FY2045 Quantum Mechanics I

Fall 2023

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Week 6

Quantization of Angular Momentum

Angular momentum operators

Define angular momentum operator $\mathbf{J}$ with components J_x, J_y, and J_z — Hermitian operators with commutation relations

$$
\begin{aligned}
[J_x, J_y] &= i\hbar J_z, \\
[J_y, J_z] &= i\hbar J_x, \\
[J_z, J_x] &= i\hbar J_y.
\end{aligned}
$$

Also define $\mathbf{J}^2 = J_x^2 + J_y^2 + J_z^2$, which satisfies

$$[\mathbf{J}^2, J_i] = 0,\ i = x, y, z.$$

Since they commute, we can find **simultaneous eigenvectors** of $\mathbf{J}^2$ and e.g. J_z.

Ø11.2, H8.1-8.2, G4.3

Eigenvalues

The eigenvectors satisfy the eigenvalue equations

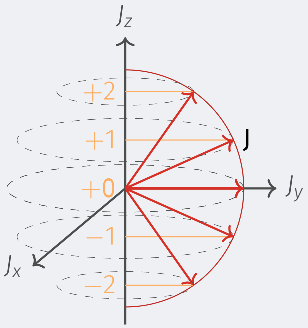
$$\mathbf{J}^2 |j, m\rangle = \hbar^2 j(j+1) |j, m\rangle \,,$$
$$J_z |j, m\rangle = \hbar m |j, m\rangle \,,$$

with

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots,$$
$$m = -j, -j+1, \ldots, j-2, j-1, j.$$

Example: $j = 2$

Orientation of $\mathbf{J}$ for different values of m:



Based on figure by Izaak Neutelings

Orbital angular momentum

From classical expression: $\mathbf{L} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$.
Simultaneous eigenvectors of $\mathbf{L}^2$ and L_z:

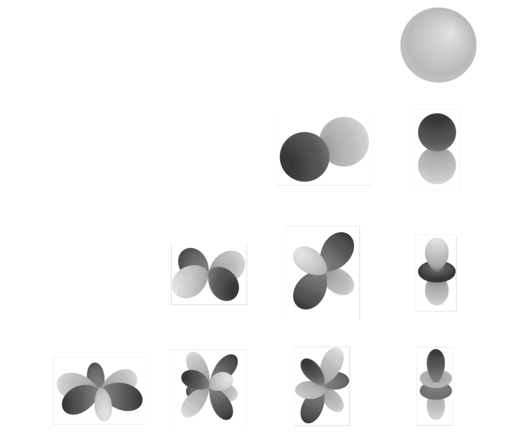
$$\mathbf{L}^2 |l, m\rangle = \hbar^2 l(l+1) |l, m\rangle ,$$
$$L_z |l, m\rangle = \hbar m |l, m\rangle .$$

Position space basis vectors in spherical coordinates, $|\mathbf{r}\rangle \equiv |r, \theta, \phi\rangle$, gives
$\psi_{lm}(r, \theta, \phi) = \langle r, \theta, \phi | l, m\rangle = R(r) Y_{lm}(\theta, \phi)$.

$Y_{lm}(\theta, \phi)$ are **spherical harmonics** .

$$Y_{lm} \propto e^{im\phi} \Rightarrow |m|, l = 0, 1, 2, \ldots .$$

Y_{lm} with $l = 0 - 3$, and $m \geq 0$



Example — Hydrogen atom

Coulomb potential is spherically symmetric:

$$\hat{H} = \left[-\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} \right) - \frac{e^2}{4\pi\epsilon_0 r} + \frac{\mathbf{L}^2}{2mr^2} \right].$$

Simultaneous eigenfunctions of $\hat{H}$, $\mathbf{L}^2$ and L_z:

$$\psi_{nlm}(r,\theta,\phi) = R_{nl}(r) Y_{lm}(\theta,\phi),$$

with eigenvalues

$$\hat{H}: \quad E_n = -\frac{13.6\text{ eV}}{n^2}, \quad n = 1,2,3,\ldots,$$

$$\mathbf{L}^2: \quad \hbar^2 l(l+1), \quad 0 \le l < n,$$

$$L_z: \quad \hbar m, \quad m = -l, -l+1, \ldots, l-1, l.$$

Energy levels

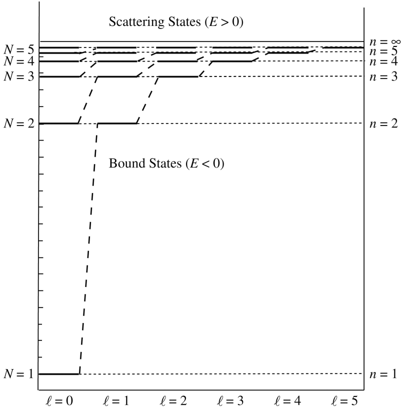
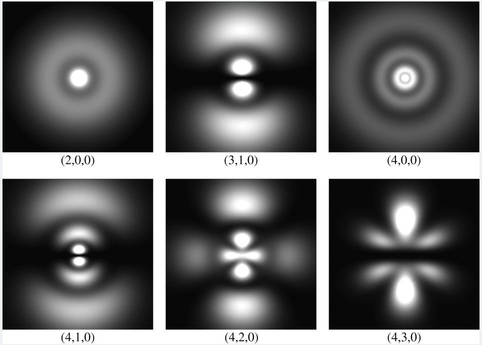
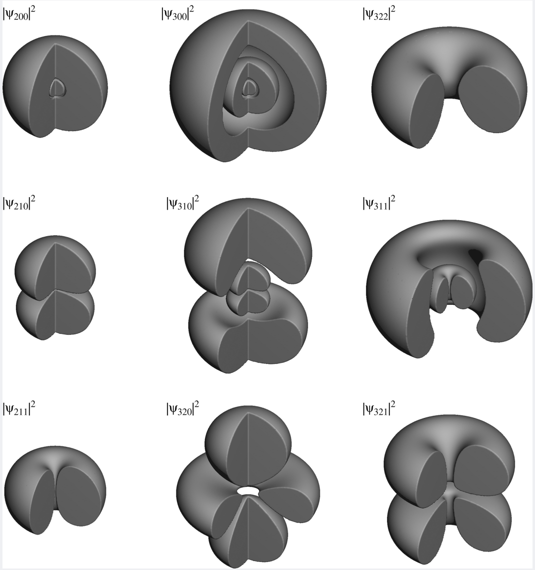


Figure from Griffiths and Schroeter, ©Cambridge University Press 2018





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Spin

Classical magnetic moment

A current loop has a magnetic (dipole) moment $\boldsymbol{\mu}$, which in a magnetic field $\mathbf{B}$ leads to a torque

$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B},$$

and force

$$\mathbf{F} = \boldsymbol{\nabla}(\boldsymbol{\mu} \cdot \mathbf{B}).$$

A particle with charge q and mass m has a magnetic moment due to the orbital motion

$$\boldsymbol{\mu}_L = \frac{q}{2m}\mathbf{L}.$$

$\boldsymbol{\mu}_L$ and $\mathbf{L}$ are not constants of motion:

Larmor precession

In a homogeneous magnetic field, the force is zero, and the torque is perpendicular to both $\boldsymbol{\mu}_L$ and $\mathbf{B}$. This leads to a precessing $\mathbf{L}$:

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\omega}_L \times \mathbf{L},$$

where $\boldsymbol{\omega}_L = -q\mathbf{B}/(2m)$ is the Larmor frequency.

QM magnetic moment due to L

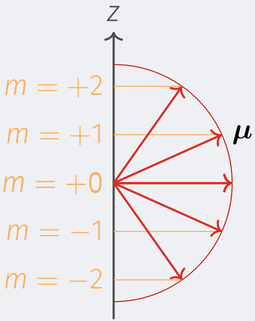
In QM $\mathbf{L}$ is quantized, meaning that $|\boldsymbol{\mu}_L|$ and $(\mu_L)_z$ can only take certain values. For electrons ($q = -e$) we get

$$|\boldsymbol{\mu}_L| = \frac{e}{2m_e}|\mathbf{L}| = \mu_B\sqrt{l(l+1)}, \quad l = 0, 1, 2\ldots,$$

$$(\mu_L)_z = -\frac{e}{2m}L_z = -m\mu_B, \quad m = 0, \pm 1, \cdots \pm l,$$

with the **Bohr magneton**

$$\mu_B = \frac{e\hbar}{2m_e}.$$



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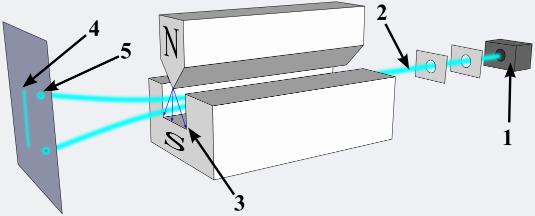
Stern-Gerlach experiment

In an inhomogeneous magnetic field, the force **F** on a magnetic moment will depend on the direction of $\boldsymbol{\mu}$.



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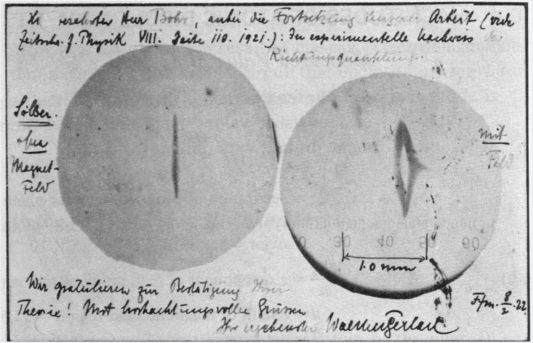


Emilio Segrè Visual Archives

See article in Physics Today for more about the struggles of Stern and Gerlach.

Spin — Intrinsic angular momentum

We define a **spin** operator $\mathbf{S}$ with components S_x, S_y and S_z, satisfying

$$[S_i, S_j] = i\hbar \sum_k \epsilon_{ijk} S_k,$$
$$[\mathbf{S}^2, S_i] = 0,$$

with eigenvalue equations

$$\mathbf{S}^2 |s, m\rangle = \hbar^2 s(s+1) |s, m\rangle, \quad s = 0, \frac{1}{2}, 1, \frac{3}{2} \ldots,$$
$$S_z |s, m\rangle = \hbar m |s, m\rangle, \quad m = -s, -s+1, \ldots, s-1, s.$$

We also have ladder operators

$$S_\pm |s, m\rangle = (S_x \pm iS_y) |s, m\rangle = \hbar\sqrt{s(s+1) - m(m \pm 1)} |s, m \pm 1\rangle.$$

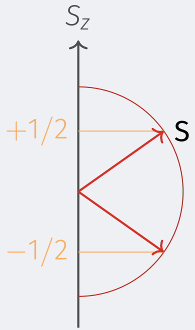
The electron spin

The electron has $s = \frac{1}{2}$, meaning it can only have $m = \pm\frac{1}{2}$, **spin up** and **spin down**.
The intrinsic magnetic moment is

$$\boldsymbol{\mu}_e = -g_e \frac{e}{2m_e}\mathbf{S} = -\frac{g_e \mu_B}{\hbar}\mathbf{S} \equiv \gamma_e \mathbf{S},$$

with electron gyromagnetic ratio γ_e, and g factor

$$g_e = 2.002\,319\,304\,362\,56 \pm 0.000\,000\,000\,000\,35.$$



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Spins of other particles

Proton: $s = 1/2$

$$\boldsymbol{\mu_p} = 5.59\frac{e}{2m_p}\mathbf{S}_p.$$

Much smaller than μ_e since $m_e \ll m_p$.

Neutron: $s = 1/2$

$$\boldsymbol{\mu_e} = -3.83\frac{e}{2m_n}\mathbf{S}_n$$

Photon: $s = 1$

Massless: can only have $m = \pm\hbar$, not $m = 0$.

Spin $\frac{1}{2}$

Abstract formulation

For $s = \frac{1}{2}$ we have two possible eigenvectors of $\mathbf{S}^2$ and S_z:

$$\left|\frac{1}{2}, +\frac{1}{2}\right\rangle \equiv |+\rangle \equiv |\uparrow\rangle\,,$$
$$\left|\frac{1}{2}, -\frac{1}{2}\right\rangle \equiv |-\rangle \equiv |\downarrow\rangle\,,$$

forming a complete set in a two-dimensional Hilbert space.

Can expand arbitrary vector $|\chi\rangle$ in the 2D Hilbert space:

$$\begin{aligned} |\chi\rangle &= \langle\uparrow|\chi\rangle\,|\uparrow\rangle + \langle\downarrow|\chi\rangle\,|\downarrow\rangle \\ &\equiv a_+\,|\uparrow\rangle + a_-\,|\downarrow\rangle\,. \end{aligned}$$

$a_\pm$: probability amplitude of measuring spin up (down) and leaving the spin in state $|\uparrow\rangle$ ($|\downarrow\rangle$).

Matrix formulation

Represent state $|\chi\rangle$ by two-element column matrix

$$\chi = \begin{pmatrix} a_+ \\ a_- \end{pmatrix},$$

called a **spinor**. The adjoint state $\langle\chi|$ is represented by the adjoint matrix

$$\chi^\dagger = (\chi^T)^* = \begin{pmatrix} a_+^* & a_-^* \end{pmatrix}.$$

Require normalization,

$$\chi^\dagger\chi = 1.$$

Eigenspinors of S_z

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ spin up,}$$

$$\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ spin down.}$$

Orthonormal:

$$\chi_\sigma^\dagger\chi_{\sigma'} = \delta_{\sigma\sigma'}$$

Matrix formulation

Operators represented by 2×2 matrices

$$\mathbf{S}^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{3}{4}\hbar^2 \mathbb{1},$$

and

$$S_x = \frac{\hbar}{2}\sigma_x, \quad S_y = \frac{\hbar}{2}\sigma_y, \quad S_z = \frac{\hbar}{2}\sigma_z.$$

Connection between matrix and abstract operator:

$$F = \begin{pmatrix} ++ & +- \\ -+ & -- \end{pmatrix} = \begin{pmatrix} \langle + | \hat{F} | + \rangle & \langle + | \hat{F} | - \rangle \\ \langle - | \hat{F} | + \rangle & \langle - | \hat{F} | - \rangle \end{pmatrix}.$$

Pauli matrices

Dimensionless matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

satisfying

$$[\sigma_i, \sigma_j] = \sigma_i\sigma_j - \sigma_j\sigma_i = 2i\epsilon_{ijk}\sigma_k,$$

$$\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\mathbb{1}.$$