FY2045 Quantum Mechanics I

Fall 2023

Henning Goa Hugdal

Week 5

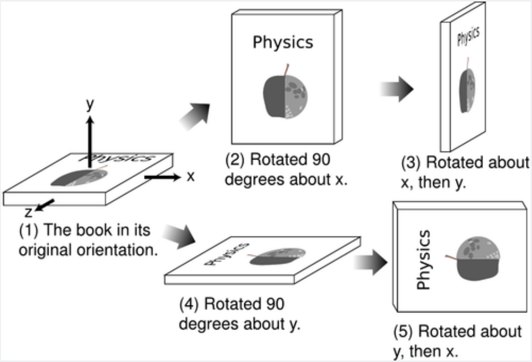
Quantization of Angular Momentum

Angular momentum operators

Define angular momentum operator **J** with components J_x, J_y, and J_z — Hermitian operators with commutation relations

$$[J_x, J_y] = i\hbar J_z,$$
$$[J_y, J_z] = i\hbar J_x,$$
$$[J_z, J_x] = i\hbar J_y.$$

Non-commutativity related to non-commutativity of rotations in 3D.[1]



Picture source: Benjamin Crowell, General Relativity, p. 270.

Ø11.2, H8.1-8.2, G4.3

[0] For more on this, see the lecture notes by Prof. Neil, on which today's lecture was based.

Angular momentum operators

Define angular momentum operator $\mathbf{J}$ with components J_x, J_y, and J_z — Hermitian operators with commutation relations

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$$[J_y, J_z] = i\hbar J_x,$$
$$[J_z, J_x] = i\hbar J_y.$$

Non-commutativity related to non-commutativity of rotations in 3D.[1]

Also define $\mathbf{J}^2 = J_x^2 + J_y^2 + J_z^2$, which satisfies

$$[\mathbf{J}^2, J_i] = 0, \; i = x, y, z.$$

Since they commute, we can find **simultaneous eigenvectors** of $\mathbf{J}^2$ and e.g. J_z.
Assume orthonormalized eigenvectors $|a, b\rangle$ such that

$$\mathbf{J}^2|a, b\rangle = a|a, b\rangle,$$
$$J_z|a, b\rangle = b|a, b\rangle.$$

Ø11.2, H8.1-8.2, G4.3
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Eigenvalues

Ladder operators

We again define ladder operators

$$J_\pm = J_x \pm iJ_y,$$

with commutation relations

$$[\mathbf{J}^2, J_\pm] = 0,$$
$$[J_z, J_\pm] = \pm\hbar J_\pm.$$

$J_\pm|a,b\rangle$ is an eigenvector of both $\mathbf{J}^2$ and J_z with eigenvalue a and $b \pm \hbar$, respectively:

$$J_\pm|a,b\rangle \propto |a, b\pm\hbar\rangle.$$

What are a and b?

Since the norm of a vector must be positive, we must have:

$$a - b(b \pm \hbar) \geq 0.$$

Must have a maximum and minimum eigenvalue of J_z such that

$$J_+|a, b_{\max}\rangle = 0 \text{ and } J_-|a, b_{\min}\rangle = 0.$$

We find $(n = 0, 1, 2, \ldots)$

$$b_{\max} = -b_{\min} = \frac{n\hbar}{2}, \text{ and } a = \frac{\hbar^2}{4}n(n+2).$$

Eigenvalues

In standard notation, we get

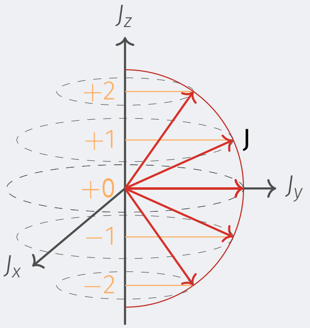
$$\mathbf{J}^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle,$$
$$J_z|j,m\rangle = \hbar m|j,m\rangle,$$

with

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots,$$
$$m = -j, -j+1, \dots, j-2, j-1, j.$$

Example: $j = 2$

Orientation of $\mathbf{J}$ for different values of m:



Based on figure by Izaak Neutelings