FY2045 Quantum Mechanics I

Fall 2023

Henning Goa Hugdal

Week 5

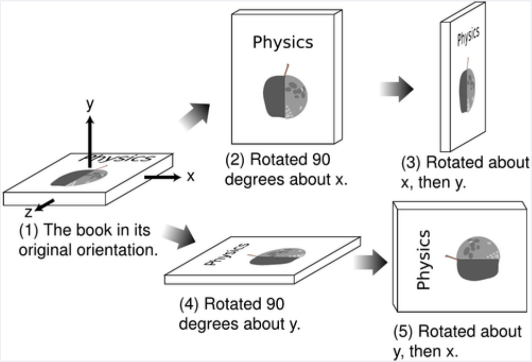
Quantization of Angular Momentum

Angular momentum operators

Define angular momentum operator **J** with components J_x, J_y, and J_z — Hermitian operators with commutation relations

$$[J_x, J_y] = i\hbar J_z,$$
$$[J_y, J_z] = i\hbar J_x,$$
$$[J_z, J_x] = i\hbar J_y.$$

Non-commutativity related to non-commutativity of rotations in 3D.[1]



Picture source: Benjamin Crowell, General Relativity, p. 270.

Ø11.2, H8.1-8.2, G4.3

[0] For more on this, see the lecture notes by Prof. Neil, on which today's lecture was based.

Angular momentum operators

Define angular momentum operator $\mathbf{J}$ with components J_x, J_y, and J_z — Hermitian operators with commutation relations

$$[J_x, J_y] = i\hbar J_z,$$
$$[J_y, J_z] = i\hbar J_x,$$
$$[J_z, J_x] = i\hbar J_y.$$

Non-commutativity related to non-commutativity of rotations in 3D.[1]

Also define $\mathbf{J}^2 = J_x^2 + J_y^2 + J_z^2$, which satisfies

$$[\mathbf{J}^2, J_i] = 0, \ i = x, y, z.$$

Since they commute, we can find **simultaneous eigenvectors** of $\mathbf{J}^2$ and e.g. J_z.
Assume orthonormalized eigenvectors $|a, b\rangle$ such that

$$\mathbf{J}^2|a, b\rangle = a|a, b\rangle,$$
$$J_z|a, b\rangle = b|a, b\rangle.$$

Ø11.2, H8.1-8.2, G4.3

[0]For more on this, see the lecture notes by Prof. Neil, on which today's lecture was based.

Eigenvalues

Ladder operators

We again define ladder operators

$$J_{\pm} = J_x \pm iJ_y,$$

with commutation relations

$$[\mathbf{J}^2, J_{\pm}] = 0,$$
$$[J_x, J_{\pm}] = \pm\hbar J_{\pm}.$$

$J_{\pm}|a, b\rangle$ is an eigenvector of both $\mathbf{J}^2$ and J_z with eigenvalue a and $b \pm \hbar$, respectively:

$$J_{\pm}|a, b\rangle \propto |a, b \pm \hbar\rangle.$$

What are a and b?

Since the norm of a vector must be positive, we must have:

$$a - b(b \pm \hbar) \geq 0.$$

Must have a maximum and minimum eigenvalue of J_z such that

$$J_{+}|a, b_{\mathrm{max}}\rangle = 0 \text{ and } J_{-}|a, b_{\mathrm{min}}\rangle = 0.$$

We find $(n = 0, 1, 2, \ldots)$

$$b_{\mathrm{max}} = -b_{\mathrm{min}} = \frac{n\hbar}{2}, \text{ and } a = \frac{\hbar^2}{4} n(n+2).$$

Eigenvalues

In standard notation, we get

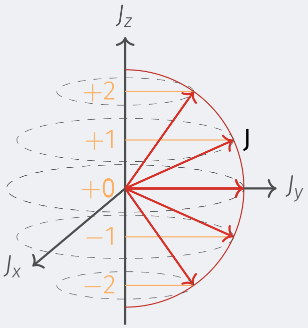
$$\mathbf{J}^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle,$$
$$J_z|j,m\rangle = \hbar m|j,m\rangle,$$

with

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots,$$
$$m = -j, -j+1, \ldots, j-2, j-1, j.$$

Example: $j = 2$

Orientation of **J** for different values of m:



Based on figure by Izaak Neutelings

Orbital angular momentum in position basis

In spherical coordinates we have the position basis vectors $|\mathbf{r}\rangle \equiv |r, \theta, \phi\rangle$. Operating on the eigenvalue equations with these basis vectors, we find the position space wavefunctions

$$\psi_{lm}(r, \theta, \phi) = \langle r, \theta, \phi | l, m\rangle = R(r) Y_{lm}(\theta, \phi),$$

where $Y_{lm}(\theta, \phi)$ are the **spherical harmonics**.

Since $Y_{lm} \propto e^{im\phi}$ and must be continuous, l and m can only take integer values.

Y_{lm} with $l = 0 - 3$, and $m \geq 0$