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FY2045 Problem set 6 fall 2023

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Problem 1[1]

One often uses the analogy that the electron spin due to the electron rotating around its own axis. The classical electron radius, assuming that the electron is a solid sphere, is given by

$$r_c = \frac{e^2}{4\pi\epsilon_0 mc^2}. \tag{1}$$

This assumes that the electron's mass is given by $E = mc^2$, and the energy is given by the energy stored in its electric field. How fast must a point on the "equator" of the electron be moving in order for the classical spin angular momentum to take the value $\hbar/2$, the spin of the electron? Does this seem like a good model for the electron spin? *Hint:* Remember that the classical spin angular momentum is given by $S = I\omega$, where I is the moment of inertia and ω is the angular velocity.

[1] Based on Problem 4.28 in Griffiths.

Problem 2 — Spin 1

For a particle with spin $s = 1$ ($|S| = \hbar\sqrt{1(1+1)} = \sqrt{2}\,\hbar$), the observable S_z can take the values $\hbar$, 0, and $-\hbar$. With these three states we can associate three abstract vectors:

$$|s=1, m=1\rangle \equiv |1,1\rangle \equiv |1\rangle \;, \tag{2}$$
$$|s=1, m=0\rangle \equiv |1,0\rangle \equiv |0\rangle \;, \tag{3}$$
$$|s=1, m=-1\rangle \equiv |1,-1\rangle \equiv |-1\rangle \;. \tag{4}$$

The most general state of this system, the vector

$$|\chi\rangle = a\;|1\rangle + b\;|0\rangle + c\;|-1\rangle\,,$$

can be represented by a column matrix

$$\chi = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \langle\chi|1\rangle \\ \langle\chi|0\rangle \\ \langle\chi|-1\rangle \end{pmatrix} \;. \tag{5}$$

a) Write down the three column matrices χ_m which represent the three state vectors $|1\rangle$, $|0\rangle$ and $|-1\rangle$, and check that the matrix operator

$$S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \;, \tag{6}$$

applied to these column matrices χ_m gives the correct eigenvalues.

b) Which eigenvalues (and measured values) do you expect to find for the component S_x of this spin?

c) Suppose that the spin is in the state

$$\chi = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \;. \tag{7}$$

Show that χ is normalized. What are the probabilities of measuring $S_z = \hbar$, 0 and $-\hbar$, and what is the expectation value of S_z, in this state?

d) From the relations $S_\pm \chi_m = \hbar\sqrt{(1 \mp m)(1+1 \pm m)}\chi_{m\pm 1}$, it is possible to show that the ladder operators $\hat{S}_+$ and $\hat{S}_-$ for spin 1 are represented by the matrices

$$S_+ = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} , \tag{8}$$

$$S_- = S_+^\dagger = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} , \tag{9}$$

so that

$$S_x = \frac{1}{2}(S_+ + S_-) = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{10}$$

$$S_y = \frac{1}{2i}(S_+ - S_-) = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} . \tag{11}$$

Show that the given state χ is an eigenstate of S_x and find the eigenvalue.

e) Find the eigenstate of S_x with eigenvalue zero. Hint: Set

$$\chi = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{12}$$

and solve the eigenvalue equation

$$S_x \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 . \tag{13}$$

Problem 3 — Spin precession

We saw in the lectures that a classical magnetic moment $\boldsymbol{\mu}$ in a homogeneous magnetic field $\mathbf{B}$ precesses around the magnetic field with a frequency ω_L, the Larmor frequency. The electron spin is related to an intrinsic magnetic moment

$$\boldsymbol{\mu}_S = -g_e \frac{e}{2m_e} \mathbf{S}, \qquad \left(\frac{g_e}{2} = 1.001159652188(\pm 4) \right) ,$$

with **gyromagnetic factor** of the electron, g_e, slightly larger than 2. The spin is governed by the Hamiltonian[2]

$$\hat{H} = -\boldsymbol{\mu} \cdot \mathbf{B}.$$

Assuming a magnetic field $\mathbf{B} = B\hat{z}$, show that the *expectation value* of the spin precesses around $\mathbf{B}$ with a frequency $\omega_S = g_e eB/(2m_e)$ by deriving the formula

$$\frac{d}{dt}\langle \mathbf{S} \rangle = \boldsymbol{\omega}_S \times \langle \mathbf{S} \rangle.$$

Hint: Use the formula for the time-development of an expectation value,

$$\frac{d}{dt}\langle F \rangle = \frac{i}{\hbar}\langle [\hat{H}, \hat{F}] \rangle. \tag{14}$$

together with the angular momentum algebra, $[S_x, S_y] = i\hbar S_z$ etc, to find expressions for $d\langle S_x \rangle/dt$ etc.

[2] This Hamiltonian is very different from the Hamiltonians we have considered in position space. However, the time-development of a system is still determined by the Schrödinger equation, and hence eq. (14) still holds.