Contents

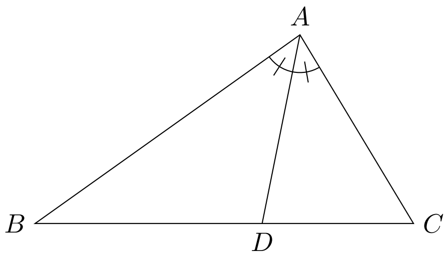
1 Lines, angles and shapes

1.10 Four centres of triangle

1.10.1 Angle bisector, perpendicular bisector, median, altitude and cevian

Angle bisector

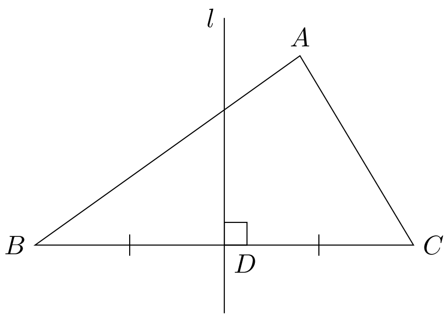
An angle bisector of a triangle is a line segment from a vertex to the opposite side such that the angle of the vertex is bisected by the line segment.



$$\because \angle BAD = \angle CAD$$
$$\therefore\ AD \text{ is the angle bisector of } \angle A \text{ in } \triangle ABC\ .$$

Perpendicular bisector

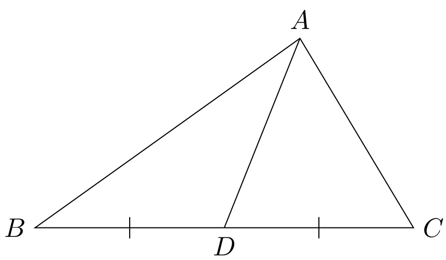
A perpendicular bisector of a triangle is the perpendicular bisector of a side of the triangle. It is a line instead of line segment.



$$\because BD = DC \text{ and } l \perp BC$$
$$\therefore\ \text{Line } l \text{ is the the perpendicular bisector of } BC \text{ in } \triangle ABC\ .$$

Median

A **median** of a triangle is a line segment from a vertex to the mid-point of the opposite side.



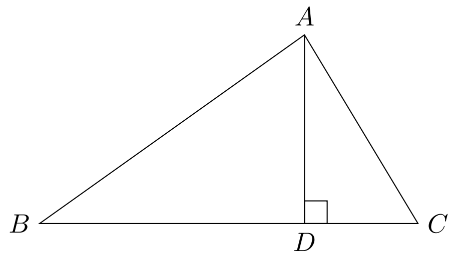
$$\because BD = DC$$
$$\therefore\ AD \text{ is a median of } \triangle ABC \text{ (that corresponds to } BC\text{).}$$

Altitude

An **altitude** of a triangle is a perpendicular line segment (or line) from a vertex to the (extended) opposite side.

It is sometimes also called the height of the triangle (but height more often refers to the length of the altitude while altitude more often refers to the line segment itself).

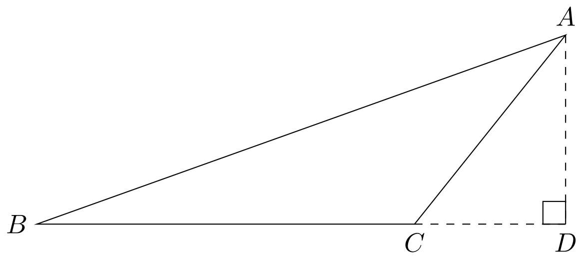
Note that the point where the altitude meet the side is called the **foot** of the altitude.



$\because AD \perp BC$
$\therefore\ AD$ is an altitude of $\triangle ABC$ (that corresponds to BC).

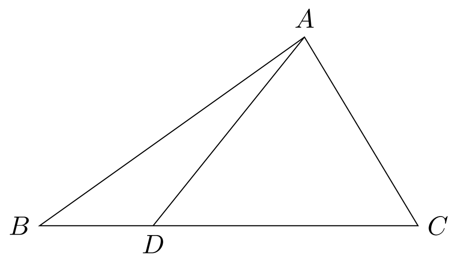
(And D is the foot of altitude AD .)

Note that this AD is also an altitude:



Cevian

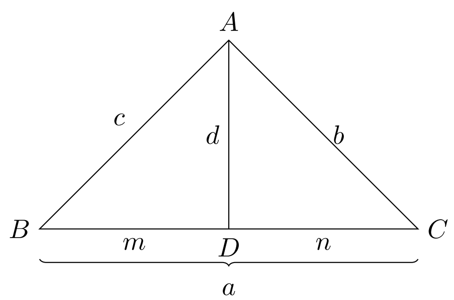
A **cevian** of a triangle is any line segment from a vertex to the opposite side. It must lie inside the triangle. All angle bisectors and medians are cevians, but not all perpendicular bisectors and altitudes are cevians.



$\because D$ lies on side BC .
$\therefore\ AD$ is a cevian of $\triangle ABC$ (that corresponds to BC).

Lengths of angle bisector, median, altitude and cevian

Here's a summary of their lengths (the formulas have appeared and been proved in previous prepositions).



Type	Condition	Formula
Cevian	D is on side BC .	$d = \sqrt{\frac{b^2m + c^2n}{m+n} - mn}$
Median	$m = n$	$d = \frac{\sqrt{2b^2 + 2c^2 - a^2}}{2}$
Angle bisector	$\angle BAD = \angle CAD$	$d = \sqrt{bc - mn} = \sqrt{bc(1 - \frac{a^2}{(b+c)^2})}$
Altitude	$AD \perp BC$	$d = \sqrt{c^2 - (\frac{a^2 + c^2 - b^2}{2a})^2} = \sqrt{b^2 - (\frac{a^2 + b^2 - c^2}{2a})^2}$

1.10.2 The four centres of triangles

Any triangle has four centres, which are **incentre**, **circumcentre**, **centroid** and **orthocentre**. By definition,

Incentre is the intersection of angle bisectors of the angles.
Circumcentre is the intersection of perpendicular bisectors of the sides.
Centroid is the intersection of the triangle's medians.
Orthocentre is the intersection of the triangle's altitudes.

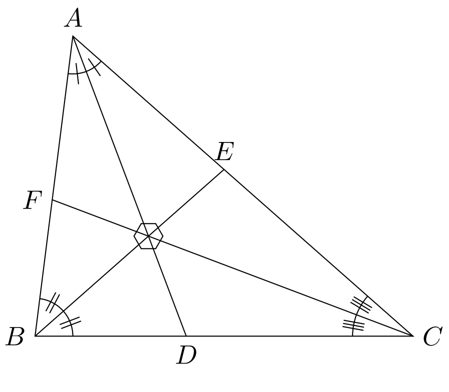
Any triangle has only one unique point for each type of centre. There cannot be two incentres in a triangle, or two centroids.

This means the angle bisectors of a triangle are concurrent, and perpendicular bisectors of triangle sides are concurrent, and medians of the triangles are concurrent, and altitudes of triangles are concurrent. We will prove these facts in the following subsubsections.

1.10.3 Incentre (and incircle)

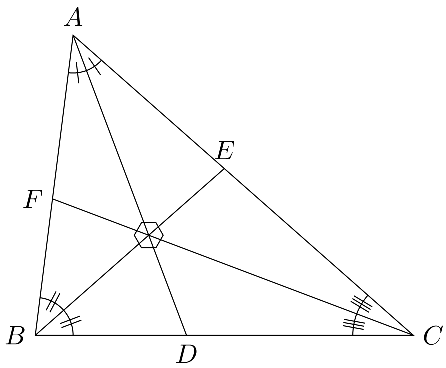
Preposition 1. The angle bisctors of a triangle are concurrent. (concurrency of $\angle$ bisctors of $\triangle$)

(The mini hexagon indicates that we don't initially know whether AD , BE and CF are concurrent.)



$$\because \angle BAD = \angle CAD \text{ , } \angle ABE = \angle CBE \text{ , } \angle ACF = \angle BCF$$
$$\therefore \ AD \text{ , } BE \text{ , } CF \text{ are concurrent.} \qquad \text{(concurrency of } \angle \text{ bisctors of } \triangle\text{)}$$

Proof. [1] Let there be $\triangle ABC$ with points D, E, F on sides BC, AC, AB respectively such that $\angle BAD = \angle CAD$, $\angle ABE = \angle CBE$, $\angle ACF = \angle BCF$.



Since AD is the angle bisector of A , by angle bisector theorem, we have

$$\frac{BD}{DC} = \frac{AB}{AC} \tag{1}$$

Similar, since BE and CF are angle bisector of $\angle B$ and $\angle C$ respectively, by angle bisector theorem, we have

$$\frac{CE}{EA} = \frac{BC}{AB} \tag{2}$$

$$\frac{AF}{FB} = \frac{AC}{BC} \tag{3}$$

Multiply (1), (2) and (3) together:

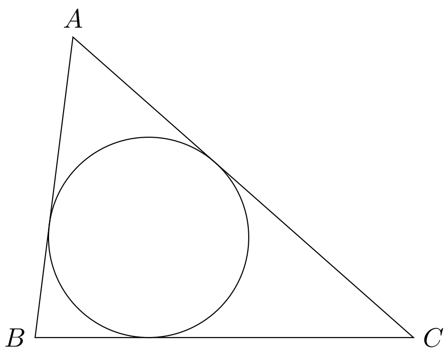
$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{AB}{AC} \cdot \frac{BC}{AB} \cdot \frac{AC}{BC}$$
$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$$

By converse of Ceva's theorem, AD, BE, CF are concurrent. □

And this point of concurrency is called incentre (usually denoted I).

Inscribed circle

The **inscribed circle** (/**incircle**) of a triangle is a circle that is tangent to all three sides of the triangle:

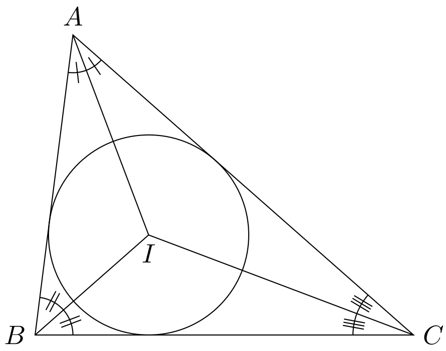


The incircle is the largest possible circle that can be contained in the triangle, and there is a unique incircle for each triangle.

And the incircle's centre is the incentre of the triangle.

The radius of the incircle is called **inradius** .

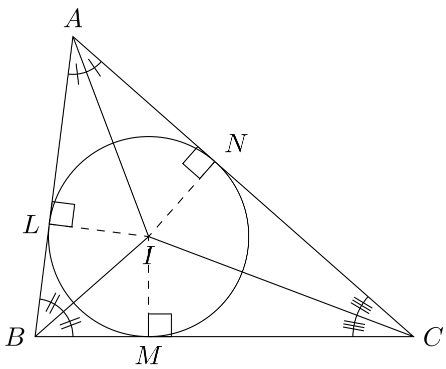
Preposition 2. The incentre of a triangle is the centre of the inscribed circle of the triangle. (prop. of incentre)



$\because I$ is the incentre of $\triangle ABC$.

$\therefore$ I is the centre of the incircle of $\triangle ABC$. (prop. of incentre)

Proof. Drop perpendicular line segments from the incentre to the sides. Namely, draw $IM \perp BC$, $IN \perp AC$, $IL \perp AB$.



Since IB is the angle bisector of ABC , we have $IL = IM$ (prop. of $\angle$ bisector).
Similarly, since IC is the angle bisector of ACB , we have $IM = IN$ (prop. of $\angle$ bisector).
By transitivity of equality, we have $IL = IM = IN$.

By definition, the incircle is tangent to sides AB, BC, AC . So the radii to the points of tangency are perpendicular to AB, BC, AC (tangent $\perp$ radius).

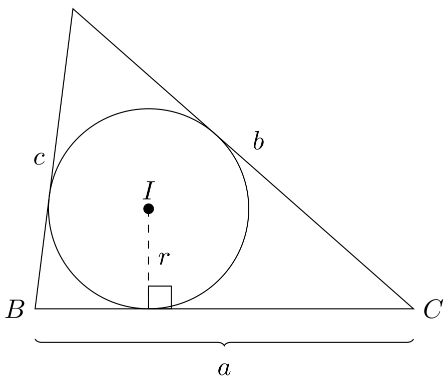
Since $IM \perp BC$, M must be a point of tangency (since there is a unique point on BC that is the perpendicular projection of I onto BC).

So M lies on the incircle. By similar reasoning, N and L also lie on the incircle.

By '3R theorem', since $IL = IM = IN$, I must be the centre of the incircle. □

Note: This preposition means the incentre is the (only) point that is equidistant from the three sides of the triangle.

Preposition 3. Given a triangle with inradius (radius of incircle) r and semi-perimeter s , the area (A) of the triangle is rs. (semi-perimeter inradius formula)

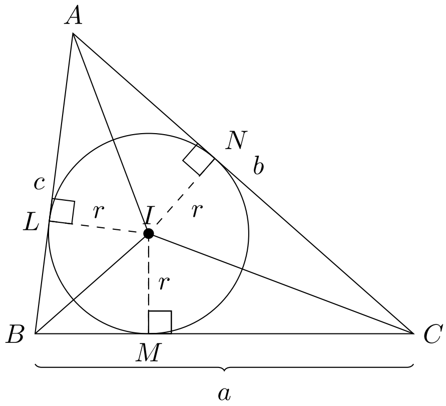


$$\underline{\text{Given:}} \quad s = \frac{a+b+c}{2}$$

$$\boxed{A = rs}$$

Proof. [2] Join IA , IB and IC . Let A be the area of $\triangle ABC$. (This A is different from the vertex A , but I am too lazy to make a new variable.)

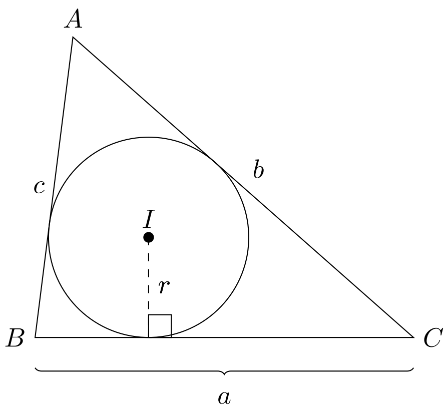
Draw $IM \perp BC$, $IN \perp AC$, $IL \perp AB$. Note that $IM = IN = IL = r$ (inradius).



$$\begin{aligned}
\text{area of } \triangle ABC &= \text{area of } \triangle AIB + \text{area of } \triangle BIC + \text{area of } \triangle AIC \\
&= \frac{AB \cdot IL}{2} + \frac{BC \cdot IM}{2} + \frac{AC \cdot IN}{2} \qquad \text{(area of } \triangle\text{)} \\
&= \frac{cr}{2} + \frac{ar}{2} + \frac{br}{2} \\
&= r(\frac{a+b+c}{2}) \\
A &= rs
\end{aligned}$$

□

Preposition 4. Given a triangle with side lenghths a, b, c , the **inradius** (radius of incircle) (r) of the triangle is $\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$, where $s = \frac{a+b+c}{2}$ is the semi-perimeter of the triangle. (inradius formula)



$$\boxed{r = \sqrt{\frac{s(s-a)(s-b)(s-c)}{s}}}$$

Proof. Let A be the area of $\triangle ABC$, r be the inradius, and s be the semi-perimeter.

By semi-perimeter inradius formula, we have $A = rs$.

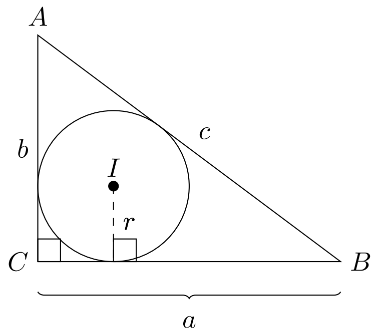
By Heron's formula, we have $A = \sqrt{s(s-a)(s-b)(s-c)}$.

Thus,

$$\begin{aligned}
rs &= \sqrt{s(s-a)(s-b)(s-c)} \\
r &= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} \\
&= \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2}} \\
&= \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}
\end{aligned}$$

□

Preposition 5. Given a right triangle with legs a, b and hypotenuse c, the inradius (r) of the triangle is $\dfrac{a+b-c}{2}$ and also $s-c$ (where s is the semi-perimeter) . (inradius formula of right triangle)

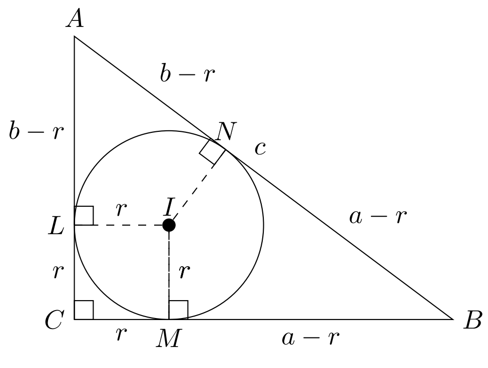


$$\underline{\text{Given:}} \quad \angle C = 90^\circ \ .$$

$$\boxed{r = \frac{a+b-c}{2}}$$

$$\boxed{r = s-c}$$

Proof. [3] Draw $IM \perp BC$, $IN \perp AC$, $IL \perp AB$.

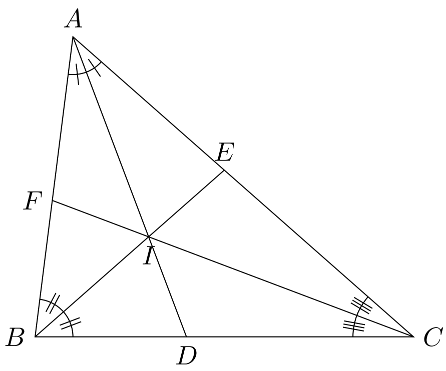


Note that $BMIL$ is a square with side length r. Then $MB = a - r$ and $AL = b - r$.
By tangent properties, we have $AN = AL = b - r$ and $NB = MB = a - r$. So

$$\begin{aligned}
AB &= AN + NB \\
c &= (b-r) + (a-r) \\
2r &= a + b - c \\
r &= \frac{a+b-c}{2}
\end{aligned}$$

And $r = \dfrac{a+b+c-2c}{2} = s - c$ □

Preposition 6. For an angle bisector of a triangle, the incentre is closer to the landing point on the opposite side than to the vertex. (position of incentre on $\angle$ bisector)

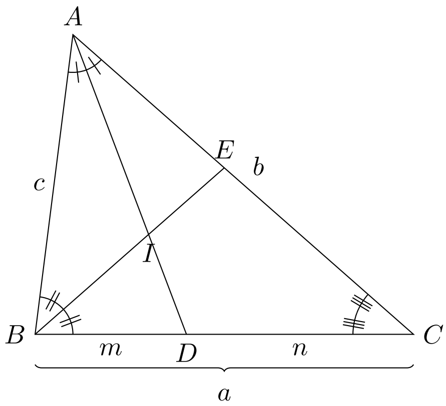


Given: I is the incentre of $\triangle ABC$.

$\therefore\ ID < IA$, $IE < IB$, $IF < IC$ (position of incentre on $\angle$ bisector)

Proof. Let $BC = a$, $AC = b$, $AB = c$. Let $BD = m$ and $DC = n$.

It is sufficient to only prove that $ID < IA$, as the rest follows a similar argument.



By angle bisector theorem, we have $\dfrac{ID}{IA} = \dfrac{m}{c}$. If $m < c$, then $\dfrac{m}{c} < 1 \Rightarrow \dfrac{ID}{IA} < 1 \Rightarrow ID < IA$. Thus, we want to show that $m < c$.

By angle bisector theorem, we have $\dfrac{m}{n} = \dfrac{c}{b}$, which means $m = \dfrac{ac}{b+c}$.

By triangle inequality, we have $a < b + c$. Multiply both sides by $\dfrac{c}{b+c}$:

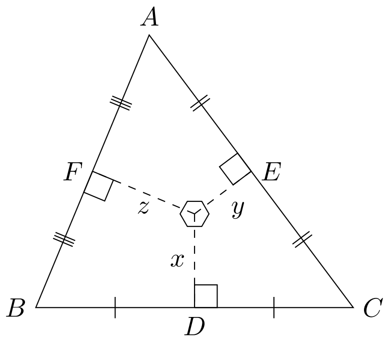
$$\begin{aligned} a\left(\frac{c}{b+c}\right) &< (b+c)\left(\frac{c}{b+c}\right) \\ \frac{ac}{b+c} &< c \\ m &< c \end{aligned}$$

This means $ID < IA$. By similar argument, we have $IE < IB$ and $IF < IC$.

□

1.10.4 Circumcentre (and circumcircle)

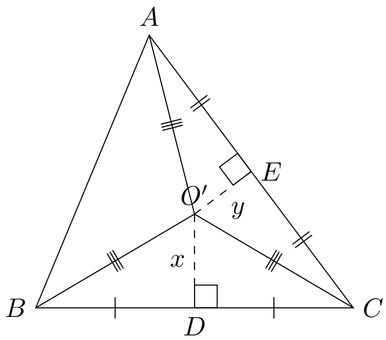
Preposition 7. The perpendicular bisectors of a triangle's sides are concurrent. (concurrency of $\perp$ bisectors of $\triangle$)



$\because x \perp BC$, $y \perp AC$, $z \perp AB$, $BD = DC$, $CE = EA$, $AF = FB$

$\therefore$ x, y, z are concurrent. (concurrency of $\perp$ bisectors of $\triangle$)

Proof. Let line x, y, z be the perpendicular bisector of BC, AC, AB respectively. Let O' be the intersection of x and y .



Since O' lies on the $\perp$ bisector of BC , we have $O'B = O'C$ (prop. of $\perp$ bisector).

Similarly, since O' lies on the $\perp$ bisector of AC , we have $O'A = O'C$ (prop. of $\perp$ bisector).

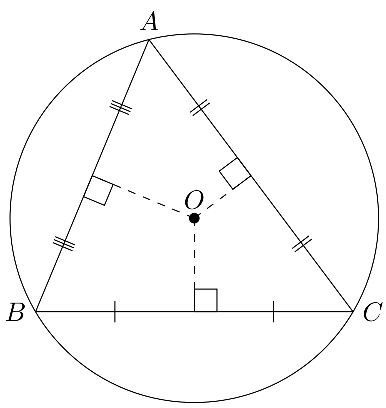
By transitivity of equality, we have $O'A = O'B$, which means O' lies on the perpendicular bisector of AB (prop. of $\perp$ bisector).

This means all three perpendicular bisectors are concurrent. $\square$

And the point of concurrency of the perpendicular bisectors are called circumcentre.

Preposition 8. The circumcentre of a triangle is the centre of the triangle's circumcircle. (prop. of circumcentre)

(Note: The **circumcircle** of a triangle is the circle that passes through all three vertices of the triangle.)



$\because O$ is the circumcentre of $\triangle ABC$.

$\therefore$ O is the centre of circumcircle of $\triangle ABC$.

Proof. Since O lies on perpendicular bisectors of all three sides, we have $OA = OB = OC$ (prop. of $\perp$ biosector).

Note that A, B, C lie on the circumcircle by definition.

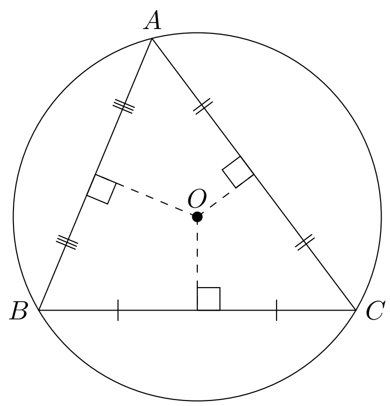
Since $OA = OB = OC$, by '3R theorem', O is the centre of the circumcircle. $\square$

Note 1: This preposition means the circumcentre is the (only) point that is equidistant from the three vertices of the triangle.

Note 2: Unlike incentre which can only lie inside a triangle, the circumcentre may lie on a triangle's side or outside the triangle. The former happens for a right triangle and the latter for an obtuse triangle.

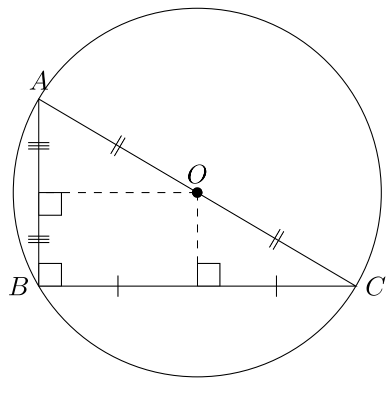
Preposition 9. The circumcentre lies inside an acute triangle, lies on the side of a right triangle, and lies outside an obtuse triangle. (position of circumcentre)

Case 1:



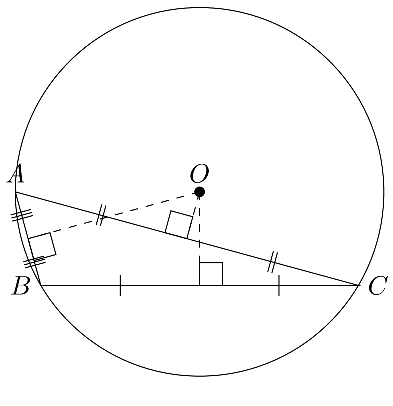
$\because \angle A < 90^\circ$, $\angle B < 90^\circ$, $\angle C < 90^\circ$.
$\therefore$ O lies inside $\triangle ABC$.

Case 2:



$\because \angle ABC = 90^\circ$
$\therefore$ O lies on side AC .

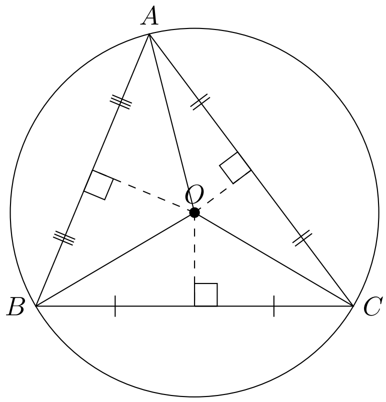
Case 3:



$$\because \angle ABC > 90^\circ$$
$$\therefore\ O \text{ lies outside } \triangle ABC\ .$$

Proof. Case 1: $\angle A < 90^\circ$, $\angle B < 90^\circ$, $\angle C < 90^\circ$

Let $\angle A = x$, $\angle B = y$, $\angle C = z$, where $x, y, z < 90^\circ$. This means $2x, 2y, 2z < 180^\circ$.



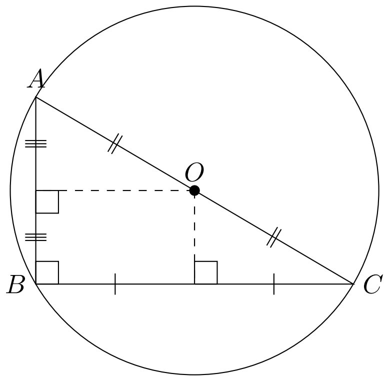
Note that anticlockwise$\angle COA = 2y$, (anticlockwise shortened to anti) anti$\angle AOB = 2z$, anti$\angle BOC = 2x$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$).

Thus anti$\angle COA < 180^\circ$, anti$\angle AOB < 180^\circ$, anti$\angle BOC < 180^\circ$.

All three conditions are satisfied only when O is inside the triangle.

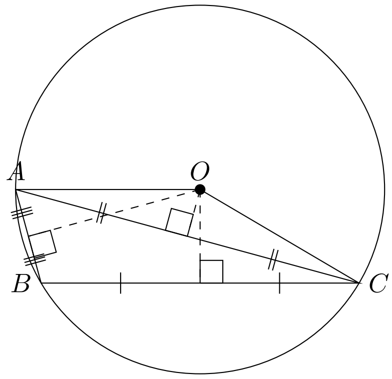
Otherwise, say O is outside $\triangle ABC$ at the right of AC . Then anti$\angle COA > 180^\circ$, but this violates the condition that anti$\angle COA < 180^\circ$. Thus it is impossible that O lies outside the triangle when $\triangle ABC$ is acute.

Case 2: $\angle ABC = 90^\circ$



By converse of $\angle$ in semi-circle, AC is a diamter of the circumcircle. Since O is the centre of the circumcircle , O must be lie on AC .

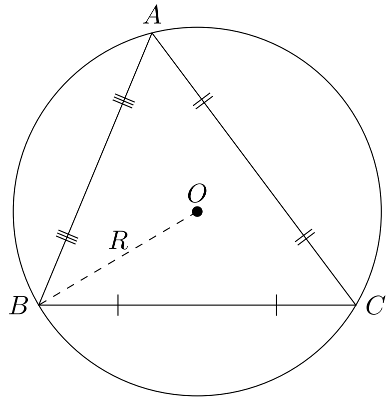
Case 3: $\angle ABC > 90^\circ$



By ($\angle$ at centre twice $\angle$ at $\odot^{ce}$), anti$\angle COA = 2\angle ABC > 180^\circ$, which means O must lie above AC , which means outside $\triangle ABC$.

□

Preposition 10. Given a triangle with side lengths a, b, c , the **circumradius** (radius of circumcircle) (R) of the triangle is $\dfrac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$, where $s = \dfrac{a+b+c}{2}$ is the semi-perimeter of the triangle. (circumradius formula)



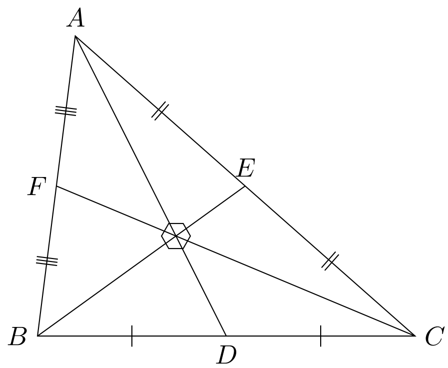
$$\boxed{R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}}$$

Proof. By 'circumradius of triangle' , we have the formula $R = \dfrac{abc}{4K}$, where K is the area of the triangle.

Since $K = \sqrt{s(s-a)(s-b)(s-c)}$ by Heron's formula, we have $R = \dfrac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$
. □

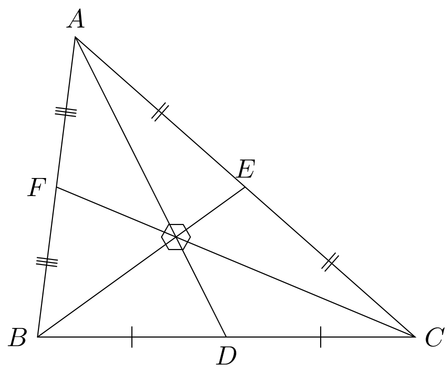
1.10.5 Centroid

Preposition 11. The medians of a triangle are concurrent. (concurrency of medians)



$$\because BD = DC \text{ , } CE = EA \text{ , } AF = FB$$
$$\therefore\ AD, BE, CF \text{ are concurrent.} \qquad \text{(concurrency of medians)}$$

Proof. .



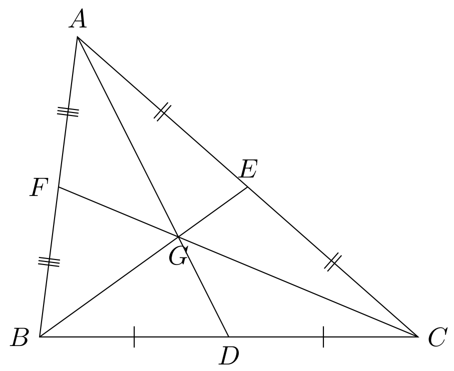
Note that

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{BD}{BD} \cdot \frac{CE}{CE} \cdot \frac{AF}{AF} = 1$$

Thus, by converse of Ceva's theorem, AD, BE, FD are concurrent. $\square$

And the point of concurrency of the medians is called centroid.

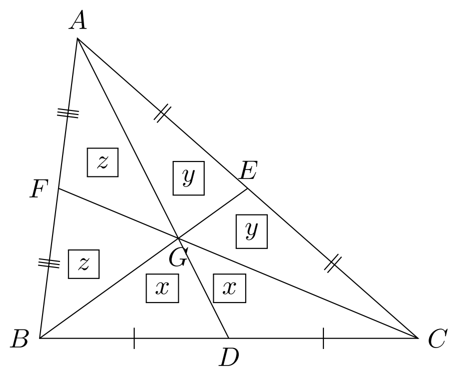
Preposition 12. The three medians of a triangle divide it into six triangles of equal area. (area property of medians)



$$\because G \text{ is centroid of } \triangle ABC$$
$$\therefore\ [\triangle GAF] = [\triangle GFB] = [\triangle GBD] = [\triangle GDC] = [\triangle GCE] = [\triangle GEA]$$
(area property of medians)

Proof. [4] Since $BD = DC$, we have $[\triangle GBD] = [\triangle GDC]$ (bases prop. to areas of $\triangle$s).
Similarly, we have $[\triangle GAF] = [\triangle GFB]$ and $[\triangle GCE] = [\triangle GEA]$.

Let $[\triangle GBD] = [\triangle GDC] = x$, $[\triangle GCE] = [\triangle GEA] = y$, $[\triangle GAF] = [\triangle GFB] = z$.



Note that $[\triangle ABD] = [\triangle ADC] = \frac{1}{2}[\triangle ABC]$ (bases prop. to areas of $\triangle$s) . So we have

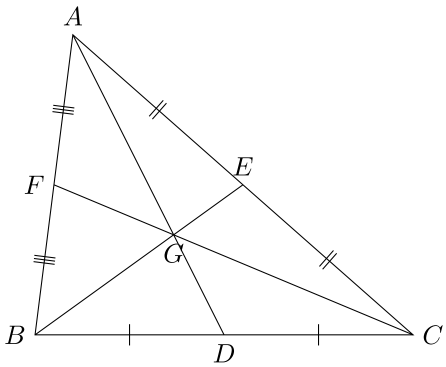
$$\begin{aligned} z + z + x &= y + y + x \\ 2z &= 2y \\ z &= y \end{aligned}$$

Similarly, since $[\triangle BCE] = [\triangle BEA]$, we have

$$\begin{aligned} x + x + y &= z + z + y \\ 2x &= 2z \\ x &= z \end{aligned}$$

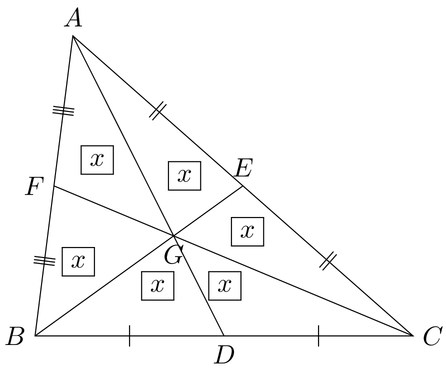
This means $x = y = z$, and $[\triangle GAF] = [\triangle GFB] = [\triangle GBD] = [\triangle GDC] = [\triangle GCE] = [\triangle GEA]$. $\square$

Preposition 13. For a median, the segment from the vertex to centroid is twice the length of the segment from the centroid to the mid-point on the opposite side. (prop. of centroid)



$$\because AF = FB \ , \ BD = DC \ , \ AE = EC$$
$$\therefore \ AG = 2DG \ , \ BG = 2EG \ , \ CG = 2FG \qquad \text{(prop. of centroid)}$$

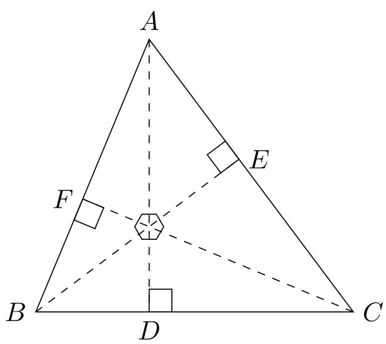
Proof. By 'area property of medians' , the medians divide $\triangle ABC$ into six triangles of equal area (denoted x).



By 'bases prop. to areas of $\triangle$s' , $\dfrac{DG}{AG} = \dfrac{[\triangle BDG]}{[\triangle BGA]} = \dfrac{x}{2x} = \dfrac{1}{2}$.
This means $AG = 2DG$.
By similarly reasoning, we have $GC = 2EG$ and $CG = 2FG$. □

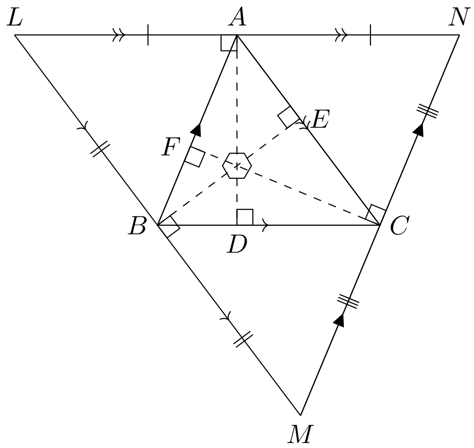
1.10.6 Orthocentre

Preposition 14. The altitudes of a triangle are concurent. (concurrency of altitudes)



$$\because AD \perp BC \ , \ BE \perp AC \ , \ CF \perp AB$$
$$\therefore \ AD, BE, CF \text{ are concurrent.} \qquad \text{(concurrency of altitudes)}$$

Proof. [5] Let there be $\triangle ABC$ with $AD \perp BC$, $BE \perp AC$, $CF \perp AB$.
Draw LM through B , LN through A , MN through C such that $LM//AC$, $LN//BC$, $MN//BA$.



Note that $\angle FCN = \angle BFC = 90°$ (alt. $\angle$s , $BA//MN$). Similarly, $\angle LAD = \angle ADC$ (alt. $\angle$s , $LN//BC$) , and $\angle EBM = \angle AEB = 90°$ (alt. $\angle$s , $AC//LM$) .

This means $AD \perp LN$, $EB \perp LM$, $FC \perp MN$.

Also, note that $LA = AN$, $LB = BM$, $MC = CN$ (prop. of being mid-pt. $\triangle$). This means AD , BE , CF are perpendicular bisector of LN , LM , MN respectively.

So AD, BE, CF must be concurrent at a point that is the circumcentre of $\triangle LMN$ (concurrency of $\perp$ bisectors of $\triangle$) .

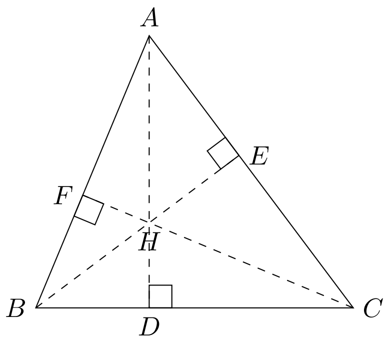
□

And this point of concurrency is the orthocentre of $\triangle ABC$.

An implication of this preposition is that any two altitudes must intersect at the orthocentre, since by definition, the orthocentre lies on all three altitudes, so if the orthocentre does not lie on the intersection of two altitudes, then it dose not lie on at least one altitude, making it not the orthocentre by definition.

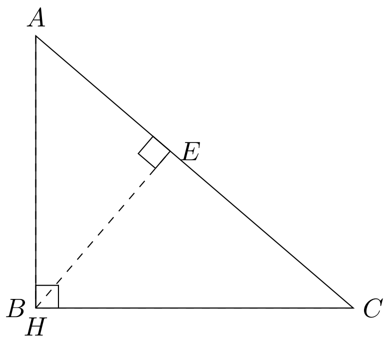
Preposition 15. The orthocentre lies inside an acute triangle, lies on the right angle vertex of a right triangle, and lies outside an obtuse triangle. (position of orthocentre)

Case 1:



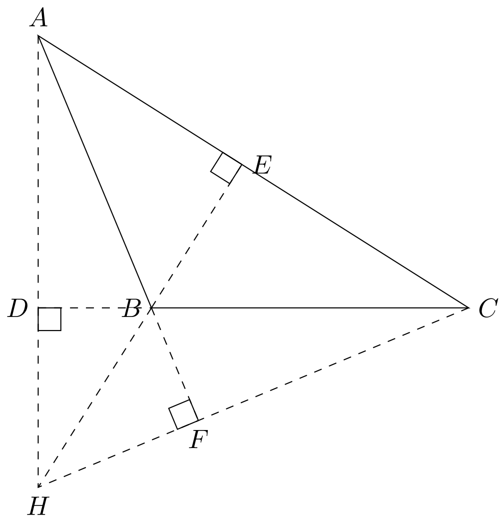
$$\because \angle A, \angle B, \angle C < 90°$$
$$\therefore \ H \text{ lies inside } \triangle ABC \ . \qquad \text{(position of orthocentre)}$$

Case 2:



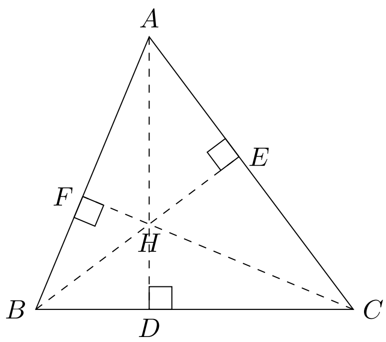
$$\because \angle B = 90^\circ$$
$$\therefore\ H \text{ lies on vertex } B\ . \qquad \text{(position of orthocentre)}$$

<u>Case 3:</u>



$$\because \angle ABC > 90^\circ$$
$$\therefore\ H \text{ lies outside } \triangle ABC\ . \qquad \text{(position of orthocentre)}$$

Proof. <u>Case 1:</u> $\angle A, \angle B, \angle C < 90^\circ$

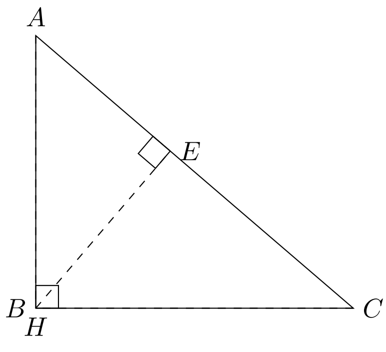


Since $\angle ABC < 90^\circ$ and $\angle ACB < 90^\circ$, A lies directly above BC (if BC is placed horizontally). So the foot of the altitude, D, must lie between B, C .

Similarly, the foots of the other two altitudes, E, F , must lie between A, C and A, B respectively.

Since the altitudes lie inside the triangle, the orthocentre must also lie inside the triangle (since orthocentre lies on altitudes) .

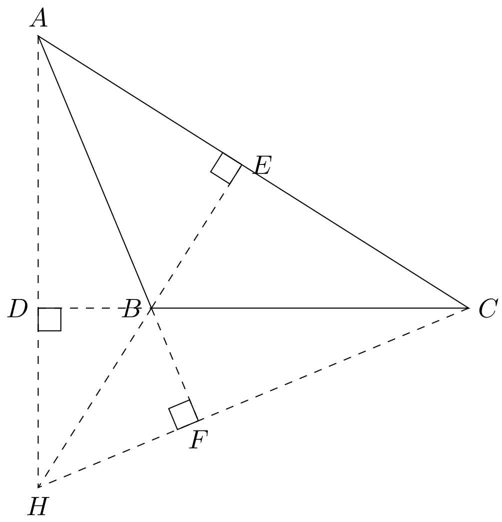
<u>Case 2:</u> $\angle B = 90^\circ$



Since $AB \perp BC$ and $AD \perp DC$ and D is a point on line BC , D must coincide with B since there is a unique point of projection from point A to line BC (prop. of $\perp$ line).

Similarly, F must coincide with B . This means B is the intersection of altitudes AD and CF , so B is the orthocentre of $\triangle ABC$.

Case 3: $\angle ABC > 90°$



Since A does not lie directly above side BC , but insteasd lie 'diagonally above', the altitude AD must lie outside $\triangle ABC$ (except point A).

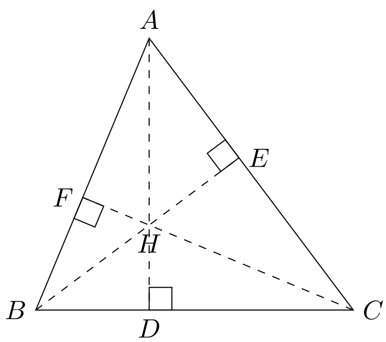
Similarly, altitude CF must lie outside $\triangle ABC$ except C.

Note that H cannot lie on a vertex since that would make the triangle a right triangle, which contradicts the assumption.

Since the orthocentre H must lie on line AD and line CF and must not lie on the vertex, H must lie outside $\triangle ABC$ too.

□

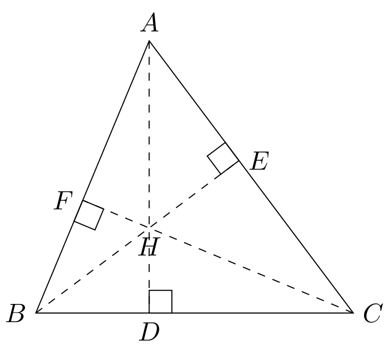
Preposition 16. The three altitudes of an acute triangle splits it into six triangles, where each pair of vertically opposite triangles are similar. (prop. of orthocentre)



Given: H is the orthocentre of $\triangle ABC$.

$\therefore$ $\triangle AFH \sim \triangle CDH$, $\triangle BFH \sim \triangle CEH$, $\triangle BDH \sim \triangle AEH$ (prop. of orthocentre)

Proof. .

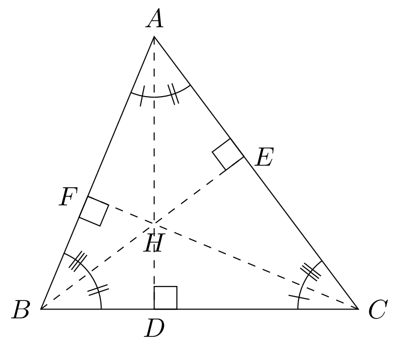


In $\triangle AFH$ and $\triangle CDH$,

$$\begin{aligned} \angle AFH &= \angle CDH && \text{(altitude)} \\ \angle AHF &= \angle CDH && \text{(altitude)} \\ \therefore\ \triangle AFH &\sim \triangle CDH && \text{(AA)} \end{aligned}$$

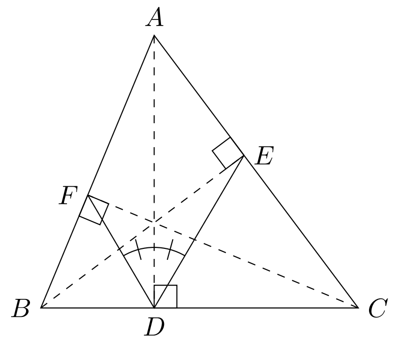
Similarly, we have $\triangle BFH \sim \triangle CEH$ and $\triangle BDH \sim \triangle AEH$. □

This preposition means for a pair of vertical opposite triangles, the corresponding angles touching the vertices of $\triangle ABC$ are equal:



$$\angle HAF = \angle HCD\ ,\ \angle HBF = \angle HCE\ ,\ \angle HBD = \angle HAE \qquad \text{(prop. of orthocentre)}$$

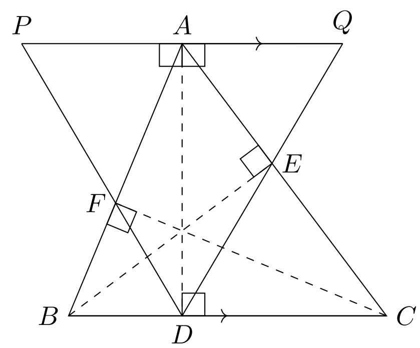
Preposition 17. In an acute triangle, if an angle is formed by connecting the three foots of the triangle's altitudes with two line segments, then the altitude that corresponds to the vertex of the angle is the angle bisector of that angle. (Blanchet's theorem)



$$\because AD \perp BC\ ,\ BE \perp AC\ ,\ CF \perp AB$$
$$\therefore\ \angle ADF = \angle ADE \qquad \text{(Blanchet's theorem)}$$

Proof. [6] Let P be a point on extended DF such that $PA \perp AD$. Let Q be a point on extended DE such that $QA \perp AD$.

Note that $PQ//BC$ (alt. $\angle$s equal).



Note that $\angle APF = \angle BDF$ (alt. $\angle$s , $PQ//BC$) and $\angle PFA = \angle DFB$ (vert. opp. $\angle$s).
Thus, we have $\triangle FAP \sim \triangle FBD$ (AA), and similarly, $\triangle EQA \sim \triangle EDC$ (AA).
Thus,

$$\frac{PA}{BD} = \frac{AF}{FB} \quad \text{(corr. sides, } \sim \triangle\text{s)}$$
$$PA = \frac{AF \cdot BD}{FB} \tag{1}$$

And

$$\frac{AQ}{DC} = \frac{AE}{EC} \quad \text{(corr. sides, } \sim \triangle\text{s)}$$
$$AQ = \frac{DC \cdot AE}{EC} \tag{2}$$

By Ceva's theorem in $\triangle ABC$,

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$
$$\frac{AF \cdot BD}{FB} = \frac{DC \cdot AE}{EC}$$

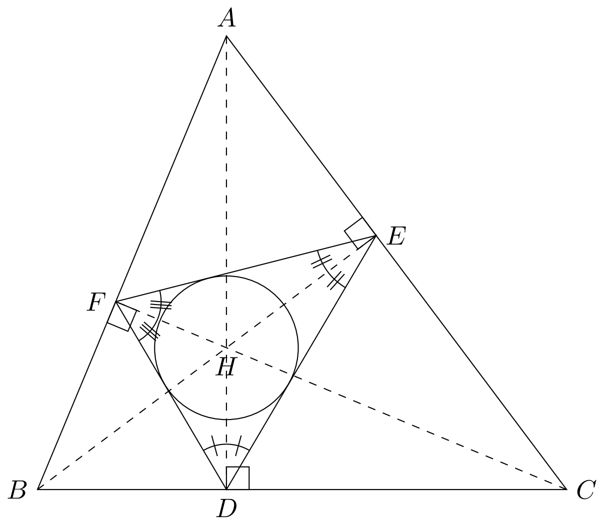
By (1) and (2):

$$PA = AQ$$

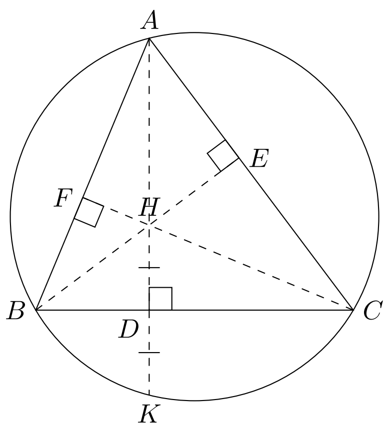
Thus, $\triangle DAP \cong \triangle DAQ$ (SAS), which means $\angle ADF = \angle ADE$ (corr. $\angle$s, $\cong \triangle$s).

□

This preposition implies that H (orthocentre of $\triangle ABC$) is the incentre of $\triangle DEF$:

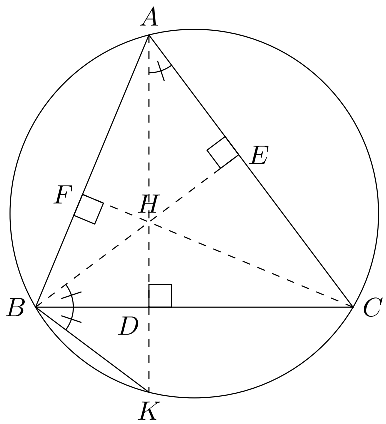


Preposition 18. In an acute triangle, the reflection of the orthocentre about a side lies on the circumcircle. (reflection of orthocentre on circumcircle)



Given: H is the orthocentre of $\triangle ABC$.
$\therefore\ HD = DK$ (reflection of orthocentre on circumcircle)

Proof. Let K be the reflection of orthocentre H about side BC . Join BK .



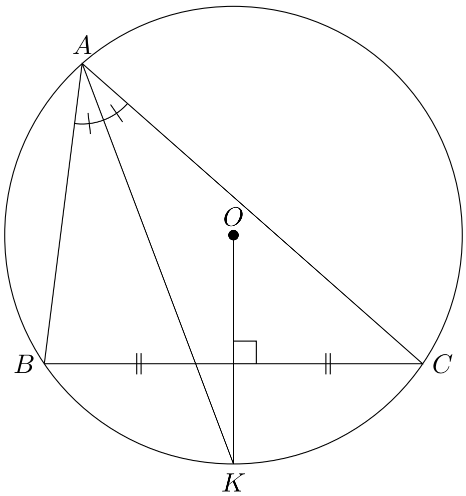
Note that $\angle KBC = \angle KAC$ ($\angle$s in the same segment). Also note that $\angle HBD = \angle KAC$ (prop. of orthocentre).

Thus, $\angle HBD = \angle KBD$. Thus $\triangle BDH \cong \triangle BDK$ (ASA).

So $HD = DK$ (corr. sides, $\cong \triangle$s). ☐

1.10.7 General properties

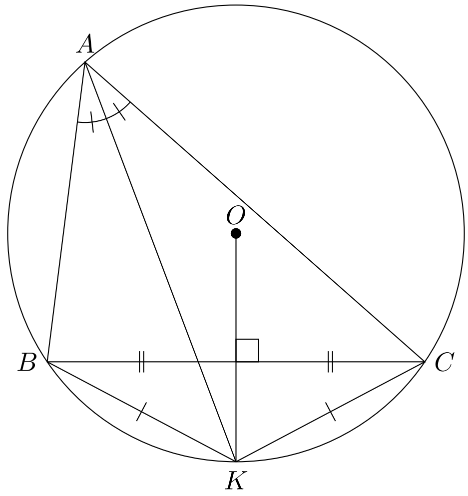
Preposition 19. In a triangle, the angle bisector of a vertex and the perpendicular bisector of the opposite side meet at a point on the circumcircle. ($\angle$ bisector meet $\perp$ bisector at circumcircle)



$\because \angle BAK = \angle CAK$, OK is the perpendicular bisector of BC .
$\therefore\ K$ lies on the circucmcircle of $\triangle ABC$.

Proof. Redefine K to be a point on the circumcircle such that $BC \perp OK$.

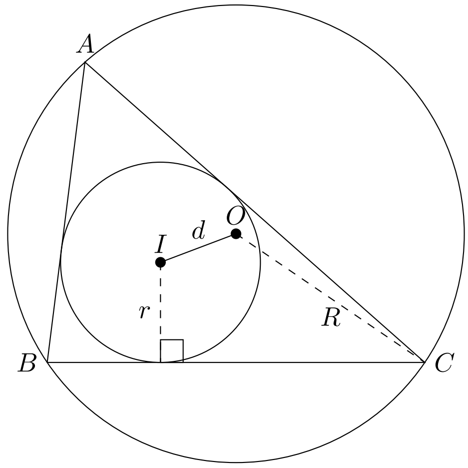
Join BK and KC .



Note that $BK = KC$ (prop. of $\perp$ bisector). Thus, we have that $\angle BAK = \angle CAK$ (equal chords, equal $\angle$s at $\odot^{ce}$) .

Thus, the angle bisector of $\triangle BAC$ and the perpendicular bisector of BC intersect at the point on the circumcircle. $\square$

Preposition 20. Given a triangle with inradius r and circumradius R, the distance (d) between the incentre and circumcentre is $\sqrt{R(R-2r)}$. (Euler's theorem / distance between incentre and circumcentre)



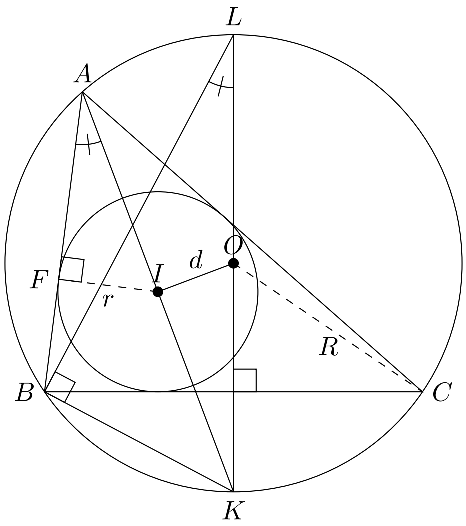
<u>Given:</u> I is incentre, O is circumcentre.

$$\therefore\ d = \sqrt{R(R-2r)}$$

Proof. [7] Let r be the inradius, R be the circumradius.

Extend AI to meet the circumcircle at K . Then $OK \perp BC$ ($\angle$ bisector meet $\perp$ bisector at circumcircle).

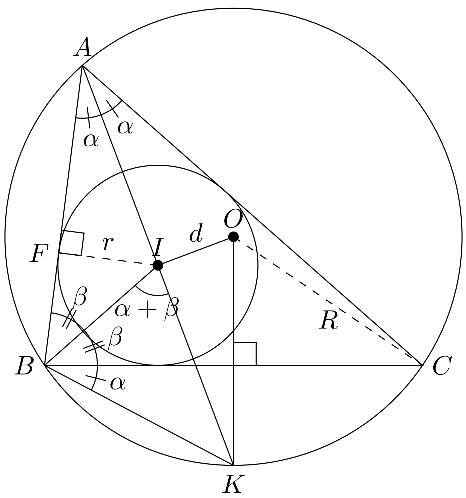
Extend KO to meet the circumcircle at L . Join $\triangle BLK$. Draw $IF \perp AB$.



Note that $\angle ABK = 90^\circ$ ($\angle$ in semi-circle).
In $\triangle AFL$ and $\triangle LBK$,

$$
\begin{aligned}
\angle FAI &= \angle BLK \qquad (\angle\text{s in the same segment})\\
\angle AFI &= \angle LBK = 90^\circ\\
\therefore\ \triangle AFI &\sim \triangle LBK \qquad (\text{AA})\\
\therefore\ \frac{FI}{BK} &= \frac{AI}{LK} \qquad (\text{corr. sides}, \sim \triangle\text{s})\\
FI \cdot LK &= AI \cdot BK\\
r(2R) &= AI \cdot BK \qquad (FI = r\ ,\ LK = 2R)\\
AI \cdot BK &= 2Rr \qquad (1)
\end{aligned}
$$

Join BK . Let $\angle A = 2\alpha$ and $\angle B = 2\beta$. Note that $\angle BAK = \angle CAK = \alpha$ and $\angle ABI = \angle CBI = \beta$ (incentre).

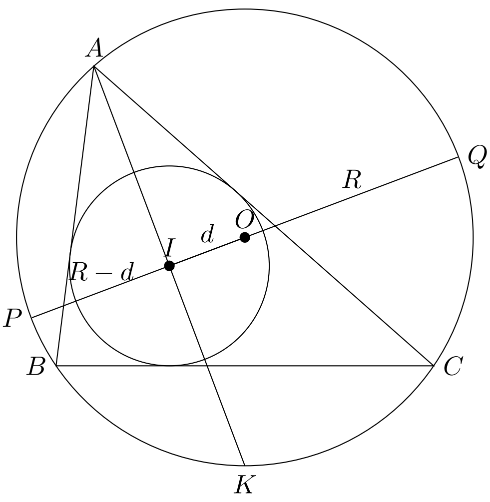


Note that $\angle CBK = \angle CAK = \alpha$ ($\angle$s in the same segment).
In $\triangle ABI$, note that $\angle BIK = \alpha + \beta$ (ext. $\angle$ of $\triangle$).
Since $\angle IBK = \angle BIK = \alpha + \beta$, we have $BK = IK$. (sides opp. equal $\angle$s).

By $AI \cdot BK = 2Rr$, we have

$$AI \cdot IK = 2Rr \qquad (2)$$

Extend IO to the circumcircle on both sides (label the points P and Q). Then $PI = R - d$ and $IQ = R + d$.
(If I and O coincide, then just take $d = 0$ and arbitrarily draw a diameter.)



By intersecting chords theorem, we have

$$\begin{aligned} AI \cdot IK &= PI \cdot IQ \\ 2Rr &= (R-d)(R+d) \\ 2Rr &= R^2 - d^2 \\ d^2 &= R^2 - 2Rr \\ d &= \sqrt{R(R-2r)} \end{aligned}$$

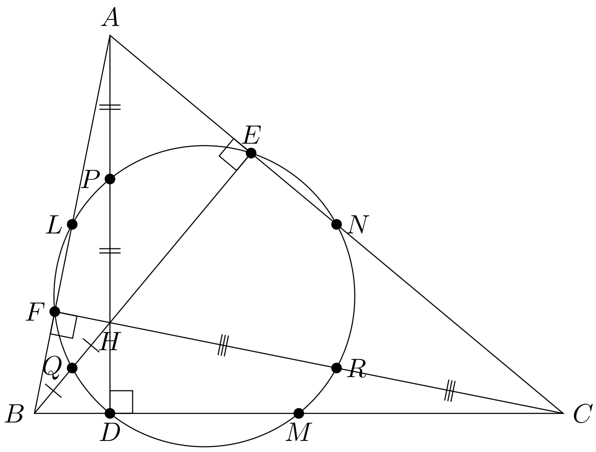
□

Preposition 21. In a triangle, the following nine points are concyclic:

- the three mid-points of the sides of the triangle.
- the three foots of the altitudes of the triangle.
- the three mid-points from the vertices to the orthocentre of the triangle.

(nine-point circle)

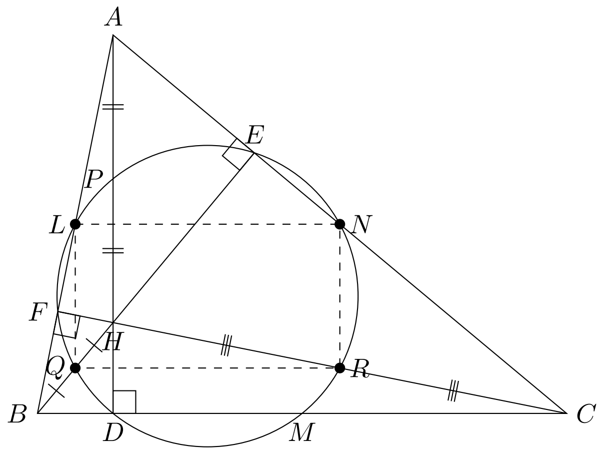
The circle passing through these nine points is called the **nine-point circle**.



Given: $AD \perp BC$, $BE \perp AC$, $CF \perp AB$, $AN = NC$, $AL = LB$, $BM = MC$, $AP = PH$, $BQ = QH$, $CR = RH$

$\therefore$ $D, E, F, M, N, L, P, Q, R$ are concyclic. (nine-point circle)

Proof. Case 1: $\angle A, \angle B, \angle C < 90°$ [8]
Join $LNRQ$.



In $\triangle ABC$, since $AL = LB$ and $AN = NC$, we have $LN//BC$ (mid-pt. theorem).
In $\triangle HBC$, since $HQ = QB$ and $HR = RC$, we have $QR//BC$ (mid-pt. theorem).
This means $LN//QR$.

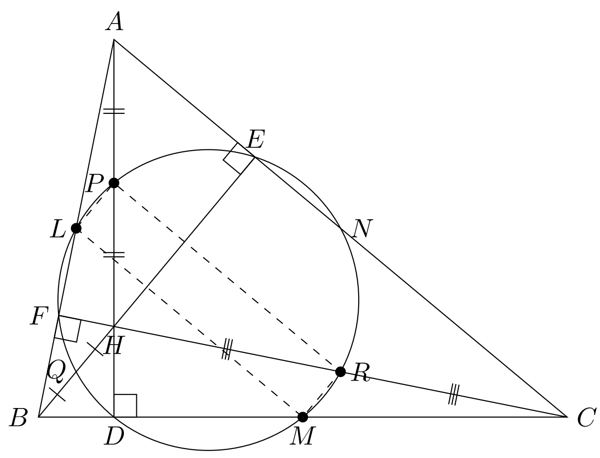
Similarly,
In $\triangle ABH$, since $AL = LB$ and $HQ = QB$, we have $LQ//AH$ (mid-pt. theorem).
In $\triangle AHC$, since $AN = NC$ and $HR = RC$, we have $NR//AH$ (mid-pt. theorem).
This means $LQ//NR$.

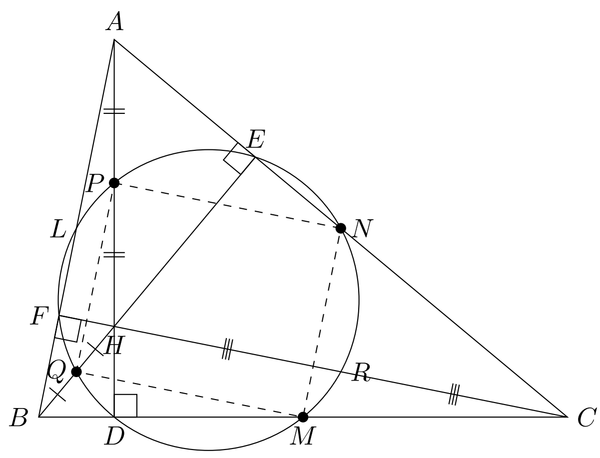
Thus, $LNRQ$ is a parallelogram.

Note that $AD \perp QR$ (corr. $\angle$s , $QR//BC$). So $LQ \perp QR$ (corr. $\angle$s , $LQ//AD$) , which means $\angle LQR = 90°$.

Thus, $LNRQ$ is a rectangle (//gram with right $\angle$) , and it is a cyclic quadrilateral (opp. $\angle$s supp.). So we can draw the circumcircle of $LNRQ$.

By similar reasoning, $LPRM$ and $PNMQ$ are rectangles as well, and they are also cyclic quadrilaterals (opp. $\angle$s supp.). Draw the circumcircle of $LPRM$ and $PNMQ$.



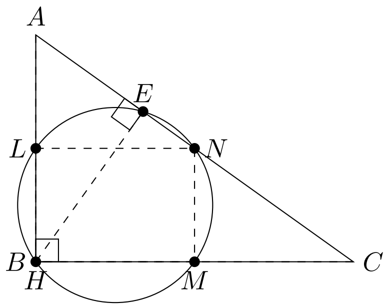


Note that both circles in the figures above have PM as the diameter (converse of $\angle$ in semi-circle). So they are in fact the same circle, which means the circle passes thorugh P, R, M, L, N, Q .

Also note that both the circumcircle of $LNRQ$ and $PNMQ$ have NQ as the diameter (converse of $\angle$ in semi-circle). So they are in fact the same circle.

So the circles in all three figures are actually the same circle, which means the circle passes thorugh $D, E, F, M, N, L, P, Q, R$.

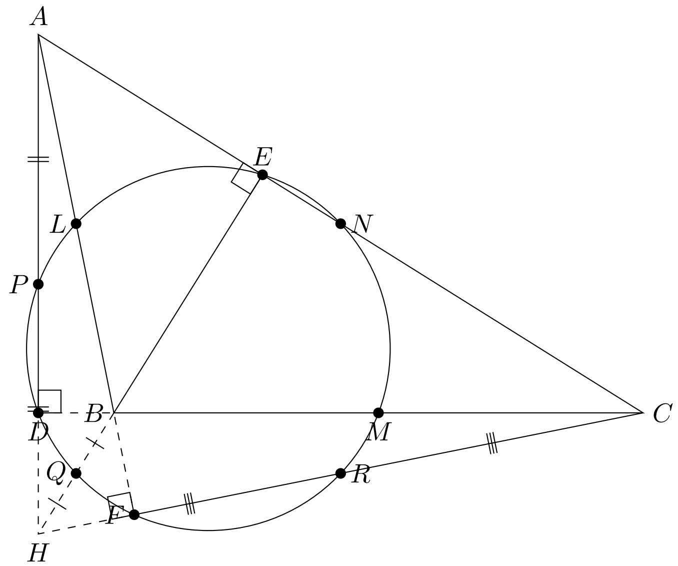
Case 2: $\angle ABC = 90°$



Then D, E, Q coincide with B, and P, L coincide, and R, M coincide.

Note that $LNMB$ is a cyclic quad. since it is a rectangle. Note that E also lies on the circumcircle of $LNMB$ since $\angle HEN = \angle BLN = 90°$ (converse of $\angle$s in the same segment). So the nine-point circle is the circumcircle of $LNMB$.

<u>Case 3:</u> $\angle ABC > 90°$



Note that P, N, R are the mid-points of AH, AC, HC respectively. And M, Q, L are the mid-points of BC, BH, BA respectively.

Note that $\angle AHC$ is an acute triangle since $\angle HAC, \angle ACH, \angle AHC$ are angles belonging to some right triangles in the figure, which means they are acute. Thus there exists a nine-point circle for $\triangle AHC$, which passes through $D, E, F, M, N, L, P, Q, R$.

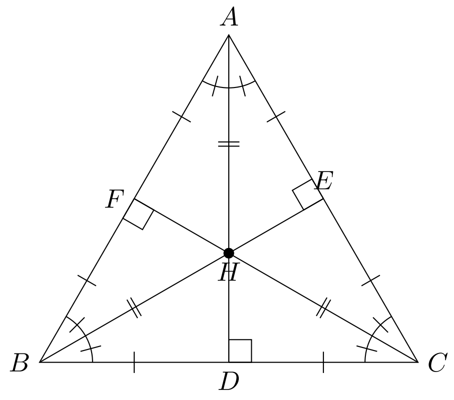
$\square$

Note: The radius of the nine-point circle is exactly half the radius of the circumcircle of $\triangle ABC$, since the nine-point circle is the circumcircle of the mid-point triangle of $\triangle ABC$, and the mid-point triangle has half the size of $\triangle ABC$.

Preposition 22. In an equilateral triangle, the incentre, circumcentre, centroid and orthocentre all lie at the same point. (centres of equil. $\triangle$)

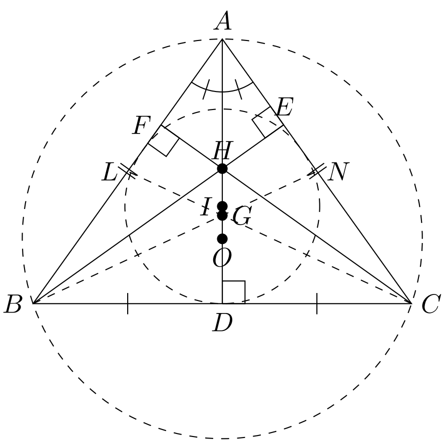
In an isosceles triangle, the incentre, circumcentre, centroid and orthocentre all lie on the perpendicular median of the triangle. (centres of isos. $\triangle$)

<u>Case 1:</u>



$\because AB = AC = BC$,
I is incentre, O is circumcentre, G is centroid, H is orthocentre.
$\therefore$ I, O, G, H lies on the same point. (centres of equil. $\triangle$)

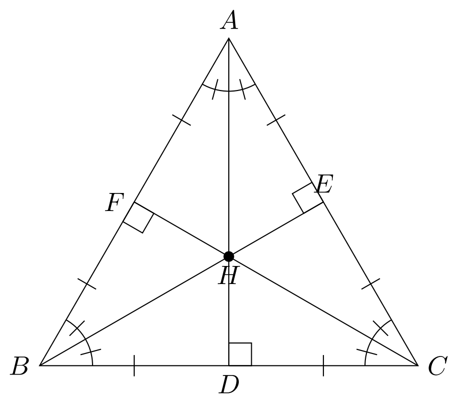
<u>Case 2:</u>



$\because AB = AC$
I is incentre, O is circumcentre, G is centroid, H is orthocentre.
$\therefore$ I, O, G, H lie on median AD . (centres of isos. $\triangle$)

Proof. <u>Case 1:</u> $AB = BC = AC$

Let $AD \perp BC$, $BE \perp AC$, $CF \perp AB$. Let H be the orthocentre of $\triangle ABC$. In other words, AD, BE, CF intersect at H .



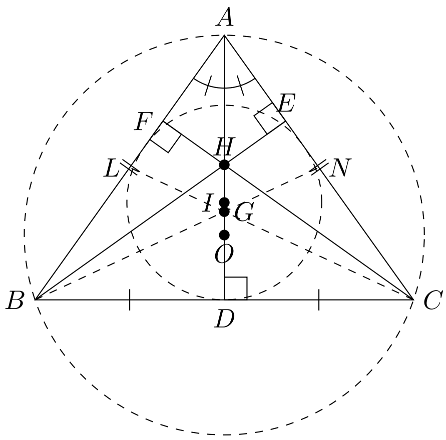
By prop. of isos. $\triangle$, we have $BD = DC$, $CE = EA$, $AF = FB$. Thus, H is also the intersection of the medians and the perpendicular bisectors of the sides.

By prop. of isos. $\triangle$, we have $\angle BAD = \angle CAD$, $\angle ABE = \angle CBE$, $\angle BCF = \angle ACF$. Thus, H is also the intersection of the angle bisectors of the triangle.

So incentre, circumcentre, centroid, orthocentre all lie on H .

<u>Case 2:</u> $AB = AC$

Let $AD \perp BC$. Then by 'prop. of isos. △', $\angle BAD = \angle CAD$ and $BD = DC$.



Then AD is an angle bisector of $\angle BAC$, and a perpendicular bisector of BC , and a median and altitude corresponding to BC .

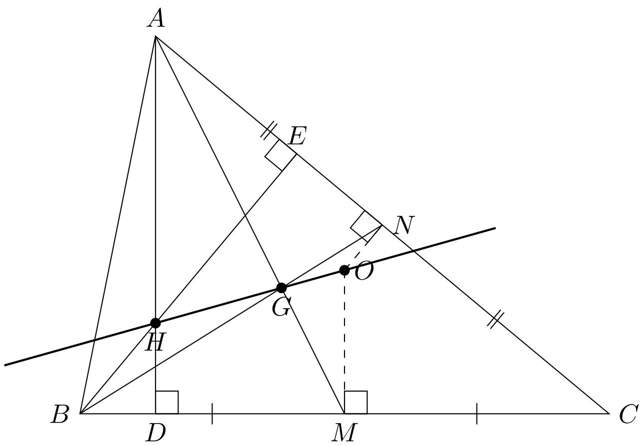
Note that by definition, incentre must lie on an angle bisector, circumcentre on perpendicular bisector, centroid on median, and orthocentre on altitude.

Thus all the centres must lie on AD .

□

Preposition 23. In a non-equilateral triangle, the circumcentre, centroid and orthocentre are collinear. (Euler line)

(Note: The line passing through these centres are called **Euler line** , and it also passes through the centre of the nine-point circle, which will be proved in the next preposition.)

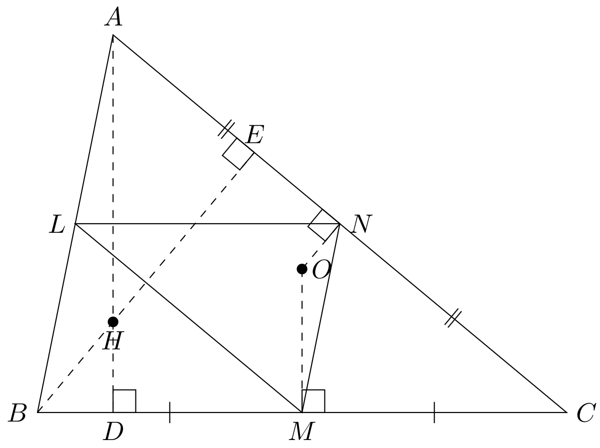


<u>Given:</u> O is circumcentre, G is centroid, H is orthocentre.
$\therefore$ O, G, H are collinear. (Euler line)

Proof. [9] Note that the triangle is non-equilateral, so there is at least one pair of non-equal sides.

Assume that $AB \neq AC$. Note that the the centroid must lie between line AD and line OM , since the centroid must lie between A, M.

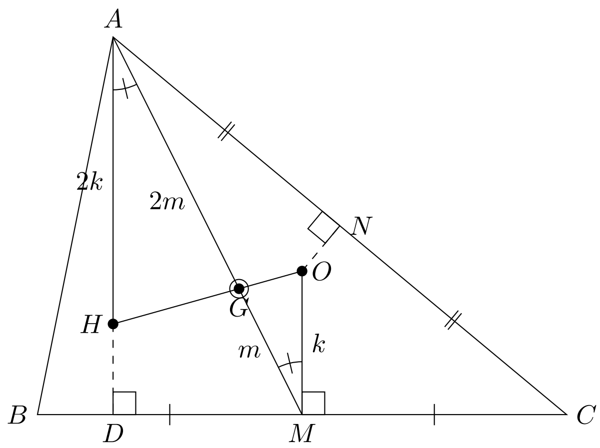
Let M, N, L be the mid-point of BC, AC, AB respectively. Join $\triangle MNL$.



Note that $\triangle MNL \sim \triangle ABC$ (prop. of mid-point $\triangle \Rightarrow$ AA) with a scale ratio $1:2$ (mid-point theorem).

Also, note that O is the orthocentre of $\triangle MNL$ (since $MO \perp LN$ and $NO \perp LM$) .

Note that the altitude segment MO in $\triangle MNL$ corresponds to the altitude segment AH in $\triangle ABC$. Thus $MO : AH = 1 : 2$. (Or we can show that $\triangle MON \sim \triangle AHB$ with $MN = \frac{1}{2}AB$, and thus $MO = \frac{1}{2}AH$ by (corr. sides, $\sim \triangle$s).)



Since AM is a median and G is the centroid, we have $GM : AG = 1 : 2$ by 'prop. of centroid' .

Moreover, note that $AD//OM$ (corr. $\angle$s equal) , so $\angle HAG = \angle OMG$ (alt. $\angle$s , $AD//OM$).

Join HG and GO . We want to show that HGO is a straight line.

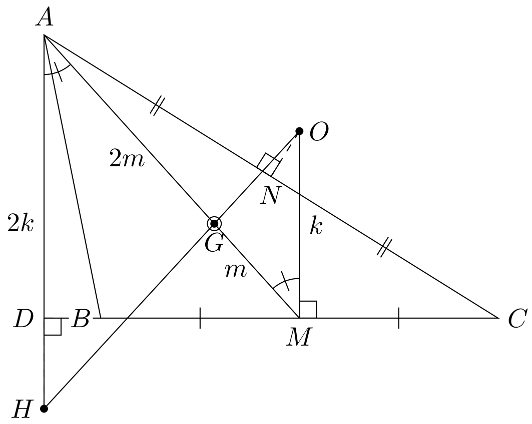
To summarize, in $\triangle OMG$ and $\triangle HAG$,

$$\begin{aligned}
\frac{OM}{AH} &= \frac{1}{2} && \text{(shown above)}\\
\angle OMG &= \angle HAG && \text{(alt. } \angle\text{s , } AD//OM)\\
\frac{GM}{AG} &= \frac{1}{2} && \text{(prop. of centroid)}\\
\therefore\ \triangle OMG &\sim \triangle HAG && \text{(ratio of 2 sides, inc. } \angle)\\
\therefore\ \angle OGM &= \angle AGH && \text{(corr. } \angle\text{s, } \sim \triangle\text{s)}
\end{aligned}$$

Thus,
$$\begin{aligned}
\angle AGH + \angle AGO &= \angle AGH + (180^\circ - \angle OGM) && \text{(adj. } \angle\text{s on st. line)}\\
&= \angle AGH + 180^\circ - \angle AGH\\
&= 180^\circ
\end{aligned}$$

Thus, HGO is a straight line (adj. $\angle$s supp.) .

Note: This proof works even when $\triangle ABC$ is an obtuse triangle:

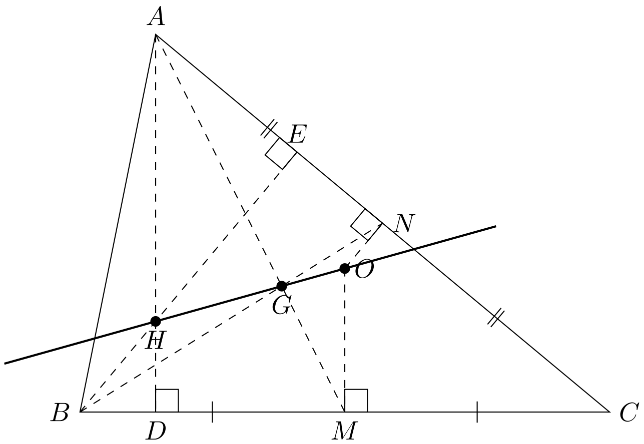


□

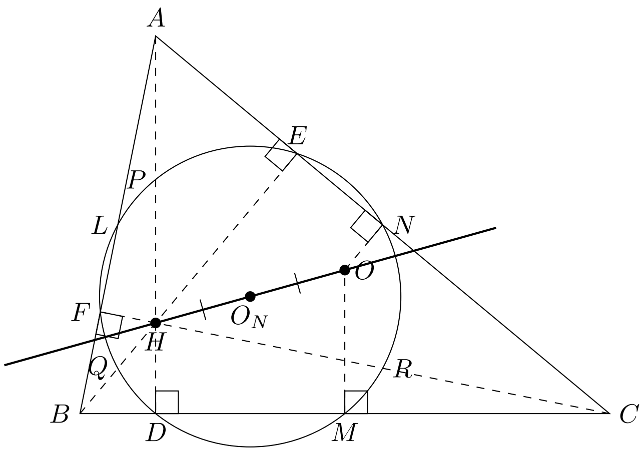
Preposition 24. In a non-equilateral triangle, the centroid lies one-third of the way from the circumcentre to the orthocentre.

Also, the centre of the nine-point circle is the mid-point of orthocentre and circumcentre.

(prop. of Euler line)

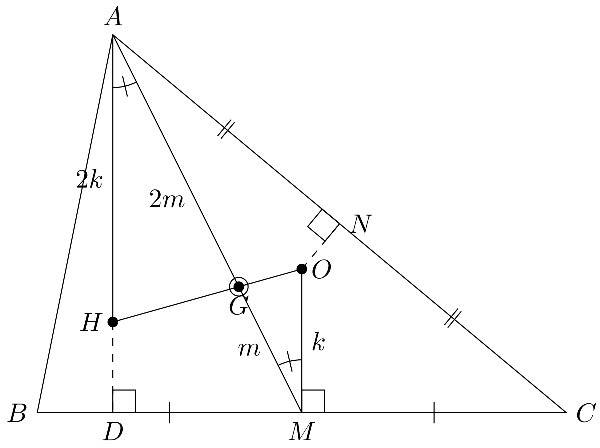


Given: O is circumcentre, G is centroid, H is orthocentre.
$\therefore\ HG = 2 \cdot OG$ (prop. of Euler line)



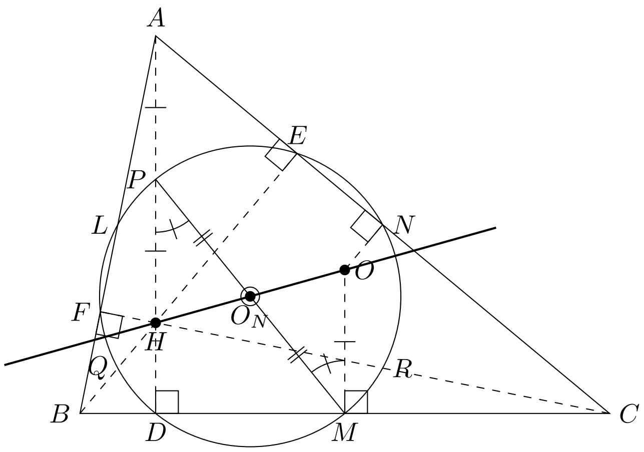
Given: O is circumcentre, O_N is centre of nine-point circle, H is orthocentre.
$\therefore\ O_N O = O_N H$ (prop. of Euler line)

Proof. Case 1:



In the proof of the Euler line, we have proven that $\triangle OMG \sim \triangle HAG$ with a scale ratio of $1:2$. So we have $OG : HG = 1:2$ (corr. sides, $\sim \triangle$s), and $\dfrac{OG}{OH} = \dfrac{1}{3}$ and $HG = 2OG$.

Case 2a: $\angle ABC \leq 90°$



Note that P is the mid-point of AH . Since $AH = 2OM$ (shown above), we have $PH = OM$.

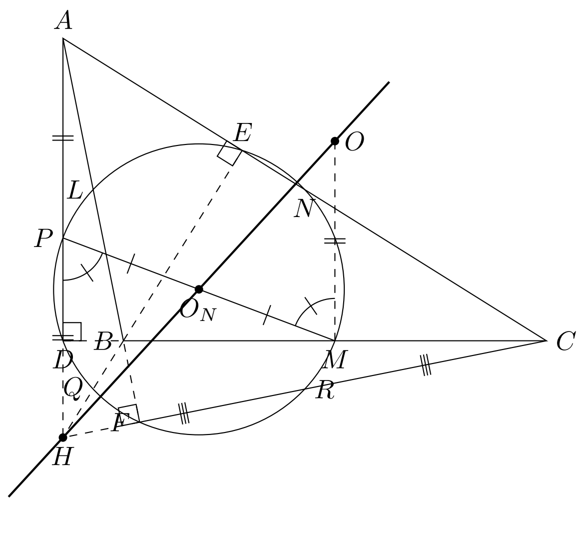
Note that $\angle PDM = 90°$, so by 'converse of $\angle$ in semi-circle', PM is the diameter of the nine-point circle and $O_NP = O_NM$ (since O_N is the centre) .

Also, note that $\angle O_NPH = \angle O_NMO$ (alt. $\angle$s , $AD//OM$).

Thus, $\triangle O_NPH \cong \triangle O_NMO$ (SAS).

Thus, $O_NH = O_NO$ (corr. sides, $\cong \triangle$s) and HO_NO is a straight line (since $\angle PO_NH = \angle OO_NM$).

Case 2b: $\angle ABC > 90°$



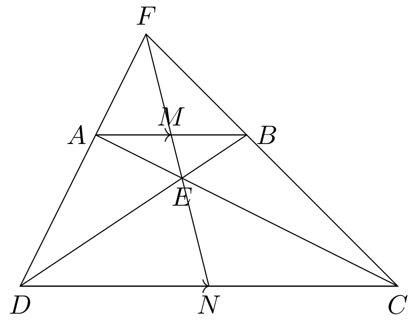
The proof is same as above. The figure just looks different.

□

1.11 Line and circle properties

1.11.1 Line properties

Preposition 25. In a proper trapezium, if a line passes through the intersection point of the diagonals and the intersection point of the extended non-parallel sides, then the line bisects the bases of the trapezium. (trapezium bisection theorem)



Given: $AB//DC$, and line FE intersects AB and DC at M and N respectively.
$\therefore$ $AM = MB$ and $DN = NC$ (trapezium bisection theorem)

Proof. By general intercept theorem, we have $\dfrac{FA}{AD} = \dfrac{FB}{BC}$, which means

$$\frac{FA}{AD} \cdot \frac{CB}{BF} = 1 \tag{1}$$

By Ceva's theorem, we have

$$\frac{FA}{AD} \cdot \frac{DN}{NC} \cdot \frac{CB}{BF} = 1 \tag{2}$$

Put (1) into (2):

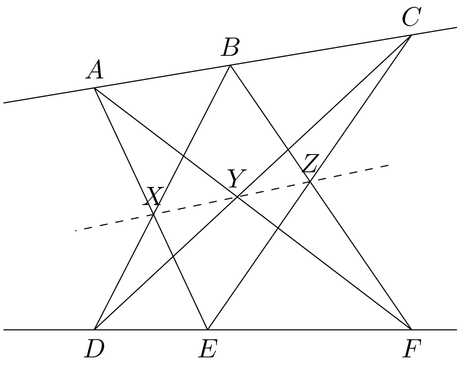
$$1 \cdot \frac{DN}{NC} = 1$$
$$DN = NC$$

Note that $\triangle FAM \sim FDN$ and $\triangle FMB \sim FNC$ (AA).
Thus $\dfrac{AM}{DN} = \dfrac{FM}{FN}$ and $\dfrac{MB}{NC} = \dfrac{FM}{FN}$ (corr. sides, $\sim \triangle$s). So

$$\frac{AM}{DN} = \frac{MB}{NC}$$
$$\frac{AM}{DN} = \frac{MB}{DN} \qquad (DN = NC)$$
$$AM = MB$$

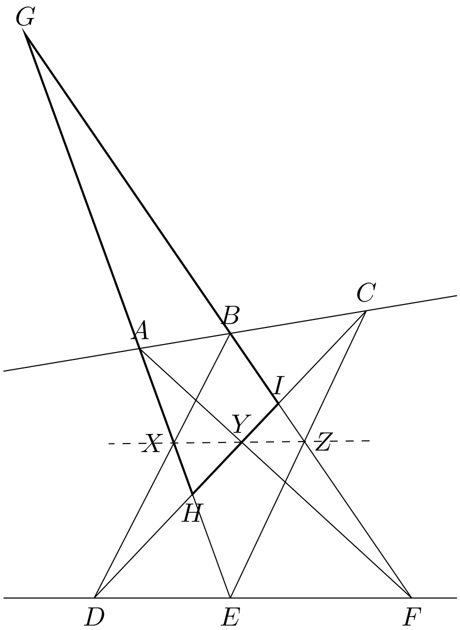
□

Preposition 26. Let A, B, C be three points on a line, and let D, E, F be three points on another line. If the lines AE intersect BD at X , AF intersect CD at Y , and BF intersect CE at Z , then the three points X, Y, Z are collinear. (Pappus' hexagon theorem)



Proof. [10] Assume the configuration of points is as depicted in the figure, and AE is not parallel to BF.

Let G be the intersection of line EA and line FB , H be intersection of AE and CD , I be intersection of BF and CD .



Apply Menelaus' theorem on $\triangle GHI$ and its five transversal lines: DXB, AYF, CZE, ABC and DEF:

Transversal	Menelau's theorem
DXB	$\frac{HX}{GX} \cdot \frac{ID}{HD} \cdot \frac{GB}{IB} = 1$
AYF	$\frac{HA}{GA} \cdot \frac{IY}{HY} \cdot \frac{GF}{IF} = 1$
CZE	$\frac{HE}{GE} \cdot \frac{IC}{HC} \cdot \frac{GZ}{IZ} = 1$
ABC	$\frac{GA}{HA} \cdot \frac{HC}{IC} \cdot \frac{IB}{GB} = 1$
DEF	$\frac{GE}{HE} \cdot \frac{HD}{ID} \cdot \frac{IF}{GF} = 1$

Multiply the five equations together:

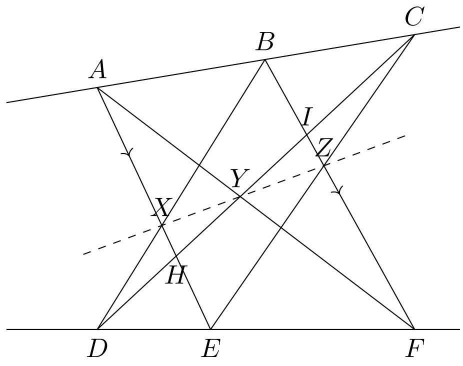
$$(\frac{HX}{GX} \cdot \frac{ID}{HD} \cdot \frac{GB}{IB}) \cdot (\frac{HA}{GA} \cdot \frac{IY}{HY} \cdot \frac{GF}{IF}) \cdot (\frac{HE}{GE} \cdot \frac{IC}{HC} \cdot \frac{GZ}{IZ}) \cdot (\frac{GA}{HA} \cdot \frac{HC}{IC} \cdot \frac{IB}{GB}) \cdot (\frac{GE}{HE} \cdot \frac{HD}{ID} \cdot \frac{IF}{GF}) = 1$$

Cancelling out like terms (fun activity to do it yourself), we get:

$$\frac{HX}{GX} \cdot \frac{IY}{HY} \cdot \frac{GZ}{IZ} = 1$$

By converse of Menelaus' theorem, X, Y, Z lies on a straight line.

Now, suppose that $AE//BF$.



Then we have these five pairs of similar triangles (by AA), and by (corr. sides, $\sim \triangle$s) rearranged:

Similar triangles	Proportions
$\triangle HDX \sim \triangle IDB$	$\frac{HX}{HD} \cdot \frac{ID}{IB} = 1$
$\triangle HAY \sim \triangle IFY$	$\frac{HA}{HY} \cdot \frac{IY}{IF} = 1$
$\triangle HEC \sim \triangle IZC$	$\frac{HE}{HC} \cdot \frac{IC}{IZ} = 1$
$\triangle HCA \sim \triangle ICB$	$\frac{HC}{HA} \cdot \frac{IB}{IC} = 1$
$\triangle HDE \sim \triangle IDF$	$\frac{HD}{HE} \cdot \frac{IF}{ID} = 1$

Multiply the five equations together:

$$(\frac{HX}{HD} \cdot \frac{ID}{IB}) \cdot (\frac{HA}{HY} \cdot \frac{IY}{IF}) \cdot (\frac{HE}{HC} \cdot \frac{IC}{IZ}) \cdot (\frac{HC}{HA} \cdot \frac{IB}{IC}) \cdot (\frac{HD}{HE} \cdot \frac{IF}{ID}) = 1$$

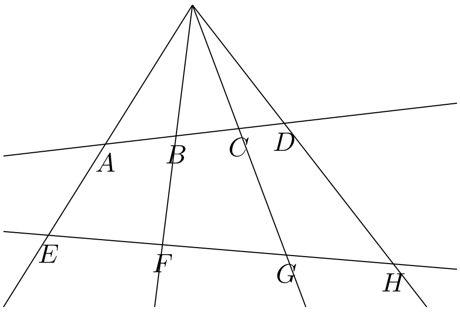
Simplifying, we get:

$$\frac{HX}{HY} \cdot \frac{IY}{IZ} = 1$$

$$\frac{HX}{HY} = \frac{IZ}{IY}$$

which means $\triangle HXY \sim IZY$ (ratio of two sides, inc. $\angle$). Thus, $\angle HYX = \angle IYZ$ (corr. $\angle$s, $\sim \triangle$s) so XYZ is a straight line (adj. $\angle$s supp.) . $\square$

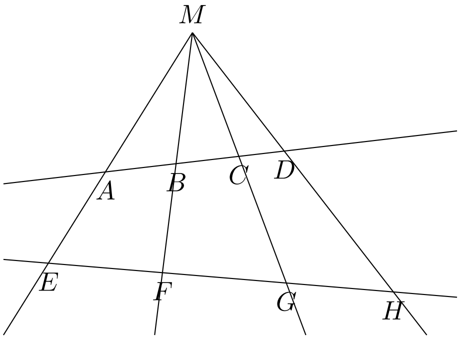
Preposition 27. Given four concurrent rays (/lines), the cross-ratio of the intercepts made by a transversal line is unchanged for any transversal lines. (cross-ratio invariance) [11]



$$\frac{AC \cdot BD}{BC \cdot AD} = \frac{EG \cdot FH}{FG \cdot EH} \qquad \text{(cross-ratio invariance)}$$

Note: $\dfrac{AC \cdot BD}{BC \cdot AD}$ is defined to be the **cross-ratio** of the four collinear points A, B, C, D . (This is the unsigned version of cross-ratio where all the values are positive.)

Proof. Let M be the point of concurrency of the four rays.



Let $[\triangle MAC]$ denote the area of $\triangle MAC$. Other triangle areas are denoted similarly.

By 'bases prop. to areas of $\triangle$s' , we have $\dfrac{AC}{BC} = \dfrac{[\triangle MAC]}{[\triangle MBC]}$ and $\dfrac{BD}{AD} = \dfrac{[\triangle MBD]}{[\triangle MAD]}$.

Multiply the two equations together:

$$\frac{AC \cdot BD}{BC \cdot AD} = \frac{[\triangle MAC] \cdot [\triangle MBD]}{[\triangle MBC] \cdot [\triangle MAD]} \tag{1}$$

Similarly, we have

$$\frac{EG \cdot FH}{FG \cdot EH} = \frac{[\triangle MEG] \cdot [\triangle MFH]}{[\triangle MFG] \cdot [\triangle MEH]} \tag{2}$$

By 'areas of $\triangle$s with common $\angle$' , we have $\dfrac{AM \cdot CM}{EM \cdot GM} = \dfrac{[\triangle MAC]}{[\triangle MEG]}$. Similarly, we have $\dfrac{BM \cdot DM}{FM \cdot HM} = \dfrac{[\triangle MBD]}{[\triangle MFH]}$, $\dfrac{BM \cdot CM}{FM \cdot GM} = \dfrac{[\triangle MBC]}{[\triangle MFG]}$, $\dfrac{AM \cdot DM}{EM \cdot HM} = \dfrac{[\triangle MAD]}{[\triangle MEH]}$.

Note that we have the identity (in which everything cancels out):

$$(\frac{AM \cdot CM}{EM \cdot GM}) \cdot (\frac{BM \cdot DM}{FM \cdot HM}) = (\frac{BM \cdot CM}{FM \cdot GM}) \cdot (\frac{AM \cdot DM}{EM \cdot HM})$$

Replace the ratio in each bracket with the ratio of areas:

$$(\frac{[\triangle MAC]}{[\triangle MEG]}) \cdot (\frac{[\triangle MBD]}{[\triangle MFH]}) = (\frac{[\triangle MBC]}{[\triangle MFG]}) \cdot (\frac{[\triangle MAD]}{[\triangle MEH]})$$

Rearranging:

$$\frac{[\triangle MAC] \cdot [\triangle MBD]}{[\triangle MBC] \cdot [\triangle MAD]} = \frac{[\triangle MEG] \cdot [\triangle MFH]}{[\triangle MFG] \cdot [\triangle MEH]}$$

By (1) and (2), we get:

$$\frac{AC \cdot BD}{BC \cdot AD} = \frac{EG \cdot FH}{FG \cdot EH}$$

□

1.11.2 Power of a point

The **power of a point** is a real number that reflects the relative distance of a given point from a given circle. [12]

If we are given a point P and a circle ω with centre O and radius r , then the power of P with respect to ω is defined by

$$\text{pow}(P, \omega) = OP^2 - r^2$$

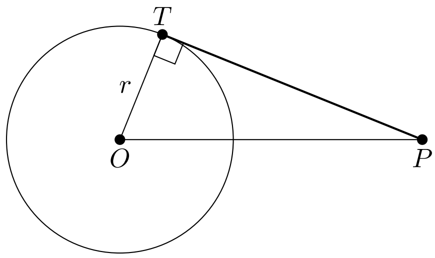
(Note that 'pow' is a function that takes in a point in the first argument and a circle in the second argument, and outputs a real number which is the power of that point. If there is only one circle involved, then the second argument can be omitted.)

This implies that

- points inside the circle have negative power;
- points on the circle have zero power;
- points outside the circle have positive power.

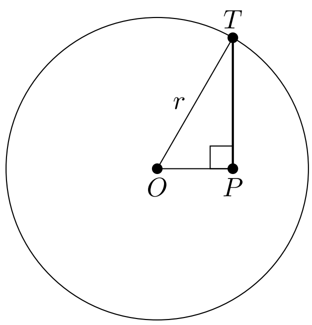
Geometric interpretations

Due to the Pythagoras theorem, the power of a point outside a circle is equal to the squared length of the tangent from the point to the circle:



$$\text{pow}(P) = OP^2 - r^2 = PT^2$$

The power of a point inside a circle is the negative of squared length of half the chord perpendicular to the point:



$$\text{pow}(P) = OP^2 - r^2 = -(r^2 - OP^2) = -PT^2$$

Using power of a point, the three theorems: intersecting chords theorem, tangent-secant theorem, and intersecting secants theorem, can be unified to a single theorem called power of a point theorem:

Preposition 28. (i) The power of a point outside a circle is equal to

- squared length of the tangent from the point to the circle.

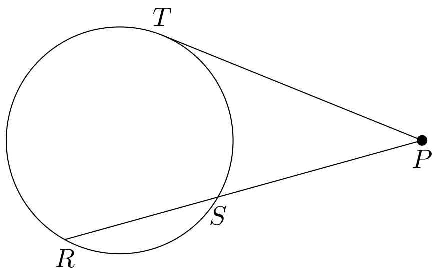
- product of the whole length of secant from the point and the length of external part of the secant.

(ii) The power of a point inside a circle is equal to the negative of the product of two chord segments separated by the point.

(iii) The power of a point on the circumference of a circle is zero.

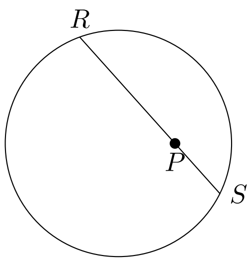
(power of a point theorem)

<u>Case 1:</u> (T is the point of tangency.)



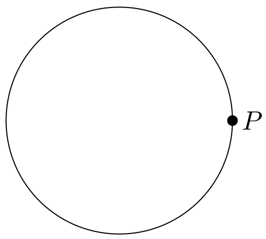
$$\text{pow}(P) = PT^2 = PR \cdot PS$$

<u>Case 2:</u>



$$\text{pow}(P) = -PR \cdot PS$$

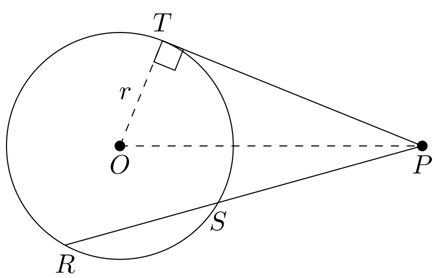
<u>Case 3:</u>



$$\text{pow}(P) = 0$$

Proof. <u>Case 1:</u> P is outside the circle.

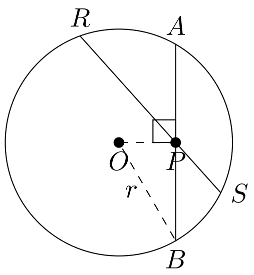
Let O be the centre of the circle. Note that $OT \perp PT$ (tangent $\perp$ radius). Join OP .



By tangent-secant theorem, we have $PT^2 = PR \cdot PS$.
By pyth. theorem in $\triangle OTP$, we have $PT^2 = OP^2 - OT^2 = \text{pow}(P)$.
Thus, $\text{pow}(P) = PT^2 = PR \cdot PS$

Case 2: P is inside the circle.

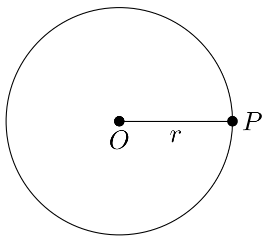
Let AB be a chord such that $OP \perp AB$. Join OP and OB .



By intersecting chords theorem, we have $PA \cdot PB = PR \cdot PS$.
By 'line from centre $\perp$ chord bisects chord' , we have $PA = PB$. Thus, $PB^2 = PR \cdot PS$.
By pyth. theorem in $\triangle OBP$, we have $PB^2 = OB^2 - OP^2 = -(OP^2 - OB^2) = -\text{pow}(P)$.
Thus, $\text{pow}(P) = -PB^2 = -PR \cdot PS$.

(Note: If P lies on O , then take OP to be 0.)

Case 3: P is on the circumference of the circle.
(Let r be the radius of the circle.)



Since $OP = r$, by definition of power of a point,

$$\text{pow}(P) = OP^2 - r^2 = 0$$

$\square$

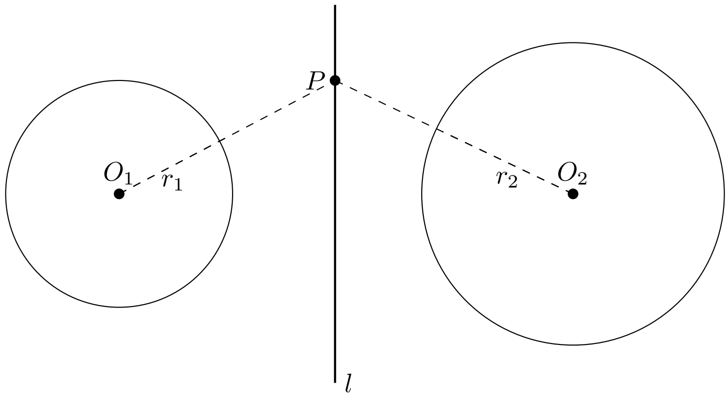
1.11.3 Radical axis

The **radical axis** is a unique line that exists when given two non-concentric [1] circles . Any point P on the radical axis has equal power with respect to the two circles involved.

Mathematically, given two non-concentric circles ω_1 and ω_2 , the radical axis l of ω_1 and ω_2 is defined to be the set of all points P such that $\text{pow}(P, \omega_1) = \text{pow}(P, \omega_2)$.

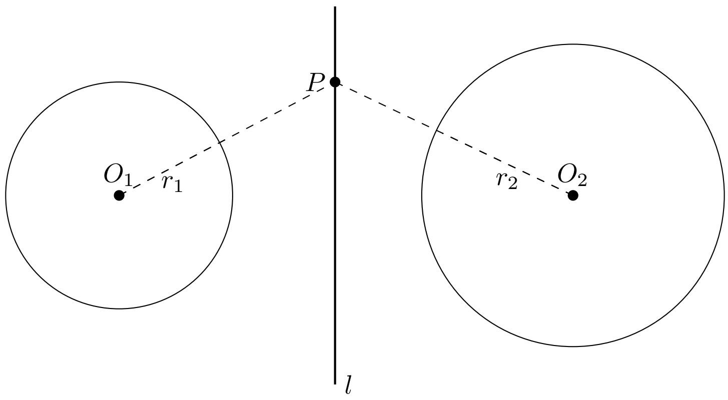
This means if ω_1 has centre O_1 and radius r_1 , and ω_2 has centre O_2 and radius r_2 , then for any point P on l, we have

$$O_1P^2 - r_1^2 = O_2P^2 - r_2^2$$



Note that the radical axis exists for any two non-concentric circles, and it can only be a line.

Preposition 29. Given two non-concentric circles ω_1 and ω_2 , there exists a unique line l such that for all point P on l , we have $\text{pow}(P, \omega_1) = \text{pow}(P, \omega_2)$. (radical axis)



$$O_1P^2 - r_1^2 = O_2P^2 - r_2^2 \qquad \text{(radical axis)}$$

Proof. Join O_1O_2 . Let O_1O_2 be placed horizontally. Let $d = O_1O_2$.

Assume that P be a point such that $O_1P^2 - r_1^2 = O_2P^2 - r_2^2$. Let x be the horizontal displacement [2] from O_1 , where $x > 0$ if P is at the right of O_1 , and $x < 0$ if P is at the left of O_1 .

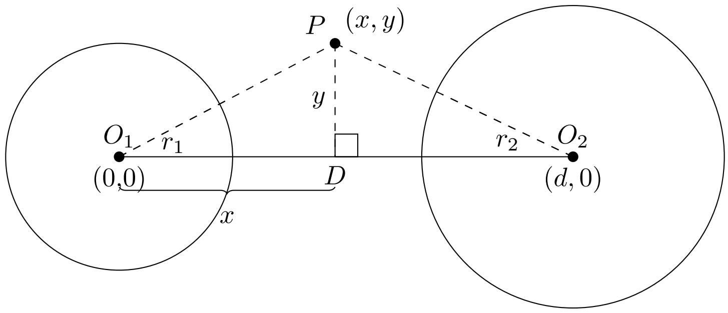
Similarly, let y be the vertical displacement from O_1 .

(This is essentially a Cartesian coordinate system, which I will introduce in later sections.)

Let D be a point on O_1O_2 such that $PD \perp O_1O_2$. Note that $|O_1D| = x$ and $O_2D = |d - x|$.

[1] Concentric means two circles share the same centre. Non-concentric means two circles do not share the same centre.

[2] Displacement is different from distance. Displacement is a vector instead of a magnitude (scalar), so it is allowed to be negative.



Starting from initial assumption, by pyth. theorem, we have (this holds no matter where P is relative to O_1O_2):

$$\begin{aligned} O_1P^2 - r_1^2 &= O_2P^2 - r_2^2 \\ (x^2 + y^2) - r_1^2 &= (d - x)^2 + y^2 - r_2 \\ &= d^2 - 2dx + x^2 + y^2 - r^2 \\ 2dx &= d^2 + r_1^2 - r_2^2 \\ x &= \frac{d^2 + r_1^2 - r_2^2}{2d} \end{aligned}$$

Note that $d > 0$ since the circles are non-concentric. Since d, r_1, r_2 all have specific values and are independent variables when given the two circles, there exists a unique horizontal displacement x and a unique point D .

And since y cancels out, the vertical displacement of P does not matter. So point P must be on a vertical line with horizontal displacement $\dfrac{d^2 + r_1^2 - r_2^2}{2d}$, and it is be necessarily unique since it passes through a unique point D .

$\square$

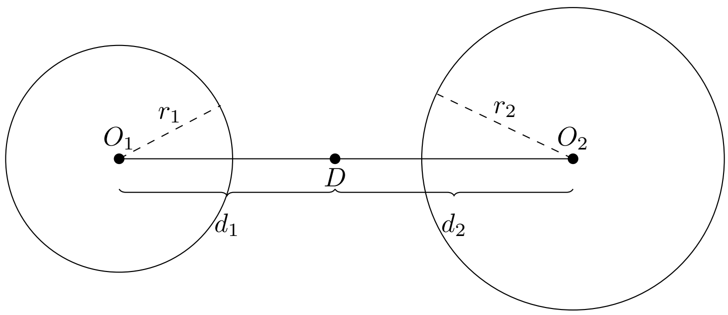
(If you think the proof is 'cheaty' because it uses a coordinate system, then try the proof below. You may skip it because it is mostly repetitive information.)

Proof. Assume that D is a point on line O_1O_2 such that $O_1D^2 - r_1^2 = O_2D^2 - r_2^2$.

Let $d = O_1O_2$, $d_1 = O_1D$ and $d_2 = O_2D$.

Let's consider several cases:

<u>Case 1:</u> D is between O_1 and O_2 .



Note that $d_1 + d_2 = d$. So $d_2 = d - d_1$ and $d_1 = d - d_2$.

By assumption, we have

$$\begin{aligned} d_1^2 - r_1^2 &= d_2^2 - r_2^2 \\ &= (d - d_1)^2 - r_2^2 \\ &= d^2 - 2dd_1 + d_1^2 - r_2^2 \\ 2dd_1 &= d^2 + r_1^2 - r_2^2 \\ d_1 &= \frac{d^2 + r_1^2 - r_2^2}{2d} \end{aligned}$$

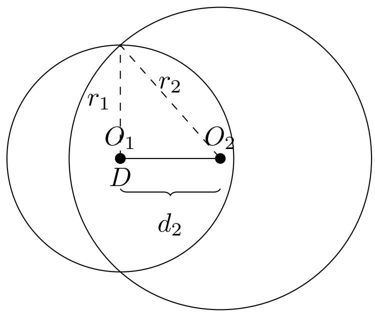
Similarly,

$$\begin{aligned}
d_2^2 - r_2^2 &= d_1^2 - r_1^2 \\
&= (d - d_2)^2 - r_1^2 \\
&= d^2 - 2dd_2 + d_2^2 - r_1^2 \\
2dd_2 &= d^2 + r_2^2 - r_1^2 \\
d_2 &= \frac{d^2 + r_2^2 - r_1^2}{2d}
\end{aligned}$$

Note that $d > 0$ since the circles are non-concentric. Since d, r_1, r_2 all have specific values and are independent variables when given the two circles, there exists a unique d_1 and d_2 , which means the position of D is also unique.

So when $d^2 + r_1^2 > r_2^2$, we have $d_1 > 0$, and when $d^2 + r_2^2 > r_1^2$, we have $d_2 > 0$. These requirements are satisfied when $d + r_1 > r_2$ and $d + r_2 > r_1$, which is possible since they are just triangle inequalities.

<u>Case 2a:</u> D is on O_1 .



l

Note that $d_1 = 0$ and $d_2 = d$. So we still have $d_2 = d - d_1$ and $d_1 = d - d_2$.

Like Case 1, we have:

$$d_1 = \frac{d^2 + r_1^2 - r_2^2}{2d}$$

And,

$$d_2 = \frac{d^2 + r_2^2 - r_1^2}{2d}$$

When $d^2 + r_1^2 = r_2^2$, we have $d_1 = 0$, and when $d^2 = r_2^2 - r_1^2$, we have $d_2 = d$. This is possible as long as $r_2 > r_1$ and $r_2 > d$.

<u>Case 2b:</u> D is on O_2 .

Note that $d_1 = d$ and $d_2 = 0$. So we still have $d_2 = d - d_1$ and $d_1 = d - d_2$.

Like Case 1, we have:

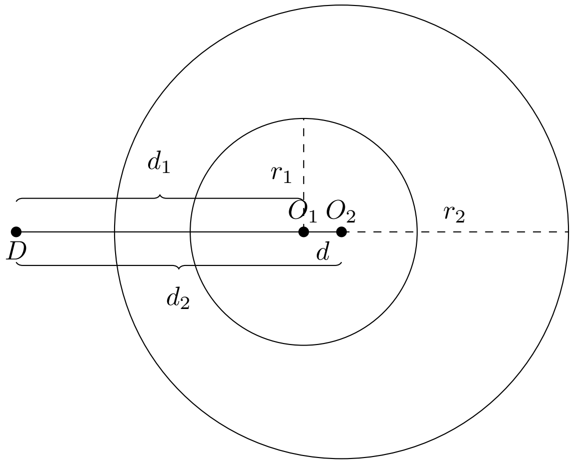
$$d_1 = \frac{d^2 + r_1^2 - r_2^2}{2d}$$

And,

$$d_2 = \frac{d^2 + r_2^2 - r_1^2}{2d}$$

When $d^2 = r_1^2 - r_2^2$, we have $d_1 = d$, and when $d^2 + r_2^2 = r_1^2$, we have $d_2 = 0$. This is possible as long as $r_1 > d$ and $r_1 > r_2$.

<u>Case 3a:</u> D is at the left of O_1 .



Note that $d_1 + d = d_2$. So $d_1 = d_2 - d$.
By assumption, we have

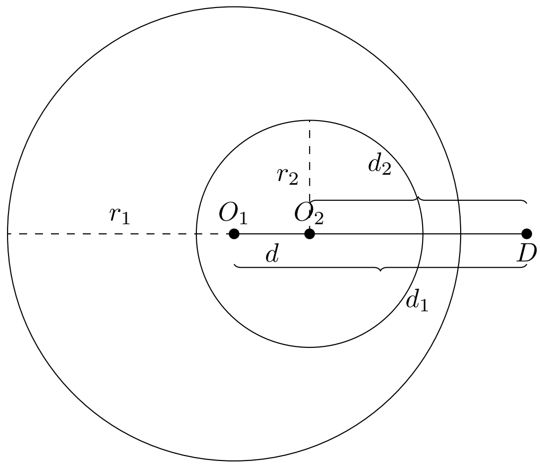
$$\begin{aligned}
d_1^2 - r_1^2 &= d_2^2 - r_2^2 \\
&= (d + d_1)^2 - r_2^2 \\
&= d^2 + 2dd_1 + d_1^2 - r_2^2 \\
-2dd_1 &= d^2 + r_1^2 - r_2^2 \\
d_1 &= \frac{d^2 + r_1^2 - r_2^2}{-2d} \\
&= \frac{r_2^2 - d^2 - r_1^2}{2d}
\end{aligned}$$

Similarly,

$$\begin{aligned}
d_2^2 - r_2^2 &= d_1^2 - r_1^2 \\
&= (d_2 - d)^2 - r_1^2 \\
&= d_2^2 - 2dd_2 + d^2 - r_1^2 \\
2dd_2 &= d^2 + r_2^2 - r_1^2 \\
d_2 &= \frac{d^2 + r_2^2 - r_1^2}{2d}
\end{aligned}$$

So when $r_2^2 > d^2 + r_1^2$, we have $d_1 > 0$, which also means $d_2 > 0$. This holds if $r_2 > d + r_1$.

<u>Case 3b:</u> D is at the right of O_2 .



Note that $d_2 + d = d_1$. So $d_2 = d_1 - d$.

By assumption, we have

$$\begin{aligned}
d_1^2 - r_1^2 &= d_2^2 - r_2^2 \\
&= (d_1 - d)^2 - r_2^2 \\
&= d_1^2 - 2dd_1 + d^2 - r_2^2 \\
2dd_1 &= d^2 + r_1^2 - r_2^2 \\
d_1 &= \frac{d^2 + r_1^2 - r_2^2}{2d}
\end{aligned}$$

Similarly,

$$\begin{aligned}
d_2^2 - r_2^2 &= d_1^2 - r_1^2 \\
&= (d_2 + d)^2 - r_1^2 \\
&= d_2^2 + 2dd_2 + d^2 - r_1^2 \\
-2dd_2 &= d^2 + r_2^2 - r_1^2 \\
d_2 &= \frac{d^2 + r_2^2 - r_1^2}{-2d} \\
&= \frac{r_1^2 - d^2 - r_2^2}{2d}
\end{aligned}$$

So when $r_1^2 > d^2 + r_2^2$, we have $d_2 > 0$, which also means $d_1 > 0$. This holds if $r_1 > d + r_2$.

Thus, it is possible for D to be everywhere on O_1O_2 .

Lastly, note that for any positive d, r_1, r_2 , by law of trichotomy, either one of the following must hold:

1. $d^2 + r_1^2 > r_2^2$ and $d^2 + r_2^2 > r_1^2$
2. $d^2 + r_1^2 = r_2^2$
3. $d^2 + r_2^2 = r_1^2$
4. $d^2 + r_1^2 < r_2^2$ and $d_2 + r_2^2 > r_1^2$
5. $d^2 + r_1^2 > r_2^2$ and $d^2 + r_2^2 < r_1^2$

As, note that either one can be true:

1. $d^2 + r_1^2 > r_2^2$ and $d^2 + r_2^2 > r_1^2$

3. $d^2 + r_1^2 > r_2^2$ and $(\Leftarrow)$ $d^2 + r_2^2 = r_1^2$

5. $d^2 + r_1^2 > r_2^2$ and $d^2 + r_2^2 < r_1^2$

2. $d^2 + r_1^2 = r_2^2 \;\Rightarrow d^2 + r_2^2 > r_1^2$

4. $d^2 + r_1^2 < r_2^2 \;\Rightarrow d^2 + r_2^2 > r_1^2$

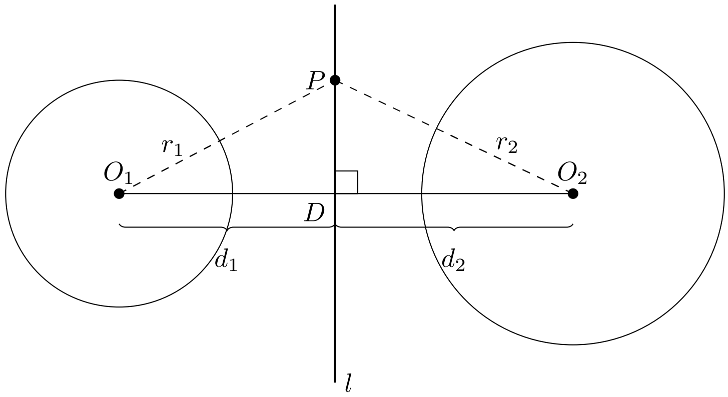
The logic is that if $d^2 + r_1^2 > r_2^2$, then either $d^2 + r_2^2 > r_1^2$ or $d^2 + r_2^2 = r_1^2$ or $d^2 + r_2^2 < r_1^2$. The first case satisfies (1), second case satisfies (3), third case satisfies (5).
If $d^2 + r_1^2 = r_2^2$, then $r_2^2 > r_1^2$, so $d^2 + r_2^2 > r_1^2$, so (2) is satisfied.
If $d^2 + r_1^2 < r_2^2$, then $r_2^2 > r_1^2$, so $d^2 + r_2^2 > r_1^2$, so (4) is satisfied.

Furthermore, note that (1), (2), (3), (4), (5) are mutually exclusive (meaning exactly one can be true). In particular, note that if both (2) and (3) were true, then we have $r_1^2 - r_2^2 = r_2^2 - r_1^2$, which means $r_1^2 = r_2^2$, which means $d = 0$, which contradicts $d > 0$.

This means each condition corresponds to each case. So every choice of d, r_1, r_2 falls into one of the five cases.

<u>Next step</u>

Draw line l through D such that $l \perp O_1O_2$.
Let P be a point on l .



Then by pyth. theorem (this holds no matter where D is relative to O_1O_2),

$$O_1P^2 = d_1^2 + PD^2 \tag{1}$$
$$O_2P^2 = d_2^2 + PD^2 \tag{2}$$

(1) − (2):

$$O_1P^2 - O_2P^2 = d_1^2 - d_2^2$$

Note that $d_1^2 - d_2^2 = r_1^2 - r_2^2$ (assumption rearranged). So

$$\begin{aligned} O_1P^2 - O_2P^2 &= r_1^2 - r_2^2 \\ O_1P^2 - r_1^2 &= O_2P^2 - r_2^2 \\ \text{pow}(P, \omega_1) &= \text{pow}(P, \omega_2) \end{aligned}$$

Thus, whenever P is vertically above/below D , we have $\text{pow}(P, \omega_1) = \text{pow}(P, \omega_2)$. Thus, every point P on l has this property.

To show that l is a unique line with this property, assume that Q is a point such that $O_1Q^2 - r_1^2 = O_2Q^2 - r_2^2$. Let E be a point on O_1O_2 such that $QE \perp O_1O_2$, and let $e_1 = O_1E$, $e_2 = O_2E$.

Then by pyth. theorem (this holds no matter where Q is relative to O_1O_2),

$$O_1Q^2 = e_1^2 + QE^2 \tag{3}$$
$$O_2Q^2 = e_2^2 + QE^2 \tag{4}$$

(3) − (4):

$$O_1Q^2 - O_2Q^2 = e_1^2 - e_2^2$$
$$O_1Q^2 - e_1^2 = O_2Q^2 - e_2^2 \tag{5}$$

Since there is a unique point E on O_1O_2 that can satisfy equation (5), and that point is D , we must have $E = D$ and $e_1 = d_1$ and $e_2 = d_2$. Since $QE \perp O_1O_2$, Q must also be on l by prop. of $\perp$ line.

Thus, a point P is on this perpendicular line l if and only if $\text{pow}(P, \omega_1) = \text{pow}(P, \omega_2)$. □

Preposition 30. (i) The radical axis of two circles are perpendicular to the line joining their centres.

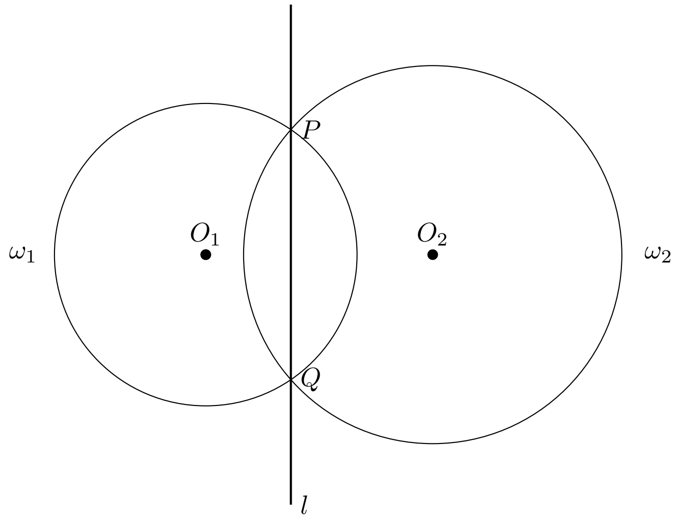
(ii) The radical axis of two intersecting circles is the line joining the points of intersection.

(iii) The radical axis of two touching circles is their common tangent.

(prop. of radical axis)

Proof. (i) Shown in the previous proof.

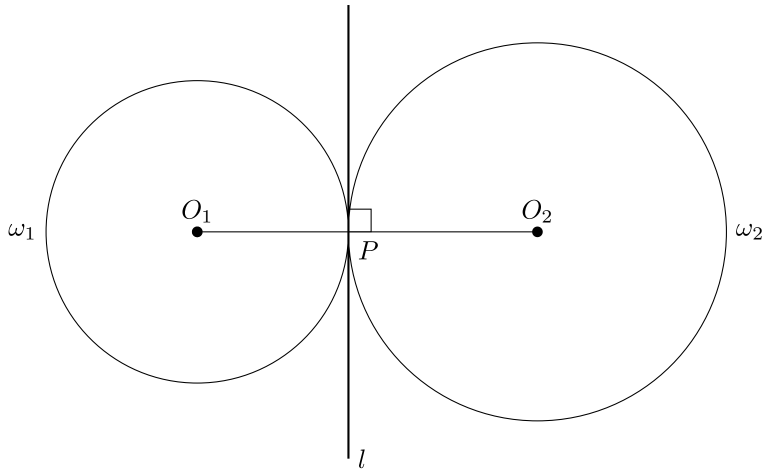
(ii) Let P , Q be the intersections of the two circles ω_1 and ω_2 .



Since P is on the circumference of both circles, we have $\text{pow}(P, \omega_1) = \text{pow}(P, \omega_2) = 0$. Similarly, $\text{pow}(Q, \omega_1) = \text{pow}(Q, \omega_2) = 0$. Thus, both P and Q are on the radical axis.

Since the radical axis of two circles is unique, the radical axis must be the line that passes through P and Q .

(iii) Let P be the point of tangency.

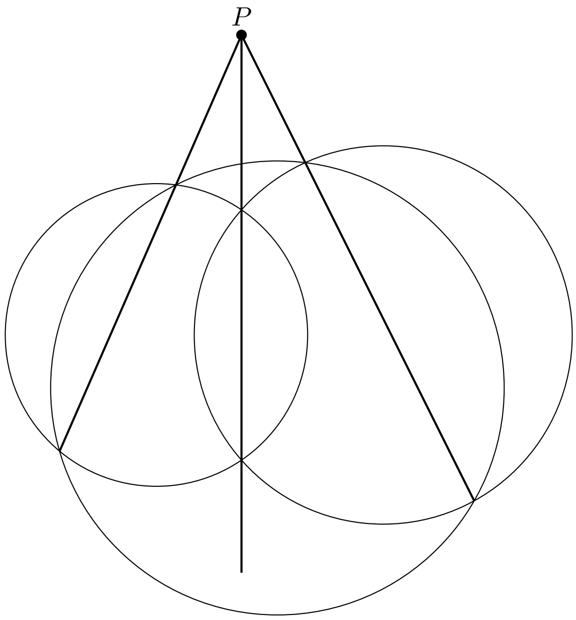


Since P is on the circumference of both circles, we have $\text{pow}(P, \omega_1) = \text{pow}(P, \omega_2) = 0$. Thus, P is on the radical axis.

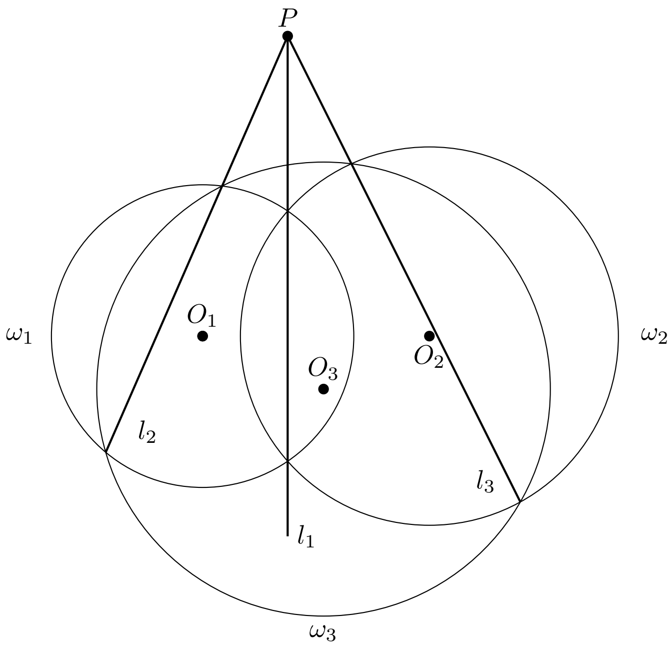
Note that the radical axis is perpendicular to O_1O_2 by property (i) of this preposition. Since $l \perp O_1T$ and $l \perp O_2T$, l is the tangent line of both ω_1 and ω_2 (converse of tangent $\perp$ radius).

$\square$

Preposition 31. Given three non-concentric circles with non-collinear centres, the radical axes ('axises') of each pair of circles are concurrent. (radical axis theorem)



Proof. [13] Label the three circles $\omega_1, \omega_2, \omega_3$.



Let the radical axes of (ω_1, ω_2) and (ω_2, ω_3) intersect at P .

Note that $\text{pow}(P, \omega_1) = \text{pow}(P, \omega_2)$ and $\text{pow}(P, \omega_2) = \text{pow}(P, \omega_3)$. By transitivity of equality, we have $\text{pow}(P, \omega_1) = \text{pow}(P, \omega_3)$.

Since a point lies on the radical axis if and only if it has equal power with respect to the two circles, P lies on the radical axis of (ω_1, ω_3) as well.

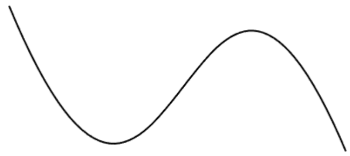
Thus, the three radical axes are concurrent.

□

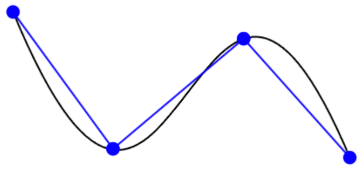
1.12 Area and circumference of circle

So far, we have avoided talking about the area and perimeter (called circumference) of a circle since it is tricky.

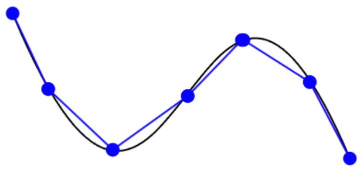
First, how do we measure the length of a smooth curve segment? (Smooth means there is no discontinuity and no jagged corners.)



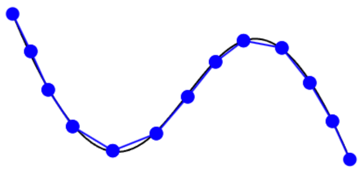
We may make some equally spaced points on the curve segment and connect them with line segments (excuse me with the blueness here):



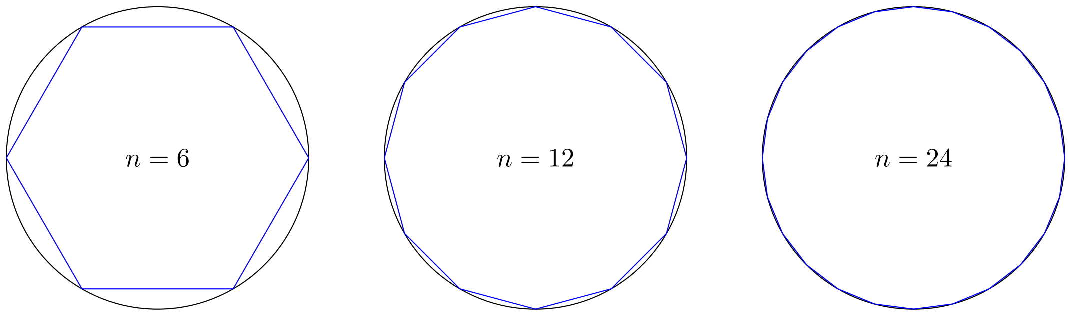
The more points we make, the more closely the line segments resemble the curve segment:



And more:



We can do the same thing to a circle. In fact, this is how we define the length of the circumference of a circle: Inscribe an n-sided regular polygon inside the circle (meaning we make the circle circumscribe the regular polygon). The perimeter of the regular polygon will approach the circumference of the circle as n becomes larger and larger (/tends towards infinity).



As $n \to \infty$, the value that the perimeter of the regular polygon approaches is defined to be the circumference of the circle. Note that circles of different radii have different circumferences.

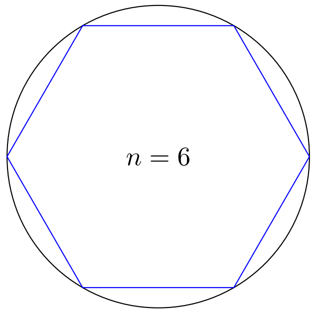
But how do we know that this circumference is a finite value? And how do we calculate it?

1.12.1 Calculation of pi

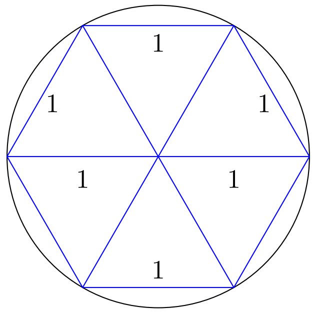
We can use **Archimedes' method**.

Inscribed polygons

To make things simple, start with a circle with radius of 1 unit, and inscribe a regular hexagon in it.

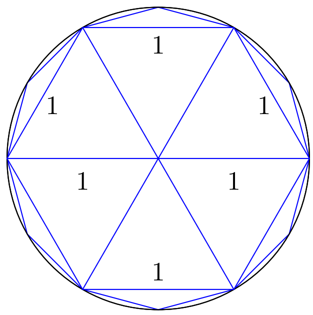


Note that the side length of the hexagon is 1 unit, since the hexagon is made up of six equilateral triangles of side length 1: (We can easily show that the point of concurrency of the diagonals is the centre of the circle by '3R theorem'.)

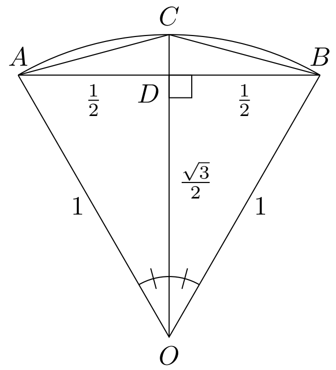


Note that the semi-perimeter of the hexagon is 3. (We use semi-perimeter instead of perimeter because it corresponds to radius better.) Then the semi-circumference of the circle must be greater than 3 since an arc is always longer than the corresponding chord.

Now inscribe a 12-sided regular polygon in the same circle.



We can find the side length of the 12-gon by focusing on a sector:



Note that C is a vertex of the 12-gon, so it is the mid-point of $\overset{\frown}{AB}$.

Thus, we have $\angle AOC = \angle BOC$ (equal arcs, equal $\angle$s) . Thus, $AD = DB = \frac{1}{2}$ and $OD \perp AB$ (prop. of isos. $\triangle$) .

Then OD is the height of equilateral triangle $\triangle OAB$, so $OD = \frac{\sqrt{3}}{2}$ by height of equil $\triangle$ formula.

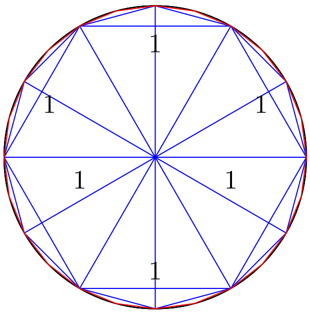
Note that $OC = 1$ (radius) , so we have $CD = 1 - \frac{\sqrt{3}}{2}$.

By pyth. theorem in $\triangle ACD$, we have the side length AC:

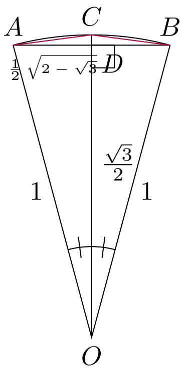
$$AC = \sqrt{(\frac{1}{2})^2 + (1 - \frac{\sqrt{3}}{2})^2} = \sqrt{2 - \sqrt{3}} \quad \approx 0.518$$

Thus, the semi-perimeter of the 12-gon is $(\frac{1}{2}) \cdot 12\sqrt{2 - \sqrt{3}} \approx 3.106$.

We can do that again by inscribing a regular 24-gon in the same circle (it is almost like a circle now):



And focusing on a sector (points relabelled):



Note that $AD = DB = \frac{1}{2}\sqrt{2 - \sqrt{3}}$.

By pyth. theorem in $\triangle AOD$, $OD = \sqrt{1 - (\frac{\sqrt{2 - \sqrt{3}}}{2})^2} = \frac{\sqrt{2 + \sqrt{3}}}{2}$.

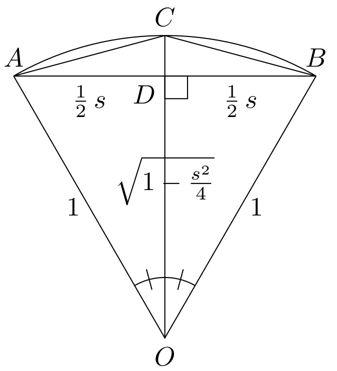
So $CD = 1 - \frac{\sqrt{2 + \sqrt{3}}}{2}$.

By pyth. theorem in $\triangle ACD$,

$$\begin{aligned} AC &= \sqrt{(\frac{\sqrt{2 - \sqrt{3}}}{2})^2 + (1 - \frac{\sqrt{2 + \sqrt{3}}}{2})^2} \\ &= \sqrt{\frac{2 - \sqrt{3}}{4} + 1 - \sqrt{2 + \sqrt{3}} + \frac{2 + \sqrt{3}}{4}} \\ &= \sqrt{2 - \sqrt{2 + \sqrt{3}}} \\ &\approx 0.261 \end{aligned}$$

So semi-perimeter of 24-gon $= 12\sqrt{2-\sqrt{2+\sqrt{3}}} \approx 3.133$.

Next we may inscribe 48-gon, 96-gon, and so on. These steps can be generalized as follows: Let $OA = OB = OC = 1$, $AB = s$.



Then $AD = DB = \dfrac{1}{2}s$, and by pyth. theorem in $\triangle AOD$, $OD = \sqrt{1-\dfrac{s^2}{4}}$.

So $CD = 1-\sqrt{1-\dfrac{s^2}{4}}$, and by pyth. theorem in $\triangle ACD$,

$$
\begin{aligned}
AC &= \sqrt{\left(\frac{1}{2}s\right)^2 + \left(1-\sqrt{1-\frac{s^2}{4}}\right)^2} \\
&= \sqrt{\frac{s^2}{4} + 1 - 2\sqrt{1-\frac{s^2}{4}} + (1-\frac{s^2}{4})} \\
&= \sqrt{2-2\sqrt{1-\frac{s^2}{4}}} \\
&= \sqrt{2-\sqrt{4-s^2}}
\end{aligned}
$$

If we iterate (repeat) these steps again, using $\sqrt{2-\sqrt{4-s^2}}$ to be the new AB , we have (replace s with $\sqrt{2-\sqrt{4-s^2}}$ in the above equation):

$$
\begin{aligned}
AC_2 &= \sqrt{2-\sqrt{4-(\sqrt{2-\sqrt{4-s^2}})^2}} \\
&= \sqrt{2-\sqrt{4-(2-\sqrt{4-s^2})}} \\
&= \sqrt{2-\sqrt{2+\sqrt{4-s^2}}}
\end{aligned}
$$

And again:

$$
\begin{aligned}
AC_3 &= \sqrt{2-\sqrt{4-(\sqrt{2-\sqrt{2+\sqrt{4-s^2}}})^2}} \\
&= \sqrt{2-\sqrt{4-2+\sqrt{2+\sqrt{4-s^2}}}} \\
&= \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{4-s^2}}}}
\end{aligned}
$$

There seems to be a pattern: the signs next to the '2' in the middle is $+$, and the signs at the leftmost and rightmost is $-$.

Let a_i be the perimeter of the regular $6(2^i)$-gon.
Initially, we have $s = 1$, so $4 - s^2 = 3$. We have the sequence for a_i:

$$
\begin{aligned}
a_0 &= 3 \\
a_1 &= 6\sqrt{2-\sqrt{3}} && \approx 3.10583 \\
a_2 &= 12\sqrt{2-\sqrt{2+\sqrt{3}}} && \approx 3.13263 \\
a_3 &= 24\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}} && \approx 3.13935 \\
a_4 &= 48\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}} && \approx 3.14103 \\
a_5 &= 96\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}} && \approx 3.14145
\end{aligned}
$$

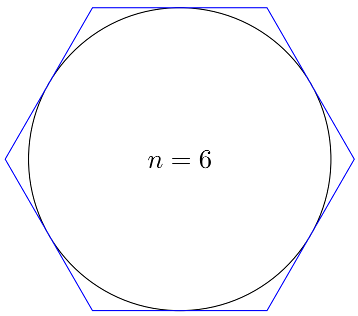
(I'll omit the equation now.)

$$
\begin{aligned}
a_6 &\approx 3.1415576 \\
a_7 &\approx 3.1415839 \\
a_8 &\approx 3.1415905 \\
a_9 &\approx 3.1415921 \\
a_{10} &\approx 3.1415925 \\
a_{11} &\approx 3.1415926
\end{aligned}
$$

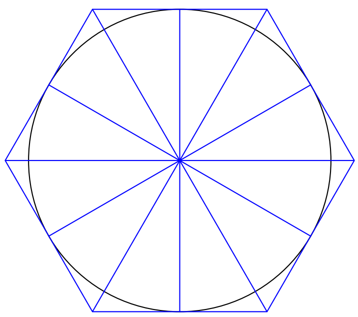
It seems to be converging to a certain value, known as π (pi).

Circumscribed polygons

Let's circumscribe a regular hexagon outside the unit circle instead and see what we can find.

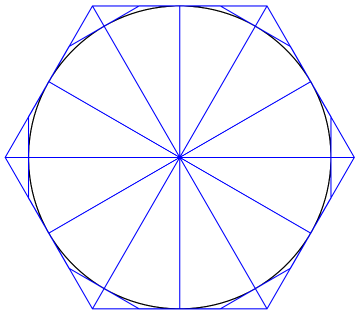


Beside diagonals, we can connect the mid-points of the sides to the centre of the circle.

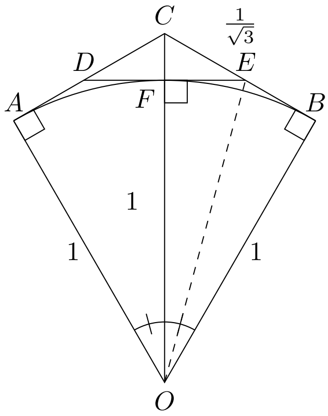


Note that each equilateral of the regular hexagon has a height of 1 unit, which is the radius of the circle. So the side length of the regular hexagon is $\dfrac{2}{\sqrt{3}}$, and its semi-perimeter is $3 \cdot \dfrac{2}{\sqrt{3}} \approx 3.464$.

Now circumscribe a 12-sided regular polygon outside the same circle: (a regular polygon can always circumscribe a circle since each side is equidistant from the centre by congruent triangles.)



Focus on a $60°$ sector. Join OE .



Note that $\triangle FOE \cong \triangle BOE$ (SSS), so $\angle FOE = \angle BOE$ (corr. $\angle$s, $\cong \triangle$s). Note that $OC = \dfrac{2}{\sqrt{3}}$ and $BC = \dfrac{1}{\sqrt{3}}$ since $\triangle OBC$ is half an equilateral triangle.

In $\triangle OBC$, by angle bisector theorem, we have

$$BE = \frac{OB \cdot BC}{OB + OC} = \frac{1(\frac{1}{\sqrt{3}})}{1 + \frac{2}{\sqrt{3}}} = \frac{1}{\sqrt{3} + 2} = 2 - \sqrt{3}$$

which means the side length of the 12-gon is $2(2-\sqrt{3})$ and the semi-perimeter of the 12-gon is $12(2-\sqrt{3}) \approx 3.215$.

Note that $OE = \sqrt{(2-\sqrt{3})^2 + 1} = \sqrt{8 - 4\sqrt{3}} = \sqrt{6 - 2\sqrt{2}\sqrt{6} + 2} = \sqrt{(\sqrt{6} - \sqrt{2})^2} = \sqrt{6} - \sqrt{2}$.

Repeat the steps for a 24-gon and relabel the points (OE becomes OC and the sector OFB becomes OAB) , we have new $OC =$ old $OE = \sqrt{6} - \sqrt{2}$.

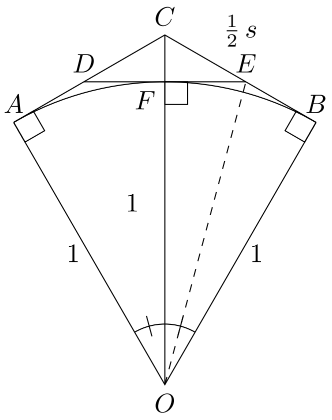
In $\triangle OBC$, by angle bisector theorem, we have

$$BE = \frac{OB \cdot BC}{OB + OC} = \frac{(1)(2-\sqrt{3})}{1 + (\sqrt{6} - \sqrt{2})} = \sqrt{6} + \sqrt{2} - \sqrt{3} - 2$$

(after some rationalization and simplifying) .

So the side length of the 24-gon is $2(\sqrt{6} + \sqrt{2} - \sqrt{3} - 2)$ and its semi-perimeter is $24(\sqrt{6} + \sqrt{2} - \sqrt{3} - 2) \approx 3.160$.

In general, let s be the side length of the $6(2^i)$-gon. Then $AC = CB = \dfrac{1}{2}s$.



Then $OC = \sqrt{1+\frac{s^2}{4}}$ by pyth. theorem.

In $\triangle OBC$, by angle bisector theorem, we have

$$
\begin{aligned}
BE = \frac{OB \cdot BC}{OB+OC} &= \frac{(1)(\frac{1}{2}\,s)}{1+\sqrt{1+\frac{s^2}{4}}} \\
&= (\frac{1}{2}) \cdot \frac{s(1-\sqrt{1+\frac{s^2}{4}})}{(1+\sqrt{1+\frac{s^2}{4}})(1-\sqrt{1+\frac{s^2}{4}})} \\
&= (\frac{1}{2}) \cdot \frac{s(1-\sqrt{1+\frac{s^2}{4}})}{(1-(1+\frac{s^2}{4})} \\
&= \frac{2(\sqrt{1+\frac{s^2}{4}}-1)}{s} \\
&= \frac{\sqrt{4+s^2}-2}{s} \\
&= \sqrt{\frac{4}{s^2}+1}-\frac{2}{s}
\end{aligned}
$$

So the side length of the $6(2^{i+1})$-gon is $\frac{2(\sqrt{4+s^2}-2)}{s}$, and its semi-perimeter is $\frac{6(2^i)(\sqrt{4+s^2}-2)}{s}$.

Let b_i be the perimeter of the regular $6(2^i)$-gon. Initially, we have $s = \frac{2}{\sqrt{3}}$. We have the sequence for b_i :

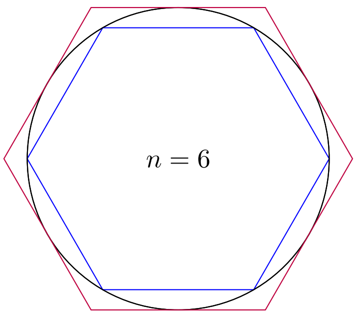
$$
\begin{aligned}
b_0 &= 2\sqrt{3} && \approx 3.46410 \\
b_1 &= 12(2-\sqrt{3}) && \approx 3.21539 \\
b_2 &= 24(\sqrt{6}+\sqrt{2}-\sqrt{3}-2) && \approx 3.15966
\end{aligned}
$$

I can't find any patterns in the formula...

$$
\begin{aligned}
b_3 &\approx 3.14609 \\
b_4 &\approx 3.14271 \\
b_5 &\approx 3.14187 \\
b_6 &\approx 3.1416627 \\
b_7 &\approx 3.1416102 \\
b_8 &\approx 3.1415970 \\
b_9 &\approx 3.1415937 \\
b_{10} &\approx 3.1415929 \\
b_{11} &\approx 3.1415927
\end{aligned}
$$

But it also seems to converge to the same value as a_i .

Finally, note that for the same number of sides, the circumscribed polygon is always larger than the inscribed polygon:



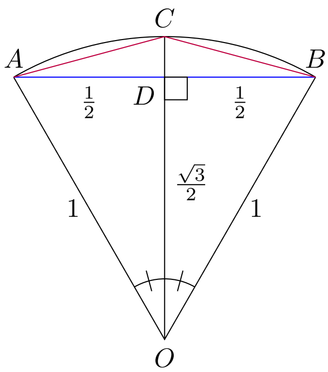
This is because the inscribed polygon can be rotated such that it joins the mid-points of adjacent sides of the circumscribed polygon. Then the perimeter of the circumscribed polygon forms some triangles sticking out of the inscribed polygon (like in the 12-gon figure).

So by triangle inequality, the perimeter of the circumscribed polygon is always longer than the perimeter of the inscribed polygon.

This means $b_i > a_i$ for all non-negative integer i .

(For a more concrete proof, see [14] but it uses trigonometry which is not introduced in here yet.)

Moreover, not that the sequence $\{a_i\}$ is an strictly increasing sequence, meaning if $m > n$, then $a_m > a_n$. This is also because of triangle inequality. Each doubling of the number of sides must form some triangles with the sides of the original polygon (like $\triangle ACB$ in the figure):



In the figure, AB is a side of original polygon (corresponding to a term, say a_k), and AC and AB are sides of the successive polygon a_{k+1}. We have $AC + CB > AB$ by triangle inequality. Since these are regular polygons, the perimeter of the successive polygon a_{k+1} must also be larger than the perimeter of the original polygon a_k.

By similar argument using triangle inequality, the sequence $\{b_i\}$ is an strictly decreasing sequence, meaning if $m > n$, then $b_m < b_n$.

And since $b_i > a_i$ for all of the terms, $\{a_i\}$ is bounded from above (meaning the term can never be greater than upper bound), as $a_i < b_i \leq b_0$, since b_0 must be the largest term of $\{b_i\}$.

Similarly, $\{b_i\}$ is bounded from below, meaning it has a lower bound, as $b_i > a_i \geq a_0$, since a_0 must be the smallest term of $\{a_i\}$.

In Calculus, we have the **monotone convergence theorem**, which states that if a sequence is bounded and increasing/decreasing, then it **converges** [3] to a fixed value. Thus, the sequences $\{a_i\}$ and $\{b_i\}$ must converge to a fixed value. (This fixed value is called the **limit** of the sequence.)

In fact,$\{a_i\}$ and $\{b_i\}$ converge to the same value (but I won't bother to show it). And let's call this value π (pi). I'm sure this will be a super useful number:

$$\pi \approx 3.1415926$$

[3]Converging to a value (informally) means the terms of the sequence settles to a value instead of fluctuating around or exploding to infinity.

And if we generate other sequences starting with any other regular polygons like a square or a pentagon, instead of a regular hexagon, then the perimeter will still converge to π . Why? The proof is left as an exercise (because I don't know yet).

Now that we have shown that $\{a_i\}$ converges, let's summarize the formula for a_i and π . Let's look at the square roots of the a_i terms where there are + signs inside.

$$
\begin{aligned}
a_0 &= 3 \\
a_1 &= 6\sqrt{2-\sqrt{3}} && \approx 3.10583 \\
a_2 &= 12\sqrt{2-\sqrt{2+\sqrt{3}}} && \approx 3.13263 \\
a_3 &= 24\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}} && \approx 3.13935 \\
a_4 &= 48\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}} && \approx 3.14103 \\
a_5 &= 96\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}} && \approx 3.14145
\end{aligned}
$$

Definition 1. Let d_n be a sequence with $d_1 = \sqrt{3}$ and the recurrence relation

$$d_{n+1} = \sqrt{2 + d_n}$$

. For $n \geq 1$, let

$$a_n = 3(2^n)\sqrt{2 - d_n}$$

Then the definition of π is:

$$\pi = \lim_{n\to\infty} \{a_n\}$$

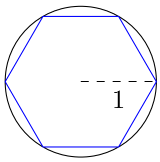
($\lim_{n\to\infty} \{a_n\}$ means the **limit** [4] of sequence $\{a_n\}$ [15]. I use n instead of i for the index because it looks better.)

1.12.2 Circumference of circle

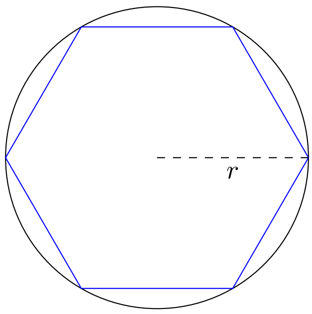
We know that π is the semi-circumference of a circle with radius 1.

What is the semi-circumference of a circle with radius 2 then? Intuition tells us that it is 2π , and we are correct.

First, note that all n-sided regular polygons are similar. This means their perimeter is proportional to their radius (/radius of circumcircle) .



Semi-perimeter= 3



Semi-perimeter= $3r$

[4] The precise definition of the limit of a sequence is $\lim_{n\to\infty} \{a_n\} = L$ if for every $\epsilon > 0$, there exists a positive integer M such that if $n > M$, then $|a_n - L| < \epsilon$. This is called the **epsilon-delta definition** of limits.

So if there is a circle with radius r , then we can inscribe a regular hexagon of radius r . Then its semi-perimeter will be $3r$. If we inscribe a regular 12-gon, then its semi-perimeter will be $6\sqrt{2-\sqrt{3}}\,r$, etc.

As the number of sides of the regular polygon tends towards infinity, the limit of the perimeter of the polygon, $\lim_{n\to\infty}\{ra_n\}$, will be the semi-circumference of the circle of radius r .

And since by limit laws, we have

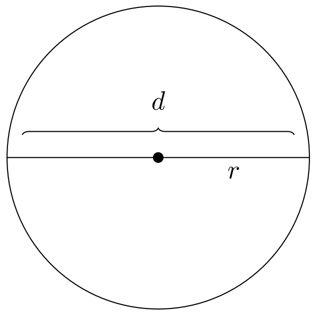
$$\lim_{n\to\infty}\{ra_n\} = r\lim_{n\to\infty}\{a_n\} = r\pi$$

, the semi-circumference of the circle will be πr .

And thus, the circumference of the circle will be $2\pi r$. Since the diameter (d) of the circle is $2r$, we also have circumference$=\pi d$.

Preposition 32. (i) The circumference (C) of a circle of radius r is $2\pi r$.
(ii) π is the ratio of the circumference (C) to the diameter (d) of a circle.

(circumference of circle)



$$\boxed{C = 2\pi r}$$

$$\boxed{\pi = \frac{C}{d}}$$

Proof. (i) Shown above.
(ii) Note that $d = 2r$. Start with the first formula:

$$C = 2\pi r$$
$$\pi = \frac{C}{2r}$$
$$\pi = \frac{C}{d}$$

□

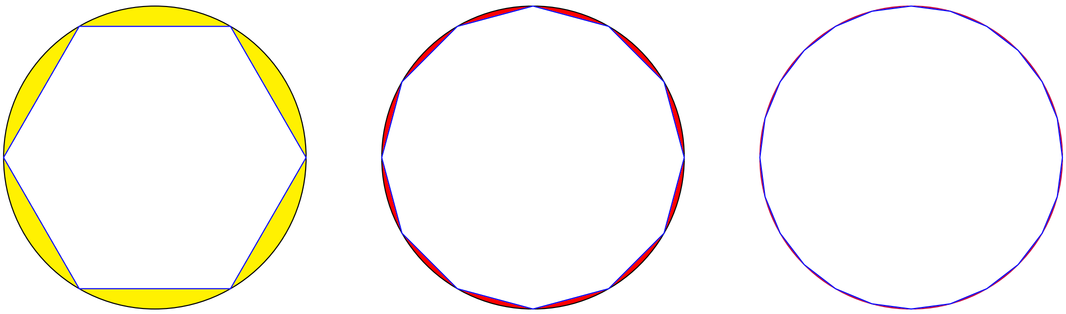
Note: π is usually defined to be the ratio of circumference to the diameter of a circle. But if I don't how the circumference is measured, then I can't know what π really is.

1.12.3 Area of circle

Similar to the circumference of circle, we can also define the area of a circle by the limit of the area of the inscribed regular polygon as its number of sides tends to infinity.

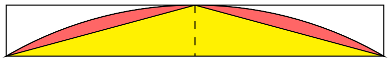
But how do we know that this limit of area of polygon actually approaches the area of a circle (instead of always less than the area of circle by some fixed value)?

Let there be a circle of radius r . Starting by inscribing a hexagon. Then a 12-gon, 24-gon, and so on.

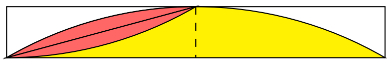


Note that the (shaded) area of segments outside the polygon is the difference between the area of polygon and area of circle. And it gets smaller and smaller each time the number of polygon sides doubles.

Let's focus on a segment. Draw a rectangle around it so that the segment touches the top side. Note that the figure is symmetric about the dotted line segment.



Then we can flip the right red segment vertically and move it to align with the left red segment:

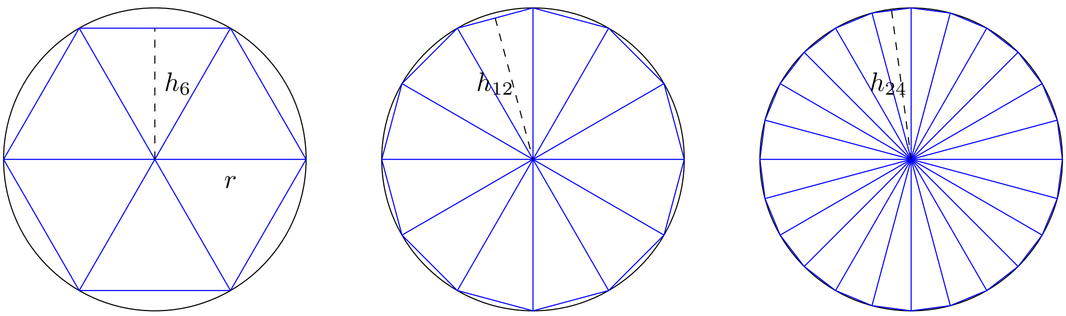


It will fit completely inside the rectangle since it is tangent to the bottom side. Note that the red area is less than half of the big yellow segment, as it only fills part of the left part of yellow segment.

This means that after each doubling of number of sides, the total area of segments outside the polygon becomes less than half of the previous total area of segment outside the polygon. For example, the red area outside the 12-gon is less than half of the yellow area outside the hexagon.

Since the shaded area gets less than halved each time, the shaded area approaches zero. This would mean that the area of polygon really gets arbitrarily close to the area of the circle,

Now split the polygons into triangles.



Let p_n and s_n be the perimeter and side length of the regular n-gon respectively. Note that $p_n = ns_n$.

Note that an n-gon is made up of n isosceles triangles, Let their height be h_n . Then the area of the inscribed n-gon is $n \cdot \frac{s_n h_n}{2} = \frac{p_n h_n}{2}$.

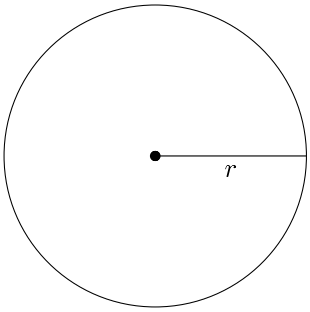
As n tends to infinity, p tends to the circumference of the circle $2\pi r$, and h tends to r (by property of cosines). Let A be the area that the inscribed polygon tends to. By multiplicative law of limits,

$$A = \lim_{n\to\infty}\left(\frac{p_n h_n}{2}\right) = \frac{\lim_{n\to\infty}(p_n)\lim_{n\to\infty}(h_n)}{2} = \frac{(2\pi r)(r)}{2} = \pi r^2$$

From above, we see that the area of circle of radius r is πr^2 .

(For a more concrete proof, see [16].)

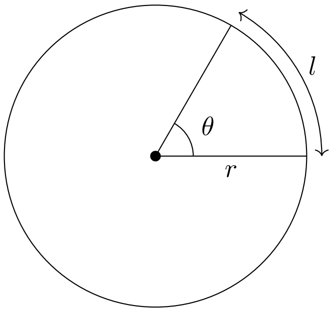
Preposition 33. The area (A) of a circle of radius r is πr^2 .



$$\boxed{A = \pi r^2}$$

1.12.4 Arc length and area of sector

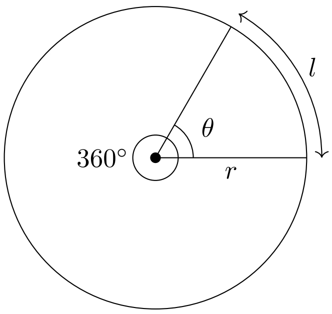
Preposition 34. The length of an arc that subtends angle θ (in degrees) at the centre of the circle is $\pi r(\frac{\theta}{180°})$. (arc length formula [degree version])



$$\boxed{l = \pi r(\frac{\theta}{180°})}$$

Proof. Let l be the length of the arc that subtends θ at the centre.

Note that an arc that subtends 360° at the centre is just the circumference of the circle, so its length is $2\pi r$.

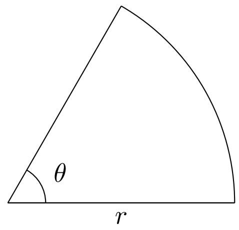


By 'arcs prop. to ∠s at centre' , we have

$$\begin{aligned}
\frac{l}{2\pi r} &= \frac{\theta}{360°} \\
l &= 2\pi r(\frac{\theta}{360°}) \\
&= \pi r(\frac{\theta}{180°})
\end{aligned}$$

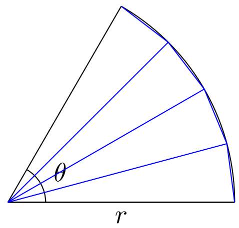
□

Preposition 35. The area (A) of a sector of radius r and angle θ (in degrees) is $\pi r^2(\frac{\theta}{360^\circ})$. (area of sector [degree version])



$$\boxed{A = \pi r^2(\frac{\theta}{360^\circ})}$$

Proof. First option: Let there be a sector of radius r and angle θ , and let A be its area.



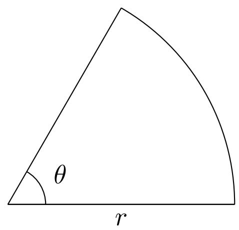
If we inscribe a bunch of congruent triangles sharing a vertex with the sector, and increase the number of triangles like how we increase the number of sides of regular polygons, then the height of these triangles will approach r and the sum of bases of these triangles will approach the arc length $2\pi r(\frac{\theta}{360^\circ})$.

Thus, the sum of area of triangles will approach $(\frac{1}{2})(r)(2\pi r(\frac{\theta}{360^\circ}))$, which is the area of the sector.

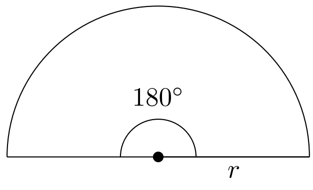
Simplifying, we have $A = \pi r^2(\frac{\theta}{360^\circ})$.

□

Preposition 36. The perimeter (P) of a sector of radius r and angle θ (in degrees) is is $2r(1+\frac{\pi\theta}{360^\circ})$, and the perimeter of a semi-cricle is $r(\pi+2)$. (perimeter of sector & perimeter of semi-circle)



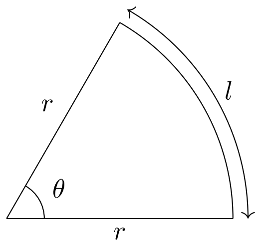
$$\boxed{P = 2r(1 + \frac{\pi\theta}{360^\circ})}$$ (perimeter of sector)



$$\boxed{P = r(\pi + 2)}$$ (perimeter of semi-circle)

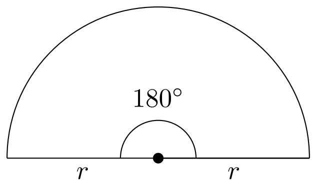
Proof. <u>Case 1:</u>

The perimeter of a sector is the sum of its bounderies, which consists of two radii r and an arc length l .



$$
\begin{aligned}
P &= r + r + l \\
&= 2r + 2\pi r(\frac{\theta}{360°}) \\
&= 2r(1 + \frac{\pi\theta}{360°})
\end{aligned}
$$

<u>Case 2:</u>



$$
\begin{aligned}
P &= r + r + \pi r \\
&= r(\pi + 2)
\end{aligned}
$$

□

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