Lecture notes 12

12 Magnetic moments. Spin

Written by Ingjald Øverbø

The hypothesis about the electron spin (intrinsic angular momentum) was published already in 1925 by Uhlenbeck and Goudsmit, almost at the same time as the discovery by Heisenberg and Schrödinger of quantum mechanics and Pauli's formulation of the exclusion principle. The main experimental clues leading to the spin hypothesis were

- the fine structure of optical spectra
- the Zeeman effect
- the Stern–Gerlach experiment

All these effects involve the behaviour of a particle with spin and/or orbital angular momentum in a magnetic field.

In section 12.1 in these notes, we begin by considering the magnetic moment due to the orbital motion of a particle, both classically and quantum-mechanically. We then review the experiment of Stern and Gerlach, which clearly reveals the intrinsic angular momentum (spin) of the electron. We also give a survey of the spins of other particles.

In section 12.2 we establish a formalism for spin $\frac{1}{2}$, based on the general discussion of angular momenta in Lecture notes 11.

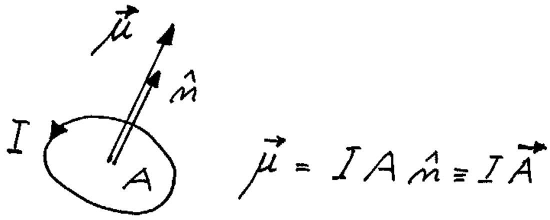
12.1 Magnetic moments connected with orbital angular momentum and spin

12.1.a Classical magnetic moment

(Hemmer p 178, 1.5 in B&J)

First a little bit of classical electrodynamics.

Small update September 2022 by H.G. Hugdal.



In electromagnetic theory (see e.g. D.J. Griffiths, *Introduction to Electrodynamics*, chapter 6) one learns that an infinitesimal current loop, placed in a magnetic field, experiences a torque

$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B} \tag{T12.1}$$

and a force

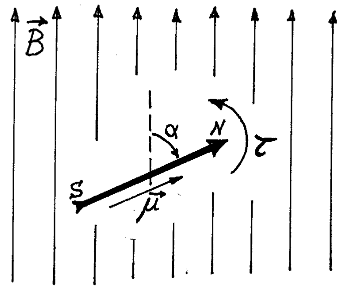
$$\mathbf{F} = -\boldsymbol{\nabla}(-\boldsymbol{\mu}\cdot\mathbf{B}). \tag{T12.2}$$

Here $\boldsymbol{\mu}$ is the **magnetic (dipole) moment** of the infinitesimal current loop. For a planar loop encircling an area A this magnetic moment is

$$\boldsymbol{\mu} = I\,A\,\hat{\mathbf{n}} \equiv I\mathbf{A},$$

where I is the current and $\hat{\mathbf{n}}$ is the unit vector perpendicular to the loop plane.

The same effect is observed when a magnetic needle is placed in a megnetic field.



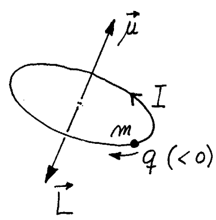
The needle experiences a torque

$$\tau(\alpha) = -\mu B\sin\alpha = -\frac{\partial}{\partial\alpha}(-\mu B\cos\alpha) = -\frac{\partial}{\partial\alpha}(-\boldsymbol{\mu}\cdot\mathbf{B}). \tag{T12.3}$$

Note that this relation is analogous to $F_x(x) = -\partial V/\partial x$. Equations (T12.2) and (T12.3) show that the magnetic moment placed in the magnetic field $\mathbf{B}$ corresponds to the interaction energy

$$\boxed{E_\mu = -\boldsymbol{\mu}\cdot\mathbf{B}.} \tag{T12.4}$$

This potential energy is by choice equal to zero when E_μ is perpendicular to $\boldsymbol{\mu} \perp \mathbf{B}$, and is maximal (and positive) when $\boldsymbol{\mu}$ and $\mathbf{B}$ are antiparallel. The energy has its lowest value when $\boldsymbol{\mu}$ is parallel to $\mathbf{B}$; this is the orientation preferred by the needle.



Another example: The figure shows a particle with mass m and charge q (< 0) which is kept moving in a classical circular orbit by a central field $V(r)$. This constitutes a current loop. With the radius r, velocity v and revolution frequency $\nu = v/(2\pi r)$ the current is $I = q\nu$. This results in a magnetic moment

$$|\boldsymbol{\mu}| = I\,A = q\,\frac{v}{2\pi r}\,\pi r^2 = \tfrac{1}{2}qrv = \tfrac{1}{2}q\,|\mathbf{r}\times\mathbf{v}|.$$

Thus the magnetic moment due to the motion of the charge is proportional to the orbital angular momentum $\mathbf{L} = m\,\mathbf{r}\times\mathbf{v}$:

$$\boldsymbol{\mu}_L = \frac{q}{2m}\,\mathbf{L}. \qquad \text{(T12.5)}$$

The ratio $q/2m$ between these two quantities is known as the **gyromagnetic ratio**.

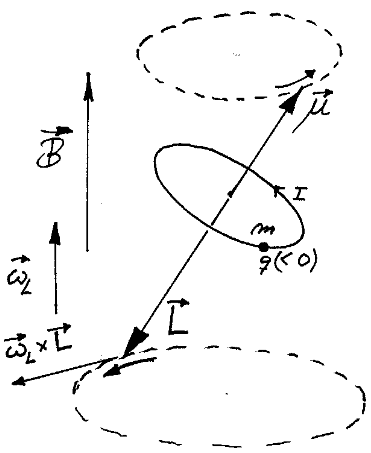
Let us add that for this kind of classical motion the angular momentum $\mathbf{L}$ and the magnetic moment $\boldsymbol{\mu}_L$ are not constants of motion when $\mathbf{B}$ differs from zero. According to Newtons 2. law and equation (T12.1) we have that

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau} = \boldsymbol{\mu}_L \times \mathbf{B} = -\frac{q}{2m}\,\mathbf{B}\times\mathbf{L} \equiv \boldsymbol{\omega}_L \times \mathbf{L}. \qquad \text{(T12.6)}$$

From this equation it follows that $\mathbf{L}$ (and hence $\boldsymbol{\mu}_L$) **precess**. The precession frequency,

$$\boldsymbol{\omega}_L \equiv -\frac{q}{2m}\,\mathbf{B}, \qquad \text{(T12.7)}$$

is known as the **Larmor frequency**.



With these classical considerations in mind, let us see what quantum mechanics has to say about these things.

12.1.b Magnetic moment due to orbital motion — quantum-mechanical treatment

From the classical relation (T12.5) we see that the quantum-mechanical operator corresponding to the observable $\boldsymbol{\mu}_L$ (the magnetic moment due to the orbital motion) must be

$$\widehat{\boldsymbol{\mu}}_L = \frac{q}{2m}\,\widehat{\mathbf{L}} = \frac{q}{2m}\,\mathbf{r}\times\widehat{\mathbf{p}} \qquad \text{(T12.8)}$$

for a particle with mass m and charge q. As an example we may consider the "doughnut" state

$$\psi_{211} = -(64\pi a_0^5)^{-1/2}\, r\, e^{-r/2a_0} \sin\theta\, e^{i\phi}$$

for the hydrogen atom. This is an eigenstate of $\widehat{L}_z$ with eigenvalue $\hbar$. According to (T12.8) it is then also an eigenstate of

$$(\widehat{\boldsymbol{\mu}}_L)_z = \frac{-e}{2m_e}\,\widehat{L}_z,$$

with eigenvalue $-e\hbar/(2m_e)$. (In this example we neglect the difference between the electron mass and the reduced mass.)

It is instructive to consider this example more closely. The probability distribution resembles a doughnut and is of course time independent, as is the case for all stationary states,

$$\rho_{211} = \frac{1}{64\pi a_0^5}\, r^2 e^{r/a_0} \sin^2\theta.$$

Inside this doughnut there is a probability current. This can be calculated from the formula for the probability-current density:

$$\mathbf{j} = \Re\mathrm{e}\left[\psi^* \frac{\hbar}{im}\,\boldsymbol{\nabla}\psi\right]; \qquad \boldsymbol{\nabla} = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\mathbf{e}}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}.$$

Here we observe that the components of the gradient in the r- and θ directions give imaginary contributions to the expression inside the brackets, and hence no contribution to $\mathbf{j}$. Thus the probability in the doughnut is flowing in the ϕ direction. We find that

$$\mathbf{j} = \hat{\mathbf{e}}_\phi \frac{\hbar/m_e}{64\pi a_0^5}\, r\, e^{-r/a_0} \sin\theta = \hat{\mathbf{e}}_\phi\, \rho_{211} \cdot \frac{\hbar}{m_e}\frac{1}{r\sin\theta}.$$

Defining a velocity by the relation $\mathbf{j} = \rho_{211}\mathbf{v}$, we can calculate the local velocity of the probability current:

$$v = \frac{|\mathbf{j}|}{\rho_{211}} = \frac{\hbar}{m_e}\frac{1}{r\sin\theta} = \alpha c\, \frac{a_0}{r\sin\theta}.$$

Here, $r\sin\theta$ — the distance from the z axis — is of course typically of the order of the Bohr radius a_0, so that the velocity is of the order of αc, as could be expected. Note, however, that this velocity is variable; the probability distribution is not rotating around the z axis as a "solid" doughnut.

The probability ρd^3r in the volume element d^3r corresponds to a momentum contribution $\rho d^3r \cdot m\mathbf{v} = m\mathbf{j}d^3r$ and an electric-current contribution $-e\mathbf{j}d^3r$. Since $\langle\, \mathbf{p}\, \rangle$ is real, the former also follows from

$$\langle\, \mathbf{p}\, \rangle = \int \psi^* \frac{\hbar}{i}\,\boldsymbol{\nabla}\psi\, d^3r = m\int \Re\mathrm{e}\left[\psi^* \frac{\hbar}{im}\,\boldsymbol{\nabla}\psi\right] d^3r = \int m\mathbf{j}\, d^3r.$$

The expectation values of L_z and $(\mu_L)_z$ can be calculated in a similar manner, and the results are as expected: Because $(\mathbf{r}\times\hat{\mathbf{e}}_\phi)_z = r\sin\theta$, we have

$$\begin{aligned}\langle\, L_z\, \rangle_{211} &= \int \psi_{211}^* \left(\mathbf{r}\times\frac{\hbar}{i}\boldsymbol{\nabla}\right)_z \psi_{211} d^3r = \int (\mathbf{r}\times m_e\mathbf{j})_z d^3r \\ &= \int r m_e j \sin\theta\, d^3r = \int \hbar\rho_{211} d^3r = \hbar,\end{aligned}$$

and

$$\langle\,(\mu_L)_z\,\rangle_{211} = \tfrac{1}{2}\int [\mathbf{r}\times(-e\mathbf{j})]_z d^3r = -\frac{e\hbar}{2m_e}\int \rho_{211}d^3r = -\frac{e\hbar}{2m_e}, \quad \text{q.e.d.}$$

This example illustrates the connection between the orbital angular momentum and the magnetic moment of a moving particle.

As illustrated by this example, the operator relation (T12.8) implies that the magnetic moment $\boldsymbol{\mu}_L$ due to the orbital motion is quantized in the same way as the orbital angular momentum. This means that the size and one of the components can have sharp values simultaneously. For an electron (with $q = -e$) this implies that the size can take the values

$$|\boldsymbol{\mu}_L| = \frac{e}{2m_e}|\mathbf{L}| = \mu_B\sqrt{l(l+1)}\;; \qquad l = 0, 1, 2, \cdots, \tag{T12.9}$$

while for example the z component can take the values

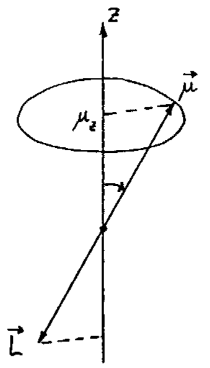
$$(\mu_L)_z = -\frac{e}{2m_e}L_z = -m\mu_B\;; \quad m = 0, \pm1, \pm2, \cdots, \pm l. \tag{T12.10}$$

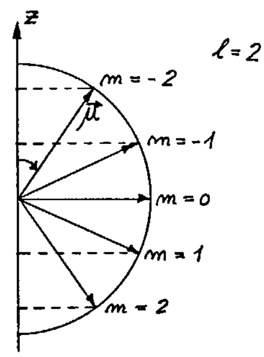
Here the quantity

$$\mu_B \equiv \frac{e\hbar}{2m_e} \qquad \text{(1 Bohr magneton)} \tag{T12.11}$$

is the natural unit for the magnetic moment for the elctron, just as $\hbar$ is the natural unit for angular momenta. [$1\,\mu_B = 5.788\cdot10^{-5}$ eV/T(esla).]

We note also that the quantization of L_z and $(\mu_L)_z$ corresponds to a so-called space quantization, that is, quantized values of the angle between the vector $\boldsymbol{\mu}_L$ and the z axis.



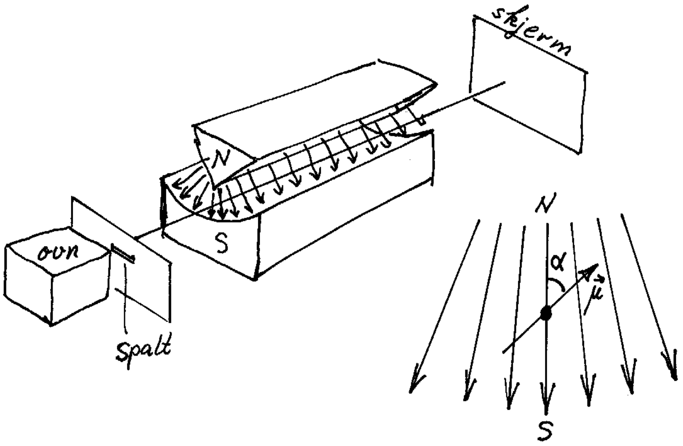


Note also that because $\mathbf{L}$ and $\boldsymbol{\mu}_L$ are not *classical* constants of motion in a magnetic field, they also are not **quantum-mechanical constants of motion**. With this statement we mean that the *expectation values* $\langle\,\mathbf{L}\,\rangle$ and $\langle\,\boldsymbol{\mu}_L\,\rangle$ are not constant. These vector-valued expectation values will in fact precess in the same manner as the classical quantities.

12.1.c The Stern–Gerlach experiment and the electron spin

(Section 4.4 in Griffiths, 1.5 in B&J.)

In a *homogeneous* magnetic field the magnetic moment $\boldsymbol{\mu}$ experiences a torque $\boldsymbol{\tau} = \boldsymbol{\mu}\times\mathbf{B}$; the force $\mathbf{F} = -\boldsymbol{\nabla}(-\boldsymbol{\mu}\cdot\mathbf{B})$ then is equal to zero, and the magnetic moment only *precesses* around $\mathbf{B}$, as discussed above. In 1921 Stern and Gerlachs had the following idea:



Let us try to measure the magnetic moment of an atom by sending a *beam* of these atoms (collimated by a slit) through an *inhomogeneous* magnetic field in order to measure the *deflection.* If the beam is positioned such that it passes through a vertical field, it should be deflected vertically (in the z direction). We can show this by assuming an inhomogeneous field given by

$$\mathbf{B} = ax\hat{\mathbf{e}}_x - (B_0 + az)\hat{\mathbf{e}}_z,$$

where $a = -\partial B_z/\partial z$ is a suitable constant. You can easily check that this field is divergence free ($\boldsymbol{\nabla}\cdot\mathbf{B} = 0$), as is the case for all magnetic fields. The force becomes

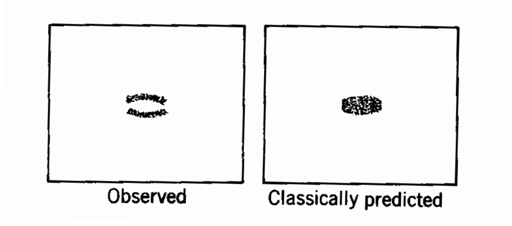
$$\mathbf{F} = \boldsymbol{\nabla}(\boldsymbol{\mu}\cdot\mathbf{B}) = \boldsymbol{\nabla}(\mu_x ax - \mu_z(B_0 + az)) = a(\mu_x\hat{\mathbf{e}}_x - \mu_z\hat{\mathbf{e}}_z). \tag{T12.12}$$

Here we must now remember that $\boldsymbol{\mu}$ precesses rapidly around the $\mathbf{B}$ field, that is, about the z axis. The x-component μ_x therefore averages to zero and gives no deflection in the x direction. Thus we get a deflection in the positive or negative z direction, depending on the force $F_z = -a\mu_z$. The measurement of the deflection therefore is a measurement of μ_z. (Note that μ_z is constant during the precession.)

In 1921 (before quantum mechanics) Stern and Gerlach expected that the directions of the magnetic moments $|\boldsymbol{\mu}|$ of the atoms entering the magnetic field should be randomly distributed, corresponding to a continuous variation of μ_z between $-|\boldsymbol{\mu}|$ and $+|\boldsymbol{\mu}|$. They were hoping to measure the maximal deflection up and down (corresponding to $\alpha = 0$ and $\alpha = \pi$), making it possible to calculate $|\boldsymbol{\mu}|$, using the velocity of the atoms, the atomic mass, the length of the magnet, and the parameter a.

After 1925, quantum mechanics has taught us that a measurement of μ_z must give one of its eigenvalues (according to the measurement postulate). Since the direction of $\boldsymbol{\mu}$ and hence μ_z are quantized according to (T12.10), we must therefore expect to observe a *quantized deflection.* If the total angular momentum of the atom is "integral" (given by an integral angular-momentum quantum number l), we should then expect to find $2l + 1$ discrete deflections, that is, an odd number of pictures of the slit on the screen in the figure above.

The experiment, however, showed something else: Stern and Gerlach (1922) used a gas of silver atoms ($Z = 47$) heated in a furnace. The result was *two* distinct pictures of the slit on the screen.



A similar experiment with hydrogen atoms, conducted by Phipps and Taylor in 1927, also resulted in two pictures of the slit.

Spin and magnetic moment of the electron

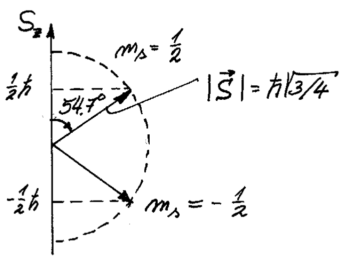
(8.3.3 in Hemmer)

The explanation was found by Uhlenbeck and Goudsmit in 1925, and is most easily understood in the hydrogen case: Even with $l = 0$ the electron in the hydrogen atom has a magnetic moment $\boldsymbol{\mu}_S$, which causes the deflection of the orbit of each atom. This magnetic moment is connected with an *intrinsic* angular momentum of the electron, the so-called **spin angular momentum S**, simply called the **spin**. As all other angular momenta, the spin can be characterized by an angular-momentum quantum number which is usually denoted by s, so that $|\mathbf{S}| = \hbar\sqrt{s(s+1)}$, and such that the z component can take the values

$$S_z = m_s\hbar, \qquad \text{where} \quad m_s = -s,\, -s+1,\, -s+2, \cdots,\, +s.$$

This is analogous to $m = L_z/\hbar$ taking the values $-l,\, -l+1,\, -l+2, \cdots,\, +l$ for a given orbital angular-momentum quantum number l. From the general discussion of angular momenta in Lecture notes 11, it follows that a spin quantum number must in general take one of the values $s = 0,\, \frac{1}{2},\, 1,\, \frac{3}{2},\, 2,\, \cdots$. The number of m_s values is $2s+1$, in analogy with the $2l+1$ values of L_z.

The fact that we observe *two* discrete deflections in the Stern–Gerlach experiment then leads to the interpretation that in this case $2s+1$ is equal to 2; that is, the spin quantum number s of the electron is equal to $\frac{1}{2}$. Thus the electron has spin "one half", as we use to express it. This corresponds to $|\mathbf{S}| = \hbar\sqrt{s(s+1)} = \hbar\sqrt{3/4} = 0.866\,\hbar$. The two possible values of the magnetic quantum number of the electron spin then are $m_s = \pm\frac{1}{2}$, corresponding to $S_z = \pm\frac{1}{2}\hbar$. These two spin states are commonly denoted by **spin up** and **spin down**.



Experiments show that the intrinsic magnetic moment connected with the spin is

$$\boldsymbol{\mu}_S = g_e \frac{-e}{2m_e}\,\mathbf{S}. \tag{T12.13}$$

Here the factors in front of $\mathbf{S}$ is the product of the **gyromagnetic ratio** we found for the orbital motion, and g_e which is a dimensionless *factor.* Very accurate measurements show that this factor is

$$g_e = 2 \times 1.001159652187\ (\pm 4) \qquad (\textbf{gyromagnetic factor} \text{ of the electron}), \qquad \text{(T12.14)}$$

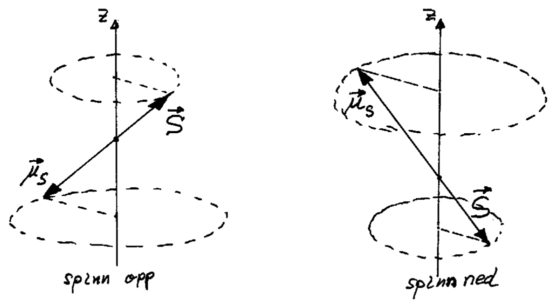
where the uncertainty (± 4) is in the last digit.

To sum up, we may state that the two discrete deflections observed in the SG-experiment are due to the fact that there are only two possible values of μ_z, S_z and F_z:

$$m_s = +\tfrac{1}{2}: \qquad S_z = +\tfrac{1}{2}\hbar, \quad (\mu_S)_z = -\tfrac{1}{2}g_e\,\frac{e\hbar}{2m_e} = -\tfrac{1}{2}g_e\mu_B, \quad F_z = -a\mu_z = \tfrac{1}{2}g_e\,a\mu_B,$$

$$m_s = -\tfrac{1}{2}: \qquad S_z = -\tfrac{1}{2}\hbar, \quad (\mu_S)_z = +\tfrac{1}{2}g_e\mu_B, \quad F_z = -a\mu_z = -\tfrac{1}{2}g_e\,a\mu_B.$$

With $a > 0$ we see that the upper beam emerging from the SG magnet corresponds to measuring $S_z = +\frac{1}{2}\hbar$ (spin up), while the lower beam corresponds to $S_z = -\frac{1}{2}\hbar$ (spin down).[1]



12.1.d Spins of other particles

(8.3 in Hemmer)

The proton and the neutron

What about the proton, which is the *nucleus* of the hydrogen atom — doesn't also this particle have spin $\frac{1}{2}$ and a magnetic moment? The answer is yes, but it turns out that that the magnetic moment of the proton is much smaller than that of the electron;

$$\boldsymbol{\mu}_p = 5.59\,\frac{e}{2m_p}\,\mathbf{S}_p \qquad (g_p = 5.59). \qquad \text{(T12.15)}$$

Thus, for the proton the natural unit for the magnetic moment is

$$\mu_N \equiv \frac{e\hbar}{2m_p} = 3.1524515 \cdot 10^{-8}\ \text{eV/T(esla)} \qquad (1 \text{ nuclear magneton}). \qquad \text{(T12.16)}$$

[1]With hydrogen atoms or silver atoms from a furnace, the SG experiment gives a fifty-fifty distributions of atoms in the upper and lower beams; the probabilities of measuring spin up or spin down are both equal to $\frac{1}{2}$. (This holds also if the SG magnet is turned so that another component of the spin is measured.) One then says that the beam emerging from the furnace is unpolarized. On the other hand, after the passage through the inhomogeneous field, the atoms on the upper beam are all left in the state "spin up", they constitute an ensemble of completely spin-polarized atoms. Thus a Stern–Gerlach device, characterized essentially by the direction of the inhomogeneous field, can be used both to measure a spin component and to prepare an ensemble in a well-defined spin state.

We note that the gyromagnetic factor $g_p = 5.59$ for the proton deviates quite much from the factor $g_e \approx 2$ for the electron. This has to do with the fact that the proton is a composite particle, consisting of two u-quarks with charge $q = 2e/3$ and one d-quark with $q = -e/3$. The neutron is also a composite particle, consisting of one u-quark and two d-quarks, and then it is perhaps not so surprising that it has a non-zero magnetic moment,

$$\boldsymbol{\mu}_n = -3.83 \, \frac{e}{2m_p} \, \mathbf{S}_p \qquad (g_n = -3.83), \tag{T12.17}$$

even if it is a neutral particle.

It is important to note that μ_p and μ_n are a factor ~ 1000 smaller than μ_e. Therefore the behavior of atoms in a magnetic field is largely determined by the electrons. [2]

Why do we measure only two discrete deflections also for the silver atoms? The answer lies in the fact that 46 out of the 47 silver electrons are in a quantum-mechanical state in which both the total orbital momentum and the total spin are equal to zero. Both the orbital and the spin angular momenta then are determined by electron number 47. This electron is in an s-state, with $l = 0$. Thus the total magnetic moment of the atom simply is the intrinsic magnetic moment of electron no 47 (as for a free electron). (Here we may neglect a possible magnetic moment of the nucleus, because this will anyhow be about a factor 1000 smaller than that of the electron.)

Spins of other particles

The electron and its antiparticle, the positron, are only two of several so-called **leptons** (particles that do not interact strongly), which all have spin $\frac{1}{2}$. Among these are the muons $\mu^\pm$ (with mass $m_\mu = 105.66$ MeV/c^2) and $\tau^\pm$ ($m_\tau = 1777$ MeV/c^2), which in a way are heavier relatives of the electron. Thus the g factor of the muon is also very close to 2:

$$\boldsymbol{\mu}_{\mu^-} = g_\mu \, \frac{-e}{2m_\mu} \, \mathbf{S}_\mu, \tag{T12.18}$$

$$g_\mu = 2 \times 1.0011659160\ (\pm 6) \qquad (\textbf{gyromagnetic factor} \text{ of the muon}). \tag{T12.19}$$

Related to these charged leptons there is a set of almost massless *neutral* leptons which interact only through the **weak force**. These are the electron **neutrino** ν_e and the electron **anti-neutrino** $\bar{\nu}_e$. Related to the muon there is a corresponding pair (ν_μ and $\bar{\nu}_\mu$), and related to the tau lepton there is ν_τ and $\bar{\nu}_\tau$. All these neutrinos are spin-one-half particles.

The photon is somewhat special, because the component of the spin along the direction of motion can only have the two values $\pm\hbar$, corresponding to spin 1. This is due to the fact that the photon is massless. Related to the photon are the massive **vector bosons**, $W^\pm$ ($M_W \approx 80.4$ MeV/c^2) and Z^0 ($M_Z \approx 91.2$ MeV/c^2), which have spin 1, and which "mediate" the **electro-weak interactions** together with the photon. Spin 1 is also found in the gluons, which mediate the **strong forces**. The **graviton**, the carrier of the gravitational force, is believed to have spin 2.

All the remaining particle species are so-called **hadrons** (which all interact strongly). These can be divided into two groups, one of which is the **baryons** (p, n, Λ, Ω^-, $\cdots$), which

[2] There are exceptions: In nuclear magnetic resonance (NMR) it is an important point that the coupling term $-\boldsymbol{\mu}\cdot\mathbf{B}$ between the magnetic field and the magnetic moments of the nuclei are so small that they correspond to radio-frequency energies.

are all three-quark systems. Because the quarks have spin $\frac{1}{2}$, it follows from the rules for the "addition of angular momenta" (which we shall treat later) that the baryons can only have "half-integral" spins (1/2, 3/2, etc). The other group consists of the **mesons** $(\pi, K, \rho, \omega, \cdots)$, which are quark-antiquark systems, and which therefore can only have integral spins (e.g. 0 for the π mesons and 1 for the ρ mesons).

12.2 Formalism for spin $\frac{1}{2}$

(Hemmer 8.3, Griffiths 4.4, B&J 6.7–8)

What the spin is *not*

Uhlenbeck and Goudsmit based their spin hypothesis (in 1925) on the classical notion of a rotating electron, with a certain mass and charge distribution. Modern scattering experiments have shown, however, that the size of the electron, if it differs at all from zero, must be smaller than 10^{-18} m. It has also turned out to be impossible to construct a classical model with a mass and charge distribution that reproduces the spin and the magnetic moment of the electron. Thus the electron behaves as a point particle, and we have to state that the spin and the magnetic moment (with $g_e \approx 2$) of this particle can not be understood as the result of any kind of "material rotation" which can be pictured classically and which can be described in terms of a wave equation and a wave function. The latter is only possible for the orbital angular momentum, for which l can take only integer values, while the spin quantum number s can also take half-integral values, depending on which particle we are looking at. Note that for a given particle species s is completely fixed. Thus for the electron the intrinsic angular momentum has *no choice*, it *has* to be $|\widehat{\mathbf{S}}| = \hbar\sqrt{3/4}$, in contrast to the orbital angular momentum which can vary, even if it is quantized. Again we see that the spin does not behave as we would expect for an ordinary rotational motion.

12.2.a "Ladder" of ket vectors for spin $\frac{1}{2}$

Even if we do not quite "understand" what the spin is, we have a perfectly applicable theoretical model in the abstract ket-vector formalism for angular momenta which was developed in Lecture notes 11, in the sense that we can apply it to processes involving the spin and accurately predict the outcome of experiments.

The starting point is (T11.38) and (T11.39), which for the spin (with $\widehat{\mathbf{J}} = \widehat{\mathbf{S}}$, and with $j = s = \frac{1}{2}$) take the form

$$\begin{aligned} \widehat{\mathbf{S}}^2 \,|\tfrac{1}{2}, m\rangle &= \tfrac{3}{4}\hbar^2 \,|\tfrac{1}{2}, m\rangle, \\ \widehat{S}_z \,|\tfrac{1}{2}, m\rangle &= \hbar m \,|\tfrac{1}{2}, m\rangle \,; \quad m = \pm\tfrac{1}{2}, \end{aligned} \tag{T12.20}$$

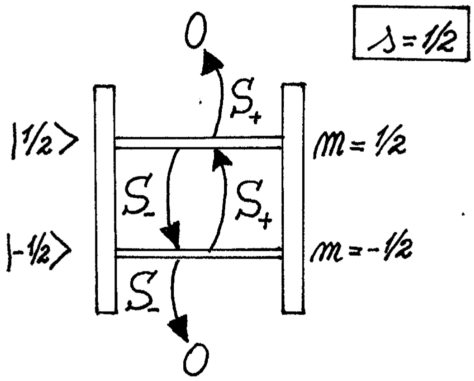
The two eigenvectors are

$$|\tfrac{1}{2}, +\tfrac{1}{2}\rangle \equiv |+\tfrac{1}{2}\rangle \equiv |+\rangle \equiv |\!\uparrow\rangle \qquad \text{(spin up)}$$

and

$$|\tfrac{1}{2}, -\tfrac{1}{2}\rangle \equiv |-\tfrac{1}{2}\rangle \equiv |-\rangle \equiv |\!\downarrow\rangle \qquad \text{(spin down)}.$$

Note that we may choose the labels as we wish. Thus we may use e.g. only m or only the *sign* of m; without stating explicitly that $s = \frac{1}{2}$. These two vectors form a *basis* of orthogonal and normalized vectors; they are the *only* eigenvectors of the Hermitian operator $\widehat{S}_z$. Thus the state space of this half-integral spin is a two-dimensional vector space, and is spanned by a ladder with only two rungs:



The ladder operators $\widehat{S}_+$ and $\widehat{S}_-$ take us up and down in the ladder: According to (T11.52) we have (with $j = s = \frac{1}{2}$)

$$\widehat{S}_\pm \, |m\rangle = \hbar\sqrt{(\tfrac{1}{2} \mp m)(\tfrac{3}{2} \pm m)} \;\; |m \pm 1\rangle, \qquad m = \pm\tfrac{1}{2},$$

or explicitly:

$$\widehat{S}_+ \, |+\tfrac{1}{2}\rangle = 0 \quad ; \qquad \widehat{S}_+ \, |-\tfrac{1}{2}\rangle = \hbar \, |+\tfrac{1}{2}\rangle \; ;$$

$$\widehat{S}_- \, |-\tfrac{1}{2}\rangle = 0 \quad ; \qquad \widehat{S}_- \, |+\tfrac{1}{2}\rangle = \hbar \, |-\tfrac{1}{2}\rangle \; . \qquad \text{(T12.21)}$$

From these formulae we see that

$$\widehat{S}_+^2 \, |\pm\tfrac{1}{2}\rangle = 0 \quad \text{and} \quad \widehat{S}_-^2 \, |\pm\tfrac{1}{2}\rangle = 0. \qquad \text{(T12.22)}$$

The last two formulae are particular for spin $\frac{1}{2}$ and are of course due to the fact that we have a ladder with only two rungs.

Note that as long as we consider only the spin, that is, do not take into account the other degrees of freedom for the particle, the two vectors $|+\frac{1}{2}\rangle$ and $|-\frac{1}{2}\rangle$ are a complete set; the spin space is a two-dimensional sub-space of the Hilbert space. The completeness relation of this set is (cf (T10.29))

$$\sum_{m=\pm\frac{1}{2}} |m\rangle\langle m| = |+\tfrac{1}{2}\rangle\langle +\tfrac{1}{2}| \; + \; |-\tfrac{1}{2}\rangle\langle -\tfrac{1}{2}| = \mathbb{1}. \qquad \text{(T12.23)}$$

Applying this unit operator we may expand an arbitrary vector $|\chi\rangle$ in the two-dimensional spin space:

$$\begin{aligned} |\chi\rangle &= \sum_m |m\rangle\langle m|\chi\rangle = \sum_m \langle m|\chi\rangle \, |m\rangle = \langle +\tfrac{1}{2}|\chi\rangle \, |+\tfrac{1}{2}\rangle + \langle -\tfrac{1}{2}|\chi\rangle \, |-\tfrac{1}{2}\rangle \\ &\equiv a_+ \, |+\tfrac{1}{2}\rangle + a_- \, |-\tfrac{1}{2}\rangle. \end{aligned}$$

The normalization condition for the vector $|\chi\rangle$ is

$$\langle\chi|\chi\rangle \equiv \langle\chi|\cdot|\chi\rangle = \left(a_+^*\langle+\tfrac{1}{2}| + a_-^*\langle-\tfrac{1}{2}| \right) \cdot \left(a_+|+\tfrac{1}{2}\rangle + a_-|-\tfrac{1}{2}\rangle \right) = |a_+|^2 + |a_-|^2 = 1. \tag{T12.24}$$

Note that the coefficient (or projection) $a_+ = \langle+\frac{1}{2}|\chi\rangle$ is the probability amplitude of measuring spin up ($S_z = \frac{1}{2}\hbar$) and leaving the spin in the state $|+\frac{1}{2}\rangle$. In the same manner, $a_- = \langle-\frac{1}{2}|\chi\rangle$ is the amplitude of "measuring spin down".

12.2.b Matrix formulation. The Pauli matrices

(8.3 in Hemmer, 6.8 in B&J)

Ket- and bra-vectors are represented by column and row vectors

It is customary, both for spin $\frac{1}{2}$ and higher angualar momenta, to use a matrix representation of vectors and operators, with the $2j+1$ vectors $|j,m\rangle$ (where m goes from j to $-j$) as basis. In the spin-$\frac{1}{2}$ case the general vector $|\chi\rangle$ is then represented by the column matrix

$$\begin{pmatrix} \langle+\frac{1}{2}|\chi\rangle \\ \langle-\frac{1}{2}|\chi\rangle \end{pmatrix} = \begin{pmatrix} a_+ \\ a_- \end{pmatrix} \equiv \chi, \tag{T12.25}$$

which we denote simply by χ, and which is called a **spinor**. The basis vector$|+\frac{1}{2}\rangle$ is represented by [3]

$$\begin{pmatrix} \langle+\frac{1}{2}|+\frac{1}{2}\rangle \\ \langle-\frac{1}{2}|+\frac{1}{2}\rangle \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \chi_+ \tag{T12.26}$$

and the spin-down vector $|-\frac{1}{2}\rangle$ is represented by

$$\begin{pmatrix} \langle+\frac{1}{2}|-\frac{1}{2}\rangle \\ \langle-\frac{1}{2}|-\frac{1}{2}\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \chi_-. \tag{T12.27}$$

These are known as the **Pauli spinors**, after Pauli, who invented the theory for spin $\frac{1}{2}$ in 1925. We note that these spinors are orthonormal,

$$\chi_{\epsilon'}^\dagger \chi_\epsilon = \delta_{\epsilon'\epsilon}.$$

The adjoint of the ket vector $|\chi\rangle = a_+|+\frac{1}{2}\rangle + a_-|-\frac{1}{2}\rangle$, which is $(|\chi\rangle)^\dagger = \langle\chi|$, is represented by the **adjoint matrix** (transpose and complex conjugate), that is, by the row matrix

$$\chi^\dagger = (a_+^* \;\; a_-^*).$$

The normalization condition may then be written as

$$\chi^\dagger\chi = (a_+^* \;\; a_-^*) \begin{pmatrix} a_+ \\ a_- \end{pmatrix} = |a_+|^2 + |a_-|^2 = 1,$$

[3]Note: In the literature it is not uncommon to use an equality sign between the vector $|\chi\rangle$ and the column matrix or spinor χ:

$$|\chi\rangle = \chi \equiv \begin{pmatrix} a_+ \\ a_- \end{pmatrix}.$$

The first equality sign should be read as "represented by".

Operators are represented by 2×2-matrices

Since we are using the eigenvectors of $\widehat{\mathbf{S}}^2$ and $\widehat{S}_z$ as a basis, these operators must be represented by diagonal matrices; cf the discussion of matrix mechanics in Lecture notes 10 (section 10.4): From (T12.20) we find that

$$\langle m'|\widehat{\mathbf{S}}^2|m\rangle = \tfrac{3}{4}\hbar^2\,\delta_{m'm} \qquad \text{and} \qquad \langle m'|\widehat{S}_z|m\rangle = \hbar m\,\delta_{m'm},$$

or

$$\boxed{\mathbf{S}^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \equiv \frac{3}{4}\hbar^2\,\mathbb{1} \qquad \text{and} \qquad \boxed{S_z = \tfrac{1}{2}\hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \equiv \tfrac{1}{2}\hbar\sigma_z. \quad \text{(T12.28)}$$

Note that the matrix elements are "numbered" acording to the scheme $\begin{pmatrix} ++ & +- \\ -+ & -- \end{pmatrix}$, and that $\mathbb{1}$ here stands for the unit matrix. Note also that the diagonal elements are identical to the eigenvalues, and hence to the possible results for measurements of $\mathbf{S}^2$ and S_z; a measurement of S_z can only give $+\frac{1}{2}\hbar$ or $-\frac{1}{2}\hbar$, no matter which state the spin is prior to the measurement.

To find the matrices representing the remaining operators we take (T12.21) as the starting point. Projecting these relations on $|+\frac{1}{2}\rangle$ and $|-\frac{1}{2}\rangle$ we get for example

$$(S_+)_{++} = \langle +\tfrac{1}{2}|\widehat{S}_+|+\tfrac{1}{2}\rangle = 0 \qquad \text{and} \qquad (S_+)_{+-} = \langle +\tfrac{1}{2}|\widehat{S}_+|-\tfrac{1}{2}\rangle = \hbar.$$

In this manner we find that the ladder operators $\widehat{S}_+ = \widehat{S}_x + i\widehat{S}_y$ and $\widehat{S}_- = \widehat{S}_x - i\widehat{S}_y$ are represented by the matrices

$$\boxed{S_+ = \tfrac{1}{2}\hbar \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}} \qquad \text{and} \qquad \boxed{S_- = \tfrac{1}{2}\hbar \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}.} \qquad \text{(T12.29)}$$

By adding and subtracting these we find at last the matrices for $\widehat{S}_x$ and $\widehat{S}_y$:

$$\boxed{S_x = \tfrac{1}{2}\hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \equiv \tfrac{1}{2}\hbar\sigma_x \qquad \text{and} \qquad \boxed{S_y = \tfrac{1}{2}\hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}} \equiv \tfrac{1}{2}\hbar\sigma_y. \qquad \text{(T12.30)}$$

Here we observe that the matrices S_x and S_y become non-diagonal, which was of course to be expected, since $\widehat{S}_x$, $\widehat{S}_y$ and $\widehat{S}_z$ do not commute, and since we have chosen the eigenvectors of $\widehat{S}_z$ as our basis. To gain some experience with these matrices you should now check the eigenvalue equations

$$S_z\,\chi_\pm = \pm\tfrac{1}{2}\hbar\,\chi_\pm, \qquad \text{(T12.31)}$$

which correspond to (T12.20), and also the relations

$$S_+\,\chi_+ = 0, \qquad S_+\,\chi_- = \hbar\,\chi_+, \quad \text{etc},$$

which are (T12.21) on matrix form.

A small exercise: Check that these matrices satisfy $[S_x, S_y] = i\hbar S_z$, that is, the angular-momentum algebra.

Another exercise: Show that the matrix squares S_x^2, S_y^2 and S_z^2 are equal. What does this tell us about the possible eigenvalues of S_x and S_y, and hence about possible measurement results?

Solution: If you square the matrices (T12.30), you will find that

$$S_x^2 = S_y^2 = \frac{\hbar^2}{4} \cdot \mathbb{1} = S_z^2. \tag{T12.32}$$

This implies that the eigenvalues of S_x and S_y are $\pm\frac{1}{2}\hbar$, just as for S_z. Thus the possible measured values for the x- and y-components of $\mathbf{S}$ are the same as for S_z. This should of course be expected from symmetry considerations; we are free to choose the orientation of our coordinate system, and then there can be no difference between the possible measured values for S_x and S_z.

Pauli matrices. Rules of calculation

To simplify the notation it is customary to use the dimensionless matrices

$$\boxed{\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},} \tag{T12.33}$$

which are called the **Pauli matrices**. The matrix representation of the spin operator may then be written as

$$\mathbf{S} = \tfrac{1}{2}\hbar\boldsymbol{\sigma}; \qquad \boldsymbol{\sigma} \equiv \hat{\mathbf{e}}_x\sigma_x + \hat{\mathbf{e}}_y\sigma_y + \hat{\mathbf{e}}_z\sigma_z.$$

The eigenvalue equation (T12.20) takes the form

$$\sigma_z\,\chi_+ = 1 \cdot \chi_+ \,, \qquad \sigma_z\,\chi_- = -1 \cdot \chi_-. \tag{T12.34}$$

For future use we include some rules of calculation for the Pauli matrices. The angular-momentum algebra $[S_x, S_y] = i\hbar S_z$ takes the form

$$[\sigma_x, \sigma_y] = 2i\sigma_z, \;\; \text{etc}, \tag{T12.35}$$

which can be written as [4]

$$\sigma_i\sigma_j - \sigma_j\sigma_i = 2i\,\varepsilon_{ijk}\,\sigma_k. \tag{T12.36}$$

[4]**We use Einstein's summation convention**: When a latin index as e.g. k occurs twice in a term, it means that this index is summed over, from $k = 1$ to $k = 3$. Example:

$$a_k b_k \equiv \sum_{k=1}^{3} a_k b_k.$$

(If an index occurs more than twice in a term we have made a mistake).

Here we use Einstein's summatiuon convention. The Levi-Civita tensor is defined by

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{for } ijk = 123,\ 231,\ 312, \\ -1 & \text{for } ijk = 132,\ 321,\ 213, \\ 0 & \text{otherwise.} \end{cases} \tag{T12.37}$$

By direct calculation (or from (T12.32)) we see that

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv 1\!\!1. \tag{T12.38}$$

It is also straightforward to show that the matrices *anticommute*;

$$\sigma_x\sigma_y + \sigma_y\sigma_x = 0, \quad \text{etc.}$$

See the relations (T12.22) which imply that S_x and S_y satisfy the equation

$$S_x^2 - S_y^2 + i(S_xS_y + S_yS_x) = 0.$$

These relations, which can be collected in the formula

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\, 1\!\!1, \tag{T12.39}$$

are special for spin $\frac{1}{2}$.

By combining (T12.36) and (T12.39) we find that

$$\sigma_i\sigma_j = \delta_{ij}\, 1\!\!1 + i\,\varepsilon_{ijk}\,\sigma_k. \tag{T12.40}$$

From this we can derive the following relation which holds for arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$:

$$(\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{b}) = \sigma_i a_i\ \sigma_j b_j = \delta_{ij}a_ib_j 1\!\!1 + i\,\varepsilon_{ijk}\,a_ib_j\sigma_k,$$

that is,

$$\boxed{(\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{b}\, 1\!\!1 + i\,\boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{b}).} \tag{T12.41}$$

In particular we have for $\mathbf{a} = \mathbf{b} = \hat{\mathbf{n}}$ (where $\hat{\mathbf{n}}$ is a unit vector):

$$(\boldsymbol{\sigma}\cdot\hat{\mathbf{n}})(\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}) = \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}\, 1\!\!1 = 1\!\!1. \tag{T12.42}$$

Note that the relations (T12.38) are special cases of this formula, and that it corresponds to

$$(\mathbf{S}\cdot\hat{\mathbf{n}})^2 = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar^2}{4}\cdot 1\!\!1. \tag{T12.43}$$

This formula is a generalization of (T12.32), and had to be expected considering the symmetry argument above; no matter which component $\mathbf{S}\cdot\hat{\mathbf{n}}$ of the spin we choose to measure, the result must be $\frac{1}{2}\hbar$ or $-\frac{1}{2}\hbar$.

12.2.c The spin direction

(B&J p 311)

Even if the formalism above was based on abstract Hilbert vectors (without a wave-function representation), we can use it to calculate all the relevant physical quantities which can be measured experimentally. This means that we have a well-functioning *physical theory* for the spin. Thus we can calculate e.g. the expectation value $\langle S_z \rangle$ with the following recipe (which we may as usual call the "sandwich" or "burger" recipe):

$$\langle S_z \rangle_\chi = \langle \chi | \widehat{S}_z | \chi \rangle = \chi^\dagger S_z \chi. \qquad \text{(T12.44)}$$

A small exercise: Check the last step above, using

$$\langle \chi | \widehat{S}_z | \chi \rangle = \langle \chi | \, 1\!\!1 \cdot \widehat{S}_z \cdot 1\!\!1 \, | \chi \rangle = \sum_{mn} \langle \chi | m \rangle \langle m | \widehat{S}_z | n \rangle \langle n | \chi \rangle.$$

[Hint: Set $\langle m | \chi \rangle = a_m$, so that $\langle \chi | m \rangle = a_m^*$.]

As an example, we may choose the state $\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which gives [5]

$$\begin{aligned}
\langle S_z \rangle &= \chi_+^\dagger \, S_z \, \chi_+ = \chi_+^\dagger \, \tfrac{1}{2}\hbar \, \chi_+ = \tfrac{1}{2}\hbar, \\
\langle S_x \rangle &= \chi_+^\dagger \, S_x \, \chi_+ = (1,\, 0)\tfrac{1}{2}\hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0, \\
\langle S_y \rangle &= \chi_+^\dagger \, S_y \, \chi_+ = (1,\, 0)\tfrac{1}{2}\hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0.
\end{aligned}$$

Thus, for the state $\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we find that

$$\langle \mathbf{S} \rangle = \tfrac{1}{2}\hbar \, \langle \boldsymbol{\sigma} \rangle = \tfrac{1}{2}\hbar \, \hat{\mathbf{e}}_z.$$

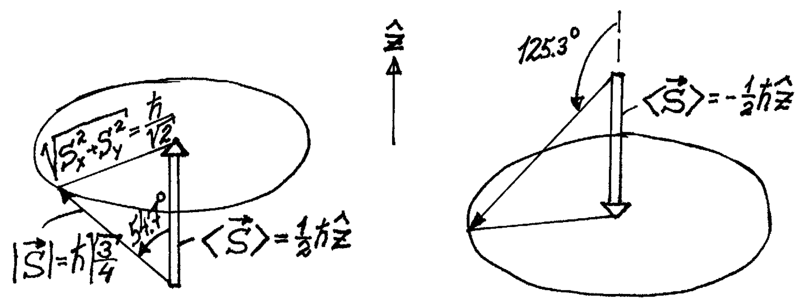
This is a good reason to call this state **spin up.** The *direction* of $\langle \mathbf{S} \rangle$, which is $\langle \boldsymbol{\sigma} \rangle = \hat{\mathbf{e}}_z \equiv \hat{\mathbf{z}}$ in this case, may be called the **spin direction**. Therefore we also use the notation

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \chi_{\hat{\mathbf{z}}} \, , \qquad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \chi_{-\hat{\mathbf{z}}}, \qquad \text{(T12.45)}$$

where the direction $\langle \boldsymbol{\sigma} \rangle$ is used as an index.

spin up: $\chi_{\hat{\mathbf{z}}}$ $\quad$ spin down: $\chi_{-\hat{\mathbf{z}}}$

$\langle \boldsymbol{\sigma} \rangle = \hat{\mathbf{z}}$ $\quad$ $\langle \boldsymbol{\sigma} \rangle = -\hat{\mathbf{z}}.$



[5] $\langle S_z \rangle$ becomes equal to $\frac{1}{2}\hbar$ because χ_+ is an eigenspinor with sharp $S_z = \frac{1}{2}\hbar$. That $\langle S_x \rangle$ and $\langle S_y \rangle$ become equal to zero is not strange. It *would* be strange if $\langle \mathbf{S} \rangle$ were to point in another direction than $\hat{\mathbf{z}}$ after the *preparation* of the state χ_+.

These states are illustrated in the figure, which is equivalent to that used for the orbital angular momentum in Lecture notes 5. This picture is of limited value. It gives the correct $|\mathbf{S}|$ $(=\hbar\sqrt{3/4}\ )$ and $S_z = \pm\frac{1}{2}\hbar$, and also $S_x^2 + S_y^2 = \mathbf{S}^2 - S_z^2 = \frac{1}{2}\hbar^2$. The picture may also (barely) remind us that the expectation values of S_x and S_y are equal to zero for both the two states $\chi_\pm$,

$$\langle\, S_x \,\rangle = \langle\, S_y \,\rangle = 0.$$

The drawback of this illustration is that it can mislead us to believe that the *direction* of $\mathbf{S}$ is an observable, which it is not, because the components are not compatible; they can not have sharp values simultaneously. The angle $\arccos(\frac{1}{2}/\sqrt{\frac{3}{4}}) = 54.7°$ or $125.3°$ between $\mathbf{S}$ and the z axis thus is only another way to illustrate the fact that the two possible values of S_z are $\pm\frac{1}{2}\hbar$, compared with the fixed value $|\mathbf{S}| = \hbar\sqrt{\frac{3}{4}}$. This is the same type of **space quantization** that was found for the orbital angular momentum in Lecture notes 5. When the particle is sent through the Stern–Gerlach magnet in section 12.1.c, the spin has to choose one of these two states (no matter what state it is in before the "measurement"). This choice determines whether the force on the magnetic moment acts upwards or downwards, and decides whether the particle goes into the upper or lower beam.

However, a Stern–Gerlach magnet can also be rotated. Suppose that we rotate it 90 degrees so that it measures S_x instead of S_z. What kind of states do we *then* get for the ensembles in the "upper" and "lower" beam? According to the measurement postulate the answer is that the spin must then choose between the eigenstates $\chi_{\pm\hat{\mathbf{x}}}$ of S_x.

> **A small exercise**: Set $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$ and solve the eigenvalue problem
>
> $$S_x \begin{pmatrix} a \\ b \end{pmatrix} = e \begin{pmatrix} a \\ b \end{pmatrix},$$
>
> where e is the eigenvalue.

The solution is that the eigenvalues are $\pm\frac{1}{2}\hbar$ (the same as for S_z) and that the eigenstates with spin directions $\hat{\mathbf{x}}$ and $-\hat{\mathbf{x}}$ are respectively

$$\chi_{\hat{\mathbf{x}}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \chi_{-\hat{\mathbf{x}}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{T12.46}$$

In the same manner one finds that

$$\chi_{\hat{\mathbf{y}}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \text{and} \qquad \chi_{-\hat{\mathbf{y}}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}, \tag{T12.47}$$

with the same eigenvalues. In these spinors we have chosen the phases in such a way that the upper components are real and positive.

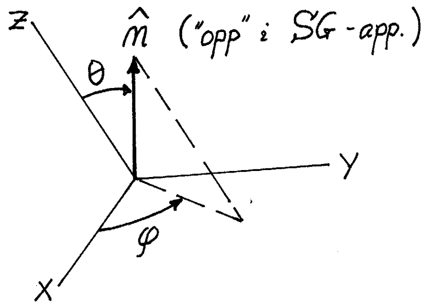
General spin-$\frac{1}{2}$ states

(Hemmer p 178, B&J p 309)

The physical rotation of the Stern–Gerlach magnet by 90 degrees may be called an **active rotation**. Consider now instead a so-called **passive rotation,** where the Stern–Gerlach device keeps the original physical orientation, while we imagine that the *coordinate system* is rotated instead. The "up" direction of the device (the original z direction) in the new coordinate system then becomes a unit vector

$$\hat{\mathbf{n}} \equiv \hat{\mathbf{x}}\,\sin\theta\cos\phi + \hat{\mathbf{y}}\,\sin\theta\sin\phi + \hat{\mathbf{z}}\,\cos\theta, \tag{T12.48}$$

characterized by the angles θ and ϕ:



In this coordinate system, the spin directions in the upper/lower beam are $\langle\,\boldsymbol{\sigma}\,\rangle = \pm\hat{\mathbf{n}}$; the spin directions are physical and measurable and remain fixed together with the device.

Viewed from the new coordinate system we are now measuring the spin component $\mathbf{S}\cdot\hat{\mathbf{n}}$. The two possible measured values are of course unchanged, and so are the physical states, but the *matrix representations* of these states are changed, because the new coordinate system corresponds to a change of basis; the Pauli spinors

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

now correspond to spin up and spin down related to the new $\hat{\mathbf{z}}$ direction.

These matrix representations, which correspond to spin "up" and spin "down" in the $\hat{\mathbf{n}}$ direction, are determined by the eigenvalue equations

$$\mathbf{S}\cdot\hat{\mathbf{n}}\,\chi_{\pm\hat{\mathbf{n}}} = \pm\tfrac{1}{2}\hbar\,\chi_{\pm\hat{\mathbf{n}}} \qquad \Longleftrightarrow \qquad \boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\,\chi_{\pm\hat{\mathbf{n}}} = \pm\,\chi_{\pm\hat{\mathbf{n}}}. \tag{T12.49}$$

It is sufficient to solve the equation $\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\,\chi_{\hat{\mathbf{n}}} = \chi_{\hat{\mathbf{n}}}$. We set $\chi_{\hat{\mathbf{n}}} = \begin{pmatrix} a \\ b \end{pmatrix}$. This spinor satisfies the eigenvalue equation

$$\begin{aligned} (\boldsymbol{\sigma}\cdot\hat{\mathbf{n}} - 1\!\!1)\,\chi_{\hat{\mathbf{n}}} &= (\sigma_x n_x + \sigma_y n_y + \sigma_z n_z - 1\!\!1)\begin{pmatrix} a \\ b \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta - 1 & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta - 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0. \end{aligned} \tag{T12.50}$$

As you can easily check, both the upper and the lower components of this equation are satisfied when

$$(\cos\theta - 1)a + \sin\theta\, e^{-i\phi}\, b = 0,$$

that is, for

$$\frac{b}{a} = \frac{1-\cos\theta}{\sin\theta}\, e^{i\phi} = \frac{\sin\frac{1}{2}\theta\, e^{i\phi}}{\cos\frac{1}{2}\theta}\,. \tag{T12.51}$$

This complex ratio between the lower and upper components is in fact all we need to know about the spinor. Admittedly, the two complex numbers a and b contain *four* real parameters, but the normalization condition $|a|^2 + |b|^2 = 1$ reduces the number to three, and one parameter may be allowed to be free, in the form of a common phase factor which is of no physical significance. Explicitly we have from the normalization condition

$$\frac{1}{|a|^2} = 1 + \left|\frac{b}{a}\right|^2 = 1 + \frac{\sin^2\frac{1}{2}\theta}{\cos^2\frac{1}{2}\theta} = \frac{1}{\cos^2\frac{1}{2}\theta},$$

that is,

$$a = e^{i\alpha}\cos\tfrac{1}{2}\theta,$$

where the phase α can be chosen freely. Hence

$$b = a\cdot\frac{b}{a} = e^{i\alpha}\sin\tfrac{1}{2}\theta\, e^{i\phi},$$

so that

$$\boxed{\chi_{\hat{\mathbf{n}}} = e^{i\alpha}\begin{pmatrix}\cos\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta\, e^{i\phi}\end{pmatrix},} \tag{T12.52}$$

where the arbitrary phase α is unimportant.

A small exercise: Show that the other state $\chi_{-\hat{\mathbf{n}}}$ (with *spin down* compared to the direction $\hat{\mathbf{n}}$) may be written as

$$\boxed{\chi_{-\hat{\mathbf{n}}} = e^{i\alpha}\begin{pmatrix}\sin\frac{1}{2}\theta \\ -\cos\frac{1}{2}\theta\, e^{i\phi}\end{pmatrix} = e^{i\alpha'}\begin{pmatrix}-\sin\frac{1}{2}\theta\, e^{-i\phi} \\ \cos\frac{1}{2}\theta\end{pmatrix}.} \tag{T12.53}$$

There are several ways to do this: You may (i) change the sign of the eigenvalue in (T12.49) and (T12.50) and proceed as above, or (ii) use that $\chi_{-\hat{\mathbf{n}}}$ must be orthogonal to $\chi_{\hat{\mathbf{n}}}$. A third method is (iii) to notice that this state has *spin up* in the direction $\hat{\mathbf{n}}' = -\hat{\mathbf{n}}$, which corresponds to the angles $\theta' = \pi - \theta$ and $\phi' = \phi + \pi$.

Some other exercises:

a) Show that for a general normalized spinor $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$ we have

$$\boxed{\begin{aligned}\langle\,\sigma_x\,\rangle &= \chi^\dagger\sigma_x\chi = \Re\mathrm{e}(2a^*b),\\ \langle\,\sigma_y\,\rangle &= \Im\mathrm{m}(2a^*b),\\ \langle\,\sigma_z\,\rangle &= |a|^2 - |b|^2,\end{aligned}} \tag{T12.54}$$

giving the spin direction

$$\boxed{\langle\,\boldsymbol{\sigma}\,\rangle_\chi = \hat{\mathbf{x}}\,\Re\mathrm{e}(2a^*b) + \hat{\mathbf{y}}\,\Im\mathrm{m}(2a^*b) + \hat{\mathbf{z}}\,(|a|^2 - |b|^2).} \tag{T12.55}$$

b) Show that the length of the real vector $\langle \boldsymbol{\sigma} \rangle$ is equal to 1. [Hint: $[\Re\mathrm{e}(a^*b)]^2 + [\Im\mathrm{m}(a^*b)]^2 = |a^*b|^2 = |a|^2|b|^2$.]

c) Show that the *direction* of $\langle \boldsymbol{\sigma} \rangle_\chi$ is determined uniquely by the complex ratio b/a. [Hint: Take a factor $|a|^2 = a^*a$ outside.]

d) Check that $\langle \boldsymbol{\sigma} \rangle = \hat{\mathbf{n}}$ for the state $\chi_{\hat{\mathbf{n}}}$, by inserting a and b from (T12.52).

e) Find $\langle \mathbf{S} \rangle$ for the state $\chi_{\hat{\mathbf{y}}}$ using the formula (T12.54).

As a conclusion of the discussions above, we can state that an arbitrary spinor $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$ corresponds to a well-defined spin direction $\langle \boldsymbol{\sigma} \rangle$, and that χ is an eigenspinor of the $\langle \boldsymbol{\sigma} \rangle$ component of $\mathbf{S}$:

$$\langle \boldsymbol{\sigma} \rangle \cdot \mathbf{S}\, \chi = \tfrac{1}{2}\hbar\, \chi.$$

Note that both $\langle \boldsymbol{\sigma} \rangle$ and χ are uniquely determined by the complex ratio b/a (modulo a phase factor for χ).

12.2.d Precession in homogeneous magnetic field

(8.3.5 in Hemmer, 4.3 in Griffiths)

A *homogeneous* magnetic field $\mathbf{B} = B\hat{\mathbf{e}}_z$ will not split a particle beam the way a Stern–Gerlach magnet does, but it will still affect the spin state, via the interaction term $-\widehat{\boldsymbol{\mu}}\cdot\mathbf{B}$. If we disregard the orbital motion of the electron and all other interactions, that is, if we consider only the behaviour of the spin, then the Hamiltonian reduces to this interaction term:

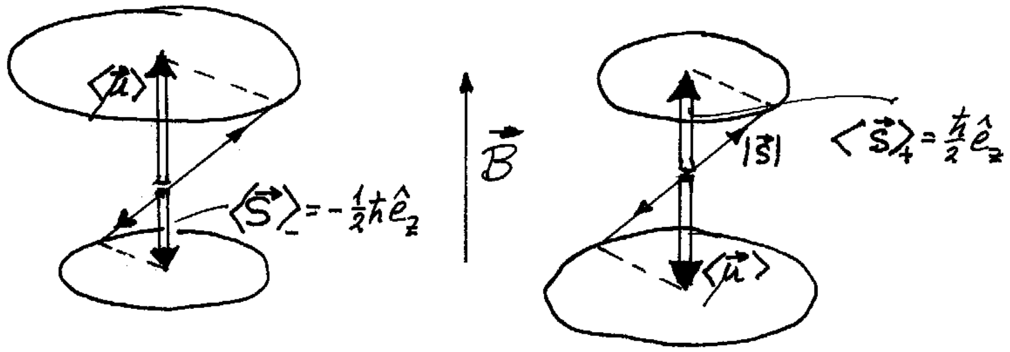
$$\widehat{H} = -\mathbf{B}\cdot\widehat{\boldsymbol{\mu}} = -\mathbf{B}\cdot\left(\frac{-g_e e}{2m_e}\,\mathbf{S}\right) \equiv \boldsymbol{\omega}\cdot\mathbf{S} = \omega S_z, \qquad \text{(T12.56)}$$

where

$$\boldsymbol{\omega} \equiv \frac{g_e e B}{2m_e}\,\hat{\mathbf{e}}_z.$$

We start by noting that since the Hamiltonian is proportional to S_z, the spin eigenfunctions $\chi_{\pm\hat{\mathbf{z}}}$ of S_z now become energy eigenstates, with the energies

$$E_\pm = \pm\tfrac{1}{2}\hbar\omega.$$



ground state $\chi_{-\hat{\mathbf{z}}}$ 1. excited state $\chi_{\hat{\mathbf{z}}}$

The state $\chi_{-\hat{\mathbf{z}}}$ has $\langle\,\boldsymbol{\mu}\,\rangle$ parallell to the $\mathbf{B}$ field, and therefore is the ground state, with the energy $E_- = -\frac{1}{2}\hbar\omega$. The first (and only) excited state has spin up and therefore $\langle\,\boldsymbol{\mu}\,\rangle$ antiparallell to the $\mathbf{B}$ field.

Each of these energy eigenstates corresponds to a stationary state,

$$\chi_\pm(t) = e^{-iE_\pm t/\hbar}\chi_{\pm\hat{\mathbf{z}}} = e^{\mp i\omega t/2}\chi_{\pm\hat{\mathbf{z}}}. \tag{T12.57}$$

In these stationary states nothing "happens", so $\langle\,\boldsymbol{\mu}\,\rangle$ stays parallell or antiparallell to $\mathbf{B}$ the whole time.

On the other hand, if we consider the superposition

$$\chi(t) = a_0\chi_+(t) + b_0\chi_-(t) = \begin{pmatrix} a_0 e^{-i\omega t/2} \\ b_0 e^{i\omega t/2} \end{pmatrix} \equiv \begin{pmatrix} a \\ b \end{pmatrix}, \tag{T12.58}$$

then this is a non-stationary state where things "happen". If we assume for simplicity that $a_0^* b_0$ is real, we have

$$a^* b = a_0^* b_0\, e^{i\omega t}.$$

Inserting into (T12.55) we then find that

$$\langle\,\mathbf{S}\,\rangle_\chi = \tfrac{1}{2}\hbar\,\langle\,\boldsymbol{\sigma}\,\rangle = \tfrac{1}{2}\hbar\left[\hat{\mathbf{e}}_x(2a_0^* b_0)\cos\omega t + \hat{\mathbf{e}}_y(2a_0^* b_0)\sin\omega t + \hat{\mathbf{e}}_z(|a_0|^2 - |b_0|^2)\right]. \tag{T12.59}$$

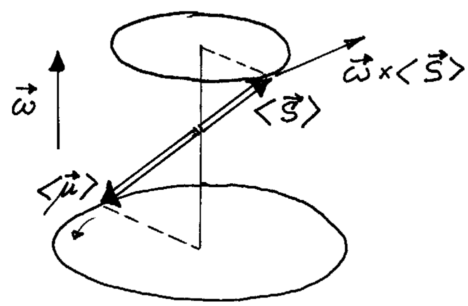
Here we see that $\langle\,S_z\,\rangle$ is constant (independent of t), while

$$\frac{d}{dt}\langle\,S_x\,\rangle = -\omega\,\langle\,S_y\,\rangle \qquad \text{and} \qquad \frac{d}{dt}\langle\,S_y\,\rangle = \omega\,\langle\,S_x\,\rangle\,. \tag{T12.60}$$

This shows that $\langle\,\mathbf{S}\,\rangle$ (and hence also $\langle\,\boldsymbol{\mu}\,\rangle$) precess around the z axis with angular frequency ω:

$$\frac{d}{dt}\langle\,\mathbf{S}\,\rangle = \boldsymbol{\omega}\times\langle\,\mathbf{S}\,\rangle\;, \qquad \frac{d}{dt}\langle\,\boldsymbol{\mu}\,\rangle = \boldsymbol{\omega}\times\langle\,\boldsymbol{\mu}\,\rangle\,. \tag{T12.61}$$

Note that this is quite analogous to the classical precession that we found for the magnetic moment on page 3 [see equation (T12.6)].



A small challenge: Use the formula ((4.19) in Hemmer) for the time development of expectation values,

$$\frac{d}{dt}\langle\,F\,\rangle = \frac{i}{\hbar}\left\langle\,[\widehat{H},\widehat{F}]\,\right\rangle,$$

to derive the formulae (T12.60).