FY2045 Quantum Mechanics I

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Week 2

Measurement of a degenerate eigenvalue

D: The measurement postulate

(i) The only possible result of a precise measurement of an observable F is **one of the eigenvalues** f_n of the corresponding linear operator $\hat{F}$.

(ii) Immediately after the measurement of the eigenvalue f_n, the system is in an eigenstate of $\hat{F}$, namely, the eigenstate ψ_n corresponding to the measured eigenvalue f_n.

∅7.2

Non-degenerate case

$$\hat{F}\psi_n = f_n\psi_n$$

has only one solution ψ_n for eigenvalue f_n.

If a measurement of observable F gives f_n, the system is in state ψ_n immediately after the measurement.

Measurement of a degenerate eigenvalue

$$\hat{F}\psi_{ni} = f_n\psi_{ni}, \quad i = 1, 2, \ldots g_n.$$

Complete set, expand general state as

$$\Psi = \sum_n \Psi_n = \sum_n \sum_{i=1}^{g_n} c_{ni}\psi_{ni},$$

resulting in probability of measuring f_n

$$P_n = \sum_{i=1}^{g_n} |c_{ni}|^2.$$

D (ii) — degenerate case

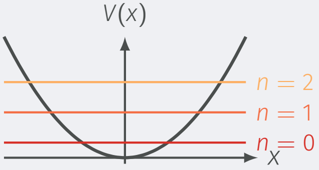
Immediately after a measurement of eigenvalue f_n, the system is in the normalized state

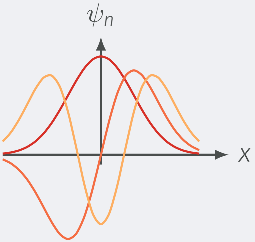
$$\frac{\Psi_n}{||\Psi_n||} = \frac{\sum_{i=1}^{g_n} c_i\psi_{ni}}{||\sum_{i=1}^{g_n} c_i\psi_{ni}||},$$

with $||\Psi_n||$ the norm of Ψ_n.

Example — 3D isotropic harmonic oscillator

1D case





$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2,$$

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \qquad n = 0, 1, 2\ldots$$

$$\psi_n = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} \frac{e^{-m\omega x^2/2\hbar}}{\sqrt{2^n n!}} H_n\left(x\sqrt{\frac{m\omega}{\hbar}}\right),$$

3D case

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega^2(x^2 + y^2 + z^2),$$

$$\psi_{n_x n_y n_z} = \psi_{n_x}(x)\psi_{n_y}(y)\psi_{n_z}(z),$$

$$\begin{aligned} E_{n_x n_y n_z} &= \hbar\omega\left(n_x + n_y + n_z + \frac{3}{2}\right) \\ &= \hbar\omega\left(N + \frac{3}{2}\right) \equiv E_N. \end{aligned}$$

H3.5 H5.1, G2.3

Eigenfunctions of continuous variables

Momentum eigenfunctions

$$\hat{p}\psi_p(x) = p\psi_p(x) \quad \Rightarrow \quad \psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}.$$

Position eigenfunctions

$$\hat{x}\psi_y(x) = y\psi_y(x) \quad \Rightarrow \quad \psi_y(x) = \delta(x-y).$$

Normalization

For continuous case,

$$\int d\tau\, \Psi^*_{f'}\Psi_f = \delta(f-f'),$$

compared to

$$\int d\tau\, \Psi^*_{n'}\Psi_n = \delta_{nn'},$$

in discrete case.

Physical interpretation of the continuous case

Discrete case

The probability that a measurement of F gives the result f_n, when the system is in the state Ψ, is

$$|c_n|^2 = \left| \int d\tau \; \Psi_n^* \Psi \right|^2 ,$$

where Ψ_n is the eigenstate corresponding to f_n.

Continuous case

The probability that a measurement of F gives a result in the interval $(f, f + df)$ when the system is in the state Ψ, is

$$|c(f)|^2 df = \left| \int d\tau \; \Psi_f^* \Psi \right|^2 df,$$

where Ψ_f is the eigenstate corresponding to the value f.

Ø7.3

Momentum-space representation

	Position-space formulation	Momentum-space formulation
Wavefunction	$\Psi(x, y, z, t)$	$\Phi(p_x, p_y, p_z, t)$
Operator $\hat{x}_i$	x_i	$-\frac{\hbar}{i}\frac{\partial}{\partial p_i}$
Operator $\hat{p}_i$	$\frac{\hbar}{i}\frac{\partial}{\partial x_i}$	p_i
Wave equation	$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}(\hat{x}_i, \hat{p}_i)\Psi$	$i\hbar\frac{\partial\Phi}{\partial t} = \hat{H}(\hat{x}_i, \hat{p}_i)\Phi$

Ø7.4