Lecture notes 11

11 Harmonic oscillator and angular momentum — via operator algebra

Written by Ingjald Øverbø

In Lecture notes 3 and in 4.7 in Bransden & Joachain you will find a comprehensive wave-mechanical treatment of the harmonic oscillator. We shall now show that the energy spectrum (and the eigenstates) can be found more easily by the use of **operator algebra**. In this method we use the properties of the Hilbert-space operators directly, without the use of a particular representation, that is, without projecting on a particular basis.

Similar methods can be applied to a whole range of problems. Here we shall use such a method to quantize angular momentum. Then it turns out that the abstract operator algebra not only reproduces the results for orbital angular momenta, but also provides a description of half-integral angular momenta (e.g. spin $\frac{1}{2}$), which can not be discribed in terms of wave mechanics. Thus, employing the Dirac notation and operator algebra, we are able to formulate a more general theory of angular momenta than that encountered in the position representation.

We start by attacking the one-dimensional oscillator, in order to gain some experience with the algebraic technique.

11.1 Harmonic oscillator

The so-called **algebraic** method or the **operator method** is explained in section 6.6 in Hemmers book; see also section 2.3 in J.J. Sakurai, *Modern Quantum Mechanics*. (In the book by Griffiths, section 2.3.1, you can see that the method may also be applied in the position representation.)

The ladder operators a and $a^\dagger$

The starting point is the Hamiltonian operator

$$\widehat{H} = \frac{\widehat{p}^2}{2m} + \tfrac{1}{2}m\omega^2\widehat{q}^2, \tag{T11.1}$$

where the abstract Hilbert-space operators $\widehat{p}$ and $\widehat{q}$ satisfy the commutator relation

$$[\widehat{q}, \widehat{p}] = i\hbar. \tag{T11.2}$$

Small update September 2022 by H.G. Hugdal.

In addition to the Hermitian operators $\widehat{q}$ and $\widehat{p}$ it is convenient to introduce the dimensionless operator

$$a = \sqrt{\frac{m\omega}{2\hbar}}\,\widehat{q} + \frac{i\widehat{p}}{\sqrt{2m\hbar\omega}}. \tag{T11.3}$$

This operator is non-Hermitian; it differs from its own adjoint, which is[1]

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\,\widehat{q} - \frac{i\widehat{p}}{\sqrt{2m\hbar\omega}}. \tag{T11.4}$$

We note in passing that expressed in terms of a and $a^\dagger$ the position and momentum operators take the form

$$\begin{aligned} \widehat{q} &= \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^\dagger), \\ \widehat{p} &= \frac{1}{i}\sqrt{\frac{m\hbar\omega}{2}}(a - a^\dagger). \end{aligned} \tag{T11.5}$$

We proceed to calculate the operator products $a^\dagger a$ and $aa^\dagger$. Here we must remember that $\widehat{q}$ and $\widehat{p}$ do not commute; $\widehat{q}\widehat{p} - \widehat{p}\widehat{q} = [\widehat{q}, \widehat{p}] = i\hbar$. We then get

$$\begin{aligned} a^\dagger a &= \left(\sqrt{\frac{m\omega}{2\hbar}}\,\widehat{q} - \frac{i\widehat{p}}{\sqrt{2m\hbar\omega}}\right)\left(\sqrt{\frac{m\omega}{2\hbar}}\,\widehat{q} + \frac{i\widehat{p}}{\sqrt{2m\hbar\omega}}\right) \\ &= \frac{m\omega}{2\hbar}\,\widehat{q}^2 + \frac{\widehat{p}^2}{2m\hbar\omega} + \frac{i}{2\hbar}(\widehat{q}\widehat{p} - \widehat{p}\widehat{q}) = \frac{\widehat{H}}{\hbar\omega} - \tfrac{1}{2}. \end{aligned} \tag{T11.6}$$

Here $\widehat{H} = \widehat{p}^2/2m + \frac{1}{2}m\omega^2\widehat{q}^2$ is the Hamiltonian operator for the oscillator. Reversing the order of the operators we find that the last term changes sign:

$$aa^\dagger = \cdots = \frac{\widehat{H}}{\hbar\omega} - \frac{i}{2\hbar}(\widehat{q}\widehat{p} - \widehat{p}\widehat{q}) = \frac{\widehat{H}}{\hbar\omega} + \tfrac{1}{2}. \tag{T11.7}$$

Thus the two operators do not commute, but satisfy the important commutation relation

$$\boxed{aa^\dagger - a^\dagger a = [a, a^\dagger] = 1.} \tag{T11.8}$$

Energy spectrum and eigenvectors

We shall see that this **algebraic** relation is *all we need* to solve the eigenvalue problem for the Hamiltonian (cf (T11.6))

$$\boxed{\widehat{H} = \hbar\omega(a^\dagger a + \tfrac{1}{2}),} \tag{T11.9}$$

that is, to find the eigenvalues E_n and the eigenvectors $|n\rangle$ of the eigenvalue equation

$$\widehat{H}\,|n\rangle = \hbar\omega(a^\dagger a + \tfrac{1}{2})\,|n\rangle = E_n\,|n\rangle. \tag{T11.10}$$

[1] The non-Hermitian operators a and $a^\dagger$ do not correspond to physical obervables, and it is customary not to use the usual operator symbol $\widehat{\ }$ on top of these; it is understood that they are operators.

By writing the eigenvalues as

$$E_n = \hbar\omega(n + \tfrac{1}{2}),$$

where the dimensionless numbers n are unknown, we can write this eigenvalue equation on the form

$$\widehat{N}\,|n\rangle = n\,|n\rangle, \qquad (\widehat{N} = a^\dagger a), \tag{T11.11}$$

As you have probably guessed we want to show that the only possible eigenvalues of the operator $\widehat{N} = a^\dagger a$ are $n = 0, 1, 2, \cdots$. Therefore this operator $\widehat{N} = a^\dagger a$ is called the **number operator**.

How do we *solve* the eigenvalue problem (T11.11) (where the unknown eigenvalues n of the number operator $\widehat{N}$ are used to label the eigenvectors)?

We start by using the commutator $aa^\dagger - a^\dagger a = 1$ (which is the only thing we need) to show that

$$\widehat{N}a = (a^\dagger a)a = (aa^\dagger - 1)a = a\widehat{N} - a = a(\widehat{N} - 1) \tag{T11.12}$$

and

$$\widehat{N}a^\dagger = (a^\dagger a)a^\dagger = a^\dagger(1 + a^\dagger a) = a^\dagger(\widehat{N} + 1). \tag{T11.13}$$

The last relation can be used to show that

> Given that $|n\rangle$ is an eigenvector of $\widehat{N}$ with eigenvalue n,
>
> $$\widehat{N}|n\rangle = n|n\rangle,$$
>
> then also the vector $a^\dagger|n\rangle$ is an eigenvector of $\widehat{N}$, with eigenvalue $n + 1$.

(T11.14)

The proof is simple: Using (T11.13) and (T11.11) we have that

$$\widehat{N}(\underline{a^\dagger|n\rangle}) = a^\dagger(\widehat{N} + 1)|n\rangle = a^\dagger(n + 1)|n\rangle = (n + 1)(\underline{a^\dagger|n\rangle}), \tag{T11.15}$$

and this is precisely what we wanted to show. In a similar manner we can use (T11.12) to show that $a|n\rangle$ is an eigenvector with eigenvalue $n - 1$:

$$\widehat{N}(a|n\rangle) = a(\widehat{N} - 1)|n\rangle = a(n - 1)|n\rangle = (n - 1)(a|n\rangle), \quad \text{q.e.d.} \tag{T11.16}$$

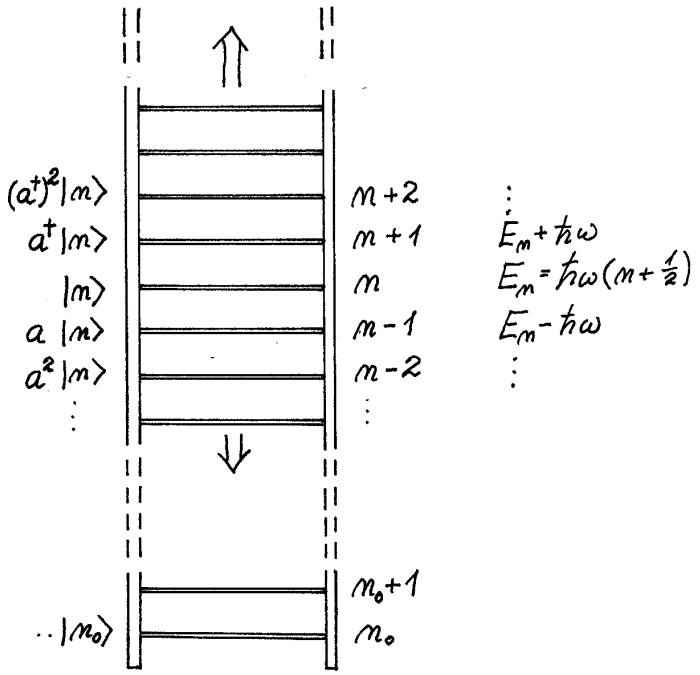
Thus if you provide me with one (normalized) eigenvector $|n\rangle$, so that

$$\widehat{N}\,|n\rangle = n\,|n\rangle, \qquad \langle n|n\rangle = 1,$$

then I can use the operators $a^\dagger$ and a repeatedly to create a whole bunch of new eigenvectors:

$$a^\dagger\,|n\rangle,\ (a^\dagger)^2|n\rangle, \cdots \qquad \text{and} \qquad a\,|n\rangle,\ a^2|n\rangle, \cdots .$$

For this reason these operators are called **ladder operators**; they allow us to "climb" up and down in energy. We may also call $a^\dagger$ a **raising operator** and a a **lowering operator**:



By using the raising operator $a^\dagger$ we can obviously create an *infinite* number of states with eigenvalues $n+1$, $n+2$, etc, but the same is not the case for the lowering operator a. As indicated in the figure, there must be a **lowest rung** in the **ladder**, with a lowest eigenvalue n_0. This is because by taking a look at the squared norm of the vector $a|n\rangle$ we find that the eigenvalue n has to be a non-negative number:

$$0 \leq ||a\,|n\rangle||^2 = (a\,|n\rangle)^\dagger (a\,|n\rangle) = \langle n|a^\dagger a|n\rangle = \langle n|\widehat{N}|n\rangle = n. \tag{T11.17}$$

With this little piece of algebra we have shown that if $|n\rangle$ is an eigenvector of $\widehat{N}$ with eigenvalue n [and of $\widehat{H}$ with eigenvalue $\hbar\omega(n+\frac{1}{2})$], then n has to be ≥ 0. This holds for all eigenstates $|n\rangle$, including the lowest rung in the ladder, $|n_0\rangle$, which is the ground state. We have therfore shown that n_0 is non-negative.

The calculation above also shows that the **norm** (the "length") of the vector $a|n\rangle$ is

$$||a|n\rangle|| = \sqrt{n}, \tag{T11.18}$$

(provided that the vector $|n\rangle$ you gave me is normalized, as was supposed above). With a similar calculation you should now check that the norm of the vector $a^\dagger|n\rangle$ is

$$||a^\dagger|n\rangle|| = \sqrt{n+1}. \tag{T11.19}$$

We note that the states $a|n\rangle$, $a^2|n\rangle$, $a^\dagger|n\rangle$, $(a^\dagger)^2|n\rangle$ etc are not normalized, but that will be taken care of below.

Let us first see what happens if we apply the lowering operator a to the lowest rung of the ladder, $|n_0\rangle$. Equation (T11.16) then takes the form

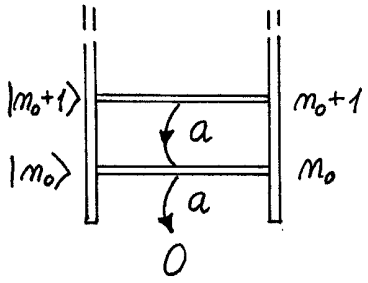
$$\widehat{N}(a|n_0\rangle) = (n_0 - 1)(a|n_0\rangle). \tag{T11.20}$$

The logic of this equation, which we have to recognize, is as follows: If the vector $a|n_0\rangle$ differs from zero (is normalizable), then this equation tells us that is an eigenvector of $\widehat{N}$ with

eigenvalue $n_0 - 1$. But since $|n_0\rangle$ is the lowest rung, there does not *exist* such an additional vector. Therefor we have to conclude that the vector $a|n_0\rangle$ is identically equal to zero (and hence is not normalizable):

$$\boxed{a|n_0\rangle = 0.} \tag{T11.21}$$

We can conclude that when the lowering operator a acts on the ground state, we get the **null vector**:



This equation (T11.21) allows us to solve the whole mystery. Using (T11.17), $||a|n\rangle||^2 = n$, we can state that

$$n_0 = ||a|n_0\rangle||^2 = 0, \qquad \text{and} \qquad E_{n_0} = E_0 = \tfrac{1}{2}\hbar\omega, \tag{T11.22}$$

as expected.

So if you give me one eigenstate $|n\rangle$, then I can use the lowering operator a to climb down an integer number of steps until I arrive at the ground state $|n_0\rangle = |0\rangle$ (apart from a normalization factor). And I may climb up again to the state (rung) you gave me with the same number of steps. This means of course that the eigenvalue n of the vector you gave me must be a non-negative integer.

We can conclude that the only possible energy eigenvalues are

$$E_n = (n + \tfrac{1}{2})\hbar\omega, \quad \text{med} \quad n = 0, 1, 2, \cdots,$$

and that there is only one eigenvector $|n\rangle$ for each E_n.

It is also easy to take care of the normalization. Since $a^\dagger|n\rangle$ has the eigenvalue $n+1$ and the norm $\sqrt{n+1}$, it follows that

$$\boxed{a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle,} \tag{T11.23}$$

where $|n+1\rangle$ is now normalized (and where we have made a *choice of phase*). Similarly it follows from (T11.16) and (T11.18) that [2]

$$\boxed{a|n\rangle = \sqrt{n}\,|n-1\rangle,} \qquad (= 0 \text{ for } n = 0), \tag{T11.24}$$

[2]Note that a removes an energy quantum $\hbar\omega$ from the oscillator, while $a^\dagger$ *adds* a quantum. The operator a is therefore called an **annihilation operator** and $a^\dagger$ is a **creation operator**.

where $|n-1\rangle$ is normalized. The first of these important formulae shows that the entire infinite ladder of states can be constructed starting from the ground state $|0\rangle$:

$$\begin{aligned} |1\rangle &= \frac{a^\dagger}{\sqrt{1}}|0\rangle, \\ |2\rangle &= \frac{a^\dagger}{\sqrt{2}}|1\rangle = \frac{(a^\dagger)^2}{\sqrt{1\cdot 2}}|0\rangle, \\ &\vdots \\ &\vdots \\ |n\rangle &= \frac{a^\dagger}{\sqrt{n}}|n-1\rangle = \cdots = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle. \end{aligned}$$

Thus the ground state and the excited states are determined by the equations

$$\boxed{a\,|0\rangle = 0 \qquad \text{and} \qquad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\,|0\rangle.} \tag{T11.25}$$

The position representation

The above results for the energy spectrum and the eigenvectors of the number operator $\widehat{N} = a^\dagger a$ (and hence also of $\widehat{H} = \hbar\omega(\widehat{N} + \frac{1}{2})$) were based exclusively on the operator relation (or "algebra") $[a, a^\dagger] = 1$, without projecting the Hilbert ket vectors onto any basis.

As we have seen before a choice of a particular basis will give us operators and states with a more concrete form. As an example we shall here use the position basis $|q\rangle$. We start by checking that the relation in (T11.25),

$$a|0\rangle = 0,$$

really determines the ground state $|0\rangle$ and the corresponding wave function. By projecting the ground state $|0\rangle$ onto the position vector $|q\rangle$ and using the expression (T11.3) for a we have:

$$\langle q|\left(\sqrt{\frac{m\omega}{2\hbar}}\,\widehat{q} + \frac{i\widehat{p}}{\sqrt{2m\hbar\omega}}\right)|0\rangle = 0. \tag{T11.26}$$

We may now use the "tools" (T10.62) and (T10.63), here on the form

$$\langle q|\,\widehat{q} = q\langle q| \qquad \text{and} \qquad \langle q|\widehat{p} = \frac{\hbar}{i}\frac{\partial}{\partial q}\langle q|,$$

which give

$$\left(\sqrt{\frac{m\omega}{2\hbar}}\,q + \sqrt{\frac{\hbar}{2m\omega}}\,\frac{\partial}{\partial q}\right)\,\langle q|0\rangle = 0. \tag{T11.27}$$

Here $\langle q|0\rangle$ is the wave function $\psi_0(q)$ for the ground state, and the expression in parentheses is the operator a in the position representation (cf the discussion in section 10.3.b). (Note in passing that the corresponding expression for $a^\dagger$ has a minus sign in front of the last term). The relation $a|0\rangle = 0$ thus gives us the differential equation

$$\frac{d}{dq}\psi_0(q) = -\frac{m\omega}{\hbar}\,q\,\psi_0(q)\,,$$

which is easily integrated:

$$\ln\psi_0 = -\frac{m\omega}{2\hbar}\,q^2 + \ln C_0 \quad \Longrightarrow$$
$$\psi_0(q) = C_0\, e^{-m\omega q^2/2\hbar}. \tag{T11.28}$$

We recognize this as the wave function of the ground state. [$C_0 = (m\omega/\pi\hbar)^{1/4}$ gives a normalized wave function.]

Similarly, the wave functions for the excited states can be found from the formula for $|n\rangle$:

$$\begin{aligned}\psi_n(q) &= \langle q|n\rangle = \frac{1}{\sqrt{n!}}\langle q|\left(\sqrt{\frac{m\omega}{2\hbar}}\,\widehat{q} - \frac{i\widehat{p}}{\sqrt{2m\hbar\omega}}\right)^n|0\rangle \\ &= \frac{1}{\sqrt{n!\,2^n}}\left(\sqrt{\frac{m\omega}{\hbar}}\,q - \sqrt{\frac{\hbar}{m\omega}}\,\frac{\partial}{\partial q}\right)^n \psi_0(q).\end{aligned}$$

With the dimensionless variable $x = q\sqrt{m\omega/\hbar}$ which is commonly used for the harmonic oscillator we thus have

$$\begin{aligned}\psi_n(x) &= \frac{C_0}{\sqrt{2^n\,n!}}\left(x - \frac{d}{dx}\right)^n e^{-x^2/2} \\ &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}\,H_n(x)\,e^{-x^2/2} \qquad (x = q\sqrt{m\omega/\hbar}).\end{aligned} \tag{T11.29}$$

[The last step has not been shown here, but can be taken for granted. You can of course also check it for a few values of n.]

These results show *some* of the power of the algebraic method. In the next section we shall use a similar method to find the eigenstates and the eigenvalues for angular momenta.

11.2 General quantization of angular momentum

Introductory remarks

In the preceding section we saw that the quantization of the number operator $N = a^\dagger a$ could be based exclusively on the **algebra** $[a, a^\dagger] = 1$ for the operators a and $a^\dagger$.

We shall now use a similar algebraic method to find eigenstates and eigenvalues for a generalized angular momentum denoted by $\vec{J}$. As mentioned in the introduction, it then turns out, firstly, that we are able to reproduce the results for orbital angular momenta, with integer quantum numbers l and m (see Lecture notes 5, 5.4 in Hemmer or 6.1 in Bransden & Joachain). Secondly, the algebraic method furnishes a set of angular momentum states with *half-integral* quantum numbers, which do not describe orbital angular motion. These states provide a description of spin degrees of freedom, which can not be described by the ordinary wave-function formalism. This is a triumph for our new Hilbert-space formulation of quantum mechanics, and for the algebraic method.

Before we attempt to formulate this theory of angular momentum, it is instructive to see how the non-Hermitian operators

$$\widehat{L}_\pm \equiv \widehat{L}_x \pm i\widehat{L}_y = \pm\hbar\, e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\,\cot\theta\,\frac{\partial}{\partial\phi}\right) \tag{T11.30}$$

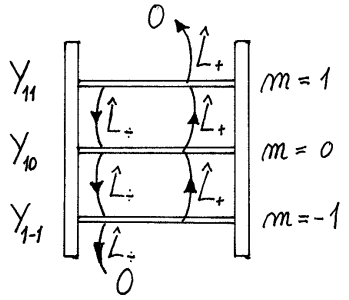
act on the "triplet" Y_{1m}, that is, on the three angular functions

$$Y_{10} = \sqrt{\frac{3}{4\pi}}\,\cos\theta, \qquad Y_{1\pm1} = \mp\sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{\pm i\phi}.$$

As you can easily check, the operators $\widehat{L}_+$ and $\widehat{L}_-$ then act as **ladder operators**, that is, they change the magnetic quantum number m by $+1$ and -1, respectively:

$$\begin{aligned} \widehat{L}_+\, Y_{1-1} &= \hbar\sqrt{2}\, Y_{10}, & \widehat{L}_-\, Y_{11} &= \hbar\sqrt{2}\, Y_{10}, \\ \widehat{L}_+\, Y_{10} &= \hbar\sqrt{2}\, Y_{11}, & \widehat{L}_-\, Y_{10} &= \hbar\sqrt{2}\, Y_{1-1}, \\ \widehat{L}_+\, Y_{11} &= 0, & \widehat{L}_-\, Y_{1-1} &= 0. \end{aligned} \tag{T11.31}$$

This is illustrated in the figure, where we see that the two operators take us respectively one step up and one step down in the triplet "ladder".



We note in particular that $\widehat{L}_+$ applied to the top rung gives zero, and similarly when $\widehat{L}_-$ is applied to the bottom rung (in analogy with $a|0\rangle = 0$).

The angular-momentum algebra and some additional operator relations

We assume that we have a general angular momentum $\vec{J}$ which corresponds to the Hilbert-space operators J_x, J_y and J_z. Our only assumptions are that these operators are Hermitian and satisfy the fundamental angular-momentum algebra [3]

$$\begin{aligned} [J_x, J_y] &= i\hbar J_z, \\ [J_y, J_z] &= i\hbar J_x, \\ [J_z, J_x] &= i\hbar J_y, \end{aligned} \tag{T11.32}$$

which is the same algebra that is found for the orbital angular momentum $\vec{L} = \vec{r}\times\vec{p}$ (cf Lecture notes 5 and B&J).

The example above indicates that it may be a good idea to define the operators J_+ and J_-, which are each others adjoint:

$$J_\pm \equiv J_x \pm i J_y; \qquad (J_+)^\dagger = J_-, \quad (J_-)^\dagger = J_+. \tag{T11.33}$$

[3] You can find a very clear discussion of this matter in Hemmer's chapter 8, and also in chapter 1 in J.J. Sakurai, *Modern Quantum Mechanics*. In this section we follow these authors and drop the "hats" $\widehat{\ }$ over the operators.

In the same manner as for the orbital angular momentum, it is easy to show that $\vec{J}^2$ commutes with all the components of $\vec{J}$;

$$[\vec{J}^2, J_x] = [\vec{J}^2, J_y] = [\vec{J}^2, J_z] = 0. \tag{T11.34}$$

It follows that

$$[\vec{J}^2, J_\pm] = 0. \tag{T11.35}$$

Furthermore, by considering the commutators $[J_z, J_\pm]$, one finds that

$$J_z J_\pm = J_\pm J_z \pm \hbar J_\pm. \tag{T11.36}$$

In addition we note that

$$\begin{aligned} J_+ J_- &= \vec{J}^2 - J_z^2 + \hbar J_z, \\ J_- J_+ &= \vec{J}^2 - J_z^2 - \hbar J_z. \end{aligned} \tag{T11.37}$$

Simultaneous eigenvectors

Since $\vec{J}^2$ and J_z commute, we may seek to find simultaneous eigenvectors of $\vec{J}^2$ and J_z:

$$\vec{J}^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle, \tag{T11.38}$$

$$J_z|j,m\rangle = \hbar m|j,m\rangle. \tag{T11.39}$$

Here, we have written the eigenvalues as $\hbar^2 j(j+1)$ and $\hbar m$, and we are using the numbers j and m as labels.[4] So far the only thing we know about j and m is that

$$|m| \leq \sqrt{j(j+1)}. \tag{T11.40}$$

This is because quadratic operators like J_z^2, J_x^2 etc in general have non-negative expectation values:

$$\left\langle J_z^2 \right\rangle_\psi = \langle\psi|J_z^2|\psi\rangle = \langle\psi|J_z^\dagger J_z|\psi\rangle = \|J_z|\psi\rangle\|^2 \geq 0 \quad \Longrightarrow \quad \left\langle \vec{J}^2 \right\rangle \geq \left\langle J_z^2 \right\rangle. \tag{T11.41}$$

This implies that the eigenvalue of $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$ can not be negative, and is the reason that we may write the eigenvalue as $\hbar^2 j(j+1)$, where j is a non-negative number;

$$j \geq 0.$$

We also note that eigenvectors with differing eigenvalues are orthogonal as usual. It is also convenient to require that they are normalized, so that

$$\langle j', m'|j, m\rangle = \delta_{jj'}\delta_{mm'}. \tag{T11.42}$$

[4] $j(j+1)$, with $j \geq 0$, so far is only a fancy way of writing a non-negative number.

Finite ladder

Suppose now that there exists one vector $|j,m\rangle$ satisfying the eigenvalue equations (T11.38) and (T11.39), with quantum numbers j and m that we do not yet know. Following the example above, we may then apply the ladder operators $J_\pm$ on the vector $|j,m\rangle$, and consider the properties of the resulting vectors $J_+|j,m\rangle$ and $J_-|j,m\rangle$.

We must first check that $J_\pm$ really *are* ladder operators, as we expect. Since $\vec{J}^2$ commutes with $J_\pm$, we see at once that $J_\pm$ does not change the angular-momentum quantum number j:

$$\vec{J}^2(J_\pm|j,m\rangle) = J_\pm\,\hbar^2 j(j+1)|j,m\rangle = \hbar^2 j(j+1)(J_\pm|j,m\rangle). \tag{T11.43}$$

From (T11.36) we proceed to show that $J_\pm$ as expected raises/lowers the magnetic quantum number by ± 1:

$$J_z(J_\pm|j,m\rangle) = (J_\pm J_z \pm \hbar J_\pm)|j,m\rangle = \hbar(m\pm 1)(J_\pm|j,m\rangle). \tag{T11.44}$$

From these equations it appears that $J_+|j,m\rangle$ is an eigenstate of J_z with eigenvalue $\hbar(m+1)$, and that $J_-|j,m\rangle$ is an eigenstate of J_z with eigenvalue $\hbar(m-1)$. In the same manner, $J_+^2|j,m\rangle$ will be an eigenvector with eigenvalue $\hbar(m+2)$, etc. Starting with the given existence of the eigenstate $|j,m\rangle$ we can thus create new "rungs in a ladder" of states by repeated use of J_+ and J_-. (In analogy with (T11.31) and with the oscillator case, these new vectors $J_+|j,m\rangle$ etc will not be normalized, but that can be corrected by dividing by the norm, as we shall see below.)

Because $|m|$ can not exceed $\sqrt{j(j+1)}$, this process of creating new rungs above and below the given rung $|j,m\rangle$ has to stop, with a **top rung** $|j,m_1\rangle$ and a **bottom rung** $|j,m_0\rangle$. This requires that

$$J_+|j,m_1\rangle = 0. \tag{T11.45}$$

If this is not the case, equation (T11.44) tells us that $J_+|j,m_1\rangle$ is an eigenvector of J_z with eigenvalue $\hbar(m_1+1)$, and that simply is not possible when $|m_1\rangle$ is the top rung. In the same manner

$$J_-|j,m_0\rangle = 0, \tag{T11.46}$$

because $|j,m_0\rangle$ is the bottom rung. Note that none of this comes as a surprise, in view of the example with the triplet set of wave functions in (T11.31).

The top and bottom rungs

Our experience in (T11.31) leads us to suspect that $m_1 = j$ and $m_0 = -j$, and this turns out to be the case. The key lies in the square of the norm of $J_-|j,m\rangle$, which is readily calculated:

$$\begin{aligned} 0 \quad &\leq \quad \|J_-|j,m\rangle\|^2 = (J_-|j,m\rangle)^\dagger\; J_-|j,m\rangle = \langle j,m|J_+J_-|j,m\rangle \\ &\overset{\text{(T11.37)}}{=} \quad \langle j,m|\vec{J}^2 - J_z^2 + \hbar J_z|j,m\rangle = \langle j,m|\;\hbar^2[j(j+1)-m^2+m]\;|j,m\rangle \\ &= \quad \hbar^2(j+m)(j+1-m). \end{aligned} \tag{T11.47}$$

Using J_+ we find in the same manner:

$$0 \leq \|J_+|j,m\rangle\|^2 = \cdots = \hbar^2(j-m)(j+1+m). \tag{T11.48}$$

In (T11.48) the square $||J_+|j,m\rangle||^2$ is either positive or equal to zero. If it is positive, it follows from (T11.44) that

$$\frac{J_+|j,m\rangle}{||J_+|j,m\rangle||} = \frac{J_+|j,m\rangle}{\hbar\sqrt{(j-m)(j+1+m)}} \equiv |j,m+1\rangle \tag{T11.49}$$

is a normalized eigenvector of J_z with eigenvalue $\hbar(m+1)$, which constitutes a new rung in the ladder, as discussed above. However, this possibility is excluded when J_+ is applied to the top rung. Hence the only possibility left is that the square of the norm is equal to zero for $J_+|j,m_1\rangle$:

$$||J_+|j,m_1\rangle||^2 = \hbar^2(j-m_1)(j+1+m_1) = 0.$$

The conclusion (which no longer comes as a surprise) is that the quantum number m for the top rung is

$$m_1 = j. \tag{T11.50}$$

(m can not be equal to $-j-1$, because of (T11.40).) In the same manner we find for the bottom rung that

$$||J_-|j,m_0\rangle||^2 = \hbar^2(j+m_0)(j+1-m_0) = 0,$$

giving

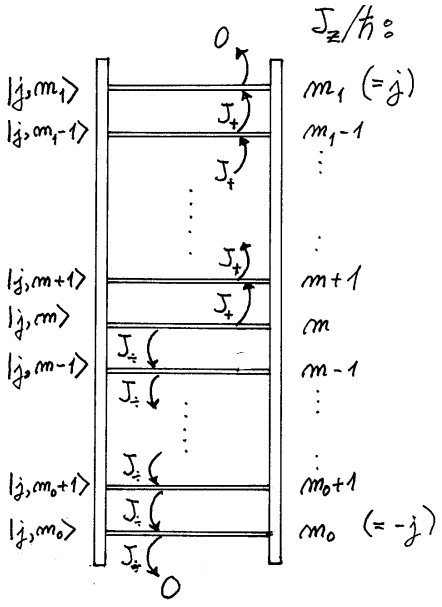
$$m_0 = -j. \tag{T11.51}$$

Conclusion

Starting with the angular-momentum algebra and the supposed existence of the normalized state $|j,m\rangle$, we have now used the relations

$$\boxed{J_\pm\,|j,m\rangle = \hbar\sqrt{(j\mp m)(j+1\pm m)}\,|j,m\pm 1\rangle} \tag{T11.52}$$

to construct a complete ladder of normalized states,



where the magnetic quantum number varies in steps of 1 from the minimal value $m_0 = -j$ to the maximal value $m_1 = j$:

$$\boxed{m = -j,\, -j+1,\, \cdots,\, j-1,\, j.} \tag{T11.53}$$

This means that the distance $2j$ between the top and bottom rungs is an integer. What is surprising with this result is of course that it allows for *half-integral* angular-momentum quantum numbers (when $2j$ is an odd number),

$$j = 1/2,\ 3/2,\ 5/2, \quad \text{etc}, \tag{T11.54}$$

in addition to the integer quantum numbers (when $2j$ is an even number),

$$j = 0,\ 1,\ 2, \quad \text{etc}, \tag{T11.55}$$

which we know from the treatment of orbital angular momenta. Now we can state that the supposed state $|j, m\rangle$ which was our starting point must either have half-integral j and m or integral j and m. We also know that m must have one of the values in (T11.53). These results are of course very promising, because such a theory allowing for only integral or half-integral quantum numbers is precisely what we are looking for.

Does this mean that we can have half-integral orbital angular momenta? The answer is no. This is because if we set $\vec{J} = \widehat{\vec{L}} = \vec{r} \times \hat{\vec{p}}$, and $\widehat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$, then we know that the solutions $e^{im\phi}$ of the eigenvalue equation for $\widehat{L}_z$ become continuous only for integer values of m and hence of l.

11.3 Orbital angular momentum, position representation

The starting point for the derivation above was the algebra (T11.32), originally found for the orbital angular momentum $\vec{L} = \vec{r} \times \vec{p}$. This means of course that the formalism above will reproduce the earlier results for the orbital angular momentum $\vec{L}$ (cf Lecture notes 5, or B&J), when we project the abstract vectors onto the position basis

$$|\vec{r}\rangle \equiv |x, y, z\rangle \equiv |r, \theta, \phi\rangle.$$

It is instructive to see how this works, even if nothing essentially new comes out of it.

The Hilbert-space operator $\vec{L} = \vec{r} \times \vec{p}$ is in the position formulation represented by the well-known operator

$$\vec{L} = \vec{r} \times \frac{\hbar}{i}\boldsymbol{\nabla}.$$

This follows from the "tools" (T10.62 – 63), which give

$$\begin{aligned} \langle r, \theta, \phi|\, \vec{r} &= \vec{r}\, \langle r, \theta, \phi|, \\ \langle r, \theta, \phi|\, \vec{p} &= \frac{\hbar}{i}\boldsymbol{\nabla}_{\vec{r}} \langle r, \theta, \phi|. \end{aligned} \tag{T11.56}$$

It follows that

$$\langle r,\theta,\phi|\ L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}\langle r,\theta,\phi|, \tag{T11.57}$$

$$\langle r,\theta,\phi|\ L_x = \frac{\hbar}{i}\left(-\sin\phi\,\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\,\frac{\partial}{\partial\phi}\right)\langle r,\theta,\phi|, \tag{T11.58}$$

$$\langle r,\theta,\phi|\ L_y = \frac{\hbar}{i}\left(\cos\phi\,\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\,\frac{\partial}{\partial\phi}\right)\langle r,\theta,\phi|, \tag{T11.59}$$

$$\langle r,\theta,\phi|\ \vec{L}^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\phi^2}\right)\langle r,\theta,\phi|, \tag{T11.60}$$

$$\langle r,\theta,\phi|\ L_\pm = \frac{\hbar}{i}\,e^{\pm i\phi}\left(\pm i\frac{\partial}{\partial\theta} - \cot\theta\,\frac{\partial}{\partial\phi}\right)\langle r,\theta,\phi|, \tag{T11.61}$$

where on the right we recognize all the well-known operators from the position representation.

Combining (T11.57) with (T11.39) we have

$$\frac{\hbar}{i}\frac{\partial}{\partial\phi}\,\langle r,\theta,\phi|l,m\rangle = \hbar m\,\langle r,\theta,\phi|l,m\rangle,$$

which is the well-known eigenvalue equation

$$\frac{\hbar}{i}\frac{\partial}{\partial\phi}\,\psi_{lm}(r,\theta,\phi) = \hbar m\,\psi_{lm}(r,\theta,\phi) \tag{T11.62}$$

for the wave function

$$\psi_{lm}(r,\theta,\phi) = \langle r,\theta,\phi|l,m\rangle. \tag{T11.63}$$

In a similar manner we may combine (T11.60) with (T11.39), and (T11.61) with (T11.52). Because the angular-momentum operators in these equations depend only on angles, the equations are separable, that is, we can write

$$\psi_{lm}(r,\theta,\phi) = R(r)\,Y_{lm}(\theta,\phi), \tag{T11.64}$$

where the angular functions $Y_{lm}(\theta,\phi)$ satisfy the same equations as ψ_{lm}, as e.g.

$$\frac{\hbar}{i}\frac{\partial}{\partial\phi}\,Y_{lm}(\theta,\phi) = \hbar m\,Y_{lm}(\theta,\phi). \tag{T11.65}$$

The radial function $R(r)$ thus far is arbitrary, as long as we only require that $|l,m\rangle$ is an eigenvector of $\vec{L}^2$ and L_z. [5]

Equation (T11.65) has the well-known solution (cf Lecture notes 5 or B&J):

$$Y_{lm}(\theta,\phi) = \Theta(\theta)\,e^{im\phi}. \tag{T11.66}$$

[5] We *could* have obtained a more complete specification of the ket vector $|l,m\rangle$ (and hence of the radial function $R(r)$) by requiring that it also be an eigenvector of a rotationally invariant Hamiltonian $\widehat{H}$, with an eigenvalue E_n. This eigenvalue could then be used as a third label of the vector: ($|E_n,l,m\rangle$ or $|n,l,m\rangle$). However, here we are primarily interested in the properties of the angular functions, the spherical harmonics $Y_{lm}(\theta,\phi)$.

As you probably remember, Y_{lm} becomes a continuous function of the azimuthal angle ϕ only if m is an integer. Half-integral values of m, which are allowed according to the results of the *algebraic* derivation above, give a discontinuity for $\phi = 0$,

$$Y(\theta, 2\pi) = -Y(\theta, 0).$$

Thus the function does *not* "bite its own tail" at this point, as it must do in order to be continuous. For half-integral m the left-hand side of (T11.65) then becomes a delta function, while the right-hand side is finite, and that is not allowed.

We therefore arrive at the following conclusion: Any rotational motion which can be described classically, in terms of an orbital angular momentum $\vec{L} = \vec{r} \times \vec{p}$, is quantized in terms of integral angular-momentum quantum numbers.

This means that half-integral spin states, like e.g. the electron spin $\vec{S}$, can *not* be understood classically, in terms of some kind of orbital angular momentum of the form $\vec{L} = \vec{r} \times \vec{p}$. The lack of such a formula (and a corresponding classical model) for the electron spin $\vec{S}$ means that this spin can not be described in the position representation of quantum mechanics. Thus for this spin there is no formula for e.g. $\langle \vec{r}|S_z$ corresponding to (T11.57). We therefore have to be content with a more abstract description based on the vectors $|j, m\rangle$ (or $|s, m\rangle$), which will be used in Lecture notes 12.

Let us return briefly to the orbital angular momentum. Here the algebraic method of course does not give any new results. It must reproduce the results which are obtained with the differential-equation method (see Lecture notes 5, or B&J). However, the algebraic method has some technical advatages: We may start with (T11.45). This gives

$$L_+|l, l\rangle = 0, \tag{T11.67}$$

which in the position representation has the form (cf (T11.61))

$$\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right) Y_{ll} = 0. \tag{T11.68}$$

By the use of (T11.66) it turns out that the solution is

$$Y_{ll}(\theta, \phi) = C_{ll}\sin^l\theta\, e^{il\phi}, \tag{T11.69}$$

where the normalization constant can be set equal to [6]

$$C_{ll} = \frac{(-1)^l}{2^l\, l!}\sqrt{\frac{(2l+1)!}{4\pi}}. \tag{T11.70}$$

Now we only need to use the ladder operator L_- to calculate $Y_{l,l-1}$, $Y_{l,l-2}$, etc, down to $Y_{l,-l}$. This is because by combining (T11.61) with (T11.52) we have:

$$Y_{l,m-1} = \frac{1}{\sqrt{(l+m)(l+1-m)}}\, e^{-i\phi}\left(-\frac{\partial}{\partial\theta} - \cot\theta\,\frac{1}{i}\frac{\partial}{\partial\phi}\right) Y_{l,m}. \tag{T11.71}$$

Here we may replace $\frac{1}{i}\frac{\partial}{\partial\phi}$ by m, according to (T11.66).

Note that the ladder operators L_+ and L_- and all the other operators on the right-hand sides in (T11.57) – (T11.61) satisfy the same algebraic relations as the abstract operators

[6] The normalization in (T11.70) corresponds to the convention used for Y_{lm} in Lecture notes 5.

J_+, J_-, etc. Thus instead of the differential equations, one can use the algebraic technique to obtain the spectra and eigenfunctions of $\vec{L}^2$ and L_z. The same goes for the oscillator problem (Lecture notes 3), which can be solved using the position representations of the operators a and $a^\dagger$, that is, with an algebraic method corresponding to that used in section 11.1. (See Griffiths.)