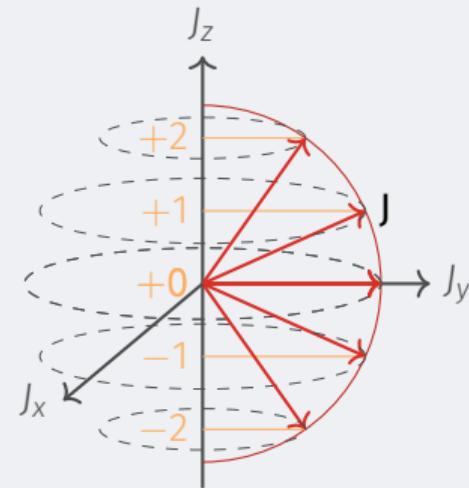
with

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots,$$

$$m = -j, -j+1, \dots, j-2, j-1, j.$$

Example: $j = 2$

Orientation of \mathbf{J} for different values of m :



Based on figure by Izaak Neutelings

Orbital angular momentum

From classical expression: $\mathbf{L} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$.

Simultaneous eigenvectors of \mathbf{L}^2 and L_z :

$$\mathbf{L}^2 |l, m\rangle = \hbar^2 l(l+1) |l, m\rangle ,$$

$$L_z |l, m\rangle = \hbar m |l, m\rangle .$$

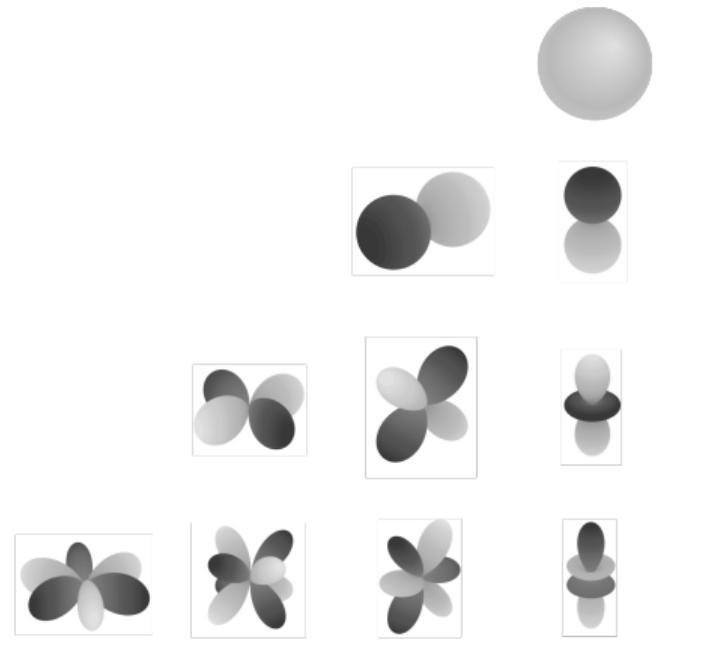
Position space basis vectors in spherical coordinates, $|\mathbf{r}\rangle \equiv |r, \theta, \phi\rangle$, gives

$$\psi_{lm}(r, \theta, \phi) = \langle r, \theta, \phi | l, m \rangle = R(r) Y_{lm}(\theta, \phi).$$

$Y_{lm}(\theta, \phi)$ are **spherical harmonics**.

$$Y_{lm} \propto e^{im\phi} \Rightarrow |m|, l = 0, 1, 2, \dots$$

Y_{lm} with $l = 0 - 3$, and $m \geq 0$



Example – Hydrogen atom

Coulomb potential is spherically symmetric:

$$\hat{H} = \left[-\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) - \frac{e^2}{4\pi\epsilon_0 r} + \frac{\mathbf{L}^2}{2mr^2} \right].$$

Simultaneous eigenfunctions of \hat{H} , \mathbf{L}^2 and L_z :

$$\psi_{nlm}(r, \theta, \phi) = R_{nl}(r)Y_{lm}(\theta, \phi),$$

with eigenvalues

$$\hat{H} : \quad E_n = -\frac{13.6 \text{ eV}}{n^2}, \quad n = 1, 2, 3, \dots,$$

$$\mathbf{L}^2 : \quad \hbar^2 l(l+1), \quad 0 \leq l < n,$$

$$L_z : \quad \hbar m, \quad m = -l, -l+1, \dots, l-1, l.$$

Energy levels

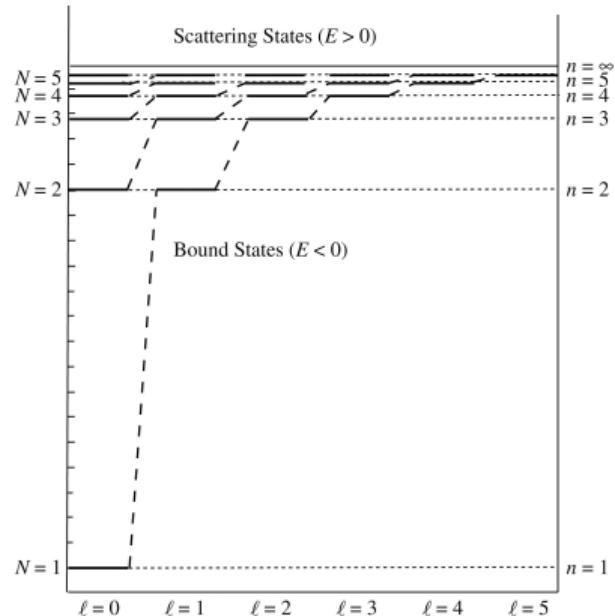
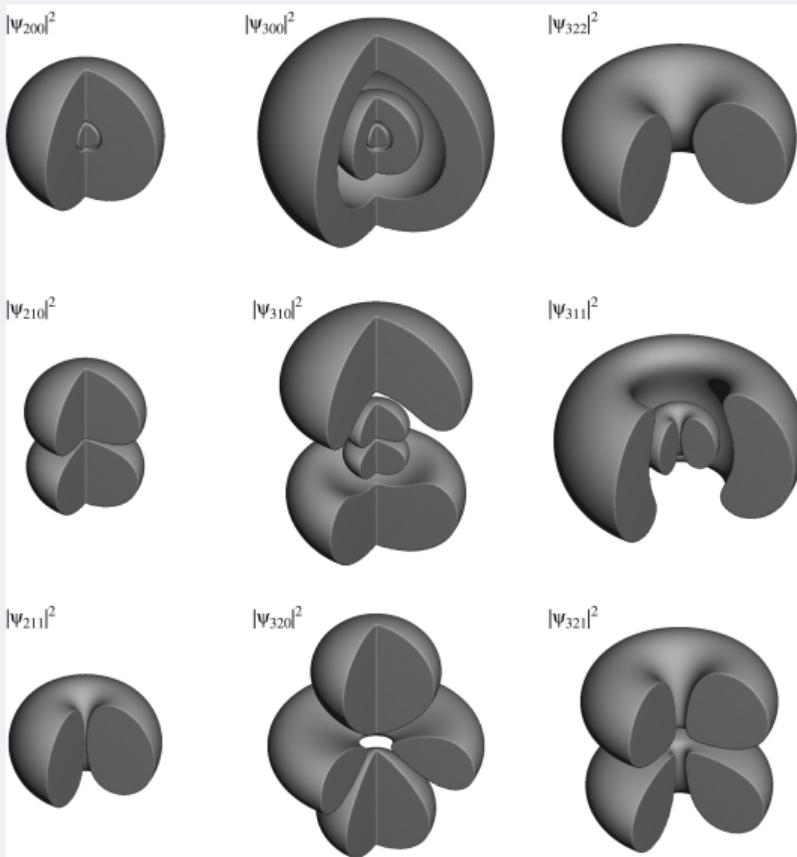
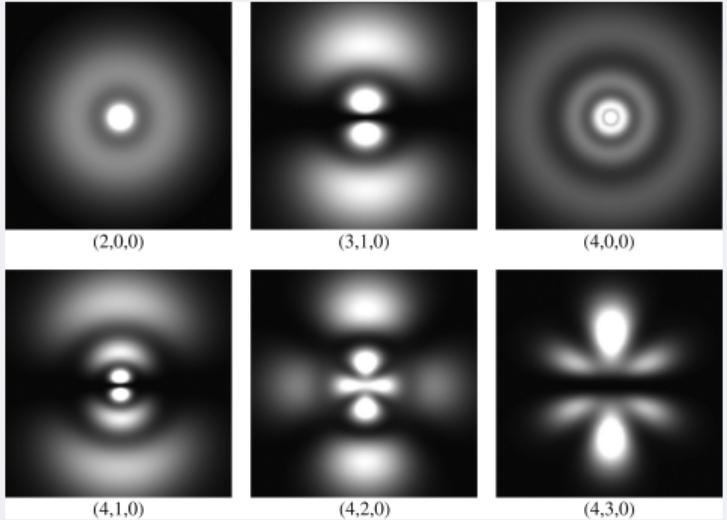


Figure from Griffiths and Schroeter, ©Cambridge University Press 2018



Figures from Griffiths and Schroeter, ©Cambridge University Press 2018

Spin

Classical magnetic moment

A current loop has a magnetic (dipole) moment μ , which in a magnetic field B leads to a torque

$$\tau = \mu \times B,$$

and force

$$F = \nabla(\mu \cdot B).$$

A particle with charge q and mass m has a magnetic moment due to the orbital motion

$$\mu_L = \frac{q}{2m} L.$$

μ_L and L are not constants of motion:

Larmor precession

In a homogeneous magnetic field, the force is zero, and the torque is perpendicular to both μ_L and B . This leads to a precessing L :

$$\frac{dL}{dt} = \omega_L \times L,$$

where $\omega_L = -qB/(2m)$ is the Larmor frequency.

QM magnetic moment due to L

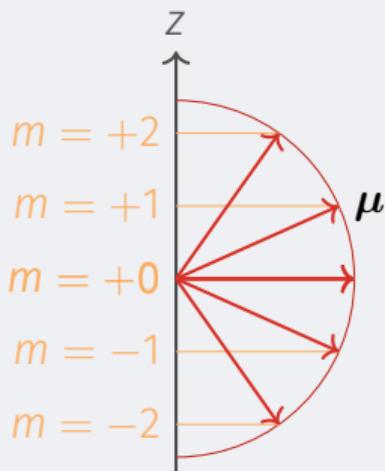
In QM \mathbf{L} is quantized, meaning that $|\boldsymbol{\mu}_L|$ and $(\mu_L)_z$ can only take certain values. For electrons ($q = -e$) we get

$$|\boldsymbol{\mu}_L| = \frac{e}{2m_e} |\mathbf{L}| = \mu_B \sqrt{l(l+1)}, \quad l = 0, 1, 2, \dots,$$

$$(\mu_L)_z = -\frac{e}{2m} L_z = -m\mu_B, \quad m = 0, \pm 1, \dots, \pm l,$$

with the Bohr magneton

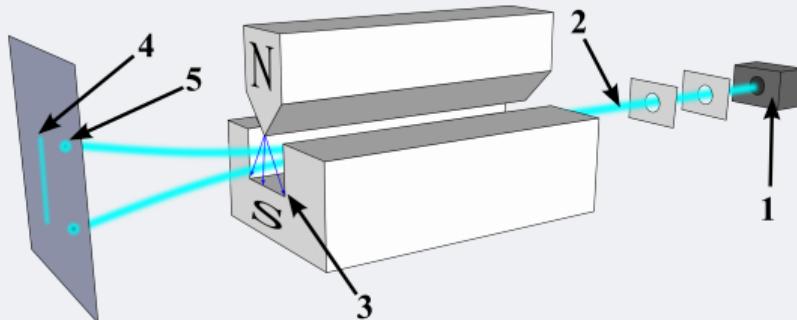
$$\mu_B = \frac{e\hbar}{2m_e}.$$



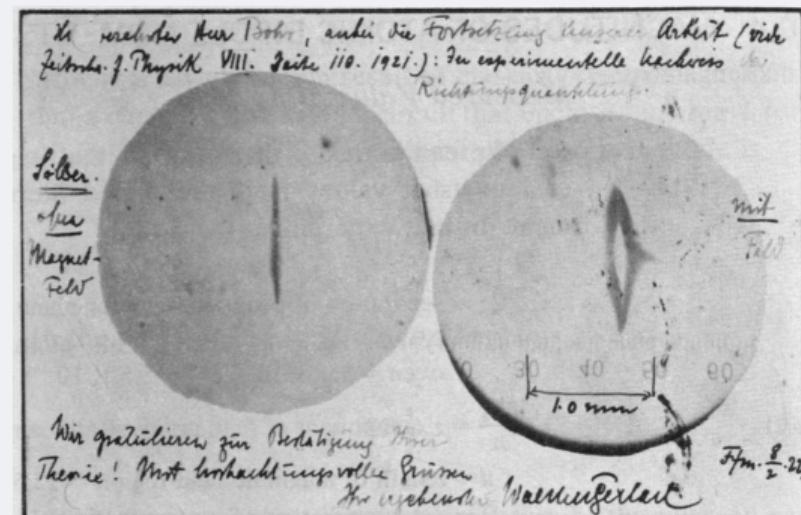
Based on figure by Izaak Neutelings

Stern-Gerlach experiment

In an inhomogeneous magnetic field, the force \mathbf{F} on a magnetic moment will depend on the direction of μ .



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Emilio Segrè Visual Archives

See [article in Physics Today](#) for more about the struggles of Stern and Gerlach.

Spin – Intrinsic angular momentum

We define a **spin** operator \mathbf{S} with components S_x, S_y and S_z , satisfying

$$[S_i, S_j] = i\hbar \sum_k \epsilon_{ijk} S_k,$$

$$[\mathbf{S}^2, S_i] = 0,$$

with eigenvalue equations

$$\mathbf{S}^2 |s, m\rangle = \hbar^2 s(s+1) |s, m\rangle, \quad s = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots,$$

$$S_z |s, m\rangle = \hbar m |s, m\rangle, \quad m = -s, -s+1, \dots, s-1, s.$$

We also have ladder operators

$$S_{\pm} |s, m\rangle = (S_x \pm iS_y) |s, m\rangle = \hbar \sqrt{s(s+1) - m(m \pm 1)} |s, m \pm 1\rangle.$$

The electron spin

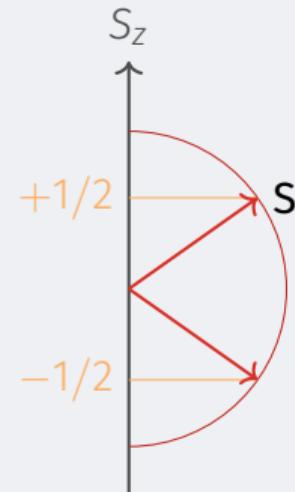
The electron has $s = \frac{1}{2}$, meaning it can only have $m = \pm \frac{1}{2}$, **spin up** and **spin down**.

The intrinsic magnetic moment is

$$\mu_e = -g_e \frac{e}{2m_e} \mathbf{S} = -\frac{g_e \mu_B}{\hbar} \mathbf{S} \equiv \gamma_e \mathbf{S},$$

with electron gyromagnetic ratio γ_e , and g factor

$$g_e = 2.002\,319\,304\,362\,56 \pm 0.000\,000\,000\,000\,35.$$



Based on figure by Izaak Neutelings

Spins of other particles

Proton: $s = 1/2$

$$\mu_p = 5.59 \frac{e}{2m_p} S_p.$$

Much smaller than μ_e
since $m_e \ll m_p$.

Neutron: $s = 1/2$

$$\mu_e = -3.83 \frac{e}{2m_n} S_n$$

Photon: $s = 1$

Massless: can only have $m = \pm \hbar$,
not $m = 0$.

Nobel prize in physics 2022: “for experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science”¹

¹For more info see e.g. [the summary from the Nobel foundation](#).

Spin $\frac{1}{2}$

Abstract formulation

For $s = \frac{1}{2}$ we have two possible eigenvectors of S^2 and S_z :

$$\left| \frac{1}{2}, +\frac{1}{2} \right\rangle \equiv |+\rangle \equiv |\uparrow\rangle,$$

$$\left| \frac{1}{2}, -\frac{1}{2} \right\rangle \equiv |-\rangle \equiv |\downarrow\rangle,$$

forming a complete set in a two-dimensional Hilbert space.

Can expand arbitrary vector $|\chi\rangle$ in the 2D Hilbert space:

$$\begin{aligned} |\chi\rangle &= \langle \uparrow | \chi \rangle |\uparrow\rangle + \langle \downarrow | \chi \rangle |\downarrow\rangle \\ &\equiv a_+ |\uparrow\rangle + a_- |\downarrow\rangle. \end{aligned}$$

a_{\pm} : probability amplitude of measuring spin up (down) and leaving the spin in state $|\uparrow\rangle$ ($|\downarrow\rangle$).