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FY2045 Solutions Problem set 8 fall 2023

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Problem 1

a) Since $l = 1$, we have $m_l = 0, \pm 1$. We also have $m_s = \pm\frac{1}{2}$, and so there are $(2l+1)(2s+1) = 3 \times 2 = 6$ states.

b) Let us consider the commutator $[\hat{\mathbf{J}}^2, \hat{\mathbf{L}}^2]$,

$$[\hat{\mathbf{J}}^2, \hat{\mathbf{L}}^2] = [\hat{\mathbf{L}}^2, \hat{\mathbf{L}}^2] + [\hat{\mathbf{S}}^2, \hat{\mathbf{L}}^2] + 2[\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}, \hat{\mathbf{L}}^2]. \tag{1}$$

The first commutator is trivially zero, while the second vanishes since the two operators act on different subspaces. The last commutator vanishes since $[\hat{\mathbf{L}}, \hat{\mathbf{L}}^2] = 0$ and since the spin and orbital-angular momentum operators act on different subspaces. Similar arguments can be applied to the other commutators involving angular-momentum operators. Moreover, we find

$$[\hat{\mathbf{J}}^2, \hat{H}] = [\hat{\mathbf{L}}^2, \hat{H}] + [\hat{\mathbf{S}}^2, \hat{H}] + 2[\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}, \hat{H}]. \tag{2}$$

The first commutator vanishes since $[\hat{\mathbf{L}}, \hat{H}] = 0$ for a spherically symmetric potential, and the second vanishes since $\hat{H}$ is independent of spin. The last commutator vanishes by the very same arguments combined. Finally, $[\hat{J}_z, \hat{H}] = 0$ can be shown in the same manner.

c) $l = 1$ and $s = \frac{1}{2}$ yields $j = \frac{1}{2}$ or $j = \frac{3}{2}$. The first case yields $2j+1 = 2$ states, and the second yields $2j+1 = 4$ states. Thus we have one quadruplet and one doublet and a total of 6 states. This is the same as in a) as it must be.

d) The state with $m_l = 1$ and $m_s = \frac{1}{2}$ has $m_j = \frac{3}{2}$ and there is only one way to obtain this. Thus this state is also an eigenstate of $\hat{\mathbf{J}}^2$. We can therefore write

$$\underline{\left|n1\tfrac{3}{2}\tfrac{3}{2}\right\rangle_J = \left|n11\tfrac{1}{2}\right\rangle_{LS}.} \tag{3}$$

We can now use the ladder operator $J_- = L_- + S_-$ to construct the other rungs of the ladder with $j = \frac{3}{2}$. Using the general formula $J_- \left|jm\right\rangle = \hbar\sqrt{(j+m)(j-m+1)}\left|jm-1\right\rangle$, we find

$$J_- \left|n1\tfrac{3}{2}\tfrac{3}{2}\right\rangle_J = \hbar\sqrt{3}\left|n1\tfrac{3}{2}\tfrac{1}{2}\right\rangle_J, \tag{4}$$

$$(L_- + S_-)\left|n11\tfrac{1}{2}\right\rangle_{LS} = \hbar\sqrt{2}\left|n10\tfrac{1}{2}\right\rangle_{LS} + \hbar\left|n11-\tfrac{1}{2}\right\rangle_{LS}. \tag{5}$$

Since the left hand sides of the two above equations are equal, we obtain

$$\underline{\left|n1\tfrac{3}{2}\tfrac{1}{2}\right\rangle_J = \sqrt{\tfrac{2}{3}}\left|n10\tfrac{1}{2}\right\rangle_{LS} + \sqrt{\tfrac{1}{3}}\left|n11-\tfrac{1}{2}\right\rangle_{LS}.} \tag{6}$$

By repeated use of J_-, we find

$$\underline{\left|n1\tfrac{3}{2}-\tfrac{1}{2}\right\rangle_J = \sqrt{\tfrac{1}{3}}\left|n1-1\tfrac{1}{2}\right\rangle_{LS} + \sqrt{\tfrac{2}{3}}\left|n10-\tfrac{1}{2}\right\rangle_{LS},} \tag{7}$$

$$\underline{\left|n1\tfrac{3}{2}-\tfrac{3}{2}\right\rangle_J = \left|n1-1-\tfrac{1}{2}\right\rangle_{LS}.} \tag{8}$$

This completes the quadruplet. The second ladder has $j = \frac{1}{2}$ and the top rung has $m_j = \frac{1}{2}$. This state must be orthogonal to the state $\left|n1\frac{3}{2}\frac{1}{2}\right\rangle_J$, i.e. we require ${}_J\langle n1\frac{3}{2}\frac{1}{2}|n1\frac{1}{2}\frac{1}{2}\rangle_J = 0$. Since $m_j = m_l + m_s$, we must have

$$\left|n1\tfrac{1}{2}\tfrac{1}{2}\right\rangle_J = A\left|n10\tfrac{1}{2}\right\rangle_{LS} + B\left|n11-\tfrac{1}{2}\right\rangle_{LS},$$

with constants A and B, since the two states on the right hand side are the only "old" states with $m_l + m_s = \frac{1}{2}$. From the orthogonality requirement, we get

$$\begin{aligned} 0 &= {}_J\langle n1\tfrac{3}{2}\tfrac{1}{2}|n1\tfrac{1}{2}\tfrac{1}{2}\rangle_J = \left[\sqrt{\tfrac{2}{3}}\left\langle n10\tfrac{1}{2}\right|_{LS} + \sqrt{\tfrac{1}{3}}\left\langle n11-\tfrac{1}{2}\right|_{LS}\right]\left[A\left|n10\tfrac{1}{2}\right\rangle_{LS} + B\left|n11-\tfrac{1}{2}\right\rangle_{LS}\right] \\ &= \sqrt{\tfrac{2}{3}}A + \sqrt{\tfrac{1}{3}}B, \end{aligned}$$

or $B = -\sqrt{2}A$. We determine A by normalizing the state, choosing A real and positive. This yields

$$\underline{\left|n1\tfrac{1}{2}\tfrac{1}{2}\right\rangle_J = \sqrt{\tfrac{1}{3}}\left|n10\tfrac{1}{2}\right\rangle_{LS} - \sqrt{\tfrac{2}{3}}\left|n11-\tfrac{1}{2}\right\rangle_{LS}.} \tag{9}$$

The second rung is found by using J_- (or by requiring orthogonality to $\left|n1\frac{3}{2}-\frac{1}{2}\right\rangle_J$ which is the only other state with $m_j = -\frac{1}{2}$), resulting in

$$\underline{\left|n1\tfrac{1}{2}-\tfrac{1}{2}\right\rangle_J = \sqrt{\tfrac{2}{3}}\left|n1-1\tfrac{1}{2}\right\rangle_{LS} - \sqrt{\tfrac{1}{3}}\left|n10-\tfrac{1}{2}\right\rangle_{LS}.} \tag{10}$$

Note: We could also have used a table of Clebsch-Gordan coefficients for $j_1 = 1, j_2 = \frac{1}{2}$ to construct these states.

Problem 2

a) — Matrix formulation The eigenspinors $\chi_{i\pm}$ along x, y and z for spin $\frac{1}{2}$ can be found by solving the eigenvalue equations

$$\frac{\hbar}{2}\sigma_i\chi_{i\pm} = \pm\frac{\hbar}{2}\chi_{i\pm}, \tag{11}$$

where the subscripts $\pm$ denote spin up and down along direction $i = x, y, z$, with Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{12}$$

The matrix notation for spin $\frac{1}{2}$ is defined by projecting a general spin state $|\chi\rangle$ onto the spin up and spin down eigenstates in the z direction, $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$,

$$\chi = \begin{pmatrix} a_+ \\ a_- \end{pmatrix} \equiv \begin{pmatrix} \langle\uparrow_z|\chi\rangle \\ \langle\downarrow_z|\chi\rangle \end{pmatrix}. \tag{13}$$

Solving the eigenvalue equations, eq. (11), for all three directions and relating the results to the above definition of the matrix notation, we get

$$\chi_{z+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \langle\uparrow_z|\uparrow_z\rangle \\ \langle\downarrow_z|\uparrow_z\rangle \end{pmatrix}, \qquad \chi_{z-} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \langle\uparrow_z|\downarrow_z\rangle \\ \langle\downarrow_z|\downarrow_z\rangle \end{pmatrix}, \tag{14}$$

$$\chi_{x+} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \langle\uparrow_z|\uparrow_x\rangle \\ \langle\downarrow_z|\uparrow_x\rangle \end{pmatrix}, \qquad \chi_{x-} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \langle\uparrow_z|\downarrow_x\rangle \\ \langle\downarrow_z|\downarrow_x\rangle \end{pmatrix}, \tag{15}$$

$$\chi_{y+} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} \langle\uparrow_z|\uparrow_y\rangle \\ \langle\downarrow_z|\uparrow_y\rangle \end{pmatrix}, \qquad \chi_{y-} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} \langle\uparrow_z|\downarrow_y\rangle \\ \langle\downarrow_z|\downarrow_y\rangle \end{pmatrix}. \tag{16}$$

From the above, we can draw out the following relations between the states by writing out the states using a completeness relation, $|\chi\rangle = \langle\uparrow_z|\chi\rangle|\uparrow_z\rangle + \langle\downarrow_z|\chi\rangle|\downarrow_z\rangle = a_+|\uparrow_z\rangle + a_-|\downarrow_z\rangle$, leading to

$$|\uparrow_x\rangle = \frac{|\uparrow_z\rangle + |\downarrow_z\rangle}{\sqrt{2}}, \qquad |\downarrow_x\rangle = \frac{|\uparrow_z\rangle - |\downarrow_z\rangle}{\sqrt{2}}, \tag{17}$$

$$|\uparrow_y\rangle = \frac{|\uparrow_z\rangle + i|\downarrow_z\rangle}{\sqrt{2}}, \qquad |\downarrow_y\rangle = \frac{|\uparrow_z\rangle - i|\downarrow_z\rangle}{\sqrt{2}}. \tag{18}$$

By adding or subtracting different combinations of the above equations, we get the relations given in the problem set. We could also have used the fact that $a_+^* = \langle\uparrow_z|\chi\rangle^* = \langle\chi|\uparrow_z\rangle$ and $a_-^* = \langle\downarrow_z|\chi\rangle^* = \langle\chi|\downarrow_z\rangle$, to more directly get

$$|\uparrow_z\rangle = \langle\uparrow_x|\uparrow_z\rangle|\uparrow_x\rangle + \langle\downarrow_x|\uparrow_z\rangle|\downarrow_x\rangle = \langle\uparrow_z|\uparrow_x\rangle^*|\uparrow_x\rangle + \langle\uparrow_z|\downarrow_x\rangle^*|\downarrow_x\rangle = \frac{|\uparrow_x\rangle + |\downarrow_x\rangle}{\sqrt{2}}, \tag{19}$$

etc.

Abstract formulation It's also possible to solve the problem using the abstract formulation, so we show this here for completeness.

The ladder operators are related to the x and y components of the spin operator by $S_\pm = S_x \pm iS_y$, or

$$S_x = \frac{S_+ + S_-}{2}, \tag{20}$$

$$S_y = \frac{S_+ - S_-}{2i}. \tag{21}$$

We now look for eigenstates of S_x and S_y expressed as linear combinations of $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$. Since we know that the spin up and spin down states in z direction are not eigenstates of S_x and S_y separately, we try linear combinations of the type $[|\uparrow_z\rangle + e^{i\phi}|\downarrow_z\rangle]$, which should be normalized by an overall constant. We then get

$$S_x[|\uparrow_z\rangle + e^{i\phi}|\downarrow_z\rangle] = \frac{S_+ + S_-}{2}[|\uparrow_z\rangle + e^{i\phi}|\downarrow_z\rangle] = \frac{\hbar|\downarrow_z\rangle + \hbar e^{i\phi}|\uparrow_z\rangle}{2} = \frac{\hbar e^{i\phi}}{2}\left[|\uparrow_z\rangle + e^{-i\phi}|\downarrow_z\rangle\right].$$

In order to get the same state on both sides of the equation, we must either have $\phi_{x+} = 0$, with corresponding eigenvalue $+\frac{\hbar}{2}$, or $\phi_{x-} = \pi$, with eigenvalue $-\frac{\hbar}{2}$. Doing the same for S_y we get

$$S_y[|\uparrow_z\rangle + e^{i\phi}|\downarrow_z\rangle] = \frac{S_+ - S_-}{2i}[|\uparrow_z\rangle + e^{i\phi}|\downarrow_z\rangle] = \frac{-\hbar|\downarrow_z\rangle + \hbar e^{i\phi}|\uparrow_z\rangle}{2i} = \frac{\hbar e^{i\phi}}{2i}\left[|\uparrow_z\rangle - e^{-i\phi}|\downarrow_z\rangle\right].$$

Since the spin is an observable, the eigenvalue must be real. We therefore use $1/i = e^{-i\pi/2}$, leading to

$$S_y[|\uparrow_z\rangle + e^{i\phi}|\downarrow_z\rangle] = \frac{\hbar e^{i(\phi-\pi/2)}}{2}\left[|\uparrow_z\rangle - e^{-i\phi}|\downarrow_z\rangle\right].$$

For this to be an eigenvalue equation with real eigenvalues, we must have $\phi_{y+} = \pi/2$, with corresponding eigenvalue $+\frac{\hbar}{2}$, or $\phi_{y-} = -\pi/2$, with eigenvalue $-\frac{\hbar}{2}$. Hence, the normalized eigenstates of S_x and S_y are

$$|\uparrow_x\rangle = \frac{|\uparrow_z\rangle + |\downarrow_z\rangle}{\sqrt{2}}, \qquad |\downarrow_x\rangle = \frac{|\uparrow_z\rangle - |\downarrow_z\rangle}{\sqrt{2}}, \tag{22}$$

$$|\uparrow_y\rangle = \frac{|\uparrow_z\rangle + i\,|\downarrow_z\rangle}{\sqrt{2}}, \qquad |\downarrow_y\rangle = \frac{|\uparrow_z\rangle - i\,|\downarrow_z\rangle}{\sqrt{2}}. \tag{23}$$

By adding or subtracting different combinations of the above equations, we get the relations given in the problem set.

b) For the state $|10\rangle$ we find

$$
\begin{aligned}
|10\rangle_z &= \frac{1}{\sqrt{2}}[|\uparrow_z\rangle\,|\downarrow_z\rangle + |\downarrow_z\rangle\,|\uparrow_z\rangle] \\
&= \frac{|\uparrow_x\rangle\,|\uparrow_x\rangle - |\downarrow_x\rangle\,|\downarrow_x\rangle}{\sqrt{2}} = \frac{|11\rangle_x - |1{-}1\rangle_x}{\sqrt{2}} \qquad (24) \\
&= \frac{|\uparrow_y\rangle\,|\uparrow_y\rangle - |\downarrow_y\rangle\,|\downarrow_y\rangle}{\sqrt{2}i} = \frac{|11\rangle_y - |1{-}1\rangle_y}{\sqrt{2}i}. \qquad (25)
\end{aligned}
$$

Hence, if we have a state prepared in $|10\rangle$, with zero z component of the total spin, and we measure the component of the total spin along the x or y direction, we can get $m = \pm 1$ with equal probability.

For the state $|00\rangle$ we find

$$
|00\rangle_z = |00\rangle_x = |00\rangle_y\,. \qquad (26)
$$

No matter in which direction we measure the component of the total spin, we get 0. This must be the case since $s = 0$, meaning that $m = 0$ for any direction we choose to measure along.[1]

[1]The difference between the spin-triplet state with $m = 0$ and the spin-singlet state with $m = 0$ has some interesting consequences for instance in the field of superconducting spintronics, where one of the aims is to make electron pairs with spins pointing in the same direction. In most superconductors the electron pairs are in the spin-singlet state, and therefore always have opposite spins, independent of which direction one measures along. Therefore, the electron pairs have to be converted to spin-triplet electron pairs in order to allow for the possibility of electron pairs with parallel spins. For more information, see e.g. Spin-polarized supercurrents for spintronics by M. Eschrig or Superconducting spintronics by J. Linder and J.W.A. Robinson.