Euclidea Solutions Explained

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Abstract

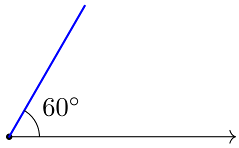
Euclidea is a puzzle game in which the player has to construct geometrical figures using compass and straight edge only. It is probably the most difficult puzzle game I've played. The complexity of Euclidean Geometry gives rise to so many possible constructions, and it is difficult to brute force solving (at least I don't know how). On top of that, the player has to use the minimum number of moves possible to pass the level in order to get 3 stars and unlock the next level pack. There is just no way I can come up with the solutions myself, so I've cheated by looking up the solutions online. Nonetheless, it is an interesting math-related game that is one of a kind.

Contents

1 Alpha

1.1 Angle of 60 deg

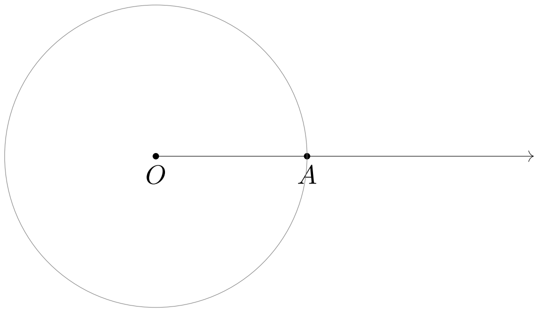
Task 1.1. Construct an angle of 60° with the given side.
(3L, 3E, 2V)



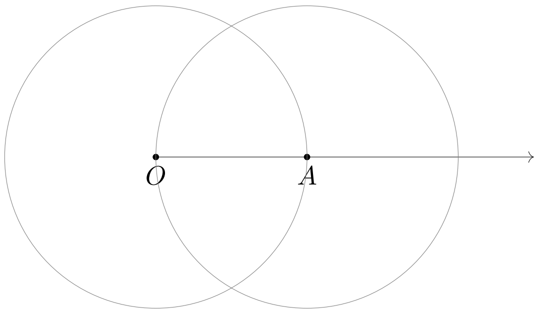
(Arrowhead means the line is infinitely long.)

Solution 1.1. (3L, 5E)

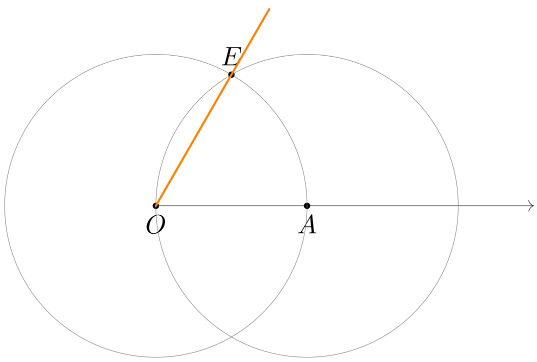
1. Let O be the endpoint of the given ray. Label an arbitrary point A on the given ray. Draw circle centered O through A.



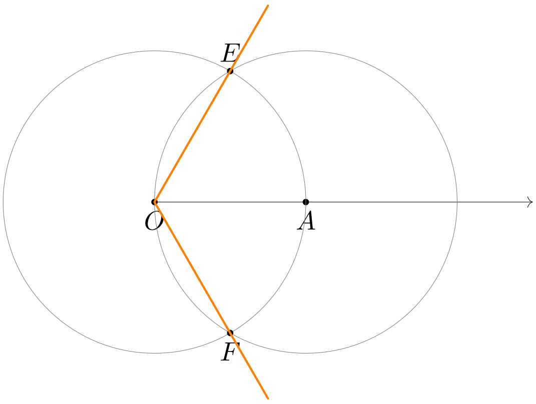
2. Draw circle centered A through O.



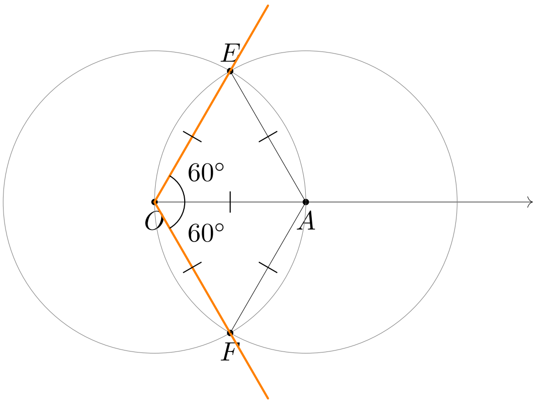
3. Let E be one of the intersections of the two circles. Draw line OE. We get the desired 60° angle.



(2V: Extra solutions) Let F be another intersections of the two circles. Draw line OF. We get another 60° angle.



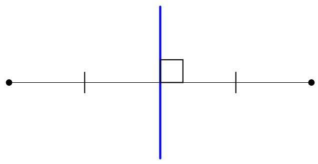
Proof. To see why $\angle AOE$ and $\angle AOF$ are $60°$ angles, first note that the two circles have the same radii since they share the same segment OA. Thus OA, OE, AE, OF, AF all have lengths equal to the radii of the circles, so $\triangle OAE$ and $\triangle OAF$ are equilateral triangles.



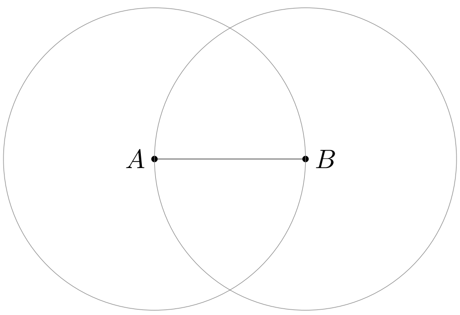
By "prop. of equil. $\triangle$", all the interior angles of equilateral triangle is $60°$, meaning $\angle AOE = \angle AOF = 60°$. □

1.2 Perpendicular bisector

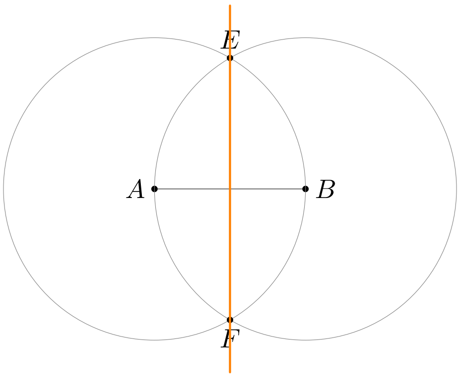
Task 1.2. Construct the perpendicular bisector of the segment.
(3L, 3E)



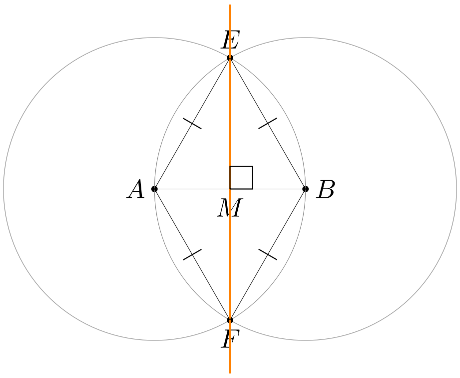
Solution 1.2. **1, 2.** Let A and B be the endpoints of the given segment. Draw circle centered A through B. Draw another circle centered B through A.



3. Let E, F be intersection of the two circles. Draw line EF. We get the desired perpendicular bisector.



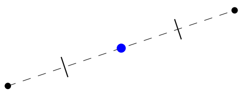
Proof. Let AB and EF intersect at M. Since $AE = BE = AF = BF$, $AEBF$ is a rhombus. By property of rhombus, the diagonals AB and EF are perpendicular to each other.



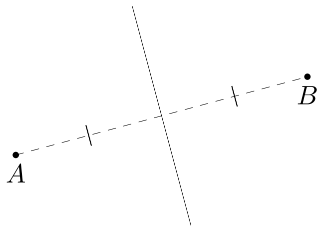
Moreover, since $AEBF$ is a rhombus, $AEBF$ is a parallelogram. By "diags. of //gram", the diagonals AB and EF bisector each other, giving $AM = MB$. Thus, EF is the perpendicular bisector of AB. □

1.3 Midpoint

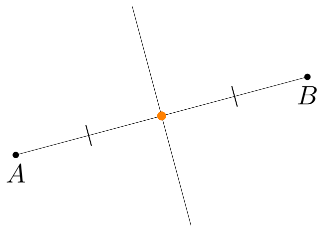
Task 1.3. Construct the midpoint of the segment defined by two points.
(2L, 4E)



Solution 1.3. 1. Let A, B be the endpoints of the given segment. Draw the perpendicular bisector of AB.



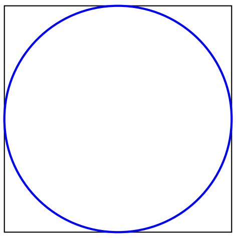
2. Draw line AB. The intersection of AB and the perpendicular bisector is the desired midpoint.



Proof. By definition, perpendicular bisector bisects AB. So the intersection of AB and the perpendicular bisector is the midpoint of AB. □

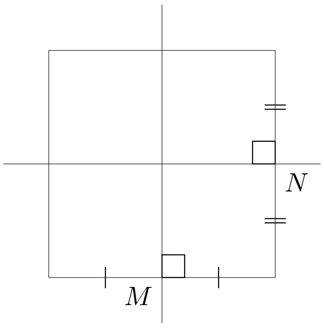
1.4 Circle in square

Task 1.4. Inscribe a circle in the square.
(3L, 5E)

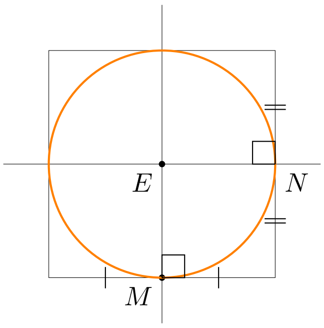


Solution 1.4. (3L)

1, 2. Draw perpendicular bisectors of two adjacent sides of the square. Let M, N be midpoints of these two sides.



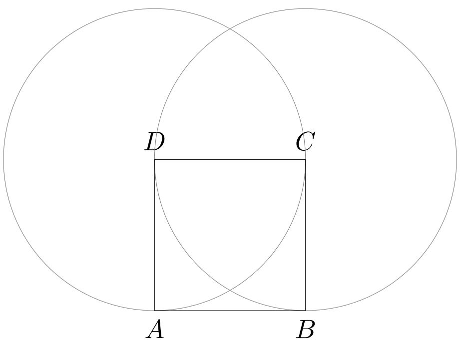
3. Let E be the intersection of perpendicular bisectors. Draw circle centered E through M (or N). We get the desired circle.



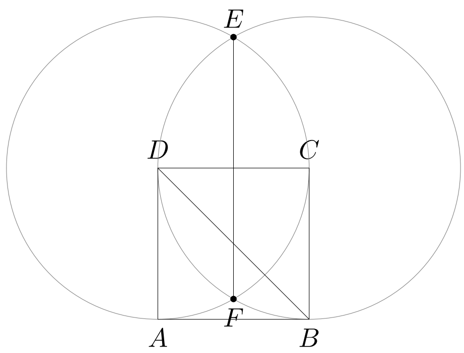
Proof. Note that the perpendicular bisectors divide the big square into four smaller squares of the same side length, so the circle with radius EM passes through all the midpoints of the sides of big square. By 'converse of tangent $\perp$ radius", the circle is tangent to the four sides of the big square, which means it is inscribed in the square. □

(5E)

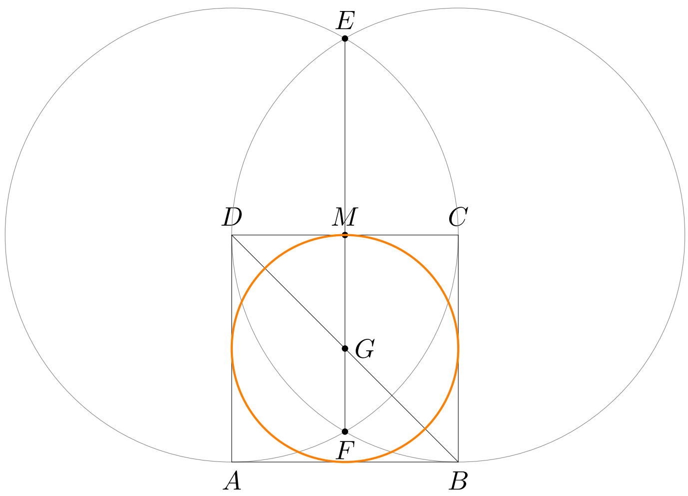
1, 2. Let vertices of square A, B, C, D. Draw circle centered D through C, and draw circle centered C through D.



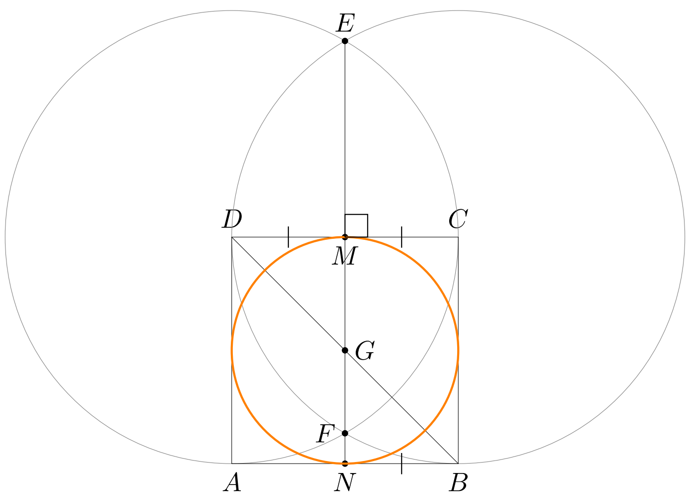
3, 4. Draw line BD. Let the intersections of the circles be E, F. Draw line EF.



5. Let G be the intersection of BD and EF, and let M be the intersection of CD and EF. Draw circle centered G through M. We get the desired circle.



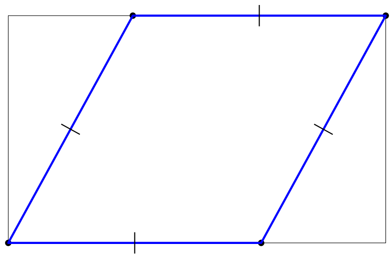
Proof. Note that EF is the perpendicular bisector of CD (by Task 1.2), so $DM = MC$. Extend MF to meet AB at N. We also have $DM = NB$ since MN divides square $ABCD$ into two congruent rectangles. Also note that $\angle GDM = \angle GBN$ (alt. $\angle$s, $DC // AB$).



Thus $\triangle DMG \cong \triangle BNG$ (AAS), so G is the midpoint of MN (corr. sides, $\cong$ $\triangle$s). This means G is the center of the square (same point as "E" in previous 3L solution), so the circle centered G through M is the inscribed circle of the square. $\square$

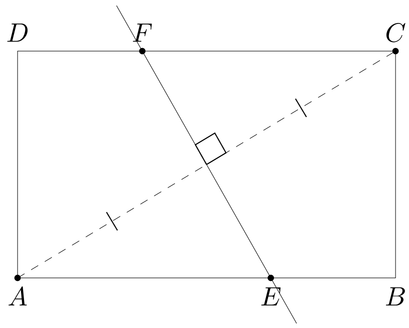
1.5 Rhombus in rectangle

Task 1.5. Inscribe a rhombus in the rectangle so that they share a diagonal.
(3L, 5E, 2V)

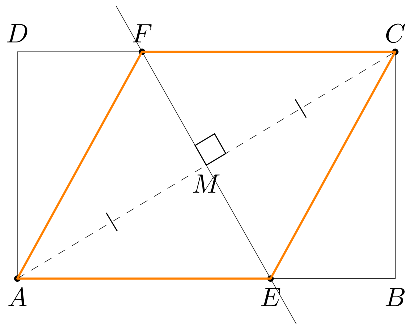


Solution 1.5. (3L, 5E)

1. Let the given rectangle be $ABCD$. Draw perpendicular bisector of AC, and let it intersect AB and CD at E and F respectively.



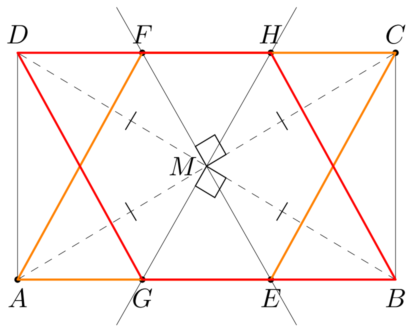
2, 3. Draw AF and EC. We get the desired rhombus $AECF$.



Proof. Let M be the midpoint of AC. Note that $AM = CM$. Also, $\angle MFC = \angle MEA$ (alt. $\angle$s, $FC//AE$). Thus $\triangle MFC \sim \triangle MEA$ (AAS), and $CF = AE$ (corr. sides, $\cong$ $\triangle$s)

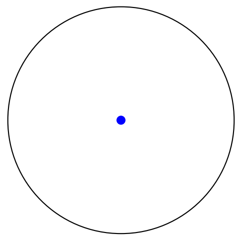
Note that $AF = CF$ and $AE = CE$ by property of perpendicular bisector. Combined with $CF = AE$, we have $AE = CE = AF = CF$, which means $AECF$ is a rhombus. $\square$

(2V). Similarly argument but flipped horizontally.



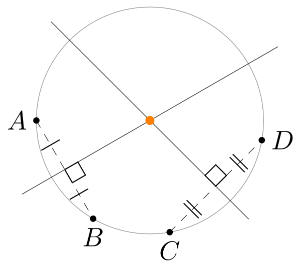
1.6 Circle center

Task 1.6. Construct the center of the circle.
(2L, 5E)



Solution 1.6. (2L)

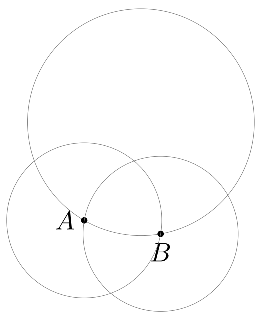
1, 2. Label two pairs of arbitrary points on the circle, and draw the perpendicular bisector of each pair of point. The intersection of the perpendicular bisectors is the desired center of circle.



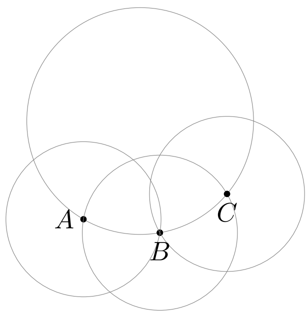
Proof. Perpendicular bisector of any chord passes through the center of a circle ("$\perp$ bisector of chord passes through center"). This means the center of circle lies on both perpendicular bisectors of AB and CD, so it must be their point of intersection. □

(5E)

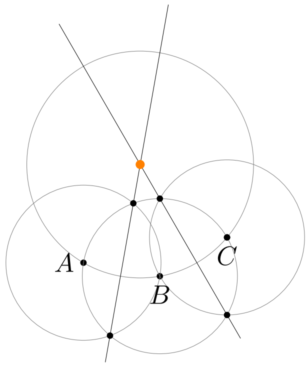
1, 2. Label two arbitrary points A and B. Draw circle centered A through B, and draw circle centered B through A.



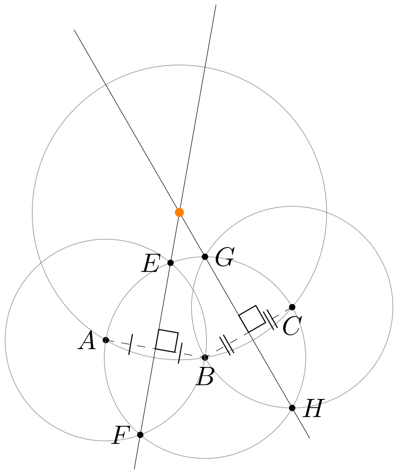
3. (Let (B, A) denote the circle centered B through A.) Let circle (B, A) intersect the given circle at another point C. Draw circle centered C through B.



4, 5. Draw line through the intersections of circles (A, B) and (B, A), and draw line through the intersections of circles (B, C) and (C, B). The intersection of these two lines is the desired center.



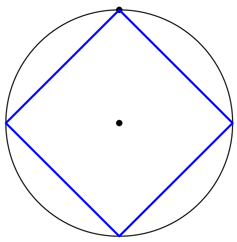
Proof. Note that EF and GH are perpendicular bisectors of chords AB and BC respectively. So they intersect at the center of the circle.



□

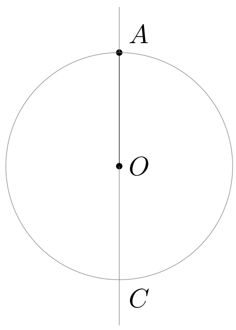
1.7 Inscribed square

Task 1.7. Inscribe a square in the circle. One vertex of the square is given. (The circle center is also given.)
(6L, 7E)

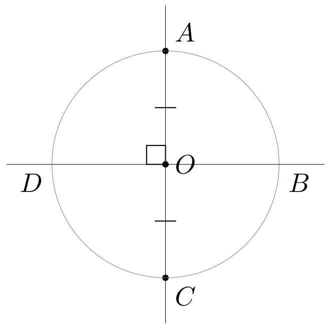


Solution 1.7. (6L)

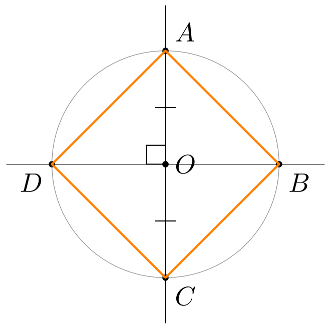
1. Let O be center of circle and A be the given vertex. Draw line AO. Let AO intersect the circle at C.



2. Draw the perpendicular bisector of AC. Let it intersect the circle at B and D.



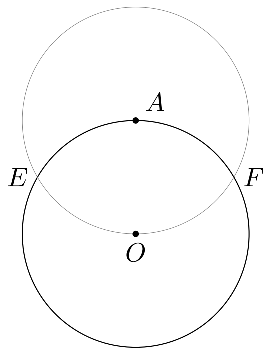
3, 4, 5, 6. Draw lines AB, BC, CD, DA. We get the desired square.



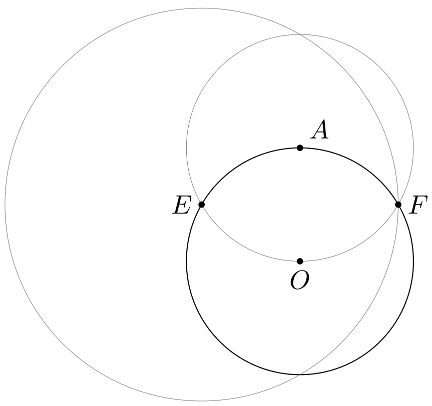
Proof. Note that the perpendicular bisector of AC passes through circle center O. So we have $OA = OB = OC = OD$. Since the diagonals of $ABCD$ are perpendicular and bisect each other, $ABCD$ is a square (con. of square). □

(7E)

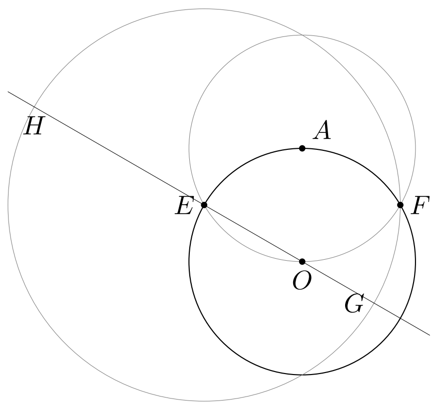
1. Draw circle centered A through O. Let the intersections of two circles be E and F.



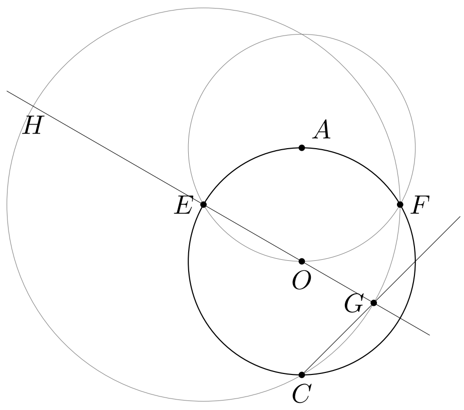
2. Draw circle centered E through F.



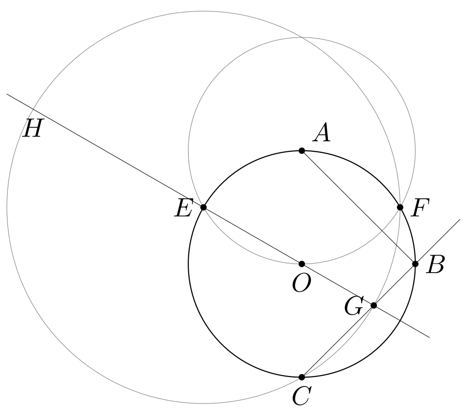
3. Draw line EO. Let EO intersect circle (E, F) at G and H, where G lies inside the given circle and H lies outside.



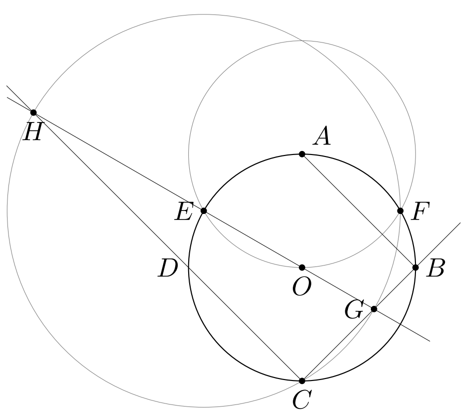
4. Let C be another intersection of (E, F) and the given circle. Draw line CG.



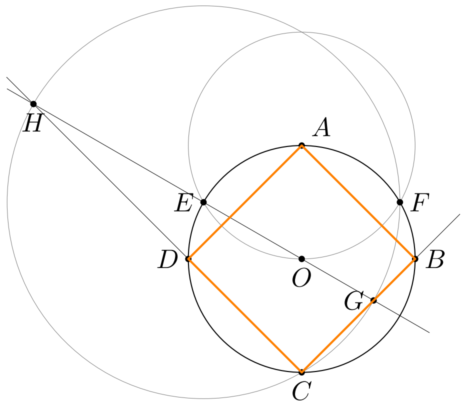
5. Let CG intersect given circle at another point B. Draw line AB.



6. Draw line CH. Let CH intersect given circle at D.

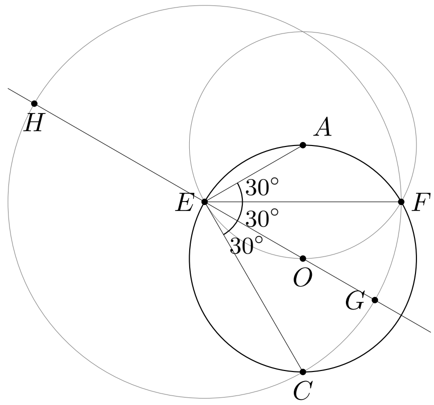


7. Draw line AD. $ABCD$ is the desired square.



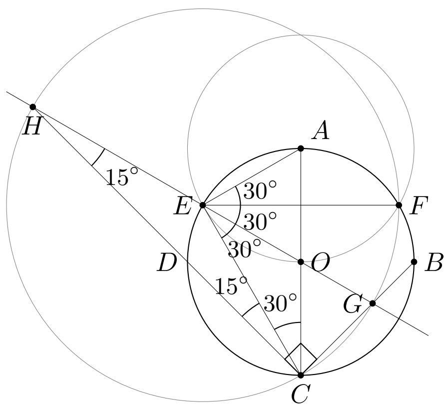
Proof. **1-3.** Note that EF bisects $\angle OEA$ since EF is the diagonal of rhombus $AEOF$ which is made up of two equilateral triangles $\triangle OAE$ and $\triangle OAF$. Thus $\angle AEF = \angle OEF = 60°/2 = 30°$.

Also, note that $\angle OEC = \angle OEF = 30°$ since $\triangle OEC \sim \triangle OEF$ (SSS). Thus $\angle AEC = 30° + 30° + 30° = 90°$. By "converse of $\angle$ in semi-circle", AC is the diameter of given circle, which means A, O, C are collinear.

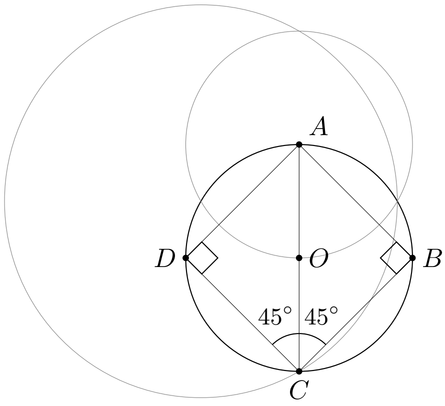


4-7. Note that GH is a diameter of circle (E, G), so $\angle HCG = 90°$ ($\angle$ in semi-circle).

Note that $EH = EC$ (radii), so $\angle ECH = \angle EHC = 30°/2 = 15°$ (base $\angle$s, isos. $\triangle$)& (ext. $\angle$ of $\triangle$). Also, $\angle OCE = \angle OEC = 30°$ (base $\angle$s, isos. $\triangle$).



Thus, $\angle OCD = 30° + 15° = 45°$, and $\angle OCB = \angle OCG = 90° - 45° = 45°$.

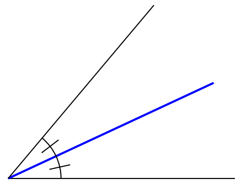


Let's focus on points A, B, C, D. Note that $\angle ADC = \angle ABC = 90°$ ($\angle$ in semi-circle), $\angle ACD = \angle ACB = 45°$, and $AC = AC$. Thus $\triangle ADC \cong \triangle ABC$ (AAS) and $BC = CD$ (corr. sides, $\cong \triangle$s). Since $ABCD$ has four right angles and adjacent sides are equal, $ABCD$ is a square (con. of square), as desired. $\square$

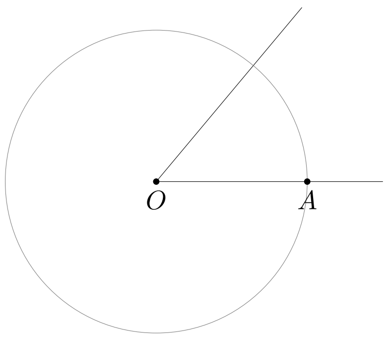
2 Beta

2.1 Angle bisector

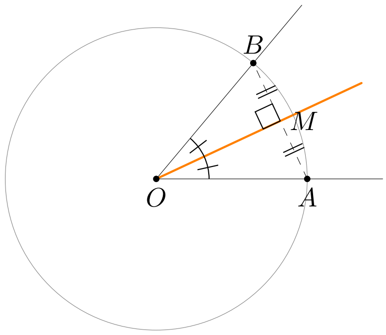
Task 2.1. Construct the line that bisects the given angle.
(2L, 4E)



Solution 2.1. **1.** Let O be the vertex of the given angle. Label an arbitrary point A on one of the given rays. Draw circle (O, A).



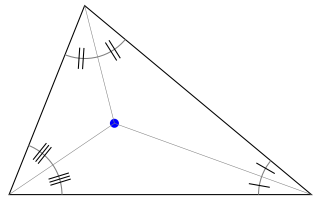
2. Let B be the intersection of the circle and the other ray. Draw perpbi AB (perpendicular bisector of A, B), which is the desired angle bisector.



Proof. Note that $\triangle OAB$ is an isosceles triangle since $OA = OB$ (radii). Let M be the midpoint of AB. Since $OM \perp AB$, by "prop. of isos. $\triangle$", we have $\angle AOM = \angle BOM$. $\square$

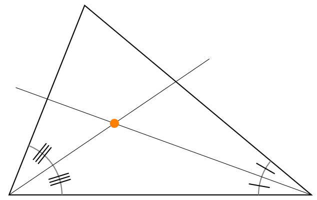
2.2 Intersection of angle bisectors

Task 2.2. Construct the point where the angle bisectors of the triangle are intersected. (2L, 6E)



Solution 2.2. (2L)

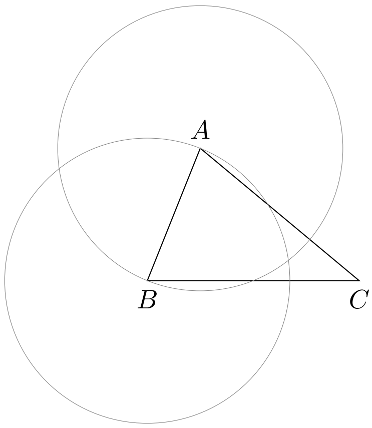
1, 2. Draw angle bisectors of two of the vertices of the triangle. Their intersection is the desired point.



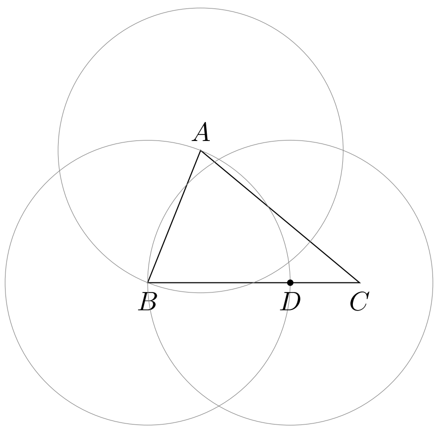
Proof. Note that the three angle bisectors of a triangle are concurrent (prop. of $\angle$ bisector). So we only need to find the intersection of two of them. $\square$

(6E)

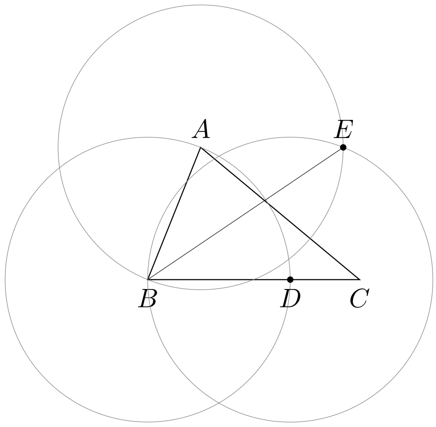
1, 2. Let the vertices of triangle be A, B, C. Draw circle (A, B) and circle (B, A).



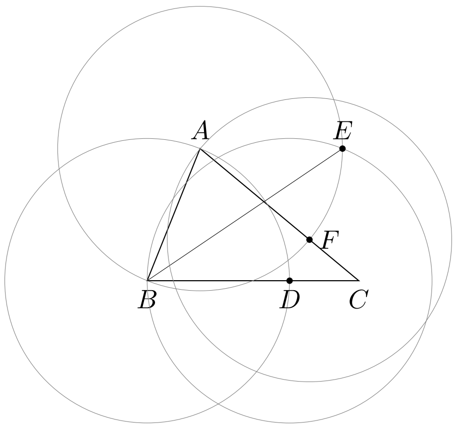
3. Let circle (B, A) intersect side BC at D. Draw circle (D, B).



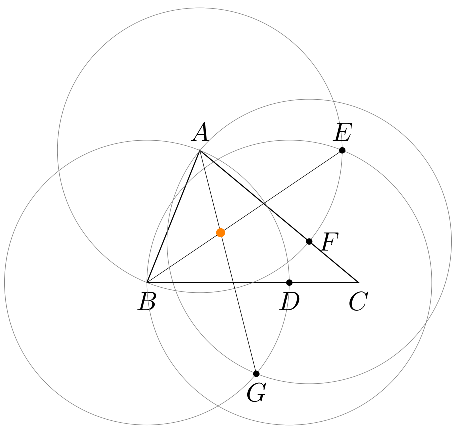
4. Let (D, B) and (A, B) intersect at another point E. Draw line BE.



5. Let (A, B) intersect side AC at F. Draw circle (F, A).

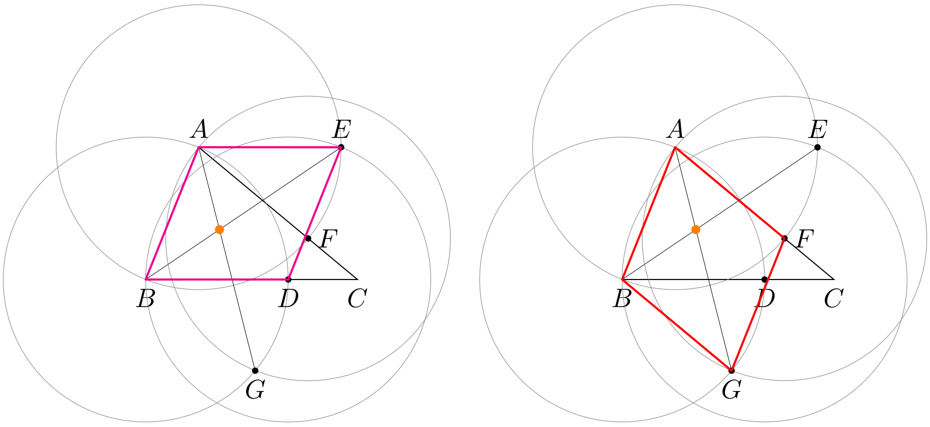


6. Let (F, A) and (B, A) intersect at another point G. Draw line AG. The intersection of BE and AG is the desired point.



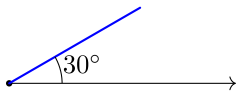
Proof. **1-4.** Let r be the length of AB. Note that $AE = AB = BD = DE$ since they are all radii of circles with radius r. So $ABDE$ is a rhombus. Since BE is a diagonal of the rhombus, BE bisects $\angle B$ (prop. of rhombus).

5-6. Similarly, since $AB = BG = FG = FA$, $ABGF$ is a rhombus of side length r. Since AG is a diagonal of rhombus $ABGF$, AG bisects $\angle A$.

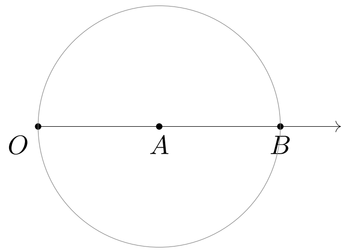


□

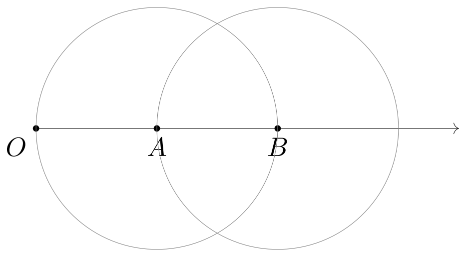
2.3 Angle of 30 deg

> **Task 2.3.** Construct an angle of 30° with the given side.
> (3L, 3E, 2V)
>
> 
>

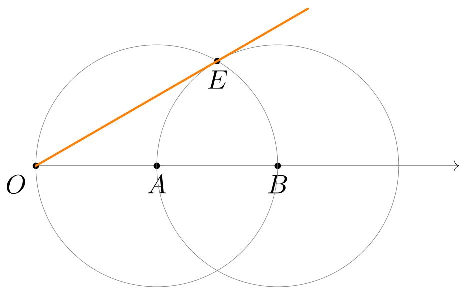
Solution 2.3. (3L, 3E) 1. Let O be the endpoint of the given ray, and A be an arbitrary point on the given ray. Draw circle (A, O), intersecting given ray at B.



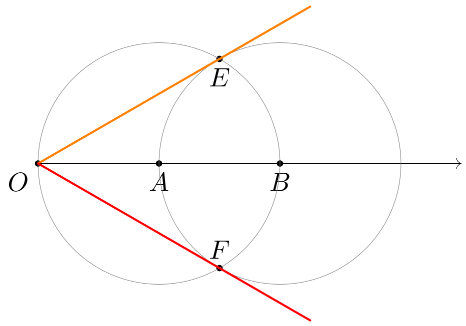
2. Draw circle (B, A).



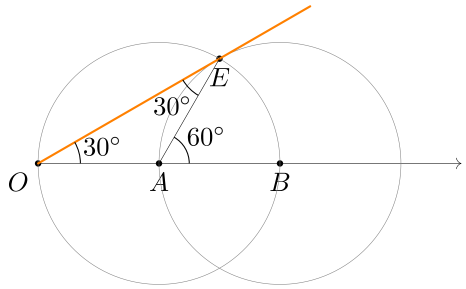
3. Let E be one intersection of the two circles. Draw line OE, which is the desired line.



(2V) 4. Let F be another intersection of the two circles. Draw line OF.



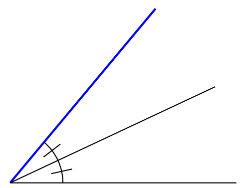
Proof. Note that $\angle EAB = 60°$ by construction. Also, $AO = AE$ (radii) so $\angle AOE = \angle AEO$ (base $\angle$s, isos. $\triangle$). So $\angle EOA = 60°/2 = 30°$ (ext. $\angle$ of $\triangle$). Similar argument for the other line.



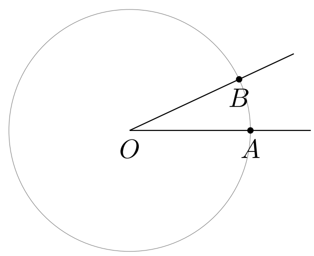
□

2.4 Double angle

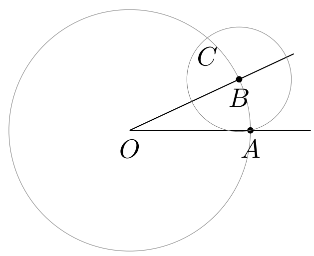
Task 2.4. Construct an angle equal to the given one so that they share one side.



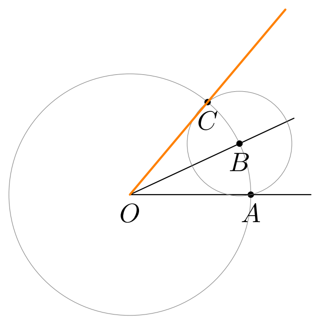
Solution 2.4. **1.** Let O be the vertex of given angle, and A be an arbitrary point on one ray. Draw circle (O, A), intersecting the other ray at B.



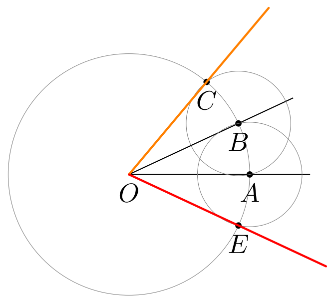
2. Draw circle (B, A), intersecting (O, A) at another point C.



3. Draw line OC, which is the desired line.



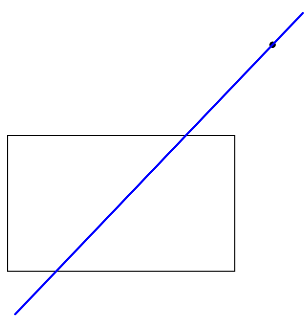
(2V) Draw circle (A, B), intersecting (O, A) at D. Draw line OD.



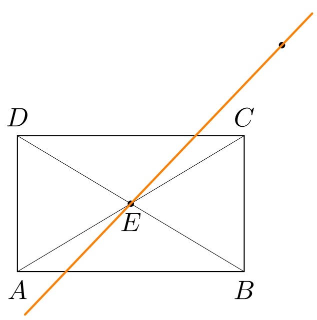
Proof. Note that $BA = BC$ (radii), so $\angle BOC = \angle BOA$ (equal chord, equal $\angle$). Similar argument for line OE. $\square$

2.5 Cut rectangle

Task 2.5. Construct a line through the given point that cuts the rectangle into two parts of equal area.
(3L, 3E)



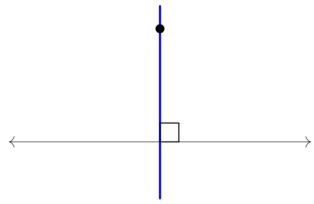
Solution 2.5. Let the rectangle be $ABCD$. **1, 2.** Draw diagonals AC and BD, intersecting at E.
3. Draw line through given point and E.



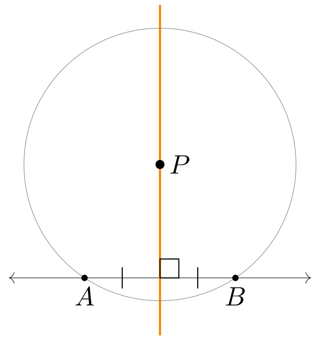
Proof. Let the orange line intersect the rectangle sides CD and AB at G and H respectively. Note that $\triangle EAH \cong \triangle ECG$. So we can move $\triangle EAH$ to $\triangle ECG$ and vice versa to create $\triangle ADC$ and $\triangle CBA$ which are equal in area (because they are half of rectangle). $\square$

2.6 Drop a perpendicular

Task 2.6. Drop a perpendicular from the point to the line.
(2L, 3E)

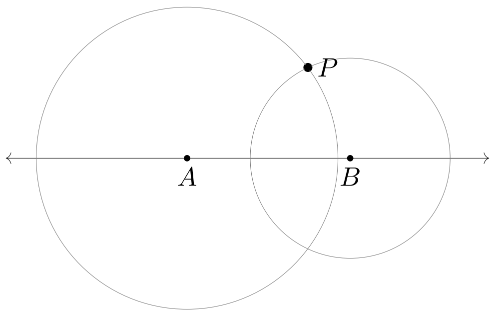


Solution 2.6. (2L) Let the given point be P, and A be an arbitrary point on given line.
1. Draw circle (P, A), intersecting the line on B.
2. Draw perpbi AB.

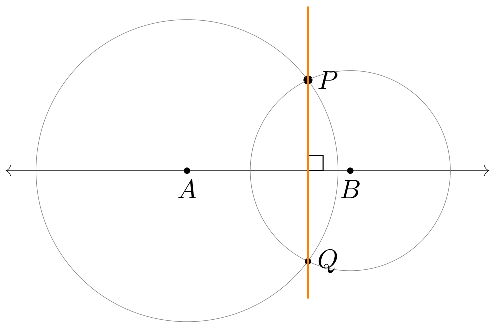


Proof. AB is a chord of the circle, so the perpendicular bisector of AB passes through center P. This means we have constructed a line through P that is perpendicular to line AB. □

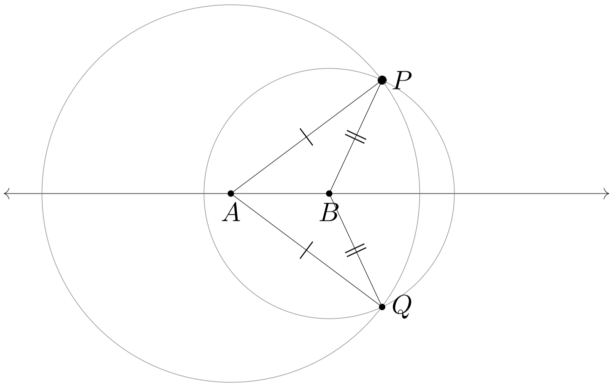
(3E) 1, 2. Label two arbitrary points A, B. Draw circles (A, P) and circle (B, P).



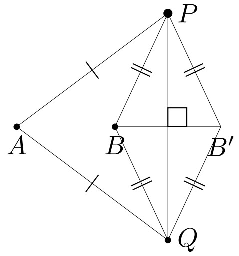
3. Draw line through the intersection of the two circles, which is the desired line.



Proof. Let Q be the other intersection of the two circles. Note that $AP = AQ$ and $BP = BQ$ (radii), so $APBQ$ is either a kite or a dart. If $APBQ$ is a kite, then by "prop. of kite", the diagonals of the kite are perpendicular to each other, meaning $PQ \perp AB$.



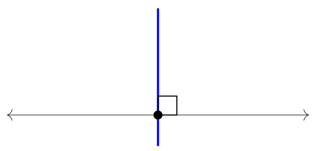
If $APBQ$ is a dart with B being the concave point, then reflect B about line PQ to get B'. Note that $PQ \perp BB'$ and $BPB'Q$ is a rhombus (by reflection). Since $APB'Q$ is a kite, we also have $PQ \perp AB'$. Thus AB' and BB' are parallel, but they share the same point B', so A, B, B' must lie on the same line. This means $PQ \perp AB$, our desired result.



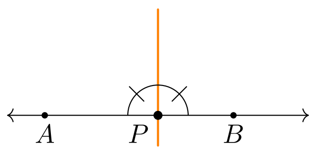
□

2.7 Erect a perpendicular

Task 2.7. Erect a perpendicular from the point on the line.
(1L, 3E)

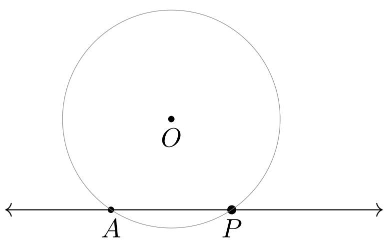


Solution 2.7. (1L) Let P be the given point. Let A be an arbitrary point to the left of P and B be an arbitrary point to the right of P. Draw the angle bisector of $\angle AOB$, which is the desired line.

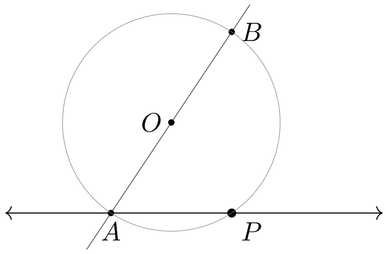


Proof. Since A, O, P are on a straight line, $\angle AOP = 180°$, so the angle bisector makes two angles of $90°$, which means the angle bisector is perpendicular to line AOB. □

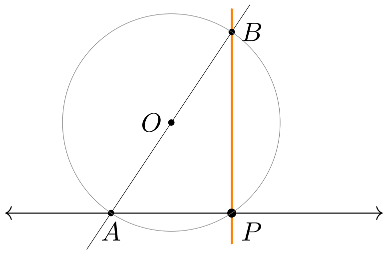
(3E) 1. Label an arbitrary point O not on the given line. Draw circle (O, P), intersecting the given line at another point A.



2. Draw line AO. Let it intersect the circle at B.



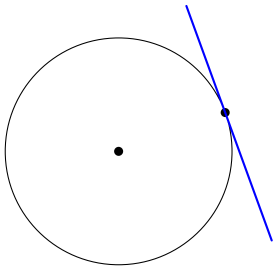
3. Draw line BP, which is the desired line.



Proof. Note that AB is the diameter of the circle, so $\angle APB = 90^\circ$ ($\angle$ in semi-circle), which means $BP \perp AP$. $\square$

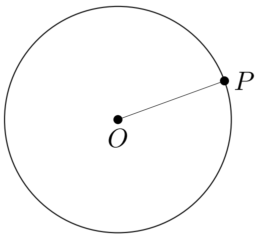
2.8 Tangent to circle at point

Task 2.8. Construct a tangent to the circle at the given point.
(2L, 3E)

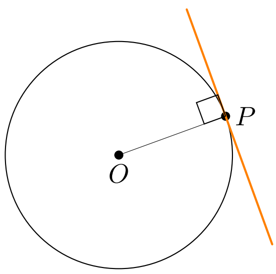


Solution 2.8. Let O be the center of circle and P be the given point on the circle.

(2L) 1. Draw line OP.

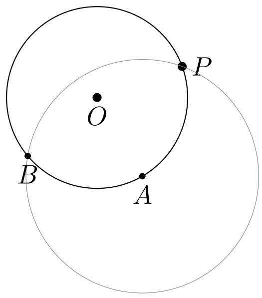


2. Draw the perpendicular line of OP at P.

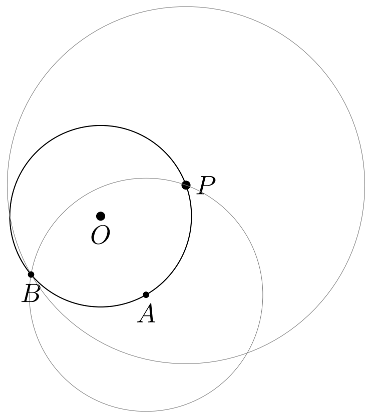


Proof. Since OP is a radius of the circle and is perpendicular to the orange line, by "converse of tangent $\perp$ radius", the orange line is tangent to the circle at P. $\square$

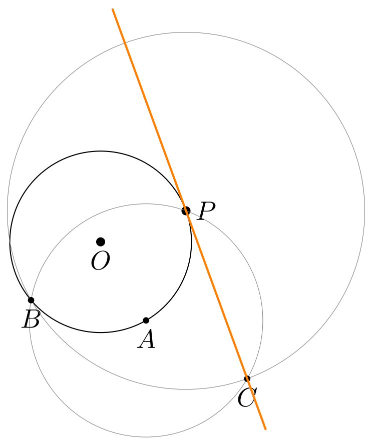
(3E) 1. Let A be an arbitrary point on the given circle. Draw circle (A, P), intersecting the given circle at B.



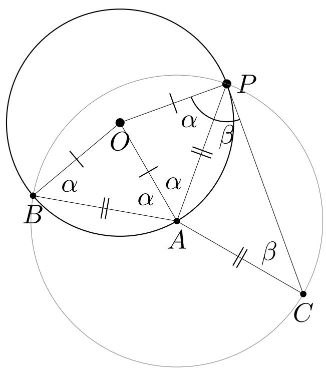
2. Draw circle (P, B).



3. Let (P, B) intersect (A, P) at another point C. Draw line PC, the desired line.



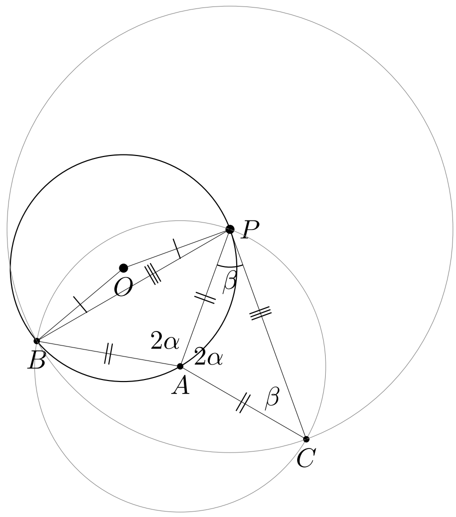
Proof. Let $\angle OPA = \alpha$ and $\angle APC = \beta$. We want to show that $\alpha + \beta = 90°$, which will prove that PC is the tangent to the given circle at P.



$$\begin{aligned} OA &= OP && \text{(radii)} \\ \therefore \angle OAP &= \angle OPA = \alpha && \text{(base } \angle\text{s, isos. } \triangle) \end{aligned}$$

$$\triangle OBA \cong \triangle OPA \qquad \text{(SSS)}$$
$$\therefore\ \angle OBA = \angle OPA = \alpha \ \text{ and } \ \angle OAB = \angle OAB = \alpha \ \ \text{(corr. } \angle\text{s, } \cong \triangle\text{s).}$$

Now consider $\triangle APC$ and $\triangle APB$.

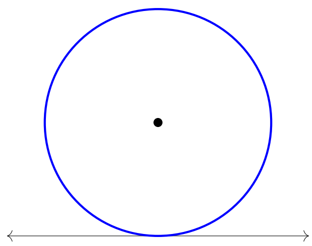


$$AC = AP \qquad \text{(radii)}$$
$$\therefore \angle ACP = \angle APC = \beta \qquad \text{(base } \angle\text{s, isos. } \triangle\text{)}$$
$$PB = PC \qquad \text{(radii of biggest circle)}$$
$$\therefore\ \triangle APC \cong \triangle APB \qquad \text{(SSS)}$$
$$\therefore\ \angle PAC = \angle PAB = 2\alpha \qquad \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}$$

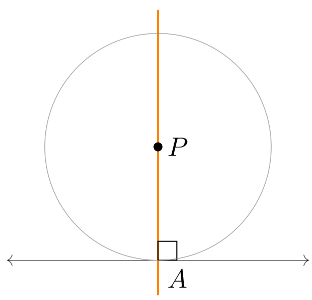
In $\triangle APC$, we have $2\alpha + \beta + \beta = 180°$ ($\angle$ sum of $\triangle$), giving $\alpha + \beta = 90°$, as desired. ☐

2.9 Circle tangent to line

Task 2.9. Construct a circle with the given center that is tangent to the given line. (2L, 4E)



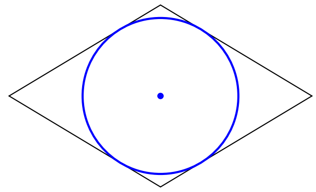
Solution 2.9. **1.** Draw line perpendicular to the given line passing through given point P.
2. Draw circle centered P through the intersection of the two lines A.



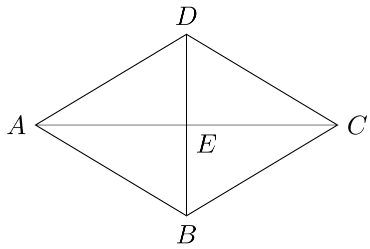
Proof. PA is tangent to the given line by "converse of tangent $\perp$ radius". ☐

2.10 Circle in rhombus

Task 2.10. Inscribe a circle in the rhombus.
(4L, 6E)

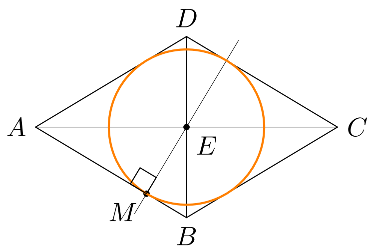


Solution 2.10. **1, 2.** Let the rhombus be $ABCD$. Draw diagonals AC and BD. Let them intersect at E.



3. Draw $ME \perp AB$ (i.e. line perpendicular to AB passing through E, intersecting AB at M).

4. Draw circle (E, M).

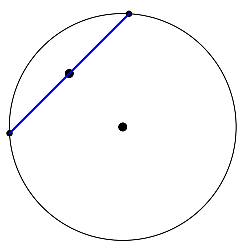


Proof. Note that the diagonals divide the rhombus into four congruent triangles (prop. of rhombus), so they have the same height. This means sides AB, BC, CD, DA have the same perpendicular distance from E. Thus, a circle tangent to one of the sides must be tangent to all of them. □

3 Gamma

3.1 Chord midpoint

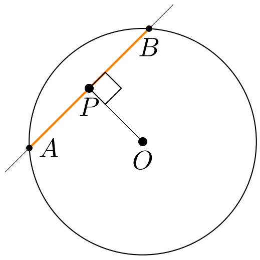
Task 3.1. Construct a chord whose midpoint is given.
(2L, 4E)



Solution 3.1. Let O be center of given circle and P be given point.

1. Draw line OP.

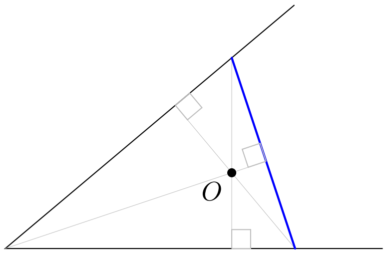
2. Draw $OP \vdash P$ (i.e. line perpendicular to OP passing through P), intersecting the circle at A and B. AB is the desired chord.



Proof. Since $OP \perp AB$, we have $AP = PB$ by "line from center $\perp$ chord bisects chord". □

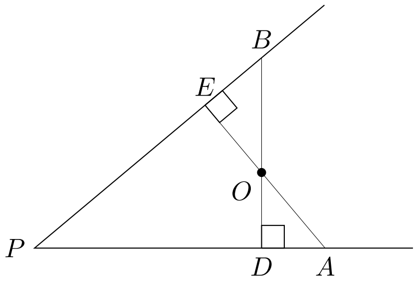
3.2 Triangle by angle and orthocenter

Task 3.2. Construct a segment connecting the sides of the angle to get a triangle whose orthocenter is in the point O.
(3L, 6E)

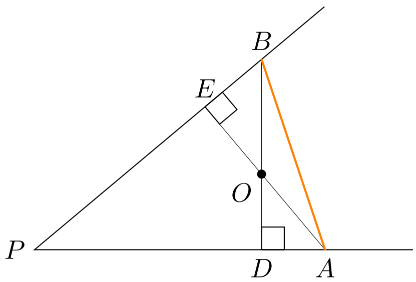


Solution 3.2. Let P be the vertex of given angle.

(3L) 1, 2. Draw lines perpendicular to the given rays passing through O. Let D, E be the feet of the perpendicular lines, and let EO and DO meet the given rays at A and B respectively.



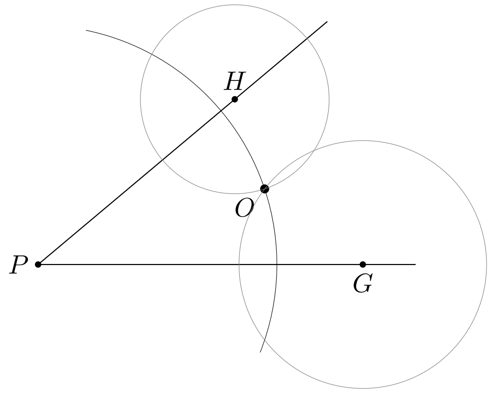
3. Draw line AB, the desired line.



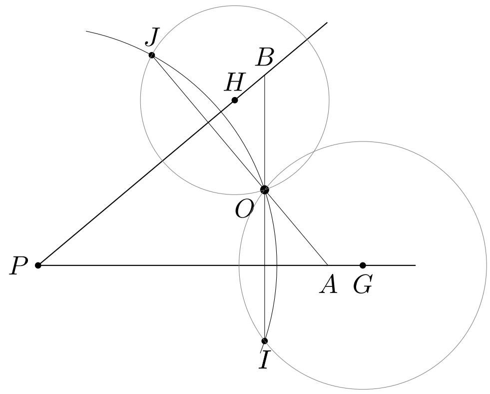
Proof. Note that O is the orthocenter of $\triangle PAB$ since it is the intersection of two altitudes. And any two altitudes intersect at the orthocenter because the three altitudes of a triangle are concurrent. □

(6E) 1. Draw circle (P, O).

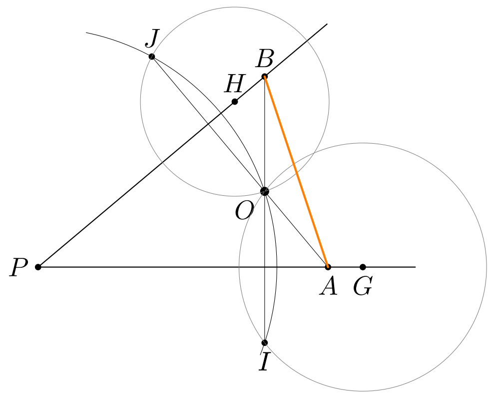
2, 3. Let G, H be two points (arbitrary or on intersection, doesn't matter) on each of the given ray. Draw circles (G, O) and (H, O).



4, 5. Let (P, O) intersect (G, O) and (H, O) at the other point I and J respectively. Draw line IO, meeting PH at B. Draw line JO, meeting PG at A.



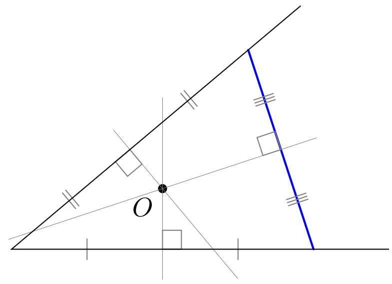
6. Draw line AB, the desired line.



Proof. Note that $OI \perp PG$ since $POGI$ forms a kite. Similarly, $OJ \perp PH$ since $POHJ$ forms a kite. Thus line OI and OJ are altitudes of $\triangle PAB$, so O is the orthocenter of $\triangle PAB$. $\square$

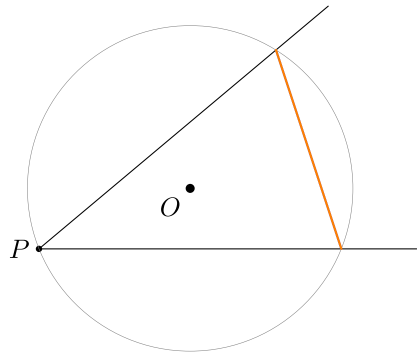
3.3 Intersection of perpendicular bisectors

Task 3.3. Construct a segment connecting the sides of the angle to get a triangle whose perpendicular bisectors are intersected in the point O.
(2L, 2E)

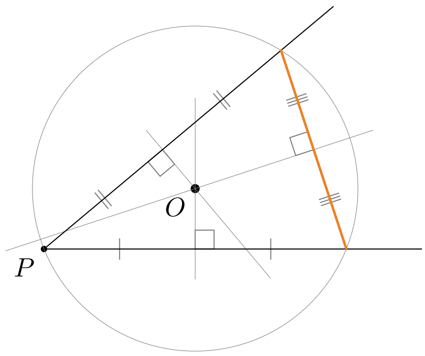


Solution 3.3. Let P be the vertex of the given angle.

1. Draw circle (O, P), intersecting the given rays at A and B respectively.
2. Draw line AB.



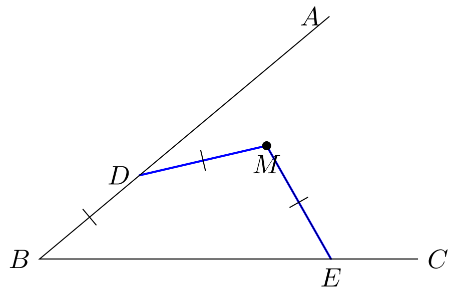
Proof. Note that O is the circumcenter of $\triangle PAB$. And the perpendicular bisectors of sides of $\triangle PAB$ intersect at the circumcenter by "prop. of circumcenter".



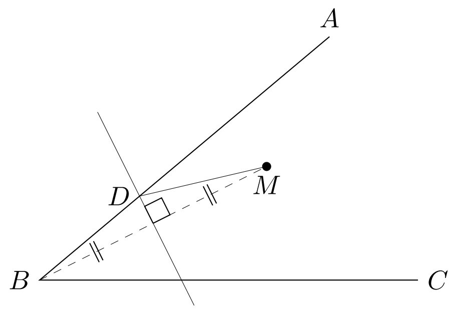
$\square$

3.4 Three equal segments - 1

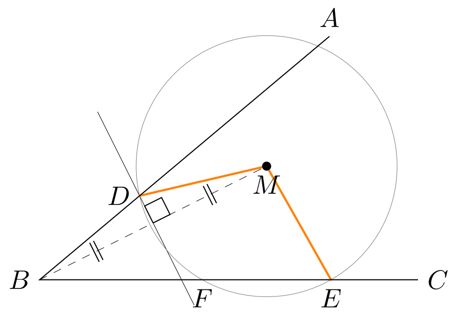
Task 3.4. Given an angle ABC and a point M inside it, find points D on BA and E on BC and construct segments DM and ME such that $BD = DM = ME$.
(4L, 6E, 2V)



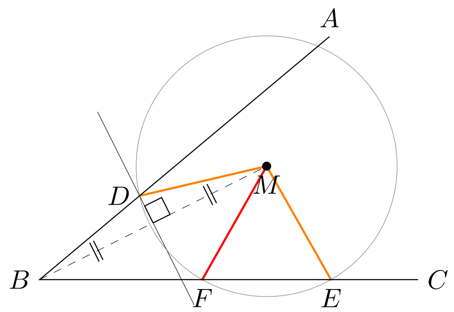
Solution 3.4. (4L, 6E) 1. Draw perpbi BM, intersecting AB at D.
2. Draw line MD.



3. Draw circle (M, D), intersecting line BC at E and F.
4. Draw line ME (or MF).



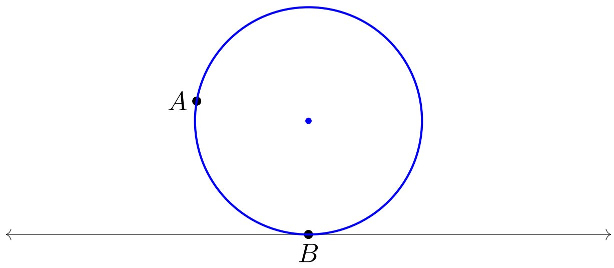
(2V) Draw line MF (or ME).



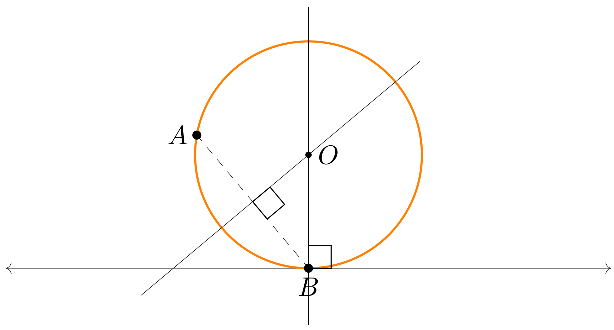
Proof. $BD = DM$ since D lies on the perpendicular bisector of BM. $DM = ME = MF$ since D, E and F lie on the circle centered M. Thus $BD = DM = ME = MF$. □

3.5 Circle through point tangent to line

Task 3.5. Construct a circle through the point A that is tangent to the given line at the point B.
(3L, 6E)

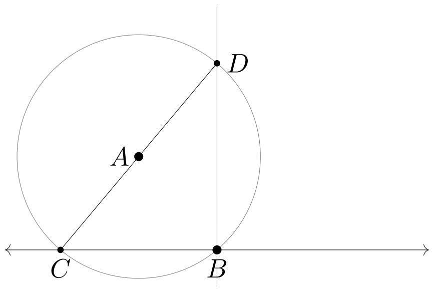


Solution 3.5. (3L)
1, 2. Draw perpbi AB. Draw perpendicular line to given line through B. Let the two drawn lines intersect at O.
3. Draw OB.

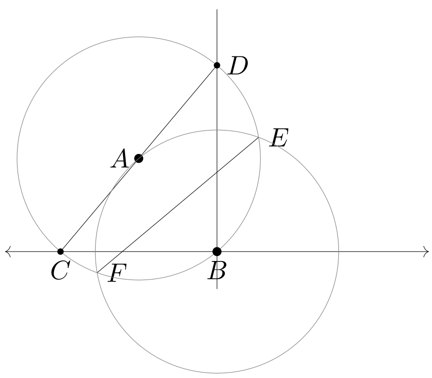


Proof. Since the circle passes through both A and B, center O must lie on the perpendicular bisector of AB (prop. of $\perp$ bisector). Since O is tangent to give line, OB must be perpendicular to given line (tangent $\perp$ radius). Thus O lies on the intersection of the two drawn lines. □

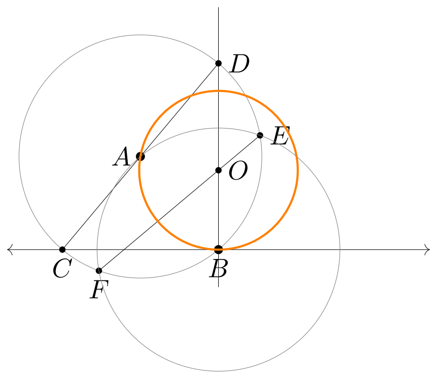
(6E) 1. Draw circle (A, B), intersecting given line at C.
2. Draw line CA, meeting circle (A, B) at D.
3. Draw line BD.



4. Draw circle (B, A), intersecting (A, B) at E and F.
5. Draw line EF, intersecting BD at O.



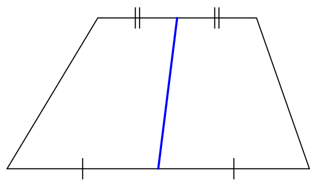
6. Draw circle (O, B).



Proof. Note that BD is perpendicular to given line by Task 2.7E, and EF is the perpendicular bisector of AB by Task 1.2. So O is the same point as the (3L) part of this level. ☐

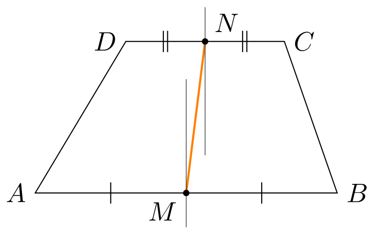
3.6 Midpoints of trapezoid bases

Task 3.6. Construct a line passing through the midpoints of the trapezoid bases.
(3L, 5E)



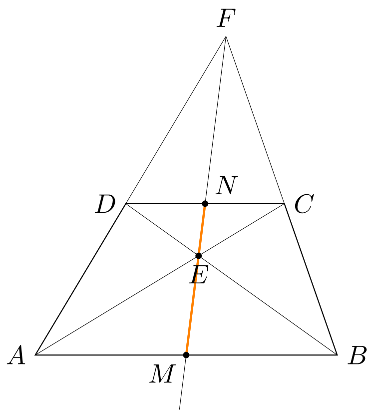
Solution 3.6. (3L)

1, 2. Draw perpbi AB and draw perpbi CD. Let the midpoints of the sides be M and N.
3. Draw line MN.



Proof. $AM = MB$ and $DN = NC$ by perpendicular bisector construction. ☐

(5E) 1, 2. Draw the diagonals of the trapezoid. Let them intersect at E.
3, 4. Extend the non-parallel sides to meet at F.
5. Draw line FE, which is the desired line.



Proof. Let FE intersect sides AB and CD at M and N respectively. We want to show that $AM = MB$ and $DN = NC$.

By Ceva's theorem, we have

$$\frac{AM}{MB} \cdot \frac{BC}{CF} \cdot \frac{FD}{DA} = 1 \tag{1}$$

Since $AB//CD$, by intercept theorem, we also have

$$\begin{aligned} \frac{BC}{CF} &= \frac{AD}{DF} \\ \iff \qquad \frac{BC}{CF} \cdot \frac{FD}{DA} &= 1 \end{aligned} \tag{2}$$

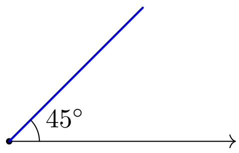
Put (2) into (1).

$$\begin{aligned} \frac{AM}{MB} \cdot (1) &= 1 \\ AM &= MB \end{aligned}$$

Note that $\triangle FDN \sim \triangle FAM$ and $\triangle FNC \sim \triangle FMB$ (AAA). So $\dfrac{DN}{AM} = \dfrac{FN}{FM} = \dfrac{NC}{MB}$ (corr. sides, $\sim \triangle$s). Since $AM = MB$, this gives $\dfrac{DN}{AM} = \dfrac{NC}{AM}$, and thus $DN = NC$, as desired. □

3.7 Angle of 45 deg

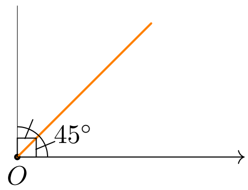
Task 3.7. Construct an angle of 45° with the given side.
(2L, 5E, 2V)



Solution 3.7. Let O be the endpoint of the given ray.

(2L) 1. Draw line perpendicular to given line through O.

2. Draw the angle bisector of the two lines.

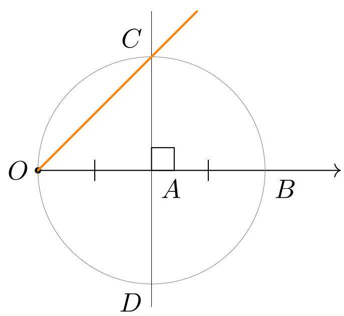


Proof. The angle between the two perpendicular lines is 90°, and the angle bisector makes $90°/2 = 45°$. □

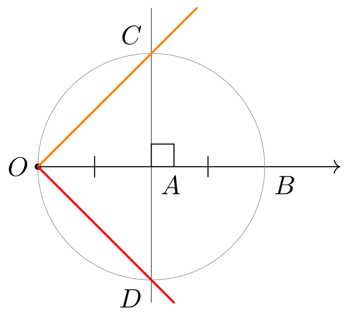
(5E) 1. Let A be an arbitrary point on the given ray. Draw circle (A, O), intersecting the ray again at B.

2. Draw perpbi OB, intersecting the circle at C and D.

3. Draw line OC, the desired line.



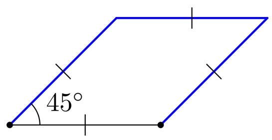
(2V)



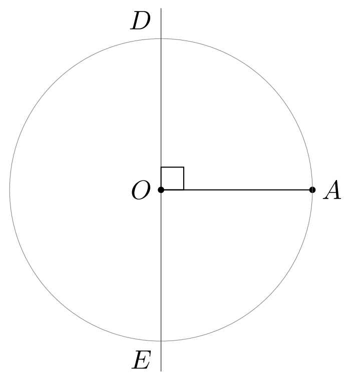
Proof. Since $AO = AC$ and $CA \perp OB$, $\triangle OAC$ is an isosceles right triangle, so its acute angles are $45°$, which means $\angle AOC = 45°$. Same for the other line OD. $\square$

3.8 Lozenge

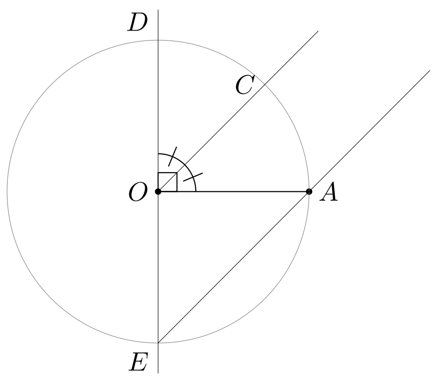
Task 3.8. Construct a rhombus with the given side and an angle of $45°$ in a vertex. (5L, 7E, 4V)



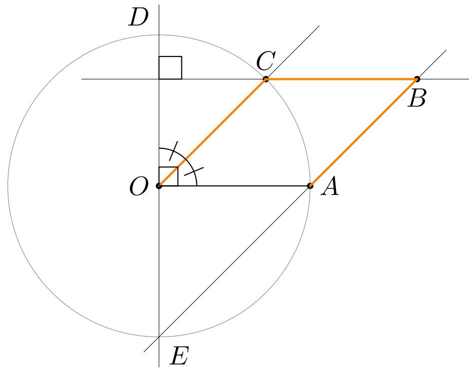
Solution 3.8. Let O and A be the endpoints of the given line segment.
(5L) 1. Draw $OA \vdash O$.
2. Draw circle (O, A), intersecting the vertical line at D and E (where D above E).



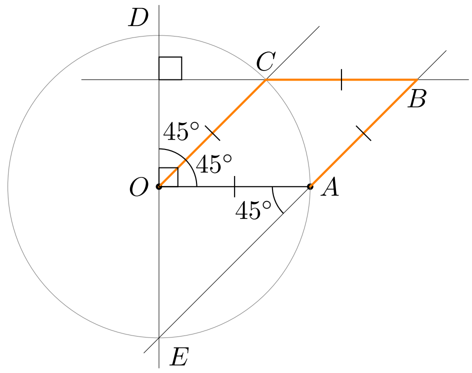
3. Draw angbi DOA (angle bisector of $\angle DOA$), intersecting (O, A) at C.
4. Draw line EA.



5. Draw $OD \vdash C$, intersecting EA at B. $OABC$ is the desired rhombus.

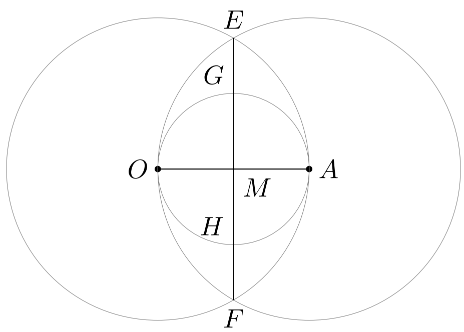


Proof. Note that $CB//OA$ since they are both perpendicular to DO. Note that $\angle AOC = 45°$ (since it is half of right angle), and $\angle OAE = 45°$ since $\triangle OAE$ is an isosceles right triangle. Thus $OC//EB$ (alt. $\angle$s equal). This means $OABC$ is a parallelogram.

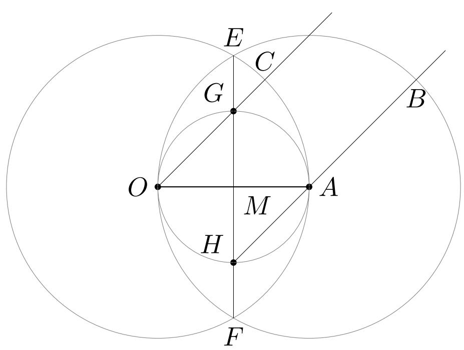


Since $OA = OC$ (radii), $OABC$ is a parallelogram with adjacent sides equal, so $OABC$ is a rhombus. Along with $\angle AOC = 45°$, $OABC$ is the desired rhombus. □

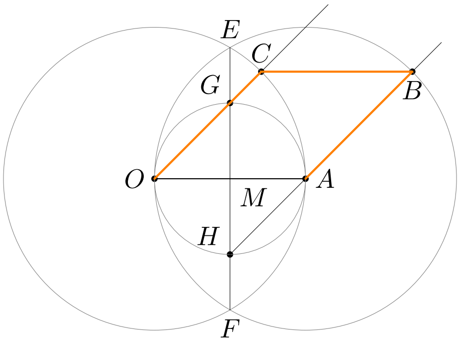
(7E) 1, 2. Draw circle (O, A) and (A, O), intersecting at E and F.
3. Draw line EF, intersect OA at M.
4. Draw circle (M, O), intersecting EF at G and H (G above H).



5, 6. Draw lines OG and HA. Let OG intersect (O, A) at C, and let HA intersect (A, O) at B (where both points are on the same side of OA).



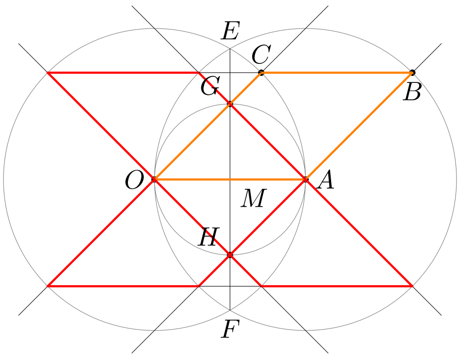
7. Draw line BC.



Proof. Note that $\triangle MOG$ and $\triangle MAH$ are isosceles right triangles, so $\angle MOG = \angle MAH = 45°$ and $OC//HB$ (alt. $\angle$s equal). Moreover, note that $OC = AB$ since they lie on circles of the same radius. Thus $OABC$ is a parallelogram (opp. sides equal and //).

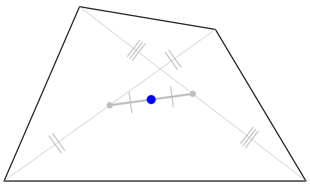
And since $OA = OC$ (radii), $OABC$ has adjacent sides equal, so it is a rhombus. □

(4V) Draw line GA and OH. Connect the intersections of the lines and the big circles to form a symmetric figure.

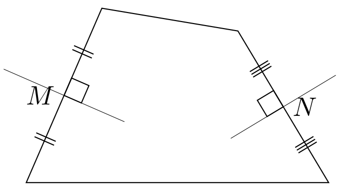


3.9 Center of quadrilateral

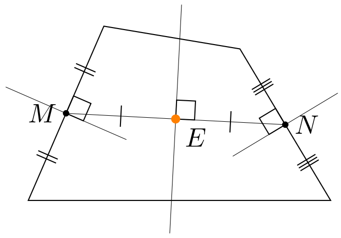
Task 3.9. Construct the midpoint of the segment that connects the midpoints of the diagonals of the quadrilateral.
(4L, 10E)



Solution 3.9. 1, 2. Draw the perpendicular bisectors of the two non-parallel sides. Let the midpoints of the non-parallel sides be M and N.



3, 4. Draw MN. Draw the perpendicular bisector of MN. The midpoint of MN is the desired point.

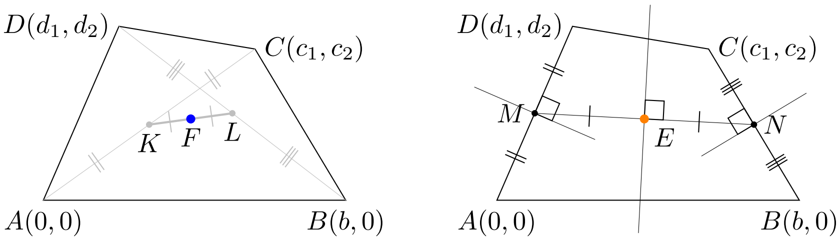


Proof. We use Cartesian coordinates. Let the quadrilateral be $ABCD$, and let

$A = (0, 0)$, $B = (b, 0)$, $C = (c_1, c_2)$, $D = (d_1, d_2)$.

Let K, L be the midpoints of AC and BD respectively. Then $K = (\frac{c_1}{2}, \frac{c_2}{2})$ and $L = (\frac{d_1 + b}{2}, \frac{d_2}{2})$ (mid-pt. coordinate formula). Let F be the midpoint of KL (the desired point).

Then $F = (\frac{c_1 + d_1 + b}{4}, \frac{c_2 + d_2}{4})$.

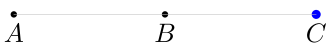


Now we show that E has the same coordinates as F. Since M and N are midpoints of AD and BC, we have $M = (\frac{d_1}{2}, \frac{d_2}{2})$ and $N = (\frac{c_1 + b}{2}, \frac{c_2}{2})$. And E is the midpoint of MN, so $E = (\frac{d_1 + c_1 + b}{4}, \frac{d_2 + c_2}{4})$, which is the same as F. So we conclude $E = F$. □

4 Delta

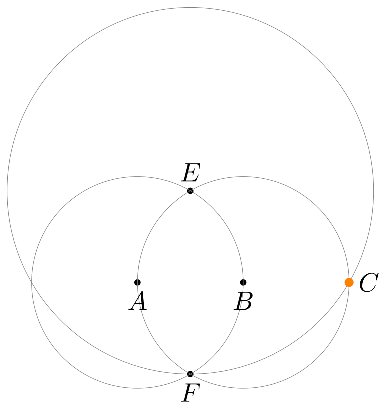
4.1 Double segment

Task 4.1. Construct a point C on the line AB such that $|AC| = 2|AB|$ using only a compass. (3L, 3E, 2V)



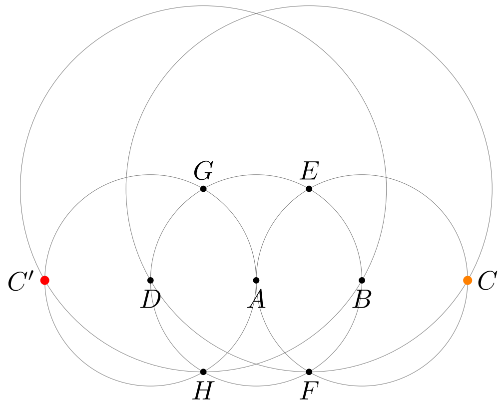
Solution 4.1. (3L, 3E) 1, 2. Draw circle (A, B) and (B, A), intersecting at E and F, intersecting (B, A) again at desired point C,

3. Draw circle (E, F).



Proof. Note that $\angle AEB = 60°$ and $\angle BEC = \angle BEF = 30°$ by congruent triangles. So $\angle AEC = 60° + 30° = 90°$, and so A, B, C are collinear by "converse of $\angle$ in semi-circle". Also $AC = 2AB$ by radii. □

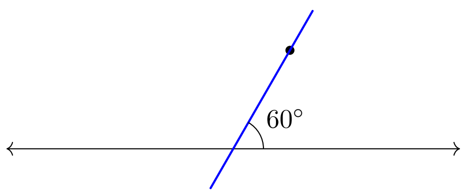
(2V) 4. Let (E, F) intersect (A, B) at D. Draw (D, A), intersecting (A, B) at G and H.
5. Draw circle (G, H), intersecting (D, A) again at desired point C'.



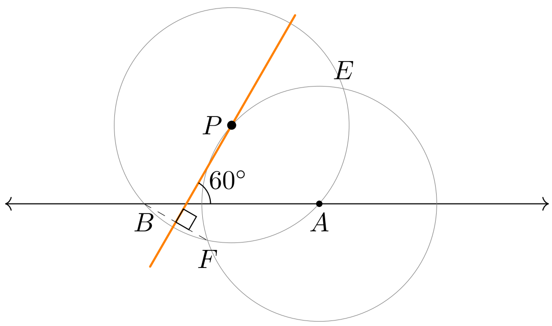
Proof. The same argument works for C' since the figure is symmetric (and because we have D lying on line AB using the same argument as above). □

4.2 Angle of 60 deg - 2

Task 4.2. Construct a straight line through the given point that makes an angle of 60° with the given line.
(3L, 4E, 2V)



Solution 4.2. Let given point be P. Let A be an arbitrary point on given line (such that the angle formed by PA and given line is less than 60°).
(3L) 1. Draw circle (P, A), intersecting the give line again at B.
2. Draw circle (A, P), intersecting (P, A) at E and F.
3. Draw perpbi BF, the desired line.



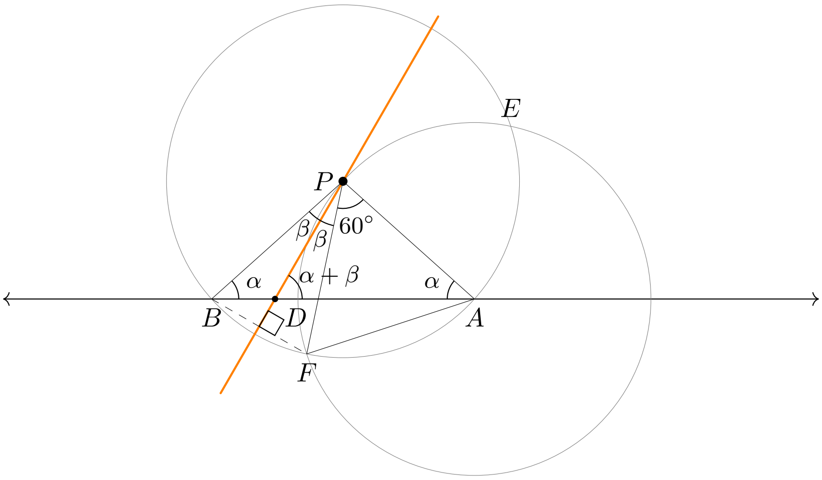
Proof. Let D be the landing point of orange line. First, note that the perpendicular bisector of BF passes through P because BF is a chord of circle centered P.

Since $PB = PA$ (radii), $\angle PBA = \angle PAB$ (base $\angle$s, isos. $\triangle$). Since DP is the perpendicular bisector of BF, $\angle BPD = \angle FPD$ (SAS) & (corr. $\angle$s, $\cong \triangle$s). Also, note that $\angle FPA = 60°$ since $\triangle FPA$ is equilateral triangle.

Let $\angle PBA = \angle PAB = \alpha$ and $\angle BPD = \angle FPD = \beta$.
In $\triangle APB$,

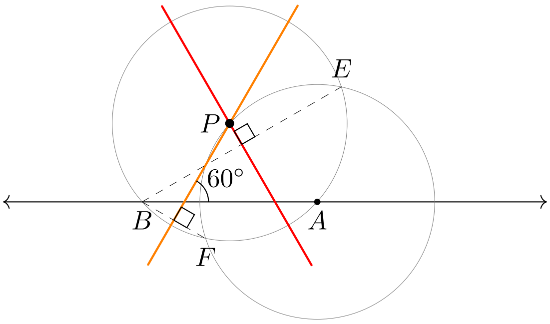
$$\begin{aligned} \angle BPA + \angle PBA + \angle PAB &= 180° \qquad (\angle \text{ sum of } \triangle) \\ (2\beta + 60°) + \alpha + \alpha &= 180° \\ \alpha + \beta &= 60° \end{aligned}$$

Since $\angle PDA = \alpha + \beta$ (ext. $\angle$ of $\triangle$), $\angle PDA = 60°$. This means the orange line makes an angle of $60°$ with the given line.



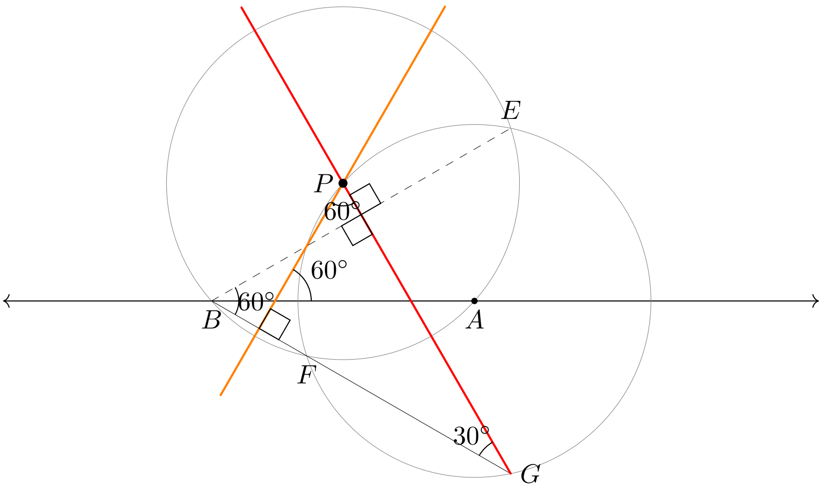
□

(2V) 4. Draw perpbi BE. We get the extra solution.



Proof. Extend BF to meet the red line at G. Note that $\angle FPE = 120°$, so $\angle FBE = 120°/2 = 60°$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$). So $\angle PGB = 90° - 60° = 30°$, and the angle between orange and red line is $90° - 30° = 60°$ ($\angle$ sum of $\triangle$).

Thus the red line makes an angle of $180° - 60° - 60° = 60°$ ($\angle$ sum of $\triangle$).

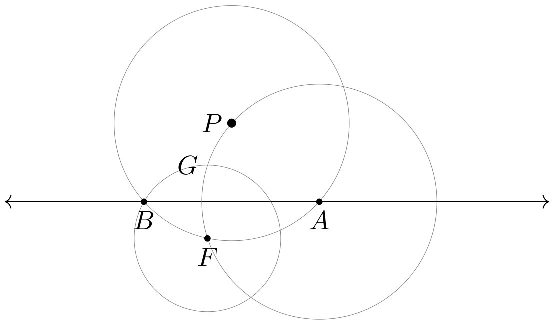


□

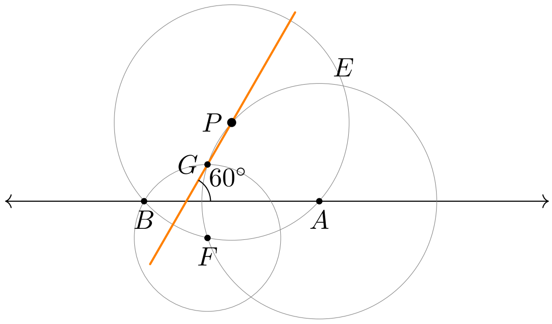
(4E) 1. Draw circle (P, A), intersecting the give line again at B.

2. Draw circle (A, P), intersecting (P, A) at E and F.

3. Draw circle (F, B), intersecting (A, P) above given lien at G.



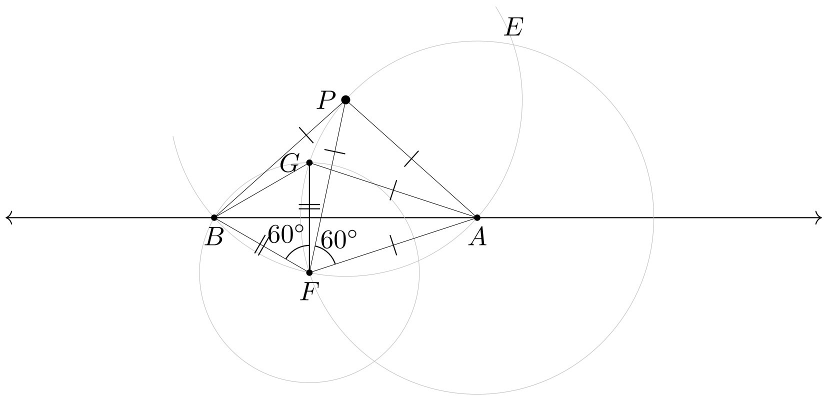
4. Draw line GP, the desired line.



Proof. Let the radius of (P, A) and (A, P) be r, and the radius of (F, B) be s.

Note that $\triangle PBF \cong \triangle AGF$ (SSS), since $PB = AG = r$, $PF = AF = r$, $BF = GF = s$.

Thus $\angle BFP = \angle GFA$ (corr. $\angle$s, $\cong$ $\triangle$s)

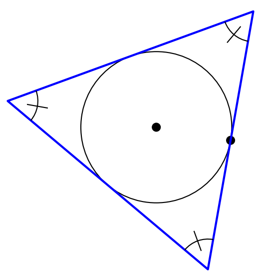


Note that $\angle PFA = 60°$ since $\triangle PFA$ is equilateral. Thus $\angle BFG = \angle BFP - \angle GFP = \angle GFA - \angle GFP = 60°$.

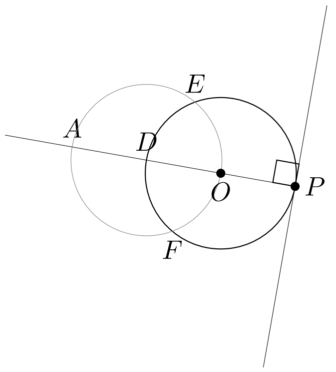
Since $BF = GF$ and $\angle BFG = 60°$, $\triangle GBF$ is equilateral triangle (con. of equil. $\triangle$). This means G lies on the perpendicular bisector of BF, so GP must be the same line as the orange line of (3L). $\square$

4.3 Circumscribed equilateral triangle

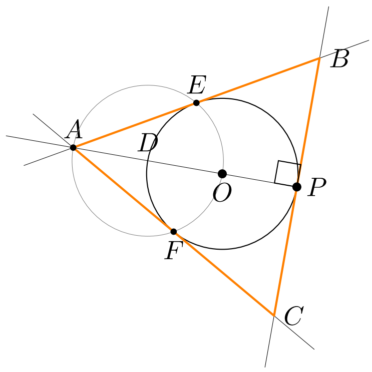
Task 4.3. Construct an equilateral triangle that is circumscribed about the circle and contains the given point.



Solution 4.3. Let P be the given point on circle and O be the given circle center.
(5L) 1. Draw line OP, intersecting given circle again at D.
  2. Draw $OP \vdash P$.
  3. Draw circle (D, O), intersecting given circle at E and F, and intersecting OD at A.



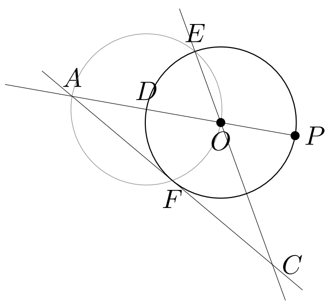
4, 5. Draw lines AE and AF, intersecting $OP \vdash P$ at B and C respectively. $\triangle ABC$ is the desired triangle.



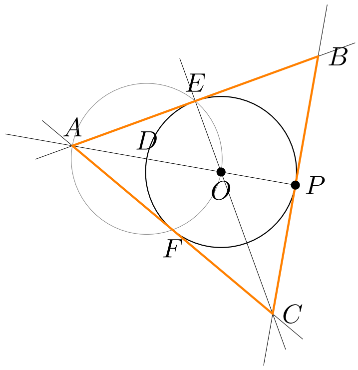
Proof. Note that $\angle AEO = \angle AFO = 90°$ ($\angle$ in semi-circle), so AB and AF are tangent to the given circle ‘ (converse of tangent $\perp$ radius).

Moreover, $\angle OAE = 30°$ since $\dfrac{OE}{OA} = \dfrac{1}{2}$. Similarly $\angle OAF = 30°$. This means $\angle BAC = 60°$. Since the figure is reflectional symmetric about line AP, we have $AB = AC$, and thus $\triangle ABC$ is an equilateral triangle (con. of equil. $\triangle$). □

(6E) 1. Draw line OP, intersecting given circle again at D.
  2. Draw circle (D, O), intersecting given circle at E and F, and intersecting OD at A.
  3. Draw line EO.
  4. Draw line AF, intersecting EO at C.



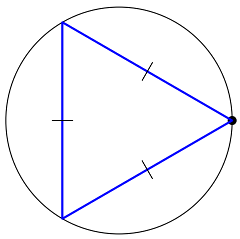
  5. Draw line AE.
  6. Draw line CP, intersecting AE at B. $\triangle ABC$ is the desired triangle.



Proof. Note that in the (5L) figure, E, O, C is collinear since the orthocenter and incenter of an equilateral triangle coincide. Note that point E and line AF in the (6E) figure is the same as that of (5L), so point C in (6E) must be the same as point C in (5L), so $\triangle ABC$ is the same triangle as (5L). □

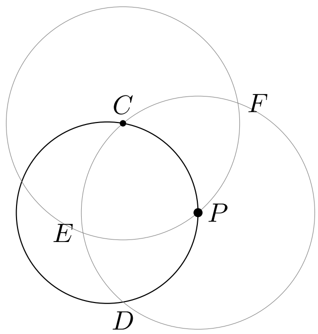
4.4 Equilateral triangle in circle

Task 4.4. Inscribe an equilateral triangle in the circle using the given point as a vertex. The center of the circle is not given.
(5L, 6E)

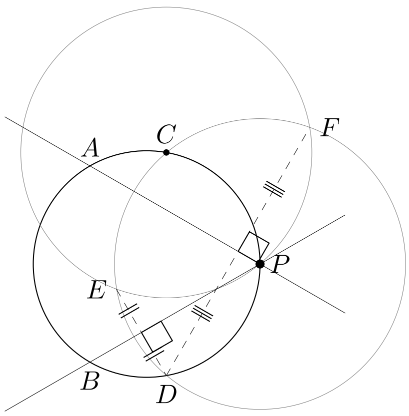


Solution 4.4. Let P be the given point on circle. Let C be an arbitrary point on the circle.

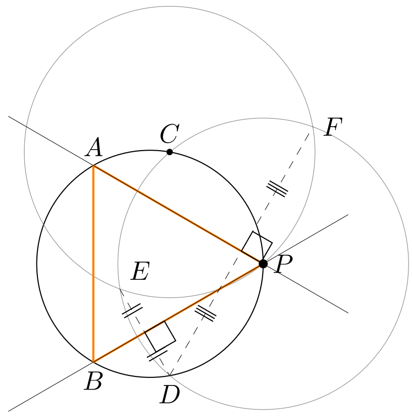
(5L) 1, 2. Draw circles (P, C) and (C, P), intersecting at E and F (where E is nearer to the center of given circle). Let (P, C) intersect given circle again at D.



3, 4. Draw perpbi ED and perpbi FD, intersecting the given circle again at A and B respectively.



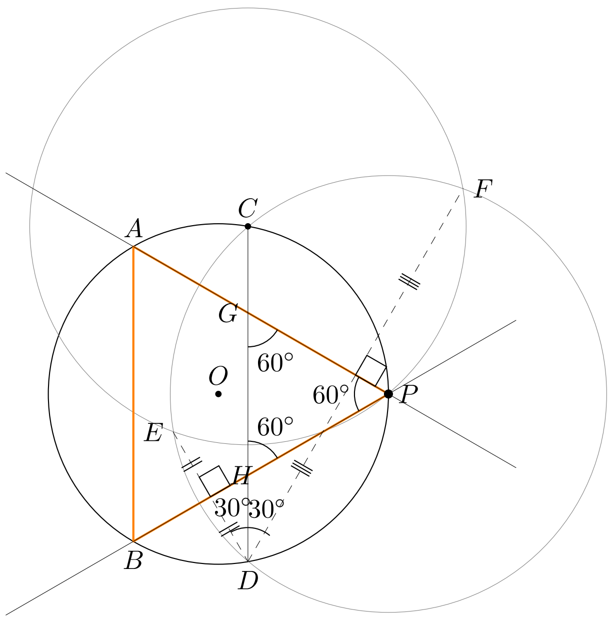
5. Draw line AB. $\triangle PAB$ is the desired triangle.



Proof. Consider angles subtended by arc $\widehat{ECF}$ in circle (P, C). Note that $\angle EPF = 120°$ (since $\triangle ECP$ and $\triangle FCP$ are two equilateral triangles). Thus $\angle EDF = 120°/2 = 60°$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$).

Since $EC = CF$, we have $\angle EDC = \angle CDF = 60°/2 = 30°$ (equal chord, equal $\angle$ at $\odot^{ce}$)

Let CD cut AP and BP at G and H. Since $FD \perp GP$ and $ED \perp HP$, by some angle chasing, we find that $\angle PHG = \angle PGH = 60°$, so $\angle APB = 180° - 60° - 60° = 60°$ ($\angle$ sum of $\triangle$).

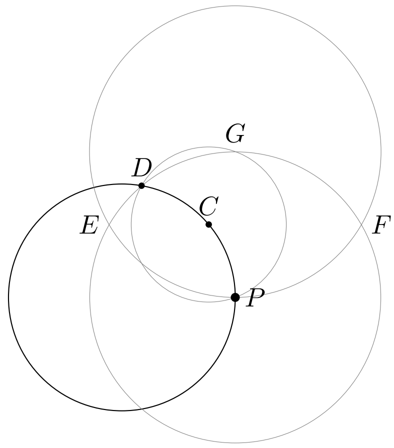


Let O be the center of given circle, and M be the midpoint of CD. Note that $OP \perp CD$ since $ODPC$ forms a kite. Thus PO bisects $\angle GPH$ (prop. of isos. $\triangle$). This means $\triangle PAB$ is reflectional symmetric about line OP, giving $PA = PB$, and thus $\triangle PAB$ is an equilateral triangle. $\square$

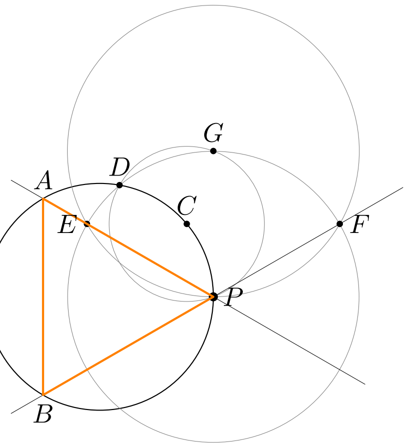
(6E) 1. Let C be an arbitrary point on the circle. Draw circle (C, P), intersecting given circle again at D.

2. Draw circle (P, D), intersecting (C, P) again at G.

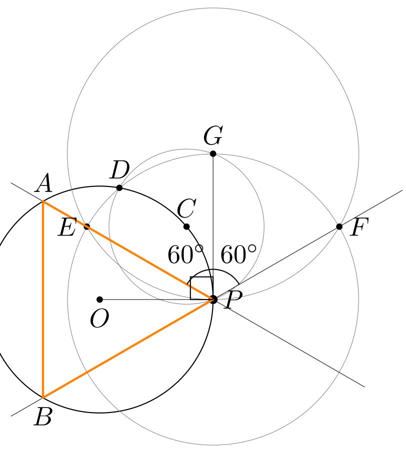
3. Draw circle (G, P), intersecting (P, D) at E and F (E left, F right).



4, 5. Draw line PE and PF, intersecting given circle again at A and B.
6. Draw line AB. $\triangle PAB$ is the desired triangle.



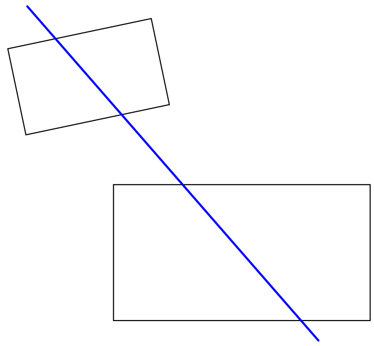
Proof. Let O be center of given circle. Note that PG is tangent to given circle at P by Task 2.8E (tangent to circle at point). Thus $OP \perp GP$. Since $\angle EPG = 60°$, $\angle OPA = 90° - 60° = 30°$ and $\angle OPB = 180° - 90° - 60° = 30°$ (adj. $\angle$s on st. line). Thus OP bisects $\angle APB$, so $\triangle PAB$ is same as (5L) of this level, which means it is equilateral.



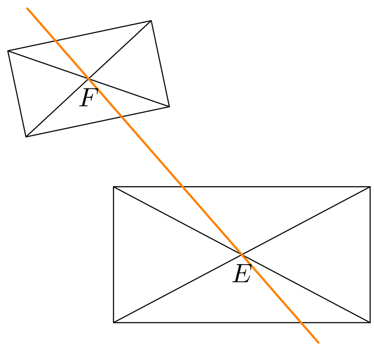
□

4.5 Cut two rectangles

Task 4.5. Construct a line that cuts each of the rectangles into two parts of equal area. (5L, 5E)



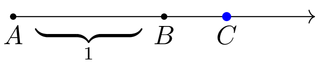
Solution 4.5. 1-5. Draw the diagonals of the two given rectangles and let them intersect at E and F. Draw line EF, the desired line.



Proof. By Task 2.5, a line through the center of a rectangle cuts it into two parts of equal area. Since the orange line passes through the centers of both rectangles, it divides both rectangles into parts of equal area. □

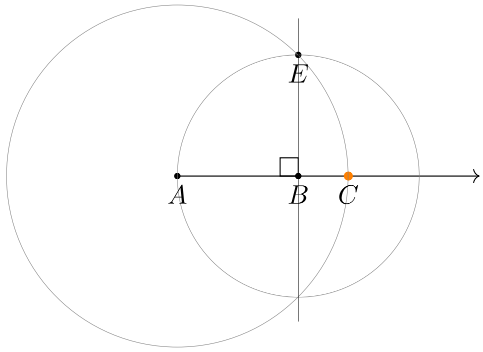
4.6 Square root of 2

Task 4.6. Let $|AB| = 1$. Construct a point C on the ray AB such that the length of AC is equal to $\sqrt{2}$.



$$|AC| = \sqrt{2}$$

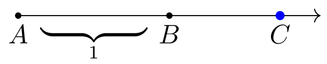
Solution 4.6. 1. Draw circle (B, A).
2. Draw $AB \vdash B$, intersecting (B, A) at one of the points E.
3. Draw circle (A, E), intersecting the given ray at C. C is the desired point.



Proof. Note that $\triangle ABE$ is an isosceles right triangle, so $AE = \sqrt{2}$ (Pyth. thm), thus $AC = AE = \sqrt{2}$ (radii). □

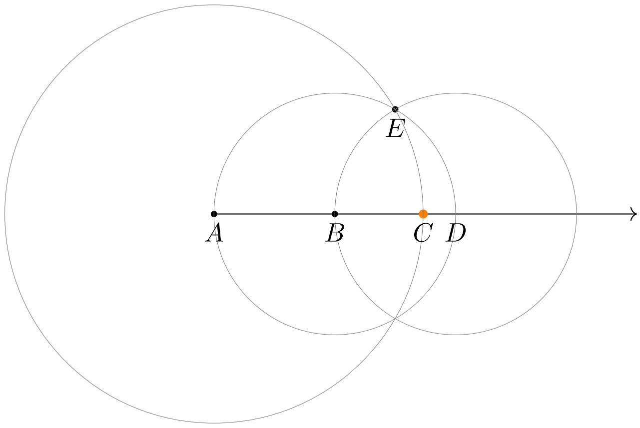
4.7 Square root of 3

Task 4.7. Let $|AB| = 1$. Construct a point C on the ray AB such that the length of AC is equal to $\sqrt{3}$.



$$|AC| = \sqrt{3}$$

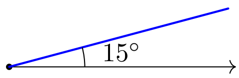
Solution 4.7. **1.** Draw circle (B, A), intersecting given ray again at D.
2. Draw circle (D, B), intersecting (B, A) at one of intersections E.
3. Draw circle (A, E), intersecting given ray at C. C is the desired point.



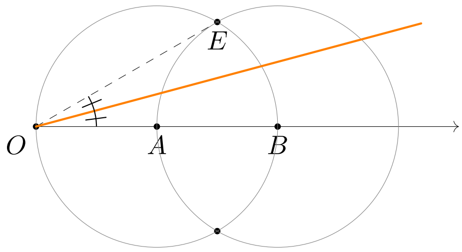
Proof. Since $AB = BE = 1$ and $\angle ABE = 180° - 60° = 120°$ (adj. $\angle$s on st. line), we have $AE = \sqrt{1 + 1 - 2(1)(1)\cos(120°)} = \sqrt{3}$ (law of cosines). So $AC = AE = \sqrt{3}$ (radii). □

4.8 Angle of 15 deg

Task 4.8. Construct an angle of 15° with the given side.
(3L, 5E, 2V)

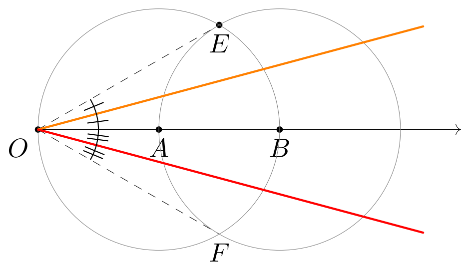


Solution 4.8. Let O be the endpoint of the given ray, and A be an arbitrary point on the ray.
(3L) 1. Draw circle (O, A), intersecting given ray at B.
2. Draw circle (B, A), intersecting (O, A) at E and F.
3. Draw angbi BOE, the desired line.



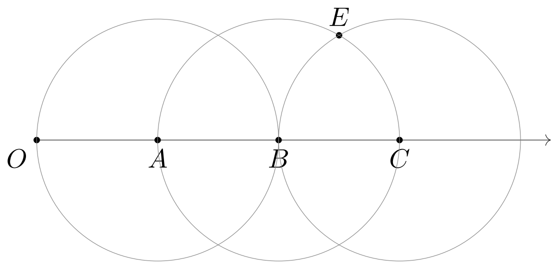
Proof. Note that $\angle OEB = 90°$ ($\angle$ in semi-circle) and $\angle ABE = 60°$, so $\angle BOE = 180° - 90° - 60° = 30°$ ($\angle$ sum of $\triangle$). Thus, bisecting $\angle BOE$ gives a 15° angle. □

(2V) Draw angbi BOF, the extra solution.

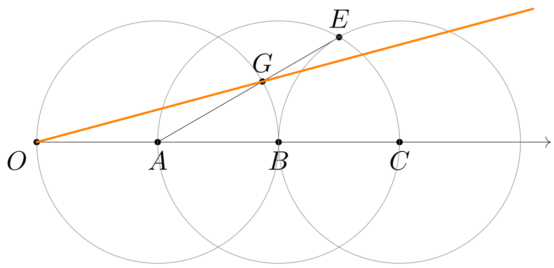


Proof. Similar argument as (3L). □

(5E) 1. Draw circle (O, A), intersecting given ray at B.
2. Draw circle (B, A), intersecting given ray at C.
3. Draw circle (C, A), intersecting (B, A) at one of intersections E.



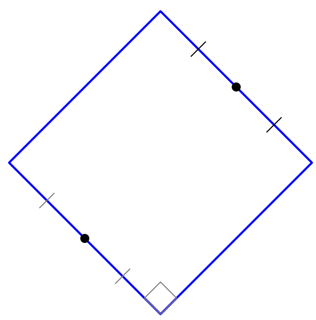
4. Draw segment AE, intersecting (A, O) at G.
5. Draw line OG, the desired line.



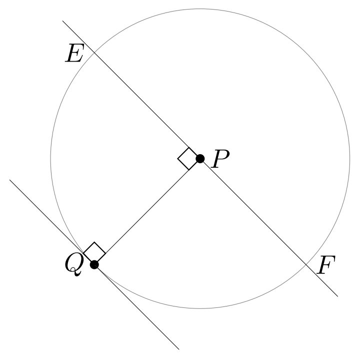
Proof. Note that $\angle EAC = 30°$ (similar to $\angle BOE$ in (3L)). Since $AO = AG$, we have $\angle AOG = 30°/2 = 15°$ (base $\angle$s, isos. $\triangle$)& (ext. $\angle$ of $\triangle$). □

4.9 Square by opposite midpoints

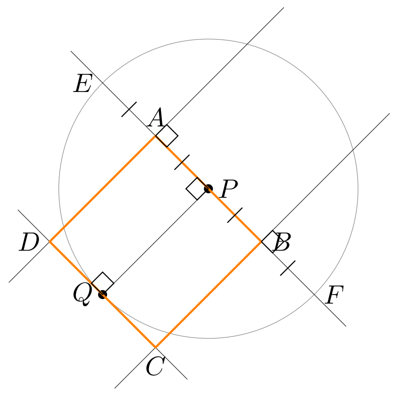
Task 4.9. Construct a square, given two midpoints of opposite sides.
(6L, 10E)



Solution 4.9. Let the given points be P and Q.
(6L) 1, 2. Draw circle (P, Q). Draw line PQ.
3, 4. Draw $PQ \vdash Q$. Draw $PQ \vdash P$, intersecting (P, Q) at E, F.

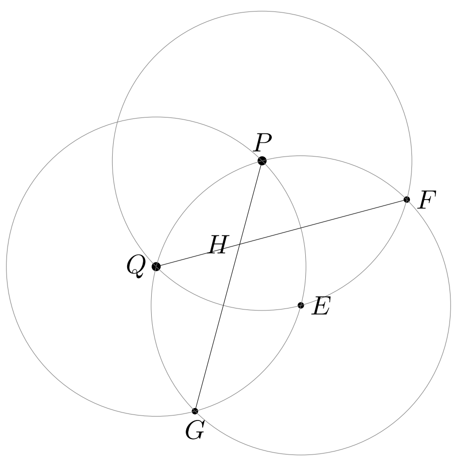


5, 6. Draw perpbi EP and perpbi PF. We get the desired square $ABCD$.

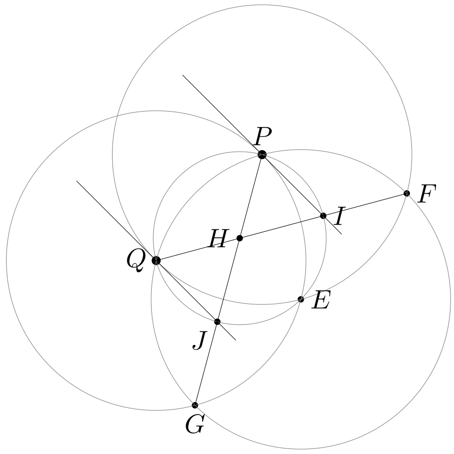


Proof. Note that PQ is equal to the side length of the square. So $AP = \frac{1}{2}EP = \frac{1}{2}PQ$. Similarly, $PB = \frac{1}{2}PQ$, so $AB = PQ$. Also $AD = BC = PQ$, so $ABCD$ is a square. □

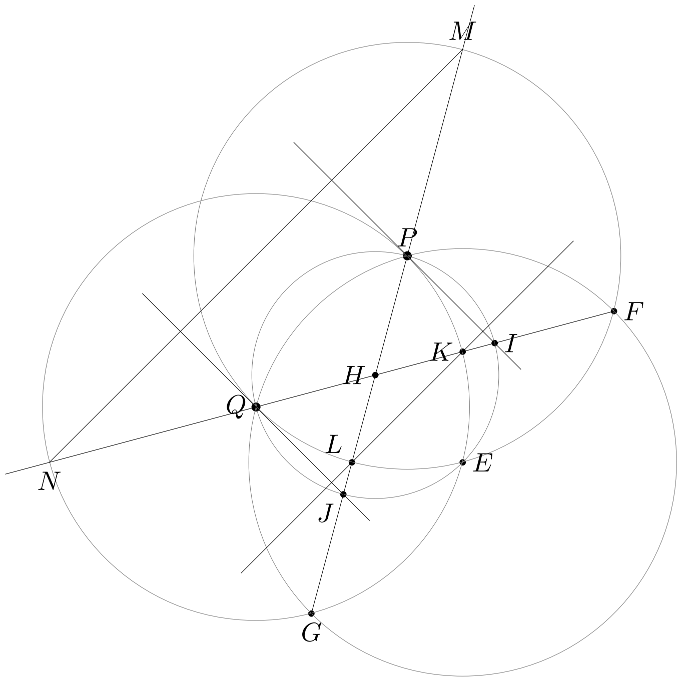
(10E) 1, 2. Draw circle (P, Q) and (Q, P). Let E be one of their intersections.
3. Draw circle (E, P), intersecting (P, Q) and (Q, P) at new points F and G.
4, 5. Draw lines PG and QF, intersecting at H.



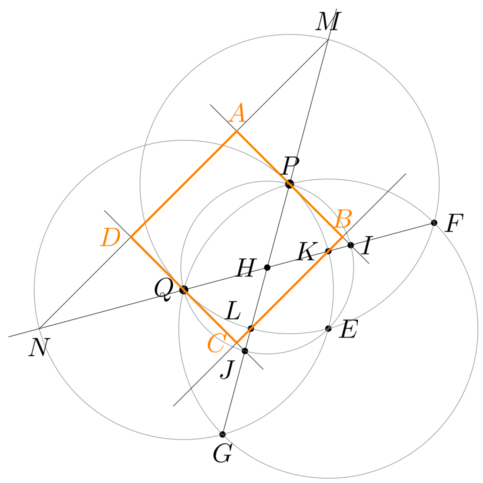
6. Draw circle (H, P), intersecting segments QF and PG at I and J respectively.
7, 8. Draw line PI and QJ.



9. Let segments QF and PG intersect (Q, P) and (P, Q) at K and L respectively. Draw line KL.
10. Extend segment GP to meet (P, Q) at M. Extend segment FQ to meet (Q, P) at N (doesn't cost anything). Draw line MN.

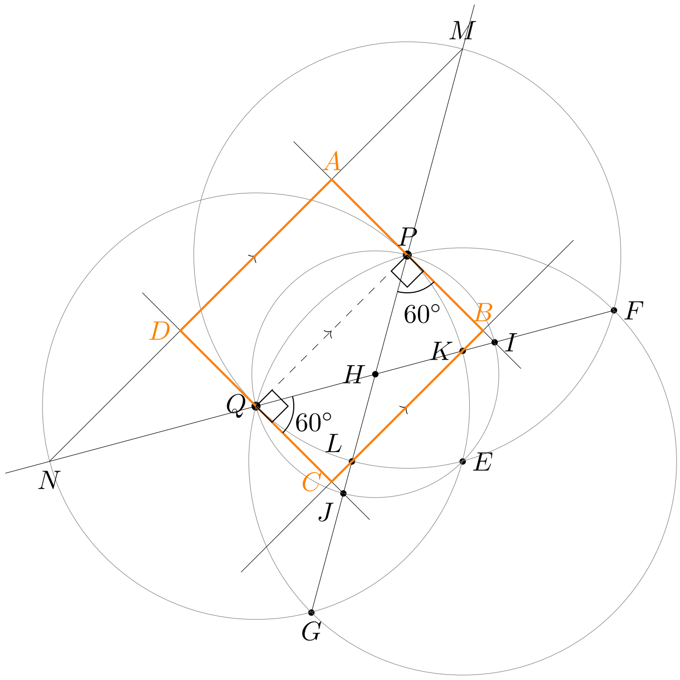


The desired square is the area enclosed by the four lines.



Proof. (Let $ABCD$ be the vertices of the orange quadrilateral.)

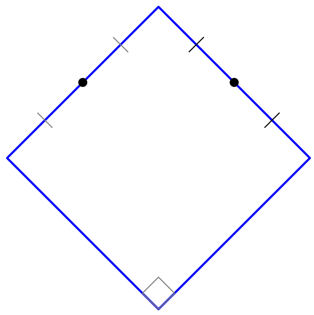
First, note that $PI \perp PQ$ and $QJ \perp PQ$ by "$\angle$ in semi-circle" for circle (H, P). Also, $KL // PQ$ since $\triangle HPQ \sim HLK$ (ratio of 2 sides, inc. $\angle$), and $MN // PQ$ since $\triangle HPQ \sim \triangle HMN$ (ratio of 2 sides, inc. $\angle$). This means $ABCD$ is a rectangle.



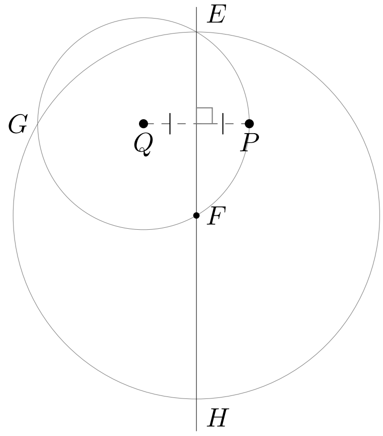
Now note that $\angle HPI = 60°$ (since $\triangle HPI$ is equil.), so $PB = \cos(60°)PL = \frac{1}{2}PL$. And note that $\triangle PAM \cong \triangle PBC$ (AAS by $\angle APM = \angle BPL$, $\angle MAP = \angle LBP$, $PM = PL$), giving $AP = PB = \frac{1}{2}PL$ (corr. sides, $\cong \triangle$s). This means $AB = PL = PQ$ (radii) $= BC$. So $ABCD$ is a rectangle with adjacane sides equal, i.e. a square, as desired. $\square$

4.10 Square by adjacent midpoints

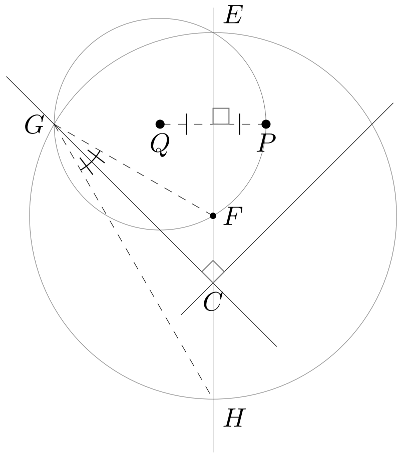
Task 4.10. Construct a square, given two midpoints of adjacent sides.
(7L, 10E, 2V)



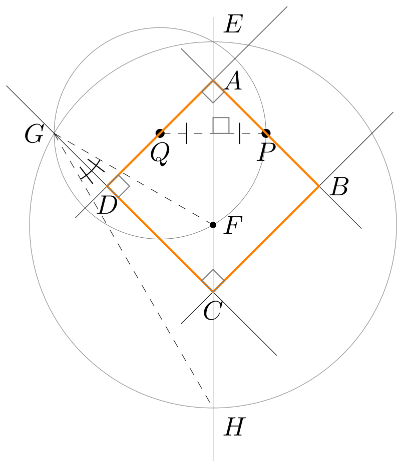
Solution 4.10. Let P, Q be the given points.
(7L) 1. Draw circle (Q, P).
2. Draw perpbi PQ, intersecting (Q, P) at E and F
3. Draw circle (F, E), intersecting (Q, P) again at G, and EF again at H.



4. Draw angbi FGH, intersecting FH at C.
5. Draw $GC \vdash C$.

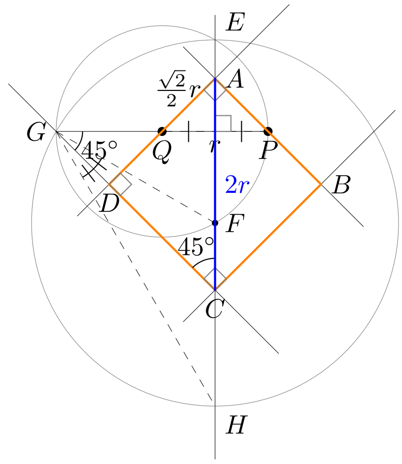


6. Draw $GC \vdash Q$, intersecting EF at A.
7. Draw $QA \vdash P$. The shape $ABCD$ enclosed by the lines is the desired square.



Proof. Let the length of QP be r, and M be midpoint of QP. We want to show that rectangle $ABCD$ is a square and that $AQ = QD = AP = PB = \frac{\sqrt{2}}{2}r$.

Note that $\angle EGH = 90°$ ($\angle$ in semi-circle) and $\angle AGF = 60°$, so $\angle FGH = 30°$. Since GC is the angle bisector of $\angle FGH$, we have $\angle FGC = 15°$, so $\angle QGC = 45°$ and thus $\angle GCA = 45°$. Since the diagonal of rectangle $ABCD$ makes $45°$ with a side, $ABCD$ is a square.



To show that Q, P are midpoints of the sides, note that $AQ = QP\cos 45° = \dfrac{\sqrt{2}}{2}r$. Also, $GP = AC$ because $GM = MC$ and $MP = AM$. Since GP is a diameter of circle (Q, P), $GP = 2r$ and thus $AC = 2r$.

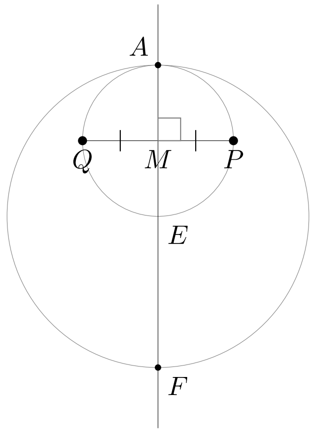
So $AD = \dfrac{2r}{\cos 45°} = \sqrt{2}\,r$, and $QD = AD - AQ = \dfrac{2}{2}r = AQ$.

Then we also have $AP = PB$ by intercept theorem (because $QP//DB$). We get everything desired. ☐

(10E) 1, 2. Draw perpbi PQ. Draw line PQ. Let M be midpoint of PQ.

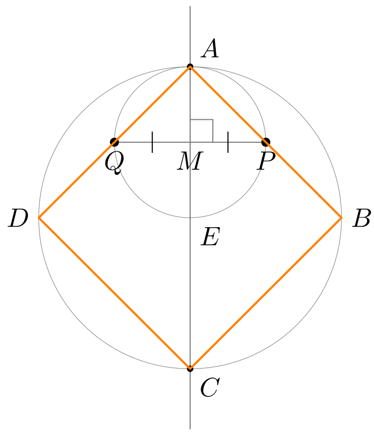
3. Draw circle (M, P), intersecting perpbi PQ at A at top and E at bottom.

4. Draw circle (E, A), intersecting perpbi PQ again at C (at bottom).



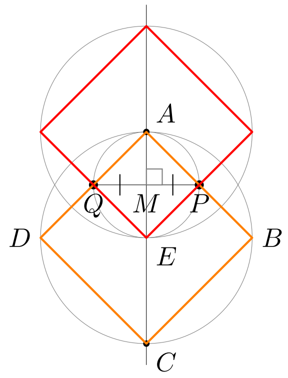
4, 5. Draw line AP and AQ, intersecting big circle (E, A) at B and D.

6, 7. Draw line BC and DC. $ABCD$ is the desired square.



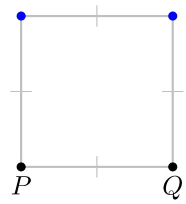
Proof. Note that $\triangle AMQ$ and $\triangle AMP$ are isosceles right triangles, so we have $\angle AQM = \angle APM = 45°$. Thus $\angle BAD = 90°$. Also, $\angle ADC = \angle ABC = 90°$ ($\angle$ in semi-circle). Thus, $ABCD$ is a rectangle with diagonal making $45°$ with the sides, so $ABCD$ is a square (con. of square). ☐

(2V) Draw circle (A, E) and draw the lines similarly.

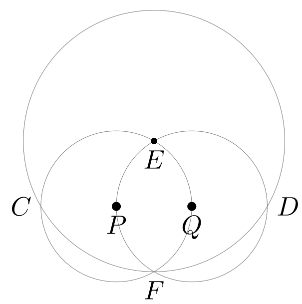


4.11 Square by two vertices

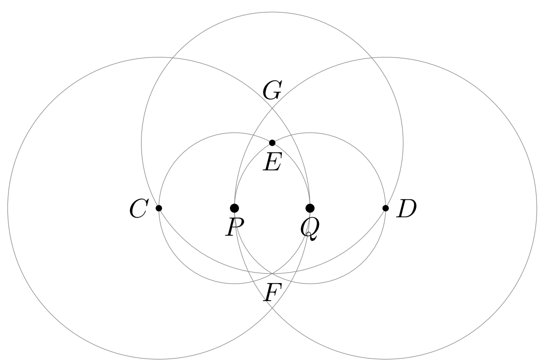
Task 4.11. Given two vertices of a square. Construct the two other vertices using only a compass.
(7L, 7E, 3V)



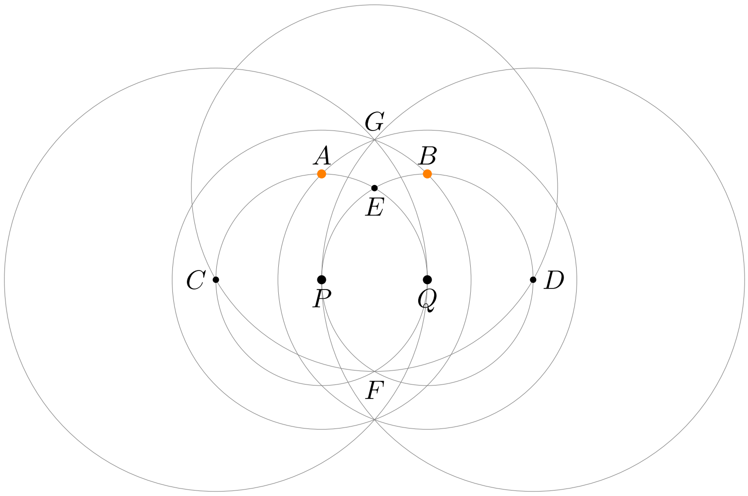
Solution 4.11. Let the given points be P and Q.
(7L, 7E) 1, 2. Draw circles (P, Q) and (Q, P), intersecting at E and F.
3. Draw circle (E, F), intersecting (P, Q) and (Q, P) again at C and D.



4, 5. Draw circles (C, Q) and (D, P). Let intersection at top be G.

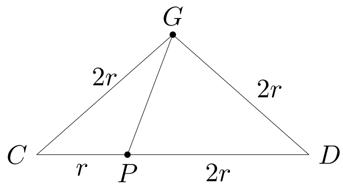


6, 7. Draw circles (P, G) and (Q, G), intersecting (Q, P) and (P, Q) at top at B and A respectively. A and B are the desired points.



Proof. Let the distance between PQ be r. Note that C and D lie on circle (E, F), so C, P, Q, D are collinear and $CP = PQ = QD = r$ (see Task 1.7E for proof). We also have $GC = GD = 2$ since G lie on circles (C, Q) and (D, P).

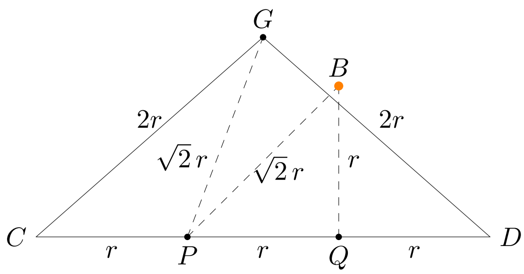
To find PG, let's focus on $\triangle GCP$ and $\triangle GPD$:



By Stewart's theorem, we have

$$\begin{aligned} GC^2 \cdot PD + GD^2 \cdot CP &= (CP + PD)(PG^2 + CP \cdot PD) \\ (2r)^2(2r) + (2r)^2(r) &= (2r + r)(PG^2 + (r)(2r)) \\ 4r^2 &= PG^2 + 2r^2 \\ PG &= \sqrt{2}\, r \end{aligned}$$

Let's add points B and Q to the figure. We have $PB = PG = \sqrt{2}$ and $QB = QP = 1$ (radii).

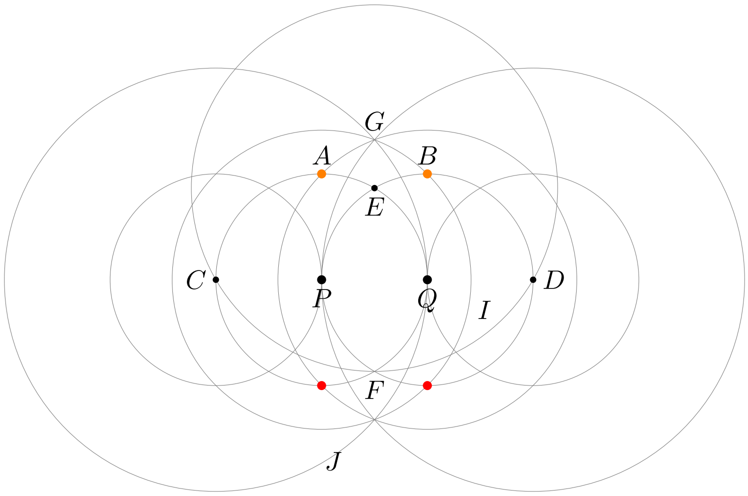


Since $BQ^2 + PQ^2 = 2r^2 = BP^2$, by converse of Pythagoras theorem in $\triangle BPQ$, we have $\angle BQP = 90°$. By symmetry, we have $\angle APQ = 90°$.

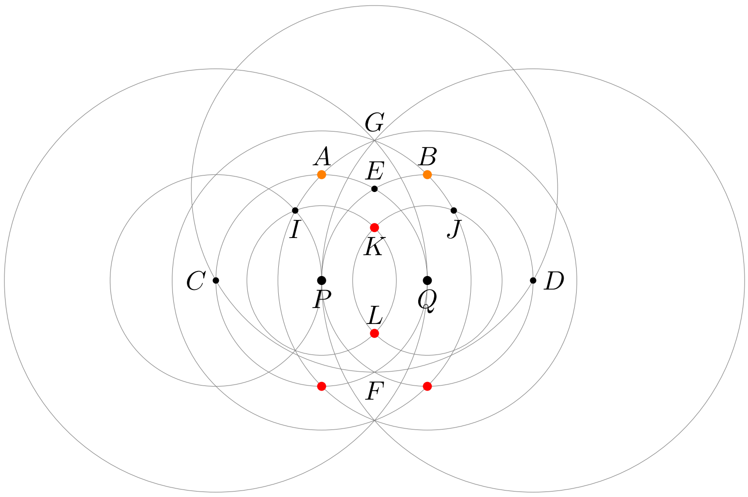
Since $PQBA$ has three sides equal ($AP = PQ = BQ$) and two right angles ($\angle APQ = \angle BQP = 90°$), $ABCD$ is a square (con. of square). □

(3V) 2nd solution: Intersection of (P, G) and (Q, P) at bottom, and intersection of (G, P) and (P, Q) at bottom.

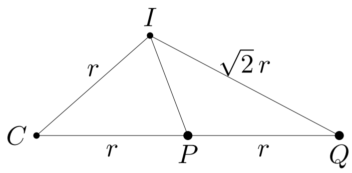
3rd solution: 8, 9. Draw circle (C, P), intersecting (Q, G) at I (top). Draw circle (D, Q), intersecting (P, G) at J (top).



10, 11. Draw circles (P, I) and (Q, J), intersecting at K and L at the middle. K and L are the two desired points.



Proof. Note that $CI = r$ and $QI = QG = \sqrt{2}\,r$. Let's focus on $\triangle ICP$ and $\triangle IPQ$:



By Stewart's theorem,

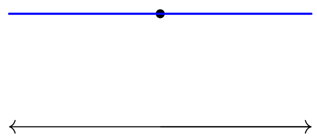
$$r^2(r) + (\sqrt{2}r)^2 r = (r + r)(IP^2 + r(r))$$
$$\frac{3}{2}r^2 = IP^2 + r^2$$
$$IP = \frac{\sqrt{2}}{2}r$$

This means $PK = \frac{\sqrt{2}}{2}r$. By symmetry, we also have $KQ = PL = LQ = \frac{\sqrt{2}}{2}r$. By Pyth. thm in $\triangle KPQ$, $\angle PKQ = 90°$. Thus, $PLQK$ is a rhombus with a right angle, i.e. a square, as desired. $\square$

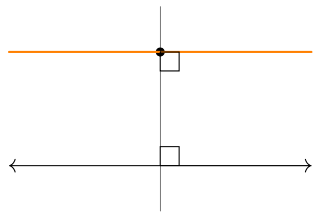
5 Epsilon

5.1 Parellel line

Task 5.1. Construct a line parallel to the given line through the given point.
(2L, 4E)

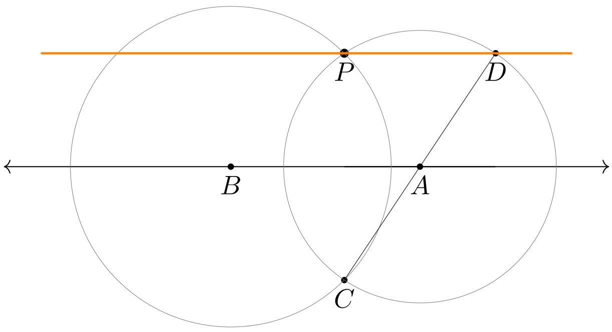


Solution 5.1. **(2L)** **1.** Draw line perpendicular to given line through given point.
2. Draw line perpendicular to the drawn line through given point.



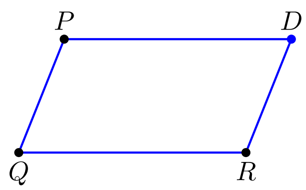
Proof. Since $90° + 90° = 180°$, the interior angles are supplementary , so the orange line is parallel to given line (int. $\angle$s supp.). □

(4E) Let P be given point, A, B be two arbitrary points on given line.
1. Draw circles (A, P) and (B, P), intersecting at P and C.
3. Draw line CA, meeting circle (A, P) at D.
4. Draw line PD, the desired line.

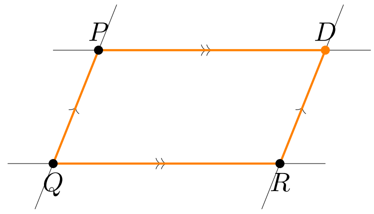


5.2 Parallelogram by three vertices

Task 5.2. Construct a parallelogram whose three or four vertices are given.



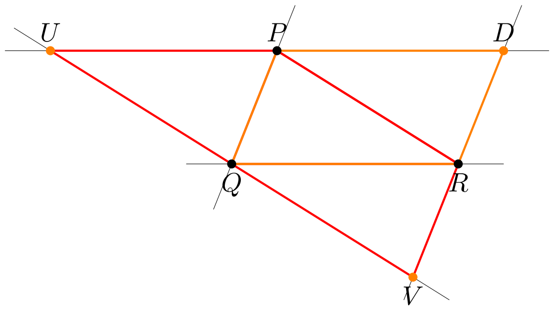
Solution 5.2. **(4L)** Let given points be P, Q, R.
1, 2. Draw line PQ and QR.
3. Draw $QR \rightleftharpoons P$ (line parallel to QR through P).
4. Draw $PQ \rightleftharpoons R$.



Proof. By definition. □

(3V) 5. Draw line PR.

6. Draw $PR \rightleftharpoons Q$, intersecting PD and DR at U and V respectively. The extra solutions are $PUQR$ and $PQVR$.



References