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FY2045 Solutions Problem set 6 fall 2023

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Problem 1

The moment of inertia for a sphere with radius r is

$$I = \frac{2}{5}mr^2, \tag{1}$$

and the angular velocity

$$\omega = \frac{v}{r}, \tag{2}$$

where v is the speed at the "equator". We therefore have the angular momentum

$$L = I\omega = \frac{2}{5}mvr, \tag{3}$$

which we require to be equal to the spin angular momentum of the electron $\hbar/2$, for an electron with radius r_c. We therefore get

$$\begin{aligned} v = \frac{5\hbar}{4mr_c} = \frac{5\pi\hbar\epsilon_0 c^2}{e^2} = \frac{5\pi\hbar}{\mu_0 e^2} &= \frac{5\pi \times 1.055 \times 10^{-34}\,\mathrm{J\,s}}{(1.602 \times 10^{-19}\,\mathrm{C})^2 \times 1.257 \times 10^{-6}\,\mathrm{H\,m^{-1}}} \\ &= \frac{5\pi \times 1.055}{1.602^2 \times 1.257} \times 10^{-34+38+6} \times \frac{\mathrm{kg\,m^2\,s^{-1}}}{(\mathrm{A\,s})^2 \times \mathrm{kg\,m\,s^{-2}\,A^{-2}}} = \underline{5.14 \times 10^{10}\,\mathrm{m/s}.} \end{aligned} \tag{4}$$

This is more than 100 times the speed of light! So, no, this is not a realistic model for the spin. The problem becomes even worse when considering that the actual electron radius (a topic of debate) is many orders of magnitude smaller than the classical estimate for the electron radius.

Problem 2 — Spin 1

a) The three column matrices representing the three states

$$
\begin{aligned}
|s=1, m=1\rangle &\equiv |1,1\rangle \equiv |1\rangle \ , &(5)\\
|s=1, m=0\rangle &\equiv |1,0\rangle \equiv |0\rangle &(6)\\
|s=1, m=-1\rangle &\equiv |1,-1\rangle \equiv |-1\rangle \ , &(7)
\end{aligned}
$$

are

$$
\chi_+ = \underline{\underline{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}} \ , \qquad \chi_0 = \underline{\underline{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}} \ , \qquad \chi_- = \underline{\underline{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}} \ . \tag{8}
$$

The results of operating on these eigenvectors with the operator S_z, may be read off from the following formulas:

$$
\begin{aligned}
S_z\chi_+ &= \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \hbar\chi_+ &(9)\\
S_z\chi_0 &= \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0 = 0 \cdot \chi_0 \ , &(10)\\
S_z\chi_- &= \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -\hbar\chi_- \ . &(11)
\end{aligned}
$$

b) There is nothing special about the z-direction, so we must expect to find the same eigenvalues (and measured values) for S_x (and for any other component of $\boldsymbol{S}$) as for S_z, namely, $\hbar$, 0 and $-\hbar$.

c) The three components of the state

$$
\chi = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \ , \tag{12}
$$

are the probability *amplitudes* of measuring $S_z = \hbar$, 0 and $-\hbar$, respectively. The probabilities are the squares of these amplitudes, that is, 1/4, 1/2 and 1/4, respectively. Note that these squares sum up to 1, verifying that the given state is normalised. The expectation value of S_z in this state becomes

$$
S_z = \hbar \cdot P_{(S_z=\hbar)} + 0 \cdot P_{(S_z=0)} + (-\hbar) \cdot P_{(S_z=-\hbar)} = \hbar(1/4 - 1/4) = \underline{\underline{0}} \ . \tag{13}
$$

d) We find

$$S_x\chi = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} = \hbar\begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} = \underline{\underline{\hbar\chi}}\,, \tag{14}$$

verifying one of the eigenvalues of S_x.

e) The eigenvalue equation for S_x with eigenvalue equal to zero is

$$0 = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} b \\ a+c \\ b \end{pmatrix}\,. \tag{15}$$

Thus we must have $b = 0$ and $c = -a$. Normalization and standard choice of phase give

$$\chi_{(S_x=0)} \quad = \quad \underline{\underline{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}}\,. \tag{16}$$

Problem 3 — Spin precession

Using the given expression for the magnetic moment and magnetic field, we get the Hamiltonian

$$\hat{H} = \frac{g_e e}{2m_e}\mathbf{S}\cdot\mathbf{B} = \frac{g_e e B}{2m_e}S_z.$$

To find the expression for the time-derivative of $\mathbf{S}$ we use the given expression for the time-development of expectation values:

$$\frac{d}{dt}\langle S_i\rangle = \frac{i}{\hbar}\langle[\hat{H}, S_i]\rangle = \frac{i g_e e B}{2m_e\hbar}\langle[S_z, S_i]\rangle = -\frac{g_e e B}{2m_e}\epsilon_{zik}\langle S_k\rangle,$$

where we have used $[S_j, S_i] = i\hbar\epsilon_{jik}S_k$, with $j = z$. Again we use the Einstein sum convention discussed in Problem 2.

We now use the antisymmetric property of the Levi-Cevita symbol, and the expression for the vector product in terms of the Levi-Cevita symbol given in Problem 2:

$$\epsilon_{zik}\langle S_k\rangle = -\epsilon_{izk}\langle S_k\rangle = -(\hat{z}\times\langle\mathbf{S}\rangle)_i,$$

resulting in

$$\frac{d}{dt}\langle S_i\rangle = \frac{g_e e B}{2m_e}(\hat{z}\times\langle\mathbf{S}\rangle)_i.$$

Defining

$$\boldsymbol{\omega}_S = \frac{g_e e B}{2m_e}\hat{z},$$

and combining the equations for the components S_i, we get

$$\frac{d}{dt}\langle \mathbf{S} \rangle = \boldsymbol{\omega}_S \times \langle \mathbf{S} \rangle.$$

Hence, the *expectation value* of the spin — the spin direction — precesses around **B** exactly like the classical spin.