FY2045 Quantum Mechanics I

Fall 2023

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Week 8

Two-particle Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla_1^2 - \frac{\hbar^2}{2m}\nabla_2^2 + V(\mathbf{r}_1, \mathbf{r}_2),$$

Non-interacting particles

If $V(\mathbf{r}_1, \mathbf{r}_2) = V(\mathbf{r}_1) + V(\mathbf{r}_2)$, we can write $\hat{H} = \hat{H}_1 + \hat{H}_2$ and use separation of variables:

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2),$$

with single-particle states $\psi_{a/b}$ satisfying $\hat{H}_1\psi_a = E_a\psi_a$ and $\hat{H}_2\psi_b = E_b\psi_b$.

Indistinguishable particles

Identical particles not distinguishable in quantum mechanics — must have

$$|\Psi(\mathbf{r}_1, \mathbf{r}_2, t)|^2 = |\Psi(\mathbf{r}_2, \mathbf{r}_1, t)|^2.$$

In 3D we have two possibilities:

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, t) = \begin{cases} +\Psi(\mathbf{r}_2, \mathbf{r}_1, t) & \text{for bosons,} \\ -\Psi(\mathbf{r}_2, \mathbf{r}_1, t) & \text{for fermions.} \end{cases}$$

In 2D it is possible to have particle with in-between statistics — *anyons*. See, e.g., The story behind the mysterious anyon particles.

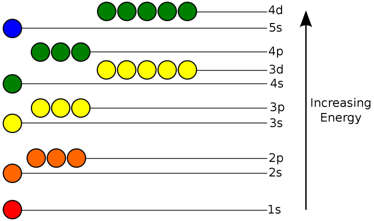
Two non-interacting particles

$$\psi(\mathrm{r}_1, \mathrm{r}_2) = \begin{cases} \frac{1}{\sqrt{2}}[\psi_a(\mathrm{r}_1)\psi_b(\mathrm{r}_2) + \psi_b(\mathrm{r}_1)\psi_a(\mathrm{r}_2)], & \text{bosons} \\ \frac{1}{\sqrt{2}}[\psi_a(\mathrm{r}_1)\psi_b(\mathrm{r}_2) - \psi_b(\mathrm{r}_1)\psi_a(\mathrm{r}_2)], & \text{fermions} \end{cases}$$

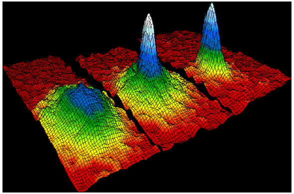
Pauli exclusion principle

Fermions cannot be in the same single-particle state.

Electrons in an atom



Bose-Einstein condensation



Figures by Richard Parsons and National Institute of Standards and Technology (NIST)

Spin and statistics

Spin statistics theorem

Particles with **integer spin** are **bosons**, and particles with **half-integer spin** are **fermions.**

Bosons

Spin 0: Higgs

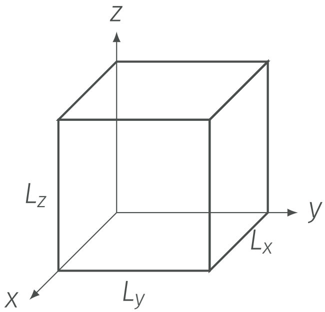
Spin 1: Photon

Fermions

Spin $\frac{1}{2}$: electron, proton, neutron, ...

Three-dimensional box

3D infinite square well



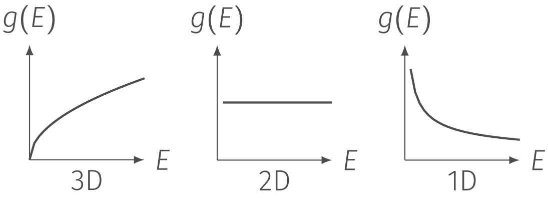
$$\psi_{n_x n_y n_z}(x, y, z) = \psi_{n_x}(x)\psi_{n_y}(y)\psi_{n_z}(z)$$
$$E_{n_x n_y n_z} = E_{n_x} + E_{n_y} + E_{n_z}$$

Density of states

$$g(E) = \frac{\text{Number of quantum states}}{\text{Energy interval}} = \frac{dN}{dE}.$$

For a particle in a box

$$g(E) \propto \begin{cases} \sqrt{E}, & 3\text{D}, \\ \text{const.}, & 2\text{D}, \\ 1/\sqrt{E}, & 1\text{D}. \end{cases}$$



Periodic boundary conditions

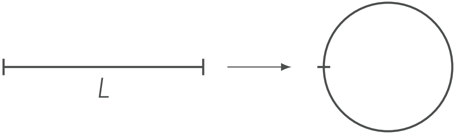
For a box of length L_i in direction i, we connect the edges at 0 and L_i.
Uniqueness of wavefunction:

$$\psi(x) = \psi(x + L).$$

For free particles $\psi \propto e^{ikx}$, we get

$$k = \frac{2\pi n}{L}, \quad n \in \mathbb{Z}$$

1D



2D

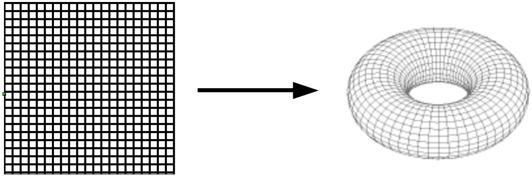


Figure from http://complex.upf.es/ josep/CA