Make A Sequence Walkthrough

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0 Introduction

Make The Sequence is a maths puzzle game I created using Pygame. The goal of this game is to make the sequence given in each level by inputting an appropriate formula with a limited number of characters. It is preferred that the solution uses as few characters as possible.

1 Levels

Sequence 1. 1, 2, 3, 4, 5, ...

Solution 1. a(n) = $\boxed{n}$

Sequence 2. 2, 4, 6, 8, 10, ...

Solution 2. a(n) = $\boxed{2n}$

Sequence 3. 4, 5, 6, 7, 8, ...

Solution 3. a(n) = $\boxed{n+3}$

Sequence 4. 1, 1, 1, 1, 1, ...

Solution 4. a(n) = $\boxed{1}$

Sequence 5. 1, 3, 5, 7, 9, ...

Solution 5. a(n) = $\boxed{2n-1}$

Sequence 6. 5, 9, 13, 17, 21, ...

Solution 6. a(n) = $\boxed{4n+1}$

Sequence 7. 13, 23, 33, 43, 53, ...

Solution 7. a(n) = $\boxed{10n+3}$

Sequence 8. 38, 31, 24, 17, 10, ...

Solution 8. Note that this is an arithmetic sequence with common difference -7. So a(n)= $38 - 7(n-1)$= $\boxed{45 - 7n}$.

Sequence 9. 1, 4, 9, 16, 25, ...

Solution 9. a(n) = $n**2 = \boxed{nn}$.

Sequence 10. 1, 3, 6, 10, 15, ...

Solution 10. Note that this is the sequence of **triangular numbers**.

So a(n) = $\boxed{n(n+1)/2}$.

Sequence 11. 1, 8, 27, 64, 125, ...

Solution 11. a(n) = $n**3 = \boxed{nnn}$.

Sequence 12. 1, 2, 4, 8, 16, ...

Solution 12. a(n) = $\boxed{2**(n-1)}$.

Sequence 13. 1, 2, 4, 7, 11, ...

Solution 13. Note that the first difference of the sequence is 1, 2, 3, 4 , ... and the second difference is 1, 1, 1, ... Let $\Delta a(n)$ denote the first difference sequence and $\Delta^2 a(n)$ denote the second difference sequence.

We have $\Delta a(n+1) - \Delta a(n) = \Delta^2 a(n)$.

Suppose we want to find $\Delta a(n)$ given $\Delta^2 a(n) = 1$. Putting $\Delta^2 a(n) = 1$ into the above equation:

$$\Delta a(n+1) - \Delta a(n) = 1$$

Since $\Delta^2 a(n)$ is a constant sequence, we guess that $\Delta a(n)$ is a linear sequence, so $\Delta a(n) = cn + d$ for some c and d. Putting $\Delta a(n) = cn + d$:

$$c(n+1) + d - (cn + d) = 1$$
$$c = 1$$

From our observations above, we know that $\Delta a(1) = c(1) + d = 1$, so $d = 0$, and $\Delta a(n) = n$.

Now we want to find $a(n)$ given $\Delta a(n) = n$. We have $a(n+1) - a(n) = \Delta a(n) = n$. Since $\Delta a(n)$ is a linear sequence, we guess that $a(n)$ is a quadratic sequence, so we let $a(n) = An^2 + Bn + C$. Putting $a(n) = An^2 + Bn + C$:

$$A(n+1)^2 + B(n+1) + C - (An^2 + Bn + C) = n$$
$$A((n+1)^2 - n^2) + B((n+1) - n) + C - C = n$$
$$A(2n+1) + B = n \qquad \ldots(*)$$

Since the difference equation holds true for all positive integer n , putting any positive integer value of n will work. Putting $n = 1$ and $n = 2$ into (*):

$$\begin{cases} A(2(1)+1) + B = 1 \\ A(2(2)+1) + B = 2 \end{cases}$$

Solving, $A = \dfrac{1}{2}$ and $B = -\dfrac{1}{2}$.

So $a(n) = \dfrac{1}{2}n^2 - \dfrac{1}{2}n + C$. From the sequence given at the beginning, we know that $a(1) = 1$. Putting $a(1) = 1$:

$$a(1) = \frac{1}{2}(1)^2 - \frac{1}{2}(1) + C = 1$$
$$C = 1$$

Thus, a(n) = $\dfrac{1}{2}n^2 - \dfrac{1}{2}n + 1 = \boxed{n(n-1)/2 + 1}$.

Sequence 14. 3, 9, 27, 81, 243, ...

Solution 14. Note that this is a geometric sequence with first term = 3 and common ratio = 3 . More succinctly, this is a sequence of powers of 3.

a(n) = $\boxed{3 * * n}$.

Sequence 15. 48, 72, 108, 162, 243, ...

Solution 15. Note that this is a geometric sequence with first term = 48 and common ratio = 3/2 .

a(n) $= 48(3/2)**(n-1) = \boxed{32(3/2)**n}$.

Sequence 16. 1, 11, 111, 1111, 11111, ...

Solution 16. Note that
$a(1) = 1$,
$a(2) = 10^2 + 1$,
$a(3) = 10^3 + 10^2 + 1$,
$a(4) = 10^4 + 10^3 + 10^2 + 10 + 1$,
$a(5) = 10^5 + 10^4 + 10^3 + 10^2 + 10 + 1$.
Each term is the sum of geometric sequence with first term = 1 and common ration = 10 . Using the sum of geometric sequence formula $S(n) = \dfrac{A(r^n - 1)}{r - 1}$ where first term = A , common ratio = r and number of terms = n ,
We get a(n) $= \dfrac{10^n - 1}{10 - 1} = \boxed{(10**n-1)/9}$.

Sequence 17. 3, -6, 9, -12, 15, ...

Solution 17. Note that the absolute sequence 3, 6, 9, 12, 15, ... is given by $b(n) = 3n$. The corresponding alternating sequence can be obtained by adding $(-1)^n$ to the formula of the absolute sequence. However, in this alternating sequence, the odd-n th term is positive but the even-n th term is negative, so we add one more negative sign to the formula.
a(n) $= \boxed{-3n(-1)**n}$.

Sequence 18. 5, 13, 25, 41, 61, ...

Solution 18. Note that the sequence of $\Delta a(n)$ is 8, 12, 16, 20, ... and $\Delta^2 a(n) = 4$. By inspection, we see that $\Delta a(n) = 4n + 4$. Let $a(n) = An^2 + Bn + C$. Then

$$A(2n+1) + B = 4n + 4 \qquad \ldots(*)$$

Putting $n = 1$ and $n = 2$ into (*):

$$\begin{cases} A(2(1)+1) + B = 4(1) + 4 \\ A(2(2)+1) + B = 4(2) + 4 \end{cases}$$

Solving, $A = 2$ and $B = 2$. Thus $a(n) = 2n^2 + 2n + C$. Since $a(1) = 2(1)^2 + 2(1)^2 + C = 5$, we have $C = 1$.
a(n) $= \boxed{2nn + 2n + 1}$.

Discussion 18. Note that $a(n) = n^2 + (n+1)^2$.

Sequence 19. 1, 5, 12, 22, 35, ...

Solution 19. Note that the sequence of $\Delta a(n)$ is 4, 7, 10, 13, ... and $\Delta^2 a(n) = 3$. By inspection, we see that $\Delta a(n) = 3n + 1$. Let $a(n) = An^2 + Bn + C$. Then

$$A(2n+1) + B = 3n + 1 \qquad \ldots (*)$$

Putting $n = 1$ and $n = 2$ into (*):

$$\begin{cases} A(2(1)+1) + B = 3(1) + 1 \\ A(2(2)+1) + B = 3(2) + 1 \end{cases}$$

Solving, $A = \dfrac{3}{2}$ and $B = -\dfrac{1}{2}$. Thus $a(n) = \dfrac{3}{2}n^2 - \dfrac{1}{2}n + C$. Since $a(1) = \dfrac{3}{2}(1)^2 - \dfrac{1}{2}(1) + C = 1$, we have $C = 0$.

a(n) = $\boxed{n(3n-1)/2}$.

Discussion 19. This sequence is the sequence of **pentagonal numbers** .

Sequence 20. 1, 5, 14, 30, 55, ...

Solution 20. Note that the sequence of $\Delta a(n)$ is 4, 9, 16, 25, ... , so $\Delta a(n) = (n+1)^2$, which is quadratic. So we guess that $a(n)$ is a cubic sequence. Let $a(n) = An^3 + Bn^2 + Cn + D$. Then

$$a(n+1) - a(n) = \Delta a(n)$$
$$A(n+1)^3 + B(n+1)^2 + C(n+1) + D - (An^3 + Bn^2 + Cn + D) = (n+1)^2$$
$$A(3n^2 + 3n + 1) + B(2n+1) + C = (n+1)^2 \qquad \ldots (*)$$

Putting $n = 1$, $n = 2$ and $n = 3$ into (*):

$$\begin{cases} 7A + 3B + C = 4 \\ 19A + 5B + C = 9 \\ 37A + 7B + C = 16 \end{cases}$$

Solving, $A = \dfrac{1}{3}$, $B = \dfrac{1}{2}$ and $C = \dfrac{1}{6}$. Thus $a(n) = \dfrac{1}{3}n^3 + \dfrac{1}{2}n^2 + \dfrac{1}{6}n + D$. Since $a(1) = \dfrac{1}{3}(1)^3 + \dfrac{1}{2}(1)^2 + \dfrac{1}{6}(1) + D = 1$, we have $D = 0$.

a(n) = $\dfrac{1}{3}n^3 - \dfrac{1}{2}n^2 + \dfrac{1}{6}n = \boxed{n(n+1)(2n+1)/6}$.

Discussion 20. This formula is the sum of the first n square number, i.e. $n(n+1)(2n+1)/6 = \sum_{k=1}^{n} k^2$

Sequence 21. 1, 0, 1, 0, 1, ...

Solution 21. Start with the powers of (-1). The formula is $b(n) = (-1)^n$ and the sequence is: -1, 1, -1, 1, -1, ... Let $c(n) = 1 - b(n)$. The sequence is 2, 0, 2, 0, 2, ... Now divide by 2 to get $a(n)$.

a(n) = $\boxed{(1 - (-1) * *n)/2}$

Sequence 22. 23, 45, 89, 177, 353, ...

Solution 22. Note that the sequence satisfies the recurrence relation $a(n + 1) = 2 * a(n) - 1$ with initial condition $a(1) = 23$.

Let b_n be a sequence that also satisfies the recurrence relation of $a(n)$ (but with different initial conditions) , so that $b_{n+1} = 2b_n - 1$. As the recurrence relation is in the form $a(n + 1) = ka(n) + f(n)$ where f is a constant function, we guess that the general formula of b_n is also a constant function, say $b_n = d$. Putting $b_n = d$ and $b_{n+1} = d$ into the recurrence relation:

$$d = 2d - 1$$

$$d = 1$$

So $b_n = 1$ for all positive integers n . We want to somehow transform b_n into $a(n)$, so let's find some help.

Let h_n be a sequence with homogeneous recurrence relation $h_{n+1} = 2h_n$. This is a geometric sequence with common ration = 2. Let A be the first term (initial condition) . Then the general formula is $h_n = A(2^{n-1})$.

Let's sum these two sequences to make a new sequence a_n and see what happens. Define $a_n = h_n + b_n$ for all positive integers n. We start with the recurrence relations :

$$h_{n+1} = 2h_n \tag{1}$$

$$b_{n+1} = 2b_n - 1 \tag{2}$$

(1) + (2):

$$h_{n+1} + b_{n+1} = 2h_n + 2b_n - 1$$

$$a_{n+1} = 2a_n - 1$$

We get the recurrence relation of $a(n)$, so we are in the right direction.

Let $a(n) = a_n$ and $a(n) = h_n + b_n = A(2^{n-1}) + 1$. We can find the constant A by considering $a(1)$:

$$a(1) = A(2^{1-1}) + 1 = 23$$

$$A = 22$$

a(n) = $22(2^{n-1}) + 1 = \boxed{2 * *n * 11 + 1}$

Sequence 23. 1, 4, 9, 18, 35, ...

Solution 23. Note that the sequence of $\Delta a(n)$ is 3, 5, 9, 17, ... , and the sequence of $\Delta^2 a(n)$ is 2, 4, 8, ... This suggests that the general formula of $\Delta a(n)$ has the component of 2^n . By inspection, we find that $\Delta a(n) = 2^n + 1$. This suggests that $a(n)$ is in a similar form. Let $a(n) = A(2^n) + Bn + C$. Then

$$a(n+1) - a(n) = \Delta a(n)$$

$$A(2^{n+1}) + B(n+1) + C - (A(2^n) + Bn + C) = 2^n + 1$$
$$A(2^n)(2-1) + B = 2^n + 1$$

Comparing both sides, we get $A = 1$ and $B = 1$. So $a(n) = (2^n) + n + C$. Since $a(1) = (2^1) + 1 + C = 1$, we have $C = -2$.
a(n) = $\boxed{2 * *n + n - 2}$

Sequence 24. 1, 9, 36, 100, 225, ...

Solution 24. Note that the sequence is $1^2, 3^2, 6^2, 10^2, 15^2$, ..., which is the square of triangular numbers. The formula of triangular number is $\dfrac{n(n+1)}{2}$, so
a(n) = $\boxed{(n(n+1)/2) * *2}$

Discussion 24. $\left(\dfrac{n(n+1)}{2}\right)^2 = \displaystyle\sum_{k=1}^{n} k^3$

Sequence 25. 17, 69, 17, 69, 17, ...

Solution 25. Note that the sequence of $1 + (-1)^n$ is 0, 2, 0, 2, 0, ... Multiply by 26 to get 0, 52, 0, 52, 0 , ..., then add 17 to get 17, 69, 17, 69, 17, ...
a(n) = $26(1 + (-1)^n) + 17 = \boxed{26(-1) * *n + 43}$

Sequence 26. 4, 16, 64, 256, 1024, ...

Solution 26. Note that the sequence is $2^2, 2^4, 2^6, 2^8, 2^{10}, \ldots$.
a(n) = $\boxed{2 * *(2n)}$

Sequence 27. 2, 16, 512, 65536, 33554432, ...

Solution 27. Note that the sequence is $2^1, 2^4, 2^9, 2^{16}, 2^{25}, \ldots$.
a(n) = $\boxed{2 * *n * *2}$

Sequence 28. 1, 4, 27, 256, 3125, ...

Solution 28. Note that the sequence is $1^1, 2^2, 3^3, 4^4, 5^5, \ldots$.
a(n) = $\boxed{n**n}$

Sequence 29. 2, 4, 16, 256, 65536, ...

Solution 29. Note that the sequence is $2^1, 2^2, 2^4, 2^8, 2^{16}, \ldots$.
a(n) = $\boxed{2**2**(n-1)}$

Sequence 30. 1, 12, 108, 864, 6480, ...

Solution 30. Let $\delta\, a(n) = a(n+1)/a(n)$ denote the first ratio of $a(n)$ (the ratio between two successive terms) . Note that the first ratio is:

$$\begin{aligned}
\delta\, a(1) &= a(2)/a(1) = 12 \\
\delta\, a(2) &= a(3)/a(2) = 9 \\
\delta\, a(3) &= a(4)/a(3) = 8 \\
\delta\, a(4) &= a(5)/a(4) = 7.5
\end{aligned}$$

Let $\delta^2 a(n) = \delta\, a(n+1)/\delta\, a(n)$ denote the second ratio of $a(n)$ (the ratio between two successive first ratios). The second ratio is:

$$\begin{aligned}
\delta^2\, a(1) &= \frac{\delta\, a(2)}{\delta\, a(1)} = \frac{3}{4} \\
\delta^2\, a(2) &= \frac{\delta\, a(3)}{\delta\, a(2)} = \frac{8}{9} \\
\delta^2\, a(3) &= \frac{\delta\, a(4)}{\delta\, a(3)} = \frac{15}{16} \\
\delta^2\, a(n) &= \frac{\delta\, a(n+1)}{\delta\, a(n)} = 1 - \frac{1}{(n+1)^2}
\end{aligned}$$

Multiply both sides by $\delta\, a(n)$:

$$\delta\, a(n+1) = \delta\, a(n)\left(1 - \frac{1}{(n+1)^2}\right)$$

We know that $\delta\, a(1) = 12$. Writing out the following terms:

$$\begin{aligned}
&\delta\, a(2) = 12\,(1 - \frac{1}{2^2}) \\
&\delta\, a(3) = \delta\, a(2)\,(1 - \frac{1}{3^2}) = 12(1 - \frac{1}{2^2})(1 - \frac{1}{3^2})
\end{aligned}$$

$$\delta\, a(4) = \delta\, a(3)\,(1 - \frac{1}{4^2}) = 12(1 - \frac{1}{2^2})(1 - \frac{1}{3^2})(1 - \frac{1}{4^2})$$

$$\vdots$$

We are interested in finding the general formula of this product for $n \geq 2$: (LHS is a just shorthand for the product, not a general formula.)

$$\prod_{k=2}^{n}(1 - \frac{1}{k^2}) = (1 - \frac{1}{2^2})(1 - \frac{1}{3^2})\cdots(1 - \frac{1}{n^2})$$

We can write: [1]

$$\begin{aligned}
\prod_{k=2}^{n}(1 - \frac{1}{k^2}) = \prod_{k=2}^{n}(\frac{k^2-1}{k^2}) &= \prod_{k=2}^{n}(\frac{(k+1)(k-1)}{k^2}) \\
&= \left(\prod_{k=2}^{n}(k+1)\right)\left(\prod_{k=2}^{n}(k-1)\right)\left(\prod_{k=2}^{n}\frac{1}{k^2}\right) \\
&= \left(\prod_{k=3}^{n+1}k\right)\left(\prod_{k=1}^{n-1}k\right)\left(\prod_{k=2}^{n}\frac{1}{k}\right)^2 \\
&= (\frac{(n+1)!}{2!})(n-1)!(\frac{1}{n!})^2 \\
&= \frac{n+1}{2n}
\end{aligned}$$

Thus, $\delta\, a(n) = 12(\frac{n+1}{2n}) = 6\,(\frac{n+1}{n})$. Now we find the formula for $a(n)$. Recall that

$$\delta\, a(n) = \frac{a(n+1)}{a(n)} = 6\,(\frac{n+1}{n})$$

$$a(n+1) = 6\, a(n)(\frac{n+1}{n})$$

We know that $a(1) = 1$. Writing out the following terms:

$$\begin{aligned}
a(2) &= 6(1)(\frac{1+1}{1}) \\
a(3) &= 6\, a(2)\cdot(\frac{2+1}{2}) = 6\left(6(\frac{1+1}{1})\right)\cdot(\frac{2+1}{2}) \\
a(4) &= 6\, a(3)\cdot(\frac{3+1}{3}) = 6\left(6\left(6(\frac{1+1}{1})\right)\cdot(\frac{2+1}{2})\right)\cdot(\frac{3+1}{3}) \\
&\vdots \\
a(n) &= 6^{n-1}\frac{n!}{(n-1)!} \\
a(n) &= n(6^{n-1})
\end{aligned}$$

a(n) = $\boxed{n6**(n-1)}$

Sequence 31. 1, 1, 2, 2, 3, ...

Solution 31. By inspection, we see that the formula of the sequence is $\left\lfloor \dfrac{n+1}{2} \right\rfloor$. Note that the definition of the floor division operator // is:

$$a\,//\,b = \left\lfloor \frac{a}{b} \right\rfloor$$

a(n) = $\boxed{(n+1)//2}$

Sequence 32. 1, 3, 4, 6, 7, ...

Solution 32. We start with the sequence 1, 2, 3, 4, 5 , ..., and add another sequence 0, 1, 1, 2, 2 , ..., which can be created by the formula $n//2$.

a(n) = $\boxed{n+n//2}$

Sequence 33. 1, 2, 3, 4, 4, ...

Solution 33. Consider the sequence 0, 0, 0, 0, -1 . This can be created by the formula $-n//5$. Now add this to the sequence 1, 2, 3, 4, 5 to get the desired sequence.

a(n) = $\boxed{n-n//5}$

Sequence 34. 1, 2, 0, 1, 2, ...

Solution 34. We start with the sequence 1, 2, 3, 4, 5 , ... We need to add another sequence 0, 0, -3, -3, -3, ..., which can be created by the formula $-n//3*3$. Adding the sequences to together, we get

a(n) = $\boxed{n-n//3*3}$

Sequence 35. 1, 2, 3, 2, 1, ...

Solution 35. We start with the sequence 1, 2, 3, 4, 5 , ... We need to add another sequence that has 0 for the first 3 terms and -2 for the 4th term, which can be created by the formula $-n//4*2$. So the sequence becomes 1, 2, 3, 2, 3, ... We need to add another sequence that has 0 for the first 4 terms and -2 for the 5th term, which can be created by the formula $-n//5*2$.

Adding the three sequences to together, we get a(n) = $\boxed{n - n//4 * 2 - n//5 * 2}$

Sequence 36. 1, 0, 0, 0, 0, ...

Solution 36. a(n) = $\boxed{1//n}$

Sequence 37. 1, 2, 3, 5, 9, ...

Solution 37. We start with the sequence of powers of 2 with shifted index: 1/2, 1, 2, 4, 8 , ..., which is 2^{n-2} . Taking the floor of each term, the sequence becomes 0, 1, 2, 4, 8 and the formula becomes $2^{n-2}//1$. Adding 1 to each term, we get the desired sequence 1, 2, 3, 5, 9, ...

a(n) = $\boxed{2 * *(n-2)//1 + 1}$

Sequence 38. 4, 7, 15, 29, 59, ...

Solution 38. We start with the sequence of powers of 2 with shifted index: 1/2, 1, 2, 4, 8 , ..., which is 2^{n-2} . Taking the floor of each term, the sequence becomes 0, 1, 2, 4, 8 and the formula becomes $2^{n-2}//1$. Adding 1 to each term, we get the desired sequence 1, 2, 3, 5, 9, ...

a(n) = $\boxed{2 * *(n-2)//1 + 1}$

Sequence 39. 7, 5, 8, 4, 9, ...

Solution 39. Note that the sequence of $\Delta a(n)$ is -2, 3, -4, 5, ... , so $\Delta a(n) = (n+1)(-1)^n$. Let $a(n) = An^2 + Bn + C$. Then

$$A(2n+1) + B = (n+1)(-1)^n \qquad \ldots(*)$$

Putting $n = 1$ and $n = 2$ into (*):

$$\begin{cases} A(2(1)+1) + B = (2)(-1)^1 \\ A(2(2)+1) + B = (3)(-1)^2 \end{cases}$$

Solving, $A = \frac{5}{2}$ and $B = -\frac{19}{2}$. Thus $a(n) = \frac{5}{2}n^2 - \frac{19}{2}n + C$. Since $a(1) = \frac{5}{2}(1)^2 - \frac{19}{2}(1) + C = 7$, we have $C = 14$.

a(n) = $\frac{5}{2}n^2 - \frac{19}{2}n + 14 = \boxed{(1 - (-1) * *n)/2}$

References

$\boxed{\text{Œ}}$ [1]S. Robot, "Establish a formula for $(1 - 1/4)(1 - 1/9) \ldots (1 - 1/n^2)$," 2015. [Online]. Available: https://www.stumblingrobot.com/2015/07/02/establish-a-formula-for-1-141-19-1-1n2/