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FY2045 Problem set 5 fall 2023

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Problem 1 — Coherent states

In the lectures we showed that it's possible to construct eigenvectors of the lowering operator, satisfying the eigenvalue equation

$$a\,|\alpha\rangle = e^{-i\omega t}\alpha\,|\alpha\rangle\,, \tag{1}$$

with eigenvalue α. Expanded in terms of the energy eigenvectors $|n\rangle$ of the harmonic oscillator, we found

$$|\alpha\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle\,, \tag{2}$$

where c_0 is an overall constant.

a) Determine the constant c_0 by requiring that $|\alpha\rangle$ is normalized.

b) Calculate the time-dependence of the expectation values $\langle q\rangle$, $\langle q^2\rangle$, $\langle p\rangle$, and $\langle p^2\rangle$ for the coherent state using operator algebra. *Hint:* Express the position and momentum operators in terms of the raising and lowering operators a and $a^\dagger$. Use $\alpha = |\alpha|e^{i\Theta}$ to simplify the resulting expressions.

c) Show that the coherent state corresponds to a minimal uncertainty state, i.e. $\Delta q \Delta p = \hbar/2$, where $\Delta q = \sqrt{\langle (q - \langle q \rangle)^2 \rangle}$ and $\Delta p = \sqrt{\langle (p - \langle p \rangle)^2 \rangle}$.

d) This exercise demonstrates that the results above can be obtained without explicit knowledge of the wavefunction $\Psi(q,t) = \langle q|\alpha(t)\rangle$ and the probability density $|\Psi(q,t)|^2$. Of course, this does not mean that the latter are not interesting. From the solution of exercise 3, it follows that the probability density of a state with minimal uncertainty product ($\Delta q \Delta p = \hbar/2$) must be Gaussian:

$$|\langle q|\alpha\rangle|^2 = |\Psi(q,t)|^2 \ \propto \ |C|^2 \exp\left[-\frac{(q - \langle q \rangle)^2}{2(\Delta q)^2}\right].$$

Insert into this formula the above results for $\langle q \rangle$ and Δq and show that the probability density is the same as that given in the lectures, describing an oscillating wavepacket.

Problem 2 — The Levi-Cevita symbol and Pauli matrices

The Levi-Cevita symbol ϵ_{ijk} is a completely antisymmetric quantity, meaning that interchanging any two neighboring indexes results in a relative negative sign, $\epsilon_{ijk} = -\epsilon_{jik}$, and that if any two indexes are the same we must have $\epsilon_{iik} = 0$ etc. The indexes i, j, k can take the values $1, 2, 3$ or equivalently x, y, z, where by definition $\epsilon_{123} = \epsilon_{xyz} = 1$. In other words, we have

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } (ijk) = (xyz), (yzx), (zxy), \\ -1, & \text{if } (ijk) = (yxz), (xzy), (zyx), \\ 0 & \text{if two or more indexes are the same.} \end{cases}$$

a) Convince yourself that the vector product $\mathbf{A} \times \mathbf{B}$ can be written as

$$\mathbf{A} \times \mathbf{B} = \epsilon_{ijk} \hat{e}_i A_j B_k, \tag{3}$$

where $\hat{e}_i$ is the unit vector in direction i, and we use the Einstein sum convention: any index which appears twice is summed over, i.e. $c_i d_i \equiv \sum_{i=x,y,z} c_i d_i$. Using the fact that ϵ_{ijk} is antisymmetric, show that $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$.

b) When working with the matrix representation of spin $\frac{1}{2}$, we will frequently use the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show, by performing the matrix multiplications, that the matrices satisfy the commutation relations

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k.$$

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c) We can also define the *anticommutator* between two quantities A and B,

$$\{A, B\} = AB + BA.$$

Show that for the Pauli matrices we have

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} I,$$

where I denotes the 2×2 identity matrix,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

d) Use the commutation and anticommutation relations for the Pauli matrices to prove the formula

$$\sigma_i \sigma_j = I\delta_{ij} + i\epsilon_{ijk}\sigma_k.$$

Problem 3 — Spin $\frac{1}{2}$

An electron is in the spin state

$$\chi = A \begin{pmatrix} 1 - 2i \\ 2 \end{pmatrix}.$$

a) Determine the constant A by normalizing χ.

b) If you measured S_z on this electron, what values could you get and what is the probability of each? What is the expectation value of S_z?

Hint: In the abstract formulation the probability of measuring the eigenvalue corresponding to the state $|m\rangle$ when the system is in the state $|\chi\rangle$ before the measurement is $|\langle m|\chi\rangle|^2$. What is the corresponding expression in the matrix formulation?

c) If you instead measured S_y on this electron, what values could you get and what is the probability of each? What is the expectation value of S_y?

d) What is the spin direction[1]

$$\langle \boldsymbol{\sigma} \rangle = \langle \sigma_x \rangle \hat{x} + \langle \sigma_y \rangle \hat{y} + \langle \sigma_z \rangle \hat{z},$$

of the state χ?

e) We define a unit vector $\mathbf{n}$ which points in the spin direction, i.e. $\mathbf{n} \propto \langle \boldsymbol{\sigma} \rangle$. Write down the matrix $\mathbf{n} \cdot \mathbf{S}$, and show that the state χ is an eigenvector of the operator matrix $\mathbf{n} \cdot \mathbf{S}$ with eigenvalue $\frac{\hbar}{2}$.

[1]We will sometimes also define the spin direction as the direction of $\langle \mathbf{S} \rangle$, which is equivalent to this definition due to the relation $\mathbf{S} = \hbar\boldsymbol{\sigma}/2$ for spin $\frac{1}{2}$.