ADDITION OF ANGULAR MOMENTA [ØB, H8.4, G4.4.3]

In a system with more than one contribution to the angular momentum, we need to know how to treat addition of angular momentum. Examples can be an electron with both orbital and spin angular momentum,

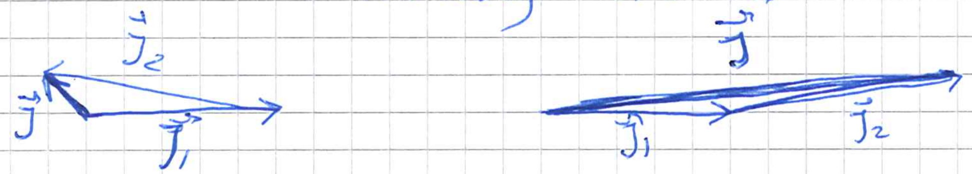
$$\vec{J} = \hat{L} + \hat{S},$$

or a system consisting of many particles with spin, such as e.g. the hydrogen atom.

Classically, the total angular momentum of two systems with angular momenta $\vec{J}_1$ and $\vec{J}_2$ is $\vec{J} = \vec{J}_1 + \vec{J}_2$. The size $J = |\vec{J}|$ must satisfy

$$|J_1 - J_2| \leq J \leq J_1 + J_2,$$

and varies continuously between.



In quantum mechanics it is more complicated since we add operators with quantized eigenvalues. Two questions arise in the general case of calulating $\hat{\vec{J}}_1 + \hat{\vec{J}}_2 = \hat{\vec{J}}$:

i) What quantum numbers j can we have for $\vec{J}$, when $\hat{J}_1$ and $\hat{J}_2$ have quantum numbers j_1 and j_2.

ii) How do we express the eigenstates of $\vec{J}$ in terms of the eigenstates of $\vec{J}_1$ and $\vec{J}_2$?

First of all, we need to determine if $\vec{J}$ satisfies the angular momentum algebra,

$$[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k. \qquad \text{(Einstein sum conv.)}$$

The key thing to note is that, the two angular momenta are compatible observables, i.e. they commute

$$[J_{1i}, J_{2j}] = 0 \quad \forall\ i,j \in \{x,y,z\},$$

then we have:

$$\begin{aligned}
[J_i, J_j] &= [J_{1i} + J_{2i}, J_{1j} + J_{2j}] \\
&= [J_{1i}, J_{1j}] + [J_{2i}, J_{2j}] \\
&= i\hbar \varepsilon_{ijk}[J_{1k} + J_{2k}] = i\hbar \varepsilon_{ijk} J_k,
\end{aligned}$$

and $\vec{J}$ satisfies the angular momentum algebra.

To proceed with answering questions (i) and (ii), we will look at the simplest nontrivial example.

Addition of two spin ½ particles

We consider two particles with spin $\frac{1}{2}$, $s_1 = \frac{1}{2}$, $s_2 = \frac{1}{2}$, e.g. the electron and the proton in the ground state of hydrogen. The first is in the state $|s_1\, m_1\rangle$, the second in $|s_2\, m_2\rangle$, and we have the combined state $|s_1\, s_2\, m_1 m_2\rangle$.

In the general case, we have

$$\vec{S}_1^{\,2} |s_1\, s_2\, m_1\, m_2\rangle = \hbar^2 s_1(s_1+1)\, |s_1\, s_2\, m_1\, m_2\rangle$$
$$\vec{S}_2^{\,2} |s_1\, s_2\, m_1\, m_2\rangle = \hbar^2 s_2(s_2+1)\, |s_1\, s_2\, m_1\, m_2\rangle$$
$$S_{1z} |s_1\, s_2\, m_1\, m_2\rangle = \hbar m_1\, |s_1\, s_2\, m_1\, m_2\rangle$$
$$S_{2z} |s_1\, s_2\, m_1\, m_2\rangle = \hbar m_2\, |s_1\, s_2\, m_1\, m_2\rangle .$$

Hence, the total z-component m is given by

$$S_z |s_1\, s_2\, m_1\, m_2\rangle = (S_{1z} + S_{2z}) |s_1\, s_2\, m_1\, m_2\rangle$$
$$= \hbar(m_1 + m_2) |s_1\, s_2\, m_1\, m_2\rangle = \hbar m |s_1\, s_2\, m_1 m_2\rangle ,$$

that is

$$m = m_1 + m_2 .$$

Specializing to $s_1 = s_2 = \frac{1}{2}$, we get four possible states:

$$|\uparrow\uparrow\rangle = |\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle \quad \text{with} \quad m = 1$$
$$|\uparrow\downarrow\rangle = |\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,-\tfrac{1}{2}\rangle \quad m = 0$$
$$|\downarrow\uparrow\rangle = |\tfrac{1}{2}\,\tfrac{1}{2}\,-\tfrac{1}{2}\,\tfrac{1}{2}\rangle \quad m = 0$$
$$|\downarrow\downarrow\rangle = |\tfrac{1}{2}\,\tfrac{1}{2}\,-\tfrac{1}{2}\,-\tfrac{1}{2}\rangle \quad m = -1 .$$

This is a bit strange, since m should have integer steps between $-s$ and s, but here we have <u>two</u> $m=0$ states, meaning we have one "extra" $m=0$ state.

To untangle this problem, we operate with $S_- = S_{1-} + S_{2-}$ on $|\uparrow\uparrow\rangle$:

$$S_- |\uparrow\uparrow\rangle = (S_{1-}|\uparrow\rangle)|\uparrow\rangle + |\uparrow\rangle(S_{2-}|\uparrow\rangle)$$
$$= \hbar|\downarrow\rangle|\uparrow\rangle + |\uparrow\rangle\hbar|\downarrow\rangle$$
$$= \hbar\left[|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle\right].$$

If we apply S_- again, we get:

$$S_-\left[|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle\right]$$
$$= (S_{1-}|\downarrow\rangle)|\uparrow\rangle + |\downarrow\rangle(S_{2-}|\uparrow\rangle) + (S_{1-}|\uparrow\rangle)|\downarrow\rangle + |\uparrow\rangle(S_{2-}|\downarrow\rangle)$$
$$= 2\hbar|\downarrow\downarrow\rangle$$

Evidently, we have three states with $s=1$:

$$\left.\begin{aligned} |1\ 1\rangle &= |\uparrow\uparrow\rangle \\ |1\ 0\rangle &= \frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right] \\ |1\ {-1}\rangle &= |\downarrow\downarrow\rangle \end{aligned}\right\}\quad s=1\ \text{(triplet)}.$$

the triplet combination, with states $|s, m\rangle$ with $s=1$, $m=-1,0,1$. The remaining state has $s=0$, $m=0$

$$|0\ 0\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right]\ \Big\}\quad s=0\ \text{(singlet)}$$

the singlet state, which is orthogonal to the triplets.

We therefore claim that the combination of two spin $\frac{1}{2}$ particles can carry a total spin of 1 or 0, depending on whether they occupy the triplet or singlet configuration. However, in order to confirm this, we need to show that the triplets and singlet are eigenvectors of $\vec{S}^2$ with eigenvalues $2\hbar^2$ and 0 respectively, i.e:

$$\vec{S}^2 |1\ m\rangle = \hbar^2 s(s+1)|1\ m\rangle = 2\hbar^2 |1\ m\rangle$$
$$\vec{S}|0\ 0\rangle = 0 \cdot |0\ 0\rangle.$$

We find an expression for $\vec{S}^2$:

$$\vec{S}^2 = (\vec{S}_1 + \vec{S}_2^2) = \vec{S}_1^2 + \vec{S}_2^2 + 2\vec{S}_1 \cdot \vec{S}_2$$
$$= \vec{S}_1^2 + \vec{S}_2^2 + 2 S_{1z} S_{2z} + S_{1+} S_{2-} + S_{1-} S_{2+},$$

where we have expressed S_{ix} and S_{iy} in terms of $S_{i\pm}$.

For the triplet states we then get:

$$\vec{S}^2 |1\ 1\rangle = \left(\tfrac{3\hbar^2}{4}|\uparrow\rangle\right)|\uparrow\rangle + |\uparrow\rangle\left(\tfrac{3\hbar^2}{4}|\uparrow\rangle\right)$$
$$+ 2\left(\tfrac{\hbar}{2}|\uparrow\rangle\right)\left(\tfrac{\hbar}{2}|\uparrow\rangle\right) + 0 + 0$$
$$= 2\hbar^2 |\uparrow\uparrow\rangle = 2\hbar^2 |1\ 1\rangle$$

$$\vec{S}^2 |1\ -1\rangle = 2\hbar^2 |1\ -1\rangle$$

and

$$\vec{S}^2 |1\,0\rangle = \frac{1}{\sqrt{2}} \vec{S}^2 \left[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \right]$$

$$= \frac{1}{\sqrt{2}} \left\{ \left(\frac{3\hbar^2}{4}|\uparrow\rangle\right)|\downarrow\rangle + |\uparrow\rangle\left(\frac{3\hbar^2}{4}|\downarrow\rangle\right) \right.$$

$$+ \left(\frac{3\hbar^2}{4}|\downarrow\rangle\right)|\uparrow\rangle + |\downarrow\rangle\left(\frac{3\hbar^2}{4}|\uparrow\rangle\right)$$

$$+ 2\left(\frac{\hbar}{2}|\uparrow\rangle\right)\left(-\frac{\hbar}{2}|\downarrow\rangle\right) + 2\left(-\frac{\hbar}{2}|\downarrow\rangle\right)\left(\frac{\hbar}{2}|\uparrow\rangle\right)$$

$$+ \left(S_{1+}|\uparrow\rangle\right)\left(S_{2-}|\downarrow\rangle\right) + \left(S_{1+}|\downarrow\rangle\right)\left(S_{2-}|\uparrow\rangle\right)$$

$$\left. + \left(S_{1-}|\uparrow\rangle\right)\left(S_{2+}|\downarrow\rangle\right) + \left(S_{1-}|\downarrow\rangle\right)\left(S_{2+}|\uparrow\rangle\right) \right\}$$

$$= \frac{1}{\sqrt{2}} \left\{ \frac{3}{2}\hbar^2 \left[|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle \right] \right.$$

$$- \frac{1}{2}\hbar^2 \left[|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle \right]$$

$$\left. + \hbar^2 \left[|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle \right] \right\}$$

$$= 2\hbar^2 |1\,0\rangle. \qquad \text{OK.}$$

Hence, all the triplet states have the correct eigenvalue.

Performing the same calculation for the singlet, we get:

$$\vec{S}^2 |0\,0\rangle = \frac{1}{\sqrt{2}} \left\{ \frac{3\hbar^2}{2} \left[|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle \right] \right.$$

$$- \frac{1}{2}\hbar^2 \left[|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle \right]$$

$$\left. - \hbar^2 \left[|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle \right] \right\}$$

$$= 0. \qquad \text{OK.}$$

Hence, combining two spins $\frac{1}{2}$, we get $S=1$ or $S=0$.

We now have two possible basis sets for the two spin system. The "old" basis

$$|\uparrow\uparrow\rangle, \ |\uparrow\downarrow\rangle, \ |\downarrow\uparrow\rangle, \ |\downarrow\downarrow\rangle$$

and the "new" basis

$$\left.\begin{matrix} |1\ 1\rangle \\ |1\ 0\rangle \\ |1\ {-1}\rangle \end{matrix}\right\} \text{triplets} \qquad |0\ 0\rangle \Big\} \text{ singlet.}$$

In the "old" basis, S_{1z} and S_{2z} have sharp values, as does S_z. However, $\vec{S}^2$ is not sharp, except for $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$, since $\vec{S}^2$ does not commute with S_{1z} and S_{2z} separately.

In the "new" basis S_z and $\vec{S}^2$ are sharp, and S_{1z} and S_{2z} are unsharp, except for $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$, i.e $|1\,1\rangle$ and $|1\,{-1}\rangle$.

To prepare old states, we can measure S_{1z} and S_{2z} separately; to prepare new states we can measure S_z and $|\vec{S}|$ for the total spin.

We also note that the triplets are symmetric under exchange of spins, while the singlet is antisymmetric. Impotant for identical fermions

General addition of angular momenta

We can generalize the above treatment: If we add two angular momenta, described by

$$|j_1, m_1\rangle \qquad m_1 = -j_1, -j_1+1, \ldots, j_1$$
$$|j_2, m_2\rangle \qquad m_2 = -j_2, -j_2+1, \ldots j_2$$

being eigenstates of $\vec{J}_i^2$ and J_{iz}, $i = 1,2$, we have a set of $(2j_1+1)(2j_2+1)$ "old" states $|j_1\, m_1\rangle|j_2\, m_2\rangle$.

The possible values for the total quantum number j is

$$j = j_1+j_2,\ j_1+j_2-1,\ j_1+j_2-2, \ldots |j_1-j_2|.$$

In order to generate the "new" states, we can begin with $|j_1+j_2, j_1+j_2\rangle$ and operate with J_- until we reach the state $|j_1+j_2, -j_1-j_2\rangle$. In this way we find all states with $j = j_1+j_2$, $-j \leq m \leq j$: the new states $|j_1+j_2, m\rangle$.

In order to find the states with $j = j_1+j_2-1$, we have to find a linear combination of states $|j_1\, m_1\rangle|j_2\, m_2\rangle$ with $m_1+m_2 = j_1+j_2-1$ which is orthogonal to $|j_1+j_2, j_1+j_2-1\rangle$. Operating with J_- will give all states $|j_1+j_2-1, m\rangle$, $|m| \leq j_1+j_2-1$ and so on.

For a given set of j_1 and j_2, we therefore get

$$N = j_1 + j_2 - \sqrt{(j_1 - j_2)^2} + 1$$

ladders corresponding to new states with

$$j_1 + j_2 \geq j \geq |j_1 - j_2|,$$

with j taking integer steps.

The combined ("new") state $|j\ m\rangle$ with total angular momentum j and total z component m can be written in terms of the composite states $|j_1\ j_2\ m_1\ m_2\rangle$:

$$|j\ m\rangle = \sum_{m_1+m_2=m}{}^{'} C^{j_1 j_2 j}_{m_1 m_2 m} |j_1\ j_2\ m_1\ m_2\rangle,$$

where the coefficients $C^{j_1 j_2 j}_{m_1 m_2 m}$ are called Clebsch–Gordan coefficients.

Example — Addition of orbital angular momentum and spin

One important example, is the addition of orbital angular momentum and spin, e.g. for the electron in a hydrogen atom:

$$\vec{J} = \vec{L} + \vec{S}.$$

The possible values for j are thus

$$\left| \ell - \frac{1}{2} \right| \leq j \leq \ell + \frac{1}{2}.$$

If $\ell = 0$, we have simply $j = s = \frac{1}{2}$.

If $\ell > 0$, we have two possibilities, $j = \begin{cases} \ell - \frac{1}{2} \\ \ell + \frac{1}{2} \end{cases}$

If $\ell = 1$, what would the state with $j = \frac{3}{2}$ and $m = \frac{1}{2}$ look like in terms of the "old" states? We can use the Clebsch-Gordan coefficients:

$$\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle = C^{1\,\frac{1}{2}\,\frac{3}{2}}_{1\,-\frac{1}{2}\,\frac{1}{2}} \left|1\ \tfrac{1}{2}\ 1\ -\tfrac{1}{2}\right\rangle + C^{1\,\frac{1}{2}\,\frac{3}{2}}_{0\,\frac{1}{2}\,\frac{1}{2}} \left|1\ \tfrac{1}{2}\ 0\ \tfrac{1}{2}\right\rangle$$

$$= \frac{1}{\sqrt{3}} \left|1\ \tfrac{1}{2}\ 1\ -\tfrac{1}{2}\right\rangle + \sqrt{\frac{2}{3}} \left|1\ \tfrac{1}{2}\ 0\ \tfrac{1}{2}\right\rangle.$$

If we were to measure the spin of the electron in this state, we could get $\frac{\hbar}{2}$ with probability $2/3$, and $-\frac{\hbar}{2}$ with probability $1/3$.

We could also ask "what is the old states expressed in the new?" We can then use

$$|j_1\ j_2\ m_1\ m_2\rangle = \sum_j C^{j_1 j_2 j}_{m_1 m_2 m} |j\ m\rangle\ , \quad (m_1 + m_2 = m).$$

For the state $\left|1\ \tfrac{1}{2}\ 0\ \tfrac{1}{2}\right\rangle$ we get

$$\left|1\ \tfrac{1}{2}\ 0\ \tfrac{1}{2}\right\rangle = \sqrt{\frac{2}{3}} \left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle - \frac{1}{\sqrt{3}} \left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle.$$

If we know our electron is in the state with $m_\ell = 0$ and $m_s = \frac{1}{2}$, there is a $2/3$ probability of measuring the total angular momentum $j = \frac{3}{2}$, and $1/3$ of measuring $j = \frac{1}{2}$.

Addition of several angular momenta

If we want to add several angular momenta

$$\vec{J} = \vec{J}_1 + \vec{J}_2 + \vec{J}_3 + \dots$$

We can do it in a stepwise manner.

First combine $\vec{J}_1$ and $\vec{J}_2$:

$$\vec{J}_{12} = \vec{J}_1 + \vec{J}_2,$$

with possible quantum numbers

$$|j_1 - j_2| \le j_{12} \le j_1 + j_2.$$

Then we combine $\vec{J}_{12}$ with $\vec{J}_3$, and so on.

Example Total angular momentum of hydrogen

The hydrogen atom consists of (in the simplest case) an electron with spin $\frac{1}{2}$ and orbital angular momentum ℓ, and a proton with spin $\frac{1}{2}$. Hence:

$$\vec{J} = \vec{S}_e + \vec{L}_e + \vec{S}_p.$$

We already considered the electron, and found for $\vec{J}_e = \vec{S}_e + \vec{L}_e$, that

$$j_e = \ell - \frac{1}{2},\ \ell + \frac{1}{2}.$$

If we now consider $\vec{J}$, we get

$$|j_e - \frac{1}{2}| \le j \le j_e + \frac{1}{2}$$

$$\Rightarrow j = |\ell - 1|,\ \ell,\ \ell + 1.$$