Cubic Formula

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December 29, 2023

Abstract

Let's learn about the cubic formula. I wonder how people come up with this since I will never solve this in a million lifetimes.

Contents

1 Quadratic formula review

A quadratic equation is of the form

$$ax^2 + bx + c = 0$$

where solutions for x are sought after.

First, it is good practice to divide the equation by a.

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Then we can make the x term go away by letting $x = y - \frac{b}{2a}$:

$$\begin{aligned}
(y - \frac{b}{2a})^2 + \frac{b}{a}(y - \frac{b}{2a}) + \frac{c}{a} &= 0 \\
y^2 - \frac{b}{a}y + \frac{b^2}{4a^2} + \frac{b}{a}y + \frac{c}{a} &= 0 \\
y^2 + \frac{b^2}{4a^2} + \frac{c}{a} &= 0 \\
y &= \pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} \\
\text{Put back } x: \qquad x &= \frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} \\
x &= \frac{b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

We end up with the familiar quadratic formula.

2 Stuff about complex numbers

Since the cubic formula involves complex numbers, we should acquaint ourselves with some of their properties.

2.1 Basics

Square root of -1

The **imaginary unit** i is defined as $\sqrt{-1}$. We have

$$\begin{aligned} i^2 &= -1 \\ i^3 &= -i \\ i^4 &= 1 \\ i^5 &= i \\ &\vdots \\ i^k &= i^{k \bmod 4} \qquad \text{(k is integer)} \end{aligned}$$

An **imaginary number** is a real number multiplied by the imaginary unit i.
In general, for real number $D > 0$, $\sqrt{-D} = \sqrt{D}\, i$ is an imaginary number.

Note that for negative numbers $a, b < 0$, the property $\sqrt{ab} = \sqrt{a}\sqrt{b}$ is no longer true, as

$$\sqrt{ab} = \sqrt{(-|a|)(-|b|)} = \sqrt{|a||b|} = \sqrt{ab}$$
$$\sqrt{a}\sqrt{b} = \sqrt{-|a|}\sqrt{-|b|} = \sqrt{a}\, i\sqrt{b}\, i = -\sqrt{ab}$$

Complex number

A **complex number** z is expressed as $a + bi$, where a, b are real numbers. a is called **real part**, denoted by $\mathrm{Re}(z)$. b is called the **imaginary part**, denoted by $\mathrm{Im}(z)$.

In general, we treat i as a variable, like x, but with the special property that $i^2 = -1$.
Each complex number is associated with a unique pair of a and b. Namely, if $a + bi = c + di$, then $a = c$ and $b = d$.

For operations of two complex numbers $z_1 = a + bi$ and $z_2 = c + di$, we have

$$\begin{aligned} z_1 + z_2 &= (a + bi) + (c + di) = (a + c) + (b + d)i \\ z_1 - z_2 &= (a + bi) - (c + di) = (a - c) + (b - d)i \\ z_1 z_2 &= (a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (bc + ad)i \\ \frac{z_1}{z_2} &= \frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{ac - adi + bci - bdi^2}{c^2 - d^2 i^2} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} \end{aligned}$$

A complex number can be interpreted as a point or a vector on the complex plane. A complex plane is like a Cartesian plane, but the x-axis is replaced by the real axis, and the y-axis is replaced by the imaginary axis. And we can think of adding complex numbers as adding vectors.

Absolute value

The **absolute value / magnitude** of $z = a + bi$, denoted by $|z|$, is defined as $\sqrt{a^2 + b^2}$. The magnitude of a complex number must be a non-negative real number, i.e. $|z| \geq 0$.

For two complex numbers z_1, z_2, the absolute value of their product is the product of their absolute values. Namely,

$$|z_1 z_2| = |z_1||z_2|$$

Proof.

$$\begin{aligned} |z_1 z_2| &= |(a + bi)(c + di)| \\ &= |(ac + bd) + (bc - ad)i| \\ &= \sqrt{(ac + bd)^2 + (bc - ad)^2} \\ &= \sqrt{a^2c^2 + 2abcd + b^2d^2 + b^2c^2 - 2abcd + a^2d^2} \\ &= \sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2} \end{aligned}$$

$$
\begin{aligned}
|z_1||z_2| &= |a+bi||c+di| \\
&= \sqrt{a^2+b^2}\sqrt{c^2+d^2} \\
&= \sqrt{a^2c^2+a^2d^2+b^2c^2+c^2d^2} \\
&= |z_1z_2|
\end{aligned}
$$

□

Conjugates

For a complex number $z = a + bi$, its **conjugate**[1], denoted by $\bar{z}$ or z', is $a - bi$.

Every complex number has a conjugate, and the conjugate of z' is z itself. z and z' are called a conjugate pair, and they have the same magnitude. The geometric interpretation of taking conjugate of z is reflecting z about the real axis.

Note that the sum or product of a conjugate pair must be a real number:

$$
\begin{aligned}
z + z' &= (a+bi) + (a-bi) = 2a \\
zz' &= (a+bi)(a-bi) = a^2 - b^2 i = a^2 + b^2
\end{aligned}
$$

Note that the product of a conjugate pair is the square of their magnitude:

$$zz' = |z|^2 = a^2 + b^2$$

Note that taking conjugates of complex numbers before and after an operation is the same:

$$
\begin{aligned}
(z_1 + z_2)' &= z_1' + z_2' \\
(z_1 - z_2)' &= z_1' - z_2' \\
(z_1 z_2)' &= z_1' z_2' \\
(\frac{z_1}{z_2})' &= \frac{z_1'}{z_2'}
\end{aligned}
$$

Proof.

$$
\begin{aligned}
(z_1 + z_2)' &= ((a+c) + (b+d)i)' = (a+c) - (b+d)i \\
z_1' + z_2' &= a - bi + c - di = (a+c) - (b+d)i
\end{aligned}
$$

$$
\begin{aligned}
(z_1 - z_2)' &= ((a-c) + (b-d)i)' = (a-c) - (b-d)i \\
z_1' - z_2' &= (a-bi) - (c-di) = (a-c) - (b-d)i
\end{aligned}
$$

$$
\begin{aligned}
(z_1 z_2)' &= ((ac-bd) + (bc+ad)i)' = (ac-bd) - (bc+ad)i \\
z_1' z_2' &= (a-bi)(c-di) = (ac-bd) - (bc+ad)i
\end{aligned}
$$

$$
\begin{aligned}
(\frac{z_1}{z_2})' &= \left(\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}\right)' = \frac{(ac+bd)-(bc-ad)i}{c^2+d^2} \\
\frac{z_1'}{z_2'} &= \frac{a-bi}{c-di} = \frac{(a-bi)(c+di)}{(c-di)(c+di)} = \frac{(ac+bd)-(bc-ad)i}{c^2+d^2}
\end{aligned}
$$

□

Polar form

A complex number $z = a + bi$ can be expressed in the polar form:

$$z = r(\cos\theta + i\sin\theta)$$

[1] Also commonly called **complex conjugate**, but this is more verbose so we omit the word 'complex'.

where $r = \sqrt{a^2 + b^2}$ is the magnitude of z, and θ is the angle between the $x-$axis and the point of z in the complex plane (measured anticlockwise). This is analogous to the conversion from rectangular coordinates to polar coordinates.

Note: To compact the notation, we can write $z = r\,\text{cis}(\theta)$. To express θ in terms of z, we write $\theta = \arg(z)$. [2]

If we restrict θ so that $0 \leq \theta < 2\pi$, then each complex number (except 0) is associated with a unique pair of r and θ. Namely, if

$$r_1(\cos\theta_1 + i\sin\theta_1) = r_2(\cos\theta_2 + i\sin\theta_2)$$

, then $r_1 = r_2$ and $\theta_1 = \theta_2$.

Proof. Without loss of generality, assume $\theta_1 \leq \theta_2$. By comparing both sides, we have

$$r_1\cos\theta_1 = r_2\cos\theta_2 \tag{1}$$
$$r_1\sin\theta_1 = r_2\sin\theta_2 \tag{2}$$

(2) / (1):

$$\begin{aligned} \tan\theta_1 &= \tan\theta_2 \\ \theta_1 = \theta_2 \quad &\text{or} \quad \theta_2 = \theta_1 + \pi \end{aligned}$$

If $\theta_1 = \theta_2$ then it's clear that $r_1 = r_2$. (Just choose either equation such that the sin/cos value is non-zero and divide both sides by that). If $\theta_2 = \theta_1 + \pi$, then from (1),

$$\begin{aligned} r_1\cos\theta_1 &= r_2\cos(\theta_1 + \pi) \\ &= -r_2\cos\theta_1 \qquad \text{(ASTC rule in trigonometry)} \\ (r_1 + r_2)\cos\theta_1 &= 0 \end{aligned}$$

And from (2),

$$\begin{aligned} r_1\sin\theta_1 &= r_2\sin(\theta_1 + \pi) \\ &= -r_2\sin\theta_1 \qquad \text{(ASTC rule in trigonometry)} \\ (r_1 + r_2)\sin\theta_1 &= 0 \end{aligned}$$

Since $\cos\theta_1$ and $\sin\theta_1$ can't be both 0, it must be that $r_1 + r_2 = 0$. But $r_1, r_2 \geq 0$ because they are magnitudes, so $r_1 = r_2 = 0$, and the complex number is 0, which is a special case that can be associated with whatever θ value is.

□

Multiplication of complex numbers

We want to convert complex numbers into polar form because it has a nice property: To obtain the product of two complex numbers in polar form, we multiply their magnitudes and sum their angles. Namely,

Theorem 2.1.1. If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then the product of these numbers is given as:

$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

Proof.

$$\begin{aligned} z_1 z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2(\cos\theta_1\cos\theta_1 + i(\cos\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2) + i^2\sin\theta_1\sin\theta_2) \\ &= r_1 r_2(\cos\theta_1\cos\theta_1 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)) \\ &= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \qquad \text{(compound angle formula)} \end{aligned}$$

□

[2]Depending on the author's definition, $\arg(z)$ may be multi-valued or may refer to the principal value, where θ lies in the interval $[0, 2\pi)$ or $(-\pi, \pi]$.

Similarly, to obtain the quotient of two complex numbers in polar form, we divide their magnitudes and subtract the angle of divisor from dividend.

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$

To justify this, note that

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \\
&= \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)} \\
&= \frac{r_1(\cos\theta_1\cos\theta_1 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2))}{r_2(\cos^2\theta_2 - (-1)\sin^2\theta_2)} \\
&= \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)) \qquad \text{(compound angle formula and pyth. identity)}
\end{aligned}$$

2.2 De Moivre's formula

We can find a complex number raised to a power using **De Moivre's formula**.

Theorem 2.2.1. If $z = r(\cos\theta + i\sin\theta)$ and n is an integer, then

$$z^n = (r^n)(\cos(n\theta) + i\sin(n\theta))$$

This can be obtained by repeatedly applying the property of multiplication of complex numbers.

2.3 Complex roots

Roots of unity

Theorem 2.3.1. Suppose we have an equation in z and complex roots are sought for (where n is integer):

$$z^n = 1$$

Then there are n solutions. Namely, $z_0 = 1$, $z_1 = \text{cis}(1 \cdot \frac{2\pi}{n})$, $z_2 = \text{cis}(2 \cdot \frac{2\pi}{n})$, ... , $z_{n-1} = \text{cis}((n-1) \cdot \frac{2\pi}{n})$.

These are called the n-th roots of unity. Note that they all lie on the unit circle with a magnitude of 1, their positions forming a n-sided regular polygon.

For example, for the equation $z_3 = 1$, we have the 3rd roots of unity:
$z_0 = 1$, $z_1 = \text{cis}(\frac{2\pi}{3}) = -\frac{1}{2} + i\,\frac{\sqrt{3}}{2}$, $z_2 = \text{cis}(\frac{4\pi}{3}) = -\frac{1}{2} - i\,\frac{\sqrt{3}}{2}$.

Proof. To see why, first note that the magnitude of z must be 1, since only roots of magnitude 1 raised to a power will have a magnitude of 1. Express z in polar form to use De Moivre's formula, and note that 1 is basically $\cos(2k\pi) + i\sin(2k\pi)$ (for any integer k). We want to find the solutions to θ:

$$\begin{aligned}
(\cos\theta + i\sin\theta)^n &= 1 \\
\cos(n\theta) + i\sin(n\theta) &= \cos(2k\pi) + i\sin(2k\pi)
\end{aligned}$$

Comparing both sides, we have $n\theta = 2k\pi \Rightarrow \theta = k \cdot \frac{2\pi}{n}$. Putting k from 0 to $n-1$, we get n distinct values of z as roots, but for $k \geq n$, then θ makes a full revolution (i.e. $\theta \geq 2\pi$), so the values of z starts repeating.

To elaborate, let $k = mn + r$ where $0 \le r < n$ and m is integer. Then

$$\begin{aligned} z &= \cos((mn+r) \cdot \frac{2\pi}{n}) + i\sin((mn+r) \cdot \frac{2\pi}{n})) \\ &= \cos(2m\pi + r \cdot \frac{2\pi}{n}) + i\sin(2m\pi + r \cdot \frac{2\pi}{n}) \\ &= \cos(r \cdot \frac{2\pi}{n}) + i\sin(r \cdot \frac{2\pi}{n}) \end{aligned}$$

which must be one of the n values appeared before θ made a full revolution. Thus there are only n distinct roots for z. $\square$

Now, if we have $z^n = R$ for some real number R, then there are still n solutions, just scaled to a factor: $z_0 = \sqrt[n]{R}$, $z_1 = \sqrt[n]{R}\operatorname{cis}(1 \cdot \frac{2\pi}{n})$, $z_2 = \sqrt[n]{R}\operatorname{cis}(2 \cdot \frac{2\pi}{n})$, ... , $z_{n-1} = \sqrt[n]{R}\operatorname{cis}((n-1) \cdot \frac{2\pi}{n})$.

Roots of complex number

Now, what if we have $z^n = w$ for some given complex number $w = s(\cos\alpha + i\sin\alpha)$? Note that like the unity situation, w can also be $s(\cos(\alpha + 2k\pi) + i\sin(\alpha + 2k\pi))$.

Let $z = r\cos\theta + i\sin\theta$ and we seek for values of θ. Then

$$\begin{aligned} z^n &= w \\ r^n(\cos\theta + i\sin\theta)^n &= s(\cos(\alpha + 2k\pi) + i\sin(\alpha + 2k\pi)) \\ r^n(\cos(n\theta) + i\sin(n\theta)) &= s(\cos(\alpha + 2k\pi) + i\sin(\alpha + 2k\pi)) \end{aligned}$$

Comparing both sides, $r^n = s$ and $n\theta = \alpha + 2k\pi$, which gives us $r = \sqrt[n]{s}$ and

$$\theta = \frac{\alpha + 2k\pi}{n}$$

Like above, We will get n distinct values of θ for $k = 0, 1, 2, \ldots, n-1$, which will be all our solutions, as any higher k will make a full revolution and we get the same z values as before.

The z values we get are called n-th roots of the complex number.

Theorem 2.3.2. [1] Let n be an integer. The n-th roots of the complex number $z = r(\cos\theta + i\sin\theta)$ are given by

$$\sqrt[n]{r}(\cos(\frac{\theta + 2\pi k}{n}) + \sin(\frac{\theta + 2\pi k}{n}))$$

for $k = 0, 1, 2, \ldots, n-1$.

(Notice that the variable names are 'shifted' back to make it consistent with the source I copied from.)

Note: If we let $k = 0, -1, -2, \ldots, -n+1$ instead, we will still get the same set of n roots, since $k = -n+1$ can be made into $k = 2$, $k = -n+2$ to $k = 2$, and so on.

Also note: To interpret this geometrically, the roots are evenly spaced points on the circle with radius $\sqrt[n]{r}$. If we find one of the roots, we can find the other roots by inscribing a regular polygon in the circle with a vertex being the found root.

Principal root

The root of z whose real part is the largest, is called the **principal root**, and we will denote it $\sqrt[n]{z}$. If there are two roots with the same real part, then the root with the positive imaginary part is the principal root.

In this article, whenever we have the n-th radical $\sqrt[n]{z}$ where z is a complex number, we always mean the principal root. However, in other places, people may use $\sqrt[n]{z}$ to denote n possible values.

Note that the principal roots of a pair of complex conjugates are also conjugates. This arises from the following theorem:

Theorem 2.3.3. Let $z = r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \leq \pi$. Then the principal root of z is

$$\sqrt[n]{z} = \sqrt[n]{r}(\cos(\frac{\theta}{n}) + i\sin(\frac{\theta}{n}))$$

Proof. Let $n \geq 2$ be an integer. Let $z_0 = \sqrt[n]{r}(\cos(\frac{\theta}{n})+i\sin(\frac{\theta}{n}))$. Then the real part of z_0 is $\sqrt[n]{r}\cos(\frac{\theta}{n})$. We want to show that $\sqrt[n]{r}\cos(\frac{\theta}{n}) \geq \sqrt[n]{r}\cos(\frac{\theta+2\pi k}{n})$ for any other k in $(0, 1, 2, \ldots, n-1)$.

Suppose $0 \leq \theta \leq \pi$.

To show the above statement, it is sufficient to show that $\cos(\frac{\theta}{n}) \geq \cos\alpha$ for any $\alpha \in [\frac{\theta}{n}, \frac{\theta+2\pi(n-1)}{n}]$

From the bounds of θ, we have $0 \leq \frac{2\theta}{n} \leq \frac{2\pi}{n}$, so $\frac{\theta}{n} - \frac{2\theta}{n} \geq \frac{\theta}{n} - \frac{2\pi}{n}$.

Now, since $n \geq 2$, , we have $\frac{2\pi}{n} \leq \pi$, so $\frac{\theta}{n} - \frac{2\pi}{n} \geq \frac{\theta}{n} - \pi$, and since $\frac{\theta}{n} \geq 0$ we have $\frac{\theta}{n} - \pi \geq -\pi$. This means $\frac{\theta}{n} - \frac{2\pi}{n}$, $\frac{\theta}{n} - \frac{2\theta}{n}$ are in $[-\pi, 0]$.

Now, note that $\cos x$ is increasing in the interval $[-\pi, 0]$. This means if $x_1 \geq x_2$ then $\cos(x_1) \geq \cos(x_2)$ for $x_1, x_2 \in [0, \pi]$. So we have $\cos(\frac{\theta}{n} - \frac{2\theta}{n}) \geq \cos(\frac{\theta}{n} - \frac{2\pi}{n})$.

After simplifying LHS and adding 2π to the argument of RHS we have

$$\cos(\frac{\theta}{n}) \geq \cos(2\pi + (\frac{\theta}{n} - \frac{2\pi}{n}))$$
$$\cos(\frac{\theta}{n}) \geq \cos(\frac{\theta+2\pi(n-1)}{n}) \tag{3}$$

Finally, note that $\frac{\theta}{n}$ lies in the first quadrant while $\frac{\theta+2\pi(n-1)}{n}$ lies in the third or fourth quadrant (shown above, just added 2π). Since $\cos x$ is decreasing in the first and second quadrant, we have $\cos\alpha \leq \cos(\frac{\theta}{n})$ for any $\alpha \in [\frac{\theta}{n}, \pi]$. Similarly, since $\cos x$ is increasing in the third and fourth quadrant, we have $\cos\alpha \leq \cos(\frac{\theta+2\pi(n-1)}{n})$ for any $\alpha \in [\pi, \cos(\frac{\theta+2\pi(n-1)}{n})]$.

Combining this with inequality (3), we have $\cos\alpha \leq \cos(\frac{\theta}{n})$ for any $\alpha \in [\pi, \cos(\frac{\theta+2\pi(n-1)}{n})]$.

Thus, $\cos(\frac{\theta}{n}) \geq \cos\alpha$ for any $\alpha \in [\frac{\theta}{n}, \frac{\theta+2\pi(n-1)}{n}]$.

Now, suppose $-\pi < \theta < 0$. The situation is similar to above, just flipped about the real axis. Now $\frac{\theta}{n}$ is in the fourth quadrant, and we go clockwise for each decrement of k (because θ is negative). And so $\frac{\theta+2\pi(-n+1)}{n}$ is in the first or second quadrant.

I am worried about a situation: what if $-2\pi - \frac{\theta}{n} = \frac{\theta+2\pi(-n+1)}{n}$? Then $-2\theta = 2\pi \Rightarrow \theta = -\pi$. But this is impossible since θ is out of bounds. Thus we can safely assume that $\frac{\theta+2\pi(-n+1)}{n} > -2\pi - \frac{\theta}{n}$.

Since the first and second quadrant is decreasing, we have

$$\cos(\frac{\theta+2\pi(-n+1)}{n}) < \cos(-2\pi - \frac{\theta}{n}) = \cos(\frac{\theta}{n})$$

. Repeating the argument above we can show that $\cos(\frac{\theta}{n}) > \cos\alpha$ for any $\alpha \in [\frac{\theta+2\pi(-n+1)}{n}, \frac{\theta}{n})$. Thus $\sqrt[n]{r}\cos(\frac{\theta}{n}) > \sqrt[n]{r}\cos(\frac{\theta+2\pi k}{n})$ for any other k in $(0, -1, -2, \ldots, -n+1)$, and $\sqrt[n]{r}(\cos(\frac{\theta}{n}) + i\sin(\frac{\theta}{n}))$ is the principal root.

$\square$

This theorem is useful for showing the partially applicable radical distribution rule for complex numbers.

Theorem 2.3.4. Let n be an integer. For complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, if $-\pi < \theta_1,\ \theta_2 \le \pi$ and $-\pi < \theta_1 + \theta_2 \le \pi$, then

$$\sqrt[n]{z_1 z_2} = \sqrt[n]{z_1}\,\sqrt[n]{z_2}$$

Note: This rule doesn't work when $\theta_1 + \theta_2$ is out of bounds.

Proof. By the previous theorem, $\sqrt[n]{z_1} = \sqrt[n]{r_1}(\cos(\frac{\theta_1}{n}) + i\sin(\frac{\theta_1}{n}))$ and $\sqrt[n]{z_2} = \sqrt[n]{r_2}(\cos(\frac{\theta_2}{n}) + i\sin(\frac{\theta_2}{n}))$. And

$$\begin{aligned}
\sqrt[n]{z_1}\,\sqrt[n]{z_2} &= \sqrt[n]{r_1}(\cos(\frac{\theta_1}{n}) + i\sin(\frac{\theta_1}{n}))\,\sqrt[n]{r_2}(\cos(\frac{\theta_2}{n}) + i\sin(\frac{\theta_2}{n})) \\
&= \sqrt[n]{r_1}\,\sqrt[n]{r_2}(\cos(\frac{\theta_1+\theta_2}{n}) + i\sin(\frac{\theta_1+\theta_2}{n})) \\
\sqrt[n]{z_1 z_2} &= \sqrt[n]{r_1(\cos\theta_1 + i\sin\theta_1) r_2(\cos\theta_2 + i\sin\theta_2)} \\
&= \sqrt[n]{r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))} \\
&= \sqrt[n]{r_1}\,\sqrt[n]{r_2}(\cos(\frac{\theta_1+\theta_2}{n}) + i\sin(\frac{\theta_1+\theta_2}{n}))
\end{aligned}$$

$\square$

From this theorem, we can see that the radical distribution rule always works for a pair of complex conjugates, since the sum of their angles must be 0.

Theorem 2.3.5. Let n be an integer and z be a complex number. Then

$$(\sqrt[n]{z})' = \sqrt[n]{z'}$$

In other words, the conjugate of a number's principal root is the principal root of the original number's conjugate.

Proof. Let $z = r(\cos\theta + i\sin\theta)$ where $-\pi < \theta < \pi$. Then $z' = r(\cos\theta - i\sin\theta) = r(\cos\theta + i\sin(-\theta))$ and

$$\begin{aligned}
(\sqrt[n]{z})' &= \sqrt[n]{r}\,(\cos(\frac{\theta}{n})) + i\sin(\frac{\theta}{n}))' \\
&= \sqrt[n]{r}\,(\cos(\frac{\theta}{n})) - i\sin(\frac{\theta}{n})) \\
\sqrt[n]{z'} &= \sqrt[n]{r}\,(\cos(\frac{\theta}{n})) + i\sin(\frac{-\theta}{n})) \\
&= \sqrt[n]{r}\,(\cos(\frac{\theta}{n})) - i\sin(\frac{\theta}{n}))
\end{aligned}$$

If $\theta = \pi$ then $\sin\theta = 0$, so $z = z' = r\cos\theta$ and

$$\begin{aligned}
(\sqrt[n]{z})' &= \sqrt[n]{r}(\cos(\frac{\theta}{n}))' \\
&= \sqrt[n]{r}\cos(\frac{\theta}{n}) \\
&= \sqrt[n]{z'}
\end{aligned}$$

$\square$

3 Cubic formula

A cubic equation is of the form

$$ax^3 + bx^2 + cx + d = 0$$

where solutions for x are sought for.

3.1 Derivation

[2] First, it is a good practice to divide the equation by a.

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$$

Then we can make the x^2 term go away by letting $x = y - \frac{b}{3a}$:

$$\begin{aligned}
(y - \frac{b}{3a})^3 + \frac{b}{a}(y - \frac{b}{3a})^2 + \frac{c}{a}(y - \frac{b}{3a}) + \frac{d}{a} &= 0 \\
y^3 - 3(\frac{b}{3a})y^2 + 3(\frac{b}{3a})^2 y - (\frac{b}{3a})^3 + \frac{b}{a}(y^2 - \frac{2b}{3a}y + \frac{b^2}{9a^2}) + \frac{c}{a}y - \frac{bc}{3a^2} + \frac{d}{a} &= 0 \\
y^3 + (-\frac{b}{a} + \frac{b}{a})y^2 + (\frac{b^2}{3a^2} - \frac{2b^2}{3a^2} + \frac{c}{a})y + (-\frac{b^3}{27a^3} + \frac{b^3}{9a^3} - \frac{bc}{3a^2} + \frac{d}{a}) &= 0 \\
y^3 + (\frac{c}{a} - \frac{b^2}{3a^2})y + (\frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a}) &= 0
\end{aligned}$$

Let $p = \frac{c}{a} - \frac{b^2}{3a^2}$, $q = \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a}$. Then

$$y^3 + py + q = 0$$

This is known as a **depressed cubic** (as it gives me depression).

Now use Vieta's substitution to get rid of the y term. Let $y = z - \frac{p}{3z}$.

$$\begin{aligned}
(z - \frac{p}{3z})^3 + p(z - \frac{p}{3z}) + q &= 0 \\
z^3 - 3z^2\frac{p}{3z} + 3z(\frac{p}{3z})^2 - (\frac{p}{3z})^3 + pz - \frac{p^2}{3z} + q &= 0 \\
z^3 - \frac{p^3}{27z^3} + q &= 0 \\
(z^3)^2 + qz^3 - \frac{p^3}{27} &= 0 \\
z^3 &= -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \qquad (4)
\end{aligned}$$

The cube root of a number has 3 possible complex values. Together with the $\pm$ sign, we have like, six choices in total for the value of z. Let's pick the $+$ sign for now. But we'll eventually see that no matter what we choose for the $\pm$ sign, there can only be three distinct y values. Consider the principal root first.

Take

$$z_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

Then

$$y_1 = z_1 - \frac{p}{3z_1}$$

$$y_1 = \sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} - \frac{p}{3\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}}$$

$$= \sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} - \frac{p\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}}{3\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}}$$

In above, we rationalize by multiplying both sides of fraction by the conjugate of the denominator.

Since the denominator is a conjugate pair, we can use the radical distribution rule:

$$y_1 = \sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} - \frac{p\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}}{3\sqrt[3]{\left(-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right)\left(-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right)}}$$

$$= \sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} - \frac{p\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}}{3\sqrt[3]{\frac{q^2}{4}-\left(\frac{q^2}{4}+\frac{p^3}{27}\right)}} \qquad \text{by}(a+b)(a-b)=a^2-b^2$$

$$= \sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} - \frac{p\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}}{3(-\frac{p}{3})}$$

$$= \sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$$

Now consider other two roots of z^3. Recall that we have picked the positive sign for equation (4):

$$z^3 = -\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}$$

By Theorem 2.3.2, z_2 and z_3 are just z_1 rotated by $120°$ ($\frac{2\pi}{3}$ rad) and $240°$ ($\frac{4\pi}{3}$ rad) in the complex plane respectively. To rotate z_1 by $120°$, we can multiply z_1 by $\cos 120° + i\sin 120°$, which is $-\frac{1}{2}+\frac{i\sqrt{3}}{2}$. So

$$z_2 = (-\frac{1}{2}+\frac{i\sqrt{3}}{2})z_1 = (-\frac{1}{2}+\frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$$

Similarly, to rotate z_1 by $240°$, multiply z_1 by $\cos 240° + i\sin 240° = -\frac{1}{2}-\frac{i\sqrt{3}}{2}$. So

$$z_3 = (-\frac{1}{2}-\frac{i\sqrt{3}}{2})z_1 = (-\frac{1}{2}-\frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$$

Then $y_2 = z_2 - \frac{p}{3z_2}$ and $y_3 = z_3 - \frac{p}{3z_3}$.

$$y_2 = (-\frac{1}{2} + \frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \frac{p}{3(-\frac{1}{2} + \frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}}$$

We have to rationalize to get rid of both the cube root and the complex number in the denominator.

$$\begin{aligned}
y_2 &= (-\frac{1}{2} + \frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \frac{p(-\frac{1}{2} - \frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}}{3(-\frac{1}{2} + \frac{i\sqrt{3}}{2})(-\frac{1}{2} - \frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}} \\
&= (-\frac{1}{2} + \frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \frac{p(-\frac{1}{2} - \frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}}{3(\frac{1}{4} + \frac{3}{4})\sqrt[3]{\frac{q^2}{4} - \left(\frac{q^2}{4} + \frac{p^3}{27}\right)}} \\
&= (-\frac{1}{2} + \frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + (-\frac{1}{2} - \frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}
\end{aligned}$$

And similarly for y_3,

$$\begin{aligned}
y_3 &= (-\frac{1}{2} - \frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \frac{p}{3(-\frac{1}{2} - \frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}} \\
&= (-\frac{1}{2} - \frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \frac{p(-\frac{1}{2} + \frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}}{3(-\frac{1}{2} - \frac{i\sqrt{3}}{2})(-\frac{1}{2} + \frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}} \\
&= (-\frac{1}{2} - \frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \frac{p(-\frac{1}{2} + \frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}}{3(\frac{1}{4} + \frac{3}{4})\sqrt[3]{\frac{q^2}{4} - \left(\frac{q^2}{4} + \frac{p^3}{27}\right)}} \\
&= (-\frac{1}{2} - \frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + (-\frac{1}{2} + \frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}
\end{aligned}$$

Now, if we choose the $-$ sign for equation (4) instead, then we have

$$z_1 = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

$$z_2 = (-\frac{1}{2} + \frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

$$z_3 = (-\frac{1}{2} - \frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

And now calculate for y_1,

$$
\begin{aligned}
y_1 &= \sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}-\frac{p}{3\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}} \\
&= \sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}-\frac{p\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}}{3\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}} \\
&= \sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}-\frac{p\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}}{3\sqrt[3]{\frac{q^2}{4}-\left(\frac{q^2}{4}+\frac{p^3}{27}\right)}} \qquad \text{by}(a+b)(a-b)=a^2-b^2 \\
&= \sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}
\end{aligned}
$$

We get the same y_1 value as $+$ sign equation. What about the other two roots?

$$
\begin{aligned}
y_2 &= (-\frac{1}{2}+\frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}-\frac{p}{3(-\frac{1}{2}+\frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}\frac{p^3}{27}}}} \\
&= (-\frac{1}{2}+\frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+(-\frac{1}{2}-\frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}
\end{aligned}
$$

Interestingly, we get the value of y_3 from $+$ sign equation.

And now calculate the new y_3,

$$
\begin{aligned}
y_3 &= (-\frac{1}{2}-\frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}-\frac{p}{3(-\frac{1}{2}-\frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}\frac{p^3}{27}}}} \\
&= (-\frac{1}{2}-\frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+(-\frac{1}{2}+\frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}
\end{aligned}
$$

We get the value of y_2 from $+$ sign equation. So choosing $+$ or $-$ will give the same three roots, just in different order.

After finding y, we can find x very easily by adding $-\dfrac{b}{3a}$.

And there we have it, the cubic formula.

Theorem 3.1.1 (Cubic Formula)**.** For a general cubic equation

$$ax^3 + bx^2 + cx + d = 0$$

let $p = \dfrac{c}{a} - \dfrac{b^2}{3a^2}$, $q = \dfrac{2b^3}{27a^3} - \dfrac{bc}{3a^2} + \dfrac{d}{a}$. Then the solution to the equation is

$$x_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \frac{b}{3a}$$

$$x_2 = (-\frac{1}{2} + \frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + (-\frac{1}{2} - \frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \frac{b}{3a}$$

$$x_3 = (-\frac{1}{2} - \frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + (-\frac{1}{2} + \frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \frac{b}{3a}$$

If we don't use p and q, the formulas are even more ugly:

$$x_1 = \sqrt[3]{-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a} + \sqrt{(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a})^2 + (\frac{c}{a} - \frac{b^2}{3a^2})^2}} + \sqrt[3]{-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a} - \sqrt{(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a})^2 + (\frac{c}{a} - \frac{b^2}{3a^2})^2}} - \frac{b}{3a}$$

$$x_2 = (-\frac{1}{2} + \frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a} + \sqrt{(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a})^2 + (\frac{c}{a} - \frac{b^2}{3a^2})^2}} + (-\frac{1}{2} - \frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a} - \sqrt{(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a})^2 + (\frac{c}{a} - \frac{b^2}{3a^2})^2}} - \frac{b}{3a}$$

$$x_3 = (-\frac{1}{2} - \frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a} + \sqrt{(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a})^2 + (\frac{c}{a} - \frac{b^2}{3a^2})^2}} + (-\frac{1}{2} + \frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a} - \sqrt{(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a})^2 + (\frac{c}{a} - \frac{b^2}{3a^2})^2}} - \frac{b}{3a}$$

However, after we obtained these formulas, we are not done yet. The formulas are very ugly, and perhaps there are things we can do to simplify the result.

3.2 Further processing

Now, there are three cases depending on the value of $\Delta = \dfrac{q^2}{4} + \dfrac{p^3}{27}$, which I'll call the cubic determinant:

<u>Case 1:</u> $\dfrac{q^2}{4} + \dfrac{p^3}{27} < 0$

Let $w = z^3 = -\dfrac{q}{2} + \sqrt{\dfrac{q^2}{4} + \dfrac{p^3}{27}} = -\dfrac{q}{2} + i\sqrt{-\dfrac{q^2}{4} - \dfrac{p^3}{27}}$.

Then w is a complex number with the absolute value

$$|w| = \sqrt{\frac{q^2}{4} + (-\frac{q^2}{4} - \frac{p^3}{27})} = \sqrt{-\frac{p^3}{27}}$$

And $y_1 = \sqrt[3]{w} + \sqrt[3]{w'}$. From Theorem 2.3.5 we know that $\sqrt[3]{w} + \sqrt[3]{w'} = \sqrt[3]{w} + (\sqrt[3]{w})' = z_1 + z_1'$ is also a pair of conjugate. This means their imaginary part must cancel out, leaving only a real number.

And thus $y_1 = 2\,\mathrm{Re}(z_1)$.

To find $\mathrm{Re}(z_1)$, we can first convert w into polar form: $w = |w|(\cos\theta + i\sin\theta)$ where $0 < \theta \leq \pi$. This θ can be obtained by $\cos\theta = \dfrac{-\frac{q}{2}}{\sqrt{-\frac{p^3}{27}}} \Rightarrow \theta = \arccos(\dfrac{-\frac{q}{2}}{\sqrt{-\frac{p^3}{27}}})$

By Theorem 2.3.3, $z_1 = \sqrt[3]{w} = \sqrt[3]{\sqrt{-\dfrac{p^3}{27}}(\cos(\dfrac{\theta}{3}) + i\sin(\dfrac{\theta}{3}))} = \sqrt{-\dfrac{p}{3}}(\cos(\dfrac{\theta}{3}) + i\sin(\dfrac{\theta}{3}))$

And so $\mathrm{Re}(z_1) = \sqrt{-\dfrac{p}{3}}\cos(\dfrac{\theta}{3}) = \sqrt{-\dfrac{p}{3}}\cos\left(\dfrac{1}{3}\arccos(\dfrac{-\frac{q}{2}}{\sqrt{-\frac{p^3}{27}}})\right)$.

To find the imaginary part of z_1, we can do a similar process, and we will have

$$\mathrm{Im}(z_1) = \sqrt{-\frac{p}{3}}\sin(\frac{\theta}{3}) = \sqrt{-\frac{p}{3}}\sin\left(\frac{1}{3}\arccos(\frac{-\frac{q}{2}}{\sqrt{-\frac{p^3}{27}}})\right)$$

For convenience, let's denote $\mathrm{Re}(z_1)$ by R and denote $\mathrm{Im}(z_1)$ by I, so $z_1 = R + Ii$.

Now, express y_2 and y_3 in terms of R and I:

$$\begin{aligned}
y_2 &= (-\frac{1}{2} + \frac{i\sqrt{3}}{2})\, z_1 + (-\frac{1}{2} - \frac{i\sqrt{3}}{2})\, z_1' \\
&= (-\frac{1}{2} + \frac{i\sqrt{3}}{2})(R + Ii) + (-\frac{1}{2} - \frac{i\sqrt{3}}{2})\,(R - Ii) \\
&= -R - I\sqrt{3}
\end{aligned}$$

$$\begin{aligned}
y_3 &= (-\frac{1}{2} - \frac{i\sqrt{3}}{2})\, z_1 + (-\frac{1}{2} + \frac{i\sqrt{3}}{2})\, z_1' \\
&= (-\frac{1}{2} - \frac{i\sqrt{3}}{2})(R + Ii) + (-\frac{1}{2} + \frac{i\sqrt{3}}{2})\,(R - Ii) \\
&= -R + I\sqrt{3}
\end{aligned}$$

Note that all three roots are real numbers. That means the cubic has three real solutions.

<u>Case 2:</u> $\frac{q^2}{4} + \frac{p^3}{27} = 0$

The formula above applies to this case, but there is a simpler formula.

Then $w = -\frac{q}{2}$ does not have an imaginary part and is a real number. So

$z_1 = \sqrt[3]{w} = \sqrt[3]{-\frac{q}{2}}$

$$\begin{aligned}
y_1 &= 2\sqrt[3]{-\frac{q}{2}} = -2\sqrt[3]{\frac{q}{2}} \\
y_2 &= -\sqrt[3]{-\frac{q}{2}} = \sqrt[3]{\frac{q}{2}} \\
y_3 &= -\sqrt[3]{-\frac{q}{2}} = \sqrt[3]{\frac{q}{2}}
\end{aligned}$$

We see that we have repeated root $y_2 = y_3$.

<u>Case 3:</u> $\frac{q^2}{4} + \frac{p^3}{27} > 0$

Then $w = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$ and $w' = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$ are real numbers, so we can simply calculate the cube root normally.

$$y_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

Note that $z_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$ is a real number, and $\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$ (which ceases to be the complex conjugate of z_1) is also a real number, so y_1 is also a real number.

For y_2 and y_3, we have

$$
\begin{aligned}
y_2 &= (-\frac{1}{2}+\frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+(-\frac{1}{2}-\frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} \\
&= -\frac{1}{2}\left(\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}\right) \\
&+\left(\frac{\sqrt{3}}{2}\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}-\frac{\sqrt{3}}{2}\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}\right)i
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
y_3 &= (-\frac{1}{2}-\frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+(-\frac{1}{2}+\frac{i\sqrt{3}}{2})\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} \\
&= -\frac{1}{2}\left(\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}\right) \\
&+\left(-\frac{\sqrt{3}}{2}\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\frac{\sqrt{3}}{2}\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}\right)i
\end{aligned}
$$

Since $\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} \neq \sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$, the imaginary part will never be 0, so y_2 and y_3 will always be complex numbers. This means we have only one real root y_1.

3.3 Process of finding roots of depressed cubic

To summarize, the steps to find the roots of the depressed cubic $y^3+py+q=0$ are: [3]

1. Calculate the cubic determinant $\Delta = \frac{q^2}{4}+\frac{p^3}{27}$ and check whether it is negative (case a), zero (case b) or positive (case c).

2(a). If $\Delta < 0$, let $w = -\frac{q}{2}+i\sqrt{-\frac{q^2}{4}-\frac{p^3}{27}}$.

3(a). Let $z_1 = \sqrt[3]{w}$ be the principal cube root of w. Let R be the real part and I be the imaginary part of z_1. We have

$$R = \sqrt{-\frac{p}{3}}\cos\left(\frac{1}{3}\arccos(\frac{-\frac{q}{2}}{\sqrt{-\frac{p^3}{27}}})\right)$$

$$I = \sqrt{-\frac{p}{3}}\sin\left(\frac{1}{3}\arccos(\frac{-\frac{q}{2}}{\sqrt{-\frac{p^3}{27}}})\right)$$

4(a). The three real roots are $-2R$, $-R+\sqrt{3}I$ and $-R-\sqrt{3}I$.

2(b) If $\Delta = 0$, then the three real roots are $-2\sqrt[3]{\frac{q}{2}}$, $\sqrt[3]{\frac{q}{2}}$, $\sqrt[3]{\frac{q}{2}}$, with $\sqrt[3]{\frac{q}{2}}$ being the repeated root.

2(c) If $\Delta > 0$, then there is only one real root $\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$.

The two complex roots are

$$-\frac{1}{2}\left(\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}\right)+\left(\frac{\sqrt{3}}{2}\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}-\frac{\sqrt{3}}{2}\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}\right)i$$

and

$$-\frac{1}{2}\left(\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}\right)+\left(-\frac{\sqrt{3}}{2}\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\frac{\sqrt{3}}{2}\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}\right)i$$

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That's it. Now I can rest in peace knowing how to derive the cubic formula. And I can finally understand the calculator program for finding roots of cubic equation.

References

[1] Ted Sundstrom & Steven Schlicker, "Demoivre's theorem and powers of complex numbers." [Online]. Available: https://math.libretexts.org/Bookshelves/Precalculus/Book%3A_Trigonometry_(Sundstrom_and_Schlicker)/05%3A_Complex_Numbers_and_Polar_Coordinates/5.03%3A_DeMoivres_Theorem_and_Powers_of_Complex_Numbers

[2] blackpenredpen, "Cubic equation (full version)." [Online]. Available: https://www.youtube.com/watch?v=ULsPhWmqhyc&ab_channel=blackpenredpen

[3] Kevin Wortman, "The cubic formula." [Online]. Available: https://www.math.utah.edu/~wortman/1060text-tcf.pdf