NTNU – Trondheim
Norwegian University of
Science and Technology

NTNU, Department of Physics

FY2045 Mandatory problem set fall 2023

Henning G. Hugdal

October 1, 2023

Due date: October 6th, 11:59 PM.
Grade: Pass/fail. Must be passed in order to access the final exam.
Submit your answers in LaTeX or pictures of handwritten calculations on Blackboard.
Include your reasoning, not just the final answers to the problems.

Problem 1

You may find the following integrals useful when you are solving the problems below.

$$I_0(\alpha) \equiv \int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}, \tag{1}$$

$$I_2(\alpha) \equiv \int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = -\frac{d}{d\alpha} I_0(\alpha) = \frac{1}{2}\sqrt{\pi}\,\alpha^{-3/2}\,, \tag{2}$$

$$J(A,B) \equiv \int_{-\infty}^{\infty} e^{-Ay^2+By} dy = e^{B^2/4A}\sqrt{\frac{\pi}{A}}\,, \qquad (\mathrm{Re}(A) > 0)\,. \tag{3}$$

We have seen that the position of a particle cannot be measured with *zero* uncertainty. It is, however, in principle possible to prepare an ensemble of particles in a state such that the uncertainty is arbitrarily small (but finite). Imagine that we carry out such a measurement on an ensemble, leaving it in a state that is immediately afterwards described by the normalized wave function

$$\psi(x) = \left(\frac{2\beta}{\pi}\right)^{1/4} e^{-\beta(x-a)^2} \,, \tag{4}$$

where $\beta \in \mathbb{R}$ is large.

a) Find, without calculation, the expectation value $\langle x \rangle$ in the state given by Eq. (4).

b) Find the uncertainty Δx in the position, expressed in terms of β. Remember that $(\Delta x)^2 = \langle (x - \langle x \rangle)^2 \rangle$. You may find it convenient to introduce $x' = x - a$ as a new integration variable. Show also that the Gaussian probability distribution $|\psi(x)|^2$ may be written in the form

$$|\psi(x)|^2 = \frac{1}{\sqrt{2\pi(\Delta x)^2}} \exp\left[-\frac{(x-a)^2}{2(\Delta x)^2}\right] \,. \tag{5}$$

Conclusion: For a Gaussian probability density, we can read off the uncertainty from the exponent.

c) Assume that we choose β very large, in order to make Δx very small, that is, in order to prepare the ensemble in a state with a very well-defined position. Calculate the expectation values $\langle p_x \rangle$ and $\langle p_x^2 \rangle$, and show that there is a penalty in a very large uncertainty Δp_x in the momentum. Check also that the results for Δx and Δp_x agree with Heisenberg's uncertainty relation.

d) Express the expectation value of the kinetic energy in terms of the uncertainty Δx. What happens if you insist on letting Δx go to zero?

Problem 2

An infinite square well of length L has energy eigenvectors $|n\rangle$,

$$\hat{H} |n\rangle = E_n |n\rangle \,, \tag{6}$$

where the energy eigenvalues are

$$E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2}, \quad n = 1, 2, 3, \ldots \tag{7}$$

The position space wavefunctions are

$$\langle x|n\rangle \equiv \psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin \frac{\pi n x}{L}, & 0 < x < L, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

a) Is the energy eigenstate also an eigenstate of the momentum operator $\hat{p}$? *Hint:* You can use the position representation expressions $\hat{p} = -i\hbar\frac{d}{dx}$ and $\psi_n(x)$.

b) In the lectures we saw that the wavefunctions $\psi_n(x)$ are orthonormal:

$$\int_0^L dx\ \psi_n^*(x)\psi_m(x) = \delta_{nm}.$$

Use this, along with the completeness relation

$$1 = \int_{-\infty}^{\infty} dx\ \ |x\rangle \langle x| \,, \tag{9}$$

to show that the energy eigenvectors $|n\rangle$ are orthonormal,

$$\langle m|n\rangle = \delta_{mn}. \tag{10}$$

c) Use the completeness relation in eq. (9) to find an expression for the *momentum space wavefunction* $\langle p|n\rangle \equiv \phi_n(p)$ written in terms of $\langle p|x\rangle$ and $\langle x|n\rangle$.

d) By inserting the plane wave solutions

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}, \tag{11}$$

show that

$$\phi_n(p) = \frac{1}{\sqrt{\pi\hbar L}} \int_0^L dx\ \ \sin\frac{\pi n x}{L} e^{-ipx/\hbar}. \tag{12}$$

Calculate the integral, and plot the probability density in momentum space $|\phi_n(p)|^2$ for $n = 1, 2, 3$.

Problem 3

The ladder operators for the harmonic oscillator are defined as

$$a = \frac{1}{\sqrt{2\hbar m\omega}}[i\hat{p} + m\omega\hat{x}], \tag{13a}$$

$$a^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}[-i\hat{p} + m\omega\hat{x}], \tag{13b}$$

where $\hat{p}$ and $\hat{x}$ are the momentum and position operators. The ladder operators can be used to effectively calculate expectation values using the relations

$$a\,|n\rangle = e^{-i\omega t}\sqrt{n}\,|n-1\rangle\,, \tag{14a}$$

$$a^\dagger\,|n\rangle = e^{i\omega t}\sqrt{n+1}\,|n+1\rangle\,, \tag{14b}$$

where we have included the time-dependence in the energy eigenstates $|n\rangle$.

a) Show that the momentum and position expectation values are both zero for all states $|n\rangle$. *Hint:* Express $\hat{p}$ and $\hat{x}$ in terms of a and $a^\dagger$.

b) Consider a system in the state

$$|\psi\rangle = A\,|n\rangle + B\,|m\rangle\,, \tag{15}$$

where $n \neq m$ are integers. What is the dual vector $\langle\psi|$? What condition must A and B fulfill in order for $|\psi\rangle$ to be normalized?

c) Calculate the expectation value $\langle x\rangle$ for the state $|\psi\rangle$ for any n and m. What values must m take in order for the expectation value $\langle x\rangle$ to be non-zero?