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FY2045 Problem set 3 fall 2023

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Problem 1 — Working with ket and bra vectors

Let $|1\rangle$, $|2\rangle$ and $|3\rangle$ be three orthonormalized vectors spanning a three-dimensional vector space (a subspace of "Hilbert" space).

a) Write down relations expressing that the three vectors are normalized and orthogonal.

b) Write down the completeness relation for this set of vectors, and use this relation to expand an arbitrary vector $|\psi\rangle$ (in the three-dimensional space) in terms of the basis set $|1\rangle$, $|2\rangle$ and $|3\rangle$.

c) Show that when the operator $P_1 \equiv |1\rangle\langle 1|$ acts on $|\psi\rangle$, then it "projects out" the component of $|\psi\rangle$ in the "$|1\rangle$-direction". Show also that $P_1^2 = P_1$ and that P_1 is Hermitian. (Consider the adjoint; see task **f** below.) Operators of this type are generally called projection operators. Show that also

$$P_{12} \equiv |1\rangle\langle 1| + |2\rangle\langle 2| \tag{1}$$

is a projection operator, i.e., that P_{12} has the properties $P_{12}^2 = P_{12}$ and is Hermitian.

d) Set $|a\rangle = |1\rangle$ and $|b\rangle = (1+i)|1\rangle$. What are then $\langle b|$, $\langle a|b\rangle$, $\langle b|a\rangle$, and $\langle b|b\rangle$?

e) Set $|\psi\rangle = 3^{-1/2}|1\rangle + c_1|2\rangle$. Which condition must $|c_1|^2$ satisfy if $|\psi_1\rangle$ is to be normalized? Choose c_1 real and positive. Set $|\psi_2\rangle = c_2|1\rangle + c_3|2\rangle$. Choose c_3/c_2 such that $|\psi_2\rangle$ becomes orthogonal to $|\psi_1\rangle$, and choose c_2 real and positive such that $|\psi_2\rangle$ becomes normalized. Find a third linear combination of the three vectors $|1\rangle$, $|2\rangle$ and $|3\rangle$ which is orthogonal to both $|\psi_1\rangle$ and $|\psi_2\rangle$.

f) Use the definition of the adjoint (or Hermitian conjugate)

$$\langle a|\hat{A}^\dagger|b\rangle = \langle b|\hat{A}|a\rangle^*, \tag{2}$$

to show that

$$(|c\rangle\langle d|)^\dagger = |d\rangle\langle c|. \tag{3}$$

Problem 2

We have seen that there is a one-to-one correspondence between the abstract vectors of the Hilbert space and the good old wave functions:

$$\psi_a(x) \longleftrightarrow |\psi_a\rangle \equiv |a\rangle\,,$$
$$\psi_b(x) \longleftrightarrow |\psi_b\rangle \equiv |b\rangle\,, \text{ etc,}$$

where we are free to choose whichever labels we wish to use. We have also seen that the new scalar products are identical to the old ones:

$$\langle\psi_a|\,\psi_b\rangle = \int d\tau\ \psi_a^*\psi_b. \tag{4}$$

a) In the good old position representation of quantum mechanics, the eigenvalue equation for the operator $\hat{x} = x$,

$$\hat{x}\psi_{x'}(x) = x'\psi_{x'}(x), \tag{5}$$

has the solution

$$\psi_{x'}(x) = \delta(x - x'). \tag{6}$$

To this wavefunction there corresponds a vector $|\psi_{x'}\rangle \equiv |x'\rangle$ in Hilbert space. You may also recall that the eigenvalue equation

$$\hat{p}_x\psi_p(x) = p\psi_p(x), \tag{7}$$

for the momentum operator has the solution

$$\psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}. \tag{8}$$

To this momentum eigenfunction there corresponds a vector $|\psi_p\rangle \equiv |p\rangle$ in Hilbert space. What are $\langle x'|\,p\rangle$ and $\langle p|\,x'\rangle$?

b) The operator $\hat{x}$ representing the observable x is in the Dirac formalism defined by the eigenvalue equation

$$\hat{x}\left|x'\right\rangle = x'\left|x'\right\rangle, \quad \text{where } \hat{x}^\dagger = \hat{x}, \quad \text{so that } \left\langle x'\right|\hat{x} = x'\left\langle x'\right|.$$

What is then $\langle p|\,\hat{x}\,|x'\rangle$? And what is $\langle x'|\,\hat{x}\,|p\rangle$?

c) Consider a general state vector $|\psi\rangle$. Show that

$$|\psi\rangle = \int dx\ \psi(x)\,|x\rangle\,, \tag{9}$$

and that a special case of this formula is

$$|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int dx\ e^{ipx/\hbar}\,|x\rangle\,. \tag{10}$$

d) In a similar manner, find $|x\rangle$ expressed in terms of the $|p\rangle$ basis.

Problem 3

In the lectures we argued that the existence of the scalar product in Hilbert space was guaranteed by the Schwartz inequality. You will now prove this inequality.

The norm $||\,|v\rangle\,||$ of a vector $|v\rangle$ in Hilbert space is defined as

$$||\,|v\rangle\,||^2 = \langle v|\,v\rangle \geq 0.$$

For two arbitrary vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ in Hilbert space, prove the Schwarz inequality:

$$|\,\langle\psi_1|\,\psi_2\rangle\,|^2 \leq \langle\psi_1|\,\psi_1\rangle\,\langle\psi_2|\,\psi_2\rangle\,.$$

(The two vectors do not have to be normalized.) *Hint:* Consider the vector

$$|h\rangle = |\psi_1\rangle - |\psi_2\rangle\,\frac{\langle\psi_2|\,\psi_1\rangle}{\langle\psi_2|\,\psi_2\rangle}.$$

Note that the scalar product $\langle\psi_2|\,\psi_2\rangle$ is real and positive, while $\langle\psi_2|\,\psi_1\rangle$ is a complex number. Be aware to use the correct formula for $\langle h|$, and consider $\langle h|\,h\rangle \geq 0$.