Toddler Geometry (Problem set)

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Abstract

Geometry problems are harder than they seem.

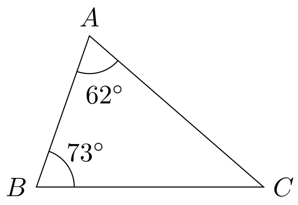
Contents

1 Lines, angles and shapes

After all the preposition stating, let's try some practical problems. (The diagrams in the problems are not necessarily to scale.)

1.1 Basic properties

Problem 1. In $\triangle ABC$, $\angle A = 62°$ and $\angle B = 73°$. What is $\angle C$?

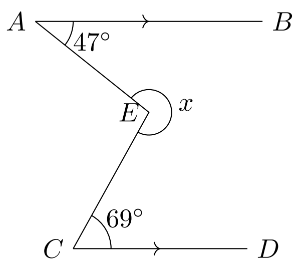


(Difficulty: 1 [Beginner])

Solution 1.

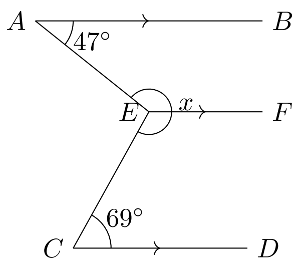
$$\begin{aligned} \angle C &= 180° - \angle A - \angle B \qquad (\angle \text{ sum of } \triangle) \\ &= 180° - 62° - 73° \\ &= \boxed{45°} \end{aligned}$$

Problem 2. In the figure, $AB//CD$, and E is a point between line AB and line CD . $\angle BAE = 47°$ and $\angle DCE = 69°$. What is x ?



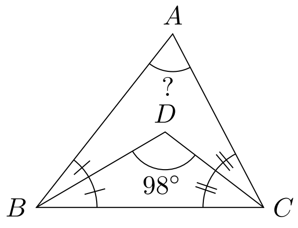
(Difficulty: 3 [Easy])

Solution 2. Draw $EF//AB//CD$.



$$\begin{aligned} \angle AEF + 47° &= 180° \qquad (\text{alt. } \angle\text{s , } AB//EF) \\ \angle AEF &= 133° \\ \angle CEF + 69° &= 180° \qquad (\text{alt. } \angle\text{s , } EF//CD) \\ \angle CEF &= 111° \\ x &= \angle AEF + \angle CEF \\ &= 133° + 111° \\ &= \boxed{244°} \end{aligned}$$

Problem 3. D is a point inside $\triangle ABC$ such that $\angle ABD = \angle DBC$ and $\angle ACD = \angle DCB$, $\angle BDC = 98°$. What is $\angle BAC$?



(Difficulty: 3)

Solution 3. Let $\angle ABD = \angle DBC = x$ and $\angle ACD = \angle DCB = y$. In $\triangle DBC$,

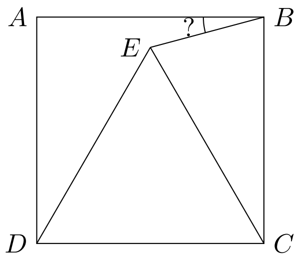
$$\begin{aligned} x + y + 98° &= 180° \qquad (\angle \text{ sum of } \triangle) \\ x + y &= 82° \end{aligned}$$

In $\triangle ABC$,

$$\begin{aligned} \angle BAC + 2x + 2y &= 180° \qquad (\angle \text{ sum of } \triangle) \\ \angle BAC &= 180° - 2(x+y) \\ &= 180° - 2(82°) \\ &= \boxed{16°} \end{aligned}$$

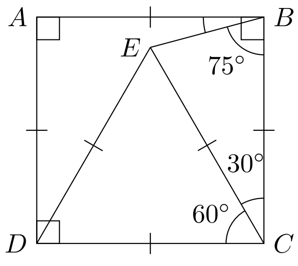
1.3 Triangle properties

Problem 4. $ABCD$ is a square. E is a point inside $ABCD$ such that $\triangle ECD$ is an equilateral triangle. Join BE. What is $\angle ABE$?



(Difficulty: 3 [Easy])

Solution 4. .

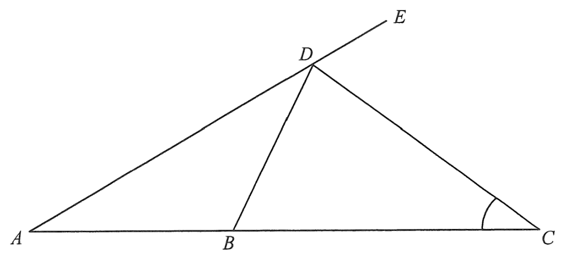


$$\begin{aligned} &\angle DCB = \angle CBA = 90° \qquad (ABCD \text{ is square.}) \\ &\angle ECD = 60° \qquad (\text{prop. of equil } \triangle) \\ &\angle ECB = 90° - 60° = 30° \end{aligned}$$

Note that $EC = BC$.

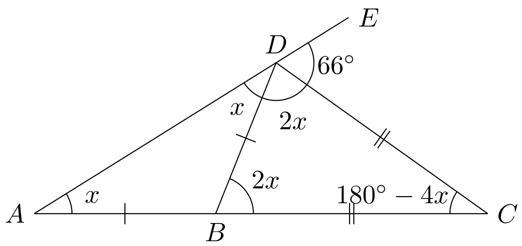
$$\begin{aligned} \therefore\ &\angle CBE = \angle CEB \qquad (\text{base } \angle\text{s, isos. } \triangle) \\ &\angle CBE = (180° - 30°)/2 = 75° \qquad (\angle \text{ sum of } \triangle) \\ &\angle ABE = 90° - 75° = \boxed{15°} \end{aligned}$$

Problem 5. In the figure, ABC and ADC are straight lines. It is given that $AB = BD$ and $BC = CD$. If $\angle CDE = 66°$, then $\angle ACD =$?



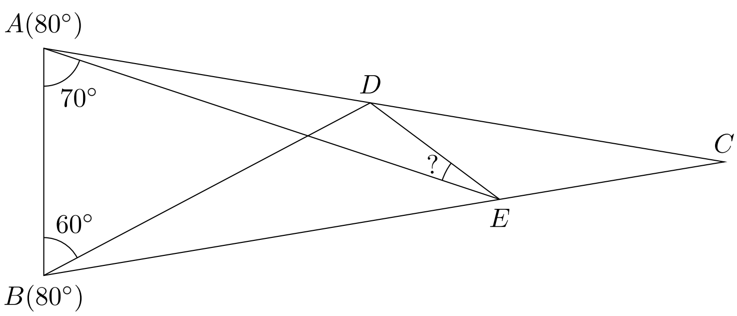
(Difficulty: 3) (2019 DSE Paper 2 Q17)

Solution 5. Let $\angle BAD = x$.



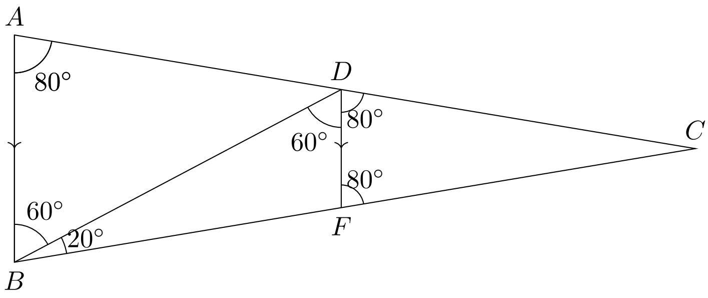
$$
\begin{aligned}
\angle BAD = \angle BDA = x & \qquad \text{(base } \angle\text{s, isos. } \triangle) \\
\angle CBD = 2x & \qquad \text{(ext. } \angle \text{ of } \triangle) \\
\angle CDB = \angle CBD = 2x & \qquad \text{(base } \angle\text{s, isos. } \triangle) \\
\angle BCD = 180° - 2x - 2x = 180° - 4x & \qquad (\angle \text{ sum of } \triangle) \\
\angle DAC + \angle ACD = x + (180° - 4x) = 66° & \qquad \text{(ext. } \angle \text{ of } \triangle) \\
x = 38° & \\
\angle ACD = 180° - 4(38°) = \boxed{28°} &
\end{aligned}
$$

Problem 6. [1] In $\triangle ABC$, $\angle BAC = \angle ABC = 80°$. Let D be a point on side AC such that $\angle ABD = 60°$. Let E be a point on side BC such that $\angle BAE = 70°$. Join DE. What is $\angle AED$?

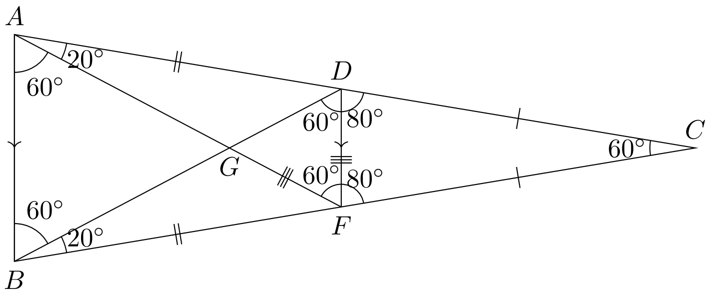


(Difficulty: 6 [Very hard])

Solution 6. Let F be a point on side BC such that $AB//DF$. Hide point E to make the figure tidier. Note that $\angle DBC = 80° - 60° = 20°$.



$$\angle CDF = \angle CAB = 80^\circ \qquad (\text{corr. } \angle\text{s , } DF//AB)$$
$$\angle CFD = \angle CBA = 80^\circ \qquad (\text{corr. } \angle\text{s , } DF//AB)$$
$$\angle BDF = 80^\circ - 20^\circ = 60^\circ \qquad (\text{ext. } \angle \text{ of } \triangle)$$

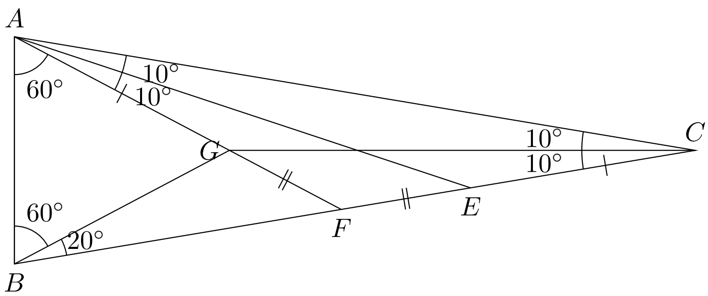


Note that $CD = CF$ and $CA = CB$ (sides opp. equal $\angle$s). Thus $AD = BF$.

Join AF, and let AF and BD intersect at G . In $\triangle ADF$ and $\triangle BFD$, $AD = BF$, $\angle ADF = \angle BFD = 110^\circ$ (adj. $\angle$s on st. line), $DF = DF$. Thus $\triangle ADF \cong \triangle BFD$ (SAS). Thus $\angle DAF = \angle FBD = 20^\circ$ (corr. $\angle$s, $\cong \triangle$s). Also, $\angle AFD = \angle BDF = 60^\circ$ (corr. $\angle$s, $\cong \triangle$s). Thus $\triangle GDF$ is an equilateral triangle (con. of equil. $\triangle$), which means $GF = DF$.

Note that $\angle ACF = 180^\circ - 80^\circ - 80^\circ = 20^\circ$ ($\angle$ sum of $\triangle$). Since $\angle CAF = \angle ACF = 20^\circ$, we have $AF = FC$ (base $\angle$s, isos. $\triangle$).

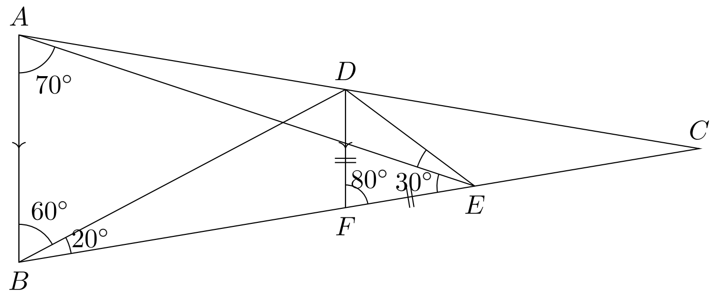
Show point E again and hide GD and DF. Join CG .



Note that $\angle CAE = \angle EAF = 10^\circ$. Also note that GC bisects ACB (because G is in the middle), so $\angle ACG = \angle GCF = 10^\circ$.

Note that $\triangle GAC \cong \triangle ECA$ (ASA), so $AG = EC$ (corr. sides, $\cong \triangle$s). Since $AF = FC$, we have $GF = FE$.

Show D again and hide AF.

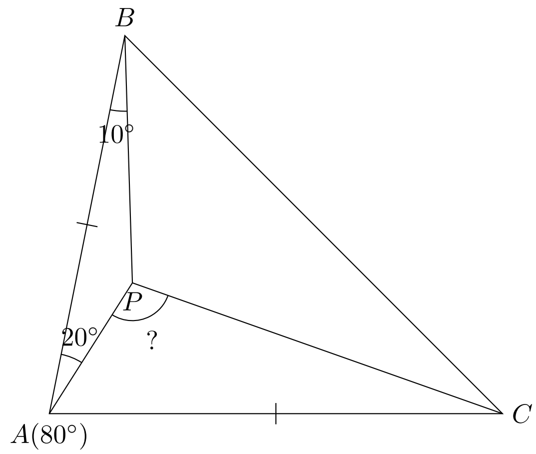


We have shown that $GF = DF$ and $GF = FE$. Thus $DF = FE$. In $\triangle FDE$, $\triangle FDE = \triangle FED$ (base $\angle$s, isos. $\triangle$). So $\angle FED = (180° - 80°)/2 = 50°$ ($\angle$ sum of $\triangle$).

Note that $\angle AEB = 180° - 80° - 70° = 30°$ ($\angle$ sum of $\triangle$).

So $\angle AED = \angle FED - \angle AEB = 50° - 30° = \boxed{20°}$.

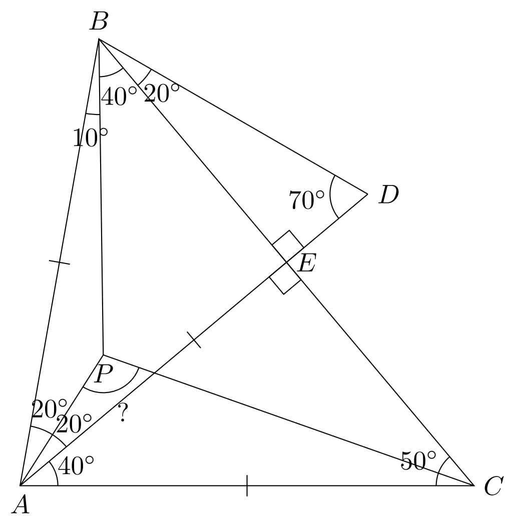
Problem 7. [2] In $\triangle ABC$, $AB = AC$ and $\angle BAC = 80°$. Let P be a point inside $\triangle ABC$ such that $\angle BAP = 20°$ and $\angle ABP = 10°$. What is $\angle APC$?



(Difficulty: 6)

Solution 7. Since $AB = AC$, we have $\angle ABC = \angle ACB$ (base $\angle$s, isos. $\triangle$), so $\angle ABC = \angle ACB = (180° - 80°)/2 = 50°$. So $\angle PBC = 50° - 10° = 40°$.

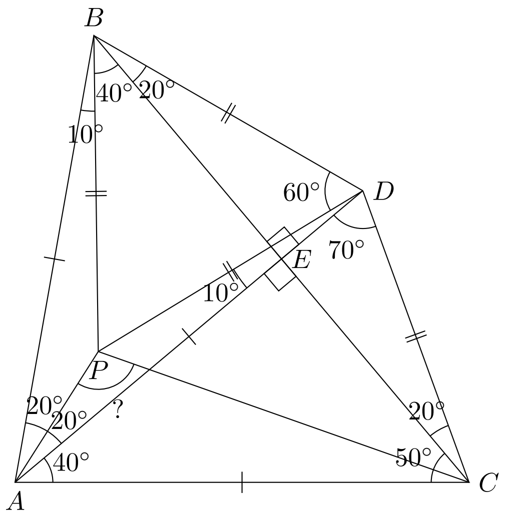
Draw AD between $\angle BAC$ such that $AD = AB$ and $\angle DAC = 40°$. Note that $\angle PAD = 80° - 20° - 40° = 20°$.



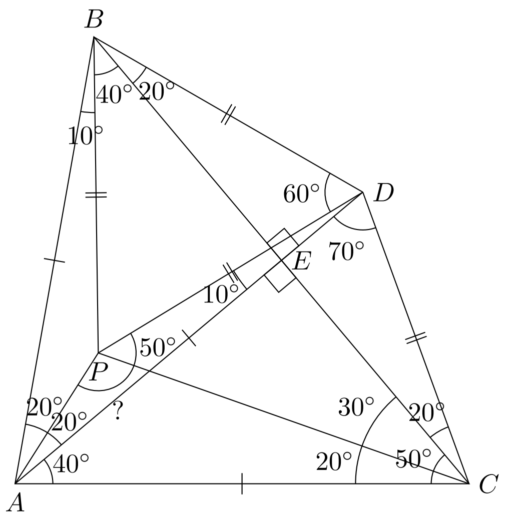
Mark E as the intersection of AD and BC. In $\triangle AEC$, $\angle AEC = 180° - 40° - 50° = 90°$ ($\angle$ sum of $\triangle$).

Join BD . Since $AB = AD$, we have $\angle ABD = \angle ADB = (180° - 40°)/2 = 70°$ (base $\angle$s, isos. $\triangle$)& ($\angle$ sum of $\triangle$). Note that $\angle BED = 90°$ (vert. opp. $\angle$s), so $\angle DBE = 180° - 70° - 90° = 20°$ ($\angle$ sum of $\triangle$).

Join DC and PD. Note that $\triangle DAB \cong \triangle DAC$ (SAS), so $BD = DC$ and $\angle ADC = \angle ADB = 70°$. Since $BD = DC$, we have $\angle DCB = \angle DBC = 20°$ (base $\angle$s, isos. $\triangle$).



Note that $\triangle BAP \cong \triangle DAP$ (SAS), so $\angle PDA = \angle PBA = 10°$ (corr. $\angle$s, $\cong$ $\triangle$s). Thus $\angle PDB = 70° - 10° = 60°$. Note that in $\triangle BPD$, $\angle PBD = \angle PDB = 60°$. Thus $\triangle BPD$ is an equil. $\triangle$ (con. of equil. $\triangle$), so $BP = DP = BD$. Since $BD = DC$, we have $DP = DC$.



Since $\triangle DPC$ is an isos. $\triangle$ with $DP = DC$, we have $\angle DPC = \angle DCP = (180° - 80°)/2 = 50°$ (base $\angle$s, isos. $\triangle$)& ($\angle$ sum of $\triangle$). Thus $\angle ECP = 50° - 20° = 30°$. So $\angle PCA = 50° - 30° = 20°$.

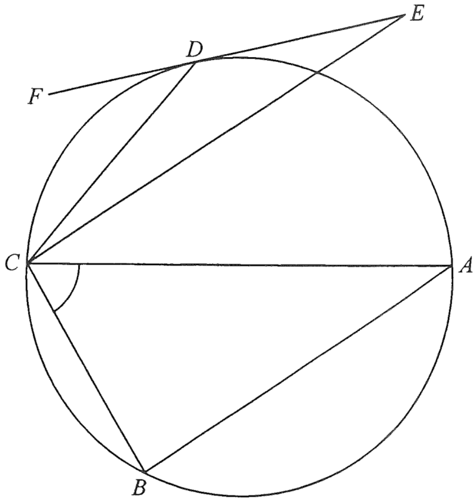
Finally, in $\triangle APC$, $\angle APC = 180° - (20° + 40°) - 20° = \boxed{100°}$.

1.6 Circle properties

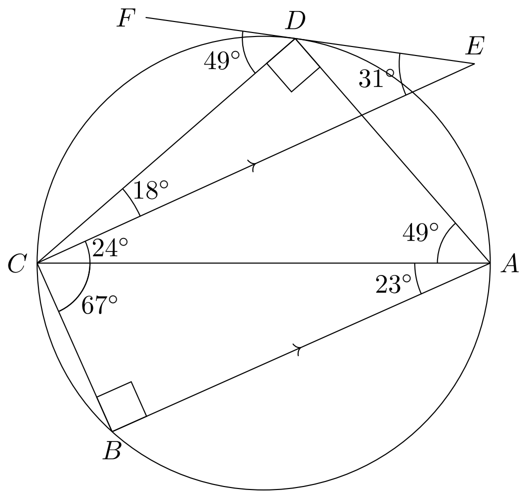
(Problem solving tips: try to use all the information given in the problem.)

Problem 8. In the figure, AC is a diameter of the circle $ABCD$. EF is the tangent to the circle at D such that $AB//EC$. If $\angle CDF = 49°$ and $\angle CED = 31°$, then $\angle ACB =$?

(Difficulty: 4 [Medium]) (2021 DSE Paper 2 Q39)

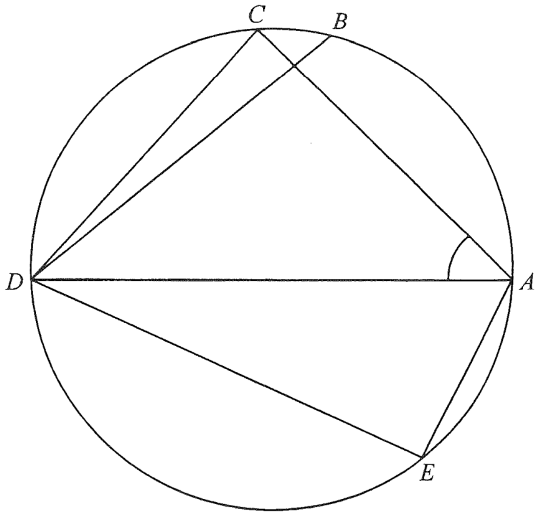


Solution 8. (Diagram adjusted for accuracy.) Join DA .



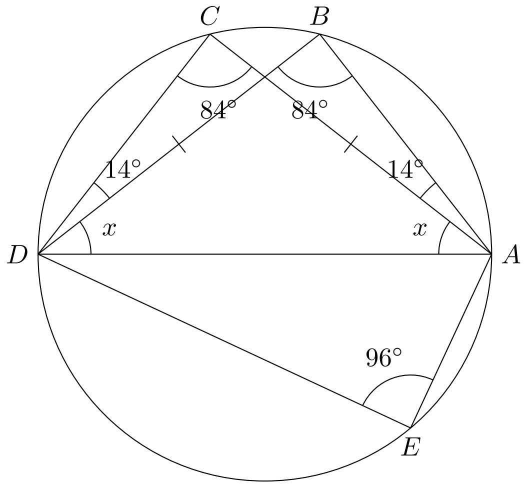
$$\begin{aligned}
\angle CDA, \angle ABC &= 90^\circ && (\angle \text{ in semi-circle}) \\
\angle CAD &= 49^\circ && (\angle \text{ in alt. segment}) \\
\angle DCA &= 90^\circ - 49^\circ = 41^\circ && (\angle \text{ sum of } \triangle) \\
\angle DCE &= 49^\circ - 31^\circ = 18^\circ && (\text{ext. } \angle \text{ of } \triangle) \\
\angle ACE &= 41^\circ - 18^\circ = 23^\circ && \\
\angle BAC &= \angle ACE = 23^\circ && (\text{alt. } \angle\text{s}\ ,\ AB//EC) \\
\angle ACB &= 90^\circ - 23 = \boxed{67^\circ} && (\angle \text{ sum of } \triangle)
\end{aligned}$$

Problem 9. In the figure, $ABCDE$ is a circle. If $AC = BD$, $\angle AED = 96^\circ$ and $\angle BDC = 14^\circ$, then $\angle CAD =$?



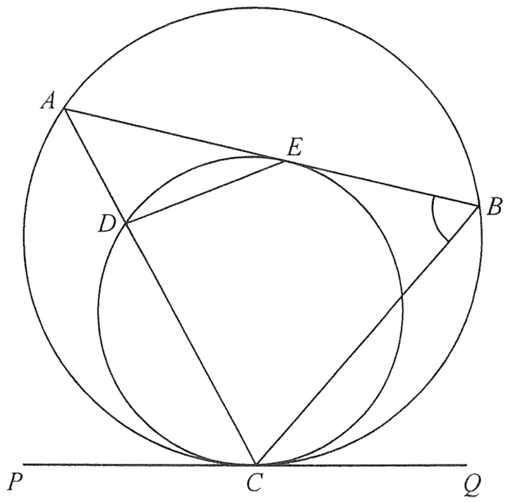
(Difficulty: 4) (2021 DSE Paper 2 Q22)

Solution 9. Join AB . Let $\angle CAD = x$.



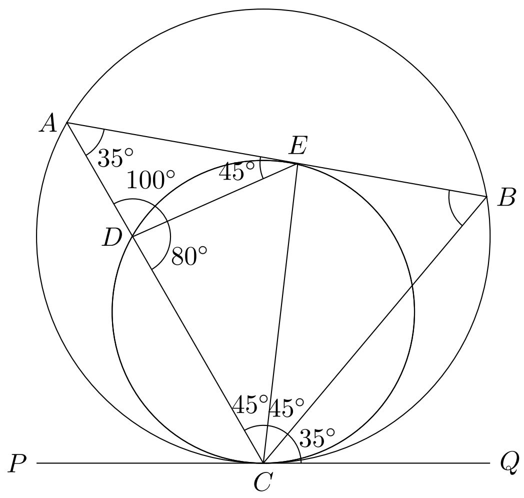
$$
\begin{aligned}
\angle DCA, \angle DBA &= 180 - 96^\circ = 84^\circ \qquad \text{(opp. } \angle\text{s , cyclic quad.)} \\
\angle BAC &= 14^\circ \qquad (\angle\text{s in the same segment}) \\
\angle CDA = \angle BAD &= x + 14^\circ \qquad \text{(equal chords, equal } \angle\text{s at } \odot^{ce}) \\
\angle BDA &= x \\
84^\circ + 14^\circ + 2x &= 180^\circ \qquad \text{(ext. } \angle \text{ of } \triangle)\& \, (\angle \text{ sum of } \triangle) \\
x &= \boxed{41^\circ}
\end{aligned}
$$

Problem 10. In the figure, ABC and CDE are circles such that ADC is a straight line. PQ is the common tangent to the two circles at C . AB is the tangent to the circle CDE at E . If $\angle ADE = 100^\circ$ and $\angle BCQ = 35^\circ$, then $\angle ABC =$?



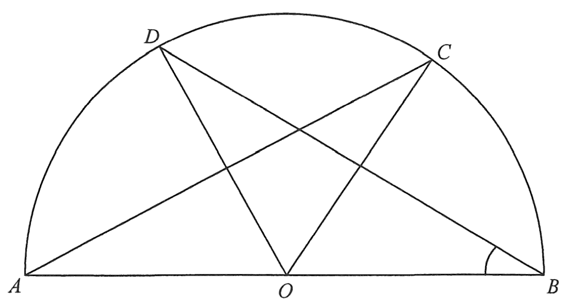
(Difficulty: 4) (2020 DSE Paper 2 Q39)

Solution 10. Join EC .



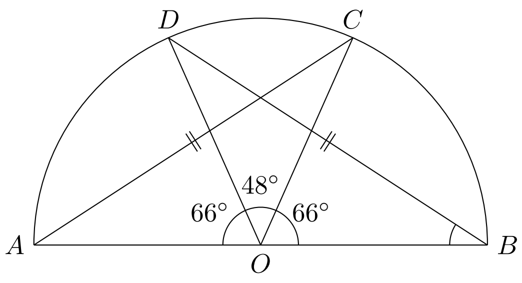
$$\begin{aligned}
&\angle CAB = 35^\circ \qquad (\angle \text{ in alt. segment}) \\
&\angle AED = 180^\circ - 35^\circ - 100^\circ = 45^\circ \qquad (\angle \text{ sum of } \triangle) \\
&\angle DCE = 45^\circ \qquad (\angle \text{ in alt. segment}) \\
&\angle EDC = 180^\circ - 100^\circ = 80^\circ \qquad (\text{adj. } \angle\text{s on st. line}) \\
&\angle ECQ = \angle EDC = 80^\circ \qquad (\angle \text{ in alt. segment}) \\
&\angle ECB = 80^\circ - 35^\circ = 45^\circ \\
&\angle ABC = 180^\circ - 35^\circ - (45^\circ + 45^\circ) = \boxed{55^\circ} \qquad (\angle \text{ sum of } \triangle)
\end{aligned}$$

Problem 11. In the figure, O is the centre of the semi-circle $ABCD$. If $AC = BD$ and $\angle COD = 48^\circ$, then $\angle ABD =$?



(Difficulty: 3) (2019 DSE Paper 2 Q21)

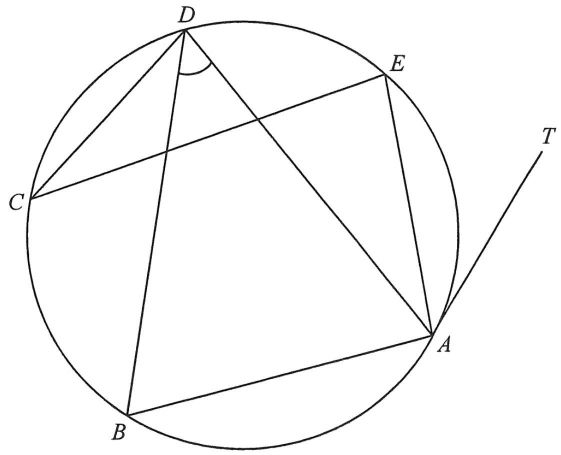
Solution 11. .



Note that $\triangle OAC \cong \triangle OBD$ (SSS) . This means $\angle AOC = \angle DOB$ (corr. sides, $\cong$ $\triangle$s), and thus $\angle AOD = \angle BOC = (180° - 48°)/2 = 66°$ (adj. $\angle$s on st. line). In $\triangle OBD$,

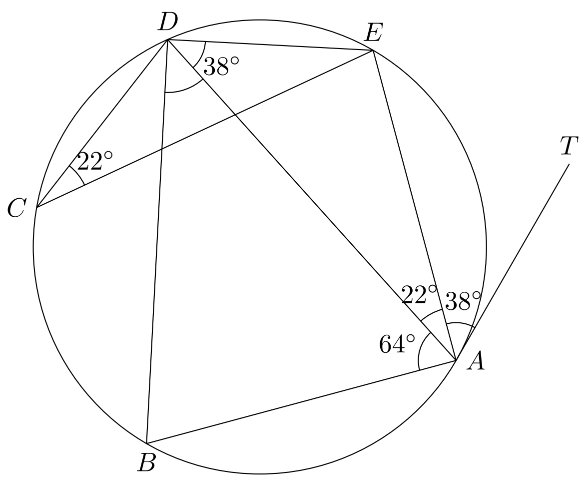
$$\angle ABD = (180° - 48° - 66°)/2 = \boxed{33°} \qquad (\angle \text{ sum of } \triangle)$$

Problem 12. In the figure, TA is the tangent to the circle $ABCDE$ at point A . If $\angle BAD = 64°$, $\angle EAT = 38°$ and $\angle DCE = 22°$, then $\angle ADB =$?



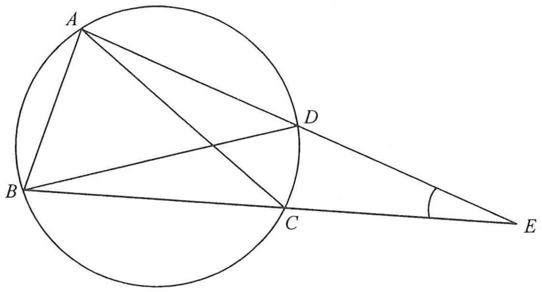
(Difficulty: 3) (2019 DSE Paper 2 Q39)

Solution 12. Join DE .



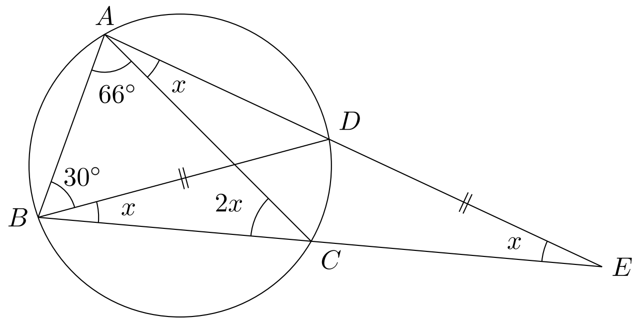
$$\angle ADE = 38^\circ \qquad (\angle \text{ in alt. segment})$$
$$\angle EAD = 22^\circ \qquad (\angle\text{s in the same segment})$$
$$\angle ADB = 180^\circ - 64^\circ - 22^\circ - 38^\circ = \boxed{56^\circ}$$

Problem 13. In the figure, $ABCD$ is a circle. AD produced and BC produced meet at the point E . It is given that $BD = DE$, $\angle BAC = 66^\circ$ and $\angle ABD = 30^\circ$. Find $\angle CED$.



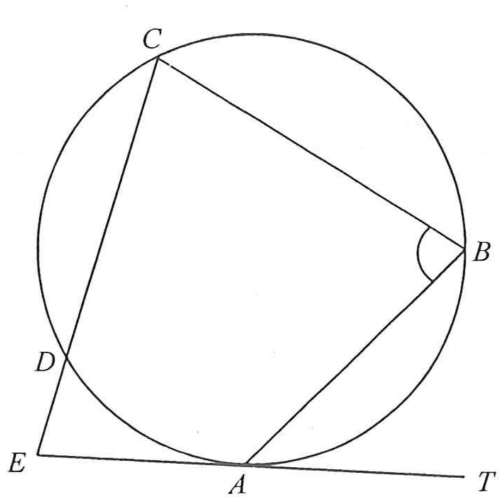
(Difficulty: 3) (2018 DSE Paper 2 Q22)

Solution 13. Let $\angle CED = x$.



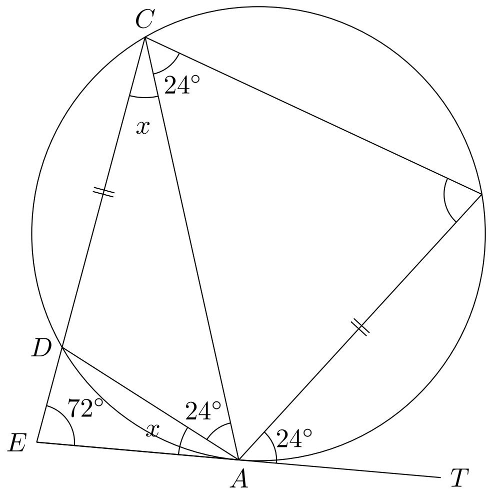
$$\angle DBE = x \qquad (\text{base } \angle\text{s, isos. } \triangle)$$
$$\angle CAD = \angle CBD = x \qquad (\angle\text{s in the same segment})$$
$$\angle ACB = \angle CED + \angle CAD = 2x \qquad (\text{ext. } \angle \text{ of } \triangle)$$
In $\triangle ABC$, $$66^\circ + (30^\circ + x) + 2x = 180^\circ \qquad (\angle \text{ sum of } \triangle)$$
$$x = \boxed{28^\circ}$$

Problem 14. In the figure, TA is the tangent to the circle $ABCD$ at the point A . CD produced and TA produced meet at the point E . It is given that $AB = CD$, $\angle BAT = 24^\circ$ and $\angle AED = 72^\circ$. Find $\angle ABC$.



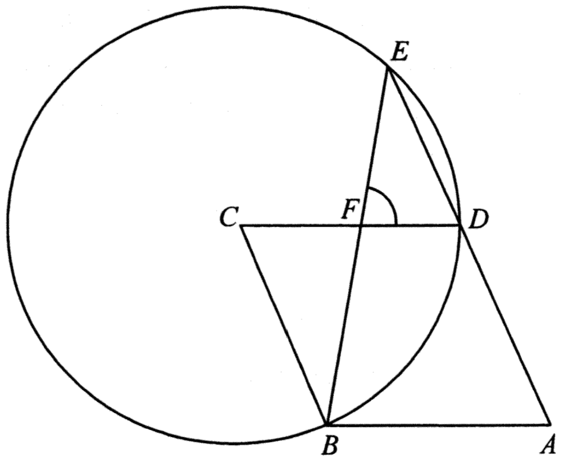
(Difficulty: 4) (2018 DSE Paper 2 Q39)

Solution 14. Join AD and AC. Let $\angle EAD = x$.



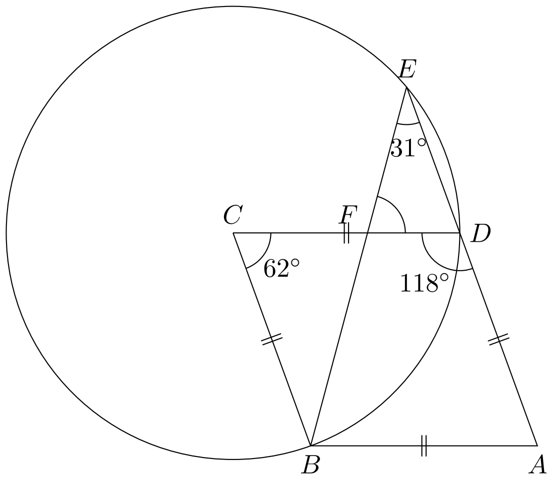
$$\begin{aligned}
&\angle ACB = 24^\circ && (\angle \text{ in alt. segment})\\
&\angle CAD = \angle ACB = 24^\circ && (\text{equal chords, equal } \angle\text{s at } \odot^{ce})\\
&\angle DCA = \angle EAD = x && (\angle \text{ in alt. segment})\\
\text{In } \triangle CEA \text{ ,}\quad &72^\circ + x + (x + 24^\circ) = 180^\circ && (\angle \text{ sum of } \triangle)\\
&x = 42^\circ\\
&\angle ABC = \angle EAC = 42^\circ + 24^\circ && (\angle \text{ in alt. segment})\\
&= \boxed{66^\circ}
\end{aligned}$$

Problem 15. In the figure, $ABCD$ is a rhombus. C is the centre of the circle BDE and ADE is a straight line. BE and CD intersect at F . If $\angle ADC = 118^\circ$, then $\angle DFE =$?



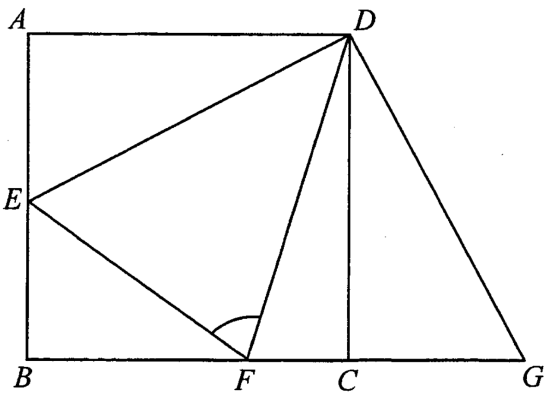
(Difficulty: 3) (2016 DSE Paper 2 Q22)

Solution 15. .



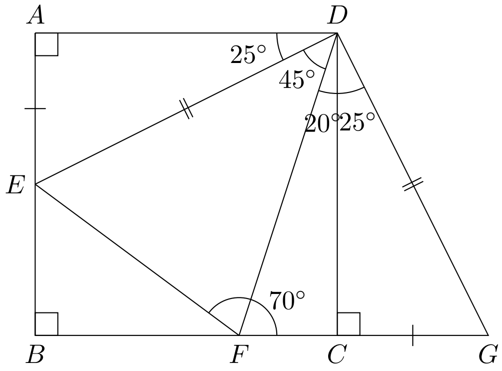
$$CB//DA \qquad \text{(prop. of rhombus)}$$
$$\angle C = 180^\circ - 118^\circ = 62^\circ \qquad (\text{int. } \angle\text{s }, CB//DA)$$
$$\angle FED = 62^\circ/2 = 31^\circ \qquad (\angle \text{ at centre twice } \angle \text{ at } \odot^{ce})$$
$$\angle DFE = 118^\circ - 31^\circ = \boxed{87^\circ} \qquad (\text{ext. } \angle \text{ of } \triangle)$$

Problem 16. In the figure, $ABCD$ is a square. BC is produced to G such that $\angle CDG = 25^\circ$. E is a point lying on AB such that $AE = CG$. If F is a point lying on BC such that $\angle CDF = 20^\circ$, then $\angle DFE = ?$



(Difficulty: 4) (2014 DSE Paper 2 Q16)

Solution 16. .

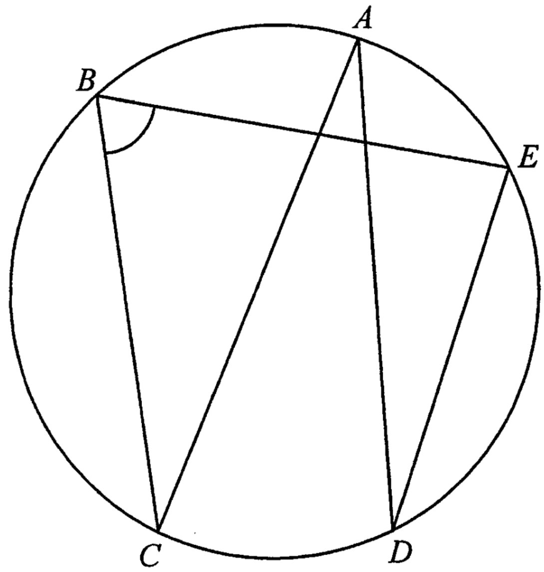


Note that $\triangle DAE \cong \triangle DCG$ (SAS) , so we have $\angle ADE = \angle CDG = 25°$ (corr. sides, $\cong \triangle$s).
Note that $\angle EDF = 90° - 25° - 20° = 45°$.

In $\triangle DFE$ and $\triangle DFG$,

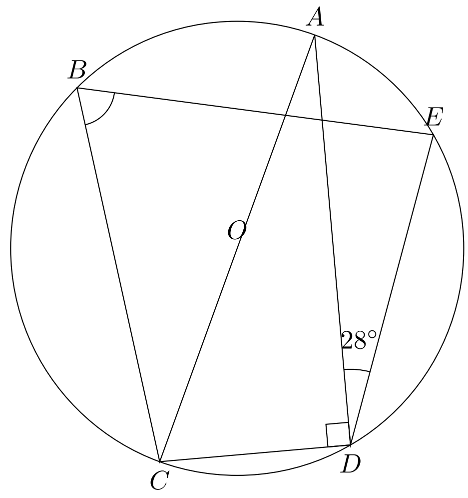
$$\begin{aligned}
DE &= DG \qquad \text{(corr. sides, } \cong \triangle\text{s)} \\
\angle EDF &= \angle FDG = 45° \\
DF &= DF \qquad \text{(common side)} \\
\therefore\ \triangle DFE &\cong \triangle DFG \qquad \text{(SAS)} \\
\therefore\ \angle DFE &= \angle DFG \qquad \text{(corr. } \angle\text{s, } \cong \triangle\text{s)} \\
&= 90° - 20° = \boxed{70°} \qquad (\angle \text{ sum of } \triangle)
\end{aligned}$$

Problem 17. In the figure, AC is a diameter of the circle $ABCDE$. If $\angle ADE = 28°$, then $\angle CBE =$?



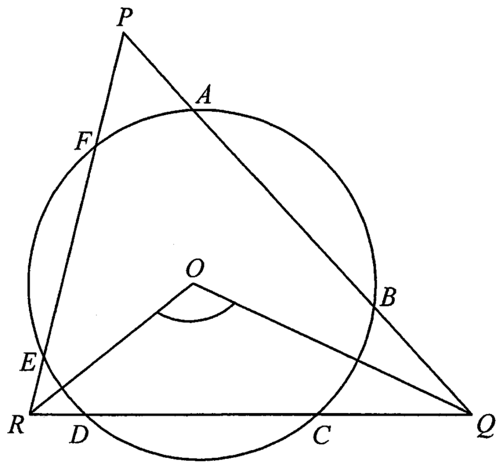
(Difficulty: 3) (2014 DSE paper 2 Q20)

Solution 17. Join CD .



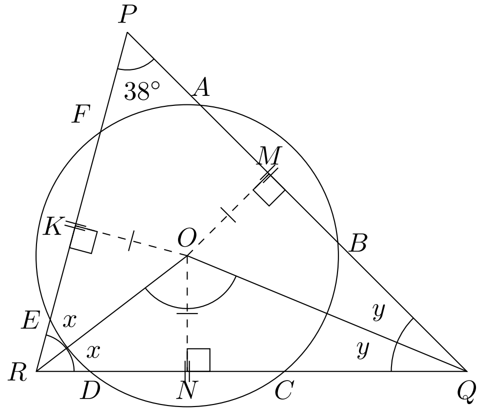
$$\angle ADC = 90^\circ \qquad (\angle \text{ in semi-circle})$$
$$\angle CDE = 90^\circ + 28^\circ = 118^\circ$$
$$\angle CBE = 180^\circ - 118^\circ = \boxed{62^\circ} \qquad (\text{opp. } \angle\text{s , cyclic quad.})$$

Problem 18. In the figure, O is the centre of the circle $ABCDEF$. $\triangle PQR$ intersects the circle at A, B, C, D, E and F . If $\angle QPR = 38^\circ$ and $AB = CD = EF$, then $\angle QOR =$?



(Difficulty: 4) (2014 DSE Paper 2 Q21)

Solution 18. Draw $OM \perp AB$, $ON \perp DC$, $OK \perp FE$.



Note that $OM = ON = OK$ (equal chords, equidistant from centre) . Thus, $\angle ORK = \angle ORN$ and $\angle OQN = \angle OQM$ (prop. of $\angle$ bisector) .

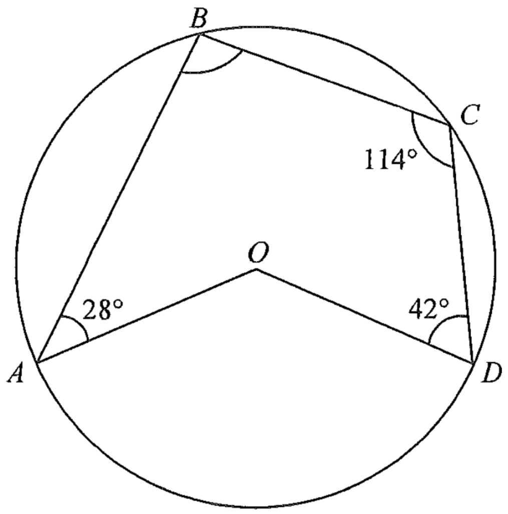
Let $\angle ORK = \angle ORN = x$ and $\angle OQN = \angle OQM = y$. In $\triangle PQR$,

$$\begin{aligned} 38^\circ + 2x + 2y &= 180^\circ \qquad (\angle \text{ sum of } \triangle) \\ x + y &= 71^\circ \end{aligned}$$

In $\triangle ORQ$,

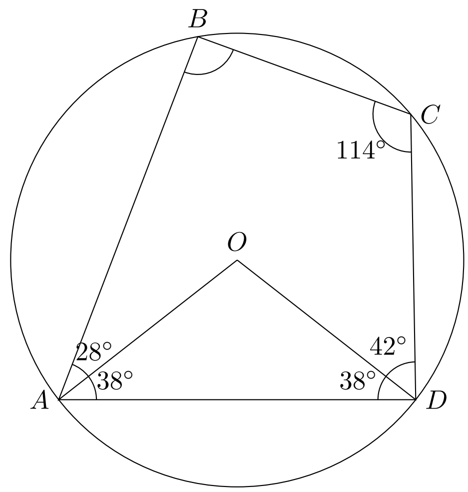
$$\begin{aligned} x + y + \angle QOR &= 180^\circ \qquad (\angle \text{ sum of } \triangle) \\ \angle QOR &= 180^\circ - 71^\circ = \boxed{109^\circ} \end{aligned}$$

Problem 19. In the figure, O is the centre of the circle $ABCD$. If $\angle BAO = 28^\circ$, $\angle BCD = 114^\circ$ and $\angle CDO = 42^\circ$, then $\angle ABC =$?



(Difficulty: 3) (2012 DSE Paper 2 Q20)

Solution 19. Join AD .

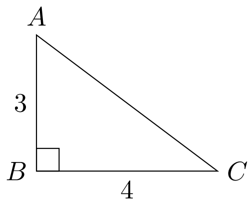


$$\begin{aligned} &\angle BAD = 180^\circ - 114^\circ = 66^\circ \qquad (\text{opp. } \angle\text{s , cyclic quad.}) \\ &\angle OAD = 66^\circ - 28^\circ = 38^\circ \\ &\angle ODA = 38^\circ \qquad (\text{base } \angle\text{s, isos. } \triangle) \\ &\angle ABC = 180^\circ - (38^\circ + 42^\circ) = \boxed{100^\circ} \qquad (\text{opp. } \angle\text{s , cyclic quad.}) \end{aligned}$$

1.7 Area and perimeter

1.7.1 Pythagoras theorem

Problem 20. $\triangle ABC$ has $\angle B = 90°$, $AB = 3$ and $BC = 4$. What is AC?

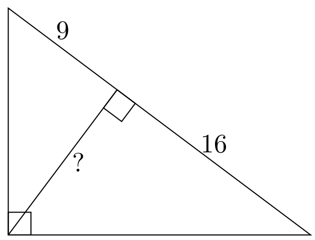


(Difficulty: 1 [Beginner])

Solution 20. Since $\triangle ABC$ is a right triangle, we can apply Pythagoras theorem:

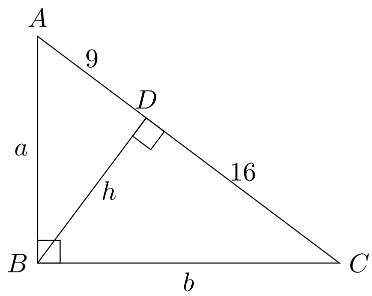
$$\begin{aligned} AB^2 + BC^2 &= AC^2 \qquad \text{(Pyth. theorem)} \\ AC^2 &= 3^2 + 4^2 \\ AC &= \sqrt{3^2 + 4^2} \\ &= \boxed{5} \end{aligned}$$

Problem 21. In a right triangle, the perpendicular line segment dropped from the vertex of the right angle upon the hypotenuse divides it into two segments of 9 and 16 units respectively. What is the length of this perpendicular line segment?



(Difficulty: 3) [3]

Solution 21. Let h be the length of the perpendicular line segment, and a , b be the two legs (non-hypotenuse sides) of the triangle.

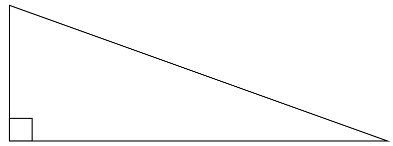


In $\triangle ABC$, $a^2 + b^2 = (9 + 16)^2$ (Pyth. theorem).
In $\triangle ADB$, $h^2 + 9^2 = a^2$ (Pyth. theorem).
In $\triangle CDB$, $h^2 + 16^2 = b^2$ (Pyth. theorem).

Substituting the 2nd and 3rd equation into the 1st equation:

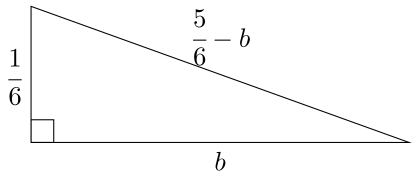
$$\begin{aligned} (h^2 + 9^2) + (h^2 + 16^2) &= (9 + 16)^2 \\ 2h^2 &= 625 - 337 \\ h^2 &= 144 \\ h &= \boxed{12} \end{aligned}$$

Problem 22. A leg of a right triangle is equal to 1/5 the sum of the other two sides. The triangle has a perimeter of 1. What is the triangle's area?



(Difficulty: 4) [4]

Solution 22. Let k be the length of the leg. Then considering the perimeter of the triangle, we have $k + 5k = 1$, so $k = \frac{1}{6}$.



Let b be the length of the other leg. Then the hypotenuse is $1 - \frac{1}{6} - b = \frac{5}{6} - b$. By pyth. theorem,

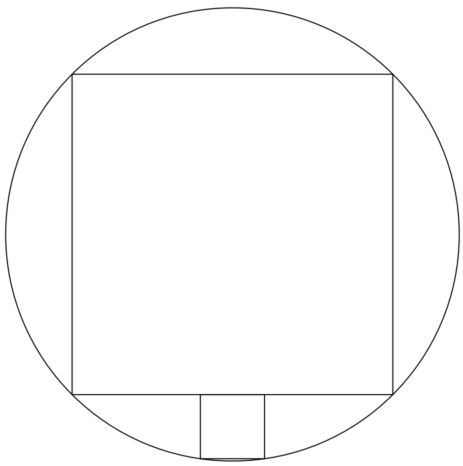
$$b^2 + (\frac{1}{6})^2 = (\frac{5}{6} - b)^2$$
$$b^2 + \frac{1}{36} = \frac{25}{36} - \frac{5b}{3} + b^2$$
$$b = \frac{2}{5}$$

Area of triangle $= \frac{1}{2}(\frac{1}{6})(\frac{2}{5}) = \boxed{\frac{1}{30}}$

Problem 23. A square is inscribed in a circle.

A smaller square is drawn. It shares side with the inscribed square and its other two corners touch the circle.

What is the ratio of the larger square's area to the smaller square's area?

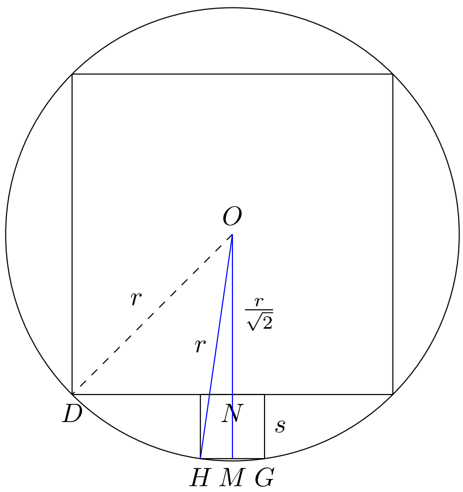


(Difficulty: 5 [Hard]) [5]

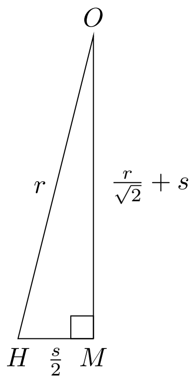
Solution 23. Let r be the radius of the circle, and s be the side length of the small square.

Draw a radius of the circle to a a corner of the small square.

Drop a perpendicular from the centre of the circle to the bottom side of the small square. Note that it bisects the bottom side of both squares (line from centre $\perp$ chord bisects chord). Thus, $HM = \frac{1}{2}s$.



Since $\triangle ODN$ is a right isosceles triangle, we have $ON = \frac{r}{\sqrt{2}}$. Let's focus on $\triangle OMH$. Note that $OM = \frac{r}{\sqrt{2}} + s$.



By pyth. theorem, we have

$$\begin{aligned}
(\frac{r}{\sqrt{2}} + s)^2 + (\frac{s}{2})^2 &= r^2 \\
\frac{r^2}{2} + \sqrt{2}rs + s^2 + \frac{s^2}{4} &= r^2 \\
5s^2 + 4\sqrt{2}rs - 2r^2 &= 0
\end{aligned}$$

Using **quadratic formula** on s :

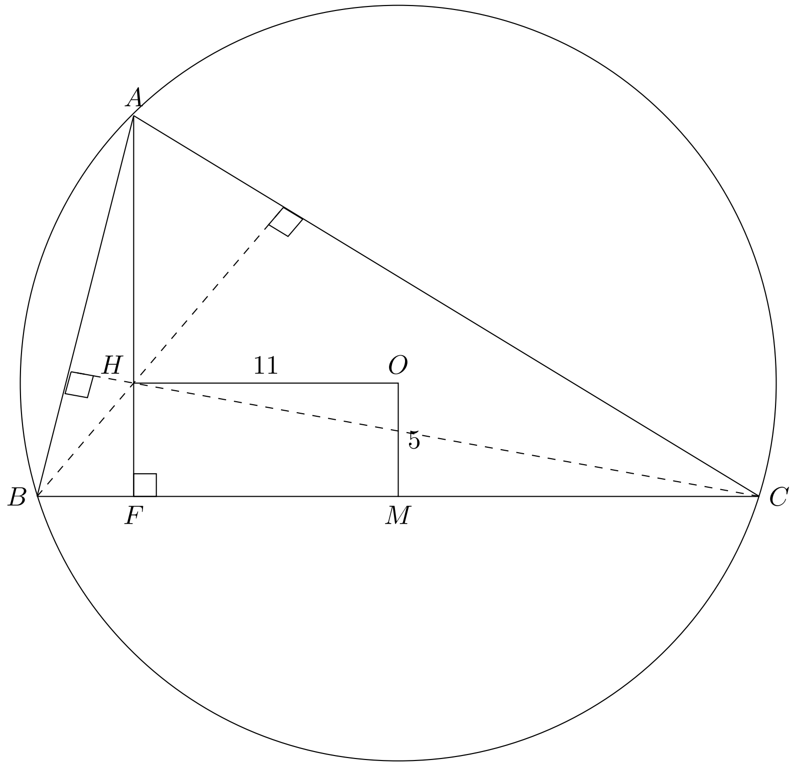
$$\begin{aligned}
s &= \frac{-4\sqrt{2}r + \sqrt{(4\sqrt{2}r)^2 - 4(5)(-2r^2)}}{2(5)} \\
&= (\frac{-4\sqrt{2} + \sqrt{72}}{10})r \\
&= (\frac{\sqrt{2}}{5})r
\end{aligned}$$

Since the side length of the large square is $r\sqrt{2}$, the area of the large square is $2r^2$.

Thus, $\frac{\text{area of larger square}}{\text{area of smaller square}} = \frac{2r^2}{s^2} = \frac{2r^2}{((\frac{\sqrt{2}}{5})r)^2} = \frac{2r^2}{(\frac{2}{25})r^2} = \boxed{25}$.

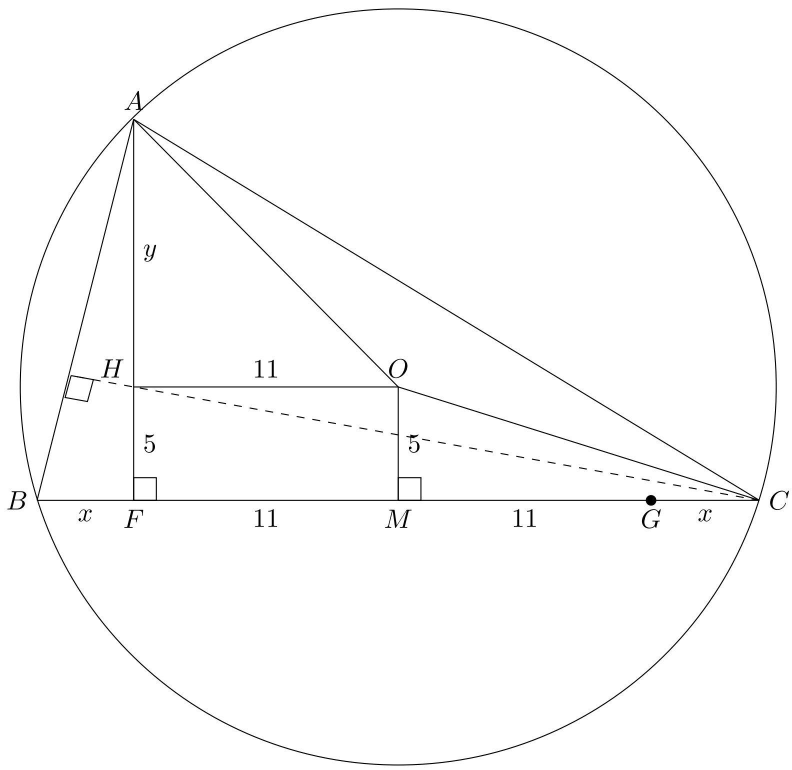
1.8 Proportions and similar triangles

Problem 24. A rectangle, $HOMF$, has sides $HO = 11$ and $OM = 5$. A triangle ABC has H as the intersection of the altitudes, O the centre of the circumscribed circle, M the midpoint of BC , and F the foot of the altitude from A . What is the length of BC ?



(Difficulty: 6) (Putnam 1997 A1) [6]

Solution 24. Let $BF = x$ and $AH = y$. Let G be a point on BC such that $GC = BF = x$. Then $MG = FM = 11$.



Note that $OA = OC$. Considering $\triangle AHO$ and $\triangle OMC$, we have $OA^2 = y^2 + 11^2$ and $OC^2 = 5^2 + (11 + x)^2$ by pyth. theorem, so we have

$$y^2 + 11^2 = 5^2 + (11 + x)^2 \tag{1}$$
$$y^2 + 121 = 25 + 121 + 22x + x^2 \tag{2}$$

Also note that $\angle HCF = 90^\circ - \angle ABC = \angle BAF$ ($\angle$ sum of $\triangle$). Thus $\triangle AFB \sim \triangle CFH$ (AA).

So we have

$$\begin{aligned} \frac{AF}{BF} &= \frac{CF}{HF} \qquad \text{(corr. sides, } \sim \triangle\text{s)} \\ \frac{y+5}{x} &= \frac{x+11+11}{5} \\ 5y + 25 &= x^2 + 22x \end{aligned}$$

Note that $x^2 + 22x$ appears in both equation (2) and (3). Putting (3) into (2):

$$\begin{aligned} y^2 + 121 &= 25 + 121 + 5y + 25 \\ y^2 - 5y - 50 &= 0 \\ (y - 10)(y + 5) &= 0 \\ y &= 10 \text{ or } y = -5 \qquad \text{(rej.)} \end{aligned}$$

Put $y = 10$ into (1):

$$\begin{aligned} 10^2 + 11^2 &= 5^2 + (11 + x)^2 \\ 196 &= (11 + x)^2 \\ 14 &= 11 + x \\ x &= 3 \\ \therefore\ BC &= 3 + 11 + 11 + 3 = \boxed{28} \end{aligned}$$

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