SPIN [Φ.12, H8.3, G4.4]

In classical mechanics, a rigid object admits two kinds of angular momentum

- orbital: $\vec{L} = \vec{r} \times \vec{p}$
- spin: $\vec{S} = I\,\vec{\omega}$

moment of inertia — rotation

The first is associated with motion of the center of mass [earth around sun], the second with motion about the center of mass [earth's rotation about the north-south axis]. Classically, the distinction is a bit artificial: the spin is just the sum of the "orbital" angular momentum of the matter making up the earth when circling the axis.

In quantum mechanics, we have already seen that particles can carry orbital angular momentum, e.g. electron circling a nucleus. However, there is also another form of angular momentum, which is not due to motion in space. Elementary particles carry <u>intrinsic</u> angular momentum — analogously called spin, $\vec{S}$ — in addition to the <u>extrinsic</u> angular momentum $\vec{L}$.

The hypothesis about the spin of the electron was put forth in 1925 by Uhlenbeck and Goudsmit. The experimental background for attributing an intrinsic angular momentum to the electron was mainly:

- the fine structure of atomic spectra
- the Zeeman effect
- the Stern-Gerlach experiment.

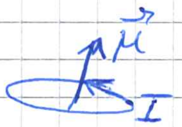
We will/might come back to the first two at a later point, but will now take a look at how angular momentum leads to a magnetic moment, which in turn affects the motion of a particle through a magnetic field, allowing us to discuss the Stern-Gerlach experiment.

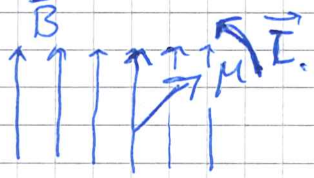
Magnetic moments

Classically, a current-carrying loop has a magnetic dipole moment $\vec{\mu}$. When placed in a magnetic field $\vec{B}$, it experiences a torque

$$\vec{\tau} = \vec{\mu} \times \vec{B},$$

which will tend to align the magnetic dipole parallel to the field.





This is what happens with a compass needle. The energy associated with the torque is

$$H = -\vec{\mu} \cdot \vec{B}.$$

H is minimized when $\vec{\mu} \parallel \vec{B}$.

For a particle with mass m and charge $q<0$ following a classical circular orbit, we get the magnetic moment

$$|\vec{\mu}| = \underbrace{I}_{\text{current}} \cdot \underbrace{A}_{\text{area of loop}} = q \frac{v}{2\pi r} \cdot \pi r^2 = \frac{1}{2} q v r.$$

with velocity v, and radius r. Identifying $\vec{L} = m \vec{r} \times \vec{v}$, and $rv = |\vec{r} \times \vec{v}|$, we get

$$\vec{\mu}_L = \frac{q}{2m} \vec{L} \equiv \gamma \vec{L},$$

with gyromagnetic ratio γ.

When $\vec{B} \neq 0$, $\vec{L}$ and $\vec{\mu}_L$ are not constants of motion:

From Newton's second law, we have

$$\frac{d\vec{L}}{dt} = \vec{\tau} = \vec{\mu}_L \times \vec{B} = -\frac{q}{2m}\vec{B} \times \vec{L} \equiv \vec{\omega}_L \times \vec{L},$$

with

$$\vec{\omega}_L = -\frac{q}{2m}\vec{B}.$$

This equation tells us that $\vec{L}$ (and hence $\vec{\mu}_L$) precesses around the direction of $\vec{B}$:

The precession frequency, $|\vec{\omega}_L| = \left|-\frac{q}{2m}\vec{B}\right|$ is called the Larmor frequency.

Finally, we note that the magnetic moment can experience a force

$$\vec{F} = \nabla(\vec{\mu} \cdot \vec{B}),$$

corresponding to the negative gradient of the energy H, when $\vec{B}$ is inhomogeneous. This will become relevant when discussing the Stern-Gerlach experiment.

Quantum mechanical treatment

From the classical relation between $\vec{\mu}_L$ and $\vec{L}$, we see that the QM operator corresponding to the observable $\vec{\mu}_L$ must be

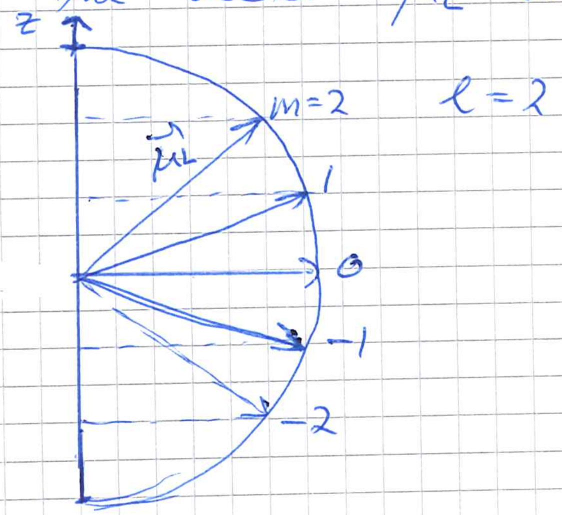
$$\hat{\vec{\mu}}_L = \frac{q}{2m}\hat{\vec{L}}.$$

However, we know that $\vec{L}$ cannot take any value, as it can classically. In fact, we must have

$$|\vec{\mu}_L| = \frac{q\hbar}{2m}\sqrt{\ell(\ell+1)}, \qquad \ell = 0, 1, 2 \ldots$$

$$(\mu_L)_z = \frac{q\hbar}{2m}m \qquad m = 0, \pm 1, \pm 2 \ldots \pm \ell,$$

meaning there are quantized values of angle between the vector $\vec{\mu}_L$ and the z axis.



Since $\vec{L}$ and $\vec{\mu}_L$ are not classical constants of motion in a magnetic field, they are also not quantum-mechanical constants of motion, meaning that expectation values $\langle\vec{L}\rangle$ and $\langle\vec{\mu}_L\rangle$ are not c[onstant].
They will precess, as we will see

The Stern-Gerlach experiment

In a homogeneous magnetic field, we saw that $\vec{L}$ and $\vec{\mu}$ precess around $\vec{B}$ due to the torque $\vec{\tau} = \vec{\mu} \times \vec{B}$. In an inhomogeneous magnetic field, we also have a finite force

$$\vec{F} = \nabla(\vec{\mu} \cdot \vec{B}).$$

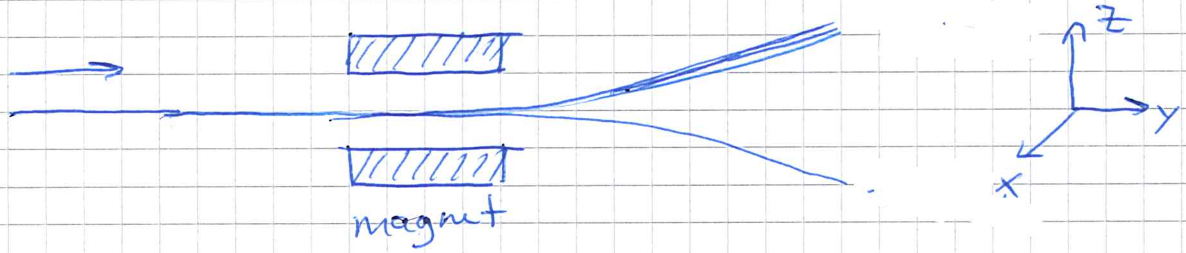
Particles with different $\vec{\mu}$ will therefore experience different forces, making it possible to separate particles with different orientations of $\vec{\mu}$.

This is the idea behind the Stern-Gerlach experiment: send a beam of atoms through an inhomogeneous magnetic field, and measure the deflection in order to get information about the magnetic moment.

Imagine a beam of neutral atoms traveling in the y-direction through a static, inhomogeneous magnetic field

$$\vec{B} = a x \hat{x} - (B_0 + a z)\hat{z}, \qquad [\nabla \cdot \vec{B} = 0, \text{ OK}]$$

where $B_0 \gg a$ describes the strong, uniform part of the field.



The force on the magnetic moment becomes

$$\vec{F} = a\left[\mu_x \hat{x} - \mu_z \hat{z}\right].$$

Since we have a strong uniform component in the z-direction, μ_x precesses rapidly around the z-axis, averaging to zero. Hence there is no deflection in the x direction, and we are left with

$$F_z = -a\mu_z,$$

giving a deflection in the z-direction depending on the size and sign of μ_z. Note that μ_z is constand during the precession since $\vec{B}_{uniform} = B_0 \hat{z}$.

In 1921, Stern and Gerlach expected to find a continuous deflection between a maximum and minimum value, corresponding to a continuous variation of μ_z between $-|\vec{\mu}|$ and $+|\vec{\mu}|$.

However, using silver atoms, they found that the beam split in two. After the development of quantum theory in the late 1920s, this result makes sense.

Since the direction of $\vec{\mu}$ is quantized, we expect a quantized deflection.
However if the angular quantum number j is an integer, and we have $2j+1$ possible values of the quantum number m, the beam should split in an odd number of beams, but they got two beams.

<u>The theory of spin</u>

This was explained by Uhlenbeck and Goudsmit in 1925. In a silver atom, which has 47 protons and electrons, 46 of the electrons are paired up in such a way that their angular momenta cancel. Hence, the net angular momentum of the atom is simply that of the outermost, unpaired electron, which is in a state with $(n, l, m) = (5, 0, 0)$, i.e. it has no orbital angular momentum.
Hence, for the beam to split in two, the electron must have an intrinsic angular momentum quantum number s such that $2s+1 = 2$

$$\Rightarrow s = \frac{1}{2},$$

with a corresponding magnetic moment $\vec{\mu}_s$.
This is the spin angular momentum $\vec{S}$, or simply the <u>spin</u>.

Using our earlier algebraic theory for angular momentum, we can therefore define an algebraic theory for spin:

The operator $\vec{S}$ has components S_x, S_y, S_z, satisfying commutation relations

$$[S_i, S_j] = i\hbar \sum_k \varepsilon_{ijk} S_k, \qquad i,j,k \in \{x,y,z\}$$

and

$$[\vec{S}^2, S_i] = 0, \qquad i = x,y,z,$$

and with eigenvalue equations

$$\vec{S}^2 |s,m\rangle = \hbar^2 s(s+1) |s,m\rangle$$
$$S_z |s,m\rangle = \hbar m |s,m\rangle \qquad \text{(some use } m_s\text{)}$$

with ladder operators

$$S_\pm |s,m\rangle = (S_x \pm i S_y)|s,m\rangle = \hbar\sqrt{s(s+1)-m(m\pm1)}\,|s,m\pm1\rangle.$$

The eigenvectors are not spherical harmonics, and actually not functions of θ and ϕ at all! As we have seen, we need to allow for half-integer values of s:

$$s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2} \ldots$$

$$m = -s, -s+1, \ldots s-1, s,$$

giving $2s+1$ possible values of m.

The electron spin

Since there were two discrete deflections in the Stern-Gerlach experiment, we concluded that the electron has spin quantum number $s = \frac{1}{2}$, meaning that there are two possible values of m: $m = \pm\frac{1}{2}$, corresponding to $S_z = \pm\frac{\hbar}{2}$. These two states are usually called spin up and spin down.

The intrinsic magnetic moment is

$$\hat{\mu}_s = -g_e \frac{e}{2m_e} \vec{S} \equiv \gamma_e \vec{S},$$

where γ_e is the gyromagnetic ratio of the electron, and

$$g_e \approx 2 \times 1.001159\ldots$$

is the gyromagnetic factor (or g-factor) of the electron. Notice that $\vec{\mu}_s$ points in the opposite direction of $\vec{S}$.

Spins of other particles

The proton and neutron also have spin $\frac{1}{2}$. Why didn't they contribute to the SG experiment? The short answer is that

$$\vec{\mu} \propto \frac{\vec{S}}{m},$$

and since $m_p, m_n \gg m_e$, $|\vec{\mu}_p|, |\vec{\mu}_n| \ll |\vec{\mu}_e|$, and the behaviour of atoms in magnetic fields is largely determined by the electrons.

SPIN $\frac{1}{2}$ [Ø12.2, H8.3.2, G4.4.1]

For spin $s=\frac{1}{2}$, we get the two possible eigenvectors of $\vec{S}^2$ and S_z:

$$\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle \equiv |\uparrow\rangle \equiv |+\rangle$$

$$\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle \equiv |\downarrow\rangle \equiv |-\rangle.$$

Our Hilbert space for spin $\frac{1}{2}$ is two-dimensional. Operating with our ladder operators, we get:

$$S_+|\uparrow\rangle = 0 \qquad S_+|\downarrow\rangle = \hbar|\uparrow\rangle$$

$$S_-|\downarrow\rangle = 0 \qquad S_-|\uparrow\rangle = \hbar|\downarrow\rangle.$$

Since the two states $|\uparrow\rangle, |\downarrow\rangle$ form a complete, orthonormalized set

$$1 = |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|,$$

an arbitrary vector in 2D spin space can be expanded:

$$|\chi\rangle = \langle\uparrow|\chi\rangle|\uparrow\rangle + \langle\downarrow|\chi\rangle|\downarrow\rangle$$
$$\equiv a_+|\uparrow\rangle + a_-|\downarrow\rangle.$$

Requiring normalization leads to:

$$\langle\chi|\chi\rangle = \left(\langle\uparrow|a_+^* + \langle\downarrow|a_-^*\right)\left(a_+|\uparrow\rangle + a_-|\downarrow\rangle\right)$$
$$= |a_+|^2 + |a_-|^2 = 1.$$

a_+: probability amplitude of measuring spin up ($S_z = \frac{\hbar}{2}$) and leaving the spin in state $|\uparrow\rangle$.

a_-: amplitude for spin down.

Matrix formulation

Since our vector-space is two-dimensional, it is convenient to express the theory in terms of two-component vectors and 2×2 matrices.

We can represent the state $|\chi\rangle$ by the two-element column matrix

$$\chi = \begin{pmatrix} a_+ \\ a_- \end{pmatrix},$$

called a <u>spinor</u>. For a spin up or spin down state, we have $a_+ = 1$, $a_- = 0$ and $a_+ = 0$, $a_- = 1$, meaning the states can be represented by the spinors

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{spin up}$$

$$\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{spin down},$$

meaning that $\chi = a_+\chi_+ + a_-\chi_-$. These spinors are orthonormal

$$\chi_\sigma^\dagger \chi_{\sigma'} = \delta_{\sigma\sigma'}, \qquad \sigma, \sigma' \in \{+,-\}.$$

The adjoint state $\langle\chi|$ is represented by the adjoint matrix — transpose and complex conjugate —

$$\chi^\dagger = (a_+^* \quad a_-^*),$$

a row matrix. The normalization requirement is then $\chi^\dagger\chi = |a_+|^2 + |a_-|^2 = 1$, as before.

Operators are represented by 2x2 matrices.

From the eigenvalue equations we require

$$* \quad \vec{S}^2 \chi_\pm = \frac{3\hbar^2}{4} \chi_\pm$$

$$\Rightarrow \underline{\vec{S}^2 = \frac{3\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} = \frac{3\hbar^2}{4} \mathbb{1} = \frac{3\hbar^2}{4} \sigma_0$$

and

$$* \quad S_z \chi_\pm = \pm \frac{\hbar}{2} \chi_\pm$$

$$\Rightarrow \underline{S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \equiv \frac{\hbar}{2} \sigma_z .$$

Connection between matrix and operator: $\langle m | \hat{S}_i | n \rangle = S_i$, $m,n = \pm$.

From the equations for the ladder operators, we have

$$S_+ \chi_- = \hbar \chi_+ \qquad S_+ \chi_+ = 0$$

$$S_- \chi_+ = \hbar \chi_- \qquad S_- \chi_- = 0$$

$$\Rightarrow S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Since $S_\pm = S_x \pm i S_y$, we have

$$S_x = \frac{S_+ + S_-}{2} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv \frac{\hbar}{2} \sigma_x$$

$$S_y = \frac{S_+ - S_-}{2i} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \equiv \frac{\hbar}{2} \sigma_y .$$

Pauli matrices

Above, we have defined the dimensionless Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

allowing us to write the spin operator as

$$\vec{S} = \frac{\hbar}{2} \vec{\sigma}$$

with Pauli vector $\vec{\sigma} = \sigma_x \hat{x} + \sigma_y \hat{y} + \sigma_z \hat{z}$.
The Pauli matrices satisfy the relations

$$[\sigma_i, \sigma_j] = \sigma_i \sigma_j - \sigma_j \sigma_i = 2i \varepsilon_{ijk} \sigma_k$$
$$\{\sigma_i, \sigma_j\} = \sigma_i \sigma_j + \sigma_j \sigma_i = 2 \delta_{ij} \mathbb{1}$$
$$\sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i \varepsilon_{ijk} \sigma_k,$$

where we use the Einstein sum convention.
See exercise 7 for more details.

The spin direction

We now have a well-functioning physical theory for the spin, meaning we should be able to calculate expectation values. Using $\langle S_z \rangle$ as an example, we get the following recipe:

$$\underline{\langle S_z \rangle_\chi} = \langle \chi | \hat{S}_z | \chi \rangle = \langle \chi | \mathbb{1} \hat{S}_z \mathbb{1} | \chi \rangle$$
$$= \sum_{m,n} \langle \chi | m \rangle \langle m | \hat{S}_z | n \rangle \langle n | \chi \rangle$$
$$= \sum_{m,n} a_m^* (S_z)_{mn} a_n = \underline{\chi^\dagger S_z \chi}.$$

Example:

Calculate $\langle S_i \rangle$, $i = x, y, z$ for the state $\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

We get:

$$\langle S_z \rangle = \chi_+^\dagger S_z \chi_+ = \frac{\hbar}{2}(1 \quad 0)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{\hbar}{2}(1 \quad 0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}.$$

$$\langle S_x \rangle = \chi_+^\dagger S_x \chi_+ = \frac{\hbar}{2}(1 \quad 0)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{\hbar}{2}(0 \quad 1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0.$$

$$\langle S_y \rangle = \chi_+^\dagger S_y \chi_+ = \frac{\hbar}{2}(1 \quad 0)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{\hbar}{2}(0 \quad -i)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0.$$

Hence,

$$\langle \vec{S} \rangle = \frac{\hbar}{2}\hat{z},$$

the expectation value points upwards, which is why we call it spin up. We often call the direction of $\langle \vec{S} \rangle$ the <u>spin direction</u>. However, we must remember that the direction of $\vec{S}$ is <u>not</u> an observable, since the components S_i do not commute and therefore cannot have sharp values simultaneously.

We have already seen that if we have a general spin state

$$\chi = \begin{pmatrix} a_+ \\ a_- \end{pmatrix},$$

and we measure S_z, we would get $+\frac{\hbar}{2}$ with probability $|a_+|^2$, and $-\frac{\hbar}{2}$ with probability $|a_-|^2$. But what if we measure S_x instead? What are the possible results, and probabilities for each result?

We need to know the eigenvalues and eigenvectors of S_x:

Eigenvalues: What I mean by this is "We require"

$$|S_x - \lambda \mathbb{1}| \stackrel{!}{=} 0$$

$$\Rightarrow \left| \begin{pmatrix} -\lambda & \hbar/2 \\ \hbar/2 & -\lambda \end{pmatrix} \right| = \lambda^2 - \left(\frac{\hbar}{2}\right)^2 \Rightarrow \lambda = \pm\frac{\hbar}{2}$$

Eigenvectors:

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\Rightarrow \alpha = \pm\beta$$

Hence, the normalized eigenspinors of S_x are

$$\chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \left(\text{eigenvalue } +\frac{\hbar}{2}\right)$$

$$\chi_-^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \left(\text{eigenvalue } -\frac{\hbar}{2}\right)$$

where we have chosen the first element real and positive.

They are eigenvectors of a Hermitian matrix, and thus span the space, meaning that the generic spinor χ can be expressed

$$\chi = \frac{a_+ + a_-}{\sqrt{2}} \chi_+^{(x)} + \frac{a_+ - a_-}{\sqrt{2}} \chi_-^{(x)}.$$

Hence, if you measure S_x, the probability of getting $+\frac{\hbar}{2}$ is $\frac{|a_+ + a_-|^2}{2}$, and the probability of getting $-\frac{\hbar}{2}$ is $\frac{|a_+ - a_-|^2}{2}$.

The eigenspinors of S_y are

$$\chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \text{eigenvalue } +\frac{\hbar}{2}$$

$$\chi_-^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad \text{eigenvalue } -\frac{\hbar}{2}.$$

Can also describe spin states along general vector $\hat{n}$.

Example:

Suppose we have a spin $\frac{1}{2}$ particle in the state

$$\chi = \frac{1}{\sqrt{6}}\begin{pmatrix} 1+i \\ 2 \end{pmatrix}.$$

What are the probabilities of measuring $\pm\frac{\hbar}{2}$ if you measure S_z and S_x?

For S_z:

$$P_+^{(z)} = |a_+|^2 = \frac{|1+i|^2}{6} = \frac{1}{3}, \qquad P_-^{(z)} = |a_-|^2 = \frac{2}{3}.$$

For S_x:

$$P_+^{(x)} = \frac{|a_+ + a_-|^2}{2} = \frac{|3+i|^2}{12} = \frac{5}{6}, \qquad P_-^{(x)} = \frac{|a_+ - a_-|^2}{2} = \frac{1}{6}.$$

A small comment on measuring spins

If we have a particle in the state χ_+, we know that if we measure S_z, we will get $+\frac{\hbar}{2}$. Every time (granted we perform the measurement correctly).

But, what if we then ask:
"What is the x-component of the spin angular momentum?" The only answer we can give is that if we measure S_x we have a 50% chance of gettin $+\frac{\hbar}{2}$, and 50% chance of getting $-\frac{\hbar}{2}$. This does not mean that we do not know the state of the particle – the particle is in the state χ_+. But S_x and S_z can't both be well-defined, that would violate the uncertainty principle.

Let's say we then measure S_x and get $+\frac{\hbar}{2}$. This does not mean we now know both S_z and S_x! Because we have changed the state of the particle – it is now in $\chi_+^{(x)}$.
If we now were to measure S_z again, there is 50% chance of measuring $+\frac{\hbar}{2}$, and 50% chance of gettin $-\frac{\hbar}{2}$ along the z-direction.

The moral is: if you take turns measuring S_x, S_y, S_z, the eigenstates change!

Electron in a magnetic field
– Larmor precession revisited

We have already seen that the magnetic dipole moment of the electron due to spin is

$$\vec{\mu} = \gamma_e \vec{S}, \qquad (\gamma_e < 0).$$

and the Hamiltonian if at rest in a field $\vec{B}$ is

$$\hat{H} = -\vec{\mu} \cdot \vec{B}.$$

Assuming a field $\vec{B} = B_0 \hat{z}$, we get the Hamiltonian matrix

$$H = -\gamma_e B_0 S_z = -\frac{\gamma_e B_0 \hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with eigenstates

$$\chi_+, \quad \text{with energy} \quad E_+ = -\frac{\gamma_e B_0 \hbar}{2}$$

$$\chi_-, \quad \text{with energy} \quad E_- = +\frac{\gamma_e B_0 \hbar}{2}.$$

The lowest energy corresponds to the magnetic moment being parallel to the field.

H is time-independent, and a general solution to the SE

$$i\hbar \frac{\partial \chi}{\partial t} = H\chi,$$

is therefore

$$\chi(t) = a_+ \chi_+ e^{-iE_+ t/\hbar} + a_- \chi_- e^{-iE_- t/\hbar}.$$

where a_+ and a_- are determined by initial cond.

Let's calculate the expectation value of the components of $\vec{S}$. For notational simplicity, we define

$$\omega = \gamma B_0 ,$$

such that

$$\chi(t) = \begin{pmatrix} a_+ e^{i\omega t/2} \\ a_- e^{-i\omega t/2} \end{pmatrix}.$$

Hence, we get:

$$\langle S_x \rangle = \frac{\hbar}{2}\left(a_+^* e^{-\frac{i\omega t}{2}} \quad a_-^* e^{\frac{i\omega t}{2}}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_+ e^{\frac{i\omega t}{2}} \\ a_- e^{-\frac{i\omega t}{2}} \end{pmatrix}$$
$$= \frac{\hbar}{2}\left[a_-^* a_+ e^{i\omega t} + a_+^* a_- e^{-i\omega t}\right]$$

If we assume $a_+, a_- \in \mathbb{R}$, we get

$$\langle S_x \rangle = \hbar a_+ a_- \cos \omega t .$$

Similarily we get

$$\langle S_y \rangle = -\hbar a_+ a_- \sin \omega t$$

and

$$\langle S_z \rangle = \hbar \left[|a_+|^2 - |a_-|^2\right].$$

Hence, we see that as long as $a_+, a_- > 0$, $\langle \vec{S} \rangle$ is tilted at some angle α to the z-axis, determined by a_+, a_-, and precesses about the field at the frequency

$$\omega = \gamma B_0 ,$$

the Larmor frequency. [Alternative approach in set 7].