Toddler Geometry (Problem set)

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Abstract

Geometry problems are harder than they seem.

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1 Lines, angles and shapes

After all the preposition stating, let's try some practical problems. (The diagrams in the problems are not necessarily to scale.)

Rules and assumptions:

1. The geometric figures are all valid when given all the information in a problem. There won't be a triangle with side lengths 3, 5, 9, which would violate triangle inequality.

2. When we consider things case by case, it is allowed to suppose something that the problem doesn't state. However we need to cover all possibilities.

3. Otherwise, do not assume what the problem doesn't state without proving it. If the problem doesn't state that M is the mid-point of AB , even if the figure looks like it, we cannot assume M is the mid-point of AB unless we can actually prove it. (But if the assumption is true, then skipping some steps to prove it is allowed.)

4. If there is an **invariant** [1] in a problem, then in the solution, we cannot only assume specific values to solve the problem. Otherwise, the solution is incomplete. We need to prove how the invariant is an invariant if the problem doesn't explicitly state that the invariant is an invariant.

5. In a solution, we need to consider edge cases. For example, if there is a quadrilateral with at least one pair of opposite side parallel (i.e. a trapezium), then we need to consider both proper trapezium and parallelogram. If the solution requires finding the intersection of two opposite sides, then it is incomplete.

6. If the problem does not request an approximation for the answer like 'cor. to 3 sig. fig.' , then the answer must be in exact value.

7. Clear steps must be shown in the solution. Using a calculator or computer to skip some computational / arithmetic steps is allowed, but an answer reached by using calculators to calculate approximate numerical values is not a complete solution.

   For example of the former, we can skip the steps to calculate that

   $$(-284\ 650\ 292\ 555\ 885)^3 + 66\ 229\ 832\ 190\ 556^3 + 283\ 450\ 105\ 697\ 727^3 = 74.$$

   For example of the latter, if we use a calculator to calculate that

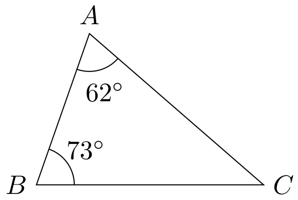
   $$\cos(\frac{\pi}{7}) + \cos(\frac{3\pi}{7}) + \cos(\frac{5\pi}{7}) = \frac{1}{2}$$

   , then the solution is incomplete even if the answer is correct. For a complete solution, we need to show steps.

[1] An invariant means a value that remains the same when the values of other objects change. For example, for a given semi-circle, the sum of area of two squares side-by-side inscribed in the semi-circle is the same for different side lengths of the squares.

1.1 Basic properties

Problem 1. In $\triangle ABC$, $\angle A = 62°$ and $\angle B = 73°$. What is $\angle C$?

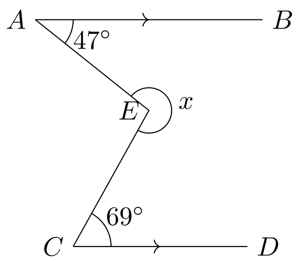


(Difficulty: 1 [Beginner])

Solution 1.

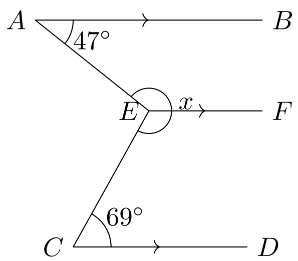
$$\begin{aligned}\angle C &= 180° - \angle A - \angle B \qquad (\angle \text{ sum of } \triangle)\\ &= 180° - 62° - 73°\\ &= \boxed{45°}\end{aligned}$$

Problem 2. In the figure, $AB//CD$, and E is a point between line AB and line CD . $\angle BAE = 47°$ and $\angle DCE = 69°$. What is x ?



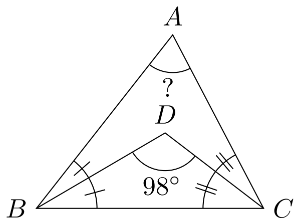
(Difficulty: 3 [Easy])

Solution 2. Draw $EF//AB//CD$.



$$\begin{aligned}\angle AEF + 47° &= 180° \qquad (\text{alt. } \angle\text{s , } AB//EF)\\ \angle AEF &= 133°\\ \angle CEF + 69° &= 180° \qquad (\text{alt. } \angle\text{s , } EF//CD)\\ \angle CEF &= 111°\\ x &= \angle AEF + \angle CEF\\ &= 133° + 111°\\ &= \boxed{244°}\end{aligned}$$

Problem 3. D is a point inside $\triangle ABC$ such that $\angle ABD = \angle DBC$ and $\angle ACD = \angle DCB$, $\angle BDC = 98°$. What is $\angle BAC$?



(Difficulty: 3)

Solution 3. Let $\angle ABD = \angle DBC = x$ and $\angle ACD = \angle DCB = y$. In $\triangle DBC$,

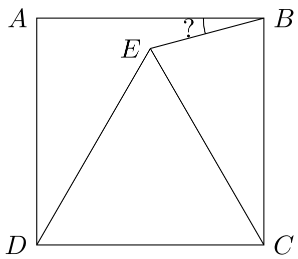
$$\begin{aligned} x + y + 98^\circ &= 180^\circ \qquad (\angle \text{ sum of } \triangle) \\ x + y &= 82^\circ \end{aligned}$$

In $\triangle ABC$,

$$\begin{aligned} \angle BAC + 2x + 2y &= 180^\circ \qquad (\angle \text{ sum of } \triangle) \\ \angle BAC &= 180^\circ - 2(x+y) \\ &= 180^\circ - 2(82^\circ) \\ &= \boxed{16^\circ} \end{aligned}$$

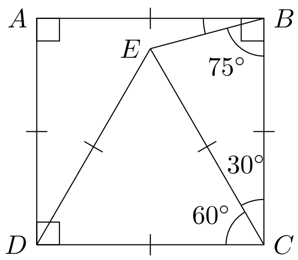
1.3 Triangle properties

Problem 4. $ABCD$ is a square. E is a point inside $ABCD$ such that $\triangle ECD$ is an equilateral triangle. Join BE. What is $\angle ABE$?



(Difficulty: 3 [Easy])

Solution 4. .

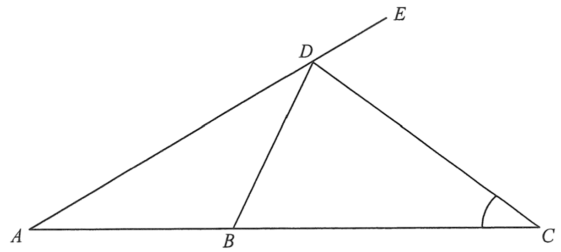


$$\begin{aligned} \angle DCB &= \angle CBA = 90^\circ \qquad (ABCD \text{ is square.}) \\ \angle ECD &= 60^\circ \qquad (\text{prop. of equil } \triangle) \\ \angle ECB &= 90^\circ - 60^\circ = 30^\circ \end{aligned}$$

Note that $EC = BC$.

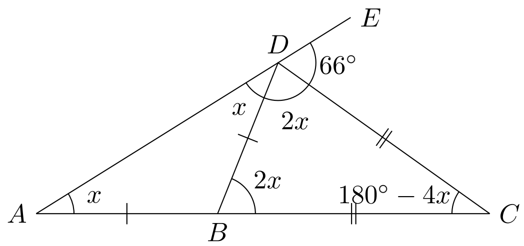
$$\begin{aligned} \therefore\ \angle CBE &= \angle CEB \qquad (\text{base } \angle\text{s, isos. } \triangle) \\ \angle CBE &= (180^\circ - 30^\circ)/2 = 75^\circ \qquad (\angle \text{ sum of } \triangle) \\ \angle ABE &= 90^\circ - 75^\circ = \boxed{15^\circ} \end{aligned}$$

Problem 5. In the figure, ABC and ADE are straight lines. It is given that $AB = BD$ and $BC = CD$. If $\angle CDE = 66°$, then $\angle ACD =$?



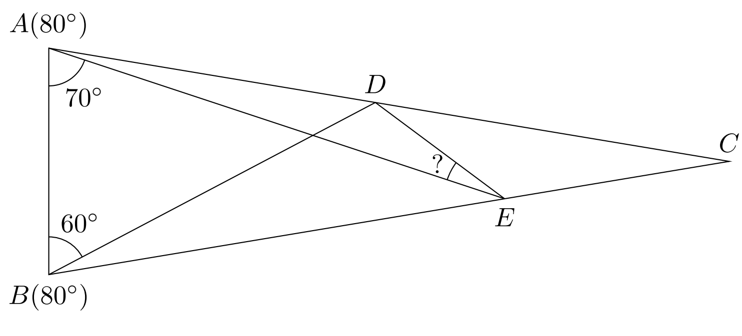
(Difficulty: 3) (2019 DSE Paper 2 Q17)

Solution 5. Let $\angle BAD = x$.



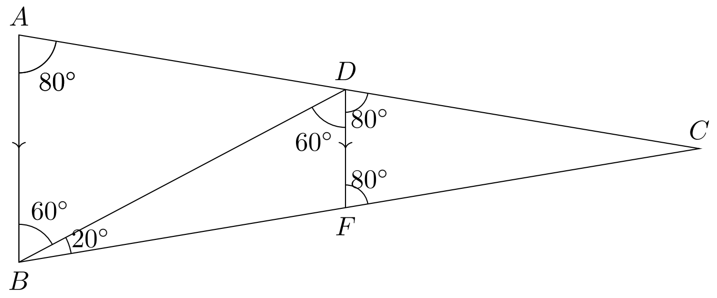
$$
\begin{aligned}
\angle BAD = \angle BDA = x &\qquad \text{(base } \angle\text{s, isos. } \triangle) \\
\angle CBD = 2x &\qquad \text{(ext. } \angle \text{ of } \triangle) \\
\angle CDB = \angle CBD = 2x &\qquad \text{(base } \angle\text{s, isos. } \triangle) \\
\angle BCD = 180° - 2x - 2x = 180° - 4x &\qquad (\angle \text{ sum of } \triangle) \\
\angle DAC + \angle ACD = x + (180° - 4x) = 66° &\qquad \text{(ext. } \angle \text{ of } \triangle) \\
x = 38° & \\
\angle ACD = 180° - 4(38°) = \boxed{28°} &
\end{aligned}
$$

Problem 6. [1] In $\triangle ABC$, $\angle BAC = \angle ABC = 80°$. Let D be a point on side AC such that $\angle ABD = 60°$. Let E be a point on side BC such that $\angle BAE = 70°$. Join DE. What is $\angle AED$?

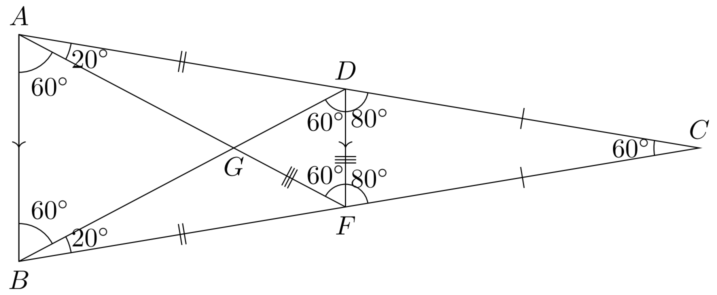


(Difficulty: 7 [Insane])

Solution 6. Let F be a point on side BC such that $AB//DF$. Hide point E to make the figure tidier. Note that $\angle DBC = 80° - 60° = 20°$.



$$\angle CDF = \angle CAB = 80^\circ \qquad \text{(corr. } \angle\text{s , } DF//AB)$$
$$\angle CFD = \angle CBA = 80^\circ \qquad \text{(corr. } \angle\text{s , } DF//AB)$$
$$\angle BDF = 80^\circ - 20^\circ = 60^\circ \qquad \text{(ext. } \angle \text{ of } \triangle)$$

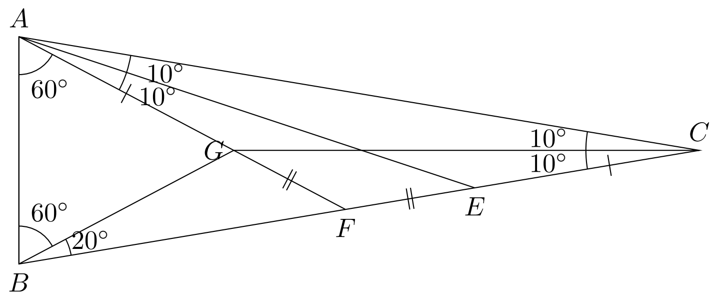


Note that $CD = CF$ and $CA = CB$ (sides opp. equal $\angle$s). Thus $AD = BF$.

Join AF, and let AF and BD intersect at G . In $\triangle ADF$ and $\triangle BFD$, $AD = BF$, $\angle ADF = \angle BFD = 110^\circ$ (adj. $\angle$s on st. line), $DF = DF$. Thus $\triangle ADF \cong \triangle BFD$ (SAS). Thus $\angle DAF = \angle FBD = 20^\circ$ (corr. $\angle$s, $\cong \triangle$s). Also, $\angle AFD = \angle BDF = 60^\circ$ (corr. $\angle$s, $\cong \triangle$s). Thus $\triangle GDF$ is an equilateral triangle (con. of equil. $\triangle$), which means $GF = DF$.

Note that $\angle ACF = 180^\circ - 80^\circ - 80^\circ = 20^\circ$ ($\angle$ sum of $\triangle$). Since $\angle CAF = \angle ACF = 20^\circ$, we have $AF = FC$ (base $\angle$s, isos. $\triangle$).

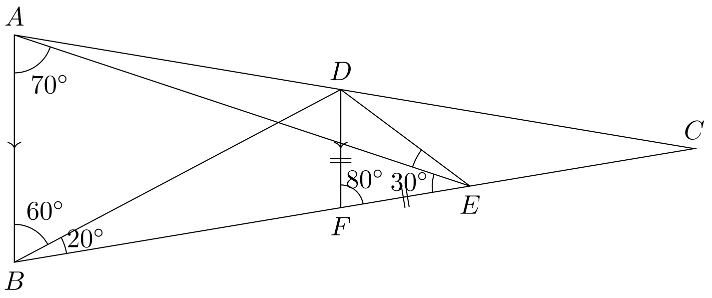
Show point E again and hide GD and DF. Join CG .



Note that $\angle CAE = \angle EAF = 10^\circ$. Also note that GC bisects ACB (because G is in the middle), so $\angle ACG = \angle GCF = 10^\circ$.

Note that $\triangle GAC \cong \triangle ECA$ (ASA), so $AG = EC$ (corr. sides, $\cong \triangle$s). Since $AF = FC$, we have $GF = FE$.

Show D again and hide AF.

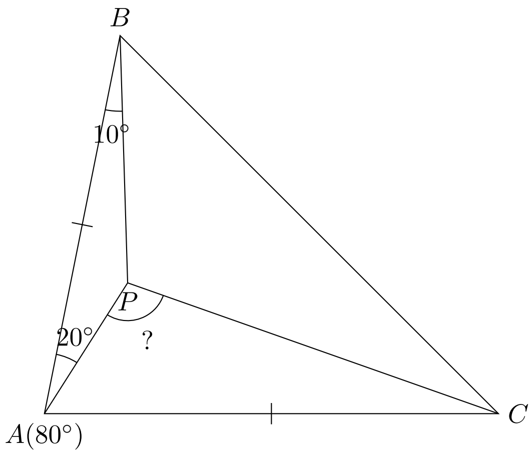


We have shown that $GF = DF$ and $GF = FE$. Thus $DF = FE$. In $\triangle FDE$, $\triangle FDE = \triangle FED$ (base $\angle$s, isos. $\triangle$). So $\angle FED = (180° - 80°)/2 = 50°$ ($\angle$ sum of $\triangle$).

Note that $\angle AEB = 180° - 80° - 70° = 30°$ ($\angle$ sum of $\triangle$).

So $\angle AED = \angle FED - \angle AEB = 50° - 30° = \boxed{20°}$.

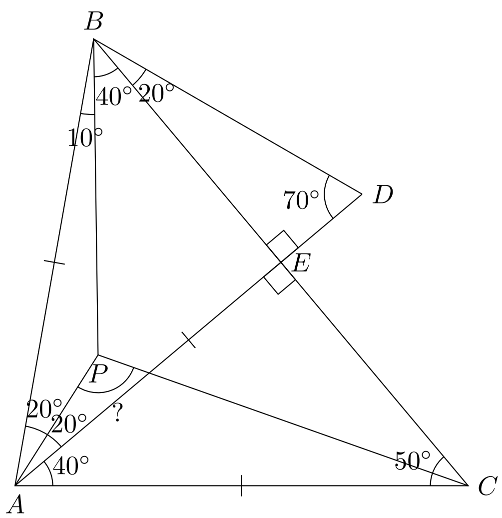
Problem 7. [2] In $\triangle ABC$, $AB = AC$ and $\angle BAC = 80°$. Let P be a point inside $\triangle ABC$ such that $\angle BAP = 20°$ and $\angle ABP = 10°$. What is $\angle APC$?



(Difficulty: 7)

Solution 7. Since $AB = AC$, we have $\angle ABC = \angle ACB$ (base $\angle$s, isos. $\triangle$), so $\angle ABC = \angle ACB = (180° - 80°)/2 = 50°$. So $\angle PBC = 50° - 10° = 40°$.

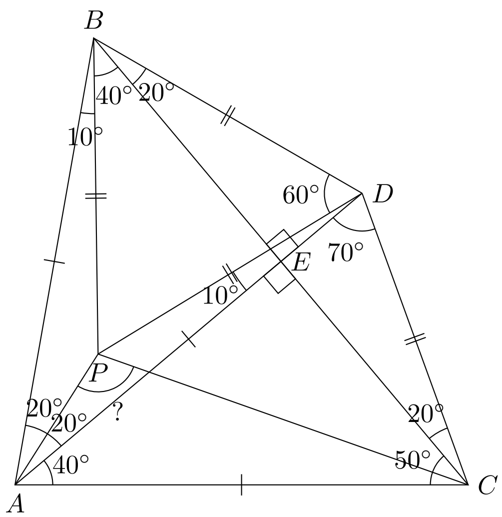
Draw AD between $\angle BAC$ such that $AD = AB$ and $\angle DAC = 40°$. Note that $\angle PAD = 80° - 20° - 40° = 20°$.



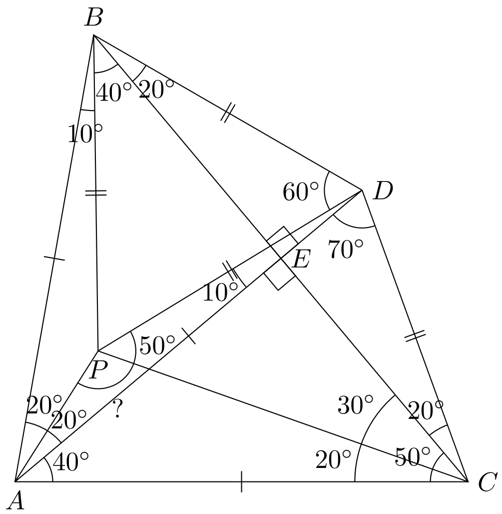
Mark E as the intersection of AD and BC. In $\triangle AEC$, $\angle AEC = 180° - 40° - 50° = 90°$ ($\angle$ sum of $\triangle$).

Join BD . Since $AB = AD$, we have $\angle ABD = \angle ADB = (180° - 40°)/2 = 70°$ (base $\angle$s, isos. $\triangle$)& ($\angle$ sum of $\triangle$). Note that $\angle BED = 90°$ (vert. opp. $\angle$s), so $\angle DBE = 180° - 70° - 90° = 20°$ ($\angle$ sum of $\triangle$).

Join DC and PD. Note that $\triangle DAB \cong \triangle DAC$ (SAS), so $BD = DC$ and $\angle ADC = \angle ADB = 70°$. Since $BD = DC$, we have $\angle DCB = \angle DBC = 20°$ (base $\angle$s, isos. $\triangle$).



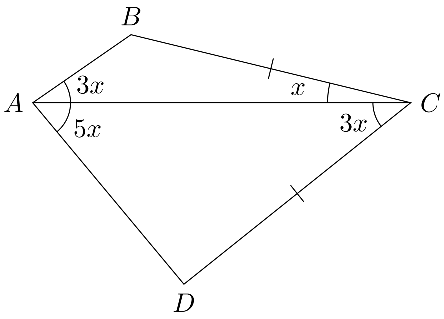
Note that $\triangle BAP \cong \triangle DAP$ (SAS), so $\angle PDA = \angle PBA = 10°$ (corr. $\angle$s, $\cong$ $\triangle$s). Thus $\angle PDB = 70° - 10° = 60°$. Note that in $\triangle BPD$, $\angle PBD = \angle PDB = 60°$. Thus $\triangle BPD$ is an equil. $\triangle$ (con. of equil. $\triangle$), so $BP = DP = BD$. Since $BD = DC$, we have $DP = DC$.



Since $\triangle DPC$ is an isos. $\triangle$ with $DP = DC$, we have $\angle DPC = \angle DCP = (180° - 80°)/2 = 50°$ (base $\angle$s, isos. $\triangle$)& ($\angle$ sum of $\triangle$). Thus $\angle ECP = 50° - 20° = 30°$. So $\angle PCA = 50° - 30° = 20°$.

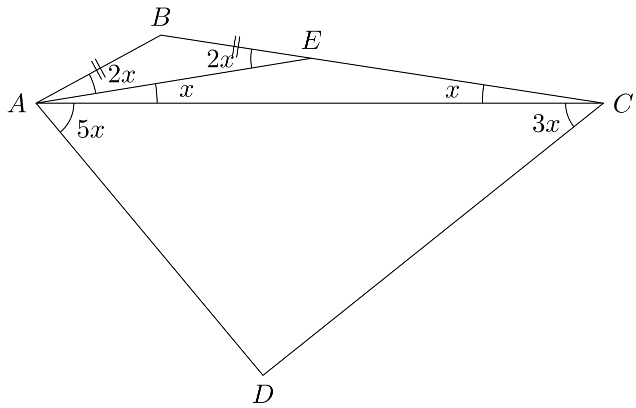
Finally, in $\triangle APC$, $\angle APC = 180° - (20° + 40°) - 20° = \boxed{100°}$.

Problem 8. In quadrilateral $ABCD$, $BC = CD$, $\angle BAC = 3x$, $\angle BCA = x$, $\angle CAD = 5x$ and $\angle ACD = 3x$. What is x ?



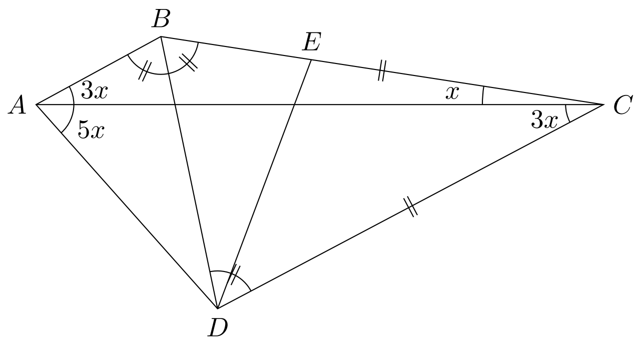
(Difficulty: 6 [Very Hard]) [3]

Solution 8. Let E be on BC such that $EAC = x$.



Then $EA = EC$ (sides opp. equal $\angle$s), $\angle BEA = 2x$ (ext. $\angle$ of $\triangle$), and $\angle BAE = 3x - x = 2x$. Thus $BA = BE$ (sides opp. equal $\angle$s).

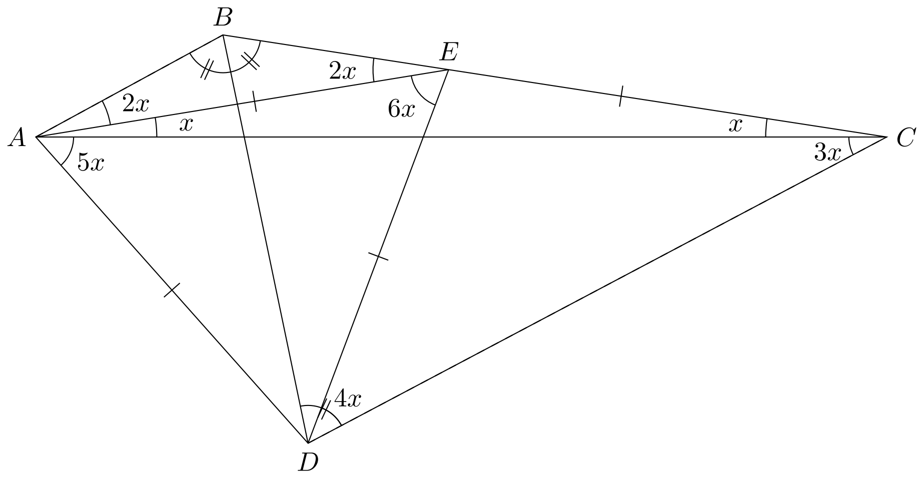
Join BD . Note that $AB//DC$ (alt. $\angle$s equal) , so $\angle ABD = \angle BDC$ (alt. $\angle$s, AB DC). Also, since $CD = CB$ (given), we have $\angle BDC = \angle DBC$. Thus $\angle ABD = \angle CBD$.



In $\triangle ABD$ and $\triangle EBD$, $AB = BE$, $\angle ABD = \angle EBD$, $BD = BD$. Thus $\triangle ABD \cong \triangle EBD$ (SAS).

So $AD = ED$ (corr. sides, $\cong \triangle$s).

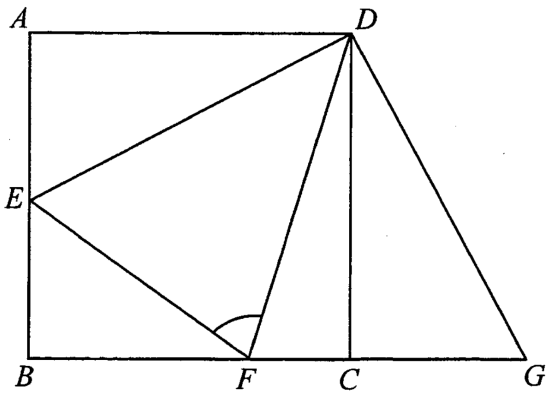
Also $\angle BED = \angle BAD = 8x$ (corr. $\angle$s, $\cong$ $\triangle$s), and $\angle EDC = 8x - 4x = 4x$ (ext. $\angle$ of $\triangle$). Since $\angle EDC = \angle ECD = 4x$, we have $ED = EC$. Since $EA = EC$, we have $EA = AD = ED$, which means $\triangle AED$ is an equil. triangle.



Thus $\angle AED = 60°$ (prop. of equil. $\triangle$) . Since $\angle AED = 8x - 2x = 6x$, we have $6x = 60°$ and $x = \boxed{10°}$.

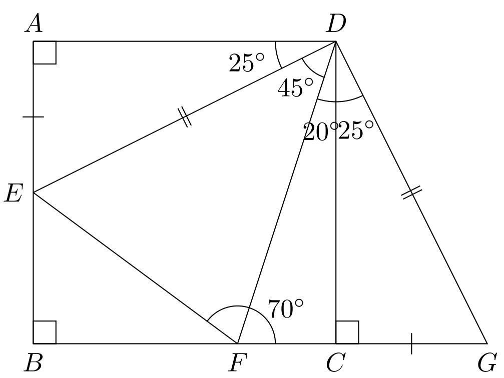
1.4 Quadrilateral properties

Problem 9. In the figure, $ABCD$ is a square. BC is produced to G such that $\angle CDG = 25°$. E is a point lying on AB such that $AE = CG$. If F is a point lying on BC such that $\angle CDF = 20°$, then $\angle DFE =$?



(Difficulty: 4) (2014 DSE Paper 2 Q16)

Solution 9. .



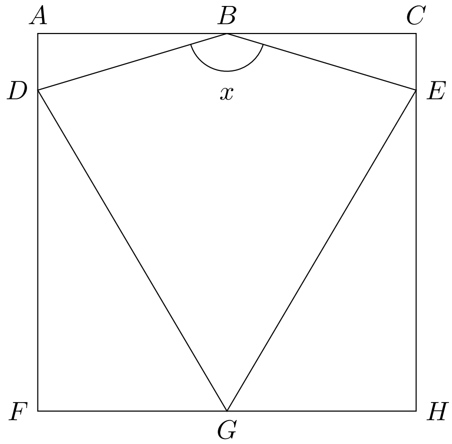
Note that $\triangle DAE \cong \triangle DCG$ (SAS) , so we have $\angle ADE = \angle CDG = 25°$ (corr. sides, $\cong \triangle$s). Note that $\angle EDF = 90° - 25° - 20° = 45°$.

In $\triangle DFE$ and $\triangle DFG$,

$$\begin{aligned} DE &= DG \qquad \text{(corr. sides, } \cong \triangle\text{s)} \\ \angle EDF &= \angle FDG = 45^\circ \\ DF &= DF \qquad \text{(common side)} \\ \therefore\ \triangle DFE &\cong \triangle DFG \qquad \text{(SAS)} \\ \therefore\ \angle DFE &= \angle DFG \qquad \text{(corr. } \angle\text{s, } \cong \triangle\text{s)} \\ &= 90^\circ - 20^\circ = \boxed{70^\circ} \qquad (\angle \text{ sum of } \triangle) \end{aligned}$$

Problem 10. The kite $GDBE$ is inscribed in the square $ACHF$. $DG = GB = EG$.

Calculate the size, x, of $\angle DBE$.

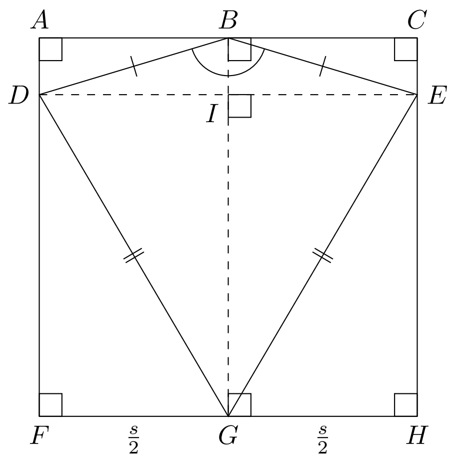


(Note: Do not assume that G must be the mid-point of FH . Otherwise, the solution is not complete.)

(Difficulty: 6) [4]

Solution 10. Let s be the side length of the square. Join BG and DE , and let I be their intersection. Note that $BG \perp DE$ (diags of kite).

Suppose that G is the mid-point of FH (lol, you can't tell me what to do) .



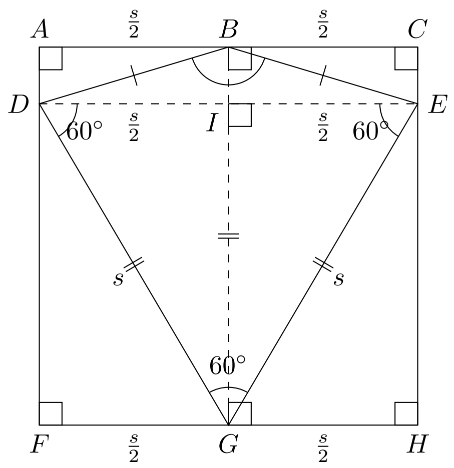
Then $\triangle GFD \cong \triangle GHE$ (RHS) , so $DF = EH$ (corr. sides, $\cong \triangle$s). So $DEHF$ is a rectangle (1 equal pair, 2 right $\angle$s) .

Since $DE//FH$ (prop. of rectangle) and $BG \perp DE$ (diags of kite), we also have $BG \perp FH$ and $BG \perp AC$ (int. $\angle$s , $DE//FH//AC$) .

Thus $ABGF$ and $BCHG$ are rectangles (3 right $\angle$s) . Thus $AB = BC = \frac{s}{2}$ (opp. sides of rectangles), and B is also the mid-point of AC .

Similarly, $DIGF$ and $IEHG$ are rectangles (3 right $\angle$s), so $DI = IE = \frac{s}{2}$ (opp. sides of rectangles) .

Updated figure:



Note that $DG = BG = EG = s$ (given). Since $DE = DG = EG = s$, $\triangle DEG$ is an equilateral triangle, so $\angle DGE = \angle GDE = \angle GED = 60°$ (prop. of equil. $\triangle$) .

Note that $\triangle GDB \cong \triangle GEB$ (SSS) . So we have $\angle DGI = \angle EGI = 30°$ (corr. $\angle$s, $\cong$ $\triangle$s).

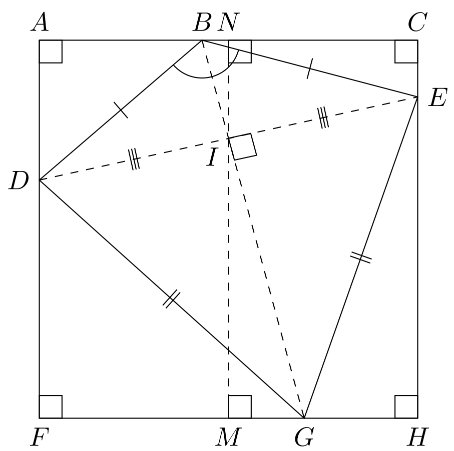
Note that $\triangle GDB$ and $\triangle GEB$ are isos. triangles, so we have $\angle GBD = (180° - 30°)/2 = 75°$ (base $\angle$s, isos. $\triangle$)&($\angle$ sum of $\triangle$). Similarly, $\angle GBE = 75°$, which means $x = \angle DBE = 75° + 75° = \boxed{150°}$.

Wait. We are not done yet. (Skip this part if you want to live in blissful ignorance.) Now we suppose that G is not the mid-point of FH . First, we need to show that such a kite is possible to exist.

Let M be the mid-point of FH and N be the mid-point of AC. Suppose that G is at the right of M .

Let there be quadrilateral $GDBE$ inscribed in the square as in the figure, where $BG \perp DE$. Let I be the intersection of BG and DE .

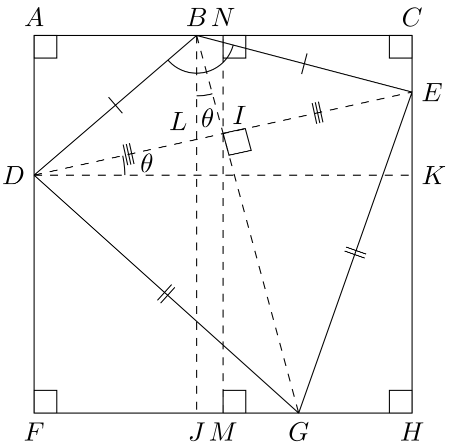
To make $GDBE$ a kite, we want to make $DI = IE$, which can only happen when I lies on MN (intercept theorem). Thus, B must be lying to the left of AC , so that BG and MN intersect inside the square.



Since BG is the perpendicular bisector of DE , we have $BD = BE$ and $GD = GE$ (prop. of $\perp$ bisector) , which means $GDBE$ is a kite. So it is possible that the kite is tilted inside the square.

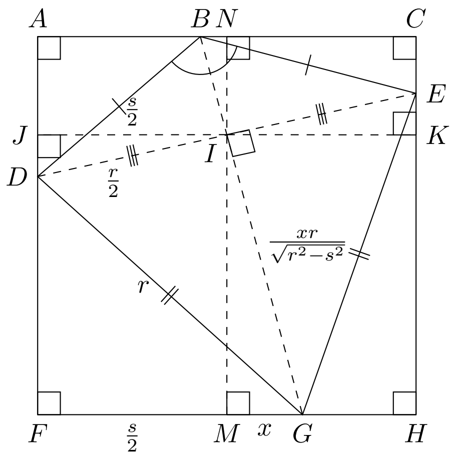
Note that $BG = DE$, explained as follows: Let $BJ \perp FH$ and $DK \perp CH$, and DE and BJ intersect at L. Note that $\angle EDK = 90° - \angle DLJ = 90° - \angle BLE = \angle JBG$ ($\angle$ sum of $\triangle$)&(vert. opp. $\angle$s).

In $\triangle BJG$ and $\triangle DKE$, we have $\angle BJG = \angle DKE = 90°$, $\angle JBG = \angle EDK$, $BJ = DK$. Thus $\triangle BJG \cong \triangle DKE$ (AAS) , so $BG = DE$ (corr. sides, $\cong \triangle$s).



But don't forget that we need one more condition given in the problem: $DG = BG = EG$. Is it still possible that the kite is tilted? First suppose that $DG = BG = EG = r$.

Let $MG = x$ and the side length of the square be s . Let $IJ \perp AF$ and $IK \perp CH$.



Note that $\angle JID = \angle MIG$ since $\angle DIG = \angle JIM = 90°$. Thus $\angle DJI \sim \angle GMI$ (AA) .

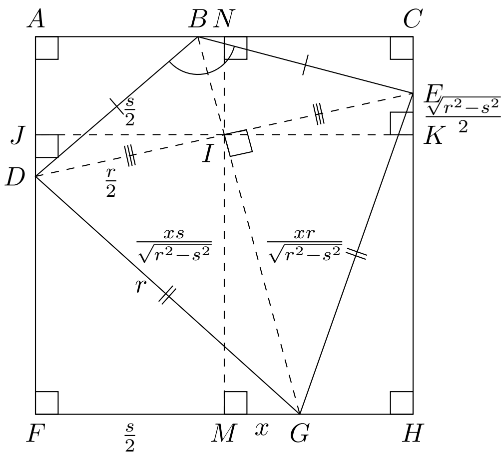
Note that $DI = \frac{r}{2}$ since $DI = BG = r$. Then $JD = \sqrt{(\frac{r}{2})^2 - (\frac{s}{2})^2} = \dfrac{\sqrt{r^2 - s^2}}{2}$ (pyth. theorem).

So $\dfrac{IG}{MG} = \dfrac{ID}{JD} = \dfrac{r}{\sqrt{r^2 - s^2}}$ (corr. sides, $\sim \triangle$s), which means $IG = \dfrac{xr}{\sqrt{r^2 - s^2}}$.

In $\triangle DIG$, we have by pyth. theorem:

$$\begin{aligned}
(\frac{r}{2})^2 + (\frac{xr}{\sqrt{r^2 - s^2}})^2 &= r^2 \\
\frac{r^2}{4} + \frac{x^2 r^2}{r^2 - s^2} &= r^2 \\
\frac{x^2}{r^2 - s^2} &= \frac{3}{4} \\
\frac{4}{3} x^2 &= r^2 - s^2 \\
r &= \sqrt{\frac{4}{3}\, x^2 + s^2}
\end{aligned}$$

So there is a specific value of r when given an x . But we still need to show that E can lie on side CH .



Note that $EK = JD = \dfrac{\sqrt{r^2 - s}}{2}$ (corr. sides, $\triangle JID \cong \triangle KIE$) . Also note that $IM = \dfrac{xs}{\sqrt{r^2 - s^2}}$ by similar triangles. Then the position of E above H is $IM + EK$. For E to lie on side CH , we must have $IM + EK < s$. Thus we have

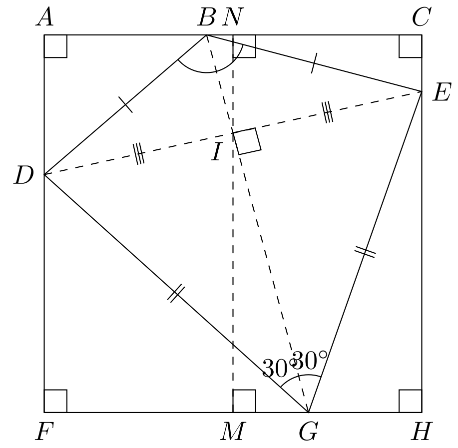
$$\frac{xs}{\sqrt{r^2 - s^2}} + \frac{\sqrt{r^2 - s}}{2} < s$$

Put $r = \sqrt{\frac{4}{3}\, x^2 + s^2}$:

$$\begin{aligned}
\frac{xs}{\sqrt{\frac{4}{3}\, x^2 + s^2 - s^2}} + \frac{\sqrt{\frac{4}{3}\, x^2 + s^2 - s}}{2} &< s \\
\frac{xs}{\frac{2}{\sqrt{3}}\, x} + \frac{\frac{2}{\sqrt{3}}\, x}{2} &< s \\
\frac{1}{\sqrt{3}}\, x &< s - \frac{\sqrt{3}}{2}\, s \\
x &< s(\sqrt{3} - \frac{3}{2})
\end{aligned}$$

Since $\sqrt{3} - \frac{3}{2} \approx 0.232$, it is possible for the kite to be inscribed in the square if $x \approx< 0.232s$ while satisfying the requirements given in the problem.

The rest of the solution proceeds like the case where G is the mid-point of FH . We have $\triangle DEG$ being an equil. $\triangle$, so $\angle DGB = \angle BGE = 30^\circ$, and $\angle DBE = (180^\circ - 30^\circ)/2 \times 2 = \boxed{150^\circ}$.

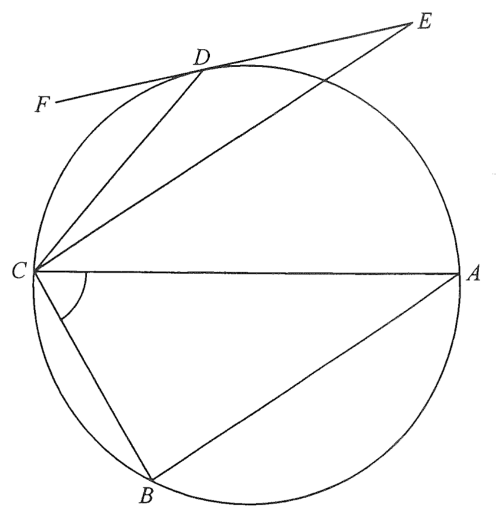


1.6 Circle properties

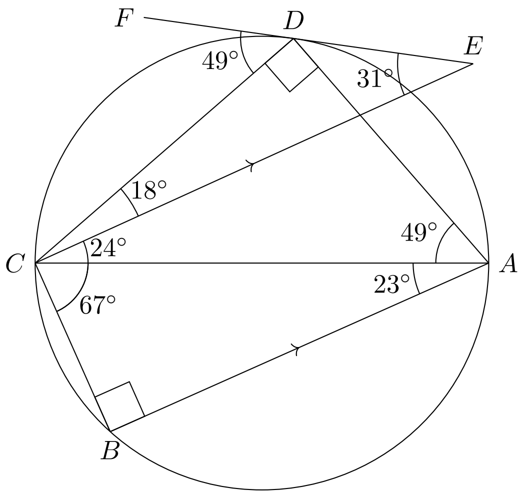
(Problem solving tips: try to use all the information given in the problem.)

Problem 11. In the figure, AC is a diameter of the circle $ABCD$. EF is the tangent to the circle at D such that $AB//EC$. If $\angle CDF = 49°$ and $\angle CED = 31°$, then $\angle ACB =$?

(Difficulty: 4 [Medium]) (2021 DSE Paper 2 Q39)

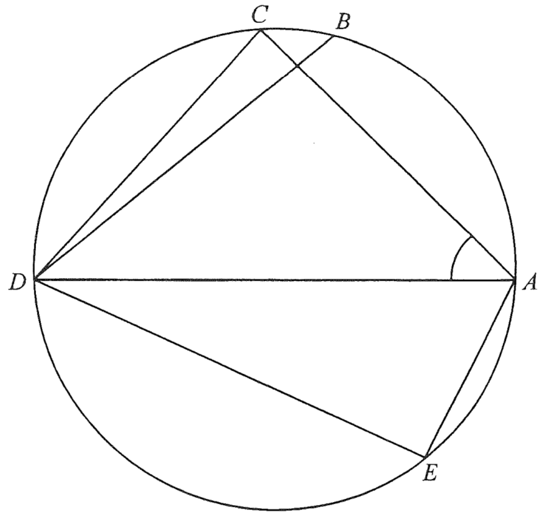


Solution 11. (Diagram adjusted for accuracy.) Join DA .



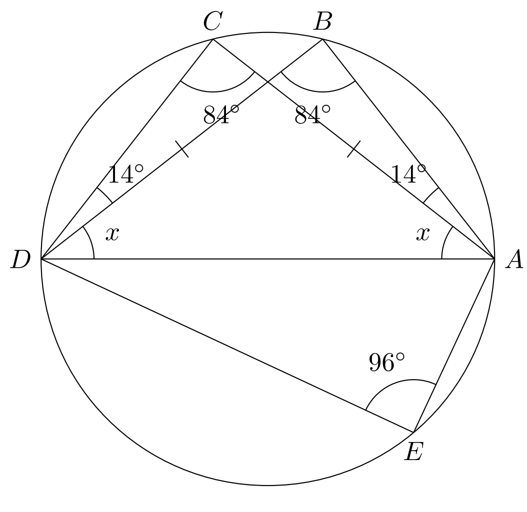
$$\begin{aligned}
\angle CDA, \angle ABC &= 90^\circ \qquad (\angle \text{ in semi-circle}) \\
\angle CAD &= 49^\circ \qquad (\angle \text{ in alt. segment}) \\
\angle DCA &= 90^\circ - 49^\circ = 41^\circ \qquad (\angle \text{ sum of } \triangle) \\
\angle DCE &= 49^\circ - 31^\circ = 18^\circ \qquad (\text{ext. } \angle \text{ of } \triangle) \\
\angle ACE &= 41^\circ - 18^\circ = 23^\circ \\
\angle BAC &= \angle ACE = 23^\circ \qquad (\text{alt. } \angle\text{s , } AB//EC) \\
\angle ACB &= 90^\circ - 23 = \boxed{67^\circ} \qquad (\angle \text{ sum of } \triangle)
\end{aligned}$$

Problem 12. In the figure, $ABCDE$ is a circle. If $AC = BD$, $\angle AED = 96^\circ$ and $\angle BDC = 14^\circ$, then $\angle CAD =$?



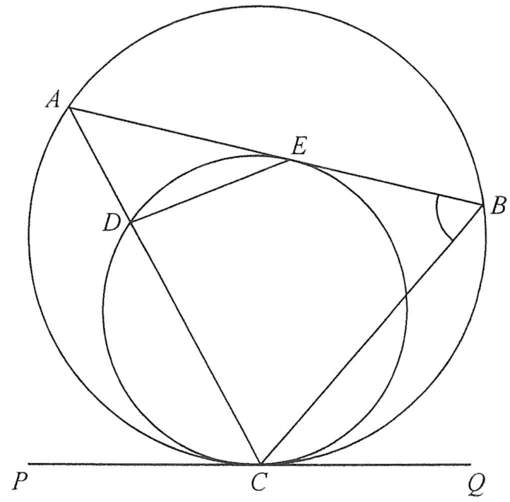
(Difficulty: 4) (2021 DSE Paper 2 Q22)

Solution 12. Join AB . Let $\angle CAD = x$.



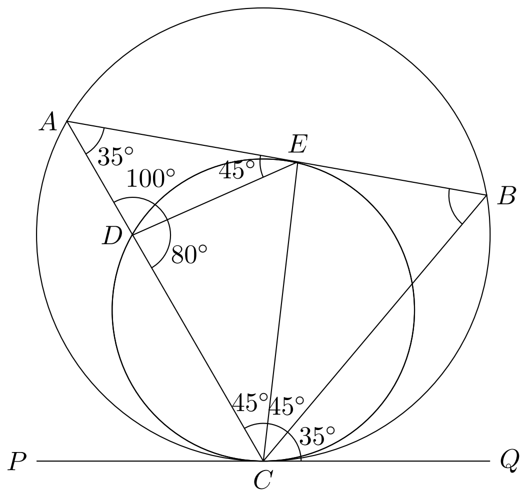
$$\begin{aligned}
\angle DCA, \angle DBA &= 180 - 96^\circ = 84^\circ \qquad (\text{opp. } \angle\text{s , cyclic quad.}) \\
\angle BAC &= 14^\circ \qquad (\angle\text{s in the same segment}) \\
\angle CDA &= \angle BAD = x + 14^\circ \qquad (\text{equal chords, equal } \angle\text{s at } \odot^{ce}) \\
\angle BDA &= x \\
84^\circ + 14^\circ + 2x &= 180^\circ \qquad (\text{ext. } \angle \text{ of } \triangle)\& \ (\angle \text{ sum of } \triangle) \\
x &= \boxed{41^\circ}
\end{aligned}$$

Problem 13. In the figure, ABC and CDE are circles such that ADC is a straight line. PQ is the common tangent to the two circles at C . AB is the tangent to the circle CDE at E . If $\angle ADE = 100°$ and $\angle BCQ = 35°$, then $\angle ABC =$?



(Difficulty: 4) (2020 DSE Paper 2 Q39)

Solution 13. Join EC .



$\angle CAB = 35°$ (∠ in alt. segment)

$\angle AED = 180° - 35° - 100° = 45°$ (∠ sum of △)

$\angle DCE = 45°$ (∠ in alt. segment)

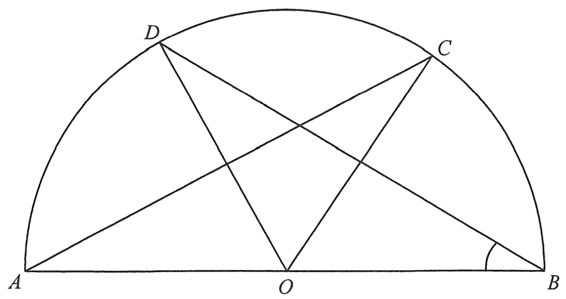
$\angle EDC = 180° - 100° = 80°$ (adj. ∠s on st. line)

$\angle ECQ = \angle EDC = 80°$ (∠ in alt. segment)

$\angle ECB = 80° - 35° = 45°$

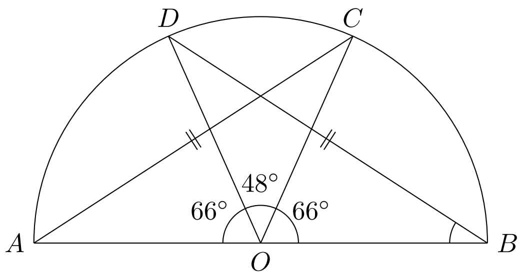
$\angle ABC = 180° - 35° - (45° + 45°) = \boxed{55°}$ (∠ sum of △)

Problem 14. In the figure, O is the centre of the semi-circle $ABCD$. If $AC = BD$ and $\angle COD = 48°$, then $\angle ABD =$?



(Difficulty: 3) (2019 DSE Paper 2 Q21)

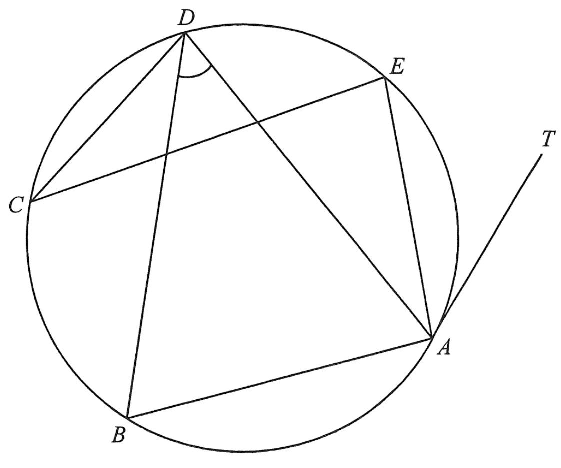
Solution 14. .



Note that $\triangle OAC \cong \triangle OBD$ (SSS) . This means $\angle AOC = \angle DOB$ (corr. sides, $\cong \triangle$s), and thus $\angle AOD = \angle BOC = (180° - 48°)/2 = 66°$ (adj. $\angle$s on st. line). In $\triangle OBD$,

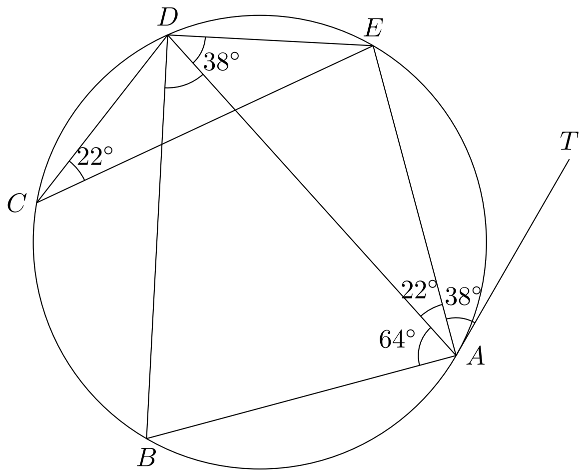
$$\angle ABD = (180° - 48° - 66°)/2 = \boxed{33°} \qquad (\angle \text{ sum of } \triangle)$$

Problem 15. In the figure, TA is the tangent to the circle $ABCDE$ at point A . If $\angle BAD = 64°$, $\angle EAT = 38°$ and $\angle DCE = 22°$, then $\angle ADB =$?



(Difficulty: 3) (2019 DSE Paper 2 Q39)

Solution 15. Join DE .

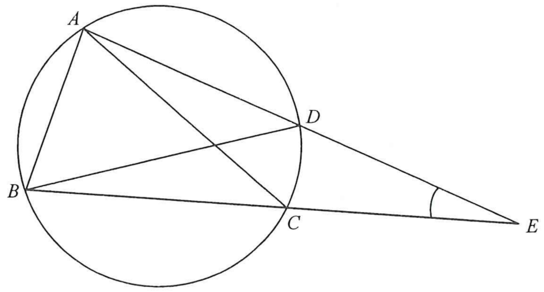


$\angle ADE = 38°$ ($\angle$ in alt. segment)

$\angle EAD = 22°$ ($\angle$s in the same segment)

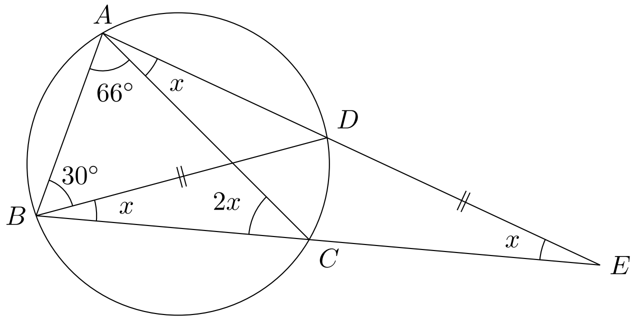
$\angle ADB = 180° - 64° - 22° - 38° = \boxed{56°}$

Problem 16. In the figure, $ABCD$ is a circle. AD produced and BC produced meet at the point E . It is given that $BD = DE$, $\angle BAC = 66°$ and $\angle ABD = 30°$. Find $\angle CED$.



(Difficulty: 3) (2018 DSE Paper 2 Q22)

Solution 16. Let $\angle CED = x$.



$\angle DBE = x$ (base $\angle$s, isos. $\triangle$)

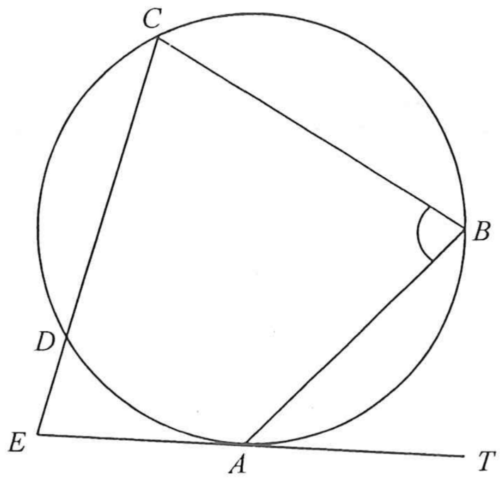
$\angle CAD = \angle CBD = x$ ($\angle$s in the same segment)

$\angle ACB = \angle CED + \angle CAD = 2x$ (ext. $\angle$ of $\triangle$)

In $\triangle ABC$, $66° + (30° + x) + 2x = 180°$ ($\angle$ sum of $\triangle$)

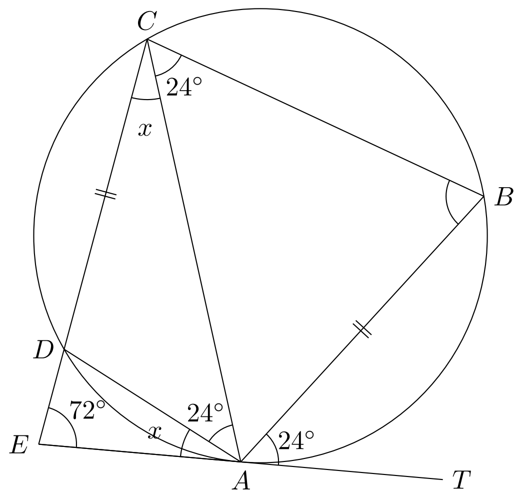
$x = \boxed{28°}$

Problem 17. In the figure, TA is the tangent to the circle $ABCD$ at the point A . CD produced and TA produced meet at the point E . It is given that $AB = CD$, $\angle BAT = 24°$ and $\angle AED = 72°$. Find $\angle ABC$.



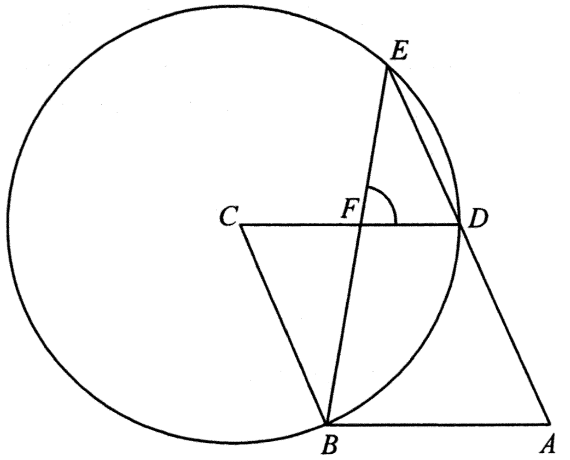
(Difficulty: 4) (2018 DSE Paper 2 Q39)

Solution 17. Join AD and AC. Let $\angle EAD = x$.



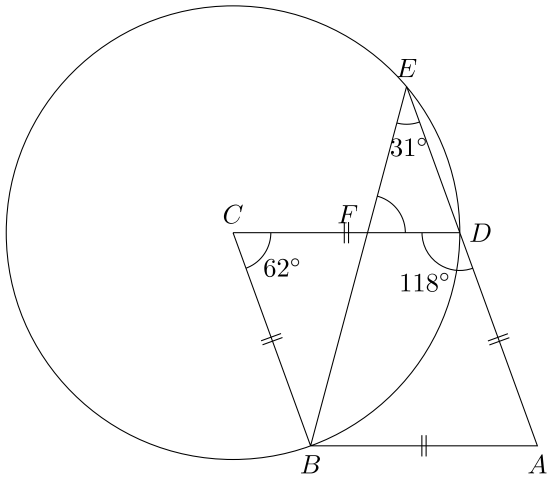
$$\angle ACB = 24° \qquad (\angle \text{ in alt. segment})$$
$$\angle CAD = \angle ACB = 24° \qquad (\text{equal chords, equal } \angle\text{s at } \odot^{ce})$$
$$\angle DCA = \angle EAD = x \qquad (\angle \text{ in alt. segment})$$
In $\triangle CEA$,
$$72° + x + (x + 24°) = 180° \qquad (\angle \text{ sum of } \triangle)$$
$$x = 42°$$
$$\angle ABC = \angle EAC = 42° + 24° \qquad (\angle \text{ in alt. segment})$$
$$= \boxed{66°}$$

Problem 18. In the figure, $ABCD$ is a rhombus. C is the centre of the circle BDE and ADE is a straight line. BE and CD intersect at F . If $\angle ADC = 118°$, then $\angle DFE =?$



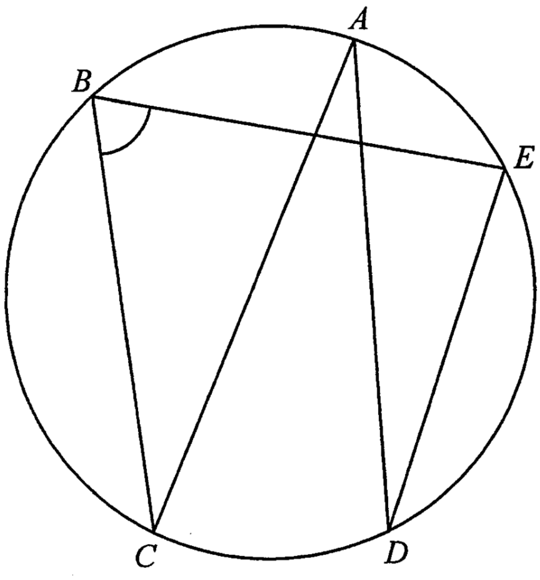
(Difficulty: 3) (2016 DSE Paper 2 Q22)

Solution 18. .



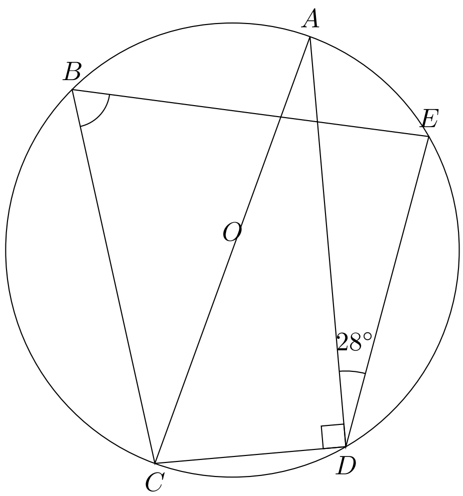
$$CB//DA \qquad \text{(prop. of rhombus)}$$
$$\angle C = 180^\circ - 118^\circ = 62^\circ \qquad (\text{int. } \angle\text{s}\ ,\ CB//DA)$$
$$\angle FED = 62^\circ/2 = 31^\circ \qquad (\angle \text{ at centre twice } \angle \text{ at } \odot^{ce})$$
$$\angle DFE = 118^\circ - 31^\circ = \boxed{87^\circ} \qquad (\text{ext. } \angle \text{ of } \triangle)$$

Problem 19. In the figure, AC is a diameter of the circle $ABCDE$. If $\angle ADE = 28^\circ$, then $\angle CBE =$?



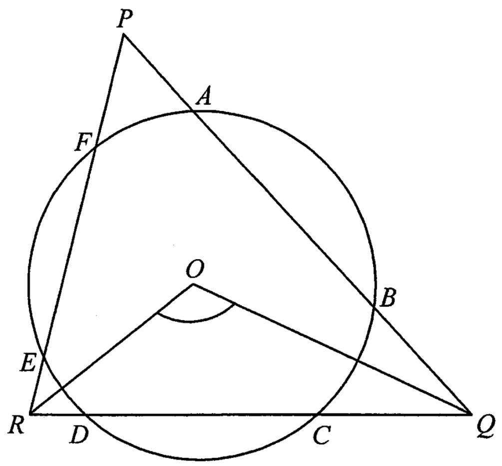
(Difficulty: 3) (2014 DSE paper 2 Q20)

Solution 19. Join CD .



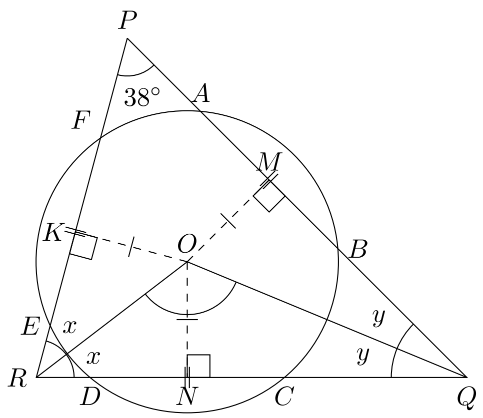
$$\angle ADC = 90^\circ \qquad (\angle \text{ in semi-circle})$$
$$\angle CDE = 90^\circ + 28^\circ = 118^\circ$$
$$\angle CBE = 180^\circ - 118^\circ = \boxed{62^\circ} \qquad (\text{opp. } \angle\text{s , cyclic quad.})$$

Problem 20. In the figure, O is the centre of the circle $ABCDEF$. $\triangle PQR$ intersects the circle at A, B, C, D, E and F . If $\angle QPR = 38^\circ$ and $AB = CD = EF$, then $\angle QOR =$?



(Difficulty: 4) (2014 DSE Paper 2 Q21)

Solution 20. Draw $OM \perp AB$, $ON \perp DC$, $OK \perp FE$.



Note that $OM = ON = OK$ (equal chords, equidistant from centre) . Thus, $\angle ORK = \angle ORN$ and $\angle OQN = \angle OQM$ (prop. of $\angle$ bisector) .

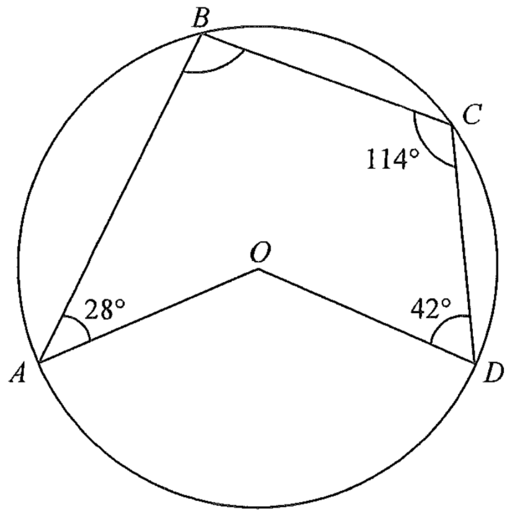
Let $\angle ORK = \angle ORN = x$ and $\angle OQN = \angle OQM = y$. In $\triangle PQR$,

$$\begin{aligned} 38^\circ + 2x + 2y &= 180^\circ \qquad (\angle \text{ sum of } \triangle) \\ x + y &= 71^\circ \end{aligned}$$

In $\triangle ORQ$,

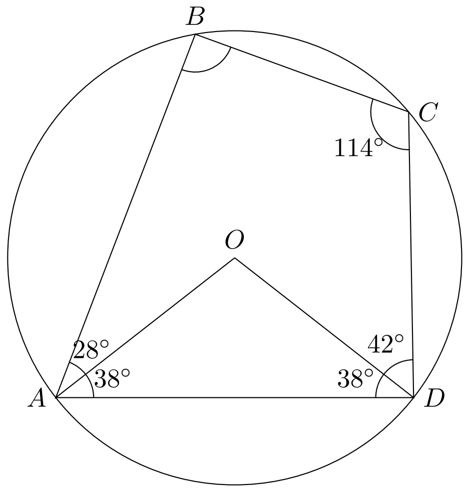
$$\begin{aligned} x + y + \angle QOR &= 180^\circ \qquad (\angle \text{ sum of } \triangle) \\ \angle QOR &= 180^\circ - 71^\circ = \boxed{109^\circ} \end{aligned}$$

Problem 21. In the figure, O is the centre of the circle $ABCD$. If $\angle BAO = 28^\circ$, $\angle BCD = 114^\circ$ and $\angle CDO = 42^\circ$, then $\angle ABC =$?



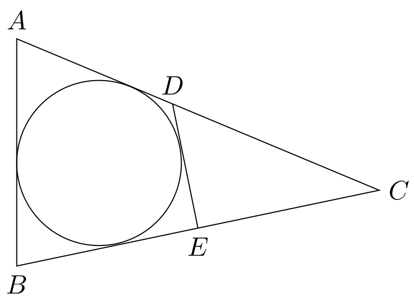
(Difficulty: 3) (2012 DSE Paper 2 Q20)

Solution 21. Join AD .



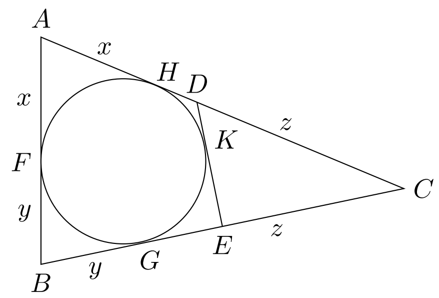
$$\angle BAD = 180^\circ - 114^\circ = 66^\circ \qquad \text{(opp. } \angle\text{s , cyclic quad.)}$$
$$\angle OAD = 66^\circ - 28^\circ = 38^\circ$$
$$\angle ODA = 38^\circ \qquad \text{(base } \angle\text{s, isos. } \triangle\text{)}$$
$$\angle ABC = 180^\circ - (38^\circ + 42^\circ) = \boxed{100^\circ} \qquad \text{(opp. } \angle\text{s , cyclic quad.)}$$

Problem 22. In $\triangle ABC$, $AB = 3$, $BC = 5$, and $CA = 6$. D is on AC and E is on BC such that DE is tangent to the inscribed circle of $\triangle ABC$. What is the perimeter of $\triangle CDE$?



(Difficulty: 4) [5]

Solution 22. Let the circle touches AB, BC, CA, DE at F, G, H, K respectively. Note and let that $AF = AH = x$, $BF = BG = y$, $CG = CH = z$ (tangent properties).



By considering the side lengths of the triangle, we have

$$x + y = 3 \quad (1)$$
$$y + z = 5 \quad (2)$$
$$x + z = 6 \quad (3)$$

$(1) + (2) + (3)$:

$$2x + 2y + 2z = 14$$
$$x + y + z = 7 \quad (4)$$

Put (1) into (4):

$$z = 7 - 3 = 4$$

Note that $DH = DK$ and $EG = EK$ (tangent properties) . Thus $CD + DK = CD + DH = CH$. Similarly, $CE + EK = CE + EG = CH$.

Thus, the perimeter of triangle $= CD + DK + CE + EK = CH + CG = 4 + 4 = \boxed{8}$.

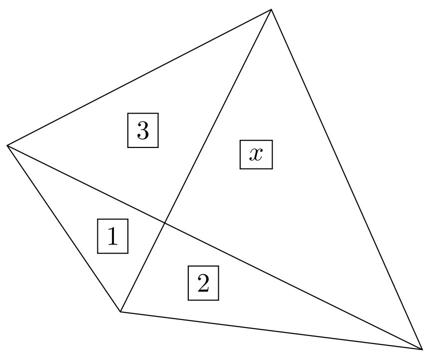
1.7 Area, perimeter and hypotenuse

1.7.1 Area

Problem 23. A convex quadrilateral is divided into four parts by its diagonals.

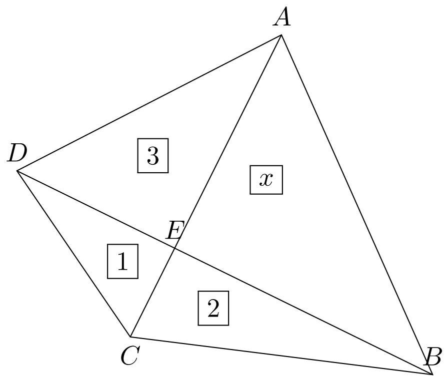
Three of the areas are 2 , 1 , and 3 as shown in the diagram.

What is the area of the fourth region denoted by x ?



(Difficulty: 2 [Very Easy]) [6]

Solution 23. Label the quadrilateral as $ABCD$, and let E be the intersection of diagonals AC and BD .



Note that $\dfrac{\text{area of } \triangle AED}{\text{area of } \triangle CED} = \dfrac{AE}{EC}$ (bases prop. to areas of $\triangle$s) .

Similarly, $\dfrac{\text{area of } \triangle AEB}{\text{area of } \triangle CEB} = \dfrac{AE}{EC}$ (bases prop. to areas of $\triangle$s) .

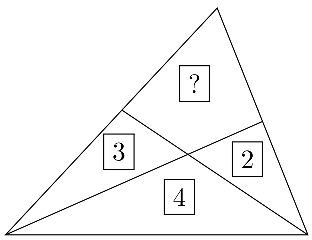
Thus, we have $\dfrac{\text{area of } \triangle AEB}{\text{area of } \triangle CEB} = \dfrac{\text{area of } \triangle AED}{\text{area of } \triangle CED}$, which means

$$\frac{x}{2} = \frac{3}{1}$$
$$x = \boxed{6}$$

Problem 24. In the figure, a triangle is divided into four parts by two cevians, in which there are three triangle parts and a quadrilateral part.

The area of the triangle parts are 3, 4, 2, where the triangle part with area 4 is vertically opposite of the quadrilateral part.

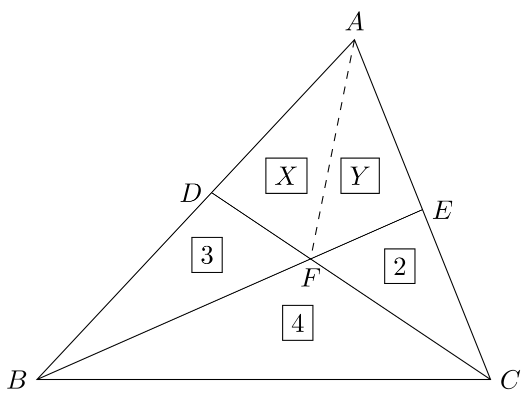
What is the area of the quadrilateral part?



(Difficulty: 4) [7]

Solution 24. Label the triangle $\triangle ABC$. Label D be on AB and E be on AC , F be the intersection of DC and BE , such that area of $\triangle DBF = 3$ and area of $\triangle ECF = 2$.

Join AF . Let area of $\triangle ADF = X$ and area of $\triangle AEF = Y$.



By considering the ratio of the bases $\dfrac{EF}{FB}$, we have by 'bases prop. to areas of △s':

$$\begin{aligned}\frac{\text{area of }\triangle AFE}{\text{area of }\triangle AFB} &= \frac{\text{area of }\triangle CFE}{\text{area of }\triangle CFB}\\ \frac{Y}{X+3} &= \frac{2}{4}\\ 4Y &= 2X+6\\ -X+2Y &= 3 \qquad (1)\end{aligned}$$

Similarly, by considering the ratio of the bases $\dfrac{DF}{FC}$, we have by 'bases prop. to areas of △s':

$$\begin{aligned}\frac{\text{area of }\triangle AFD}{\text{area of }\triangle AFC} &= \frac{\text{area of }\triangle BFD}{\text{area of }\triangle BFC}\\ \frac{X}{Y+2} &= \frac{3}{4}\\ 4X &= 3Y+6\\ 4X-3Y &= 6 \qquad (2)\end{aligned}$$

(1)×4 + (2):

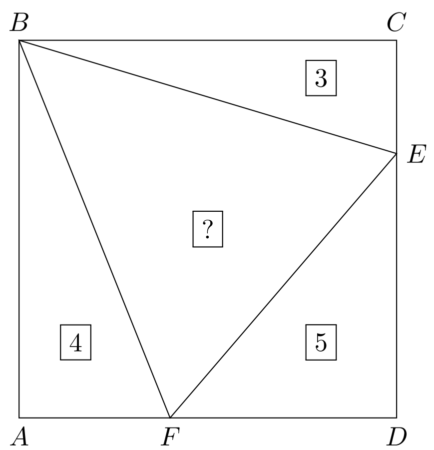
$$\begin{aligned}5Y &= 18\\ Y &= \frac{18}{5}\end{aligned}$$

Put $Y = \frac{18}{5}$ into (1):

$$X = 2(\frac{18}{5}) - 3 = \frac{21}{5}$$

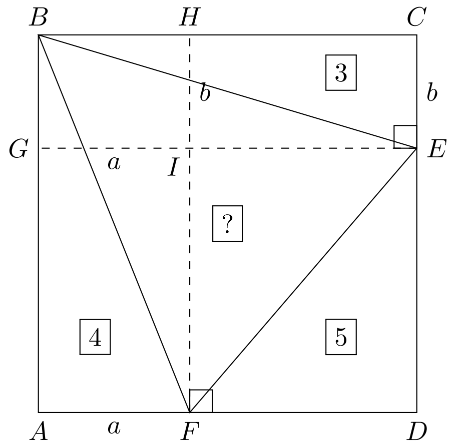
Required area $= \dfrac{21}{5} + \dfrac{18}{5} = \dfrac{39}{5} = \boxed{7.8}$

Problem 25. In square $ABCD$, E is a point on CD and F is a point on AD such that area of $\triangle BCE = 3$, area of $\triangle BAF = 4$ and area of $\triangle EFD = 5$. What is the area of $\triangle BEF$?



(Difficulty: 5 [Hard]) [8]

Solution 25. Draw $EG \perp BA$ and $FH \perp BC$. Let EG and FH intersect at I .



Let s be the side length of the square, $a = AF$ and $b = CE$. Note that s^2 is the area of the square, and ab is the area of rectangle $BHIG$. Note that the square is comprised of two pieces of each of the corner triangles minus the rectangle $BHIG$. Thus we have

$$\begin{aligned} s^2 &= 2 \cdot (3 + 4 + 5) - ab \\ s^2 &= 24 - ab \qquad (1) \end{aligned}$$

Considering the area of $\triangle BAF$ and $\triangle BCE$, we also have

$$\frac{sa}{2} = 4 \qquad (2)$$

$$\frac{sb}{2} = 3 \qquad (3)$$

(2) $\times$(3) :

$$\begin{aligned} \frac{s^2 ab}{4} &= 12 \\ ab &= \frac{48}{s^2} \qquad (4) \end{aligned}$$

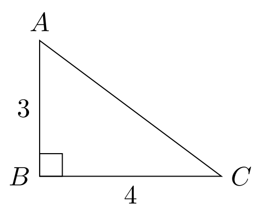
Put (4) into (1):

$$\begin{aligned} s^2 &= 24 - \frac{48}{s^2} \\ s^4 - 24s^2 + 48 &= 0 \\ s^2 &= \frac{24 \pm \sqrt{(-24)^2 - 4(48)}}{2} \\ &= \frac{24 \pm 8\sqrt{6}}{2} \\ &= 12 \pm 4\sqrt{6} \\ &\approx 21.798 \text{ or } 2.202 \text{ (rej. since } s^2 \text{ must be larger than 12)} \end{aligned}$$

Thus $s^2 = 12 + 4\sqrt{6}$, and area of $BEF = s^2 - (3 + 4 + 5) = \boxed{4\sqrt{6}}$.

1.7.2 Pythagoras theorem

Problem 26. $\triangle ABC$ has $\angle B = 90°$, $AB = 3$ and $BC = 4$. What is AC?

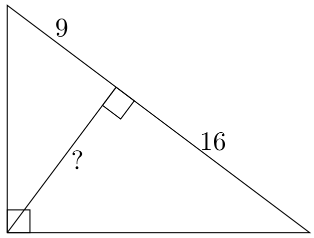


(Difficulty: 1 [Beginner])

Solution 26. Since $\triangle ABC$ is a right triangle, we can apply Pythagoras theorem:

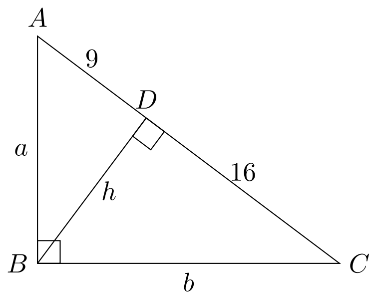
$$\begin{aligned} AB^2 + BC^2 &= AC^2 \qquad \text{(Pyth. theorem)} \\ AC^2 &= 3^2 + 4^2 \\ AC &= \sqrt{3^2 + 4^2} \\ &= \boxed{5} \end{aligned}$$

Problem 27. In a right triangle, the perpendicular line segment dropped from the vertex of the right angle upon the hypotenuse divides it into two segments of 9 and 16 units respectively. What is the length of this perpendicular line segment?



(Difficulty: 3) [9]

Solution 27. Let h be the length of the perpendicular line segment, and a , b be the two legs (non-hypotenuse sides) of the triangle.

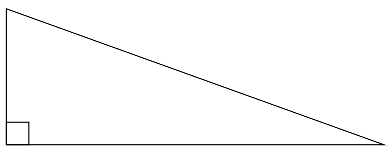


In $\triangle ABC$, $a^2 + b^2 = (9 + 16)^2$ (Pyth. theorem).
In $\triangle ADB$, $h^2 + 9^2 = a^2$ (Pyth. theorem).
In $\triangle CDB$, $h^2 + 16^2 = b^2$ (Pyth. theorem).

Substituting the 2nd and 3rd equation into the 1st equation:

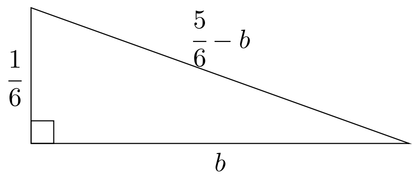
$$\begin{aligned} (h^2 + 9^2) + (h^2 + 16^2) &= (9 + 16)^2 \\ 2h^2 &= 625 - 337 \\ h^2 &= 144 \\ h &= \boxed{12} \end{aligned}$$

Problem 28. A leg of a right triangle is equal to 1/5 the sum of the other two sides. The triangle has a perimeter of 1. What is the triangle's area?



(Difficulty: 4) [10]

Solution 28. Let k be the length of the leg. Then considering the perimeter of the triangle, we have $k+5k=1$, so $k=\frac{1}{6}$.



Let b be the length of the other leg. Then the hypotenuse is $1-\frac{1}{6}-b=\frac{5}{6}-b$. By pyth. theorem,

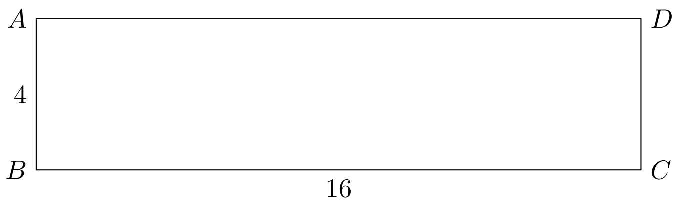
$$b^2+(\frac{1}{6})^2=(\frac{5}{6}-b)^2$$
$$b^2+\frac{1}{36}=\frac{25}{36}-\frac{5b}{3}+b^2$$
$$b=\frac{2}{5}$$

Area of triangle $=\frac{1}{2}(\frac{1}{6})(\frac{2}{5})=\boxed{\frac{1}{30}}$

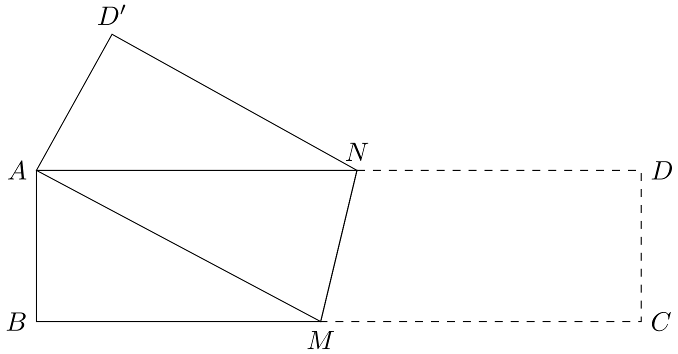
Problem 29. Rectangle $ABCD$ has $AB=4$ and $BC=16$. Fold this rectangle over line MN such that C goes to point A , as shown in the second figure. (In other words, C and D are reflected about line MN to make A and D' .)

What is the area of the resulting pentagon $ABMND'$?

Original:

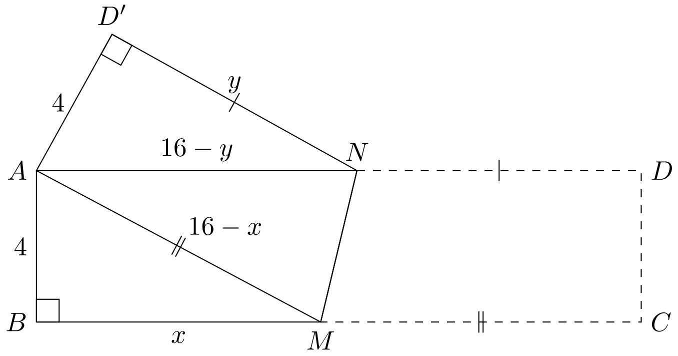


Folded:



(Difficulty: 4) [11]

Solution 29. Since MN is the axis of reflection, we have $AM=MC$ and $D'N=ND$ (reflection postulate). Let $BM=x$ and $D'N=y$.



Note that $BM + AM = BM + MC = 16$. Thus $AM = 16 - BM = 16 - x$. Then by pyth. theorem,

$$
\begin{aligned}
x^2 + 4^2 &= (16-x)^2 \\
x^2 + 16 &= 256 - 32x + x^2 \\
x &= 7.5 \\
AM &= 16 - 7.5 = 8.5
\end{aligned}
$$

Similarly, since $ND = D'N = y$, we have $AN = 16 - y$, which also yields $y = 7.5$ and $AN = 8.5$.

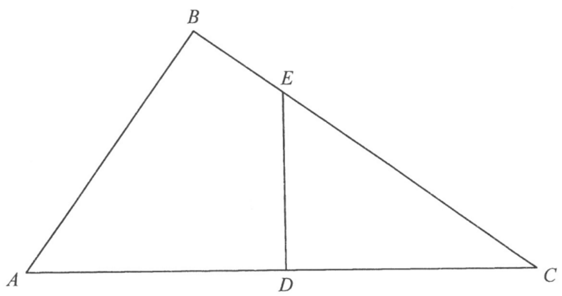
Area of $ABMND'$

$= \text{area of } \triangle AD'N + \text{area of } ABMN$

$= \dfrac{(4)(7.5)}{2} + \dfrac{(8.5+7.5)(4)}{2}$ (area of $\triangle$ & area of trapezium)

$= \boxed{47}$

Problem 30. In the figure, ABC is a right-angled triangle with $\angle ABC = 90°$. Let D and E be points lying on AC and BC respectively such that $ABED$ is a cyclic quadrilateral. If $AB = 660$ cm , $AD = 572$ cm and $BE = 275$ cm , then $CD =$?



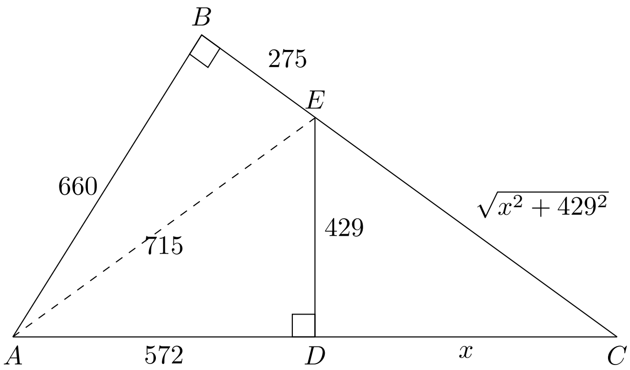
(Difficulty: 5) (2022 DSE Paper 2 Q22)

Solution 30. I didn't solve this problem in the exam, as I thought that we have to solely use similar triangles ratios to solve this 💀 .

(I'll omit the cm in the lengths since it is not important.)

Note that $\angle ADE = \angle ABC = 90°$ (opp. $\angle$s , cyclic quad.) . Thus $ED \perp AC$.

Join AE .



By pyth. theorem, $AE = \sqrt{660^2 + 275^2} = 715$, so $ED = \sqrt{715^2 - 572^2} = 429$.
Since $\angle EDC = \angle ABC = 90^\circ$ and $\angle ECD = \angle ACB$ (common $\angle$) , we have $\triangle EDC \sim \triangle ABC$ (AA).
Let $CD = x$. Then $EC = \sqrt{x^2 + 429^2}$. We have by similar triangles:

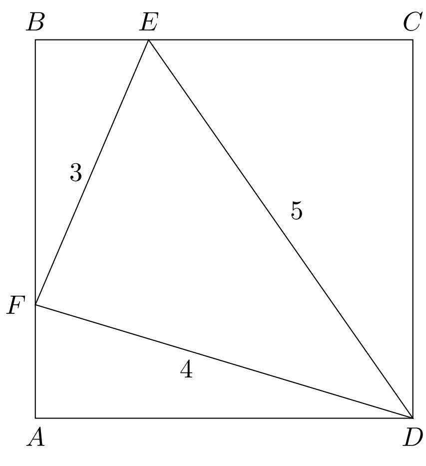
$$\begin{aligned}
\frac{CD}{CB} &= \frac{ED}{AB} \qquad \text{(corr. sides, } \sim \triangle\text{s)} \\
\frac{x}{275 + \sqrt{x^2 + 429^2}} &= \frac{429}{660} \\
660x &= 117975 + 429\sqrt{x^2 + 429^2} \\
(660x - 117975)^2 &= (429\sqrt{x^2 + 429^2})^2 \\
435\,600x^2 - 155727000x + 13\,918\,100\,625 &= 184041(x^2 + 184041)
\end{aligned}$$

$$251\,559x^2 - 155\,727\,000x - 19\,952\,989\,056 = 0$$

$$\begin{aligned}
x &= \frac{155\,727\,000 \pm \sqrt{155\,727\,000^2 - 4(251\,559)(-19\,952\,989\,056)}}{2(251\,559)} \\
&= \frac{155\,727\,000 \pm 210\,542\,904}{503\,118} \\
&= 728 \text{ or } -\frac{2288}{21} \text{ (rej.)}
\end{aligned}$$

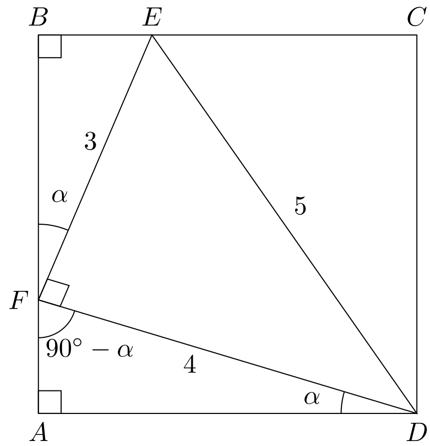
Thus, $CD = \boxed{728}$.

Problem 31. In square $ABCD$, E is a point on BC and F is a point on AB such that $EF = 3$, $FD = 4$ and $ED = 5$. What is the side length of square $ABCD$?



(Difficulty: 5) [12]

Solution 31. Since $3^2 + 4^2 = 5^2$, note that $\triangle EFD$ is a right triangle with $\angle EFD = 90^\circ$ (converse of pyth. theorem) .



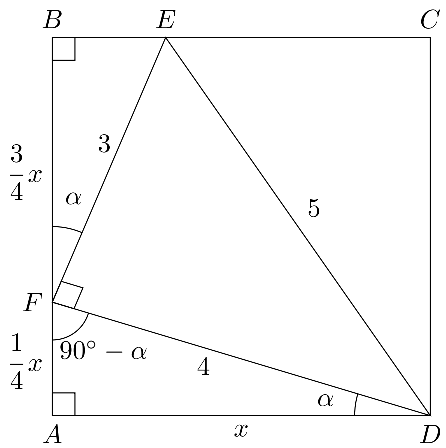
Let $\angle FDA = \alpha$. Then $\angle DFA = 90° - \alpha$ ($\angle$ sum of $\triangle$), and
$\angle BFE = 180° - 90° - \angle DFA = 180° - 90° - (90° - \alpha) = \alpha$.

Since $\angle A = \angle B$ and $\angle FDA = \angle EFB$, we have $\triangle FAD \sim \triangle EBF$ (AA) .

Let x be the side length of the square. We have

$$\begin{aligned} \frac{AD}{BF} &= \frac{FD}{EF} \qquad \text{(corr. sides, } \sim \triangle\text{s)} \\ \frac{x}{BF} &= \frac{4}{3} \\ BF &= \frac{3}{4}x \end{aligned}$$

Thus, $FA = x - \frac{3}{4}x = \frac{1}{4}x$.



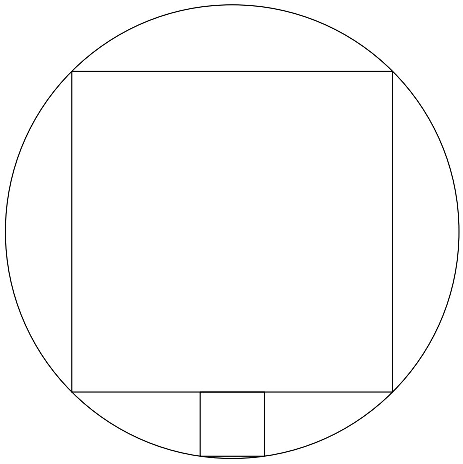
In $\triangle FAD$, solve for x by pyth. theorem:

$$\begin{aligned} (\frac{1}{4}x)^2 + x^2 &= 4^2 \\ \frac{17}{16}x^2 &= 16 \\ x &= \boxed{\frac{16}{\sqrt{17}}} \qquad (\approx 3.881) \end{aligned}$$

Problem 32. A square is inscribed in a circle.

A smaller square is drawn. It shares side with the inscribed square and its other two corners touch the circle.

What is the ratio of the larger square's area to the smaller square's area?

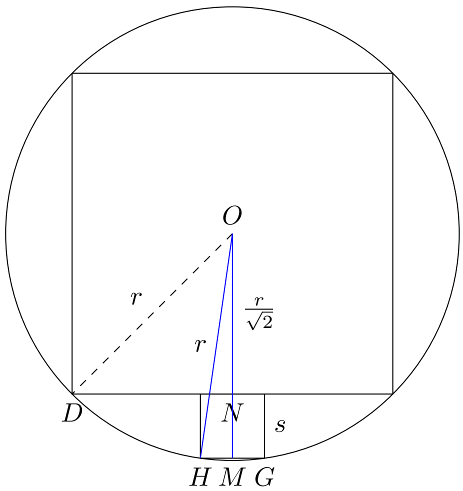


(Difficulty: 5 [Hard]) [13]

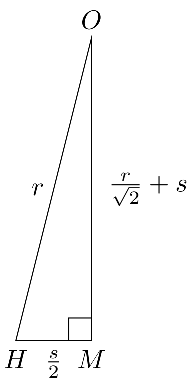
Solution 32. Let r be the radius of the circle, and s be the side length of the small square.

Draw a radius of the circle to a a corner of the small square.

Drop a perpendicular from the centre of the circle to the bottom side of the small square. Note that it bisects the bottom side of both squares (line from centre $\perp$ chord bisects chord). Thus, $HM = \frac{1}{2}s$.



Since $\triangle ODN$ is a right isosceles triangle, we have $ON = \frac{r}{\sqrt{2}}$. Let's focus on $\triangle OMH$. Note that $OM = \frac{r}{\sqrt{2}} + s$.



By pyth. theorem, we have

$$\begin{aligned}(\frac{r}{\sqrt{2}}+s)^2+(\frac{s}{2})^2&=r^2\\ \frac{r^2}{2}+\sqrt{2}rs+s^2+\frac{s^2}{4}&=r^2\\ 5s^2+4\sqrt{2}rs-2r^2&=0\end{aligned}$$

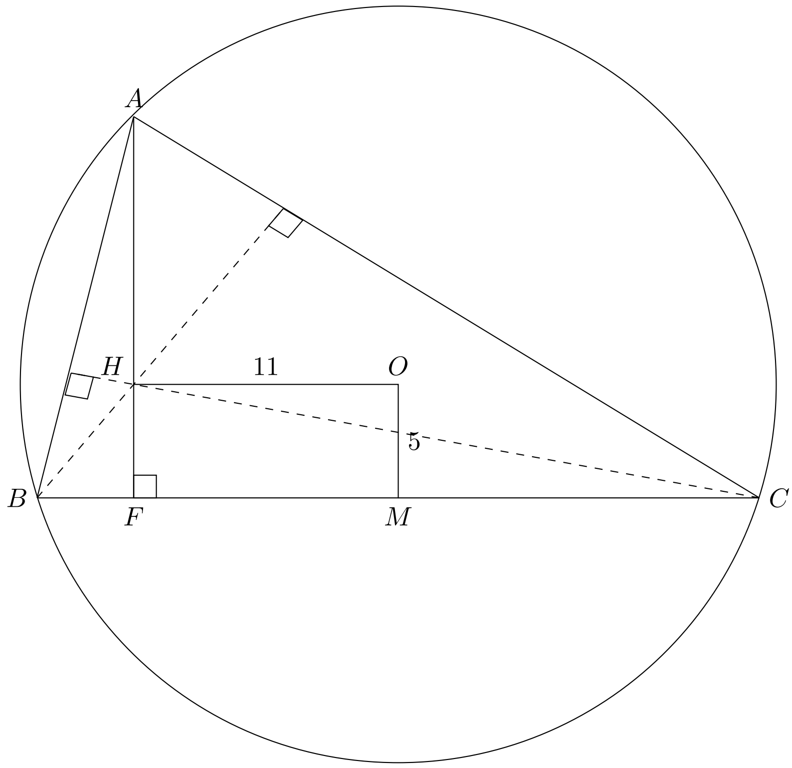
Using **quadratic formula** on s :

$$\begin{aligned}s&=\frac{-4\sqrt{2}r+\sqrt{(4\sqrt{2}r)^2-4(5)(-2r^2)}}{2(5)}\\ &=(\frac{-4\sqrt{2}+\sqrt{72}}{10})r\\ &=(\frac{\sqrt{2}}{5})r\end{aligned}$$

Since the side length of the large square is $r\sqrt{2}$, the area of the large square is $2r^2$.

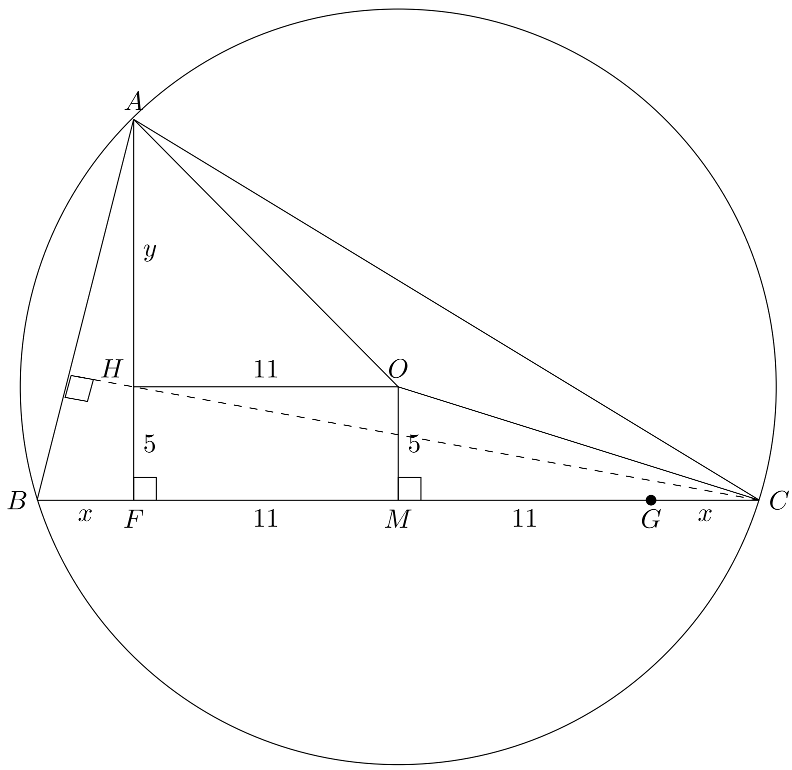
Thus, $\dfrac{\text{area of larger square}}{\text{area of smaller square}}=\dfrac{2r^2}{s^2}=\dfrac{2r^2}{((\frac{\sqrt{2}}{5})r)^2}=\dfrac{2r^2}{(\frac{2}{25})r^2}=\boxed{25}$.

Problem 33. A rectangle, $HOMF$, has sides $HO=11$ and $OM=5$. A triangle ABC has H as the intersection of the altitudes, O the centre of the circumscribed circle, M the midpoint of BC , and F the foot of the altitude from A . What is the length of BC ?



(Difficulty: 6) (Putnam 1997 A1) [14]

Solution 33. Let $BF=x$ and $AH=y$. Let G be a point on BC such that $GC=BF=x$. Then $MG=FM=11$.



Note that $OA = OC$. Considering $\triangle AHO$ and $\triangle OMC$, we have $OA^2 = y^2 + 11^2$ and $OC^2 = 5^2 + (11 + x)^2$ by pyth. theorem, so we have

$$y^2 + 11^2 = 5^2 + (11 + x)^2 \tag{5}$$

$$y^2 + 121 = 25 + 121 + 22x + x^2 \tag{6}$$

Also note that $\angle HCF = 90° - \angle ABC = \angle BAF$ ($\angle$ sum of $\triangle$). Thus $\triangle AFB \sim \triangle CFH$ (AA).

So we have

$$\begin{aligned}
\frac{AF}{BF} &= \frac{CF}{HF} \qquad \text{(corr. sides, } \sim \triangle\text{s)} \\
\frac{y+5}{x} &= \frac{x+11+11}{5} \\
5y + 25 &= x^2 + 22x
\end{aligned} \tag{7}$$

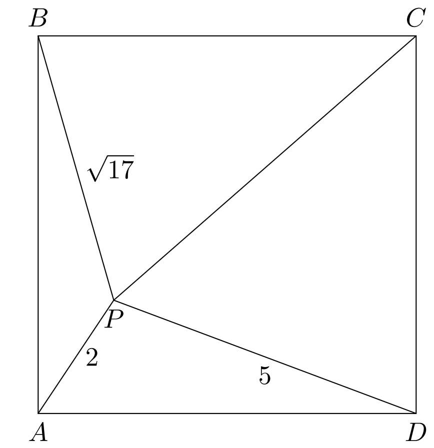
Note that $x^2 + 22x$ appears in both equation (2) and (3). Putting (3) into (2):

$$\begin{aligned}
y^2 + 121 &= 25 + 121 + 5y + 25 \\
y^2 - 5y - 50 &= 0 \\
(y-10)(y+5) &= 0 \\
y &= 10 \text{ or } y = -5 \text{ (rej.)}
\end{aligned}$$

Put $y = 10$ into (1):

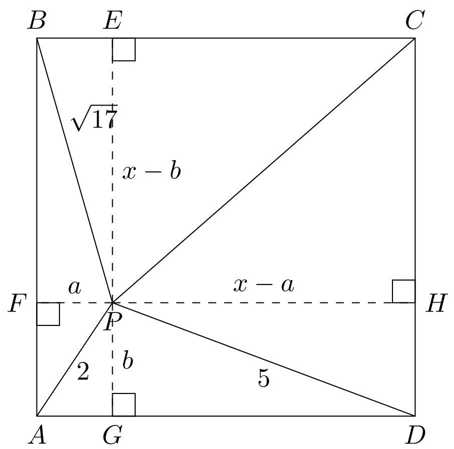
$$\begin{aligned}
10^2 + 11^2 &= 5^2 + (11 + x)^2 \\
196 &= (11 + x)^2 \\
14 &= 11 + x \\
x &= 3 \\
\therefore BC &= 3 + 11 + 11 + 3 = \boxed{28}
\end{aligned}$$

Problem 34. P is a point inside square $ABCD$ such that $PA = 2$, $PB = \sqrt{17}$ and $PD = 5$. What is the area of the square?



(Difficulty: 6) [15]

Solution 34. Let x be the side length of the square. Drop perpendiculars from P to the four sides of the square as in the figure. In particular, let $PF \perp AB$, $PE \perp BC$, $PG \perp AD$ and $PH \perp CD$.



Let $PF = a$ and $PG = b$. Then $PH = x - a$ and $PE = x - b$. By pyth. theorem,

$$\text{In } \triangle PAG, \quad a^2 + b^2 = 2^2 \tag{1}$$

$$\text{In } \triangle PGD, \quad (x-a)^2 + b^2 = 5^2 \tag{2}$$

$$\text{In } \triangle PBE, \quad a^2 + (x-b)^2 = (\sqrt{17})^2 \tag{3}$$

We can solve this system of equations since there are three variables in three equations. Expand the brackets in (2) and (3):

$$x^2 - 2ax + a^2 + b^2 = 25 \tag{2}$$

$$a^2 + x^2 - 2xb + b^2 = 17 \tag{3}$$

Put $a^2 + b^2 = 4$ into (2):

$$\begin{aligned} x^2 - 2ax + 4 &= 25 \\ x^2 - 21 &= 2ax \\ a &= \frac{x^2 - 21}{2x} \end{aligned} \tag{4}$$

Similarly, put $a^2+b^2=4$ into (3):

$$\begin{aligned} x^2-2xb+4&=17\\ x^2-13&=2xb\\ b&=\frac{x^2-13}{2x} \end{aligned}$$

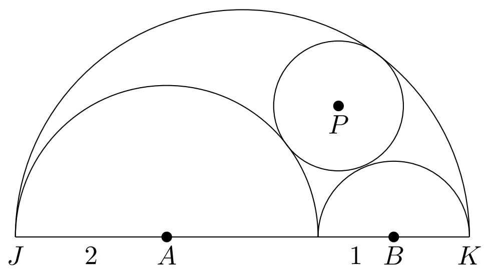
Put (4) and (5) into (1):

$$\begin{aligned} (\frac{x^2-21}{2x})^2+(\frac{x^2-13}{2x})^2&=4\\ (x^4-42x^2+441)+(x^4-26x^2+169)&=4(4x^2)\\ x^4-42x^2+305&=0\\ x^2&=\frac{42\pm\sqrt{42^2-4(305)}}{2}\\ &=21\pm2\sqrt{34}\\ &\approx 32.662 \quad \text{or} \quad 9.338 \end{aligned}$$

Note that if the square's area is $21-2\sqrt{34}\approx 9.338$, then the side length $x\approx 3.056$, which isn't big enough to fit $PD=5$ inside it (since the square's diagonal would be $3.056\sqrt{2}\approx 4.322<5$) . Thus $x^2=21-2\sqrt{34}$ is rejected, and it can only be that $x^2=21+2\sqrt{34}$.

Thus the area of the square is $\boxed{21+2\sqrt{34}}$.

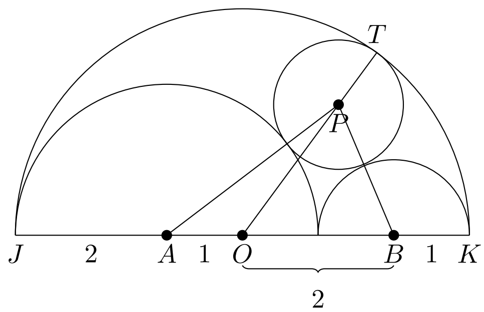
Problem 35. In the figure below, semi-circles with centers at A and B and with radii 2 and 1, respectively, are drawn in the interior of, and sharing bases with, a semi-circle with diameter JK. The two smaller semi-circles are externally tangent to each other and internally tangent to the largest semicircle. A circle centered at P is drawn externally tangent to the two smaller semi-circles and internally tangent to the largest semi-circle. What is the radius of the circle centered at P?



(Difficulty: 6) (2017 AMC 12A Problem 16) [16] [17]

Solution 35. Let O be the centre of the largest semi-circle, and let T be the point of tangency of circle P and semi-circle O . Note that the diamter of semi-circle O is $2+2+1+1=6$, so $OJ=3$ and $AO=1$. We also have $OB=3-1=2$.

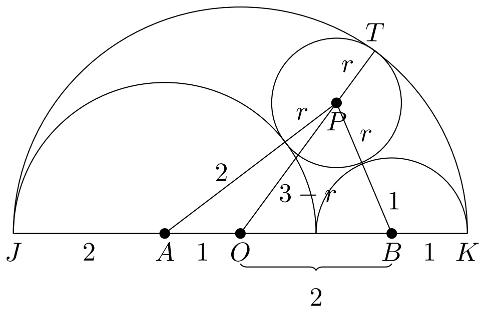
Join AP , PB and radius OP.



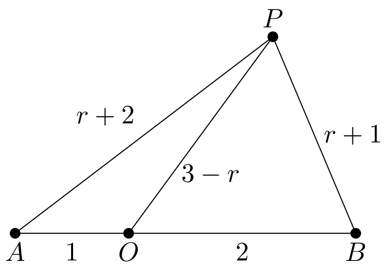
By 'property of touching circles', the centres and the point of tangency for two tangent circles are collinear, and this is true for both internal tangency and external tangency.

Thus, the points of tangency lie on the line segments drawn, and OPT is a straight line segment.

Let $PT = r$. Note that $OT = 3$ and $OP = 3 - r$. We also have $AP = r + 2$ and $PB = r + 1$.



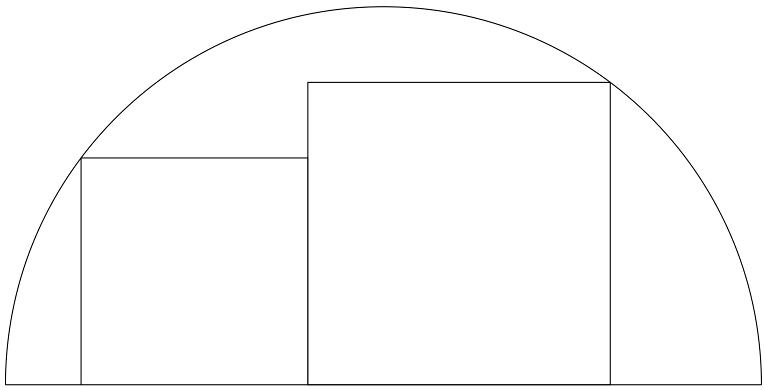
Now we can focus on $\triangle PAB$:



By Stewart's theorem, we have

$$\begin{aligned}(r+1)^2(1) + (r+2)^2(2) &= (1+2)((3-r)^2 + (1)(2)) \\ r^2 + 2r + 1 + 2r^2 + 8r + 8 &= 3(9 - 6r + r^2 + 2) \\ 3r^2 + 10r + 9 &= 33 - 18r + 3r^2 \\ 28r &= 24 \\ r &= \boxed{\frac{6}{7}}\end{aligned}$$

Problem 36. In the figure, two side-by-side squares are inscribed in a semi-circle of radius 10 . What is the total area of the two squares?

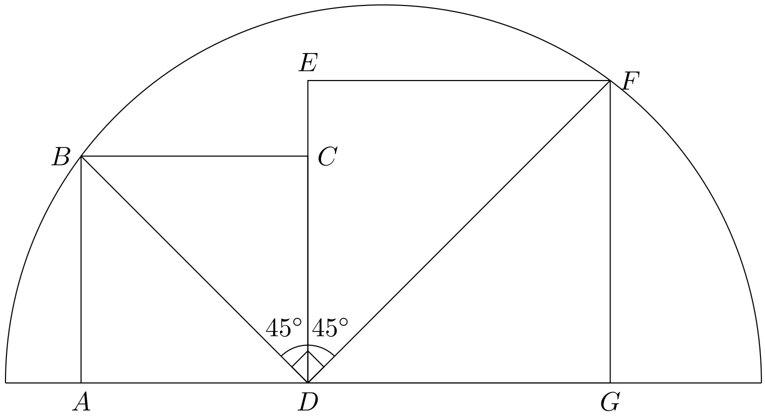


Note: The solution must show that the total area of the squares is fixed no matter the side lengths of the two squares.

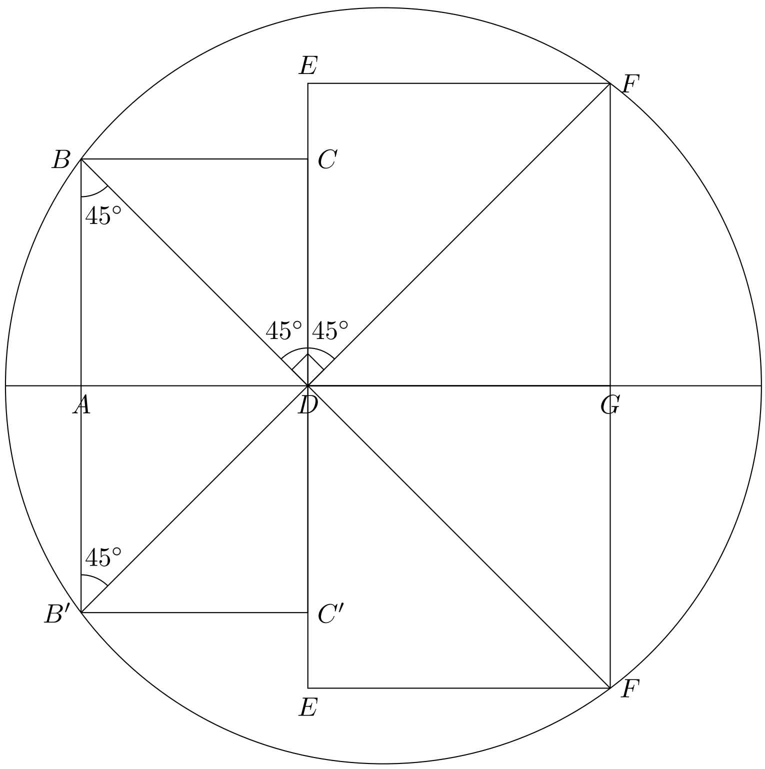
(Difficulty: 6) [18]

Solution 36. Label the squares $ABCD$ and $DEFG$.

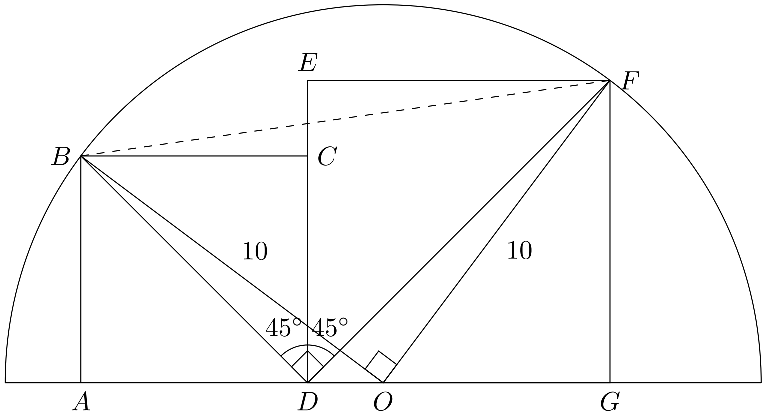
Join the diagonals of the squares with endpoint on the circumference. Note that $\angle BDC = \angle CDF = 45^\circ$ (prop. of square). Thus $\angle BDF = 90^\circ$.



Reflect the figure about the diameter to make it a full circle:



Note that $B'DC$ is a straight line segment, and we have $\angle BB'C = \angle ABD = 45^\circ$ (reflection postulate). Thus, arc $\widehat{BF}$ subtends 90° at the centre ($\angle$ at centre twice $\angle$ at $\odot^{ce}$).



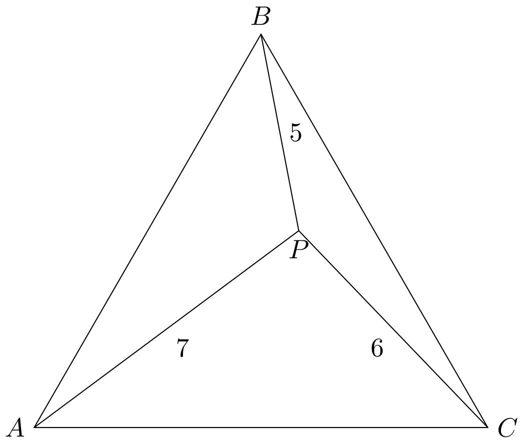
By pyth. theorem in $\triangle BOF$ and $\triangle BDF$, we have $BF^2 = 10^2 + 10^2$, and also $BF^2 = BD^2 + DF^2$.

Thus $BD^2 + DF^2 = 10^2 + 10^2 = 200$.

Note that $BD = \sqrt{2}\,AD$ and $DF = \sqrt{2}\,DG$ (diags of square). Thus

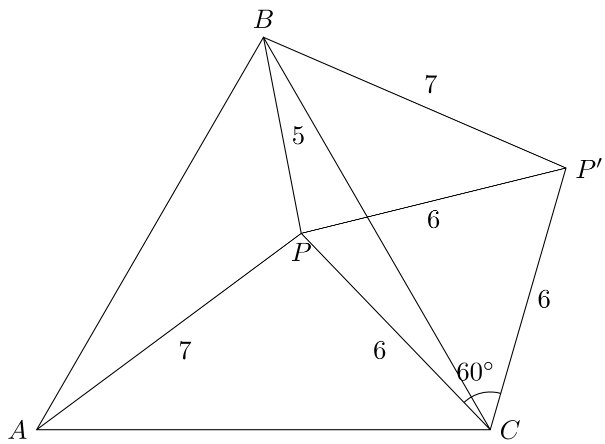
$$\begin{aligned}\text{Total area} &= AD^2 + DG^2\\ &= (\frac{BD}{\sqrt{2}})^2 + (\frac{DF}{\sqrt{2}})^2\\ &= \frac{BD^2}{2} + \frac{DF^2}{2}\\ &= \frac{200}{2}\\ &= \boxed{100}\end{aligned}$$

Problem 37. $\triangle ABC$ is an equilateral triangle. P is a point inside $\triangle ABC$ such that $AP = 7$, $BP = 5$ and $CP = 6$. What is the area of $\triangle ABC$?



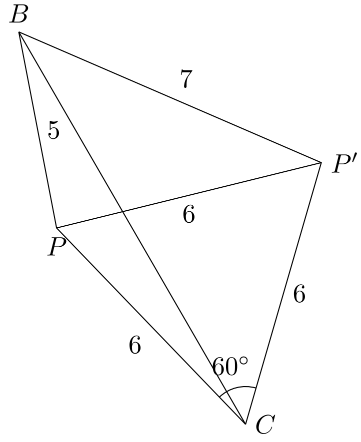
(Difficulty: 6) [19]

Solution 37. Rotate PC $60°$ clockwise about point C to make $P'C$. Note that $CP' = CP = 6$ and $\angle PCP' = 60°$. Thus $\triangle PP'C$ is a equilateral triangle, and $PP' = 6$.



In $\triangle P'CB$ and $\triangle PCA$, we have $P'C = PC$, $\angle P'CB = \angle PCA = 60° - \angle BCP$ and $BC = AC$. Thus $\triangle P'CB \cong \triangle PCA$ (SAS), and we have $BP' = AP = 7$ (corr. sides, $\cong \triangle$s).

Now focus on quadrilateral $BPCP'$.



We can find the quadrilateral's area by summing the area of $\triangle BPP'$ and $\triangle PP'C$. The semi-perimeter of $\triangle BPP'$ is $s = \dfrac{5+6+7}{2} = 9$.

$$\begin{aligned}
\text{Area of } BPCP' &= \text{area of } \triangle BPP' + \text{area of } \triangle PP'C \\
&= \sqrt{9(9-5)(9-6)(9-7)} + \frac{(6^2)\sqrt{3}}{4} \qquad \text{(Heron's formula \& area of equil. } \triangle) \\
&= \sqrt{216} + 9\sqrt{3}
\end{aligned}$$

Let $BC = p$. By **Bretschneider's formula** , we have:

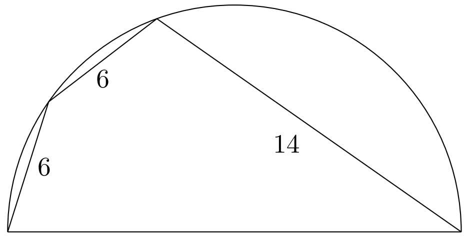
$$\begin{aligned}
\sqrt{216} + 9\sqrt{3} &= \frac{1}{4}\sqrt{4(6)^2p^2 - (6^2+7^2-6^2-5^2)^2} \\
4\sqrt{216} + 36\sqrt{3} &= \sqrt{144p^2 - 576} \\
7344 + 288\sqrt{648} &= 144p^2 - 576 \\
55 + 2(18\sqrt{2}) &= p^2 \\
p^2 &= 55 + 36\sqrt{2}
\end{aligned}$$

And finally,

$$\begin{aligned}
\text{Area of } \triangle ABC &= \frac{p^2\sqrt{3}}{4} \\
&= \frac{(55+36\sqrt{2})\sqrt{3}}{4} \\
&= \boxed{\frac{55\sqrt{3}}{4} + 9\sqrt{6}} \qquad (\approx 45.861)
\end{aligned}$$

1.8 Proportions and similar triangles

Problem 38. In a semi-circle, there are three chords with lengths 6, 6, 14 that are connected one after another, forming a quadrilateral with the diameter, as shown in the figure. What is the radius of the semi-circle?



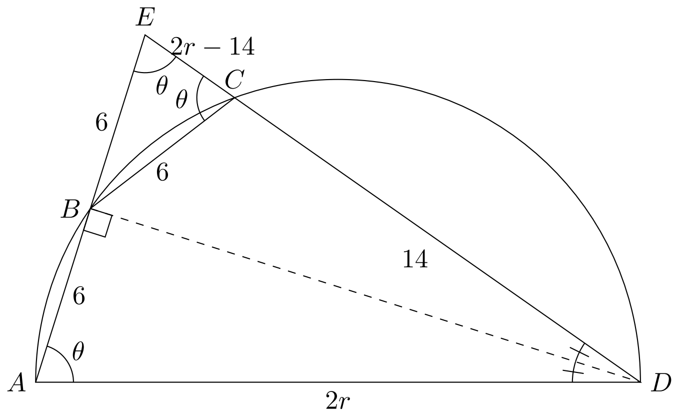
(Difficulty: 6) [20]

Solution 38. Label the points A, B, C, D where AD is the diameter, as in the figure.

Let r be the radius of the semi-circle. Then $AD = 2r$.

Join BD . Note that $\angle ABD = 90^\circ$ ($\angle$ in semi-circle) , and $\angle BDA = \angle BDE$ (equal chords, equal $\angle$s at $\odot^{ce}$) .

Extend AB and DC to meet at E .



Note that $\triangle EBD \cong \triangle ABD$ (AAS) . Thus we have $BE = AB = 6$ and $ED = AD = 2r$ (corr. sides, $\cong \triangle$s). So $EC = 2r - 14$.

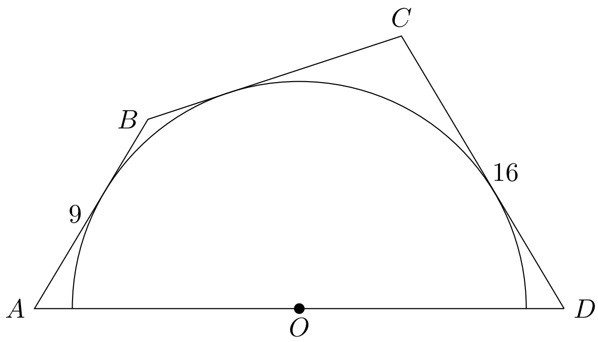
Since $ED = AD$, we have $\angle AED = \angle EAD$ (base $\angle$s, isos. $\triangle$).

Since $BE = BC$, we have $\angle BEC = \angle BCE$. Let $\angle BEC = \angle BCE = \angle EAD = \theta$. Note that $\triangle BEC \sim DEA$ (AA) . Thus we have

$$\begin{aligned} \frac{BC}{EC} &= \frac{AD}{AE} \qquad \text{(corr. sides, } \sim \triangle\text{s)} \\ \frac{6}{2r-14} &= \frac{2r}{6+6} \\ 72 &= 4r^2 - 28r \\ r^2 - 7r - 18 &= 0 \\ (r+2)(r-9) &= 0 \\ r &= 9 \quad \text{or} \quad -2 \text{ (rej.)} \end{aligned}$$

Thus, the radius of the semi-circle is $\boxed{9}$.

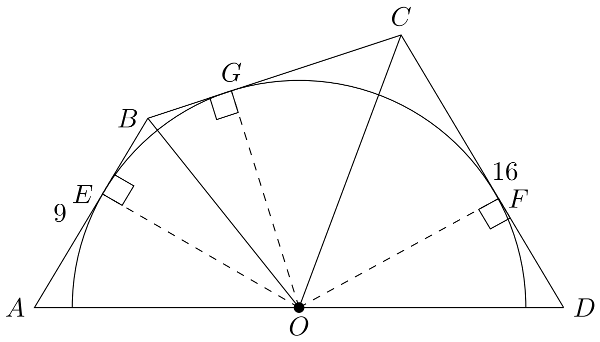
Problem 39. In quadrilateral $ABCD$, O is the mid-point of AD and also the centre of a semi-circle that is tangent to AB , BC and CD . If $AB = 9$ and $CD = 16$, what is the length of AD ?



(Difficulty: 6) [21]

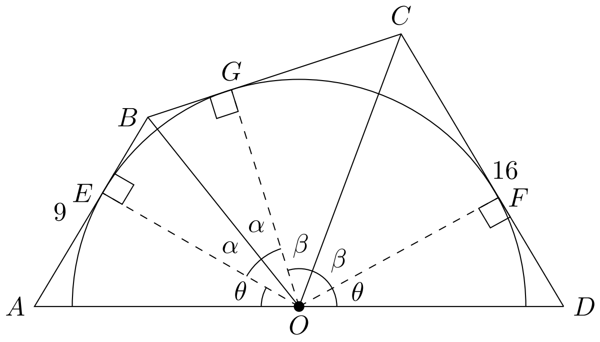
Solution 39. (Pinned comment of the source video) Let the semi-circle be tangent to AB, BC, CD at E, G, F respectively. Then $OE \perp AB$, $OG \perp BC$, $OF \perp CD$ (tangent$\perp$ radius).

Let r be the radius of the semi-circle. Join OB, OC .



Since $OA = OD$ (mid-pt.), $OE = OF$ (radii) and $\angle AEO = \angle DFO$, we have $\triangle OEA \cong \triangle OFD$. Thus let $AE = FD = x$ (corr. sides, $\cong \triangle$s)and let $\angle AOE = \angle DOF = \theta$ (corr. $\angle$s, $\cong \triangle$s).

Note that we have two more pairs of congruent triangles: $\triangle BEO \cong \triangle BGO$ and $\triangle CGO \cong CFO$ (RHS). Thus let $\angle BOE = \angle BOG = \alpha$ and $\angle COG = \angle COF = \beta$.



Note that $2\theta + 2\alpha + 2\beta = 180°$ (adj. $\angle$s on st. line)

$\Rightarrow \beta = \dfrac{180° - 2\alpha - 2\theta}{2} = 90° - \alpha - \theta$.

We have $\angle EBO = 90° - \alpha$ ($\angle$ sum of $\triangle$), and

$\angle COD = \theta + \beta = (90° - \alpha - \theta) + \theta = 90° - \alpha = \angle EBO$.

Note that we also have $\angle ODC = \angle BAO$ (corr. $\angle$s, $\cong \triangle$s).

Thus $\triangle ODC \sim \triangle BAO$ (AA), and we have

$$\begin{aligned} \frac{OD}{DC} &= \frac{BA}{AO} \qquad \text{(corr. sides, } \sim \triangle\text{s)} \\ \frac{OD}{16} &= \frac{9}{AO} \\ OD \cdot OA &= 144 \end{aligned}$$

Since $OA = OD$,

$$OD = OA = \sqrt{144} = 12$$

And $AD = 12 + 12 = \boxed{24}$.

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