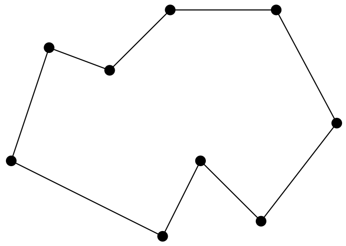
Contents

1 Lines, angles and shapes

1.5 Polygon properties

Preposition 1. A polygon has the same number of sides and vertices. (property of polygon sides and vertices)

Example



Observation: The polygon has 9 sides.
$\therefore$ It has 9 vertices. (property of polygon sides and vertices)

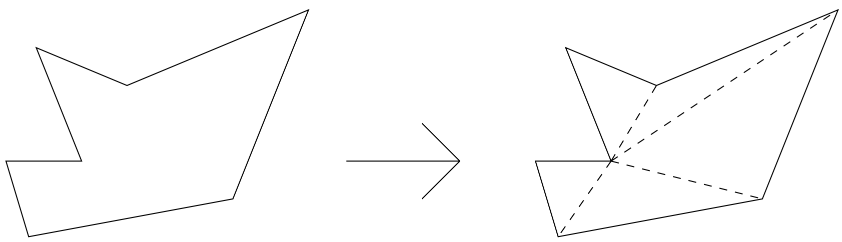
Proof. Let's rephrase the statement as: A polygon has n sides if and only if it has n vertices. Let's prove that $A \Rightarrow B$ and $B \Rightarrow A$, where A , B are the first and second half of the above statement.

($\Leftarrow$) Let's start from an arbitrary vertex and label the vertices of the n-verticed polygon clockwisely (or anticlockwisely, it doesn't matter) as $V_1, V_2, \ldots, V_n$. Note that a side has exactly two vertices as its endpoints. Also note that V_1 and V_2 make a side, V_2 and V_3 make a side, and so on, and lastly V_n and V_1 make the n-th side.

There are no other sides, since when we look at a vertex like V_2 , it is the endpoint of two sides, V_1V_2 and V_2V_3 . A third 'side' that has V_2 and V_i as the endpoints (where $i \neq 1$ and 3) will make either make the polygon into two or more enclosed space or make protruding line segments, so it is not allowed.

($\Rightarrow$) Let's start from an arbitrary side and label the vertices of the n-sided polygon clockwisely (or anticlockwisely, it doesn't matter) as $S_1, S_2, \ldots, S_n$. Note that a vertex is the common endpoint of exactly 2 sides. Note that S_1 and S_2 make a vertex, S_2 and S_3 make a vertex, and so on, and lastly S_n and S_1 make the n-th vertex. There are no other vertices for the same reason explained above. □

Preposition 2. A polygon with n sides can be split into exactly $n-2$ triangles by drawing non-intersecting diagonals that lie completely inside the polygon. (property of polygon triangulation)



Proof. [1] (We call splitting the polygon into triangles as the **triangulation** of the polygon.)

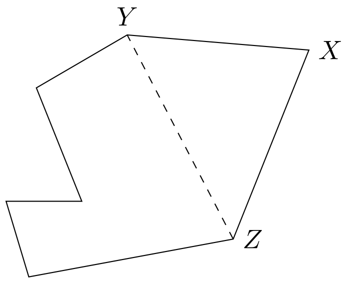
We will use proof by induction.

For the base case $n = 3$, if the polygon has 3 sides, then it is a triangle itself, so we are done.

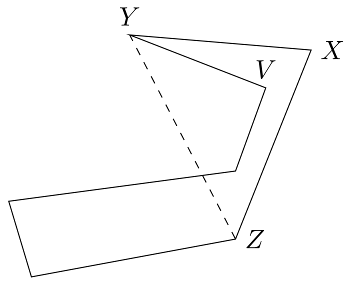
Let there be a polygon with more than 3 sides. First, we want to show that there is at least one diagonal that lies completely inside the the polygon.

Look at the rightmost vertex and name it X (if there are two leftmost vertex then randomly choose one). Let the adjacent vertices of X be Y and Z . Note that the interior angle of vertex X must be less than $180°$, or otherwise X is not an actual vertex or X is not the rightmost vertex.

Draw diagonal YZ . Then either this diagonal YZ lie completely inside the polygon , or there is at least one other vertex that lies inside $\triangle XYZ$:



Case 1



Case 2

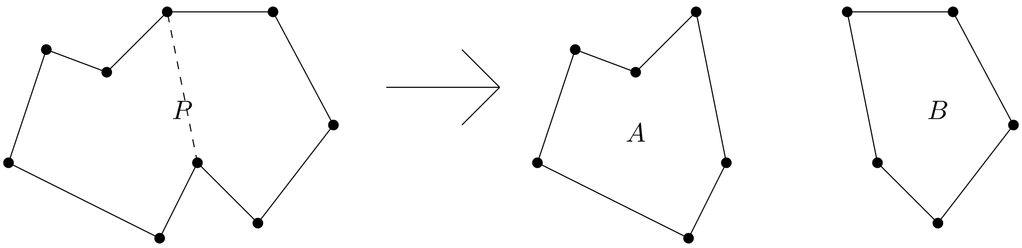
Let's look at case 2. Let V be a vertex in $\triangle XYZ$ such that perpendicular distance from V to line YZ is the furthest. (If there are several vertices with furthest distance, randomly choose one.) Then diagonal XV lies completely inside the polygon (otherwise V will not be the furthest point from line YZ).

Now let $S(n)$ be the statement that every triangulation of a polygon with n vertices consists of exactly $n-2$ triangles.

Then $S(3)$ is trivially true since a triangulation of any triangle is the triangle itself.

Next consider a polygon P with n vertices where $n > 3$, and let $t(P)$ denote the number of triangles in any triangulation of P .

Since there exists a diagonal that lies completely inside P , we can use it to split P into two polygons A and B .

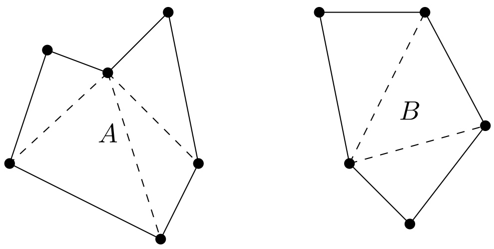


Note that after splitting the polygon, there are two more vertices in total because the original diagonal becomes two separate sides, which has 4 vertices in total instead of 2.

Let A has m vertices and B has k vertices. We have $m + k = n + 2$.

Now let's assume that for all integers x where $3 \leq x < n$, $S(x)$ is true. Then since $3 \leq m, k < n$, by assumption $S(m)$ and $S(k)$ are true, and so we have $t(A) = m - 2$ and $t(B) = k - 2$.

In other words, by assumption A can be triangulated into $m-2$ triangles and B can be triangulated into $k-2$ triangles.



Putting the two polygons together we have:

$$t(P) = t(A) + t(B) = m - 2 + k - 2 = m + k - 4$$

Since $m + k = n + 2$, we can substitute $n + 2$ for $m + k$:

$$t(P) = (n + 2) - 4 = n - 2$$

Since $t(P) = n - 2$ then $S(n)$ is true.

To recap, we have shown that:

1. Given any polygon with n vertices, if $S(x)$ is true for all $3 \leq x < n$ then $S(n)$ is also true.

2. $S(3)$ is true.

By combining (1) and (2) we get that $S(n)$ is true for all n since:

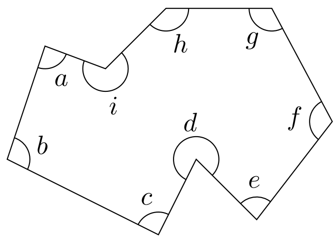
$$S(3) \Rightarrow S(4)$$
$$S(3), S(4) \Rightarrow S(5)$$
$$\vdots$$
$$S(3), \ldots, S(n-1) \Rightarrow S(n)$$

□

Therefore, by mathematical induction, every polygon with n vertices can be triangulated into exactly $n-2$ triangles.

Preposition 3. The sum of interior angles of an n-sided polygon is $(n-2) \cdot 180°$. (∠ sum of polygon) *

Example



Observation: The polygon has 9 sides.

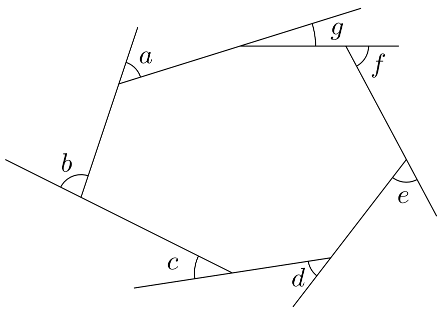
$$\therefore\ a+b+c+d+e+f+g+h+i = (9-2) \cdot 180° = 1260° \qquad (\angle \text{ sum of polygon})$$

Proof. By property of polygon triangulation, an n-sided polygon can be triangulated into exactly $n-2$ triangles. By '(∠ sum of △)' , the sum of interior angles of a triangle is $180°$.

Since the sum of interior angles of all triangles is also the sum of interior angles of the polygon, we have the interior angle sum be $(n-2) \cdot 180°$. □

Preposition 4. The sum of exterior angles of a convex polygon is $360°$. (sum of ext. ∠s of polygon)*

Example



$$a+b+c+d+e+f+g = 360° \qquad (\text{sum of ext. } \angle\text{s of polygon})$$

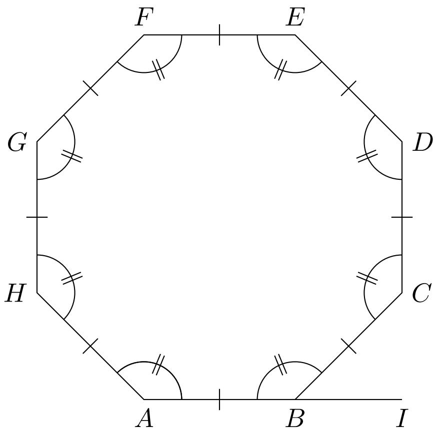
Proof. Since the polygon is convex, every interior angle is smaller than $180°$, and thus every interior angle has a corresponding exterior angle that has a positive measure.

Note that for an n-sided polygon, the sum of interior angles and exterior angles is $180° \cdot n$, since an n-sided polygon has n pairs of interior and exterior angles making up n straight angles.

Since the sum of interior angles is $(n-2) \cdot 180°$ (∠ sum of polygon), the sum of exterior angles is $180° \cdot n - (n-2) \cdot 180° = 180° \cdot n - 180° \cdot n + 2 \cdot 180° = 360°$. □

Preposition 5. Each exterior angle of an n-sided regular polygon is $360°/n$, and each interior angle of an n-sided regular polygon is $180° - 360°/n$. (prop. of regular polygon)

Example



$$\because ABCDEFGH \text{ is a regular polygon with 8 sides.}$$
$$\therefore \angle A = \angle B = \angle C = \angle D = \angle E = \angle F = \angle G = \angle H = 180° - 360°/8 = 135°$$
$$\angle CBI = 360°/8 = 45°$$
$$\text{(prop. of regular polygon)}$$

Proof. By definition, in a regular polygon, every side and interior angle is equal. Let x be the measure of an interior angle.

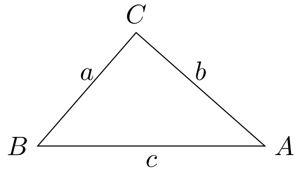
The sum of interior angle of an n-sided regular polygon is $(n-2) \cdot 180°$ ($\angle$ sum of polygon), and since there are n interior angles with the same measure, the measure of each interior angle is $\dfrac{(n-2) \cdot 180°}{n} = 180° - 360°/n$. Note that a regular polygon must be convex since $360°/n$ is positive, so an interior angle can never be larger than $180°$.

Then each exterior angle is $180° - (180° - 360°/n) = 360°/n$ (adj. $\angle$s on st. line). □

1.5.1 Triangle properties again

(This belongs to triangle properties)

Preposition 6. Given positive side lengths a, b, c , a triangle can be constructed if and only if $a + b > c$ and $|a - b| < c$. (condition of triangle construction)



$$\text{Given: } a, b, c \text{ where } a + b > c \text{ and } |a - b| < c$$
$$\therefore \triangle ABC \text{ is a valid triangle.} \qquad \text{(condition of triangle construction)}$$

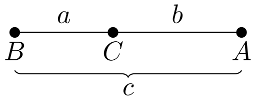
Proof. Suppose $a + b > c$ and $|a - b| < c$. There are two cases: $a < b$ and $a \geq b$.

Let's first consider the case where $a < b$. We have $a - b < 0$, which means $|a - b| = -(a - b)$. Thus $|a - b| < c \Rightarrow -(a - b) < c \Rightarrow a + c > b$.

Then we consider the case where $a \geq b$. We have $a - b \geq 0$, which means $|a - b| = a - b$. Thus $|a - b| > c \Rightarrow a - b > c \Rightarrow b + c > a$.

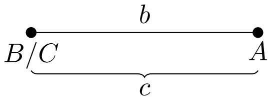
Given line segment AB with length c on the plane, let's see what happens when we place the third vertex C on different places on the plane and connect the vertices.

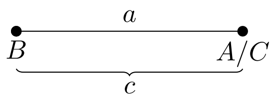
1. C is between A and B .



By segment addition postulate, we have $a+b=c$, which doesn't satisfy the requirement $a+b>c$.

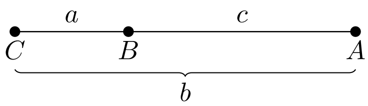
2. C coincides with A or B .

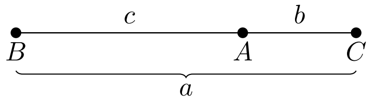




We have either $a=0$, $b=c$ or $b=0$, $a=c$. Either case leads to $a+b=c$, which does not satisfy the requirement $a+b>c$.

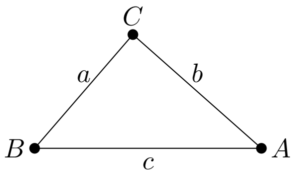
3. C lies on line AB but outside of line segment AB .





In either case, we have $|a-b|=c$, which doesn't satisfy the requirement $|a-b|<c$.

4. C is not on line AB .



Since A, B, C are not collinear, $\triangle ABC$ is a valid triangle. By triangle inequality, we have $a+b>c$, $a+c>b$ and $b+c>a$. Note that

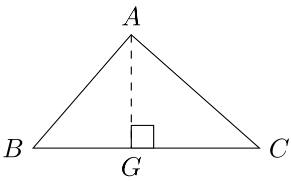
$$\begin{aligned}
&b+c>a \;\Rightarrow a-b<c \\
&a+c>b \;\Rightarrow b-a<c \;\Rightarrow -(a-b)<c
\end{aligned}$$

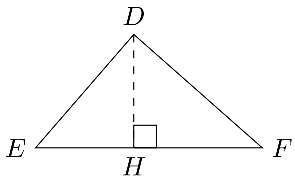
Thus, $|a-b|<c$. This case (C not lying on line AB) is the only case that satisfies both requirements ($a+b>c$ and $|a-b|<c$) .

Thus, when given side lengths a, b, c where $a+b>c$ and $|a-b|<c$, C must not lie on line AB , which means A, B, C are not collinear and is a valid triangle.

□

Preposition 7. If two triangles are congruent, then their corresponding heights are equal. (corr. heights, $\cong \triangle$s)





$$\because \triangle ABC \cong \triangle DEF \text{ , } AG \perp BC \text{ , } DH \perp EF$$
$$\therefore\ AG = DH \qquad \text{(corr. heights, } \cong \triangle\text{s)}$$

Proof. Let there be triangles $\triangle ABC$ and $\triangle DEF$ where $\triangle ABC \cong \triangle DEF$. Let's consider heights AG and DH . There are three cases: $\angle ABC < 90^\circ$, $\angle ABC = 90^\circ$, $\angle ABC > 90^\circ$.

1. $\angle ABC < 90^\circ$

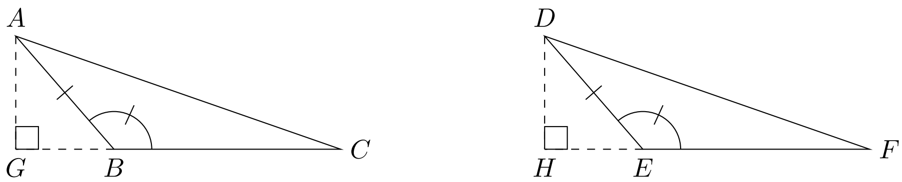
If $\angle ABC < 90°$, then AG lies inside the triangle.
In $\triangle ABG$ and $\triangle DEH$,

$$
\begin{aligned}
\angle AGB = \angle DHE = 90° &\qquad (AG \perp BC \text{ and } DH \perp EF) \\
\angle ABC = \angle DEF &\qquad (\text{corr. } \angle\text{s}, \cong \triangle\text{s}) \\
AB = DE &\qquad (\text{corr. sides}, \cong \triangle\text{s}) \\
\therefore\ \triangle ABG \cong \triangle DEH &\qquad (\text{AAS}) \\
\therefore\ AG = DH &\qquad (\text{corr. sides}, \triangle ABG \cong \triangle DEH)
\end{aligned}
$$

2. $\angle ABC = 90°$

If $\angle ABC = 90°$, then G coincides with B and H coincides with E . Thus, $AG = DH$ by (corr. sides, $\cong \triangle$s).

3. $\angle ABC > 90°$



If $\angle ABC > 90°$, then AG lies outside the triangle. Since $\angle ABC = \angle DEF$ (corr. $\angle$s, $\cong \triangle$s), we have $\angle ABG = \angle DEH = 180° - \angle ABC$ (adj. $\angle$s on st. line).
In $\triangle ABG$ and $\triangle DEH$,

$$
\begin{aligned}
\angle AGB = \angle DHE = 90° &\qquad (AG \perp GC \text{ and } DH \perp HF) \\
\angle ABG = \angle DEH &\qquad (\text{corr. } \angle\text{s}, \cong \triangle\text{s})\&(\text{adj. } \angle\text{s on st. line}) \\
AB = DE &\qquad (\text{corr. sides}, \cong \triangle\text{s}) \\
\therefore\ \triangle ABG = \triangle DEH &\qquad (\text{AAS}) \\
\therefore\ AG = DH &\qquad (\text{corr. sides}, \triangle ABG \cong \triangle DEH)
\end{aligned}
$$

The other two corresponding heights can be proved to be equal similarly. □

Preposition 8. For a triangle with two given sides, the larger the included angle is, the longer the third side is. Conversely, the longer the third side is, the larger the included angle is. (Hinge theorem) [2]

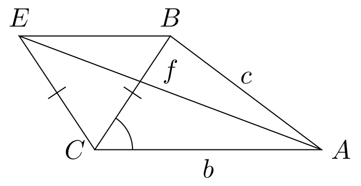
1a:

$$\because \angle EFD > \angle BCA\,,\ BC = EF\,,\ CA = FD$$
$$\therefore\ d > c$$

1b:

$$\because f > c\,,\ BC = EF\,,\ CA = FD$$
$$\therefore\ \angle EFD > \angle BCA$$

Proof. [2] **1a:** Let there be $\triangle ABC$ and $\triangle DEF$ where $AC = DF$, $BC = EF$ and $\angle EFD > BCA$. Move side FD to coincide with AC . Join EB .



Since $CE = CB$ (given), we have $\angle CEB = \angle CBE$ (base $\angle$s, isos. $\triangle$).

Note that $\angle ABE > \angle CBE = \angle CEB > \angle AEB$, which means $\angle ABE > AEB$. By 'larger $\angle$, longer side' in $\triangle ABE$, we have $AE > AB$, which means $f > c$.

1b: Let there be $\triangle ABC$ and $\triangle DEF$ where $AC = DF$, $BC = EF$ and $DE > AB$.

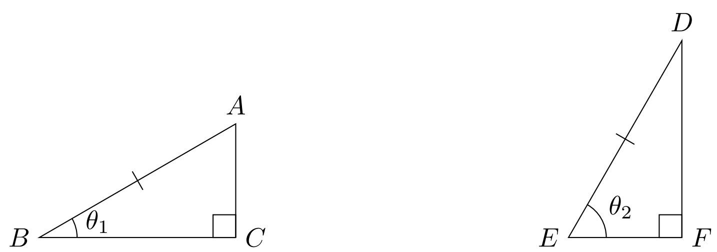
Suppose that $\angle EFD$ is not greater than $\angle BCA$.

If $\angle EFD = \angle BCA$, then since $BC = EF$, $\angle EFD = \angle BCA$, $\angle CA = FD$, we have $\triangle BCA \cong \triangle EFD$ (SAS), so $DE = AB$, which contradicts the assumption $DE > AB$. So it cannot be the case that $\angle EFD = \angle BCA$.

If $\angle EFD < \angle BCA$, then by 'larger angle, longer side', we have $AB > DE$, which contradicts the assumption $DE > AB$. So it cannot be the case that $\angle EFD < \angle BCA$.

Therefore, the only possible case is that $\angle EFD > \angle BCA$. □

Preposition 9. In a right triangle, for a hypotenuse with a given length, the longer the height, the larger its opposite angle. Conversely, the larger an acute angle, the longer its opposite side. (property of sines)



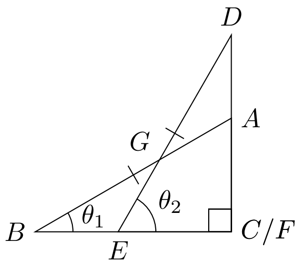
1a:

$$\because AB = DE\ ,\ \angle C = \angle F = 90^\circ\ ,\ DF > AC$$
$$\therefore\ \theta_2 > \theta_1 \qquad \text{(property of sines)}$$

1b:

$$\because AB = DE,\ \angle C = \angle F = 90^\circ\ ,\ \theta_2 > \theta_1$$
$$\therefore\ DF > AC \qquad \text{(property of sines)}$$

Proof. **1a:** Let's move angle $\angle DFE$ to coincide with $\angle ACB$ such that D is on line AC and E is on line BC (at the left of DC).

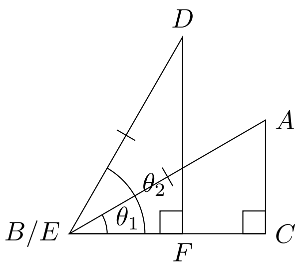


Note that E must not lie on B since that would mean $DB > AB$ by property of hypotenuse length, a contradiction.

Note that E must also not lie at the left of B. Suppose it does. Then $DE > DB$ by property of hypotenuse length, but we have $DB > AB$, which means $DE > AB$, a contradiction.

Thus, E must lie between B and C . Let AB and DE intersect at G . In $\triangle BGE$, θ_2 is the exterior angle while θ_1 is the opposite interior angle. Thus $\theta_2 > \theta_1$ by 'ext. $\angle >$ int. opp. $\angle$. of $\triangle$' .

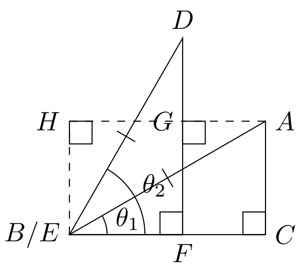
<u>**1b:**</u> Let's move vertex E to coincide with vertex B such that F lies on line BC .



Note that F must not lie on C since that would mean $DB > AB$ by property of hypotenuse length, a contradiction.

Note that F must also not lie at the right of C because then DB would be longer than if F lies at C for the same angle θ_2 , and we have shown that F can not lie on C .

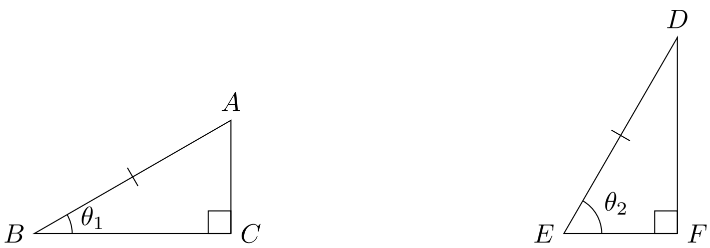
Thus, F must lie between B and C . Let's make a point G on line DF such that $AG \perp DF$. Let's make a point H on line AG such that $BH \perp BC$.



Note that $ACFG$ and $GFBH$ are rectangles (3 right $\angle$s) . Note that $GB < AB$ by property of hypotenuse length. Thus D cannot lie on G . Thus D can also not lie between GF since if so, then $BD < BG < AB$ by property of hypotenuse length. Thus D must lie above G . This means $DF > AC$ must be true.

□

Preposition 10. In a right triangle, for a given hypotenuse, the longer the base, the smaller its adjacent acute angle. Conversely, the smaller an acute angle, the longer its adjacent side. (property of cosines)



1a:

$$\because AB = DE \ , \angle C = \angle F = 90^\circ \ , BC > EF$$
$$\therefore \ \theta_1 < \theta_2 \qquad \text{(property of cosines)}$$

1b:

$$\because AB = DE, \angle C = \angle F = 90^\circ \ , \theta_1 < \theta_2$$
$$\therefore \ BC > EF \qquad \text{(property of cosines)}$$

Proof. **1a:** Since $BC > EF$ (given), we have $\angle A > \angle D$ by property of sines.

Note that $\theta_1 = 90^\circ - \angle A$ and $\theta_2 = 90^\circ - \angle D$ ($\angle$ sum of $\triangle$).

Since $\angle A > \angle D$, we have $90^\circ - \angle A < 90^\circ - \angle D$ (subtractive property of inequality), which means $\theta_1 < \theta_2$.

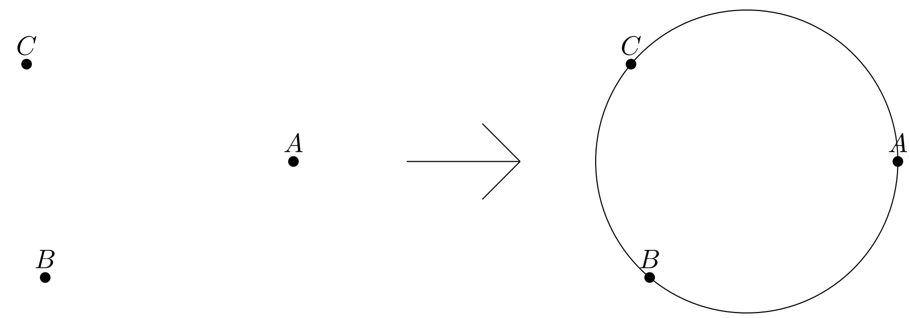
1b: Since $\theta_1 < \theta_2$ (given) and $\angle A = 90^\circ - \theta_1$ and $\angle D = 90^\circ - \theta_2$, we have $\angle A > \angle D$ (subtractive property of inequality).

By property of sines, we have $BC > EF$. □

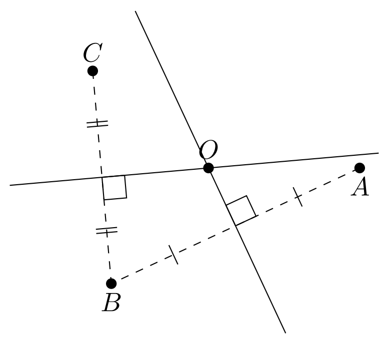
1.6 Circle properties

1.6.1 Basic properties

Preposition 11. A unique circle can be defined by any three points that it passes through. (3-point theorem)



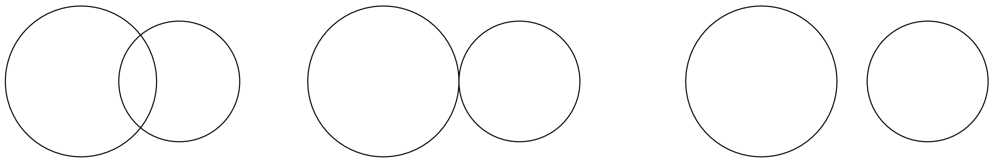
Proof. Let O be the centre of the circle. Then $OA = OB = OC$.



Since $OA = OB$, O must lie on the perpendicular bisector of AB (prop. of $\perp$ bisector). Similarly, since $OC = OB$, O must lie on the perpendicular bisector of CB (prop. of $\perp$ bisector). Since CB and BA are non-collinear line segments, the two perpendicular bisectors must intersect at a point, which is the centre of the circle.

□

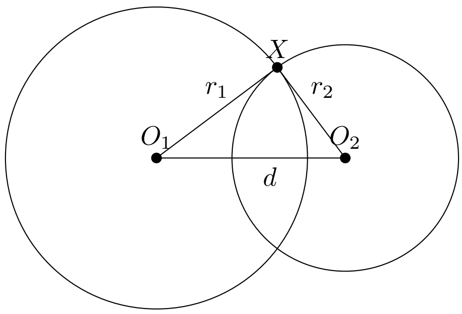
Preposition 12. Two circles can intersect at two points at most, and zero points at minimum. (property of circle intersection)



2 intersections 1 intersection 0 intersection

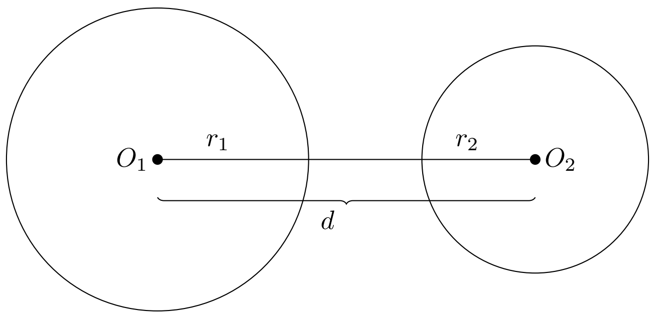
Proof. Let there be two circles with radius r_1 and r_2 , and centres O_1 and O_2 respectively. Let $d = O_1O_2$ be the distance between the centres of the two circles. Let's consider several cases:

1. $r_1 + r_2 < d$



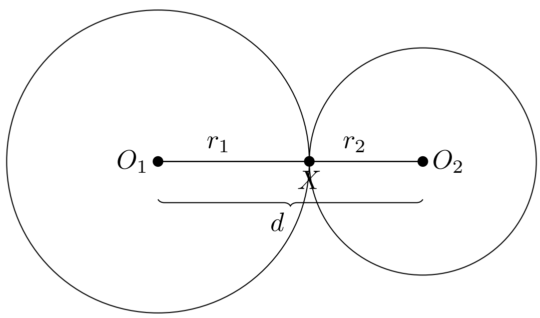
(Hypothetical figure)

If $r_1 + r_2 < d$, then the circle must not intersect. Otherwise, suppose they intersect at X . X must not lie on line segement O_1O_2 as that would imply $r_1 + r_2 = d$ (segment addition postulate), which contradicts the assumption $r_1 + r_2 < d$. If X is not on O_1O_2 , then in $\triangle XO_1O_2$, $O_1X + O_2X = r_1 + r_2 > d$ by triangle inequality, which also contradicts $r_1 + r_2 < d$. Thus, there must not exist an intersection of the two circles.



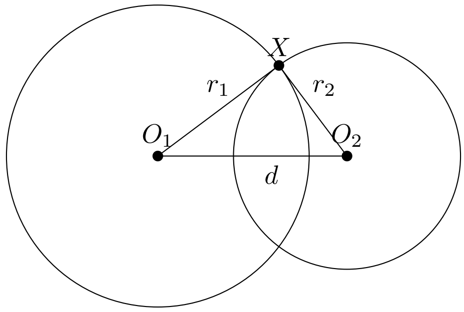
(Actual figure)

2. $r_1 + r_2 = d$



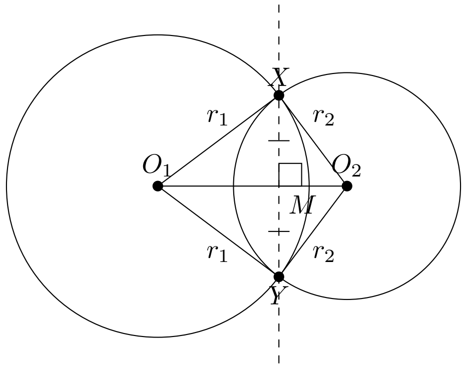
If $r_1 + r_2 = d$, then the circle must intersect at exactly one point, because we can let a point X on line segment O_1O_2 such that $O_1X = r_1$ and $O_2X = r_2$. This must be the only intersection because by ruler postulate, given a value of r_1 there is a unique point X on O_1O_2 such that $O_1X = r_1$. If there is an intersection X' that is not on line O_1O_2 , then by triangle inequality, $O_1X' + O_2X' = r_1 + r_2 > d$, which contradicts the initial assumption.

3. $r_1 + r_2 > d$ and $|r_1 - r_2| < d$



By condition of triangle construction, there exists a triangle with side lengths r_1 , r_2 and d . Let O_1 , O_2 be two of the vertices of this triangle with $O_1O_2 = d$, and let X be the third vertex. Then X must be an intersection of the two circles since $O_1X = r_1$ and $O_2X = r_2$.

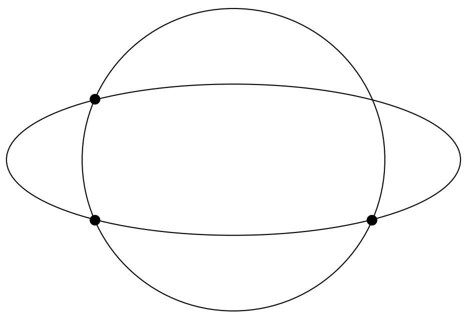
There must be another intersection Y which is the reflection of X about line O_1O_2 . This is because every point not on a line has exactly one reflection (image) about that line (left-right property). We can draw $XM \perp O_1O_2$ and let Y be the reflection of X about line O_1O_2 , which means $XM = YM$. By 'prop. of $\perp$ bisector', we have $O_1Y = O_1X = r_1$ and $O_2Y = O_2X = r_2$, which means Y must be another intersection of the two circles.



We have shown that it is possible for two circle to intersect at 0 point, 1 point and 2 points.

Now we show that it is impossible for two circles to intersect at more than two points. To see why, suppose some two circles intersects at at least 3 points. Let's arbitrarily pick three of the points of intersection. By definition of intersection, both two circles pass through these points.

But by 3-point theorem, there is a unique circle passing through these three points, which contradicts our assumption that two distinct circles pass through these three points.

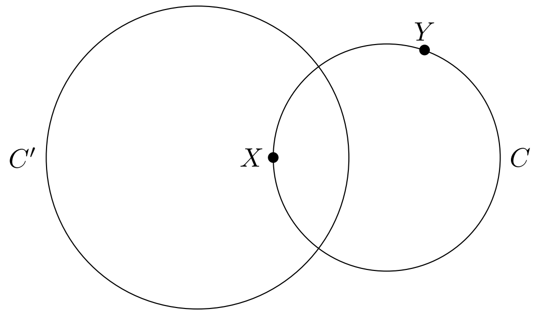


(Hmm... Something's wrong with one of the 'circles'.)

Thus, for any two circles, under any circumstances, there must not be more than two intersections.

□

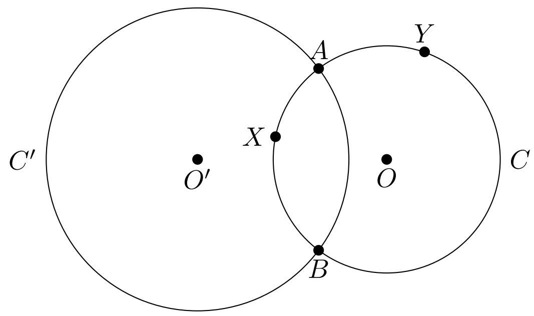
Preposition 13. If a circle C has one point inside and one point outside another circle C' , then the two circles intersect in exactly two distinct points. (circle continuity principle)



(Apparently, this ‘obvious’ principle is actually not that obvious, as it requires the previous preposition (property of circle intersection), which requires triangle inequality, which requires (base ∠s, isos. △), which requires SAS triangle congruence.)

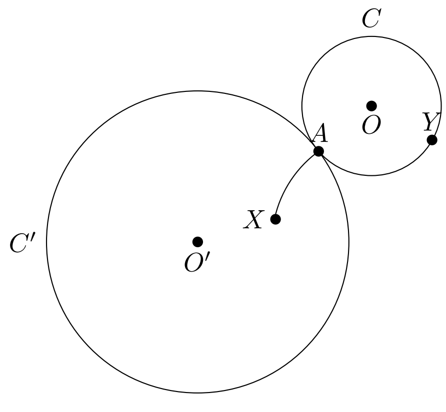
Proof. By property of circle intersection, the two circles C and C' can only intersect at two points at most.

Let’s say X is a point of C inside C' and Y is a point of C outside C' . An arc that goes from point X to Y must intersect C' at at least one point (by property of continuous path). Thus the two circles must intersect at at least one point. And in circle C, there are two arcs that go from X to Y (the major arc and minor arc). Let’s say these two arcs intersect C' at point A and B .



(Normal figure)

Suppose that A coincides with B , which means the two circles actually intersects at one point only. Then either there is a curve segment sticking out of circle C , or circle C actually encloses two spaces in the plane, which is impossible:

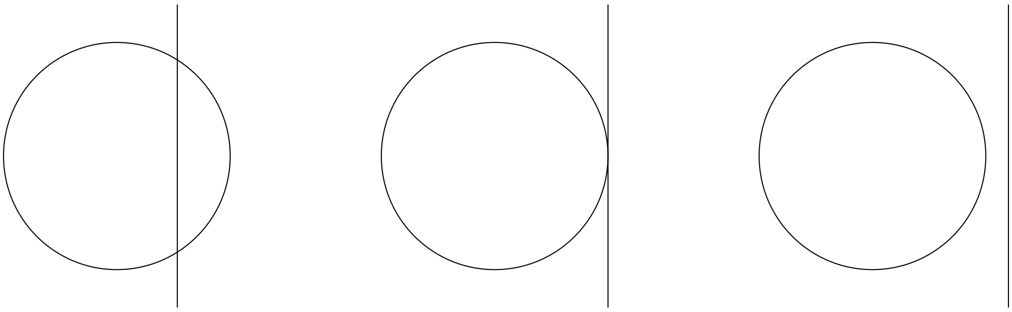


($\widehat{AX}$ is a curve segment sticking out of C , which isn’t found in a normal circle.)

Therefore C and C' can only have exactly two intersections.

□

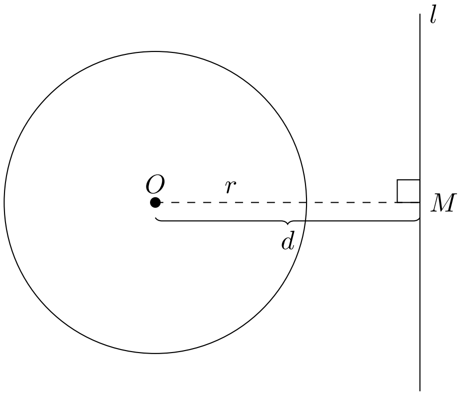
Preposition 14. A line and a circle can intersect at two points at most, and zero points at minimum. (property of line-circle intersection)



2 intersections 1 intersection 0 intersection

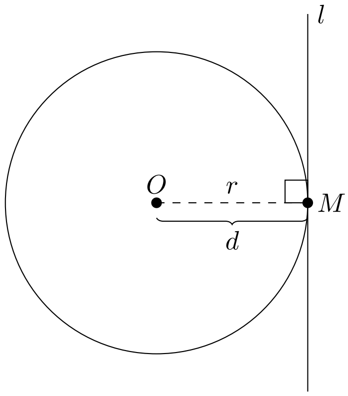
Proof. Let there be a circle with radius r and centre O . Let there be a line l , and M is a point on l such that $OM \perp l$. Let d be the perpendicular distance between O and l . Let's consider several cases:

1. $r < d$



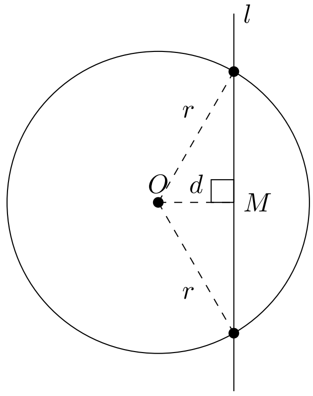
By property of hypotenuse length, $d = OM$ is the shortest distance between point O and line l . That means if there is a point P on line l , then $d \leq OP$, which means $r < OP$, which means there cannot be an intersection (which would require $r = OP$) .

2. $r = d$

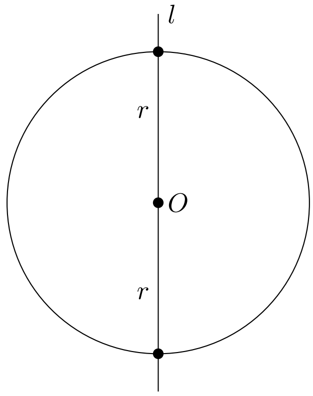


By property of hypotenuse length, OM is the only point on l such that $OM = r$, and if there is another point P on l that is not M , then $OP > r$, so P must lie outside the circle. Thus there can only be one intersection of the line and the circle, which is point M .

3. $r > d$



If l doesn't pass throught the centre of the circle O , then by the proof of RHS triangle congruence, there are exactly two points on l (labelled A and B) such that $OA = r$ and $OB = r$, which means the line and the circle intersect at exactly two points.



If l passes through O , then there are still two intersections since by left-right property, when given a point O and a radius r , there are exactly two points on l that have distance r away from O .

These are all the possible cases, so it is impossible for a line and a circle to intersect at more than two points.

□

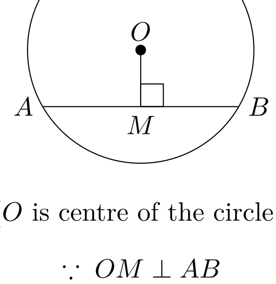
Note: A line that intersects a circle at two points is called **secant line** , and we say that the line **cuts** the circle at two points.

A line that intersects a circle at exactly one point is called **tangent line** , and we say that the line is tangent to the circle, or the line **touches** the circle at one point.

The point of intersection of a circle and a tangent line is called the **point of tangency** .

1.6.2 Chord properties

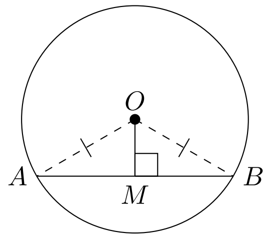
Preposition 15. A line (segment) passing through the centre of a circle and perpendicular to a chord must bisect the chord. (line from centre ⊥ chord bisects chord) *



(O is centre of the circle.)

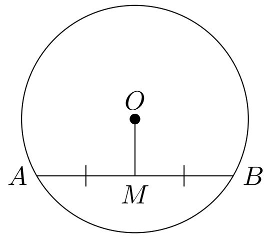
$$\because\ OM \perp AB$$
$$\therefore\ AM = MB \qquad \text{(line from centre} \perp \text{chord bisects chord)}$$

Proof. Draw OA and OB .



Note that $OA = OB$ (radii). Since $\triangle OAB$ is an isos. triangle and $OM \perp AB$ (given), we have $AM = MB$ (prop. of isos. $\triangle$). □

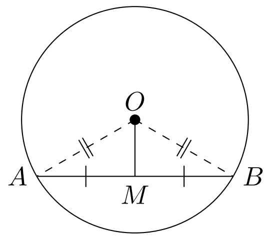
Preposition 16. A line (segment) joining the centre of a circle to the mid-point of a chord must be perpendicular to the chord. (line joining centre to mid-pt. of chord ⊥ chord) *



(O is centre of the circle.)

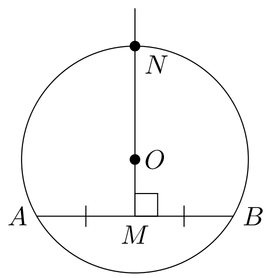
$$\because\ AM = MB$$
$$\therefore\ OM \perp AB \qquad \text{(line joining centre to mid-pt. of chord} \perp \text{chord)}$$

Proof. Draw OA and OB .



Note that $OA = OB$ (radii). Since $\triangle OAB$ is an isos. triangle and $AM = MB$ (given), we have $OM \perp AB$ (prop. of isos. $\triangle$). □

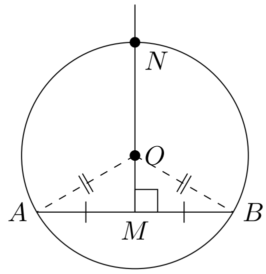
Preposition 17. The perpendicular bisector of a chord must pass through the centre of the circle. (⊥ bisector of chord passes through centre) *



(O is centre of the circle.)

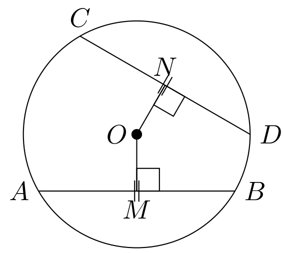
$$\because\ NM \perp AB \text{ and } AM = MB$$
$$\therefore\ O \text{ lies on } MN\ . \qquad (\perp \text{ bisector of chord passes through centre})$$

Proof. Let O be the centre of the circle. Draw OA and OB .



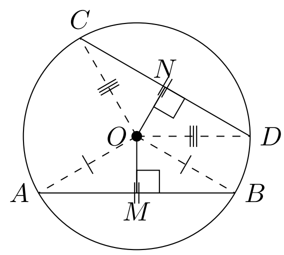
Note that $OA = OB$ (radii). By 'prop. of $\perp$ bisector', O must lie on the perpendicular bisector of AB , which is MN . In other words, the perpendicular bisector MN must pass through centre O . □

Preposition 18. Chords equal in length are equidistant from the centre of the circle. (equal chords, equidistant from centre) *



$$\because AB = CD \text{ , } OM \perp AB \text{ , } ON \perp CD$$
$$\therefore \; OM = ON \qquad \text{(equal chords, equidistant from centre)}$$

Proof. Draw OA, OB, OC and OD .

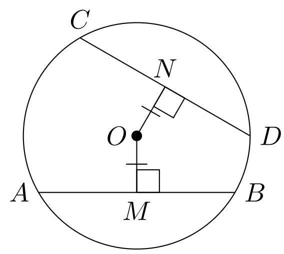


In $\triangle OAB$ and $\triangle OCD$,

$$\begin{aligned} OA &= OC && \text{(radii)} \\ OB &= OD && \text{(radii)} \\ AB &= CD && \text{(given)} \\ \therefore \; \triangle OAB &\cong \triangle OCD && \text{(SSS)} \end{aligned}$$

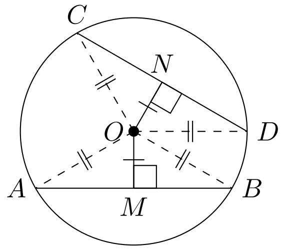
Thus, $OM = ON$ (corr. heights, $\cong \triangle$s) . □

Preposition 19. Chords equidistant from the centre of the circle are equal in length. (chords equidistant from centre are equal) *



$$\because OM = ON \text{ , } OM \perp AB \text{ , } ON \perp CD$$
$$\therefore\ AB = CD \qquad \text{(chords equidistant from centre are equal)}$$

Proof. Draw OA, OB, OC and OD .

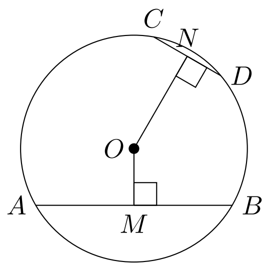


In $\triangle OAM, \triangle OBM, \triangle OCN, \triangle ODN$,

$$\angle OMA = \angle OMB = \angle ONC = \angle OND = 90° \qquad (OM \perp AB, ON \perp CD)$$
$$OA = OB = OC = OD \qquad \text{(radii)}$$
$$OM = OM = ON = ON \qquad \text{(common side \& given)}$$
$$\therefore\ \triangle OAM \cong \triangle OBM \cong \triangle OCN \cong \triangle ODN \qquad \text{(RHS)}$$
$$\therefore\ AM = MB = CN = DN \qquad \text{(corr. sides, } \cong \triangle\text{s)}$$
$$\therefore\ AB = 2AM = 2CN = CD$$

□

Preposition 20. The nearer a chord is from the centre of the circle, the longer it is. Conversely, the longer a chord is, the nearer it is from the centre. (property of chord length)



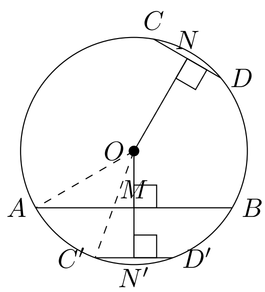
<u>1a:</u>

$$\because OM < ON \text{ , } OM \perp AB \text{ , } ON \perp CD$$
$$\therefore\ AB > CD \qquad \text{(property of chord length)}$$

<u>1b:</u>

$$\because AB > CD \text{ , } OM \perp AB \text{ , } ON \perp CD$$
$$\therefore\ OM < ON \qquad \text{(property of chord length)}$$

Proof. **<u>1a.</u>** Extend OM. Let N' be a point on line OM such that $ON' = ON$, and draw chord $C'D' \perp ON$.

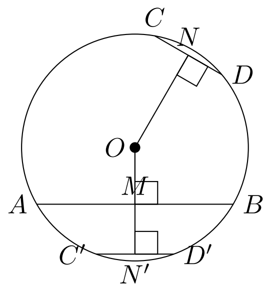


Since $ON = ON'$, we have $C'D' = CD$ (chords equidistant from centre are equal).

Focus on $\triangle OAM$ and $\triangle OC'N'$. Since $ON' > OM$ and $OC' = OA$ (radii), we have $\angle C'ON' < \angle AOM$ (property of sines). Similarly, $\angle D'ON' < \angle BOM$ (property of sines).

Thus $\angle C'OD' < \angle AOB$. By hinge theorem, we have $C'D' < AB$. Thus, $CD < AB$.

<u>1b.</u> Let N' be a point on line OM such that $ONON$ and draw $C'D' \perp ON$.



Since $ON = ON'$, we have $C'D' = CD$ (chords equidistant from centre are equal).

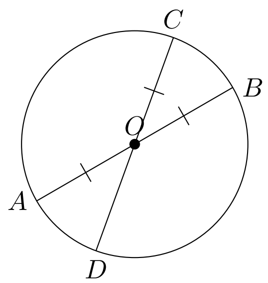
Note that N' cannot lie on M since if so, then we have $C'D' = AB$ by (chords equidistant from centre are equal), which means $CD = AB$, contradicting the assumption $AB > CD$.

Note that N' also cannot lie above M (meaning $ON' < OM$) since if so, then we have $CD > AB$ by case 1a, which contradicts the assumption $AB > CD$.

Thus, the only possible case is that N' lies below M , and $OM < ON'$, meaning $OM < ON$. ☐

1.6.3 Radius and diameter properties

Preposition 21. The intersection of two distinct diameters is the centre of the circle. (property of diameter intersection)



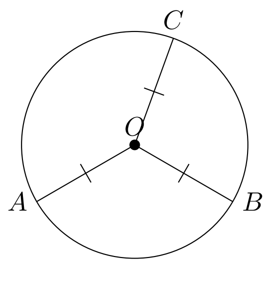
$\because$ AB, CD are diameters, and O is their intersection.

$\therefore$ O is the centre of the circle. (property of diameter intersection)

Proof. By definition, a diameter must pass through the centre of the circle. By property of line intersection, two given diameters must intersect at a unique point. In other words, there is a unique point that both diameters pass through.

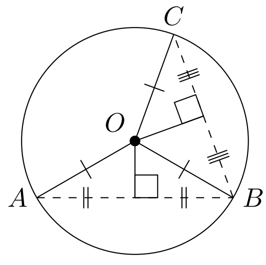
Since both diameters pass through the centre by definition, the point of intersection must be the centre of the circle. ☐

Preposition 22. If there is a point inside a circle such that the point has the same distance from three points on the circumference, then the point is the centre of the circle. (3R theorem)



$$\because OA = OB = OC$$
$$\therefore\ O \text{ is the centre of the circle.} \qquad \text{(3R theorem)}$$

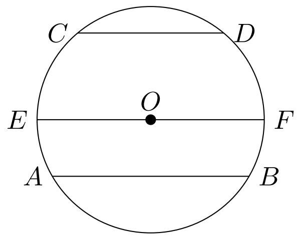
Proof. .



Since $OA = OB$, O lies on the $\perp$ bisector of AB (prop. of $\perp$ bisector) . Similarly, since $OB = OC$, O lies on the $\perp$ bisector of BC (prop. of $\perp$ bisector) .

By '$\perp$ bisector of chords passes through centre' , both of the $\perp$ bisectors intersect at the centre. Since O lies on both of the $\perp$ bisectors, O must be the centre of the circle. □

Preposition 23. The diameter is the longest possible chord of the circle. A chord is a diameter if and only if the chord length is equal to twice the circle's radius. (property of diameter length)



(Let r be the radius of the circle, and O be the centre.)

1a:

$$\because EF \text{ passes through centre } O\ .$$
$$\therefore\ EF \geq \text{any chord length.} \qquad \text{(property of diameter length)}$$

1b:

$$\because EF \geq \text{any chord length.}$$
$$\therefore\ EF \text{ passes through centre } O\ . \qquad \text{(property of diameter length)}$$

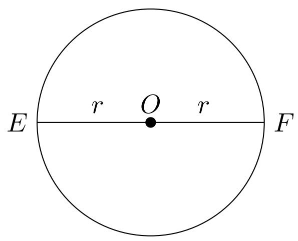
2a:

$$\because EF \text{ is a diameter.}$$
$$\therefore\ EF = 2r \qquad \text{(property of diameter length)}$$

2b:

$$\because EF = 2r$$
$$\therefore\ EF \text{ is a diameter.} \qquad \text{(property of diameter length)}$$

Proof. Let r be the radius of the circle, and O be the centre.
2a.

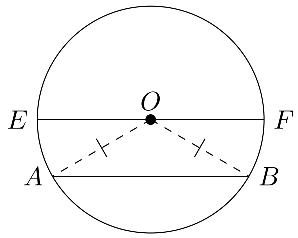


Let EF be a chord that passes through O . Note that $OE = OF = r$ (radii). Thus $EF = OE + OF = r + r = 2r$.

<u>1a.</u> Let EF be a chord that passes through O .

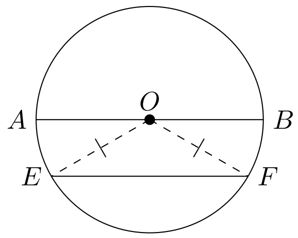
If there is any other chord that passes through the centre, then it must be of length $2r$ (by case 1a), which is equal to EF since EF is also a diameter.

Now suppose chord AB does not pass through the centre.



Note that $OA = OB = r$ (radii). In $\triangle OAB$ and $\triangle OMB$, note that $OA + OB > AB$ (triangle inequality) $\Rightarrow 2r > AB$ $\Rightarrow EF > AB$.

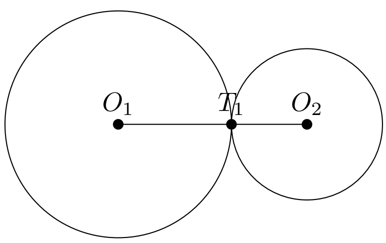
<u>1b.</u> Let's prove the contrapositive statement: if EF does not pass through centre O , then there exists a chord AB such that $EF < AB$.



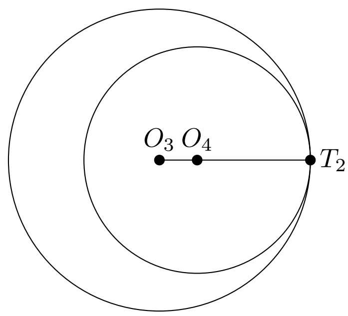
We can let AB be the diameter. By triangle inequality, $OE + OF > EF$, so $AB > EF$. So there always exists a chord longer than EF .

<u>2b.</u> Let EF be a chord with a length of $2r$. Suppose it is not a diameter, i.e. it does not pass through the centre O . Then by triangle inequality (shown above) , we have $EF < 2r$, which contradicts the assumption that $EF = 2r$. Thus, EF must be a diameter. □

Preposition 24. For two circles touching at exactly one point, their centres and their point of contact (/intersection) are collinear. (property of touching circles)



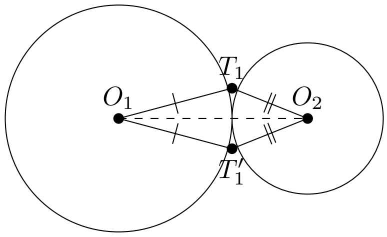
1a: Touching externally



1b: Touching internally

$\because T_1$ and T_2 are the only points of contact of the circles.
$\therefore$ O_1, T_1, O_2 are collinear, and O_3, O_4, T_2 are collinear.

Proof. **<u>1a:</u>** Suppose O_1, T_1, O_2 are not collinear. Then O_1, T_1, O_2 form a triangle. Let T_1' be the reflection of T_1 about line O_1O_2 .



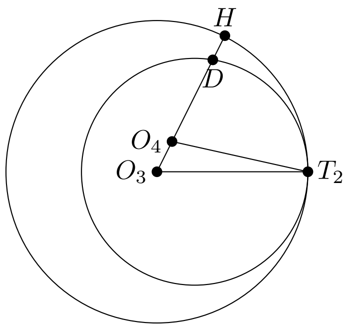
1a: Touching externally

By reflection postulate (or prop. of $\perp$ bisector), we have $O_1T_1' = O_1T_1$ and $O_2T_1' = O_2T_1$. Since O_1T_1' and O_2T_1' are the radii of the two circles, T_1' must also be a point of intersection of the two circles. But this contradicts the assumption that the circles only touch at one point.

So it must be the case that O_1, T_1, O_2 are collinear.

__1b:__ [3] Suppose O_3, O_4, T_2 are not collinear. Note that the radii of the twomcircles must not be equal, so suppose the radius of O_3 circle is larger than the radius of O_4 circle. Note that O_4 must lie inside the circle of O_3 .

Extend O_3, O_4 to the circumference of the larger circle and let that point be H . O_3H will also pass through the circumference of the smaller circle, so let that point be D .



1b: Touching internally

Join O_3T_2 and O_4T_2 . In $\triangle O_3O_4T_2$, by triangle inequality, we have $O_3O_4 + O_4T_2 > O_3T_2$.

Note that $O_3T_2 = O_3H$ since they are radii of the larger circle. So we have $O_3O_4 + O_4T_2 > O_3H$.

Subtracting O_3O_4 from both sides, we have $O_4T_2 > O_4H$. But since O_4 is the centre of the smaller circle, we have $O_4T_2 = O_4D$, and also, $O_4D < O_4H$ since D is between O_4 and H . This means $O_4T_2 < O_4H$, which is a contradiction.

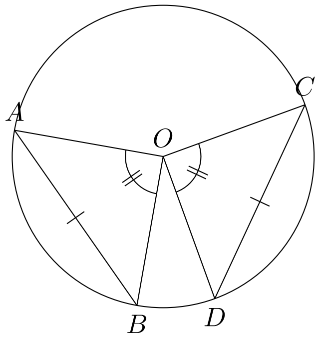
Thus, it can only be the case that O_3, O_4, T_2 are collinear.

□

1.6.4 Arc, angle and chord properties

Preposition 25. Equal chords of a circle subtend equal angles at the centre. (equal chords, equal $\angle$s) *

Conversely, equal angles at the centre are subtended by equal chords. (equal $\angle$s , equal chords) *



(O is the centre of circle.)

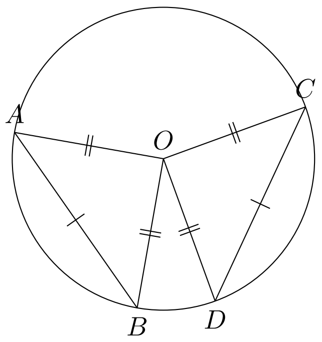
1a:

$$\because AB = CD$$
$$\therefore \angle AOB = \angle COD \qquad \text{(equal chords, equal } \angle\text{s)}$$

1b:

$$\therefore \angle AOB = \angle COD$$
$$\because AB = CD \qquad \text{(equal } \angle\text{s , equal chords)}$$

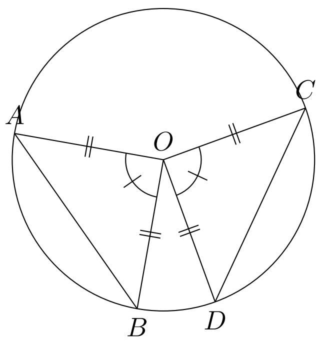
Proof. **1a:** AB=CD



In $\triangle OAB$ and $\triangle OCD$,

$$AB = CD \qquad \text{(given)}$$
$$OA = OC \qquad \text{(radii)}$$
$$OB = OD \qquad \text{(radii)}$$
$$\therefore \triangle OAB \cong \triangle OCD \qquad \text{(SSS)}$$
$$\therefore \angle AOB = \angle COD \qquad \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}$$

1b: $\angle AOB = \angle COD$

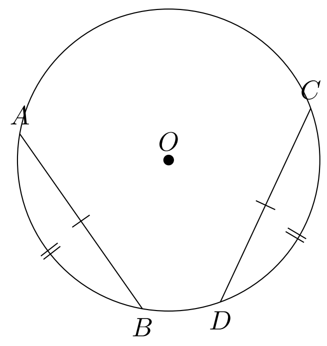


In $\triangle OAB$ and $\triangle OCD$,

$$OA = OC \qquad \text{(radii)}$$
$$\angle AOB = \angle COD \qquad \text{(given)}$$
$$OB = OD \qquad \text{(radii)}$$
$$\therefore \triangle OAB \cong \triangle OCD \qquad \text{(SAS)}$$
$$\therefore AB = CD \qquad \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}$$

□

Preposition 26. Equal chords of a circle span equal arcs. (equal chords, equal arcs) *
Conversely, equal arcs of a circle are spanned by equal chords. (equal arcs, equal chords) *



1a:

$$\because AB = CD$$
$$\therefore \overparen{AB} = \overparen{CD}$$

1b:

$$\because \overparen{AB} = \overparen{CD}$$
$$\therefore AB = CD$$

(Note that minor arc must be equal to minor arc, and major arc must be equal to major arc. So major arc is not equal to minor arc unless the chord is a diameter.)

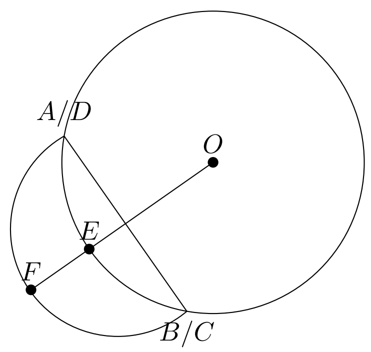
Proof. (We only consider minor arcs, as this automatically proves that the major arcs are equal since the circumference is the same.)

1a: AB=CD

Move the minor segment cut by CD such that C coincides with B and D coincides with A . Then AB coincides with DC .

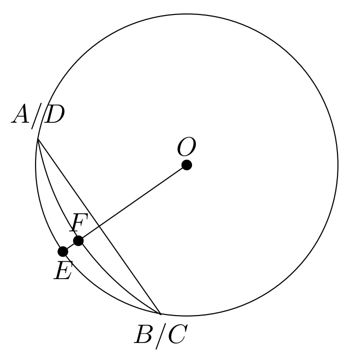
Suppose that $\overparen{CD}$ does not coincide with $\overparen{AB}$. There are three cases:

1. $\overparen{CD}$ is outside segment AB .



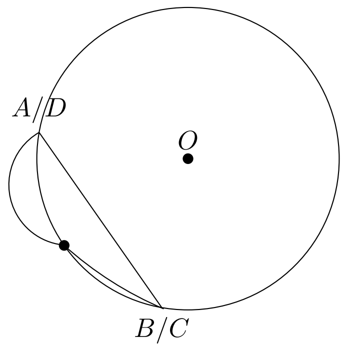
Draw a line segment OF from centre to $\overparen{CD}$ (which is a radius of the circle). Let it intersect $\overparen{AB}$ at E . Note that $OF > OE$ since $\overparen{CD}$ is outside segment AB . But this can't be true since $OF = OE$ because they are both radii of the same circle. Thus it cannot be the case that $\overparen{CD}$ is outside segment AB .

2. $\overparen{CD}$ is inside segment AB .



Draw a radius OE where E is on $\overset{\frown}{AB}$. Let OE intersect $\overset{\frown}{CD}$ at F . Note that $OE > OF$ since $\overset{\frown}{CD}$ is inside segment AB . But this can't be true since $OF = OE$ because they are both radii of the same circle. Thus it cannot be the case that $\overset{\frown}{CD}$ is inside segment AB .

3. $\overset{\frown}{CD}$ sometimes lies outside with the middle of segment AB and sometimes inside.



There must exist a point F on $\overset{\frown}{CD}$ that is inside or outside segment AB . Then when a radius OF is drawn, either $OF < OE$ or $OF > OE$, both of which contradict $OE = OF$ (radii). Thus this cannot be the case.

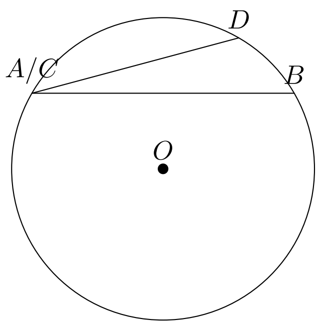
Therefore, the only possibility is that $\overset{\frown}{CD}$ completely coincides with $\overset{\frown}{AB}$. By common notion, the lengths of two completely coinciding curve segments are equal. Thus $\overset{\frown}{AB} = \overset{\frown}{CD}$.

<u>1b:</u> $\overset{\frown}{AB} = \overset{\frown}{CD}$

Let's prove the contrapositive statement: if two chords of a circle are not equal, then they span unequal arcs.

Namely, we want to prove if $AB \neq CD$, then $\overset{\frown}{AB} \neq \overset{\frown}{CD}$.

Refer to the figure. Let AB and CD be two chords of a circle where $CD < AB$. Move chord CD around the circle such that C coincides with A . This preserves the arc length $\overset{\frown}{CD}$ since 'equal chords, equal arcs' . Then D must lie on the circumference of the circle (because OD is a radius).

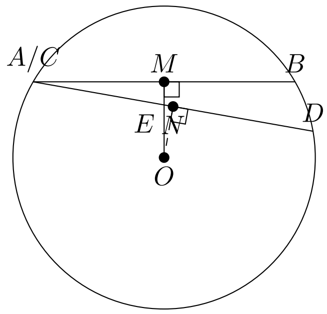


Note that D cannot coincide with B since if so, then $AB = CD$, which contradicts the (new) assumption $CD < AB$.

Thus D can only lie on either minor arc $\overset{\frown}{AB}$ or major arc $\overset{\frown}{AB}$ (denoted 'major$\overset{\frown}{AB}$') , both of which means $\overset{\frown}{AB} \neq \overset{\frown}{CD}$ (as we have either $\overset{\frown}{AD} + \overset{\frown}{DB} = \overset{\frown}{AB}$ or $\overset{\frown}{AB} + \overset{\frown}{BD} = \overset{\frown}{AD}$).

For the sake of completeness, let's show that D can only lie on minor arc $\overset{\frown}{AB}$ if chord CD is above O .

Suppose that D lies on major$\overset{\frown}{AB}$ but CD is above O. Let $OM \perp AB$, and $ON \perp CD$. Then ON and OM are the perpendicular distances of the chords from centre.



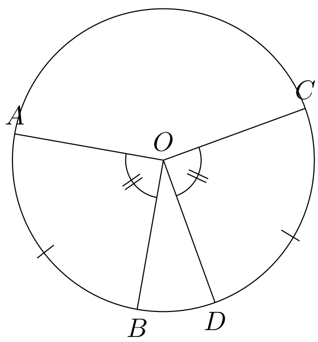
Note that OM is not perpendicular to CD . Otherwise, D will coincide with B . Let OM and CD intersect at E . Note that $OE > ON$ by property of hypotenuse length, and $OM > OE$ since N is between M and O . Thus $OM > ON$, and we have $CD > AB$ by property of chord length, which contradicts the intial assumption $CD < AB$.

Thus, the only possible case is that D lies on $\overset{\frown}{AB}$, which means $\overset{\frown}{CD} < \overset{\frown}{AB}$.

Thus we have proved that shorter chords span shorter arcs. By similar reasoning, we can also prove that longer chords span longer arcs (but I'm too lazy to show it here). □

Preposition 27. Equal arcs of a circle subtend equal angles at the centre. (equal arcs, equal $\angle$s) *

Conversely, equal angles at the centre are subtended by equal arcs. (equal $\angle$s, equal arcs) *



1a:

$$\because \overset{\frown}{AB} = \overset{\frown}{CD}$$
$$\therefore\ \angle AOB = \angle COD \qquad \text{(equal arcs, equal } \angle\text{s)}$$

1b:

$$\because \angle AOB = \angle COD$$
$$\therefore\ \overset{\frown}{AB} = \overset{\frown}{CD} \qquad \text{(equal } \angle\text{s, equal arcs)}$$

Proof. **1a:** $\overset{\frown}{AB} = \overset{\frown}{CD}$

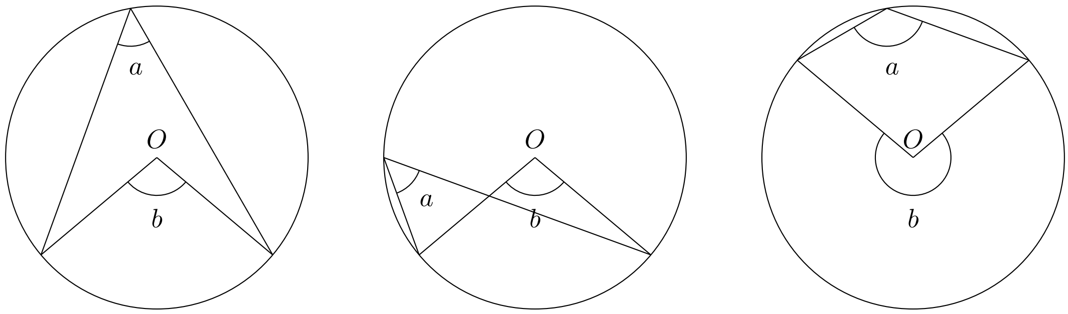
By 'equal arcs, equal chords', we have $AB = CD$. By 'equal chords, equal $\angle$s' , we have $\angle AOB = \angle COD$.

1b: $\angle AOB = \angle COD$

By 'equal $\angle$s, equal chords', we have $AB = CD$. By 'equal chords, equal arcs' , we have $\overset{\frown}{AB} = \overset{\frown}{CD}$. □

1.6.5 Inscribed angle properties

Preposition 28. For the same arc or same chord of a circle, the angle subtended at the centre is twice the angle subtended at the circumference. ($\angle$ at centre twice $\angle$ at $\odot^{ce}$) *



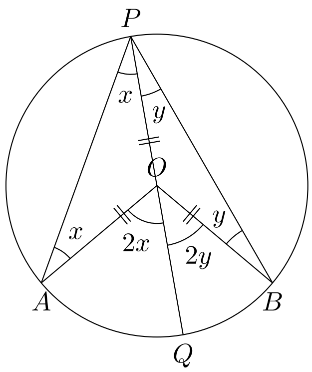
Case 1 Case 2 Case 3

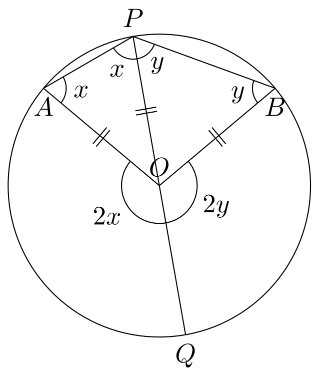
$$b = 2a \qquad (\angle \text{ at centre twice } \angle \text{ at } \odot^{ce})$$

Proof. Let A and B be the endpoints of the subtending arc, and P be vertex of angle at the circumference. Let $\angle APB = a$ and $\angle AOB = b$. Let's consider the cases:

<u>Case 1 & Case 3:</u>

Extend PO to meet the other side of the circumference at Q .



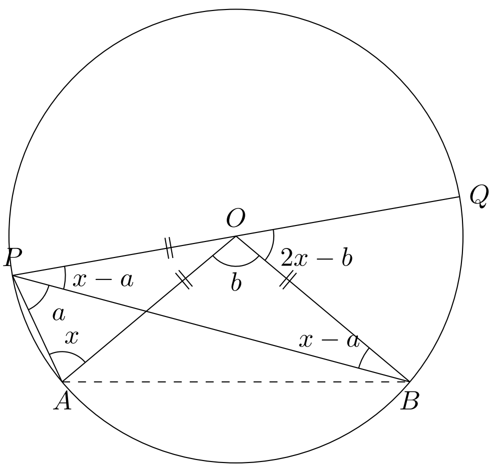


Since $OP = OA$ (radii) , we have $\angle OPA = \angle OAP$ (base $\angle$s, isos. $\triangle$).
Since $OB = OP$ (radii) , we have $\angle OBP = \angle OPB$ (base $\angle$s, isos. $\triangle$).
Let $\angle OPA = \angle OAP = x$ and $\angle OBP = \angle OPB = y$. Note that

$$\begin{aligned} \angle AOQ &= \angle OAP + \angle OPA = 2x \qquad (\text{ext. } \angle \text{ of } \triangle) \\ \angle BOQ &= \angle OBP + \angle OPB = 2y \qquad (\text{ext. } \angle \text{ of } \triangle) \\ \therefore\ AOB &= 2x + 2y = 2(x+y) = 2 \cdot \angle APB \\ \therefore\ b &= 2a \end{aligned}$$

<u>Case 2:</u>

Extend PO to meet the other side of the circumference at Q .



Since $OP = OA$ (radii) , we have $\angle OPA = \angle OAP$ (base $\angle$s, isos. $\triangle$).
Since $OB = OP$ (radii) , we have $\angle OBP = \angle OPB$ (base $\angle$s, isos. $\triangle$).
Let $\angle OPA = \angle OAP = x$. Note that $\angle OBP = \angle OPB = x - a$.
Note that in $\triangle OAP$, $\angle AOQ = \angle OAP + \angle OPA = 2x$. Thus $\angle BOQ = 2x - b$.

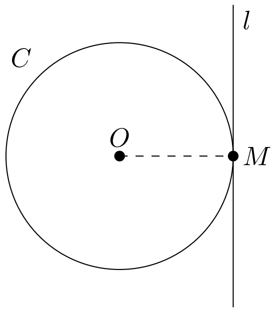
Note that in $\triangle OBP$, $\angle BOQ = \angle OBP + \angle OPB = 2(x - a)$.
Since $\angle BOQ = 2x - b = 2(x - a)$, we have

$$
\begin{aligned}
2x - b &= 2x - 2a \\
b &= 2a
\end{aligned}
$$

□

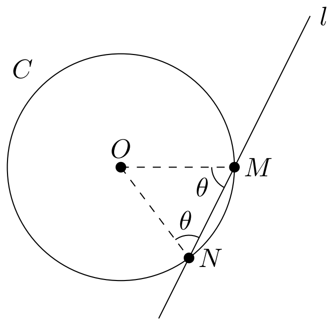
1.6.6 Tangent properties

Preposition 29. If a line is tangent to a circle, then the line segment joining the point of tangency and the centre of the circle is perpendicular to the given line. (tangent $\perp$ radius) *



Given: line l is tangent to circle C
$\therefore$ $OM \perp l$ (tangent $\perp$ radius)

Proof. Let M be a point of tangency of circle C . Suppose OM is not perpendicular to l . Let θ be the smaller angle formed between OM and l . Note that $\theta < 90°$.

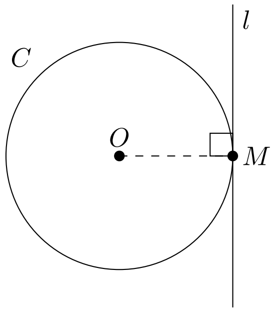


By property of falling lines, there exists one other point (call it N) on l such that the angle formed between ON and l is θ . Thus we have $OM = ON$ (sides opp. equal $\angle$s). Since OM is the radius of the circle, N must also be an intersection point of line l and circle C , which contradicts the assumption that l intersects with C at only one point M (since it is point of tangency).

Thus it can only be the case that $OM \perp l$.

□

Preposition 30. If a line and a circle intersect at some point such that the line segment joining this point and the centre of the circle is perpendicular to the given line, then the given line is tangent to the circle. (converse of tangent $\perp$ radius) *



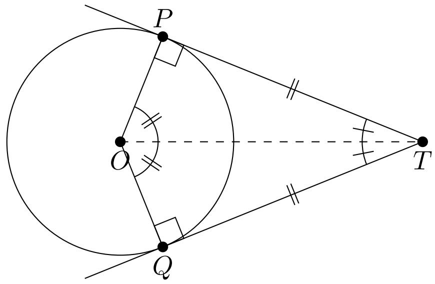
Given: $OM \perp l$
$\therefore$ M is the point of tangency, which means C and l intersect at M only.
(converse of tangent $\perp$ radius)

Proof. Let r be the radius of the circle. By property of hypotenuse length, OM is the only point on l such that $OM = r$, and if there is another point P on l (label it P) that is not M , then $OP > r$, so P must lie outside the circle. Thus there can only be one intersection of the line and the circle, so M must be the point of tangency. □

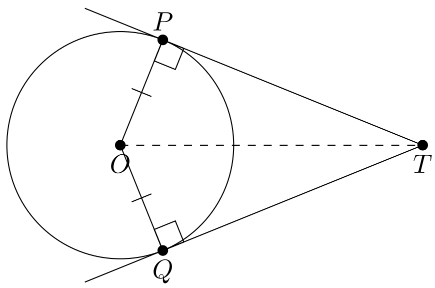
Preposition 31. If there is a circle with centre O , and T is a point outside the circle, and two tangent lines passing through T touch the circle at P and Q respectively, then the following properties are true:

1. $TP = TQ$
2. $\angle TOP = \angle TOQ$
3. $\angle OTP = \angle OTQ$

(tangent properties) *



Proof. Since P and Q are points of tangency, we have $\angle OPT = \angle OQT = 90°$ (tangent $\perp$ radius) .

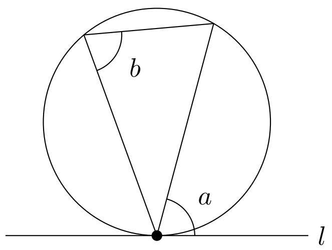


In $\triangle POT$ and $\triangle QOT$,

$$\begin{aligned}
\angle OPT &= \angle OQT && \text{(tangent } \perp \text{ radius)}\\
OT &= OT && \text{(common side)}\\
OP &= OQ && \text{(radii)}\\
\therefore \triangle POT &\cong QOT && \text{(RHS)}\\
\therefore TP &= TQ && \text{(corr. sides, } \cong \triangle\text{s)}\\
\angle TOP &= \angle TOQ && \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}\\
\angle OTP &= \angle OTQ && \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}
\end{aligned}$$

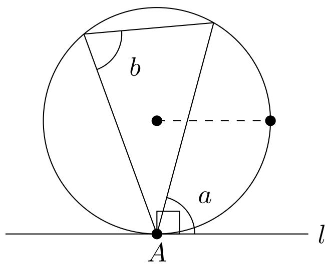
□

Preposition 32. If a tangent line and a chord of a circle intersect at the point of tangency, then the angle formed between the tangent line and the chord is equal to the angle subtended by the chord in the alternate segment. $\quad$ ($\angle$ in alt. segment) *



Given: line l is tangent to the circle.
$\therefore\ a = b \qquad (\angle \text{ in alt. segment})$

Proof. .



$\square$

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