HARMONIC OSCILLATOR [Ø 11.1, H6.6]

For those who have had intro to quantum physics, you have seen how to find the energy eigenvalues and eigenfunctions using wavemechanics.
We will now use the general formulation, and treat the problem using operator method (or algebraic method).

Ladder operators

The Hamiltonian for the 1D harmonic oscillator reads

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{q}^2 = \frac{1}{2m}\left[\hat{p}^2 + (m\omega\hat{q})^2\right],$$

where $[\hat{q},\hat{p}] = i\hbar$. If $\hat{p},\hat{q}$ were numbers, we could write

$$p^2 + (m\omega q)^2 = (ip + m\omega q)(-ip + m\omega q).$$

However, $\hat{p}, \hat{q}$ do not commute. Still, we define the operator

$$a = \frac{1}{\sqrt{2\hbar m\omega}}\left[i\hat{p} + m\omega\hat{q}\right]$$

with

$$a^+ = \frac{1}{\sqrt{2\hbar m\omega}}\left[-i\hat{p} + m\omega\hat{q}\right],$$

meaning that a is not Hermitian.

If we now calculate $a^\dagger a$, we get

$$a^\dagger a = \frac{1}{2\hbar m\omega}\left[-i\hat{p} + m\omega\hat{q}\right]\left[i\hat{p} + m\omega\hat{q}\right]$$

$$= \frac{1}{2\hbar m\omega}\left[\underbrace{\hat{p}^2 + (m\omega\hat{q})^2}_{2m\hat{H}} + im\omega\underbrace{\left[\hat{q}\hat{p} - \hat{p}\hat{q}\right]}_{=i\hbar}\right]$$

$$= \frac{\hat{H}}{\hbar\omega} - \frac{1}{2}.$$

Similarily:

$$aa^\dagger = \frac{\hat{H}}{\hbar\omega} + \frac{1}{2}.$$

Hence, we get the commutator

$$\underline{[a, a^\dagger] = 1}$$

We can also write the Hamiltonian in terms of the new operators:

$$\underline{\hat{H} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right).}$$

Finding the eigenvalues of $\hat{H}$ thus boils down to finding the eigenvalues of

$$\hat{N} = a^\dagger a.$$

If we assume that we have orthonormalized eigenvectors $|n\rangle$ of $\hat{N}$,

$$\hat{N}|n\rangle = n|n\rangle$$

we will get energy eigenvalues

$$\hat{H}|n\rangle = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle = E_n|n\rangle.$$

The operator $\hat{N}$ is called the <u>number operator</u>, for reasons which will become clear soon.

In what follows we will need the commutators of $\hat{N}$ with a and $a^\dagger$:

$$[\hat{N}, a] = a^\dagger \underbrace{[a, a]}_{=0} + \underbrace{[a^\dagger, a]}_{=-1} a = -a,$$

$$[\hat{N}, a^\dagger] = a^\dagger \underbrace{[a, a^\dagger]}_{=1} + \underbrace{[a^\dagger, a^\dagger]}_{=0} a = a^\dagger.$$

Energy spectrum

Let us now determine the eigenenergies of the Harmonic oscillator,

$$\hat{H}|n\rangle = E_n|n\rangle,$$

where $|n\rangle$ is an eigenvector of both $\hat{H}$ and $\hat{N}$.
To do this we start by looking at the properties of the vector $a|n\rangle$.
(Remember: operator · vector = new vector).

is $a|n\rangle$ an eigenvector of $\hat{H}$?

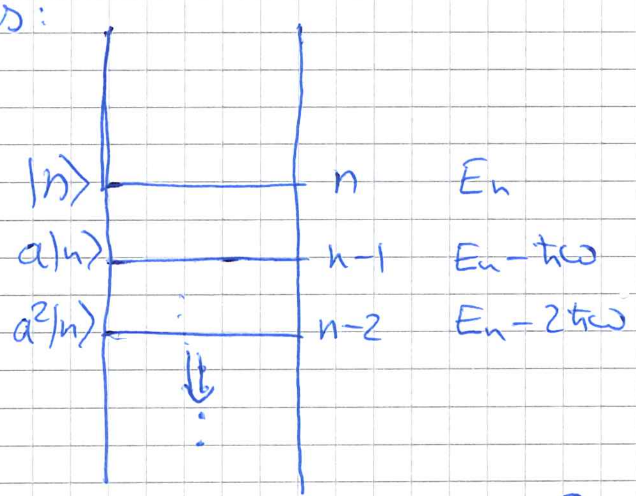
$$\hat{H}a|n\rangle = \hbar\omega\left(\hat{N}a + \frac{a}{2}\right)|n\rangle = \hbar\omega\left(a\hat{N} - a + \frac{a}{2}\right)|n\rangle$$

$$= a(\hat{H} - \hbar\omega)|n\rangle = (E_n - \hbar\omega)a|n\rangle.$$

If $a|n\rangle \neq 0$, $a|n\rangle$ is an eigenvector of $\hat{H}$ with eigenvalue $E_n - \hbar\omega$.

This is the same as saying that, if $a|n\rangle \neq 0$, $a|n\rangle$ is an eigenvector of $\hat{N}$ with eigenvalue $n-1$. For this reason, a is called a <u>lowering operator</u>.

We can repeat the above argument:
If $a^2|n\rangle \neq 0$, $a^2|n\rangle$ is an eigenvector of $\hat{H}$ $(\hat{N})$ with eigenvalue $E_n - 2\hbar\omega$ $(n-2)$, and so on. We get a downwardgoing ladder of eigenvalues:



How far down can we go?
The norm of a vector cannot be negative, meaning that the square of the norm of $a|n\rangle$ must satisfy:

$$0 \leq \| a|n\rangle \|^2 = \langle n | \underbrace{a^\dagger a}_{\hat{N}} | n \rangle = \langle n | n | n \rangle = n.$$

For this to always hold, there must exist a lowest eigenvector $|0\rangle$ such that

$$\underline{a|0\rangle = 0}.$$

The corresponding energy is the lowest energy, the ground state energy, is given by

Wavefunctions in position space

We can do a lot of calculations without knowing the wavefunctions in positions space. Let us anyway see how we can find them:

Our starting point is

$$a|0\rangle = 0.$$

We operate with $\langle q|$ from the left

$$\langle q|a|0\rangle = \int dq' \, \langle q|a|q'\rangle\langle q'|0\rangle = 0$$

$$= \int dq' \, \underbrace{\langle q| \sqrt{\frac{m\omega}{2\hbar}}\hat{q} + \frac{i\hat{p}}{\sqrt{2m\omega\hbar}} |q'\rangle}_{=\left(\sqrt{\frac{m\omega}{2\hbar}}q' + \sqrt{\frac{\hbar}{2m\omega}}\frac{d}{dq'}\right)\delta(q'-q)} \langle q'|0\rangle$$

$$= \left(\sqrt{\frac{m\omega}{2\hbar}}q + \sqrt{\frac{\hbar}{2m\omega}}\frac{d}{dq}\right)\langle q|0\rangle = 0.$$

Hence, we have a differential equation

$$\frac{d}{dq}\langle q|0\rangle = -\frac{m\omega}{\hbar} q \langle q|0\rangle$$

which has the normalized solution

$$\langle q|0\rangle = \psi_0(q) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega q^2/2\hbar}.$$

$$\hat{H}|0\rangle = \hbar\omega\left(a^{\dagger}a + \frac{1}{2}\right)|0\rangle = \frac{1}{2}\hbar\omega|0\rangle,$$

namely $E_0 = \frac{1}{2}\hbar\omega$. Since this state can be reached from an eigenvector $|n\rangle$ by applying a^n, removing n quanta of $\hbar\omega$, we conclude that the eigenenergies are

$$\underline{E_n = \hbar\omega\left(n + \frac{1}{2}\right)}.$$

We have calculated this without knowing what the $|n\rangle$s are. "It is almost magic!"

<u>Eigenvectors</u>

We now realize why $\hat{N}$ is called the number operator:

$$\hat{H}|n\rangle = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle.$$

The eigenvalue n of $\hat{N}$ counts the number of energy quanta $\hbar\omega$ the state $|n\rangle$ has above the ground state energy.

If we now look back at the equation

$$\hat{H}\,a|n\rangle = (E_n - \hbar\omega)\,a|n\rangle,$$

we realize that $E_n - \hbar\omega$ is the eigenvalue of the $(n-1)^{th}$ eigenstate, $|n-1\rangle$. Hence,

$$|n-1\rangle = C_n\,a|n\rangle,$$

where C_n is a constant.

The dual vector is

$$\langle n-1| = C_n^* \langle n| a^\dagger,$$

giving a normalization condition

$$1 = \langle n-1|n-1\rangle = |C_n|^2 \langle n|a^\dagger a|n\rangle = |C_n|^2 n$$

$$\Rightarrow \underline{a|n\rangle = \sqrt{n}\,|n-1\rangle},$$

when choosing the constant real and positive.

If we include time-dependence in our notation $|n\rangle$ for the stationary states,

$$|n\rangle \propto e^{-iE_n t/\hbar} = e^{-i(n+\frac{1}{2})\omega t},$$

we would need a time-dependent phase-factor:

$$\underline{a|n\rangle = e^{-i\omega t}\sqrt{n}\,|n-1\rangle} \qquad (*)$$

We have seen that a produces a state where the energy is lowered by $\hbar\omega$. What does $a^\dagger$ do? Operate with $a^\dagger$ on $(*)$:

$$a^\dagger a|n\rangle = e^{-i\omega t}\sqrt{n}\,a^\dagger|n-1\rangle$$

$$= n|n\rangle.$$

$$\Rightarrow \underline{a^\dagger|n\rangle = e^{i\omega t}\sqrt{n+1}\,|n+1\rangle} \overset{t=0}{=} \sqrt{n+1}\,|n+1\rangle$$

Hence, $a^\dagger$ produces a state where the energy is <u>raised</u> by $\hbar\omega$. $a^\dagger$ is therefore often called a raising operator.

a and $a^\dagger$ are also called annihilation and creation operators.

We can therefore find any state $|n\rangle$ by operating on the ground state:

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^+)^n |0\rangle$$

EXAMPLE – Expectation value of V.

To show how practical the operator method can be, we calculate the expectation value of V in the n'th excited state:

$$\langle V \rangle = \langle n | \hat{V} | n \rangle = \frac{1}{2} \langle n | m\omega^2 \hat{q}^2 | n \rangle.$$

To proceed, we need an expression for $\hat{q}$ in terms of a and a^+:

$$a = \frac{1}{\sqrt{2m\omega\hbar}} (i\hat{p} + m\omega\hat{q})$$

$$a^+ = \frac{1}{\sqrt{2m\omega\hbar}} (-i\hat{p} + m\omega\hat{q})$$

$$\Rightarrow \hat{q} = \sqrt{\frac{\hbar}{2m\omega}} (a + a^+)$$

Hence:

$$\langle V \rangle = \frac{1}{2} m\omega^2 \frac{\hbar}{2m\omega} \langle n | (a + a^+)^2 | n \rangle$$

$$= \frac{1}{4} \hbar\omega \langle n | aa + aa^+ + a^+a + a^+a^+ | n \rangle$$

$$= \frac{1}{4} \hbar\omega \left[n+1 + n \right] = \frac{E_n}{2}.$$

To get higher excited states, we can use the same method on

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n |0\rangle$$

resulting in the expression

$$\langle q|n\rangle = \psi_n = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} \left(x - \frac{d}{dx}\right)^n e^{-\frac{1}{2}x^2}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(x)\, e^{-\frac{1}{2}x^2},$$

where $H_n(x)$ is the n'th Hermite polynomial, and $x = \sqrt{\frac{m\omega}{\hbar}}\, q$.

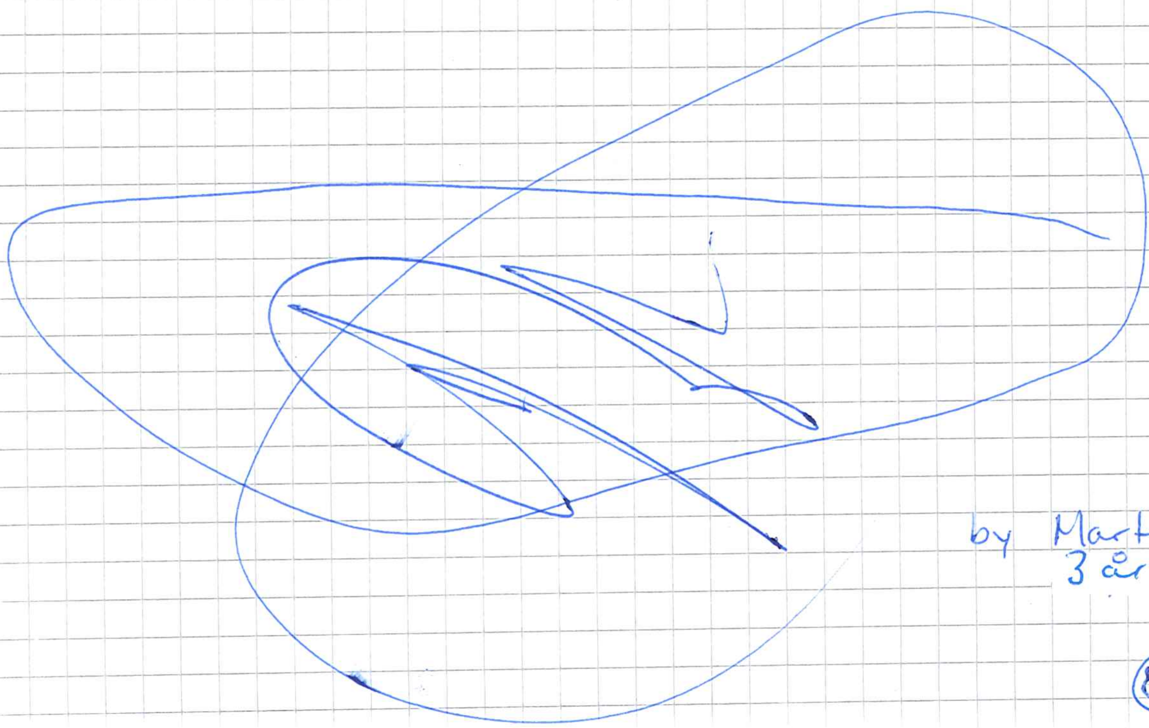
This is exactly the solution obtained in Intro. to quantum physics, and which we have also been using in the exercises.

COHERENT STATES

[H.6.6.6]

If we look at the energy eigenstates of the quantum harmonic oscillator, there is not an obvious similarity with the oscillating solutions we are used to for a classical harmonic oscillation. Even if we take the superposition of a few states, the time-dependence does not resemble a classical oscillator.

Is it possible to find a state which resembles the classical state?
[Ref. Schrödinger, 1926].



by Martha, 3 år

It turns out that, yes, it is possible to find Gaussian wave-packets with minimal uncertainty $\left(\Delta x \Delta p = \frac{\hbar}{2}\right)$, that are analogous to a classically oscillating particle, and they are defined by

$$a|\alpha\rangle \propto |\alpha\rangle,$$

they are eigenstates of the lowering operator!? Let's see how it's possible to get such a state. We expand in energy eigenstates of the harmonic oscillator

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle,$$

and operate with a:

$$a|\alpha\rangle = \sum_{n=0}^{\infty} c_n a|n\rangle = e^{-i\omega t} \sum_{n=0}^{\infty} c_n \sqrt{n}\, |n-1\rangle$$

$$= e^{-i\omega t} \sum_{n'=0}^{\infty} c_{n'+1} \sqrt{n'+1}\, |n'\rangle \overset{!}{=} e^{-i\omega t} \alpha \sum_{n=0}^{\infty} c_n |n\rangle.$$

For this to hold with some constant α, we require

$$c_{n+1}\sqrt{n+1} = \alpha c_n \quad \Rightarrow \quad c_n = \frac{c_0 \alpha^n}{\sqrt{n!}}$$

with c_0 some constand which must be determined by normalization of $|\alpha\rangle$.

Hence,

$$|\alpha\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle.$$

and we get

$$a|\alpha\rangle = e^{-i\omega t} \alpha |\alpha\rangle.$$

This gives us the following mean values:

$$\langle\alpha|a|\alpha\rangle = \alpha e^{-i\omega t}$$

$$\langle\alpha|a^+|\alpha\rangle = \langle\alpha|a|\alpha\rangle^* = \alpha^* e^{i\omega t},$$

which we can use to calculate the expectation value of the position:

$$\langle q\rangle = \langle\alpha|\hat{q}|\alpha\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle\alpha|a+a^+|\alpha\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\alpha e^{-i\omega t} + \alpha^* e^{i\omega t}\right)$$

$$= \sqrt{\frac{2\hbar}{m\omega}}|\alpha|\cos(\omega t-\theta), \equiv q_0\cos(\omega t-\theta)$$

where $\alpha = |\alpha|e^{i\theta}$, which is like a classical oscillation with amplitude

$$q_0 = |\alpha|\sqrt{\frac{2\hbar}{m\omega}}$$

determined by the eigenvalue α.

We can also calculate the probability density in position space:

$$|\langle q|\alpha\rangle|^2 = \sqrt{\frac{m\omega}{\pi\hbar}}\, e^{-m\omega[q-q_0\cos(\omega t-\theta)]^2/\hbar},$$

an oscillating wavepacket which "remains compact, and does not spread out." [Schrödinger]
For this reason, the states $|\alpha\rangle$ are called <u>coherent states</u>.

Important in quantum optics/quantum electrodynamics, where one can create coherent states of the oscillations of the electromagnetic field.
Nobel in 2005.