Toddler Geometry (Problem set)

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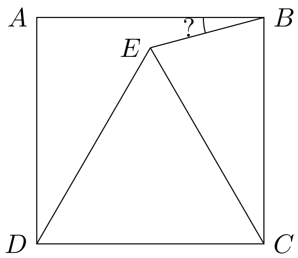
Abstract

Geometry is an ancient branch of Mathematics, dating as far back as 4000 years ago. Humanity has been fascinated and puzzled by these 'simple' lines and shapes for millennia, so it is only natural for a maths person like me to want to study Geometry and uncover its mysteries. But unlike other branches of mathematics such as Calculus and Linear Algebra, why are all the geometry theorems so useless and unapplicable in real life? I have no idea. After studying some circle theorems in high school, we don't even touch them again in University, which is doing Geometry a disservice in my opinion. So here I am, fully embracing the uselessness of Geometry and just studying for the fun of it, because it is the purest form of art.

Contents

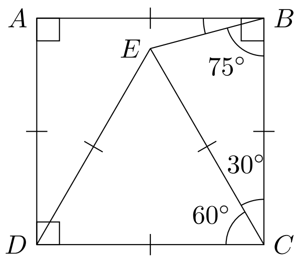
0 Triangle properties

Problem 1. $ABCD$ is a square. E is a point inside $ABCD$ such that $\triangle ECD$ is an equilateral triangle. Join BE. What is $\angle ABE$?



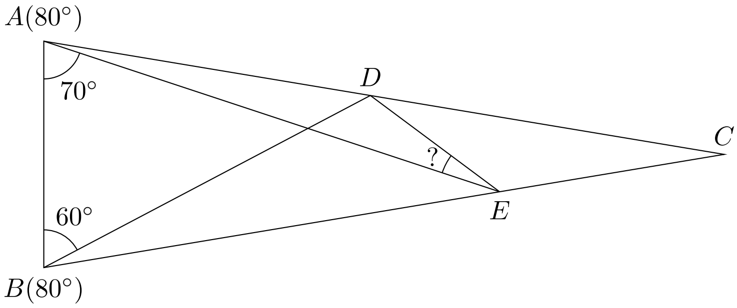
(Difficulty: 3)

Solution 1. .



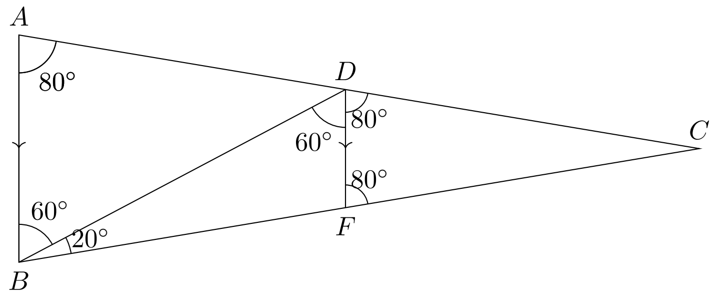
$$\begin{aligned}
&\angle DCB = \angle CBA = 90^\circ \qquad (ABCD \text{ is square.})\\
&\angle ECD = 60^\circ \qquad (\text{prop. of equil } \triangle)\\
&\angle ECB = 90^\circ - 60^\circ = 30^\circ\\
&\text{Note that } EC = BC.\\
\therefore\ &\angle CBE = \angle CEB \qquad (\text{base } \angle\text{s, isos. } \triangle)\\
&\angle CBE = (180^\circ - 30^\circ)/2 = 75^\circ \qquad (\angle \text{ sum of } \triangle)\\
&\angle ABE = 90^\circ - 75^\circ = \boxed{15^\circ}
\end{aligned}$$

Problem 2. [1] In $\triangle ABC$, $\angle BAC = \angle ABC = 80^\circ$. Let D be a point on side AC such that $\angle ABD = 60^\circ$. Let E be a point on side BC such that $\angle BAE = 70^\circ$. Join DE. What is $\angle AED$?

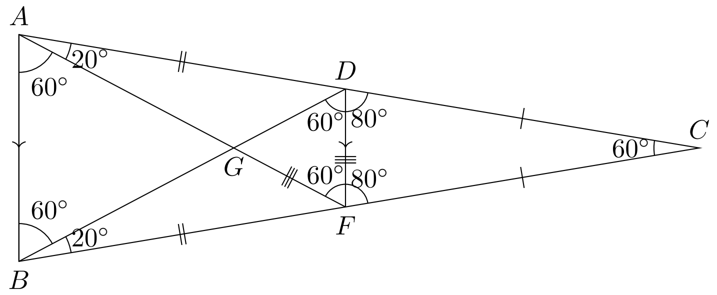


(Difficulty: 6 [Very hard])

Solution 2. Let F be a point on side BC such that $AB//DF$. Hide point E to make the figure tidier. Note that $\angle DBC = 80^\circ - 60^\circ = 20^\circ$.



$$\angle CDF = \angle CAB = 80^\circ \qquad (\text{corr. } \angle\text{s , } DF//AB)$$
$$\angle CFD = \angle CBA = 80^\circ \qquad (\text{corr. } \angle\text{s , } DF//AB)$$
$$\angle BDF = 80^\circ - 20^\circ = 60^\circ \qquad (\text{ext. } \angle \text{ of } \triangle)$$

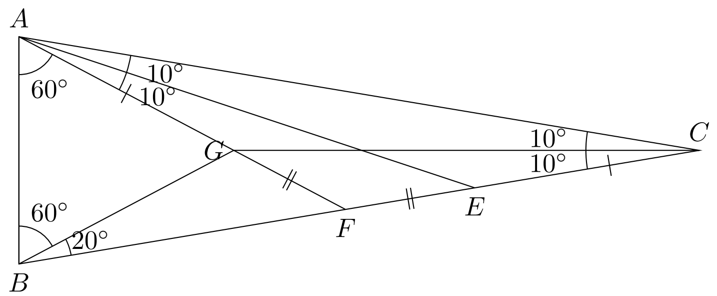


Note that $CD = CF$ and $CA = CB$ (sides opp. equal $\angle$s). Thus $AD = BF$.

Join AF, and let AF and BD intersect at G . In $\triangle ADF$ and $\triangle BFD$, $AD = BF$, $\angle ADF = \angle BFD = 110^\circ$ (adj. $\angle$s on st. line), $DF = DF$. Thus $\triangle ADF \cong \triangle BFD$ (SAS). Thus $\angle DAF = \angle FBD = 20^\circ$ (corr. $\angle$s, $\cong \triangle$s). Also, $\angle AFD = \angle BDF = 60^\circ$ (corr. $\angle$s, $\cong \triangle$s). Thus $\triangle GDF$ is an equilateral triangle (con. of equil. $\triangle$), which means $GF = DF$.

Note that $\angle ACF = 180^\circ - 80^\circ - 80^\circ = 20^\circ$ ($\angle$ sum of $\triangle$). Since $\angle CAF = \angle ACF = 20^\circ$, we have $AF = FC$ (base $\angle$s, isos. $\triangle$).

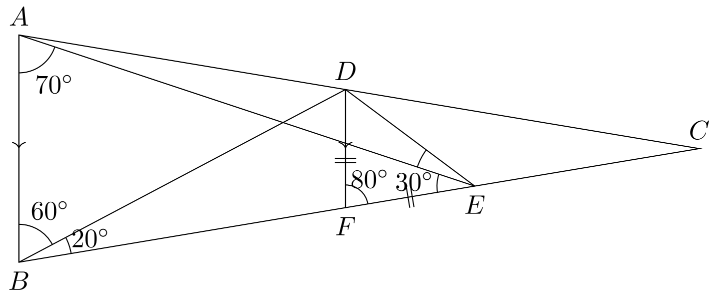
Show point E again and hide GD and DF. Join CG .



Note that $\angle CAE = \angle EAF = 10^\circ$. Also note that GC bisects ACB (because G is in the middle), so $\angle ACG = \angle GCF = 10^\circ$.

Note that $\triangle GAC \cong \triangle ECA$ (ASA), so $AG = EC$ (corr. sides, $\cong \triangle$s). Since $AF = FC$, we have $GF = FE$.

Show D again and hide AF.

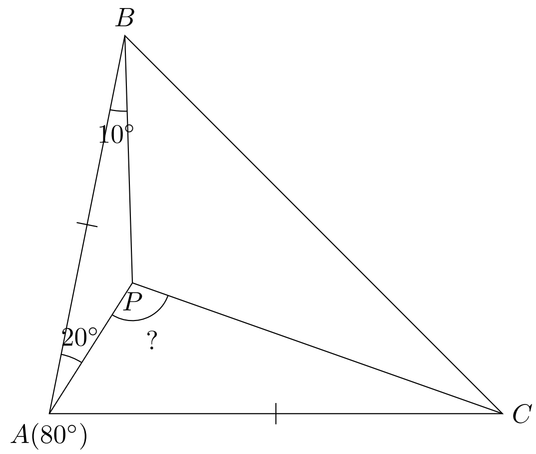


We have shown that $GF = DF$ and $GF = FE$. Thus $DF = FE$. In $\triangle FDE$, $\triangle FDE = \triangle FED$ (base $\angle$s, isos. $\triangle$). So $\angle FED = (180° - 80°)/2 = 50°$ ($\angle$ sum of $\triangle$).

Note that $\angle AEB = 180° - 80° - 70° = 30°$ ($\angle$ sum of $\triangle$).

So $\angle AED = \angle FED - \angle AEB = 50° - 30° = \boxed{20°}$.

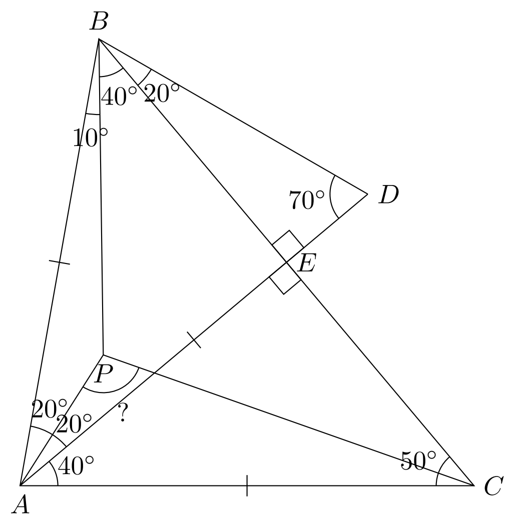
Problem 3. [2] In $\triangle ABC$, $AB = AC$ and $\angle BAC = 80°$. Let P be a point inside $\triangle ABC$ such that $\angle BAP = 20°$ and $\angle ABP = 10°$. What is $\angle APC$?



(Difficulty: 6)

Solution 3. Since $AB = AC$, we have $\angle ABC = \angle ACB$ (base $\angle$s, isos. $\triangle$), so $\angle ABC = \angle ACB = (180° - 80°)/2 = 50°$. So $\angle PBC = 50° - 10° = 40°$.

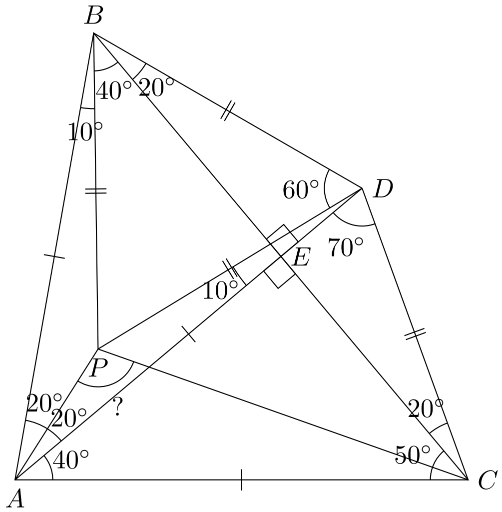
Draw AD between $\angle BAC$ such that $AD = AB$ and $\angle DAC = 40°$. Note that $\angle PAD = 80° - 20° - 40° = 20°$.



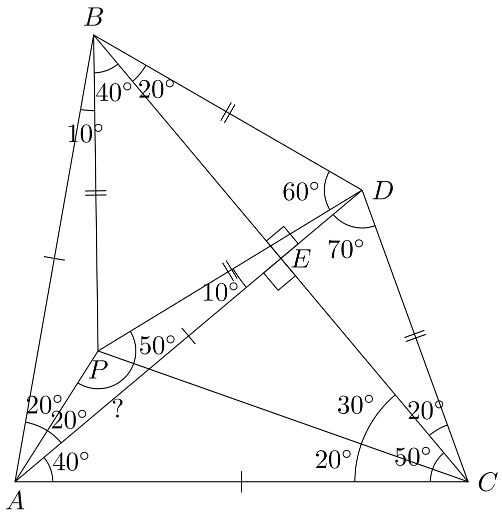
Mark E as the intersection of AD and BC. In $\triangle AEC$, $\angle AEC = 180° - 40° - 50° = 90°$ ($\angle$ sum of $\triangle$).

Join BD . Since $AB = AD$, we have $\angle ABD = \angle ADB = (180° - 40°)/2 = 70°$ (base $\angle$s, isos. $\triangle$)& ($\angle$ sum of $\triangle$). Note that $\angle BED = 90°$ (vert. opp. $\angle$s), so $\angle DBE = 180° - 70° - 90° = 20°$ ($\angle$ sum of $\triangle$).

Join DC and PD. Note that $\triangle DAB \cong \triangle DAC$ (SAS), so $BD = DC$ and $\angle ADC = \angle ADB = 70°$. Since $BD = DC$, we have $\angle DCB = \angle DBC = 20°$ (base $\angle$s, isos. $\triangle$).



Note that $\triangle BAP \cong \triangle DAP$ (SAS), so $\angle PDA = \angle PBA = 10°$ (corr. $\angle$s, $\cong$ $\triangle$s). Thus $\angle PDB = 70° - 10° = 60°$. Note that in $\triangle BPD$, $\angle PBD = \angle PDB = 60°$. Thus $\triangle BPD$ is an equil. $\triangle$ (con. of equil. $\triangle$), so $BP = DP = BD$. Since $BD = DC$, we have $DP = DC$.



Since $\triangle DPC$ is an isos. $\triangle$ with $DP = DC$, we have $\angle DPC = \angle DCP = (180° - 80°)/2 = 50°$ (base $\angle$s, isos. $\triangle$)& ($\angle$ sum of $\triangle$). Thus $\angle ECP = 50° - 20° = 30°$. So $\angle PCA = 50° - 30° = 20°$.

Finally, in $\triangle APC$, $\angle APC = 180° - (20° + 40°) - 20° = \boxed{100°}$.

References

[1] MindYourDecisions, “A classically hard geometry problem,” YouTube. [Online]. Available: https://www.youtube.com/watch?v=CFhFx4n3aH8&ab_channel=MindYourDecisions

[2] ——, “A classically hard geometry problem,” YouTube. [Online]. Available: https://www.youtube.com/watch?v=Rjo-PcrKrB0&t=272s&ab_channel=MindYourDecisions