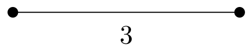
Contents

1 Lines, angles and shapes

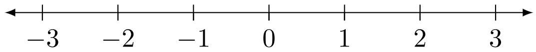
1.7 Area, perimeter and hypotenuse

1.7.1 Unit length

The length of a line segment can be some specific real number, such as 3:

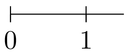


Recall the ruler postulate, which states that the points on a line can be (arbitrarily) matched one to one with the real numbers, like a number line:



The distance between the points associated with two successive integers is called a **unit length** (or simply unit), and any length can be expressed as a quantity of unit lengths. To measure the length of a line segment, we can count or calculate how many unit lengths it contains:

length = 3 units



length = 1.5 units

(For convenience, sometimes we will omit the ‘units’ after the number.)

1.7.2 Perimeter

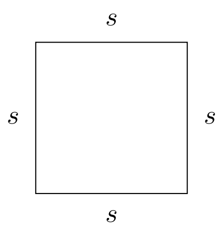
The **perimeter** of a shape is the total length of the line/curve segments that form the shape. We can imagine the line/curve segments being a string that we can straighten to become a straight line segment, whose length will be the perimeter. We will only focus on the perimeter of polygons now, since finding the perimeter of a non-polygon is complicated.

We typically use P to denote the perimeter of a shape.

Preposition 1. The perimeter of different quadrilaterals is as follows: (perimeter of //gram)

1. Square

The perimeter of a square is four times its side length (denoted s).

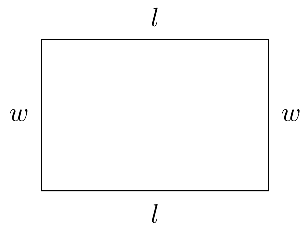


$$\boxed{P = 4s}$$

Proof. By definition, a square has four equal sides. A side length is s , and the perimeter is the sum of four side lengths, which is $s + s + s + s = 4s$. □

2. Rectangle

The perimeter of a rectangle is twice the sum of its length (l) and width (w).
(Note: Length and width can be arbitrarily assigned to the sides, but length usually refers to the longer side of the rectangle.)

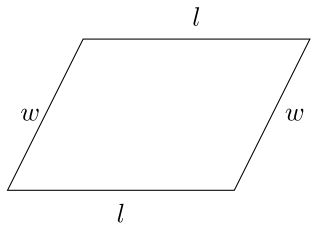


$$\boxed{P = 2(l + w)}$$

Proof. By 'opp. sides of rectangle', a rectangle has two equal pairs of opposite sides. So $P = 2l + 2w = 2(l + w)$. □

3. Parallelogram

The perimeter of a parallelogram is twice the sum of its two adjacent sides (still denoted l and w).

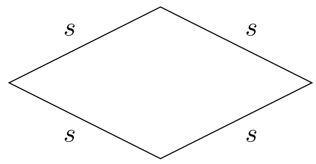


$$\boxed{P = 2(l + w)}$$

Proof. By 'opp. sides of //gram', a parallelogram has two equal pairs of opposite sides. So $P = 2l + 2w = 2(l + w)$. □

4. rhombus

The perimeter of a rhombus is four times its side length (s).



$$\boxed{P = 4s}$$

Proof. By definition, a rhombus has four equal sides. So $P = 4s$. □

As for the other shapes, just add the side lengths together to get the perimeter.

1.7.3 Area

Area is the measure of how much space a shape encloses in the plane. The larger the shape, the more area it has:

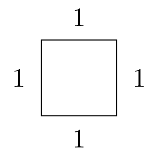
smaller area

larger area

Only shapes have area. Lines, points and angles have no area because they don't enclose any space.

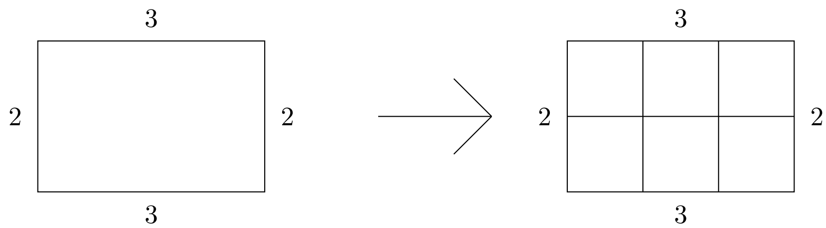
Different shapes can have the same area:

We typically measure the area of a shape by counting or calculating how many squares of a fixed size that the shape contains. This square of fixed size is called a **unit square**, which has the side length of 1 unit:



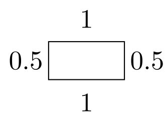
This square is said to have an area of 1 **sq. unit**. (Sometimes we will omit the 'sq. unit'.)

A rectangle can be divided into unit squares to count its area:



This rectangle has an area of $3 \times 2 = 6$ sq. units.

The area can be calculated even when there is not a whole number of squares. For example, there can be half a square:



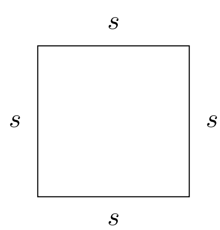
This rectangle has an area of $1 \times 0.5 = 0.5$ sq. units.

Notice that the area of a rectangle is length $\times$ width. This formula can be derived from a few postulates that I forgot to mention in the common notion part:

1. The area of a shape must be a positive number.
2. The area of a square with side length s is s^2 . (square area postulate)
3. Congruent shapes have the same area.
4. The area of a shape consisting of any amount (even countably infinite) of non-overlapping parts is the sum of the area of the individual parts.

We typically use A to denote the area of a shape.

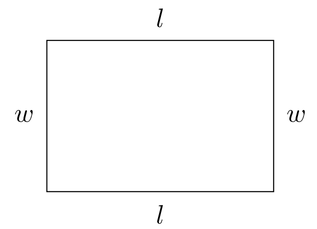
Preposition 2. The area of a square is its side length squared. (area of square)



$$\boxed{A = s^2}$$

Proof. This is automatically true by square area postulate. (I just pulled another 'because I told you so'.) ☐

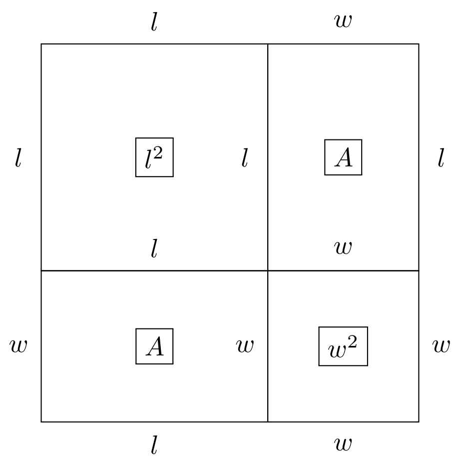
Preposition 3. The area of a rectangle is the product of its length and width. (area of rectangle)



$$\boxed{A = lw}$$

Proof. [1] Construct two squares with side lengths w and l , and another rectangle as follows:

(Boxed label is the area of the rectangle/square. The area of the original rectangle is labelled A .)



The two rectangles are congruent because they have the same length and width, so they must have the same area A .

Note that the area of the whole big square is $(l+w)^2$. Since the area of this big square is the sum of the area of its individual parts, we have the equation:

$$(l+w)^2 = l^2 + w^2 + 2A$$

Expanding the left hand side, we have:

$$l^2 + 2lw + w^2 = l^2 + w^2 + 2A$$

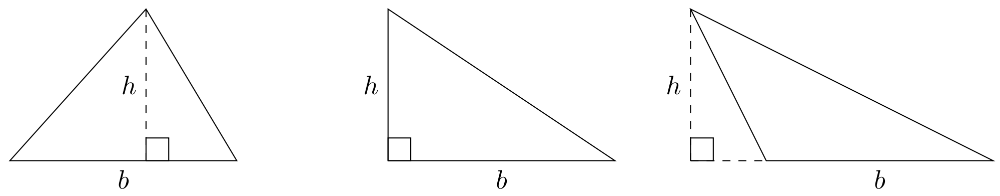
Subtracting $l^2 + w^2$ from both sides results to

$$2lw = 2A$$

$$A = lw$$

But l and w are the length and width of the rectangle, therefore, the area of any rectangle is the product of its length and its width. ☐

Preposition 4. The area of a triangle is half the product of its base (b) and height (h). (area of △)



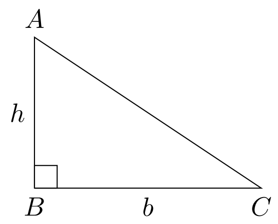
$$\boxed{A = \frac{1}{2}bh}$$

Note: Any side of the triangle can be the base, but the dotted part extending from the base in the diagram is not part of the base.

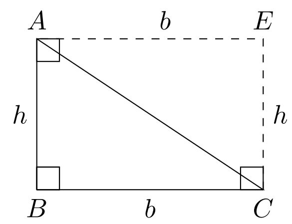
The height of a triangle that corresponds to a base is the perpendicular distance between that base and its opposite vertex.

Proof. Let there be $\triangle ABC$. Let $AD \perp BC$. AD is the height of the triangle (that corresponds to the base BC) . There are three types of triangles to consider: right triangle, acute triangle and obtuse triangle.

Case 1: $\angle ABC = 90°$



In this case, D coincides with B . Make a point E such that $EC \perp BC$ and $EA \perp AB$.



Note that $ABCE$ is a rectangle (3 right $\angle$s) . So the opposite sides are equal, namely $AE = BC$, $AB = EC$.

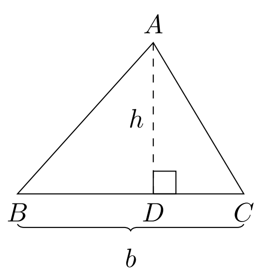
In $\triangle ABC$ and $\triangle CEA$,

$$\begin{aligned} AB &= EC && \text{(opp. sides of rectangle)} \\ BC &= AE && \text{(opp. sides of rectangle)} \\ AC &= AC && \text{(common side)} \\ \therefore\ \triangle ABC &\cong \triangle CEA && \text{(SSS)} \end{aligned}$$

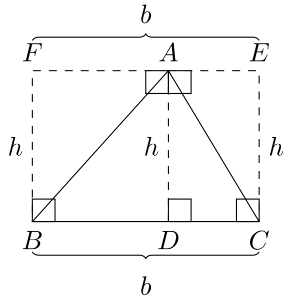
Thus, area of $\triangle ABC =$ area of $\triangle CEA$, since congruent triangles have equal areas.

By 'area of rectangle', area of $ABCE = bh$, so we have area of $\triangle ABC +$ area of $\triangle CEA = bh$, which means area of $\triangle ABC = \frac{1}{2}bh$.

Case 2: $\angle ABC < 90°$



Make a point E such that $EC \perp BC$ and $EA \perp AD$. Make a point F such that $FB \perp BC$ and $FA \perp AD$.



Note that $ADBF$ and $ADCE$ are rectangles (3 right $\angle$s) . So $FB = EC = AD = h$ and $FE = BC = b$ (opp. sides of rectangle).

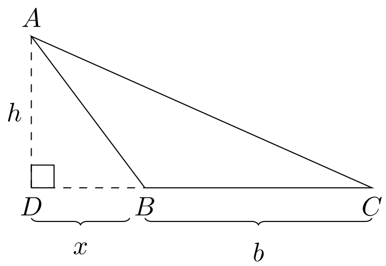
Note that area of $\triangle ABD$ is half the area of $ADBF$, and the area of $\triangle ADC$ is half the area of $ADCE$ (by case 1).

So we have:

$$\begin{aligned}
\text{area of } \triangle ABC &= \text{area of } ABD + \text{area of } ADC \\
&= \frac{1}{2}\text{area of } ADBF + \frac{1}{2}\text{area of } ADCE \\
&= \frac{1}{2}\text{area of } FECB \\
&= \frac{1}{2}bh \qquad \text{(area of rectangle)}
\end{aligned}$$

<u>Case 3:</u> $\angle ABC > 90°$

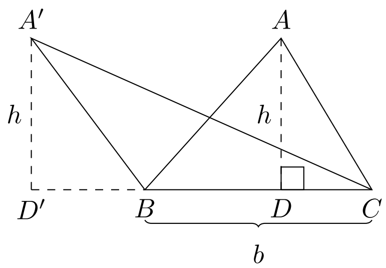
Let $BD = x$.



$$\begin{aligned}
\text{area of } ABC &= \text{area of } ADC - \text{area of } ADB \\
&= \frac{1}{2}(x+b)h - \frac{1}{2}xh \\
&= \frac{1}{2}(xh + bh - xh) \\
&= \frac{1}{2}bh
\end{aligned}$$

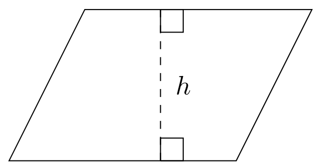
$\square$

An implication of this preposition is that two triangles with the same base and height must have the same area:



$$\text{area of } ABC = \text{area of } A'BC \qquad \text{(area of triangle)}$$

Preposition 5. The area of a parallelogram is the product of its base (b) and height (h). (area of //gram)

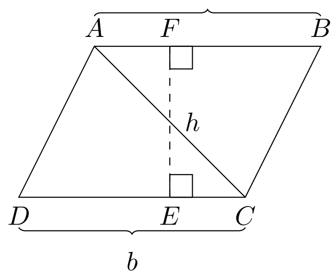


$$\boxed{A = bh}$$

Note: The height that corresponds to a base is the perpendicular distance between that base and its opposite side.

Proof. Let there be parallelogram $ABCD$. Let E be on line CD and F be on line AB such that $EF \perp AB$ and $EF \perp CD$. Then EF is the height of the parallelogram.

Join AC .

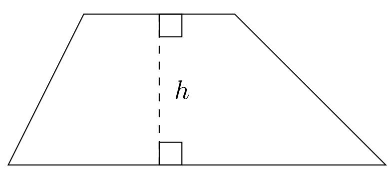


Note that $\triangle ADC \cong \triangle CBA$ (SSS). This means the area of the two triangles are equal. Note that the height of the triangles is the same as the height of the parallelogram, because parallel lines preserve perpendicular distance. So we have

$$\begin{aligned} \text{area of } ABCD &= \text{area of } ADC + \text{area of } CBA \\ &= \frac{1}{2}bh + \frac{1}{2}bh \\ &= bh \end{aligned}$$

□

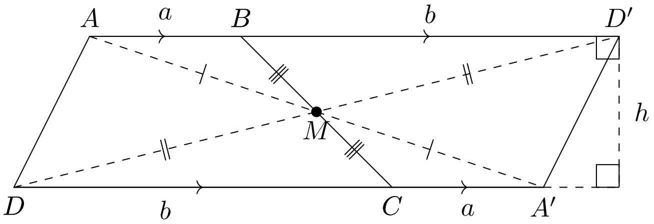
Preposition 6. The area of a trapezium is the product of its height (h) and the average of its upper base (a) and lower base (b). (area of trapezium)



$$\boxed{A = \frac{(a+b)h}{2}}$$

Note: The height of the trapezium is the perpendicular distance between its pair of parallel sides. The upper base and lower base can be arbitrarily assigned to the parallel sides, but the upper base usually refers to the shorter parallel side.

Proof. Let there be trapezium $ABCD$, where $AB//DC$. Let M be the mid point of BC . Rotate the trapezium $180°$ about point M to make an image trapezium $A'B'C'D'$. Note that C' coincides with B and B' coincides with C .



Thus we have $AM = A'M$, $DM = D'M$, $BM = CM$.

Note that the vertically opposite triangles are congruent, namely $\triangle AMB \cong \triangle A'MC$, $\triangle BMD' \cong CMD$, $\triangle AMD' \cong A'MD$, $\triangle AMD \cong \triangle A'MD'$ (SAS for all).

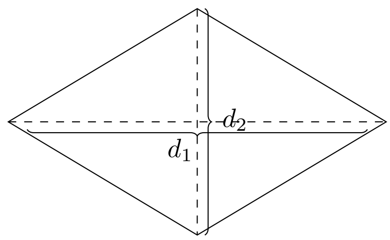
So $A'C = AB = a$ and $BA' = DC = b$.

Also note that ABD and DCA' are straight line segments. This is because we have $AB//DC$ (given) and $AB//CA'$ ($\angle ABM = \angle A'CM$) $\Rightarrow$ (alt. $\angle$s equal) . So $\angle DCM + \angle A'CM = (180° - \angle ABM) + \angle ABM = 180°$, which means DCA' is a straight line segment (adj. $\angle$s supp.). Same argument can be said for ABD' .

So $AD'A'D$ is a parallelogram (diags biset each other). And this parallelogram has an area of $(a+b)h$ by (area of //gram). Since trapezium $ABCD$ and trapezium $A'CBD'$ are congruent (because corresponding angles and sides are equal), they have equal area, which means area of $ABCD = \dfrac{(a+b)h}{2}$.

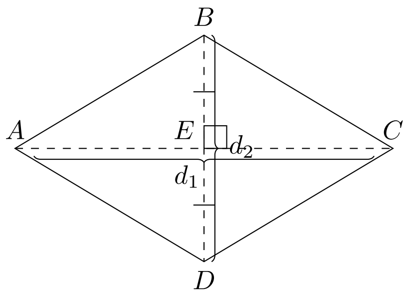
□

Preposition 7. The area (A) of a rhombus is half the product of its diagonals (d_1) , (d_2) . (area of rhombus)



$$\boxed{A = \frac{1}{2} d_1 d_2}$$

Proof. Label the rhombus $ABCD$. Note that $AC \perp BD$ (diags of rhombus) . Let AC and BD intersect at E .



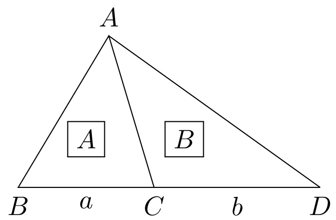
Note that $BE = ED = \dfrac{d_2}{2}$ (diags of rhombus). We have

$$\begin{aligned}\text{area of } ABCD &= \text{area of } \triangle BAC + \text{area of } DAC \\ &= \frac{1}{2} d_1 (\frac{d_2}{2}) + \frac{1}{2} d_1 (\frac{d_2}{2}) \qquad \text{(area of } \triangle) \\ &= \frac{1}{2} d_1 d_2\end{aligned}$$

□

1.7.4 Prepositions related to ratios and areas

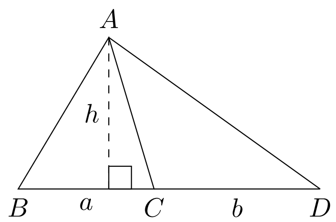
Preposition 8. For two triangles with the same height / two triangles sharing a side and with their bases on the same line, the ratio of their base lengths is equal to the ratio of their areas. (bases prop. to areas of $\triangle$s)



$$\frac{\text{area of } \triangle ABC}{\text{area of } \triangle ACD} = \frac{a}{b}$$

$$(\text{Shorter statement: } \frac{\boxed{\text{A}}}{\boxed{\text{B}}} = \frac{a}{b})$$

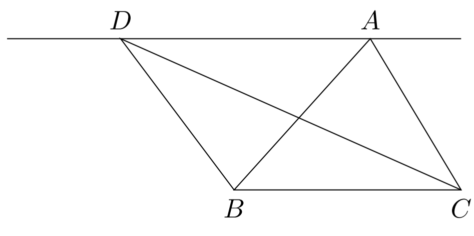
Proof. Let h be the height of the triangles.



$$\begin{aligned} \frac{\text{area of } \triangle ABC}{\text{area of } \triangle ACD} &= \frac{\frac{1}{2}ah}{\frac{1}{2}bh} \qquad \text{(area of triangle)} \\ &= \frac{a}{b} \end{aligned}$$

□

Preposition 9. For two triangles sharing the same base with the same height or same area, the line passing through the top vertices of two triangles are parallel to their base. (line joining $\triangle$ tips // base)

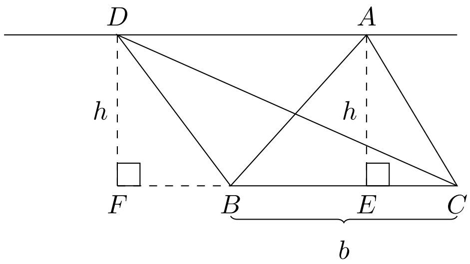


$$\because \text{area of } \triangle ABC = \text{area of } \triangle DBC$$
$$\therefore \ DA // BC \qquad \text{(line joining } \triangle \text{ tips // base)}$$

Proof. Draw $AE \perp BC$ and $DF \perp BC$.

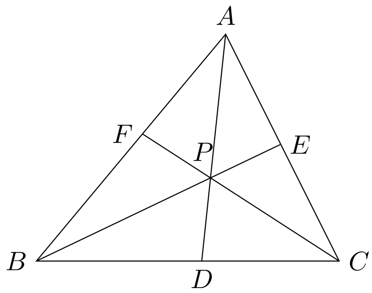
If given that $\triangle ABC$ and $\triangle DBC$ have the same height, then $DF = AE$ since they are the heights of the triangles.

If given that $\triangle ABC$ and $\triangle DBC$ have the same area instead then since $\triangle ABC$ and $\triangle DBC$ have the same base and same area, they must have the same height by 'area of $\triangle$' . Thus, $DF = AE$.



Since $DF = AE$, $\angle DFE = \angle AEF = 90°$, we have that $DAEF$ is a rectangle (1 equal pair, 2 right $\angle$s) . Thus $DA // FE$ (prop. of rectangle) . Since F and E lies on line BC , we must also have $DA // BC$. $\square$

Preposition 10. Given a triangle $\triangle ABC$ and a point P inside the triangle, if we extend AP, BP, CP to meet the sides at D, E, F respectively, then $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$. (Ceva's theorem)



$$\boxed{\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1} \qquad \text{(Ceva's theorem)}$$

Proof. [2] Let $[\triangle AFP]$ denote the area of $\triangle AFP$. Other area of triangles are denoted similarly.

Note that

$$\frac{AF}{FB} = \frac{[\triangle AFP]}{[\triangle FBP]} = \frac{[\triangle AFC]}{[\triangle FBC]}$$

by 'bases prop. to areas of $\triangle$s' .

By subtracting the triangle areas in the middle of the equation from RHS, we get:

$$\frac{AF}{FB} = \frac{[\triangle AFC] - [\triangle AFP]}{[\triangle FBC] - [\triangle FBP]}$$

$$\frac{AF}{FB} = \frac{[\triangle APC]}{[\triangle BPC]} \tag{1}$$

Similarly, we have

$$\frac{BD}{DC} = \frac{[\triangle APB]}{[\triangle APC]} \tag{2}$$

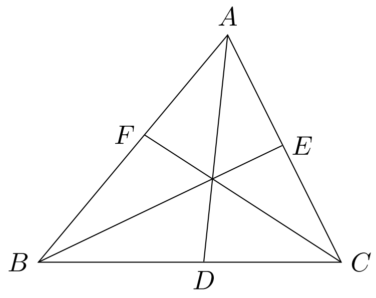
$$\frac{CE}{EA} = \frac{[\triangle BPC]}{[\triangle APB]} \tag{3}$$

Multiplying (1), (2), (3) together, we get

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{[\triangle APC]}{[\triangle BPC]} \cdot \frac{[\triangle APB]}{[\triangle APC]} \cdot \frac{[\triangle BPC]}{[\triangle APB]} = 1$$

$\square$

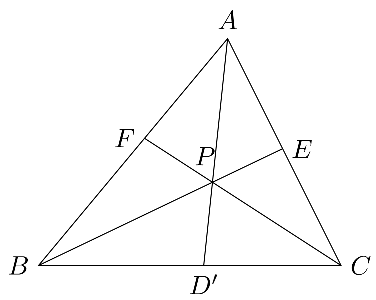
Preposition 11. Given a triangle $\triangle ABC$, let D, E, F be points on sides BC, CA, AB respectively. If $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$, then cevians AD , BE , CF are concurrent. (converse of Ceva's theorem)



$$\because \frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$

$\therefore$ AD, BE, CF are concurrent. (converse of Ceva's theorem)

Proof. [3] Assume that $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$. Let BE and CF intersect at P , and extend AP to meet side BC at a point D' . We want to show that D' coincides with D .



By Ceva's theorem, we have:

$$\frac{AF}{FB} \cdot \frac{BD'}{D'C} \cdot \frac{CE}{EA} = 1$$

By initial assumption, we have:

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$

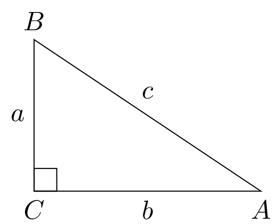
Thus

$$\begin{aligned}
\frac{AF}{FB} \cdot \frac{BD'}{D'C} \cdot \frac{CE}{EA} &= \frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} \\
\frac{BD'}{D'C} &= \frac{BD}{DC} \\
\frac{BC - D'C}{D'C} &= \frac{BC - DC}{DC} \\
\frac{BC}{D'C} - 1 &= \frac{BC}{DC} - 1 \\
D'C &= DC
\end{aligned}$$

Since $D'C$ and DC have the same length, D' and D must coincide. So A, P, D are collinear, which means BE, CF and AD are concurrent. □

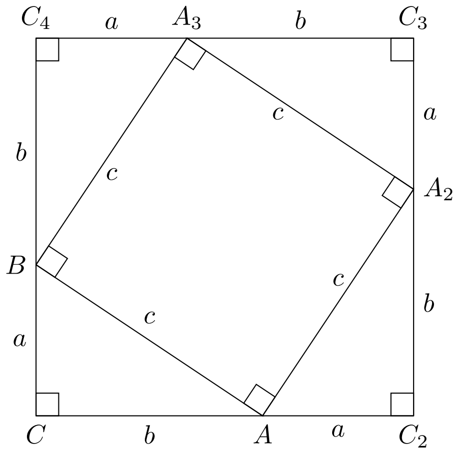
1.7.5 Pythagoras theorem and related prepositions

Preposition 12. In a right triangle, the square of hypotenuse (c) is the sum of square of the other two sides (a) , (b). (pyth. theorem) *

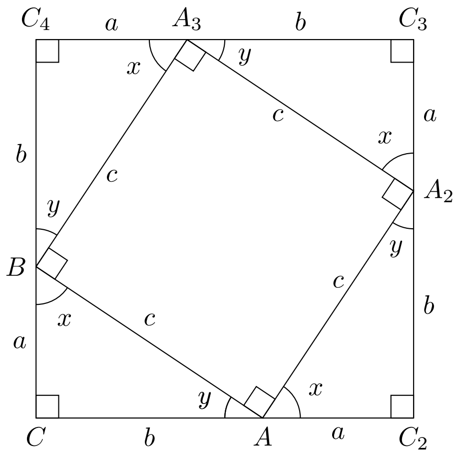


$$\because \angle C = 90°$$
$$\therefore \boxed{a^2 + b^2 = c^2} \qquad \text{(pyth. theorem)}$$

Proof. Arrange four triangles congruent to $\triangle ABC$ such that the hypotenuses enclose a square of side length c, as in the figure below:



Note that CAC_2 , $C_2A_2C_3$, $C_3A_3C_4$, C_4BC are straight lines segments by 'adj. $\angle$s supp.' :



To explain, let $\angle ABC = x$ and $\angle BAC = y$. Then $\angle A_2AC_2 = x$ (corr. sides, $\cong \triangle$s). Note that $x + y = 90°$ ($\angle$ sum of $\triangle$), so $\angle CAC_2 = x + y + 90° = 180°$, so CAC_2 is a straight line segment (adj. $\angle$s supp.). By similar argument, the other three sides are also straight line segments.

Thus, $CC_2C_3C_4$ is a square with side length $a + b$, which has an area of $(a + b)^2$.

Looking at the pieces individually, the area of each triangle is $\frac{1}{2}ab$, and the slanted square in the centre has an area of c^2 . The sum of the four triangles and the centre square must be equal to the area of the larger square. Thus we have the equation:

$$(a + b)^2 = 4(\frac{1}{2}ab) + c^2$$

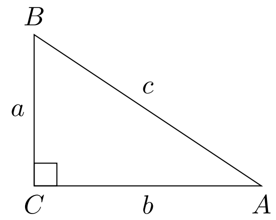
giving

$$a^2 + 2ab + b^2 = 2ab + c^2$$
$$a^2 + b^2 = c^2$$

□

Note: **Pythagoras theorem** is one of the most important theorems in Euclidean geometry, as it leads to a host of other prepositions related to the lengths of the triangles. (You'll see in the next few prepositions.)

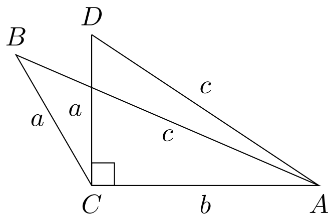
Preposition 13. In a triangle, if the square of a side is the sum of square of the other two sides, then the triangle is a right triangle, with the first side being the hypotenuse. (converse of pyth. theorem) *



$$\because a^2 + b^2 = c^2$$

$$\therefore \angle C = 90^\circ \qquad \text{(converse of pyth. theorem)}$$

Proof. Suppose $a^2 + b^2 = c^2$ but $\angle C \neq 90^\circ$. Suppose $\angle BCA > 90^\circ$. Construct another triangle $\triangle ACD$ such that $DC = BC = a$ and $\angle DCA = 90^\circ$.



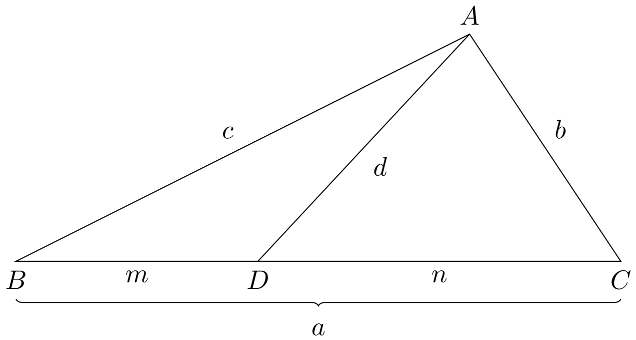
Note that in $\triangle DCA$, we have $AD^2 = a^2 + b^2$ by pyth. theorem, so $AD^2 = c^2$ and $AD = c = AB$. But by hinge theorem, since $\angle BCA > \angle DCA$, we have $AB > AD$, which is a contradiction.

Similarly, if we suppose $\angle BCA < 90^\circ$, then by hinge theorem, we have $AB < AD$, again a contradiction.

Thus, it can only be the case that $\angle BCA = 90^\circ$, and so AB is the hypotenuse.

□

Preposition 14. Given a triangle $\triangle ABC$ with side lengths a, b, c and opposite vertices A, B, C respectively. If cevian AD is drawn so that $BD = m$, $CD = n$ and $AD = d$, then we have $b^2 m + c^2 n = a(d^2 + mn)$. (Stewart's theorem)



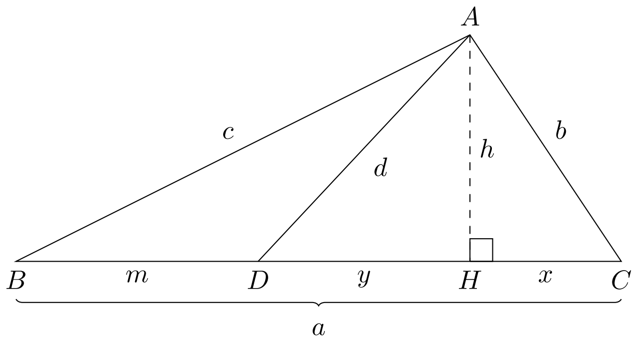
$$\boxed{b^2 m + c^2 n = a(d^2 + mn)}$$

$$\text{Rearranged:} \qquad d^2 = \frac{b^2 m + c^2 n}{m + n} - mn$$

Proof. There are several cases depending on the position of A relative to BC .

Case 1: A is directly above DC (which means $\angle ADC < 90^\circ$ and $\angle ACD < 90^\circ$) .

Let the altitude from A to BC at H . Let $AH = h$, $CH = x$, and $HD = y$.



Note that $x = n - y$

So, applying Pythagoras theorem on $\triangle AHC, \triangle AHB, \triangle AHD$, yields

$$b^2 = (n-y)^2 + h^2 \tag{4}$$
$$c^2 = (m+y)^2 + h^2 \tag{5}$$
$$d^2 = y^2 + h^2 \tag{6}$$

(1) $\times m$:

$$b^2 m = (n-y)^2 m + h^2 m$$

(2) $\times n$:

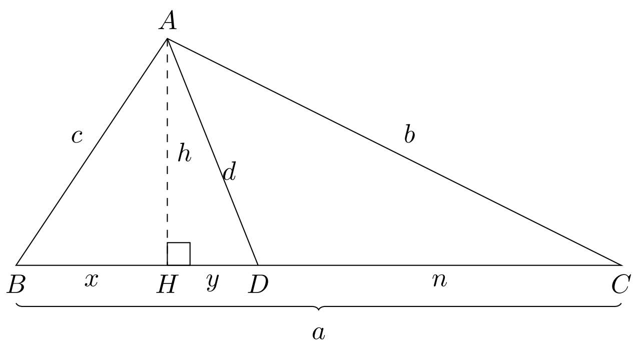
$$c^2 n = (m+y)^2 n + h^2 n$$

(1) $\times m +$ (2) $\times n$:

$$\begin{aligned} b^2 m + c^2 n &= (n-y)^2 m + h^2 m + (m+y)^2 n + h^2 n \\ &= n^2 m - 2nym + y^2 m + h^2 m + m^2 n + 2myn + y^2 n + h^2 n \\ &= h^2 m + h^2 n + m^2 n + n^2 m + y^2 m + y^2 n \\ &= h^2(m+n) + mn(m+n) + y^2(m+n) \\ &= (m+n)(h^2 + mn + y^2) \\ &= a(d^2 + mn) \end{aligned}$$

Case 2: A is directly above BD (which means $\angle ABD < 90^\circ$ and $\angle ADB < 90^\circ$) .

Let the altitude from A to BC at H . Let $AH = h$, $DH = y$, and $BH = x$.



Note that $y = m - x$

So, applying Pythagoras theorem on $\triangle AHC, \triangle AHB, \triangle AHD$, yields

$$b^2 = (n+y)^2 + h^2 \tag{1}$$
$$c^2 = (m-y)^2 + h^2 \tag{2}$$
$$d^2 = y^2 + h^2 \tag{3}$$

(1) $\times m$:

$$b^2 m = (n+y)^2 m + h^2 m$$

(2) $\times n$:

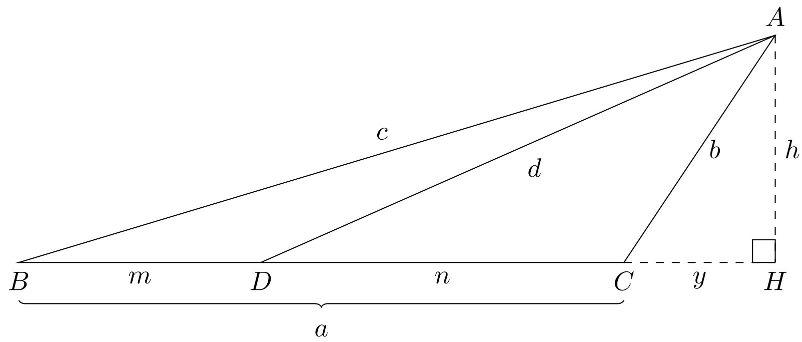
$$c^2n = (m-y)^2n + h^2n$$

(1) $\times\, m +$ (2) $\times\, n$:

$$\begin{aligned}
b^2m + c^2n &= (n+y)^2m + h^2m + (m-y)^2n + h^2n \\
&= n^2m + 2nym + y^2m + h^2m + m^2n - 2myn + y^2n + h^2n \\
&= h^2m + h^2n + m^2n + n^2m + y^2m + y^2n \\
&= h^2(m+n) + mn(m+n) + y^2(m+n) \\
&= (m+n)(h^2 + mn + y^2) \\
&= a(d^2 + mn)
\end{aligned}$$

Case 3: A is above right of BC (which means $\angle ADC > 90°$) .

Let the altitude from A to BC at H . Let $AH = h$ and $CH = y$.



So, applying Pythagoras theorem on $\triangle AHC, \triangle AHB, \triangle AHD$, yields

$$b^2 = y^2 + h^2 \tag{1}$$
$$c^2 = (m+n+y)^2 + h^2 \tag{2}$$
$$d^2 = (n+y)^2 + h^2 \tag{3}$$

(1) $\times m$:

$$b^2m = y^2m + h^2m$$

(2) $\times n$:

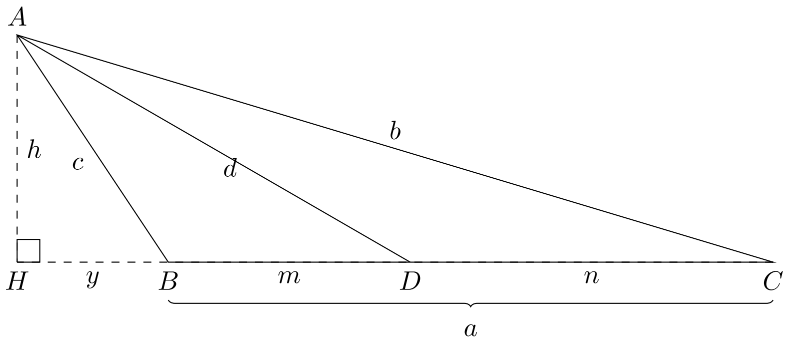
$$c^2n = (m+n+y)^2n + h^2n$$

(1) $\times\, m +$ (2) $\times\, n$:

$$\begin{aligned}
b^2m + c^2n &= y^2m + h^2m + (m+n+y)^2n + h^2n \\
&= y^2m + h^2m + m^2n + n^3 + y^2n + 2n^2m + 2n^2y + 2myn + h^2n \\
&= h^2m + h^2n + y^2m + y^2n + m^2n + 2n^2m + 2n^2y + 2myn + n^3 \\
&= h^2(m+n) + y^2(m+n) + n(m^2 + 2nm + 2ny + 2my + n^2) \\
&= h^2(m+n) + y^2(m+n) + n((m+n)^2 + 2y(m+n)) \\
&= h^2(m+n) + y^2(m+n) + n(m+n)(m+n+2y) \\
&= (m+n)(h^2 + y^2 + n(m+n+2y)) \\
&= (m+n)(h^2 + (n+y)^2 + mn) \\
&= a(d^2 + mn)
\end{aligned}$$

Case 4: A is above left of BC (which means $\angle ABC > 90°$) .

Let the altitude from A to BC at H . Let $AH = h$ and $BH = y$.



So, applying Pythagoras theorem on $\triangle AHC, \triangle AHB, \triangle AHD$, yields

$$b^2 = (m+n+y)^2 + h^2 \tag{1}$$
$$c^2 = y^2 + h^2 \tag{2}$$
$$d^2 = (m+y)^2 + h^2 \tag{3}$$

(1) $\times m$:

$$b^2 m = (m+n+y)^2 m + h^2 m$$

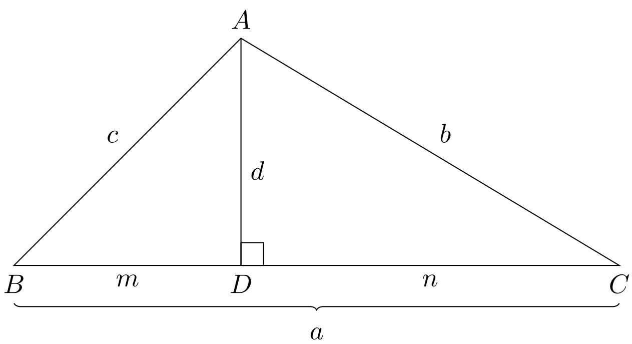
(2) $\times n$:

$$c^2 n = y^2 n + h^2 n$$

(1) $\times\, m +$ (2) $\times\, n$:

$$\begin{aligned}
b^2 m + c^2 n &= (m+n+y)^2 m + h^2 m + y^2 n + h^2 n \\
&= m^3 + mn^2 + y^2 m + 2m^2 n + 2mny + 2m^2 y + h^2 m + y^2 n + h^2 n \\
&= h^2 m + h^2 n + y^2 m + y^2 n + m^3 + mn^2 + 2m^2 n + 2mny + 2m^2 y \\
&= h^2(m+n) + y^2(m+n) + m(m^2 + n^2 + 2mn + 2ny + 2my) \\
&= h^2(m+n) + y^2(m+n) + m((m+n)^2 + 2y(m+n)) \\
&= h^2(m+n) + y^2(m+n) + m(m+n)(m+n+2y) \\
&= (m+n)(h^2 + y^2 + m(m+n+2y)) \\
&= (m+n)(h^2 + (m+y)^2 + mn) \\
&= a(d^2 + mn)
\end{aligned}$$

<u>Case 5:</u> A is vertically above D (which means $AD \perp BC$) .



Applying Pythagoras theorem on $\triangle ADB, \triangle ADC$ yields

$$b^2 = n^2 + d^2 \tag{1}$$
$$c^2 = m^2 + d^2 \tag{2}$$

(1) $\times m$:

$$b^2 m = n^2 m + d^2 m$$

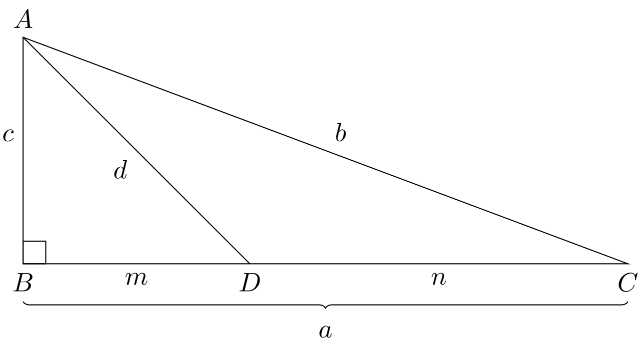
(2) $\times n$:

$$c^2 n = m^2 n + d^2 n$$

$(1) \times m + (2) \times n$:

$$\begin{aligned} b^2m + c^2n &= n^2m + d^2m + m^2n + d^2n \\ &= d^2(m+n) + mn(m+n) \\ &= a(d^2 + mn) \end{aligned}$$

<u>Case 6:</u> A is vertically above B (which means $AB \perp BC$) .



Applying Pythagoras theorem on $\triangle ADB, \triangle ACB$ yields

$$b^2 = (m+n)^2 + c^2 \quad (1)$$
$$c^2 = d^2 - m^2 \quad (2)$$

(1) $\times m$:

$$b^2m = (m+n)^2m + c^2m$$

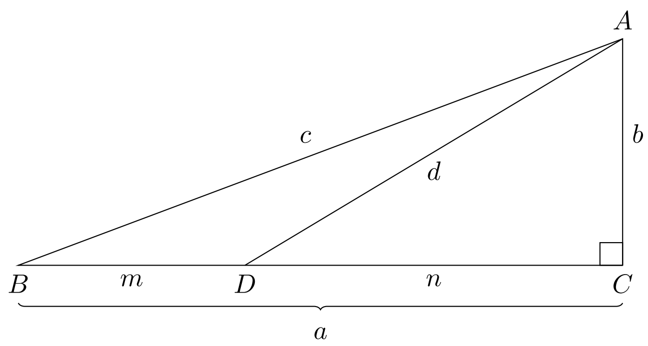
(2) $\times n$:

$$c^2n = d^2n - m^2n$$

$(1) \times m + (2) \times n$:

$$\begin{aligned} b^2m + c^2n &= (m+n)^2m + c^2m + d^2n - m^2n \\ &= (m+n)^2m + (d^2 - m^2)m + d^2n - m^2n \\ &= m^3 + 2m^2n + mn^2 + d^2m - m^3 + d^2n - m^2n \\ &= d^2m + d^2n + mn^2 + m^2n \\ &= d^2(m+n) + mn(m+n) \\ &= a(d^2 + mn) \end{aligned}$$

<u>Case 7:</u> A is vertically above C (which means $AC \perp BC$) .



Applying Pythagoras theorem on $\triangle ACB, \triangle ACD$ yields

$$b^2 = d^2 - n^2 \quad (1)$$
$$c^2 = (m+n)^2 + b^2 \quad (2)$$

(1) $\times m$:

$$b^2m = d^2m - n^2m$$

(2) $\times n$:

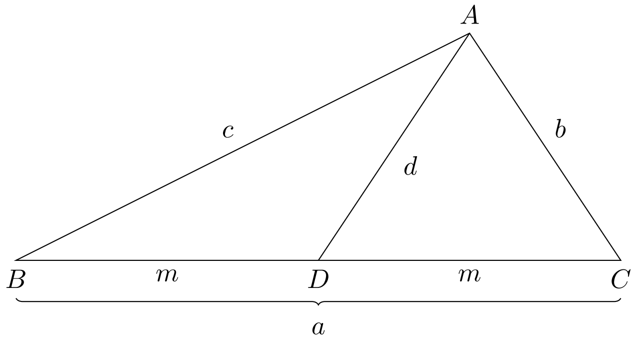
$$c^2 n = (m+n)^2 n + b^2 n$$

(1) $\times m +$ (2) $\times n$:

$$\begin{aligned}
b^2 m + c^2 n &= d^2 m - n^2 m + (m+n)^2 n + b^2 n \\
&= d^2 m - n^2 m + (m+n)^2 n + (d^2 - n^2) n \\
&= d^2 m - n^2 m + m^2 n + 2mn^2 + n^3 + d^2 n - n^3 \\
&= d^2 m + d^2 n + m^2 n + mn^2 \\
&= d^2 (m+n) + mn(m+n) \\
&= a(d^2 + mn)
\end{aligned}$$

□

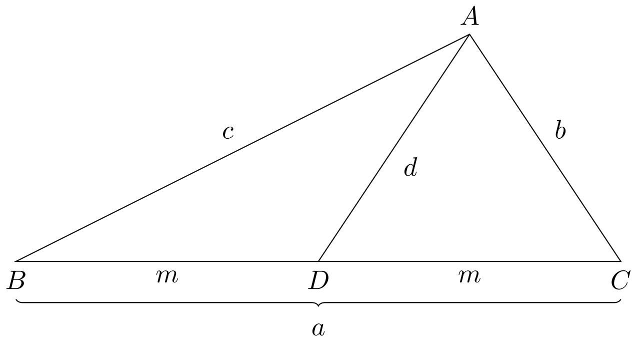
Preposition 15. Given a triangle $\triangle ABC$ with side lengths a, b, c and opposite vertices A, B, C respectively. If AD is a median of the triangle such that $AD = d$ and $BD = CD = m$, then we have $b^2 + c^2 = 2(d^2 + m^2)$. (Apollonius's theorem / length of median of $\triangle$)



$$\boxed{b^2 + c^2 = 2(d^2 + m^2)}$$

$$\text{Rearranged:} \qquad d^2 = \frac{2b^2 + 2c^2 - a^2}{4}$$

Proof. .



By Stewart's theorem, we have

$$\begin{aligned}
b^2 m + c^2 m &= (m+m)(d^2 + mm) \\
m(b^2 + c^2) &= 2m(d^2 + m^2) \\
b^2 + c^2 &= 2(d^2 + m^2)
\end{aligned}$$

Now put $m = \frac{a}{2}$ and do the rearrangement:

$$
\begin{aligned}
\frac{b^2+c^2}{2} - m^2 &= d^2 \\
d^2 &= \frac{b^2+c^2}{2} - (\frac{a}{2})^2 \\
&= \frac{2b^2+2c^2}{4} - \frac{a^2}{4} \\
d^2 &= \frac{2b^2+2c^2-a^2}{4}
\end{aligned}
$$

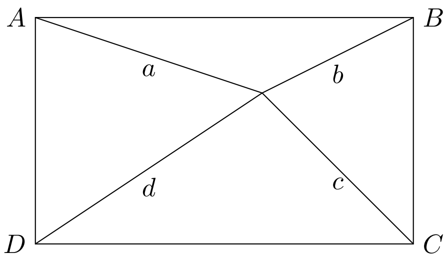
□

Preposition 16. If there is a point inside a rectangle, then the sum of squared distance from the point to a pair of opposite vertices is equal to that of the other pair of opposite vertices. (british flag theorem)

A variant of this preposition:

If the diagonals of a convex quadrilateral are perpendicular, then the sum of a pair of squared opposite side lengths is equal to that of the other pair of opposite sides. (british kite theorem)

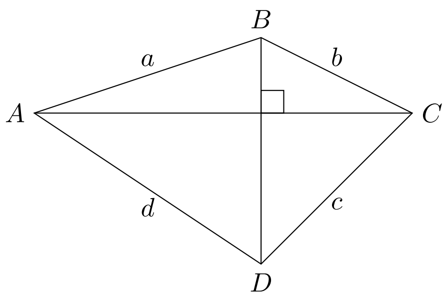
Case 1:



Given: $ABCD$ is a rectangle.

$\therefore \boxed{a^2 + c^2 = b^2 + d^2}$ (british flag theorem)

Case 2:

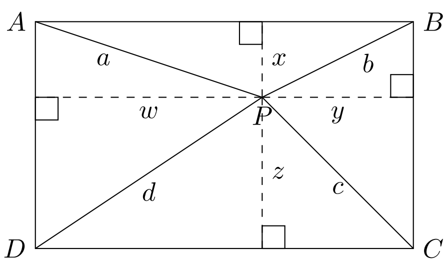


Given: $AC \perp BD$

$\therefore \boxed{a^2 + c^2 = b^2 + d^2}$ (british kite theorem)

Proof. Case 1:

Let P be the point inside the rectangle. Drop perpendicular line segments from P to the sides of the rectangle, and label them x, y, z, w . Note that the rectangle is separated into four smaller rectangles.

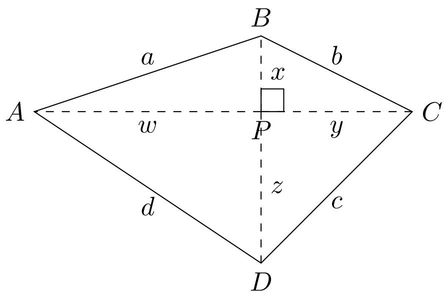


By 'Pyth. theorem' , we have $(x^2+w^2)+(y^2+z^2)=a^2+c^2$, and $(x^2+y^2)+(w^2+z^2)=b^2+d^2$.

Since $(x^2+w^2)+(y^2+z^2)=(x^2+y^2)+(w^2+z^2)$, we have $a^2+c^2=b^2+d^2$.

<u>Case 2:</u>

Let P be point of intersection of the diagonals. Let $PA=w$, $PB=x$, $PC=y$, $PD=z$.



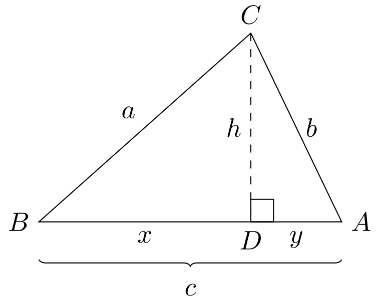
By 'Pyth. theorem' , we have $(x^2+w^2)+(y^2+z^2)=a^2+c^2$, and $(x^2+y^2)+(w^2+z^2)=b^2+d^2$.

Since $(x^2+w^2)+(y^2+z^2)=(x^2+y^2)+(w^2+z^2)$, we have $a^2+c^2=b^2+d^2$.

$\square$

Preposition 17. There are two cases:

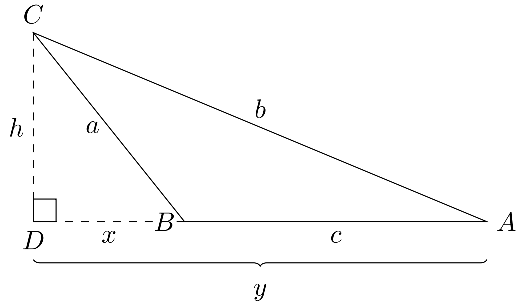
<u>Case 1.</u> Given an acute or right triangle $\triangle ABC$ with side lengths a,b,c and altitude $CD\perp AB$, we have $BD=\dfrac{a^2+c^2-b^2}{2c}$ and $AD=\dfrac{b^2+c^2-a^2}{2c}$. (Simplified law of cosines)



$$x=\frac{a^2+c^2-b^2}{2c}$$
$$y=\frac{b^2+c^2-a^2}{2c}$$

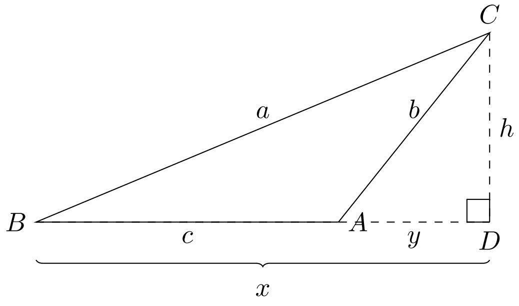
(Simplified law of cosines)

<u>Case 2a</u>. Given an obtuse triangle $\triangle ABC$ where $\angle CBA>90^\circ$ with side lengths a,b,c and altitude $CD\perp AB$, we have $BD=\dfrac{a^2+c^2-b^2}{-2c}$ and $AD=\dfrac{b^2+c^2-a^2}{2c}$. (Simplified law of cosines)



$$x = \frac{a^2 + c^2 - b^2}{-2c}$$
$$y = \frac{b^2 + c^2 - a^2}{2c}$$
(Simplified law of cosines)

Case 2b. Given an obtuse triangle $\triangle ABC$ where $\angle CAB > 90°$ with side lengths a, b, c and altitude $CD \perp AB$, we have $BD = \dfrac{a^2 + c^2 - b^2}{2c}$ and $AD = \dfrac{b^2 + c^2 - a^2}{-2c}$. (Simplified law of cosines)

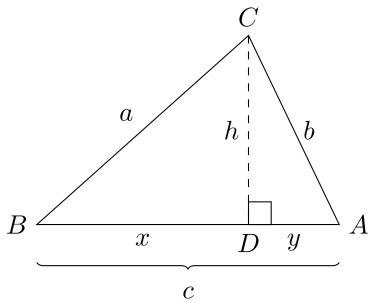


$$x = \frac{a^2 + c^2 - b^2}{2c}$$
$$y = \frac{b^2 + c^2 - a^2}{-2c}$$
(Simplified law of cosines)

Proof. Let $BD = x$ and $AD = y$.

There are several cases depending on the position of C relative to AB .

Case 1a: C is directly above AB (which means $\angle CBA < 90°$ and $\angle CAB < 90°$).



Note that $y = c - x$. In $\triangle CBD$ and $\triangle CAD$, by pyth. theorem:

$$x^2 + h^2 = a^2 \tag{1}$$
$$(c - x)^2 + h^2 = b^2 \tag{2}$$

Subtracting x^2 from both sides of (1):

$$h^2 = a^2 - x^2 \tag{3}$$

Put (3) into (2):

$$(c - x)^2 + (a^2 - x^2) = b^2$$
$$c^2 - 2cx + x^2 + a^2 - x^2 = b^2$$
$$c^2 + a^2 - b^2 = 2cx$$
$$x = \frac{a^2 + c^2 - b^2}{2c}$$

Similarly for the equation of y:

Note that $x = c - y$. In $\triangle CBD$ and $\triangle CAD$, by pyth. theorem:

$$(c-y)^2 + h^2 = a^2 \tag{4}$$
$$y^2 + h^2 = b^2 \tag{5}$$

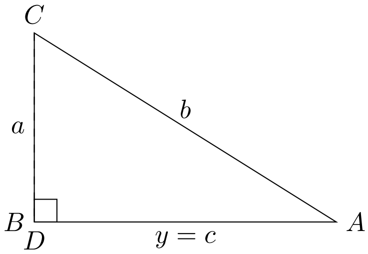
Subtracting x^2 from both sides of (5):

$$h^2 = b^2 - y^2 \tag{6}$$

Put (6) into (4):

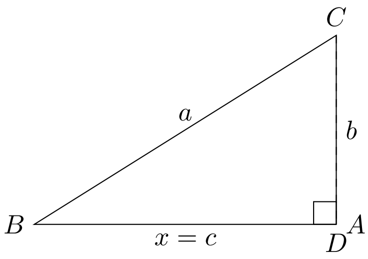
$$\begin{aligned}(c-y)^2 + (b^2 - y^2) &= a^2 \\ c^2 - 2cy + y^2 + b^2 - y^2 &= a^2 \\ c^2 + b^2 - a^2 &= 2cy \\ y &= \frac{b^2 + c^2 - a^2}{2c}\end{aligned}$$

<u>Case 1b:</u> C is vertically above B (which means $\angle CBA = 90°$ and D coincides with B).



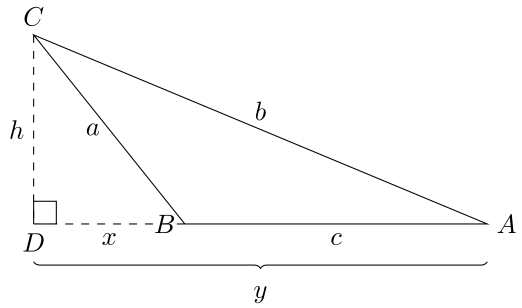
By pyth. theorem, $a^2 + c^2 = b^2$, which means $a^2 + c^2 - b^2 = 0$.
So $BD = x = \dfrac{0}{2c} = \dfrac{a^2 + c^2 - b^2}{2c}$.
For y: By pyth. theorem, $a^2 + c^2 = b^2$, which means $c^2 = b^2 - a^2$.
So $AD = y = c = \dfrac{2c^2}{2c} = \dfrac{b^2 + c^2 - a^2}{2c}$.

<u>Case 1c:</u> C is vertically above A (which means $\angle CAB = 90°$ and D coincides with A).



By pyth. theorem, $b^2 + c^2 = a^2$, which means $c^2 = a^2 - b^2$.
So $BD = x = c = \dfrac{2c^2}{2c} = \dfrac{a^2 + c^2 - b^2}{2c}$.
For y: By pyth. theorem, $b^2 + c^2 = a^2$, which means $b^2 + c^2 - a^2 = 0$.
So $AD = y = \dfrac{0}{2c} = \dfrac{b^2 + c^2 - a^2}{2c}$.

<u>Case 2a:</u> C is above left of AB (which means $\angle CBA > 90°$).



Note that $y = c + x$. In $\triangle CBD$ and $\triangle CAD$, by pyth. theorem:

$$x^2 + h^2 = a^2 \tag{1}$$
$$(c + x)^2 + h^2 = b^2 \tag{2}$$

Subtracting x^2 from both sides of (1):

$$h^2 = a^2 - x^2 \tag{3}$$

Put (3) into (2):

$$\begin{aligned}(c + x)^2 + (a^2 - x^2) &= b^2 \\ c^2 + 2cx + x^2 + a^2 - x^2 &= b^2 \\ c^2 + a^2 - b^2 &= -2cx \\ x &= \frac{a^2 + c^2 - b^2}{-2c}\end{aligned}$$

For equation of y , note that $x = y - c$. In $\triangle CBD$ and $\triangle CAD$, by pyth. theorem:

$$(y - c)^2 + h^2 = a^2 \tag{1}$$
$$y^2 + h^2 = b^2 \tag{2}$$

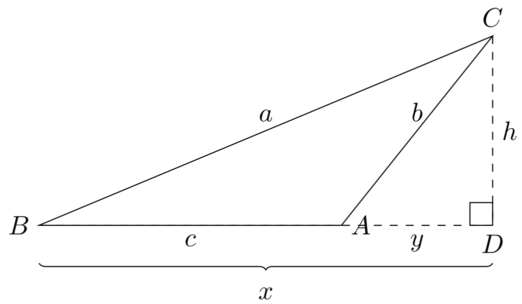
Subtracting y^2 from both sides of (2):

$$h^2 = b^2 - y^2 \tag{3}$$

Put (3) into (1):

$$\begin{aligned}(y - c)^2 + (b^2 - y^2) &= a^2 \\ c^2 - 2cy + y^2 + b^2 - y^2 &= a^2 \\ c^2 + b^2 - a^2 &= 2cy \\ y &= \frac{b^2 + c^2 - a^2}{2c}\end{aligned}$$

<u>Case 2b:</u> C is above right of AB (which means $\angle CAB > 90°$).



Note that $y = x - c$. In $\triangle CBD$ and $\triangle CAD$, by pyth. theorem:

$$x^2 + h^2 = a^2 \tag{1}$$
$$(x - c)^2 + h^2 = b^2 \tag{2}$$

Subtracting x^2 from both sides of (1):

$$h^2 = a^2 - x^2 \tag{3}$$

Put (3) into (2):

$$\begin{aligned} (x-c)^2 + (a^2 - x^2) &= b^2 \\ c^2 - 2cx + x^2 + a^2 - x^2 &= b^2 \\ c^2 + a^2 - b^2 &= 2cx \\ x &= \frac{a^2 + c^2 - b^2}{2c} \end{aligned}$$

For equation of y , note that $x = y + c$. In $\triangle CBD$ and $\triangle CAD$, by pyth. theorem:

$$(y+c)^2 + h^2 = a^2 \tag{1}$$
$$y^2 + h^2 = b^2 \tag{2}$$

Subtracting y^2 from both sides of (2):

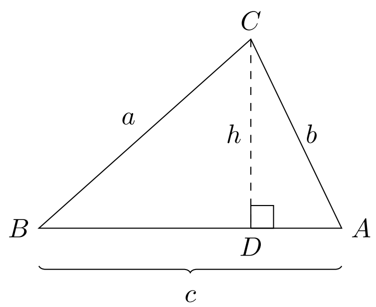
$$h^2 = b^2 - y^2 \tag{3}$$

Put (3) into (1):

$$\begin{aligned} (y+c)^2 + (b^2 - y^2) &= a^2 \\ c^2 + 2cy + y^2 + b^2 - y^2 &= a^2 \\ c^2 + b^2 - a^2 &= -2cy \\ y &= \frac{b^2 + c^2 - a^2}{-2c} \end{aligned}$$

□

Preposition 18. Given a triangle with side lengths a, b, c , the height (h) of the triangle that corresponds to side c is $\sqrt{a^2 - \left(\frac{a^2+c^2-b^2}{2c}\right)^2}$ and also $\sqrt{b^2 - \left(\frac{b^2+c^2-a^2}{2c}\right)^2}$. (triangle height formula)



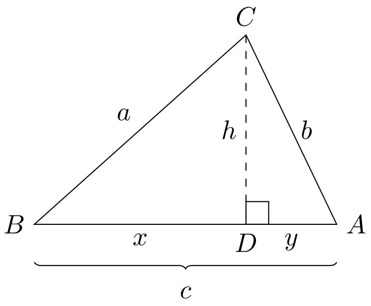
$$h = \sqrt{a^2 - \left(\frac{a^2+c^2-b^2}{2c}\right)^2} = \sqrt{b^2 - \left(\frac{b^2+c^2-a^2}{2c}\right)^2}$$

(triangle height formula)

Proof. Let $BD = x$ and $AD = y$.

There are several cases depending on the position of C relative to AB .

<u>Case 1a:</u> C is directly above AB (which means $\angle CBA < 90°$ and $\angle CAB < 90°$).

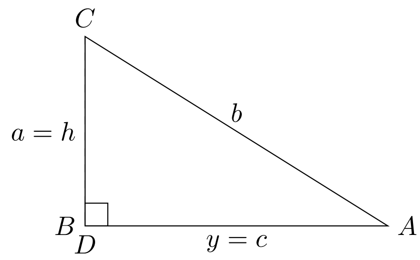


By simplified law of cosines, we have $x = \dfrac{a^2 + c^2 - b^2}{2c}$ and $y = \dfrac{b^2 + c^2 - a^2}{2c}$.

In $\triangle CBD$ and $\triangle CAD$, By pyth. theorem, we have $x^2 + h^2 = a^2$ and $y^2 + h^2 = b^2$. Thus

$$h = \sqrt{a^2 - \left(\frac{a^2 + c^2 - b^2}{2c}\right)^2} = \sqrt{b^2 - \left(\frac{b^2 + c^2 - a^2}{2c}\right)^2}$$

Case 1b: C is vertically above B (which means $\angle CBA = 90^\circ$ and D coincides with B).



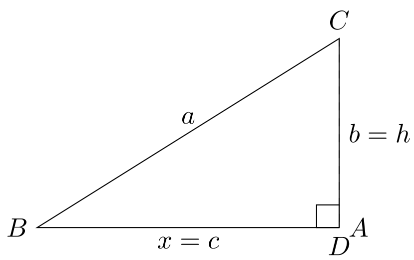
By simplified law of cosines, we have $x = \dfrac{a^2 + c^2 - b^2}{2c}$ and $y = \dfrac{b^2 + c^2 - a^2}{2c}$.

Note that pyth. theorem still applies to a degenerate right triangle (A.K.A. a line segment) with one of its side being 0.

It is because $a = h$, so we have $x^2 + h^2 = 0 + h^2 = a^2$. Like before, we also have $y^2 + h^2 = b^2$. Thus

$$h = \sqrt{a^2 - \left(\frac{a^2 + c^2 - b^2}{2c}\right)^2} = \sqrt{b^2 - \left(\frac{b^2 + c^2 - a^2}{2c}\right)^2}$$

Case 1c: C is vertically above A (which means $\angle CAB = 90^\circ$ and D coincides with A).

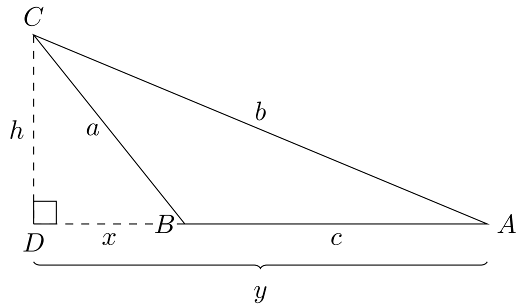


By simplified law of cosines, we have $x = \dfrac{a^2 + c^2 - b^2}{2c}$ and $y = \dfrac{b^2 + c^2 - a^2}{2c}$.

Since $b = h$, we have $y^2 + h^2 = 0 + h^2 = b^2$. Like before, we also have $x^2 + h^2 = a^2$. Thus

$$h = \sqrt{a^2 - \left(\frac{a^2 + c^2 - b^2}{2c}\right)^2} = \sqrt{b^2 - \left(\frac{b^2 + c^2 - a^2}{2c}\right)^2}$$

Case 2a: C is above left of AB (which means $\angle CBA > 90^\circ$).

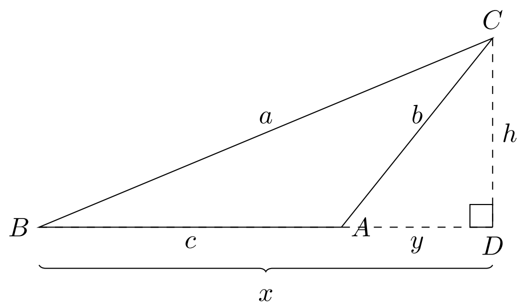


By simplified law of cosines, we have $x = \dfrac{a^2+c^2-b^2}{-2c}$ and $y = \dfrac{b^2+c^2-a^2}{2c}$.

In $\triangle CBD$ and $\triangle CAD$, By pyth. theorem, we have $x^2 + h^2 = a^2$ and $y^2 + h^2 = b^2$. Since $x^2 = (\dfrac{a^2+c^2-b^2}{-2c})^2 = (\dfrac{a^2+c^2-b^2}{2c})^2$, we have:

$$h = \sqrt{a^2 - \left(\frac{a^2+c^2-b^2}{2c}\right)^2} = \sqrt{b^2 - \left(\frac{b^2+c^2-a^2}{2c}\right)^2}$$

<u>Case 2b:</u> C is above right of AB (which means $\angle CAB > 90°$).



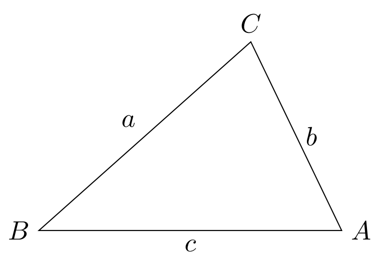
By simplified law of cosines, we have $x = \dfrac{a^2+c^2-b^2}{2c}$ and $y = \dfrac{b^2+c^2-a^2}{-2c}$.

In $\triangle CBD$ and $\triangle CAD$, By pyth. theorem, we have $x^2 + h^2 = a^2$ and $y^2 + h^2 = b^2$. Since $y^2 = (\dfrac{b^2+c^2-a^2}{-2c})^2 = (\dfrac{b^2+c^2-a^2}{2c})^2$, we have:

$$h = \sqrt{a^2 - \left(\frac{a^2+c^2-b^2}{2c}\right)^2} = \sqrt{b^2 - \left(\frac{b^2+c^2-a^2}{2c}\right)^2}$$

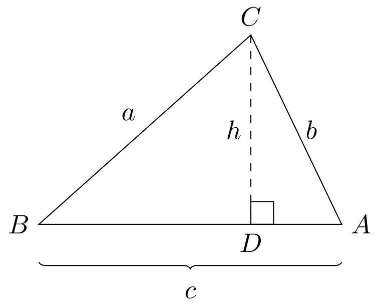
□

Preposition 19. The area (A) of a triangle with side lengths a, b, c is $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \dfrac{a+b+c}{2}$ is the **semi-perimeter** of the triangle. (Heron's formula) *



$$\boxed{A = \sqrt{s(s-a)(s-b)(s-c)}} \qquad \text{(Heron's formula)}$$

Proof. [4] Let $CD \perp AB$ and $h = CD$ be the height that corresponds to base AB .



Note that by triangle height formula, the height h is still $\sqrt{a^2 - \left(\frac{a^2+c^2-b^2}{2c}\right)^2}$ no matter whether $\triangle ABC$ is an acute, obtuse or right triangle. Thus, by the formula of area of $\triangle$, the area of $\triangle ABC$ is

$$\begin{aligned}
A &= \frac{1}{2}ch \\
&= \frac{1}{2}c\sqrt{a^2 - \left(\frac{a^2+c^2-b^2}{2c}\right)^2} \\
&= \sqrt{\frac{c^2}{4}\left(a^2 - \left(\frac{a^2+c^2-b^2}{2c}\right)^2\right)} \\
&= \sqrt{\frac{a^2c^2}{4} - \frac{c^2}{4}\cdot\frac{(a^2+c^2-b^2)^2}{4c^2}} \\
&= \sqrt{\left(\frac{ac}{2}\right)^2 - \left(\frac{a^2+c^2-b^2}{4}\right)^2} \\
&= \sqrt{\left(\frac{ac}{2} + \frac{a^2+c^2-b^2}{4}\right)\left(\frac{ac}{2} - \frac{a^2+c^2-b^2}{4}\right)} \\
&= \sqrt{\left(\frac{2ac+a^2+c^2-b^2}{4}\right)\left(\frac{2ac-a^2-c^2+b^2}{4}\right)}
\end{aligned}$$

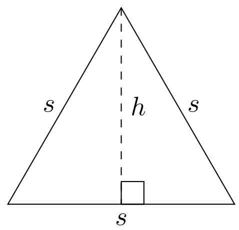
$$\begin{aligned}
&= \sqrt{\left(\frac{a^2+2ac+c^2-b^2}{4}\right)\left(\frac{b^2-(a^2-2ac+c^2)}{4}\right)} \\
&= \sqrt{\left(\frac{(a+c)^2-b^2}{4}\right)\left(\frac{b^2-(a-c)^2}{4}\right)} \\
&= \sqrt{\left(\frac{(a+c+b)(a+c-b)}{4}\right)\left(\frac{(b+a-c)(b-a+c)}{4}\right)} \\
&= \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b+c-2b}{2}\right)\left(\frac{a+b+c-2c}{2}\right)\left(\frac{a+b+c-2a}{2}\right)} \\
&= \sqrt{s(s-b)(s-c)(s-a)} \\
&= \sqrt{s(s-a)(s-b)(s-c)}
\end{aligned}$$

□

1.7.6 Special triangles

Equilateral triangle

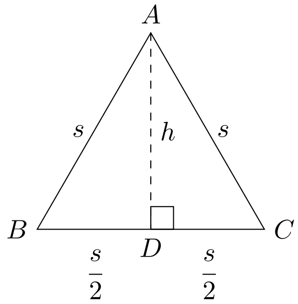
Preposition 20. The height (h) of an equilateral triangle with side length s is $\frac{\sqrt{3}}{2}s$. (height of equil. $\triangle$)



$$\boxed{h = \frac{\sqrt{3}}{2}s} \qquad \text{(height of equil. } \triangle\text{)}$$

Proof. Label the equilateral triangle $\triangle ABC$. Draw $AD \perp BC$.

Since $AB = AC$ and $AD \perp BC$, we have $BD = DC = \frac{s}{2}$ (prop. of isos. $\triangle$) .

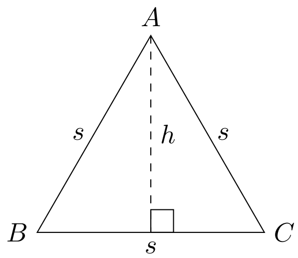


By pyth. theorem in $\triangle ABD$,

$$\begin{aligned} (\frac{s}{2})^2 + h^2 &= s^2 \\ h^2 &= s^2 - \frac{s^2}{4} \\ h &= \sqrt{\frac{3s^2}{4}} \\ &= \frac{\sqrt{3}}{2}s \end{aligned}$$

□

Preposition 21. The area (A) of an equilateral triangle with side length s is $\frac{s^2\sqrt{3}}{4}$. (area of equil. $\triangle$)



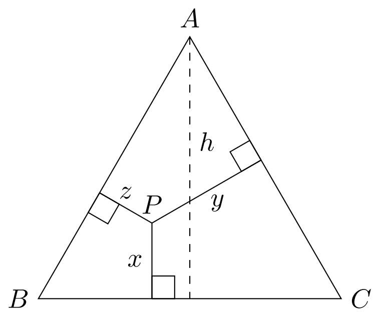
$$\boxed{A = \frac{s^2\sqrt{3}}{4}} \qquad \text{(area of equil. } \triangle\text{)}$$

Proof. Recall that a triangle's area is half its base times height. The base is s and the height h is $\frac{\sqrt{3}}{2}s$. Thus, we have

$$\begin{aligned} A &= \frac{1}{2}sh \\ &= \frac{1}{2}s(\frac{\sqrt{3}}{2}s) \\ &= \frac{s^2\sqrt{3}}{4} \end{aligned}$$

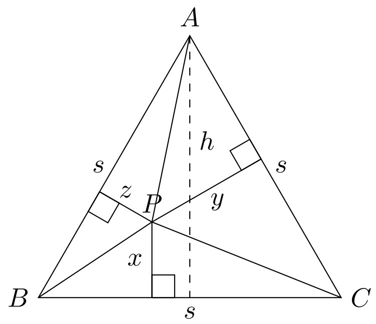
□

Preposition 22. If a point is inside an equilateral triangle, then the sum of lengths of the perpendicular line segments from the point to the sides is equal to the height (h) of the triangle. ($\perp$ segments of equil. $\triangle$)



$$\because AB = BC = AC \text{ , } x \perp BC, y \perp AC \text{ , } z \perp AB$$
$$\therefore\ h = x + y + z \qquad (\perp \text{ segments of equil. } \triangle)$$

Proof. Join AP , BP , CP . Let s be the side length of the equil. triangle $\triangle ABC$.



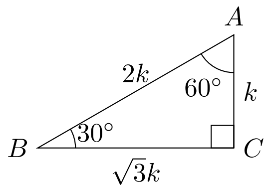
Note that

$$\begin{aligned} \text{area of } \triangle ABC &= \text{area of } \triangle PAB + \text{area of } \triangle PBC + \text{area of } \triangle PAC \\ \frac{1}{2}sh &= \frac{1}{2}sz + \frac{1}{2}sx + \frac{1}{2}sy \qquad (\text{area of } \triangle) \\ \frac{1}{2}sh &= \frac{1}{2}s(z + x + y) \\ h &= x + y + z \end{aligned}$$

□

Right triangle

Preposition 23. If and only if a triangle has angles measuring $30°, 60°, 90°$, then the ratio of its sides (opposite to angles in that order) is $1 : \sqrt{3} : 2$. (prop. of 30-60-90 $\triangle$)



<u>1a.</u>

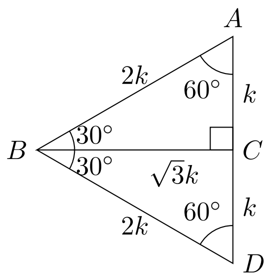
$$\because \angle B = 30^\circ\ ,\ \angle A = 60^\circ,\ \angle C = 90^\circ$$
$$\therefore\ AC : BC : AB = 1 : \sqrt{3} : 2$$

<u>1b.</u>

$$\because AC : BC : AB = 1 : \sqrt{3} : 2$$
$$\therefore\ \angle B = 30^\circ\ ,\ \angle A = 60^\circ,\ \angle C = 90^\circ$$

Proof. <u>1a.</u> Let there be $\triangle ABC$ with $\angle B = 30^\circ$, $\angle A = 60^\circ$, $\angle C = 90^\circ$.

Let D be the reflection of A about BC . Let $AC = CD = k$.



Note that $\triangle ABC \cong DBC$ (SAS) , so $\angle DBC = 30^\circ$ and $\angle BDC = 60^\circ$ (corr. $\angle$s, $\cong \triangle$s). Note that ABD is an equil. triangle (con. of equil. $\triangle$). Thus $BA = BD = AD = 2k$.

Since $BC \perp AD$, BC is the height of the triangle, and $BC = \frac{\sqrt{3}}{2}(2k) = \sqrt{3}k$ (height of equil. $\triangle$).

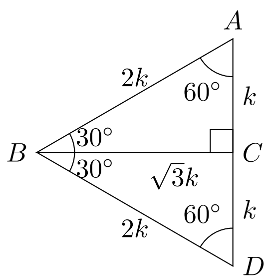
Therefore, $AC : BC : AB = k : \sqrt{3}k : 2k = 1 : \sqrt{3} : 2$.

<u>1b.</u> Let there be $\triangle ABC$ with $AC : BC : AB = 1 : \sqrt{3} : 2$.

Let $AC = k$, $BC = \sqrt{3}k$, $AC = 2k$. Note that $AC^2 + BC^2 = k^2 + (\sqrt{3}k)^2 = 4k^2$, and $AB^2 = (2k)^2 = 4k^2$.

Since $AC^2 + BC^2 = AB^2$, by converse of pyth. theorem, we have $\angle C = 90^\circ$.

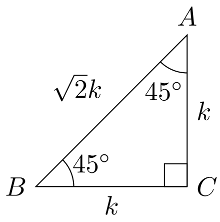
Let D be the reflection of A about BC . Then $AC = CD = k$.



Note that $\triangle ABC \cong DBC$ (SAS) , so $\angle DBC = 30^\circ$ and $\angle BDC = 60^\circ$ (corr. $\angle$s, $\cong \triangle$s). Since $BA = BD = AD = 2k$, note that $\triangle ABD$ is an equil. triangle by definition. Thus $\angle A = \angle D = \angle ABD = 60^\circ$ (prop. of equil. $\triangle$).

Note that $\angle ABC = \angle DBC$ (corr. $\angle$s, $\cong \triangle$s). Thus $\angle ABC = \angle ABD/2 = 30^\circ$. $\square$

Preposition 24. If and only if a triangle has angles measuring 45° , 45° , 90° , then the ratio of its sides (opposite to angles in that order) is $1 : 1 : \sqrt{2}$. (prop. of right isos. $\triangle$)



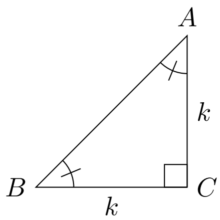
1a.

$$\because \angle A = 45^\circ \ , \angle B = 45^\circ, \angle C = 90^\circ$$
$$\therefore \ BC : AC : AB = 1 : 1 : \sqrt{2}$$

1b.

$$\because BC : AC : AB = 1 : 1 : \sqrt{2}$$
$$\therefore \ \angle A = 45^\circ \ , \angle B = 45^\circ, \angle C = 90^\circ$$

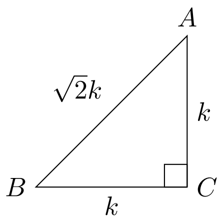
Proof. 1a. Let there be $\triangle ABC$ with $\angle A = \angle B = 45^\circ$, $\angle C = 90^\circ$.



Note that $AC = BC$ (sides opp. equal $\angle$s). Let $AB = AC = k$. Then by pyth. theorem, $AB = \sqrt{k^2 + k^2} = \sqrt{2}k$.

So $BC : AC : AB = k : k : \sqrt{2}k = 1 : 1 : \sqrt{2}$.

1b. Let there be $\triangle ABC$ with $BC : AC : AB = 1 : 1 : \sqrt{2}$.



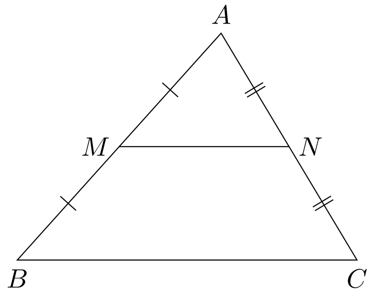
Let $BC = AC = k$. Note that $BC^2 + AC^2 = k^2 + k^2 = 2k^2$, and $AB^2 = (\sqrt{2}k)^2 = 2k^2$.
Since $BC^2 + AC^2 = AB^2$, by converse of pyth. theorem, we have $\angle C = 90^\circ$.

And since $BC = AC$, we have $\angle A = \angle B$ (base $\angle$s, isos. $\triangle$).
Thus $\angle A = \angle B = (180^\circ - 90^\circ)/2 = 45^\circ$ ($\angle$ sum of $\triangle$). □

1.8 Proportions and similar triangles

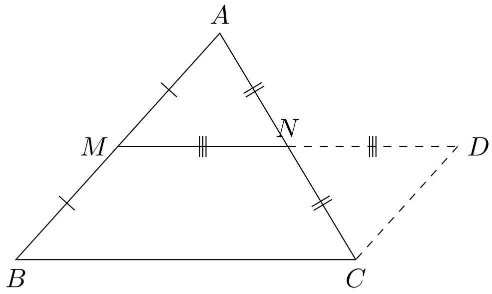
1.8.1 Proportions of side lengths

Preposition 25. In a triangle, the line segment joining the mid points of two sides is parallel to the remaining side, and is also half the length of this side. (mid-pt. theorem) *



$$\because AM = MB \text{ and } AN = NC$$
$$\therefore\ MN//BC \text{ and } MN = \frac{1}{2}BC \qquad \text{(mid-pt. theorem)}$$

Proof. Extend MN to D such that $MN = ND$.



In $\triangle ANM$ and $\triangle CND$,

$$AN = NC \qquad \text{(given)}$$
$$\angle ANM = \angle CND \qquad \text{(vert. opp. } \angle\text{s)}$$
$$MN = ND \qquad \text{(constructed)}$$
$$\therefore\ \triangle ANM \cong \angle CND \qquad \text{(SAS)}$$
$$\therefore\ AM = DC \qquad \text{(corr. sides, } \cong \triangle\text{s)}$$
$$\therefore\ MB = DC$$
$$\text{Also, } \angle AMN = \angle CDN \qquad \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}$$
$$\therefore\ AB//DC \qquad \text{(alt. } \angle\text{s supp.)}$$
$$\because MB = DC \text{ and } MB//DC$$
$$\therefore\ BCDM \text{ is a parallelogram.} \qquad \text{(opp. sides equal andd //)}$$
$$\therefore\ \text{(prop. 1) } MN//BC$$

$$\text{Also, } MD = BC \qquad \text{(opp. sides of //gram)}$$
$$\therefore\ \text{(prop. 2) } MN = ND = \frac{1}{2}MD = \frac{1}{2}BC \qquad (N \text{ is mid-pt. of } MD\ .)$$

□

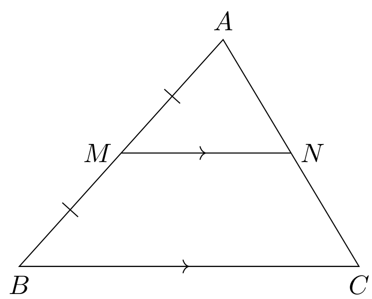
Preposition 26. There are two statements:

(i) In a triangle, if a line parallel to a side passes through the mid point of another side, then this line bisects the remaining side.

(ii) If there are three or more parallel lines and the intercepts (/segments) made by them on one transversal line are equal, then the parallel lines also make equal intercepts on any transversal line.

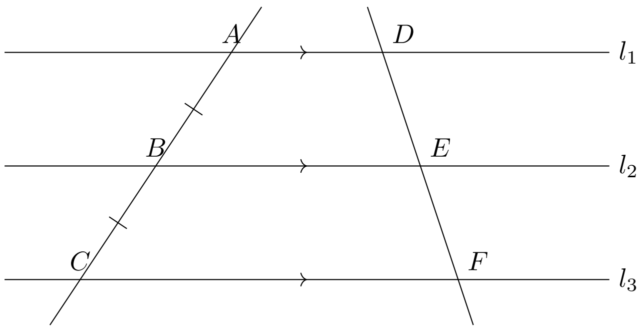
(intercept theorem) *

Case 1.



$$\because AM = MB \text{ and } MN//BC$$
$$\therefore\ AN = NC \qquad \text{(intercept theorem)}$$

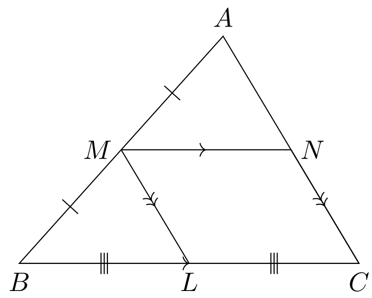
Case 2.



$$\because AD//BE//CF \text{ and } AB = BC$$
$$\therefore\ DE = EF \qquad \text{(intercept theorem)}$$

Proof. Case 1:

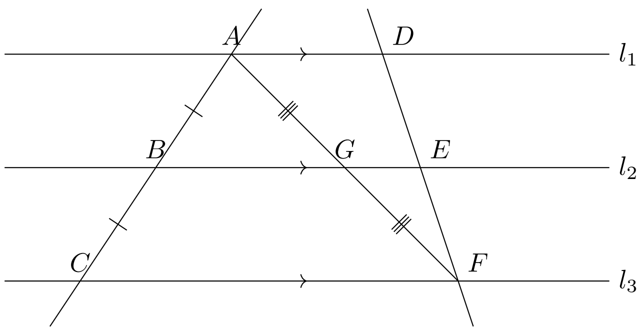
Let L be the mid-point of BC . Join ML .



By 'mid-pt. theorem' , we have $ML//AC$ and $ML = \frac{1}{2}AC$.
Since $MN//LC$ and $ML//NC$, $MNCL$ is a parallelogram. Thus, $ML = NC$ (opp. sides of //gram) , which means $NC = \frac{1}{2}AC$, which means N is the mid-point of AC , and thus $AN = NC$.

Case 2:
First consider the case where there are three parallel lines.

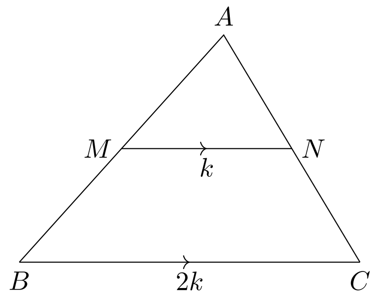
Join AF . Let AF intersect l_2 at G .



In $\triangle ACF$, we have $BG//CF$ and $AB = BC$. Thus $AG = GF$ by case 1 of intercept theorem.
In $\triangle ADF$, we have $AD//GE$ and $AG = GF$. Thus $DE = EF$ by case 1 of intercept theorem.

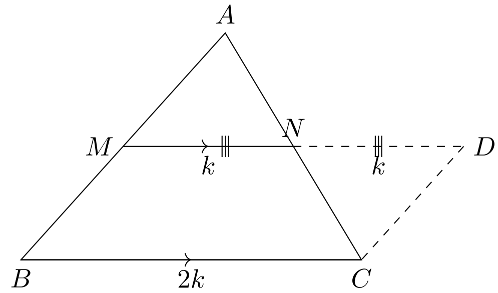
If there are more than three parallel lines, then we can prove that they all make equal intercepts on the transversal line by induction, since equality is transitive. (A.K.A. I'm too lazy to show it.) □

Preposition 27. In a triangle, if a line segment parallel to a side connects the other two sides, and is half the length of the parallel side, then the line segment joins the mid-points of the other two sides. (converse of mid-pt. theorem)



$$\because MN//BC \text{ and } MN = \frac{1}{2}BC$$
$$\therefore\ AM = MB \text{ and } AN = NC \qquad \text{(converse of mid-pt. theorem)}$$

Proof. Extend MN to D such that $MN = ND$. Then $MD = 2MN = BC$.

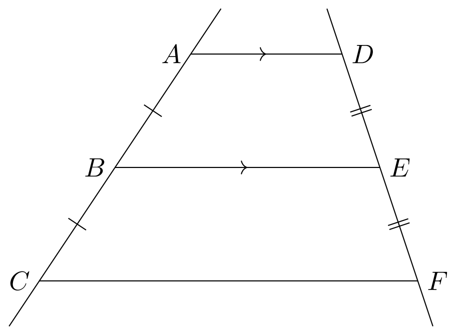


Since $MD = BC$ and $MD//BC$, $BCDM$ is a parallelogram. (opp. sides equal and //)
Thus, $MB//DC$.
In $\triangle ANM$ and $\triangle CND$,

$$\angle ANM = \angle CND \qquad \text{(vert. opp. } \angle\text{s)}$$
$$\angle MAN = \angle DCN \qquad \text{(alt. } \angle\text{s , } AB//DC\text{)}$$
$$MN = ND \qquad \text{(constructed)}$$
$$\therefore\ \triangle ANM \cong \angle CND \qquad \text{(AAS)}$$
$$\therefore\ AN = NC \qquad \text{(corr. sides, } \cong \triangle\text{s)}$$
$$\because MN//BC \text{ and } AN = NC$$
$$\therefore\ AM = MB \qquad \text{(intercept theorem)}$$

□

Preposition 28. For two lines, where on each of the lines there are three evenly spaced points, if the corresponding points of the lines are joined by line segments (one-to-one such that none intersect) such that one pair of line segments are parallel, then all three line segments are parallel. (converse of intercept theorem)

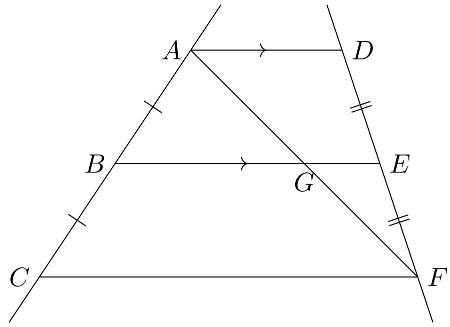


$$\because AB = BC \text{ , } DE = EF \text{ and } AD//BE$$
$$\therefore\ AD//BE//CF \qquad \text{(converse of intercept theorem)}$$

Proof. First consider the case where there are three evenly spaced points on each line.

Case 1: $AD//BE$

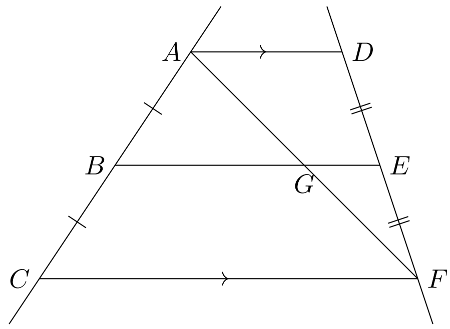
Join AF . Let AF and BE intersect at G .



Since $AD//GE$ and $DE = EF$, by (first case of) intercept theorem in $\triangle ADF$, we have $AG = GF$. Thus by mid-pt. theorem in $\triangle ACF$, we have $BG//CF$. By transitive property of parallel lines, we have $AD//BE//CF$.

Case 2: $AD//CF$

Join AF . Let AF and BE intersect at G .



Suppose G is not the mid-point of AF . Then let M be the mid-point of AF . By mid-point theorem in $\triangle ACF$ and $\triangle ADF$, we have $BM//CF$ and $AD//ME$. By transitive property of parallel lines, we have $AD//BM//CF$ and $AD//ME//CF$. Note that both BM and ME share a point M . By property of parallel lines, there can only be one unique line passing through M that is parallel to CF. Thus B, M, E all lie on the same line, and BME must be a straight line segment.

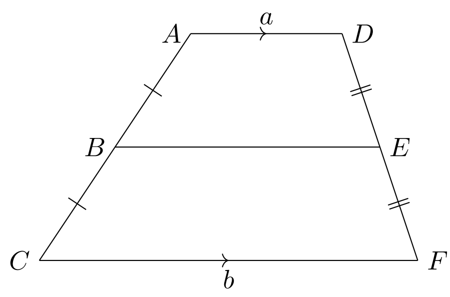
If G does not coincide with M , then B, G, E are not collinear, which is impossible since BE is a straight line segment and G lies on BE by definition (intersection of BE and AF).

Thus, G must be the mid-point of AF , and by mid-point theorem, we have $BG//CF$ and $AD//GE$, so $AD//BE//CF$ by transitivity of parallel lines.

□

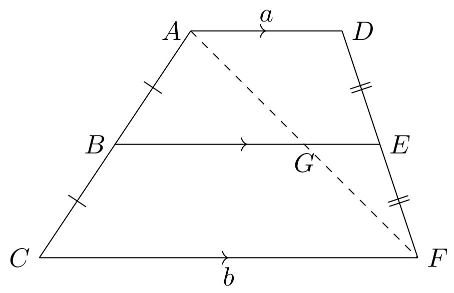
Preposition 29. The length of the median of a trapezium is the average of its upper base (a) and lower base (b). (median of trapezium)

Note: The median is the line segment joining the mid-point of the (usually) non-parallel side.



$$\because AD//CF \ , AB = BC \ , DE = EF$$
$$\therefore \ BE = \frac{a+b}{2} \qquad \text{(median of trapezium)}$$

Proof. Join AF . Let AF and BE intersect at G .



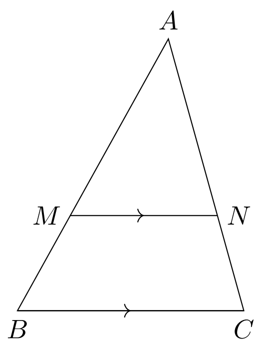
By 'converse of intercept theorem' , we have $BE//AD//CF$. By intercept theorem, we have $AG = GF$.

By 'mid-pt. theorem' , in $\triangle ACF$, we have $BG = \frac{1}{2}CF = \frac{1}{2}b$.

By 'mid-pt. theorem' , in $\triangle ACF$, we have $GE = \frac{1}{2}AD = \frac{1}{2}a$.

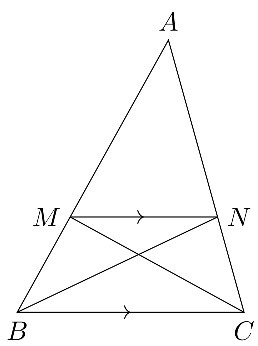
Thus, $BE = BG + GE = \frac{1}{2}b + \frac{1}{2}a = \frac{1}{2}(a+b)$. ☐

Preposition 30. In a triangle, if a line segment parallel to a side connects the other two sides, then it cuts these two sides with equal proportion. (general intercept theorem)



$$\because MN//BC$$
$$\therefore \ \frac{AM}{MB} = \frac{AN}{NC} \qquad \text{(general intercept theorem)}$$

Proof. [5] Join BN and CM .



Note that area of $\triangle BMN$ = area of $\triangle CMN$ since they have the same base and height. If we divide by area of $\triangle AMN$ on both sides, we have

$$\frac{\text{area of } \triangle BMN}{\text{area of } \triangle AMN} = \frac{\text{area of } \triangle CMN}{\text{area of } \triangle AMN}$$

Also note that by 'bases prop. to areas of $\triangle$s' ,

$$\frac{\text{area of }\triangle BMN}{\text{area of }\triangle AMN} = \frac{BM}{MA}$$

Again, by 'bases prop. to areas of $\triangle$s' ,

$$\frac{\text{area of }\triangle CMN}{\text{area of }\triangle AMN} = \frac{CN}{NA}$$

By transitive property of equality (applied twice):

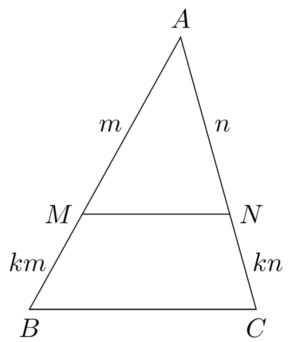
$$\frac{BM}{MA} = \frac{CN}{NA}$$

In other words:

$$\frac{AM}{MB} = \frac{AN}{NC}$$

□

Preposition 31. In a triangle, if a line segment connecting two sides cuts them with equal proportion, then the line segment is parallel to the remaining side. (general-pt. theorem)

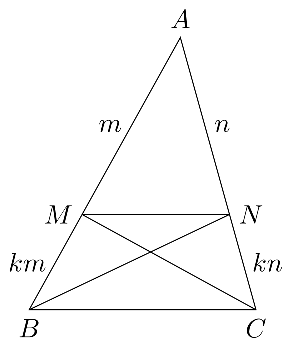


$$\because \frac{AM}{MB} = \frac{AN}{NC}$$

$$\therefore\ MN//BC \qquad \text{(general-pt. theorem)}$$

Proof. [5] (Note that $\dfrac{AM}{MB} = \dfrac{AN}{NC}$ is equivalent to $\dfrac{BM}{MA} = \dfrac{CN}{NA}$)

Join BN and CM .



By 'bases prop. to areas of $\triangle$s' , we have

$$\frac{BM}{MA} = \frac{\text{area of }\triangle BMN}{\text{area of }\triangle AMN} \qquad \text{and} \qquad \frac{CN}{NA} = \frac{\text{area of }\triangle CMN}{\text{area of }\triangle AMN}$$

Thus, by transitive property of equality :

$$\frac{\text{area of }\triangle BMN}{\text{area of }\triangle AMN} = \frac{\text{area of }\triangle CMN}{\text{area of }\triangle AMN}$$

which means

$$\text{area of } \triangle BMN = \text{area of } \triangle CMN$$

Since $\triangle BMN$ and $\triangle CMN$ have the same base and same area, by 'line joining $\triangle$ tips // base', the line joining their 'tips', which is BC , must be parallel to MN .

Thus, $MN//BC$.

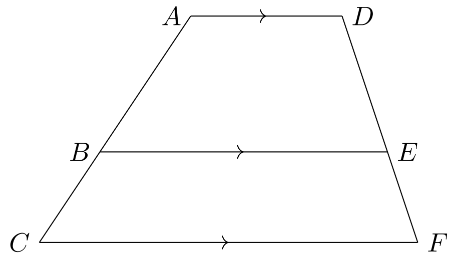
□

Preposition 32. There are two statements:

(i) In a trapezium, if a line segment parallel to the bases connects the other two sides (called the **legs**), then it cuts these two legs with equal proportion. (general intercept theorem [trapezium version])

(ii) In a trapezium, if a line segment connecting two legs cuts them with equal proportion, then the line segment is parallel to the bases. (general-pt. theorem [trapezium version])

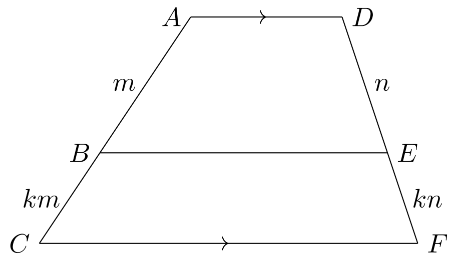
Case 1:



$$\because AD//BE//CF$$

$$\therefore \frac{AB}{BC} = \frac{DE}{EF} \qquad \text{(general intercept theorem [trapezium version])}$$

Case 2:

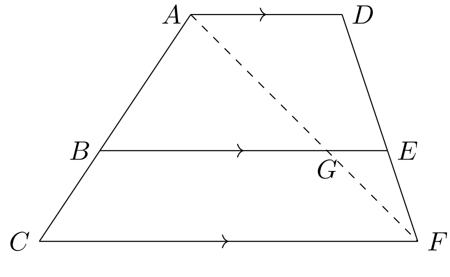


$$\because AD//CF \text{ , } \frac{AB}{BC} = \frac{DE}{EF}$$

$$\therefore BE//AD//CF \qquad \text{(general-pt. theorem [trapezium version])}$$

Proof. Join AF . Let AF and BE intersect at G .

Case 1:

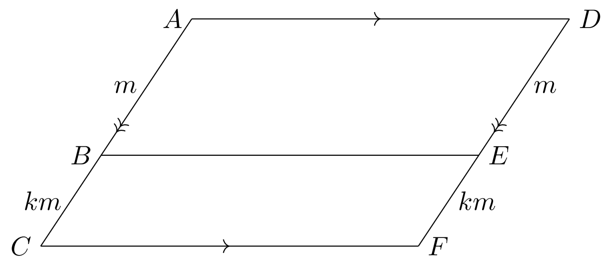


By the triangle version of general intercept theorem, in $\triangle ACF$, since $BG//CF$, we have $\frac{AB}{BC} = \frac{AG}{GF}$.

By the triangle version of general intercept theorem, in $\triangle ADF$, since $AD//GE$, we have $\frac{AG}{GF} = \frac{DE}{EF}$.

Thus, $\frac{AB}{BC} = \frac{DE}{EF}$.

<u>Case 2a:</u> $AC//DF$ (which means $ADFC$ is a parallelogram).



Note that $AC = DF$ (opp. sides of //gram) . Since $\frac{AB}{BC} = \frac{DE}{EF}$, by manipulating ratios we have $AB = DE$ and $BC = EF$.

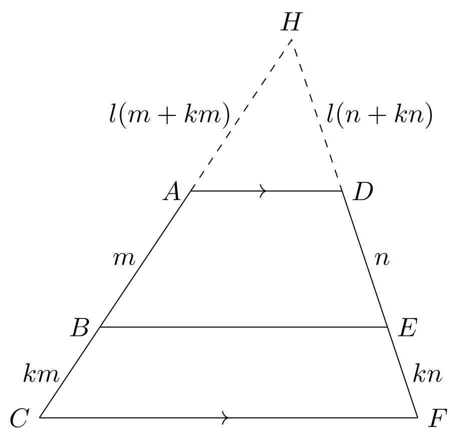
Since $AB//DE$ and $AB = DE$, note that $ADEB$ is a parallelogram (opp. sides equal and //) Similarly, $BEFC$ is a parallelogram.

By definition of parallelogram, we have $AD//BE$ and $BE//CF$. Thus $AD//BE//CF$.

<u>Case 2b:</u> $AC\neg//DF$ (which means $ADFC$ is a proper trapezium).

Let H be the intersection of line CA and FD .

Let $AB = m$, $BC = km$, $DE = n$, $EF = kn$, $HA = p$.



Consider $\triangle HCF$. Since $AD//CF$, note that $\frac{HA}{AC} = \frac{HD}{DF}$.

Let $HA = l \cdot AC = l(m + km)$ and $HD = l \cdot DF = l(n + kn)$.

Thus

$$\begin{aligned} \frac{HB}{BC} &= \frac{l(m+km)+m}{km} \\ &= \frac{m(l+k+1)}{km} \\ &= \frac{l+k+1}{k} \end{aligned}$$

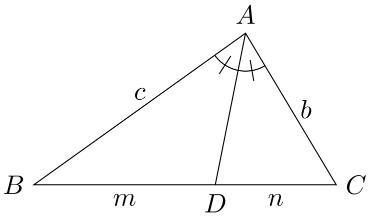
$$
\begin{aligned}
\frac{HE}{EF} &= \frac{l(n+kn)+n}{kn} \\
&= \frac{n(l+k+1)}{kn} \\
&= \frac{l+k+1}{k}
\end{aligned}
$$

Since $\dfrac{HB}{BC} = \dfrac{HE}{EF}$, we have $BE//CF$ by triangle version of general-pt. theorem.

By transitivity of parallel lines, we have $AD//BE//CF$.

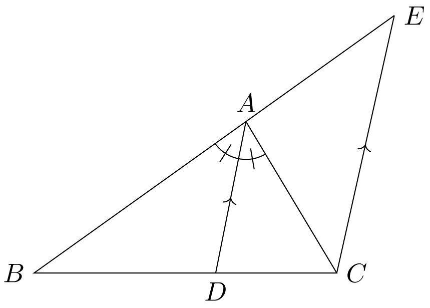
□

Preposition 33. Given $\triangle ABC$, if D is a point on BC such that AD is the angle bisector of $\angle A$, then $\dfrac{AB}{AC} = \dfrac{BD}{DC}$. (angle bisector theorem)



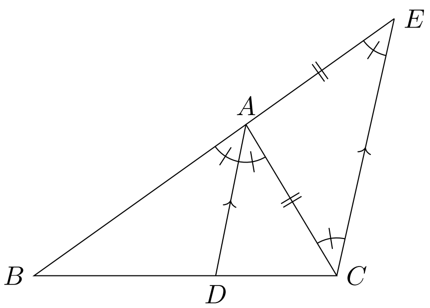
$$\because \angle BAD = \angle CAD$$
$$\therefore \frac{c}{b} = \frac{m}{n} \qquad \text{(angle bisector theorem)}$$

Proof. [6] Extend BA . Let E be a point on line BA such that $AD//EC$.



Note that $\angle ACE = \angle CAD$ (alt. $\angle$s , $AD//EC$) , and $\angle AEC = \angle BAD$ (corr. $\angle$s , $AD//EC$) .

Thus, $\angle ACE = \angle AEC$, which means $AC = AE$ (sides opp. equal $\angle$s).

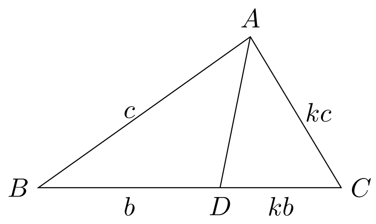


By general intercept theorem, we have $\dfrac{AB}{AE} = \dfrac{BD}{DC}$. Replace AE with AC , we have:

$$\frac{AB}{AC} = \frac{BD}{DC}$$

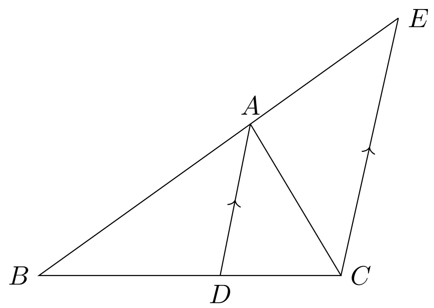
□

Preposition 34. Given $\triangle ABC$, if D is a point on BC such that $\dfrac{AB}{AC} = \dfrac{BD}{DC}$, then AD is the angle bisector of $\angle A$. (converse of angle bisector theorem)

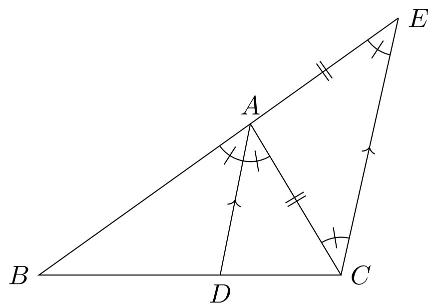


$$\because \frac{AB}{AC} = \frac{BD}{DC}$$
$$\therefore \angle BAD = \angle CAD \qquad \text{(converse of angle bisector theorem)}$$

Proof. [6] Extend BA . Let E be a point on line BA such that $AD//EC$.



By general intercept theorem, we have $\dfrac{AB}{AE} = \dfrac{BD}{DC}$.
By initial assumption, we have $\dfrac{AB}{AC} = \dfrac{BD}{DC}$.
Thus, $\dfrac{AB}{AE} = \dfrac{AB}{AC}$, which means $AC = AE$. So $\angle ACE = \angle AEC$ (base $\angle$s, isos. $\triangle$).



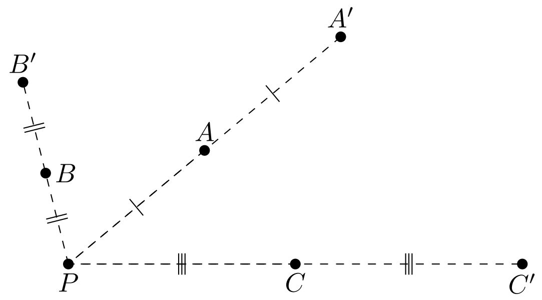
Note that $\angle CAD = \angle ACE$ (alt. $\angle$s , $AD//EC$) , and $\angle BAD = \angle AEC$ (corr. $\angle$s , $AD//EC$).
By transitive property of equality, $\angle BAD = \angle CAD$. ☐

1.8.2 Dilation

Dilation is a geometric transformation that can be used to change the size of objects.

In a dilation, there is a **point of dilation** that doesn't change position, and other points will be pushed away / pulled near that point by a constant **scale factor** to form the images.

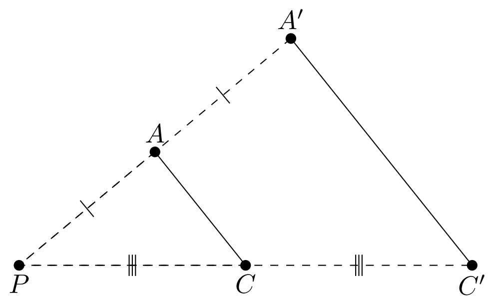
For example, if the scale factor is 2, then the dilated images of the points will be twice as far as their corresponding original points from the point of dilation (P):



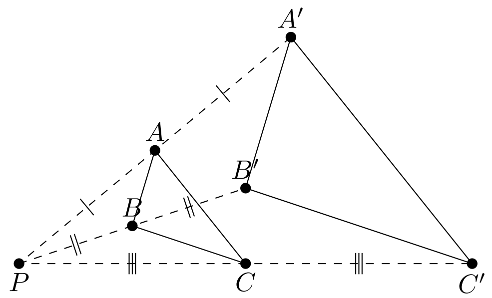
$$\because \text{Scale factor} = 2$$
$$\therefore\ PA' = 2 \cdot PA \ \ ,\ \ PB' = 2 \cdot PB \ \ ,\ \ PC' = 2 \cdot PC$$

We can dilate not only points, but also lines segments and shapes.

To dilate a line segment, both endpoints of the line segment get dilated by the scale factor. The images of the endpoints are joined to make the image line segment:

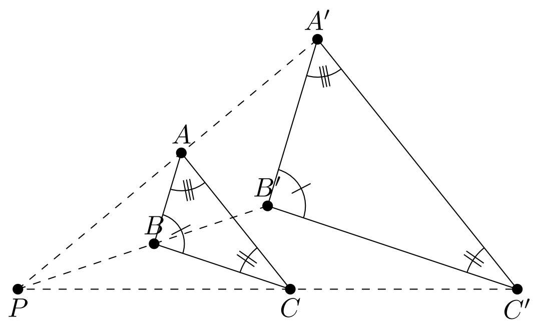


Similarly, to dilate a triangle, all vertices of the triangle get dilated by the scale factor and the image triangle is formed by connecting the image vertices:



Note that dilation preserves all angles of the triangle.

Preposition 35. When a triangle is dilated about a point, the corresponding angles of the image triangle and the original triangle are equal. (prop. of dilation)



$$\because \frac{PA'}{PA} = \frac{PB'}{PB} = \frac{PC'}{PC} = k \qquad (k = \text{scale factor})$$
$$\therefore\ \angle A = \angle A' \ ,\ \angle B = \angle B' \ ,\ \angle C = \angle C' \qquad \text{(prop. of dilation)}$$

Proof. Let P be the point of dilation. Then P may or may not lie on one of the vertices or extended sides of the triangle.

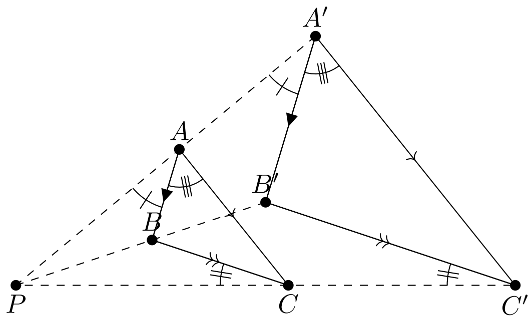
<u>Case 1:</u> Suppose that P does not lie on any vertices or any extend sides.

Since P is the point of dilation, the vertices of the triangles are pushed away / pulled near P by a constant factor. Thus we have $\dfrac{PA'}{PA} = \dfrac{PB'}{PB} = \dfrac{PC'}{PC}$.

Since $\dfrac{PA'}{PA} = \dfrac{PB'}{PB}$, we have $AB//A'B'$ (general-pt. theorem) .

Similarly, we have $BC//B'C'$ and $AC//A'C'$ (general-pt. theorem).

Thus, $\angle PAB = \angle PA'B'$ (corr. $\angle$s , $AB//A'B'$) , and $\angle PAC = \angle PA'C'$ (corr. $\angle$s , $AC//A'C'$).

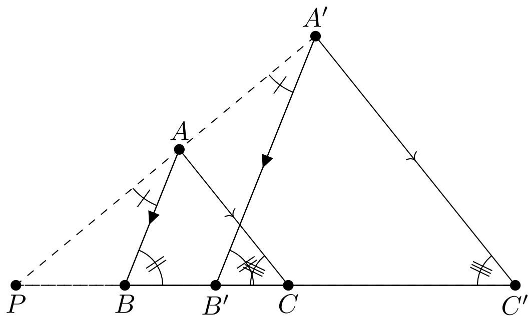


So $\angle BAC = \angle PAC - \angle PAB = \angle PA'C' - \angle PA'B' = \angle B'A'C'$.

By similar reasoning, we have $\angle BCA = \angle B'C'A'$.

Thus $\angle ABC = \angle A'B'C'$ by ($\angle$ sum of $\triangle$).

<u>Case 2:</u> Suppose that P lies line BC .

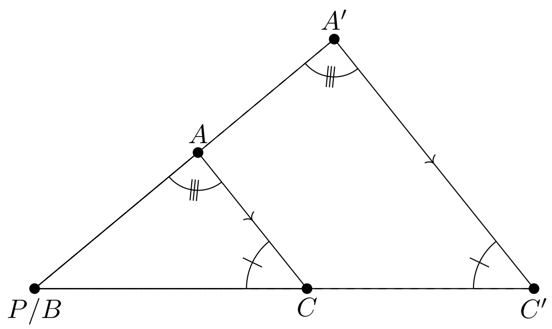


We have $\dfrac{PA'}{PA} = \dfrac{PB'}{PB} = \dfrac{PC'}{PC}$. So we still have $AB//A'B'$, $AC//A'C'$ by 'general-pt. theorem'.

Thus $\angle PAB = \angle PA'B'$, $\angle PAC = \angle PA'C'$, $\angle PCA = \angle PC'A'$, $\angle ABC = \angle A'B'C'$ by corresponding angles of parallel lines.

So $\angle BAC = \angle B'A'C'$.

<u>Case 3:</u> Suppose P lies on B .



We have $\dfrac{PA'}{PA} = \dfrac{PC'}{PC}$. So we still have $AC//A'C'$ by 'general-pt. theorem'.

Thus, $\angle PAC = \angle PA'C'$, $\angle PCA = \angle PC'A'$ by corresponding angles of parallel lines. And $\angle APC = \angle A'PC'$ by common angle.

□

1.8.3 Similar triangles

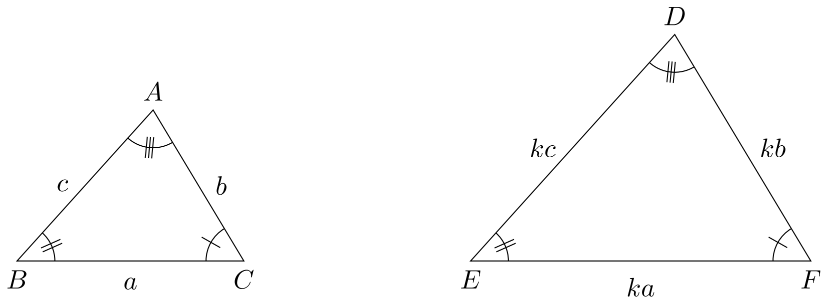
Two triangles are called **similar** if one triangle can be translated, rotated, reflected, and dilated (scaled) in any way to perfectly overlap with another triangle.

All congruent triangles are similar, but not all similar triangles are congruent, and the difference is their sizes. Two similar triangles are allowed to be in different sizes, but two congruent triangles are only allowed to be in the same size.

To denote that two triangles are similar, like $\triangle ABC$ and $\triangle DEF$, we say that $\triangle ABC \sim \triangle DEF$. Note that $\sim$ is an equivalence relation, just like congruence.

Properties

A pair of similar triangles have the corresponding angles equal, just like congruent triangles. In addition, the ratio of the corresponding sides of the similar triangles are also equal:

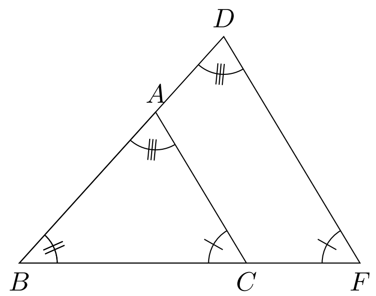


$$\because \triangle ABC \sim \triangle DEF$$
$$\therefore\ \angle A = \angle D \text{ , } \angle B = \angle E \text{ , } \angle C = \angle F \qquad \text{(corr. } \angle\text{s, } \sim \triangle\text{s)*}$$
$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} = \frac{1}{k} \qquad \text{(corr. sides, } \sim \triangle\text{s)*}$$

We will prove this later.

Conversely, if a pair of triangles have their corresponding angles equal and corresponding sides proportional, then they are similar triangles by definition.

To see why, let $\triangle ABC$ and $\triangle DEF$ have their corresponding angles equal and corresponding sides proportional. Move $\triangle DEF$ such that $\angle DEF$ coincides with $\angle ABC$.



Recall dilation, which is one of the geometric transformations. To dilate a triangle about a point, all the vertices are pushed away / pulled near the point of dilation by a constant factor.

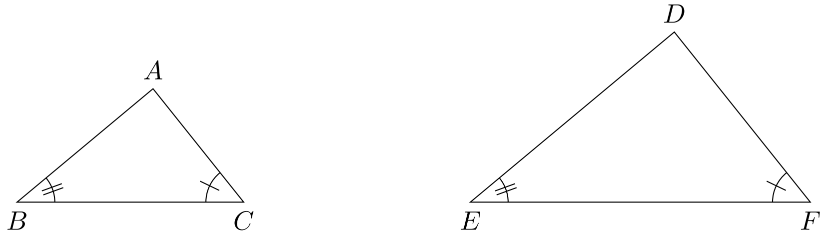
Since $\dfrac{BD}{BA}$ and $\dfrac{BF}{BC}$, $\triangle ABC$ can be dilated about point B to make $\triangle DEF$. So $\triangle ABC \sim \triangle DEF$.

1.8.4 Conditions for determining similarity

Just like congruence, there are a few minimum conditions that are sufficient to determine similarity.

1. AA (Angle-angle)

Two triangles are similar if two corresponding angles are equal.

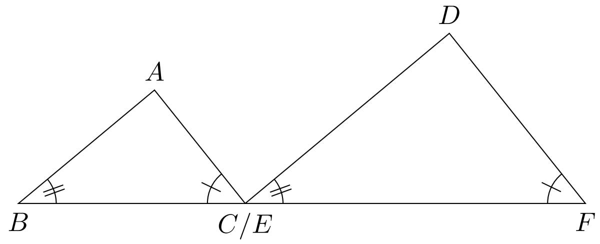


$$\because \angle B = \angle E \ , \ \angle C = \angle F$$
$$\therefore \ \triangle ABC \sim \triangle DEF \qquad \text{(AA)} *$$ [1]

Proof. [7] Note that when two corresponding angles are equal, the third corresponding angles must also be equal, because

$$\begin{aligned}\angle A &= 180° - \angle B - \angle C \qquad (\angle \text{ sum of } \triangle) \\ &= \angle 180° - \angle E - \angle F \qquad (\angle B = \angle E \ , \ \angle C = \angle F) \\ &= \angle D \qquad (\angle \text{ sum of } \triangle)\end{aligned}$$

Move vertex E to coincide with C such that B, C, F lies on a straight line.

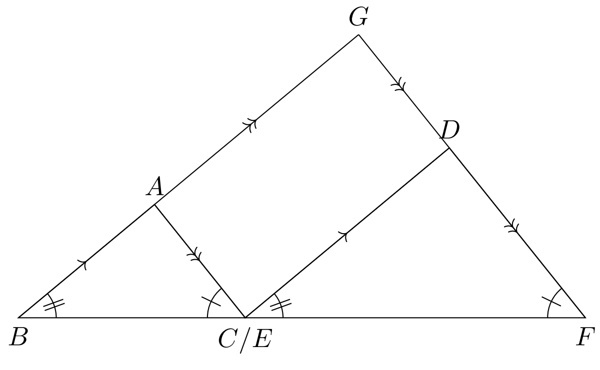


Note that $\angle ABC + \angle ACB < 180°$ ($2 \ \angle$ sum of $\triangle$) and $\angle ACB = \angle DFC$ (given), so we have $\angle ABC + \angle DFC < 180°$.

From parallel postulate, BA and FD , when extended, will meet at a point, say G .

Since $\angle ABC = \angle DCF$ (given) , we have $AB//DC$ (corr. $\angle$s equal) .

Similarly, since $\angle ACB = \angle DFC$, we have $AC//DF$ (corr. $\angle$s equal) .



Then $AGDC$ is a parallelogram by definition. Thus, $AG = CD$ and $AC = GD$ (opp. sides of //gram).

Since $AC//GF$, we have $\dfrac{BA}{AG} = \dfrac{BC}{CF}$ (general intercept theorem). Replace AG with CD , we have $\dfrac{BA}{CD} = \dfrac{BC}{CF}$.

[1](AA) is equivalent to (AAA) , of which the latter is used in HK secondary maths.

Similarly, since $BG//CD$, we have $\dfrac{BC}{CF} = \dfrac{GD}{DF}$ (general intercept theorem). Replace GD with AC , we have $\dfrac{BC}{CF} = \dfrac{AC}{DF}$.

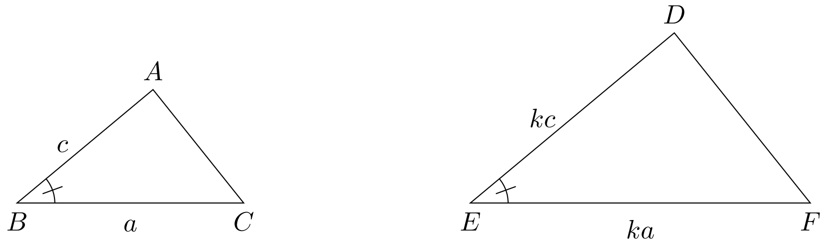
By transitive property of equality, we have $\dfrac{BA}{CD} = \dfrac{BC}{CF} = \dfrac{AC}{DF}$.

Since point C is also point E , finally we have $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF}$. So $\triangle ABC \sim DEF$.

□

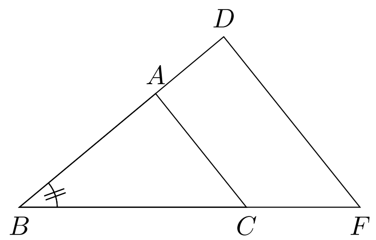
2. Ratio of two sides, included angle

Two triangles are similar if two corresponding sides are proportional and the included angles are equal.



$$\because \frac{AB}{DE} = \frac{BC}{EF} \text{ and } \angle ABC = \angle DEF$$
$$\therefore \triangle ABC \sim \triangle DEF \qquad \text{(ratio of 2 side, inc. } \angle\text{) *}$$

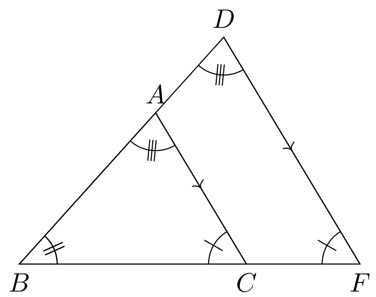
Proof. Move vertex E to coincide with B such that F lies on line BC and D lies on line BA .



Note that

$$\begin{aligned} \frac{AB}{DE} &= \frac{BC}{EF} && \text{(given)} \\ \frac{DB}{AB} &= \frac{BF}{BC} && \text{(flipping ratio \& point } B = E) \\ \frac{DB}{AB} - 1 &= \frac{BF}{BC} - 1 && \text{(subtractive property)} \\ \frac{DB}{AB} - \frac{AB}{AB} &= \frac{BF}{BC} - \frac{BC}{BC} && (\frac{k}{k} = 1) \\ \frac{DA}{AB} &= \frac{CF}{BC} && \text{(segment addition postulate)} \end{aligned}$$

Thus, $AC//DF$ by ‘general-pt. theorem’ . Thus, we have $\angle BAC = \angle BDF$ and $\angle BCA = \angle BFD$ (corr. $\angle$s , $AC//DF$) .

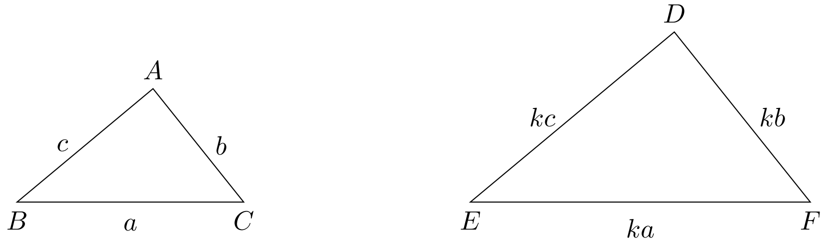


Since all three corresponding angles in $\triangle ABC$ and $\triangle DEF$ are equal, we have $\triangle ABC \sim \triangle DEF$ (AA).

$\square$

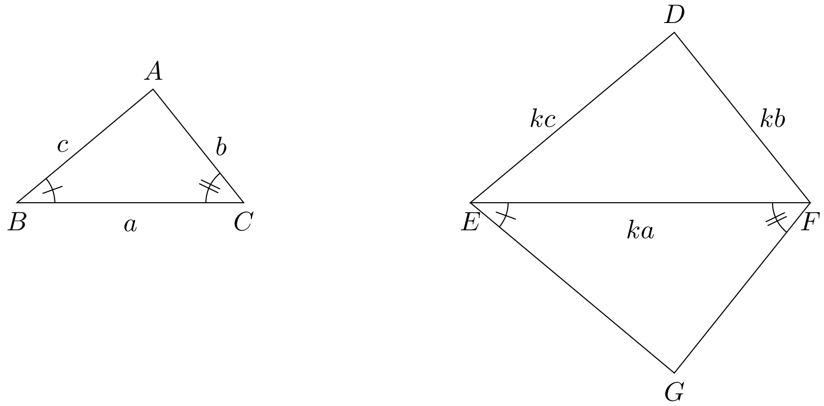
3. Three sides proportional [8]

Two triangles are similar if three corresponding sides are proportional.



$$\because \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$
$$\therefore \triangle ABC \sim \triangle DEF \qquad \text{(3 sides prop.)}\ *$$

Proof. Let $\triangle EFG$ be a triangle such that $\angle FEG = \angle ABC$ and $\angle EFG = \angle ACB$.



Note that $\angle ABC \sim \angle GEF$ (AA) . Since similar triangles have three sides proportional, we have $\dfrac{AB}{GE} = \dfrac{BC}{EF}$ (corr. sides, $\sim \triangle$s).

By initial assumption, we have $\dfrac{AB}{DE} = \dfrac{BC}{EF}$. Thus, $\dfrac{AB}{GE} = \dfrac{AB}{DE}$, which means $GE = DE$.

Similarly, we have $\dfrac{AC}{GF} = \dfrac{BC}{EF}$ (corr. sides, $\sim \triangle$s). By initial assumption, we have $\dfrac{AC}{DF} = \dfrac{BC}{EF}$. Thus, $\dfrac{AC}{GF} = \dfrac{AC}{DF}$, which means $GF = DF$.

In $\triangle DEF$ and $\triangle GEF$,

$$\begin{aligned} DE &= GE && \text{(proven above)} \\ DF &= GF && \text{(proven above)} \\ EF &= EF && \text{(common side)} \\ \therefore \triangle DEF &\cong \triangle GEF && \text{(SSS)} \end{aligned}$$

Thus, $\angle DEF = \angle GEF$, $\angle DFE = \angle GFE$, $\angle EDF = \angle EGF$ (corr. $\angle$s, $\cong \triangle$s).

Since $\angle GEF = \angle ABC$ by construction, it follows that $\angle ABC = \angle DEF$.

Similarly, since $\angle GFE = \angle ACB$ by construction, it follows that $\angle ACB = \angle DFE$.

We've shown that two corresponding angles are equal in $\triangle ABC$ and $\triangle DEF$. Thus $\angle BAC = \angle EDF$ by ($\angle$ sum of $\triangle$).

Thus all three corresponding angles are equal. So $\triangle ABC \sim DEF$. $\square$

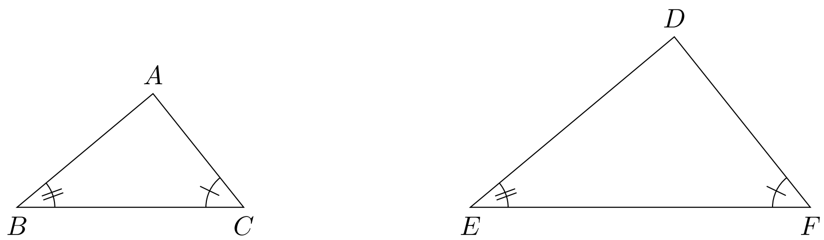
1.8.5 Prepositions related to similar triangles

Let's summarize the conditions for determining similar triangles in a preposition:

Preposition 36. Two triangles are similar if they satisfy one of the following conditions:

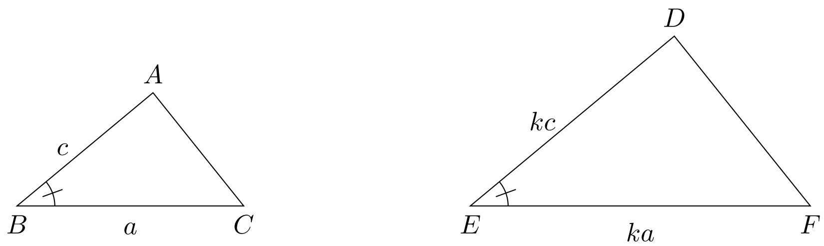
1. Two corresponding angles are equal. (AA)
2. Two corresponding sides are proportional, and the included angles are equal. (ratio of 2 sides, inc. $\angle$)
3. Three corresponding sides are proportional. (3 sides prop.)

1. AA (Angle-Angle)



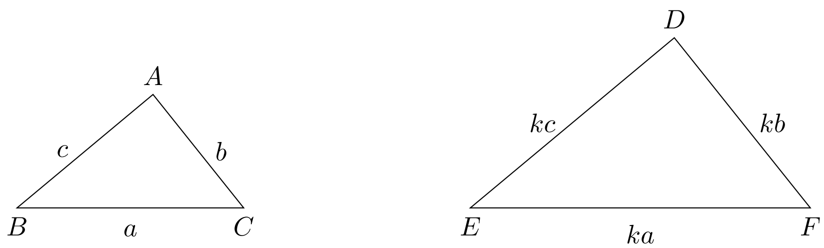
$$\because \angle B = \angle E \text{ , } \angle C = \angle F$$
$$\therefore \triangle ABC \sim \triangle DEF \qquad \text{(AA)} *$$

2. Ratio of two sides, included angle



$$\because \frac{AB}{DE} = \frac{BC}{EF} \text{ and } \angle ABC = \angle DEF$$
$$\therefore \triangle ABC \sim \triangle DEF \qquad \text{(ratio of 2 side, inc. } \angle) *$$

3. Three sides proportional

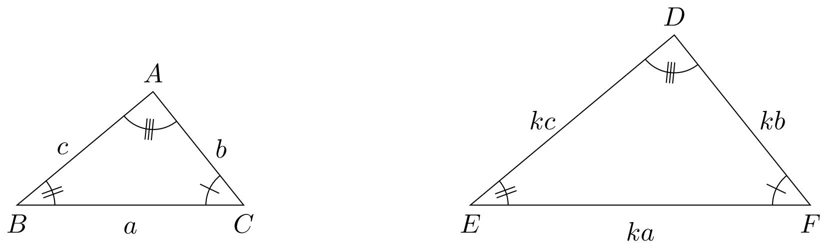


$$\because \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$
$$\therefore \triangle ABC \sim \triangle DEF \qquad \text{(3 sides prop.)} *$$

Preposition 37. If two triangles are similar, then

1. Their corresponding sides are proportional. (corr. sides, $\sim \triangle$s)

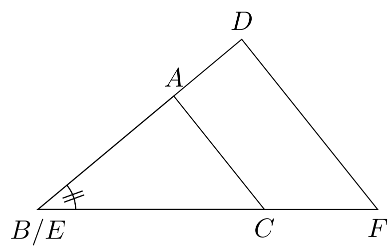
2. Their corresponding angles are equal. (corr. $\angle$s, $\sim\triangle$s)



$$\because \triangle ABC \sim \triangle DEF$$
$$\therefore\ \angle A = \angle D\ ,\ \angle B = \angle E\ ,\ \angle C = \angle F \qquad \text{(corr. } \angle\text{s, } \sim\triangle\text{s)*}$$
$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} = \frac{1}{k} \qquad \text{(corr. sides, } \sim\triangle\text{s)*}$$

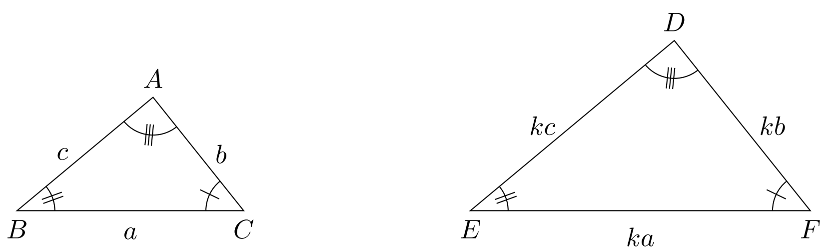
Proof. (This is just repeating the proof in 'ratio of 2 sides, inc. $\angle$' so I'll be shorter.)

Since $\triangle ABC \sim \triangle DEF$, one can be dilated and moved to complete coincide with the other. Let's dilate $\triangle ABC$ about a point, say B, to make $\triangle DEF$, where E coincides with B. Then AB and BC are scaled by a constant factor to make DE and EF . Thus we have $\frac{DE}{AB} = \frac{EF}{BC}$, which means $\frac{DA}{AB} = \frac{FC}{CB}$.



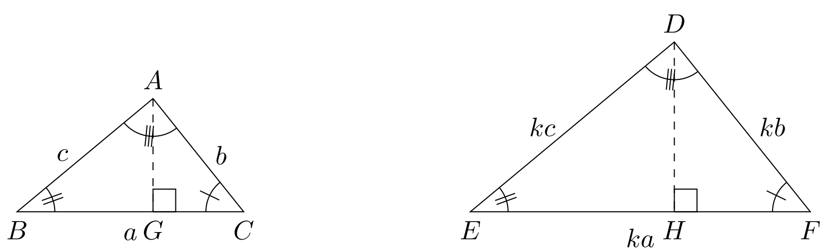
Thus $AC//DF$ by general intercept theorem, and thus we have $\angle ACB = \angle DFE$ and $\angle BAC = \angle EDF$ (alt. $\angle$s , $AC//DF$). So all three corresponding angles of the two triangles are equal, which means all three sides are proportional by (AA) . ☐

Preposition 38. The ratio of the areas of two similar triangles is the square of the ratio of their corresponding sides. (corr. areas, $\sim\triangle$s)



$$\because \triangle ABC \sim \triangle DEF$$
$$\frac{\text{area of } \triangle ABC}{\text{area of } \triangle DEF} = (\frac{AB}{DE})^2 = (\frac{BC}{EF})^2 = (\frac{AC}{DF})^2 = (\frac{1}{k})^2 \qquad \text{(corr. areas, } \sim\triangle\text{s)}$$

Proof. Let $AG \perp BC$ and $DH \perp EF$.



Since $\triangle ABC \sim \triangle DEF$ (given), we have $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF}$ (corr. sides, $\sim \triangle$s).

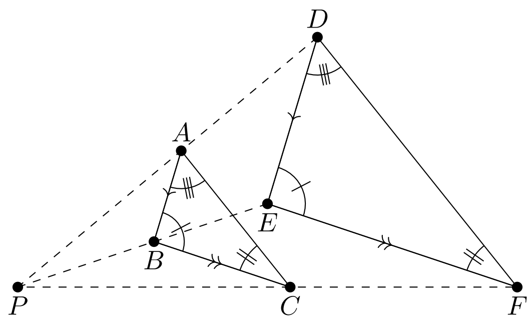
Note that $\angle B = \angle E$ (corr. $\angle$s, $\sim \triangle$s) and $\angle AGB = \angle DHE = 90°$. Thus $\triangle ABG \sim \triangle DEH$ (AA). Thus $\dfrac{AG}{DH} = \dfrac{AB}{DE}$ (corr. sides, $\sim \triangle$s).

This means the ratio of their heights is the same as the ratio of their side lengths. We have

$$\begin{aligned}
\frac{\text{area of } \triangle ABC}{\text{area of } \triangle DEF} &= \frac{\frac{1}{2}\, AG \cdot BC}{\frac{1}{2}\, DH \cdot EF} && (\text{area of } \triangle) \\
&= \frac{\frac{1}{2}\, AB \cdot BC}{\frac{1}{2}\, DE \cdot EF} && (\frac{AG}{DH} = \frac{AB}{DE}) \\
&= \frac{AB \cdot AB}{DE \cdot DE} && (\frac{AB}{DE} = \frac{BC}{EF}) \\
&= (\frac{AB}{DE})^2 \\
&= (\frac{BC}{EF})^2 = (\frac{AC}{DF})^2
\end{aligned}$$

□

Preposition 39. For two similar but not congruent triangles, if they are oriented such that all of their corresponding sides are parallel (which means they are in **homothetic position**), then the lines joining the corresponding vertices of two similar triangles are concurrent. (concurrent property of homothetic $\sim \triangle$s)



$\because \triangle ABC \sim \triangle DEF$, $AB//DE$, $BC//EF$

$\therefore$ line DA , line EB , line CF are concurrent. (concurrent property of $\sim \triangle$s)

Proof. Let there be such similar triangles $\triangle ABC \sim \triangle DEF$ where $\triangle DEF$ is larger. Since the triangles are not congruent, their corresponding side lengths are not the same, which means the lines joining the corresponding vertices are not parallel.

Note that if two corresponding sides are parallel, then the third corresponding sides will also be parallel because of corresponding angles of parallel lines being equal. So the minimum condition is two pairs of parallel sides.

Let DA and EB intersect at P_1 while EB and FC intersect at P_2 . Note that $\triangle P_1AB \sim \triangle P_1DE$ and $\triangle P_2BC \sim \triangle P_2EF$ (AA).

So in $\triangle P_1AB$ and $\triangle P_1DE$, $\dfrac{P_1B}{P_1E} = \dfrac{AB}{DE}$ (corr. sides, $\sim \triangle$s).

In $\triangle P_2BC$ and $\triangle P_2EF$, $\dfrac{P_2B}{P_2E} = \dfrac{BC}{EF}$ (corr. sides, $\sim \triangle$s).

But $\dfrac{AB}{DE} = \dfrac{BC}{EF}$ (corr. sides, $\triangle ABC \sim \triangle DEF$) .

Thus by transitivity of equality, we have $\dfrac{P_1B}{P_1E} = \dfrac{P_2B}{P_2E}$, and

$$\begin{aligned}
\frac{P_1E}{P_1B} &= \frac{P_2E}{P_2B} \\
\frac{P_1E}{P_1B} - \frac{P_1B}{P_1B} &= \frac{P_2E}{P_2B} - \frac{P_2B}{P_2B} \\
\frac{BE}{P_1B} &= \frac{BE}{P_2B} \\
P_1B &= P_2B
\end{aligned}$$

Since P_1, P_2, E, B are collinear, P_1 and P_2 must coincide, which means DA, EB, FC are concurrent. □

Other similar polygons

(Copied from Wikipedia) [9]

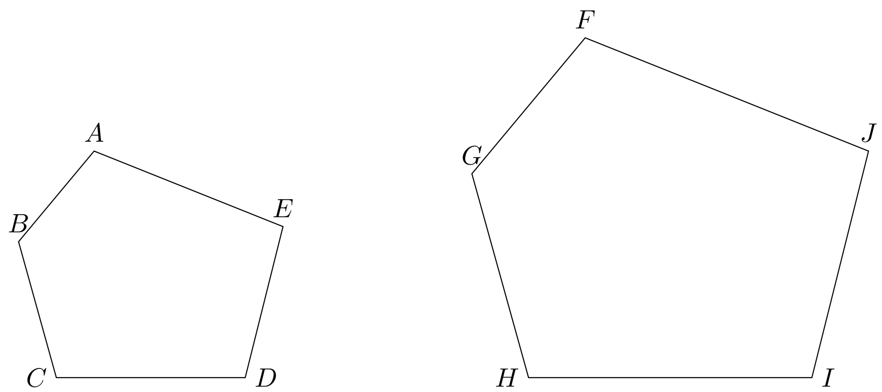
The concept of similarity extends to polygons with more than three sides. Two polygons are similar if one can be dilated and moved to perfectly overlap with another.

Given any two similar polygons, corresponding sides taken in the same sequence (even if clockwise for one polygon and counterclockwise for the other) are proportional and corresponding angles taken in the same sequence are equal in measure. However, proportionality of corresponding sides is not by itself sufficient to prove similarity for polygons beyond triangles (otherwise, for example, all rhombuses would be similar). Likewise, equality of all angles in sequence is not sufficient to guarantee similarity (otherwise all rectangles would be similar). A sufficient condition for similarity of polygons is that corresponding sides and diagonals are proportional.

For given n, all regular n-gons are similar.

Preposition 40. If two polygons are similar, then their corresponding sides are proportional, and their corresponding angles are equal. (corr. sides, $\sim$ polygons) & (corr. $\angle$s , $\sim$ polygons)

Example



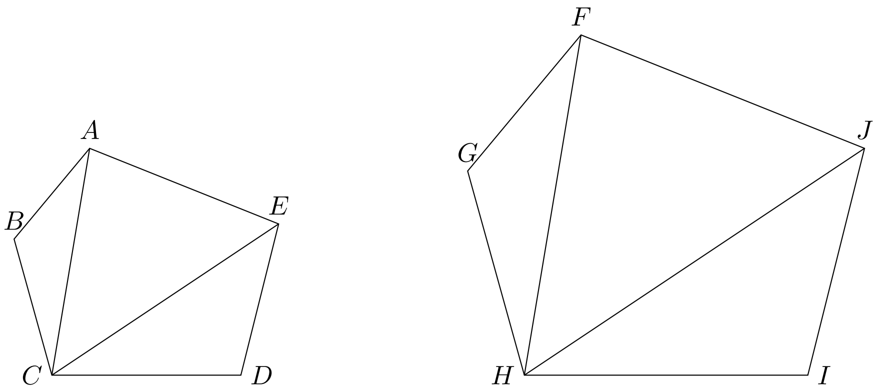
$$\because ABCDE \sim FGHIJ$$

$$\therefore \ \frac{AB}{FG} = \frac{BC}{GH} = \frac{CD}{HI} = \frac{DE}{IJ} = \frac{EA}{JF} \qquad \text{(corr. sides, } \sim \text{ polygons)}$$

Also, $\angle A = \angle F$, $\angle B = \angle G$, $\angle C = \angle H$, $\angle D = \angle I$, $\angle E = \angle J$ (corr. $\angle$s , $\sim$ polygons)

Proof. Note that every polygon can be split into triangles. Since the angles of the triangles are preserved in dilation, all the corresponding angles of the triangles are equal, so they are similar.

Refer to the figure.



Note that corresponding sides of similar triangles are proportional.

In $\triangle ABC$ and $\triangle FGH$, we have $\dfrac{AB}{FG} = \dfrac{BC}{GH} = \dfrac{AC}{FH}$.

In $\triangle ACE$ and $\triangle FHJ$, we have $\dfrac{AC}{FH} = \dfrac{AE}{FJ} = \dfrac{CE}{HI}$.

In $\triangle CDE$ and $\triangle HIJ$, we have $\dfrac{CE}{HJ} = \dfrac{CD}{HI} = \dfrac{DE}{IJ}$.

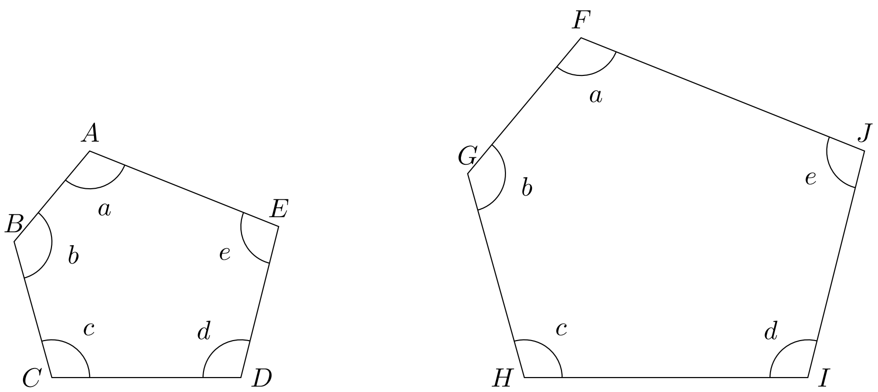
By transitivity of equality, we have $\dfrac{AB}{FG} = \dfrac{BC}{GH} = \dfrac{CD}{HI} = \dfrac{ED}{JI} = \dfrac{AE}{FJ}$.

Similar reasoning can be applied to other similar polygons, since the triangles making up the polygon must stick together (sharing a side with at least one other triangle). So equal proportions are transitive over the commons sides.

□

Preposition 41. For two polygons, if their corresponding sides taken in the same sequences are proportional, and their corresponding angles taken in the same sequences are equal, then the two polygons are similar. (corr. sides prop. and corr. ∠s equal)

Example



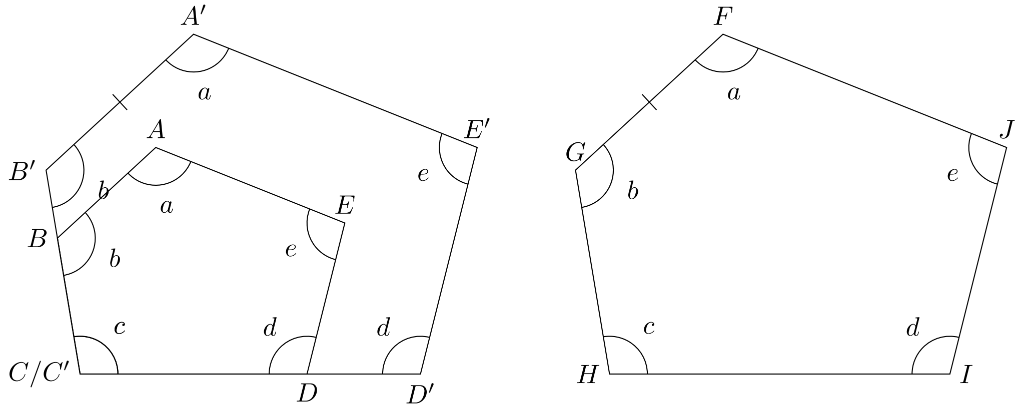
$$\because \frac{AB}{FG} = \frac{BC}{GH} = \frac{CD}{HI} = \frac{DE}{IJ} = \frac{EA}{JF}$$

$$\text{And, } \angle A = \angle F \text{ , } \angle B = \angle G \text{ , } \angle C = \angle H \text{ , } \angle D = \angle I \text{ , } \angle E = \angle J$$

$$\therefore \ ABCDE \sim FGHIJ \qquad \text{(corr. sides prop. and corr. } \angle\text{s equal)}$$

Proof. Note: it is assumed that the corresponding vertices correspond in the same direction (say, clockwise.) Otherwise, we can just flip (reflect) one of them to make that happen.

Dilate $ABCDE$ to make image polygon $A'B'C'D'E'$ such that one of the corresponding sides are equal, say $A'B' = FG$.



Note that all of the side lengths of the image polygon get multiplied by the same scale factor of the dilation, i.e. $\dfrac{A'B'}{AB}$. This is because a polygon can always be split into triangles, and dilation make the side lengths of the image triangles proportional to their original ones.

Then all the other corresponding sides of the image polygon $A'B'C'D'E'$ and $FGHIJ$ must be equal since all of the image side lengths get multiplied by the scale factor $\dfrac{FG}{AB}(=\dfrac{A'B'}{AB})$, as mentioned above. So the proportions, say, $\dfrac{B'C'}{GH}=\dfrac{BC(\frac{FG}{AB})}{GH}=\dfrac{AB}{FG}(\dfrac{FG}{AB})=1$.

Namely, $B'C'=GH$, and $C'D'=HI$, $D'E'=IJ$, $E'A'=JF$.

Note that dilation also preserves angles of polygons.

Since all corresponding sides and angles of $A'B'C'D'E'$ and $FGHIJ$ are equal, the two polygons are congruent. Thus they can be moved to overlap with each other perfectly. (Otherwise, ruler postulate and protractor postulate will be violated.)

Thus, $ABCDE$ can be dilated and moved to perfectly overlap with $FGHIJ$, which means $ABCDE \sim FGHIJ$ by definition. □

Preposition 42. There are some types of shapes that are similar to all other shapes of the same type:

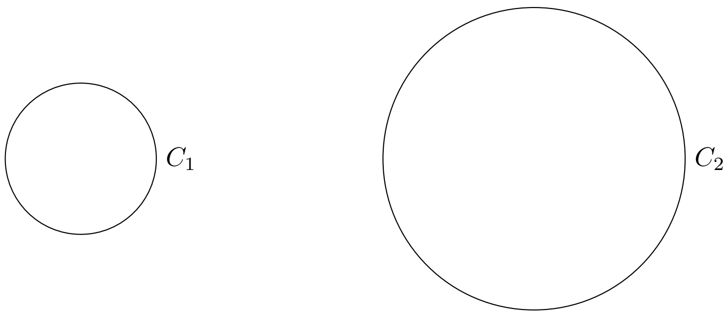
1. All circles are similar to each other.

2. All squares are similar to each other.

3. All other regular polygons with the same number of sides are similar to each other.

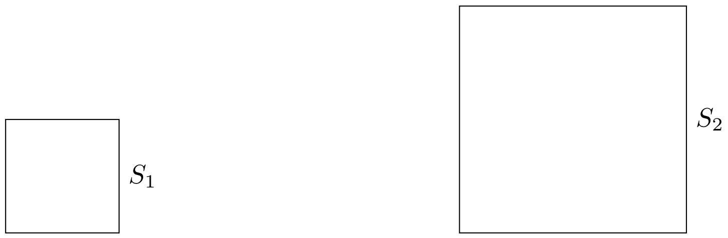
(similarity of circles and regular polygons)

1:



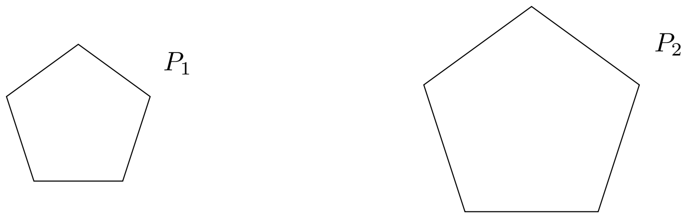
Given: C_1 and C_2 are circles.
$\therefore\ C_1 \sim C_2$

2:



<u>Given:</u> S_1 and S_2 are squares.
$\therefore\ S_1 \sim S_2$

<u>3:</u>

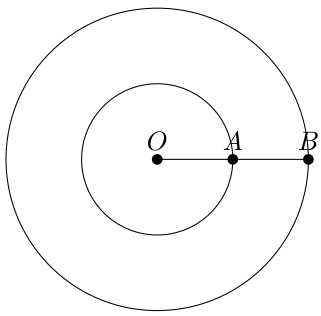


<u>Given:</u> P_1 and P_2 are regular pentagons.
$\therefore\ S_1 \sim S_2$

Proof. **<u>1:</u>** Move the centre of C_1 to coincide with the centre of C_2 . Let r_1 and r_2 be the radius of circles C_1 and C_2 respectively.

If the two circles have the same radius ($r_1 = r_2$), then they must coincide since there is a unique circle for a given radius and centre, so they must be similar.

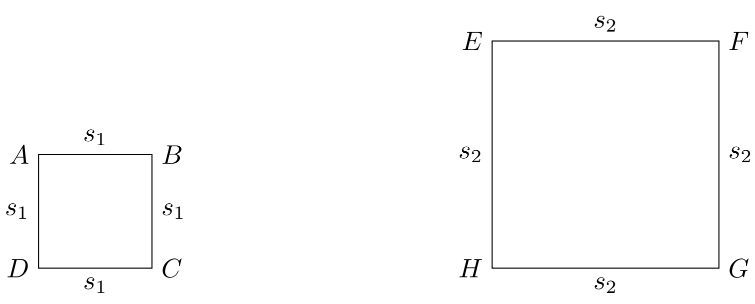
Suppose that $r_1 < r_2$.



Draw a radius for the larger circle. Let it intersects C_1 and C_2 at A and B respectively. No matter where the radius is drawn, we must have $\dfrac{OB}{OA} = \dfrac{r_2}{r_1}$ by definition of radii.

Since they have a constant scale factor from point O , C_1 can be dilated to coincide with C_2 , which means they are similar.

<u>2:</u> Let there be square $ABCD$ and square $EFGH$. Let $ABCD$ have side length s_1 and $EFGH$ have side length s_2 .



Note that $\angle A = \angle E = 90°$, $\angle B = \angle F = 90°$, $\angle C = \angle G = 90°$, $\angle D = \angle H = 90°$.

Also, $\dfrac{EF}{AB} = \dfrac{FG}{BC} = \dfrac{GH}{CD} = \dfrac{HE}{DA} = \dfrac{s_2}{s_1}$.

Since all the corresponding sides are proportional and all the corresponding angles are equal, we have $ABCD \sim EFGH$ by 'corr. sides prop. and corr. $\angle$s equal' .

<u>**3:**</u> Let there be two n-sided regular polygons.

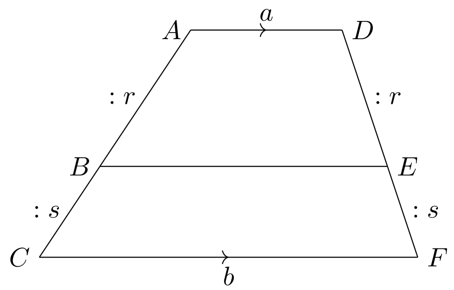
By definition, a regular polygon has equal angles and equal sides. The interior angle of an n-sided polygon must be $180° - 360°/n$, and it is the same for all n-sided regular polygons.

The ratio of side lengths must be $\dfrac{\text{side length of this regular polygon}}{\text{side length of that regular polygon}}$, so all the sides must be proportional.

Thus, for a given n , all n-sided regular polygons are similar by 'corr. sides prop. and corr. $\angle$s equal' . ☐

<u>Miscellaneous</u>

Preposition 43. In a trapezium, if a line segment connecting two legs cuts them with equal proportion, then the length of this line segment is the weighted average of the upper base and lower base, where the weight is the ratio of the lengths of the segments cut (in the order opposite to the bases). (trapezium section formula)

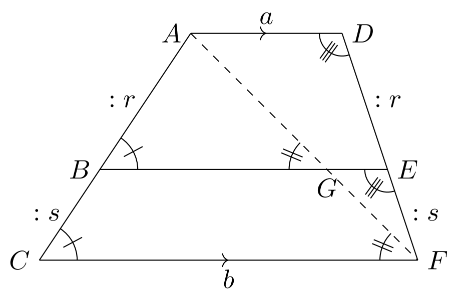


<u>Given:</u> $AD//CF$

$\because \quad AB : BC = DE : EF = r : s$

$\therefore \quad \boxed{BE = \dfrac{as + br}{r + s}}$ (trapezium section formula)

Proof. Join AF . Let AF and BE intersect at G .



Note that $BE//AD//CF$ by 'general-pt. theorem [trapezium version]' .

Thus $\angle ABE = \angle ACF$ and $\angle AGB = \angle AFC$ (corr. $\angle$s , $BE//CF$) . Thus $\triangle ABG \sim \triangle ACF$ (AA).

Similarly, $\angle ADF = \angle GEF$ and $\angle DAF = \angle EGF$ (corr. $\angle$s , $AD//BE$) . Thus $\triangle ADF \sim \triangle GEF$ (AA).

Thus,

$$\frac{BG}{CF} = \frac{AB}{AC} \qquad \text{(corr. sides, } \triangle ABG \sim \triangle ACF)$$
$$\frac{BG}{b} = \frac{r}{r+s}$$
$$BG = \frac{br}{r+s}$$

And

$$\begin{aligned}\frac{GE}{AD} &= \frac{EF}{DF} \qquad \text{(corr. sides, } \triangle GEF \sim \triangle ADF) \\ \frac{GE}{a} &= \frac{s}{r+s} \\ GE &= \frac{as}{r+s}\end{aligned}$$

So

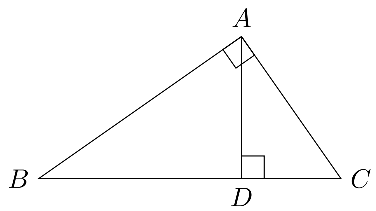
$$BE = BG + GE = \frac{br}{r+s} + \frac{as}{r+s} = \frac{as+br}{r+s}$$

□

Preposition 44. In a right triangle, a perpendicular line segment (**altitude**) drawn from the hypotenuse to the opposite vertex splits the triangle into two similar triangles that are also similar to the original triangle.

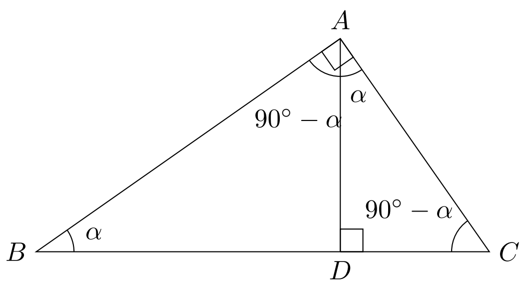
As a result, the length of the perpendicular line segment is the **geometric mean** of the two hypotenuse segments (that is split by the perpendicular line segment).

(prop. of right △)



$$\because \angle BAC = \angle 90^\circ \ , \ AD \perp BC$$
$$\therefore \ \triangle ABC \sim \triangle DBA \sim \triangle DAC \qquad \text{(prop. of right } \triangle)$$
$$\therefore \ AD^2 = BD \cdot DC \qquad \text{(prop. of right } \triangle)$$

Proof. Let $\angle ABC = \alpha$. Then $\angle ACB = 180^\circ - 90^\circ - \alpha = 90^\circ - \alpha$ ($\angle$ sum of △).



Note that $\angle BAD = 90^\circ - \alpha$ (ext. $\angle$ of △). Then $\angle CAD = 90^\circ - \angle BAD = 90^\circ - (90^\circ - \alpha) = \alpha$.

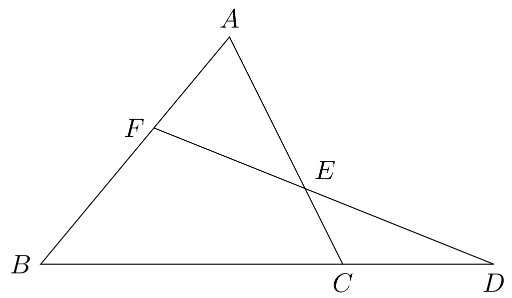
In $\triangle ABC$ and $\triangle DBA$ and $\triangle DAC$,

$$\angle BAC = \angle BDA = \angle ADC = 90^\circ \qquad \text{(given)}$$
$$\angle ABC = \angle DBA = \angle DAC = \alpha \qquad \text{(common } \angle \text{ \& shown above)}$$
$$\therefore \ \triangle ABC \sim \triangle DBA \sim \triangle DAC \qquad \text{(AA)}$$

$$\begin{aligned}\frac{DB}{DA} &= \frac{DA}{DC} \qquad \text{(corr. sides, } \triangle DBA \sim \triangle DAC) \\ DA^2 &= BD \cdot DC\end{aligned}$$

□

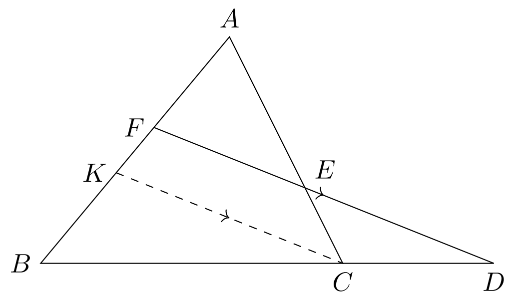
Preposition 45. Given a triangle $\triangle ABC$, if a transversal line intersects line BC, CA, AB at points D, E, F respectively (where the points of intersection do not coincide with the vertices), then $\dfrac{AF}{FB}\cdot\dfrac{BD}{DC}\cdot\dfrac{CE}{EA}=1$. (Menelaus' theorem)



$$\because F, E, D \text{ are collinear.}$$

$$\therefore \boxed{\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=1} \qquad \text{(Menelau's theorem)}$$

Proof. [10] Let K be a point on AB such that $KC//FD$.

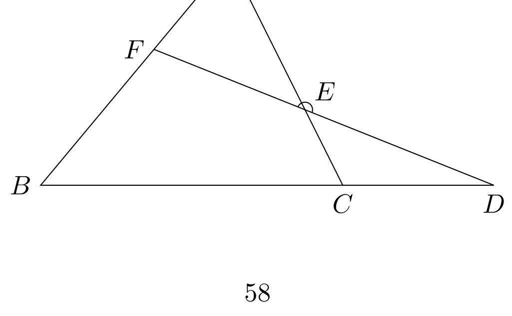


In $\triangle FBD$, note that $\dfrac{FK}{FB}=\dfrac{DC}{DB}$ (general intercept theorem with altered ratios).
In $\triangle AKC$, note that $\dfrac{AF}{FK}=\dfrac{AE}{EC}$ (general intercept theorem).
Multiply the two equations together:

$$\begin{aligned}\frac{FK}{FB}\cdot\frac{AF}{FK}&=\frac{DC}{DB}\cdot\frac{AE}{EC}\\ \frac{AF}{FB}&=\frac{DC}{BD}\cdot\frac{EA}{CE}\\ \frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}&=1\end{aligned}$$

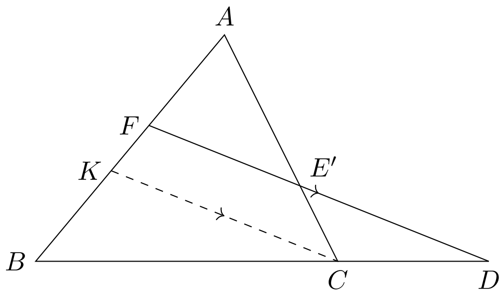
□

Preposition 46. Given a triangle $\triangle ABC$, if points D, E, F are on line BC, CA, AB respectively such that $\dfrac{AF}{FB}\cdot\dfrac{BD}{DC}\cdot\dfrac{CE}{EA}=1$ (where two of the points are on the sides), then D, E, F are collinear. (converse of Menelaus' theorem)



$$\because \frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=1$$
$$\therefore\ F,E,D \text{ are collinear.} \qquad \text{(converse of Menelau's theorem)}$$

Proof. Let E' be the intersection of FD and AC .



By Menelaus' theorem, we have $\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE'}{E'A}=1$.

By initial assumption, we have $\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=1$.

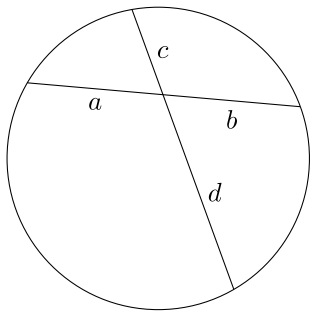
Putting the two equations together:

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE'}{E'A}=\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}$$
$$\frac{CE'}{E'A}=\frac{CE}{EA}$$

Since the ratio is the same, E must coincide with E' . Thus F,E,D are collinear. □

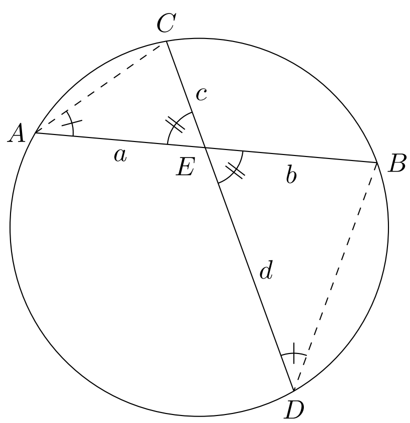
1.8.6 Circle related prepositions

Preposition 47. If there are two intersecting chords in a circle and the point of intersection split the chords into four line segments, then the product of lengths of the two segments in a chord is equal to that of the other chord. (intersecting chords theorem)



$$\boxed{ab=cd} \qquad \text{(intersecting chords theorem)}$$

Proof. Label the chords AB and CD . Let AB and CD intersect at E . Join AC and BD

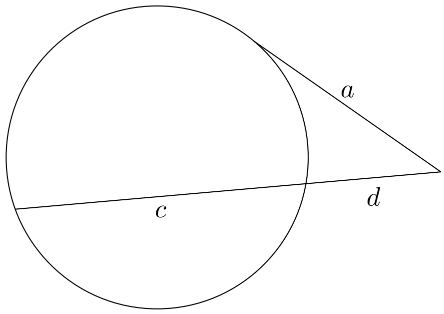


In $\triangle AEC$ and $\triangle DEB$,

$$\begin{aligned} \angle EAC &= \angle EDB && (\angle\text{s in the same segment}) \\ \angle AEC &= \angle DEB && (\text{vert. opp. } \angle\text{s}) \\ \therefore \triangle AEC &\sim \triangle DEB && (\text{AA}) \\ \therefore \frac{AE}{ED} &= \frac{CE}{EB} && (\text{corr. sides, } \sim \triangle\text{s}) \\ AE \cdot EB &= CE \cdot ED \\ ab &= cd \end{aligned}$$

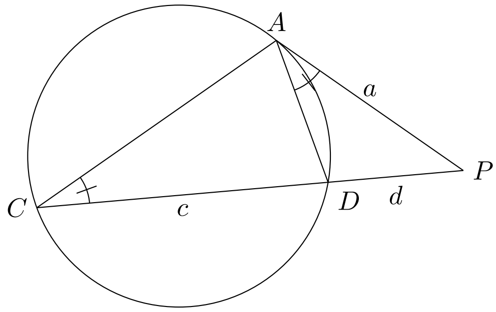
□

Preposition 48. If a tangent segment and a secant segment are drawn from a point outside a circle, then the square of the tangent segment length is equal to the product of the secant segment length and the length of the external part of the secant segment. (tangent-secant theorem)



$$\boxed{a^2 = d(c+d)} \qquad (\text{tangent-secant theorem})$$

Proof. Label the external point P , the tangent point A and the secant points C and D . Join AD and AC.

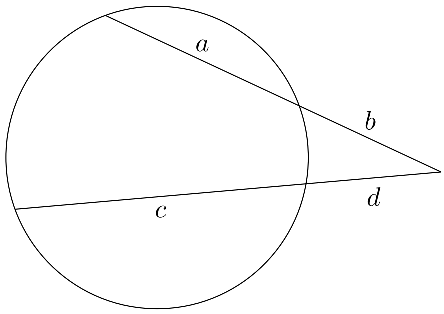


In $\triangle ADP$ and $\triangle CAP$,

$$\begin{aligned} \angle APD &= \angle CPA && (\text{common } \angle) \\ \angle DAP &= \angle ACP && (\angle \text{ in alt. segment}) \\ \therefore \triangle ADP &= \triangle CAP && (\text{AA}) \\ \therefore \frac{AP}{CP} &= \frac{DP}{AP} && (\text{corr. sides, } \sim \triangle\text{s}) \\ AP^2 &= DP \cdot CP \\ a^2 &= d(c+d) \end{aligned}$$

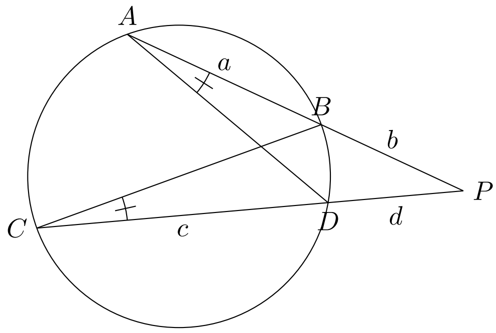
□

Preposition 49. If two secant segments are drawn from a point outside a circle, then the product of a secant segment length and the length of the external part of this secant segment is equal to the product of lengths for the other secant segment. (intersecting secants theorem)



$$\boxed{b(a+b) = d(c+d)} \qquad \text{(intersecting secants theorem)}$$

Proof. [11] Label the external P and the secant points A, B, C, D . Join AD .



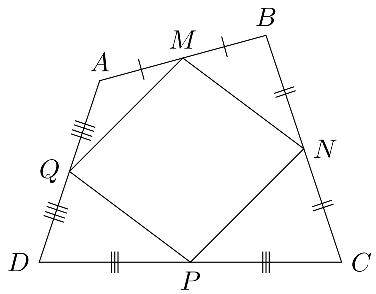
In $\triangle ADP$ and $\triangle CBP$,

$$\begin{aligned}
\angle APD &= \angle CPB && \text{(common } \angle\text{)} \\
\angle DAP &= \angle BCP && \text{(}\angle\text{s in the same segment)} \\
\therefore\ \triangle ADP &\sim \triangle CBP && \text{(AA)} \\
\therefore\ \frac{AP}{CP} &= \frac{DP}{BP} && \text{(corr. sides, } \sim \triangle\text{s)} \\
AP \cdot BP &= DP \cdot CP \\
b(a+b) &= d(c+d)
\end{aligned}$$

□

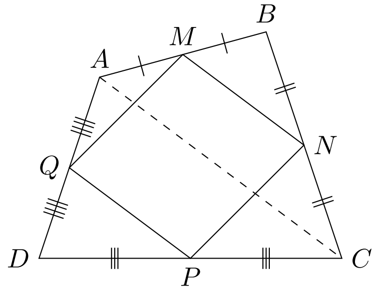
1.8.7 Additional properties of quadrilaterals

Preposition 50. The mid-points of four sides of a quadrilateral form a parallelogram that has half the area of that quadrilateral. (Varignon's theorem / mid-pts. of quad.)



$$\because AM = MB \text{ , } BN = NC \text{ , } CP = PD \text{ , } DQ = QA$$
$$\therefore\ MNPQ \text{ is a parallelogram.}$$
$$\text{Also, area of } MNPQ = \frac{1}{2}\text{area of } ABCD$$
$$\text{(mid-pts. of quad.)}$$

Proof. Join diagonal AC .



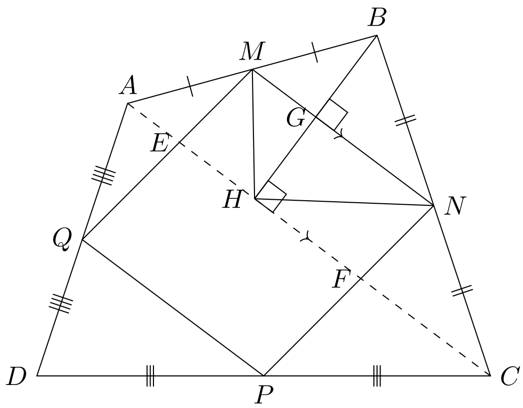
In $\triangle BAC$, $AM = MB$ and $BN = NC$ (given). Thus $MN = \dfrac{1}{2}AC$ and $MN//AC$ (mid-pt. theorem) .

In $\triangle DAC$, $DQ = QA$ and $DP = PC$ (given). Thus $QP = \dfrac{1}{2}AC$ and $QP//AC$ (mid-pt. theorem).

Thus, $MN = QP = \dfrac{1}{2}AC$, and $MN//QP$ by transitivity of parallel lines. Thus $MNPQ$ is a parallelogram (opp. sides equal and //) .

Now we prove that area of $MNPQ$ is half the area of $ABCD$.

Let $BH \perp AC$, and let BH intersect line MN at G . Also let MQ and NP intersect AC at E and F respectively.
Join MH and NH.



Since $MN//AC$, note that $BG = GH$ by intercept theorem. Thus $\triangle BMN$ and $\triangle HMN$ has the same height. Since they also have the same base, they also have the same area.
Note that $MNFE$ is a parallelogram with the same height as $\triangle HMN$. Thus area of $\triangle HMN = \dfrac{1}{2}$area of $MNFE$ (area of $\triangle$ & //gram) . Since area of $\triangle BMN =$ area of $\triangle HMN$, we have area of $\triangle BMN = \dfrac{1}{2}$area of $MNFE$.

By the same reasoning, area of $\triangle DQP = \dfrac{1}{2}$area of $PQEF$.
Also, note that

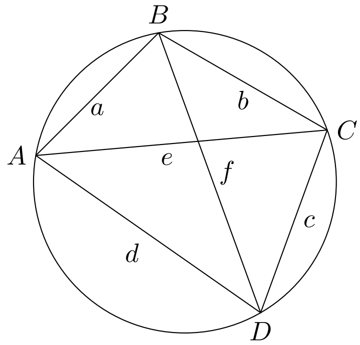
$$\begin{aligned}\text{area of } \triangle BMN &= \frac{1}{2}BG \cdot MN \\ &= \frac{1}{2}(\frac{1}{2}BH)(\frac{1}{2}AC) \\ &= \frac{1}{4}(\frac{1}{2}BH \cdot AC) \\ &= \frac{1}{4}\text{area of } \triangle BAC\end{aligned}$$

By the same reasoning, area of $\triangle DQP = \frac{1}{4}$area of $\triangle DAC$. Thus

$$
\begin{aligned}
\text{area of } MNPQ &= \text{area of } MNFE + \text{area of } PQEF \\
&= 2 \cdot \text{area of } \triangle BMN + 2 \cdot \text{area of } \triangle DQP \\
&= 2(\frac{1}{4} \text{ area of } \triangle BAC) + 2(\frac{1}{4} \text{ area of } \triangle DAC) \\
&= \frac{1}{2}(\text{area of } \triangle BAC + \text{area of } \triangle DAC) \\
&= \frac{1}{2} \text{ area of } ABCD
\end{aligned}
$$

□

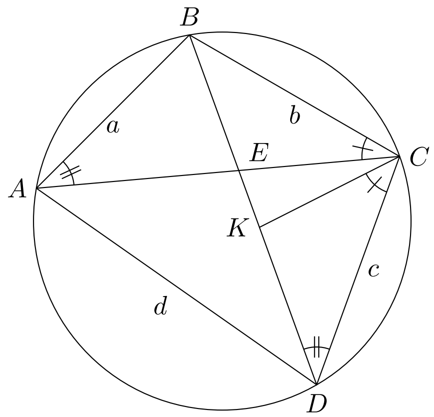
Preposition 51. In a cyclic quadrilateral, the product of its diagonals is the sum of the products of its opposites sides. (Ptolemy's theorem)



$\boxed{ef = ac + bd}$ (Ptolemy's theorem)

(Note: $e = AB$ and $f = CD$)

Proof. [12] Let AC and BD intersect at E . Let K be a point on BD such that $\angle DCK = \angle BCA$.



In $\triangle ABC$ and $\triangle DKC$,

$$
\begin{aligned}
\angle BCA &= \angle DCK && \text{(constructed)} \\
\angle BAC &= \angle KDC && (\angle\text{s in the same segment}) \\
\therefore\ \triangle ABC &\sim \triangle DKC && \text{(AA)} \\
\therefore\ \frac{AC}{DC} &= \frac{AB}{DK} && (\text{corr. sides}, \sim \triangle\text{s}) \\
AC \cdot DK &= AB \cdot DC && (1)
\end{aligned}
$$

In $\triangle BCK$ and $\triangle ACD$,

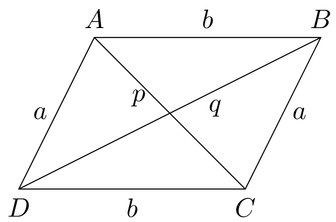
$$\angle BCK = \angle BCE + \angle ECK = \angle DCK + \angle ECK = \angle DCA \qquad \text{(angle addition postulate)}$$
$$\angle KBC = \angle DAC \qquad (\angle\text{s in the same segment})$$
$$\therefore\ \triangle BCK \sim \triangle ACD \qquad \text{(AA)}$$
$$\therefore\ \frac{BC}{AC} = \frac{BK}{AD} \qquad (\text{corr. sides}, \sim \triangle\text{s})$$
$$BC \cdot AD = BK \cdot AC$$
$$AC \cdot BK = AD \cdot BC \tag{2}$$

(1) + (2) :

$$AC \cdot DK + AC \cdot BK = AB \cdot DC + AD \cdot BC$$
$$AC(DK + BK) = AB \cdot DC + AD \cdot BC$$
$$AC \cdot BD = AB \cdot DC + AD \cdot BC$$
$$ef = ac + bd$$

□

Preposition 52. In a parallelogram, the sum of squares of two diagonals is twice the sum of squares of two adjacent sides. (diags length of //gram)



$$\boxed{p^2 + q^2 = 2(a^2 + b^2)} \qquad \text{(diags length of //gram)}$$

(Note: $p = AC$ and $q = BD$)

Proof. Let $AD = a$ and $AB = b$. Then $BC = AD = a$ and $DC = AB = b$ (opp. sides of //gram) .

There are different cases based on different sizes of $\angle ADC$.

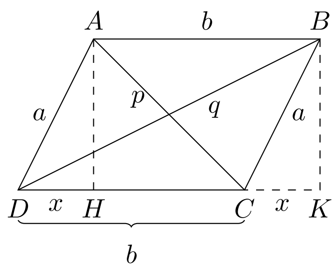
Suppose $\angle ADC = 90°$. Then $ABCD$ is a rectangle (//gram with right $\angle$) . We have $p = q$ (diags of rectangle) and $p^2 = q^2 = a^2 + b^2$ by pyth. theorem.

Thus $p^2 + q^2 = a^+ b^2 + a^+ b^2 = 2(a^2 + b^2)$.

Now suppose $\angle ADC < 90°$.

Draw $AH \perp DC$ and $BK \perp DC$. Let $DH = x$.

Note that $\triangle ADH \cong BCK$ (AAS), so $CK = DH = x$ (corr. sides, $\cong \triangle$s).



Consider $\triangle ADC$. By 'simplified law of cosines', we have

$$x = \frac{a^2 + b^2 - p^2}{2b} \tag{1}$$

Consider $\triangle BDC$. Note that it is an obtuse triangle. By 'simplified law of cosines', we have

$$x = \frac{a^2 + b^2 - q^2}{-2b} \tag{2}$$

Put (1) and (2) together:

$$\begin{aligned}
\frac{a^2+b^2-p^2}{2b} &= \frac{a^2+b^2-q^2}{-2b} \\
a^2+b^2-p^2 &= -(a^2+b^2-q^2) \\
a^2+b^2-p^2 &= -a^2-b^2+q^2 \\
2(a^2+b^2) &= p^2+q^2 \\
p^2+q^2 &= 2(a^2+b^2)
\end{aligned}$$

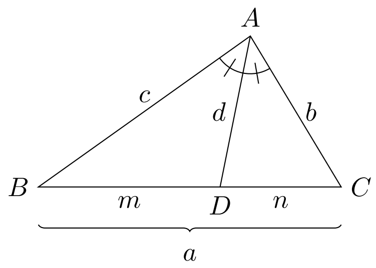
If $\angle ADC > 90°$, the formula can be arrived at by similar reasoning.

$\square$

1.9 Four centres of triangle

1.9.1 Angle bisectors

Preposition 53. Given $\triangle ABC$ with side lengths a, b, c and opposite vertices A, B, C respectively. If AD is an angle bisector of $\angle A$ in $\triangle ABC$, such that $AD = d$, $BD = m$ and $CD = n$, then $d^2 = bc - mn$ and $d^2 = bc\left(1 - \dfrac{a^2}{(b+c)^2}\right)$. (length of $\angle$ bisector of $\triangle$)



$$\because \angle BAD = \angle CAD$$
$$\therefore \boxed{d^2 = bc - mn}$$
$$\text{and} \quad \boxed{d^2 = bc\left(1 - \frac{a^2}{(b+c)^2}\right)}$$
(length of $\angle$ bisector of $\triangle$)

Proof. First we obtain the first formula.

By angle bisector theorem, we have

$$\begin{aligned}
\frac{c}{b} &= \frac{m}{n} \\
cn &= bm
\end{aligned}$$

Multiply both sides by $b - c$:

$$\begin{aligned}
(b-c)cn &= (b-c)bm \\
bcn - c^2n &= b^2m - bcm \\
bcn + bcm &= b^2m + c^2n \\
bc(m+n) &= b^2m + c^2n \\
bc &= \frac{b^2m + c^2n}{m+n} \qquad (1)
\end{aligned}$$

By Stewart's theorem, we have:

$$\begin{aligned}
b^2m + c^2n &= (m+n)(d^2+mn) \\
d^2 &= \frac{b^2m+c^2n}{m+n} - mn
\end{aligned}$$

Note that $\dfrac{b^2m + c^2n}{m+n} = bc$ by (1), so we get:

$$d^2 = bc - mn$$

Now we want to obtain the second formula. Start by the angle bisector theorem: [13]

$$\begin{aligned}
\frac{c}{b} &= \frac{m}{n} \\
\frac{c}{b} + 1 &= \frac{m}{n} + 1 \\
\frac{c+b}{b} &= \frac{m+n}{n} \\
\frac{c+b}{b} &= \frac{a}{n} \\
n &= \frac{ab}{b+c} \qquad (2)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{b}{c} &= \frac{n}{m} \\
\frac{b}{c} + 1 &= \frac{n}{m} + 1 \\
\frac{b+c}{c} &= \frac{n+m}{m} \\
\frac{b+c}{c} &= \frac{a}{m} \\
m &= \frac{ac}{b+c} \qquad (3)
\end{aligned}$$

Put (2) and (3) into $d^2 = bc - mn$:

$$\begin{aligned}
d^2 &= bc - (\frac{ac}{b+c})(\frac{ab}{b+c}) \\
&= bc - \frac{bc \cdot a^2}{(b+c)^2} \\
&= bc\,(1 - \frac{a^2}{(b+c)^2})
\end{aligned}$$

□

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