Euclidea Solutions Explained

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Abstract

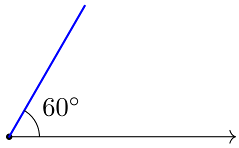
Euclidea is a puzzle game in which the player has to construct geometrical figures using compass and straight edge only. It is probably the most difficult puzzle game I've played. The complexity of Euclidean Geometry gives rise to so many possible constructions, and it is difficult to brute force solving (at least I don't know how). On top of that, the player has to use the minimum number of moves possible to pass the level in order to get 3 stars and unlock the next level pack. There is just no way I can come up with the solutions myself, so I've cheated by looking up the solutions online. Nonetheless, it is an interesting math-related game that is one of a kind.

Contents

1 Alpha

1.1 Angle of 60 deg

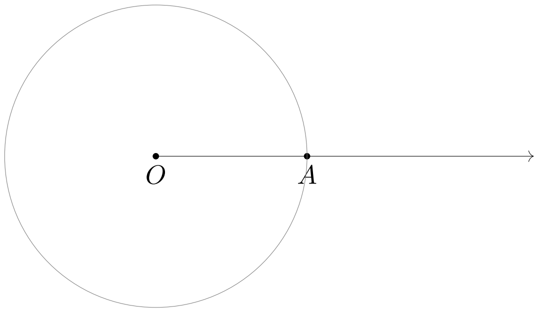
Task 1.1. Construct an angle of $60°$ with the given side.
(3L, 3E, 2V)



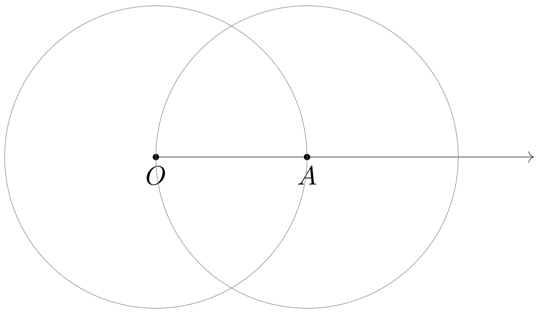
(Arrowhead means the line is infinitely long.)

Solution 1.1. (3L, 5E)

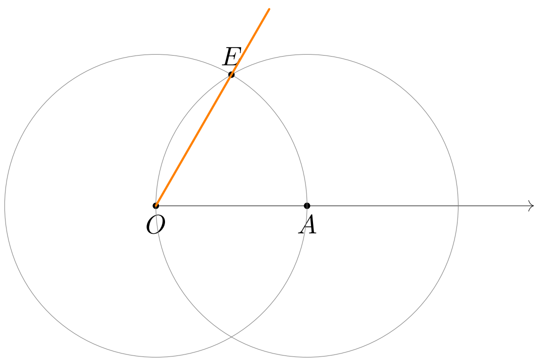
1. Let O be the endpoint of the given ray. Label an arbitrary point A on the given ray. Draw circle centered O through A.



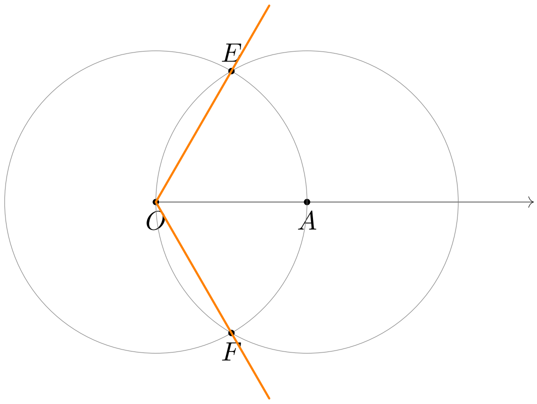
2. Draw circle centered A through O.



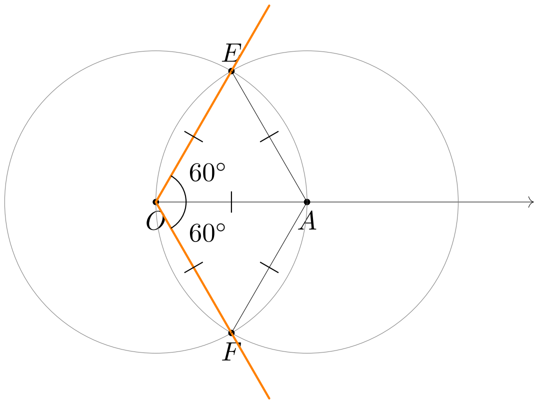
3. Let E be one of the intersections of the two circles. Draw line OE. We get the desired $60°$ angle.



(2V: Extra solutions) Let F be another intersections of the two circles. Draw line OF. We get another $60°$ angle.



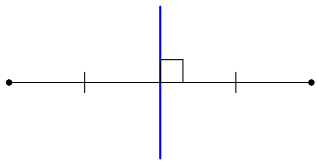
Proof. To see why $\angle AOE$ and $\angle AOF$ are $60°$ angles, first note that the two circles have the same radii since they share the same segment OA. Thus OA, OE, AE, OF, AF all have lengths equal to the radii of the circles, so $\triangle OAE$ and $\triangle OAF$ are equilateral triangles.



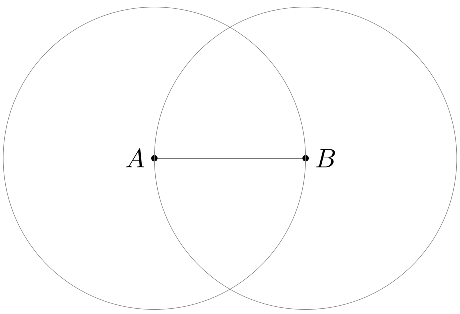
By "prop. of equil. $\triangle$", all the interior angles of equilateral triangle is $60°$, meaning $\angle AOE = \angle AOF = 60°$. □

1.2 Perpendicular bisector

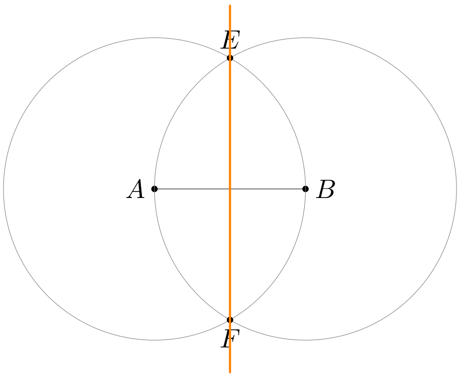
Task 1.2. Construct the perpendicular bisector of the segment.
(3L, 3E)



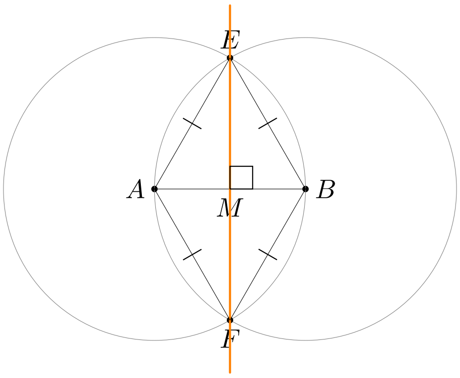
Solution 1.2. **1, 2.** Let A and B be the endpoints of the given segment. Draw circle centered A through B. Draw another circle centered B through A.



3. Let E, F be intersection of the two circles. Draw line EF. We get the desired perpendicular bisector.



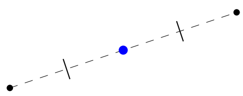
Proof. Let AB and EF intersect at M. Since $AE = BE = AF = BF$, $AEBF$ is a rhombus. By property of rhombus, the diagonals AB and EF are perpendicular to each other.



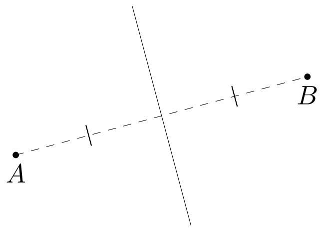
Moreover, since $AEBF$ is a rhombus, $AEBF$ is a parallelogram. By "diags. of //gram", the diagonals AB and EF bisector each other, giving $AM = MB$. Thus, EF is the perpendicular bisector of AB. □

1.3 Midpoint

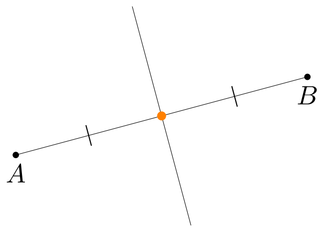
Task 1.3. Construct the midpoint of the segment defined by two points.
(2L, 4E)



Solution 1.3. 1. Let A, B be the endpoints of the given segment. Draw the perpendicular bisector of AB.



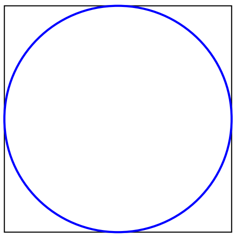
2. Draw line AB. The intersection of AB and the perpendicular bisector is the desired midpoint.



Proof. By definition, perpendicular bisector bisects AB. So the intersection of AB and the perpendicular bisector is the midpoint of AB. □

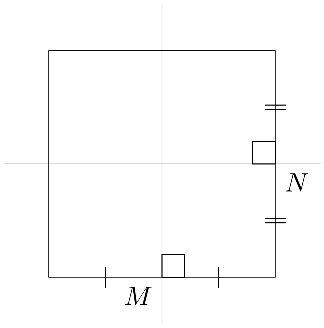
1.4 Circle in square

Task 1.4. Inscribe a circle in the square.
(3L, 5E)

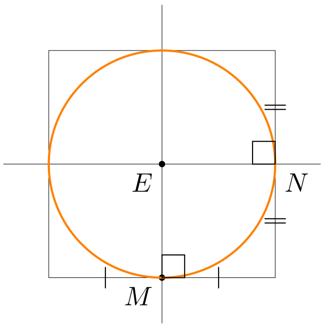


Solution 1.4. (3L)

1, 2. Draw perpendicular bisectors of two adjacent sides of the square. Let M, N be midpoints of these two sides.



3. Let E be the intersection of perpendicular bisectors. Draw circle centered E through M (or N). We get the desired circle.

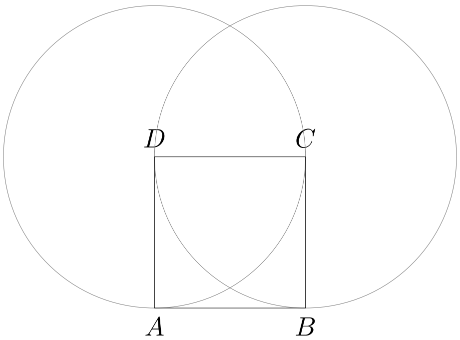


Proof. Note that the perpendicular bisectors divide the big square into four smaller squares of the same side length, so the circle with radius EM passes through all the midpoints of the sides of big square. By 'converse of tangent $\perp$ radius", the circle is tangent to the four sides of the big square, which means it is inscribed in the square.

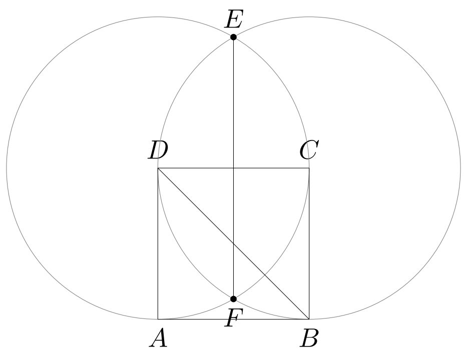
$\square$

(5E)

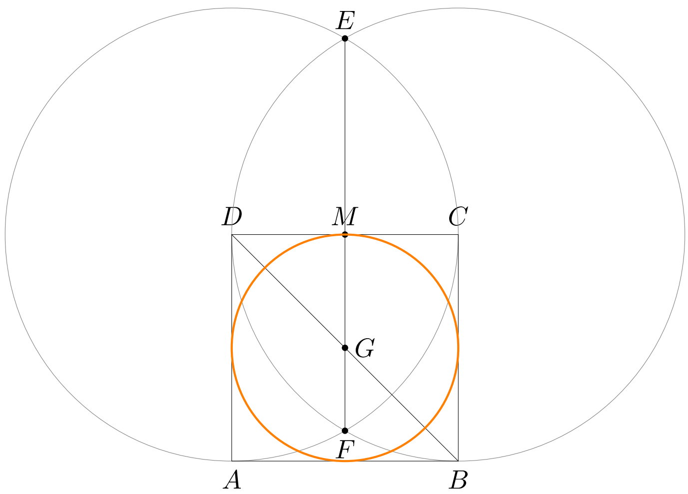
1, 2. Let vertices of square A, B, C, D. Draw circle centered D through C, and draw circle centered C through D.



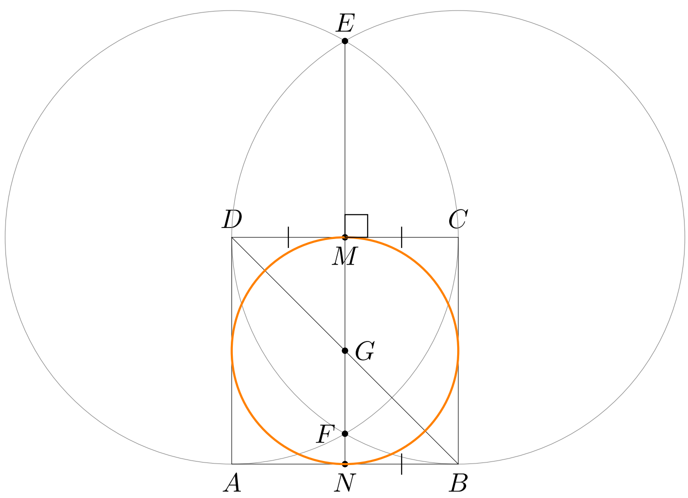
3, 4. Draw line BD. Let the intersections of the circles be E, F. Draw line EF.



5. Let G be the intersection of BD and EF, and let M be the intersection of CD and EF. Draw circle centered G through M. We get the desired circle.



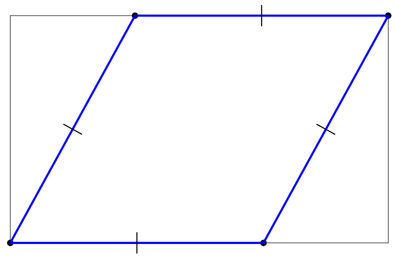
Proof. Note that EF is the perpendicular bisector of CD (by Task 1.2), so $DM = MC$. Extend MF to meet AB at N. We also have $DM = NB$ since MN divides square $ABCD$ into two congruent rectangles. Also note that $\angle GDM = \angle GBN$ (alt. $\angle$s, $DC//AB$).



Thus $\triangle DMG \cong \triangle BNG$ (AAS), so G is the midpoint of MN (corr. sides, $\cong$ $\triangle$s). This means G is the center of the square (same point as "E" in previous 3L solution), so the circle centered G through M is the inscribed circle of the square. $\square$

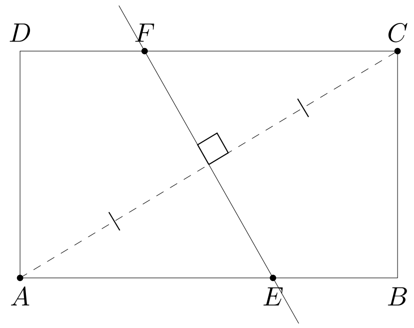
1.5 Rhombus in rectangle

Task 1.5. Inscribe a rhombus in the rectangle so that they share a diagonal.
(3L, 5E, 2V)

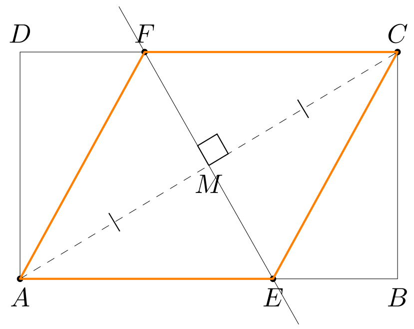


Solution 1.5. (3L, 5E)

1. Let the given rectangle be $ABCD$. Draw perpendicular bisector of AC, and let it intersect AB and CD at E and F respectively.



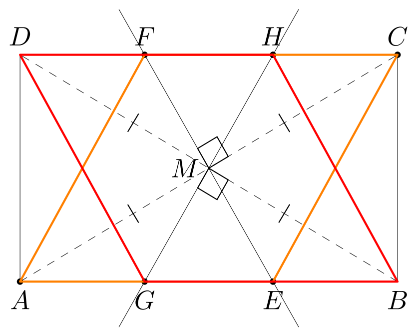
2, 3. Draw AF and EC. We get the desired rhombus $AECF$.



Proof. Let M be the midpoint of AC. Note that $AM = CM$. Also, $\angle MFC = \angle MEA$ (alt. $\angle$s, $FC // AE$). Thus $\triangle MFC \sim \triangle MEA$ (AAS), and $CF = AE$ (corr. sides, $\cong \triangle$s)

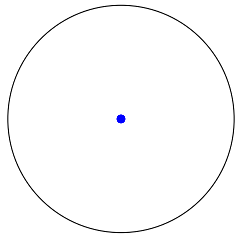
Note that $AF = CF$ and $AE = CE$ by property of perpendicular bisector. Combined with $CF = AE$, we have $AE = CE = AF = CF$, which means $AECF$ is a rhombus. $\square$

(2V). Similarly argument but flipped horizontally.



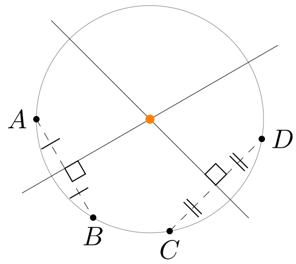
1.6 Circle center

Task 1.6. Construct the center of the circle.
(2L, 5E)



Solution 1.6. (2L)

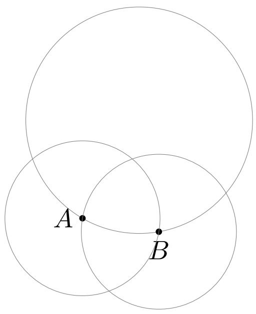
1, 2. Label two pairs of arbitrary points on the circle, and draw the perpendicular bisector of each pair of point. The intersection of the perpendicular bisectors is the desired center of circle.



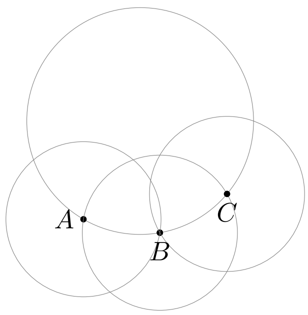
Proof. Perpendicular bisector of any chord passes through the center of a circle ("⊥ bisector of chord passes through center"). This means the center of circle lies on both perpendicular bisectors of AB and CD, so it must be their point of intersection. ☐

(5E)

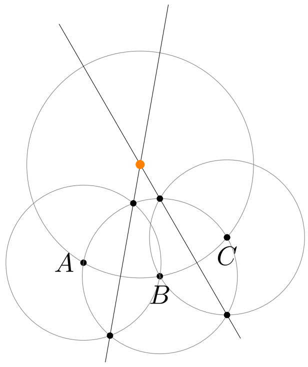
1, 2. Label two arbitrary points A and B. Draw circle centered A through B, and draw circle centered B through A.



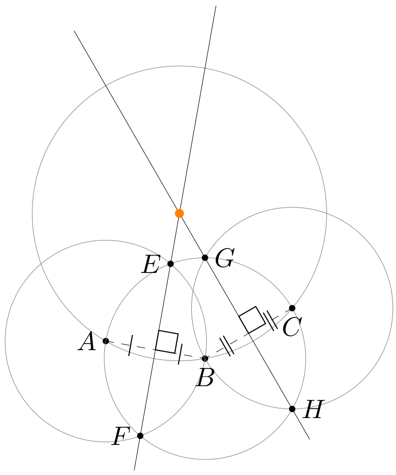
3. (Let (B, A) denote the circle centered B through A.) Let circle (B, A) intersect the given circle at another point C. Draw circle centered C through B.



4, 5. Draw line through the intersections of circles (A, B) and (B, A), and draw line through the intersections of circles (B, C) and (C, B). The intersection of these two lines is the desired center.



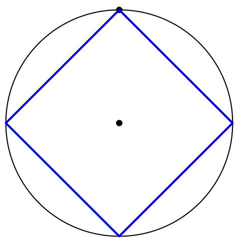
Proof. Note that EF and GH are perpendicular bisectors of chords AB and BC respectively. So they intersect at the center of the circle.



□

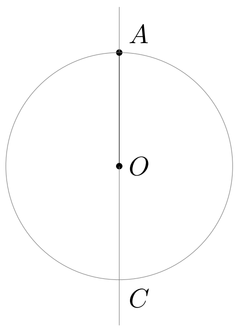
1.7 Inscribed square

Task 1.7. Inscribe a square in the circle. One vertex of the square is given. (The circle center is also given.)
(6L, 7E)

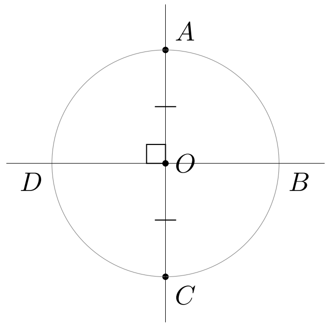


Solution 1.7. (6L)

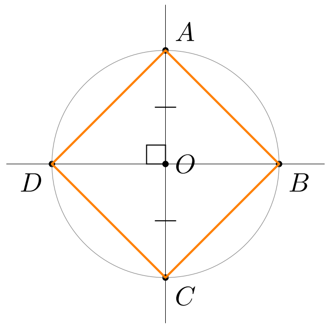
1. Let O be center of circle and A be the given vertex. Draw line AO. Let AO intersect the circle at C.



2. Draw the perpendicular bisector of AC. Let it intersect the circle at B and D.



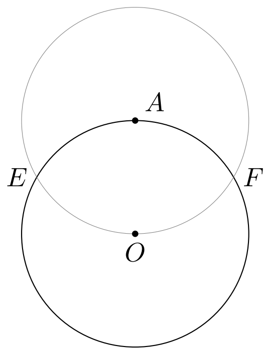
3, 4, 5, 6. Draw lines AB, BC, CD, DA. We get the desired square.



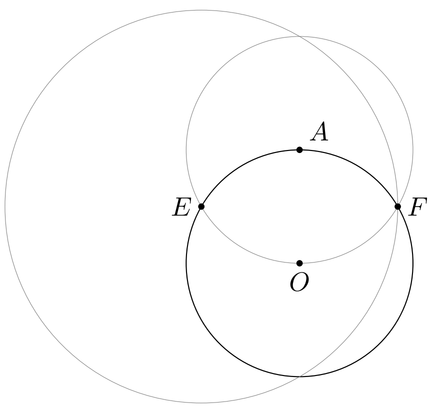
Proof. Note that the perpendicular bisector of AC passes through circle center O. So we have $OA = OB = OC = OD$. Since the diagonals of $ABCD$ are perpendicular and bisect each other, $ABCD$ is a square (con. of square). □

(7E)

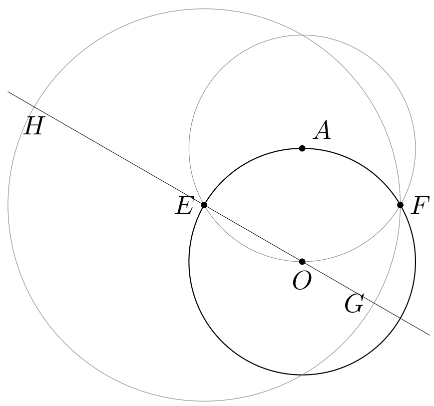
1. Draw circle centered A through O. Let the intersections of two circles be E and F.



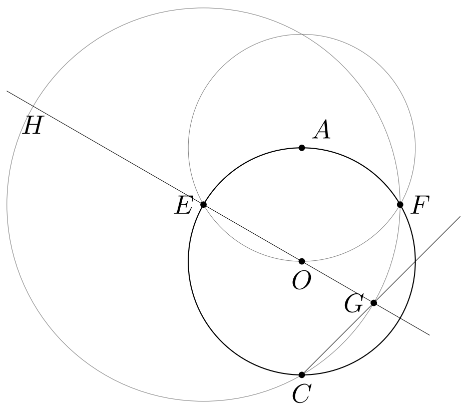
2. Draw circle centered E through F.



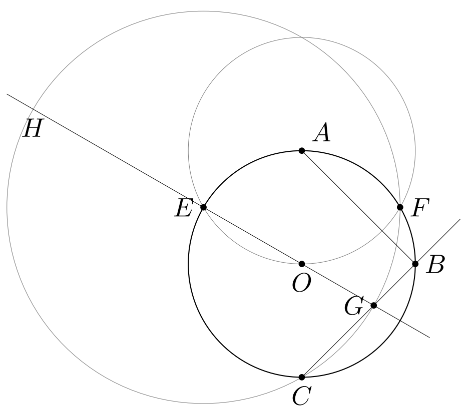
3. Draw line EO. Let EO intersect circle (E, F) at G and H, where G lies inside the given circle and H lies outside.



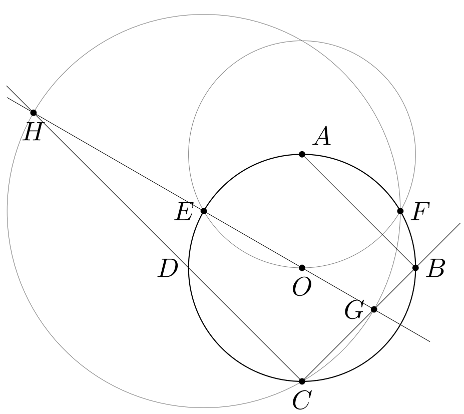
4. Let C be another intersection of (E, F) and the given circle. Draw line CG.



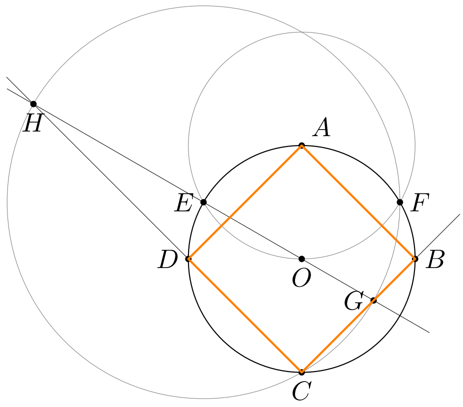
5. Let CG intersect given circle at another point B. Draw line AB.



6. Draw line CH. Let CH intersect given circle at D.

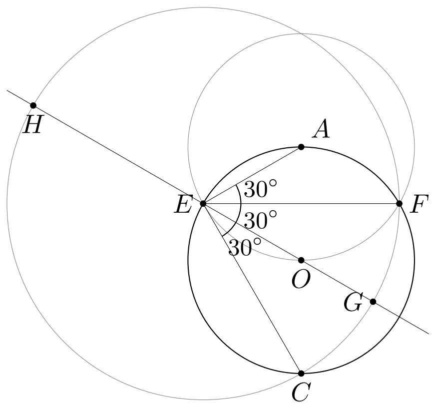


7. Draw line AD. $ABCD$ is the desired square.



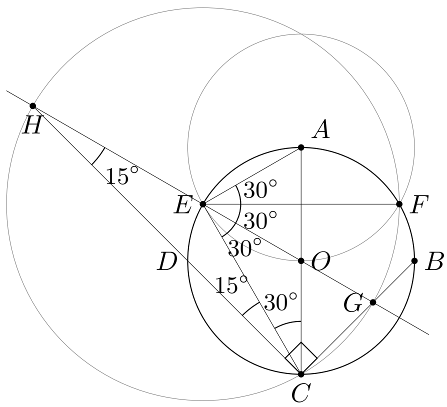
Proof. **1-3.** Note that EF bisects $\angle OEA$ since EF is the diagonal of rhombus $AEOF$ which is made up of two equilateral triangles $\triangle OAE$ and $\triangle OAF$. Thus $\angle AEF = \angle OEF = 60°/2 = 30°$.

Also, note that $\angle OEC = \angle OEF = 30°$ since $\triangle OEC \sim \triangle OEF$ (SSS). Thus $\angle AEC = 30° + 30° + 30° = 90°$. By "converse of $\angle$ in semi-circle", AC is the diameter of given circle, which means A, O, C are collinear.

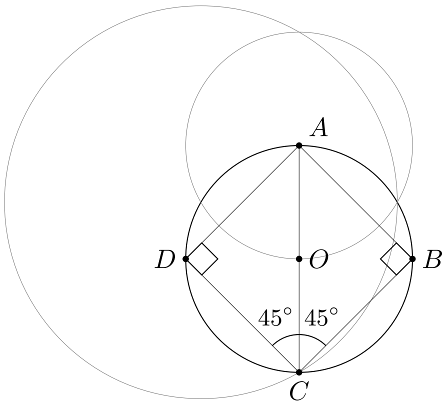


4-7. Note that GH is a diameter of circle (E, G), so $\angle HCG = 90°$ ($\angle$ in semi-circle).

Note that $EH = EC$ (radii), so $\angle ECH = \angle EHC = 30°/2 = 15°$ (base $\angle$s, isos. $\triangle$)& (ext. $\angle$ of $\triangle$). Also, $\angle OCE = \angle OEC = 30°$ (base $\angle$s, isos. $\triangle$).



Thus, $\angle OCD = 30° + 15° = 45°$, and $\angle OCB = \angle OCG = 90° - 45° = 45°$.

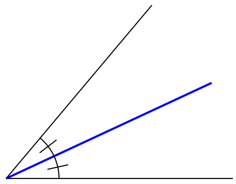


Let's focus on points A, B, C, D. Note that $\angle ADC = \angle ABC = 90°$ ($\angle$ in semi-circle), $\angle ACD = \angle ACB = 45°$, and $AC = AC$. Thus $\triangle ADC \cong \triangle ABC$ (AAS) and $BC = CD$ (corr. sides, $\cong \triangle$s). Since $ABCD$ has four right angles and adjacent sides are equal, $ABCD$ is a square (con. of square), as desired. $\square$

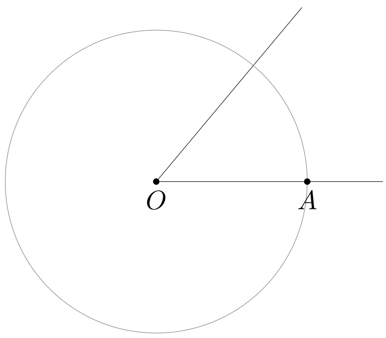
2 Beta

2.1 Angle bisector

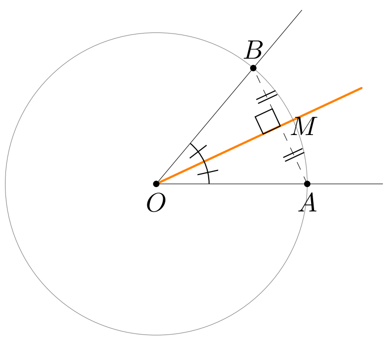
Task 2.1. Construct the line that bisects the given angle.
(2L, 4E)



Solution 2.1. 1. Let O be the vertex of the given angle. Label an arbitrary point A on one of the given rays. Draw circle (O, A).



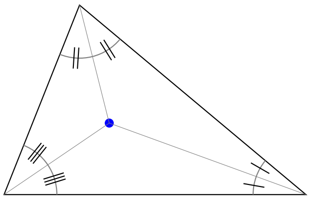
2. Let B be the intersection of the circle and the other ray. Draw perpbi AB (perpendicular bisector of A, B), which is the desired angle bisector.



Proof. Note that $\triangle OAB$ is an isosceles triangle since $OA = OB$ (radii). Let M be the midpoint of AB. Since $OM \perp AB$, by "prop. of isos. $\triangle$", we have $\angle AOM = \angle BOM$. ☐

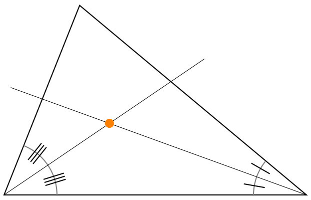
2.2 Intersection of angle bisectors

Task 2.2. Construct the point where the angle bisectors of the triangle are intersected. (2L, 6E)



Solution 2.2. (2L)

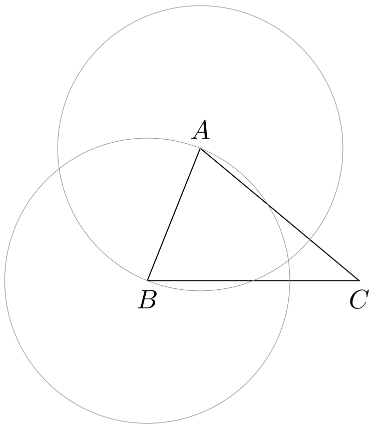
1, 2. Draw angle bisectors of two of the vertices of the triangle. Their intersection is the desired point.



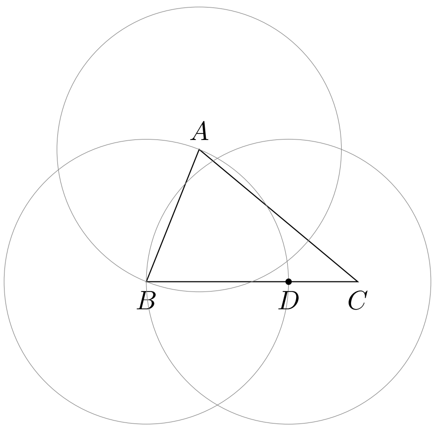
Proof. Note that the three angle bisectors of a triangle are concurrent (prop. of $\angle$ bisector). So we only need to find the intersection of two of them. ☐

(6E)

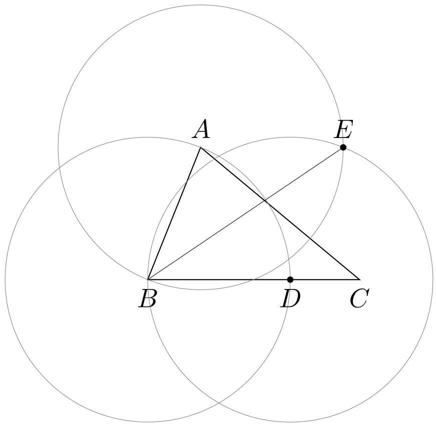
1, 2. Let the vertices of triangle be A, B, C. Draw circle (A, B) and circle (B, A).



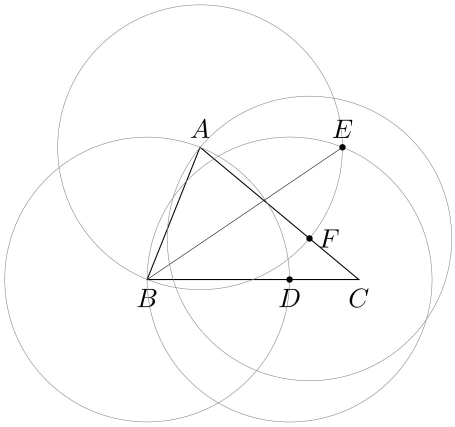
3. Let circle (B, A) intersect side BC at D. Draw circle (D, B).



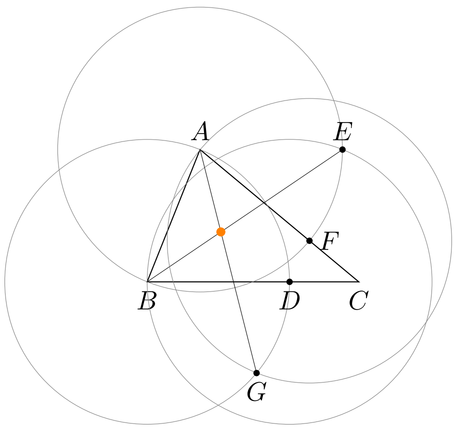
4. Let (D, B) and (A, B) intersect at another point E. Draw line BE.



5. Let (A, B) intersect side AC at F. Draw circle (F, A).

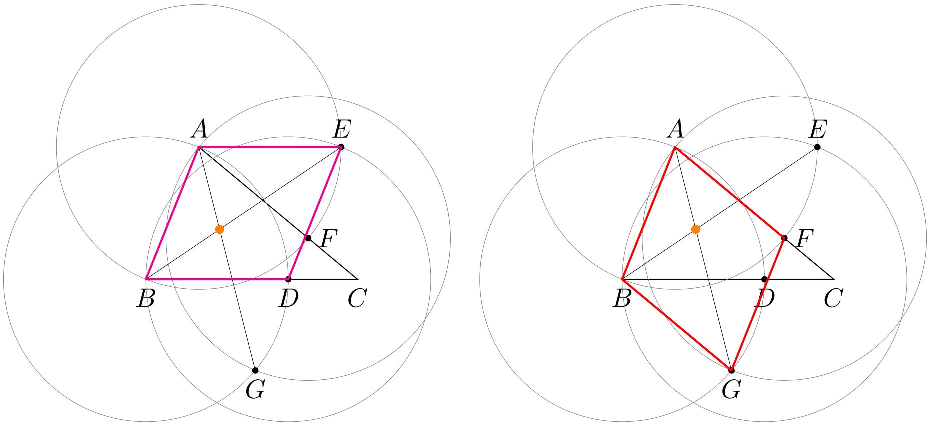


6. Let (F, A) and (B, A) intersect at another point G. Draw line AG. The intersection of BE and AG is the desired point.



Proof. **1-4.** Let r be the length of AB. Note that $AE = AB = BD = DE$ since they are all radii of circles with radius r. So $ABDE$ is a rhombus. Since BE is a diagonal of the rhombus, BE bisects $\angle B$ (prop. of rhombus).

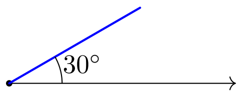
5-6. Similarly, since $AB = BG = FG = FA$, $ABGF$ is a rhombus of side length r. Since AG is a diagonal of rhombus $ABGF$, AG bisects $\angle A$.



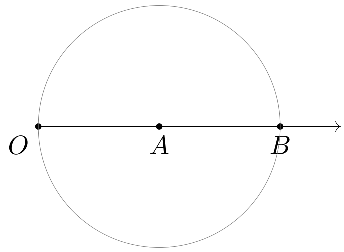
□

2.3 Angle of 30 deg

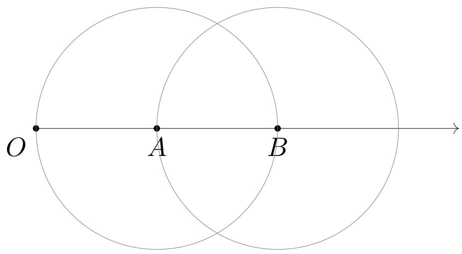
Task 2.3. Construct an angle of 30° with the given side.
(3L, 3E, 2V)



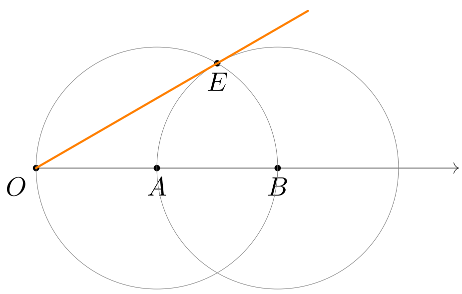
Solution 2.3. (3L, 3E) 1. Let O be the endpoint of the given ray, and A be an arbitrary point on the given ray. Draw circle (A, O), intersecting given ray at B.



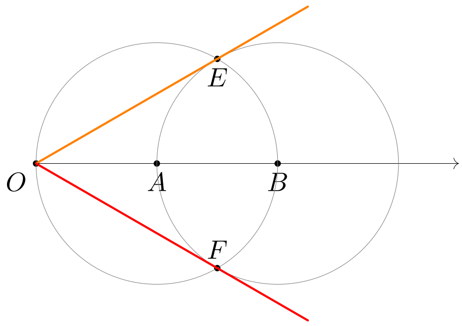
2. Draw circle (B, A).



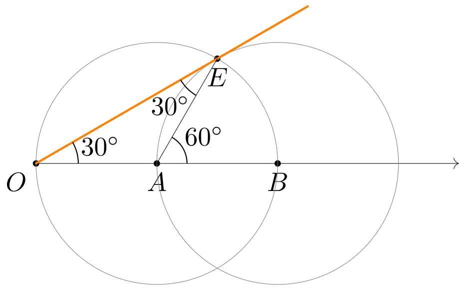
3. Let E be one intersection of the two circles. Draw line OE, which is the desired line.



(2V) 4. Let F be another intersection of the two circles. Draw line OF.



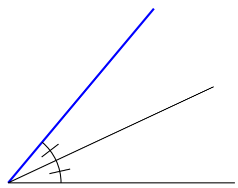
Proof. Note that $\angle EAB = 60°$ by construction. Also, $AO = AE$ (radii) so $\angle AOE = \angle AEO$ (base $\angle$s, isos. $\triangle$). So $\angle EOA = 60°/2 = 30°$ (ext. $\angle$ of $\triangle$). Similar argument for the other line.



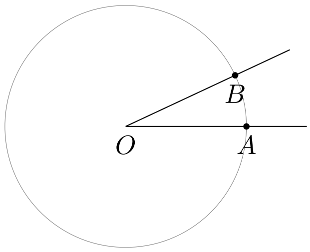
□

2.4 Double angle

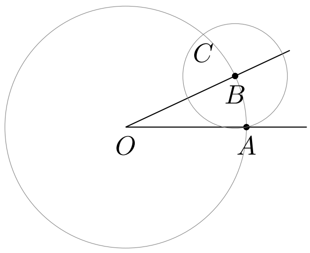
Task 2.4. Construct an angle equal to the given one so that they share one side.



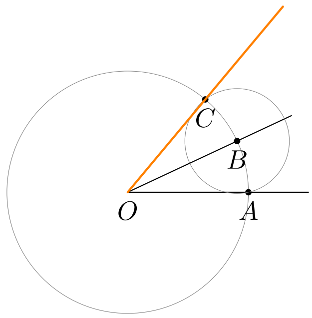
Solution 2.4. **1.** Let O be the vertex of given angle, and A be an arbitrary point on one ray. Draw circle (O, A), intersecting the other ray at B.



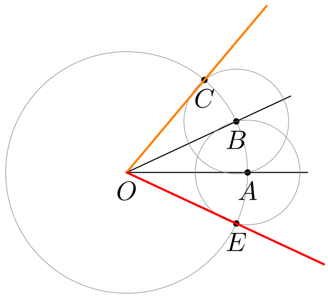
2. Draw circle (B, A), intersecting (O, A) at another point C.



3. Draw line OC, which is the desired line.



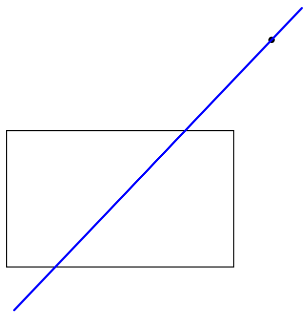
(2V) Draw circle (A, B), intersecting (O, A) at D. Draw line OD.



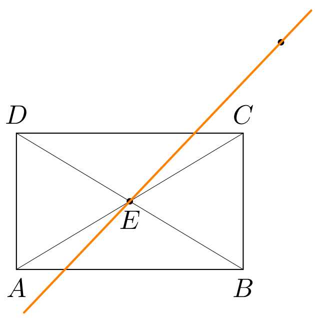
Proof. Note that $BA = BC$ (radii), so $\angle BOC = \angle BOA$ (equal chord, equal $\angle$). Similar argument for line OE. □

2.5 Cut rectangle

Task 2.5. Construct a line through the given point that cuts the rectangle into two parts of equal area.
(3L, 3E)



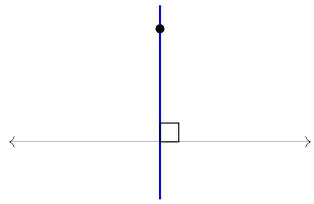
Solution 2.5. Let the rectangle be $ABCD$. **1, 2.** Draw diagonals AC and BD, intersecting at E.
3. Draw line through given point and E.



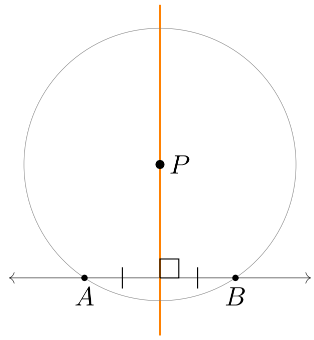
Proof. Let the orange line intersect the rectangle sides CD and AB at G and H respectively. Note that $\triangle EAH \cong \triangle ECG$. So we can move $\triangle EAH$ to $\triangle ECG$ and vice versa to create $\triangle ADC$ and $\triangle CBA$ which are equal in area (because they are half of rectangle). □

2.6 Drop a perpendicular

Task 2.6. Drop a perpendicular from the point to the line.
(2L, 3E)

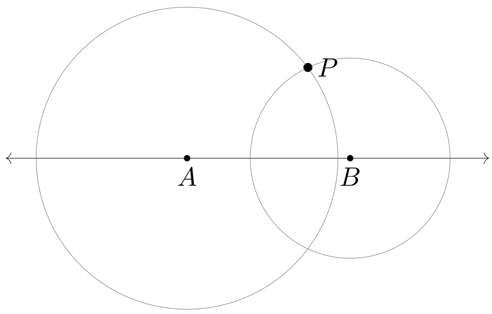


Solution 2.6. (2L) Let the given point be P, and A be an arbitrary point on given line.
1. Draw circle (P, A), intersecting the line on B.
2. Draw perpbi AB.

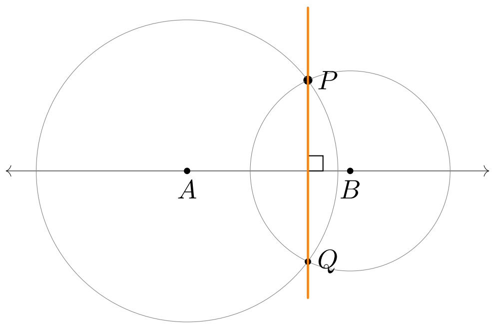


Proof. AB is a chord of the circle, so the perpendicular bisector of AB passes through center P. This means we have constructed a line through P that is perpendicular to line AB. □

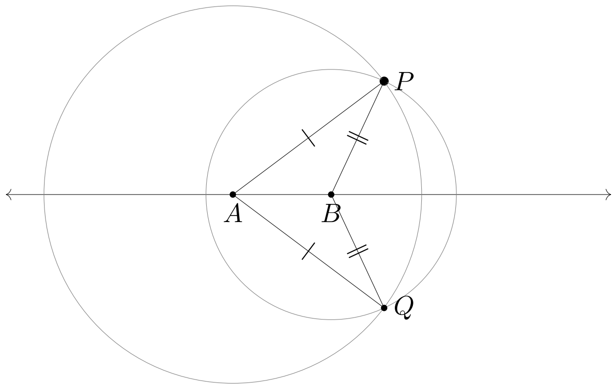
(3E) 1, 2. Label two arbitrary points A, B. Draw circles (A, P) and circle (B, P).



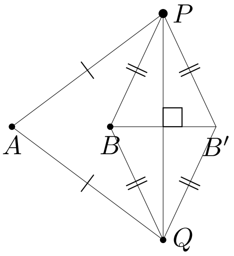
3. Draw line through the intersection of the two circles, which is the desired line.



Proof. Let Q be the other intersection of the two circles. Note that $AP = AQ$ and $BP = BQ$ (radii), so $APBQ$ is either a kite or a dart. If $APBQ$ is a kite, then by "prop. of kite", the diagonals of the kite are perpendicular to each other, meaning $PQ \perp AB$.



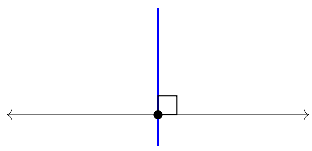
If $APBQ$ is a dart with B being the concave point, then reflect B about line PQ to get B'. Note that $PQ \perp BB'$ and $BPB'Q$ is a rhombus (by reflection). Since $APB'Q$ is a kite, we also have $PQ \perp AB'$. Thus AB' and BB' are parallel, but they share the same point B', so A, B, B' must lie on the same line. This means $PQ \perp AB$, our desired result.



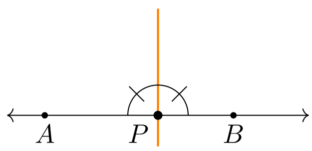
□

2.7 Erect a perpendicular

Task 2.7. Erect a perpendicular from the point on the line.
(1L, 3E)

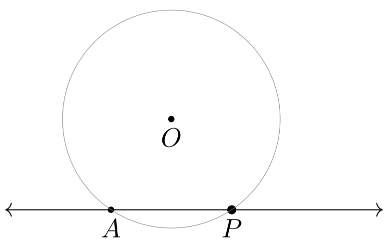


Solution 2.7. **(1L)** Let P be the given point. Let A be an arbitrary point to the left of P and B be an arbitrary point to the right of P. Draw the angle bisector of $\angle AOB$, which is the desired line.

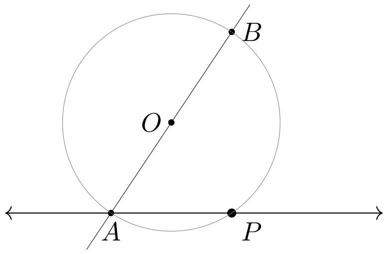


Proof. Since A, O, P are on a straight line, $\angle AOP = 180°$, so the angle bisector makes two angles of $90°$, which means the angle bisector is perpendicular to line AOB. □

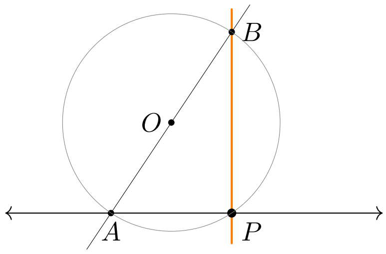
(3E) 1. Label an arbitrary point O not on the given line. Draw circle (O, P), intersecting the given line at another point A.



2. Draw line AO. Let it intersect the circle at B.



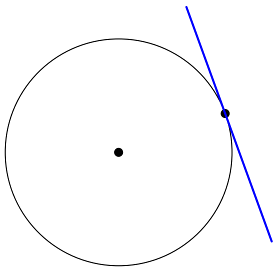
3. Draw line BP, which is the desired line.



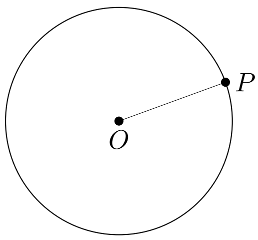
Proof. Note that AB is the diameter of the circle, so $\angle APB = 90^\circ$ ($\angle$ in semi-circle), which means $BP \perp AP$. □

2.8 Tangent to circle at point

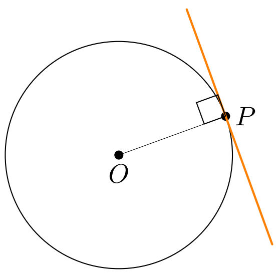
Task 2.8. Construct a tangent to the circle at the given point.
(2L, 3E)



Solution 2.8. Let O be the center of circle and P be the given point on the circle.
(2L) 1. Draw line OP.

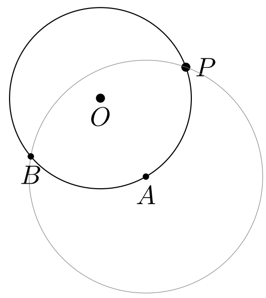


2. Draw the perpendicular line of OP at P.

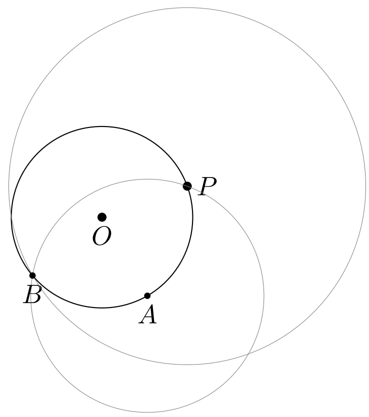


Proof. Since OP is a radius of the circle and is perpendicular to the orange line, by "converse of tangent $\perp$ radius", the orange line is tangent to the circle at P. □

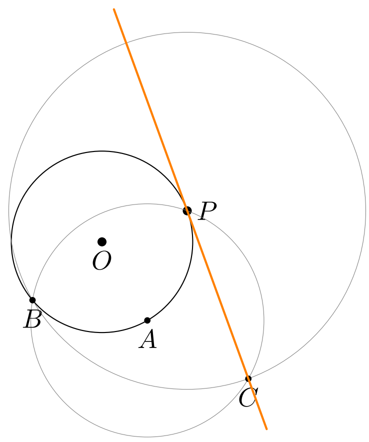
(3E) 1. Let A be an arbitrary point on the given circle. Draw circle (A, P), intersecting the given circle at B.



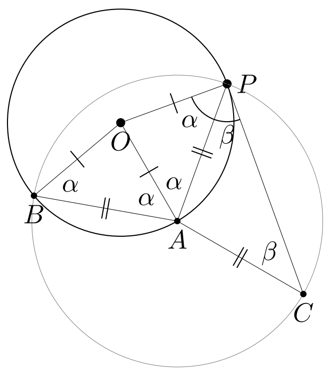
2. Draw circle (P, B).



3. Let (P, B) intersect (A, P) at another point C. Draw line PC, the desired line.



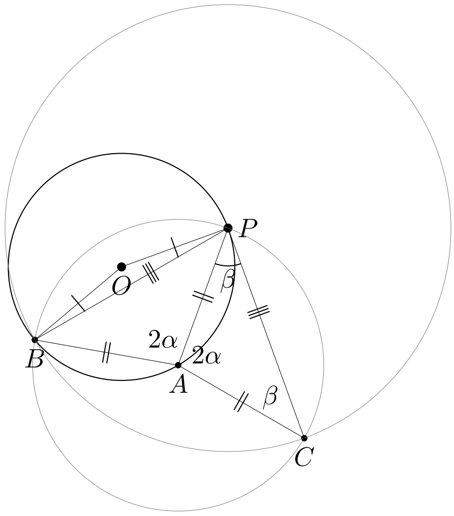
Proof. Let $\angle OPA = \alpha$ and $\angle APC = \beta$. We want to show that $\alpha + \beta = 90°$, which will prove that PC is the tangent to the given circle at P.



$$OA = OP \qquad \text{(radii)}$$
$$\therefore \angle OAP = \angle OPA = \alpha \qquad \text{(base } \angle\text{s, isos. } \triangle\text{)}$$

$$\triangle OBA \cong \triangle OPA \qquad \text{(SSS)}$$
$$\therefore\ \angle OBA = \angle OPA = \alpha \ \text{ and } \ \angle OAB = \angle OAB = \alpha \ \ \text{(corr. } \angle\text{s, } \cong \triangle\text{s).}$$

Now consider $\triangle APC$ and $\triangle APB$.

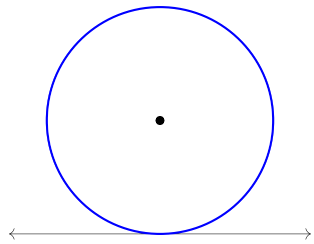


$$AC = AP \qquad \text{(radii)}$$
$$\therefore \angle ACP = \angle APC = \beta \qquad \text{(base } \angle\text{s, isos. } \triangle\text{)}$$
$$PB = PC \qquad \text{(radii of biggest circle)}$$
$$\therefore\ \triangle APC \cong \triangle APB \qquad \text{(SSS)}$$
$$\therefore\ \angle PAC = \angle PAB = 2\alpha \qquad \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}$$

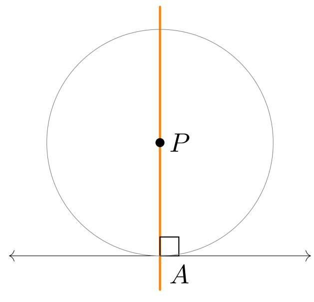
In $\triangle APC$, we have $2\alpha + \beta + \beta = 180°$ ($\angle$ sum of $\triangle$), giving $\alpha + \beta = 90°$, as desired. □

2.9 Circle tangent to line

Task 2.9. Construct a circle with the given center that is tangent to the given line. (2L, 4E)



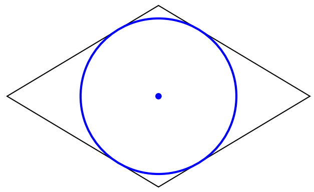
Solution 2.9. **1.** Draw line perpendicular to the given line passing through given point P.
2. Draw circle centered P through the intersection of the two lines A.



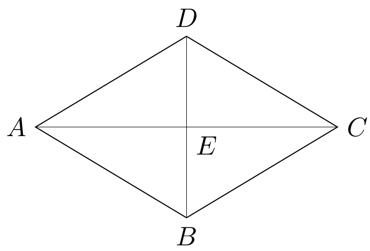
Proof. PA is tangent to the given line by "converse of tangent $\perp$ radius". □

2.10 Circle in rhombus

Task 2.10. Inscribe a circle in the rhombus.
(4L, 6E)

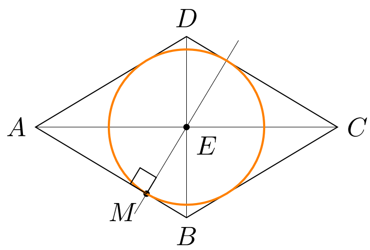


Solution 2.10. 1, 2. Let the rhombus be $ABCD$. Draw diagonals AC and BD. Let them intersect at E.



3. Draw $ME \perp AB$ (i.e. line perpendicular to AB passing through E, intersecting AB at M).

4. Draw circle (E, M).

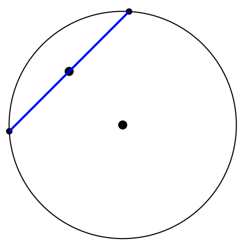


Proof. Note that the diagonals divide the rhombus into four congruent triangles (prop. of rhombus), so they have the same height. This means sides AB, BC, CD, DA have the same perpendicular distance from E. Thus, a circle tangent to one of the sides must be tangent to all of them. □

3 Gamma

3.1 Chord midpoint

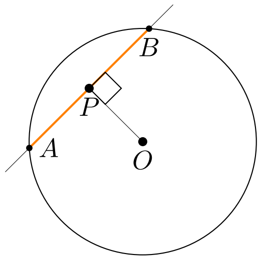
Task 3.1. Construct a chord whose midpoint is given.
(2L, 4E)



Solution 3.1. Let O be center of given circle and P be given point.

1. Draw line OP.

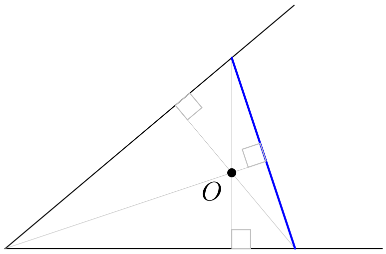
2. Draw $OP \vdash P$ (i.e. line perpendicular to OP passing through P), intersecting the circle at A and B. AB is the desired chord.



Proof. Since $OP \perp AB$, we have $AP = PB$ by "line from center $\perp$ chord bisects chord". □

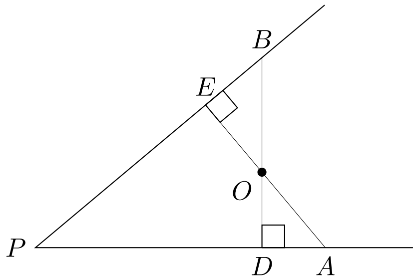
3.2 Triangle by angle and orthocenter

Task 3.2. Construct a segment connecting the sides of the angle to get a triangle whose orthocenter is in the point O.
(3L, 6E)

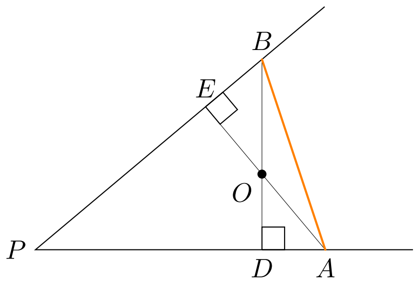


Solution 3.2. Let P be the vertex of given angle.

(3L) 1, 2. Draw lines perpendicular to the given rays passing through O. Let D, E be the feet of the perpendicular lines, and let EO and DO meet the given rays at A and B respectively.



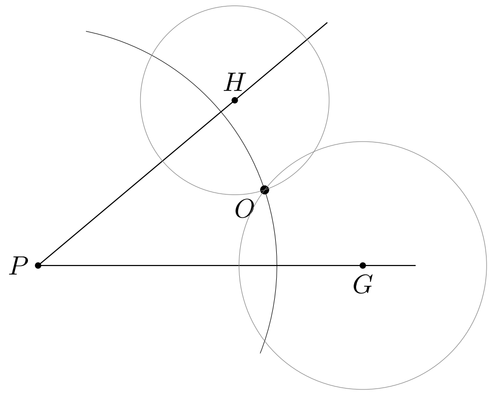
3. Draw line AB, the desired line.



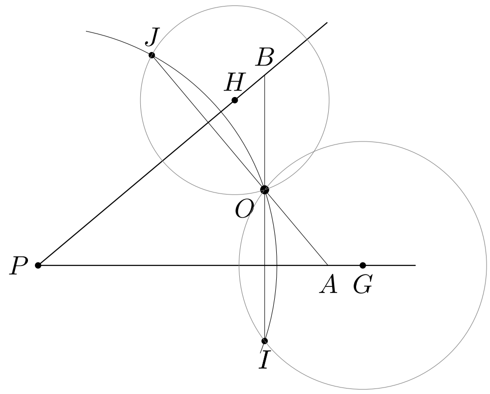
Proof. Note that O is the orthocenter of $\triangle PAB$ since it is the intersection of two altitudes. And any two altitudes intersect at the orthocenter because the three altitudes of a triangle are concurrent. □

(6E) 1. Draw circle (P, O).

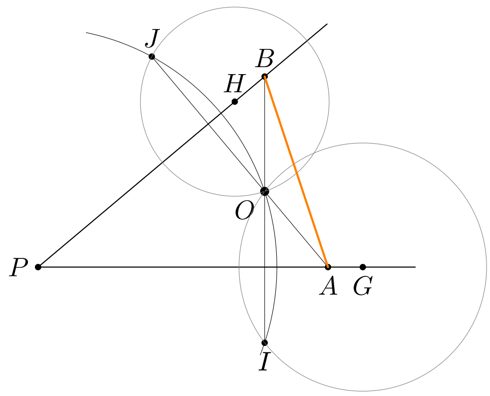
2, 3. Let G, H be two points (arbitrary or on intersection, doesn't matter) on each of the given ray. Draw circles (G, O) and (H, O).



4, 5. Let (P, O) intersect (G, O) and (H, O) at the other point I and J respectively. Draw line IO, meeting PH at B. Draw line JO, meeting PG at A.



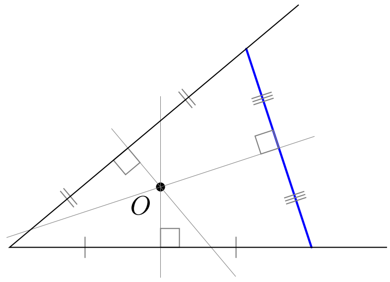
6. Draw line AB, the desired line.



Proof. Note that $OI \perp PG$ since $POGI$ forms a kite. Similarly, $OJ \perp PH$ since $POHJ$ forms a kite. Thus line OI and OJ are altitudes of $\triangle PAB$, so O is the orthocenter of $\triangle PAB$. $\square$

3.3 Intersection of perpendicular bisectors

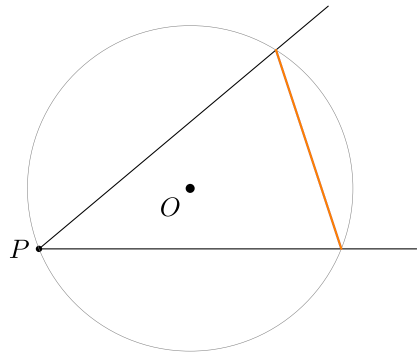
Task 3.3. Construct a segment connecting the sides of the angle to get a triangle whose perpendicular bisectors are intersected in the point O.
(2L, 2E)



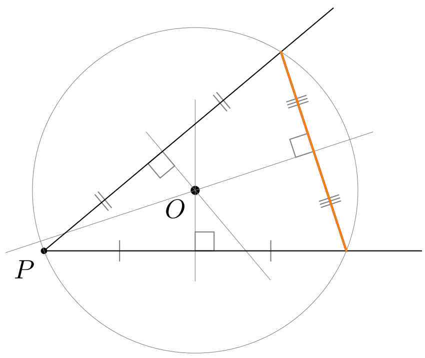
Solution 3.3. Let P be the vertex of the given angle.

1. Draw circle (O, P), intersecting the given rays at A and B respectively.

2. Draw line AB.



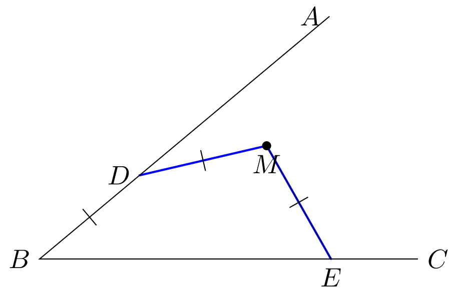
Proof. Note that O is the circumcenter of $\triangle PAB$. And the perpendicular bisectors of sides of $\triangle PAB$ intersect at the circumcenter by "prop. of circumcenter".



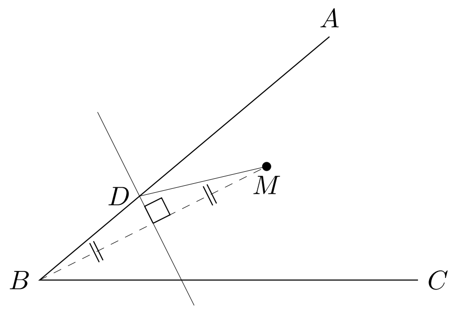
□

3.4 Three equal segments - 1

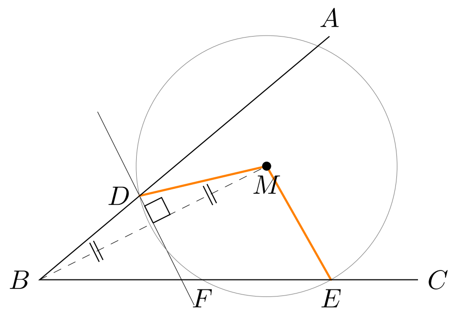
Task 3.4. Given an angle ABC and a point M inside it, find points D on BA and E on BC and construct segments DM and ME such that $BD = DM = ME$.
(4L, 6E, 2V)



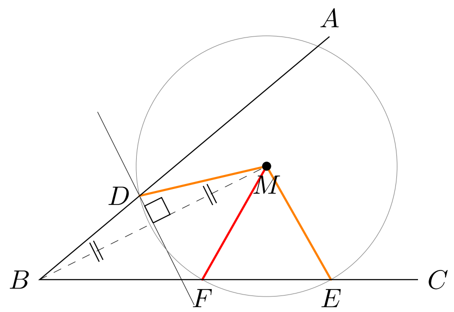
Solution 3.4. (4L, 6E) 1. Draw perpbi BM, intersecting AB at D.
2. Draw line MD.



3. Draw circle (M, D), intersecting line BC at E and F.
4. Draw line ME (or MF).



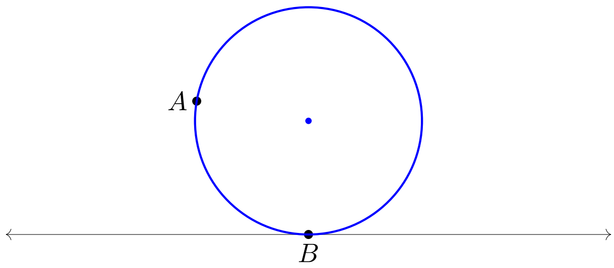
(2V) Draw line MF (or ME).



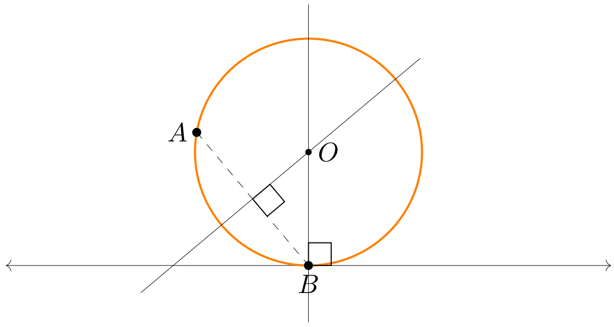
Proof. $BD = DM$ since D lies on the perpendicular bisector of BM. $DM = ME = MF$ since D, E and F lie on the circle centered M. Thus $BD = DM = ME = MF$. □

3.5 Circle through point tangent to line

Task 3.5. Construct a circle through the point A that is tangent to the given line at the point B.
(3L, 6E)

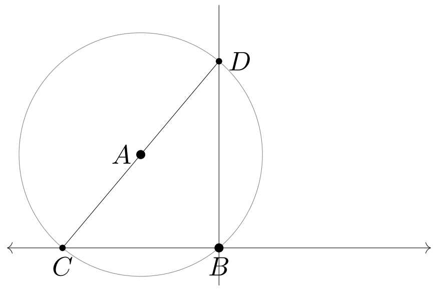


Solution 3.5. (3L)
1, 2. Draw perpbi AB. Draw perpendicular line to given line through B. Let the two drawn lines intersect at O.
3. Draw OB.

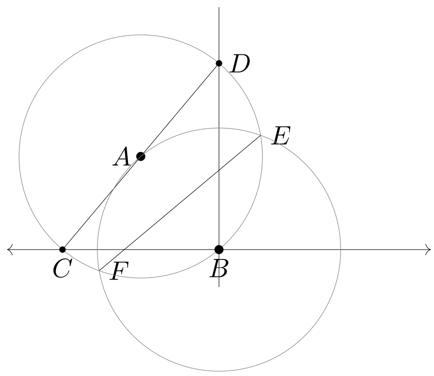


Proof. Since the circle passes through both A and B, center O must lie on the perpendicular bisector of AB (prop. of $\perp$ bisector). Since O is tangent to give line, OB must be perpendicular to given line (tangent $\perp$ radius). Thus O lies on the intersection of the two drawn lines. $\square$

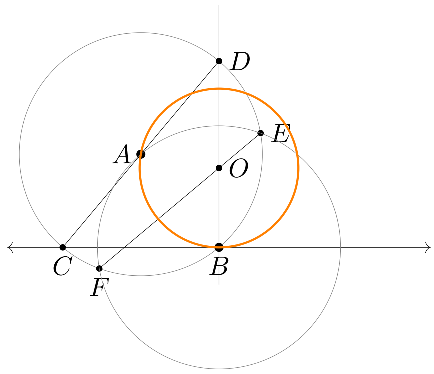
(6E) 1. Draw circle (A, B), intersecting given line at C.
2. Draw line CA, meeting circle (A, B) at D.
3. Draw line BD.



4. Draw circle (B, A), intersecting (A, B) at E and F.
5. Draw line EF, intersecting BD at O.



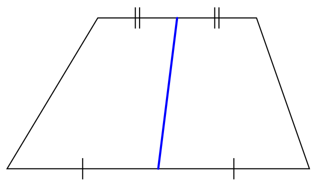
6. Draw circle (O, B).



Proof. Note that BD is perpendicular to given line by Task 2.7E, and EF is the perpendicular bisector of AB by Task 1.2. So O is the same point as the (3L) part of this level. □

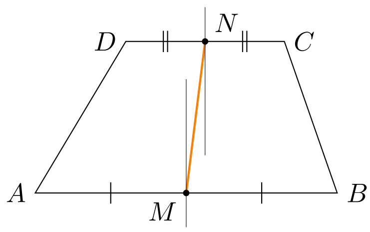
3.6 Midpoints of trapezoid bases

Task 3.6. Construct a line passing through the midpoints of the trapezoid bases.
(3L, 5E)



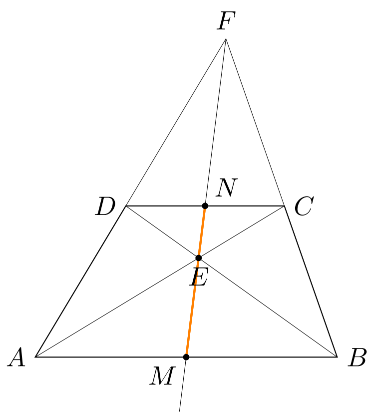
Solution 3.6. (3L)

1, 2. Draw perpbi AB and draw perpbi CD. Let the midpoints of the sides be M and N.
3. Draw line MN.



Proof. $AM = MB$ and $DN = NC$ by perpendicular bisector construction. □

(5E) 1, 2. Draw the diagonals of the trapezoid. Let them intersect at E.
3, 4. Extend the non-parallel sides to meet at F.
5. Draw line FE, which is the desired line.



Proof. Let FE intersect sides AB and CD at M and N respectively. We want to show that $AM = MB$ and $DN = NC$.

By Ceva's theorem, we have

$$\frac{AM}{MB} \cdot \frac{BC}{CF} \cdot \frac{FD}{DA} = 1 \tag{1}$$

Since $AB//CD$, by intercept theorem, we also have

$$\begin{aligned} \frac{BC}{CF} &= \frac{AD}{DF} \\ \iff \qquad \frac{BC}{CF} \cdot \frac{FD}{DA} &= 1 \end{aligned} \tag{2}$$

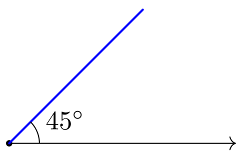
Put (2) into (1).

$$\begin{aligned} \frac{AM}{MB} \cdot (1) &= 1 \\ AM &= MB \end{aligned}$$

Note that $\triangle FDN \sim \triangle FAM$ and $\triangle FNC \sim \triangle FMB$ (AAA). So $\dfrac{DN}{AM} = \dfrac{FN}{FM} = \dfrac{NC}{MB}$ (corr. sides, $\sim \triangle$s). Since $AM = MB$, this gives $\dfrac{DN}{AM} = \dfrac{NC}{AM}$, and thus $DN = NC$, as desired. ☐

3.7 Angle of 45 deg

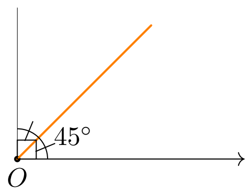
Task 3.7. Construct an angle of 45° with the given side.
(2L, 5E, 2V)



Solution 3.7. Let O be the endpoint of the given ray.

(2L) 1. Draw line perpendicular to given line through O.

2. Draw the angle bisector of the two lines.

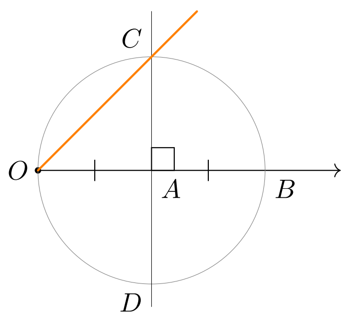


Proof. The angle between the two perpendicular lines is 90°, and the angle bisector makes $90°/2 = 45°$. ☐

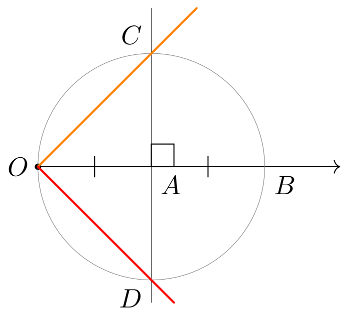
(5E) 1. Let A be an arbitrary point on the given ray. Draw circle (A, O), intersecting the ray again at B.

2. Draw perpbi OB, intersecting the circle at C and D.

3. Draw line OC, the desired line.



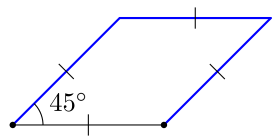
(2V)



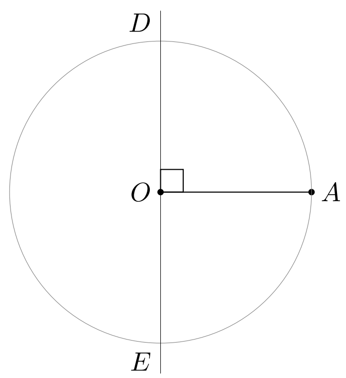
Proof. Since $AO = AC$ and $CA \perp OB$, $\triangle OAC$ is an isosceles right triangle, so its acute angles are $45°$, which means $\angle AOC = 45°$. Same for the other line OD. □

3.8 Lozenge

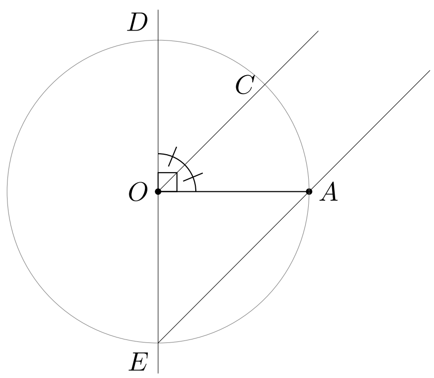
Task 3.8. Construct a rhombus with the given side and an angle of $45°$ in a vertex.
(5L, 7E, 4V)



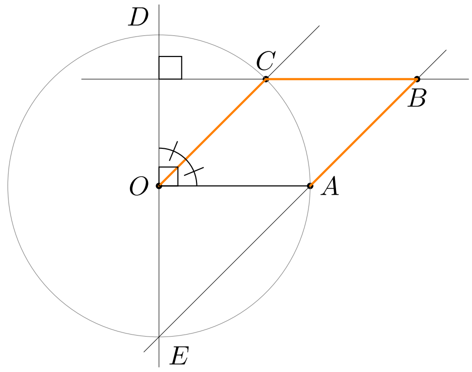
Solution 3.8. Let O and A be the endpoints of the given line segment.
(5L) 1. Draw $OA \vdash O$.
2. Draw circle (O, A), intersecting the vertical line at D and E (where D above E).



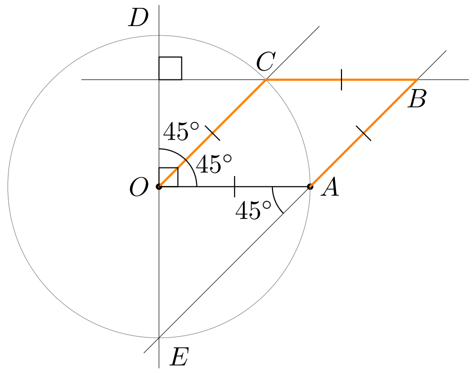
3. Draw angbi DOA (angle bisector of $\angle DOA$), intersecting (O, A) at C.
4. Draw line EA.



5. Draw $OD \vdash C$, intersecting EA at B. $OABC$ is the desired rhombus.

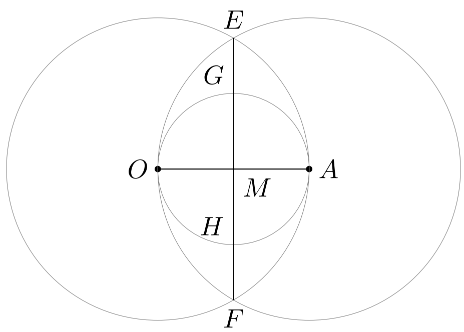


Proof. Note that $CB//OA$ since they are both perpendicular to DO. Note that $\angle AOC = 45°$ (since it is half of right angle), and $\angle OAE = 45°$ since $\triangle OAE$ is an isosceles right triangle. Thus $OC//EB$ (alt. $\angle$s equal). This means $OABC$ is a parallelogram.

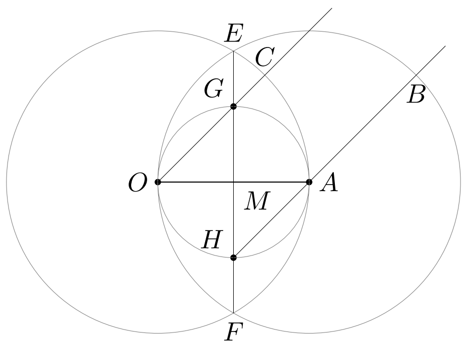


Since $OA = OC$ (radii), $OABC$ is a parallelogram with adjacent sides equal, so $OABC$ is a rhombus. Along with $\angle AOC = 45°$, $OABC$ is the desired rhombus. □

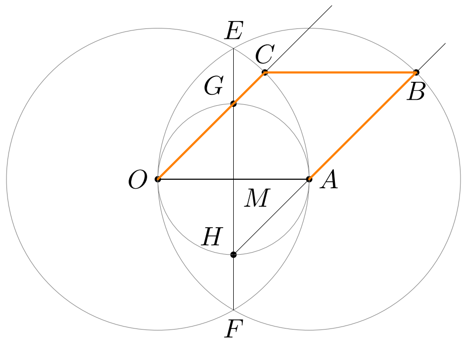
(7E) 1, 2. Draw circle (O, A) and (A, O), intersecting at E and F.
3. Draw line EF, intersect OA at M.
4. Draw circle (M, O), intersecting EF at G and H (G above H).



5, 6. Draw lines OG and HA. Let OG intersect (O, A) at C, and let HA intersect (A, O) at B (where both points are on the same side of OA).



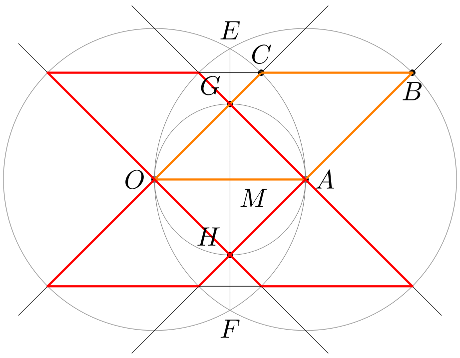
7. Draw line BC.



Proof. Note that $\triangle MOG$ and $\triangle MAH$ are isosceles right triangles, so $\angle MOG = \angle MAH = 45°$ and $OC//HB$ (alt. $\angle$s equal). Moreover, note that $OC = AB$ since they lie on circles of the same radius. Thus $OABC$ is a parallelogram (opp. sides equal and $//$).

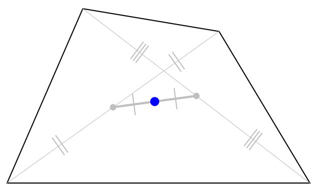
And since $OA = OC$ (radii), $OABC$ has adjacent sides equal, so it is a rhombus. □

(4V) Draw line GA and OH. Connect the intersections of the lines and the big circles to form a symmetric figure.

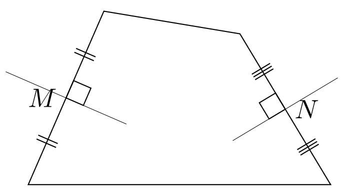


3.9 Center of quadrilateral

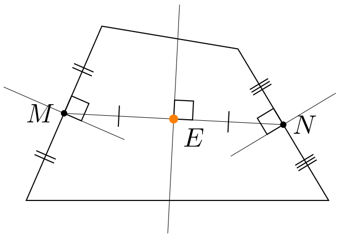
Task 3.9. Construct the midpoint of the segment that connects the midpoints of the diagonals of the quadrilateral.
(4L, 10E)



Solution 3.9. 1, 2. Draw the perpendicular bisectors of the two non-parallel sides. Let the midpoints of the non-parallel sides be M and N.



3, 4. Draw MN. Draw the perpendicular bisector of MN. The midpoint of MN is the desired point.

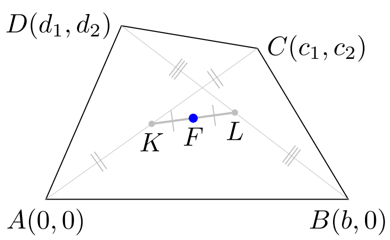


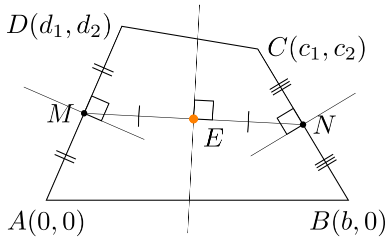
Proof. We use Cartesian coordinates. Let the quadrilateral be $ABCD$, and let

$A = (0,0)$, $B = (b,0)$, $C = (c_1, c_2)$, $D = (d_1, d_2)$.

Let K, L be the midpoints of AC and BD respectively. Then $K = (\frac{c_1}{2}, \frac{c_2}{2})$ and $L = (\frac{d_1+b}{2}, \frac{d_2}{2})$ (mid-pt. coordinate formula). Let F be the midpoint of KL (the desired point).

Then $F = (\frac{c_1+d_1+b}{4}, \frac{c_2+d_2}{4})$.



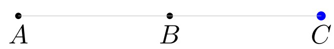


Now we show that E has the same coordinates as F. Since M and N are midpoints of AD and BC, we have $M = (\frac{d_1}{2}, \frac{d_2}{2})$ and $N = (\frac{c_1+b}{2}, \frac{c_2}{2})$. And E is the midpoint of MN, so $E = (\frac{d_1+c_1+b}{4}, \frac{d_2+c_2}{4})$, which is the same as F. So we conclude $E = F$. □

4 Delta

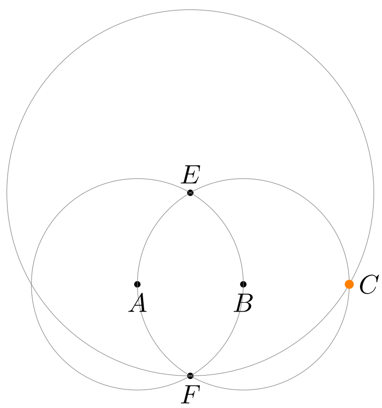
4.1 Double segment

Task 4.1. Construct a point C on the line AB such that $|AC| = 2|AB|$ using only a compass. (3L, 3E, 2V)



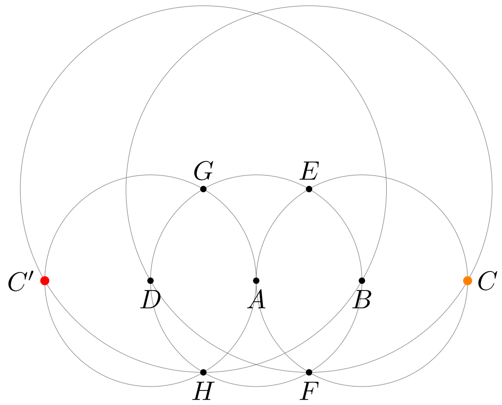
Solution 4.1. (3L, 3E) 1, 2. Draw circle (A, B) and (B, A), intersecting at E and F, intersecting (B, A) again at desired point C,

3. Draw circle (E, F).



Proof. Note that $\angle AEB = 60°$ and $\angle BEC = \angle BEF = 30°$ by congruent triangles. So $\angle AEC = 60° + 30° = 90°$, and so A, B, C are collinear by "converse of $\angle$ in semi-circle". Also $AC = 2AB$ by radii. □

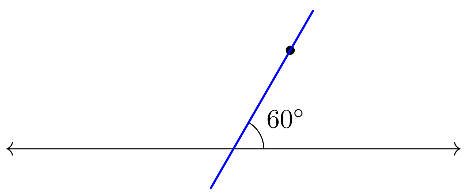
(2V) 4. Let (E, F) intersect (A, B) at D. Draw (D, A), intersecting (A, B) at G and H.
5. Draw circle (G, H), intersecting (D, A) again at desired point C'.



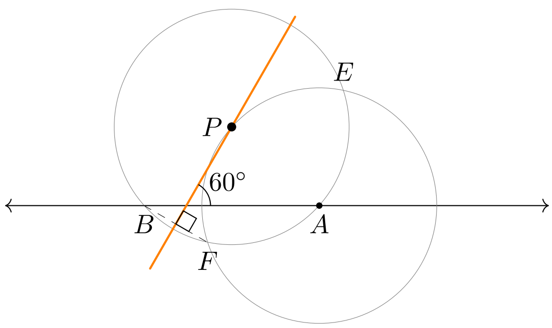
Proof. The same argument works for C' since the figure is symmetric (and because we have D lying on line AB using the same argument as above). □

4.2 Angle of 60 deg - 2

Task 4.2. Construct a straight line through the given point that makes an angle of 60° with the given line.
(3L, 4E, 2V)



Solution 4.2. Let given point be P. Let A be an arbitrary point on given line (such that the angle formed by PA and given line is less than 60°).
(3L) 1. Draw circle (P, A), intersecting the give line again at B.
2. Draw circle (A, P), intersecting (P, A) at E and F.
3. Draw perpbi BF, the desired line.



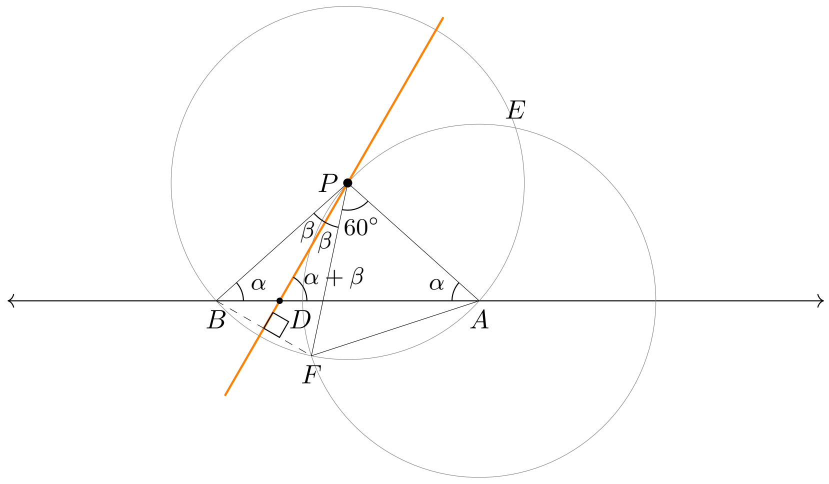
Proof. Let D be the landing point of orange line. First, note that the perpendicular bisector of BF passes through P because BF is a chord of circle centered P.

Since $PB = PA$ (radii), $\angle PBA = \angle PAB$ (base $\angle$s, isos. $\triangle$). Since DP is the perpendicular bisector of BF, $\angle BPD = \angle FPD$ (SAS) & (corr. $\angle$s, $\cong$ $\triangle$s). Also, note that $\angle FPA = 60°$ since $\triangle FPA$ is equilateral triangle.

Let $\angle PBA = \angle PAB = \alpha$ and $\angle BPD = \angle FPD = \beta$.
In $\triangle APB$,

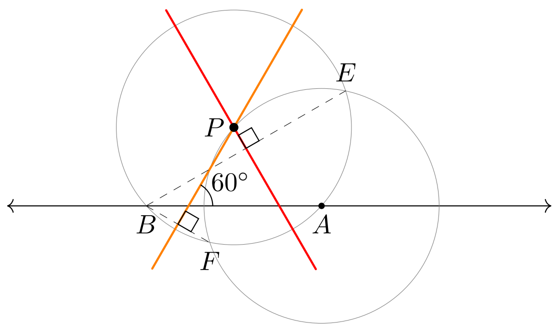
$$\begin{aligned} \angle BPA + \angle PBA + \angle PAB &= 180^\circ \qquad (\angle \text{ sum of } \triangle) \\ (2\beta + 60^\circ) + \alpha + \alpha &= 180^\circ \\ \alpha + \beta &= 60^\circ \end{aligned}$$

Since $\angle PDA = \alpha + \beta$ (ext. $\angle$ of $\triangle$), $\angle PDA = 60^\circ$. This means the orange line makes an angle of 60° with the given line.



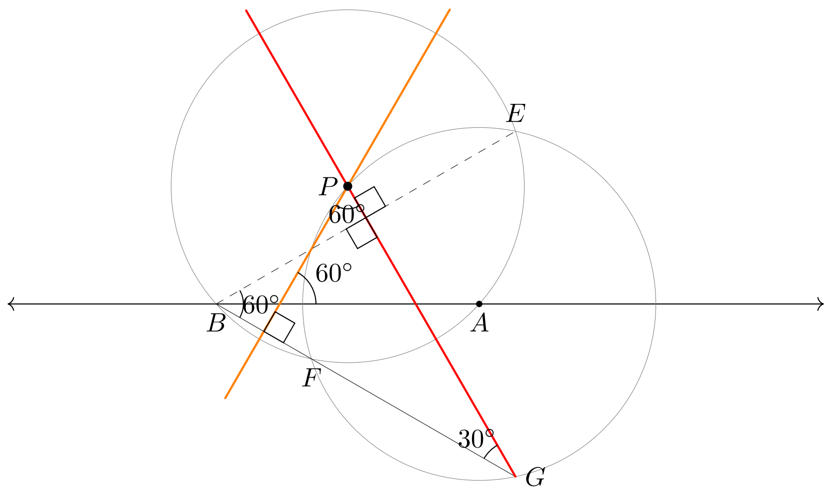
□

(2V) 4. Draw perpbi BE. We get the extra solution.



Proof. Extend BF to meet the red line at G. Note that $\angle FPE = 120^\circ$, so $\angle FBE = 120^\circ/2 = 60^\circ$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$). So $\angle PGB = 90^\circ - 60^\circ = 30^\circ$, and the angle between orange and red line is $90^\circ - 30^\circ = 60^\circ$ ($\angle$ sum of $\triangle$).

Thus the red line makes an angle of $180^\circ - 60^\circ - 60^\circ = 60^\circ$ ($\angle$ sum of $\triangle$).

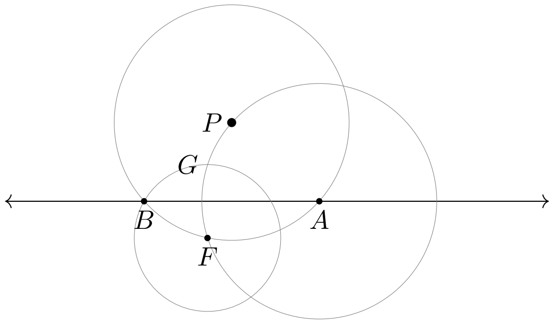


□

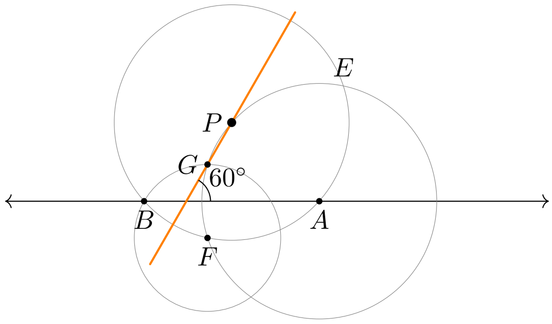
(4E) 1. Draw circle (P, A), intersecting the give line again at B.

2. Draw circle (A, P), intersecting (P, A) at E and F.

3. Draw circle (F, B), intersecting (A, P) above given lien at G.



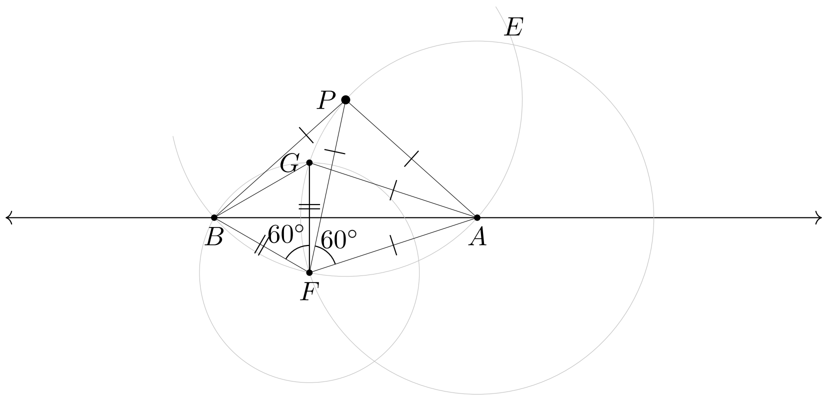
4. Draw line GP, the desired line.



Proof. Let the radius of (P, A) and (A, P) be r, and the radius of (F, B) be s.

Note that $\triangle PBF \cong \triangle AGF$ (SSS), since $PB = AG = r$, $PF = AF = r$, $BF = GF = s$.

Thus $\angle BFP = \angle GFA$ (corr. $\angle$s, $\cong \triangle$s)

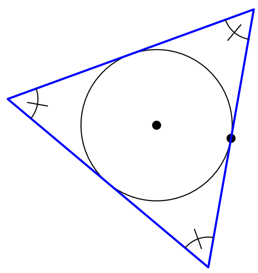


Note that $\angle PFA = 60°$ since $\triangle PFA$ is equilateral. Thus $\angle BFG = \angle BFP - \angle GFP = \angle GFA - \angle GFP = 60°$.

Since $BF = GF$ and $\angle BFG = 60°$, $\triangle GBF$ is equilateral triangle (con. of equil. $\triangle$). This means G lies on the perpendicular bisector of BF, so GP must be the same line as the orange line of (3L). □

4.3 Circumscribed equilateral triangle

Task 4.3. Construct an equilateral triangle that is circumscribed about the circle and contains the given point.

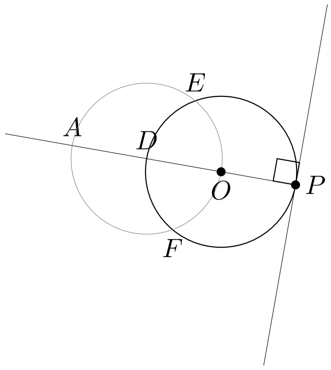


Solution 4.3. Let P be the given point on circle and O be the given circle center.

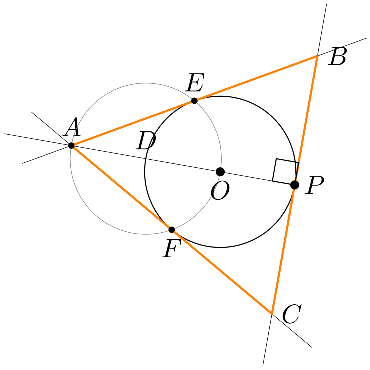
(5L) 1. Draw line OP, intersecting given circle again at D.

2. Draw $OP \vdash P$.

3. Draw circle (D, O), intersecting given circle at E and F, and intersecting OD at A.



4, 5. Draw lines AE and AF, intersecting $OP \vdash P$ at B and C respectively. $\triangle ABC$ is the desired triangle.



Proof. Note that $\angle AEO = \angle AFO = 90°$ ($\angle$ in semi-circle), so AB and AF are tangent to the given circle ‘ (converse of tangent $\perp$ radius).

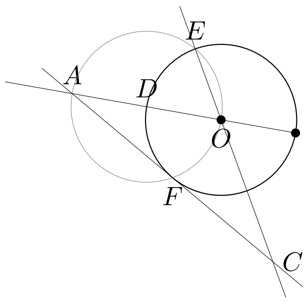
Moreover, $\angle OAE = 30°$ since $\dfrac{OE}{OA} = \dfrac{1}{2}$. Similarly $\angle OAF = 30°$. This means $\angle BAC = 60°$. Since the figure is reflectional symmetric about line AP, we have $AB = AC$, and thus $\triangle ABC$ is an equilateral triangle (con. of equil. $\triangle$). $\square$

(6E) 1. Draw line OP, intersecting given circle again at D.

2. Draw circle (D, O), intersecting given circle at E and F, and intersecting OD at A.

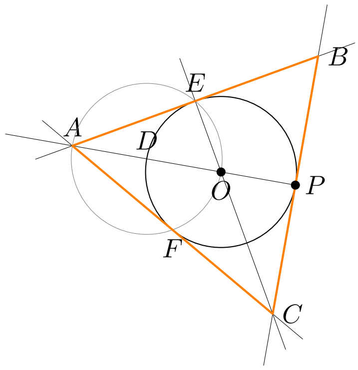
3. Draw line EO.

4. Draw line AF, intersecting EO at C.



5. Draw line AE.

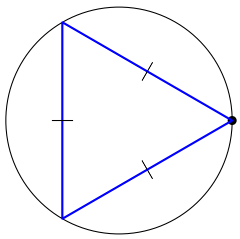
6. Draw line CP, intersecting AE at B. $\triangle ABC$ is the desired triangle.



Proof. Note that in the (5L) figure, E, O, C is collinear since the orthocenter and incenter of an equilateral triangle coincide. Note that point E and line AF in the (6E) figure is the same as that of (5L), so point C in (6E) must be the same as point C in (5L), so $\triangle ABC$ is the same triangle as (5L). □

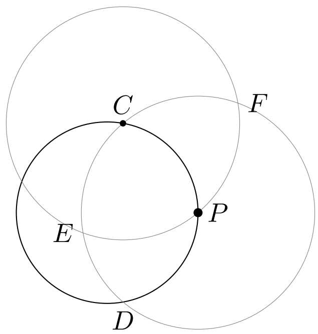
4.4 Equilateral triangle in circle

Task 4.4. Inscribe an equilateral triangle in the circle using the given point as a vertex. The center of the circle is not given.
(5L, 6E)

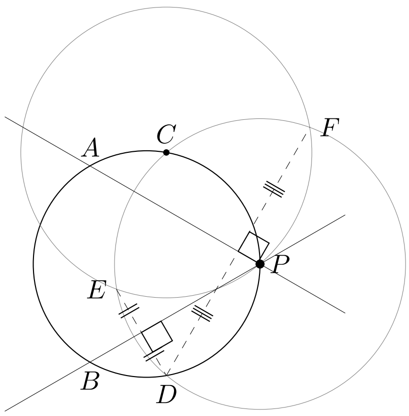


Solution 4.4. Let P be the given point on circle. Let C be an arbitrary point on the circle.

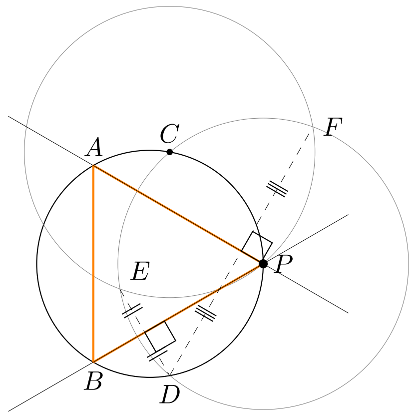
(5L) 1, 2. Draw circles (P, C) and (C, P), intersecting at E and F (where E is nearer to the center of given circle). Let (P, C) intersect given circle again at D.



3, 4. Draw perpbi ED and perpbi FD, intersecting the given circle again at A and B respectively.



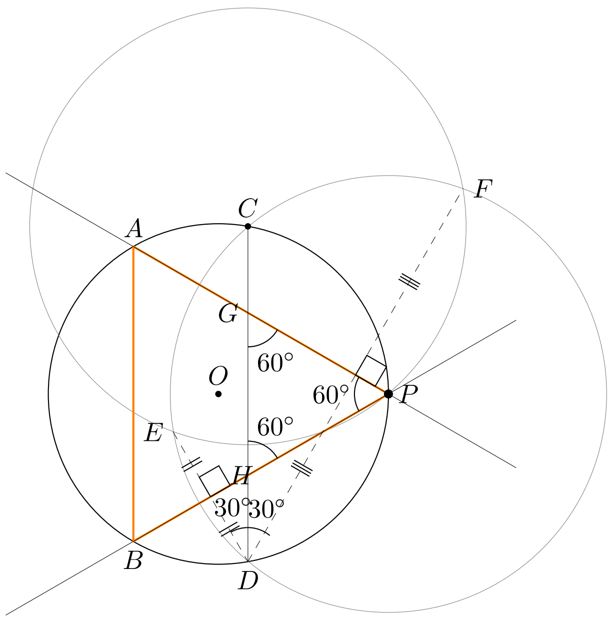
5. Draw line AB. $\triangle PAB$ is the desired triangle.



Proof. Consider angles subtended by arc $\widehat{ECF}$ in circle (P, C). Note that $\angle EPF = 120°$ (since $\triangle ECP$ and $\triangle FCP$ are two equilateral triangles). Thus $\angle EDF = 120°/2 = 60°$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$).

Since $EC = CF$, we have $\angle EDC = \angle CDF = 60°/2 = 30°$ (equal chord, equal $\angle$ at $\odot^{ce}$)

Let CD cut AP and BP at G and H. Since $FD \perp GP$ and $ED \perp HP$, by some angle chasing, we find that $\angle PHG = \angle PGH = 60°$, so $\angle APB = 180° - 60° - 60° = 60°$ ($\angle$ sum of $\triangle$).

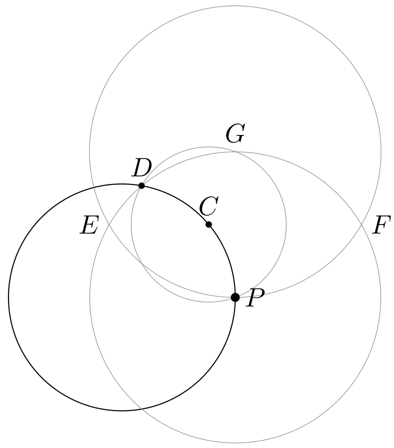


Let O be the center of given circle, and M be the midpoint of CD. Note that $OP \perp CD$ since $ODPC$ forms a kite. Thus PO bisects $\angle GPH$ (prop. of isos. $\triangle$). This means $\triangle PAB$ is reflectional symmetric about line OP, giving $PA = PB$, and thus $\triangle PAB$ is an equilateral triangle. □

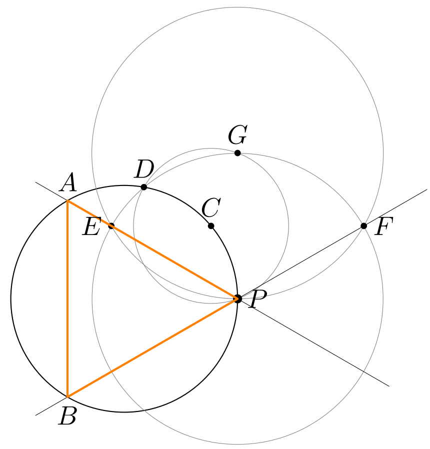
(6E) 1. Let C be an arbitrary point on the circle. Draw circle (C, P), intersecting given circle again at D.

2. Draw circle (P, D), intersecting (C, P) again at G.

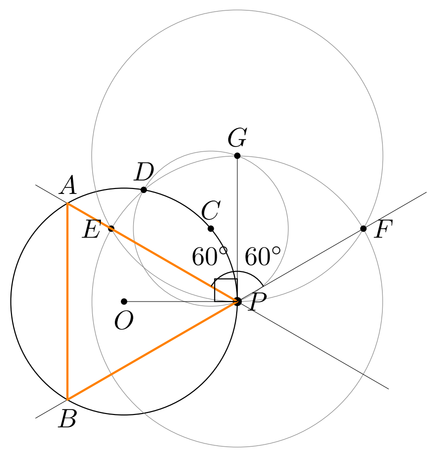
3. Draw circle (G, P), intersecting (P, D) at E and F (E left, F right).



4, 5. Draw line PE and PF, intersecting given circle again at A and B.
6. Draw line AB. $\triangle PAB$ is the desired triangle.



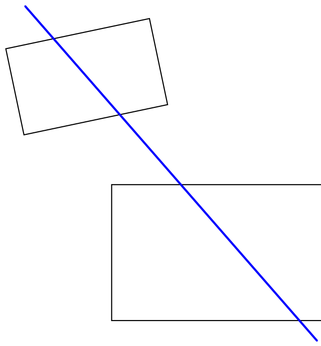
Proof. Let O be center of given circle. Note that PG is tangent to given circle at P by Task 2.8E (tangent to circle at point). Thus $OP \perp GP$. Since $\angle EPG = 60°$, $\angle OPA = 90° - 60° = 30°$ and $\angle OPB = 180° - 90° - 60° = 30°$ (adj. $\angle$s on st. line). Thus OP bisects $\angle APB$, so $\triangle PAB$ is same as (5L) of this level, which means it is equilateral.



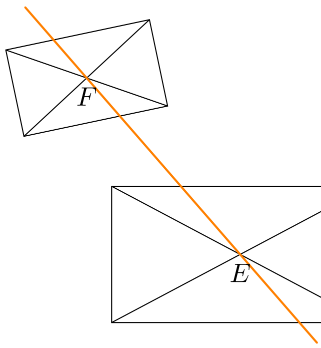
□

4.5 Cut two rectangles

Task 4.5. Construct a line that cuts each of the rectangles into two parts of equal area. (5L, 5E)



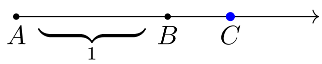
Solution 4.5. **1-5.** Draw the diagonals of the two given rectangles and let them intersect at E and F. Draw line EF, the desired line.



Proof. By Task 2.5, a line through the center of a rectangle cuts it into two parts of equal area. Since the orange line passes through the centers of both rectangles, it divides both rectangles into parts of equal area. $\square$

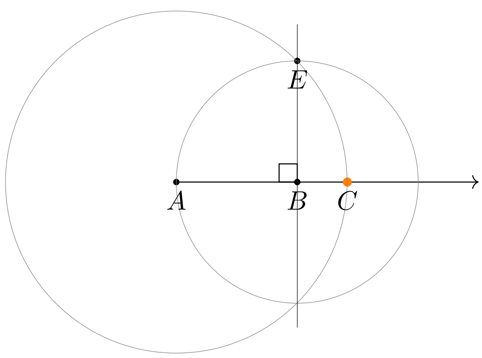
4.6 Square root of 2

Task 4.6. Let $|AB| = 1$. Construct a point C on the ray AB such that the length of AC is equal to $\sqrt{2}$.



$$|AC| = \sqrt{2}$$

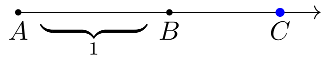
Solution 4.6. **1.** Draw circle (B, A).
2. Draw $AB \vdash B$, intersecting (B, A) at one of the points E.
3. Draw circle (A, E), intersecting the given ray at C. C is the desired point.



Proof. Note that $\triangle ABE$ is an isosceles right triangle, so $AE = \sqrt{2}$ (Pyth. thm), thus $AC = AE = \sqrt{2}$ (radii). $\square$

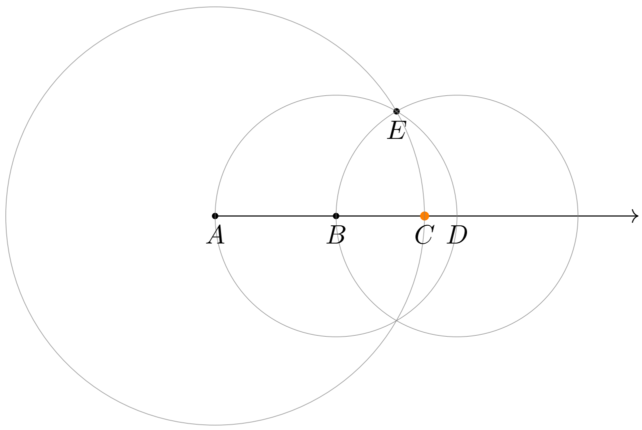
4.7 Square root of 3

Task 4.7. Let $|AB| = 1$. Construct a point C on the ray AB such that the length of AC is equal to $\sqrt{3}$.



$$|AC| = \sqrt{3}$$

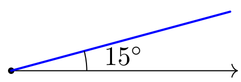
Solution 4.7. **1.** Draw circle (B, A), intersecting given ray again at D.
2. Draw circle (D, B), intersecting (B, A) at one of intersections E.
3. Draw circle (A, E), intersecting given ray at C. C is the desired point.



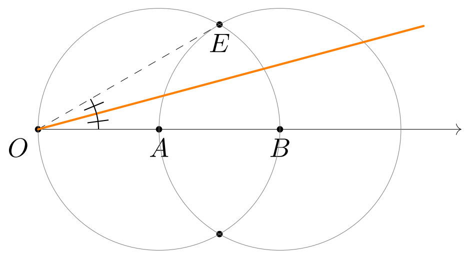
Proof. Since $AB = BE = 1$ and $\angle ABE = 180° - 60° = 120°$ (adj. $\angle$s on st. line), we have $AE = \sqrt{1 + 1 - 2(1)(1)\cos(120°)} = \sqrt{3}$ (law of cosines). So $AC = AE = \sqrt{3}$ (radii). $\square$

4.8 Angle of 15 deg

Task 4.8. Construct an angle of 15° with the given side.
(3L, 5E, 2V)

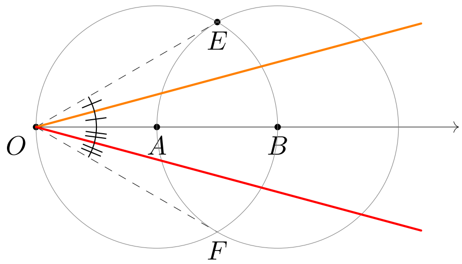


Solution 4.8. Let O be the endpoint of the given ray, and A be an arbitrary point on the ray.
(3L) 1. Draw circle (O, A), intersecting given ray at B.
2. Draw circle (B, A), intersecting (O, A) at E and F.
3. Draw angbi BOE, the desired line.



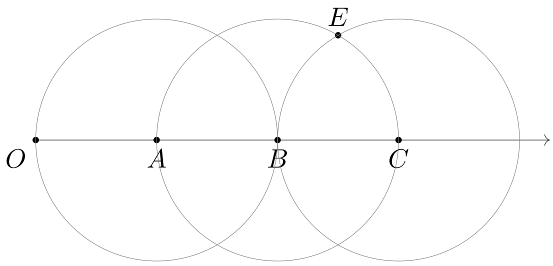
Proof. Note that $\angle OEB = 90°$ ($\angle$ in semi-circle) and $\angle ABE = 60°$, so $\angle BOE = 180° - 90° - 60° = 30°$ ($\angle$ sum of $\triangle$). Thus, bisecting $\angle BOE$ gives a 15° angle. $\square$

(2V) Draw angbi BOF, the extra solution.

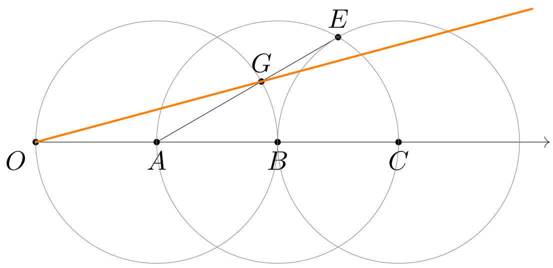


Proof. Similar argument as (3L). ☐

(5E) 1. Draw circle (O, A), intersecting given ray at B.
2. Draw circle (B, A), intersecting given ray at C.
3. Draw circle (C, A), intersecting (B, A) at one of intersections E.



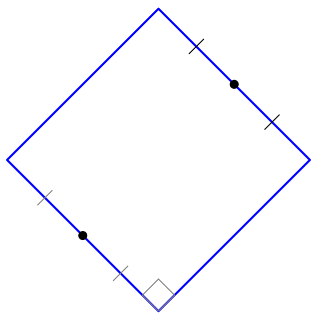
4. Draw segment AE, intersecting (A, O) at G.
5. Draw line OG, the desired line.



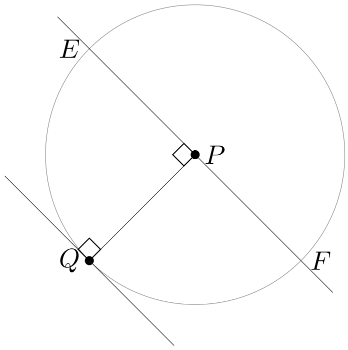
Proof. Note that $\angle EAC = 30°$ (similar to $\angle BOE$ in (3L)). Since $AO = AG$, we have $\angle AOG = 30°/2 = 15°$ (base $\angle$s, isos. $\triangle$)& (ext. $\angle$ of $\triangle$). ☐

4.9 Square by opposite midpoints

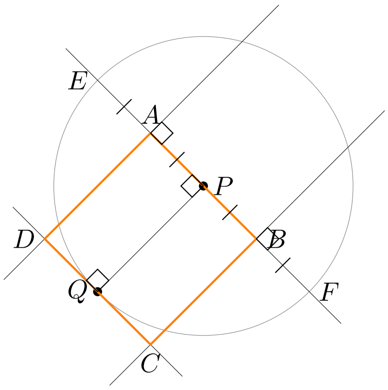
Task 4.9. Construct a square, given two midpoints of opposite sides.
(6L, 10E)



Solution 4.9. Let the given points be P and Q.
(6L) 1, 2. Draw circle (P, Q). Draw line PQ.
3, 4. Draw $PQ \vdash Q$. Draw $PQ \vdash P$, intersecting (P, Q) at E, F.

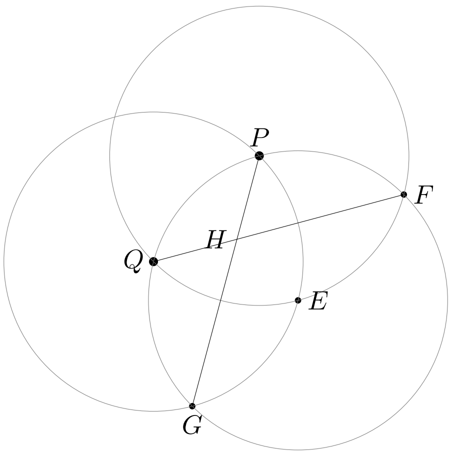


5, 6. Draw perpbi EP and perpbi PF. We get the desired square $ABCD$.

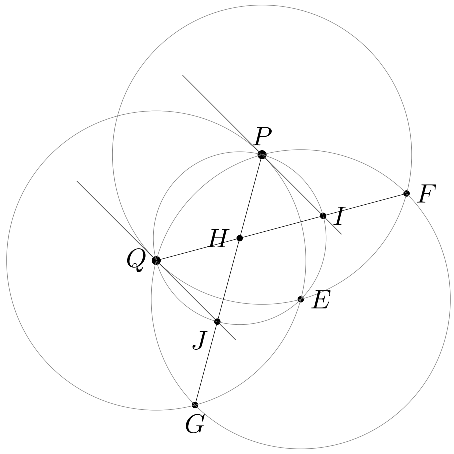


Proof. Note that PQ is equal to the side length of the square. So $AP = \frac{1}{2}EP = \frac{1}{2}PQ$. Similarly, $PB = \frac{1}{2}PQ$, so $AB = PQ$. Also $AD = BC = PQ$, so $ABCD$ is a square. □

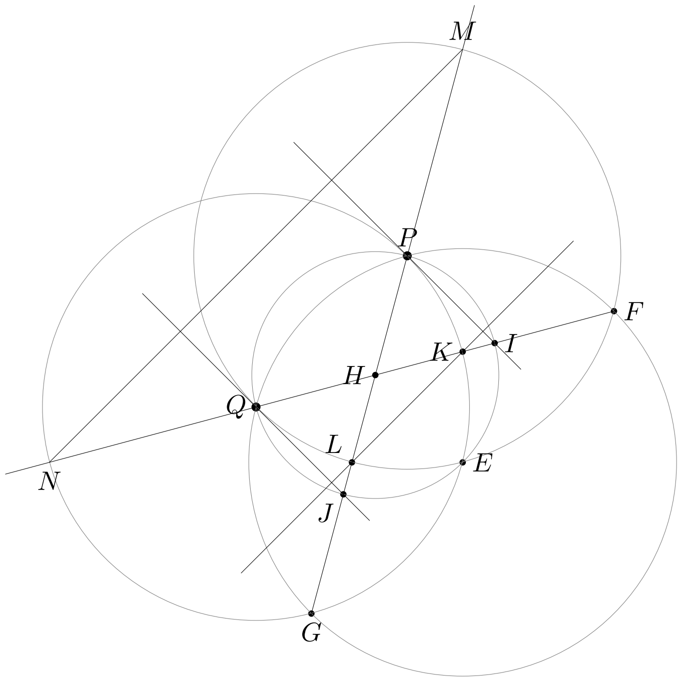
(10E) 1, 2. Draw circle (P, Q) and (Q, P). Let E be one of their intersections.
3. Draw circle (E, P), intersecting (P, Q) and (Q, P) at new points F and G.
4, 5. Draw lines PG and QF, intersecting at H.



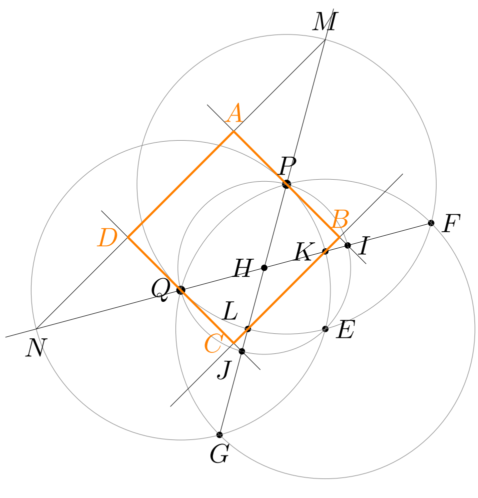
6. Draw circle (H, P), intersecting segments QF and PG at I and J respectively.
7, 8. Draw line PI and QJ.



9. Let segments QF and PG intersect (Q,P) and (P,Q) at K and L respectively. Draw line KL.
10. Extend segment GP to meet (P,Q) at M. Extend segment FQ to meet (Q,P) at N (doesn't cost anything). Draw line MN.

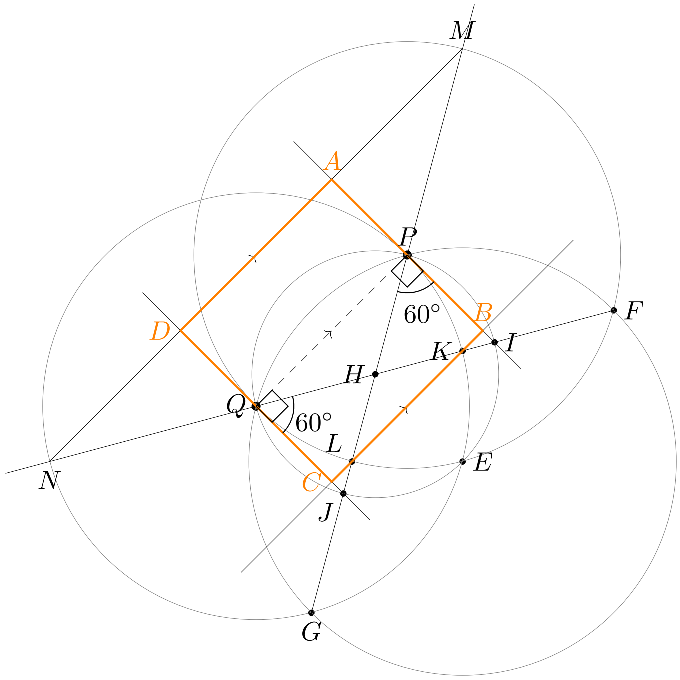


The desired square is the area enclosed by the four lines.



Proof. (Let $ABCD$ be the vertices of the orange quadrilateral.)

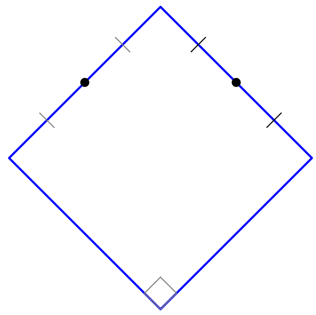
First, note that $PI \perp PQ$ and $QJ \perp PQ$ by "$\angle$ in semi-circle" for circle (H,P). Also, $KL//PQ$ since $\triangle HPQ \sim HLK$ (ratio of 2 sides, inc. $\angle$), and $MN//PQ$ since $\triangle HPQ \sim \triangle HMN$ (ratio of 2 sides, inc. $\angle$). This means $ABCD$ is a rectangle.



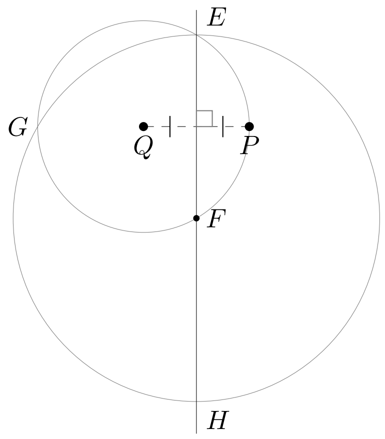
Now note that $\angle HPI = 60°$ (since $\triangle HPI$ is equil.), so $PB = \cos(60°)PL = \frac{1}{2}PL$. And note that $\triangle PAM \cong \triangle PBC$ (AAS by $\angle APM = \angle BPL$, $\angle MAP = \angle LBP$, $PM = PL$), giving $AP = PB = \frac{1}{2}PL$ (corr. sides, $\cong \triangle$s). This means $AB = PL = PQ$ (radii) $= BC$. So $ABCD$ is a rectangle with adjacane sides equal, i.e. a square, as desired. □

4.10 Square by adjacent midpoints

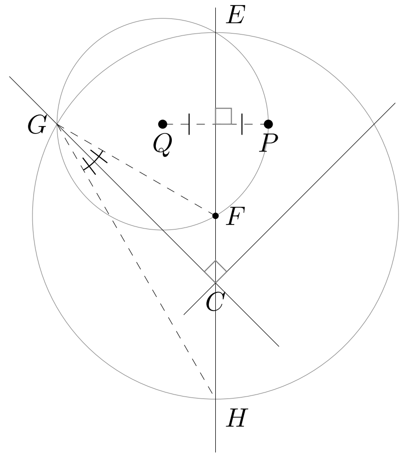
Task 4.10. Construct a square, given two midpoints of adjacent sides.
(7L, 10E, 2V)



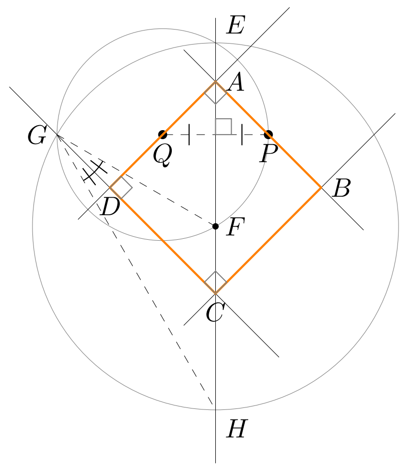
Solution 4.10. Let P, Q be the given points.
(7L) 1. Draw circle (Q, P).
2. Draw perpbi PQ, intersecting (Q, P) at E and F
3. Draw circle (F, E), intersecting (Q, P) again at G, and EF again at H.



4. Draw angbi FGH, intersecting FH at C.
5. Draw $GC \vdash C$.

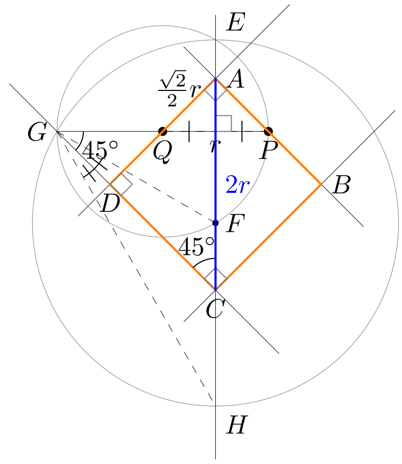


6. Draw $GC \vdash Q$, intersecting EF at A.
7. Draw $QA \vdash P$. The shape $ABCD$ enclosed by the lines is the desired square.



Proof. Let the length of QP be r, and M be midpoint of QP. We want to show that rectangle $ABCD$ is a square and that $AQ = QD = AP = PB = \frac{\sqrt{2}}{2}r$.

Note that $\angle EGH = 90°$ ($\angle$ in semi-circle) and $\angle AGF = 60°$, so $\angle FGH = 30°$. Since GC is the angle bisector of $\angle FGH$, we have $\angle FGC = 15°$, so $\angle QGC = 45°$ and thus $\angle GCA = 45°$. Since the diagonal of rectangle $ABCD$ makes $45°$ with a side, $ABCD$ is a square.



To show that Q, P are midpoints of the sides, note that $AQ = QP\cos 45° = \dfrac{\sqrt{2}}{2}r$. Also, $GP = AC$ because $GM = MC$ and $MP = AM$. Since GP is a diameter of circle (Q, P), $GP = 2r$ and thus $AC = 2r$.

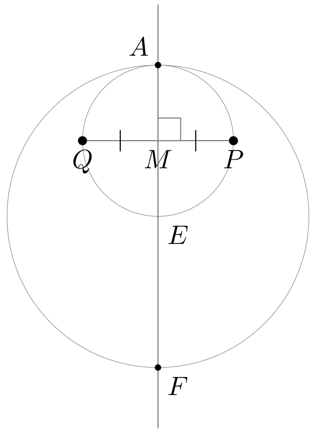
So $AD = \dfrac{2r}{\cos 45°} = \sqrt{2}\,r$, and $QD = AD - AQ = \dfrac{2}{2}r = AQ$.

Then we also have $AP = PB$ by intercept theorem (because $QP // DB$). We get everything desired. □

(10E) 1, 2. Draw perpbi PQ. Draw line PQ. Let M be midpoint of PQ.

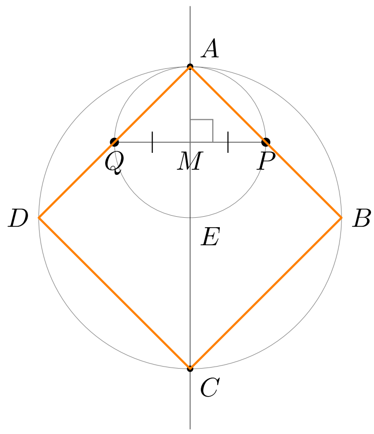
3. Draw circle (M, P), intersecting perpbi PQ at A at top and E at bottom.

4. Draw circle (E, A), intersecting perpbi PQ again at C (at bottom).



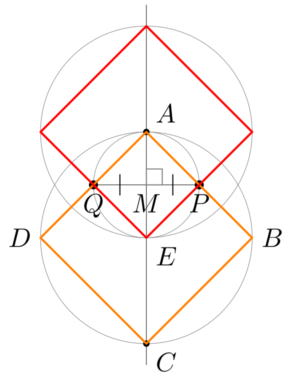
4, 5. Draw line AP and AQ, intersecting big circle (E, A) at B and D.

6, 7. Draw line BC and DC. $ABCD$ is the desired square.



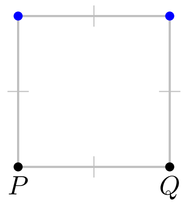
Proof. Note that $\triangle AMQ$ and $\triangle AMP$ are isosceles right triangles, so we have $\angle AQM = \angle APM = 45°$. Thus $\angle BAD = 90°$. Also, $\angle ADC = \angle ABC = 90°$ ($\angle$ in semi-circle). Thus, $ABCD$ is a rectangle with diagonal making $45°$ with the sides, so $ABCD$ is a square (con. of square). □

(2V) Draw circle (A, E) and draw the lines similarly.

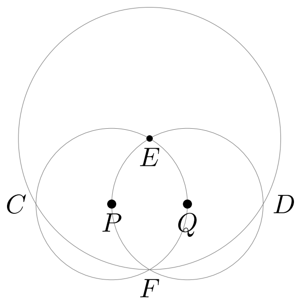


4.11 Square by two vertices

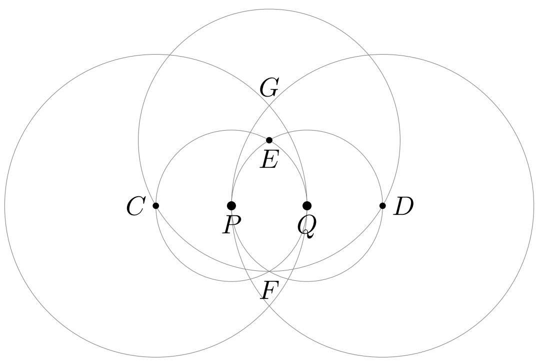
Task 4.11. Given two vertices of a square. Construct the two other vertices using only a compass.
(7L, 7E, 3V)



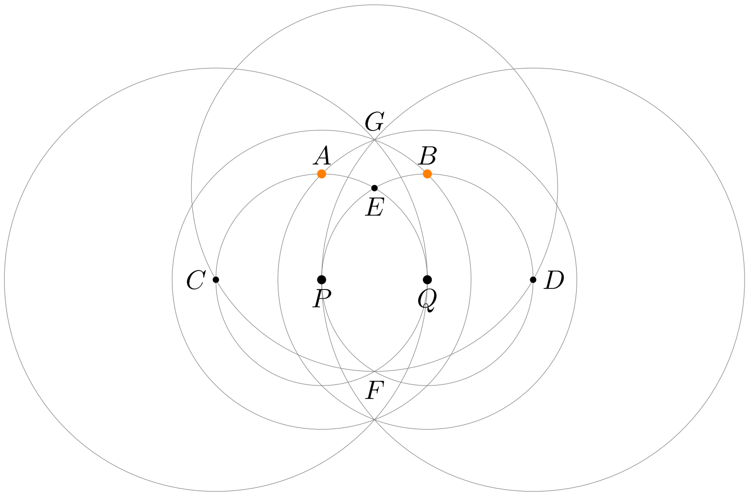
Solution 4.11. Let the given points be P and Q.
(7L, 7E) 1, 2. Draw circles (P, Q) and (Q, P), intersecting at E and F.
3. Draw circle (E, F), intersecting (P, Q) and (Q, P) again at C and D.



4, 5. Draw circles (C, Q) and (D, P). Let intersection at top be G.

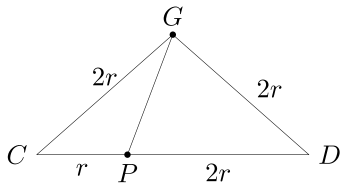


6, 7. Draw circles (P, G) and (Q, G), intersecting (Q, P) and (P, Q) at top at B and A respectively. A and B are the desired points.



Proof. Let the distance between PQ be r. Note that C and D lie on circle (E, F), so C, P, Q, D are collinear and $CP = PQ = QD = r$ (see Task 1.7E for proof). We also have $GC = GD = 2$ since G lie on circles (C, Q) and (D, P).

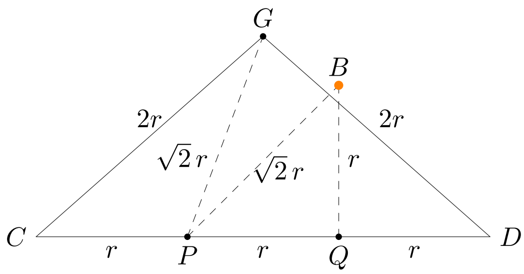
To find PG, let's focus on $\triangle GCP$ and $\triangle GPD$:



By Stewart's theorem, we have

$$\begin{aligned} GC^2 \cdot PD + GD^2 \cdot CP &= (CP + PD)(PG^2 + CP \cdot PD) \\ (2r)^2(2r) + (2r)^2(r) &= (2r + r)(PG^2 + (r)(2r)) \\ 4r^2 &= PG^2 + 2r^2 \\ PG &= \sqrt{2}\, r \end{aligned}$$

Let's add points B and Q to the figure. We have $PB = PG = \sqrt{2}$ and $QB = QP = 1$ (radii).

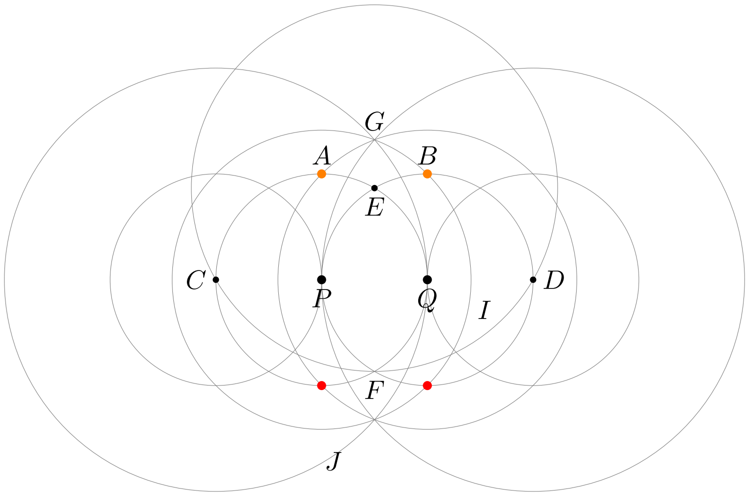


Since $BQ^2 + PQ^2 = 2r^2 = BP^2$, by converse of Pythagoras theorem in $\triangle BPQ$, we have $\angle BQP = 90°$. By symmetry, we have $\angle APQ = 90°$.

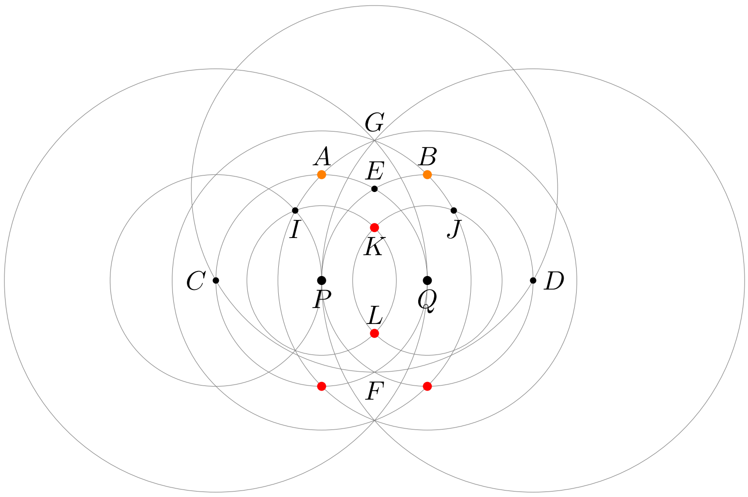
Since $PQBA$ has three sides equal ($AP = PQ = BQ$) and two right angles ($\angle APQ = \angle BQP = 90°$), $ABCD$ is a square (con. of square). □

(3V) 2nd solution: Intersection of (P, G) and (Q, P) at bottom, and intersection of (G, P) and (P, Q) at bottom.

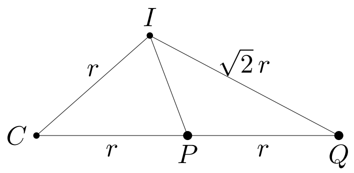
3rd solution: 8, 9. Draw circle (C, P), intersecting (Q, G) at I (top). Draw circle (D, Q), intersecting (P, G) at J (top).



10, 11. Draw circles (P, I) and (Q, J), intersecting at K and L at the middle. K and L are the two desired points.



Proof. Note that $CI = r$ and $QI = QG = \sqrt{2}\,r$. Let's focus on $\triangle ICP$ and $\triangle IPQ$:



By Stewart's theorem,

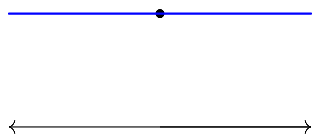
$$r^2(r) + (\sqrt{2}r)^2 r = (r + r)(IP^2 + r(r))$$
$$\frac{3}{2}r^2 = IP^2 + r^2$$
$$IP = \frac{\sqrt{2}}{2}r$$

This means $PK = \frac{\sqrt{2}}{2}r$. By symmetry, we also have $KQ = PL = LQ = \frac{\sqrt{2}}{2}r$. By Pyth. thm in $\triangle KPQ$, $\angle PKQ = 90°$. Thus, $PLQK$ is a rhombus with a right angle, i.e. a square, as desired. $\square$

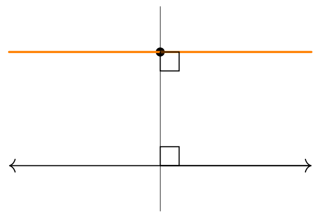
5 Epsilon

5.1 Parellel line

Task 5.1. Construct a line parallel to the given line through the given point.
(2L, 4E)

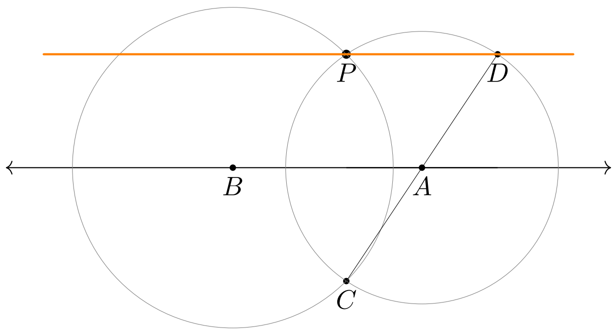


Solution 5.1. **(2L)** **1.** Draw line perpendicular to given line through given point.
2. Draw line perpendicular to the drawn line through given point.



Proof. Since $90° + 90° = 180°$, the interior angles are supplementary , so the orange line is parallel to given line (int. $\angle$s supp.). □

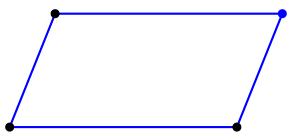
(4E) Let P be given point, A, B be two arbitrary points on given line.
1. Draw circles (A, P) and (B, P), intersecting at P and C.
3. Draw line CA, meeting circle (A, P) at D.
4. Draw line PD, the desired line.



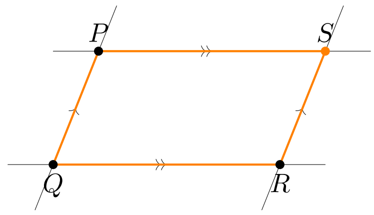
Proof. Note that $PC \perp BA$ since $PACB$ forms a kite, and $PC \perp PD$ by "$\angle$ in semi-circle. Thus $PD//BA$. □

5.2 Parallelogram by three vertices

Task 5.2. Construct a parallelogram whose three or four vertices are given.
(4L, 8E, 3V)



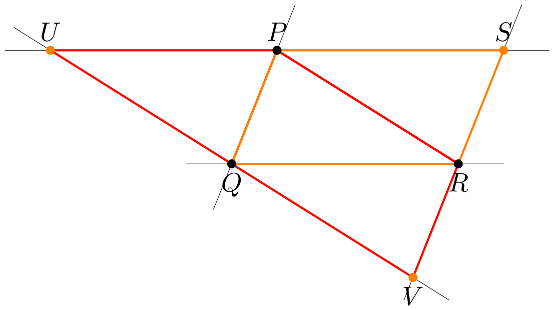
Solution 5.2. **(4L)** Let given points be P, Q, R.
1, 2. Draw line PQ and QR.
3. Draw $QR \twoheadrightarrow P$ (line parallel to QR through P).
4. Draw $PQ \twoheadrightarrow R$.



Proof. By definition. □

(3V) 5. Draw line PR.

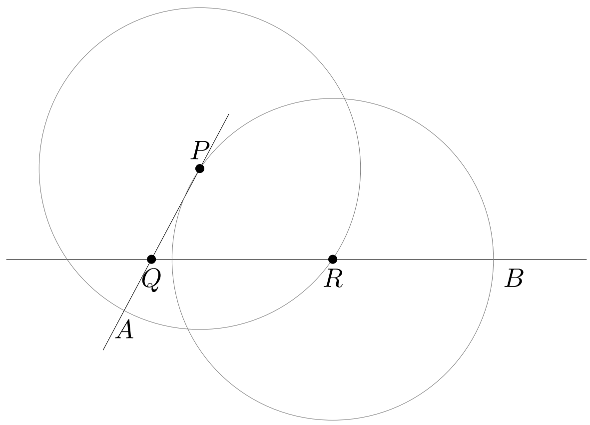
6. Draw $PR \twoheadrightarrow Q$, intersecting PS and SR at U and V respectively. The extra solutions are $PUQR$ and $PQVR$.



Proof. There are three pairs of line segments made from the given points: PQ, QR; PQ, PR; PR, QR, and each makes a parallelogram. □

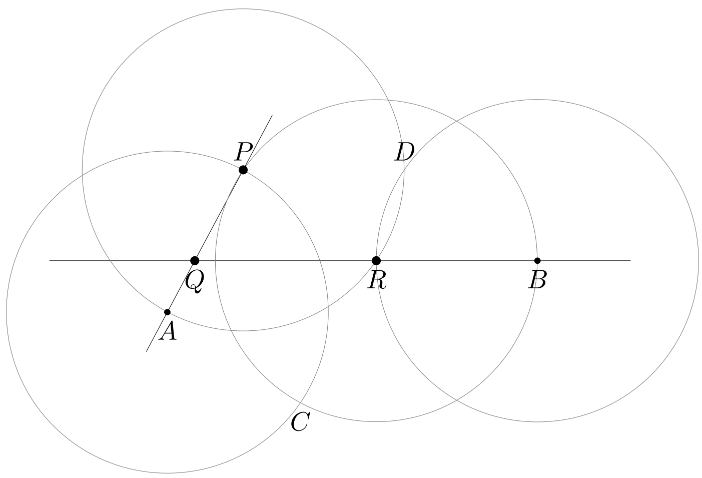
(8E) 1, 2. Draw line PQ and QR.

3, 4. Draw circles (P, R) and (R, P), intersecting line PQ and QR at A (bottom) and B (right) respectively.

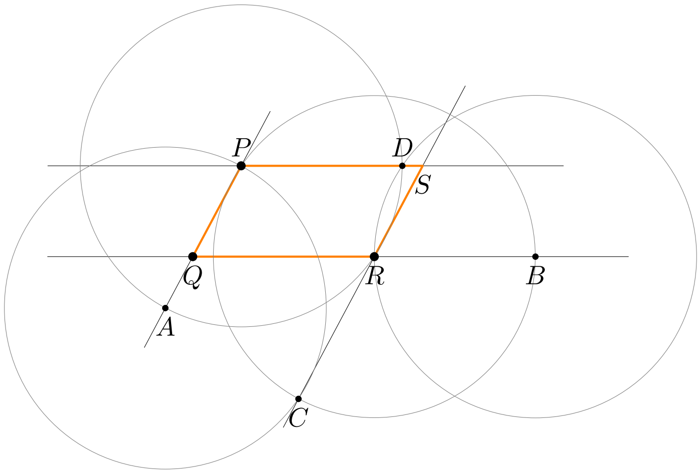


5. Draw circle (A, P), intersecting (R, P) again at C.

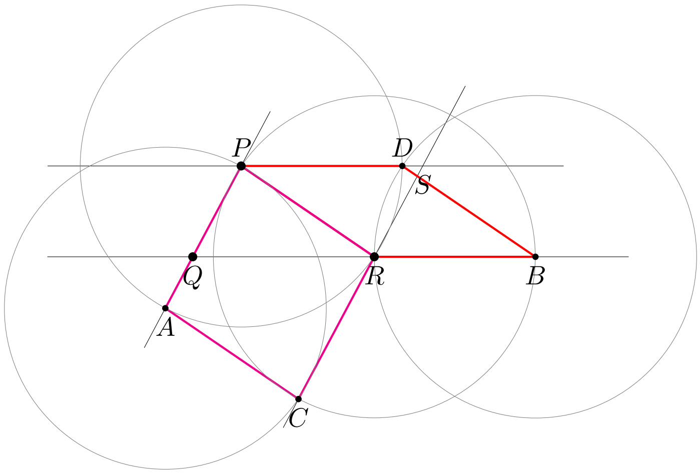
6. Draw (B, R), intersecting (P, R) again at D.



7, 8. Draw lines CR and PD, intersecting at S. $PQRS$ is the desired parallelogram.



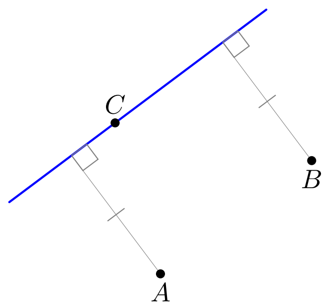
Proof. Let the length of PR be r. Note that $PRBD$ and $PRCA$ form rhombuses of side length r. This means $PD//QB$ and $PA//SC$ (prop. of rhombus), making $PQRS$ a parallelogram.



□

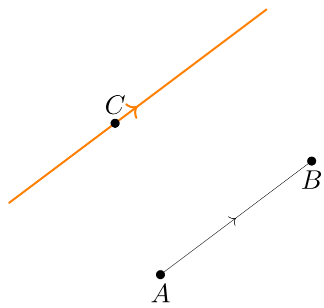
5.3 Line equidistant from two points - 1

Task 5.3. Construct a line through the point C and at equal distance from the point A and B but that does not pass between them.
(2L, 4E)

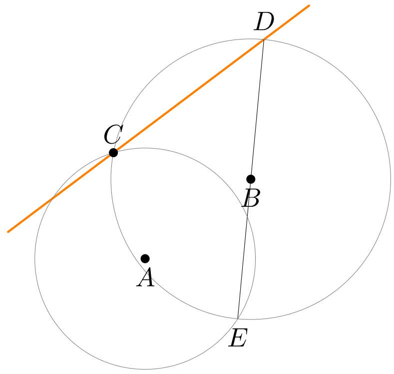


Solution 5.3. (2L) 1. Draw line AB.

2. Draw $AB \twoheadrightarrow C$.



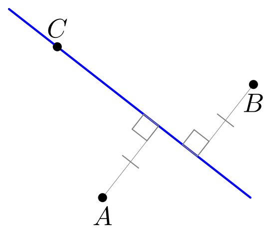
(4E) 1. Draw circles (A, C) and (B, C), intersecting again at E.
3. Draw line EB, meeting (B, C) at D.
4. Draw line CD, the desired line.



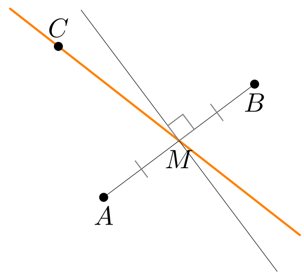
Proof. CD is parallel to AB by Task 5.1E. □

5.4 Line equidistant from two points - 2

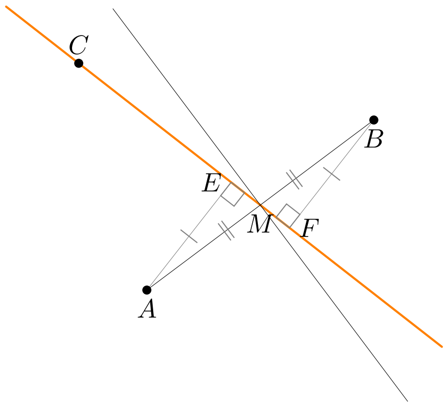
Task 5.4. Construct a line through the point C that goes between the points A and B and that is at equal distance from them.
(3L, 5E)



Solution 5.4. 1. Draw line AB.
2. Draw perpbi AB. Let M be the midpoint of AB.
3. Draw CM, the desired line.



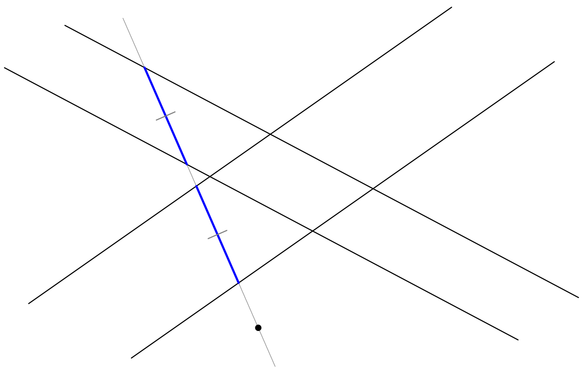
Proof. Let E and F be the projection of A and B onto line CM.



Since $\angle AME = \angle BME$ (vert. opp. $\angle$), $\angle AEM = \angle BFM$ ($AE \perp CM$ and $BE \perp CM$) and $AM = MB$, we have $\triangle AME \cong \triangle BMF$ (AAS). Thus, $AE = AF$ (corr. sides, $\cong \triangle$s), as desired. $\square$

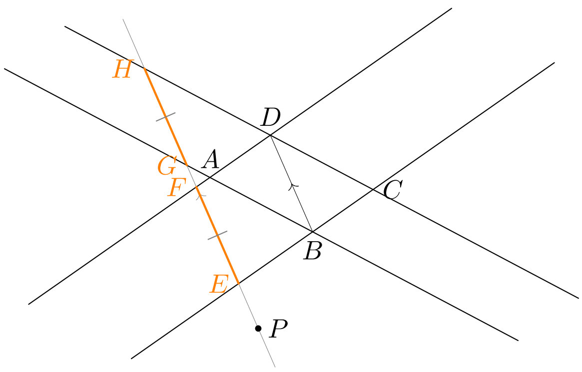
5.5 Hash

Task 5.5. Construct a line through the given point on which two pairs of parallel lines cut off equal line segments.
(2L, 4E, 2V)



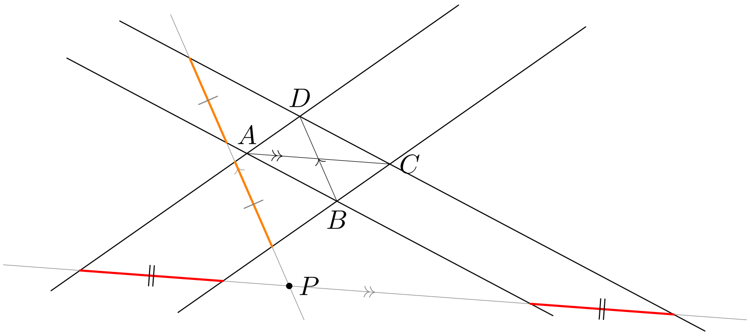
Solution 5.5. (2L) Let given point be P, and the parallelogram formed by the given lines be $ABCD$.

1. Draw line BD.
2. Draw $BD \twoheadrightarrow P$, the desired line.



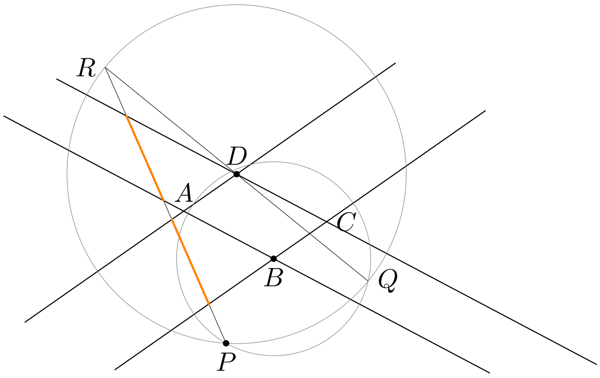
Proof. Let EF and GH be the orange line segments. Note that $EFDB$ and $GHDB$ are parallelograms by construction, so $EF = DB$ and $GH = DB$ by "opp. sides of //gram". This means $EF = GH$. $\square$

(2V) 3. Draw line AC.
4. Draw $AC \twoheadrightarrow P$, the extra solution.



Proof. Similar argument as above. □

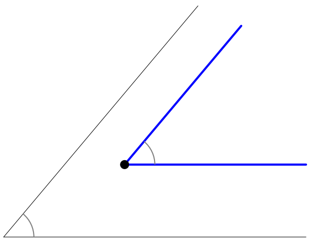
(4E) 1, 2. Draw circles (D, P) and (B, P), intersecting again at Q.
3. Draw QD, meeting (D, P) at R.
4. Draw RP, the desired line.



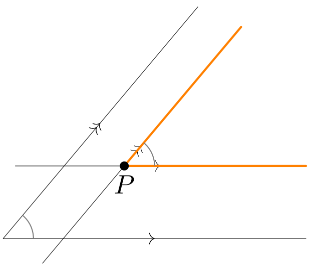
Proof. Note that $PQ \perp DB$ because $PBQD$ forms a dart. And $PQ \perp RP$ by "$\angle$ in semi-circle". Thus $DP // RP$. □

5.6 Shift angle

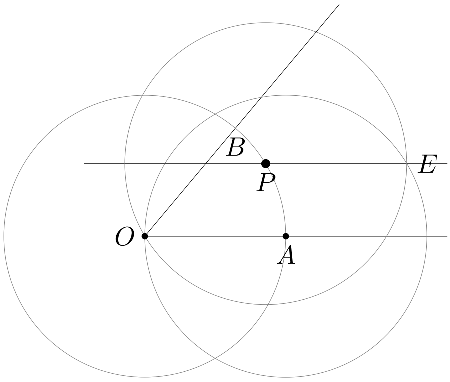
Task 5.6. Construct an angle from the given point that is equal to the given angle so that their sides are parallel.



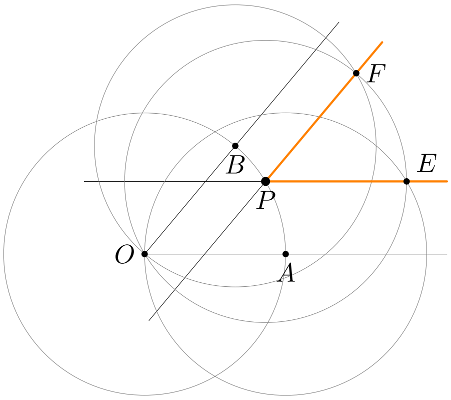
Solution 5.6. (2L) 1, 2. Draw lines parallel to given rays through given point.



(6E) Let given point be P, and vertex of angle be O.
1. Draw circle (O, P), intersecting given rays at A and B.
2, 3. Draw circles (A, O) and (P, O), intersecting again at E.
4. Draw line PE.



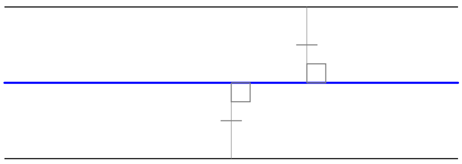
5. Draw circle (B, O), intersecting (P, O) again at F.
6. Draw line PF.



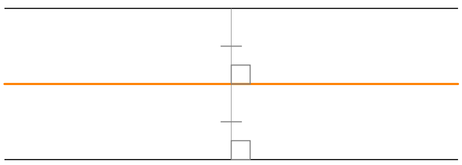
Proof. Note that all circles have the radius of OP. This means $OAEP$ and $OPFB$ are rhombuses, so $OA//PE$ and $OB//PF$. $\square$

5.7 Line equidistant from two lines

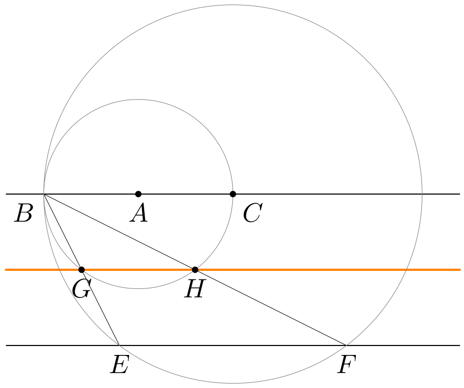
Task 5.7. Construct a straight line parallel to the given parallel lines that lies at equal distance from them.
(2L, 5E)



Solution 5.7. (2L) 1, Draw line perpendicular to a given line, intersecting the given lines at A and B.
2. Draw perpbi AB, the desired line.

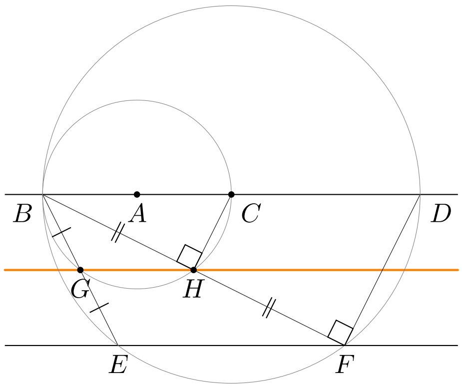


(5E) Let A, B be two arbitrary points on the top given line (such that the distance between AB is greater than the distance between two give lines).
1. Draw circle (A, B), intersecting top line again at point C.
2. Draw circle (C, B), intersecting bottom line at E and F.
3, 4. Draw lines BE and BF, intersecting (A, B) at G and H.
5. Draw line GH, the desired line.



Proof. Let D be another intersection of (C, B) and top given line. Note that $CH \perp BH$ and $DF \perp BF$ ($\angle$ in semi-circle), so $CH//DF$ (corr. $\angle$s equal). Since $CH//DF$ and $BC = CD$ (radii), by intercept theorem, we have $BH = HF$.

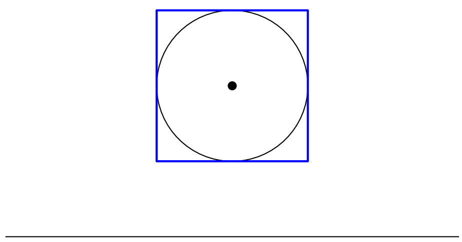
By similar argument ($CG \perp BG$ and $DE \perp BE \Rightarrow GC//ED$), we also have $BG = GE$.



Since $BH = HF$ and $BG = GE$, by midpoint theorem, we have $GH//DE$. And by intercept theorem, GH is midway between the two given lines, as desired. □

5.8 Circumscribed square

Task 5.8. Circumscribe a square about the circle. Two of its sides should be parallel to the given line.
(6L, 11E)

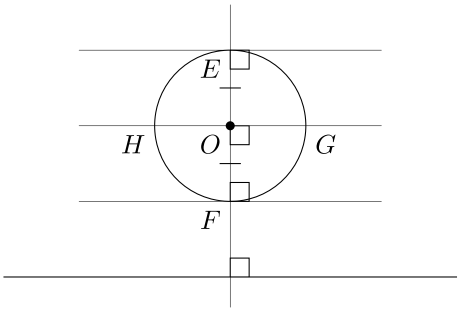


Solution 5.8. Let given circle center be O.

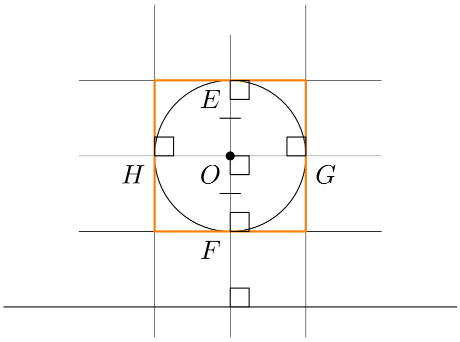
(6L) 1. Draw perpendicular of given line through O, intersecting given circle at E ad F.

2, 3. Draw $EF \vdash E$ and $EF \vdash F$.

4. Draw perpbi EF, intersecting given circle at G and H.



5, 6. Draw $GH \vdash G$ and $GH \vdash H$. We get the desired square.



Proof. Because two perpendicular lines of the same line are parallel, and converse of tangent ⊥ radius. □

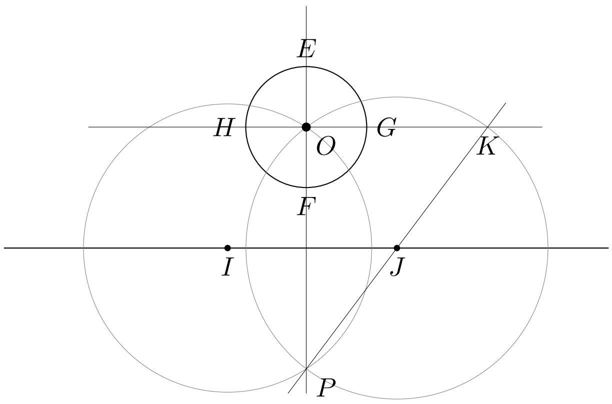
(11E) Let I, J be two arbitrary points on given line.

1, 2. Draw circles (I, O) and (J, O), intersecting at another point P.

3. Draw line PO, intersecting given circle at E and F.

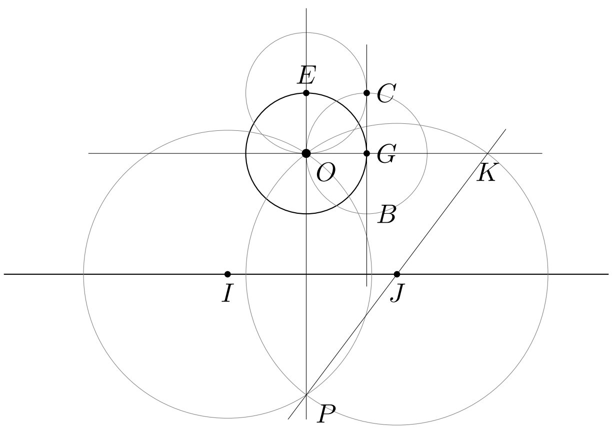
4. Draw line PJ, meeting (J, O) at K.

5. Draw line KO, intersecting given circle at G and H.

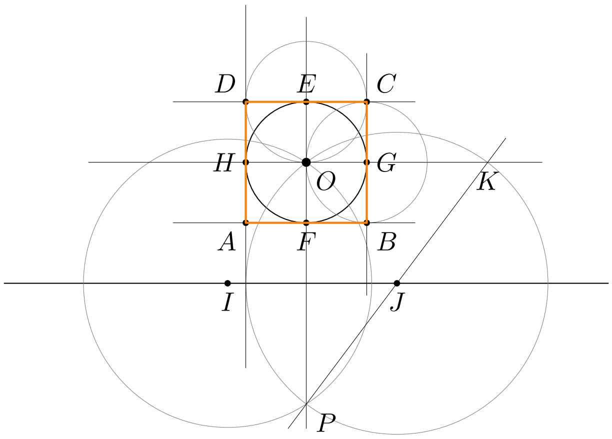


6, 7. Draw circles (E, O) and (G, O), intersecting again at C.

8. Draw line CG, meeting (G, O) at B.



9. Draw line BF.
10. Draw line CE, meeting (E, O) at D.
11. Draw line DH, intersecting BF at A. $ABCD$ is the desired square.



Proof. (Let r be radius of given circle.)

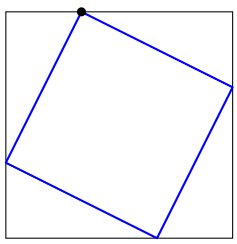
From Tasks before we know that $IJ \perp OP$ and $OK // IJ$. Note that $EOGC$ is a rhombus (with side length r) with a right angle, so it is a square.

By similar reasoning, $OFBG$ is a square of side length r. We can also easily deduce that $DHOE$ is square of side length r (3 sides equal, 2 right $\angle$s) and $HAFO$ same so (rectangle with equal adj. sides).

Thus the big square $ABCD$ formed by these four small squares is the square that circumscribes the given circle. □

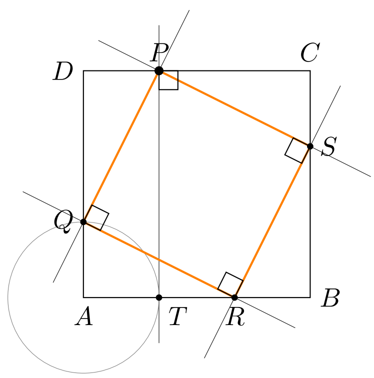
5.9 Square in square

Task 5.9. Inscribe a square in the square. A vertex is given.
(6L, 7E)



Solution 5.9. Let given square be $ABCD$ and the given point be on DC.

(6L) 1. Draw $DC \vdash P$, intersecting AB at T.
2. Draw circle (A, T), intersecting side AD at Q.
3. Draw line PQ.
4. Draw $PQ \vdash Q$, intersecting AB at R.
5. Draw $QR \vdash R$, intersecting BC at S.
6. Draw $RS \vdash P$. $PQRS$ is the desired square.

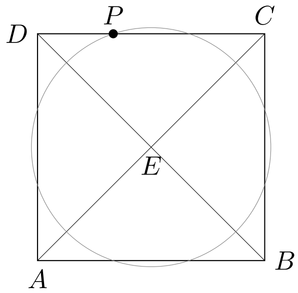


Proof. Note that $\triangle PDQ \cong \triangle QAR$ (AAS by $DP = QA$, $\angle PDQ = \angle QAR = 90°$, $\angle PQD = \angle QRA$ by angle chasing). Thus $PQ = QR$ (corr. sides, $\cong \triangle$s)

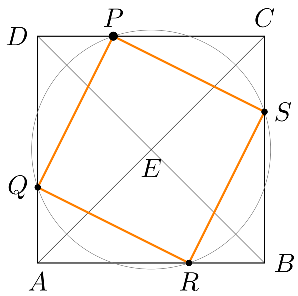
Also $PQRS$ has four right angles by construction. Since $PQRS$ is a rectangle with adjacent sides equal, $PQRS$ is a square. □

(7E) 1, 2. Draw diagonals AC and BD, intersecting at E.

3. Draw circle (E, P), making intersections with the given square sides.

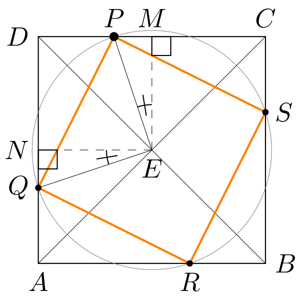


4-7. Draw four lines connecting the intersections such that they form a square.



Proof. Note that E is the center of given square since it is the intersection of the diagonals. This means E is equidistant from all of its sides.

Draw $EM \perp DC$ and $EN \perp DA$. Note that $\triangle EMP \cong \triangle ENQ$ (RHS by $EM = EN$, $\angle EMP = \angle ENQ = 90°$, $EP = EQ$ (radii)), so $\angle PEM = \angle QEN$ (corr. $\angle$s, $\cong \triangle$s).

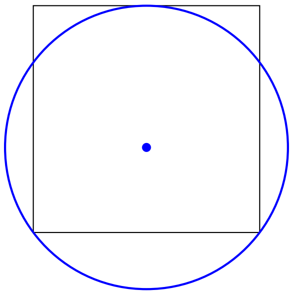


Since $\angle MEN = 90°$, we have $\angle PEQ = \angle MEN - \angle PEM + \angle QEN = 90°$.

Since $EP = EQ = ER = ES$ (radii) and $PE \perp QE$, we have that $PQRS$ is a square (diags $\perp$, equal and bisect each other). □

5.10 Circle tangent to square side

Task 5.10. Construct a circle that is tangent to a side of the square and goes through the vertices of the opposite side.
(3L, 6E, 4V)

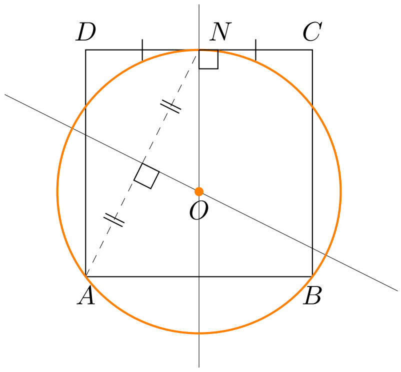


Solution 5.10. Let the given square be $ABCD$.

(3L) 1. Draw perpbi DC. Let the midpoint of DC be N.

2. Draw perpbi NA, intersecting perpbi DC at O.

3. Draw circle (O, N), the desired circle.

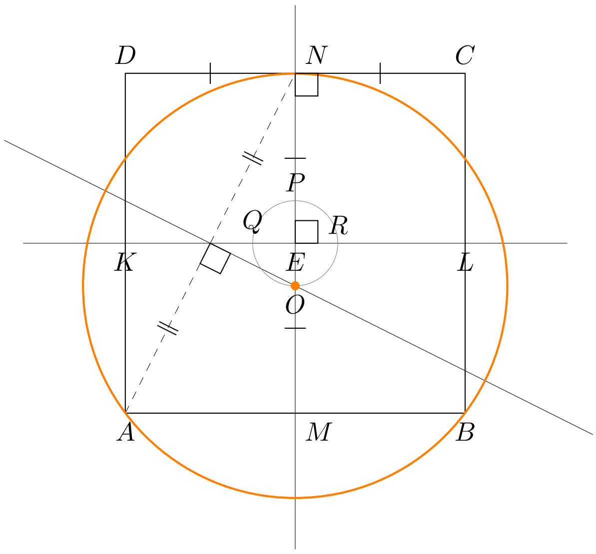


Proof. Note that circle (O, N) is tangent to DC (converse of tangent $\perp$ radius), and (O, N) passes through A and B since $ON = OA = OB$ (prop. of $\perp$ bisector). □

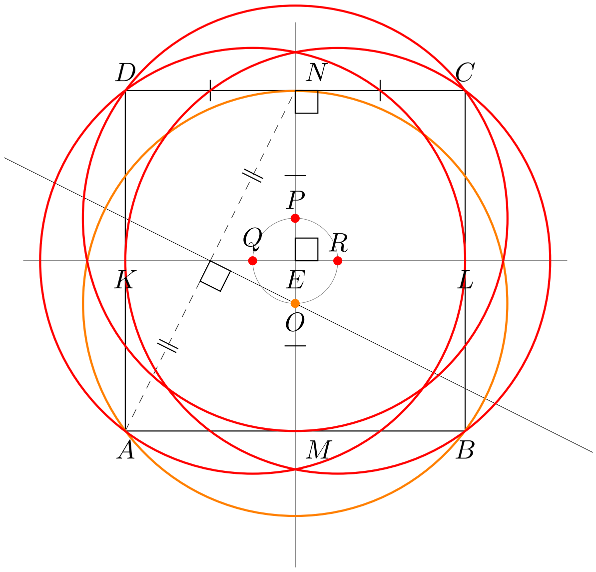
(4V) Let M be the midpoint of AB.

4. Draw perpbi MN, intersecting DA and CB at K and L. Let E be midpoint of MN.

5. Draw circle (E, O), intersecting MN at another P. Let (E, O) intersect KL at Q (left) and R (right).

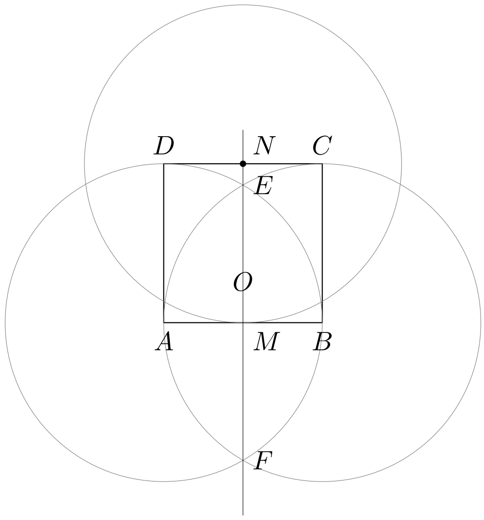


6-8. Draw circles (P, M), (Q, L) and (R, K), the extra solutions.

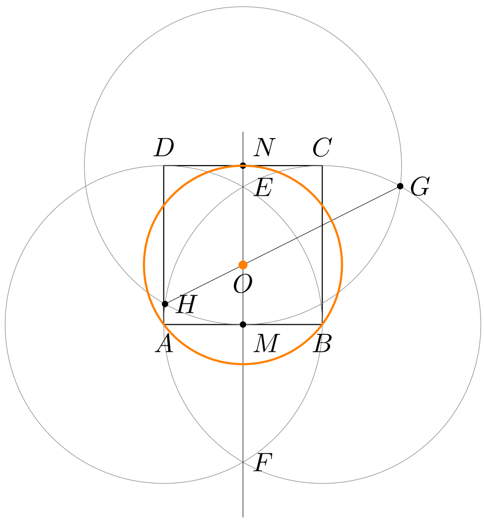


Proof. Note that E is the center of the square, and $EP = EQ = ER$ (radii). By rotational symmetry, (P, M), (Q, L) and (R, K) also satisfy the required conditions, so they are the extra solutions. □

(6E) 1, 2. Draw circles (A, B) and (B, A), intersecting at E and F.
3. Draw line EF, intersecting CD at N and AB at M.
4. Draw circle (N, M), intersecting (B, A) at G and H.



5. Draw line GH, intersecting EF at O.
6. Draw circle (O, N), the desired circle.



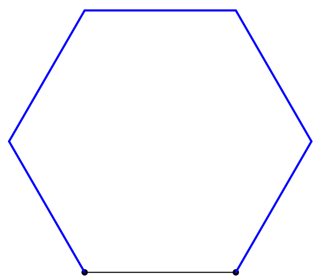
Proof. Note that N and M are midpoints of DC and AB. So $NM = BC$ ($MBCN$ being rectangle).

Thus circles (N, M) and (B, C) have the same radius, giving $GN = GB = HN = HB$. This means GH is the perpendicular bisector of NB (because it is a diagonal of rhombus $HBGN$).

Thus O is the same point as (3L). ☐

5.11 Regular hexagon

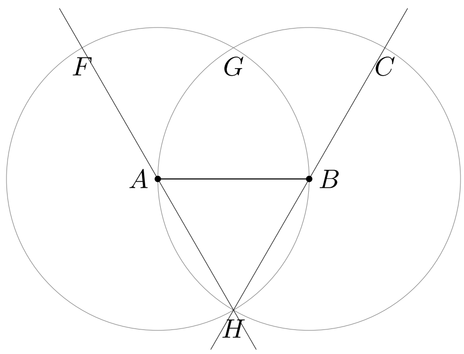
Task 5.11. Construct a regular hexagon with the given side.
(7L, 8E, 2V)



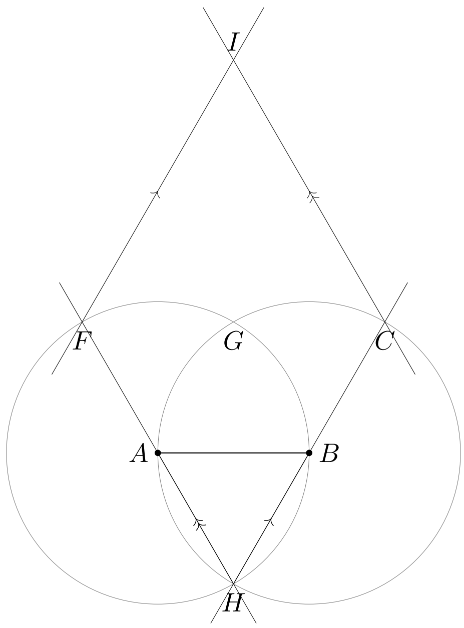
Solution 5.11. Let the given line segment be AB.

(7L) 1, 2. Draw circles (A, B) and (B, A), intersecting at G and H.

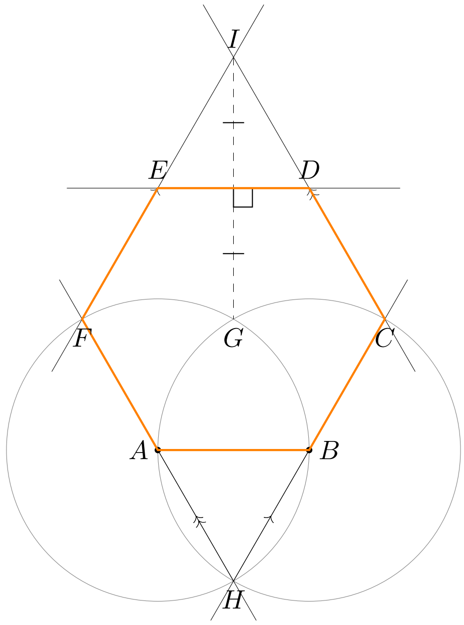
3, 4. Draw lines FA and FB, meeting (A, B) and (B, A) at F and C respectively.



5, 6. Draw $HA \twoheadrightarrow C$ and $HB \twoheadrightarrow F$.



7. Draw perpbi IG, intersecting FI and CI at E and D. $ABCDEF$ is the desired hexagon.



Proof. Let r be the length of AB.

Note that G is the midpoint of FC (because $\triangle FAG$, $\triangle GAB$ and $\triangle GBC$ are three equilateral triangles stacked together.)

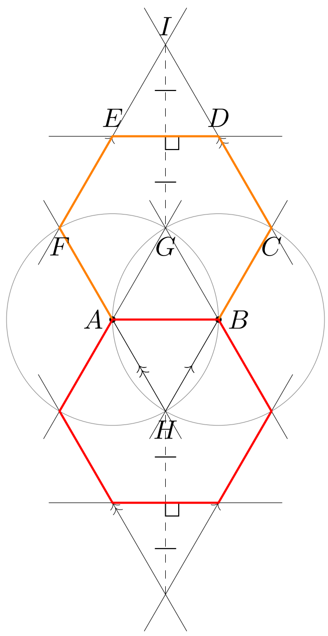
Note that $\angle AHB = 60°$, so $\angle AFI = \angle BCI = 120°$ (int. $\angle$s), giving $\angle IFC = \angle ICF = 60°$. This means $\triangle IFC$ is an equilateral triangle with side length $2r$.

Since $FG = GC$, we have $IG \perp FC$ (prop. of isos. $\triangle$). Since $IG \perp ED$ by construction, we have $ED//FC$. This gives $\angle FED = \angle CDE = 120°$ (int. $\angle$s).

By intercept theorem, $IE = EF$ and $ID = DC$, meaning $EF = DC = r$. By midpoint theorem, $ED = \frac{1}{2}FC = r$.

From there, it is easy to see that we have $AB = BC = CD = DE = EF = FA = r$ and $\angle A = \angle B = \angle C = \angle D = \angle E = \angle F = 120°$. Thus $ABCDEF$ is a regular hexagon, as desired. □

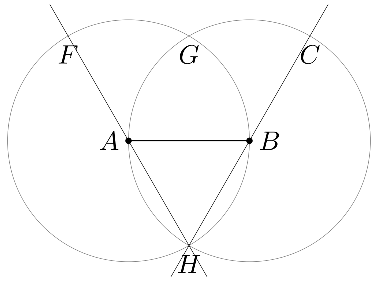
(2V) Mirror the constructions on the other side of the given segment AB.



Proof. By symmetry. □

(8E) 1, 2. Draw circles (A, B) and (B, A), intersecting at G and H.

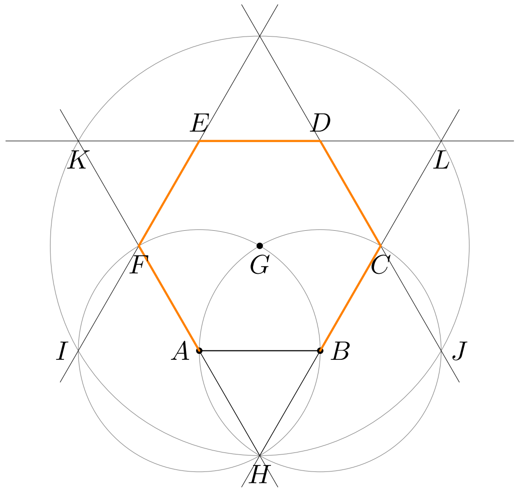
3, 4. Draw lines FA and FB, meeting (A, B) and (B, A) at F and C respectively.



5. Draw circle (G, H), intersecting (A, B) and (B, A) again at I and J respectively.

6, 7. Draw lines IF and JC.

8. Let HF and HC meet (G, H) at K and L respectively. Draw KL. We get the desired hexagon $ABCDEF$.

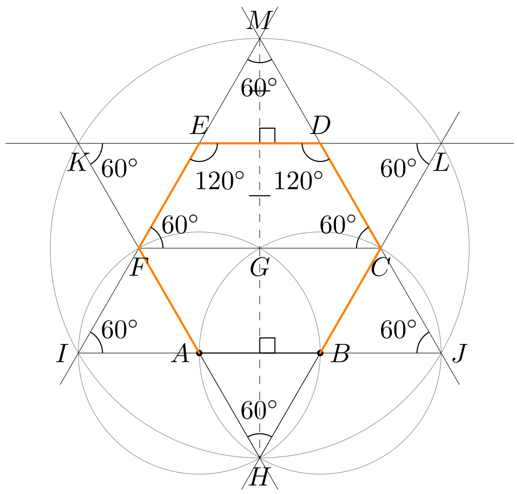


Proof. Let IF and JC meet at M. Let $AB = r$.

Note that I, J lie on line AB, and $IA = AB = BJ$. (This property is first shown in Task 1.7E. I'll call it "equil. $\triangle$ in circle" from now on.) It is not hard to see that trapezium $FIJC$ is made of five same-size equilateral triangles stacked together.

Since $\angle FIA = \angle CJB = 60°$, $\triangle MIJ$ is an equilateral triangle. Thus M lies on the circle (G, H) (since $GI = \sqrt{3}\,r$ and $GM = (\frac{3\sqrt{3}}{2} - \frac{\sqrt{3}}{2})r = \sqrt{3}\,r = GI$), and M, G, H are collinear (equil. $\triangle$ in circle).

Note that $HK = HL$ since K and L are symmetric about GH. Since we have $\angle AHB = 60°$, $\triangle HKL$ is also an equilateral triangle.



Thus, KL is a chord that subtends $120°$ at center of circle (G, H). This means KL is the perpendicular bisector of MG. Also, it can be shown that $KL // FC$ (since KL and FC are symmetric about line GH).

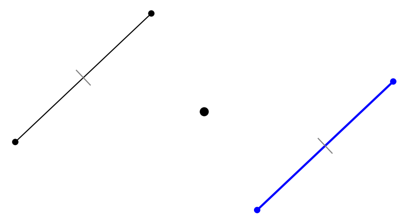
Therefore, by intercept theorem, we have $ME = EF$ and $MD = DC$. Since $\triangle MIJ$ is an equilateral triangle of side length $3r$ and $FI = CJ = r$, we have $EF = DC = r$. Moreover, $\angle FED = \angle CDE = 120°$. It is not hard to see that $ED = r$ too (by $FC = 2r \Rightarrow ED = 2r - 2(r\cos 60°) = r$).

Thus, $ABCDEF$ is a regular hexagon. ☐

6 Zeta

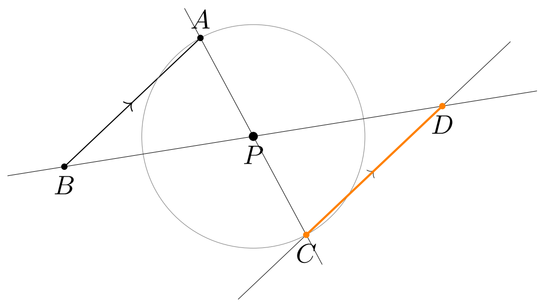
6.1 Point reflection

Task 6.1. Reflect the segment across the point.
(4L, 5E)



Solution 6.1. Let given segment be AB and given point be P.

(4L) 1. Draw circle (P, A).
2. Draw line AP, meeting (P, A) at C.
3. Draw line BP.
4. Draw $AB \twoheadrightarrow C$, intersecting BP at D. CD is the desired reflection of segment AB.

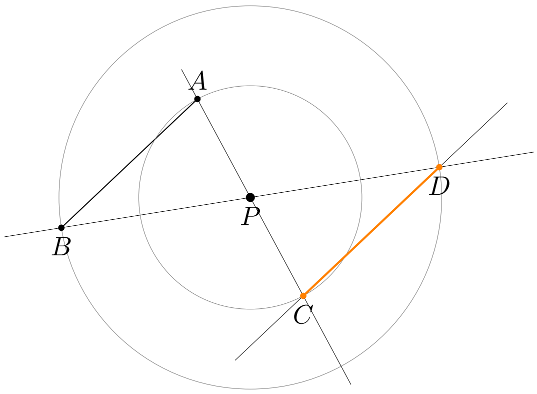


Proof.

$$
\begin{aligned}
\angle APB &= \angle CPD && \text{(vert. opp. } \angle\text{)}\\
\angle PAB &= \angle PCD && \text{(alt. } \angle\text{s, } BA // CD\text{)}\\
AP &= CP && \text{(radii)}\\
\therefore\ \triangle PAB &\cong \triangle PCD && \text{(AAS)}\\
\therefore\ BP &= DP && \text{(corr. sides, } \cong \triangle\text{s)}
\end{aligned}
$$

Since C and D are the reflection of endpoints A and B respectively, CD is the reflection of AB across point P (prop. of reflection). ☐

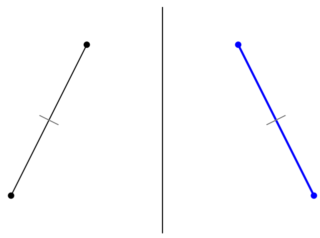
(5E) 1, 2. Draw circles (P, A) and (P, B).
3, 4. Draw lines AP and BP, meeting (P, A) and (P, B) at C and D respectively.
5. Draw CD, the desired segment.



Proof. Note that $AP = CP$ and $\angle APB = \angle CPD$ by radii, so CD is the reflection of AB across P. □

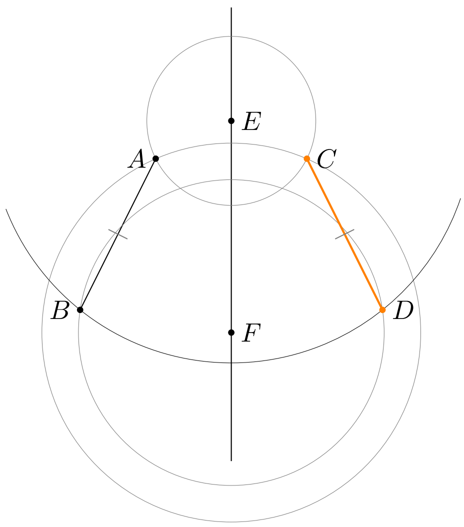
6.2 Reflection

Task 6.2. Reflect the segment across the line.
(5L, 5E)



Solution 6.2. Let AB be given segment. Let E, F be two arbitrary points on given line. (Or you can make four points, not reusing E and F at step 3, 4.)

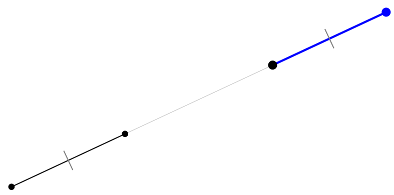
1, 2. Draw circles (E, A) and (F, A), intersecting again at C.
3, 4. Draw circles (E, B) and (F, B), intersecting again at D.
5. Draw line CD.



Proof. Note that $EAFC$ and $EBFD$ form kites, so we have $EF \perp AC$ and $EF \perp BD$. Also, from the property of kite, the perpendicular distance of A and C from EF is equal, and same for B and D. This means C and D are reflections of A and B across EF. □

6.3 Copy segment

Task 6.3. Construct a segment from the given point that is equal to the given segment and lies on the same line with it.
(3L, 4E, 2V)

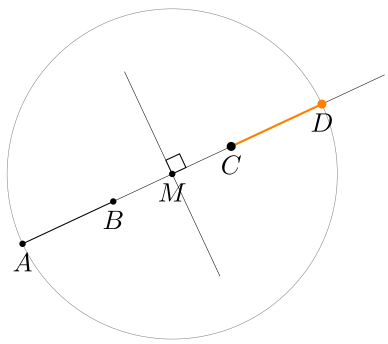


Solution 6.3. Let given segment be AB and given point be C.

(3L) 1. Draw line BC.

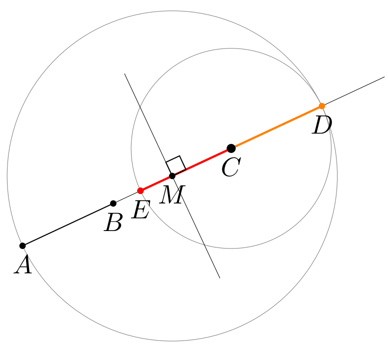
2. Draw perpbi BC. Let M be midpoint of BC.

3. Draw circle (M, A), intersecting BC at another point D. CD is the desired segment.



Proof. $AM = MD$ (radii) and $BM = MC$ by construction. So $CD = MD - MC = AM - BM = AB$. □

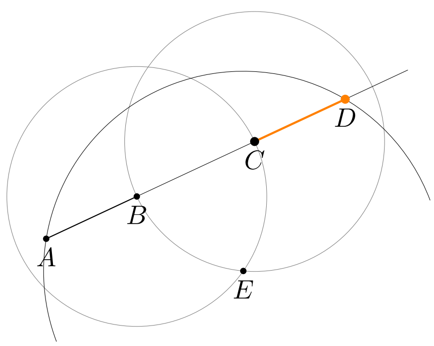
(2V) Draw circle (C, D), intersecting BC again at E. CE is the extra solution.



(4E) 1. Draw line BC.

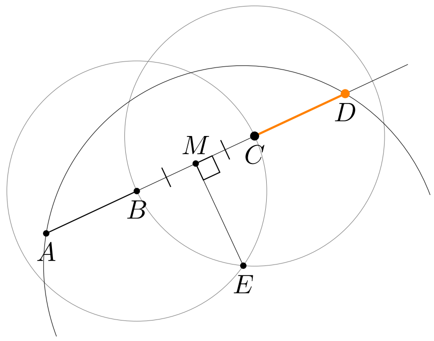
2, 3. Draw circles (B, C) and (C, B). Let one of intersections be E.

4. Draw circle (E, A), intersecting BC again at D. CD is the desired segment.



Proof. Let M be the midpoint of BC. Note that $EM \perp BC$ (prop. of isos. $\triangle$), so $AM = MD$ (line from center $\perp$ chord bisects chord).

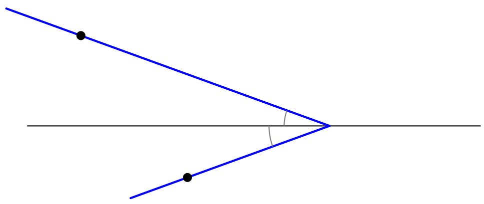
Thus, $CD = MD - MC = AM - BM = AB$, as desired.



$\square$

6.4 Given angle bisector

Task 6.4. Construct two straight lines through the two given points respectively so that the given line is a bisector of the angle that they make.
(4L, 4E)

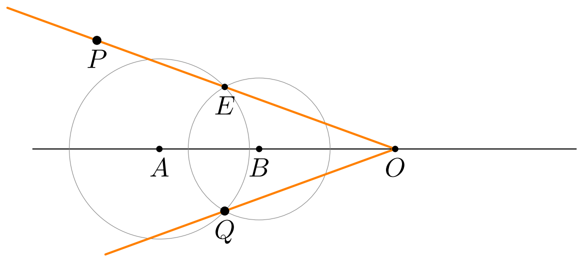


Solution 6.4. Let the given points be P, Q. Let A, B be arbitrary points on given line.

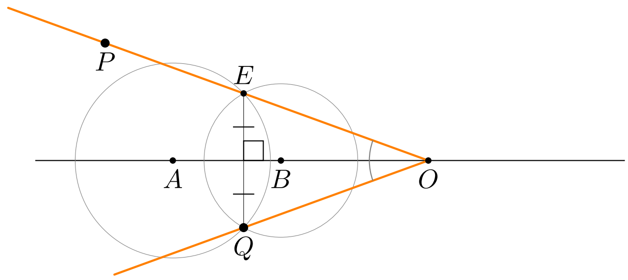
1, 2. Draw circles (A, Q) and (B, Q), intersecting at another point E.

3. Draw line PE, intersecting given line at O.

4. Draw line OQ. PE and OQ are the desired lines.



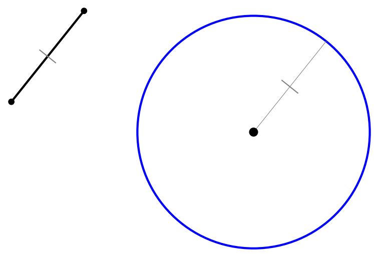
Proof. Let M be the projection of E on given line. Note that $EM = QE$ because $AQBE$ forms a kite. Thus we have $\triangle EMO \cong QMO$ by SAS, so $\angle EOM = \angle QOM$ (corr. $\angle$s, $\cong$ $\triangle$s).



$\square$

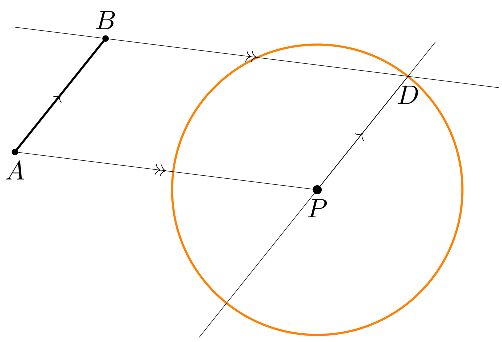
6.5 Non-collapsing compass

Task 6.5. Construct a circle with the given center and the radius equal to the length of the given segment.
(4L, 5E)



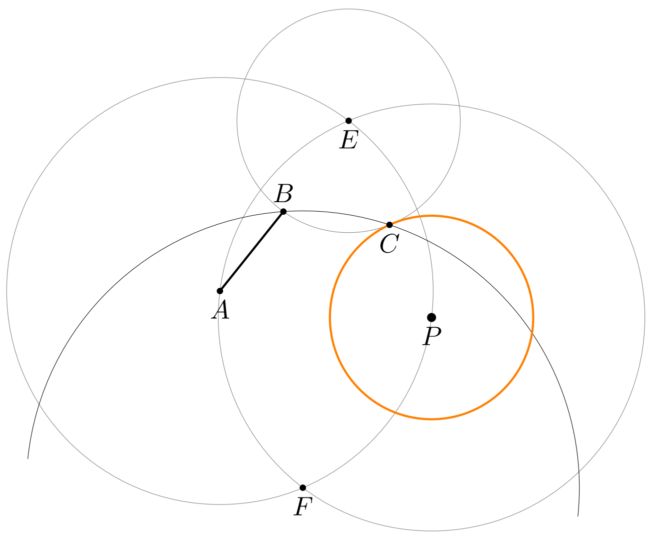
Solution 6.5. Let given segment be AB, given point be P.

(4L) 1. Draw $AB \twoheadrightarrow P$.
2. Draw line AP.
3. Draw $AP \twoheadrightarrow B$, intersecting $AB \twoheadrightarrow P$ at D.
4. Draw circle (P, D), the desired circle.



Proof. Note that $AP//BD$ and $AB//PD$ by construction, so we have $PD = AB$ (opp. sides of //gram), which means circle (P, D) has radius equal to AB. □

(5E) 1, 2. Draw circles (A, P) and (P, A), intersecting at E and F.
3, 4. Draw circles (E, B) and (F, B), intersecting at another point C.
5. Draw circle (P, C), the desired circle.



Proof. Note that $BFCE$ forms a kite (with $EB = EC$ and $FB = FC$ by radii), so C is the reflection of B across EF (prop. of kite). Similarly, since $AFPE$ is a kite, P is the reflection of A across EF.

Thus segment PC is the reflection of AB across EF. Since reflection preserves segment length, we have $PC = AB$, and so (P, C) is the desired circle. □

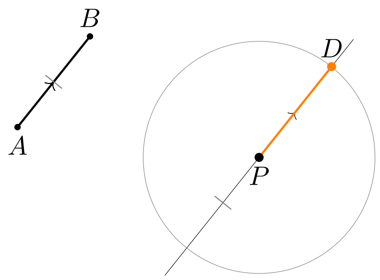
6.6 Translate segment

Task 6.6. Construct a segment from the given point parallel and equal to the given segment. (2L, 6E, 2V)

Solution 6.6. Let AB be given segment, P be given point.

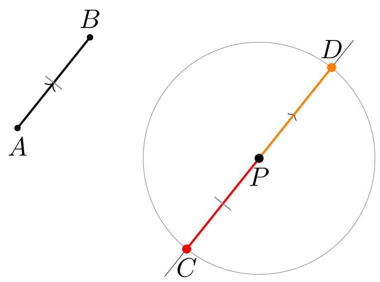
(2L) 1. Draw $AB \twoheadrightarrow P$.

2. Compass (AB, P). i.e. draw circle centered P with radius AB using non-collapsing compass tool[1]. Mark one of intersections of (AB, P) and $AB \twoheadrightarrow P$.



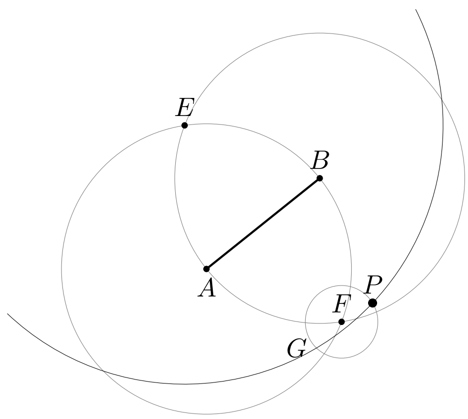
Proof. $AB = PD$ and $AB // PD$ by construction. □

(2V) Mark another intersection point of (AB, P) and $AB \twoheadrightarrow P$.



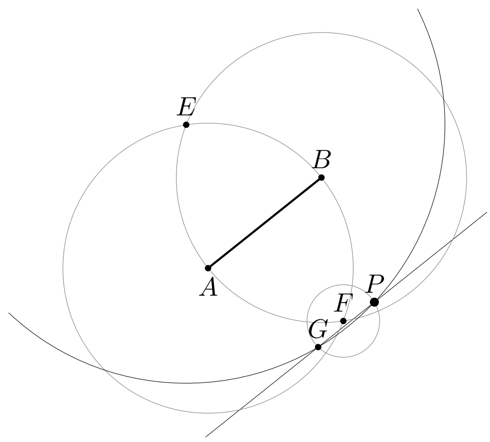
(6E) 1, 2. Draw circles (A, B) and (B, A), intersecting at E and F.

3, 4. Draw circle (E, P) and (F, P), intersecting at another point G.

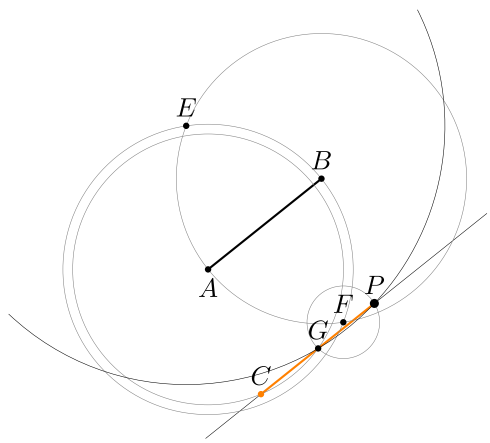


5. Draw line PG.

[1]I use (AB, P) instead of (P, AB) to follow the order of clicking the points in the game.



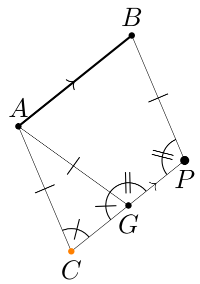
6. Draw circle (A, G), intersecting PG at another point C.



Proof. Note that $EGFP$ forms a dart, so $GP \perp EF$ (prop. of dart). Since $EF \perp AB$ (prop. of rhombus), we have $GP//AB$.

Note that $AG = BP$ since AB and GP are symmetric about line EF. Thus $AGPB$ is an isosceles trapezium. We also have $AC = AG$ by radii.

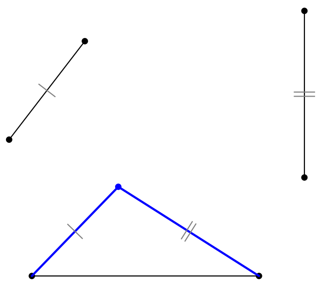
Let's focus on trapezium $AGPB$ and $\triangle ACG$. We have $\angle ACG = \angle AGC$ (base $\angle$s, isos. $\triangle$)and $\angle AGP = \angle BPG$ (prop. of isos. trapezium).



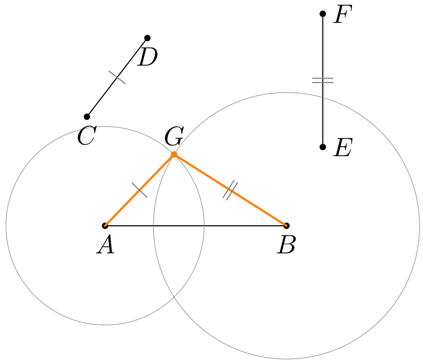
Since $\angle AGC + \angle AGP = 180°$ (adj. $\angle$s on st. line), we also have $\angle ACG + \angle BPG = 180°$, thus $AC//BP$ (int. $\angle$s supp.). This means $ABPC$ is a parallelogram, and we have $CP = AB$ (opp. sides of //gram), as desired. □

6.7 Triangle by three sides

Task 6.7. Construct a triangle with the side AB and the two other sides equal to the given segments.
(4L, 12E, 4V)

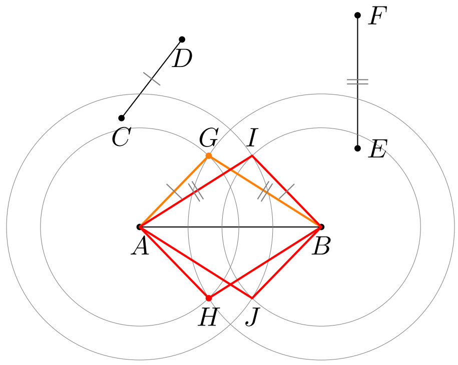


Solution 6.7. (4L, 12E) Let CD, EF be the given segments.
1, 2. Compass circles (CD, A) and (EF, B). Let one of their intersections be G.
3, 4. Draw line AG and BG.



(4V) 2nd solution: Let H be another intersection of the circles. Draw AH and BH.

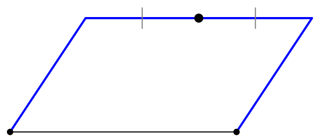
3rd & 4th solution: Compass (CD, B) and (EF, A), intersecting at I and J. Draw IA, IB, JA, JB.



Proof. By construction. □

6.8 Parallelogram

Task 6.8. Construct a parallelogram with the given side and the midpoint of the opposite side in the given point.
(5L, 8E)



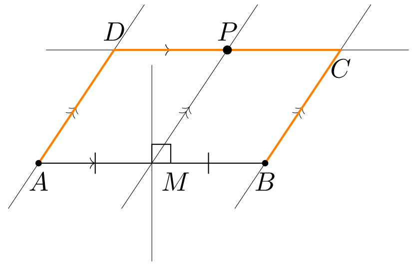
Solution 6.8. (5L) Let given segment be AB, given point be P.

1. Draw perpbi AB. Let M be midpoint of AB.

2. Draw line PM.

3. Draw $AB \twoheadrightarrow P$.

4, 5. Draw $PM \twoheadrightarrow A$ and $PM \twoheadrightarrow B$, making points D and C. $ABCD$ is the desired paralleogram.



Proof. Note that $ABCD$, $AMPD$ and $MBCP$ are parallelograms by construction. So we have $DP = AM$ and $PC = MB$ (opp. sides of //gram). Since $AM = MB$ by construction, we have $DP = PC$, which means P is the midpoint of side DC, as desired □

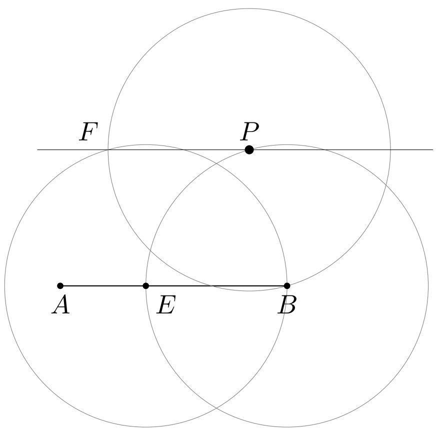
(8E)

1. Draw circle (P, B).

2. Draw circle (B, P), intersecting segment AB at E.

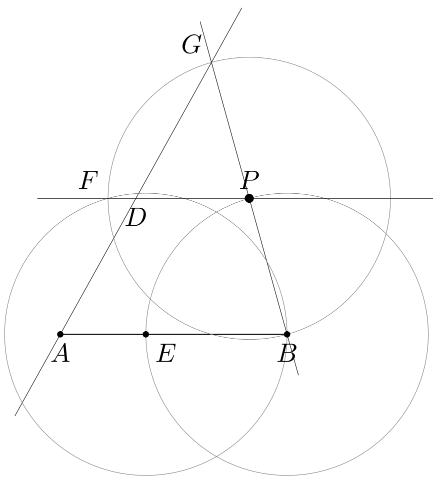
3. Draw circle (E, B), intersecting (P, B) again at F.

4. Draw line FP.



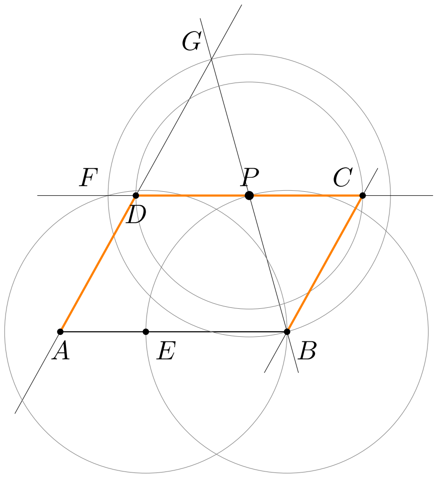
5. Draw line BP, meeting (P, B) at G.

6. Draw line AG, intersecting FP at D.



7. Draw circle (P, D), intersecting FP at another point C.

8. Draw line BC.



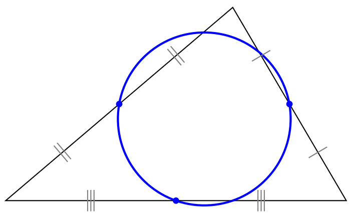
Proof. Note that $PBEF$ forms a rhombus because the circles involved have the same radius. So $FP//AB$.

Also, $GP = PB$ (radii), so $GD = DA$ by intercept theorem, and thus $DP = \frac{1}{2}AB$ by midpoint theorem.

Lastly, we have $DP = PC$ (radii), which means $DC = AB$. Thus $ABCD$ is a parallelogram by "opp. sides equal and //", as desired □

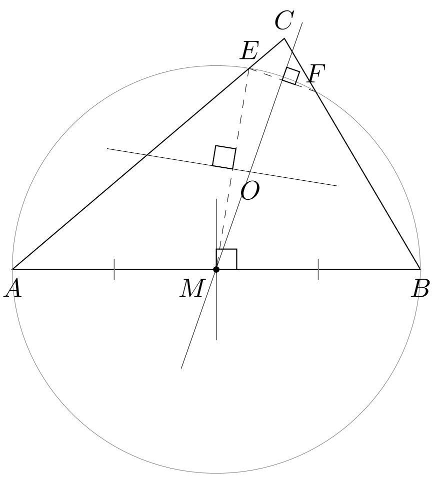
6.9 Nine point circle

Task 6.9. Construct a circle that passes through the midpoints of sides of the given acute triangle.
(5L, 9E)

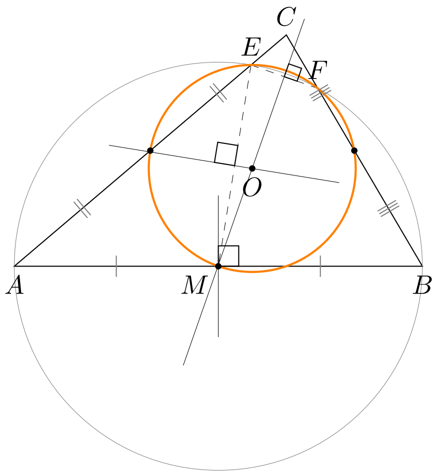


Solution 6.9. Let the given triangle be $\triangle ABC$.

(5L) 1. Draw perpbi AB. Let M be midpoint of AB.
2. Draw circle (M, A), intersecting sides AC and BC at E and F.
3. Draw perpbi EF.
4. Draw perpbi EM, intersecting perpbi EF at O.



5. Draw circle (O, M), the desired nine point circle.



Proof. The circle that passes through the midpoints of a triangle's sides is called the **nine point circle**. By "prop. of nine point circle", it also passes through the foots of altitudes of the triangle.

Note that $AE \perp EB$ and $AF \perp FB$ ($\angle$ in semi-circle). This means E and F are the foot of altitudes from B to side AC and from A to side BC respectively.

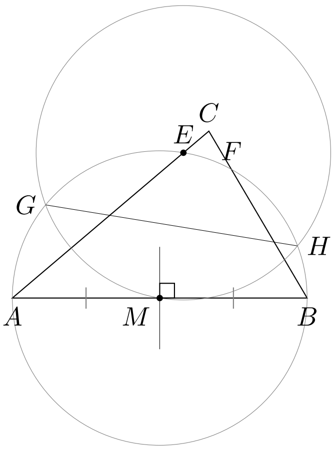
Since the nine point circle passes through E, F and M, its center O must lie on the perpendicular bisector of EF and EM (prop. of $\perp$ bisector). Thus O must be the point of intersection of the perpendicular bisectors, and (O, M) is the desired nine point circle. ☐

(9E) 1-3. Draw perpbi AB. Let M be midpoint of AB.

4. Draw circle (M, A), intersecting sides AC and BC at E and F.

5. Draw circle (E, M), intersecting (M, A) at G and H.

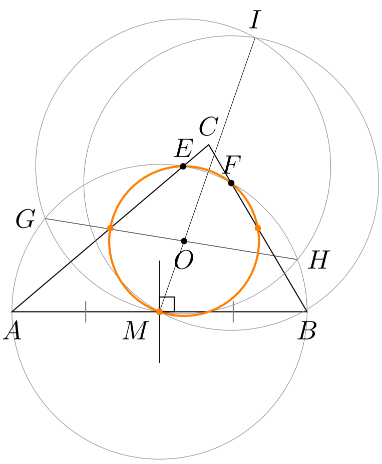
6. Draw line GH.



7. Draw circle (F, M), intersecting (E, M) at another point I.

8. Draw line IM, intersecting GH at O.

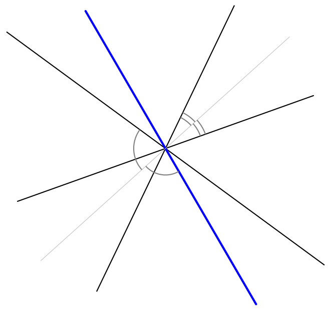
9. Draw circle (O, M), the desired nine point circle.



Proof. Note that GH is the perpendicular bisector of EM because $EGMH$ forms a rhombus by radii. Similarly, IM is the perpendicular bisector of EF because $EMFI$ forms a rhombus. Thus O is the same point as the "O" in (5L). $\square$

6.10 Symmetry of four lines

Task 6.10. Three lines are intersected in a point. Construct a line so that the set of all 4 lines is mirror symmetric.
(3L, 4E, 3V)

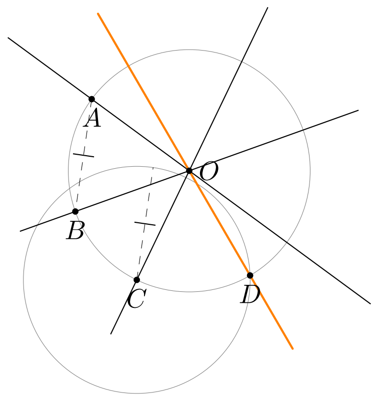


Solution 6.10. Let O be the point of intersection of the given lines.

(3L) 1. Draw a circle centered O with arbitrary radius. Let it intersect the three lines (at the left) at A, B and C.

2. Compass (AB, C), intersecting the first circle at one of points D.

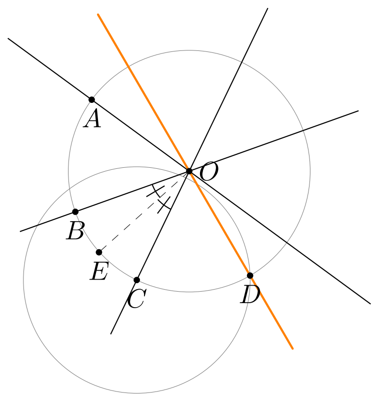
3. Draw line OD, the desired line.



Proof. Let m be the line of symmetry. Then m must pass through O. Otherwise, O would be on either side of m and the lines could never be mirror symmetric about m.

Moreover, m must be an angle bisector of two of the given lines by "prop. of $\angle$ bisector". It is also an angle bisector of the remaining two lines (one is the given line, one is the required line) in order for them to be mirror symmetric.

Let E be the midpoint of arc $\overset{\frown}{BC}$. Then $\angle BOE = \angle COE$ (equal arcs, equal $\angle$s at center), so line OE is used as the line of symmetry. We want to show that $\angle AOE = \angle DOE$.



By compass construction, we have $CD = AB$, so $\overset{\frown}{CD} = \overset{\frown}{AB}$ (equal chords, equal arcs). Thus $\overset{\frown}{DE} = \overset{\frown}{EC} + \overset{\frown}{CD} = \overset{\frown}{BE} + \overset{\frown}{AB} = \overset{\frown}{AE}$, giving $\angle AOE = \angle DOE$ (equal arcs, equal $\angle$s at center), as required.

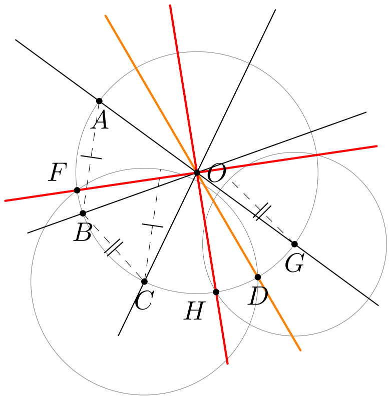
$\square$

(3V) 2nd solution: 4. Let F be another intersection of (C, D) and (O, D). Draw line OF, the 2nd solution.

3rd solution:

5. Let G be the point diametrically opposite to A on (O, A). Compass (BC, G), intersecting (O, A) at the point H that isn't passed through by line OF.

6. Draw line OH, the 3rd solution.



Proof. The 2nd solution uses the angle bisector of $\angle AOC$ as the line of symmetry, call it m_2. We have $\overset{\frown}{AB} = \overset{\frown}{CF}$ by compass construction, thus $\overset{\frown}{AF} = \overset{\frown}{BC}$. So OF is symmetric to OB about m_2.

The 3rd solution uses the angle bisector of $\angle BOG$ (or alternatively $\angle AOB$) as the line of symmetry, call it m_3. We have $\overset{\frown}{BC} = \overset{\frown}{GH}$ by construction, so OC is symmetric to OH about m_3. $\square$

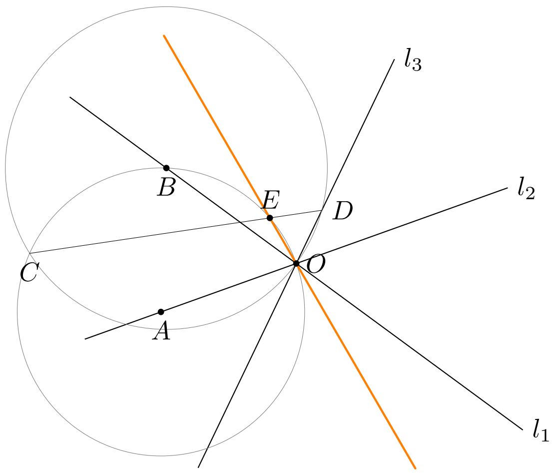
(4E) Let O be point of intersection of the given lines. Label the given lines l_1, l_2, l_3.

1. Let A be an arbitrary point on l_2. Draw circle (A, O), intersecting l_1 at another point B.

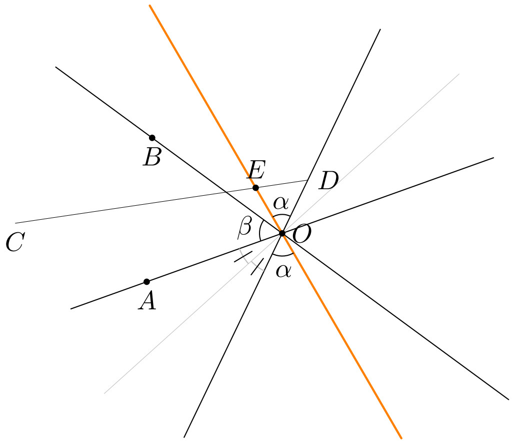
2. Draw circle (B, O), intersecting (A, O) at another point C, and intersecting l_3 at another point D.

3. Draw line CD, cutting (A, O) at E.

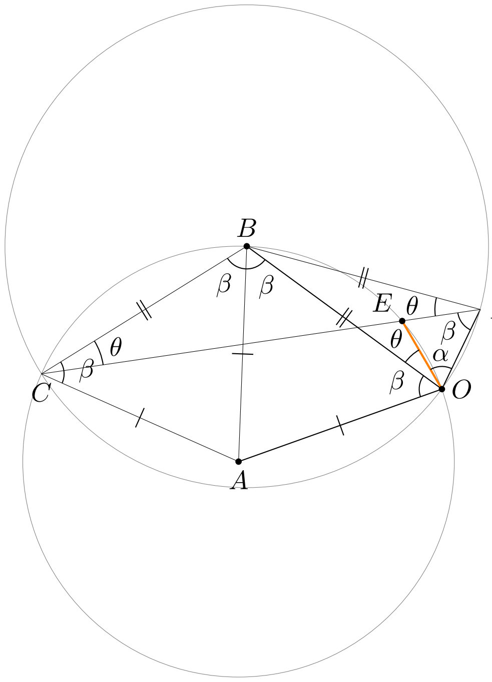
4. Draw line OE, the desired line.



Proof. The angle bisector of l_2 and l_3 is used as the line of symmetry. Refer to the figure below, we want to show that $\angle BOA = \angle EOD$. This will mean $\alpha = \beta$ and thus l_1 and the orange line are symmetric about the angle bisector of l_2 and l_3.



Let's focus on the points and segments involved in constructing OE. Let $\angle BOE = \theta$.



Note that $\triangle ACB \cong \triangle AOB$ (SSS), so $\angle ACB = \angle AOB = \beta$ (corr. $\angle$s, $\cong \triangle$s). Also, $\angle ABC = \angle ABO = \beta$ (base $\angle$s, isos. $\triangle$).

Consider circle (B, O). Note that $\angle CBO$ and $\angle CDO$ are subtended by the same arc $\overset{\frown}{CO}$. Thus $\angle CDO = \dfrac{1}{2}\angle CBO = \beta$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

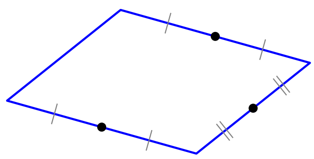
Consider circle (A, O). We have $\angle BCE = \angle BOE = \theta$ ($\angle$s in the same segment), so $\angle BDC = \angle BCD = \theta$ (base $\angle$s, isos. $\triangle$).

Consider $\triangle BOD$. Since $BO = BD$ (radii), we have $\angle BOD = \angle BDO$ (base $\angle$s, isos. $\triangle$), meaning $\alpha + \theta = \beta = \theta$, giving $\alpha = \beta$, as desired.

□

6.11 Parallelogram by three midpoints

Task 6.11. Construct a parallelogram given three of the midpoints.
(7L, 10E, 3V)

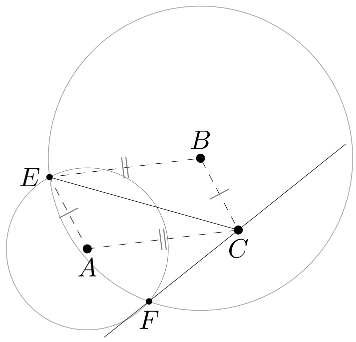


Solution 6.11. Label the given points A, B, C (as in the game).

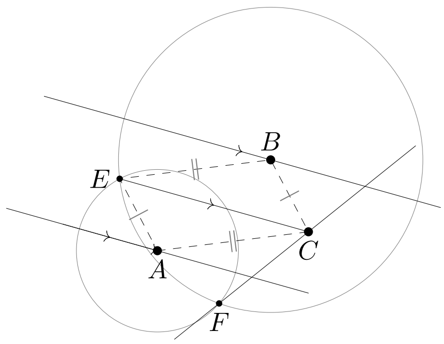
(7L) 1. Compass (BC, A).

2. Compass (AC, B), intersecting (BC, A) at E (top) and F (bottom).

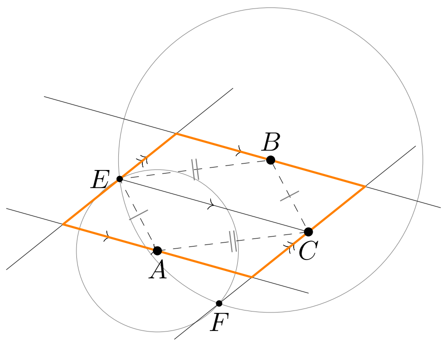
3, 4. Draw lines EC and FC.



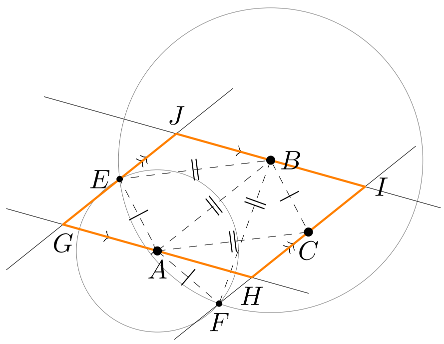
5, 6. Draw $EC \twoheadrightarrow A$ and $EC \twoheadrightarrow B$.



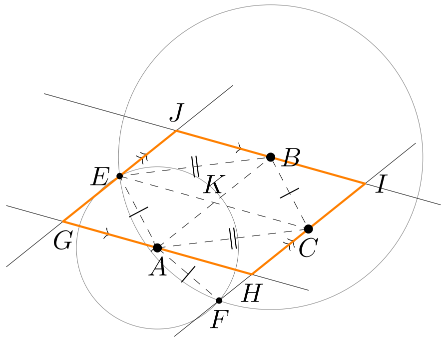
7. Draw $FC \twoheadrightarrow E$. The big parallelogram enclosed by the lines is the desired parallelogram.



Proof. Let the orange parallelogram be $GHIJ$. Let EC and AB intersect at K.



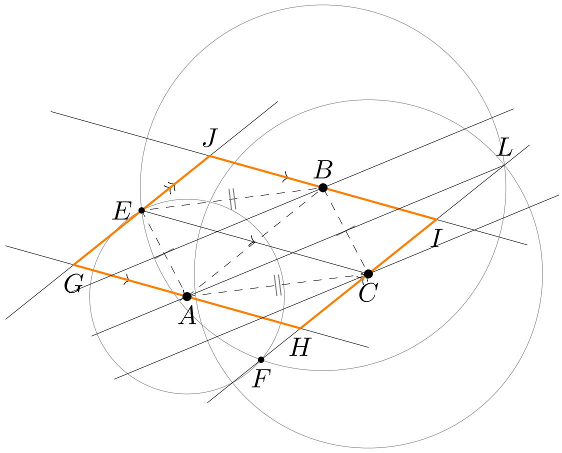
Note that $\triangle AFB \cong \triangle BCA$ (SSS), so $\triangle AFB$ and $\triangle BCA$ have the same height (with AB as the base). This means $FC // AB$. Thus we have $GJ // AB // HI$ and $JI // EC // GH$ by construction. In other words, the orange parallelogram can be divided into four smaller parallelograms ($KBJE, KCIB, KEGA, KAHC$).



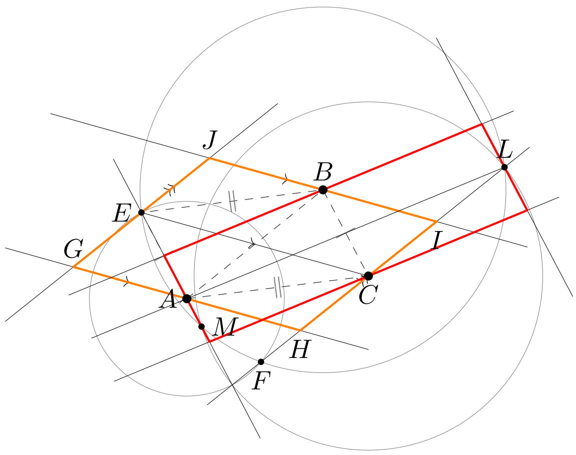
Note that $ACBE$ forms a parallelogram (opp. sides equal). Thus we have $EK = CK$ and $AK = BK$ (diags of //gram). From "opp. sides of //gram", it follows that the four smaller parallelograms are congruent to each other, meaning that A, C, B, E are midpoints of GH, HI, IJ, JG respectively. $\square$

(3V) 2nd solution:

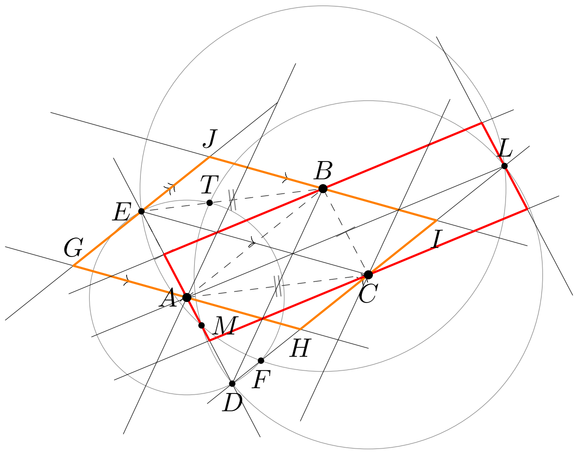
8. Compass (AB, C), intersecting (B, E) at M (left) and L (right).
9. Draw line AL.
10, 11. Draw $AL \twoheadrightarrow B$ and $AL \twoheadrightarrow C$.



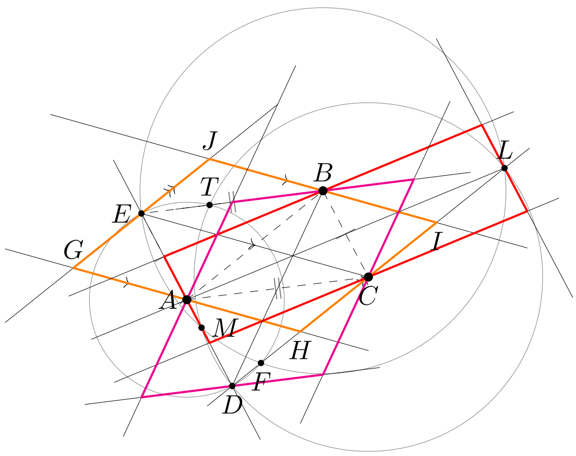
12. Draw line AM.
13. Draw $AM \twoheadrightarrow L$. We get the 2nd solution.



3rd solution: Let (A, F) and (C, M) intersect at T (top) and D (bottom).
14. Draw line BD.
15, 16. Draw $BD \twoheadrightarrow A$ and $BD \twoheadrightarrow C$.

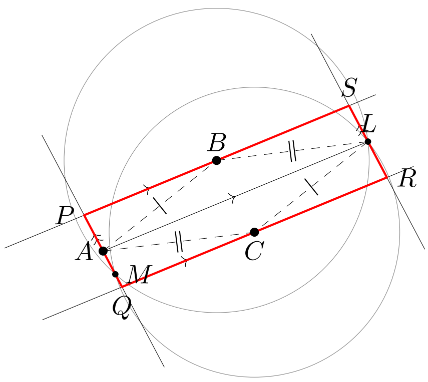


17. Draw line BT.
18. Draw $BT \twoheadrightarrow D$. We get the third solution.



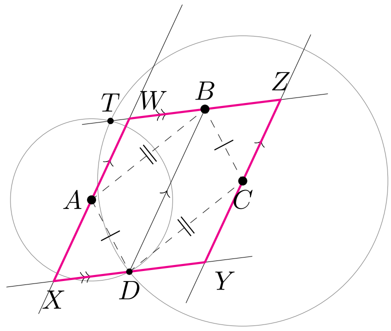
Proof. Let's look at the constructions of each solution separately.

2nd solution: Let the red parallelogram be $PQRS$.



Similarly to the proof of (7L), $ACLB$ is a parallelogram, and we have $AM//BC//SR$ and $PS//AL//QR$ by construction. It follows that $PQRS$ is a parallelogram with A, B, L, C being the midpoints of sides.

3rd solution: Let the magenta parallelogram be $WXYZ$.



Similarly, $ABCD$ is a parallelogram, and we have $WZ//AC//XY$ and $WX//BD//ZY$. It follows that $WXYZ$ is a parallelogram with A, B, C, D being the midpoints of sides.

□

(10E) 1. Draw circle (A, C).

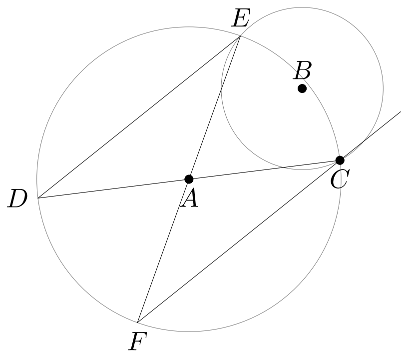
2. Draw circle (B, C), intersecting (A, C) at another point E.

3. Draw line CA, meeting (A, C) at point D.

4. Draw line DE.

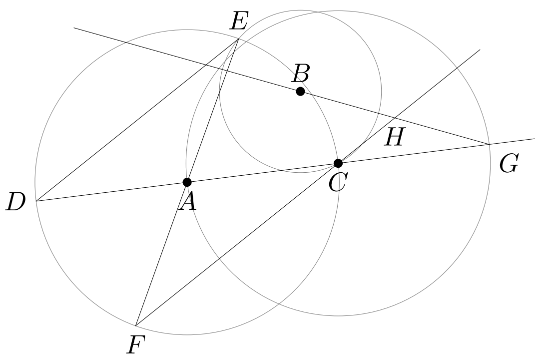
5. Draw line EA, meeting (A, C) at F.

6. Draw line FC.



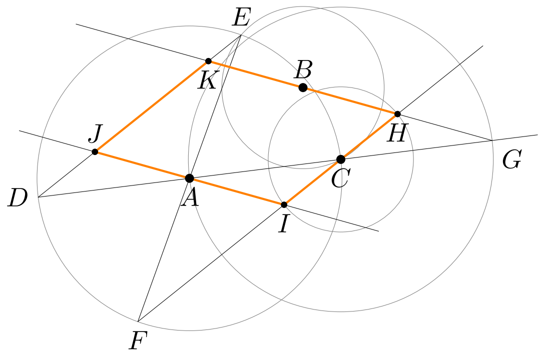
7. Draw circle (C, A), intersecting line AC again at G.

8. Draw line GB, intersecting FC at H.



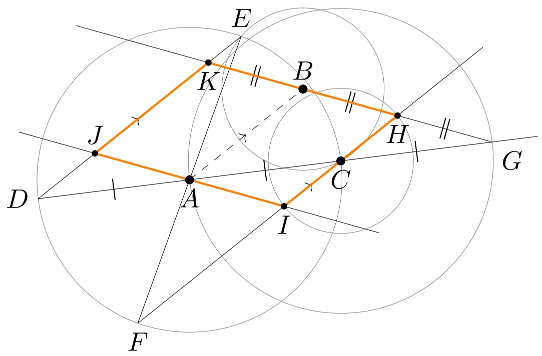
9. Draw circle (C, H), intersecting FC again at I.

10. Draw line AI. We get the desired parallelogram $HIJK$.



Proof. Note that $AEBC$ forms a kite, so $EC \perp AB$. Also, $DE \perp EC$ by "$\angle$ in semi-circle". Thus $DE//AB$. Similarly, we have $AB//FH$.

Note that $DA = AC = CG$ by radii. By intercept theorem, we have $KB = BH = HG$.



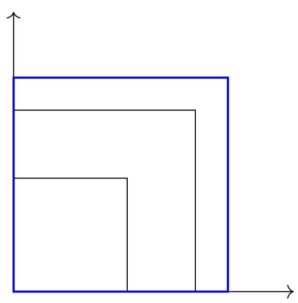
Also, $BA = 2HC$ by midpoint theorem, so $HI = 2HC = BA$. Thus $ABHI$ is a parallelogram (opp. sides equal and //). It follows that $HIJK$ is a parallelogram.

Since $KB = BH$ (shown above) and $BH = AI$ (opp. sides of //gram), we also have $JA = AI$. Thus, A, B, C are midpoints of sides JI, KH, HI respectively, as desired. □

7 Eta

7.1 Sum of areas of squares

Task 7.1. Construct a square whose area equals the sum of the areas of the two given squares and all three have the common angle.
(3L, 6E)

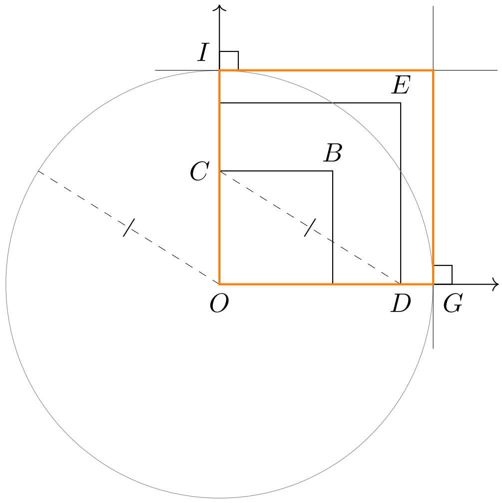


Solution 7.1. Let the given squares be $OABC$ and $ODEF$.

(3L) 1. Compass (CD, O), intersecting given horizontal ray at G, and intersecting vertical ray at I.

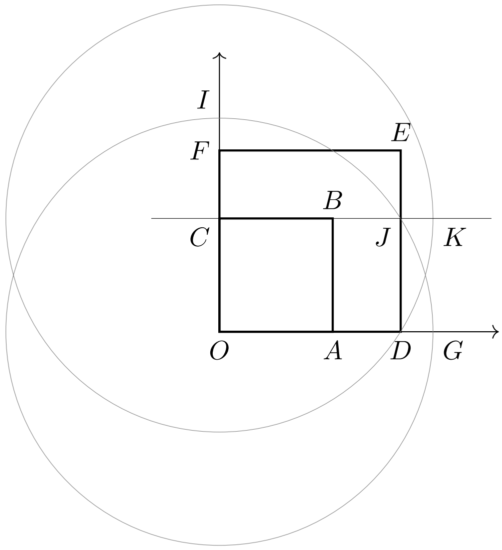
2. Draw perpendicular of horizontal line through G.

3. Draw perpendicular of vertical line through I.

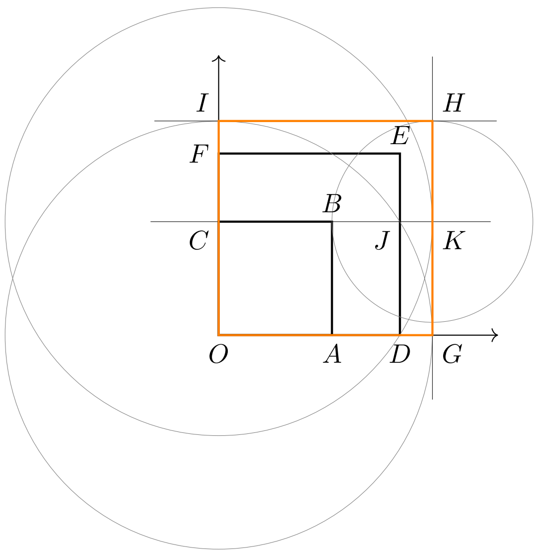


Proof. In $\triangle OCD$, by Pythagoras theorem, we have $OC^2 + OD^2 = CD^2$. Since $CD = OG$ by construction, it means that area of square OC and square OD sum up to area of square OG. $\square$

(6E) 1. Extend given side CB to be a line, intersecting side DE at J.
2. Draw circle (O, J), intersecting horizontal ray at G and vertical ray at I.
3. Draw circle (C, D), intersecting line CB at K (right).



4. Draw line GK.
5. Draw circle (K, B), intersecting GK at H (top).
6. Draw line IH. $OGHI$ is the desired square.

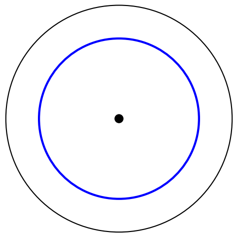


Proof. Note that $OJ = CD$ (diags of rectangle), so $CK = OG$. Thus $OGKC$ is a rectangle (1 equal pair, 2 right $\angle$s).

Thus by rectangle properties and radii, $HG = HK + KG = BK + CO = OA + AG = OG$. Thus $OGHI$ is the same square as (3L). ☐

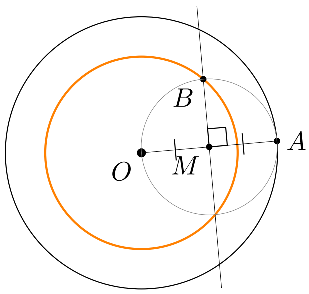
7.2 Annulus

Task 7.2. Construct a circle that is concentric with the given one and divides it into 2 parts of equal area.
(4L, 5E)



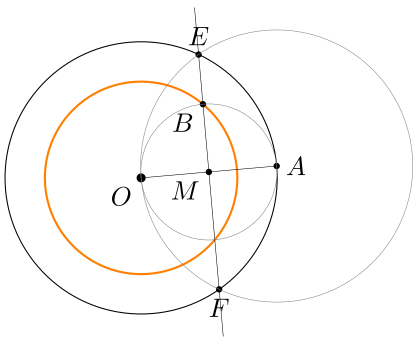
Solution 7.2. (4L) Let O be given circle center. Let A be an arbitrary point on given circle.

1. Draw line OA.
2. Draw perpbi OA. Let M be midpoint of OA.
3. Draw circle (M, O), intersecting perpbi OA at one of points B.
4. Draw circle (O, B), the desired circle.



Proof. Let the radius of given circle be r. Note that $OB = \frac{r}{\sqrt{2}}$. The area of given circle is πr^2, while the area of orange circle is $\pi(\frac{r}{\sqrt{2}})^2 = \frac{1}{2}\pi r^2$, which is half the area of given circle. Thus the orange circle divides given circle into two parts of equal area. ☐

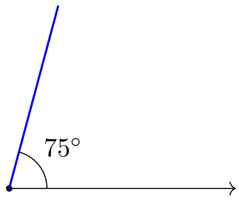
(5E) 1. Draw line OA.
2. Draw circle (A, O), intersecting given circle at E and F.
3. Draw line EF, intersecting OA at M.
4. Draw circle (M, O), intersecting EF at one point B.
5. Draw circle (O, B), the desired circle.



Proof. Note that EF is the perpendicular bisector of OA, so B is the same point as in (4L). ☐

7.3 Angle of 75 deg

Task 7.3. Construct an angle of 75° with the given side.
(3L, 5E, 2V)

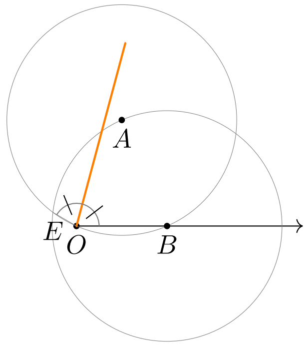


Solution 7.3. Let O be endpoint of given ray.

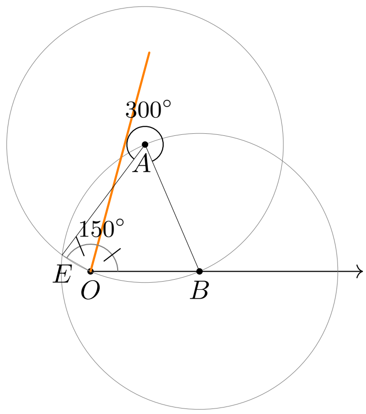
(3L) Let A be an arbitrary point directly above the given ray. **1.** Draw circle (A, O), intersecting given ray again at B.

2. Draw circle (B, O), intersecting (A, O) at E (left).

3. Draw angbi EOB, the desired line.

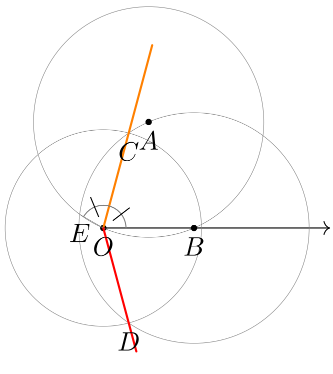


Proof. Note that $\triangle ABE$ forms an equilateral triangle, so reflex $\angle OAB = 360° - 60° = 300°$. By ($\angle$ at centre twice $\angle$ at $\odot^{ce}$), we have $\angle EOB = 150°$, so the angle bisector of $\angle EOB$ gives 75°.



□

(2V) Let orange line cut (B, A) at C. Draw circle (O, C), intersecting (B, A) again at D. Draw line OD, the extra solution.



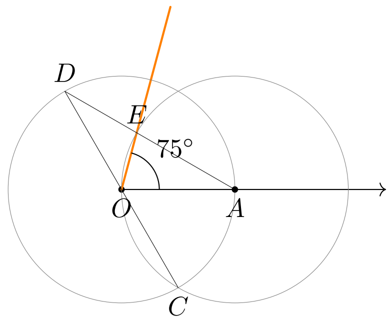
Proof. $OCBD$ forms a kite, so D is the reflection of C over given ray. □

(5E) 1, 2. Let A be arbitrary point on given ray. Draw circles (O, A) and (A, O), intersecting at C (bottom).

3. Draw line CO, meeting (O, A) at D.

4. Draw line DA, cutting (A, O) at E.

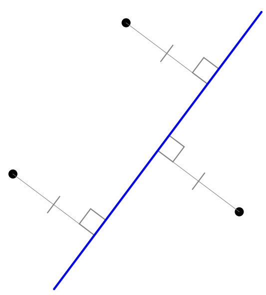
5. Draw line OE, the desired line.



Proof. Note that $\angle AOC = 60°$, so $\angle OAD = \angle ODA = 30°$ (base $\angle$s, isos. $\triangle$)& (ext. $\angle$ of $\triangle$). Look at $\triangle AOE$. Since $AO = AE$ (radii), we have $\angle AOE = \angle AEO$ (base $\angle$s, isos. $\triangle$). Thus $\angle AOE = (180° - 30°)/2 = 75°$ ($\angle$ sum of $\triangle$). □

7.4 Line equidistant from three points

Task 7.4. Construct a line that is at equal distance from the given three points.
(3L, 7E, 3V)

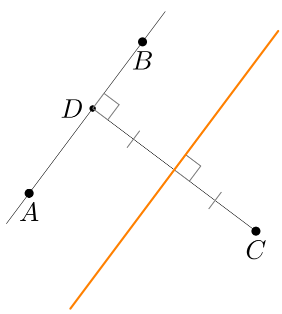


Solution 7.4. Let the given points be A, B, C.

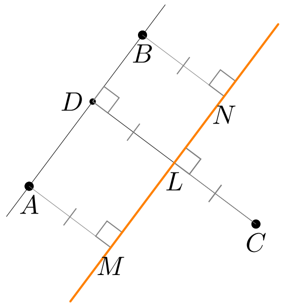
(3L, 7E) 1. Draw line AB.

2. Draw $AB \vdash C$. Let D be the foot of the perpendicular.

3. Draw perpbi DC, the desired line.

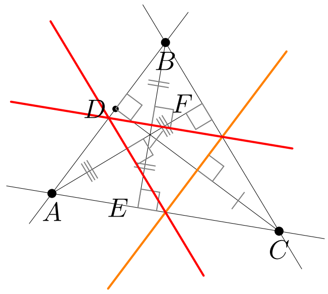


Proof. Let M, N, L be the projection of A, B, C on orange line. Note that $AM = BN = DL$ by rectangle properties, and $CL = DL$ by construction. Thus A, B, C have equal distance away from the orange line .



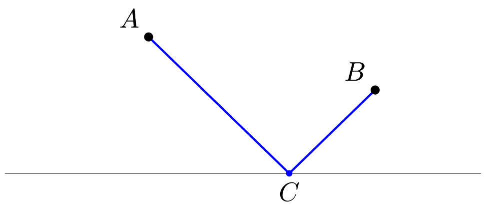
□

(3V) Do the same thing for the other two choices of initial line.



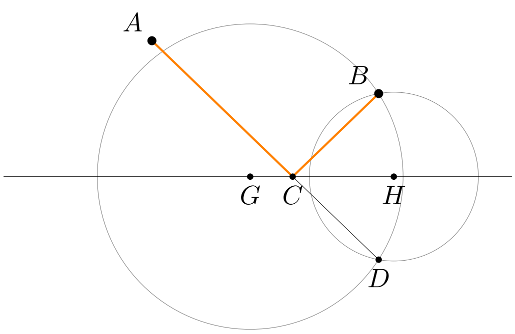
7.5 Heron's problem

Task 7.5. Construct a point C on the given line and segments AC and BC such that the sum of their length is minimal.
(4L, 4E)



$$|AC| + |CB| \to \min$$

Solution 7.5. Let G, H be two arbitrary points on given line.
1, 2. Draw circle (G, B) and (H, B), intersecting again at D.
3. Draw line AD, intersecting given line at C.
4. Draw line BC.

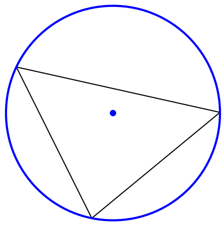


Proof. Note that D is the reflection of B over given line, so we have $BC = DC$.

The value of $AC + CD$ reaches minimum when A, C, D forms a straight line (triangle inequality), so $AC + CB$ (which $= AC + CD$) also reaches minimum. □

7.6 Circumscribed circle

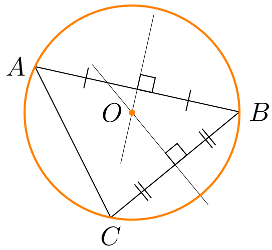
Task 7.6. Construct the circumcircle of the triangle.
(3L, 7E)



Solution 7.6. Let the given triangle be $\triangle ABC$.

1, 2. Draw perpbi AB and perpbi BC, intersecting at O.

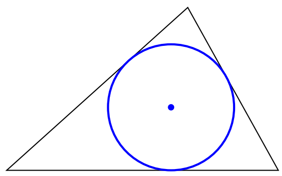
3. Draw circle (O, A), the desired circumcircle.



Proof. By property of circumcenter, the center of circumcircle of a triangle is the intersection of the perpendicular bisector of the sides. □

7.7 Inscribed circle

Task 7.7. Construct the incircle of the triangle.
(4L, 8E)

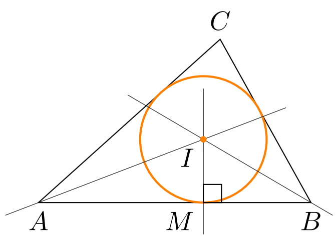


Solution 7.7. Let given triangle be $\triangle ABC$.

(4L) 1, 2. Draw angbi CAB and angbi CBA, intersecting at I.

3. Draw $AB \vdash P$. Let M be the foot of perpendicular.

4. Draw circle (I, M), the desired incircle.

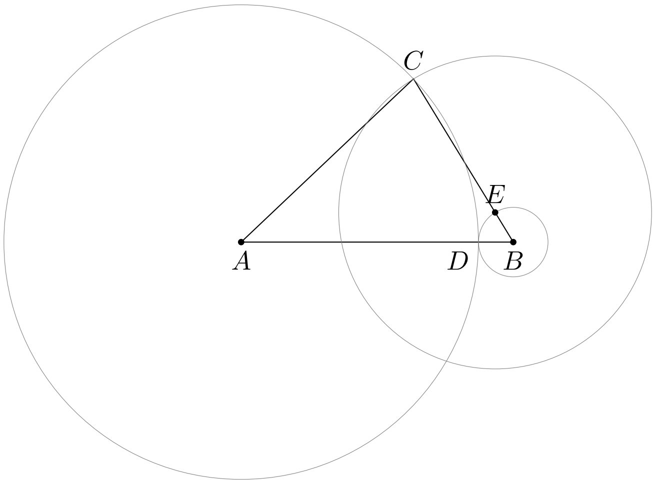


Proof. By property of incenter, the center of incircle is the intersection of angle bisector of the angles of triangle. □

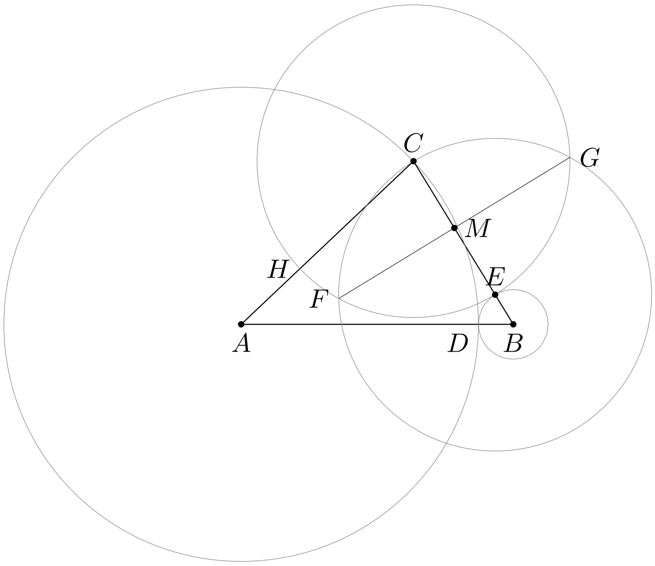
(8E) 1. Draw circle (A, C), intersecting side AB at D.

2. Draw circle (B, D), intersecting side BC at E.

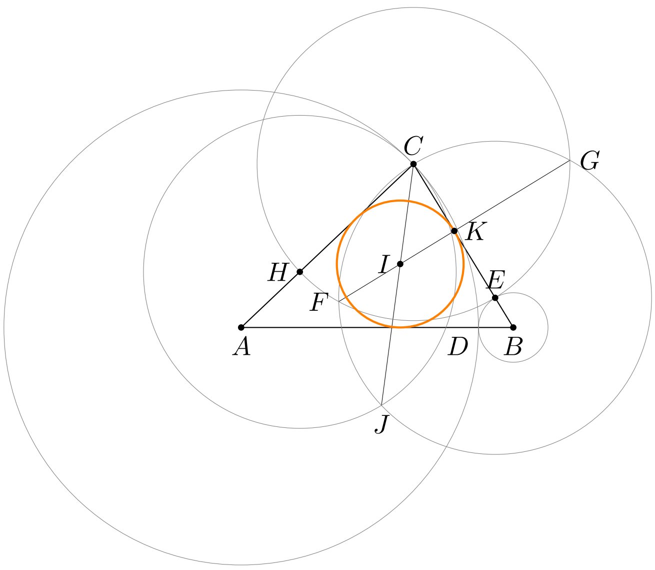
3. Draw circle (E, C).



4. Draw circle (C, E), intersecting (E, C) at F and G. Let (C, E) intersect side AC at H.
5. Draw line FG, intersecting CB at M.



6. Let (C, E) intersect side AC at H. Draw circle (H, C), intersecting (E, C) again at J.
7. Draw line CJ, intersecting FG at I.
8. Let FG and CB intersect at K. Draw circle (I, K), the desired incircle.

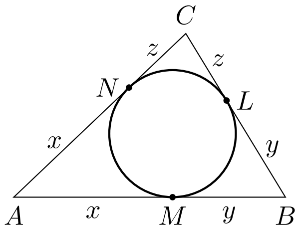


Proof. Let the true incircle's points of tangency on BC, AB, AC be L, M, N respectively. We want to show that points K and L are the same point by showing that $CK = CL$. This will help prove that the orange circle is indeed the true incircle.

Let's express CK in terms of AB, AC, BC. First we have $BE = BD = AB - AC$. Note that K bisects CE, so $CK = \dfrac{BC - BE}{2}$. This gives

$$\begin{aligned} CK &= \frac{BC - (AB - AC)}{2} \\ &= \frac{AC + BC - AB}{2} \end{aligned}$$

Let $AM = AN = x$, $BM = BK = y$, $CN = CK = z$ by tangent properties. Let's express CL in terms of AB, BC, AC.



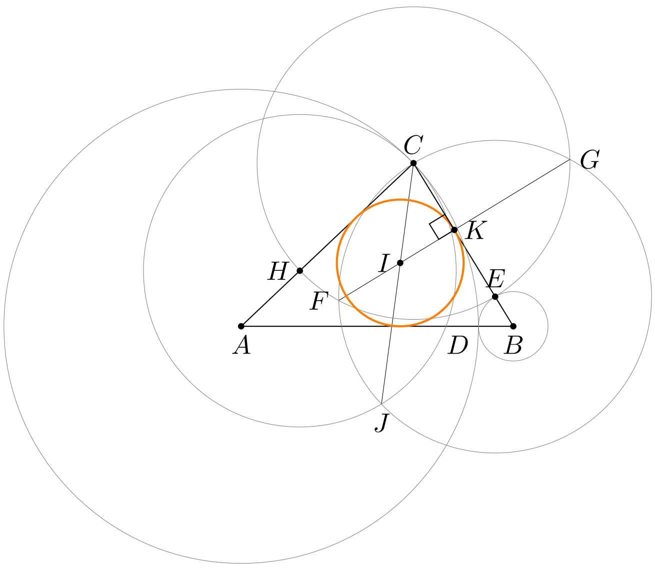
$$\begin{aligned} AB + AC + BC &= 2x + 2y + 2z \\ AB + AC + BC &= 2AB + 2z \\ \frac{AC + BC - AB}{2} &= z \\ CL &= \frac{AC + BC - AB}{2} \\ CL &= CK \end{aligned}$$

Thus K and L are the same point.

Also note that $CHJE$ forms a rhombus since $CH = HJ = JE = EC$ (radii). CJ is diagonal of the rhombus, so by property of rhombus, CJ is the angle bisector of $\angle ACB$.

Let I_t be the true incenter of $\triangle ABC$. We want to show that I is the same point as I_t. Note that $\angle ICK = \angle I_tCK$ since I and I_t both lies on the same line CJ. Also, $\angle CKI = 90°$ because GF is the perpendicular bisector of CE by construction, and $\angle CKI_t = 90°$ by "tangent $\perp$ radius".

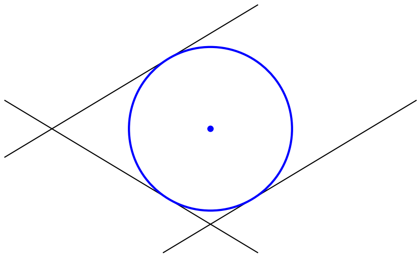
Thus, $\triangle CKI \cong \triangle CKI_t$ by (ASA), meaning that I is the same point as I_t. So (I, K) is the true incircle as desired.



$\square$

7.8 Circle tangent to three lines

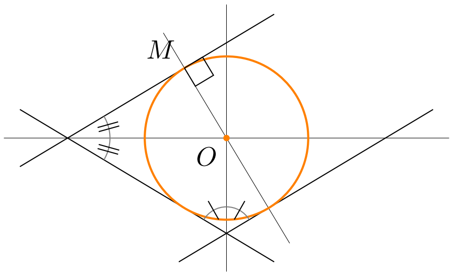
Task 7.8. Construct a circle that is tangent to the three given lines. Two of the lines are parallel.
(4L, 6E, 2V)



Solution 7.8. (4L) 1, 2. Draw the angle bisectors of the two intersections of given lines such that they intersect at point O.

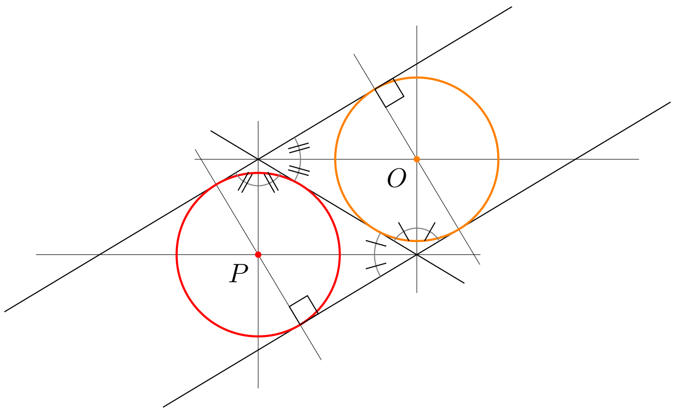
3. Draw the perpendicular of a given line through O. Let the foot of perpendicular be M.

4. Draw circle (O, M), the desired circle.



Proof. By tangent properties, the center of the circle lies on the angle bisectors of the tangent lines. By converse of tangent $\perp$ radius, M is a tangent point. Circle (O, M) will also be tangent to other given lines since O is equidistant from all the given lines (because the triangles formed by the lines are congruent by AAS.) $\square$

(2V) Make similar constructions on the other side of the given intercept line.

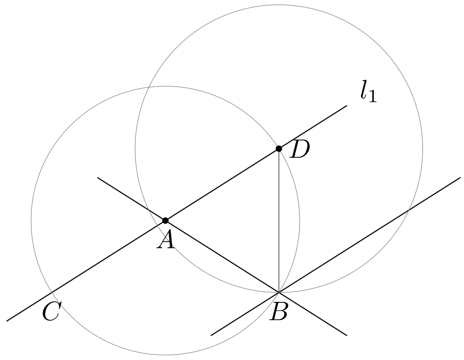


(6E) Label the top given lines l_1. Let A and B be the intersection of the given lines.

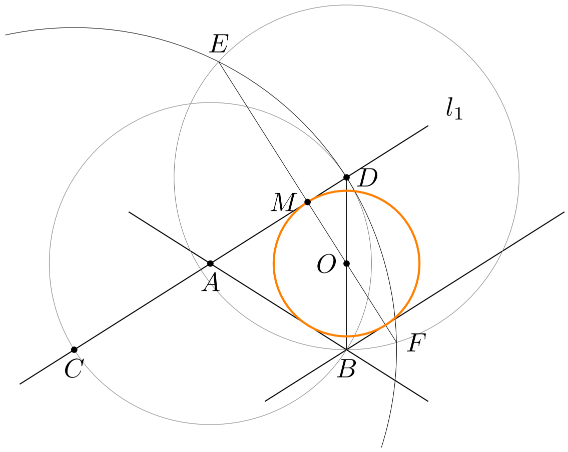
1. Draw circle (A, B), intersecting l_1 at D (right) and C (left).

2. Draw circle (D, B).

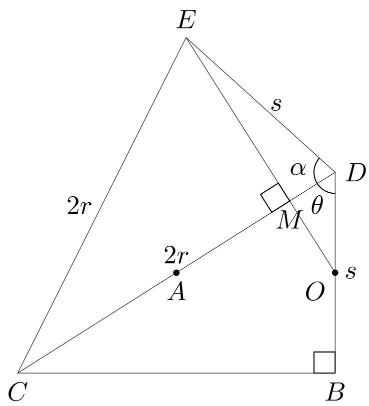
3. Draw line BD.



4. Draw circle (C, D), intersecting (D, B) at E and F.
5. Draw line EF, intersecting l_1 at M, and intersecting BD at O.
6. Draw circle (O, M), the desired circle.



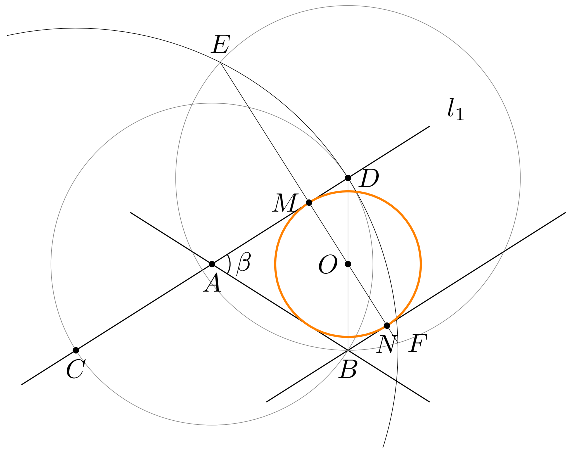
Proof. Let's focus inside $\triangle CBD$ and $\triangle CED$. Let $AD = AC = r$ and $DB = DE = s$. Then $CD = CE = 2r$. Let $\angle EDM = \alpha$ and $\angle BDM = \theta$. Note that $\angle EMD = 90°$ because $CEDF$ forms a kite, giving $EM \perp CD$.



From right angle trigonometry, we have $\cos\alpha = \dfrac{s/2}{2r}$ and $\cos\theta = \dfrac{s}{2r}$. Thus $MD = s\cos\alpha = \dfrac{s^2}{4r}$.

So $OD = \dfrac{MD}{\cos\theta} = (\dfrac{s^2}{4r})(\dfrac{2r}{s}) = \dfrac{s}{2}$. This means O is the midpoint of DB.

Let N be the point diametrically opposite from M on orange circle, and let $\angle BAD = \beta$.

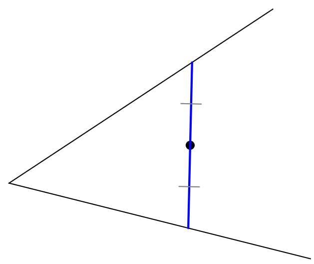


Note that $AB = AD$ (radii), so $\angle ABD = (180 - \theta)/2$. By "int. $\angle$s, $AD//BN$", we also have $\angle OBN = 180° - \beta - (180 - \theta)/2 = (180 - \theta)/2 = \angle ABD$. Thus BD is the angle bisector of $\angle ABN$. Also AO is the angle bisector of $\angle BAD$ by "prop. of isos. $\triangle$". This means O is the same point as the O in (4L), and M is also the same point, giving the same desired circle (O, M).

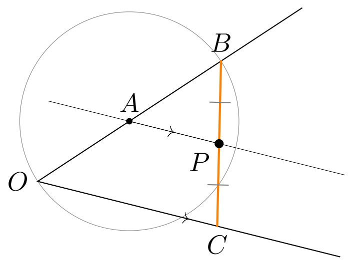
□

7.9 Segment by midpoint

Task 7.9. Construct a segment with the ends on the sides of the angle such that the given point is its midpoint.
(3L, 5E)

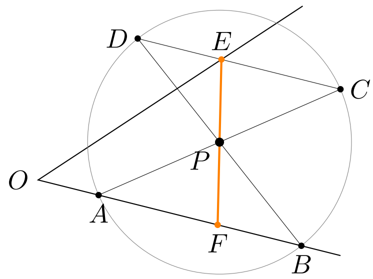


Solution 7.9. Let O be the vertex of the given angle, and P be given point.
(3L) 1. Draw line parallel to bottom ray through P, intersecting top ray at A.
2. Draw circle (A, O), intersecting top ray again at B.
3. Draw line BP, meeting bottom ray at C. Segment BC is the desired segment.



Proof. Since $AP//$ bottom ray and $OA = AB$ by radii, we have $BP = PC$ by intercept theorem. □

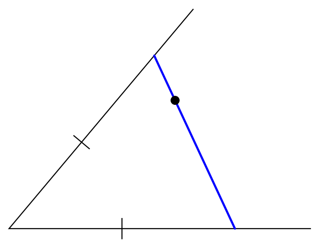
(5E) Let A be an arbitrary point on bottom ray.
1. Draw circle (P, A), intersecting bottom ray again at B.
2, 3. Draw line AP and BP, meeting (P, A) at C and D respectively.
4. Draw line CD, cutting top ray at E.
5. Draw line EP, meeting bottom ray at F. Segment EF is the desired segment.



Proof. Note that $AP = PC$ and $DP = PB$ by radii, so $ABCD$ forms a rectangle by "diags. equal and bisect each other". Thus we have $DC//AB$ by "prop. of rectangle". Since segment EF passes through P which is midway between DC and AB, by intercept theorem, we also have $EP = PF$, as desired. □

7.10 Angle isosceles

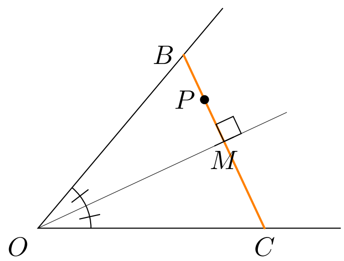
Task 7.10. Construct a line through the given point such that it cuts off equal segments on the sides of the angle.
(2L, 5E)



Solution 7.10. Let O be vertex of given angle and P be given point.

(2L) 1. Draw the angle bisector of the two given rays. Let it be l.

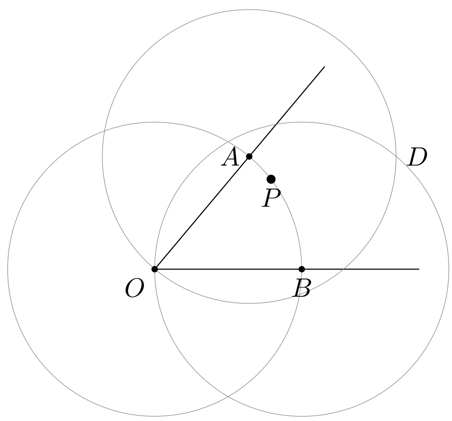
2. Draw l perpbi P, meeting top ray and bottom ray at B and C respectively. BC is the desired segment.



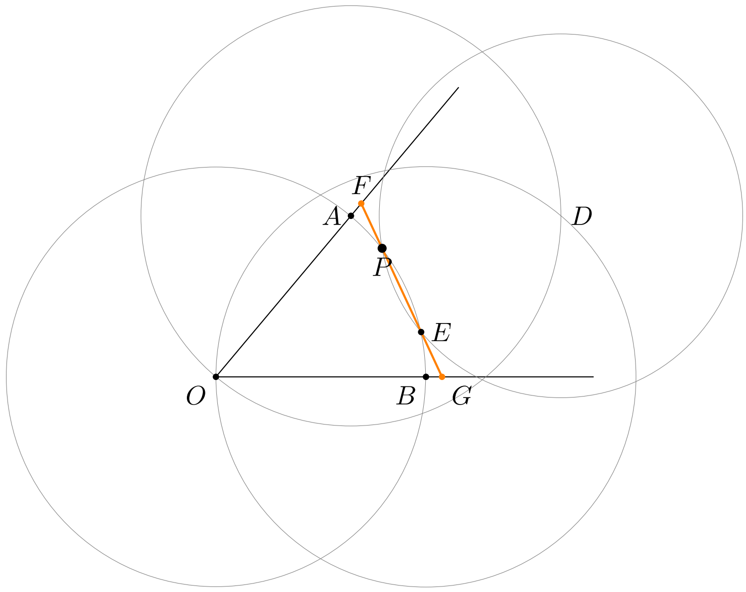
Proof. $\triangle OMC \cong \triangle OMB$ by (ASA), so $OB = OC$ (corr. sides, $\cong \triangle$s). □

(5E) 1. Draw circle (O, P), intersecting given rays at A (top) and B (bottom).

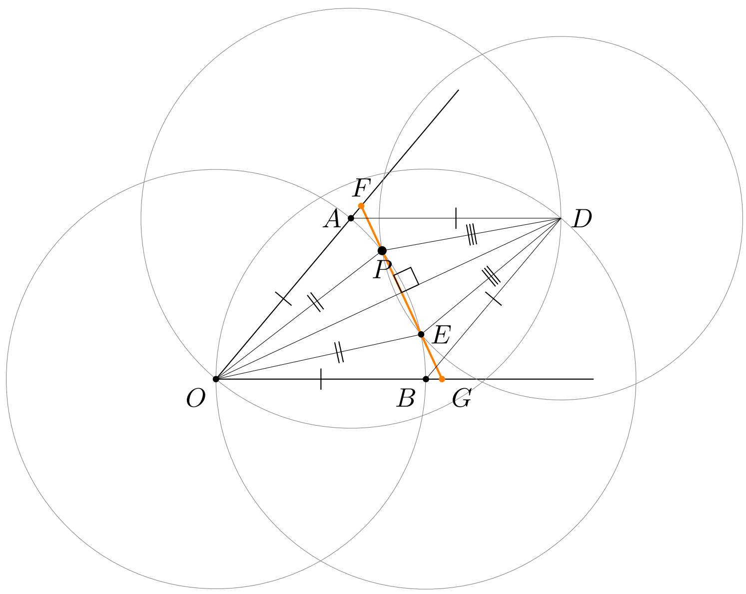
2, 3. Draw circles (A, O) and (B, O), intersecting again at D.



4. Draw circle (D, P), intersecting (O, P) again at E.
5. Draw line PE, making the desired segment FG.



Proof. Note that OD is a diagonal of rhombus $OBDA$, so OD is the angle bisector of the two given rays. Moreover, note that $OEDP$ forms a kite, giving $PE \perp OD$ (prop. of kite). Thus segment EG is the same construction as $(2L)$.



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References