Toddler Geometry (Problem set)

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Abstract

Geometry problems are harder than they seem.

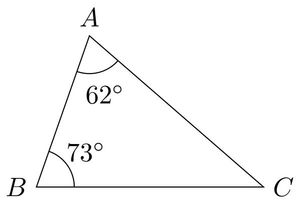
Contents

1 Lines, angles and shapes

After all the preposition stating, let's try some practical problems. (The diagrams in the problems are not necessarily to scale.)

1.1 Basic properties

Problem 1. In $\triangle ABC$, $\angle A = 62°$ and $\angle B = 73°$. What is $\angle C$?

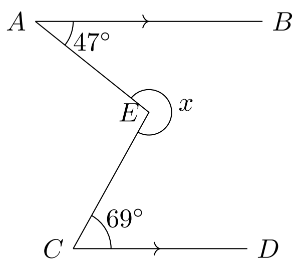


(Difficulty: 1 [Beginner])

Solution 1.

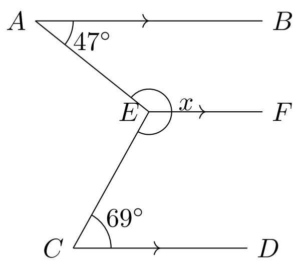
$$\begin{aligned} \angle C &= 180° - \angle A - \angle B \qquad (\angle \text{ sum of } \triangle) \\ &= 180° - 62° - 73° \\ &= \boxed{45°} \end{aligned}$$

Problem 2. In the figure, $AB//CD$, and E is a point between line AB and line CD . $\angle BAE = 47°$ and $\angle DCE = 69°$. What is x ?



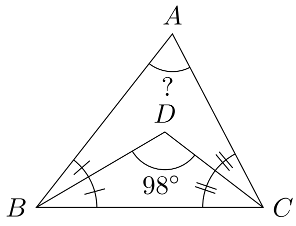
(Difficulty: 3 [Easy])

Solution 2. Draw $EF//AB//CD$.



$$\begin{aligned} \angle AEF + 47° &= 180° \qquad (\text{alt. } \angle\text{s , } AB//EF) \\ \angle AEF &= 133° \\ \angle CEF + 69° &= 180° \qquad (\text{alt. } \angle\text{s , } EF//CD) \\ \angle CEF &= 111° \\ x &= \angle AEF + \angle CEF \\ &= 133° + 111° \\ &= \boxed{244°} \end{aligned}$$

Problem 3. D is a point inside $\triangle ABC$ such that $\angle ABD = \angle DBC$ and $\angle ACD = \angle DCB$, $\angle BDC = 98°$. What is $\angle BAC$?



(Difficulty: 3)

Solution 3. Let $\angle ABD = \angle DBC = x$ and $\angle ACD = \angle DCB = y$. In $\triangle DBC$,

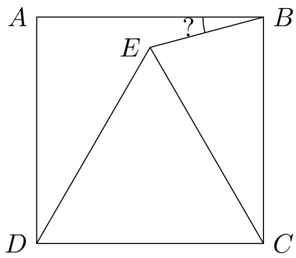
$$\begin{aligned} x + y + 98° &= 180° \qquad (\angle \text{ sum of } \triangle) \\ x + y &= 82° \end{aligned}$$

In $\triangle ABC$,

$$\begin{aligned} \angle BAC + 2x + 2y &= 180° \qquad (\angle \text{ sum of } \triangle) \\ \angle BAC &= 180° - 2(x + y) \\ &= 180° - 2(82°) \\ &= \boxed{16°} \end{aligned}$$

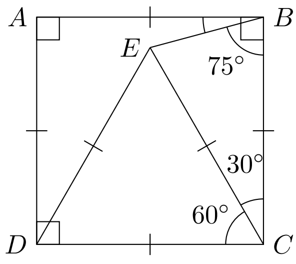
1.3 Triangle properties

Problem 4. $ABCD$ is a square. E is a point inside $ABCD$ such that $\triangle ECD$ is an equilateral triangle. Join BE. What is $\angle ABE$?



(Difficulty: 3 [Easy])

Solution 4. .

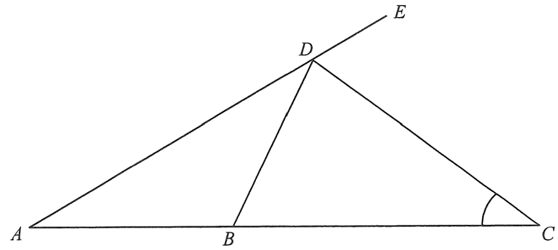


$$\begin{aligned} &\angle DCB = \angle CBA = 90° \qquad (ABCD \text{ is square.}) \\ &\angle ECD = 60° \qquad (\text{prop. of equil } \triangle) \\ &\angle ECB = 90° - 60° = 30° \end{aligned}$$

Note that $EC = BC$.

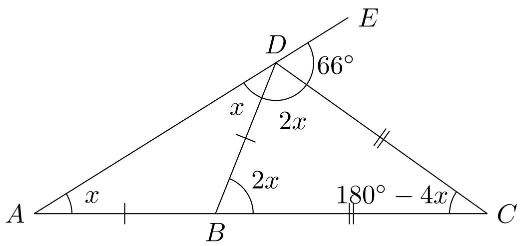
$$\begin{aligned} \therefore\ &\angle CBE = \angle CEB \qquad (\text{base } \angle\text{s, isos. } \triangle) \\ &\angle CBE = (180° - 30°)/2 = 75° \qquad (\angle \text{ sum of } \triangle) \\ &\angle ABE = 90° - 75° = \boxed{15°} \end{aligned}$$

Problem 5. In the figure, ABC and ADC are straight lines. It is given that $AB = BD$ and $BC = CD$. If $\angle CDE = 66°$, then $\angle ACD =$?



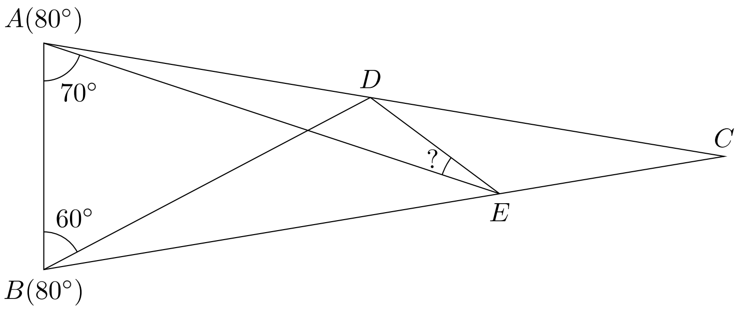
(Difficulty: 3) (2019 DSE Paper 2 Q17)

Solution 5. Let $\angle BAD = x$.



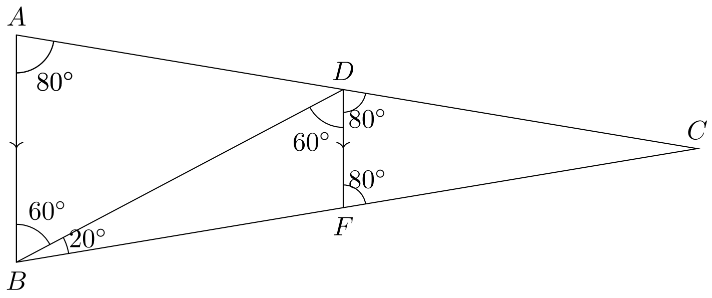
$$\begin{aligned}
\angle BAD &= \angle BDA = x \qquad \text{(base } \angle\text{s, isos. } \triangle) \\
\angle CBD &= 2x \qquad \text{(ext. } \angle \text{ of } \triangle) \\
\angle CDB &= \angle CBD = 2x \qquad \text{(base } \angle\text{s, isos. } \triangle) \\
\angle BCD &= 180° - 2x - 2x = 180° - 4x \qquad (\angle \text{ sum of } \triangle) \\
\angle DAC + \angle ACD &= x + (180° - 4x) = 66° \qquad \text{(ext. } \angle \text{ of } \triangle) \\
x &= 38° \\
\angle ACD &= 180° - 4(38°) = \boxed{28°}
\end{aligned}$$

Problem 6. [1] In $\triangle ABC$, $\angle BAC = \angle ABC = 80°$. Let D be a point on side AC such that $\angle ABD = 60°$. Let E be a point on side BC such that $\angle BAE = 70°$. Join DE. What is $\angle AED$?

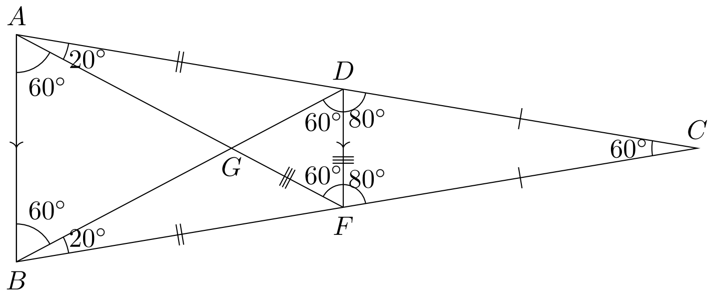


(Difficulty: 6 [Very hard])

Solution 6. Let F be a point on side BC such that $AB//DF$. Hide point E to make the figure tidier. Note that $\angle DBC = 80° - 60° = 20°$.



$$
\begin{aligned}
&\angle CDF = \angle CAB = 80^\circ \qquad (\text{corr. } \angle\text{s}, DF//AB)\\
&\angle CFD = \angle CBA = 80^\circ \qquad (\text{corr. } \angle\text{s}, DF//AB)\\
&\angle BDF = 80^\circ - 20^\circ = 60^\circ \qquad (\text{ext. } \angle \text{ of } \triangle)
\end{aligned}
$$

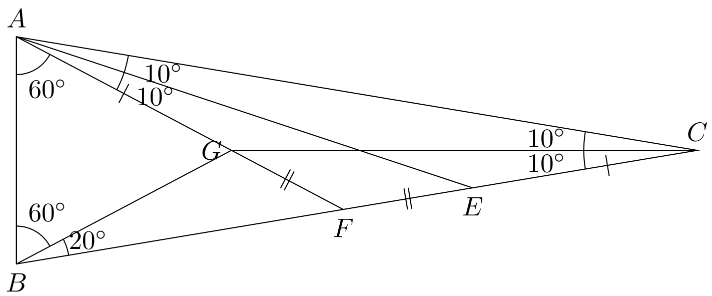


Note that $CD = CF$ and $CA = CB$ (sides opp. equal $\angle$s). Thus $AD = BF$.

Join AF, and let AF and BD intersect at G. In $\triangle ADF$ and $\triangle BFD$, $AD = BF$, $\angle ADF = \angle BFD = 110^\circ$ (adj. $\angle$s on st. line), $DF = DF$. Thus $\triangle ADF \cong \triangle BFD$ (SAS). Thus $\angle DAF = \angle FBD = 20^\circ$ (corr. $\angle$s, $\cong \triangle$s). Also, $\angle AFD = \angle BDF = 60^\circ$ (corr. $\angle$s, $\cong \triangle$s). Thus $\triangle GDF$ is an equilateral triangle (con. of equil. $\triangle$), which means $GF = DF$.

Note that $\angle ACF = 180^\circ - 80^\circ - 80^\circ = 20^\circ$ ($\angle$ sum of $\triangle$). Since $\angle CAF = \angle ACF = 20^\circ$, we have $AF = FC$ (base $\angle$s, isos. $\triangle$).

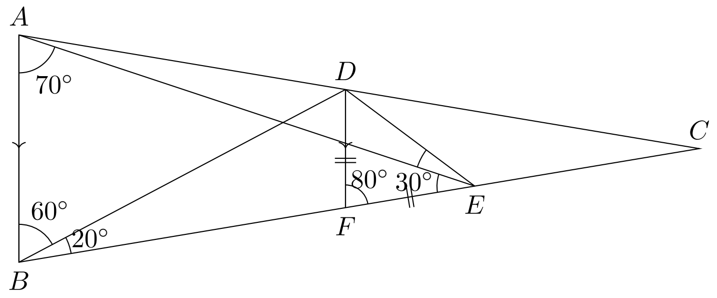
Show point E again and hide GD and DF. Join CG.



Note that $\angle CAE = \angle EAF = 10^\circ$. Also note that GC bisects ACB (because G is in the middle), so $\angle ACG = \angle GCF = 10^\circ$.

Note that $\triangle GAC \cong \triangle ECA$ (ASA), so $AG = EC$ (corr. sides, $\cong \triangle$s). Since $AF = FC$, we have $GF = FE$.

Show D again and hide AF.

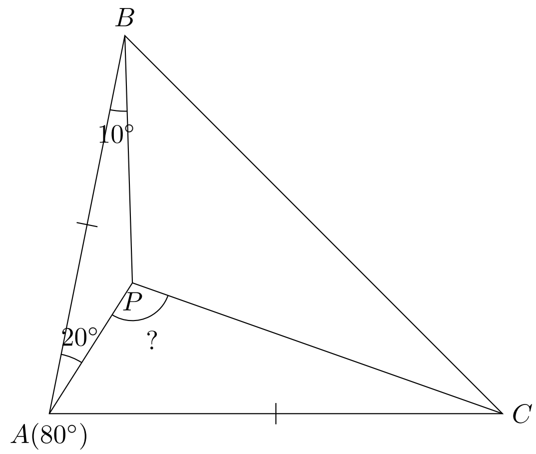


We have shown that $GF = DF$ and $GF = FE$. Thus $DF = FE$. In $\triangle FDE$, $\triangle FDE = \triangle FED$ (base $\angle$s, isos. $\triangle$). So $\angle FED = (180° - 80°)/2 = 50°$ ($\angle$ sum of $\triangle$).

Note that $\angle AEB = 180° - 80° - 70° = 30°$ ($\angle$ sum of $\triangle$).

So $\angle AED = \angle FED - \angle AEB = 50° - 30° = \boxed{20°}$.

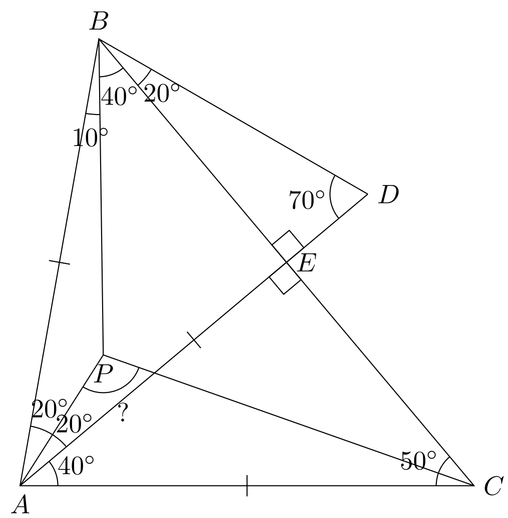
Problem 7. [2] In $\triangle ABC$, $AB = AC$ and $\angle BAC = 80°$. Let P be a point inside $\triangle ABC$ such that $\angle BAP = 20°$ and $\angle ABP = 10°$. What is $\angle APC$?



(Difficulty: 6)

Solution 7. Since $AB = AC$, we have $\angle ABC = \angle ACB$ (base $\angle$s, isos. $\triangle$), so $\angle ABC = \angle ACB = (180° - 80°)/2 = 50°$. So $\angle PBC = 50° - 10° = 40°$.

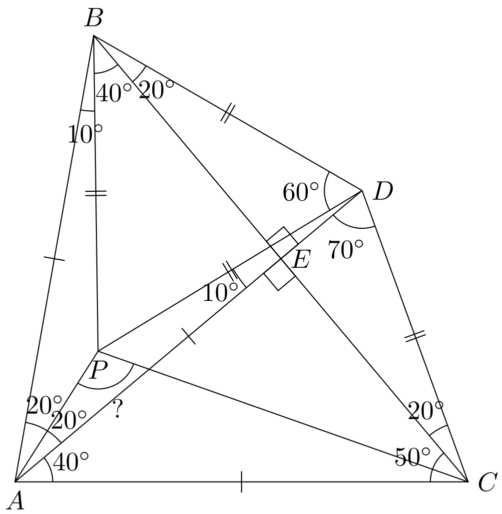
Draw AD between $\angle BAC$ such that $AD = AB$ and $\angle DAC = 40°$. Note that $\angle PAD = 80° - 20° - 40° = 20°$.



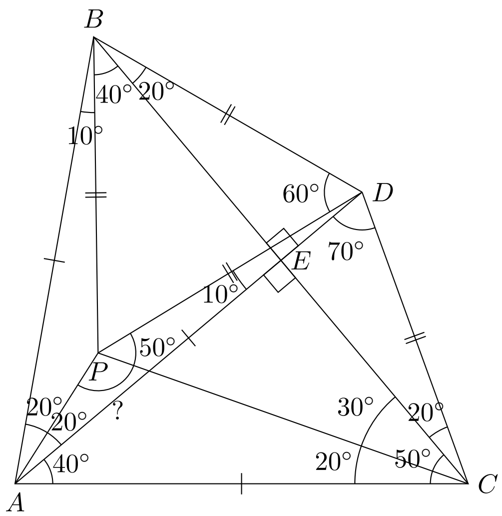
Mark E as the intersection of AD and BC. In $\triangle AEC$, $\angle AEC = 180° - 40° - 50° = 90°$ ($\angle$ sum of $\triangle$).

Join BD . Since $AB = AD$, we have $\angle ABD = \angle ADB = (180° - 40°)/2 = 70°$ (base $\angle$s, isos. $\triangle$)& ($\angle$ sum of $\triangle$). Note that $\angle BED = 90°$ (vert. opp. $\angle$s), so $\angle DBE = 180° - 70° - 90° = 20°$ ($\angle$ sum of $\triangle$).

Join DC and PD. Note that $\triangle DAB \cong \triangle DAC$ (SAS), so $BD = DC$ and $\angle ADC = \angle ADB = 70°$. Since $BD = DC$, we have $\angle DCB = \angle DBC = 20°$ (base $\angle$s, isos. $\triangle$).



Note that $\triangle BAP \cong \triangle DAP$ (SAS), so $\angle PDA = \angle PBA = 10°$ (corr. $\angle$s, $\cong$ $\triangle$s). Thus $\angle PDB = 70° - 10° = 60°$. Note that in $\triangle BPD$, $\angle PBD = \angle PDB = 60°$. Thus $\triangle BPD$ is an equil. $\triangle$ (con. of equil. $\triangle$), so $BP = DP = BD$. Since $BD = DC$, we have $DP = DC$.



Since $\triangle DPC$ is an isos. $\triangle$ with $DP = DC$, we have $\angle DPC = \angle DCP = (180° - 80°)/2 = 50°$ (base $\angle$s, isos. $\triangle$)& ($\angle$ sum of $\triangle$). Thus $\angle ECP = 50° - 20° = 30°$. So $\angle PCA = 50° - 30° = 20°$.

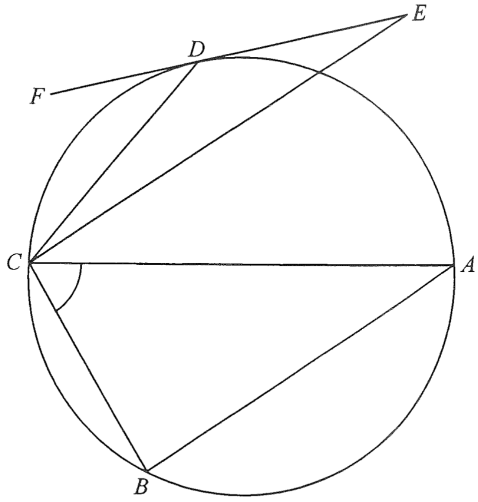
Finally, in $\triangle APC$, $\angle APC = 180° - (20° + 40°) - 20° = \boxed{100°}$.

1.6 Circle properties

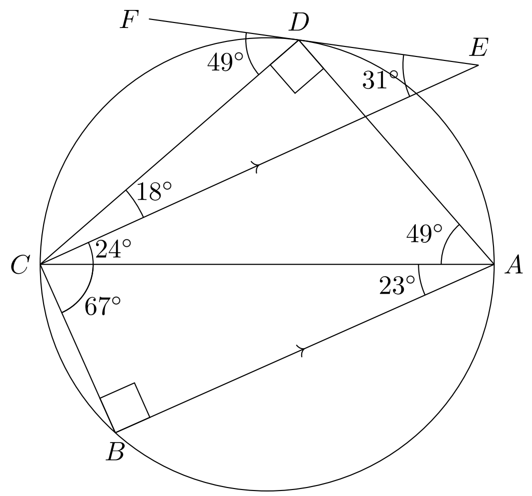
(Problem solving tips: try to use all the information given in the problem.)

Problem 8. In the figure, AC is a diameter of the circle $ABCD$. EF is the tangent to the circle at D such that $AB//EC$. If $\angle CDF = 49°$ and $\angle CED = 31°$, then $\angle ACB =$?

(Difficulty: 4 [Medium]) (2021 DSE Paper 2 Q39)

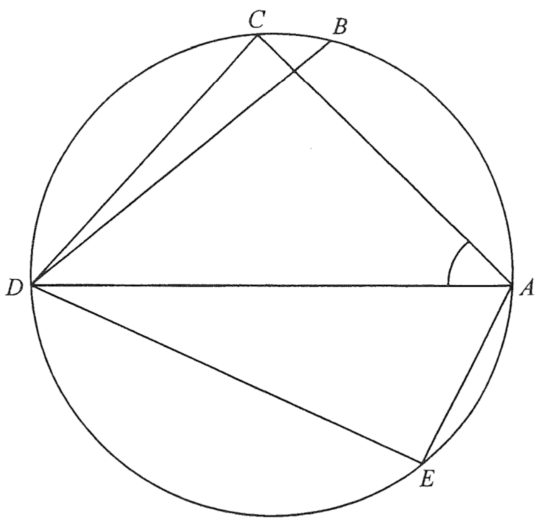


Solution 8. (Diagram adjusted for accuracy.) Join DA .



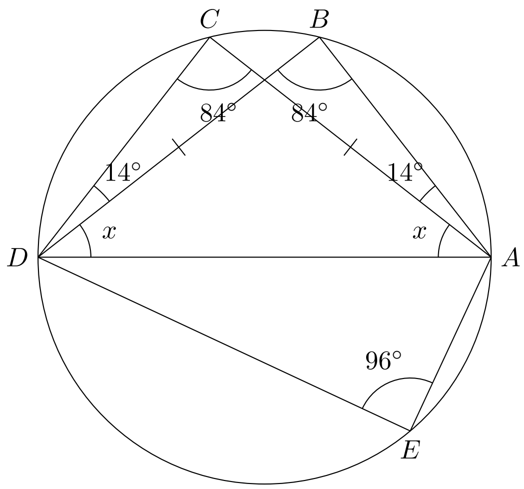
$$\begin{aligned}
\angle CDA, \angle ABC &= 90^\circ && (\angle \text{ in semi-circle}) \\
\angle CAD &= 49^\circ && (\angle \text{ in alt. segment}) \\
\angle DCA &= 90^\circ - 49^\circ = 41^\circ && (\angle \text{ sum of } \triangle) \\
\angle DCE &= 49^\circ - 31^\circ = 18^\circ && (\text{ext. } \angle \text{ of } \triangle) \\
\angle ACE &= 41^\circ - 18^\circ = 23^\circ \\
\angle BAC &= \angle ACE = 23^\circ && (\text{alt. } \angle\text{s , } AB//EC) \\
\angle ACB &= 90^\circ - 23 = \boxed{67^\circ} && (\angle \text{ sum of } \triangle)
\end{aligned}$$

Problem 9. In the figure, $ABCDE$ is a circle. If $AC = BD$, $\angle AED = 96^\circ$ and $\angle BDC = 14^\circ$, then $\angle CAD =$?



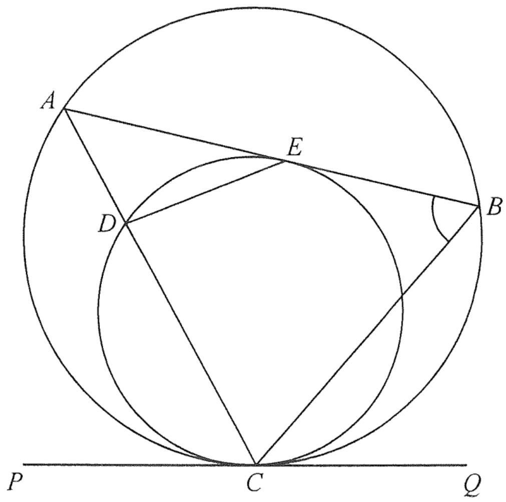
(Difficulty: 4) (2021 DSE Paper 2 Q22)

Solution 9. Join AB . Let $\angle CAD = x$.



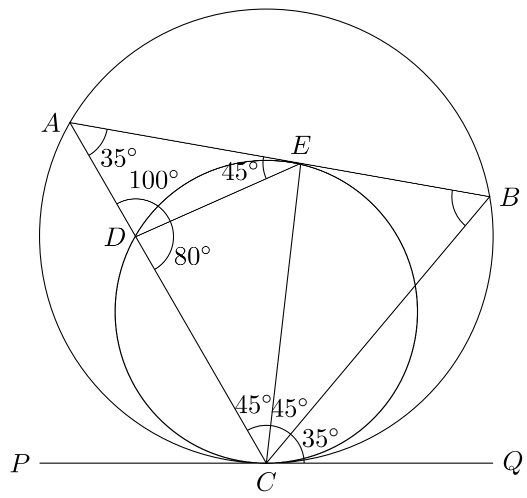
$$\begin{aligned}
\angle DCA, \angle DBA &= 180 - 96^\circ = 84^\circ \qquad \text{(opp. } \angle\text{s , cyclic quad.)} \\
\angle BAC &= 14^\circ \qquad (\angle\text{s in the same segment)} \\
\angle CDA = \angle BAD &= x + 14^\circ \qquad \text{(equal chords, equal } \angle\text{s at } \odot^{ce}) \\
\angle BDA &= x \\
84^\circ + 14^\circ + 2x &= 180^\circ \qquad \text{(ext. } \angle \text{ of } \triangle)\&\ (\angle \text{ sum of } \triangle) \\
x &= \boxed{41^\circ}
\end{aligned}$$

Problem 10. In the figure, ABC and CDE are circles such that ADC is a straight line. PQ is the common tangent to the two circles at C . AB is the tangent to the circle CDE at E . If $\angle ADE = 100^\circ$ and $\angle BCQ = 35^\circ$, then $\angle ABC =$?



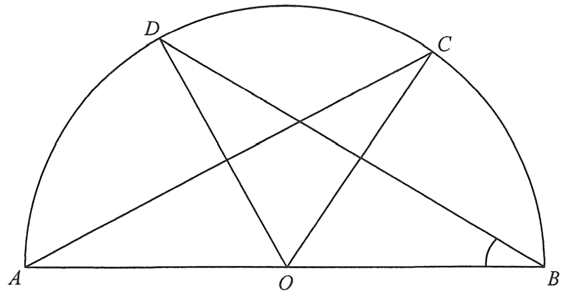
(Difficulty: 4) (2020 DSE Paper 2 Q39)

Solution 10. Join EC .



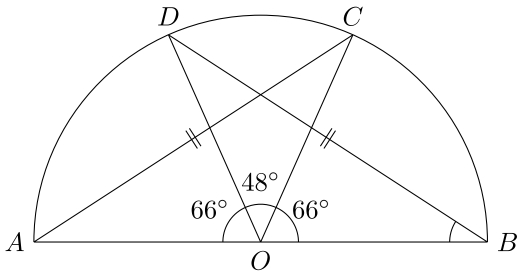
$$\angle CAB = 35^\circ \qquad (\angle \text{ in alt. segment})$$
$$\angle AED = 180^\circ - 35^\circ - 100^\circ = 45^\circ \qquad (\angle \text{ sum of } \triangle)$$
$$\angle DCE = 45^\circ \qquad (\angle \text{ in alt. segment})$$
$$\angle EDC = 180^\circ - 100^\circ = 80^\circ \qquad (\text{adj. } \angle\text{s on st. line})$$
$$\angle ECQ = \angle EDC = 80^\circ \qquad (\angle \text{ in alt. segment})$$
$$\angle ECB = 80^\circ - 35^\circ = 45^\circ$$
$$\angle ABC = 180^\circ - 35^\circ - (45^\circ + 45^\circ) = \boxed{55^\circ} \qquad (\angle \text{ sum of } \triangle)$$

Problem 11. In the figure, O is the centre of the semi-circle $ABCD$. If $AC = BD$ and $\angle COD = 48^\circ$, then $\angle ABD =$?



(Difficulty: 3) (2019 DSE Paper 2 Q21)

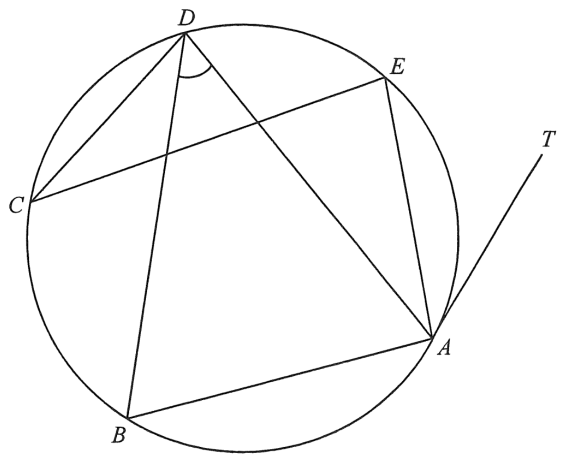
Solution 11. .



Note that $\triangle OAC \cong \triangle OBD$ (SSS) . This means $\angle AOC = \angle DOB$ (corr. sides, $\cong \triangle$s), and thus $\angle AOD = \angle BOC = (180° - 48°)/2 = 66°$ (adj. $\angle$s on st. line). In $\triangle OBD$,

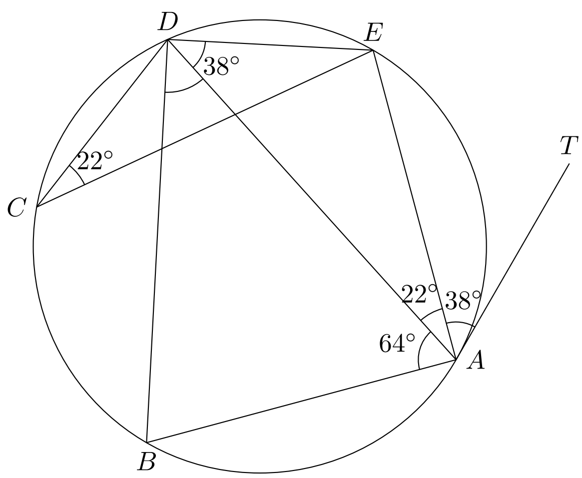
$$\angle ABD = (180° - 48° - 66°)/2 = \boxed{33°} \qquad (\angle \text{ sum of } \triangle)$$

Problem 12. In the figure, TA is the tangent to the circle $ABCDE$ at point A . If $\angle BAD = 64°$, $\angle EAT = 38°$ and $\angle DCE = 22°$, then $\angle ADB =$?



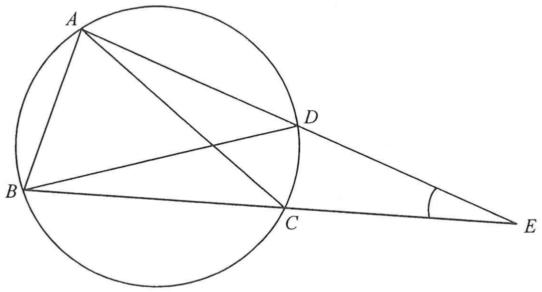
(Difficulty: 3) (2019 DSE Paper 2 Q39)

Solution 12. Join DE .



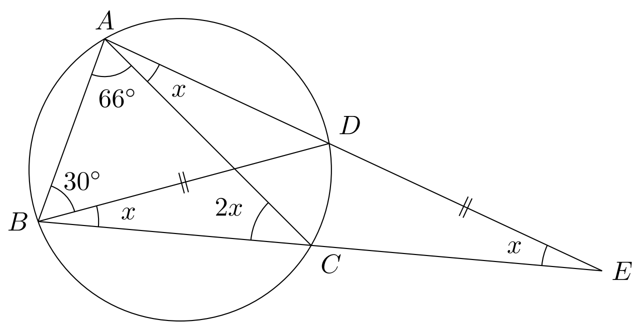
$$\angle ADE = 38^\circ \qquad (\angle \text{ in alt. segment})$$
$$\angle EAD = 22^\circ \qquad (\angle\text{s in the same segment})$$
$$\angle ADB = 180^\circ - 64^\circ - 22^\circ - 38^\circ = \boxed{56^\circ}$$

Problem 13. In the figure, $ABCD$ is a circle. AD produced and BC produced meet at the point E . It is given that $BD = DE$, $\angle BAC = 66^\circ$ and $\angle ABD = 30^\circ$. Find $\angle CED$.



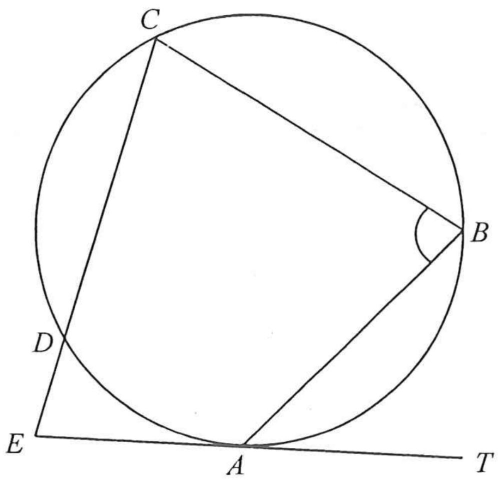
(Difficulty: 3) (2018 DSE Paper 2 Q22)

Solution 13. Let $\angle CED = x$.



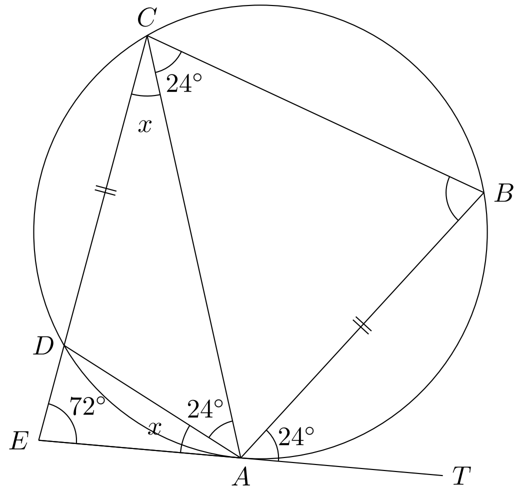
$$\angle DBE = x \qquad (\text{base } \angle\text{s, isos. } \triangle)$$
$$\angle CAD = \angle CBD = x \qquad (\angle\text{s in the same segment})$$
$$\angle ACB = \angle CED + \angle CAD = 2x \qquad (\text{ext. } \angle \text{ of } \triangle)$$
In $\triangle ABC$, $$66^\circ + (30^\circ + x) + 2x = 180^\circ \qquad (\angle \text{ sum of } \triangle)$$
$$x = \boxed{28^\circ}$$

Problem 14. In the figure, TA is the tangent to the circle $ABCD$ at the point A . CD produced and TA produced meet at the point E . It is given that $AB = CD$, $\angle BAT = 24^\circ$ and $\angle AED = 72^\circ$. Find $\angle ABC$.



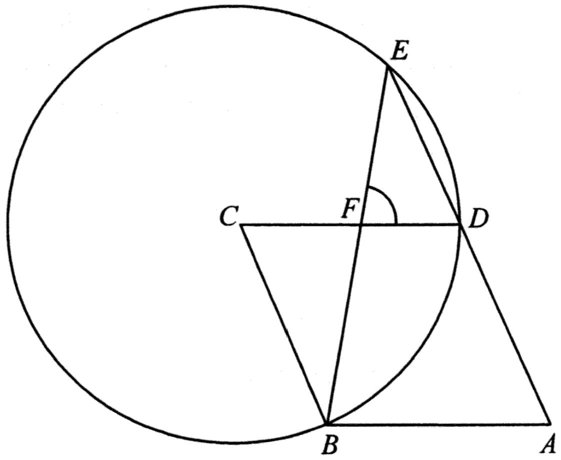
(Difficulty: 4) (2018 DSE Paper 2 Q39)

Solution 14. Join AD and AC. Let $\angle EAD = x$.



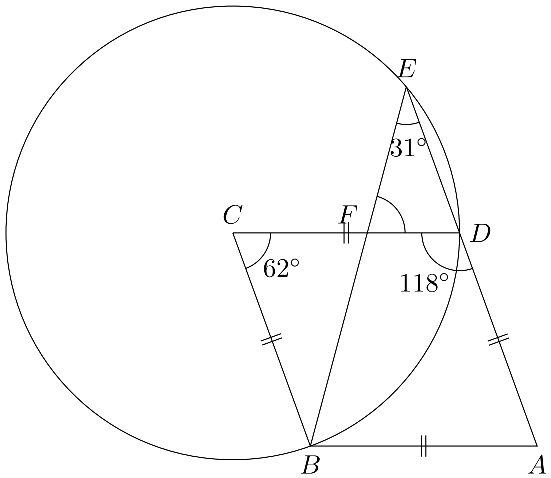
$$\begin{aligned}
\angle ACB &= 24^\circ \qquad (\angle \text{ in alt. segment})\\
\angle CAD &= \angle ACB = 24^\circ \qquad (\text{equal chords, equal } \angle\text{s at } \odot^{ce})\\
\angle DCA &= \angle EAD = x \qquad (\angle \text{ in alt. segment})\\
\text{In } \triangle CEA \text{ ,} \qquad 72^\circ + x + (x+24^\circ) &= 180^\circ \qquad (\angle \text{ sum of } \triangle)\\
x &= 42^\circ\\
\angle ABC = \angle EAC &= 42^\circ + 24^\circ \qquad (\angle \text{ in alt. segment})\\
&= \boxed{66^\circ}
\end{aligned}$$

Problem 15. In the figure, $ABCD$ is a rhombus. C is the centre of the circle BDE and ADE is a straight line. BE and CD intersect at F . If $\angle ADC = 118^\circ$, then $\angle DFE =$?



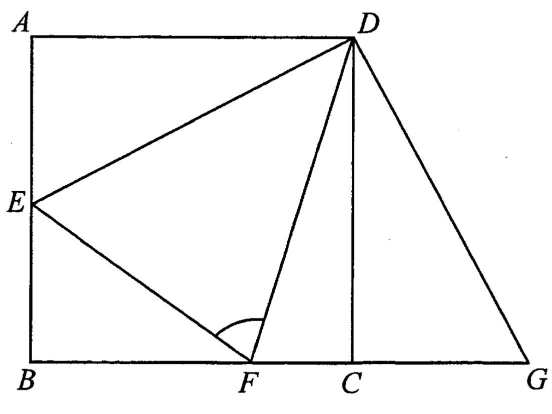
(Difficulty: 3) (2016 DSE Paper 2 Q22)

Solution 15. .



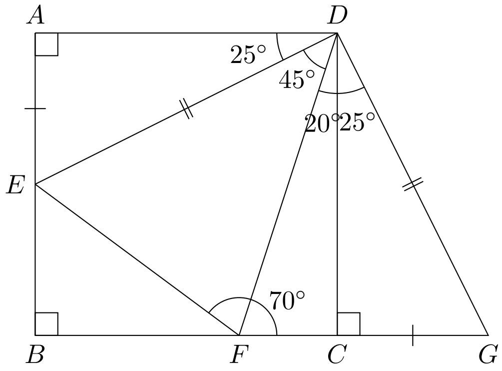
$$CB//DA \qquad \text{(prop. of rhombus)}$$
$$\angle C = 180^\circ - 118^\circ = 62^\circ \qquad (\text{int. } \angle\text{s }, CB//DA)$$
$$\angle FED = 62^\circ/2 = 31^\circ \qquad (\angle \text{ at centre twice } \angle \text{ at } \odot^{ce})$$
$$\angle DFE = 118^\circ - 31^\circ = \boxed{87^\circ} \qquad (\text{ext. } \angle \text{ of } \triangle)$$

Problem 16. In the figure, $ABCD$ is a square. BC is produced to G such that $\angle CDG = 25^\circ$. E is a point lying on AB such that $AE = CG$. If F is a point lying on BC such that $\angle CDF = 20^\circ$, then $\angle DFE =$?



(Difficulty: 4) (2014 DSE Paper 2 Q16)

Solution 16. .

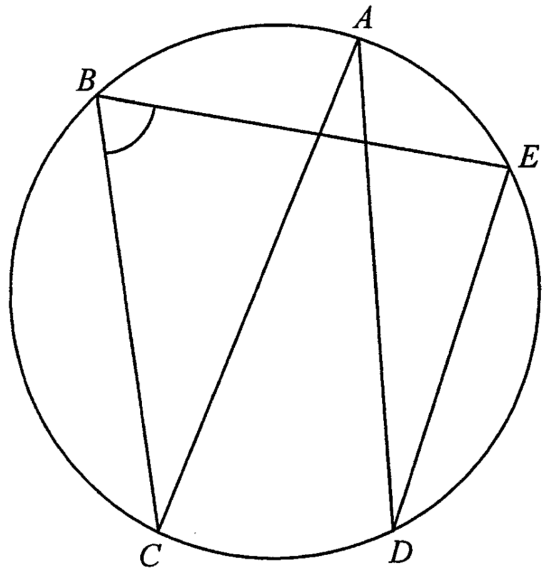


Note that $\triangle DAE \cong \triangle DCG$ (SAS) , so we have $\angle ADE = \angle CDG = 25°$ (corr. sides, $\cong$ $\triangle$s).
Note that $\angle EDF = 90° - 25° - 20° = 45°$.

In $\triangle DFE$ and $\triangle DFG$,

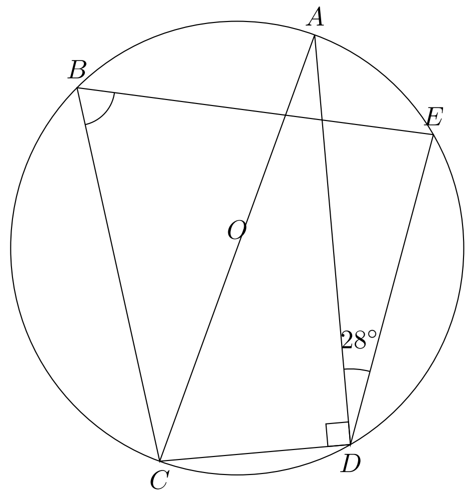
$$\begin{aligned} DE &= DG \qquad \text{(corr. sides, } \cong \triangle\text{s)} \\ \angle EDF &= \angle FDG = 45° \\ DF &= DF \qquad \text{(common side)} \\ \therefore\ \triangle DFE &\cong \triangle DFG \qquad \text{(SAS)} \\ \therefore\ \angle DFE &= \angle DFG \qquad \text{(corr. } \angle\text{s, } \cong \triangle\text{s)} \\ &= 90° - 20° = \boxed{70°} \qquad (\angle \text{ sum of } \triangle) \end{aligned}$$

Problem 17. In the figure, AC is a diameter of the circle $ABCDE$. If $\angle ADE = 28°$, then $\angle CBE =$?



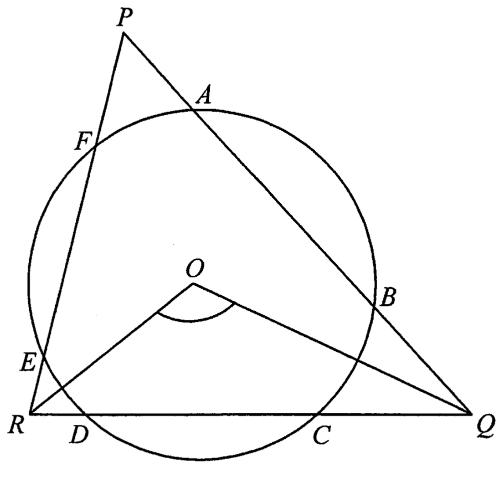
(Difficulty: 3) (2014 DSE paper 2 Q20)

Solution 17. Join CD .



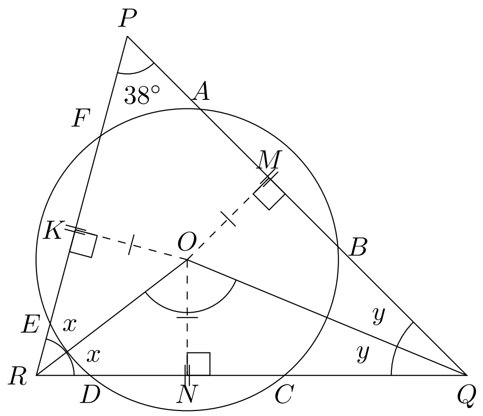
$$\angle ADC = 90^\circ \qquad (\angle \text{ in semi-circle})$$
$$\angle CDE = 90^\circ + 28^\circ = 118^\circ$$
$$\angle CBE = 180^\circ - 118^\circ = \boxed{62^\circ} \qquad (\text{opp. } \angle\text{s , cyclic quad.})$$

Problem 18. In the figure, O is the centre of the circle $ABCDEF$. $\triangle PQR$ intersects the circle at A, B, C, D, E and F . If $\angle QPR = 38^\circ$ and $AB = CD = EF$, then $\angle QOR =$?



(Difficulty: 4) (2014 DSE Paper 2 Q21)

Solution 18. Draw $OM \perp AB$, $ON \perp DC$, $OK \perp FE$.



Note that $OM = ON = OK$ (equal chords, equidistant from centre) . Thus, $\angle ORK = \angle ORN$ and $\angle OQN = \angle OQM$ (prop. of $\angle$ bisector) .

Let $\angle ORK = \angle ORN = x$ and $\angle OQN = \angle OQM = y$. In $\triangle PQR$,

$$\begin{aligned} 38^\circ + 2x + 2y &= 180^\circ \qquad (\angle \text{ sum of } \triangle) \\ x + y &= 71^\circ \end{aligned}$$

In $\triangle ORQ$,

$$\begin{aligned} x + y + \angle QOR &= 180^\circ \qquad (\angle \text{ sum of } \triangle) \\ \angle QOR &= 180^\circ - 71^\circ = \boxed{109^\circ} \end{aligned}$$

References

[1] MindYourDecisions, "A classically hard geometry problem," YouTube. [Online]. Available: https://www.youtube.com/watch?v=CFhFx4n3aH8&ab_channel=MindYourDecisions

[2] ——, "A classically hard geometry problem," YouTube. [Online]. Available: https://www.youtube.com/watch?v=Rjo-PcrKrB0&t=272s&ab_channel=MindYourDecisions