Contents

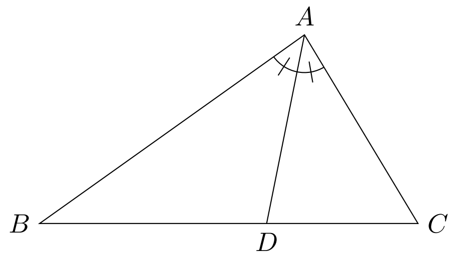
1 Lines, angles and shapes

1.10 Four centres of triangle

1.10.1 Angle bisector, perpendicular bisector, median, altitude and cevian

Angle bisector

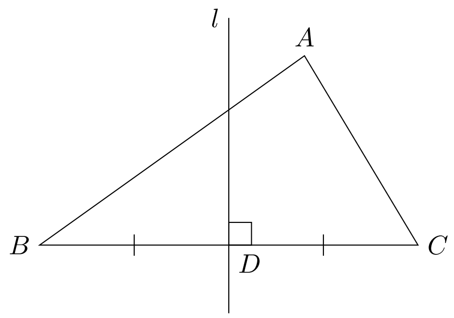
An angle bisector of a triangle is a line segment from a vertex to the opposite side such that the angle of the vertex is bisected by the line segment.



$$\because \angle BAD = \angle CAD$$
$$\therefore\ AD \text{ is the angle bisector of } \angle A \text{ in } \triangle ABC\ .$$

Perpendicular bisector

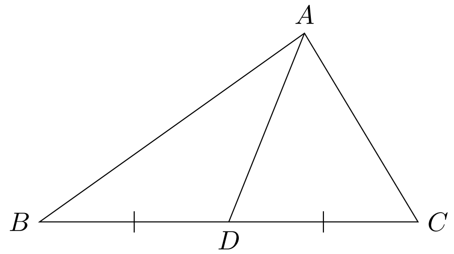
A perpendicular bisector of a triangle is the perpendicular bisector of a side of the triangle. It is a line instead of line segment.



$$\because BD = DC \text{ and } l \perp BC$$
$$\therefore\ \text{Line } l \text{ is the the perpendicular bisector of } BC \text{ in } \triangle ABC\ .$$

Median

A **median** of a triangle is a line segment from a vertex to the mid-point of the opposite side.



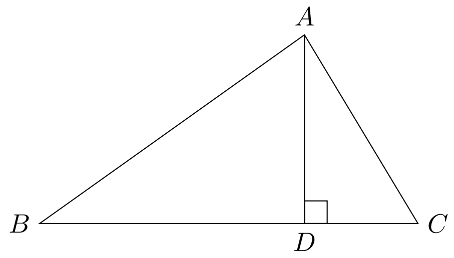
$$\because BD = DC$$
$$\therefore\ AD \text{ is a median of } \triangle ABC \text{ (that corresponds to } BC\text{).}$$

Altitude

An **altitude** of a triangle is a perpendicular line segment (or line) from a vertex to the (extended) opposite side.

It is sometimes also called the height of the triangle (but height more often refers to the length of the altitude while altitude more often refers to the line segment itself).

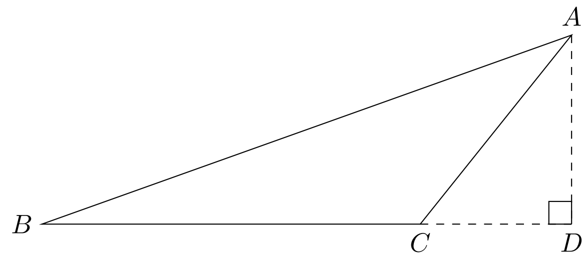
Note that the point where the altitude meet the side is called the **foot** of the altitude.



$$\because AD \perp BC$$
$$\therefore\ AD \text{ is an altitude of } \triangle ABC \text{ (that corresponds to } BC\text{).}$$

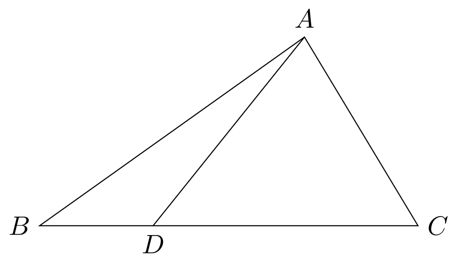
(And D is the foot of altitude AD .)

Note that this AD is also an altitude:



Cevian

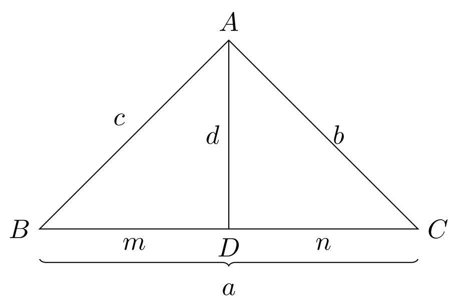
A **cevian** of a triangle is any line segment from a vertex to the opposite side. It must lie inside the triangle. All angle bisectors and medians are cevians, but not all perpendicular bisectors and altitudes are cevians.



$$\because D \text{ lies on side } BC\ .$$
$$\therefore\ AD \text{ is a cevian of } \triangle ABC \text{ (that corresponds to } BC\text{).}$$

Lengths of angle bisector, median, altitude and cevian

Here's a summary of their lengths (the formulas have appeared and been proved in previous prepositions).



Type	Condition	Formula
Cevian	D is on side BC .	$d=\sqrt{\dfrac{b^2m+c^2n}{m+n}-mn}$
Median	$m=n$	$d=\dfrac{\sqrt{2b^2+2c^2-a^2}}{2}$
Angle bisector	$\angle BAD=\angle CAD$	$d=\sqrt{bc-mn}=\sqrt{bc(1-\dfrac{a^2}{(b+c)^2})}$
Altitude	$AD\perp BC$	$d=\sqrt{c^2-(\dfrac{a^2+c^2-b^2}{2a})^2}=\sqrt{b^2-(\dfrac{a^2+b^2-c^2}{2a})^2}$

1.10.2 The four centres of triangles

Any triangle has four centres, which are **incentre**, **circumcentre**, **centroid** and **orthocentre**. By definition,

Incentre is the intersection of angle bisectors of the angles.
Circumcentre is the intersection of perpendicular bisectors of the sides.
Centroid is the intersection of the triangle's medians.
Orthocentre is the intersection of the triangle's altitudes.

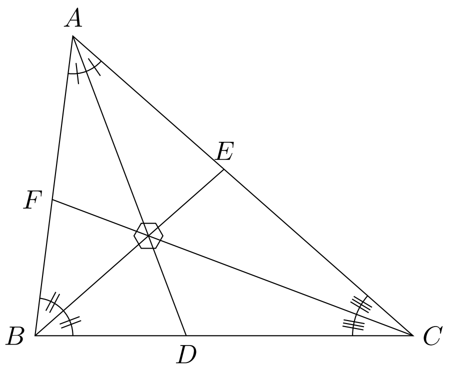
Any triangle has only one unique point for each type of centre. There cannot be two incentres in a triangle, or two centroids.

This means the angle bisectors of a triangle are concurrent, and perpendicular bisectors of triangle sides are concurrent, and medians of the triangles are concurrent, and altitudes of triangles are concurrent. We will prove these facts in the following subsubsections.

1.10.3 Incentre (and incircle)

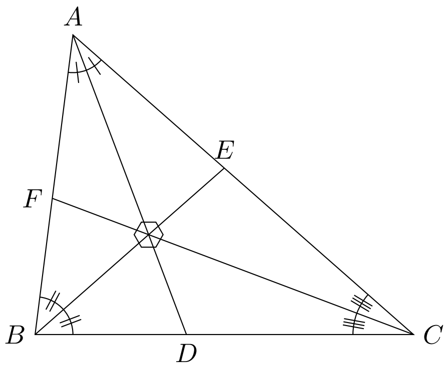
Preposition 1. The angle bisctors of a triangle are concurrent. (concurrency of $\angle$ bisctors of $\triangle$)

(The mini hexagon indicates that we don't initially know whether AD , BE and CF are concurrent.)



$$\because \angle BAD=\angle CAD\ ,\ \angle ABE=\angle CBE\ ,\ \angle ACF=\angle BCF$$
$$\therefore\ AD\ ,\ BE\ ,\ CF \text{ are concurrent.} \qquad \text{(concurrency of } \angle \text{ bisctors of } \triangle)$$

Proof. [1] Let there be $\triangle ABC$ with points D,E,F on sides BC,AC,AB respectively such that $\angle BAD=\angle CAD$, $\angle ABE=\angle CBE$, $\angle ACF=\angle BCF$.



Since AD is the angle bisector of A , by angle bisector theorem, we have

$$\frac{BD}{DC} = \frac{AB}{AC} \tag{1}$$

Similar, since BE and CF are angle bisector of $\angle B$ and $\angle C$ respectively, by angle bisector theorem, we have

$$\frac{CE}{EA} = \frac{BC}{AB} \tag{2}$$

$$\frac{AF}{FB} = \frac{AC}{BC} \tag{3}$$

Multiply (1), (2) and (3) together:

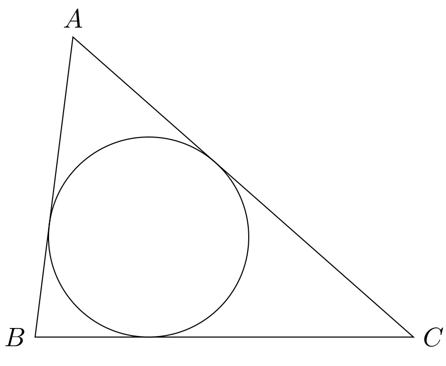
$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{AB}{AC} \cdot \frac{BC}{AB} \cdot \frac{AC}{BC}$$
$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$$

By converse of Ceva's theorem, AD, BE, CF are concurrent. □

And this point of concurrency is called incentre (usually denoted I).

Inscribed circle

The **inscribed circle** (/**incircle**) of a triangle is a circle that is tangent to all three sides of the triangle:

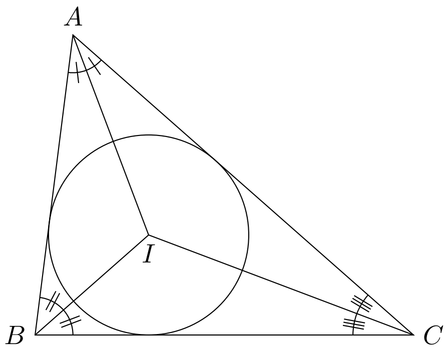


The incircle is the largest possible circle that can be contained in the triangle, and there is a unique incircle for each triangle.

And the incircle's centre is the incentre of the triangle.

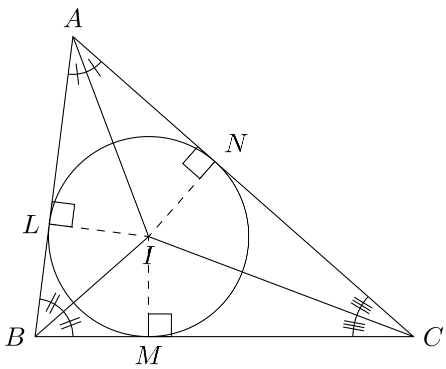
The radius of the incircle is called **inradius** .

Preposition 2. The incentre of a triangle is the centre of the inscribed circle of the triangle. (prop. of incentre)



$$\because I \text{ is the incentre of } \triangle ABC\,.$$
$$\therefore\ I \text{ is the centre of the incircle of } \triangle ABC\,. \qquad \text{(prop. of incentre)}$$

Proof. Drop perpendicular line segments from the incentre to the sides. Namely, draw $IM \perp BC$, $IN \perp AC$, $IL \perp AB$.

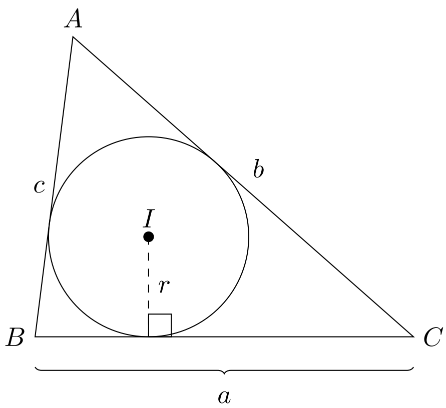


Since IB is the angle bisector of ABC , we have $IL = IM$ (prop. of $\angle$ bisector).
Similarly, since IC is the angle bisector of ACB , we have $IM = IN$ (prop. of $\angle$ bisector).
By transitivity of equality, we have $IL = IM = IN$.

By definition, the incircle is tangent to sides AB, BC, AC . So the radii to the points of tangency are perpendicular to AB, BC, AC (tangent $\perp$ radius).
Since $IM \perp BC$, M must be a point of tangency (since there is a unique point on BC that is the perpendicular projection of I onto BC).
So M lies on the incircle. By similar reasoning, N and L also lie on the incircle.
By '3R theorem', since $IL = IM = IN$, I must be the centre of the incircle. □

Note: This preposition means the incentre is the (only) point that is equidistant from the three sides of the triangle.

Preposition 3. Given a triangle with inradius (radius of incircle) r and semi-perimeter s , the area (A) of the triangle is rs. (semi-perimeter inradius formula)

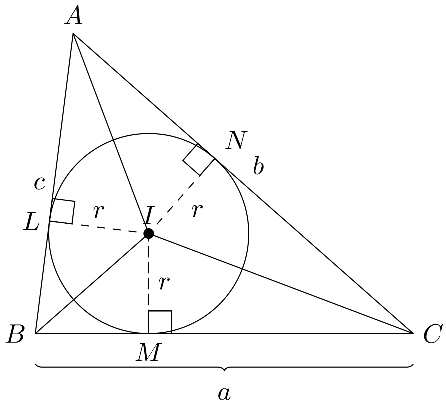


Given: $\quad s = \dfrac{a+b+c}{2}$

$$\boxed{A = rs}$$

Proof. [2] Join IA , IB and IC . Let A be the area of $\triangle ABC$. (This A is different from the vertex A , but I am too lazy to make a new variable.)

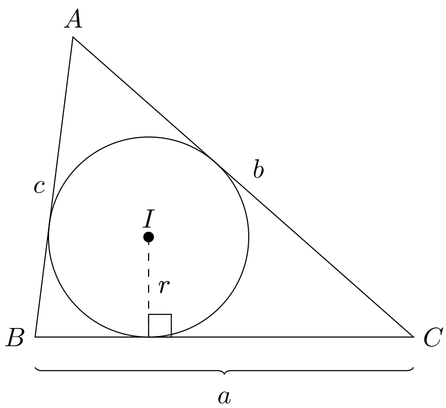
Draw $IM \perp BC$, $IN \perp AC$, $IL \perp AB$. Note that $IM = IN = IL = r$ (inradius).



$$\begin{aligned}
\text{area of } \triangle ABC &= \text{area of } \triangle AIB + \text{area of } \triangle BIC + \text{area of } \triangle AIC \\
&= \frac{AB \cdot IL}{2} + \frac{BC \cdot IM}{2} + \frac{AC \cdot IN}{2} \qquad \text{(area of } \triangle) \\
&= \frac{cr}{2} + \frac{ar}{2} + \frac{br}{2} \\
&= r(\frac{a+b+c}{2}) \\
A &= rs
\end{aligned}$$

□

Preposition 4. Given a triangle with side lenghths a, b, c , the **inradius** (radius of incircle) (r) of the triangle is $\sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$, where $s = \dfrac{a+b+c}{2}$ is the semi-perimeter of the triangle. (inradius formula)



$$\boxed{r = \sqrt{\frac{s(s-a)(s-b)(s-c)}{s}}}$$

Proof. Let A be the area of $\triangle ABC$, r be the inradius, and s be the semi-perimeter.

By semi-perimeter inradius formula, we have $A = rs$.

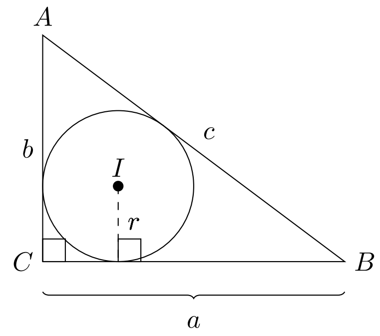
By Heron's formula, we have $A = \sqrt{s(s-a)(s-b)(s-c)}$.

Thus,

$$
\begin{aligned}
rs &= \sqrt{s(s-a)(s-b)(s-c)} \\
r &= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} \\
&= \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2}} \\
&= \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}
\end{aligned}
$$

□

Preposition 5. Given a right triangle with legs a, b and hypotenuse c, the inradius (r) of the triangle is $\dfrac{a+b-c}{2}$ and also $s-c$ (where s is the semi-perimeter) . (inradius formula of right triangle)

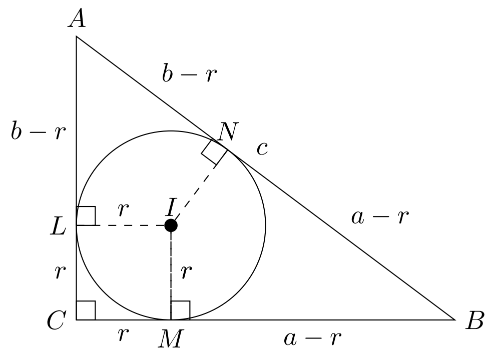


Given: $\angle C = 90°$.

$$\boxed{r = \frac{a+b-c}{2}}$$

$$\boxed{r = s-c}$$

Proof. [3] Draw $IM \perp BC$, $IN \perp AC$, $IL \perp AB$.

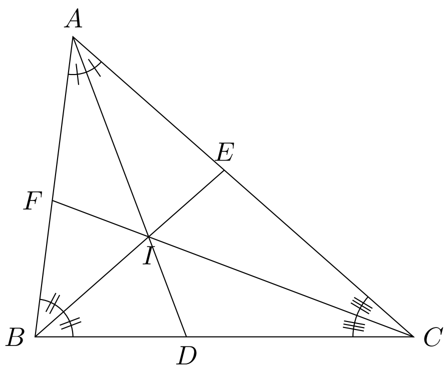


Note that $BMIL$ is a square with side length r. Then $MB = a-r$ and $AL = b-r$.
By tangent properties, we have $AN = AL = b-r$ and $NB = MB = a-r$. So

$$
\begin{aligned}
AB &= AN + NB \\
c &= (b-r) + (a-r) \\
2r &= a+b-c \\
r &= \frac{a+b-c}{2}
\end{aligned}
$$

And $r = \dfrac{a+b+c-2c}{2} = s-c$ □

Preposition 6. For an angle bisector of a triangle, the incentre is closer to the landing point on the opposite side than to the vertex. (position of incentre on $\angle$ bisector)

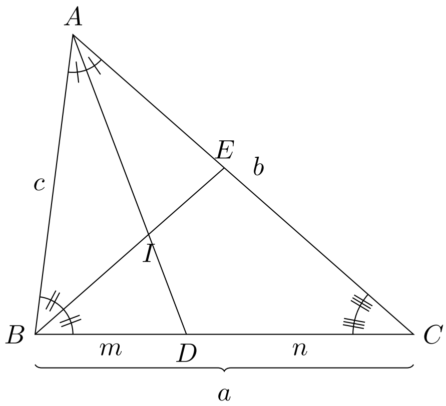


Given: I is the incentre of $\triangle ABC$.

$\therefore$ $ID < IA$, $IE < IB$, $IF < IC$ (position of incentre on $\angle$ bisector)

Proof. Let $BC = a$, $AC = b$, $AB = c$. Let $BD = m$ and $DC = n$.

It is sufficient to only prove that $ID < IA$, as the rest follows a similar argument.



By angle bisector theorem, we have $\dfrac{ID}{IA} = \dfrac{m}{c}$. If $m < c$, then $\dfrac{m}{c} < 1 \Rightarrow \dfrac{ID}{IA} < 1 \Rightarrow ID < IA$. Thus, we want to show that $m < c$.

By angle bisector theorem, we have $\dfrac{m}{n} = \dfrac{c}{b}$, which means $m = \dfrac{ac}{b+c}$.

By triangle inequality, we have $a < b + c$. Multiply both sides by $\dfrac{c}{b+c}$:

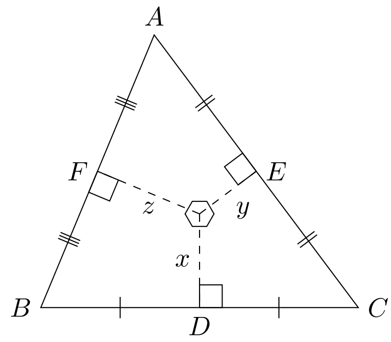
$$
\begin{aligned}
a(\frac{c}{b+c}) &< (b+c)(\frac{c}{b+c}) \\
\frac{ac}{b+c} &< c \\
m &< c
\end{aligned}
$$

This means $ID < IA$. By similar argument, we have $IE < IB$ and $IF < IC$.

$\square$

1.10.4 Circumcentre (and circumcircle)

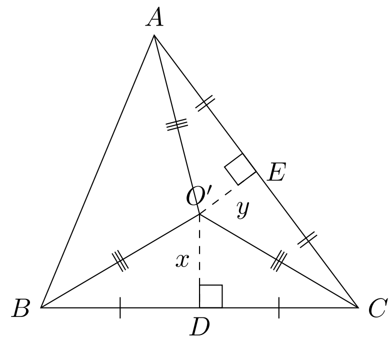
Preposition 7. The perpendicular bisectors of a triangle's sides are concurrent. (concurrency of $\perp$ bisectors of $\triangle$)



$\because x \perp BC$, $y \perp AC$, $z \perp AB$, $BD = DC$, $CE = EA$, $AF = FB$

$\therefore$ x, y, z are concurrent. (concurrency of $\perp$ bisectors of $\triangle$)

Proof. Let line x, y, z be the perpendicular bisector of BC, AC, AB respectively. Let O' be the intersection of x and y .



Since O' lies on the $\perp$ bisector of BC , we have $O'B = O'C$ (prop. of $\perp$ bisector).

Similarly, since O' lies on the $\perp$ bisector of AC , we have $O'A = O'C$ (prop. of $\perp$ bisector).

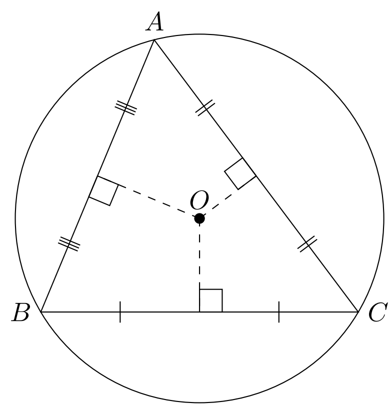
By transitivity of equality, we have $O'A = O'B$, which means O' lies on the perpendicular bisector of AB (prop. of $\perp$ bisector).

This means all three perpendicular bisectors are concurrent. □

And the point of concurrency of the perpendicular bisectors are called circumcentre.

Preposition 8. The circumcentre of a triangle is the centre of the triangle's circumcircle. (prop. of circumcentre)

(Note: The **circumcircle** of a triangle is the circle that passes through all three vertices of the triangle.)



$\because O$ is the circumcentre of $\triangle ABC$.

$\therefore$ O is the centre of circumcircle of $\triangle ABC$.

Proof. Since O lies on perpendicular bisectors of all three sides, we have $OA = OB = OC$ (prop. of $\perp$ biosector).

Note that A, B, C lie on the circumcircle by definition.

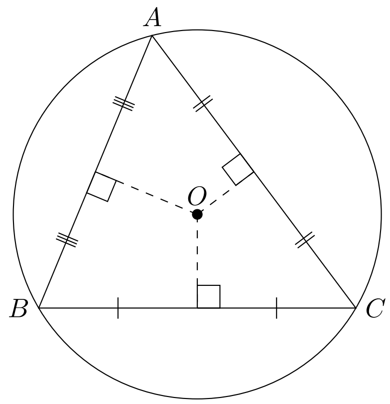
Since $OA = OB = OC$, by '3R theorem', O is the centre of the circumcircle. $\square$

Note 1: This preposition means the circumcentre is the (only) point that is equidistant from the three vertices of the triangle.

Note 2: Unlike incentre which can only lie inside a triangle, the circumcentre may lie on a triangle's side or outside the triangle. The former happens for a right triangle and the latter for an obtuse triangle.

Preposition 9. The circumcentre lies inside an acute triangle, lies on the side of a right triangle, and lies outside an obtuse triangle. (position of circumcentre)

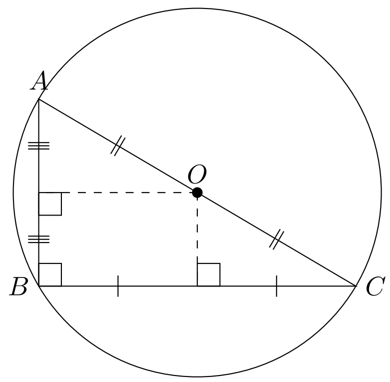
Case 1:



$\because \angle A < 90^\circ$, $\angle B < 90^\circ$, $\angle C < 90^\circ$.

$\therefore$ O lies inside $\triangle ABC$.

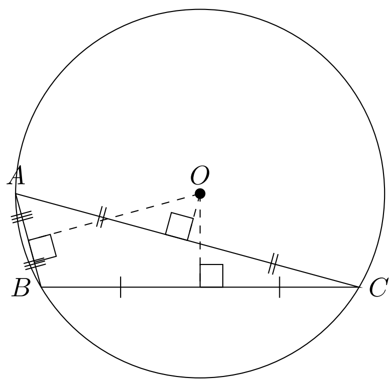
Case 2:



$\because \angle ABC = 90^\circ$

$\therefore$ O lies on side AC .

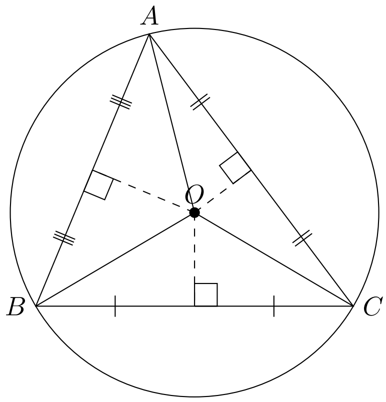
Case 3:



$$\because \angle ABC > 90^\circ$$
$$\therefore\ O \text{ lies outside } \triangle ABC\ .$$

Proof. Case 1: $\angle A < 90^\circ$, $\angle B < 90^\circ$, $\angle C < 90^\circ$

Let $\angle A = x$, $\angle B = y$, $\angle C = z$, where $x, y, z < 90^\circ$. This means $2x, 2y, 2z < 180^\circ$.



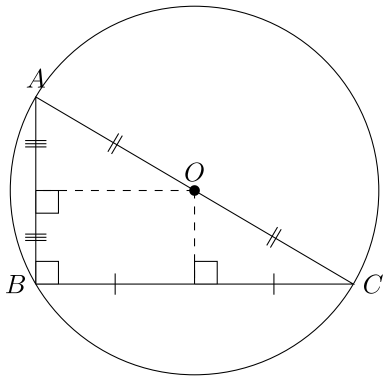
Note that anticlockwise$\angle COA = 2y$, (anticlockwise shortened to anti) anti$\angle AOB = 2z$, anti$\angle BOC = 2x$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$).

Thus anti$\angle COA < 180^\circ$, anti$\angle AOB < 180^\circ$, anti$\angle BOC < 180^\circ$.

All three conditions are satisfied only when O is inside the triangle.

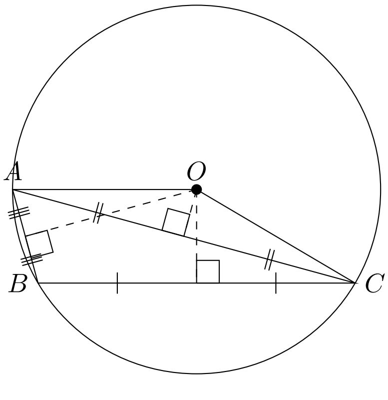
Otherwise, say O is outside $\triangle ABC$ at the right of AC . Then anti$\angle COA > 180^\circ$, but this violates the condition that anti$\angle COA < 180^\circ$. Thus it is impossible that O lies outside the triangle when $\triangle ABC$ is acute.

Case 2: $\angle ABC = 90^\circ$



By converse of $\angle$ in semi-circle, AC is a diamter of the circumcircle. Since O is the centre of the circumcircle , O must be lie on AC .

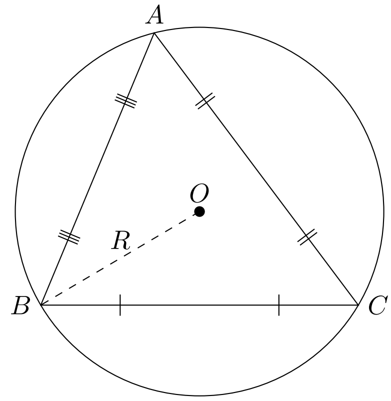
Case 3: $\angle ABC > 90^\circ$



By ($\angle$ at centre twice $\angle$ at $\odot^{ce}$), anti$\angle COA = 2\angle ABC > 180^\circ$, which means O must lie above AC , which means outside $\triangle ABC$.

$\square$

Preposition 10. Given a triangle with side lengths a, b, c , the **circumradius** (radius of circumcircle) (R) of the triangle is $\dfrac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$, where $s = \dfrac{a+b+c}{2}$ is the semi-perimeter of the triangle. (circumradius formula)



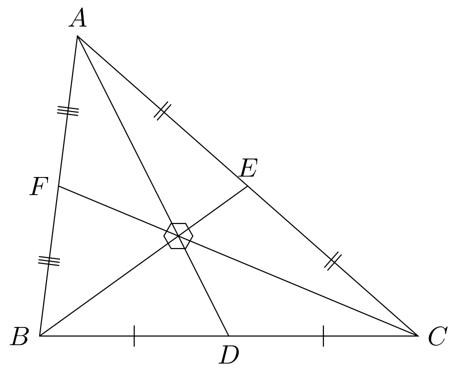
$$\boxed{R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}}$$

Proof. By 'circumradius of triangle' , we have the formula $R = \dfrac{abc}{4K}$, where K is the area of the triangle.

Since $K = \sqrt{s(s-a)(s-b)(s-c)}$ by Heron's formula, we have $R = \dfrac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$
. □

1.10.5 Centroid

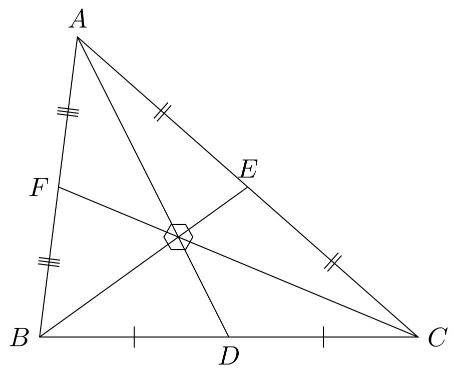
Preposition 11. The medians of a triangle are concurrent. (concurrency of medians)



$\because BD = DC$, $CE = EA$, $AF = FB$

$\therefore$ AD, BE, CF are concurrent. (concurrency of medians)

Proof. .



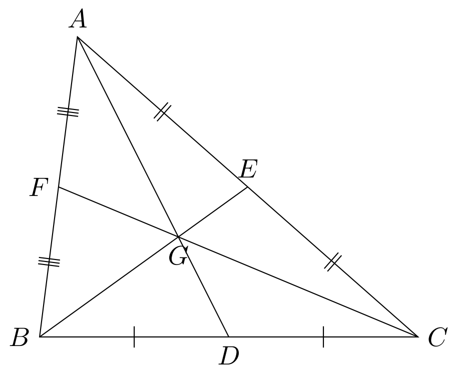
Note that

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{BD}{BD} \cdot \frac{CE}{CE} \cdot \frac{AF}{AF} = 1$$

Thus, by converse of Ceva's theorem, AD, BE, FD are concurrent. □

And the point of concurrency of the medians is called centroid.

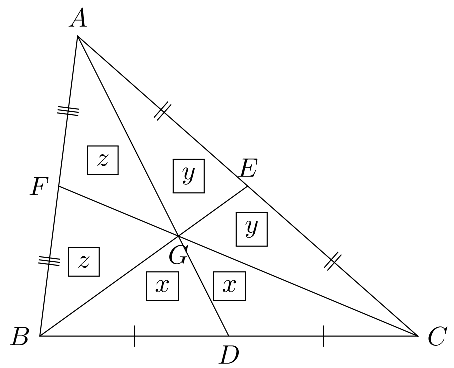
Preposition 12. The three medians of a triangle divide it into six triangles of equal area. (area property of medians)



$$\because G \text{ is centroid of } \triangle ABC$$
$$\therefore\ [\triangle GAF] = [\triangle GFB] = [\triangle GBD] = [\triangle GDC] = [\triangle GCE] = [\triangle GEA]$$
(area property of medians)

Proof. [4] Since $BD = DC$, we have $[\triangle GBD] = [\triangle GDC]$ (bases prop. to areas of △s).
Similarly, we have $[\triangle GAF] = [\triangle GFB]$ and $[\triangle GCE] = [\triangle GEA]$.

Let $[\triangle GBD] = [\triangle GDC] = x$, $[\triangle GCE] = [\triangle GEA] = y$, $[\triangle GAF] = [\triangle GFB] = z$.



Note that $[\triangle ABD] = [\triangle ADC] = \frac{1}{2}[\triangle ABC]$ (bases prop. to areas of △s) . So we have

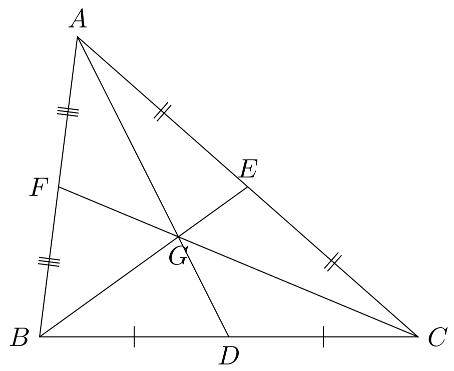
$$\begin{aligned} z + z + x &= y + y + x \\ 2z &= 2y \\ z &= y \end{aligned}$$

Similarly, since $[\triangle BCE] = [\triangle BEA]$, we have

$$\begin{aligned} x + x + y &= z + z + y \\ 2x &= 2z \\ x &= z \end{aligned}$$

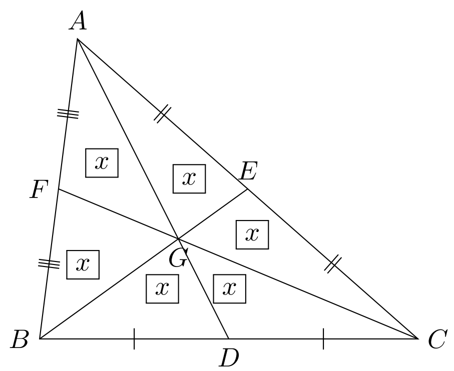
This means $x = y = z$, and $[\triangle GAF] = [\triangle GFB] = [\triangle GBD] = [\triangle GDC] = [\triangle GCE] = [\triangle GEA]$
. □

Preposition 13. For a median, the segment from the vertex to centroid is twice the length of the segment from the centroid to the mid-point on the opposite side. (prop. of centroid)



$$\because AF = FB \ , \ BD = DC \ , \ AE = EC$$
$$\therefore \ AG = 2DG \ , \ BG = 2EG \ , \ CG = 2FG \qquad \text{(prop. of centroid)}$$

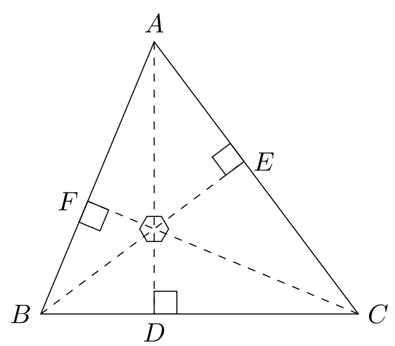
Proof. By 'area property of medians' , the medians divide $\triangle ABC$ into six triangles of equal area (denoted x).



By 'bases prop. to areas of $\triangle$s' , $\dfrac{DG}{AG} = \dfrac{[\triangle BDG]}{[\triangle BGA]} = \dfrac{x}{2x} = \dfrac{1}{2}$.
This means $AG = 2DG$.
By similarly reasoning, we have $GC = 2EG$ and $CG = 2FG$. □

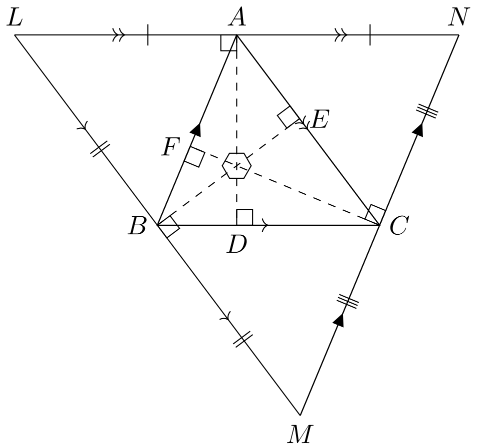
1.10.6 Orthocentre

Preposition 14. The altitudes of a triangle are concurent. (concurrency of altitudes)



$$\because AD \perp BC \ , \ BE \perp AC \ , \ CF \perp AB$$
$$\therefore \ AD, BE, CF \text{ are concurrent.} \qquad \text{(concurrency of altitudes)}$$

Proof. [5] Let there be $\triangle ABC$ with $AD \perp BC$, $BE \perp AC$, $CF \perp AB$.
Draw LM through B , LN through A , MN through C such that $LM//AC$, $LN//BC$, $MN//BA$.



Note that $\angle FCN = \angle BFC = 90^\circ$ (alt. $\angle$s , $BA//MN$). Similarly, $\angle LAD = \angle ADC$ (alt. $\angle$s , $LN//BC$) , and $\angle EBM = \angle AEB = 90^\circ$ (alt. $\angle$s , $AC//LM$) .

This means $AD \perp LN$, $EB \perp LM$, $FC \perp MN$.

Also, note that $LA = AN$, $LB = BM$, $MC = CN$ (prop. of being mid-pt. $\triangle$). This means AD , BE , CF are perpendicular bisector of LN , LM , MN respectively.

So AD, BE, CF must be concurrent at a point that is the circumcentre of $\triangle LMN$ (concurrency of $\perp$ bisectors of $\triangle$) .

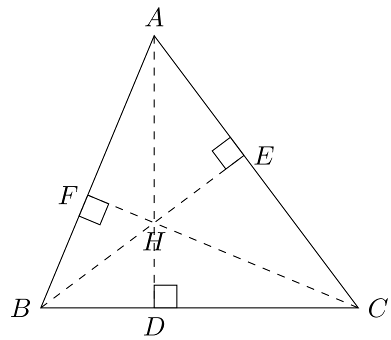
□

And this point of concurrency is the orthocentre of $\triangle ABC$.

An implication of this preposition is that any two altitudes must intersect at the orthocentre, since by definition, the orthocentre lies on all three altitudes, so if the orthocentre does not lie on the intersection of two altitudes, then it dose not lie on at least one altitude, making it not the orthocentre by definition.

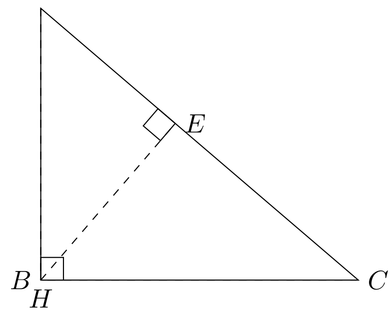
Preposition 15. The orthocentre lies inside an acute triangle, lies on the right angle vertex of a right triangle, and lies outside an obtuse triangle. (position of orthocentre)

Case 1:



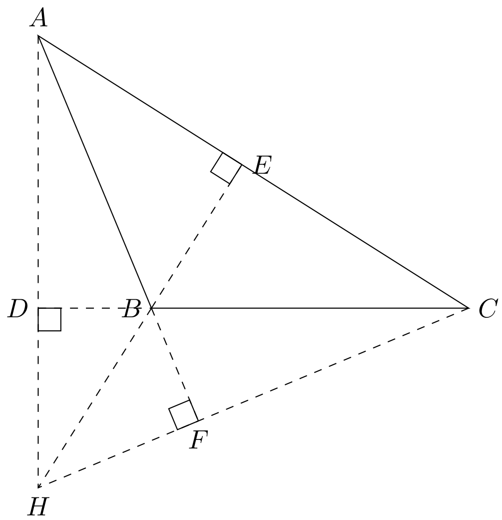
$$\because \angle A, \angle B, \angle C < 90^\circ$$
$$\therefore\ H \text{ lies inside } \triangle ABC\ . \qquad \text{(position of orthocentre)}$$

Case 2:



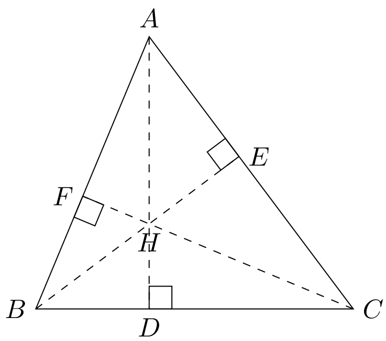
$$\because \angle B = 90^\circ$$
$$\therefore\ H \text{ lies on vertex } B\ . \qquad \text{(position of orthocentre)}$$

<u>Case 3:</u>



$$\because \angle ABC > 90^\circ$$
$$\therefore\ H \text{ lies outside } \triangle ABC\ . \qquad \text{(position of orthocentre)}$$

Proof. <u>Case 1:</u> $\angle A, \angle B, \angle C < 90^\circ$

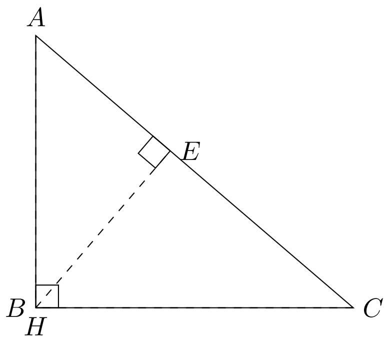


Since $\angle ABC < 90^\circ$ and $\angle ACB < 90^\circ$, A lies directly above BC (if BC is placed horizontally). So the foot of the altitude, D, must lie between B, C .

Similarly, the foots of the other two altitudes, E, F , must lie between A, C and A, B respectively.

Since the altitudes lie inside the triangle, the orthocentre must also lie inside the triangle (since orthocentre lies on altitudes) .

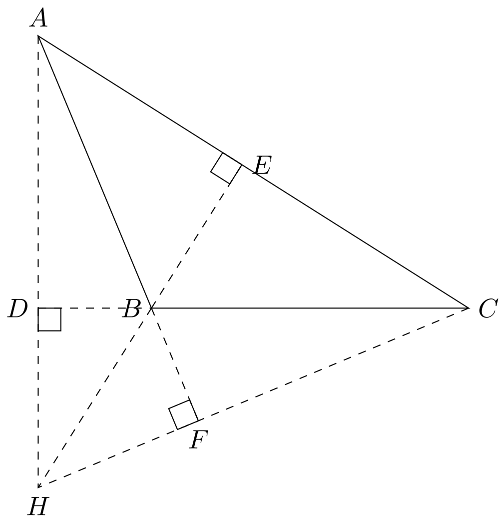
<u>Case 2:</u> $\angle B = 90^\circ$



Since $AB \perp BC$ and $AD \perp DC$ and D is a point on line BC , D must coincide with B since there is a unique point of projection from point A to line BC (prop. of $\perp$ line).

Similarly, F must coincide with B . This means B is the intersection of altitudes AD and CF , so B is the orthocentre of $\triangle ABC$.

<u>Case 3:</u> $\angle ABC > 90°$



Since A does not lie directly above side BC , but insteasd lie 'diagonally above', the altitude AD must lie outside $\triangle ABC$ (except point A).

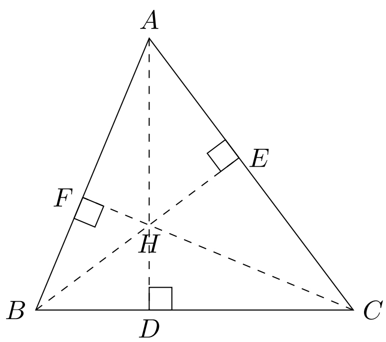
Similarly, altitude CF must lie outside $\triangle ABC$ except C.

Note that H cannot lie on a vertex since that would make the triangle a right triangle, which contradicts the assumption.

Since the orthocentre H must lie on line AD and line CF and must not lie on the vertex, H must lie outside $\triangle ABC$ too.

□

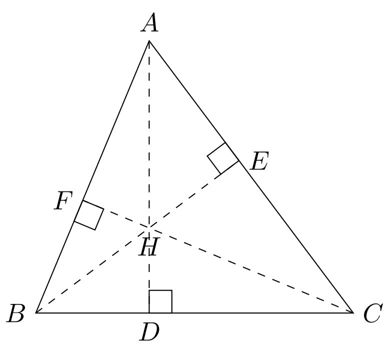
Preposition 16. The three altitudes of an acute triangle splits it into six triangles, where each pair of vertically opposite triangles are similar. (prop. of orthocentre)



<u>Given:</u> H is the orthocentre of $\triangle ABC$.

$\therefore$ $\triangle AFH \sim \triangle CDH$, $\triangle BFH \sim \triangle CEH$, $\triangle BDH \sim \triangle AEH$ (prop. of orthocentre)

Proof. .

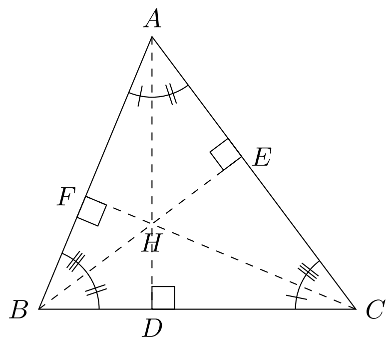


In $\triangle AFH$ and $\triangle CDH$,

$$\begin{aligned} \angle AFH &= \angle CDH && \text{(altitude)} \\ \angle AHF &= \angle CDH && \text{(altitude)} \\ \therefore \triangle AFH &\sim \triangle CDH && \text{(AA)} \end{aligned}$$

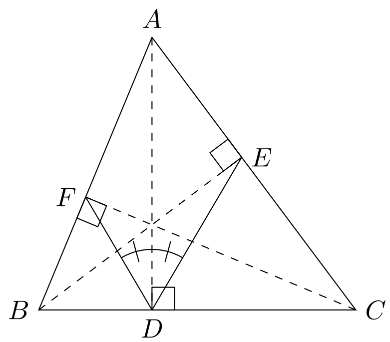
Similarly, we have $\triangle BFH \sim \triangle CEH$ and $\triangle BDH \sim \triangle AEH$. □

This preposition means for a pair of vertical opposite triangles, the corresponding angles touching the vertices of $\triangle ABC$ are equal:



$$\angle HAF = \angle HCD \text{ , } \angle HBF = \angle HCE \text{ , } \angle HBD = \angle HAE \qquad \text{(prop. of orthocentre)}$$

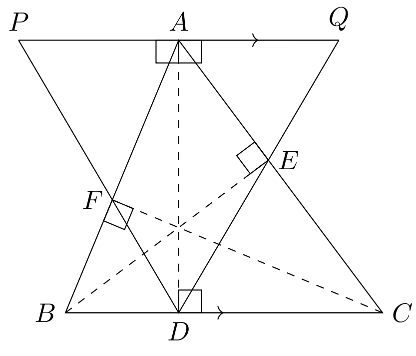
Preposition 17. In an acute triangle, if an angle is formed by connecting the three foots of the triangle's altitudes with two line segments, then the altitude that corresponds to the vertex of the angle is the angle bisector of that angle. (Blanchet's theorem)



$$\because AD \perp BC \text{ , } BE \perp AC \text{ , } CF \perp AB$$
$$\therefore \angle ADF = \angle ADE \qquad \text{(Blanchet's theorem)}$$

Proof. [6] Let P be a point on extended DF such that $PA \perp AD$. Let Q be a point on extended DE such that $QA \perp AD$.

Note that $PQ//BC$ (alt. $\angle$s equal).



Note that $\angle APF = \angle BDF$ (alt. $\angle$s , $PQ//BC$) and $\angle PFA = \angle DFB$ (vert. opp. $\angle$s).
Thus, we have $\triangle FAP \sim \triangle FBD$ (AA), and similarly, $\triangle EQA \sim \triangle EDC$ (AA).
Thus,

$$\frac{PA}{BD} = \frac{AF}{FB} \qquad \text{(corr. sides, } \sim \triangle\text{s)}$$
$$PA = \frac{AF \cdot BD}{FB} \tag{1}$$

And

$$\frac{AQ}{DC} = \frac{AE}{EC} \qquad \text{(corr. sides, } \sim \triangle\text{s)}$$
$$AQ = \frac{DC \cdot AE}{EC} \tag{2}$$

By Ceva's theorem in $\triangle ABC$,

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$
$$\frac{AF \cdot BD}{FB} = \frac{DC \cdot AE}{EC}$$

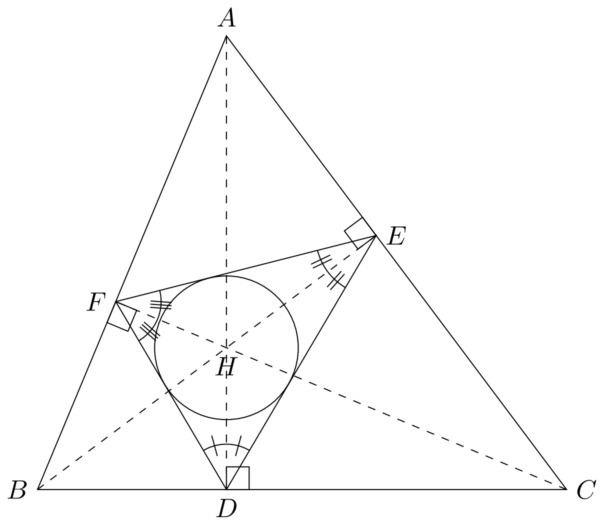
By (1) and (2):

$$PA = AQ$$

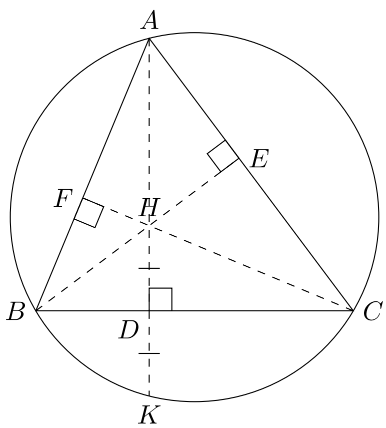
Thus, $\triangle DAP \cong \triangle DAQ$ (SAS), which means $\angle ADF = \angle ADE$ (corr. $\angle$s, $\cong \triangle$s).

□

This preposition implies that H (orthocentre of $\triangle ABC$) is the incentre of $\triangle DEF$:

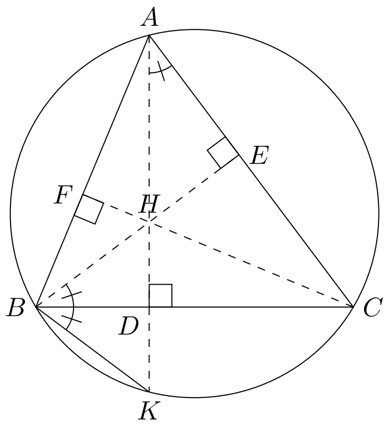


Preposition 18. In an acute triangle, the reflection of the orthocentre about a side lies on the circumcircle. (prop. of orthocentre reflection)



Given: H is the orthocentre of $\triangle ABC$.
$\therefore$ $HD = DK$ (prop. of orthocentre reflection)

Proof. Let K be the reflection of orthocentre H about side BC . Join BK .



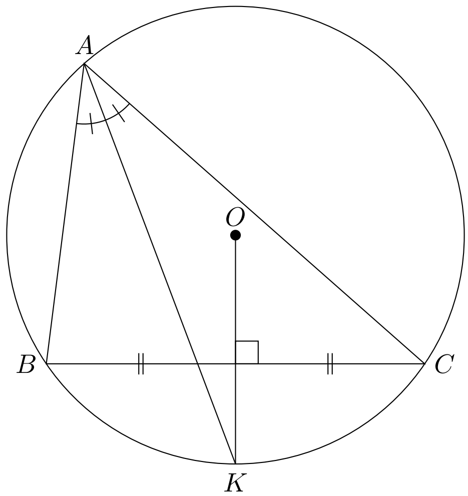
Note that $\angle KBC = \angle KAC$ ($\angle$s in the same segment). Also note that $\angle HBD = \angle KAC$ (prop. of orthocentre).

Thus, $\angle HBD = \angle KBD$. Thus $\triangle BDH \cong \triangle BDK$ (ASA).

So $HD = DK$ (corr. sides, $\cong \triangle$s). $\square$

1.10.7 General properties

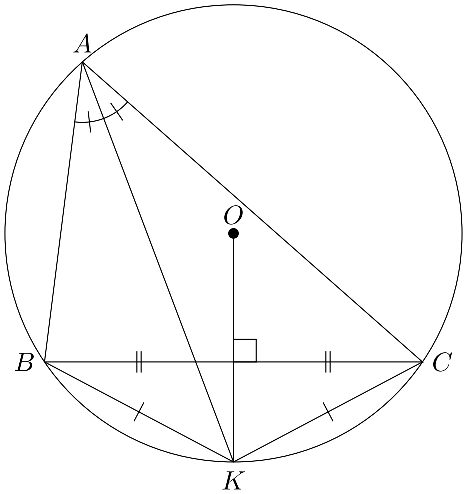
Preposition 19. In a triangle, the angle bisector of a vertex and the perpendicular bisector of the opposite side meet at a point on the circumcircle. ($\angle$ bisector meet $\perp$ bisector at circumcircle)



$\because \angle BAK = \angle CAK$, OK is the perpendicular bisector of BC .
$\therefore$ K lies on the circucmcircle of $\triangle ABC$.

Proof. Redefine K to be a point on the circumcircle such that $BC \perp OK$.

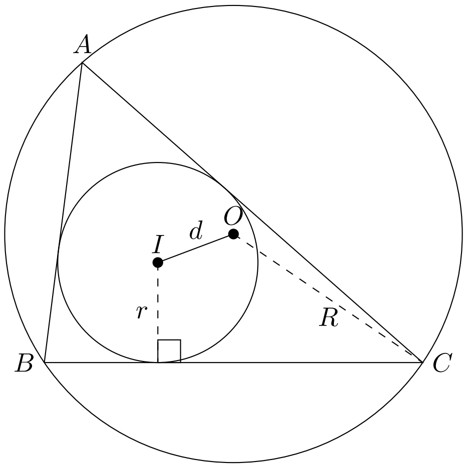
Join BK and KC .



Note that $BK = KC$ (prop. of $\perp$ bisector). Thus, we have that $\angle BAK = \angle CAK$ (equal chords, equal $\angle$s at $\odot^{ce}$) .

Thus, the angle bisector of $\triangle BAC$ and the perpendicular bisector of BC intersect at the point on the circumcircle. $\square$

Preposition 20. Given a triangle with inradius r and circumradius R, the distance (d) between the incentre and circumcentre is $\sqrt{R(R-2r)}$. (Euler's theorem / distance between incentre and circumcentre)



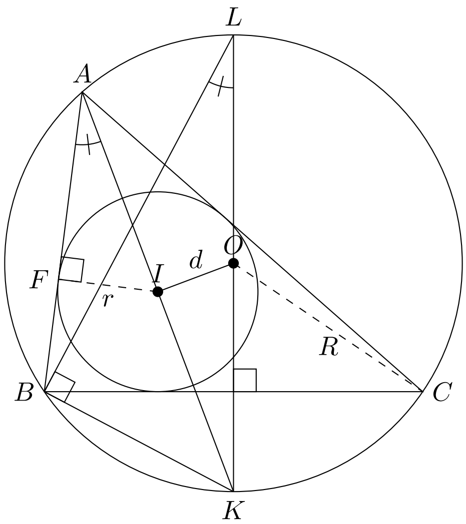
<u>Given:</u> I is incentre, O is circumcentre.

$$\therefore\ d = \sqrt{R(R-2r)}$$

Proof. [7] Let r be the inradius, R be the circumradius.

Extend AI to meet the circumcircle at K . Then $OK \perp BC$ ($\angle$ bisector meet $\perp$ bisector at circumcircle).

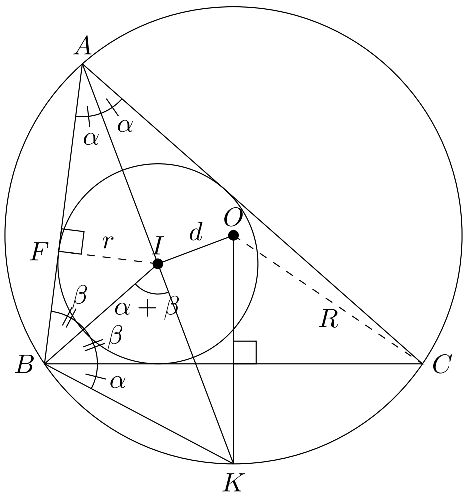
Extend KO to meet the circumcircle at L . Join $\triangle BLK$. Draw $IF \perp AB$.



Note that $\angle ABK = 90°$ ($\angle$ in semi-circle).
In $\triangle AFL$ and $\triangle LBK$,

$$
\begin{aligned}
\angle FAI &= \angle BLK \qquad (\angle\text{s in the same segment}) \\
\angle AFI &= \angle LBK = 90° \\
\therefore\ \triangle AFI &\sim \triangle LBK \qquad (\text{AA}) \\
\therefore\ \frac{FI}{BK} &= \frac{AI}{LK} \qquad (\text{corr. sides}, \sim \triangle\text{s}) \\
FI \cdot LK &= AI \cdot BK \\
r(2R) &= AI \cdot BK \qquad (FI = r\ ,\ LK = 2R) \\
AI \cdot BK &= 2Rr \qquad (1)
\end{aligned}
$$

Join BK . Let $\angle A = 2\alpha$ and $\angle B = 2\beta$. Note that $\angle BAK = \angle CAK = \alpha$ and $\angle ABI = \angle CBI = \beta$ (incentre).



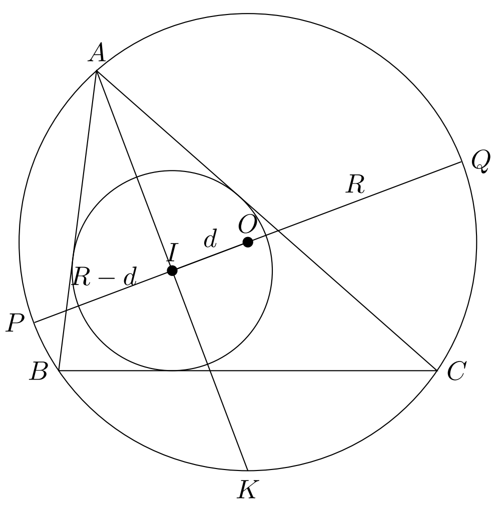
Note that $\angle CBK = \angle CAK = \alpha$ ($\angle$s in the same segment).
In $\triangle ABI$, note that $\angle BIK = \alpha + \beta$ (ext. $\angle$ of $\triangle$).
Since $\angle IBK = \angle BIK = \alpha + \beta$, we have $BK = IK$. (sides opp. equal $\angle$s).

By $AI \cdot BK = 2Rr$, we have

$$AI \cdot IK = 2Rr \qquad (2)$$

Extend IO to the circumcircle on both sides (label the points P and Q). Then $PI = R - d$ and $IQ = R + d$.

(If I and O coincide, then just take $d = 0$ and arbitrarily draw a diameter.)



By intersecting chords theorem, we have

$$\begin{aligned} AI \cdot IK &= PI \cdot IQ \\ 2Rr &= (R-d)(R+d) \\ 2Rr &= R^2 - d^2 \\ d^2 &= R^2 - 2Rr \\ d &= \sqrt{R(R-2r)} \end{aligned}$$

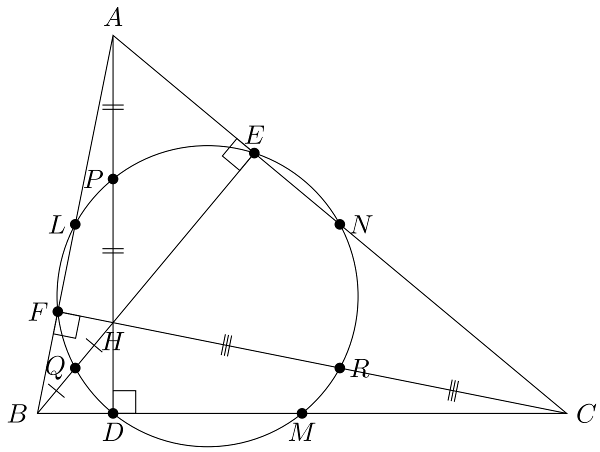
□

Preposition 21. In a triangle, the following nine points are concyclic:

- the three mid-points of the sides of the triangle.
- the three foots of the altitudes of the triangle.
- the three mid-points from the vertices to the orthocentre of the triangle.

(nine-point circle)

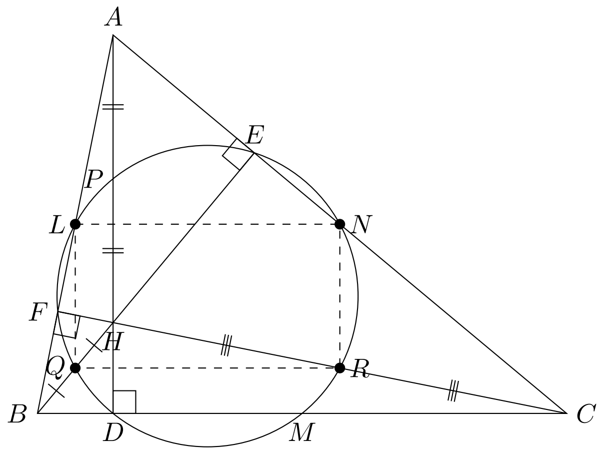
The circle passing through these nine points is called the **nine-point circle**.



Given: $AD \perp BC$, $BE \perp AC$, $CF \perp AB$, $AN = NC$, $AL = LB$, $BM = MC$, $AP = PH$, $BQ = QH$, $CR = RH$

$\therefore$ $D, E, F, M, N, L, P, Q, R$ are concyclic. (nine-point circle)

Proof. Case 1: $\angle A, \angle B, \angle C < 90°$ [8]
Join $LNRQ$.



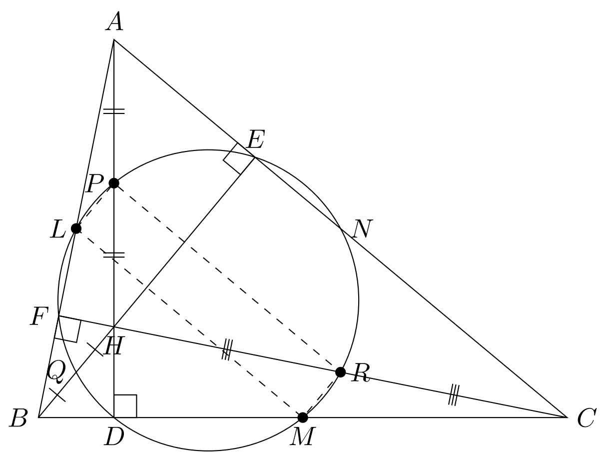
In $\triangle ABC$, since $AL = LB$ and $AN = NC$, we have $LN//BC$ (mid-pt. theorem).
In $\triangle HBC$, since $HQ = QB$ and $HR = RC$, we have $QR//BC$ (mid-pt. theorem).
This means $LN//QR$.

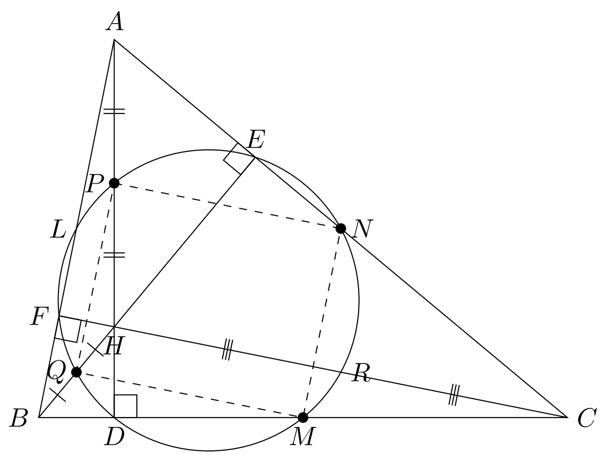
Similarly,
In $\triangle ABH$, since $AL = LB$ and $HQ = QB$, we have $LQ//AH$ (mid-pt. theorem).
In $\triangle AHC$, since $AN = NC$ and $HR = RC$, we have $NR//AH$ (mid-pt. theorem).
This means $LQ//NR$.

Thus, $LNRQ$ is a parallelogram.

Note that $AD \perp QR$ (corr. $\angle$s , $QR//BC$). So $LQ \perp QR$ (corr. $\angle$s , $LQ//AD$) , which means $\angle LQR = 90°$.
Thus, $LNRQ$ is a rectangle (//gram with right $\angle$) , and it is a cyclic quadrilateral (opp. $\angle$s supp.). So we can draw the circumcircle of $LNRQ$.

By similar reasoning, $LPRM$ and $PNMQ$ are rectangles as well, and they are also cyclic quadrilaterals (opp. $\angle$s supp.). Draw the circumcircle of $LPRM$ and $PNMQ$.

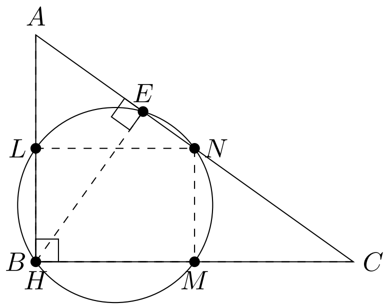




Note that both circles in the figures above have PM as the diameter (converse of $\angle$ in semi-circle). So they are in fact the same circle, which means the circle passes thorugh P, R, M, L, N, Q .

Also note that both the circumcircle of $LNRQ$ and $PNMQ$ have NQ as the diameter (converse of $\angle$ in semi-circle). So they are in fact the same circle.
So the circles in all three figures are actually the same circle, which means the circle passes thorugh $D, E, F, M, N, L, P, Q, R$.

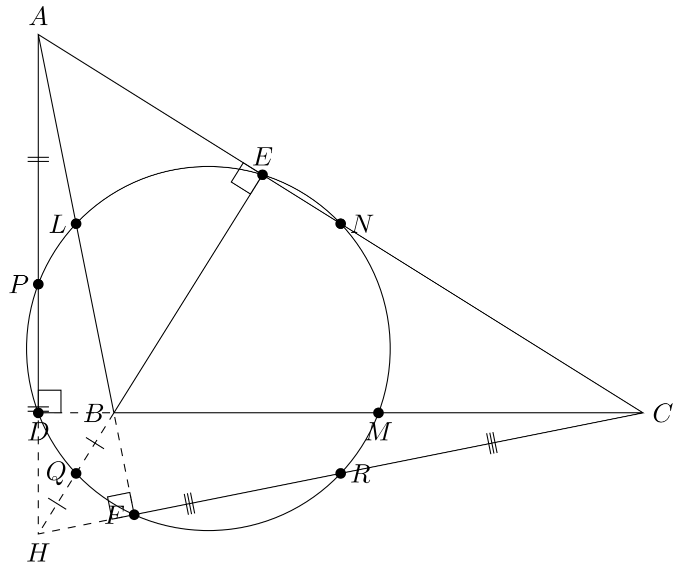
<u>Case 2:</u> $\angle ABC = 90°$



Then D, E, Q coincide with B, and P, L coincide, and R, M coincide.

Note that $LNMB$ is a cyclic quad. since it is a rectangle. Note that E also lies on the circumcircle of $LNMB$ since $\angle HEN = \angle BLN = 90°$ (converse of $\angle$s in the same segment). So the nine-point circle is the circumcircle of $LNMB$.

Case 3: $\angle ABC > 90°$



Note that P, N, R are the mid-points of AH, AC, HC respectively. And M, Q, L are the mid-points of BC, BH, BA respectively.

Note that $\angle AHC$ is an acute triangle since $\angle HAC, \angle ACH, \angle AHC$ are angles belonging to some right triangles in the figure, which means they are acute. Thus there exists a nine-point circle passing through $D, E, F, M, N, L, P, Q, R$.

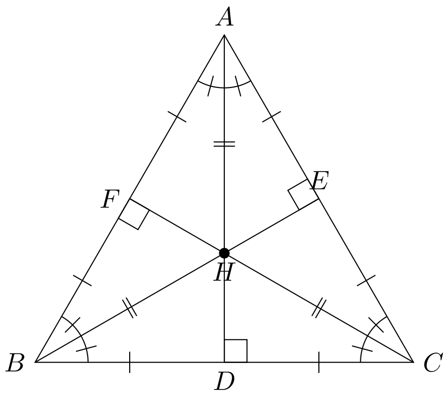
$\square$

Note: The radius of the nine-point circle is exactly half the radius of the circumcircle of $\triangle ABC$, since the nine-point circle is the circumcircle of the mid-point triangle of $\triangle ABC$, and the mid-point triangle has half the size of $\triangle ABC$.

Preposition 22. In an equilateral triangle, the incentre, circumcentre, centroid and orthocentre all lie at the same point. (centres of equil. $\triangle$)

In an isosceles triangle, the incentre, circumcentre, centroid and orthocentre all lie on the perpendicular median of the triangle. (centres of isos. $\triangle$)

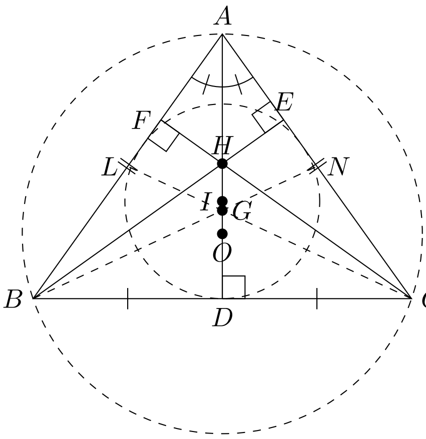
Case 1:



$\because AB = AC = BC$,
I is incentre, O is circumcentre, G is centroid, H is orthocentre.

$\therefore\ I, O, G, H$ lies on the same point. (centres of equil. $\triangle$)

Case 2:

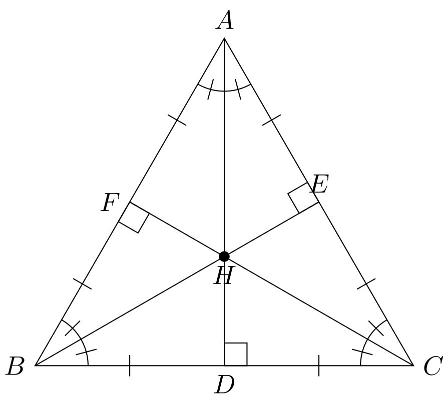


$\because AB = AC$
I is incentre, O is circumcentre, G is centroid, H is orthocentre.

$\therefore\ I, O, G, H$ lie on median AD . (centres of isos. $\triangle$)

Proof. Case 1: $AB = BC = AC$

Let $AD \perp BC$, $BE \perp AC$, $CF \perp AB$. Let H be the orthocentre of $\triangle ABC$. In other words, AD, BE, CF intersect at H .



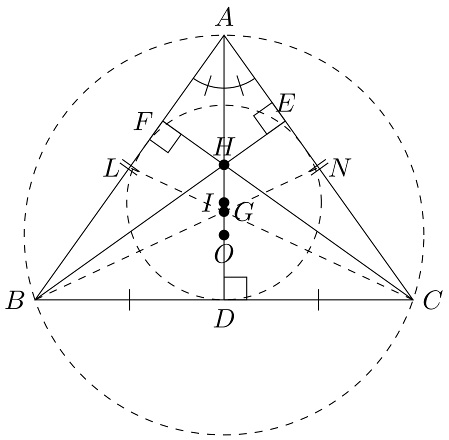
By prop. of isos. $\triangle$, we have $BD = DC$, $CE = EA$, $AF = FB$. Thus, H is also the intersection of the medians and the perpendicular bisectors of the sides.

By prop. of isos. $\triangle$, we have $\angle BAD = \angle CAD$, $\angle ABE = \angle CBE$, $\angle BCF = \angle ACF$. Thus, H is also the intersection of the angle bisectors of the triangle.

So incentre, circumcentre, centroid, orthocentre all lie on H .

<u>Case 2:</u> $AB = AC$

Let $AD \perp BC$. Then by 'prop. of isos. $\triangle$', $\angle BAD = \angle CAD$ and $BD = DC$.



Then AD is an angle bisector of $\angle BAC$, and a perpendicular bisector of BC , and a median and altitude corresponding to BC .

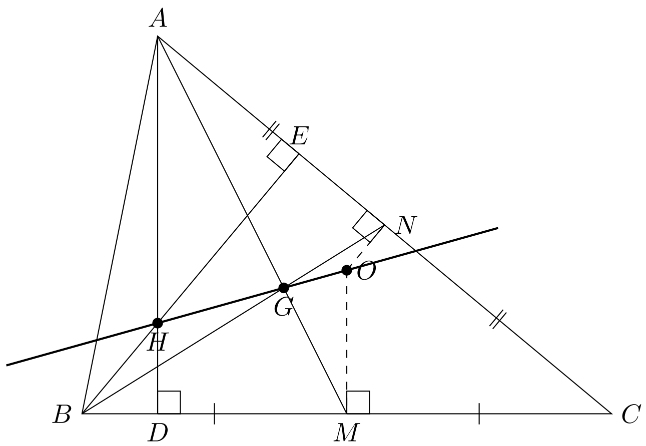
Note that by definition, incentre must lie on an angle bisector, circumcentre on perpendicular bisector, centroid on median, and orthocentre on altitude.

Thus all the centres must lie on AD .

□

Preposition 23. In a non-equilateral triangle, the circumcentre, centroid and orthocentre are collinear. (Euler line)

(Note: The line passing through these centres are called **Euler line** , and it also passes through the centre of the nine-point circle, which will be proved in the next preposition.)

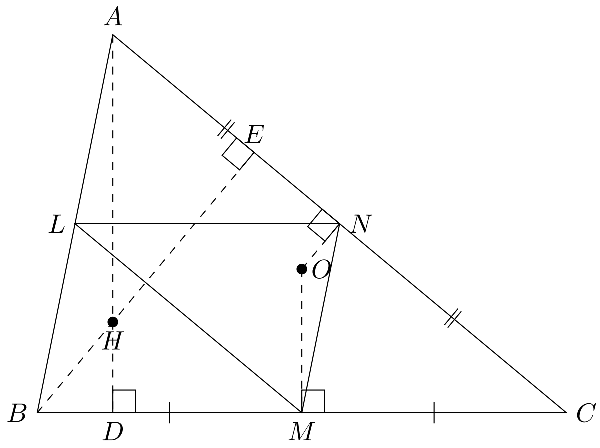


<u>Given:</u> O is circumcentre, G is centroid, H is orthocentre.
$\therefore$ O, G, H are collinear. (Euler line)

Proof. [9] Note that the triangle is non-equilateral, so there is at least one pair of non-equal sides.

Assume that $AB \neq AC$. Note that the the centroid must lie between line AD and line OM , since the centroid must lie between A, M.

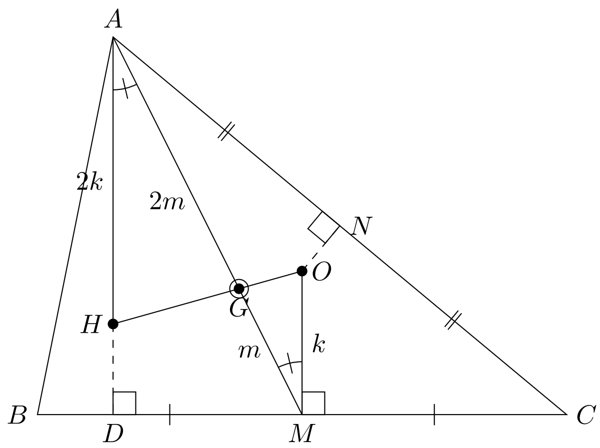
Let M, N, L be the mid-point of BC, AC, AB respectively. Join $\triangle MNL$.



Note that $\triangle MNL \sim \triangle ABC$ (prop. of mid-point $\triangle \Rightarrow$ AA) with a scale ratio $1:2$ (mid-point theorem).

Also, note that O is the orthocentre of $\triangle MNL$ (since $MO \perp LN$ and $NO \perp LM$) .

Note that the altitude segment MO in $\triangle MNL$ corresponds to the altitude segment AH in $\triangle ABC$. Thus $MO : AH = 1 : 2$. (Or we can show that $\triangle MON \sim \triangle AHB$ with $MN = \frac{1}{2}AB$, and thus $MO = \frac{1}{2}AH$ by (corr. sides, $\sim \triangle$s).)



Since AM is a median and G is the centroid, we have $GM : AG = 1 : 2$ by 'prop. of centroid' .

Moreover, note that $AD//OM$ (corr. $\angle$s equal) , so $\angle HAG = \angle OMG$ (alt. $\angle$s , $AD//OM$).

Join HG and GO . We want to show that HGO is a straight line.

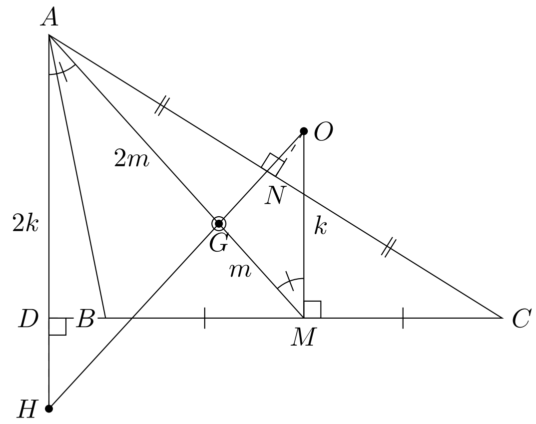
To summarize, in $\triangle OMG$ and $\triangle HAG$,

$$\begin{aligned} \frac{OM}{AH} &= \frac{1}{2} && \text{(shown above)} \\ \angle OMG &= \angle HAG && \text{(alt. } \angle\text{s , } AD//OM) \\ \frac{GM}{AG} &= \frac{1}{2} && \text{(prop. of centroid)} \\ \therefore\ \triangle OMG &\sim \triangle HAG && \text{(ratio of 2 sides, inc. } \angle) \\ \therefore\ \angle OGM &= \angle AGH && \text{(corr. } \angle\text{s, } \sim \triangle\text{s)} \end{aligned}$$

$$\text{Thus,} \quad \begin{aligned} \angle AGH + \angle AGO &= \angle AGH + (180^\circ - \angle OGM) && \text{(adj. } \angle\text{s on st. line)} \\ &= \angle AGH + 180^\circ - \angle AGH \\ &= 180^\circ \end{aligned}$$

Thus, HGO is a straight line (adj. $\angle$s supp.) .

Note: This proof works even when $\triangle ABC$ is an obtuse triangle:

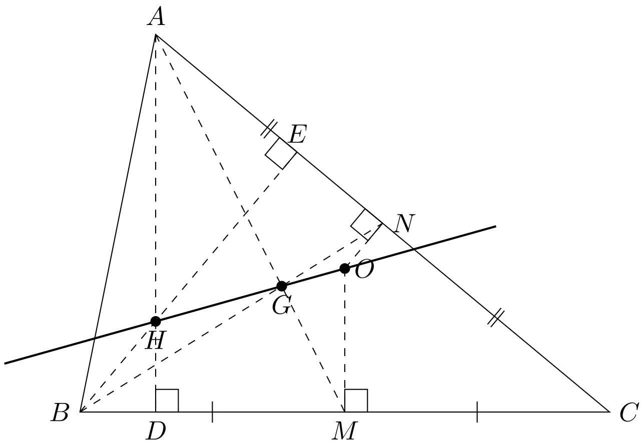


∎

Preposition 24. In a non-equilateral triangle, the centroid lies one-third of the way from the circumcentre to the orthocentre.

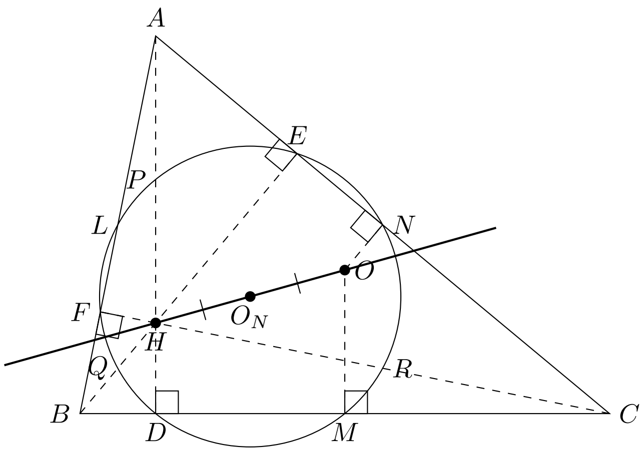
Also, the centre of the nine-point circle is the mid-point of orthocentre and circumcentre.

(prop. of Euler line)



Given: O is circumcentre, G is centroid, H is orthocentre.

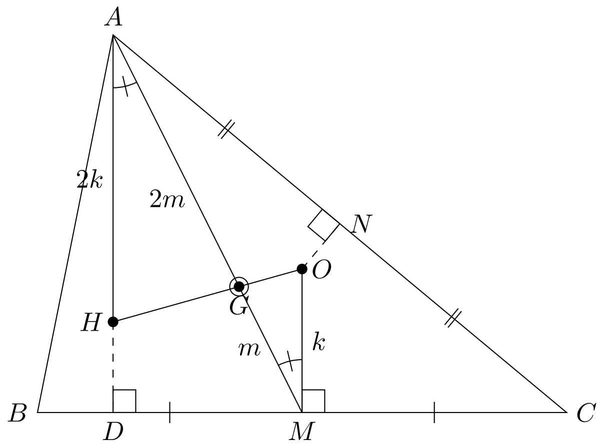
$\therefore\ HG = 2 \cdot OG$ (prop. of Euler line)



Given: O is circumcentre, O_N is centre of nine-point circle, H is orthocentre.

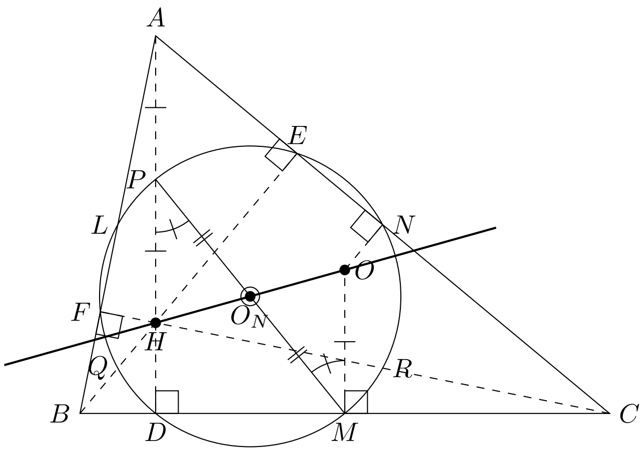
$\therefore\ O_N O = O_N H$ (prop. of Euler line)

Proof. Case 1:



In the proof of the Euler line, we have proven that $\triangle OMG \sim \triangle HAG$ with a scale ratio of $1:2$. So we have $OG : HG = 1 : 2$ (corr. sides, $\sim \triangle$s), and $\dfrac{OG}{OH} = \dfrac{1}{3}$ and $HG = 2OG$.

Case 2a: $\angle ABC \leq 90°$



Note that P is the mid-point of AH . Since $AH = 2OM$ (shown above), we have $PH = OM$.

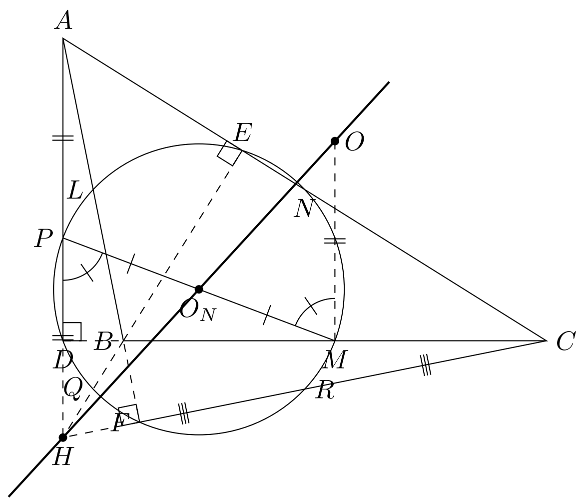
Note that $\angle PDM = 90°$, so by 'converse of $\angle$ in semi-circle', PM is the diameter of the nine-point circle and $O_NP = O_NM$ (since O_N is the centre) .

Also, note that $\angle O_NPH = \angle O_NMO$ (alt. $\angle$s , $AD//OM$).

Thus, $\triangle O_NPH \cong \triangle O_NMO$ (SAS).

Thus, $O_NH = O_NO$ (corr. sides, $\cong \triangle$s) and HO_NO is a straight line (since $\angle PO_NH = \angle OO_NM$).

Case 2b: $\angle ABC > 90°$



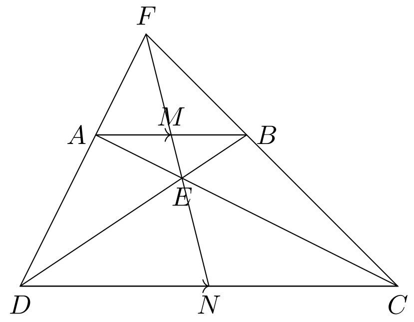
The proof is same as above. The figure just looks different.

□

1.11 Line and circle properties

1.11.1 Line properties

Preposition 25. In a proper trapezium, if a line passes through the intersection point of the diagonals and the intersection point of the extended non-parallel sides, then the line bisects the bases of the trapezium. (trapezium bisection theorem)



Given: $AB//DC$, and line FE intersects AB and DC at M and N respectively.
$\therefore$ $AM = MB$ and $DN = NC$ (trapezium bisection theorem)

Proof. By general intercept theorem, we have $\dfrac{FA}{AD} = \dfrac{FB}{BC}$, which means

$$\frac{FA}{AD} \cdot \frac{CB}{BF} = 1 \tag{1}$$

By Ceva's theorem, we have

$$\frac{FA}{AD} \cdot \frac{DN}{NC} \cdot \frac{CB}{BF} = 1 \tag{2}$$

Put (1) into (2):

$$1 \cdot \frac{DN}{NC} = 1$$
$$DN = NC$$

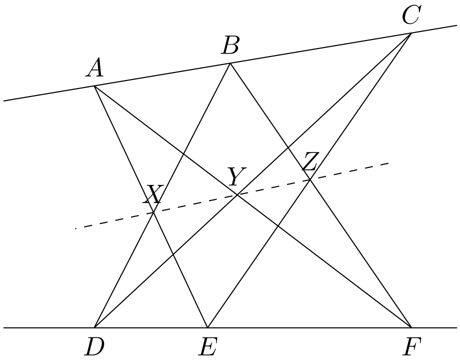
Note that $\triangle FAM \sim FDN$ and $\triangle FMB \sim FNC$ (AA).

Thus $\dfrac{AM}{DN} = \dfrac{FM}{FN}$ and $\dfrac{MB}{NC} = \dfrac{FM}{FN}$ (corr. sides, $\sim \triangle$s). So

$$\frac{AM}{DN} = \frac{MB}{NC}$$
$$\frac{AM}{DN} = \frac{MB}{DN} \qquad (DN = NC)$$
$$AM = MB$$

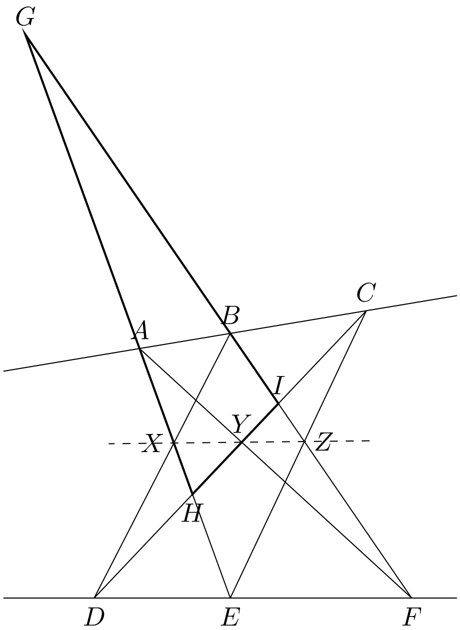
□

Preposition 26. Let A, B, C be three points on a line, and let D, E, F be three points on another line. If the lines AE intersect BD at X , AF intersect CD at Y , and BF intersect CE at Z , then the three points X, Y, Z are collinear. (Pappus' hexagon theorem)



Proof. [10] Assume the configuration of points is as depicted in the figure, and AE is not parallelt to BF.

Let G be the intersection of line EA and line FB , H be intersection of AE and CD , I be intersection of BF and CD .



Apply Menelaus' theorem on $\triangle GHI$ and its five transversal lines: DXB, AYF, CZE, ABC and DEF:

Transversal	Menelau's theorem
DXB	$\frac{HX}{GX} \cdot \frac{ID}{HD} \cdot \frac{GB}{IB} = 1$
AYF	$\frac{HA}{GA} \cdot \frac{IY}{HY} \cdot \frac{GF}{IF} = 1$
CZE	$\frac{HE}{GE} \cdot \frac{IC}{HC} \cdot \frac{GZ}{IZ} = 1$
ABC	$\frac{GA}{HA} \cdot \frac{HC}{IC} \cdot \frac{IB}{GB} = 1$
DEF	$\frac{GE}{HE} \cdot \frac{HD}{ID} \cdot \frac{IF}{GF} = 1$

Multiply the five equations together:

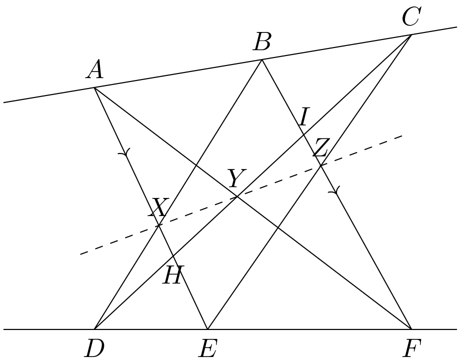
$$(\frac{HX}{GX} \cdot \frac{ID}{HD} \cdot \frac{GB}{IB}) \cdot (\frac{HA}{GA} \cdot \frac{IY}{HY} \cdot \frac{GF}{IF}) \cdot (\frac{HE}{GE} \cdot \frac{IC}{HC} \cdot \frac{GZ}{IZ}) \cdot (\frac{GA}{HA} \cdot \frac{HC}{IC} \cdot \frac{IB}{GB}) \cdot (\frac{GE}{HE} \cdot \frac{HD}{ID} \cdot \frac{IF}{GF}) = 1$$

Cancelling out like terms (fun activity to do it yourself), we get:

$$\frac{HX}{GX} \cdot \frac{IY}{HY} \cdot \frac{GZ}{IZ} = 1$$

By converse of Menelaus' theorem, X, Y, Z lies on a straight line.

Now, suppose that $AE//BF$.



Then we have these five pairs of similar triangles (by AA), and by (corr. sides, $\sim \triangle$s) rearranged:

Similar triangles	Proportions
$\triangle HDX \sim \triangle IDB$	$\frac{HX}{HD} \cdot \frac{ID}{IB} = 1$
$\triangle HAY \sim \triangle IFY$	$\frac{HA}{HY} \cdot \frac{IY}{IF} = 1$
$\triangle HEC \sim \triangle IZC$	$\frac{HE}{HC} \cdot \frac{IC}{IZ} = 1$
$\triangle HCA \sim \triangle ICB$	$\frac{HC}{HA} \cdot \frac{IB}{IC} = 1$
$\triangle HDE \sim \triangle IDF$	$\frac{HD}{HE} \cdot \frac{IF}{ID} = 1$

Multiply the five equations together:

$$(\frac{HX}{HD} \cdot \frac{ID}{IB}) \cdot (\frac{HA}{HY} \cdot \frac{IY}{IF}) \cdot (\frac{HE}{HC} \cdot \frac{IC}{IZ}) \cdot (\frac{HC}{HA} \cdot \frac{IB}{IC}) \cdot (\frac{HD}{HE} \cdot \frac{IF}{ID}) = 1$$

Simplifying, we get:

$$\frac{HX}{HY} \cdot \frac{IY}{IZ} = 1$$

$$\frac{HX}{HY} = \frac{IZ}{IY}$$

which means $\triangle HXY \sim IZY$ (ratio of two sides, inc. $\angle$). Thus, $\angle HYX = \angle IYZ$ (corr. $\angle$s, $\sim \triangle$s) so XYZ is a straight line (adj. $\angle$s supp.) . $\square$

1.11.2 Radical axis

References

[1] Shaalaa.com , “Show that the angle bisectors of a triangle are concurrent - mathematics.” [Online]. Available: https://www.shaalaa.com/question-bank-solutions/show-that-the-angle-bisectors-of-a-triangle-are-concurrent-concurrency-theorems_180102

[2] MATHalino , “Derivation of formula for radius of incircle.” [Online]. Available: https://mathalino.com/reviewer/derivation-of-formulas/derivation-of-formula-for-radius-of-incircle

[3] David Quinn , “Answer: How do you derive that the inradius in a right triangle is r=(a+b−c)/2 ?” Math StackExchange. [Online]. Available: https://math.stackexchange.com/questions/4280968/how-do-you-derive-that-the-inradius-in-a-right-triangle-is-r-fracab-c2

[4] Haresh Sagar , “Answer: What is the proof for the property of the centroid of a triangle that divides the median into 2:1?” [Online]. Available: https://www.quora.com/What-is-the-proof-for-the-property-of-the-centroid-of-a-triangle-that-divides-the-median-into-2-1

[5] Art of Problem Solving, “Orthocenter.” [Online]. Available: https://artofproblemsolving.com/wiki/index.php/Orthocenter

[6] N.gary, “Blanchet’s theorem (comments).” [Online]. Available: https://artofproblemsolving.com/community/c227495h1314069_blanchets_theorem

[7] Vedantu, “Prove that the distance between the circumcenter and the incenter of the triangle abc is $\sqrt{R^2 - 2Rr}$.” [Online]. Available: https://www.vedantu.com/question-answer/prove-that-the-distance-between-the-circumcenter-class-9-maths-cbse-5f6599c53da50302bd5d6eb9

[8] Brilliant, “Nine-point circle.” [Online]. Available: https://brilliant.org/wiki/nine-point-circle/

[9] NADI ANUSHA LIYANAPATABENDI, “Mathematical proofs euler line.” [Online]. Available: https://m500.org.uk/wp-content/uploads/2016/04/winter-sample-euler-line.pdf

[10] Cut The Knot, “Pappus’ theorem.” [Online]. Available: https://www.cut-the-knot.org/pythagoras/Pappus.shtml