Toddler Geometry (Problem set)

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Abstract

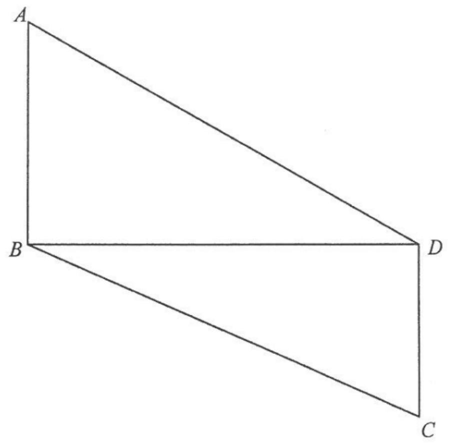
Geometry problems are harder than they seem.

Contents

1 Lines, angles and shapes

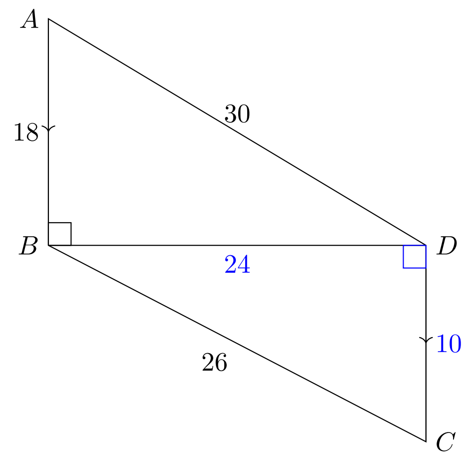
1.7 Area, perimeter and hypotenuse

Problem 1. In the figure, $ABCD$ is a trapezium with $AB//DC$ and $\angle ABD = 90°$. If $AB = 18$ cm , $BC = 26$ cm and $AD = 30$ cm , find the area of the trapezium $ABCD$.



(Difficulty: 3) (2019 DSE Paper 2 Q19)

Solution 1. Note that $\angle BDC = \angle ABD = 90°$ (alt. $\angle$s , $AB//DC$).

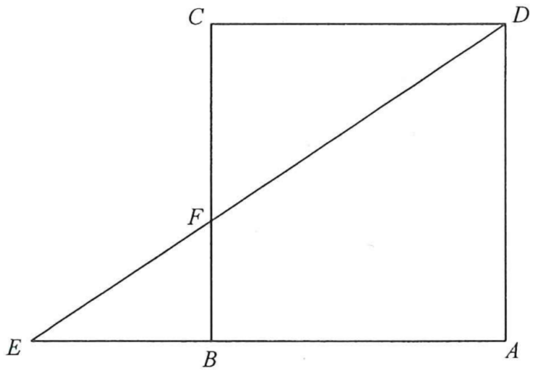


By pyth. theorem, $BD = \sqrt{30^2 - 18^2} = 24$, and $DC = \sqrt{26^2 - 24^2} = 10$.

By area of trapezium formula, area of $ABCD = \dfrac{(18+10)24}{2} = \boxed{336 \text{ cm}^2}$.

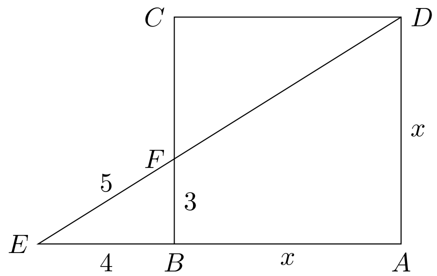
1.8 Proportions and similar triangles

Problem 2. In the figure, $ABCD$ is a square. E is a point lying on AB produced such that $BE = 4$ cm . BC and DE intersect at the point F . If $EF = 5$ cm , then $DF = ?$



(Difficulty: 3) (2018 DSE Paper 2 Q20)

Solution 2. Let $AB = AD = x$. Note that $FB = \sqrt{5^2 - 4^2} = 3$ (pyth. theorem).

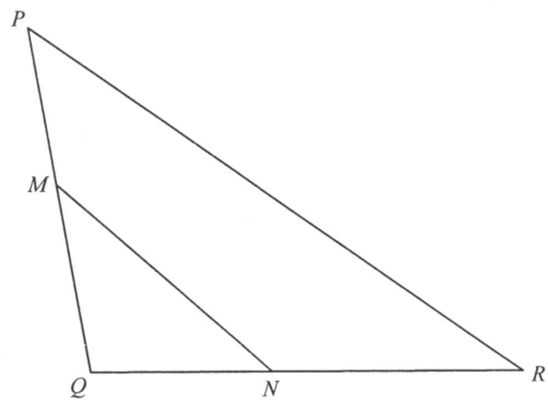


Note that $\triangle EBF \sim \triangle EAD$ (AA). So

$$\begin{aligned} \frac{FB}{AD} &= \frac{EB}{EA} \qquad \text{(corr. sides, } \sim \triangle\text{s)} \\ \frac{3}{x} &= \frac{4}{4+x} \\ 12 + 3x &= 4x \\ x &= 12 \end{aligned}$$

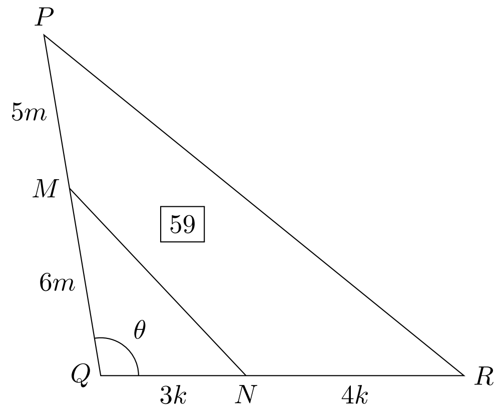
Thus, $DE = \sqrt{(4+12)^2 + 12^2} = 20$ (pyth. theorem), and $DF = 20 - 5 = \boxed{15 \text{ cm}}$.

Problem 3. In the figure, M and N are points lying on PQ and QR respectively such that $PM : MQ = 5 : 6$ and $QN : NR = 3 : 4$. If the area of the quadrilateral $MNRP$ is 59 cm^2 , then what is the area of $\triangle MNQ$?



(Difficulty: 4) (2022 DSE Paper 2 Q17)

Solution 3. Let $PM = 5m$, $MQ = 6m$ and $QN = 3k$, $NR = 4k$.



By sine formula of area of $\triangle$, note that area of $\triangle MQN = \frac{1}{2}(6m)(3k)\sin\theta = 9mk\sin\theta$.

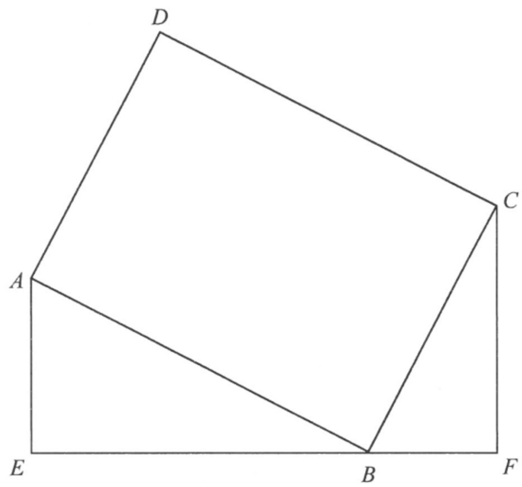
and area of $\triangle PQR = \frac{1}{2}(11m)(7k)\sin\theta = 38.5mk\sin\theta$.

We have

$$\begin{aligned} \text{area of } \triangle PQR - \text{area of } \triangle MQN &= 38.5mk\sin\theta - 9mk\sin\theta \\ 59 &= 29.5mk\sin\theta \\ mk\sin\theta &= 2 \end{aligned}$$

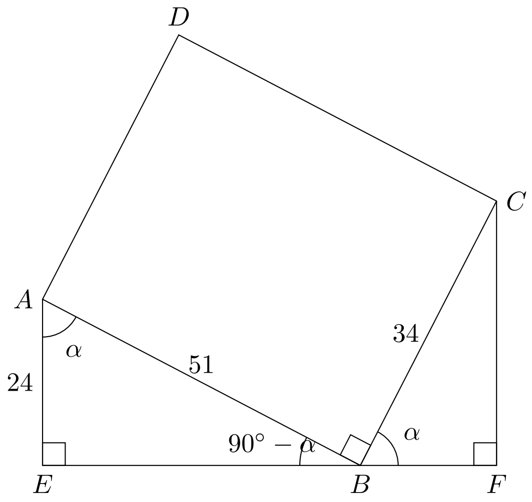
So area of $\triangle PQR = 9(2) = \boxed{18\,\mathrm{cm}^2}$.

Problem 4. In the figure, the perimeter of the rectangle $ABCD$ is 170 cm . It is given that EBF is a straight line and $\angle AEB = \angle BFC = 90°$. If $AE = 24$ cm and $BC = 34$ cm , then $EF =$?



(Difficulty: 3) (2022 DSE Paper 2 Q18)

Solution 4. Note that $\angle ABC = 90°$ (definition of rectangle).



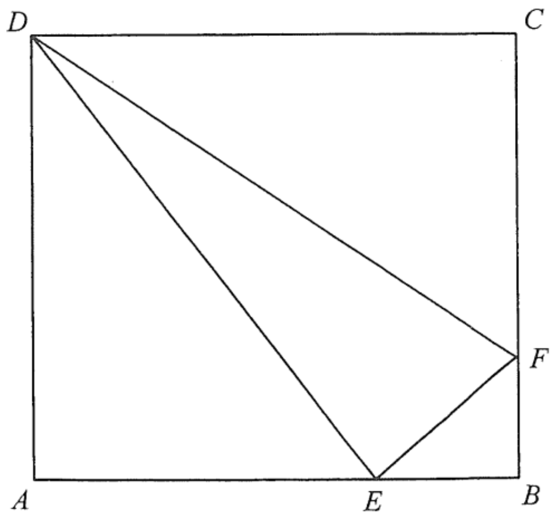
Note that $AB = 170/2 - 34 = 51$ (perimeter of rectangle).

Let $\angle EAB = \alpha$. Note that $\angle ABE = 90^\circ - \alpha$ ($\angle$ sum of $\triangle$) and $\angle CBE = 180^\circ - 90^\circ - (90^\circ - \alpha) = \alpha$. Note that also, $\angle AEB = \angle BFC = 90^\circ$. Thus $\triangle AEB \sim \triangle BFC$ (AA). So

$$\begin{aligned} \frac{BF}{AE} &= \frac{BC}{AB} \qquad \text{(corr. sides, } \sim \triangle\text{s)} \\ \frac{BF}{24} &= \frac{34}{51} \\ BF &= 16 \end{aligned}$$

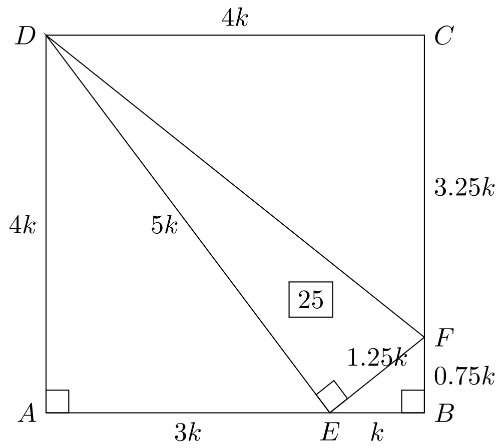
Also, $EB = \sqrt{51^2 - 24^2} = 45$ (pyth. theorem), so $EF = 45 + 16 = \boxed{61 \text{ cm}}$.

Problem 5. In the figure, $ABCD$ is a square. Let E and F be points lying on AB and BC respectively such that $AE = 3BE$ and $\angle DEF = 90^\circ$. If the area of $\triangle DEF$ is 25 cm^2 , then what is the area of $\triangle CDF$?



(Difficulty: 4) (2021 DSE Paper 2 Q20)

Solution 5. Let $AE = 3k$ and $BE = k$. Then the side length of the square is $4k$.



Note that $\triangle DAE \sim \triangle EBF$ (AA), so

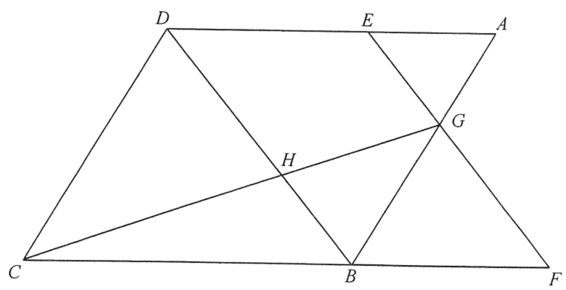
$$\begin{aligned} \frac{FB}{EB} &= \frac{AE}{AD} \qquad \text{(corr. sides, } \sim \triangle\text{s)} \\ \frac{FB}{k} &= \frac{3k}{4k} \\ FB &= 0.75k \end{aligned}$$

Then $CF = 4k - 0.75k = 3.25k$.
By pyth. theorem, $DE = \sqrt{(3k)^2 + (4k)^2} = 5k$ and $EF = \sqrt{k^2 + (0.75k)^2} = 1.25k$.
Consider the area of $\triangle DEF$:

$$\begin{aligned} \frac{1}{2}(5k)(1.25k) &= 25 \qquad \text{(area of } \triangle\text{)} \\ k^2 &= 8 \end{aligned}$$

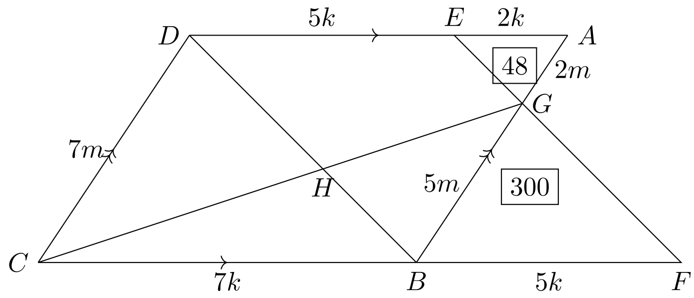
$$\begin{aligned} \text{area of } \triangle CDF &= \frac{1}{2}(4k)(3.25k) \qquad \text{(area of } \triangle\text{)} \\ &= 6.5(8) \\ &= \boxed{52 \text{ cm}^2} \end{aligned}$$

Problem 6. In the figure, $ABCD$ is a parallelogram. Let E be a point lying on AD such that $AE : ED = 2 : 5$. CB is produced to the point F such that $BF = DE$. Denote the point of intersection of AB and EF by G . It is given that BG and CG intersect at point H . If the area of $\triangle AEG$ is 48 cm^2 , then what is the area of $\triangle CDH$?



(Difficulty: 4) (2020 DSE Paper 2 Q18)

Solution 6. Let $BF = DE = 5k$ and $AE = 2k$. Then $CB = 7k$ (opp. $\angle$s of //gram).



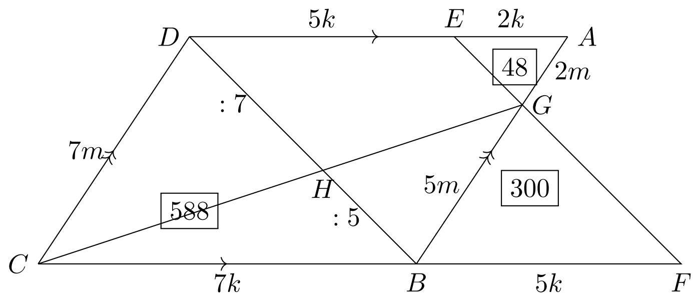
Note that $\angle EAG = \angle FBG$ (alt. $\angle$s , $EA//BF$), $\angle AGE = \angle FGB$ (vert. opp. $\angle$s). So $\triangle AGE \sim \triangle BGE$ (AA).

So area of $\triangle BGF = 48\,(\frac{5}{2})^2 = 300$ (corr. areas, $\sim \triangle$s).

Also, $\frac{AG}{GB} = \frac{EA}{BF} = \frac{2}{5}$ (corr. sides, $\sim \triangle$s). Let $AG = 2m$ and $GB = 5m$. Then $DC = AB = 7m$ (opp. sides of //gram).

In $\triangle DCB$ and $\triangle GBF$, we have $\angle DCB = \angle GBF$ (corr. $\angle$s , $DC//AB$) , $\frac{DC}{GB} = \frac{CB}{BF} = \frac{7}{5}$. so $\triangle DCB \sim \triangle GBF$ (ratio of 2 sides, inc. $\angle$)

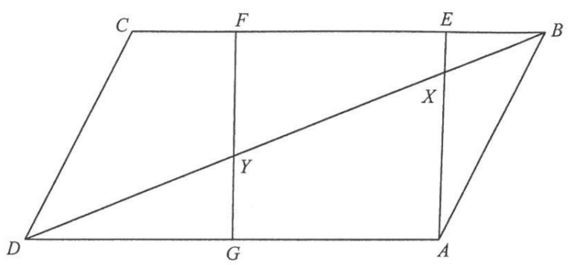
So area of $\triangle DCB = 300\,(\frac{7}{5})^2 = 588$ (corr. areas, $\sim \triangle$s).



Note that $\triangle DCH \sim \triangle GBH$ (AA). So $DH : HB = DC : GB = 7 : 5$.

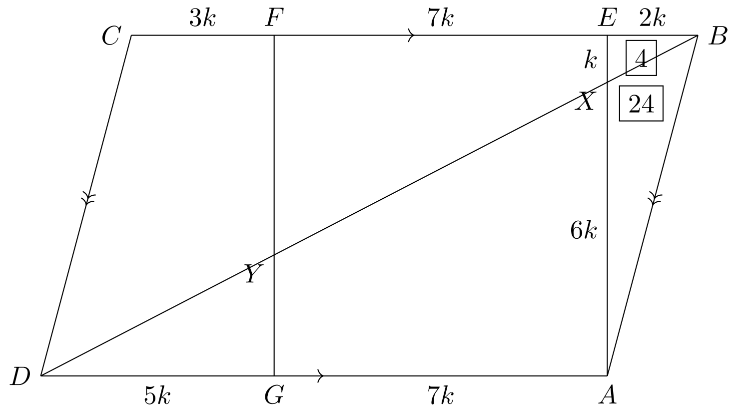
By 'bases prop. to areas of $\triangle$s' , we have area of $\triangle CDH$ = area of $\triangle DCB \cdot (\frac{7}{7+5}) = 588(\frac{7}{12}) = \boxed{343 \text{ cm}^2}$.

Problem 7. In the figure, $ABCD$ is a parallelogram and $AEFG$ is a square. It is given that $BE : EF : FC = 2 : 7 : 3$. BD cuts AE and FG at the points X and Y respectively. If the area of $\triangle ABX$ is 24 cm^2 , then what is the area of the quadrilateral $CDYF$?



(Difficulty: 4) (2019 DSE Paper 2 Q16)

Solution 7. Let $BF = 2k$, $EF = 7k$, $FC = 3k$. Then $DA = CB = 12k$ (opp. sides of //gram).



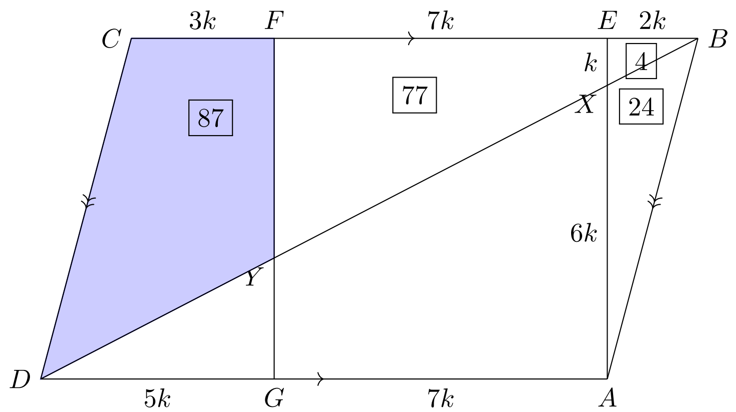
Note that $EA = GA = FE = 7k$ since $AEFG$ is a square.

Note that $\triangle BEX \sim DAX$ (AA) . So $EX : AX = EB : DA = 2 : 12 = 1 : 6$. This means $EX = k$ and $AX = 6k$.

By 'bases prop. to areas of $\triangle$s' , area of $\triangle BEX = 24(\frac{1}{6}) = 4$, and area of $\triangle AEB = 24 + 4 = 28$.

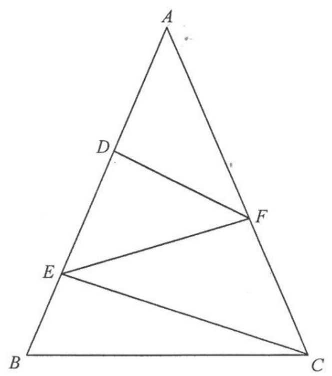
Note that $\triangle AEB$ and $\triangle DCB$ have the same height, so 'bases prop. to areas of $\triangle$s' applies to them. We have area of $\triangle DCB = 28(\frac{12}{2}) = 168$.

Note that $\triangle BFY \sim \triangle BEX$ (AA), so area of $\triangle BFY =$ area of $\triangle BEX \cdot (\frac{BF}{BE})^2 = 4(\frac{9}{2})^2 = 81$.



So area of $CDYF =$ area of $\triangle DCB -$ area of $\triangle BFY = 168 - 81 = \boxed{87 \text{ cm}^2}$.

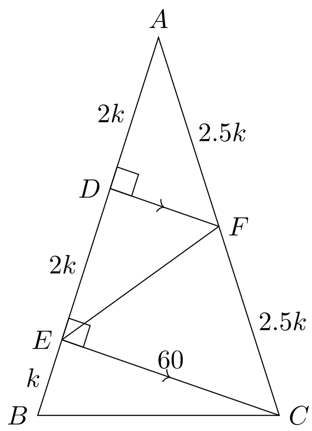
Problem 8. In the figure, ABC is an isosceles triangle with $AB = AC$, D and E are points lying on AB such that $AD = DE = 2EB$ while F is a point lying on AC such that $DF // EC$. If $\angle ADF = 90°$ and $CE = 60$ cm , then $EF =$?



(Difficulty: 4) (2019 DSE Paper 2 Q18)

Solution 8. Let $AD = DE = 2k$, $EB = k$. Then $AC = 5k$.

Note that $\angle AEC = 90°$ (alt. $\angle$s , $DF//EC$).

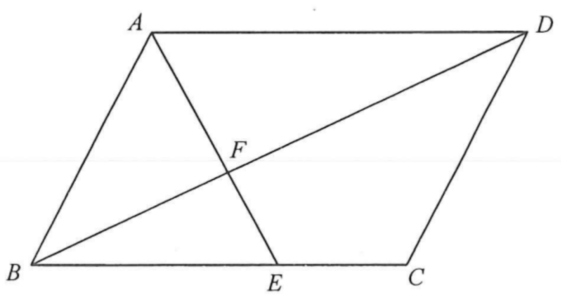


Since $AD = DE$ and $DF//EC$, we have $AF = FC$ (intercept theorem). So $AF = FC = 2.5k$.

In $\triangle AEC$, by pyth. theorem, $(4k)^2 + 60^2 = (5k)^2$, so $k = 20$.

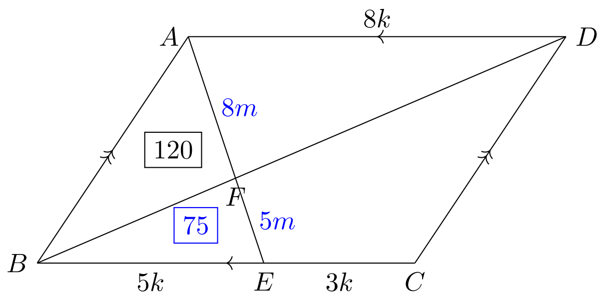
Note that AC is the diameter of the circumcircle of $\triangle AEC$ (converse of pyth. theorem), so F is the centre since it is the mid-point of AC . This means $EF = AF = FC = 2.5k = 2.5(20) = \boxed{50 \text{ cm}}$.

Problem 9. In the figure, $ABCD$ is a parallelogram. E is a point lying on BC such that $BE : EC = 5 : 3$. AE and BD intersect at the point F . If the area of $\triangle ABF$ is 120 cm^2 , then what is the area of the quadrilateral $CDFE$?



(Difficulty: 4) (2018 DSE Paper 2 Q16)

Solution 9. Let $BE = 5k$, $EC = 3k$. Then $AD = 8k$ (opp. $\angle$s , $AD//BC$) .



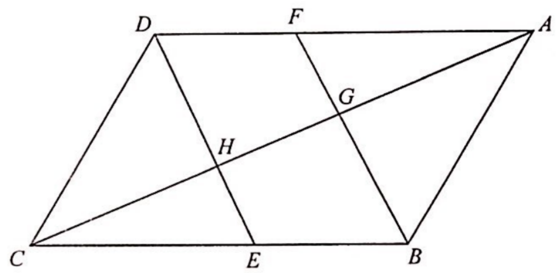
Note that $\triangle FBE \sim \triangle FDA$ (AA), so we can let $AF = 8m$ and $FE = 5m$.

Then area of $\triangle BFE = 120(\frac{5}{8}) = 75$ (bases prop. to areas of $\triangle$s) , which means area of $\triangle ABE = 120 + 75 = 195$.

By 'bases prop. to areas of $\triangle$s' , area of $\triangle DBC = 195(\frac{BC}{BE}) = 195(\frac{8}{5}) = 312$.

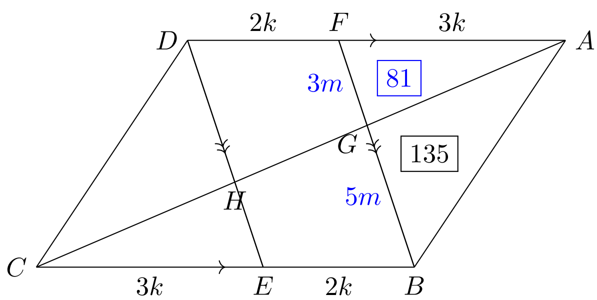
So area of $CDFE = 312 - 75 = \boxed{237 \text{ cm}^2}$.

Problem 10. In the figure, $ABCD$ and $BEDF$ are parallelograms. E is a point lying on BC such that $BE : EC = 2 : 3$. AC cuts BF and DE at G and H respectively. If the area of $\triangle ABG$ is 135 cm^2 , then what is the area of the quadrilateral $DFGH$?



(Difficulty: 4) (2017 DSE Paper 2 Q16)

Solution 10. Let $BE = 2k$, $EC = 3k$. Then $DF = BE = 2k$ and $DA = CB = 5k$ (opp. sides of //gram) , so $FA = 3k$.

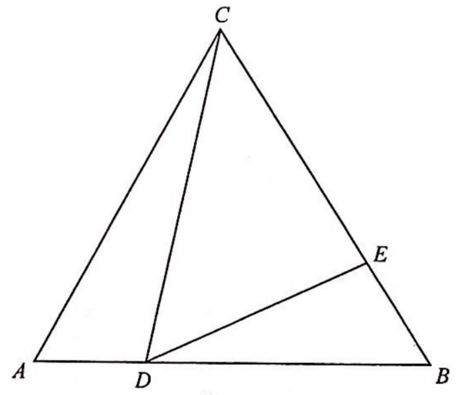


Note that $\triangle AGF \sim \triangle CGB$ (AA) . Let $FG = 3m$ and $GB = 5m$.

Then area of $\triangle AFG = 135(\frac{3}{5}) = 81$ (bases prop. to areas of $\triangle$s) .

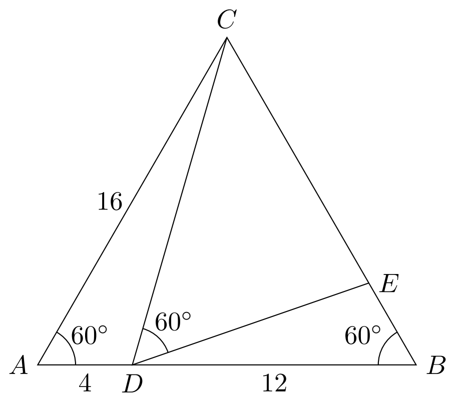
Note that $\triangle AFG \sim \triangle ADH$ (AA). So area of $\triangle ADH = 81(\frac{5}{3})^2 = 225$ (corr. areas, $\sim \triangle$s), so area of $DFGH = 225 - 81 = \boxed{144 \text{ cm}^2}$.

Problem 11. In the figure, ABC is an equilateral triangle of side 16 cm . D and E are points lying on AB and BC respectively such that $AD = 4$ cm and $\angle CDE = 60°$. Find CE .



(Difficulty: 4) (2017 DSE Paper 2 Q17)

Solution 11. Note that $DB = 16 - 4 = 12$.



Note that $\triangle CDB = \angle ACD + 60°$ (ext. $\angle$ of $\triangle$), and $\angle CDB = \angle BDE + 60°$ (angle addition postulate). So $\angle ACD = \angle BDE$.

In $\triangle CAD$ and $\triangle DBE$,

$$\begin{aligned} &\angle ACD = \angle BDE \qquad \text{(shown above)} \\ &\angle CAD = \angle DBE = 60° \qquad \text{(prop. of equi. } \triangle) \\ \therefore\ &\triangle CAD \sim \triangle DBE \qquad \text{(AA)} \\ \therefore\ &\frac{EB}{DB} = \frac{AD}{CA} \qquad \text{(corr. sides, } \sim \triangle\text{s)} \\ \therefore\ &\frac{EB}{12} = \frac{4}{16} \\ &EB = 3 \\ &CE = 16 - 3 = \boxed{13 \text{ cm}} \end{aligned}$$

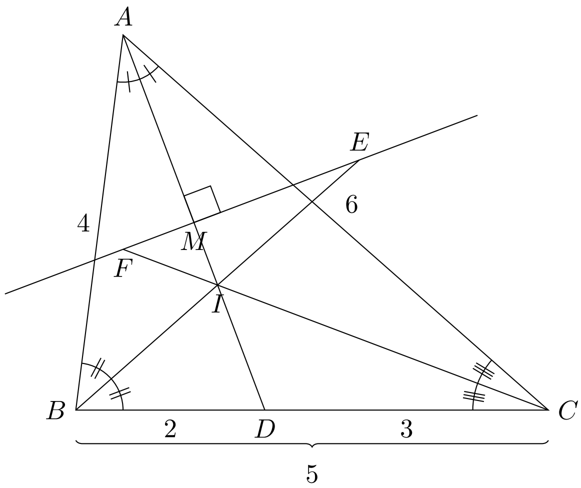
1.10 Four centres of triangle

Problem 12. Point D lies on side BC of $\triangle ABC$ so that AD bisects $\angle BAC$. The perpendicular bisector of AD intersects the bisectors of $\triangle ABC$ and $\triangle ACB$ in points E and F , respectively. Given that $AB = 4$, $BC = 5$, and $CA = 6$, find the area of $\triangle AEF$.

(Note: No figure is provided for this problem.)

(Difficulty: 7 [Insane]) (2020 AIME I Problem 13)

Solution 12. Let M be the mid-point of AD , and I be the incentre of $\triangle ABC$.



Note that I is below M on AD by 'prop. of $\angle$ bisector of $\triangle$' . This means E is at the right of M , and F is at the left of M. (Whether E lies outside the triangle is not relevant, even though it does.)

By angle bisector theorem in $\triangle ABC$, we have $BD : DC = AB : AC = 4 : 6$.

So $BD = 5 \cdot (\dfrac{4}{4+6}) = 2$, and $DC = 5 - 2 = 3$.

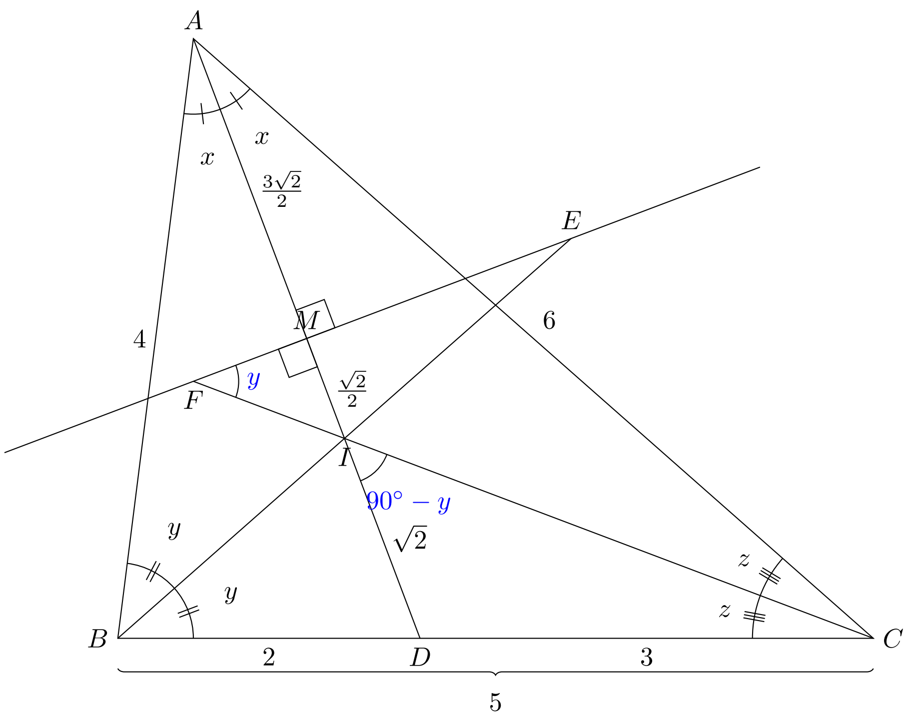
By 'length of $\angle$ bisector of $\triangle$' ($\triangle ABC$) , we have $AD = \sqrt{(4)(6) - (2)(3)} = \sqrt{18} = 3\sqrt{2}$.

Since M is the mid-point of AD , we have $AM = MD = \dfrac{3\sqrt{2}}{2}$.

By angle bisector theorem in $\triangle ABD$, we have $ID : IA = 2 : 4 = 1 : 2$. So $ID = \dfrac{1}{3}AD = \sqrt{2}$.

Thus $IM = MD - ID = \dfrac{3\sqrt{2}}{2} - \sqrt{2} = \dfrac{\sqrt{2}}{2}$.

Let $\angle A = 2x$, $\angle B = 2y$ and $\angle C = 2z$. Then $x + y + z = 90°$ ($\angle$ sum of $\triangle$)

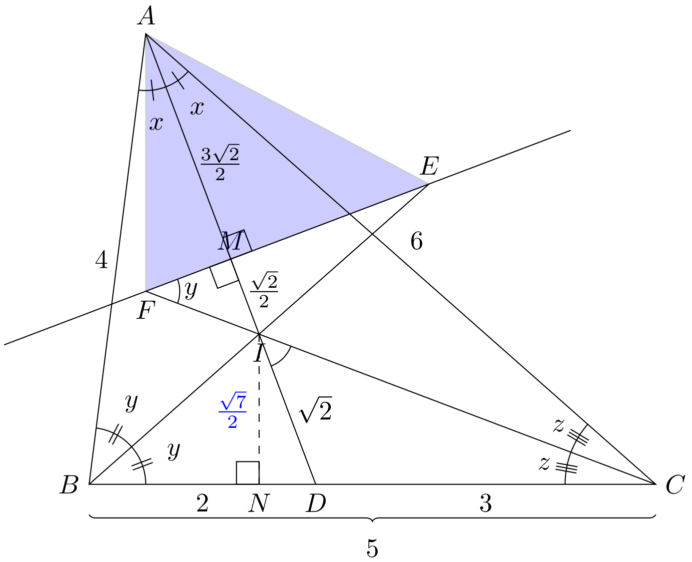


Note that $\angle CID = x + z$ (ext. $\angle$ of $\triangle AIC$) $= 90° - y$ (since $x + y + z = 90°$).

So $\angle FIM = \angle CID = 90° - y$ (vert. opp. $\angle$s), so $\angle EFI = 90° - (90° - y) = y$ ($\angle$ sum of $\triangle$).

In $\triangle IFE$ and $\triangle IBC$, $\angle EFI = \angle CBI = y$, $\angle EIF = \angle CIB$ (vert. opp. $\angle$s). Thus $\triangle IFE \sim \triangle IBC$ (AA).

Draw $IN \perp BC$. Note that IN is the inradius of $\triangle ABC$ (prop. of incentre).



Let $s = \dfrac{4+5+6}{2} = \dfrac{15}{2}$. Then by inradius formula, $IN = \sqrt{\dfrac{(s-4)(s-5)(s-6)}{s}} = \dfrac{\sqrt{7}}{2}$.

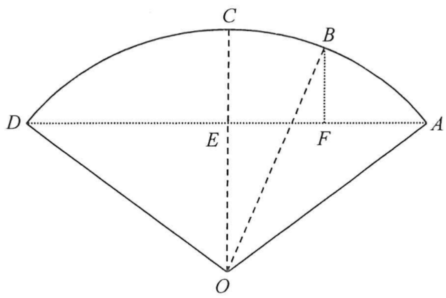
Note that in $\triangle IFE$ and $\triangle IBC$,

$$\frac{FE}{BC} = \frac{MI}{IN} \qquad \text{(heights prop. to sides of} \sim \triangle\text{s)}$$
$$\frac{FE}{5} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{7}}{2}}$$
$$FE = 5 \cdot \sqrt{\frac{2}{7}}$$

Area of $\triangle AEF = \dfrac{1}{2} AM \cdot FE = \dfrac{1}{2}(\dfrac{3\sqrt{2}}{2})(5\sqrt{\dfrac{2}{7}}) = \boxed{\dfrac{15\sqrt{7}}{14}}$.

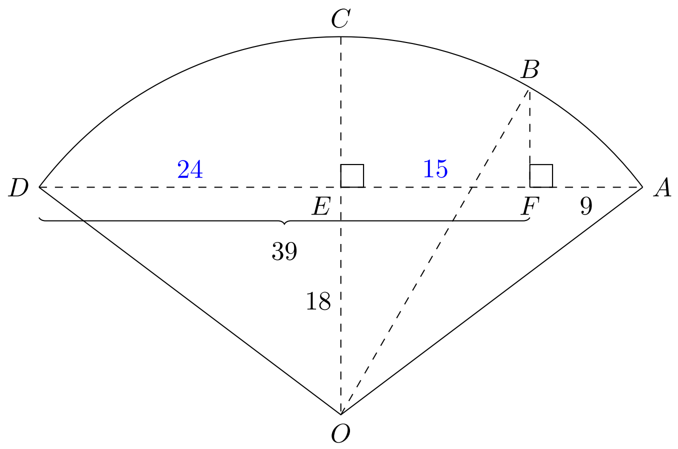
1.11 Area and perimeter of circle

Problem 13. In the figure, O is the centre of the sector $OABCD$. AD and OC are perpendicular to each other and intersect at the point E. F is a point lying on AD such that BF is perpendicular to AD. If $AF = 9$ cm, $DF = 39$ cm and $OE = 18$ cm, then what is the area of the sector OBC?



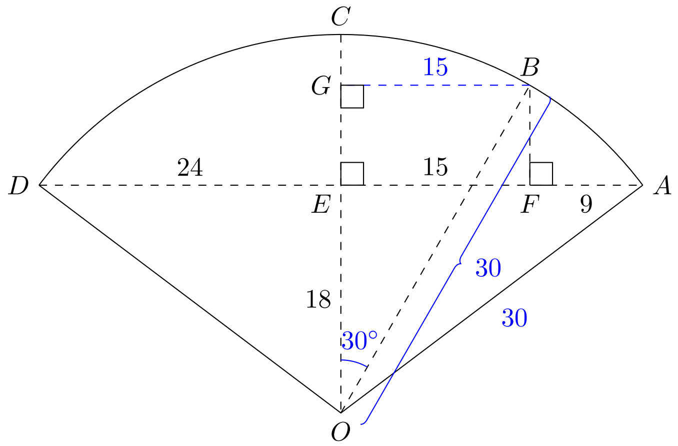
(Difficulty: 4) (2018 DSE Paper 2 Q17)

Solution 13. We have $AD = 39 + 9 = 48$. Note that $DE = EA$ (line from centre $\perp$ chord bisects chord).



This means $DE = EA = 48/2 = 24$, and $EF = 24 - 9 = 15$.
By pyth. theorem in $\triangle OEA$, $OA = \sqrt{18^2 + 24^2} = 30$. So $OB = OA = 30$ (radii).

Draw $BG \perp OC$.



Then $OG = 15$ since $GBFE$ is a rectangle. Note that $\angle BOG = \arcsin(\frac{15}{30}) = 30°$.

So, area of sector $OBC = \pi(30^2)(\frac{30°}{360°}) = \boxed{75\pi \text{ cm}^2}$

References