

# Euclidea Solutions Explained

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## Abstract

Euclidea is a puzzle game in which the player has to construct geometrical figures using compass and straight edge only. It is probably the most difficult puzzle game I've played. The complexity of Euclidean Geometry gives rise to so many possible constructions, and it is difficult to brute force solving (at least I don't know how). On top of that, the player has to use the minimum number of moves possible to pass the level in order to get 3 stars and unlock the next level pack. There is just no way I can come up with the solutions myself, so I've cheated by looking up the solutions online. Nonetheless, it is an interesting math-related game that is one of a kind.

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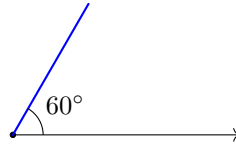
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# 1 Alpha

## 1.1 Angle of 60 deg

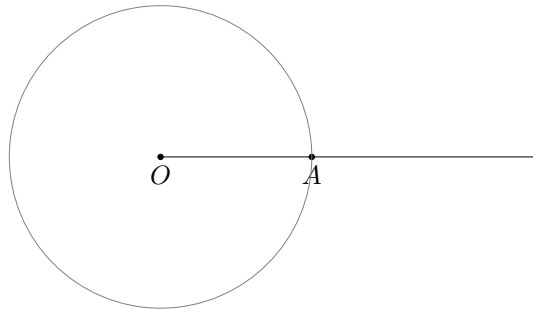
**Task 1.1.** Construct an angle of  $60^\circ$  with the given side.  
(3L, 3E, 2V)



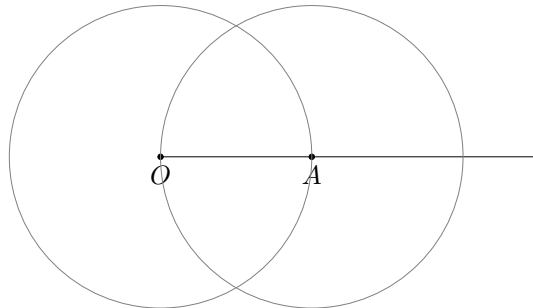
(Arrowhead means the line is infinitely long.)

**Solution 1.1.** (3L, 5E)

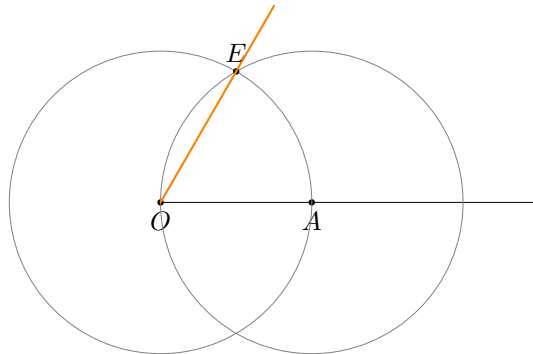
1. Let  $O$  be the endpoint of the given ray. Label an arbitrary point  $A$  on the given ray. Draw circle centered  $O$  through  $A$ .



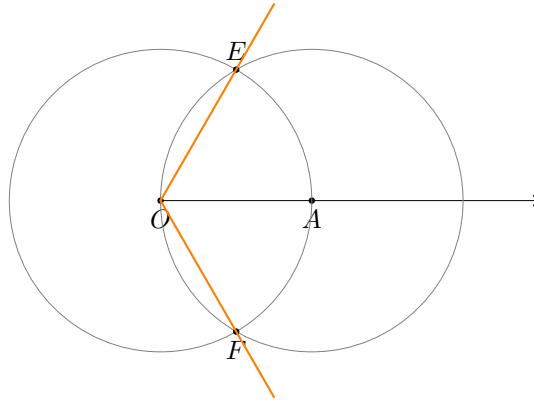
2. Draw circle centered  $A$  through  $O$ .



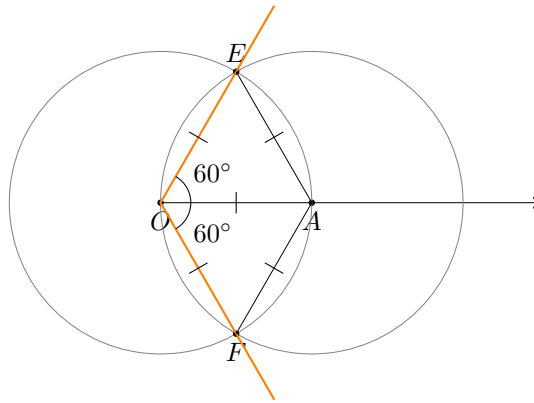
3. Let  $E$  be one of the intersections of the two circles. Draw line  $OE$ . We get the desired  $60^\circ$  angle.



(2V: Extra solutions) Let  $F$  be another intersections of the two circles. Draw line  $OF$ . We get another  $60^\circ$  angle.



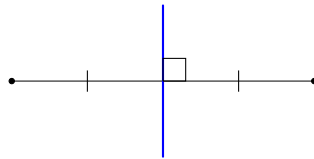
*Proof.* To see why  $\angle AOE$  and  $\angle AOF$  are  $60^\circ$  angles, first note that the two circles have the same radii since they share the same segment  $OA$ . Thus  $OA, OE, AE, OF, AF$  all have lengths equal to the radii of the circles, so  $\triangle OAE$  and  $\triangle OAF$  are equilateral triangles.



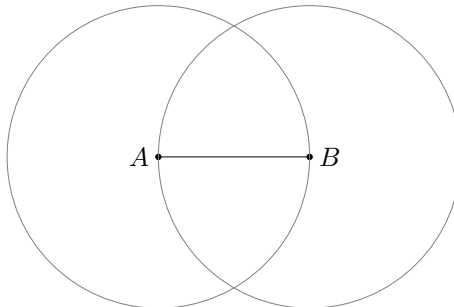
By “prop. of equil.  $\triangle$ ”, all the interior angles of equilateral triangle is  $60^\circ$ , meaning  $\angle AOE = \angle AOF = 60^\circ$ .  $\square$

## 1.2 Perpendicular bisector

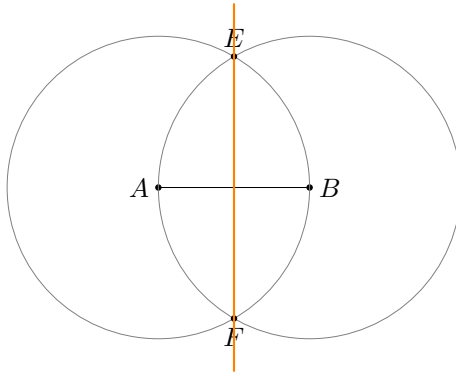
**Task 1.2.** Construct the perpendicular bisector of the segment.  
(3L, 3E)



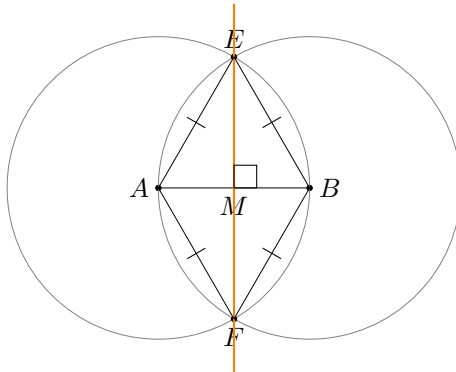
**Solution 1.2. 1, 2.** Let  $A$  and  $B$  be the endpoints of the given segment. Draw circle centered  $A$  through  $B$ . Draw another circle centered  $B$  through  $A$ .



**3.** Let  $E, F$  be intersection of the two circles. Draw line  $EF$ . We get the desired perpendicular bisector.



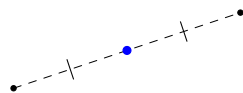
*Proof.* Let  $AB$  and  $EF$  intersect at  $M$ . Since  $AE = BE = AF = BF$ ,  $AEBF$  is a rhombus. By property of rhombus, the diagonals  $AB$  and  $EF$  are perpendicular to each other.



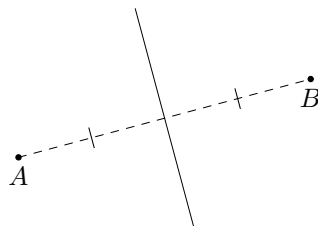
Moreover, since  $AEBF$  is a rhombus,  $AEBF$  is a parallelogram. By “diags. of //gram”, the diagonals  $AB$  and  $EF$  bisect each other, giving  $AM = MB$ . Thus,  $EF$  is the perpendicular bisector of  $AB$ .  $\square$

### 1.3 Midpoint

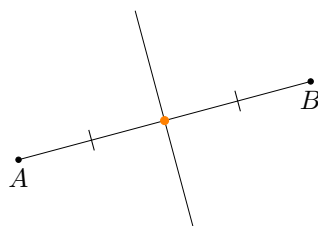
**Task 1.3.** Construct the midpoint of the segment defined by two points.  
(2L, 4E)



**Solution 1.3.** 1. Let  $A, B$  be the endpoints of the given segment. Draw the perpendicular bisector of  $AB$ .



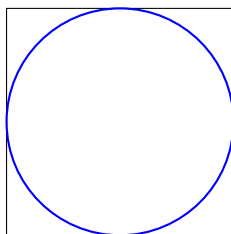
2. Draw line  $AB$ . The intersection of  $AB$  and the perpendicular bisector is the desired midpoint.



*Proof.* By definition, perpendicular bisector bisects  $AB$ . So the intersection of  $AB$  and the perpendicular bisector is the midpoint of  $AB$ .  $\square$

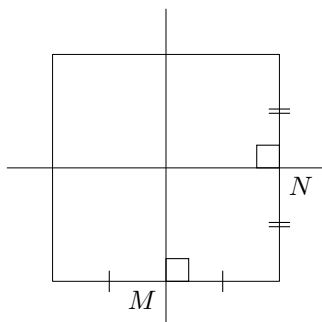
## 1.4 Circle in square

**Task 1.4.** Inscribe a circle in the square.  
(3L, 5E)

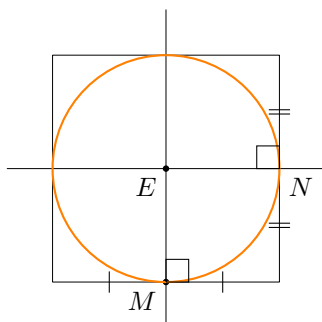


**Solution 1.4. (3L)**

**1, 2.** Draw perpendicular bisectors of two adjacent sides of the square. Let  $M, N$  be midpoints of these two sides.



**3.** Let  $E$  be the intersection of perpendicular bisectors. Draw circle centered  $E$  through  $M$  (or  $N$ ). We get the desired circle.

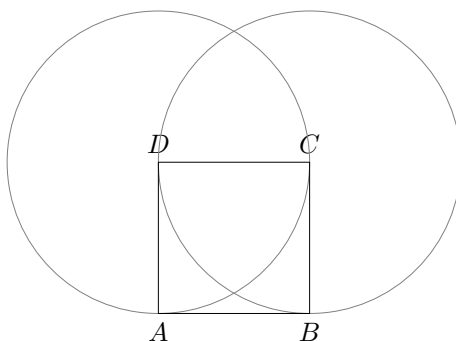


*Proof.* Note that the perpendicular bisectors divide the big square into four smaller squares of the same side length, so the circle with radius  $EM$  passes through all the midpoints of the sides of big square. By ‘converse of tangent  $\perp$  radius’, the circle is tangent to the four sides of the big square, which means it is inscribed in the square.

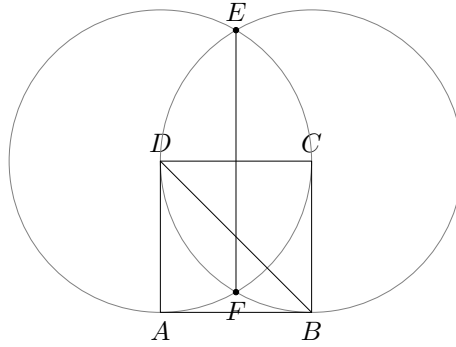
□

**(5E)**

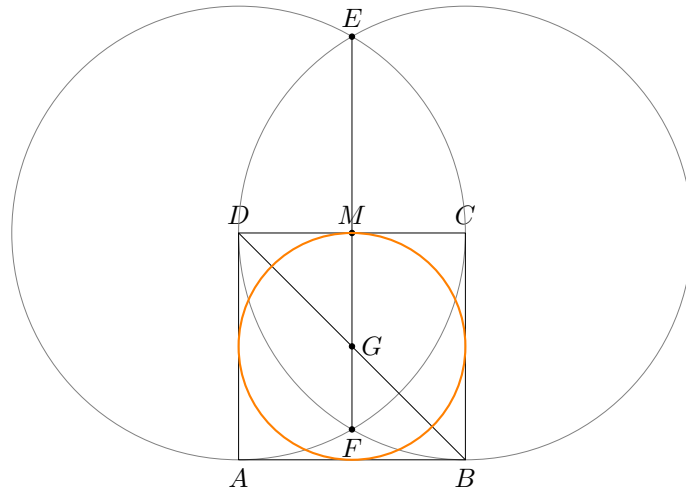
**1, 2.** Let vertices of square  $A, B, C, D$ . Draw circle centered  $D$  through  $C$ , and draw circle centered  $C$  through  $D$ .



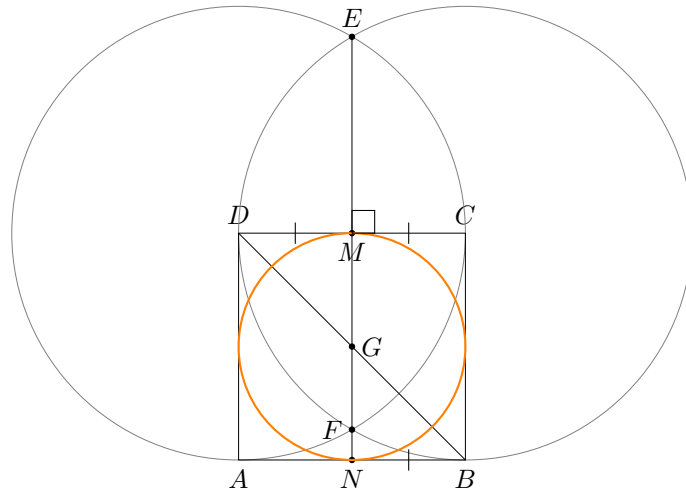
3, 4. Draw line  $BD$ . Let the intersections of the circles be  $E, F$ . Draw line  $EF$ .



5. Let  $G$  be the intersection of  $BD$  and  $EF$ , and let  $M$  be the intersection of  $CD$  and  $EF$ . Draw circle centered  $G$  through  $M$ . We get the desired circle.



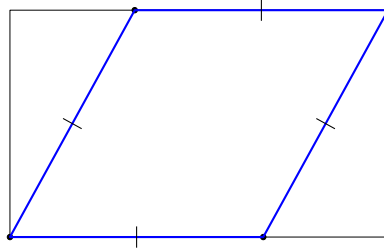
*Proof.* Note that  $EF$  is the perpendicular bisector of  $CD$  (by Task 1.2), so  $DM = MC$ . Extend  $MF$  to meet  $AB$  at  $N$ . We also have  $DM = NB$  since  $MN$  divides square  $ABCD$  into two congruent rectangles. Also note that  $\angle GDM = \angle GBN$  (alt.  $\angle$ s,  $DC \parallel AB$ ).



Thus  $\triangle DMG \cong \triangle BNG$  (AAS), so  $G$  is the midpoint of  $MN$  (corr. sides,  $\cong \triangle$ s). This means  $G$  is the center of the square (same point as “ $E$ ” in previous 3L solution), so the circle centered  $G$  through  $M$  is the inscribed circle of the square.  $\square$

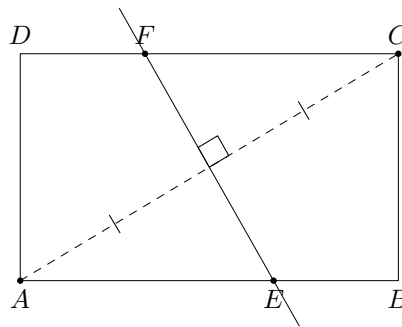
## 1.5 Rhombus in rectangle

**Task 1.5.** Inscribe a rhombus in the rectangle so that they share a diagonal.  
(3L, 5E, 2V)

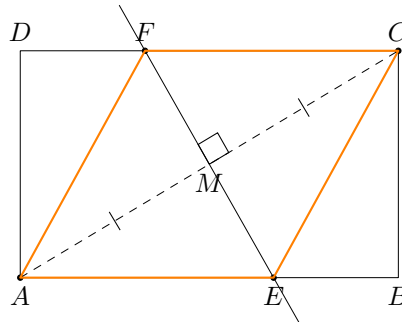


**Solution 1.5.** (3L, 5E)

1. Let the given rectangle be  $ABCD$ . Draw perpendicular bisector of  $AC$ , and let it intersect  $AB$  and  $CD$  at  $E$  and  $F$  respectively.



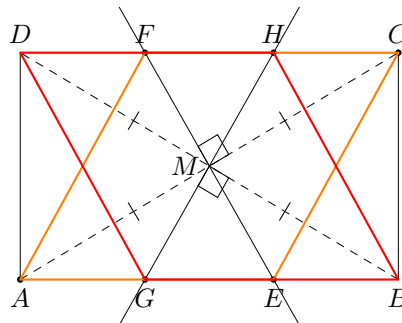
2, 3. Draw  $AF$  and  $EC$ . We get the desired rhombus  $AECF$ .



*Proof.* Let  $M$  be the midpoint of  $AC$ . Note that  $AM = CM$ . Also,  $\angle MFC = \angle MEA$  (alt.  $\angle$ s,  $FC \parallel AE$ ). Thus  $\triangle MFC \sim \triangle MEA$  (AAS), and  $CF = AE$  (corr. sides,  $\cong \triangle$ s)

Note that  $AF = CF$  and  $AE = CE$  by property of perpendicular bisector. Combined with  $CF = AE$ , we have  $AE = CE = AF = CF$ , which means  $AECF$  is a rhombus.  $\square$

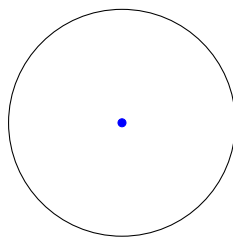
(2V). Similarly argument but flipped horizontally.





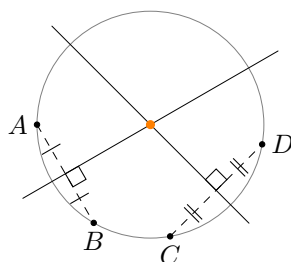
## 1.6 Circle center

**Task 1.6.** Construct the center of the circle.  
(2L, 5E)



**Solution 1.6. (2L)**

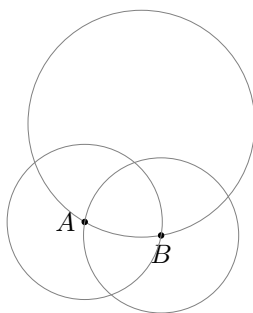
**1, 2.** Label two pairs of arbitrary points on the circle, and draw the perpendicular bisector of each pair of point. The intersection of the perpendicular bisectors is the desired center of circle.



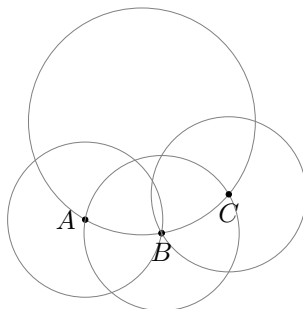
*Proof.* Perpendicular bisector of any chord passes through the center of a circle (“ $\perp$  bisector of chord passes through center”). This means the center of circle lies on both perpendicular bisectors of  $AB$  and  $CD$ , so it must be their point of intersection.  $\square$

**(5E)**

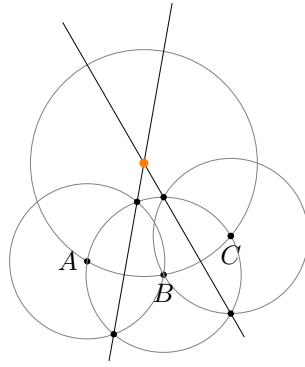
**1, 2.** Label two arbitrary points  $A$  and  $B$ . Draw circle centered  $A$  through  $B$ , and draw circle centered  $B$  through  $A$ .



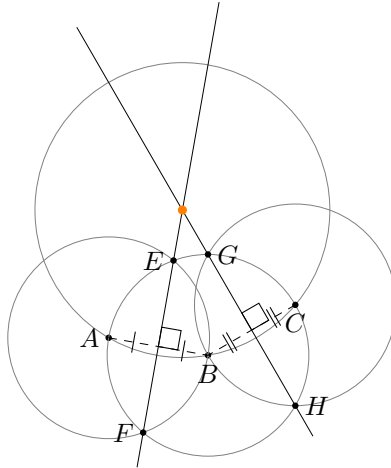
**3.** (Let  $(B, A)$  denote the circle centered  $B$  through  $A$ .) Let circle  $(B, A)$  intersect the given circle at another point  $C$ . Draw circle centered  $C$  through  $B$ .



**4, 5.** Draw line through the intersections of circles  $(A, B)$  and  $(B, A)$ , and draw line through the intersections of circles  $(B, C)$  and  $(C, B)$ . The intersection of these two lines is the desired center.



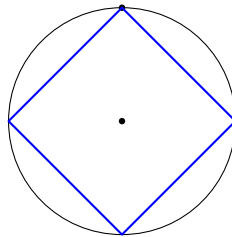
*Proof.* Note that  $EF$  and  $GH$  are perpendicular bisectors of chords  $AB$  and  $BC$  respectively. So they intersect at the center of the circle.



□

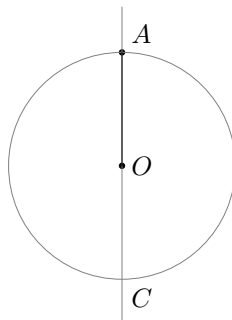
## 1.7 Inscribed square

**Task 1.7.** Inscribe a square in the circle. One vertex of the square is given. (The circle center is also given.)  
(6L, 7E)

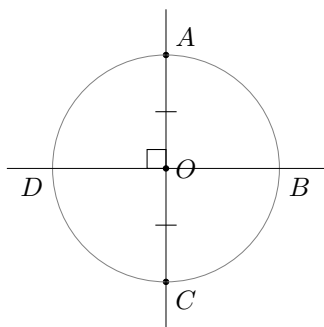


**Solution 1.7. (6L)**

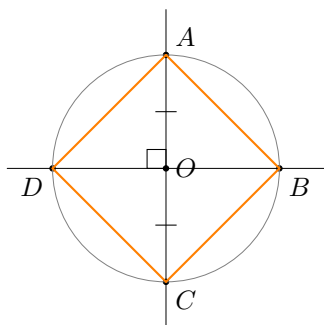
1. Let  $O$  be center of circle and  $A$  be the given vertex. Draw line  $AO$ . Let  $AO$  intersect the circle at  $C$ .



2. Draw the perpendicular bisector of  $AC$ . Let it intersect the circle at  $B$  and  $D$ .



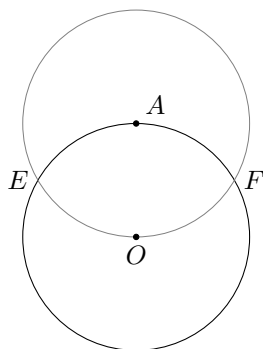
- 3, 4, 5, 6. Draw lines  $AB, BC, CD, DA$ . We get the desired square.



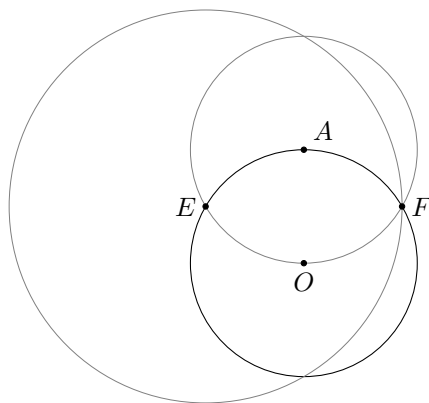
*Proof.* Note that the perpendicular bisector of  $AC$  passes through circle center  $O$ . So we have  $OA = OB = OC = OD$ . Since the diagonals of  $ABCD$  are perpendicular and bisect each other,  $ABCD$  is a square (con. of square).  $\square$

(7E)

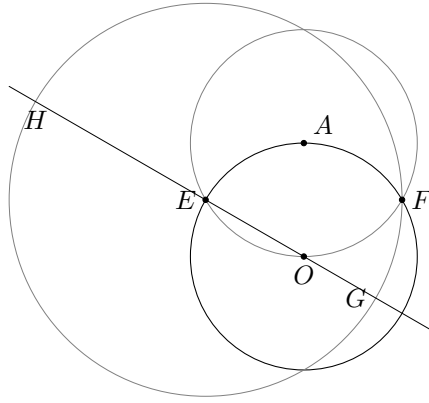
1. Draw circle centered  $A$  through  $O$ . Let the intersections of two circles be  $E$  and  $F$ .



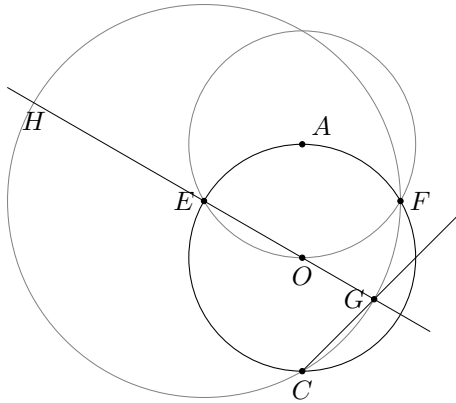
2. Draw circle centered  $E$  through  $F$ .



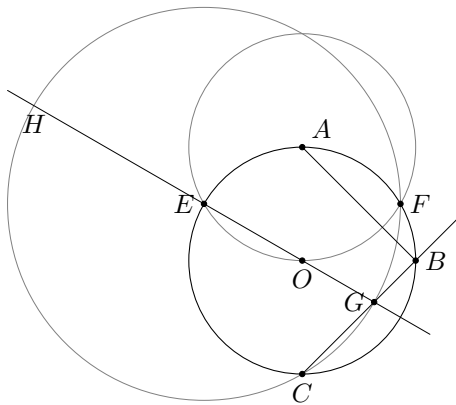
3. Draw line  $EO$ . Let  $EO$  intersect circle  $(E, F)$  at  $G$  and  $H$ , where  $G$  lies inside the given circle and  $H$  lies outside.



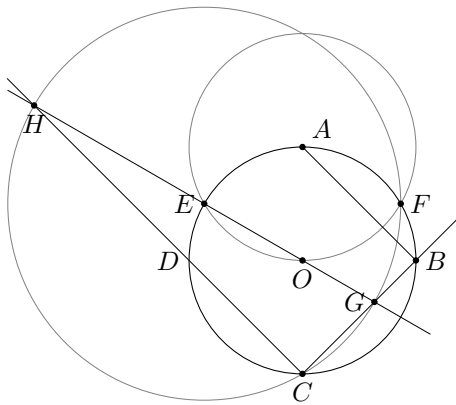
4. Let  $C$  be another intersection of  $(E, F)$  and the given circle. Draw line  $CG$ .



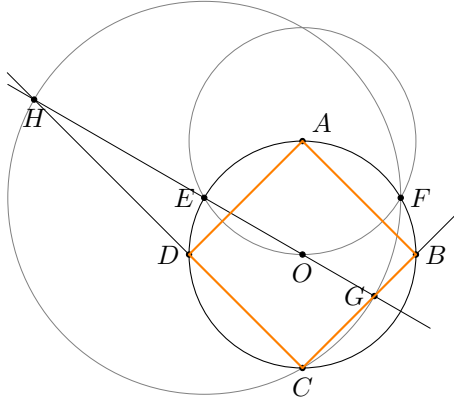
5. Let  $CG$  intersect given circle at another point  $B$ . Draw line  $AB$ .



6. Draw line  $CH$ . Let  $CH$  intersect given circle at  $D$ .

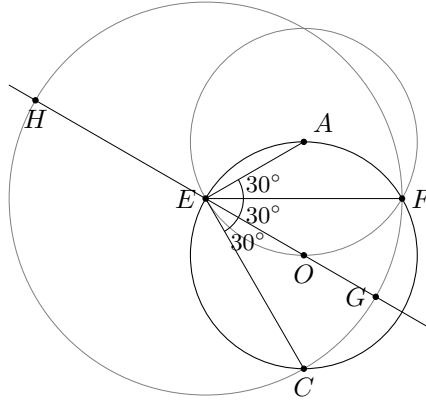


7. Draw line  $AD$ .  $ABCD$  is the desired square.



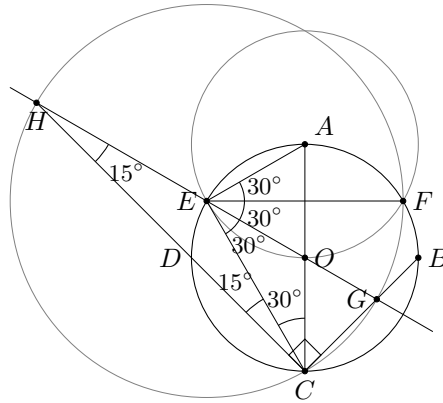
*Proof. 1-3.* Note that  $EF$  bisects  $\angle OEA$  since  $EF$  is the diagonal of rhombus  $AEOF$  which is made up of two equilateral triangles  $\triangle OAE$  and  $\triangle OAF$ . Thus  $\angle AEF = \angle OEF = 60^\circ/2 = 30^\circ$ .

Also, note that  $\angle OEC = \angle OEF = 30^\circ$  since  $\triangle OEC \sim \triangle OEF$  (SSS). Thus  $\angle AEC = 30^\circ + 30^\circ + 30^\circ = 90^\circ$ . By “converse of  $\angle$  in semi-circle”,  $AC$  is the diameter of given circle, which means  $A, O, C$  are collinear.

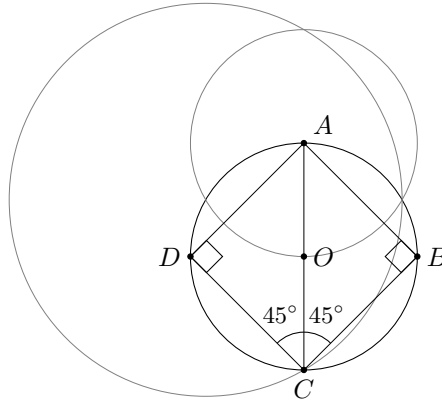


**4-7.** Note that  $GH$  is a diameter of circle  $(E, G)$ , so  $\angle HCG = 90^\circ$  ( $\angle$  in semi-circle).

Note that  $EH = EC$  (radii), so  $\angle ECH = \angle EHC = 30^\circ/2 = 15^\circ$  (base  $\angle$ s, isos.  $\triangle$ )& (ext.  $\angle$  of  $\triangle$ ). Also,  $\angle OCE = \angle OEC = 30^\circ$  (base  $\angle$ s, isos.  $\triangle$ ).



Thus,  $\angle OCD = 30^\circ + 15^\circ = 45^\circ$ , and  $\angle OCB = \angle OCG = 90^\circ - 45^\circ = 45^\circ$ .

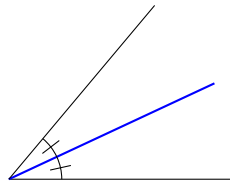


Let's focus on points  $A, B, C, D$ . Note that  $\angle ADC = \angle ABC = 90^\circ$  ( $\angle$  in semi-circle),  $\angle ACD = \angle ACB = 45^\circ$ , and  $AC = AC$ . Thus  $\triangle ADC \cong \triangle ABC$  (AAS) and  $BC = CD$  (corr. sides,  $\cong \triangle$ s). Since  $ABCD$  has four right angles and adjacent sides are equal,  $ABCD$  is a square (con. of square), as desired.  $\square$

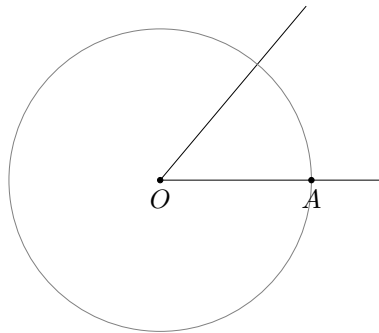
## 2 Beta

### 2.1 Angle bisector

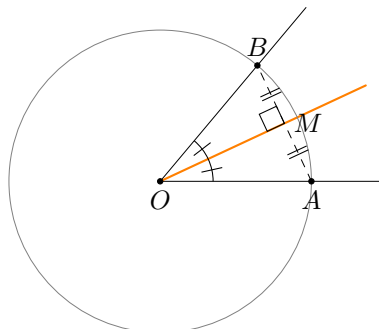
**Task 2.1.** Construct the line that bisects the given angle.  
(2L, 4E)



**Solution 2.1.** 1. Let  $O$  be the vertex of the given angle. Label an arbitrary point  $A$  on one of the given rays. Draw circle  $(O, A)$ .



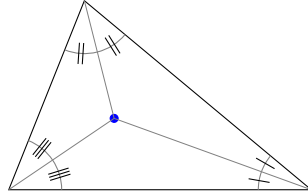
2. Let  $B$  be the intersection of the circle and the other ray. Draw perpbi  $AB$  (perpendicular bisector of  $A, B$ ), which is the desired angle bisector.



*Proof.* Note that  $\triangle OAB$  is an isosceles triangle since  $OA = OB$  (radii). Let  $M$  be the midpoint of  $AB$ . Since  $OM \perp AB$ , by “prop. of isos.  $\triangle$ ”, we have  $\angle AOM = \angle BOM$ .  $\square$

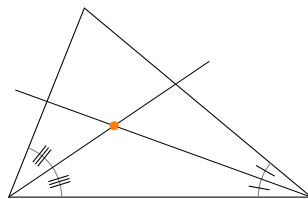
## 2.2 Intersection of angle bisectors

**Task 2.2.** Construct the point where the angle bisectors of the triangle are intersected.  
(2L, 6E)



**Solution 2.2. (2L)**

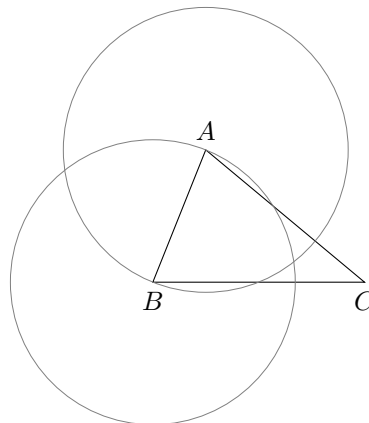
**1, 2.** Draw angle bisectors of two of the vertices of the triangle. Their intersection is the desired point.



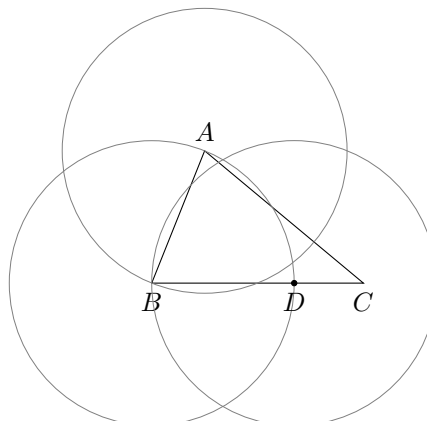
*Proof.* Note that the three angle bisectors of a triangle are concurrent (prop. of  $\angle$  bisector). So we only need to find the intersection of two of them.  $\square$

**(6E)**

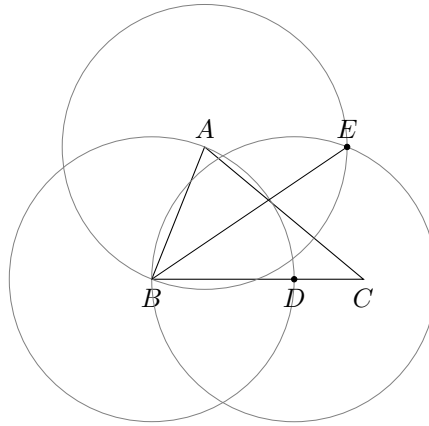
**1, 2.** Let the vertices of triangle be  $A, B, C$ . Draw circle  $(A, B)$  and circle  $(B, A)$ .



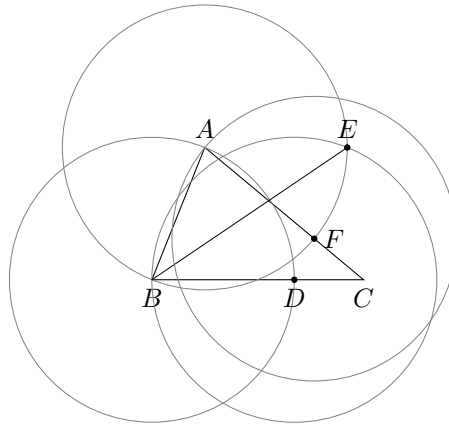
**3.** Let circle  $(B, A)$  intersect side  $BC$  at  $D$ . Draw circle  $(D, B)$ .



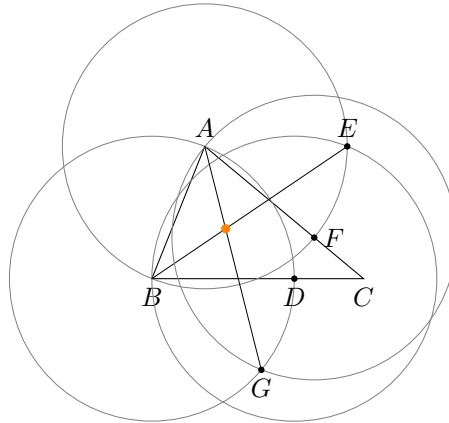
4. Let  $(D, B)$  and  $(A, B)$  intersect at another point  $E$ . Draw line  $BE$ .



5. Let  $(A, B)$  intersect side  $AC$  at  $F$ . Draw circle  $(F, A)$ .



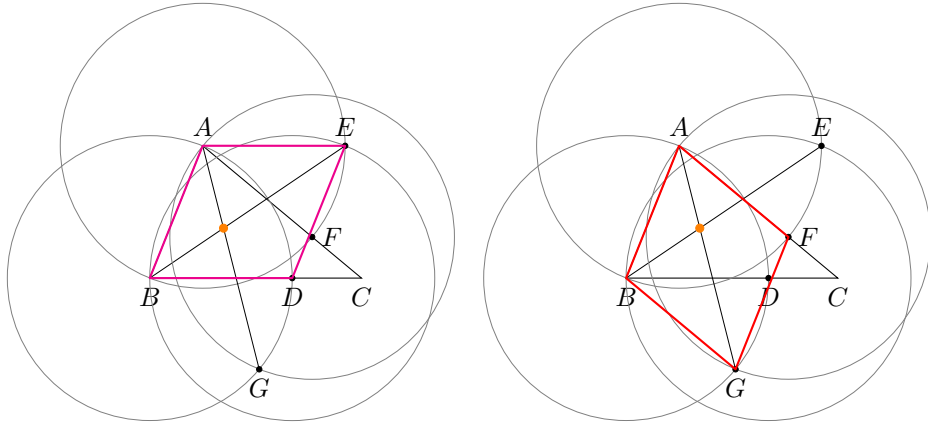
6. Let  $(F, A)$  and  $(B, A)$  intersect at another point  $G$ . Draw line  $AG$ . The intersection of  $BE$  and  $AG$  is the desired point.



*Proof. 1-4.* Let  $r$  be the length of  $AB$ . Note that  $AE = AB = BD = DE$  since they are all radii of circles with radius  $r$ . So  $ABDE$  is a rhombus. Since  $BE$  is a diagonal of the rhombus,  $BE$  bisects  $\angle B$  (prop. of rhombus).

**5-6.** Similarly, since  $AB = BG = FG = FA$ ,  $ABGF$  is a rhombus of side length  $r$ . Since  $AG$  is a diagonal of rhombus  $ABGF$ ,  $AG$  bisects  $\angle A$ .

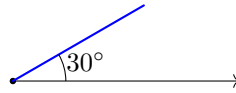




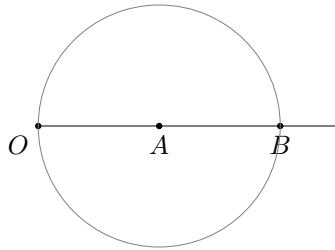
□

## 2.3 Angle of 30 deg

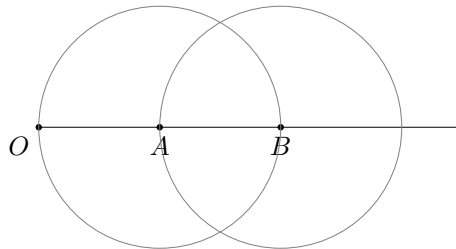
**Task 2.3.** Construct an angle of  $30^\circ$  with the given side.  
(3L, 3E, 2V)



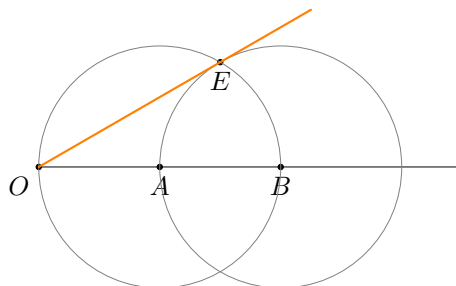
**Solution 2.3. (3L, 3E)** 1. Let  $O$  be the endpoint of the given ray, and  $A$  be an arbitrary point on the given ray. Draw circle  $(A, O)$ , intersecting given ray at  $B$ .



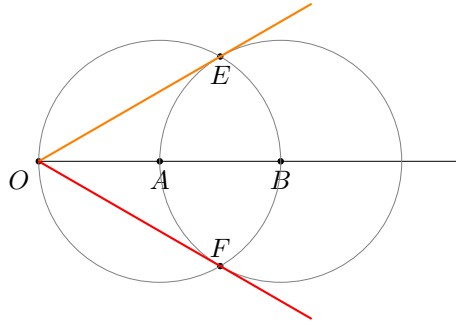
2. Draw circle  $(B, A)$ .



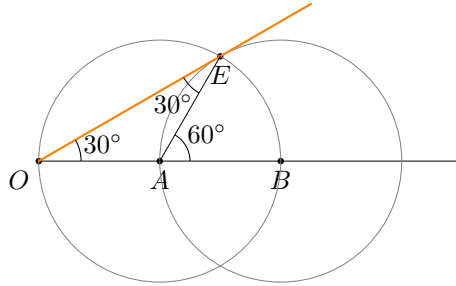
3. Let  $E$  be one intersection of the two circles. Draw line  $OE$ , which is the desired line.



(2V) 4. Let  $F$  be another intersection of the two circles. Draw line  $OF$ .



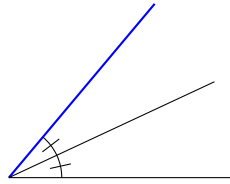
*Proof.* Note that  $\angle EAB = 60^\circ$  by construction. Also,  $AO = AE$  (radii) so  $\angle AOE = \angle AEO$  (base  $\angle$ s, isos.  $\triangle$ ). So  $\angle EOA = 60^\circ/2 = 30^\circ$  (ext.  $\angle$  of  $\triangle$ ). Similar argument for the other line.



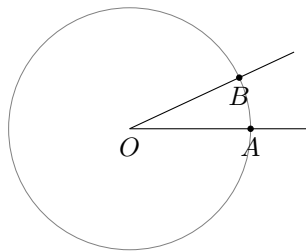
□

## 2.4 Double angle

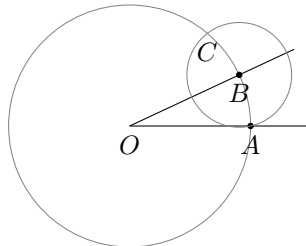
**Task 2.4.** Construct an angle equal to the given one so that they share one side.



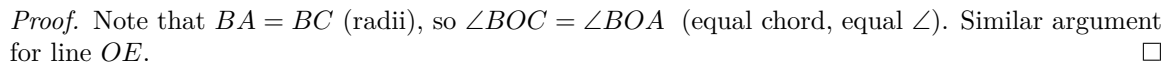
**Solution 2.4. 1.** Let  $O$  be the vertex of given angle, and  $A$  be an arbitrary point on one ray. Draw circle  $(O, A)$ , intersecting the other ray at  $B$ .



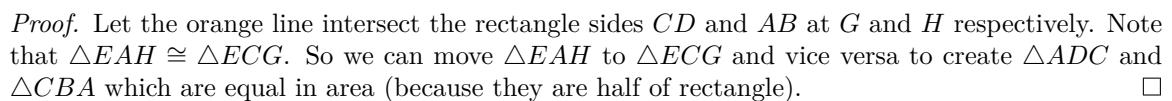
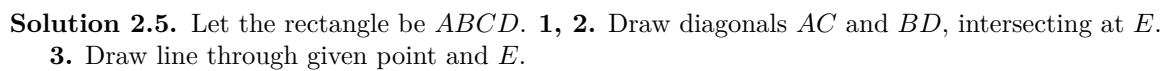
**2.** Draw circle  $(B, A)$ , intersecting  $(O, A)$  at another point  $C$ .



**3.** Draw line  $OC$ , which is the desired line.

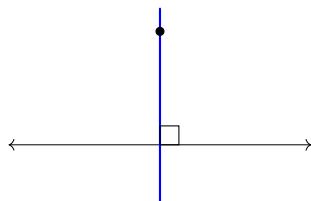


**Task 2.5.** Construct a line through the given point that cuts the rectangle into two parts of equal area.  
(3L, 3E)



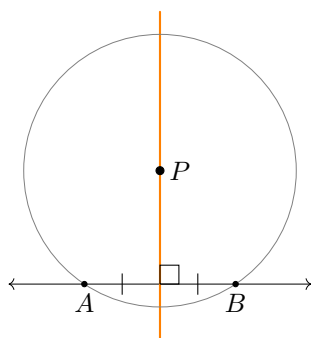
## 2.6 Drop a perpendicular

**Task 2.6.** Drop a perpendicular from the point to the line.  
(2L, 3E)



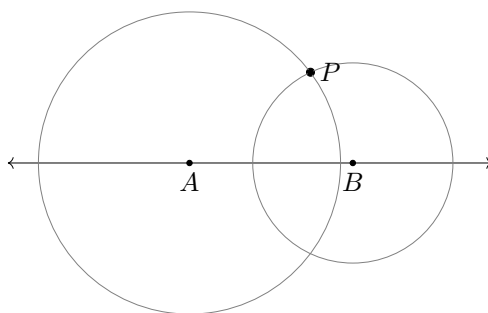
**Solution 2.6. (2L)** Let the given point be  $P$ , and  $A$  be an arbitrary point on given line.

1. Draw circle  $(P, A)$ , intersecting the line on  $B$ .
2. Draw perpbi  $AB$ .

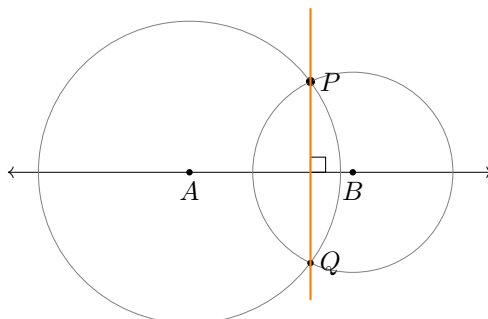


*Proof.*  $AB$  is a chord of the circle, so the perpendicular bisector of  $AB$  passes through center  $P$ . This means we have constructed a line through  $P$  that is perpendicular to line  $AB$ .  $\square$

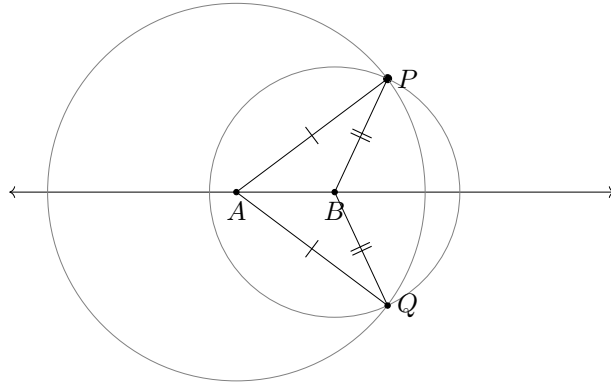
**(3E) 1, 2.** Label two arbitrary points  $A, B$ . Draw circles  $(A, P)$  and circle  $(B, P)$ .



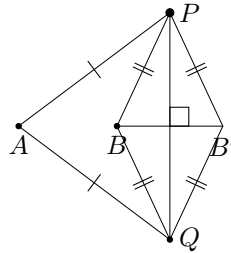
3. Draw line through the intersection of the two circles, which is the desired line.



*Proof.* Let  $Q$  be the other intersection of the two circles. Note that  $AP = AQ$  and  $BP = BQ$  (radii), so  $APBQ$  is either a kite or a dart. If  $APBQ$  is a kite, then by “prop. of kite”, the diagonals of the kite are perpendicular to each other, meaning  $PQ \perp AB$ .



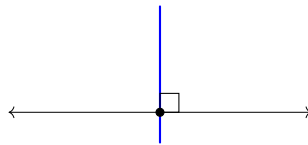
If  $APBQ$  is a dart with  $B$  being the concave point, then reflect  $B$  about line  $PQ$  to get  $B'$ . Note that  $PQ \perp BB'$  and  $BPB'Q$  is a rhombus (by reflection). Since  $APB'Q$  is a kite, we also have  $PQ \perp AB'$ . Thus  $AB'$  and  $BB'$  are parallel, but they share the same point  $B'$ , so  $A, B, B'$  must lie on the same line. This means  $PQ \perp AB$ , our desired result.



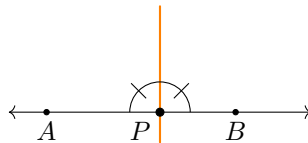
□

## 2.7 Erect a perpendicular

**Task 2.7.** Erect a perpendicular from the point on the line.  
(1L, 3E)

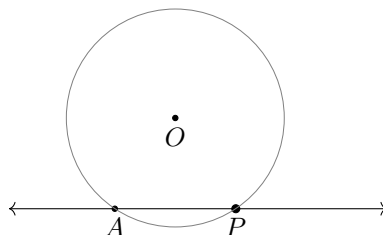


**Solution 2.7. (1L)** Let  $P$  be the given point. Let  $A$  be an arbitrary point to the left of  $P$  and  $B$  be an arbitrary point to the right of  $P$ . Draw the angle bisector of  $\angle AOB$ , which is the desired line.

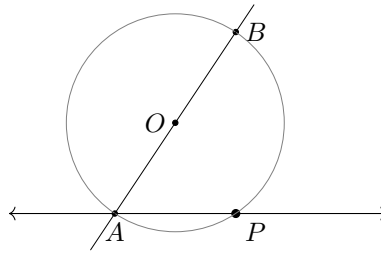


*Proof.* Since  $A, O, P$  are on a straight line,  $\angle AOP = 180^\circ$ , so the angle bisector makes two angles of  $90^\circ$ , which means the angle bisector is perpendicular to line  $AOB$ . □

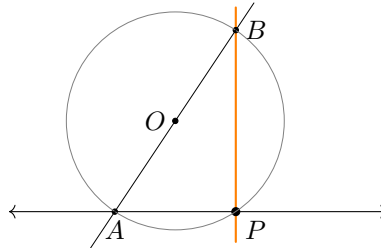
**(3E) 1.** Label an arbitrary point  $O$  not on the given line. Draw circle  $(O, P)$ , intersecting the given line at another point  $A$ .



2. Draw line  $AO$ . Let it intersect the circle at  $B$ .



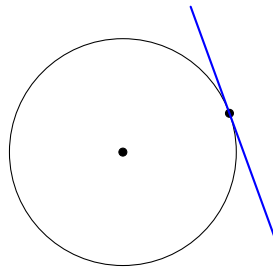
3. Draw line  $BP$ , which is the desired line.



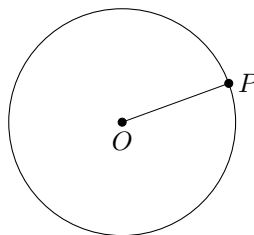
*Proof.* Note that  $AB$  is the diameter of the circle, so  $\angle APB = 90^\circ$  ( $\angle$  in semi-circle), which means  $BP \perp AP$ .  $\square$

## 2.8 Tangent to circle at point

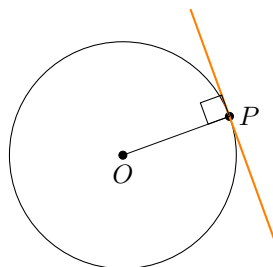
**Task 2.8.** Construct a tangent to the circle at the given point.  
(2L, 3E)



**Solution 2.8.** Let  $O$  be the center of circle and  $P$  be the given point on the circle.  
(2L) 1. Draw line  $OP$ .

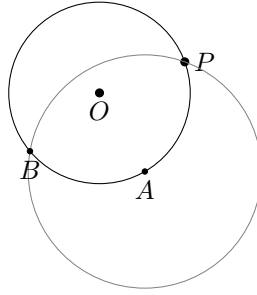


2. Draw the perpendicular line of  $OP$  at  $P$ .

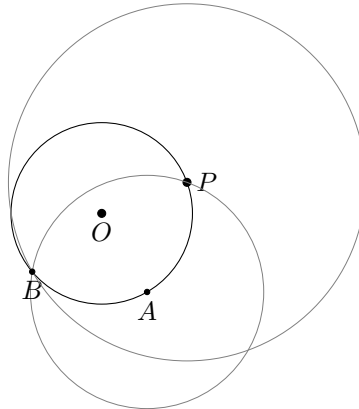


*Proof.* Since  $OP$  is a radius of the circle and is perpendicular to the orange line, by “converse of tangent  $\perp$  radius”, the orange line is tangent to the circle at  $P$ .  $\square$

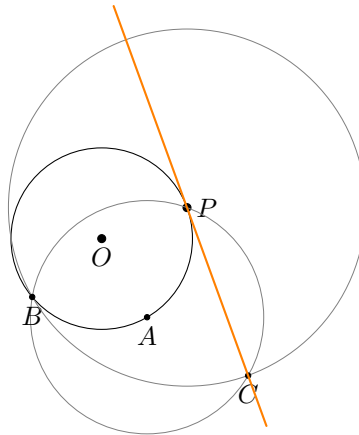
**(3E) 1.** Let  $A$  be an arbitrary point on the given circle. Draw circle  $(A, P)$ , intersecting the given circle at  $B$ .



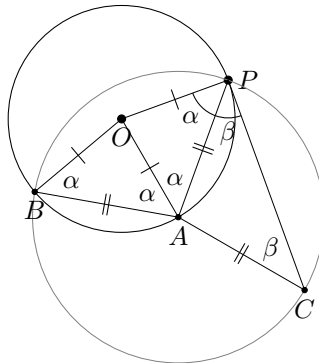
**2.** Draw circle  $(P, B)$ .



**3.** Let  $(P, B)$  intersect  $(A, P)$  at another point  $C$ . Draw line  $PC$ , the desired line.



*Proof.* Let  $\angle OPA = \alpha$  and  $\angle APC = \beta$ . We want to show that  $\alpha + \beta = 90^\circ$ , which will prove that  $PC$  is the tangent to the given circle at  $P$ .

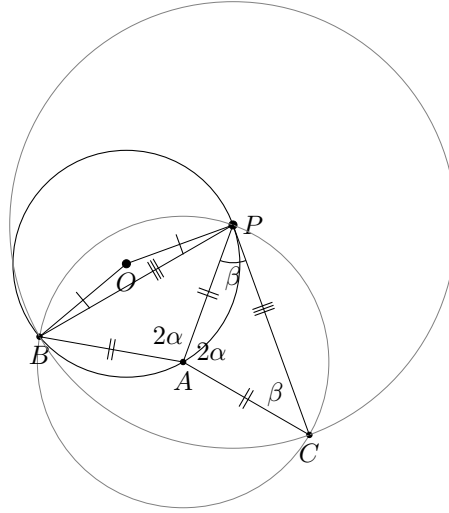


$$\begin{aligned} OA &= OP && \text{(radii)} \\ \therefore \angle OAP &= \angle OPA = \alpha && \text{(base } \angle\text{s, isos. } \triangle) \end{aligned}$$

$$\triangle OBA \cong \triangle OPA \quad (\text{SSS})$$

$$\therefore \angle OBA = \angle OPA = \alpha \text{ and } \angle OAB = \angle OAP = \alpha \text{ (corr. } \angle\text{s, } \cong \triangle\text{s)}.$$

Now consider  $\triangle APC$  and  $\triangle APB$ .



$$AC = AP \quad (\text{radii})$$

$$\therefore \angle ACP = \angle APC = \beta \quad (\text{base } \angle\text{s, isos. } \triangle)$$

$$PB = PC \quad (\text{radii of biggest circle})$$

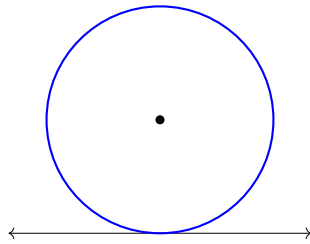
$$\therefore \triangle APC \cong \triangle APB \quad (\text{SSS})$$

$$\therefore \angle PAC = \angle PAB = 2\alpha \quad (\text{corr. } \angle\text{s, } \cong \triangle\text{s})$$

In  $\triangle APC$ , we have  $2\alpha + \beta + \beta = 180^\circ$  ( $\angle$  sum of  $\triangle$ ), giving  $\alpha + \beta = 90^\circ$ , as desired.  $\square$

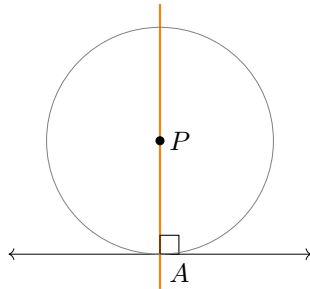
## 2.9 Circle tangent to line

**Task 2.9.** Construct a circle with the given center that is tangent to the given line.  
(2L, 4E)



**Solution 2.9.** 1. Draw line perpendicular to the given line passing through given point  $P$ .

2. Draw circle centered  $P$  through the intersection of the two lines  $A$ .

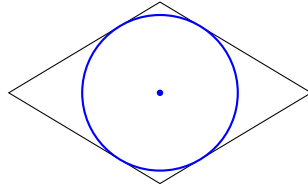


*Proof.*  $PA$  is tangent to the given line by “converse of tangent  $\perp$  radius”.  $\square$

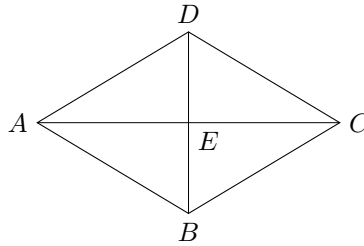


## 2.10 Circle in rhombus

**Task 2.10.** Inscribe a circle in the rhombus.  
(4L, 6E)

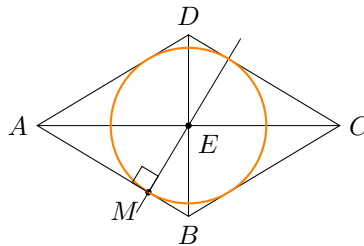


**Solution 2.10.** 1, 2. Let the rhombus be  $ABCD$ . Draw diagonals  $AC$  and  $BD$ . Let them intersect at  $E$ .



3. Draw  $ME \perp AB$  (i.e. line perpendicular to  $AB$  passing through  $E$ , intersecting  $AB$  at  $M$ ).

4. Draw circle  $(E, M)$ .

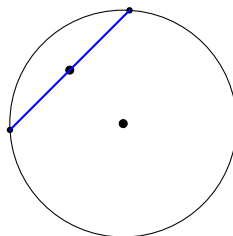


*Proof.* Note that the diagonals divide the rhombus into four congruent triangles (prop. of rhombus), so they have the same height. This means sides  $AB, BC, CD, DA$  have the same perpendicular distance from  $E$ . Thus, a circle tangent to one of the sides must be tangent to all of them.  $\square$

## 3 Gamma

### 3.1 Chord midpoint

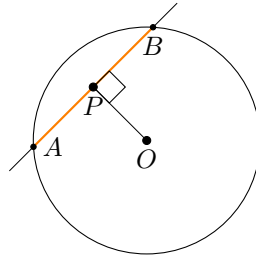
**Task 3.1.** Construct a chord whose midpoint is given.  
(2L, 4E)



**Solution 3.1.** Let  $O$  be center of given circle and  $P$  be given point.

1. Draw line  $OP$ .

2. Draw  $OP \perp P$  (i.e. line perpendicular to  $OP$  passing through  $P$ ), intersecting the circle at  $A$  and  $B$ .  $AB$  is the desired chord.

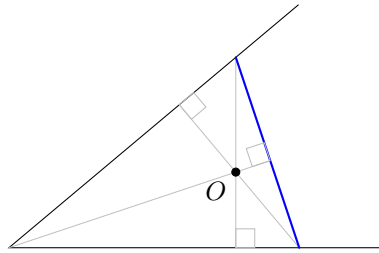


*Proof.* Since  $OP \perp AB$ , we have  $AP = PB$  by “line from center  $\perp$  chord bisects chord”.  $\square$

### 3.2 Triangle by angle and orthocenter

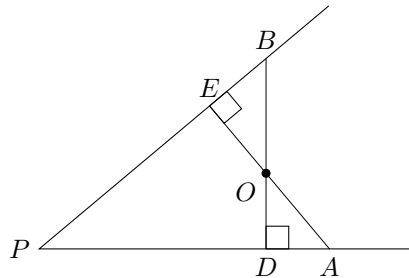
**Task 3.2.** Construct a segment connecting the sides of the angle to get a triangle whose orthocenter is in the point  $O$ .

(3L, 6E)

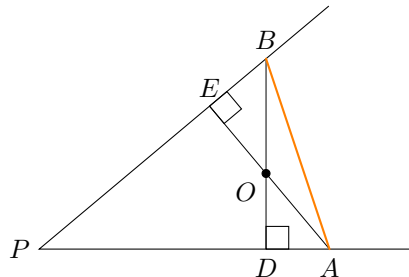


**Solution 3.2.** Let  $P$  be the vertex of given angle.

(3L) **1, 2.** Draw lines perpendicular to the given rays passing through  $O$ . Let  $D, E$  be the feet of the perpendicular lines, and let  $EO$  and  $DO$  meet the given rays at  $A$  and  $B$  respectively.



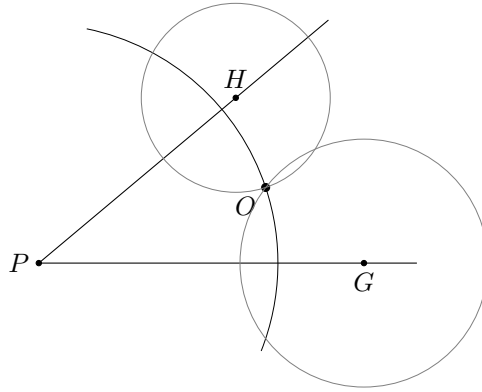
**3.** Draw line  $AB$ , the desired line.



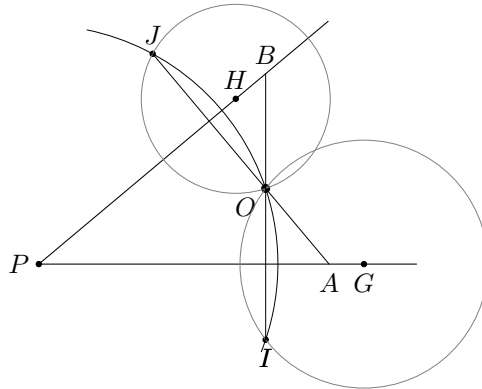
*Proof.* Note that  $O$  is the orthocenter of  $\triangle PAB$  since it is the intersection of two altitudes. And any two altitudes intersect at the orthocenter because the three altitudes of a triangle are concurrent.  $\square$

(6E) **1.** Draw circle  $(P, O)$ .

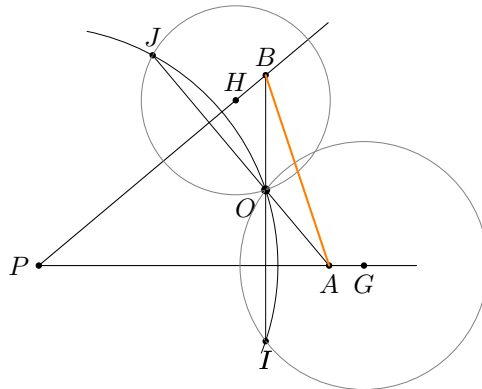
**2, 3.** Let  $G, H$  be two points (arbitrary or on intersection, doesn't matter) on each of the given ray. Draw circles  $(G, O)$  and  $(H, O)$ .



4, 5. Let  $(P, O)$  intersect  $(G, O)$  and  $(H, O)$  at the other point  $I$  and  $J$  respectively. Draw line  $IO$ , meeting  $PH$  at  $B$ . Draw line  $JO$ , meeting  $PG$  at  $A$ .



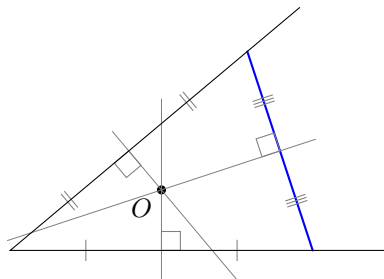
6. Draw line  $AB$ , the desired line.



*Proof.* Note that  $OI \perp PG$  since  $POGI$  forms a kite. Similarly,  $OJ \perp PH$  since  $POHJ$  forms a kite. Thus line  $OI$  and  $OJ$  are altitudes of  $\triangle PAB$ , so  $O$  is the orthocenter of  $\triangle PAB$ .  $\square$

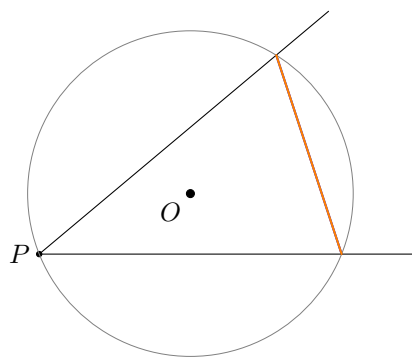
### 3.3 Intersection of perpendicular bisectors

**Task 3.3.** Construct a segment connecting the sides of the angle to get a triangle whose perpendicular bisectors are intersected in the point  $O$ .  
(2L, 2E)

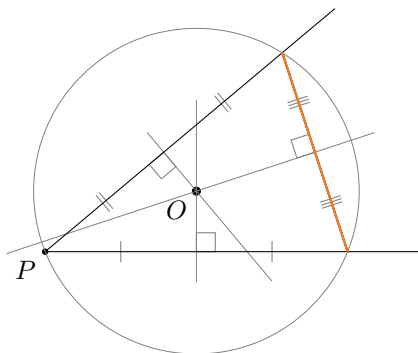


**Solution 3.3.** Let  $P$  be the vertex of the given angle.

1. Draw circle  $(O, P)$ , intersecting the given rays at  $A$  and  $B$  respectively.
2. Draw line  $AB$ .



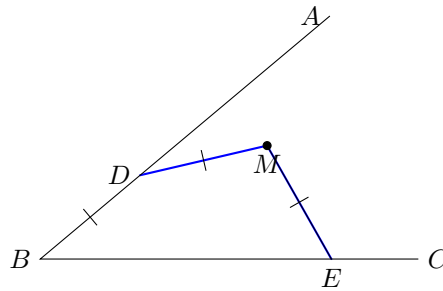
*Proof.* Note that  $O$  is the circumcenter of  $\triangle PAB$ . And the perpendicular bisectors of sides of  $\triangle PAB$  intersect at the circumcenter by “prop. of circumcenter”.



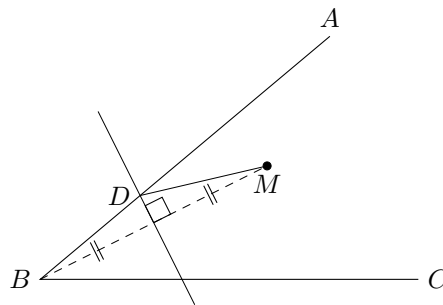
□

### 3.4 Three equal segments - 1

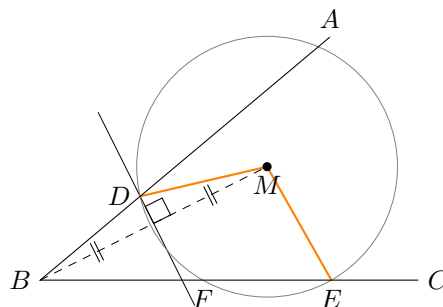
**Task 3.4.** Given an angle  $ABC$  and a point  $M$  inside it, find points  $D$  on  $BA$  and  $E$  on  $BC$  and construct segments  $DM$  and  $ME$  such that  $BD = DM = ME$ .  
(4L, 6E, 2V)



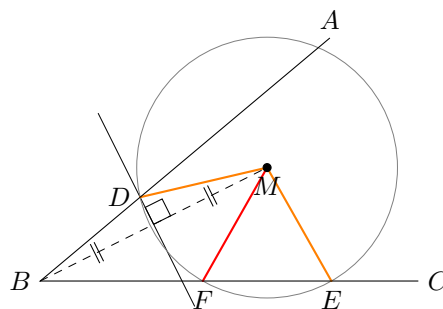
**Solution 3.4.** (4L, 6E) 1. Draw perpbi  $BM$ , intersecting  $AB$  at  $D$ .  
2. Draw line  $MD$ .



3. Draw circle  $(M, D)$ , intersecting line  $BC$  at  $E$  and  $F$ .  
4. Draw line  $ME$  (or  $MF$ ).



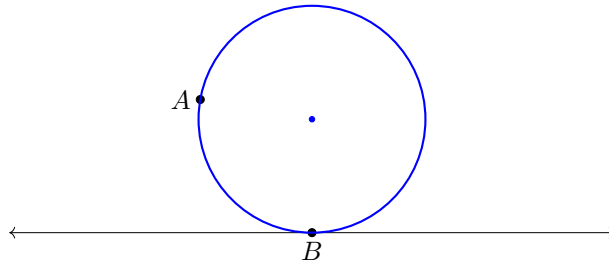
(2V) Draw line  $MF$  (or  $ME$ ).



*Proof.*  $BD = DM$  since  $D$  lies on the perpendicular bisector of  $BM$ .  $DM = ME = MF$  since  $D$ ,  $E$  and  $F$  lie on the circle centered  $M$ . Thus  $BD = DM = ME = MF$ .  $\square$

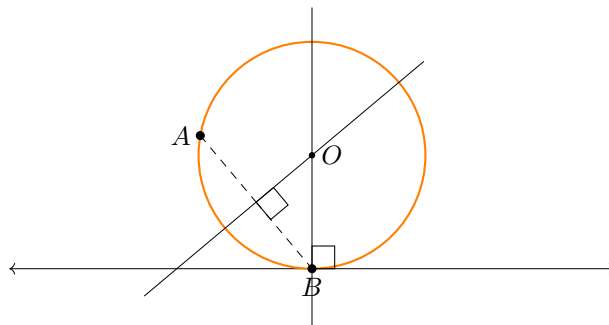
### 3.5 Circle through point tangent to line

**Task 3.5.** Construct a circle through the point  $A$  that is tangent to the given line at the point  $B$ .  
(3L, 6E)



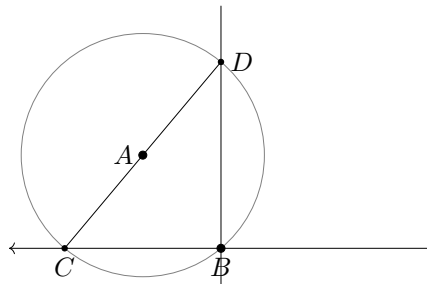
**Solution 3.5. (3L)**

- 1, 2. Draw perpbi  $AB$ . Draw perpendicular line to given line through  $B$ . Let the two drawn lines intersect at  $O$ .
3. Draw  $OB$ .

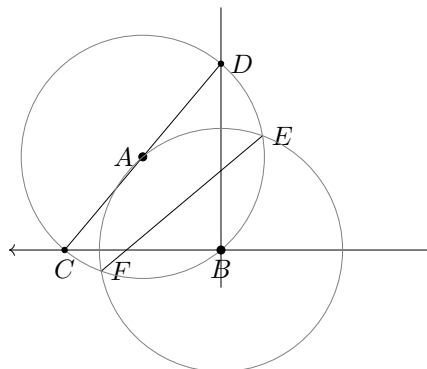


*Proof.* Since the circle passes through both  $A$  and  $B$ , center  $O$  must lie on the perpendicular bisector of  $AB$  (prop. of  $\perp$  bisector). Since  $O$  is tangent to give line,  $OB$  must be perpendicular to given line (tangent  $\perp$  radius). Thus  $O$  lies on the intersection of the two drawn lines.  $\square$

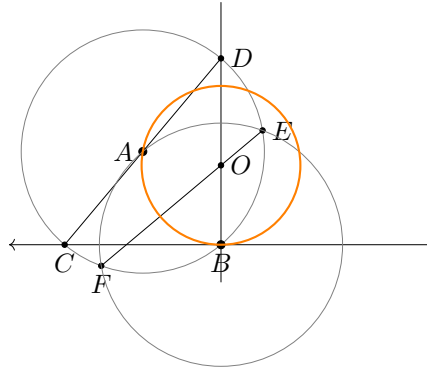
- (6E)**
1. Draw circle  $(A, B)$ , intersecting given line at  $C$ .
  2. Draw line  $CA$ , meeting circle  $(A, B)$  at  $D$ .
  3. Draw line  $BD$ .



4. Draw circle  $(B, A)$ , intersecting  $(A, B)$  at  $E$  and  $F$ .
5. Draw line  $EF$ , intersecting  $BD$  at  $O$ .



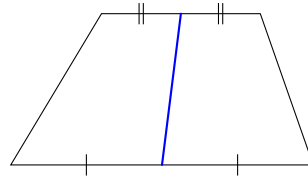
6. Draw circle  $(O, B)$ .



*Proof.* Note that  $BD$  is perpendicular to given line by Task 2.7E, and  $EF$  is the perpendicular bisector of  $AB$  by Task 1.2. So  $O$  is the same point as the (3L) part of this level.  $\square$

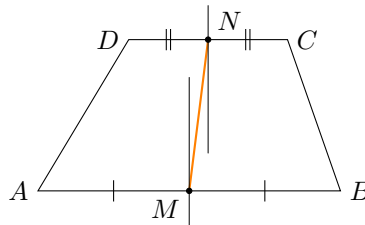
### 3.6 Midpoints of trapezoid bases

**Task 3.6.** Construct a line passing through the midpoints of the trapezoid bases.  
(3L, 5E)



**Solution 3.6. (3L)**

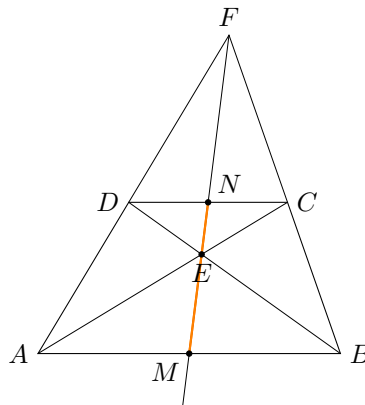
- 1, 2. Draw perpbi  $AB$  and draw perpbi  $CD$ . Let the midpoints of the sides be  $M$  and  $N$ .
3. Draw line  $MN$ .



*Proof.*  $AM = MB$  and  $DN = NC$  by perpendicular bisector construction.  $\square$

(5E) 1, 2. Draw the diagonals of the trapezoid. Let them intersect at  $E$ .

- 3, 4. Extend the non-parallel sides to meet at  $F$ .
5. Draw line  $FE$ , which is the desired line.



*Proof.* Let  $FE$  intersect sides  $AB$  and  $CD$  at  $M$  and  $N$  respectively. We want to show that  $AM = MB$  and  $DN = NC$ .

By Ceva's theorem, we have

$$\frac{AM}{MB} \cdot \frac{BC}{CF} \cdot \frac{FD}{DA} = 1 \quad (1)$$

Since  $AB \parallel CD$ , by intercept theorem, we also have

$$\begin{aligned} \frac{BC}{CF} &= \frac{AD}{DF} \\ \Leftrightarrow \frac{BC}{CF} \cdot \frac{FD}{DA} &= 1 \end{aligned} \quad (2)$$

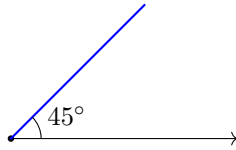
Put (2) into (1).

$$\begin{aligned} \frac{AM}{MB} \cdot (1) &= 1 \\ AM &= MB \end{aligned}$$

Note that  $\triangle FDN \sim \triangle FAM$  and  $\triangle FNC \sim \triangle FMB$  (AAA). So  $\frac{DN}{AM} = \frac{FN}{FM} = \frac{NC}{MB}$  (corr. sides,  $\sim \triangle$ s). Since  $AM = MB$ , this gives  $\frac{DN}{AM} = \frac{NC}{AM}$ , and thus  $DN = NC$ , as desired.  $\square$

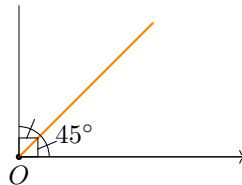
### 3.7 Angle of 45 deg

**Task 3.7.** Construct an angle of  $45^\circ$  with the given side.  
(2L, 5E, 2V)



**Solution 3.7.** Let  $O$  be the endpoint of the given ray.

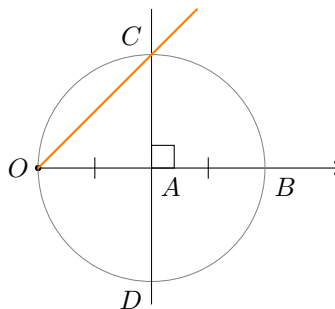
- (2L) 1. Draw line perpendicular to given line through  $O$ .
2. Draw the angle bisector of the two lines.



*Proof.* The angle between the two perpendicular lines is  $90^\circ$ , and the angle bisector makes  $90^\circ/2 = 45^\circ$ .  $\square$

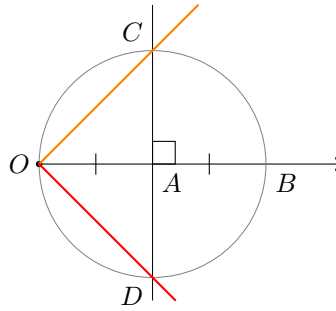
(5E) 1. Let  $A$  be an arbitrary point on the given ray. Draw circle  $(A, O)$ , intersecting the ray again at  $B$ .

2. Draw perpbi  $OB$ , intersecting the circle at  $C$  and  $D$ .
3. Draw line  $OC$ , the desired line.





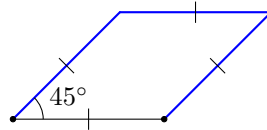
(2V)



*Proof.* Since  $AO = AC$  and  $CA \perp OB$ ,  $\triangle OAC$  is an isosceles right triangle, so its acute angles are  $45^\circ$ , which means  $\angle AOC = 45^\circ$ . Same for the other line  $OD$ .  $\square$

### 3.8 Lozenge

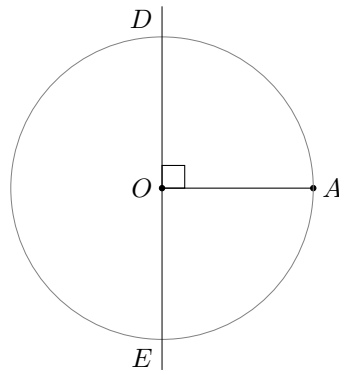
**Task 3.8.** Construct a rhombus with the given side and an angle of  $45^\circ$  in a vertex.  
(5L, 7E, 4V)



**Solution 3.8.** Let  $O$  and  $A$  be the endpoints of the given line segment.

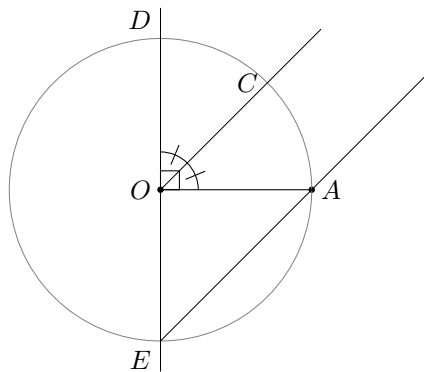
(5L) 1. Draw  $OA \perp O$ .

2. Draw circle  $(O, A)$ , intersecting the vertical line at  $D$  and  $E$  (where  $D$  above  $E$ ).

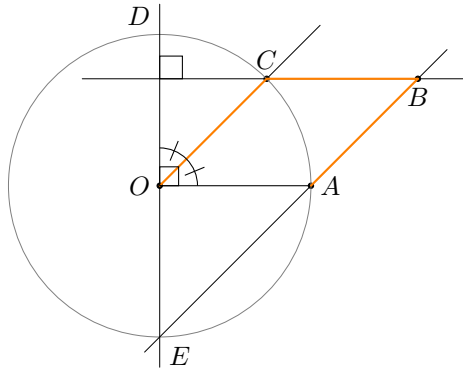


3. Draw angle bisector  $\angle DOA$  (angle bisector of  $\angle DOA$ ), intersecting  $(O, A)$  at  $C$ .

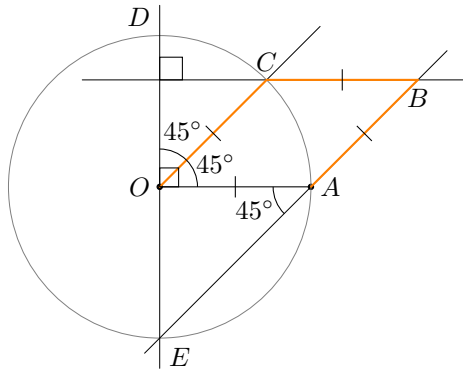
4. Draw line  $EA$ .



5. Draw  $OD \perp C$ , intersecting  $EA$  at  $B$ .  $OABC$  is the desired rhombus.

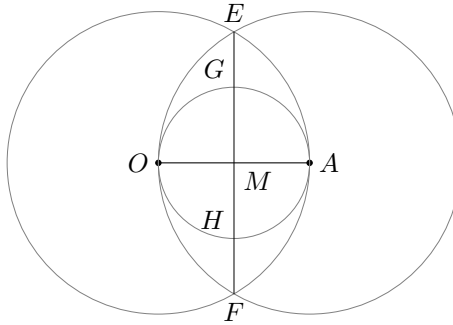


*Proof.* Note that  $CB \parallel OA$  since they are both perpendicular to  $DO$ . Note that  $\angle AOC = 45^\circ$  (since it is half of right angle), and  $\angle OAE = 45^\circ$  since  $\triangle OAE$  is an isosceles right triangle. Thus  $OC \parallel EB$  (alt.  $\angle$ s equal). This means  $OACB$  is a parallelogram.

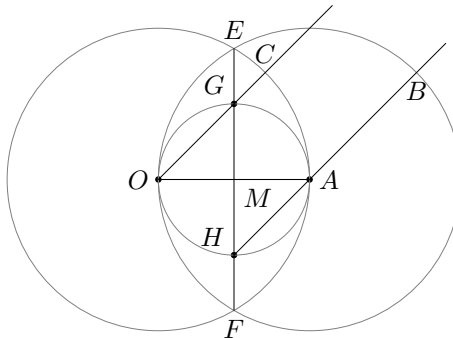


Since  $OA = OC$  (radii),  $OACB$  is a parallelogram with adjacent sides equal, so  $OACB$  is a rhombus. Along with  $\angle AOC = 45^\circ$ ,  $OACB$  is the desired rhombus.  $\square$

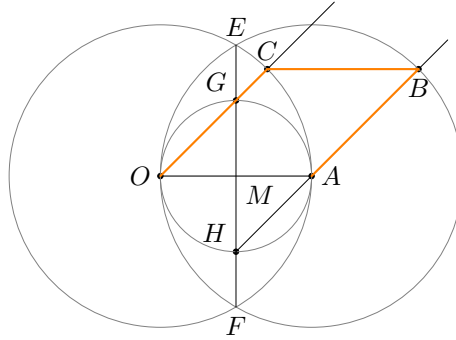
- (7E) 1, 2. Draw circle  $(O, A)$  and  $(A, O)$ , intersecting at  $E$  and  $F$ .  
 3. Draw line  $EF$ , intersect  $OA$  at  $M$ .  
 4. Draw circle  $(M, O)$ , intersecting  $EF$  at  $G$  and  $H$  ( $G$  above  $H$ ).



- 5, 6. Draw lines  $OG$  and  $HA$ . Let  $OG$  intersect  $(O, A)$  at  $C$ , and let  $HA$  intersect  $(A, O)$  at  $B$  (where both points are on the same side of  $OA$ ).



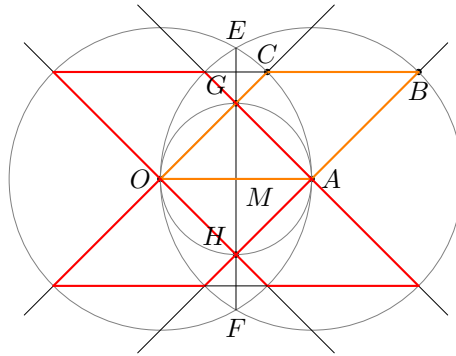
7. Draw line  $BC$ .



*Proof.* Note that  $\triangle MOG$  and  $\triangle MAH$  are isosceles right triangles, so  $\angle MOG = \angle MAH = 45^\circ$  and  $OC \parallel HB$  (alt.  $\angle$ s equal). Moreover, note that  $OC = AB$  since they lie on circles of the same radius. Thus  $OACB$  is a parallelogram (opp. sides equal and  $\parallel$ ).

And since  $OA = OC$  (radii),  $OACB$  has adjacent sides equal, so it is a rhombus.  $\square$

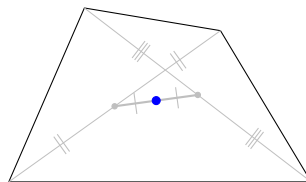
(4V) Draw line  $GA$  and  $OH$ . Connect the intersections of the lines and the big circles to form a symmetric figure.



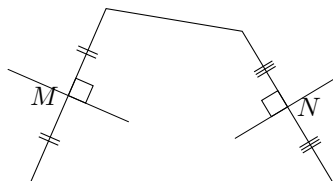
### 3.9 Center of quadrilateral

**Task 3.9.** Construct the midpoint of the segment that connects the midpoints of the diagonals of the quadrilateral.

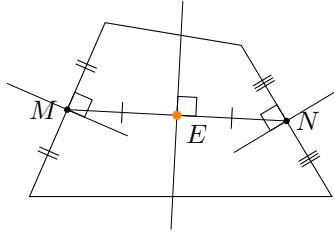
(4L, 10E)



**Solution 3.9. 1, 2.** Draw the perpendicular bisectors of the two non-parallel sides. Let the midpoints of the non-parallel sides be  $M$  and  $N$ .



**3, 4.** Draw  $MN$ . Draw the perpendicular bisector of  $MN$ . The midpoint of  $MN$  is the desired point.

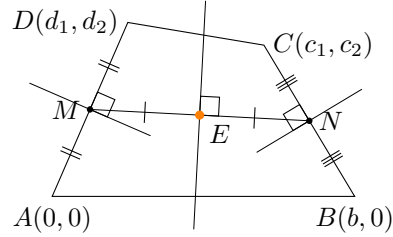
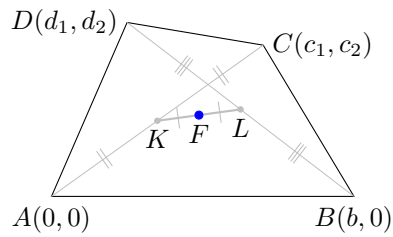


*Proof.* We use Cartesian coordinates. Let the quadrilateral be  $ABCD$ , and let

$$A = (0, 0), B = (b, 0), C = (c_1, c_2), D = (d_1, d_2).$$

Let  $K, L$  be the midpoints of  $AC$  and  $BD$  respectively. Then  $K = (\frac{c_1}{2}, \frac{c_2}{2})$  and  $L = (\frac{d_1 + b}{2}, \frac{d_2}{2})$  (mid-pt. coordinate formula). Let  $F$  be the midpoint of  $KL$  (the desired point).

$$\text{Then } F = (\frac{c_1 + d_1 + b}{4}, \frac{c_2 + d_2}{4}).$$

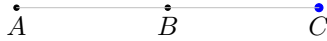


Now we show that  $E$  has the same coordinates as  $F$ . Since  $M$  and  $N$  are midpoints of  $AD$  and  $BC$ , we have  $M = (\frac{d_1}{2}, \frac{d_2}{2})$  and  $N = (\frac{c_1 + b}{2}, \frac{c_2}{2})$ . And  $E$  is the midpoint of  $MN$ , so  $E = (\frac{d_1 + c_1 + b}{4}, \frac{d_2 + c_2}{4})$ , which is the same as  $F$ . So we conclude  $E = F$ .  $\square$

## 4 Delta

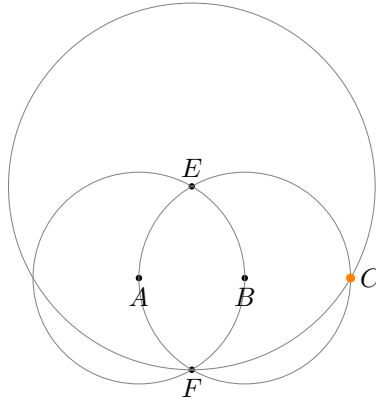
### 4.1 Double segment

**Task 4.1.** Construct a point  $C$  on the line  $AB$  such that  $|AC| = 2|AB|$  using only a compass. (3L, 3E, 2V)



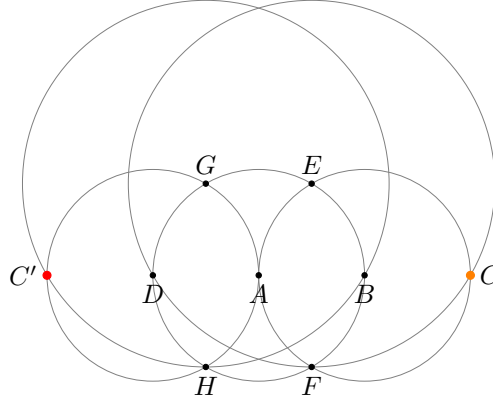
**Solution 4.1. (3L, 3E) 1, 2.** Draw circle  $(A, B)$  and  $(B, A)$ , intersecting at  $E$  and  $F$ , intersecting  $(B, A)$  again at desired point  $C$ ,

**3.** Draw circle  $(E, F)$ .



*Proof.* Note that  $\angle AEB = 60^\circ$  and  $\angle BEC = \angle BEF = 30^\circ$  by congruent triangles. So  $\angle AEC = 60^\circ + 30^\circ = 90^\circ$ , and so  $A, B, C$  are collinear by “converse of  $\angle$  in semi-circle”. Also  $AC = 2AB$  by radii.  $\square$

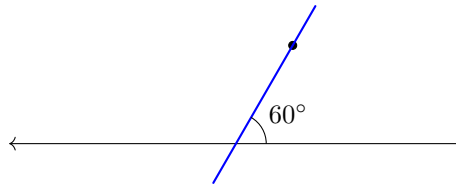
- (2V) 4. Let  $(E, F)$  intersect  $(A, B)$  at  $D$ . Draw  $(D, A)$ , intersecting  $(A, B)$  at  $G$  and  $H$ .  
 5. Draw circle  $(G, H)$ , intersecting  $(D, A)$  again at desired point  $C'$ .



*Proof.* The same argument works for  $C'$  since the figure is symmetric (and because we have  $D$  lying on line  $AB$  using the same argument as above).  $\square$

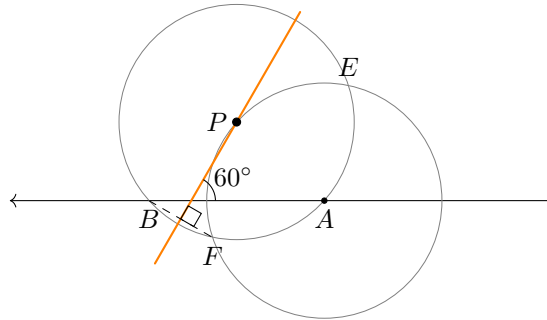
## 4.2 Angle of 60 deg - 2

**Task 4.2.** Construct a straight line through the given point that makes an angle of  $60^\circ$  with the given line.  
 (3L, 4E, 2V)



**Solution 4.2.** Let given point be  $P$ . Let  $A$  be an arbitrary point on given line (such that the angle formed by  $PA$  and given line is less than  $60^\circ$ ).

- (3L) 1. Draw circle  $(P, A)$ , intersecting the give line again at  $B$ .  
 2. Draw circle  $(A, P)$ , intersecting  $(P, A)$  at  $E$  and  $F$ .  
 3. Draw perpbi  $BF$ , the desired line.



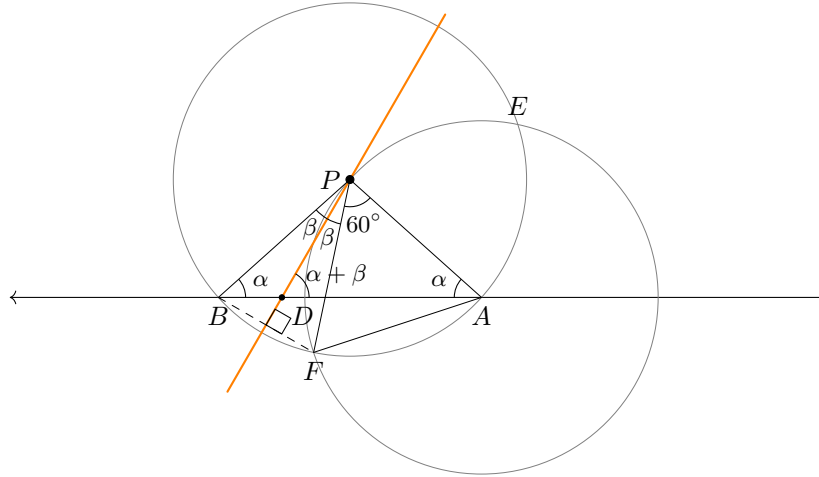
*Proof.* Let  $D$  be the landing point of orange line. First, note that the perpendicular bisector of  $BF$  passes through  $P$  because  $BF$  is a chord of circle centered  $P$ .

Since  $PB = PA$  (radii),  $\angle PBA = \angle PAB$  (base  $\angle$ s, isos.  $\triangle$ ). Since  $DP$  is the perpendicular bisector of  $BF$ ,  $\angle BPD = \angle FPD$  (SAS) & (corr.  $\angle$ s,  $\cong \triangle$ s). Also, note that  $\angle FPA = 60^\circ$  since  $\triangle FPA$  is equilateral triangle.

Let  $\angle PBA = \angle PAB = \alpha$  and  $\angle BPD = \angle FPD = \beta$ .  
 In  $\triangle APB$ ,

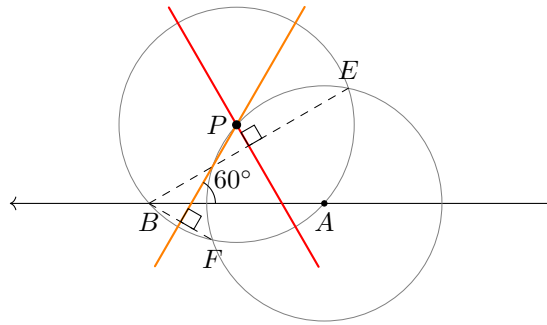
$$\begin{aligned}\angle BPA + \angle PBA + \angle PAB &= 180^\circ & (\angle \text{ sum of } \triangle) \\ (2\beta + 60^\circ) + \alpha + \alpha &= 180^\circ \\ \alpha + \beta &= 60^\circ\end{aligned}$$

Since  $\angle PDA = \alpha + \beta$  (ext.  $\angle$  of  $\triangle$ ),  $\angle PDA = 60^\circ$ . This means the orange line makes an angle of  $60^\circ$  with the given line.



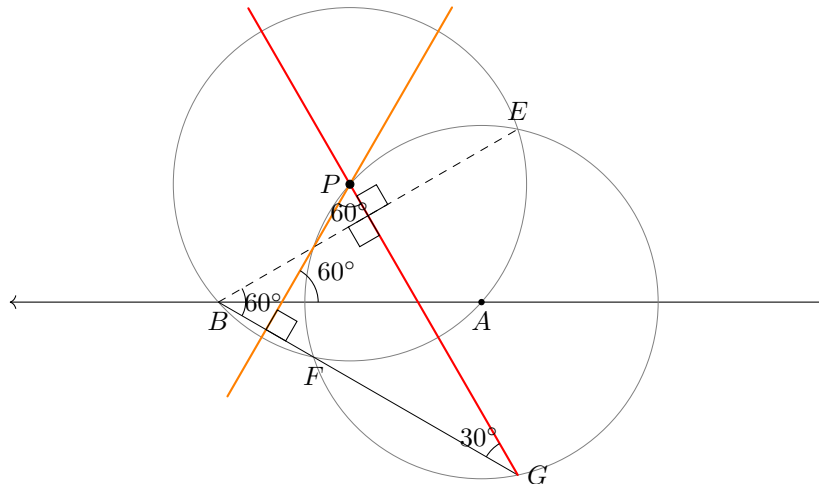
□

(2V) 4. Draw perpbi  $BE$ . We get the extra solution.



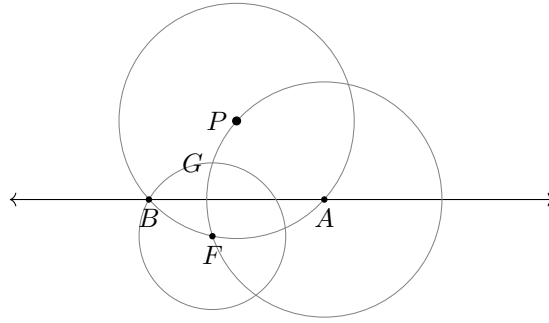
*Proof.* Extend  $BF$  to meet the red line at  $G$ . Note that  $\angle FPE = 120^\circ$ , so  $\angle FBE = 120^\circ/2 = 60^\circ$  ( $\angle$  at centre twice  $\angle$  at  $\odot^{ce}$ ). So  $\angle PGB = 90^\circ - 60^\circ = 30^\circ$ , and the angle between orange and red line is  $90^\circ - 30^\circ = 60^\circ$  ( $\angle$  sum of  $\triangle$ ).

Thus the red line makes an angle of  $180^\circ - 60^\circ - 60^\circ = 60^\circ$  ( $\angle$  sum of  $\triangle$ ).

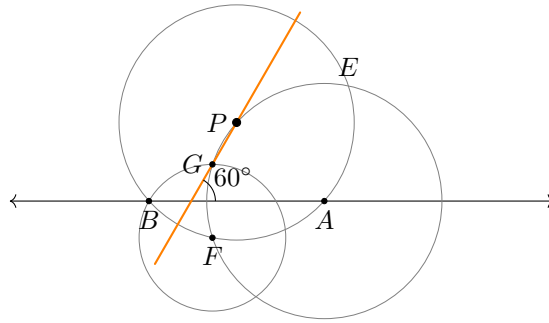


□

- (4E) 1. Draw circle  $(P, A)$ , intersecting the give line again at  $B$ .  
 2. Draw circle  $(A, P)$ , intersecting  $(P, A)$  at  $E$  and  $F$ .  
 3. Draw circle  $(F, B)$ , intersecting  $(A, P)$  above given lien at  $G$ .



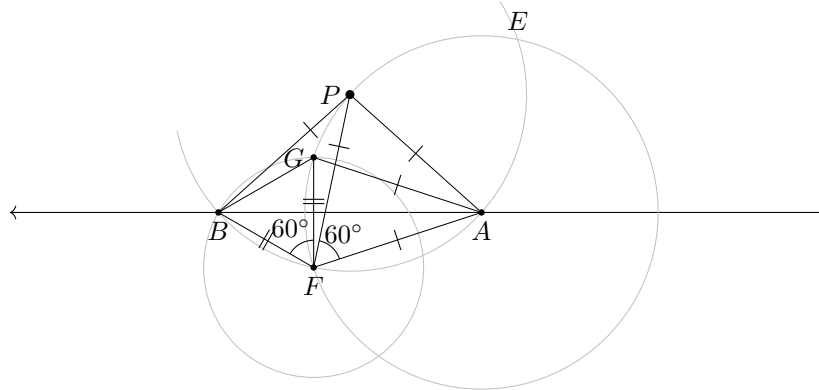
4. Draw line  $GP$ , the desired line.



*Proof.* Let the radius of  $(P, A)$  and  $(A, P)$  be  $r$ , and the radius of  $(F, B)$  be  $s$ .

Note that  $\triangle PBF \cong \triangle AGF$  (SSS), since  $PB = AG = r$ ,  $PF = AF = r$ ,  $BF = GF = s$ .

Thus  $\angle BFP = \angle GFA$  (corr.  $\angle$ s,  $\cong \triangle$ s)

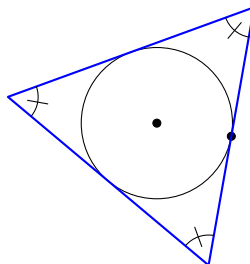


Note that  $\angle PFA = 60^\circ$  since  $\triangle PFA$  is equilateral. Thus  $\angle BFG = \angle BFP - \angle GFP = \angle GFA - \angle GFP = 60^\circ$ .

Since  $BF = GF$  and  $\angle BFG = 60^\circ$ ,  $\triangle GBF$  is equilateral triangle (con. of equil.  $\triangle$ ). This means  $G$  lies on the perpendicular bisector of  $BF$ , so  $GP$  must be the same line as the orange line of (3L).  $\square$

### 4.3 Circumscribed equilateral triangle

**Task 4.3.** Construct an equilateral triangle that is circumscribed about the circle and contains the given point.

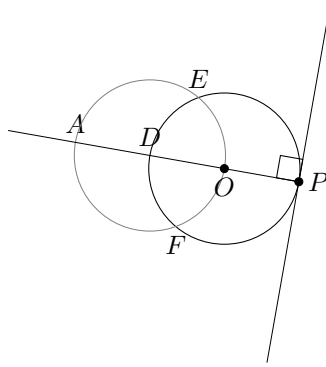


**Solution 4.3.** Let  $P$  be the given point on circle and  $O$  be the given circle center.

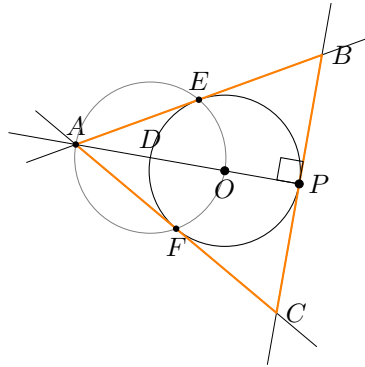
(5L) 1. Draw line  $OP$ , intersecting given circle again at  $D$ .

2. Draw  $OP \perp P$ .

3. Draw circle  $(D, O)$ , intersecting given circle at  $E$  and  $F$ , and intersecting  $OD$  at  $A$ .



4, 5. Draw lines  $AE$  and  $AF$ , intersecting  $OP \perp P$  at  $B$  and  $C$  respectively.  $\triangle ABC$  is the desired triangle.



*Proof.* Note that  $\angle AEO = \angle AFO = 90^\circ$  ( $\angle$  in semi-circle), so  $AB$  and  $AF$  are tangent to the given circle (converse of tangent  $\perp$  radius).

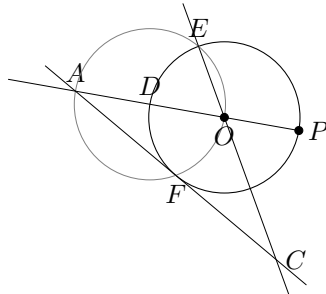
Moreover,  $\angle OAE = 30^\circ$  since  $\frac{OE}{OA} = \frac{1}{2}$ . Similarly  $\angle OAF = 30^\circ$ . This means  $\angle BAC = 60^\circ$ . Since the figure is reflectional symmetric about line  $AP$ , we have  $AB = AC$ , and thus  $\triangle ABC$  is an equilateral triangle (con. of equil.  $\triangle$ ).  $\square$

(6E) 1. Draw line  $OP$ , intersecting given circle again at  $D$ .

2. Draw circle  $(D, O)$ , intersecting given circle at  $E$  and  $F$ , and intersecting  $OD$  at  $A$ .

3. Draw line  $EO$ .

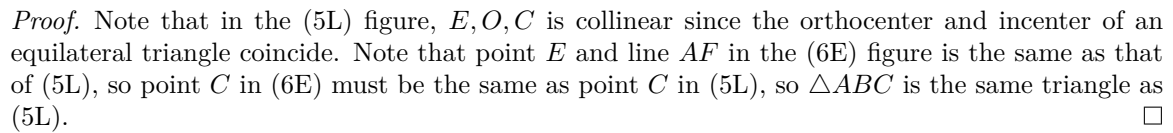
4. Draw line  $AF$ , intersecting  $EO$  at  $C$ .



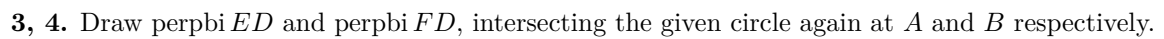
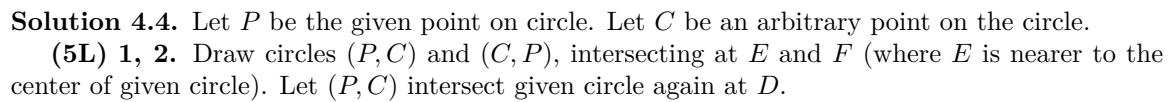
5. Draw line  $AE$ .

6. Draw line  $CP$ , intersecting  $AE$  at  $B$ .  $\triangle ABC$  is the desired triangle.

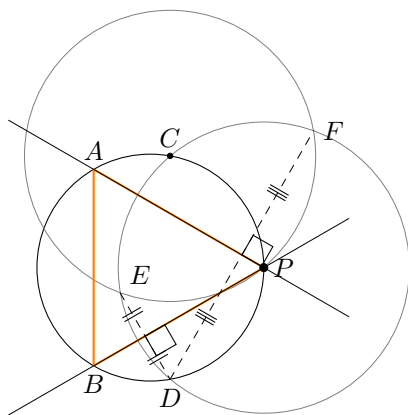




**Task 4.4.** Inscribe an equilateral triangle in the circle using the given point as a vertex. The center of the circle is not given.  
(5L, 6E)



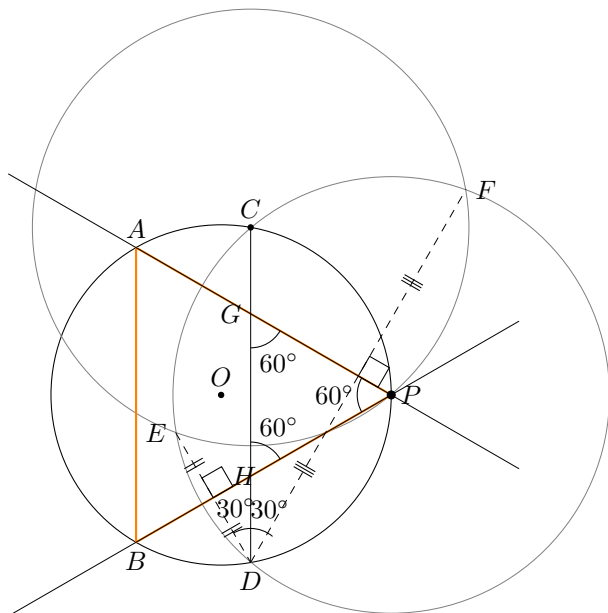
5. Draw line  $AB$ .  $\triangle PAB$  is the desired triangle.



*Proof.* Consider angles subtended by arc  $\widehat{ECF}$  in circle  $(P, C)$ . Note that  $\angle EPF = 120^\circ$  (since  $\triangle ECP$  and  $\triangle FCP$  are two equilateral triangles). Thus  $\angle EDF = 120^\circ/2 = 60^\circ$  ( $\angle$  at centre twice  $\angle$  at  $\odot^{ce}$ ).

Since  $EC = CF$ , we have  $\angle EDC = \angle CDF = 60^\circ/2 = 30^\circ$  (equal chord, equal  $\angle$  at  $\odot^{ce}$ )

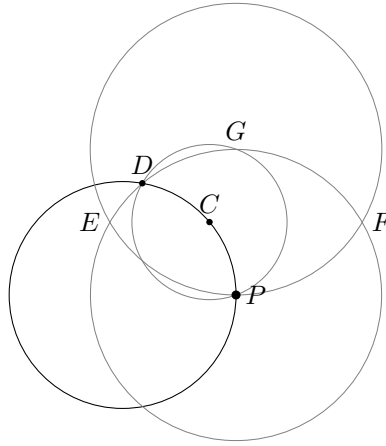
Let  $CD$  cut  $AP$  and  $BP$  at  $G$  and  $H$ . Since  $FD \perp GP$  and  $ED \perp HP$ , by some angle chasing, we find that  $\angle PHG = \angle PGH = 60^\circ$ , so  $\angle APB = 180^\circ - 60^\circ - 60^\circ = 60^\circ$  ( $\angle$  sum of  $\triangle$ ).



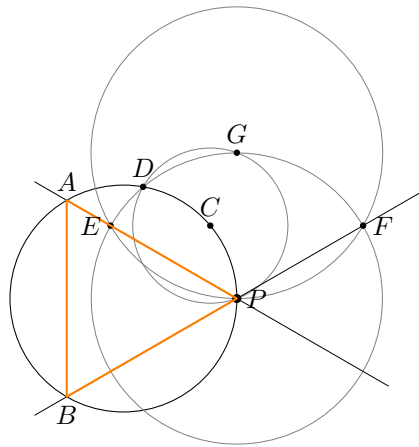
Let  $O$  be the center of given circle, and  $M$  be the midpoint of  $CD$ . Note that  $OP \perp CD$  since  $ODPC$  forms a kite. Thus  $PO$  bisects  $\angle GPH$  (prop. of isos.  $\triangle$ ). This means  $\triangle PAB$  is reflectional symmetric about line  $OP$ , giving  $PA = PB$ , and thus  $\triangle PAB$  is an equilateral triangle.  $\square$

**(6E) 1.** Let  $C$  be an arbitrary point on the circle. Draw circle  $(C, P)$ , intersecting given circle again at  $D$ .

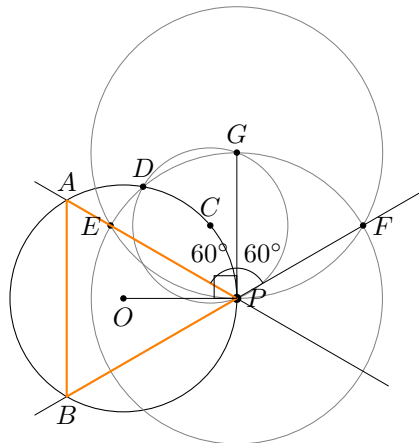
- 2.** Draw circle  $(P, D)$ , intersecting  $(C, P)$  again at  $G$ .
- 3.** Draw circle  $(G, P)$ , intersecting  $(P, D)$  at  $E$  and  $F$  ( $E$  left,  $F$  right).



- 4, 5. Draw line  $PE$  and  $PF$ , intersecting given circle again at  $A$  and  $B$ .  
 6. Draw line  $AB$ .  $\triangle PAB$  is the desired triangle.



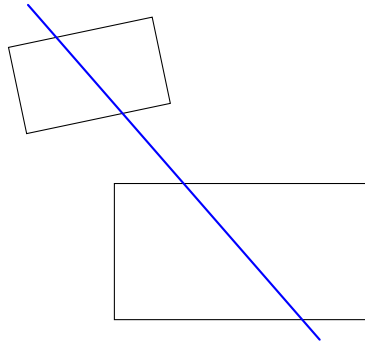
*Proof.* Let  $O$  be center of given circle. Note that  $PG$  is tangent to given circle at  $P$  by Task 2.8E (tangent to circle at point). Thus  $OP \perp GP$ . Since  $\angle EPG = 60^\circ$ ,  $\angle OPA = 90^\circ - 60^\circ = 30^\circ$  and  $\angle OPB = 180^\circ - 90^\circ - 60^\circ = 30^\circ$  (adj.  $\angle$ s on st. line). Thus  $OP$  bisects  $\angle APB$ , so  $\triangle PAB$  is same as (5L) of this level, which means it is equilateral.



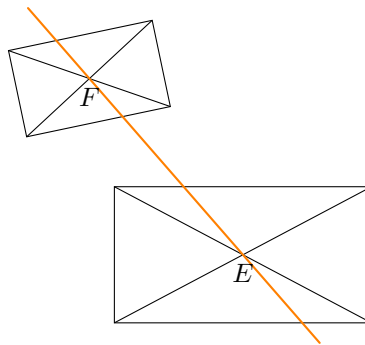
□

## 4.5 Cut two rectangles

**Task 4.5.** Construct a line that cuts each of the rectangles into two parts of equal area.  
(5L, 5E)



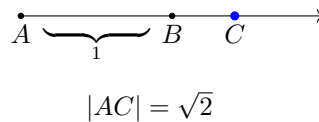
**Solution 4.5. 1-5.** Draw the diagonals of the two given rectangles and let them intersect at  $E$  and  $F$ . Draw line  $EF$ , the desired line.



*Proof.* By Task 2.5, a line through the center of a rectangle cuts it into two parts of equal area. Since the orange line passes through the centers of both rectangles, it divides both rectangles into parts of equal area.  $\square$

## 4.6 Square root of 2

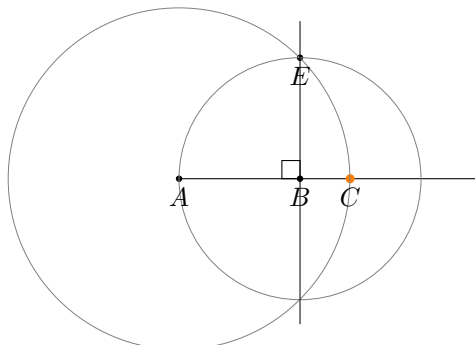
**Task 4.6.** Let  $|AB| = 1$ . Construct a point  $C$  on the ray  $AB$  such that the length of  $AC$  is equal to  $\sqrt{2}$ .



**Solution 4.6. 1.** Draw circle  $(B, A)$ .

**2.** Draw  $AB \perp B$ , intersecting  $(B, A)$  at one of the points  $E$ .

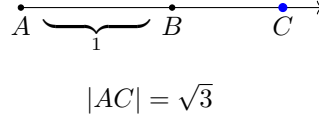
**3.** Draw circle  $(A, E)$ , intersecting the given ray at  $C$ .  $C$  is the desired point.



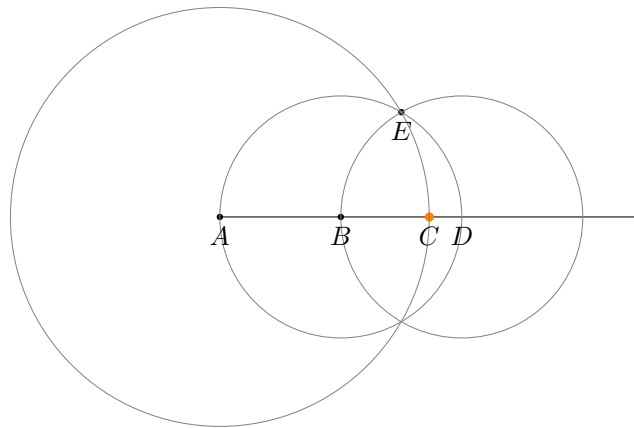
*Proof.* Note that  $\triangle ABE$  is an isosceles right triangle, so  $AE = \sqrt{2}$  (Pyth. thm), thus  $AC = AE = \sqrt{2}$  (radii).  $\square$

## 4.7 Square root of 3

**Task 4.7.** Let  $|AB| = 1$ . Construct a point  $C$  on the ray  $AB$  such that the length of  $AC$  is equal to  $\sqrt{3}$ .



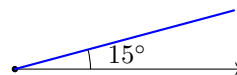
**Solution 4.7.** 1. Draw circle  $(B, A)$ , intersecting given ray again at  $D$ .  
 2. Draw circle  $(D, B)$ , intersecting  $(B, A)$  at one of intersections  $E$ .  
 3. Draw circle  $(A, E)$ , intersecting given ray at  $C$ .  $C$  is the desired point.



*Proof.* Since  $AB = BE = 1$  and  $\angle ABE = 180^\circ - 60^\circ = 120^\circ$  (adj.  $\angle$ s on st. line), we have  $AE = \sqrt{1 + 1 - 2(1)(1)\cos(120^\circ)} = \sqrt{3}$  (law of cosines). So  $AC = AE = \sqrt{3}$  (radii).  $\square$

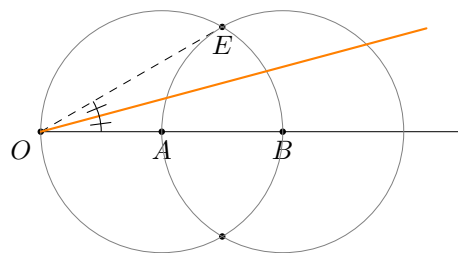
## 4.8 Angle of 15 deg

**Task 4.8.** Construct an angle of  $15^\circ$  with the given side.  
 (3L, 5E, 2V)



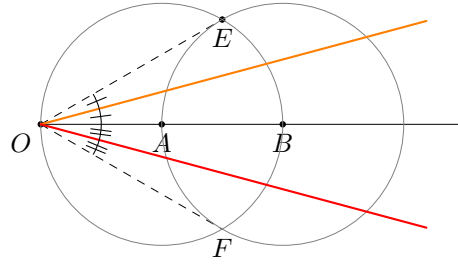
**Solution 4.8.** Let  $O$  be the endpoint of the given ray, and  $A$  be an arbitrary point on the ray.

- (3L) 1. Draw circle  $(O, A)$ , intersecting given ray at  $B$ .  
 2. Draw circle  $(B, A)$ , intersecting  $(O, A)$  at  $E$  and  $F$ .  
 3. Draw angbi  $BOE$ , the desired line.



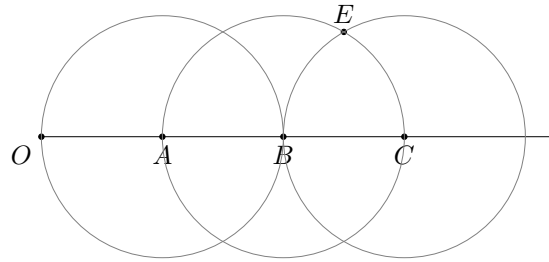
*Proof.* Note that  $\angle OEB = 90^\circ$  ( $\angle$  in semi-circle) and  $\angle ABE = 60^\circ$ , so  $\angle BOE = 180^\circ - 90^\circ - 60^\circ = 30^\circ$  ( $\angle$  sum of  $\triangle$ ). Thus, bisecting  $\angle BOE$  gives a  $15^\circ$  angle.  $\square$

(2V) Draw angbi  $BOF$ , the extra solution.

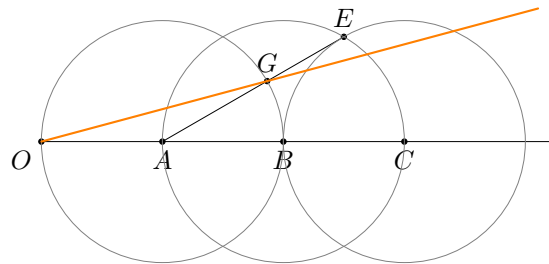


*Proof.* Similar argument as (3L). □

- (5E) 1. Draw circle  $(O, A)$ , intersecting given ray at  $B$ .  
 2. Draw circle  $(B, A)$ , intersecting given ray at  $C$ .  
 3. Draw circle  $(C, A)$ , intersecting  $(B, A)$  at one of intersections  $E$ .



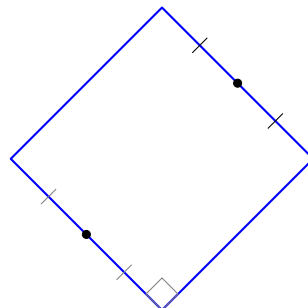
4. Draw segment  $AE$ , intersecting  $(A, O)$  at  $G$ .  
 5. Draw line  $OG$ , the desired line.



*Proof.* Note that  $\angle EAC = 30^\circ$  (similar to  $\angle BOE$  in (3L)). Since  $AO = AG$ , we have  $\angle AOG = 30^\circ/2 = 15^\circ$  (base  $\angle$ s, isos.  $\triangle$ ) & (ext.  $\angle$  of  $\triangle$ ). □

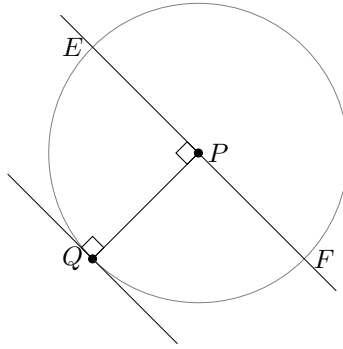
## 4.9 Square by opposite midpoints

**Task 4.9.** Construct a square, given two midpoints of opposite sides.  
 (6L, 10E)

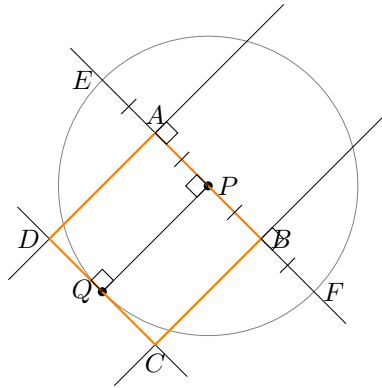


**Solution 4.9.** Let the given points be  $P$  and  $Q$ .

- (6L) 1, 2. Draw circle  $(P, Q)$ . Draw line  $PQ$ .  
 3, 4. Draw  $PQ \perp Q$ . Draw  $PQ \perp P$ , intersecting  $(P, Q)$  at  $E, F$ .



**5, 6.** Draw perpbi  $EP$  and perpbi  $PF$ . We get the desired square  $ABCD$ .

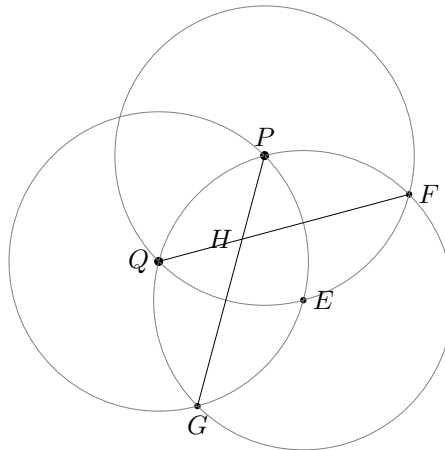


*Proof.* Note that  $PQ$  is equal to the side length of the square. So  $AP = \frac{1}{2}EP = \frac{1}{2}PQ$ . Similarly,  $PB = \frac{1}{2}PQ$ , so  $AB = PQ$ . Also  $AD = BC = PQ$ , so  $ABCD$  is a square.  $\square$

**(10E) 1, 2.** Draw circle  $(P, Q)$  and  $(Q, P)$ . Let  $E$  be one of their intersections.

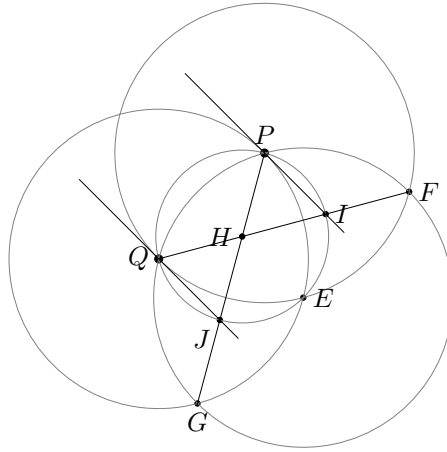
**3.** Draw circle  $(E, P)$ , intersecting  $(P, Q)$  and  $(Q, P)$  at new points  $F$  and  $G$ .

**4, 5.** Draw lines  $PG$  and  $QF$ , intersecting at  $H$ .

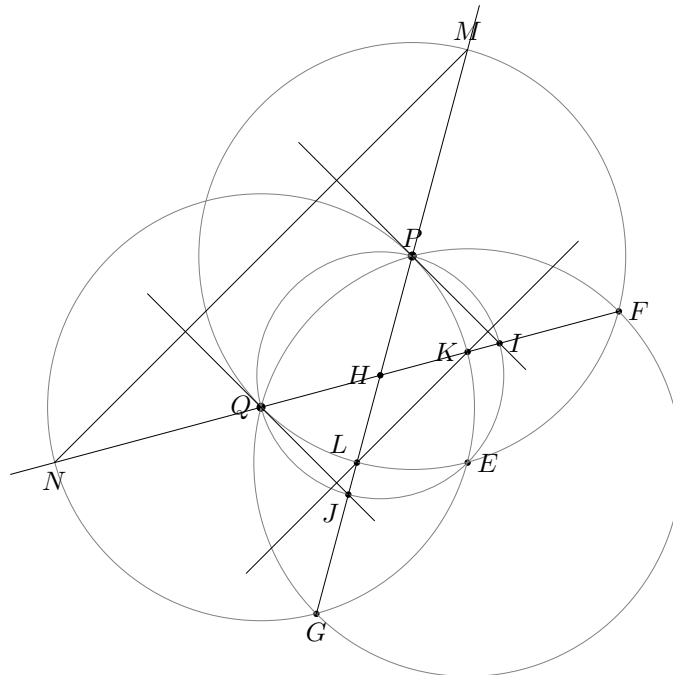


**6.** Draw circle  $(H, P)$ , intersecting segments  $QF$  and  $PG$  at  $I$  and  $J$  respectively.

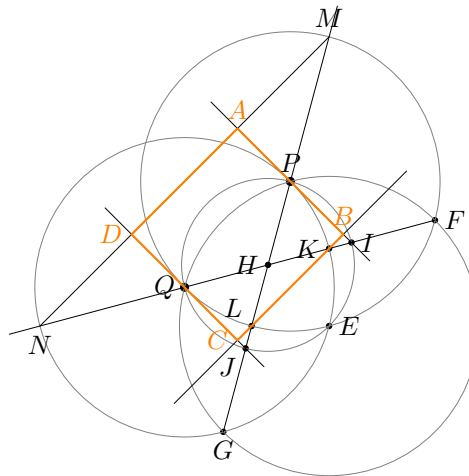
**7, 8.** Draw line  $PI$  and  $QJ$ .



9. Let segments  $QF$  and  $PG$  intersect  $(Q, P)$  and  $(P, Q)$  at  $K$  and  $L$  respectively. Draw line  $KL$ .  
 10. Extend segment  $GP$  to meet  $(P, Q)$  at  $M$ . Extend segment  $FQ$  to meet  $(Q, P)$  at  $N$  (doesn't cost anything). Draw line  $MN$ .



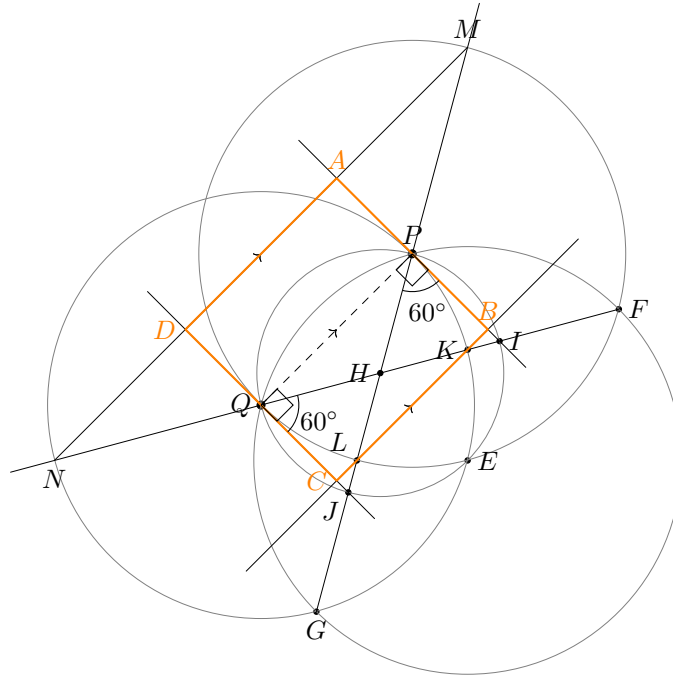
The desired square is the area enclosed by the four lines.



*Proof.* (Let  $ABCD$  be the vertices of the orange quadrilateral.)

First, note that  $PI \perp PQ$  and  $QJ \perp PQ$  by “ $\angle$  in semi-circle” for circle  $(H, P)$ . Also,  $KL \parallel PQ$  since  $\triangle HPQ \sim \triangle HLK$  (ratio of 2 sides, inc.  $\angle$ ), and  $MN \parallel PQ$  since  $\triangle HPQ \sim \triangle HMN$  (ratio of 2 sides, inc.  $\angle$ ). This means  $ABCD$  is a rectangle.

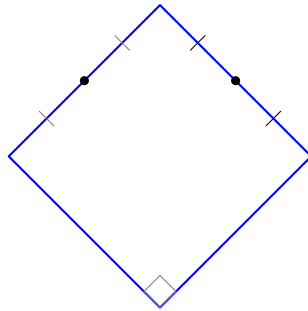




Now note that  $\angle HPI = 60^\circ$  (since  $\triangle HPI$  is equil.), so  $PB = \cos(60^\circ)PL = \frac{1}{2}PL$ . And note that  $\triangle PAM \cong \triangle PBC$  (AAS by  $\angle APM = \angle BPL$ ,  $\angle MAP = \angle LBP$ ,  $PM = PL$ ), giving  $AP = PB = \frac{1}{2}PL$  (corr. sides,  $\cong \triangle$ s). This means  $AB = PL = PQ$  (radii)  $= BC$ . So  $ABCD$  is a rectangle with adjacent sides equal, i.e. a square, as desired.  $\square$

#### 4.10 Square by adjacent midpoints

**Task 4.10.** Construct a square, given two midpoints of adjacent sides.  
(7L, 10E, 2V)

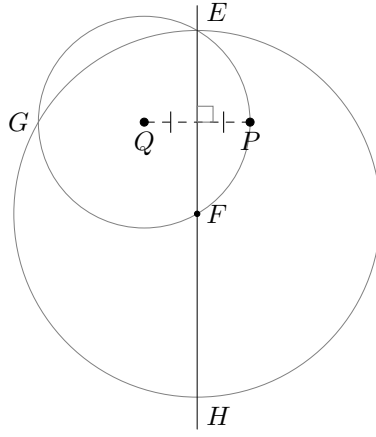


**Solution 4.10.** Let  $P, Q$  be the given points.

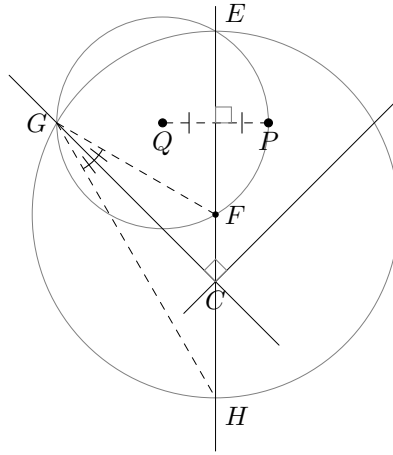
**(7L) 1.** Draw circle  $(Q, P)$ .

**2.** Draw perpbi  $PQ$ , intersecting  $(Q, P)$  at  $E$  and  $F$

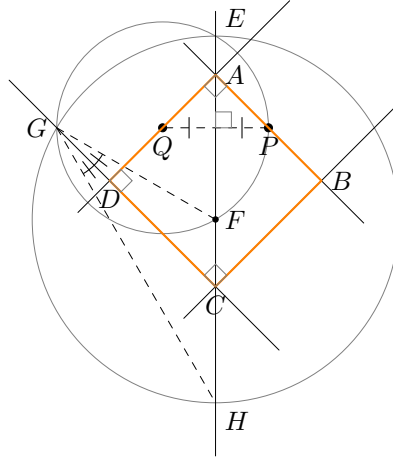
**3.** Draw circle  $(F, E)$ , intersecting  $(Q, P)$  again at  $G$ , and  $EF$  again at  $H$ .



4. Draw angbi  $FGH$ , intersecting  $FH$  at  $C$ .
5. Draw  $GC \perp C$ .

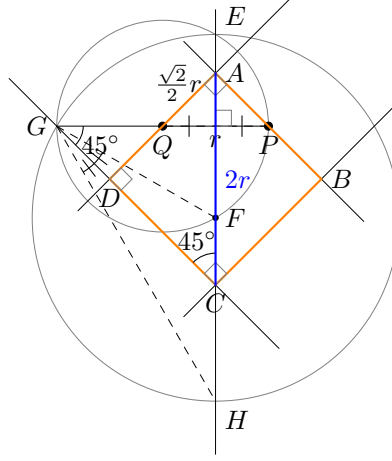


6. Draw  $GC \perp Q$ , intersecting  $EF$  at  $A$ .
7. Draw  $QA \perp P$ . The shape  $ABCD$  enclosed by the lines is the desired square.



*Proof.* Let the length of  $QP$  be  $r$ , and  $M$  be midpoint of  $QP$ . We want to show that rectangle  $ABCD$  is a square and that  $AQ = QD = AP = PB = \frac{\sqrt{2}}{2}r$ .

Note that  $\angle EGH = 90^\circ$  ( $\angle$  in semi-circle) and  $\angle AGF = 60^\circ$ , so  $\angle FGH = 30^\circ$ . Since  $GC$  is the angle bisector of  $\angle FGH$ , we have  $\angle FGC = 15^\circ$ , so  $\angle QGC = 45^\circ$  and thus  $\angle GCA = 45^\circ$ . Since the diagonal of rectangle  $ABCD$  makes  $45^\circ$  with a side,  $ABCD$  is a square.



To show that  $Q, P$  are midpoints of the sides, note that  $AQ = QP \cos 45^\circ = \frac{\sqrt{2}}{2}r$ . Also,  $GP = AC$  because  $GM = MC$  and  $MP = AM$ . Since  $GP$  is a diameter of circle  $(Q, P)$ ,  $GP = 2r$  and thus  $AC = 2r$ .

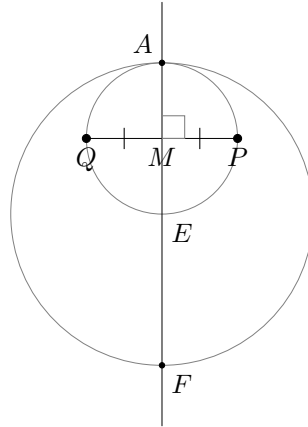
So  $AD = \frac{2r}{\cos 45^\circ} = \sqrt{2}r$ , and  $QD = AD - AQ = \frac{2}{2}r = AQ$ .

Then we also have  $AP = PB$  by intercept theorem (because  $QP \parallel DB$ ). We get everything desired.  $\square$

**(10E) 1, 2.** Draw perpbi  $PQ$ . Draw line  $PQ$ . Let  $M$  be midpoint of  $PQ$ .

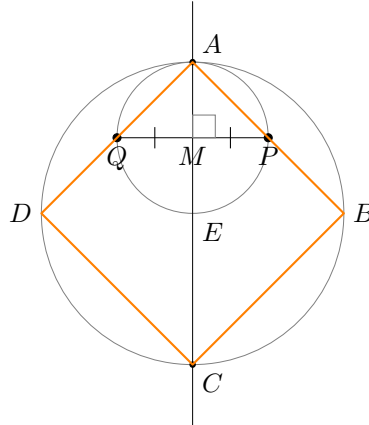
**3.** Draw circle  $(M, P)$ , intersecting perpbi  $PQ$  at  $A$  at top and  $E$  at bottom.

**4.** Draw circle  $(E, A)$ , intersecting perpbi  $PQ$  again at  $C$  (at bottom).



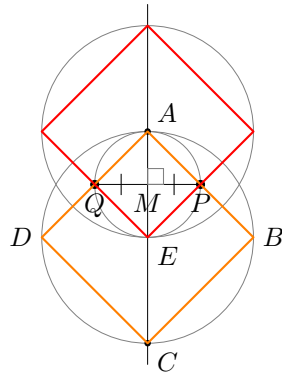
**4, 5.** Draw line  $AP$  and  $AQ$ , intersecting big circle  $(E, A)$  at  $B$  and  $D$ .

**6, 7.** Draw line  $BC$  and  $DC$ .  $ABCD$  is the desired square.



*Proof.* Note that  $\triangle AMQ$  and  $\triangle AMP$  are isosceles right triangles, so we have  $\angle AQM = \angle APM = 45^\circ$ . Thus  $\angle BAD = 90^\circ$ . Also,  $\angle ADC = \angle ABC = 90^\circ$  ( $\angle$  in semi-circle). Thus,  $ABCD$  is a rectangle with diagonal making  $45^\circ$  with the sides, so  $ABCD$  is a square (con. of square).  $\square$

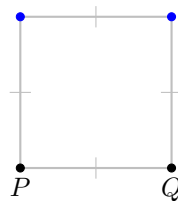
(2V) Draw circle  $(A, E)$  and draw the lines similarly.



#### 4.11 Square by two vertices

**Task 4.11.** Given two vertices of a square. Construct the two other vertices using only a compass.

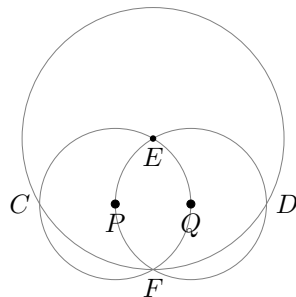
(7L, 7E, 3V)



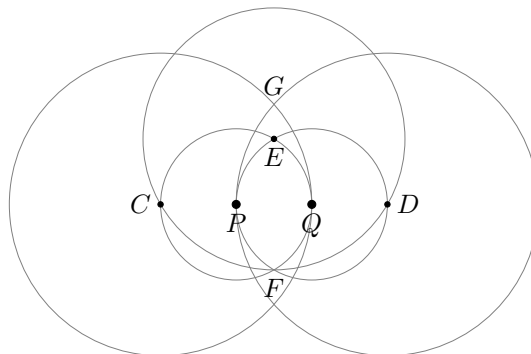
**Solution 4.11.** Let the given points be  $P$  and  $Q$ .

(7L, 7E) **1, 2.** Draw circles  $(P, Q)$  and  $(Q, P)$ , intersecting at  $E$  and  $F$ .

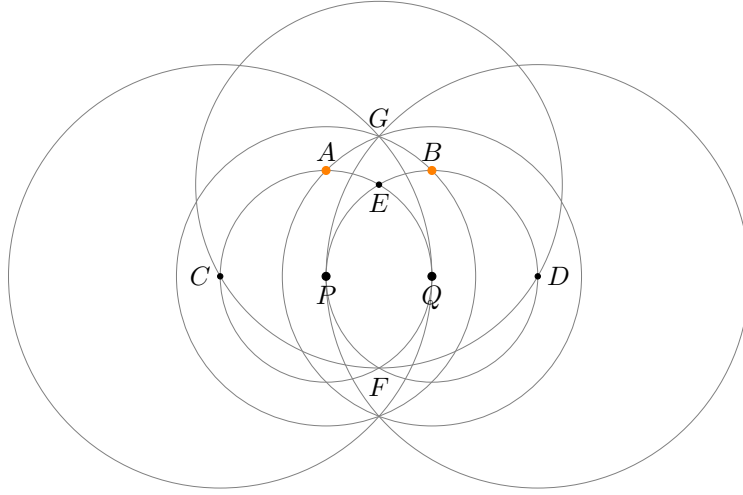
**3.** Draw circle  $(E, F)$ , intersecting  $(P, Q)$  and  $(Q, P)$  again at  $C$  and  $D$ .



**4, 5.** Draw circles  $(C, Q)$  and  $(D, P)$ . Let intersection at top be  $G$ .

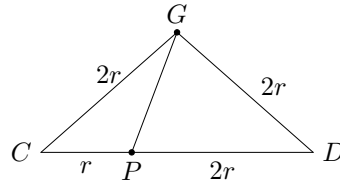


**6, 7.** Draw circles  $(P, G)$  and  $(Q, G)$ , intersecting  $(Q, P)$  and  $(P, Q)$  at top at  $B$  and  $A$  respectively.  $A$  and  $B$  are the desired points.



*Proof.* Let the distance between  $PQ$  be  $r$ . Note that  $C$  and  $D$  lie on circle  $(E, F)$ , so  $C, P, Q, D$  are collinear and  $CP = PQ = QD = r$  (see Task 1.7E for proof). We also have  $GC = GD = 2$  since  $G$  lie on circles  $(C, Q)$  and  $(D, P)$ .

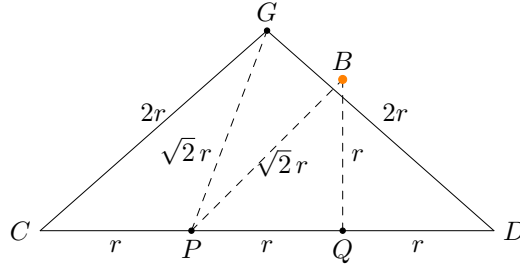
To find  $PG$ , let's focus on  $\triangle GCP$  and  $\triangle GPD$ :



By Stewart's theorem, we have

$$\begin{aligned} GC^2 \cdot PD + GD^2 \cdot CP &= (CP + PD)(PG^2 + CP \cdot PD) \\ (2r)^2(2r) + (2r)^2(r) &= (2r + r)(PG^2 + (r)(2r)) \\ 4r^2 &= PG^2 + 2r^2 \\ PG &= \sqrt{2}r \end{aligned}$$

Let's add points  $B$  and  $Q$  to the figure. We have  $PB = PG = \sqrt{2}$  and  $QB = QP = 1$  (radii).

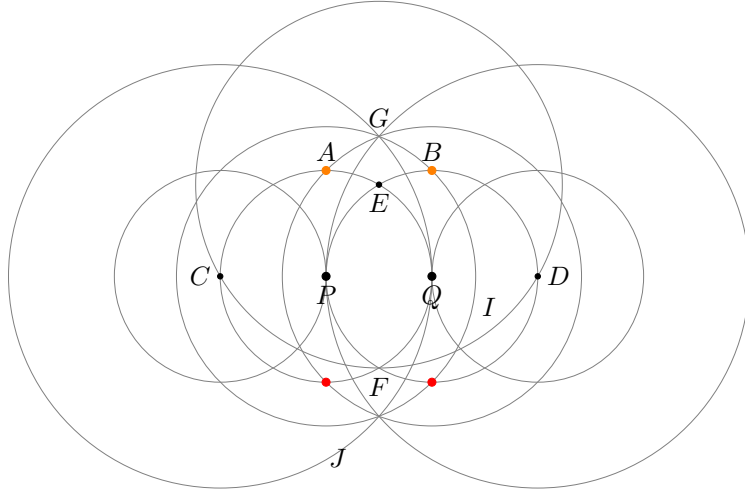


Since  $BQ^2 + PQ^2 = 2r^2 = BP^2$ , by converse of Pythagoras theorem in  $\triangle BPQ$ , we have  $\angle BQP = 90^\circ$ . By symmetry, we have  $\angle APQ = 90^\circ$ .

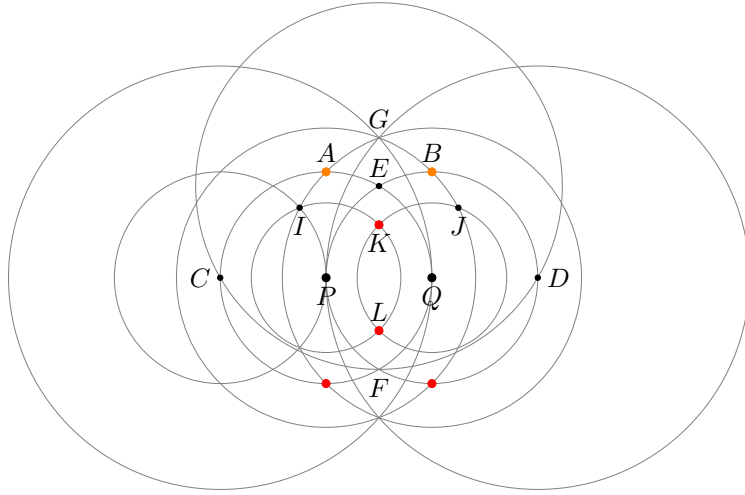
Since  $PQBA$  has three sides equal ( $AP = PQ = BQ$ ) and two right angles ( $\angle APQ = \angle BQP = 90^\circ$ ),  $ABCD$  is a square (con. of square).  $\square$

**(3V) 2nd solution:** Intersection of  $(P, G)$  and  $(Q, P)$  at bottom, and intersection of  $(G, P)$  and  $(P, Q)$  at bottom.

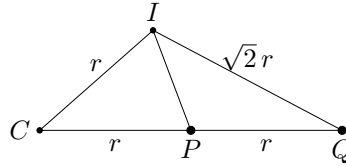
**3rd solution: 8, 9.** Draw circle  $(C, P)$ , intersecting  $(Q, G)$  at  $I$  (top). Draw circle  $(D, Q)$ , intersecting  $(P, G)$  at  $J$  (top).



**10, 11.** Draw circles  $(P, I)$  and  $(Q, J)$ , intersecting at  $K$  and  $L$  at the middle.  $K$  and  $L$  are the two desired points.



*Proof.* Note that  $CI = r$  and  $QI = QG = \sqrt{2}r$ . Let's focus on  $\triangle ICP$  and  $\triangle IPQ$ :



By Stewart's theorem,

$$r^2(r) + (\sqrt{2}r)^2r = (r+r)(IP^2 + r(r))$$

$$\frac{3}{2}r^2 = IP^2 + r^2$$

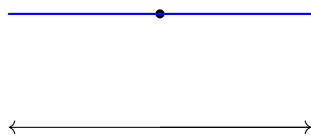
$$IP = \frac{\sqrt{2}}{2}r$$

This means  $PK = \frac{\sqrt{2}}{2}r$ . By symmetry, we also have  $KQ = PL = LQ = \frac{\sqrt{2}}{2}r$ . By Pyth. thm in  $\triangle KPQ$ ,  $\angle PKQ = 90^\circ$ . Thus,  $PLQK$  is a rhombus with a right angle, i.e. a square, as desired.  $\square$

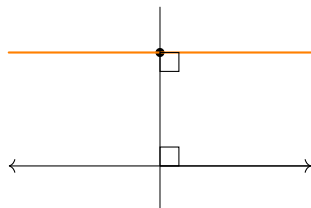
## 5 Epsilon

### 5.1 Parallel line

**Task 5.1.** Construct a line parallel to the given line through the given point.  
(2L, 4E)



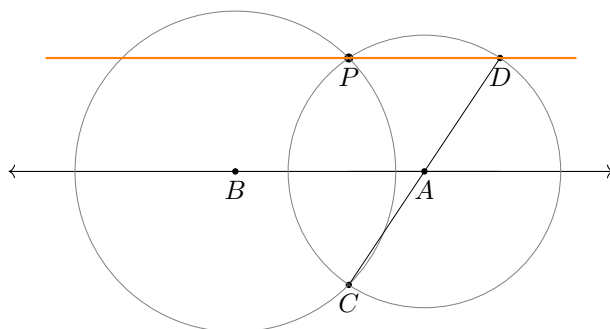
**Solution 5.1. (2L) 1.** Draw line perpendicular to given line through given point.  
**2.** Draw line perpendicular to the drawn line through given point.



*Proof.* Since  $90^\circ + 90^\circ = 180^\circ$ , the interior angles are supplementary, so the orange line is parallel to given line (int.  $\angle$ s supp.).  $\square$

**(4E)** Let  $P$  be given point,  $A, B$  be two arbitrary points on given line.

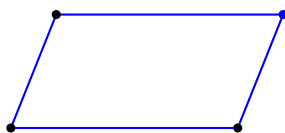
1. Draw circles  $(A, P)$  and  $(B, P)$ , intersecting at  $P$  and  $C$ .
3. Draw line  $CA$ , meeting circle  $(A, P)$  at  $D$ .
4. Draw line  $PD$ , the desired line.



*Proof.* Note that  $PC \perp BA$  since  $PACB$  forms a kite, and  $PC \perp PD$  by “ $\angle$  in semi-circle”. Thus  $PD \parallel BA$ .  $\square$

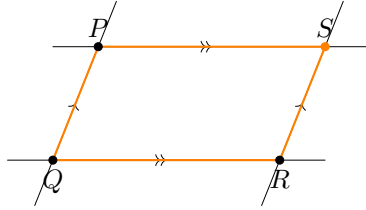
### 5.2 Parallelogram by three vertices

**Task 5.2.** Construct a parallelogram whose three or four vertices are given.  
(4L, 8E, 3V)



**Solution 5.2. (4L)** Let given points be  $P, Q, R$ .

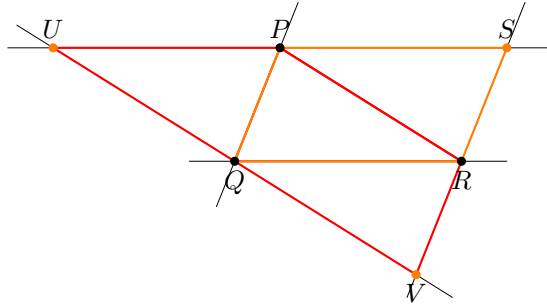
- 1, 2. Draw line  $PQ$  and  $QR$ .
3. Draw  $QR \rightarrow P$  (line parallel to  $QR$  through  $P$ ).
4. Draw  $PQ \rightarrow R$ .



*Proof.* By definition. □

**(3V) 5.** Draw line  $PR$ .

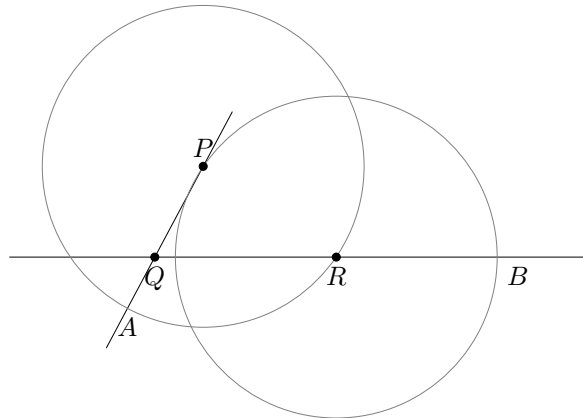
**6.** Draw  $PR \rightarrow Q$ , intersecting  $PS$  and  $SR$  at  $U$  and  $V$  respectively. The extra solutions are  $PUQR$  and  $PQVR$ .



*Proof.* There are three pairs of line segments made from the given points:  $PQ, QR$ ;  $PQ, PR$ ;  $PR, QR$ , and each makes a parallelogram. □

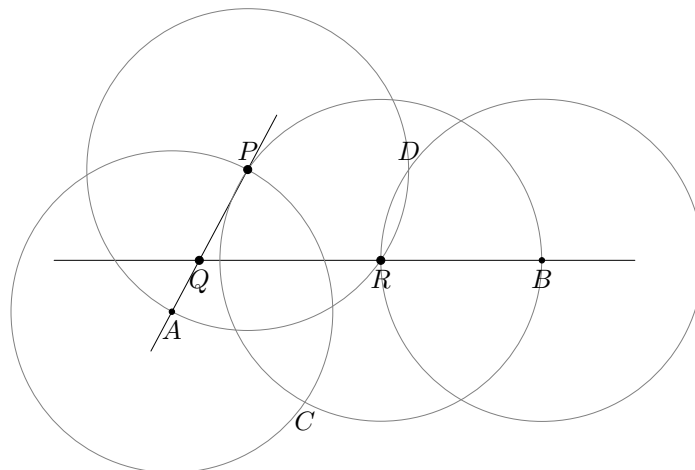
**(8E) 1, 2.** Draw line  $PQ$  and  $QR$ .

**3, 4.** Draw circles  $(P, R)$  and  $(R, P)$ , intersecting line  $PQ$  and  $QR$  at  $A$  (bottom) and  $B$  (right) respectively.



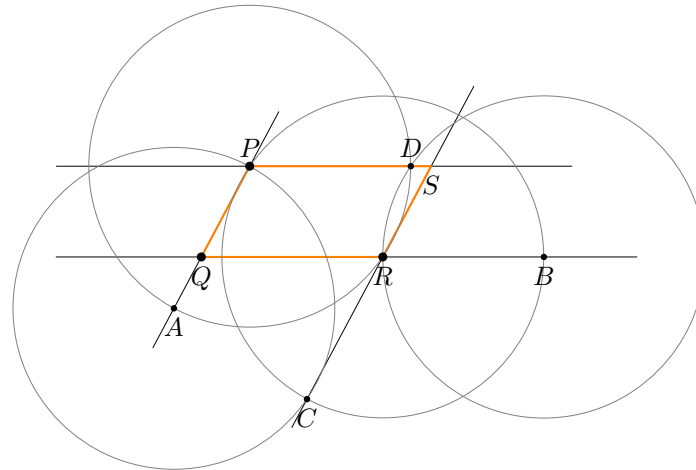
**5.** Draw circle  $(A, P)$ , intersecting  $(R, P)$  again at  $C$ .

**6.** Draw  $(B, R)$ , intersecting  $(P, R)$  again at  $D$ .

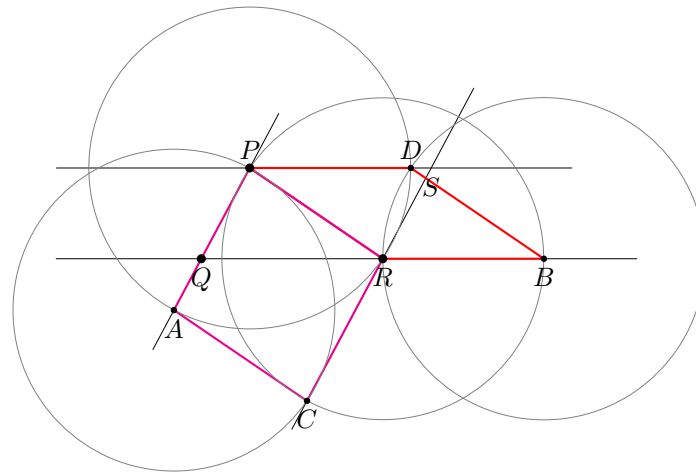




**7, 8.** Draw lines  $CR$  and  $PD$ , intersecting at  $S$ .  $PQRS$  is the desired parallelogram.



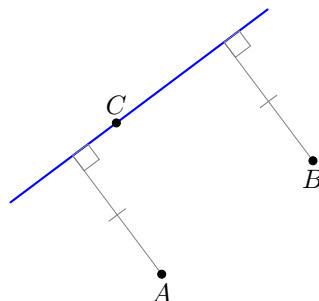
*Proof.* Let the length of  $PR$  be  $r$ . Note that  $PRBD$  and  $PRCA$  form rhombuses of side length  $r$ . This means  $PD \parallel QB$  and  $PA \parallel SC$  (prop. of rhombus), making  $PQRS$  a parallelogram.



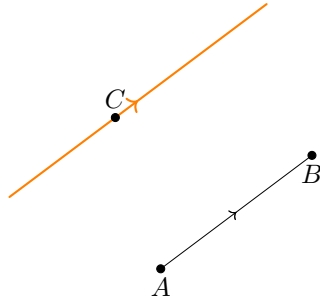
□

### 5.3 Line equidistant from two points - 1

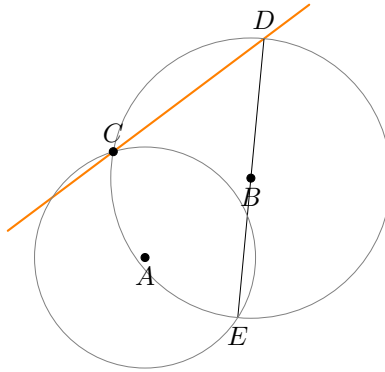
**Task 5.3.** Construct a line through the point  $C$  and at equal distance from the point  $A$  and  $B$  but that does not pass between them.  
(2L, 4E)



**Solution 5.3. (2L)** 1. Draw line  $AB$ .  
2. Draw  $AB \rightarrow C$ .



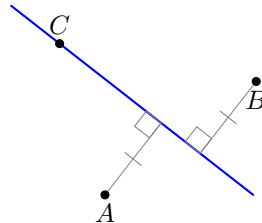
- (4E) 1. Draw circles  $(A, C)$  and  $(B, C)$ , intersecting again at  $E$ .  
 3. Draw line  $EB$ , meeting  $(B, C)$  at  $D$ .  
 4. Draw line  $CD$ , the desired line.



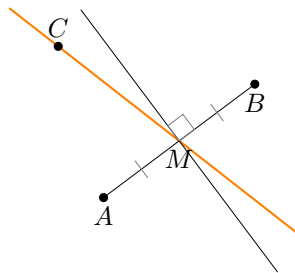
*Proof.*  $CD$  is parallel to  $AB$  by Task 5.1E. □

## 5.4 Line equidistant from two points - 2

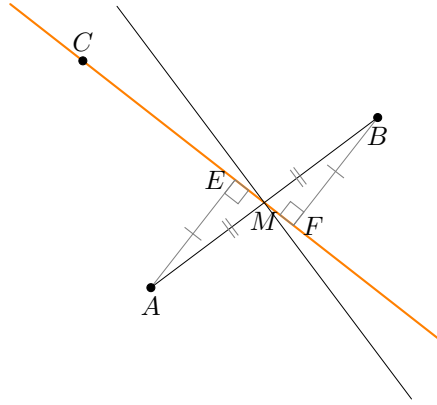
**Task 5.4.** Construct a line through the point  $C$  that goes between the points  $A$  and  $B$  and that is at equal distance from them.  
 (3L, 5E)



- Solution 5.4.** 1. Draw line  $AB$ .  
 2. Draw  $\text{perpbi } AB$ . Let  $M$  be the midpoint of  $AB$ .  
 3. Draw  $CM$ , the desired line.



*Proof.* Let  $E$  and  $F$  be the projection of  $A$  and  $B$  onto line  $CM$ .

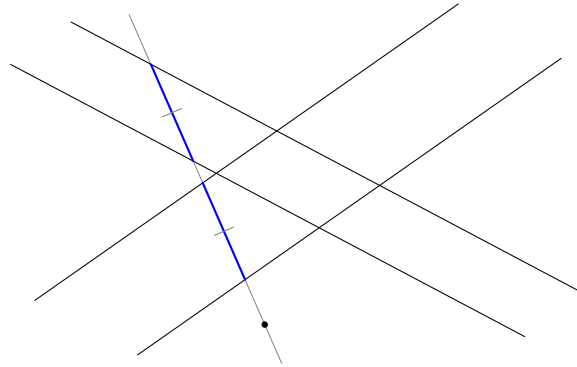


Since  $\angle AME = \angle BME$  (vert. opp.  $\angle$ ),  $\angle AEM = \angle BFM$  ( $AE \perp CM$  and  $BE \perp CM$ ) and  $AM = MB$ , we have  $\triangle AME \cong \triangle BMF$  (AAS). Thus,  $AE = BF$  (corr. sides,  $\cong \triangle$ s), as desired.  $\square$

## 5.5 Hash

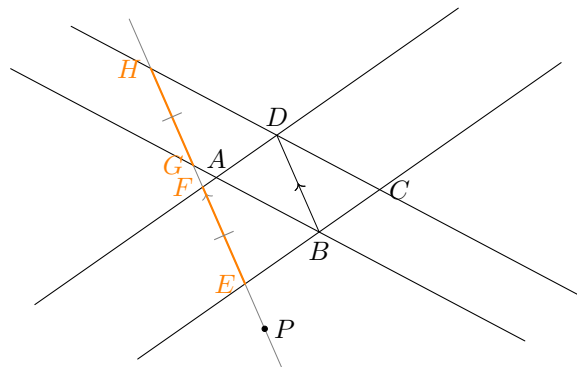
**Task 5.5.** Construct a line through the given point on which two pairs of parallel lines cut off equal line segments.

(2L, 4E, 2V)



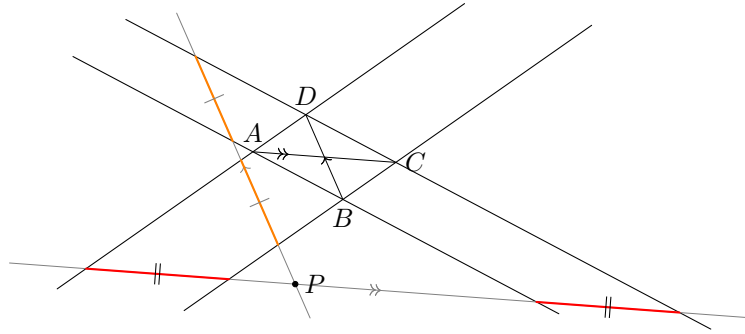
**Solution 5.5. (2L)** Let given point be  $P$ , and the parallelogram formed by the given lines be  $ABCD$ .

1. Draw line  $BD$ .
2. Draw  $BD \rightarrow P$ , the desired line.



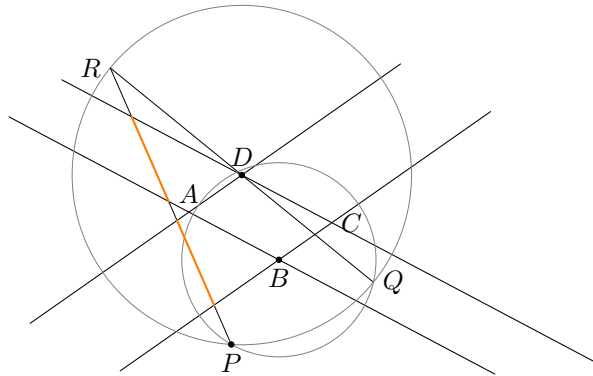
*Proof.* Let  $EF$  and  $GH$  be the orange line segments. Note that  $EFDB$  and  $GHDB$  are parallelograms by construction, so  $EF = DB$  and  $GH = DB$  by “opp. sides of //gram”. This means  $EF = GH$ .  $\square$

- (2V) 3. Draw line  $AC$ .
4. Draw  $AC \rightarrow P$ , the extra solution.



*Proof.* Similar argument as above. □

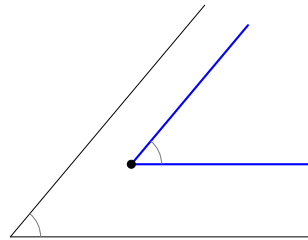
- (4E) 1, 2. Draw circles  $(D, P)$  and  $(B, P)$ , intersecting again at  $Q$ .  
 3. Draw  $QD$ , meeting  $(D, P)$  at  $R$ .  
 4. Draw  $RP$ , the desired line.



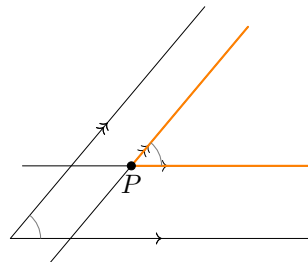
*Proof.* Note that  $PQ \perp DB$  because  $PBQD$  forms a dart. And  $PQ \perp RP$  by “ $\angle$  in semi-circle”. Thus  $DP \parallel RP$ . □

## 5.6 Shift angle

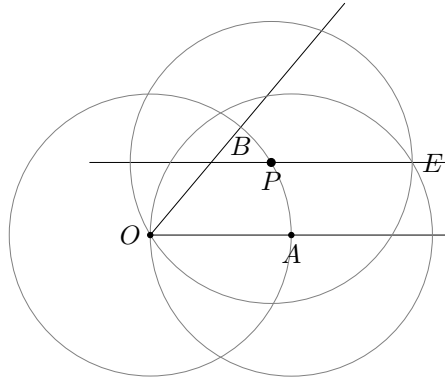
**Task 5.6.** Construct an angle from the given point that is equal to the given angle so that their sides are parallel.



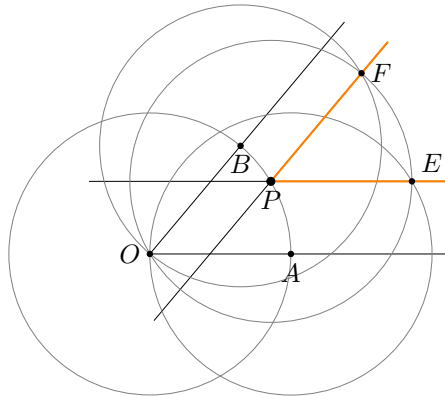
**Solution 5.6.** (2L) 1, 2. Draw lines parallel to given rays through given point.



- (6E) Let given point be  $P$ , and vertex of angle be  $O$ .  
 1. Draw circle  $(O, P)$ , intersecting given rays at  $A$  and  $B$ .  
 2, 3. Draw circles  $(A, O)$  and  $(P, O)$ , intersecting again at  $E$ .  
 4. Draw line  $PE$ .



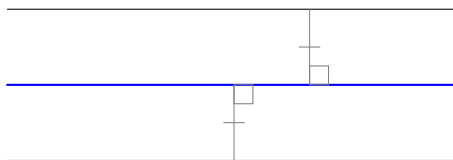
5. Draw circle  $(B, O)$ , intersecting  $(P, O)$  again at  $F$ .
6. Draw line  $PF$ .



*Proof.* Note that all circles have the radius of  $OP$ . This means  $OAEP$  and  $OPFB$  are rhombuses, so  $OA \parallel PE$  and  $OB \parallel PF$ .  $\square$

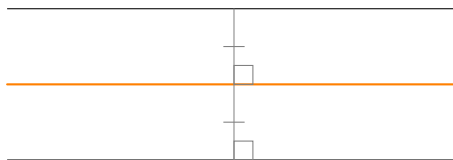
## 5.7 Line equidistant from two lines

**Task 5.7.** Construct a straight line parallel to the given parallel lines that lies at equal distance from them.  
(2L, 5E)



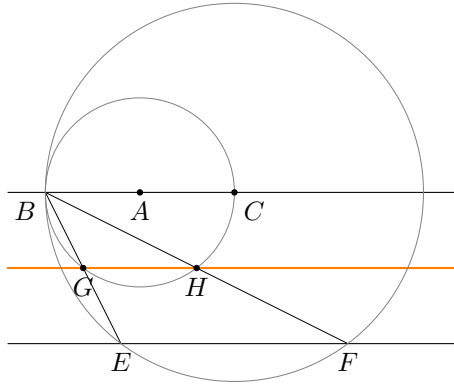
**Solution 5.7. (2L)** 1, Draw line perpendicular to a given line, intersecting the given lines at  $A$  and  $B$ .

2. Draw perpbi  $AB$ , the desired line.



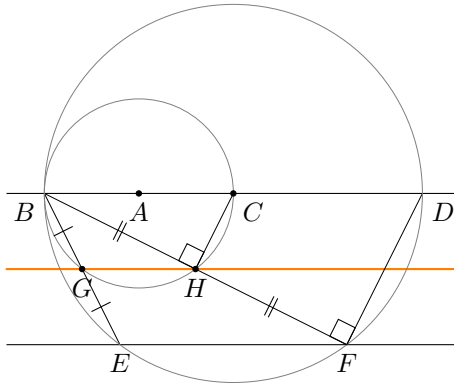
**(5E)** Let  $A, B$  be two arbitrary points on the top given line (such that the distance between  $AB$  is greater than the distance between two give lines).

1. Draw circle  $(A, B)$ , intersecting top line again at point  $C$ .
2. Draw circle  $(C, B)$ , intersecting bottom line at  $E$  and  $F$ .
- 3, 4. Draw lines  $BE$  and  $BF$ , intersecting  $(A, B)$  at  $G$  and  $H$ .
5. Draw line  $GH$ , the desired line.



*Proof.* Let  $D$  be another intersection of  $(C, B)$  and top given line. Note that  $CH \perp BH$  and  $DF \perp BF$  ( $\angle$  in semi-circle), so  $CH \parallel DF$  (corr.  $\angle$ s equal). Since  $CH \parallel DF$  and  $BC = CD$  (radii), by intercept theorem, we have  $BH = HF$ .

By similar argument ( $CG \perp BG$  and  $DE \perp BE \Rightarrow GC \parallel ED$ ), we also have  $BG = GE$ .

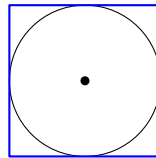


Since  $BH = HF$  and  $BG = GE$ , by midpoint theorem, we have  $GH \parallel DE$ . And by intercept theorem,  $GH$  is midway between the two given lines, as desired.  $\square$

## 5.8 Circumscribed square

**Task 5.8.** Circumscribe a square about the circle. Two of its sides should be parallel to the given line.

(6L, 11E)

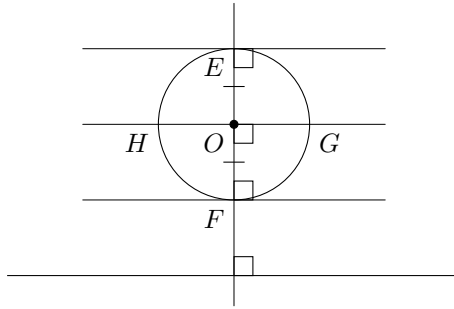


**Solution 5.8.** Let given circle center be  $O$ .

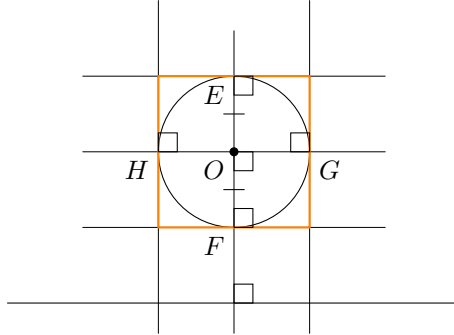
(6L) 1. Draw perpendicular of given line through  $O$ , intersecting given circle at  $E$  and  $F$ .

2, 3. Draw  $EF \perp E$  and  $EF \perp F$ .

4. Draw perpbi  $EF$ , intersecting given circle at  $G$  and  $H$ .



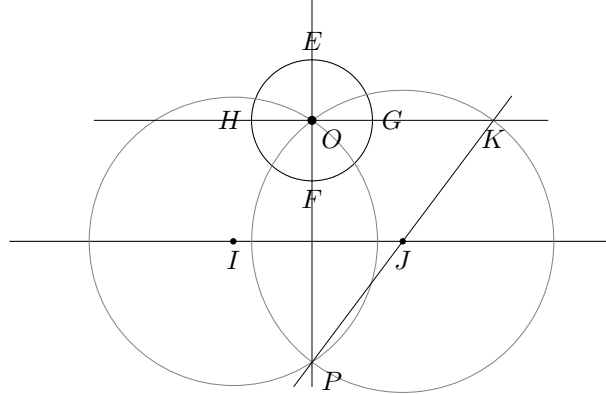
5, 6. Draw  $GH \perp G$  and  $GH \perp H$ . We get the desired square.



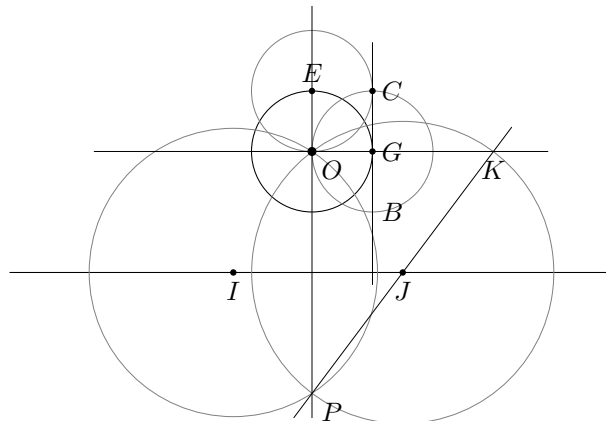
*Proof.* Because two perpendicular lines of the same line are parallel, and converse of tangent  $\perp$  radius.  $\square$

(11E) Let  $I, J$  be two arbitrary points on given line.

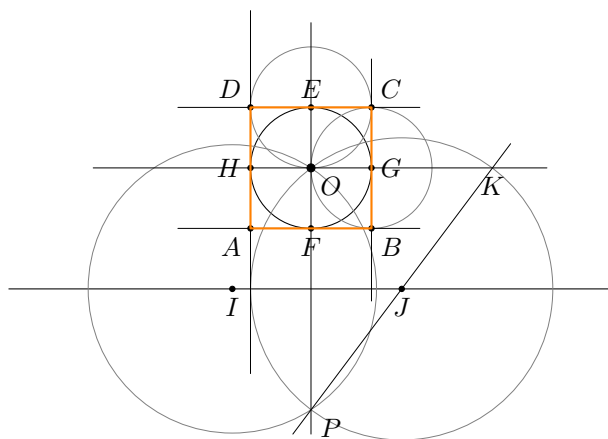
- 1, 2. Draw circles  $(I, O)$  and  $(J, O)$ , intersecting at another point  $P$ .
3. Draw line  $PO$ , intersecting given circle at  $E$  and  $F$ .
4. Draw line  $PJ$ , meeting  $(J, O)$  at  $K$ .
5. Draw line  $KO$ , intersecting given circle at  $G$  and  $H$ .



- 6, 7. Draw circles  $(E, O)$  and  $(G, O)$ , intersecting again at  $C$ .
8. Draw line  $CG$ , meeting  $(G, O)$  at  $B$ .



9. Draw line  $BF$ .
10. Draw line  $CE$ , meeting  $(E, O)$  at  $D$ .
11. Draw line  $DH$ , intersecting  $BF$  at  $A$ .  $ABCD$  is the desired square.



*Proof.* (Let  $r$  be radius of given circle.)

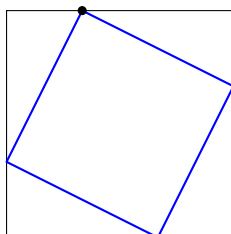
From Tasks before we know that  $IJ \perp OP$  and  $OK \parallel IJ$ . Note that  $EOGC$  is a rhombus (with side length  $r$ ) with a right angle, so it is a square.

By similar reasoning,  $OFBG$  is a square of side length  $r$ . We can also easily deduce that  $DHOE$  is square of side length  $r$  (3 sides equal, 2 right  $\angle$ s) and  $HAFB$  same so (rectangle with equal adj. sides).

Thus the big square  $ABCD$  formed by these four small squares is the square that circumscribes the given circle.  $\square$

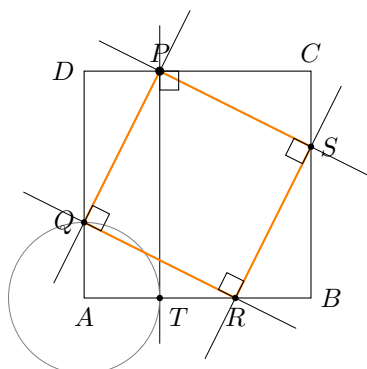
## 5.9 Square in square

**Task 5.9.** Inscribe a square in the square. A vertex is given.  
(6L, 7E)



**Solution 5.9.** Let given square be  $ABCD$  and the given point be on  $DC$ .

- (6L) 1. Draw  $DC \perp P$ , intersecting  $AB$  at  $T$ .
2. Draw circle  $(A, T)$ , intersecting side  $AD$  at  $Q$ .
3. Draw line  $PQ$ .
4. Draw  $PQ \perp Q$ , intersecting  $AB$  at  $R$ .
5. Draw  $QR \perp R$ , intersecting  $BC$  at  $S$ .
6. Draw  $RS \perp P$ .  $PQRS$  is the desired square.



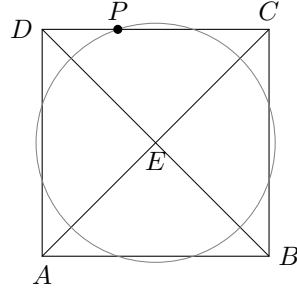


*Proof.* Note that  $\triangle PDQ \cong \triangle QAR$  (AAS by  $DP = QA$ ,  $\angle PDQ = \angle QAR = 90^\circ$ ,  $\angle PQD = \angle QRA$  by angle chasing). Thus  $PQ = QR$  (corr. sides,  $\cong \triangle$ s)

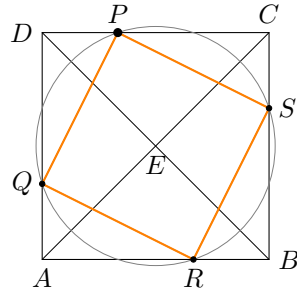
Also  $PQRS$  has four right angles by construction. Since  $PQRS$  is a rectangle with adjacent sides equal,  $PQRS$  is a square.  $\square$

(7E) 1, 2. Draw diagonals  $AC$  and  $BD$ , intersecting at  $E$ .

3. Draw circle  $(E, P)$ , making intersections with the given square sides.

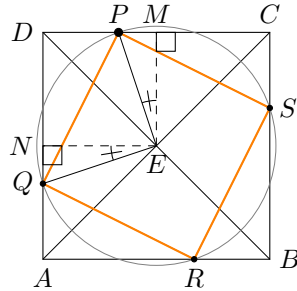


4-7. Draw four lines connecting the intersections such that they form a square.



*Proof.* Note that  $E$  is the center of given square since it is the intersection of the diagonals. This means  $E$  is equidistant from all of its sides.

Draw  $EM \perp DC$  and  $EN \perp DA$ . Note that  $\triangle EMP \cong \triangle ENQ$  (RHS by  $EM = EN$ ,  $\angle EMP = \angle ENQ = 90^\circ$ ,  $EP = EQ$  (radii)), so  $\angle PEM = \angle QEN$  (corr.  $\angle$ s,  $\cong \triangle$ s).



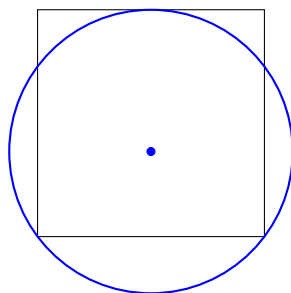
Since  $\angle MEN = 90^\circ$ , we have  $\angle PEQ = \angle MEN - \angle PEM + \angle QEN = 90^\circ$ .

Since  $EP = EQ = ER = ES$  (radii) and  $PE \perp QE$ , we have that  $PQRS$  is a square (diags  $\perp$ , equal and bisect each other).  $\square$

## 5.10 Circle tangent to square side

**Task 5.10.** Construct a circle that is tangent to a side of the square and goes through the vertices of the opposite side.

(3L, 6E, 4V)

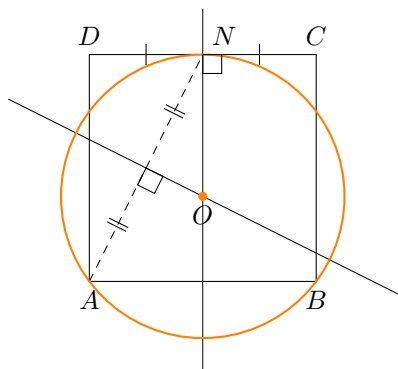


**Solution 5.10.** Let the given square be  $ABCD$ .

(3L) 1. Draw perpbi  $DC$ . Let the midpoint of  $DC$  be  $N$ .

2. Draw perpbi  $NA$ , intersecting perpbi  $DC$  at  $O$ .

3. Draw circle  $(O, N)$ , the desired circle.

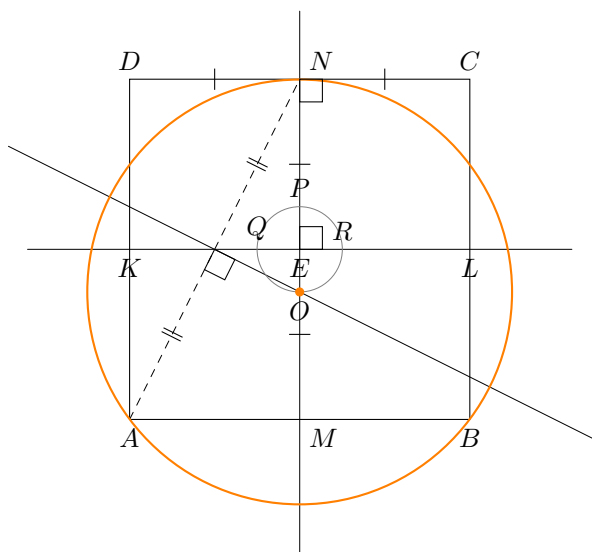


*Proof.* Note that circle  $(O, N)$  is tangent to  $DC$  (converse of tangent  $\perp$  radius), and  $(O, N)$  passes through  $A$  and  $B$  since  $ON = OA = OB$  (prop. of  $\perp$  bisector).  $\square$

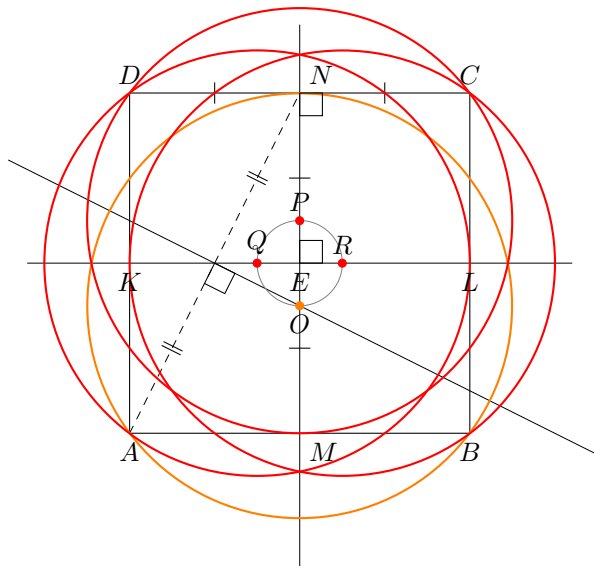
(4V) Let  $M$  be the midpoint of  $AB$ .

4. Draw perpbi  $MN$ , intersecting  $DA$  and  $CB$  at  $K$  and  $L$ . Let  $E$  be midpoint of  $MN$ .

5. Draw circle  $(E, O)$ , intersecting  $MN$  at another  $P$ . Let  $(E, O)$  intersect  $KL$  at  $Q$  (left) and  $R$  (right).



6-8. Draw circles  $(P, M)$ ,  $(Q, L)$  and  $(R, K)$ , the extra solutions.

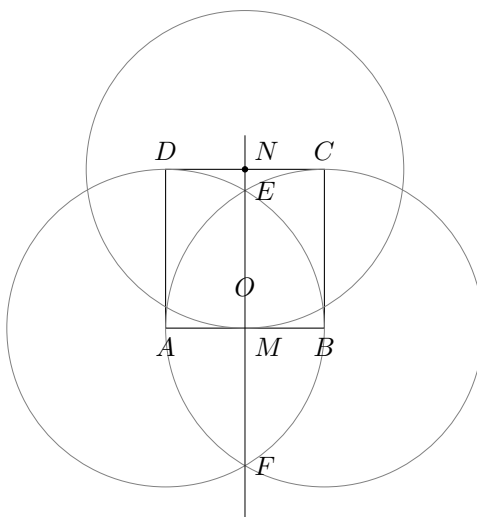


*Proof.* Note that  $E$  is the center of the square, and  $EP = EQ = ER$  (radii). By rotational symmetry,  $(P, M)$ ,  $(Q, L)$  and  $(R, K)$  also satisfy the required conditions, so they are the extra solutions.  $\square$

**(6E) 1, 2.** Draw circles  $(A, B)$  and  $(B, A)$ , intersecting at  $E$  and  $F$ .

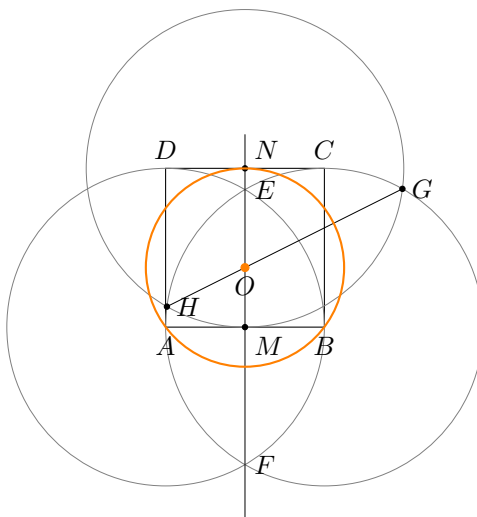
**3.** Draw line  $EF$ , intersecting  $CD$  at  $N$  and  $AB$  at  $M$ .

**4.** Draw circle  $(N, M)$ , intersecting  $(B, A)$  at  $G$  and  $H$ .



**5.** Draw line  $GH$ , intersecting  $EF$  at  $O$ .

**6.** Draw circle  $(O, N)$ , the desired circle.



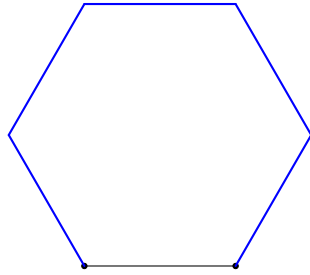
*Proof.* Note that  $N$  and  $M$  are midpoints of  $DC$  and  $AB$ . So  $NM = BC$  ( $MBCN$  being rectangle).

Thus circles  $(N, M)$  and  $(B, C)$  have the same radius, giving  $GN = GB = HN = HB$ . This means  $GH$  is the perpendicular bisector of  $NB$  (because it is a diagonal of rhombus  $HBGN$ ).

Thus  $O$  is the same point as (3L). □

## 5.11 Regular hexagon

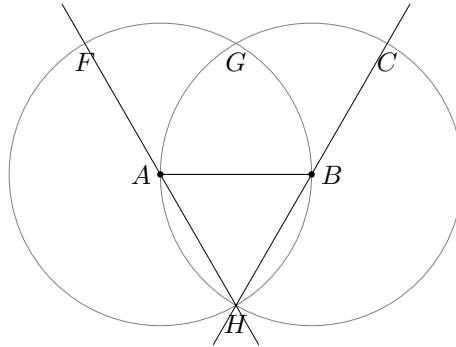
**Task 5.11.** Construct a regular hexagon with the given side.  
(7L, 8E, 2V)



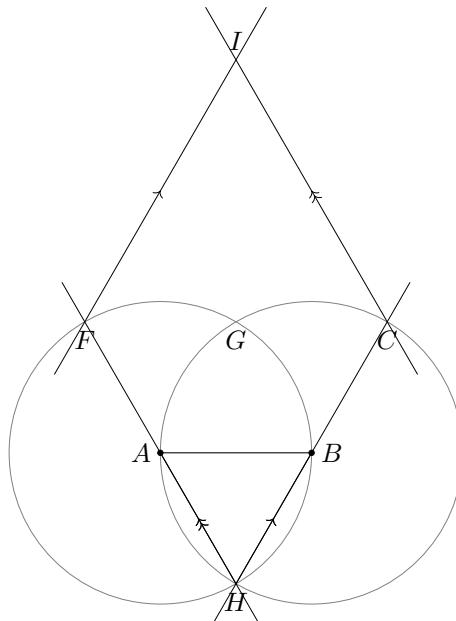
**Solution 5.11.** Let the given line segment be  $AB$ .

(7L) **1, 2.** Draw circles  $(A, B)$  and  $(B, A)$ , intersecting at  $G$  and  $H$ .

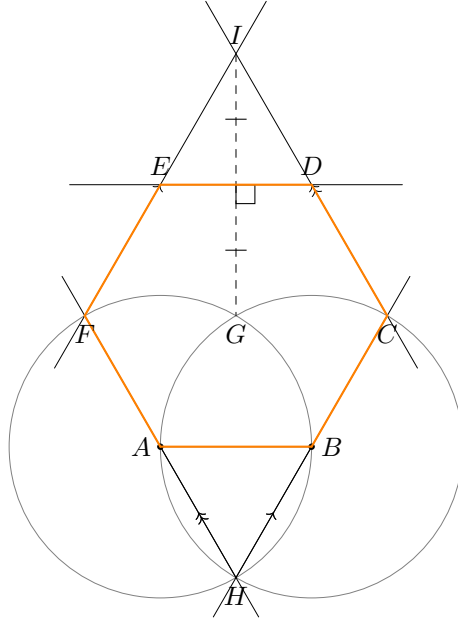
**3, 4.** Draw lines  $FA$  and  $FB$ , meeting  $(A, B)$  and  $(B, A)$  at  $F$  and  $C$  respectively.



**5, 6.** Draw  $HA \rightarrow C$  and  $HB \rightarrow F$ .



**7.** Draw perpbi  $IG$ , intersecting  $FI$  and  $CI$  at  $E$  and  $D$ .  $ABCDEF$  is the desired hexagon.



*Proof.* Let  $r$  be the length of  $AB$ .

Note that  $G$  is the midpoint of  $FC$  (because  $\triangle FAG$ ,  $\triangle GAB$  and  $\triangle GBC$  are three equilateral triangles stacked together.)

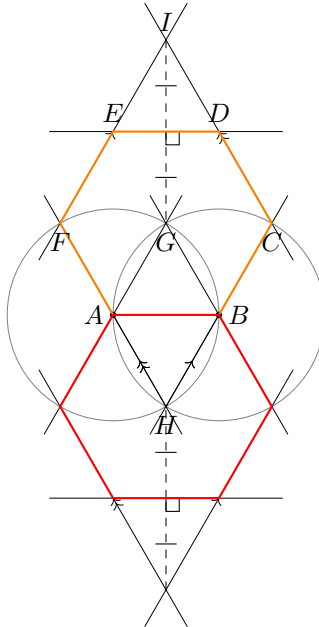
Note that  $\angle AHB = 60^\circ$ , so  $\angle AFI = \angle BCI = 120^\circ$  (int.  $\angle$ s), giving  $\angle IFC = \angle ICF = 60^\circ$ . This means  $\triangle IFC$  is an equilateral triangle with side length  $2r$ .

Since  $FG = GC$ , we have  $IG \perp FC$  (prop. of isos.  $\triangle$ ). Since  $IG \perp ED$  by construction, we have  $ED \parallel FC$ . This gives  $\angle FED = \angle CDE = 120^\circ$  (int.  $\angle$ s).

By intercept theorem,  $IE = EF$  and  $ID = DC$ , meaning  $EF = DC = r$ . By midpoint theorem,  $ED = \frac{1}{2} FC = r$ .

From there, it is easy to see that we have  $AB = BC = CD = DE = EF = FA = r$  and  $\angle A = \angle B = \angle C = \angle D = \angle E = \angle F = 120^\circ$ . Thus  $ABCDEF$  is a regular hexagon, as desired.  $\square$

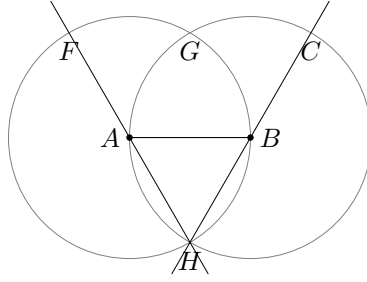
**(2V)** Mirror the constructions on the other side of the given segment  $AB$ .



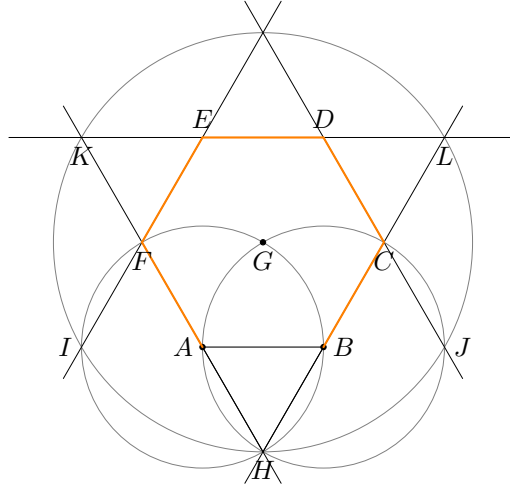
*Proof.* By symmetry.  $\square$

**(8E) 1, 2.** Draw circles  $(A, B)$  and  $(B, A)$ , intersecting at  $G$  and  $H$ .

**3, 4.** Draw lines  $FA$  and  $FB$ , meeting  $(A, B)$  and  $(B, A)$  at  $F$  and  $C$  respectively.



5. Draw circle  $(G, H)$ , intersecting  $(A, B)$  and  $(B, A)$  again at  $I$  and  $J$  respectively.
- 6, 7. Draw lines  $IF$  and  $JC$ .
8. Let  $HF$  and  $HC$  meet  $(G, H)$  at  $K$  and  $L$  respectively. Draw  $KL$ . We get the desired hexagon  $ABCDEF$ .

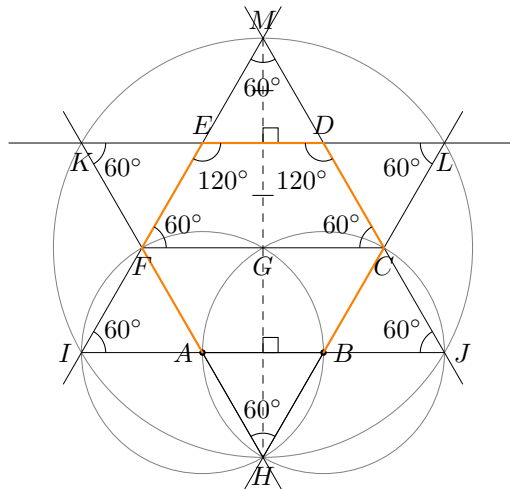


*Proof.* Let  $IF$  and  $JC$  meet at  $M$ . Let  $AB = r$ .

Note that  $I, J$  lie on line  $AB$ , and  $IA = AB = BJ$ . (This property is first shown in Task 1.7E. I'll call it "equil.  $\triangle$  in circle" from now on.) It is not hard to see that trapezium  $F I J C$  is made of five same-size equilateral triangles stacked together.

Since  $\angle F I A = \angle C J B = 60^\circ$ ,  $\triangle M I J$  is an equilateral triangle. Thus  $M$  lies on the circle  $(G, H)$  (since  $G I = \sqrt{3} r$  and  $G M = (\frac{3\sqrt{3}}{2} - \frac{\sqrt{3}}{2}) r = \sqrt{3} r = G I$ ), and  $M, G, H$  are collinear (equil.  $\triangle$  in circle).

Note that  $H K = H L$  since  $K$  and  $L$  are symmetric about  $G H$ . Since we have  $\angle A H B = 60^\circ$ ,  $\triangle H K L$  is also an equilateral triangle.



Thus,  $KL$  is a chord that subtends  $120^\circ$  at center of circle  $(G, H)$ . This means  $KL$  is the perpendicular bisector of  $MG$ . Also, it can be shown that  $KL \parallel FC$  (since  $KL$  and  $FC$  are symmetric about line  $G H$ ).

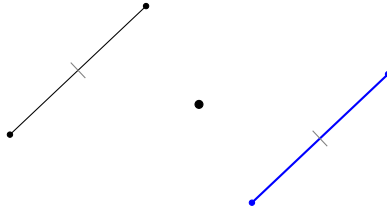
Therefore, by intercept theorem, we have  $ME = EF$  and  $MD = DC$ . Since  $\triangle MIJ$  is an equilateral triangle of side length  $3r$  and  $FI = CJ = r$ , we have  $EF = DC = r$ . Moreover,  $\angle FED = \angle CDE = 120^\circ$ . It is not hard to see that  $ED = r$  too (by  $FC = 2r \Rightarrow ED = 2r - 2(r \cos 60^\circ) = r$ ).

Thus,  $ABCDEF$  is a regular hexagon.  $\square$

## 6 Zeta

### 6.1 Point reflection

**Task 6.1.** Reflect the segment across the point.  
(4L, 5E)



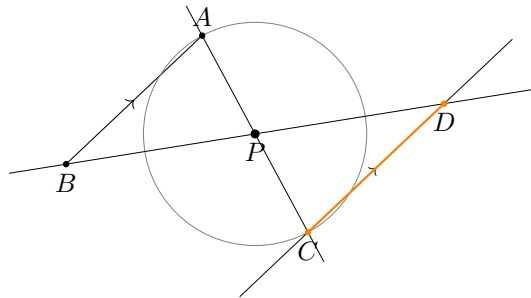
**Solution 6.1.** Let given segment be  $AB$  and given point be  $P$ .

(4L) 1. Draw circle  $(P, A)$ .

2. Draw line  $AP$ , meeting  $(P, A)$  at  $C$ .

3. Draw line  $BP$ .

4. Draw  $AB \Rightarrow C$ , intersecting  $BP$  at  $D$ .  $CD$  is the desired reflection of segment  $AB$ .



*Proof.*

$$\angle APB = \angle CPD \quad (\text{vert. opp. } \angle)$$

$$\angle PAB = \angle PCD \quad (\text{alt. } \angle\text{s, } BA \parallel CD)$$

$$AP = CP \quad (\text{radii})$$

$$\therefore \triangle PAB \cong \triangle PCD \quad (\text{AAS})$$

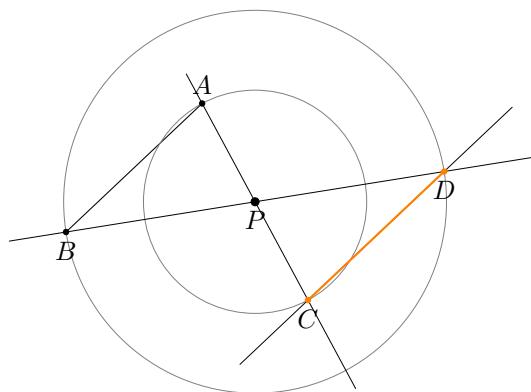
$$\therefore BP = DP \quad (\text{corr. sides, } \cong \triangle\text{s})$$

Since  $C$  and  $D$  are the reflection of endpoints  $A$  and  $B$  respectively,  $CD$  is the reflection of  $AB$  across point  $P$  (prop. of reflection).  $\square$

(5E) 1, 2. Draw circles  $(P, A)$  and  $(P, B)$ .

3, 4. Draw lines  $AP$  and  $BP$ , meeting  $(P, A)$  and  $(P, B)$  at  $C$  and  $D$  respectively.

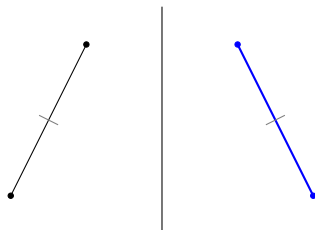
5. Draw  $CD$ , the desired segment.



*Proof.* Note that  $AP = CP$  and  $\angle APB = \angle CPD$  by radii, so  $CD$  is the reflection of  $AB$  across  $P$ .  $\square$

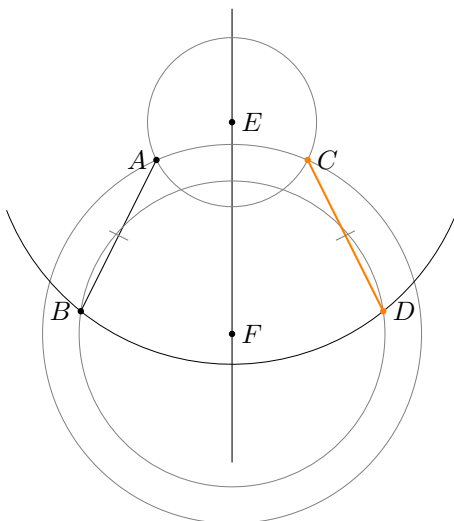
## 6.2 Reflection

**Task 6.2.** Reflect the segment across the line.  
(5L, 5E)



**Solution 6.2.** Let  $AB$  be given segment. Let  $E, F$  be two arbitrary points on given line. (Or you can make four points, not reusing  $E$  and  $F$  at step 3, 4.)

- 1, 2. Draw circles  $(E, A)$  and  $(F, A)$ , intersecting again at  $C$ .
- 3, 4. Draw circles  $(E, B)$  and  $(F, B)$ , intersecting again at  $D$ .
5. Draw line  $CD$ .

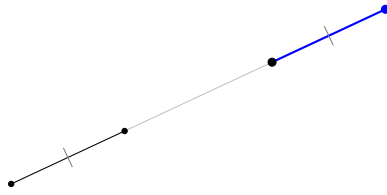


*Proof.* Note that  $EAFB$  and  $EACB$  form kites, so we have  $EF \perp AC$  and  $EF \perp BD$ . Also, from the property of kite, the perpendicular distance of  $A$  and  $C$  from  $EF$  is equal, and same for  $B$  and  $D$ . This means  $C$  and  $D$  are reflections of  $A$  and  $B$  across  $EF$ .  $\square$



## 6.3 Copy segment

**Task 6.3.** Construct a segment from the given point that is equal to the given segment and lies on the same line with it.  
(3L, 4E, 2V)

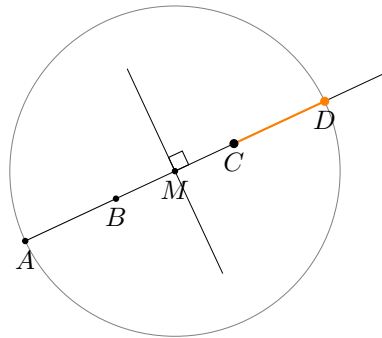


**Solution 6.3.** Let given segment be  $AB$  and given point be  $C$ .

(3L) 1. Draw line  $BC$ .

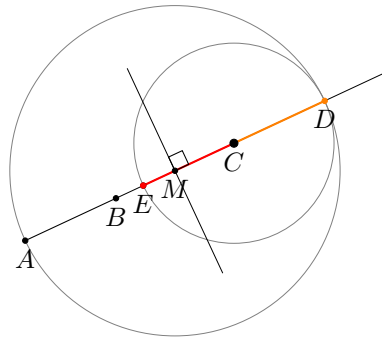
2. Draw perpbi  $BC$ . Let  $M$  be midpoint of  $BC$ .

3. Draw circle  $(M, A)$ , intersecting  $BC$  at another point  $D$ .  $CD$  is the desired segment.



*Proof.*  $AM = MD$  (radii) and  $BM = MC$  by construction. So  $CD = MD - MC = AM - BM = AB$ .  $\square$

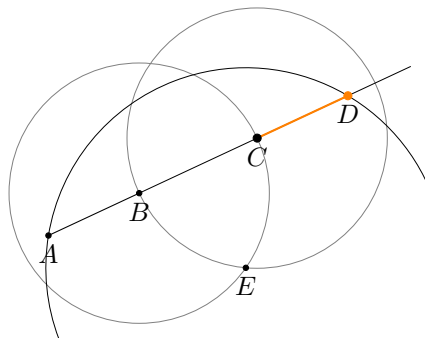
(2V) Draw circle  $(C, D)$ , intersecting  $BC$  again at  $E$ .  $CE$  is the extra solution.



(4E) 1. Draw line  $BC$ .

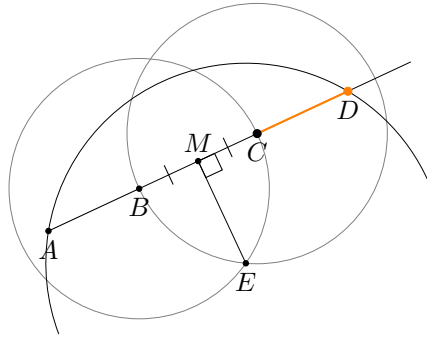
2, 3. Draw circles  $(B, C)$  and  $(C, B)$ . Let one of intersections be  $E$ .

4. Draw circle  $(E, A)$ , intersecting  $BC$  again at  $D$ .  $CD$  is the desired segment.



*Proof.* Let  $M$  be the midpoint of  $BC$ . Note that  $EM \perp BC$  (prop. of isos.  $\triangle$ ), so  $AM = MD$  (line from center  $\perp$  chord bisects chord).

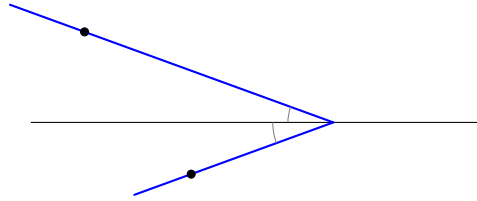
Thus,  $CD = MD - MC = AM - BM = AB$ , as desired.



□

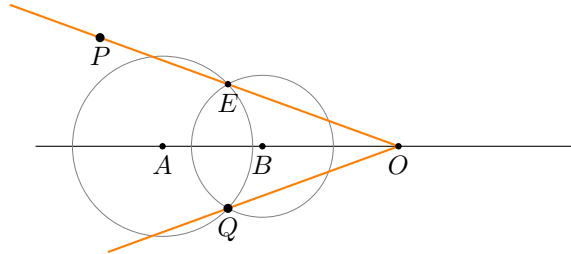
## 6.4 Given angle bisector

**Task 6.4.** Construct two straight lines through the two given points respectively so that the given line is a bisector of the angle that they make.  
(4L, 4E)

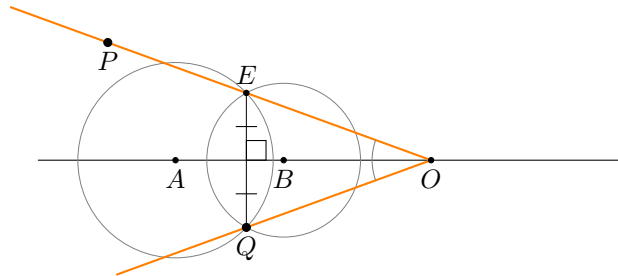


**Solution 6.4.** Let the given points be  $P, Q$ . Let  $A, B$  be arbitrary points on given line.

- 1, 2. Draw circles  $(A, Q)$  and  $(B, P)$ , intersecting at another point  $E$ .
3. Draw line  $PE$ , intersecting given line at  $O$ .
4. Draw line  $OQ$ .  $PE$  and  $OQ$  are the desired lines.



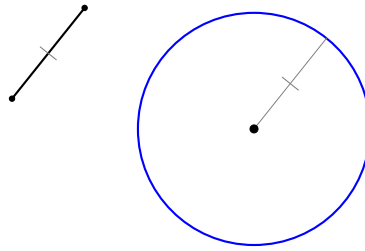
*Proof.* Let  $M$  be the projection of  $E$  on given line. Note that  $EM = QE$  because  $AQBE$  forms a kite. Thus we have  $\triangle EMO \cong \triangle QMO$  by SAS, so  $\angle EOM = \angle QOM$  (corr.  $\angle$ s,  $\cong \triangle$ s).



□

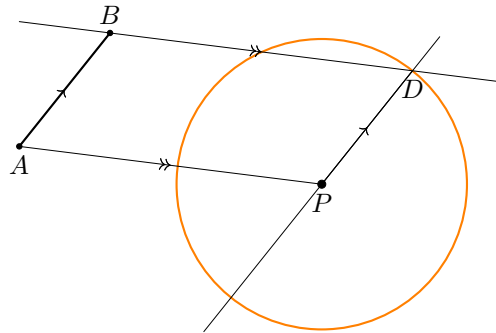
## 6.5 Non-collapsing compass

**Task 6.5.** Construct a circle with the given center and the radius equal to the length of the given segment.  
(4L, 5E)



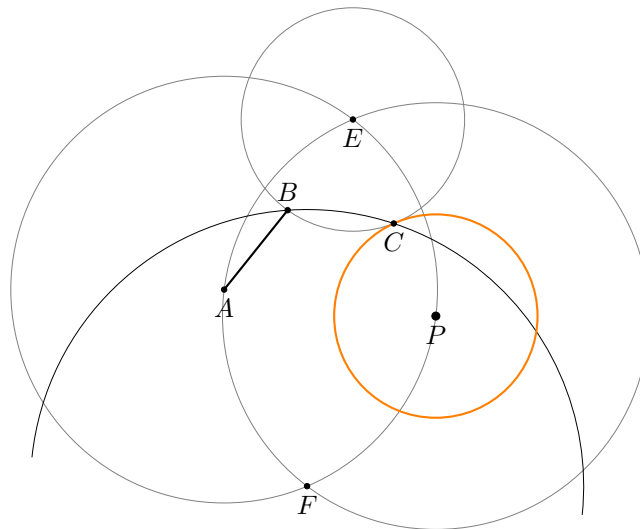
**Solution 6.5.** Let given segment be  $AB$ , given point be  $P$ .

- (4L) 1. Draw  $AB \rightarrow P$ .
2. Draw line  $AP$ .
3. Draw  $AP \rightarrow B$ , intersecting  $AB \rightarrow P$  at  $D$ .
4. Draw circle  $(P, D)$ , the desired circle.



*Proof.* Note that  $AP \parallel BD$  and  $AB \parallel PD$  by construction, so we have  $PD = AB$  (opp. sides of  $\parallel$ gram), which means circle  $(P, D)$  has radius equal to  $AB$ .  $\square$

- (5E) 1, 2. Draw circles  $(A, P)$  and  $(P, A)$ , intersecting at  $E$  and  $F$ .
- 3, 4. Draw circles  $(E, B)$  and  $(F, B)$ , intersecting at another point  $C$ .
5. Draw circle  $(P, C)$ , the desired circle.



*Proof.* Note that  $BFCE$  forms a kite (with  $EB = EC$  and  $FB = FC$  by radii), so  $C$  is the reflection of  $B$  across  $EF$  (prop. of kite). Similarly, since  $AFPE$  is a kite,  $P$  is the reflection of  $A$  across  $EF$ .

Thus segment  $PC$  is the reflection of  $AB$  across  $EF$ . Since reflection preserves segment length, we have  $PC = AB$ , and so  $(P, C)$  is the desired circle.  $\square$

## 6.6 Translate segment

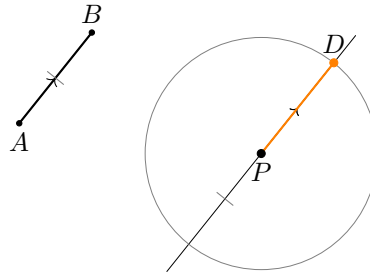
**Task 6.6.** Construct a segment from the given point parallel and equal to the given segment.  
(2L, 6E, 2V)



**Solution 6.6.** Let  $AB$  be given segment,  $P$  be given point.

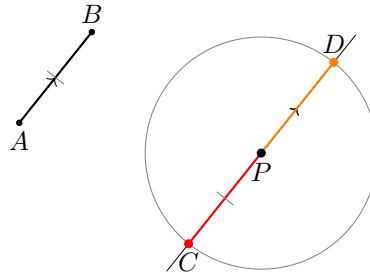
(2L) 1. Draw  $AB \rightarrow P$ .

2. Compass  $(AB, P)$ . i.e. draw circle centered  $P$  with radius  $AB$  using non-collapsing compass tool<sup>1</sup>. Mark one of intersections of  $(AB, P)$  and  $AB \rightarrow P$ .



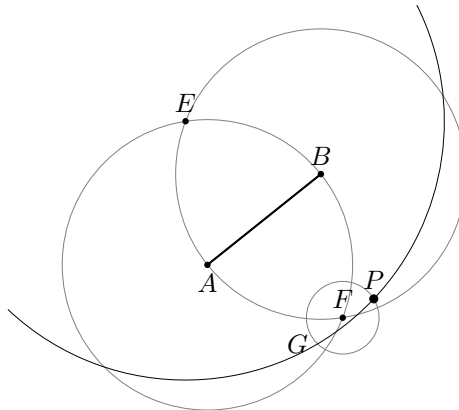
*Proof.*  $AB = PD$  and  $AB \parallel PD$  by construction. □

(2V) Mark another intersection point of  $(AB, P)$  and  $AB \rightarrow P$ .



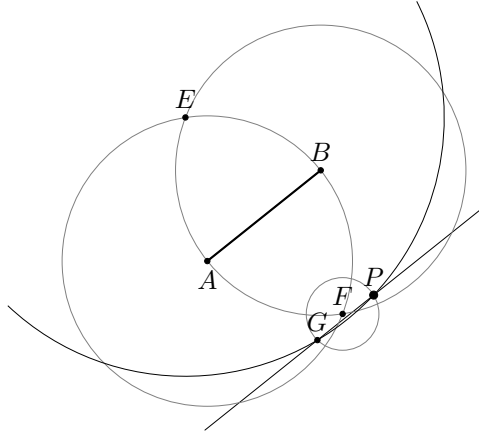
(6E) 1, 2. Draw circles  $(A, B)$  and  $(B, A)$ , intersecting at  $E$  and  $F$ .

3, 4. Draw circle  $(E, P)$  and  $(F, P)$ , intersecting at another point  $G$ .

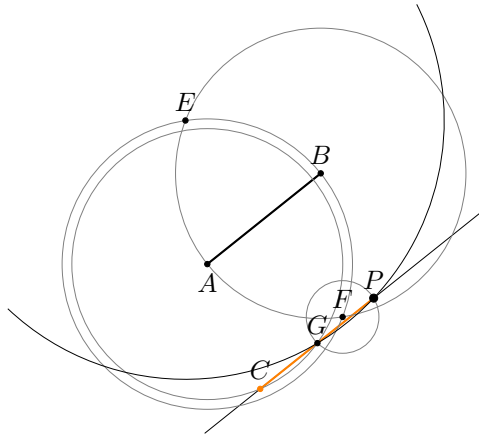


5. Draw line  $PG$ .

<sup>1</sup>I use  $(AB, P)$  instead of  $(P, AB)$  to follow the order of clicking the points in the game.



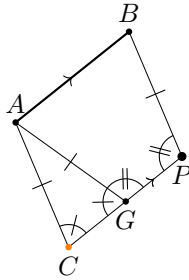
6. Draw circle  $(A, G)$ , intersecting  $PG$  at another point  $C$ .



*Proof.* Note that  $EGFP$  forms a dart, so  $GP \perp EF$  (prop. of dart). Since  $EF \perp AB$  (prop. of rhombus), we have  $GP \parallel AB$ .

Note that  $AG = BP$  since  $AB$  and  $GP$  are symmetric about line  $EF$ . Thus  $AGPB$  is an isosceles trapezium. We also have  $AC = AG$  by radii.

Let's focus on trapezium  $AGPB$  and  $\triangle ACG$ . We have  $\angle ACG = \angle AGC$  (base  $\angle$ s, isos.  $\triangle$ ) and  $\angle AGP = \angle BPG$  (prop. of isos. trapezium).

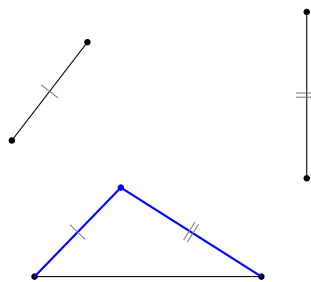


Since  $\angle AGC + \angle AGP = 180^\circ$  (adj.  $\angle$ s on st. line), we also have  $\angle ACG + \angle BPG = 180^\circ$ , thus  $AC \parallel BP$  (int.  $\angle$ s supp.). This means  $ABPC$  is a parallelogram, and we have  $CP = AB$  (opp. sides of  $\parallel$ gram), as desired.  $\square$

## 6.7 Triangle by three sides

**Task 6.7.** Construct a triangle with the side  $AB$  and the two other sides equal to the given segments.

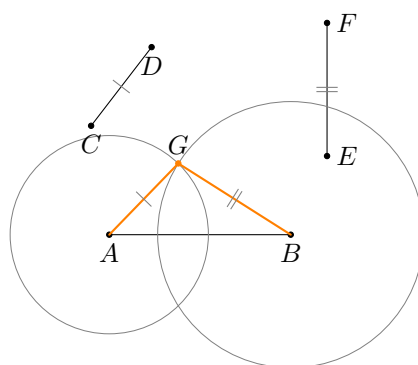
(4L, 12E, 4V)



**Solution 6.7. (4L, 12E)** Let  $CD, EF$  be the given segments.

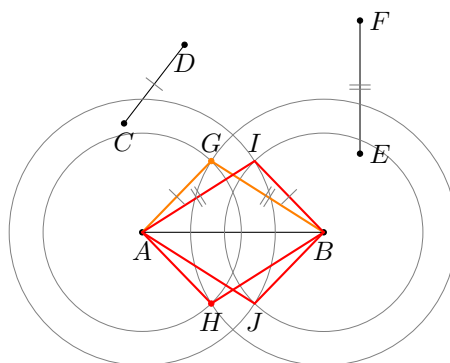
1, 2. Compass circles  $(CD, A)$  and  $(EF, B)$ . Let one of their intersections be  $G$ .

3, 4. Draw line  $AG$  and  $BG$ .



(4V) **2nd solution:** Let  $H$  be another intersection of the circles. Draw  $AH$  and  $BH$ .

**3rd & 4th solution:** Compass  $(CD, B)$  and  $(EF, A)$ , intersecting at  $I$  and  $J$ . Draw  $IA, IB, JA, JB$ .

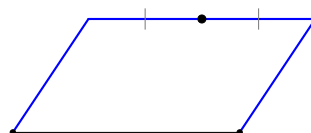


*Proof.* By construction. □

## 6.8 Parallelogram

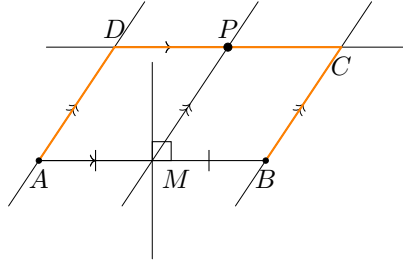
**Task 6.8.** Construct a parallelogram with the given side and the midpoint of the opposite side in the given point.

(5L, 8E)



**Solution 6.8. (5L)** Let given segment be  $AB$ , given point be  $P$ .

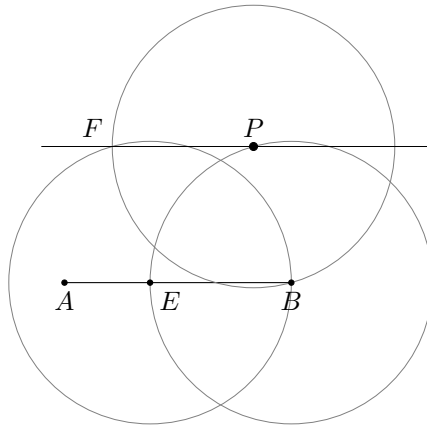
1. Draw perpbi  $AB$ . Let  $M$  be midpoint of  $AB$ .
2. Draw line  $PM$ .
3. Draw  $AB \rightarrow P$ .
- 4, 5. Draw  $PM \rightarrow A$  and  $PM \rightarrow B$ , making points  $D$  and  $C$ .  $ABCD$  is the desired parallelogram.



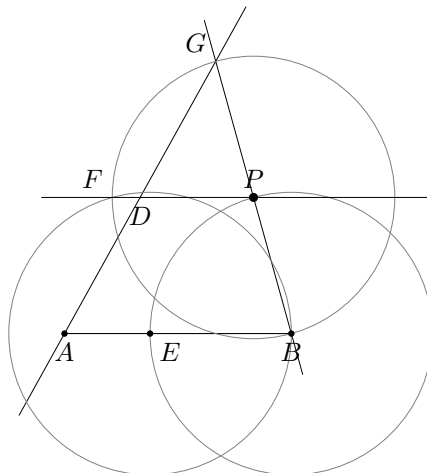
*Proof.* Note that  $ABCD$ ,  $AMPD$  and  $MBCP$  are parallelograms by construction. So we have  $DP = AM$  and  $PC = MB$  (opp. sides of //gram). Since  $AM = MB$  by construction, we have  $DP = PC$ , which means  $P$  is the midpoint of side  $DC$ , as desired  $\square$

**(8E)**

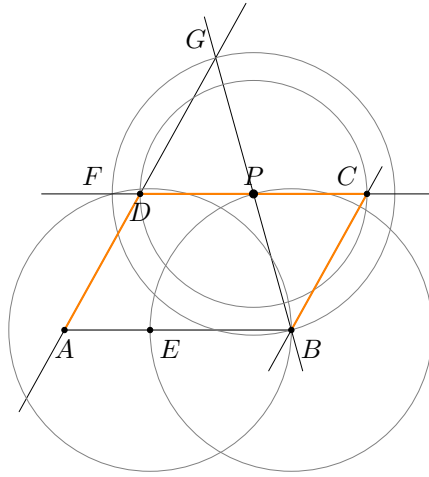
1. Draw circle  $(P, B)$ .
2. Draw circle  $(B, P)$ , intersecting segment  $AB$  at  $E$ .
3. Draw circle  $(E, B)$ , intersecting  $(P, B)$  again at  $F$ .
4. Draw line  $FP$ .



5. Draw line  $BP$ , meeting  $(P, B)$  at  $G$ .
6. Draw line  $AG$ , intersecting  $FP$  at  $D$ .



7. Draw circle  $(P, D)$ , intersecting  $FP$  at another point  $C$ .
8. Draw line  $BC$ .



*Proof.* Note that  $PBEF$  forms a rhombus because the circles involved have the same radius. So  $FP \parallel AB$ .

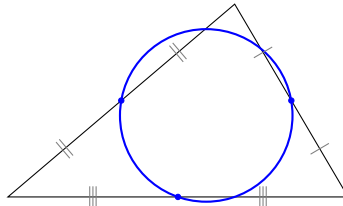
Also,  $GP = PB$  (radii), so  $GD = DA$  by intercept theorem, and thus  $DP = \frac{1}{2}AB$  by midpoint theorem.

Lastly, we have  $DP = PC$  (radii), which means  $DC = AB$ . Thus  $ABCD$  is a parallelogram by “opp. sides equal and  $\parallel$ ”, as desired  $\square$

## 6.9 Nine point circle

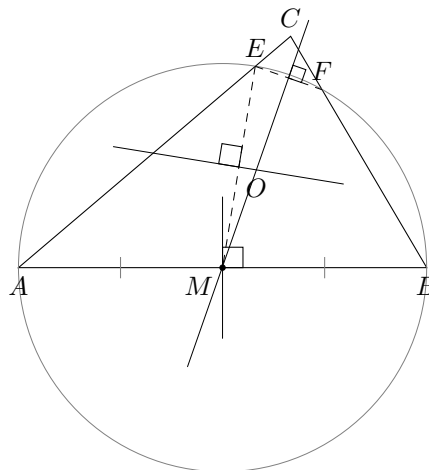
**Task 6.9.** Construct a circle that passes through the midpoints of sides of the given acute triangle.

(5L, 9E)



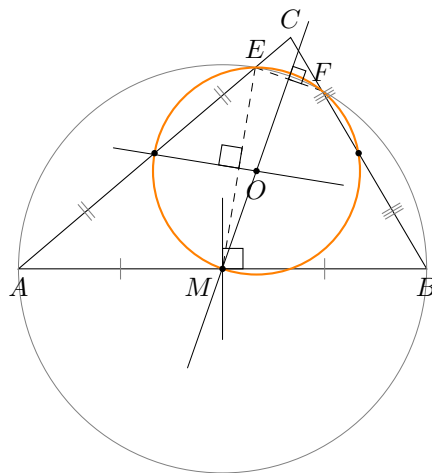
**Solution 6.9.** Let the given triangle be  $\triangle ABC$ .

- (5L) 1. Draw perpbi  $AB$ . Let  $M$  be midpoint of  $AB$ .
2. Draw circle  $(M, A)$ , intersecting sides  $AC$  and  $BC$  at  $E$  and  $F$ .
3. Draw perpbi  $EF$ .
4. Draw perpbi  $EM$ , intersecting perpbi  $EF$  at  $O$ .



5. Draw circle  $(O, M)$ , the desired nine point circle.





*Proof.* The circle that passes through the midpoints of a triangle's sides is called the **nine point circle**. By “prop. of nine point circle”, it also passes through the foots of altitudes of the triangle.

Note that  $AE \perp EB$  and  $AF \perp FB$  ( $\angle$  in semi-circle). This means  $E$  and  $F$  are the foot of altitudes from  $B$  to side  $AC$  and from  $A$  to side  $BC$  respectively.

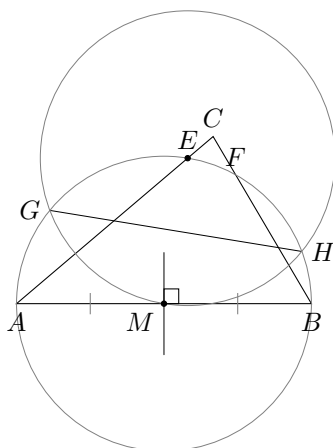
Since the nine point circle passes through  $E$ ,  $F$  and  $M$ , its center  $O$  must lie on the perpendicular bisector of  $EF$  and  $EM$  (prop. of  $\perp$  bisector). Thus  $O$  must be the point of intersection of the perpendicular bisectors, and  $(O, M)$  is the desired nine point circle.  $\square$

**(9E) 1-3.** Draw perpbi  $AB$ . Let  $M$  be midpoint of  $AB$ .

4. Draw circle  $(M, A)$ , intersecting sides  $AC$  and  $BC$  at  $E$  and  $F$ .

5. Draw circle  $(E, M)$ , intersecting  $(M, A)$  at  $G$  and  $H$ .

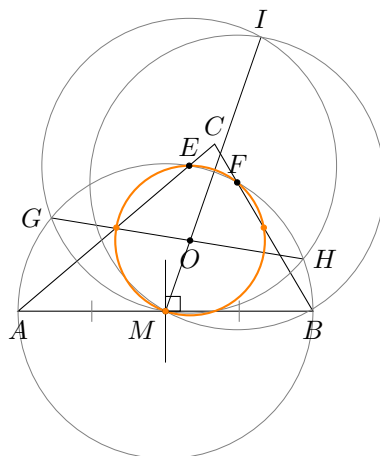
6. Draw line  $GH$ .



7. Draw circle  $(F, M)$ , intersecting  $(E, M)$  at another point  $I$ .

8. Draw line  $IM$ , intersecting  $GH$  at  $O$ .

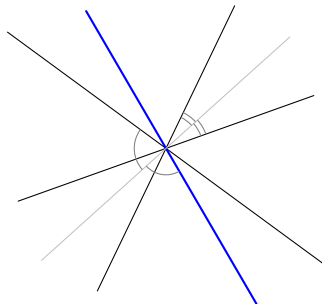
9. Draw circle  $(O, M)$ , the desired nine point circle.



*Proof.* Note that  $GH$  is the perpendicular bisector of  $EM$  because  $EGMH$  forms a rhombus by radii. Similarly,  $IM$  is the perpendicular bisector of  $EF$  because  $EMFI$  forms a rhombus. Thus  $O$  is the same point as the “O” in (5L).  $\square$

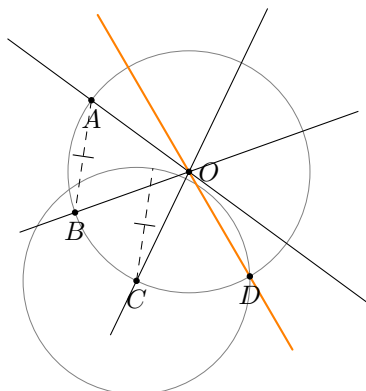
## 6.10 Symmetry of four lines

**Task 6.10.** Three lines are intersected in a point. Construct a line so that the set of all 4 lines is mirror symmetric.  
(3L, 4E, 3V)



**Solution 6.10.** Let  $O$  be the point of intersection of the given lines.

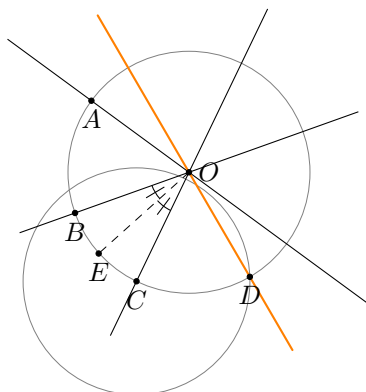
- (3L) 1. Draw a circle centered  $O$  with arbitrary radius. Let it intersect the three lines (at the left) at  $A$ ,  $B$  and  $C$ .  
2. Compass  $(AB, C)$ , intersecting the first circle at one of points  $D$ .  
3. Draw line  $OD$ , the desired line.



*Proof.* Let  $m$  be the line of symmetry. Then  $m$  must pass through  $O$ . Otherwise,  $O$  would be on either side of  $m$  and the lines could never be mirror symmetric about  $m$ .

Moreover,  $m$  must be an angle bisector of two of the given lines by “prop. of  $\angle$  bisector”. It is also an angle bisector of the remaining two lines (one is the given line, one is the required line) in order for them to be mirror symmetric.

Let  $E$  be the midpoint of arc  $\widehat{BC}$ . Then  $\angle BOE = \angle COE$  (equal arcs, equal  $\angle$ s at center), so line  $OE$  is used as the line of symmetry. We want to show that  $\angle AOE = \angle DOE$ .



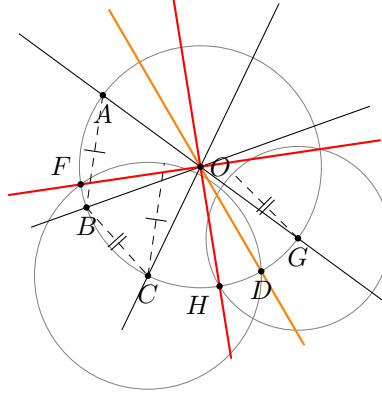
By compass construction, we have  $CD = AB$ , so  $\widehat{CD} = \widehat{AB}$  (equal chords, equal arcs). Thus  $\widehat{DE} = \widehat{EC} + \widehat{CD} = \widehat{BE} + \widehat{AB} = \widehat{AE}$ , giving  $\angle AOE = \angle DOE$  (equal arcs, equal  $\angle$ s at center), as required.  $\square$

**(3V) 2nd solution:** 4. Let  $F$  be another intersection of  $(C, D)$  and  $(O, D)$ . Draw line  $OF$ , the 2nd solution.

**3rd solution:**

5. Let  $G$  be the point diametrically opposite to  $A$  on  $(O, A)$ . Compass  $(BC, G)$ , intersecting  $(O, A)$  at the point  $H$  that isn't passed through by line  $OF$ .

6. Draw line  $OH$ , the 3rd solution.

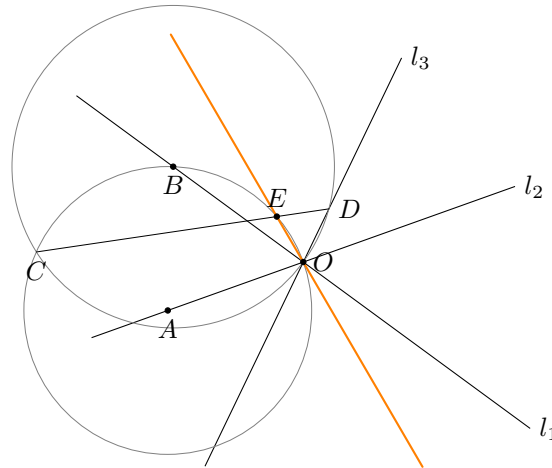


*Proof.* The 2nd solution uses the angle bisector of  $\angle AOC$  as the line of symmetry, call it  $m_2$ . We have  $\widehat{AB} = \widehat{CF}$  by compass construction, thus  $\widehat{AF} = \widehat{BC}$ . So  $OF$  is symmetric to  $OB$  about  $m_2$ .

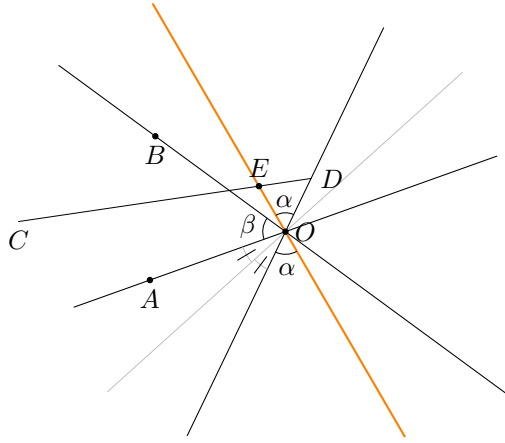
The 3rd solution uses the angle bisector of  $\angle BOG$  (or alternatively  $\angle AOB$ ) as the line of symmetry, call it  $m_3$ . We have  $\widehat{BC} = \widehat{GH}$  by construction, so  $OC$  is symmetric to  $OH$  about  $m_3$ .  $\square$

**(4E)** Let  $O$  be point of intersection of the given lines. Label the given lines  $l_1, l_2, l_3$ .

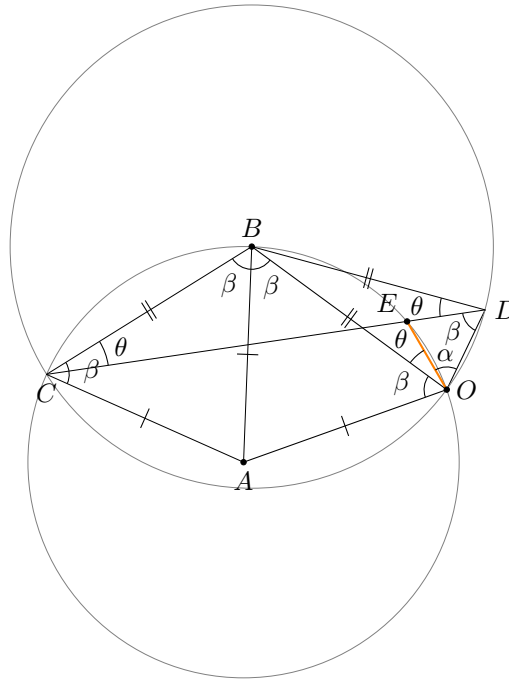
1. Let  $A$  be an arbitrary point on  $l_2$ . Draw circle  $(A, O)$ , intersecting  $l_1$  at another point  $B$ .
2. Draw circle  $(B, O)$ , intersecting  $(A, O)$  at another point  $C$ , and intersecting  $l_3$  at another point  $D$ .
3. Draw line  $CD$ , cutting  $(A, O)$  at  $E$ .
4. Draw line  $OE$ , the desired line.



*Proof.* The angle bisector of  $l_2$  and  $l_3$  is used as the line of symmetry. Refer to the figure below, we want to show that  $\angle BOA = \angle EOD$ . This will mean  $\alpha = \beta$  and thus  $l_1$  and the orange line are symmetric about the angle bisector of  $l_2$  and  $l_3$ .



Let's focus on the points and segments involved in constructing  $OE$ . Let  $\angle BOE = \theta$ .



Note that  $\triangle ACB \cong \triangle AOB$  (SSS), so  $\angle ACB = \angle AOB = \beta$  (corr.  $\angle$ s,  $\cong \triangle$ s). Also,  $\angle ABC = \angle ABO = \beta$  (base  $\angle$ s, isos.  $\triangle$ ).

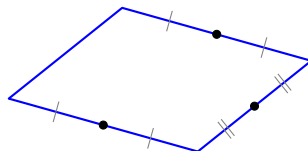
Consider circle  $(B, O)$ . Note that  $\angle CBO$  and  $\angle CDO$  are subtended by the same arc  $\widehat{CO}$ . Thus  $\angle CDO = \frac{1}{2} \angle CBO = \beta$  ( $\angle$  at centre twice  $\angle$  at  $\odot^{ce}$ )

Consider circle  $(A, O)$ . We have  $\angle BCE = \angle BOE = \theta$  ( $\angle$ s in the same segment), so  $\angle BDC = \angle BCD = \theta$  (base  $\angle$ s, isos.  $\triangle$ ).

Consider  $\triangle BOD$ . Since  $BO = BD$  (radii), we have  $\angle BOD = \angle BDO$  (base  $\angle$ s, isos.  $\triangle$ ), meaning  $\alpha + \theta = \beta = \theta$ , giving  $\alpha = \beta$ , as desired.  $\square$

## 6.11 Parallelogram by three midpoints

**Task 6.11.** Construct a parallelogram given three of the midpoints.  
(7L, 10E, 3V)

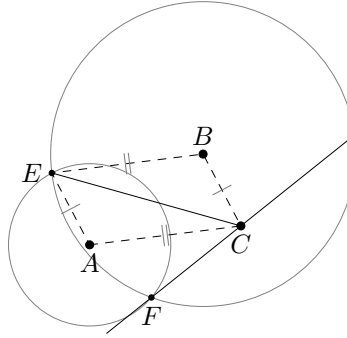


**Solution 6.11.** Label the given points  $A, B, C$  (as in the game).

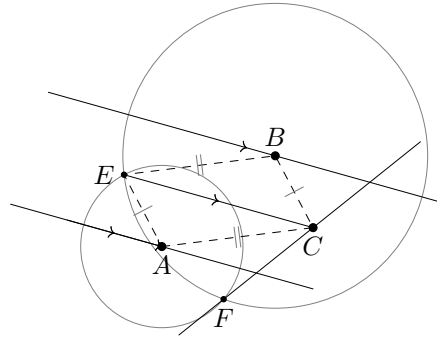
(7L) 1. Compass  $(BC, A)$ .

2. Compass  $(AC, B)$ , intersecting  $(BC, A)$  at  $E$  (top) and  $F$  (bottom).

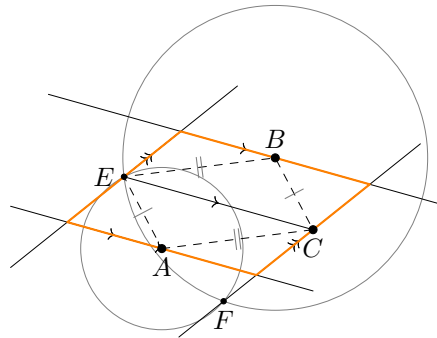
3, 4. Draw lines  $EC$  and  $FC$ .



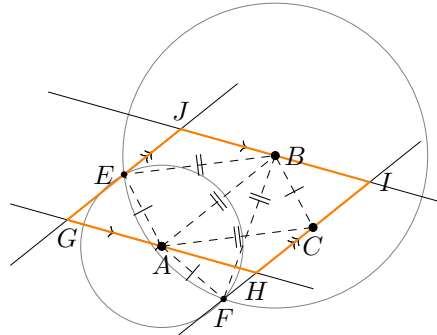
5, 6. Draw  $EC \rightarrow A$  and  $EC \rightarrow B$ .



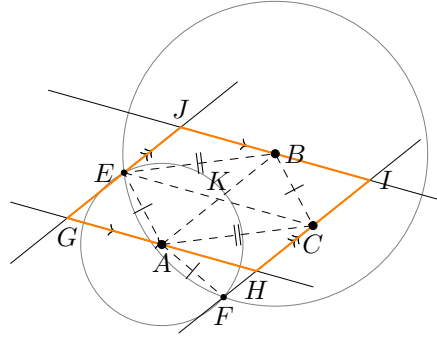
7. Draw  $FC \rightarrow E$ . The big parallelogram enclosed by the lines is the desired parallelogram.



*Proof.* Let the orange parallelogram be  $GHIJ$ . Let  $EC$  and  $AB$  intersect at  $K$ .



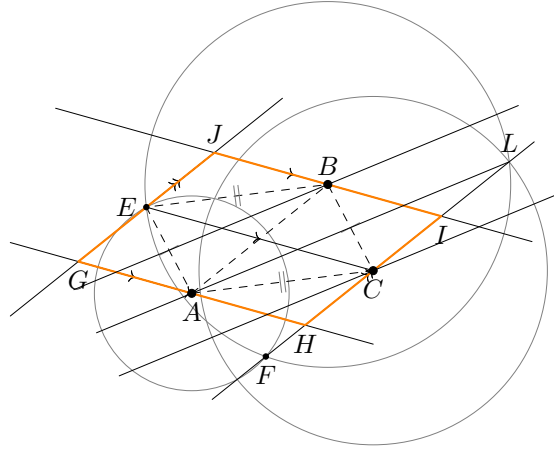
Note that  $\triangle AFB \cong \triangle BCA$  (SSS), so  $\triangle AFB$  and  $\triangle BCA$  have the same height (with  $AB$  as the base). This means  $FC \parallel AB$ . Thus we have  $GJ \parallel AB \parallel HI$  and  $JI \parallel EC \parallel GH$  by construction. In other words, the orange parallelogram can be divided into four smaller parallelograms ( $KBJE, KCIB, KEGA, KAH C$ ).



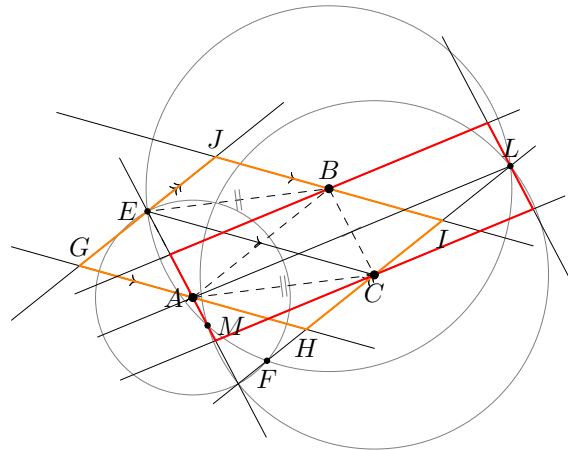
Note that  $ACBE$  forms a parallelogram (opp. sides equal). Thus we have  $EK = CK$  and  $AK = BK$  (diags of //gram). From “opp. sides of //gram”, it follows that the four smaller parallelograms are congruent to each other, meaning that  $A, C, B, E$  are midpoints of  $GH, HI, IJ, JG$  respectively.  $\square$

**(3V) 2nd solution:**

8. Compass  $(AB, C)$ , intersecting  $(B, E)$  at  $M$  (left) and  $L$  (right).
9. Draw line  $AL$ .
- 10, 11. Draw  $AL \rightarrow B$  and  $AL \rightarrow C$ .



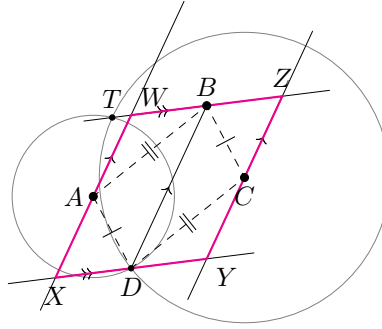
12. Draw line  $AM$ .
13. Draw  $AM \rightarrow L$ . We get the 2nd solution.



**3rd solution:** Let  $(A, F)$  and  $(C, M)$  intersect at  $T$  (top) and  $D$  (bottom).

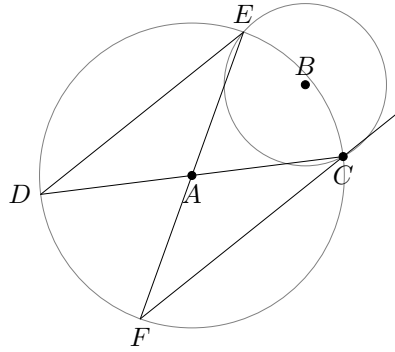
14. Draw line  $BD$ .
- 15, 16. Draw  $BD \rightarrow A$  and  $BD \rightarrow C$ .



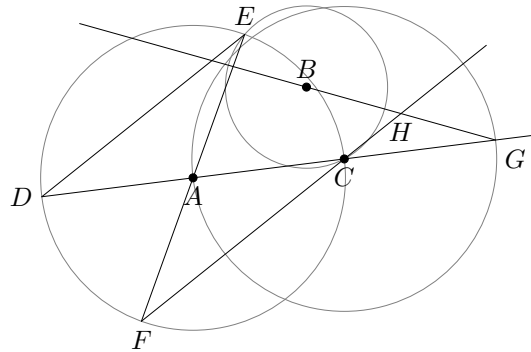


Similarly,  $ABCD$  is a parallelogram, and we have  $WZ \parallel AC \parallel XY$  and  $WX \parallel BD \parallel ZY$ . It follows that  $WXYZ$  is a parallelogram with  $A, B, C, D$  being the midpoints of sides. □

- (10E) 1.** Draw circle  $(A, C)$ .  
**2.** Draw circle  $(B, C)$ , intersecting  $(A, C)$  at another point  $E$ .  
**3.** Draw line  $CA$ , meeting  $(A, C)$  at point  $D$ .  
**4.** Draw line  $DE$ .  
**5.** Draw line  $EA$ , meeting  $(A, C)$  at  $F$ .  
**6.** Draw line  $FC$ .

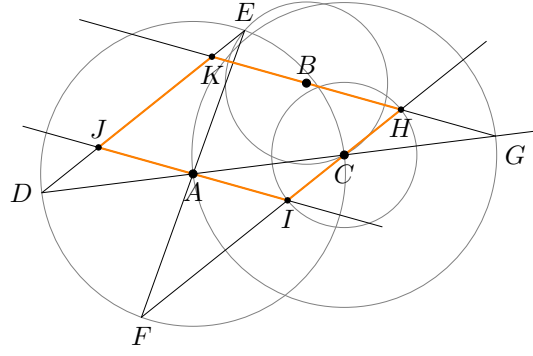


- 7.** Draw circle  $(C, A)$ , intersecting line  $AC$  again at  $G$ .  
**8.** Draw line  $GB$ , intersecting  $FC$  at  $H$ .



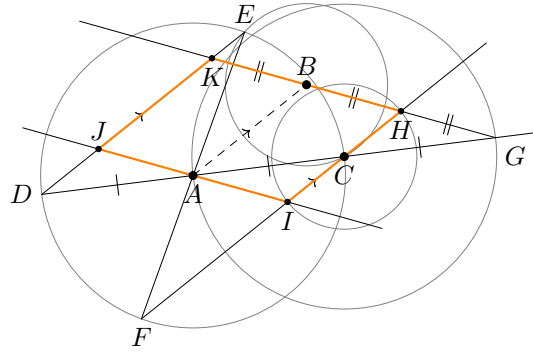
- 9.** Draw circle  $(C, H)$ , intersecting  $FC$  again at  $I$ .  
**10.** Draw line  $AI$ . We get the desired parallelogram  $HIJK$ .





*Proof.* Note that  $AEBC$  forms a kite, so  $EC \perp AB$ . Also,  $DE \perp EC$  by “ $\angle$  in semi-circle”. Thus  $DE \parallel AB$ . Similarly, we have  $AB \parallel FH$ .

Note that  $DA = AC = CG$  by radii. By intercept theorem, we have  $KB = BH = HG$ .



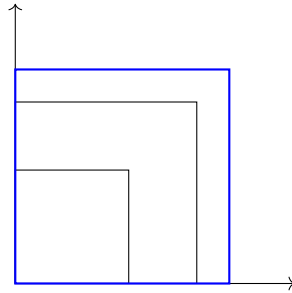
Also,  $BA = 2HC$  by midpoint theorem, so  $HI = 2HC = BA$ . Thus  $ABHI$  is a parallelogram (opp. sides equal and  $\parallel$ ). It follows that  $HIJK$  is a parallelogram.

Since  $KB = BH$  (shown above) and  $BH = AI$  (opp. sides of  $\parallel$ gram), we also have  $JA = AI$ . Thus,  $A, B, C$  are midpoints of sides  $JI, KH, HI$  respectively, as desired.  $\square$

## 7 Eta

### 7.1 Sum of areas of squares

**Task 7.1.** Construct a square whose area equals the sum of the areas of the two given squares and all three have the common angle.  
(3L, 6E)

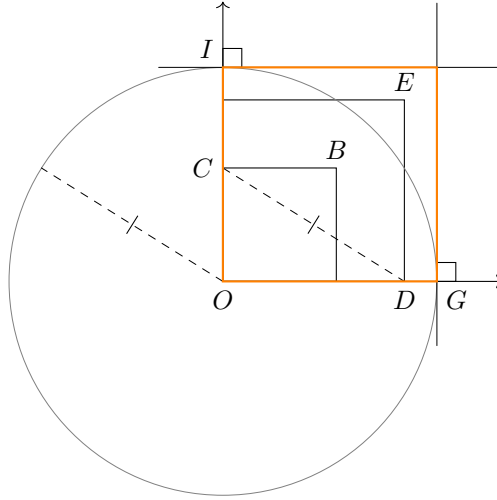


**Solution 7.1.** Let the given squares be  $OABC$  and  $ODEF$ .

(3L) 1. Compass  $(CD, O)$ , intersecting given horizontal ray at  $G$ , and intersecting vertical ray at  $I$ .

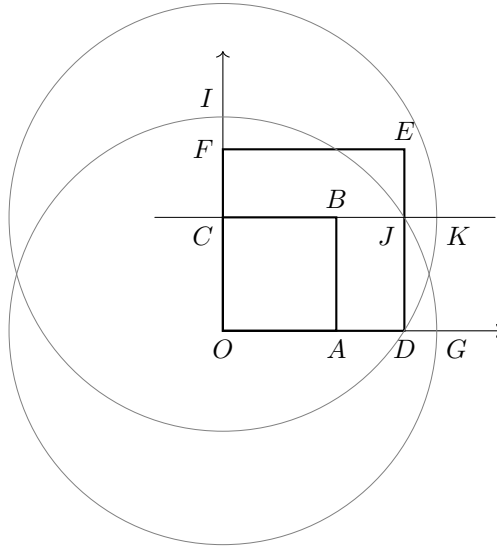
2. Draw perpendicular of horizontal line through  $G$ .

3. Draw perpendicular of vertical line through  $I$ .

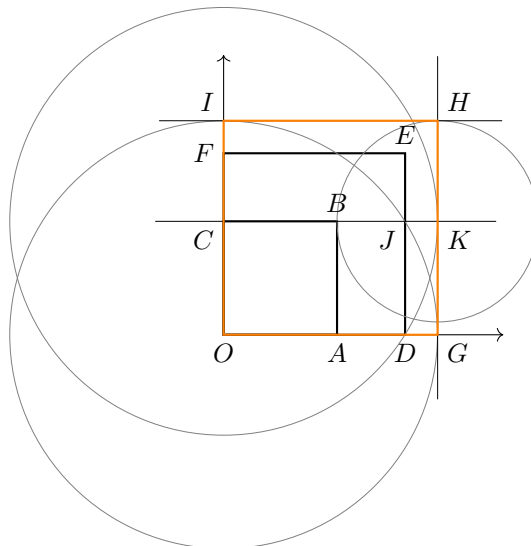


*Proof.* In  $\triangle OCD$ , by Pythagoras theorem, we have  $OC^2 + OD^2 = CD^2$ . Since  $CD = OG$  by construction, it means that area of square  $OC$  and square  $OD$  sum up to area of square  $OG$ .  $\square$

- (6E)** 1. Extend given side  $CB$  to be a line, intersecting side  $DE$  at  $J$ .  
 2. Draw circle  $(O, J)$ , intersecting horizontal ray at  $G$  and vertical ray at  $I$ .  
 3. Draw circle  $(C, D)$ , intersecting line  $CB$  at  $K$  (right).



4. Draw line  $GK$ .  
 5. Draw circle  $(K, B)$ , intersecting  $GK$  at  $H$  (top).  
 6. Draw line  $IH$ .  $OGHI$  is the desired square.



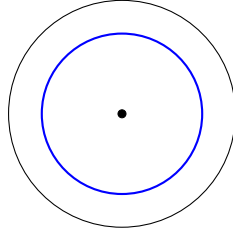
*Proof.* Note that  $OJ = CD$  (diags of rectangle), so  $CK = OG$ . Thus  $OGKC$  is a rectangle (1 equal pair, 2 right  $\angle$ s).

Thus by rectangle properties and radii,  $HG = HK + KG = BK + CO = OA + AG = OG$ . Thus  $OGHI$  is the same square as (3L).  $\square$

## 7.2 Annulus

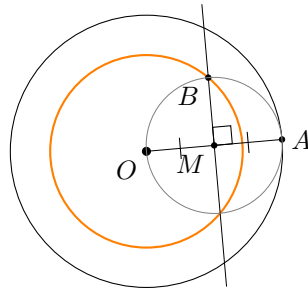
**Task 7.2.** Construct a circle that is concentric with the given one and divides it into 2 parts of equal area.

(4L, 5E)



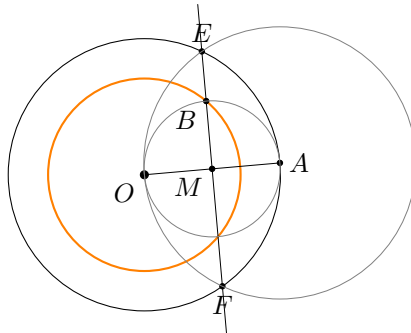
**Solution 7.2. (4L)** Let  $O$  be given circle center. Let  $A$  be an arbitrary point on given circle.

1. Draw line  $OA$ .
2. Draw perpbi  $OA$ . Let  $M$  be midpoint of  $OA$ .
3. Draw circle  $(M, O)$ , intersecting perpbi  $OA$  at one of points  $B$ .
4. Draw circle  $(O, B)$ , the desired circle.



*Proof.* Let the radius of given circle be  $r$ . Note that  $OB = \frac{r}{\sqrt{2}}$ . The area of given circle is  $\pi r^2$ , while the area of orange circle is  $\pi(\frac{r}{\sqrt{2}})^2 = \frac{1}{2}\pi r^2$ , which is half the area of given circle. Thus the orange circle divides given circle into two parts of equal area.  $\square$

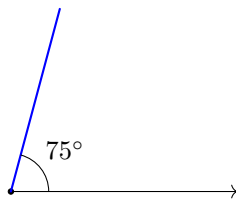
- (5E) 1.** Draw line  $OA$ .
2. Draw circle  $(A, O)$ , intersecting given circle at  $E$  and  $F$ .
  3. Draw line  $EF$ , intersecting  $OA$  at  $M$ .
  4. Draw circle  $(M, O)$ , intersecting  $EF$  at one point  $B$ .
  5. Draw circle  $(O, B)$ , the desired circle.



*Proof.* Note that  $EF$  is the perpendicular bisector of  $OA$ , so  $B$  is the same point as in (4L).  $\square$

### 7.3 Angle of 75 deg

**Task 7.3.** Construct an angle of  $75^\circ$  with the given side.  
(3L, 5E, 2V)

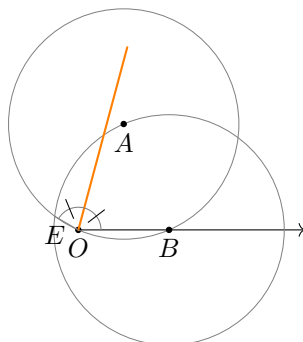


**Solution 7.3.** Let  $O$  be endpoint of given ray.

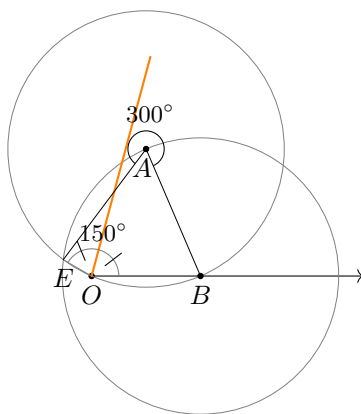
(3L) Let  $A$  be an arbitrary point directly above the given ray. **1.** Draw circle  $(A, O)$ , intersecting given ray again at  $B$ .

**2.** Draw circle  $(B, O)$ , intersecting  $(A, O)$  at  $E$  (left).

**3.** Draw angle bisector  $EOB$ , the desired line.

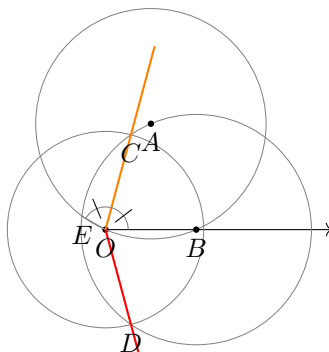


*Proof.* Note that  $\triangle ABE$  forms an equilateral triangle, so reflex  $\angle OAB = 360^\circ - 60^\circ = 300^\circ$ . By ( $\angle$  at centre twice  $\angle$  at  $\odot^{ce}$ ), we have  $\angle EOB = 150^\circ$ , so the angle bisector of  $\angle EOB$  gives  $75^\circ$ .



□

(2V) Let orange line cut  $(B, A)$  at  $C$ . Draw circle  $(O, C)$ , intersecting  $(B, A)$  again at  $D$ . Draw line  $OD$ , the extra solution.



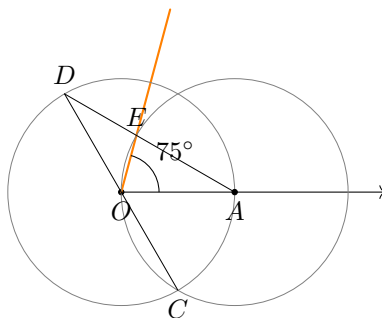
*Proof.*  $OCBD$  forms a kite, so  $D$  is the reflection of  $C$  over given ray. □

**(5E) 1, 2.** Let  $A$  be arbitrary point on given ray. Draw circles  $(O, A)$  and  $(A, O)$ , intersecting at  $C$  (bottom).

**3.** Draw line  $CO$ , meeting  $(O, A)$  at  $D$ .

**4.** Draw line  $DA$ , cutting  $(A, O)$  at  $E$ .

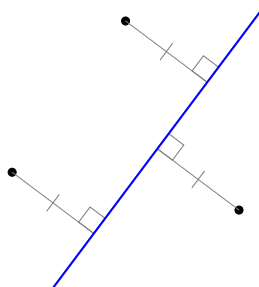
**5.** Draw line  $OE$ , the desired line.



*Proof.* Note that  $\angle AOC = 60^\circ$ , so  $\angle OAD = \angle ODA = 30^\circ$  (base  $\angle$ s, isos.  $\triangle$ ) & (ext.  $\angle$  of  $\triangle$ ). Look at  $\triangle AOE$ . Since  $AO = AE$  (radii), we have  $\angle AOE = \angle AEO$  (base  $\angle$ s, isos.  $\triangle$ ). Thus  $\angle AOE = (180^\circ - 30^\circ)/2 = 75^\circ$  ( $\angle$  sum of  $\triangle$ ). □

## 7.4 Line equidistant from three points

**Task 7.4.** Construct a line that is at equal distance from the given three points.  
(3L, 7E, 3V)

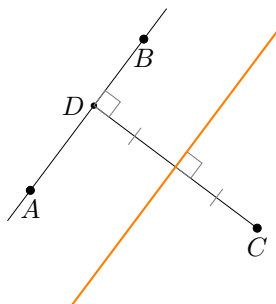


**Solution 7.4.** Let the given points be  $A, B, C$ .

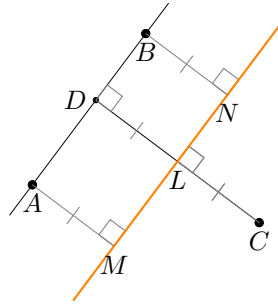
**(3L, 7E) 1.** Draw line  $AB$ .

**2.** Draw  $AB \perp DC$ . Let  $D$  be the foot of the perpendicular.

**3.** Draw perpbi  $DC$ , the desired line.

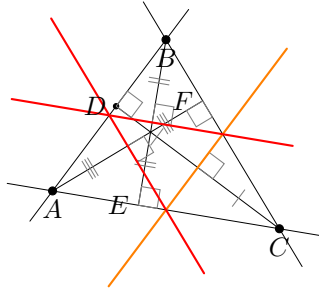


*Proof.* Let  $M, N, L$  be the projection of  $A, B, C$  on orange line. Note that  $AM = BN = DL$  by rectangle properties, and  $CL = DL$  by construction. Thus  $A, B, C$  have equal distance away from the orange line.



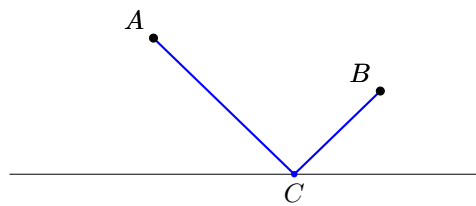
□

(3V) Do the same thing for the other two choices of initial line.



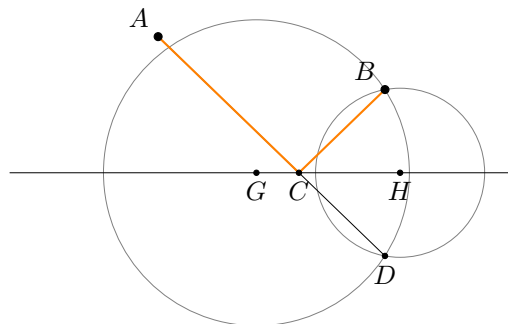
## 7.5 Heron's problem

**Task 7.5.** Construct a point  $C$  on the given line and segments  $AC$  and  $BC$  such that the sum of their length is minimal.  
(4L, 4E)



**Solution 7.5.** Let  $G, H$  be two arbitrary points on given line.

- 1, 2. Draw circle  $(G, B)$  and  $(H, B)$ , intersecting again at  $D$ .
3. Draw line  $AD$ , intersecting given line at  $C$ .
4. Draw line  $BC$ .

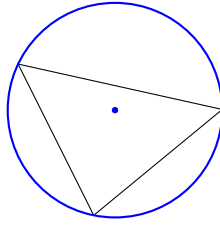


*Proof.* Note that  $D$  is the reflection of  $B$  over given line, so we have  $BC = DC$ .

The value of  $AC + CD$  reaches minimum when  $A, C, D$  forms a straight line (triangle inequality), so  $AC + CB$  (which =  $AC + CD$ ) also reaches minimum. □

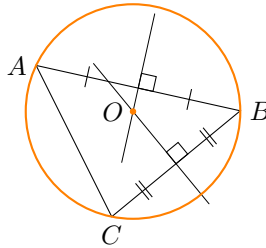
## 7.6 Circumscribed circle

**Task 7.6.** Construct the circumcircle of the triangle.  
(3L, 7E)



**Solution 7.6.** Let the given triangle be  $\triangle ABC$ .

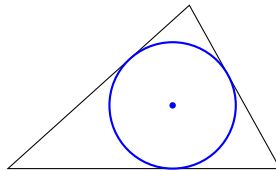
- 1, 2. Draw perpbi  $AB$  and perpbi  $BC$ , intersecting at  $O$ .
3. Draw circle  $(O, A)$ , the desired circumcircle.



*Proof.* By property of circumcenter, the center of circumcircle of a triangle is the intersection of the perpendicular bisector of the sides.  $\square$

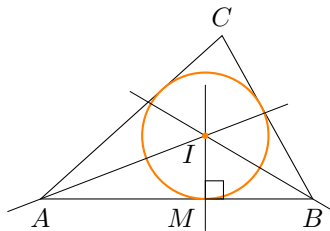
## 7.7 Inscribed circle

**Task 7.7.** Construct the incircle of the triangle.  
(4L, 8E)



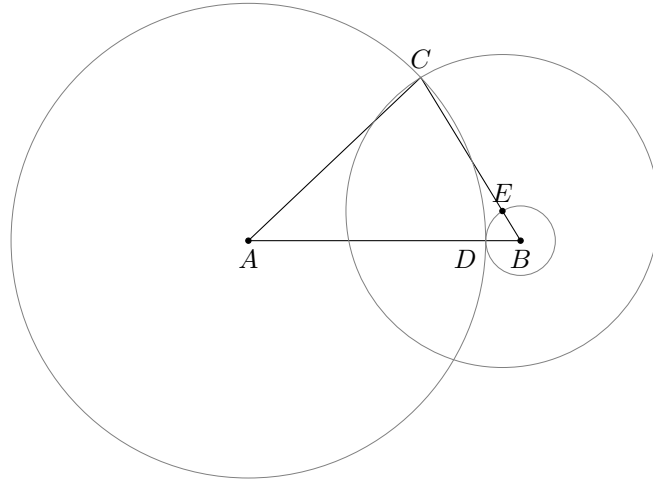
**Solution 7.7.** Let given triangle be  $\triangle ABC$ .

- (4L) 1, 2. Draw angbi  $CAB$  and angbi  $CBA$ , intersecting at  $I$ .
3. Draw  $AB \vdash P$ . Let  $M$  be the foot of perpendicular.
4. Draw circle  $(I, M)$ , the desired incircle.

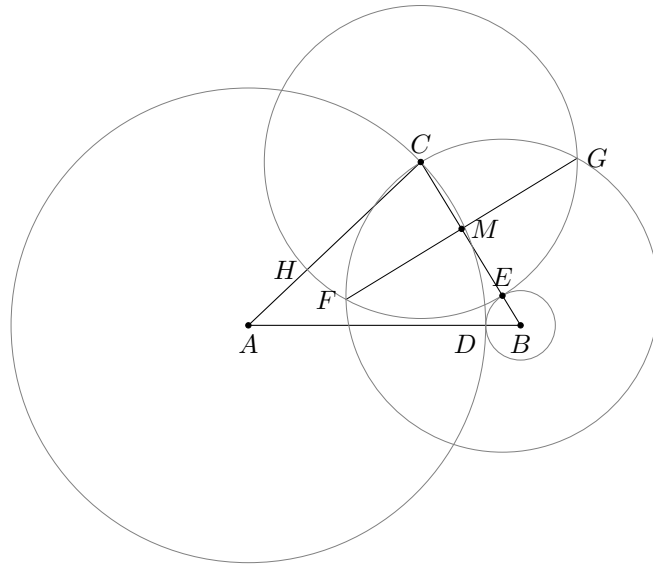


*Proof.* By property of incenter, the center of incircle is the intersection of angle bisector of the angles of triangle.  $\square$

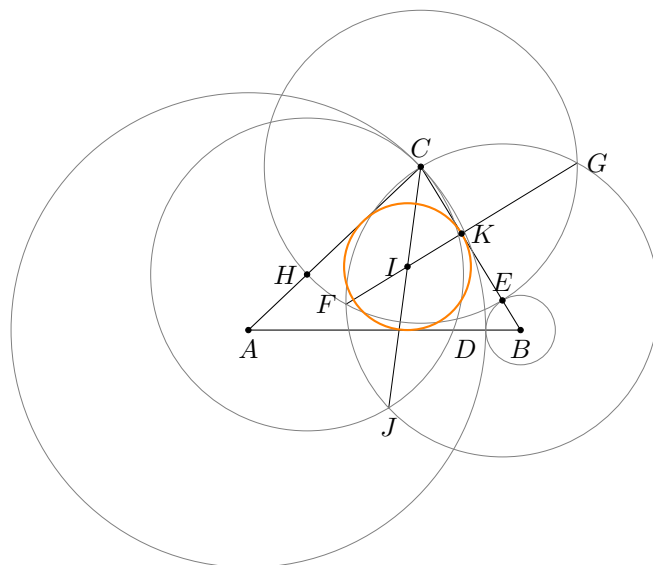
- (8E) 1. Draw circle  $(A, C)$ , intersecting side  $AB$  at  $D$ .
2. Draw circle  $(B, D)$ , intersecting side  $BC$  at  $E$ .
3. Draw circle  $(E, C)$ .



4. Draw circle  $(C, E)$ , intersecting  $(E, C)$  at  $F$  and  $G$ . Let  $(C, E)$  intersect side  $AC$  at  $H$ .
5. Draw line  $FG$ , intersecting  $CB$  at  $M$ .



6. Let  $(C, E)$  intersect side  $AC$  at  $H$ . Draw circle  $(H, C)$ , intersecting  $(E, C)$  again at  $J$ .
7. Draw line  $CJ$ , intersecting  $FG$  at  $I$ .
8. Let  $FG$  and  $CB$  intersect at  $K$ . Draw circle  $(I, K)$ , the desired incircle.



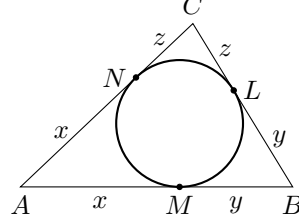
*Proof.* Let the true incircle's points of tangency on  $BC, AB, AC$  be  $L, M, N$  respectively. We want to show that points  $K$  and  $L$  are the same point by showing that  $CK = CL$ . This will help prove that the orange circle is indeed the true incircle.



Let's express  $CK$  in terms of  $AB, AC, BC$ . First we have  $BE = BD = AB - AC$ . Note that  $K$  bisects  $CE$ , so  $CK = \frac{BC - BE}{2}$ . This gives

$$\begin{aligned} CK &= \frac{BC - (AB - AC)}{2} \\ &= \frac{AC + BC - AB}{2} \end{aligned}$$

Let  $AM = AN = x$ ,  $BM = BK = y$ ,  $CN = CK = z$  by tangent properties. Let's express  $CL$  in terms of  $AB, BC, AC$ .



$$AB + AC + BC = 2x + 2y + 2z$$

$$AB + AC + BC = 2AB + 2z$$

$$\frac{AC + BC - AB}{2} = z$$

$$CL = \frac{AC + BC - AB}{2}$$

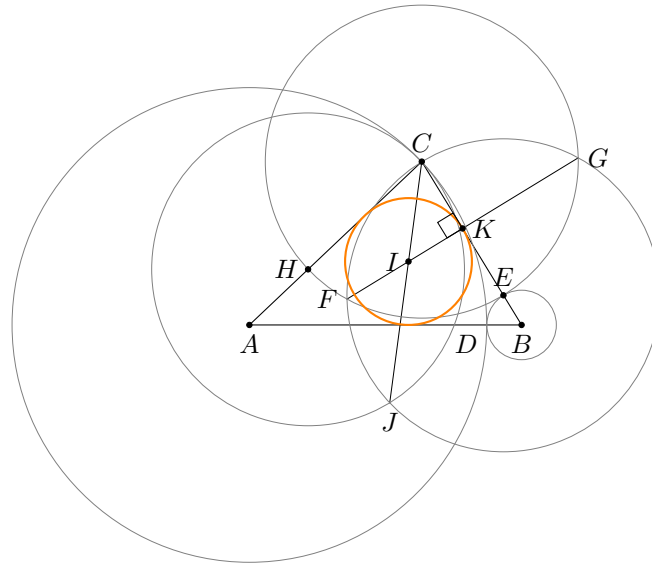
$$CL = CK$$

Thus  $K$  and  $L$  are the same point.

Also note that  $CHJE$  forms a rhombus since  $CH = HJ = JE = EC$  (radii).  $CJ$  is diagonal of the rhombus, so by property of rhombus,  $CJ$  is the angle bisector of  $\angle ACB$ .

Let  $I_t$  be the true incenter of  $\triangle ABC$ . We want to show that  $I$  is the same point as  $I_t$ . Note that  $\angle ICK = \angle I_t CK$  since  $I$  and  $I_t$  both lie on the same line  $CJ$ . Also,  $\angle CKI = 90^\circ$  because  $GF$  is the perpendicular bisector of  $CE$  by construction, and  $\angle CKI_t = 90^\circ$  by "tangent  $\perp$  radius".

Thus,  $\triangle CKI \cong \triangle CKI_t$  by (ASA), meaning that  $I$  is the same point as  $I_t$ . So  $(I, K)$  is the true incircle as desired.

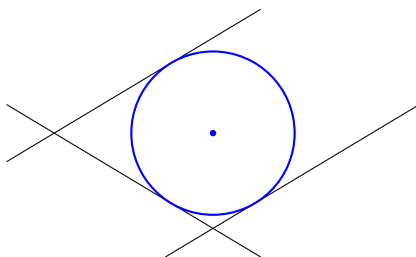


□

## 7.8 Circle tangent to three lines

**Task 7.8.** Construct a circle that is tangent to the three given lines. Two of the lines are parallel.

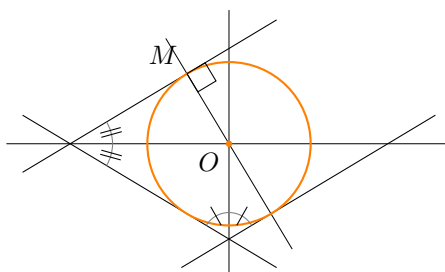
(4L, 6E, 2V)



**Solution 7.8.** (4L) **1, 2.** Draw the angle bisectors of the two intersections of given lines such that they intersect at point  $O$ .

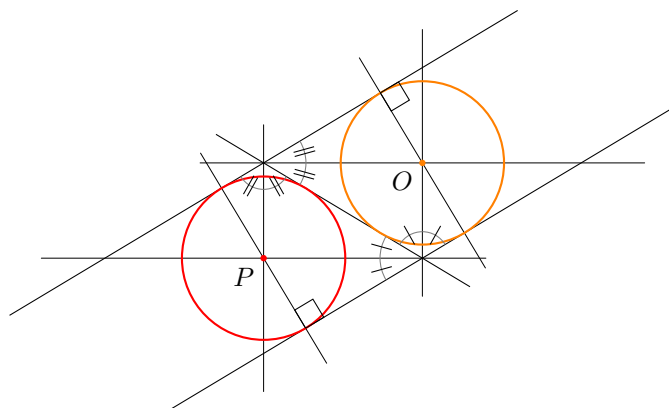
**3.** Draw the perpendicular of a given line through  $O$ . Let the foot of perpendicular be  $M$ .

**4.** Draw circle  $(O, M)$ , the desired circle.



*Proof.* By tangent properties, the center of the circle lies on the angle bisectors of the tangent lines. By converse of tangent  $\perp$  radius,  $M$  is a tangent point. Circle  $(O, M)$  will also be tangent to other given lines since  $O$  is equidistant from all the given lines (because the triangles formed by the lines are congruent by AAS.)  $\square$

(2V) Make similar constructions on the other side of the given intercept line.

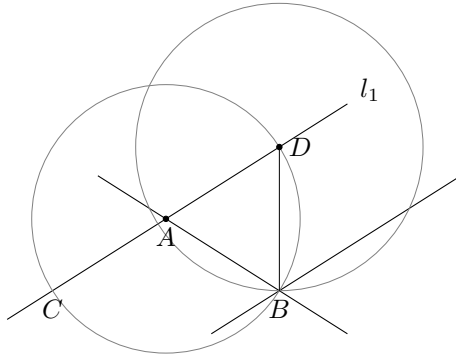


(6E) Label the top given lines  $l_1$ . Let  $A$  and  $B$  be the intersection of the given lines.

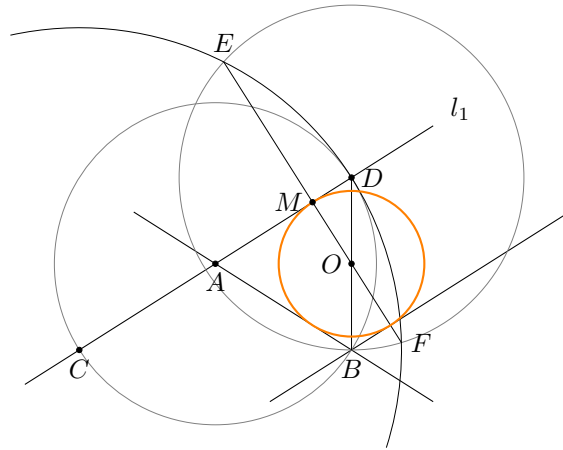
**1.** Draw circle  $(A, B)$ , intersecting  $l_1$  at  $D$  (right) and  $C$  (left).

**2.** Draw circle  $(D, B)$ .

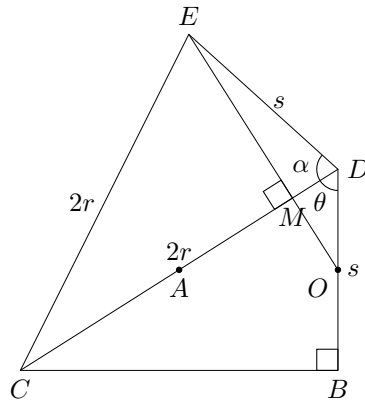
**3.** Draw line  $BD$ .



4. Draw circle  $(C, D)$ , intersecting  $(D, B)$  at  $E$  and  $F$ .
5. Draw line  $EF$ , intersecting  $l_1$  at  $M$ , and intersecting  $BD$  at  $O$ .
6. Draw circle  $(O, M)$ , the desired circle.



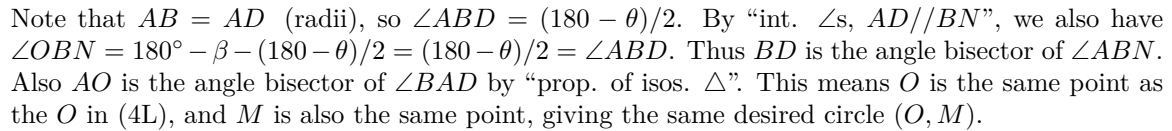
*Proof.* Let's focus inside  $\triangle CBD$  and  $\triangle CED$ . Let  $AD = AC = r$  and  $DB = DE = s$ . Then  $CD = CE = 2r$ . Let  $\angle EDM = \alpha$  and  $\angle BDM = \theta$ . Note that  $\angle EMD = 90^\circ$  because  $CEDF$  forms a kite, giving  $EM \perp CD$ .



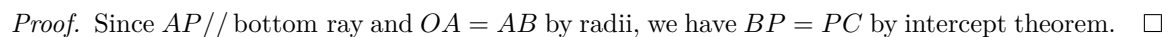
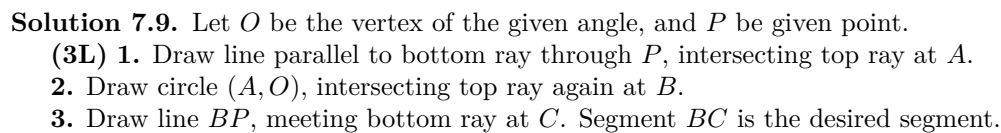
From right angle trigonometry, we have  $\cos \alpha = \frac{s/2}{2r}$  and  $\cos \theta = \frac{s}{2r}$ . Thus  $MD = s \cos \alpha = \frac{s^2}{4r}$ .

So  $OD = \frac{MD}{\cos \theta} = \left(\frac{s^2}{4r}\right)\left(\frac{2r}{s}\right) = \frac{s}{2}$ . This means  $O$  is the midpoint of  $DB$ .

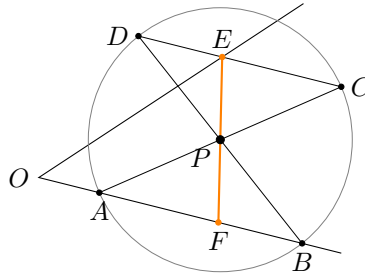
Let  $N$  be the point diametrically opposite from  $M$  on orange circle, and let  $\angle BAD = \beta$ .



**Task 7.9.** Construct a segment with the ends on the sides of the angle such that the given point is its midpoint.  
(3L, 5E)



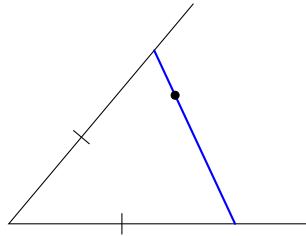
- 100



*Proof.* Note that  $AP = PC$  and  $DP = PB$  by radii, so  $ABCD$  forms a rectangle by “diags. equal and bisect each other”. Thus we have  $DC \parallel AB$  by “prop. of rectangle”. Since segment  $EF$  passes through  $P$  which is midway between  $DC$  and  $AB$ , by intercept theorem, we also have  $EP = PF$ , as desired.  $\square$

## 7.10 Angle isosceles

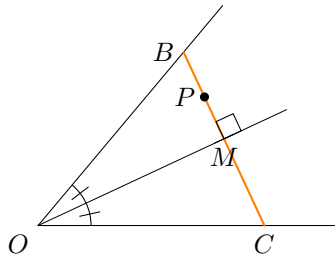
**Task 7.10.** Construct a line through the given point such that it cuts off equal segments on the sides of the angle.  
(2L, 5E)



**Solution 7.10.** Let  $O$  be vertex of given angle and  $P$  be given point.

(2L) 1. Draw the angle bisector of the two given rays. Let it be  $l$ .

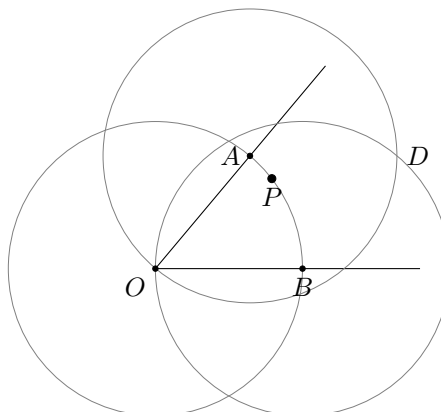
2. Draw  $l$  perpbi  $P$ , meeting top ray and bottom ray at  $B$  and  $C$  respectively.  $BC$  is the desired segment.



*Proof.*  $\triangle OMC \cong \triangle OMB$  by (ASA), so  $OB = OC$  (corr. sides,  $\cong \triangle$ s).  $\square$

(5E) 1. Draw circle  $(O, P)$ , intersecting given rays at  $A$  (top) and  $B$  (bottom).

2, 3. Draw circles  $(A, O)$  and  $(B, O)$ , intersecting again at  $D$ .

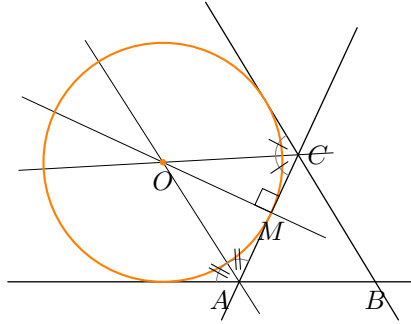




**Solution 7.11.** Let the triangle formed by given lines be  $\triangle ABC$ .

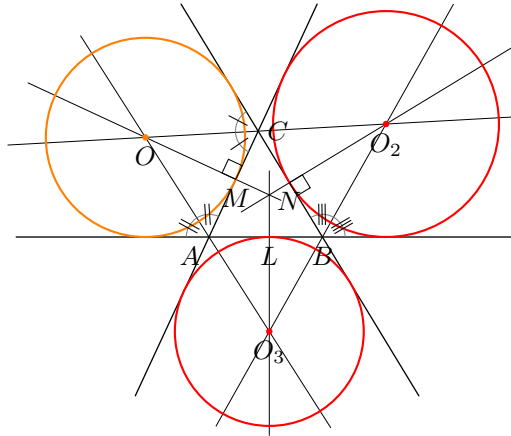
(4L) Let  $D$  and  $E$  be points on two given lines as below.

- 1, 2. Draw angbi  $DAC$  and angbi  $ACE$ , intersecting at  $O$ .
3. Draw  $AC \perp O$ . Let  $M$  be the foot of perpendicular.
4. Draw circle  $(O, M)$ , the desired circle.



*Proof.* By property of  $\angle$  bisector, point  $O$ , which is the intersection of the two angle bisectors of lines, is equidistant from the three lines. This means the circle that touches one of the line will also touch the other two lines.  $\square$

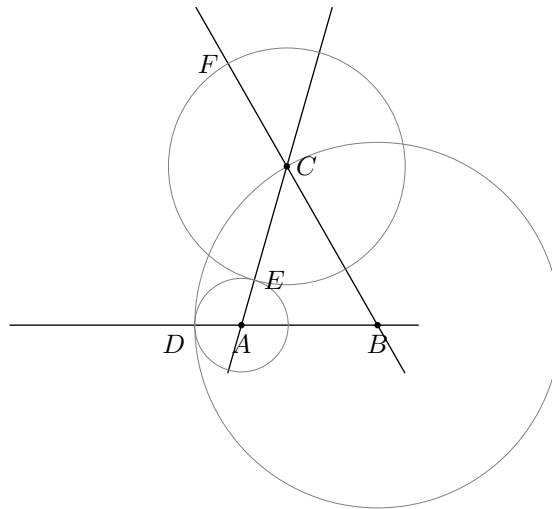
(3V) Make similar constructions in the other two exteriors of the triangle.



(8E) 1. Draw circle  $(B, C)$ , intersecting line  $AB$  at  $D$ .

2. Draw circle  $(A, D)$ , intersecting  $AC$  at  $E$ .

3. Draw circle  $(C, E)$ , intersecting line  $CB$  at  $F$  (top).

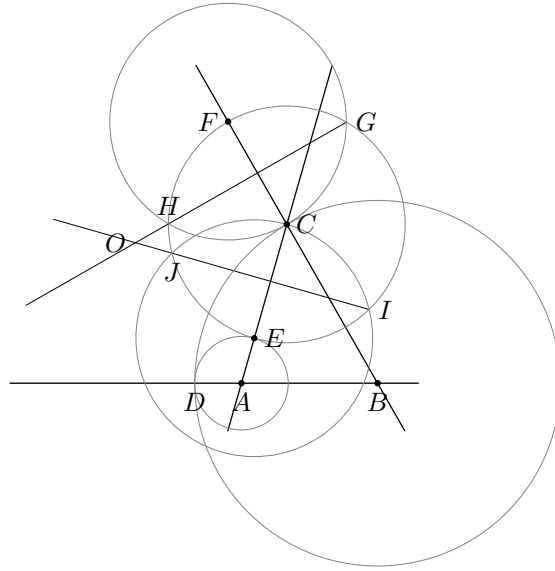


4. Draw circle  $(F, C)$ , intersecting  $(C, F)$  at  $G$  (right) and  $H$  (left).

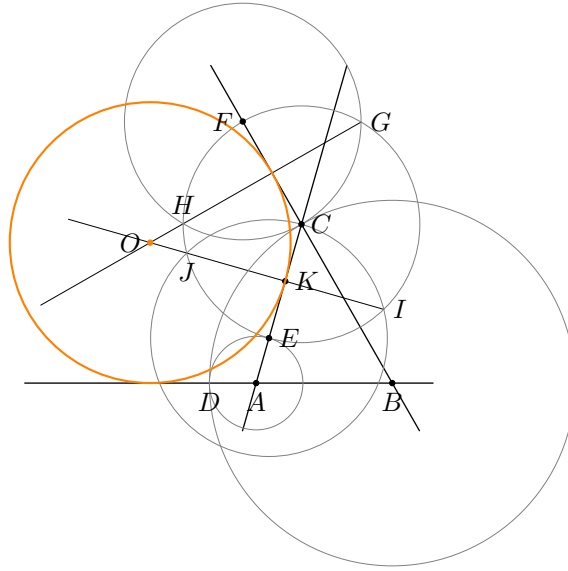
5. Draw line  $GH$ .

6. Draw circle  $(E, C)$ , intersecting  $(C, E)$  at  $I$  (right) and  $J$  (left).

7. Draw line  $IJ$ , intersecting  $GH$  at  $O$ .



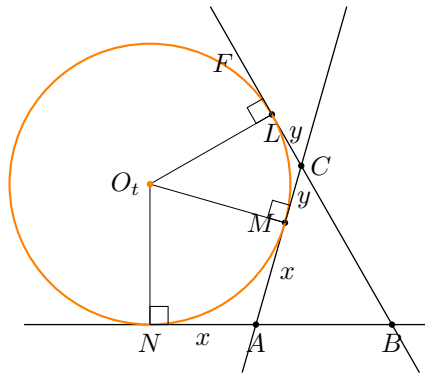
8. Let  $IJ$  and  $AC$  intersect at  $K$ . Draw circle  $(O, K)$ , the desired circle.



*Proof.* Note that  $AE = AD = BC - AB$  and  $EK = CK$ , so  $CK = \frac{AC - AE}{2} = \frac{AB + AC - BC}{2}$ .

Let the true excircle of the triangle touch line  $AC, AB, BC$  at  $M, N, L$  respectively. We want to show that  $K$  and  $M$  are the same point.

Let  $AN = AM = x$  and  $CM = CL = y$  by tangent properties. We will express  $y$  in terms of  $AB, AC$  and  $BC$  to show that  $CK = CM$ .



Note that  $BN = BL$  by tangent properties, which means  $AB + x = BC + y$ . Adding  $y$  to both sides



gives:

$$AB + x + y = BC + 2y$$

$$AB + AC = BC + 2y$$

$$y = \frac{AB + AC - BC}{2}$$

$$CM = CK$$

Thus  $M$  is the same point as  $K$ , and it is easy to see that the true excenter  $O_t$  is the same point as  $O$ .  $\square$

## References