Toddler Geometry

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Abstract

Geometry is an ancient branch of Mathematics, dating as far back as 4000 years ago. Humanity has been fascinated and puzzled by these 'simple' lines and shapes for millennia, so it is only natural for a maths person like me to want to study Geometry and uncover its mysteries. But unlike other branches of mathematics such as Calculus and Linear Algebra, why are all the geometry theorems so useless and unapplicable in real life? I have no idea. After studying some circle theorems in high school, we don't even touch them again in University, which is doing Geometry a disservice in my opinion. So here I am, fully embracing the uselessness of Geometry and just studying for the fun of it, because it is the purest form of art.

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0 Introduction

In this article, we will mainly focus on **Euclidean Geometry**, which is the geometry used by the universe that we live in. The world that contains all the geometric objects is called the **Euclidean space**. It has infinite size (so you can go in a direction and walk forever), and can have different dimensions.

Three-dimensional Euclidean space is the space we live in, and as a result we can move in 6 directions (up/down, left/right, forward/backward).

Two-dimensional Euclidean space, called **Euclidean plane**, is like a flat piece of paper (or floor or wall) which can only contain flat objects. Objects inside the plane can only move in 4 directions (up/down, left/right).

One-dimensional Euclidean space is essentially a line, which is not very interesting. Object inside can only move in two directions (left/right).

Since three-dimensional Euclidean space is too complicated and one-dimensional Euclidean space is too simple, we will mainly focus on two-dimensional Euclidean space (/Euclidean plane), as it has the right amount of complexity to be interesting but not too much complexity to be incredibly frustrating to study.

There are many geometric objects that can exist in Euclidean space, such as points, lines, curves, angles, shapes, planes, solids, and so on. The study of Geometry seeks to find the properties of these objects and how they interact with each other.

0.1 Points and lines

0.1.1 Points lying on lines

There are two basic elements of Euclidean plane, which is a **point** and a **line**: (The line is necessarily a straight line, and we don't consider curvy 'lines' now.)

A point A line

Note that a point isn't actually a circle with a positive radius, but is actually some kind of 'position marker' with zero width, zero length and zero size. The black dot is just the rendering of the point so that we can actually see it. Similarly, a line has zero thickness, but it has infinite length (just that we do not render it fully). There can be more than one point and one line on the same plane, and there can even be infinite points and infinite lines. A point can lie on a line:

It can also lie on somewhere else:

Let's briefly introduce the concept of **distance**, which is a non-negative real number that measures how far apart two objects are on the plane. The further apart the objects, the larger the distance. A point that does not lie on a line has a non-zero distance from the line. Similarly, two **distinct** points [1] on the plane have a non-zero distance between them. The distance between two points is the **length** of the line segment connecting them. And length measures how long an object is.

A **line segment** is a part of a line between two end points:

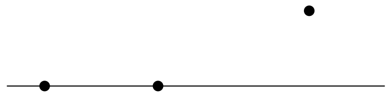
Note that a line segment must have a finite length. For any two distinct points, we can uniquely draw a line segment that connects the two points, and if we extend the line segment indefinitely, we get a line. Thus, any two points uniquely define a line and a line segment.

In fact, a line or a line segment is made of infinite points, and there are infinite points between any two distinct points on the line / line segment, as space is infinitely divisible.

When there are three points, they may or may not all lie on the same line. If the three points lie on the same line, the three points are **collinear**:

[1] 'Two distinct points' means the two points do not overlap / they are at different positions. Similarly, two distinct lines do not lie exactly on top of each other / do not coincide with each other. When there are two points or two lines on the plane, they are assumed to be distinct unless stated otherwise.

Collinear

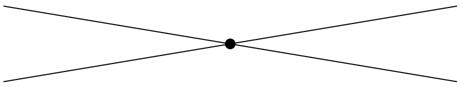


Not collinear

One property of the straight line segment is that it is the shortest path that connects the two end points.

0.1.2 Intersecting lines and parallel lines

When there are two distinct lines in the plane, they may intersect at exactly one point (they can never intersect at more than one point):

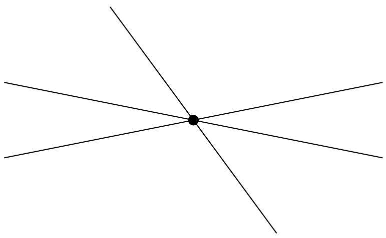


Or they may also never intersect:

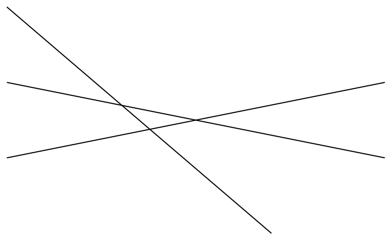
A pair of lines that never intersect is called **parallel lines**. Any pair of parallel lines must point in the same direction. If a line is horizontal and another line is parallel to this horizontal line, then we know that the other line is also horizontal. The distance between a pair of parallel lines is unchanged throughout the plane, but the distance between a pair of intersecting line varies throughout the plane.

If line m is parallel to line l , we write $m \,//\, l$ to denote this.

When there are three lines in the plane, they may or may not intersect at exactly one point. If they intersect at one point, the three lines are **concurrent**:



Concurrent



Not concurrent

0.1.3 Rays

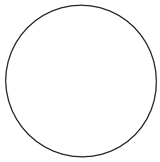
A ray is a part of a line that starts from a point and runs indefinitely:

A ray is defined by an initial point and an additional point that determines the ray's direction: (the unfilled dot is the additional point)

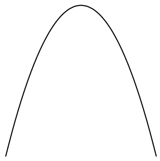
Note that these two points are not interchangeable (unlike two points on a regular line or line segment). Swapping the positions of the two points will make the ray point in the opposite direction:

0.2 Curves

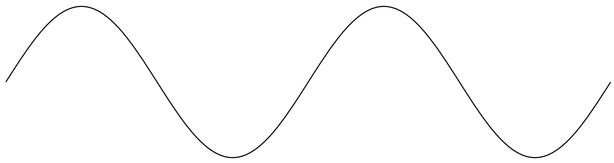
A ‘line’ that is not straight is called a **curve**. There are many types of curves, such as a circle:



Or a **parabola**:

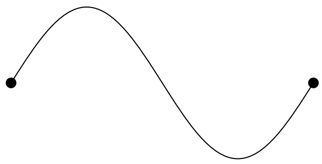


Or a sine wave:



You get the idea. Curves and lines are both **continuous**, meaning (intuitively) that we can draw them without lifting our pens. But unlike lines, curves can have infinite length or a finite length. A circle is a curve that has a finite length, but a sine wave is a curve that has infinite length

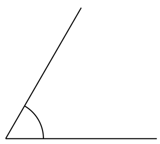
Similar to line segment, a curve segment is a part of a curve between two end points:



Note that a curve segment must have a longer length than the straight line connecting the same two end points. Measuring the length of a curve segment is much more tricky than a line segment, but we’ll worry about that later.

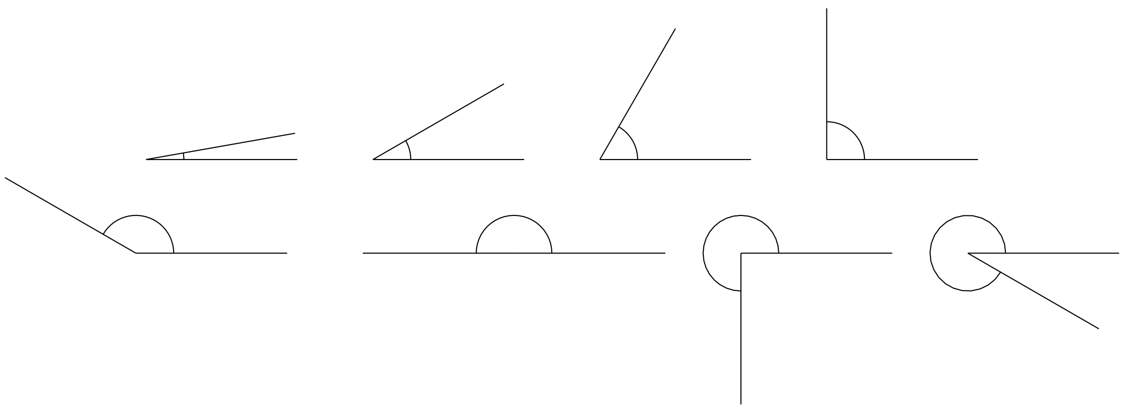
0.3 Angles

When two rays (or two lines or two line segments) intersects at a point, they form an **angle** (denoted by an arc of a mini-circle at the corner):



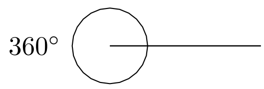
The point at the corner is called **vertex**.

Angles can have different sizes. The larger the angle, the wider the gap between the two rays:

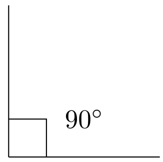


0.3.1 Types of angles

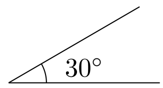
The common unit for measuring the size of angle is **degree** (°), and a full revolution is 360°:



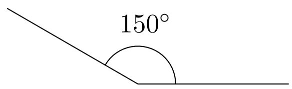
A quarter (1/4) of revolution, which is 90° , is called **right angle**. The angle notation is a mini-square to indicate that it is a right angle:



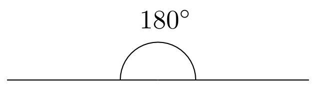
An angle smaller than 90° is called an **acute angle**:



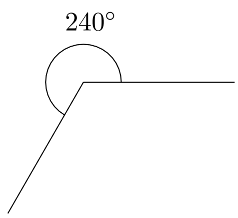
An angle larger than 90° but smaller than 180° is called an **obtuse angle**:



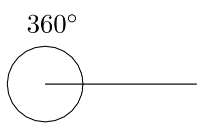
Half a revolution, which is 180° , is called a **straight angle** (because it appears as a straight line):



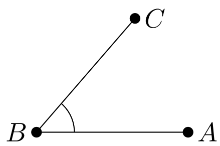
An angle larger than 180° is called a **reflex angle**:



A full revolution, which is 360°, is called a full angle (this term is rarely used):

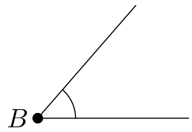


An angle can be uniquely defined by three points that are not interchangeable (I'll called these 'ordered points'): an initial point, a vertex, and an end point. To 'measure' the angle, starts from a point on the line segment of initial point and vertex, and moves anti-clockwise around the vertex, until the measurer hits the line segment (or line) of vertex and endpoint.



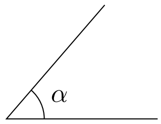
In the figure above, there are three points labelled A, B, C (Points are typically labelled with uppercase letters). A is the initial point; B is the vertex; and C is the endpoint. This angle constructed by A, B, C is denoted $\angle ABC$ or $\angle CBA$. Note that the vertex point B must be the second letter written in the angle notation, and $\angle ACB$ is different from $\angle ABC$. However, the first and third letter can be swapped to mean the same angle. (This swappability makes it easier to write the notation.)

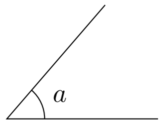
Alternatively, we can only write the vertex point in angle notation if the initial point and endpoint are not that important:



This angle can be written as $\angle B$. Note that $\angle B$ is the same thing as $\angle ABC$.

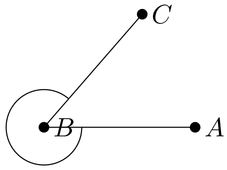
If we don't even care about the vertex point, we can just use a lowercase letter or a greek letter to denote the angle to make it even more simple:



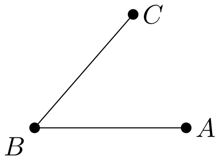


When we are referring to the angle, we can just say α or a like it is a variable.

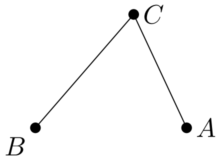
Note that if an angle is denoted $\angle ABC$ or $\angle B$, then it must be smaller or equal to $180°$.
If we want to refer to the reflex angle of $\angle ABC$, we write ' reflex$\angle ABC$ ' or ' r$\angle ABC$ ':



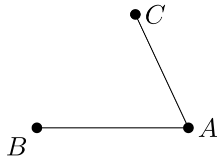
If the positions of initial vertex and initial point / endpoint are swapped around, we can create three different angles:



$\angle ABC$



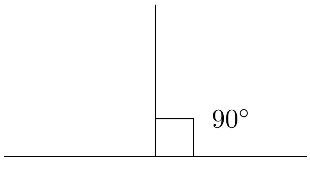
$\angle BCA$



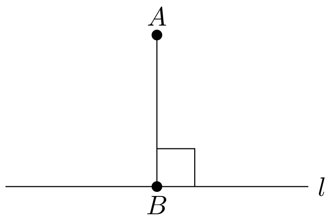
$\angle BAC$

0.3.2 Perpendicular lines

Two lines are **perpendicular** if they form a right angle:



If line m is perpendicular to line l, we write $m \perp l$ to denote this. If instead a line segment AB is perpendicular to line l , then we can say $AB \perp l$:

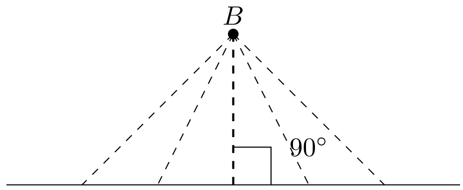


In the figure, we say that point B is the **projection** of point A on line l .

0.3.3 Distance between points and lines

Point and line

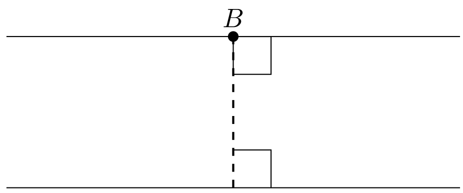
The shortest distance between a point and a line is a line segment that passes through the point and is perpendicular to the line:



Thus, when we are saying the distance between a point and a line, we are referring to the length of the perpendicular line segment.

Parallel lines

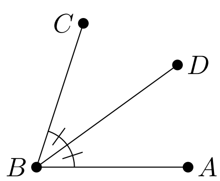
It is only meaningful to talk about the distance between a pair of parallel lines, since it stays the same throughout the plane. The distance between two parallel lines is the (unique) length of a line segment that is perpendicular to both of the lines:



0.3.4 Angle bisector and perpendicular bisector

Angle bisector

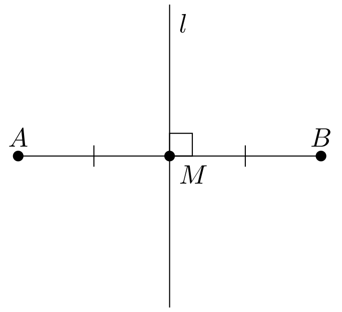
The **angle bisector** of an angle is the line or line segment that splits the angles into two equal parts:



In the figure, line segment BD is the angle bisector of $\angle ABC$ since it splits $\angle ABC$ into two equal parts: $\angle ABD$ and $\angle CBD$. We have $\angle ABD = \angle CBD = \frac{1}{2}\angle ABC$.

Perpendicular bisector

The **perpendicular bisector** of a line segment is the perpendicular line that passes through the midpoint of the two endpoints of the line segment:



In the figure, M is the **midpoint** of line segment AB , which means M is a point on AB that divides AB into two shorter line segments equal in length (which means $AM = MB = \frac{1}{2}AB$).

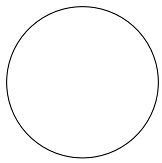
Line l is a point passing through M that is perpendicular to line segment AB , so l is the perpendicular bisector of AB .

0.4 Shapes

There are many geometric objects besides points, lines and angles, such as shapes. A **shape** is an enclosure of curves or line segments, which separates the plane into two parts: the part outside the shape (exterior) and the part inside the shape (interior). It forms a nice loop without some curves or line segments sticking out.

0.4.1 Circles

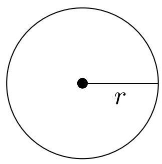
The most basic shape in Geometry is a **circle**, which is a round symmetric shape:



Circle is one of the most important shapes of Geometry, second only to triangles. Circular shapes are wide used in real life, such as the wheels of a car.

Centre and radius

A circle has two defining characteristics: a **centre** and a **radius**. The centre is a point that determines the position of the circle on the plane, and the radius is a number (or magnitude) that determines the size of the circle. In other words, we can uniquely draw a circle given a centre and a radius: (the radius r is represented by a line segment with legnth r)

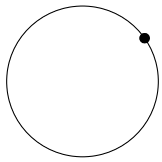


Defined more precisely, a circle is a shape consisting of all points in a plane that are at a given distance (=radius) from a given point (=centre). If a line segment is rotated about a fixed endpoint, then since rotation preserves segment length, the moving endpoint will trace out a circle with the fixed endpoint as its centre and the length of the line segment as its radius.

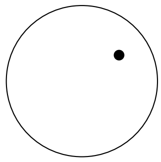
Sometimes, radius also refers to the line segment that has one endpoint being the centre and another point on the circumference of the circle.

Circumference

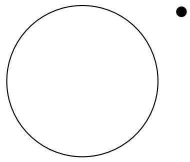
The **circumference** of a circle refers to the curve part of the circle. A point lying on the circumference of the circle:



A point not lying on the circumference of the circle:



Inside the circle



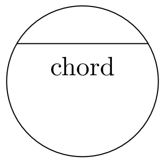
Outside the circle

The circumference can also refer to the total length of the circle curve. If C is the circumference, then $C = 2\pi r$, where r is the radius, and 2π is the ratio of $\frac{C}{r}$.

Arcs, chords and diameter

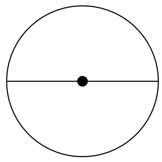
An **arc** is a part of the circumference of the circle:

A **chord** is a line segment with both of its endpoints on the circumference:



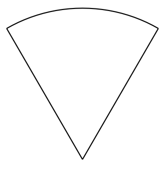
The **diameter** of the circle can refer to the value that is twice the circle's radius. It is the length of the longest possible chord of the circle.

Diameter can also refer to the longest possible chord itself. A diameter must pass through the centre of the circle:



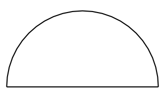
Sectors, segments and semi-circle

A **sector** is a slice of pie shape that is made of two radii and an arc of a circle:



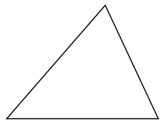
A **circle segment** (or simply segment) is the shape cut out by a chord of the circle:

A **semi-circle** is a segment that is half a circle:



0.4.2 Triangles

Shapes that are enclosed by only straight line segments are called **polygons**, and a **triangle** is the simplest polygon, enclosed by only three line segments, which is the minimum possible:

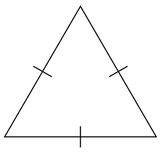


Each line segment of the triangle is called a **side**, and the 3 sides form 3 angles of the triangle (also called interior angles). A triangle must have 3 sides and 3 angles.

The point where two sides meet is called **vertex**, and a triangle must have three non-collinear and non-concurrent vertices. (Otherwise, it is just a line segment or a point.)

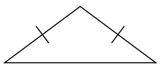
There are many types of triangles: (the marks on the sides indicate that two sides are equal in length. The 'in length' can be omitted to mean the same thing.)

<u>Equilateral triangle</u>



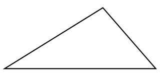
An **equilateral triangle** has three equal sides. All of its angles are 60°.

<u>Isosceles triangle</u>



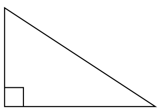
An **isosceles triangle** has two equal sides. It also has two equal angles (sharing the non-equal side).

<u>Scalene triangle</u>



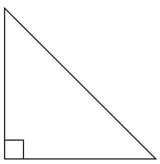
A **scalene triangle** has no equal sides. In other words, all of its sides are of different lengths.

<u>Right triangle</u>



A **right triangle** has one of its angles measuring 90°. It can be scalene or isosceles.

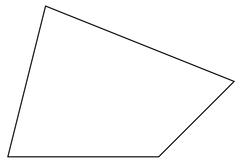
<u>Isosceles right triangle</u>



An isosceles right triangle has one angle measuring 90° and the other two angles measuring 45° .

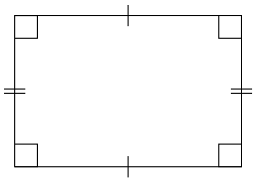
0.4.3 Quadrilaterals

Quadrilaterals are polygons that have four sides. In other words, they are enclosed by four line segments. A quadrilaterals must also have four angles: (By now, we've figured that a polygon must have the same number of sides and angles.)



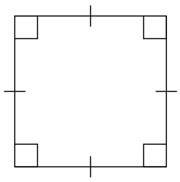
There are many types of quadrilaterals:

Rectangles



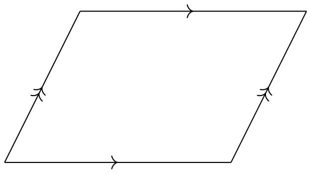
A **rectangle** has four right angles. Consequently, its opposite sides are equal but adjacent sides (neighbouring sides) are not necessarily equal. Its opposite sides are also parallel. A rectangle is a very important type of quadrilateral because it looks nice and can be tiled together neatly.

Squares



A **square** has four right angles and four equal sides. It is a special type of rectangle, but looks even nicer than non-square rectangles, as it has 4 axes of symmetry.

Parallelograms



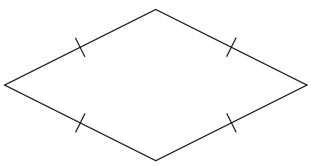
(The arrow marks on the segment

A **parallelogram** has two pairs of parallel sides. In other words, its opposite sides are necessarily parallel. Consequently, its opposite sides and opposite angles are equal.

Parallelograms looks like slanted rectangles, but they can still be tiled together nicely to form bigger parallelograms.

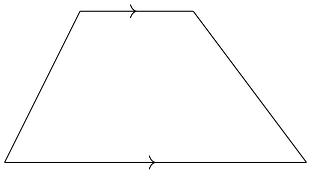
A rectangle is a special type of parallelogram, since it has parallel opposite sides.

Rhombus



A **rhombus** has four equal sides. Consequently, its opposite sides are parallel, so it is a special type of parallelogram. It inherits most properties of a parallelogram.

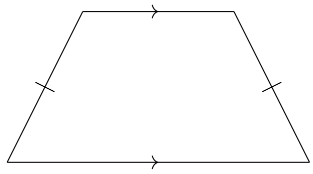
Trapeziums



A **trapezium** has at least one pair of parallel side. If it has only one pair of parallel sides, then I'll call it 'proper trapezium' (a term not generally used). If it has two pairs of parallel sides, then it is a parallelogram. Thus, a parallelogram is also a trapezium.

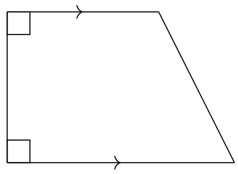
There are three common types of proper trapeziums:

- Isosceles trapeziums

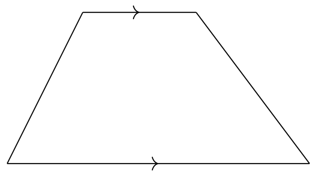
  An isosceles trapezium is a proper trapezium that has a pair of non-parallel sides with equal length. It has two adjacent pairs of equal angles.

- Right trapeziums

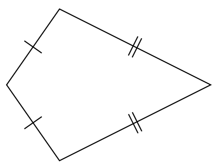
  A right trapezium is a proper trapezium that has two right angles.

- Irregular trapeziums

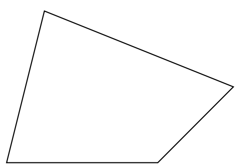
  An irregular trapezium is a proper trapezium that is neither isosceles trapezium nor right trapezium.

<u>Kite</u>



A **kite** has two adjacent pairs of equal sides. Squares, rhombuses are special types of kite.

<u>Scalene quadrilateral</u>



A scalene quadrilateral has no equal sides. A scalene quadrilateral can be an irregular trapezium or a right trapezium, but not an isosceles trapezium.

The classification of quadrilaterals is summarized in the diagram below:

quadrilaterals.png

I'll call each type of quadrilateral a class. For classes linked by solid lines, the quadrilaterals in each child class must belong to its parent class, but quadrilaterals in a parent class may or may not belong to its child class. For example, if a quadrilateral is a rectangle, then it must belong to its parent class, a parallelogram. But a parallelogram isn't necessarily a rectangle. (In other words, a child class is a proper **subset** of its parent class.)

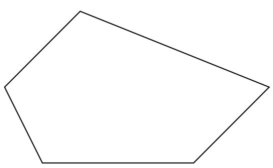
The quadrilaterals in a child class inherits most properties from quadrilaterals in a parent class. For example, a rectangle has four right angles, two opposite pairs of parallel and equal sides. So its child class, a square, also has these properties. But a rectangle has the property of two adjacent sides not necessarily equal, while a square does not. However, quadrilaterals in a child class must inherit the defining characteristics and the characteristics' consequential properties of the parent class. For example, a rectangle is defined by its four right angles, and consequently, it has two opposite pairs of parallel and equal sides. A square, being the child class of rectangle, must also have these properties (four right angless, two opposite pairs of parallel and equal sides).

As for classes linked by dotted lines, quadrilaterals in the child class may or may not belong to the parent class, and quadrilaterals in the parent class may or may not belong to the child class. So a scalene quadrilateral may or may not be a right trapezium, and a right trapezium may or may not be a scalene quadrilateral.

0.4.4 Other polygons

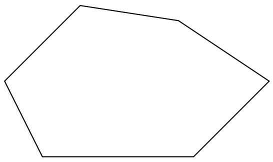
Polygons with more than 4 sides have too many types, so we won't bother to classify them within each number of sides.

<u>Pentagons</u>



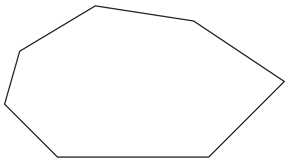
A **pentagon** is a polygon with five sides.

Hexagons



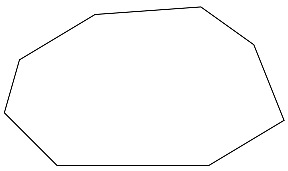
A **hexagon** is a polygon with six sides.

Heptagons



A **heptagon** is a polygon with seven sides.

Octagons



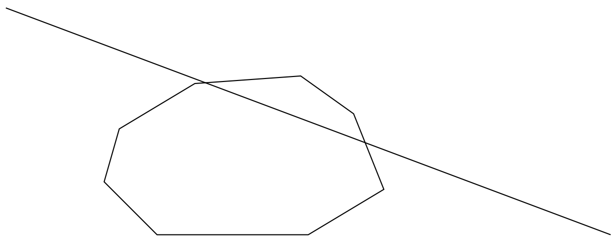
An **octagon** is a polygon with eight sides.

There can be polygons with more sides but we don't bother to name them anything specific (for our convenience), so we just call a k-sided polygon a 'k-gon' for $k > 8$.

Convex polygons

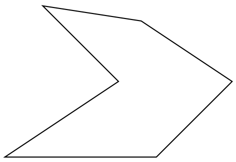
A polygon is **convex** if all of its interior angles are smaller than $180°$. All of the polygons shown above are convex polygons. Note that a triangle must be a convex polygon, but a polygon with four sides or more isn't necessarily convex.

A property of convex polygon is that every line that does not coincide with any side intersects the convex polygon in at most two points:



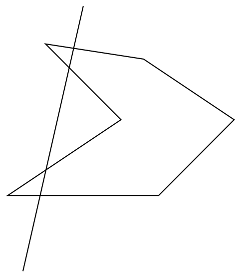
Concave polygons

A polygon is **concave** if at least one interior angle is larger than $180°$ (a reflex angle):



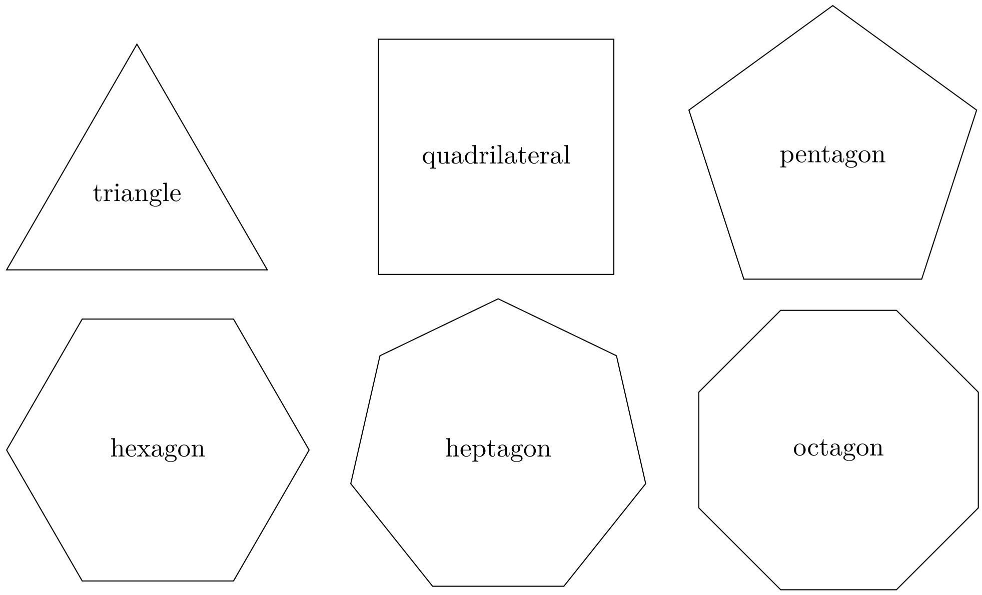
A polygon is either convex or concave.

The line property for concave polygon is similar: there exists some line that intersects the concave polygon in more than two points:



Regular polygons

A **regular polygon** is a polygon that has all sides equal and all interior angles equal. It is a nice symmetric shape. A regular triangle is called an equilateral triangle. A regular quadrilateral is called a square. For pentagon or polygons with more sides, they are just called regular pentagon or regular *whatever*-gon.



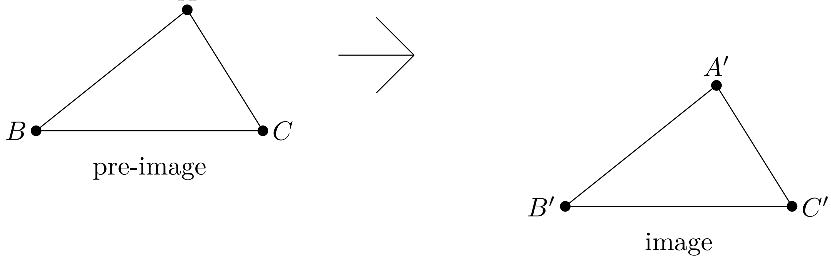
Note that as the number of sides of the regular polygon increases, it looks more and more like a circle.

0.5 Transformations of objects

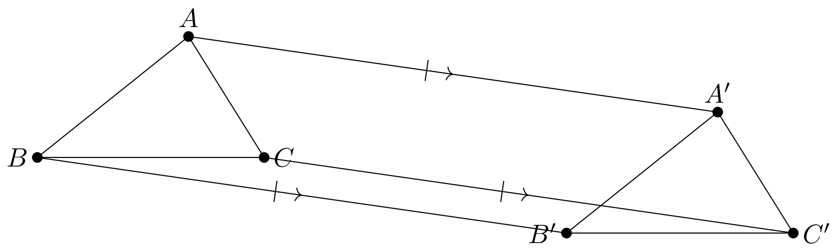
Transformations of an object can come in many forms, such as **translation** (moving with a fixed orientation), **rotation** (turning), **reflection** (flipping), and **dilation** (scaling).

Any objects can undergo transformations. The transformed object or point is called **image** and the original object or point is called **pre-image** . If a point A is in an original object, then after transformation, the image of point A is usually called A'. Same goes for other points.

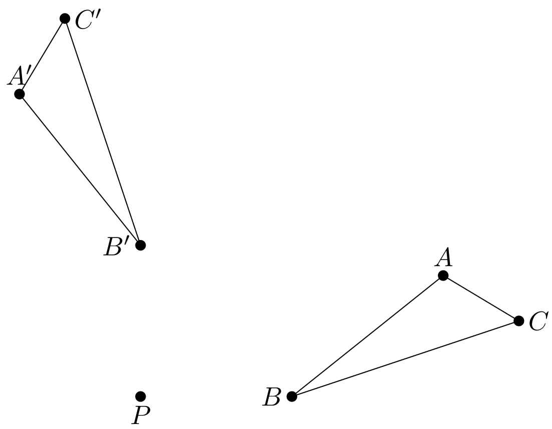
0.5.1 Translation



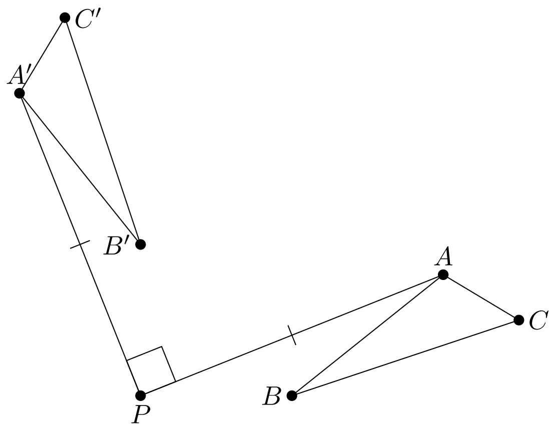
In a translation, every point of the object moves along in the same direction for the same distance. If we trace out the path travelled by the vertices of the triangle, we will find that they are parallel and equal in length:



0.5.2 Rotation



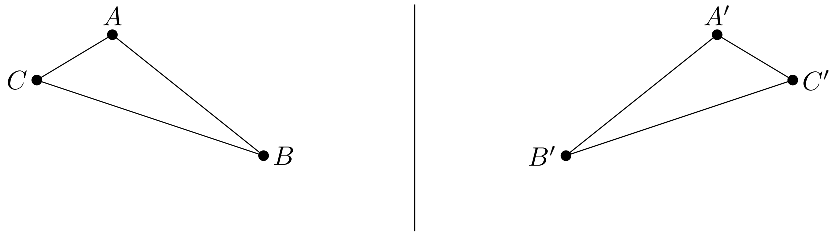
In a rotation, there is usually a point that the object revolves around during the rotation, called the **pivot**. (P is the pivot in the figure.) Every point in the object has been rotated about the pivot for the same angle, with the distance from the pivot unchanged. In the figure, we see that $\angle APA' = \angle BPB' = \angle CPC' = 90°$, and $PA = PA', PB = PB', PC = PC'$: (only one right angle is shown to avoid messiness)



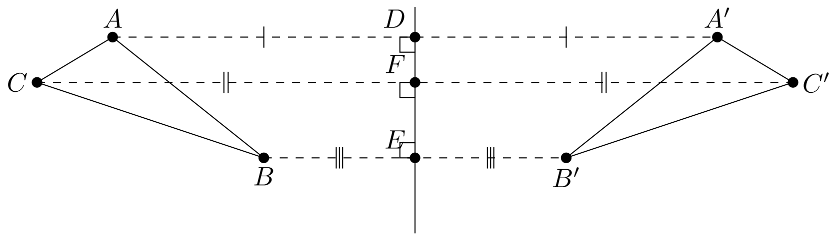
If we follow an arbitrary point M that is on a side of the triangle instead, $\angle MPM'$ will still be $90°$ and $PM = PM'$ is also true.

Note that rotation changes orientation, but preserves size, angles and lengths of the triangle.

0.5.3 Reflection

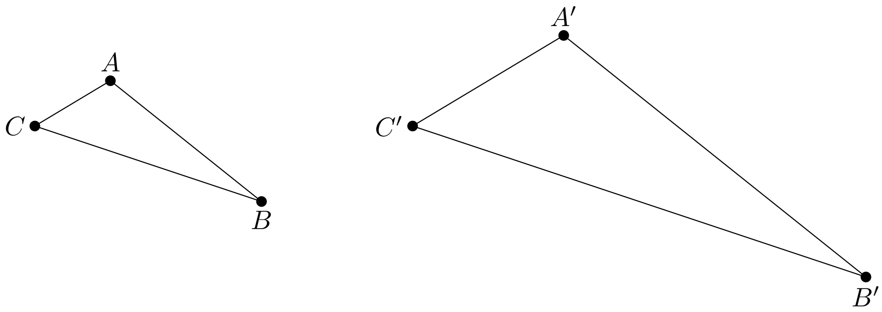


In a reflection, the object is usually reflected about a line called the **axis of reflection**. For each point of the object, the pre-image and image are the same distance away from the axis of reflection, and they are on the opposite sides of the axis:



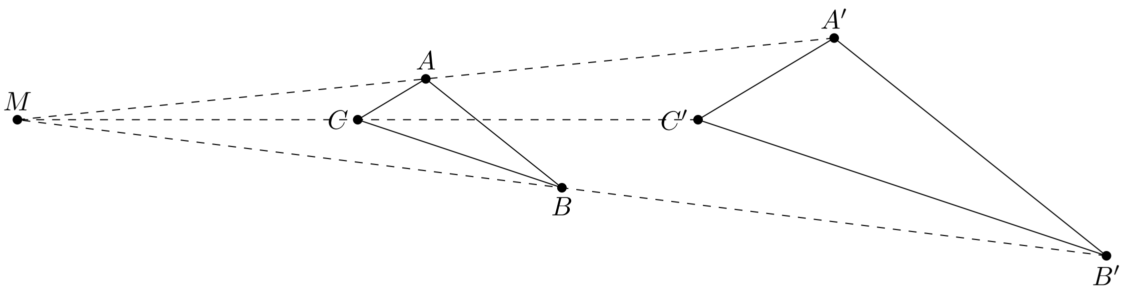
Reflection preserves size, angles and lengths of the triangle, but a non-symmetric reflected triangle cannot overlap with the original triangle using only rotation and translation. It has to be reflected again to do so.

0.5.4 Dilation



In a dilation, the scale (/size) of the object is changed, but orientation, angles, and ratio of side lengths are preserved. For a triangle / polygon, the ratio of the image side length to the corresponding original side length is the **scale factor** of the dilation. Each dilation has only one scale factor.

There is a unique point of dilation (no matter where the two objects are placed relative to each other), which, in this case, is found by joining the vertices and their corresponding images. The intersection of the lines is the point of dilation:



0.6 Axioms of Euclidean Geometry

0.6.1 Euclid's postulates

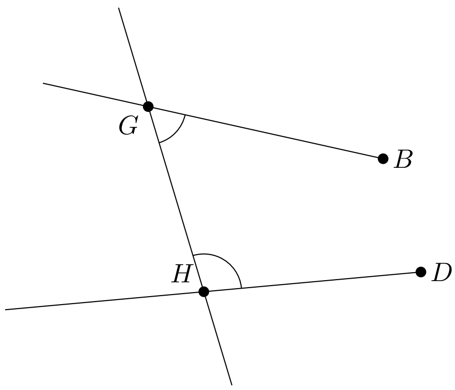
We have briefly explored some properties of points, lines, angles and shapes. The smart Geometry guy in ancient times, Euclid, has formulated five postulates (/axioms) [1] for Euclidean Geometry in his famous book titled *Elements* : [2]

1. For any two distinct points, there is a unique line that passes through them.
2. Any line segment can be extended indefinitely in a line.
3. A circle can be drawn with any centre and any radius.
4. All right angles are equal to one another.
5. If two lines are drawn which intersect a third in such a way that the sum of the interior angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

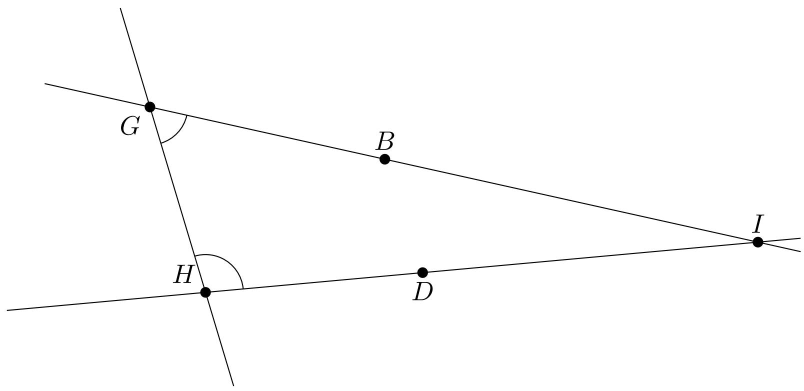
Postulate 1 implies that a line is uniquely defined by two points that it passes through. Postulate 2 allows the indefinite extension of a line segment. Postulate 3 allows the construction of a unique circle given a radius and a centre, and also allows the construction of a line segment with any possible length. Postulate 4 says that all right angles are equal, so if we move the vertex of one right angle to the vertex of another and orient it correctly, the two right angles will perfectly overlap.

Postulate 5 (called the parallel postulate) is interesting as it is a much longer statement than the other four. Without this postulate, we can derive other systems of geometry, called **non-Euclidean geometry**, but we will not go into that.

Postulate 5 visualized:



In the figure, the sum of the interior angles, $\angle BGH$ and GHD, is less than $180°$. The two slanted horizontal lines pass throught G, B and H, D respectively. Typically, we denote a line or line segment by the name of the points it passes through (generally the initial point is written first, but sometimes the order that the points are written does not matter). So we can call the two lines GB and HD respectively. If we extend GB and HD , we see that they eventually meet at a point I:



Line GH divides the plane into two sides (/parts): the left side and the right side. We see that point I is on the same side as the two interior angles, as they are all on the right side.

An implication of the parallel postulate is that if two lines never intersect each other (meaning they are parallel), then the sum of the interior angles is exactly $180°$. This is the **contrapositive** [2] of the parallel postulate.

Given axiom 1 to axiom 4, an equivalent statement of axiom 5 is :

- Given a line and a point not on it, at most one line parallel to the given line can be drawn through the point.

This is known as **Playfair's axiom**. Right now, we will not show how they are equivalent because it is too complicated. We will first derive some basic properties of lines and angles from these five Euclid's postulates (and other common notions) in the following section.

0.6.2 Common notions

Common notions are axioms / postulates and properties that are generally assumed to be true, and may involve stuff outside of Geometry. [3]

Algebra

We have a binary relation [3] called **equality** (=) and a binary operation [4] called **addition** (+) that satisfy the following axioms:

1. For each x, $x = x$ (reflexive property)
2. If $x = y$, then $y = x$ (symmetric property)
3. If $x = y$ and $y = z$, then $x = z$ (transitive property)
4. For each x, y and z , $(x + y) + z = x + (y + z)$ (associative property)
5. For each x and y, $x + y = y + x$ (commutative property)
6. If $x = y$, then $x + z = y + z$, and $z + x = z + y$ (additive property)

Now we can define order (**inequality**) in terms of addition (assuming all the following variables are positive numbers). Define a binary relation *less than* (<). Then $x < y$ means that there is some z such that $x + z = y$. And let *greater than* just have the opposite order, that is, $x > y$ means $y < x$. A number of properties of inequality and equality can be easily proved:

7. If $x = y$ and $x + d = z$, then $y + d = z$ (substitution of equals)

   Proof. $x = y$ and $x + d = z \Rightarrow x + d = y + d$ and $x + d = z$ (additive property) $\Rightarrow y + d = z$ (transitive property) □

8. If $x = y$ and $w = z$, then $x + w = y + z$ (additive property (another version))

   Proof. Assume $x = y$ and $w = z$. Then $x + w = y + w$ (additive property) $\Rightarrow x + w = y + z$ (substitution of equals) □

9. If $x < y$ and $y = z$, then $x < z$ (substitution of equals [for inequality])

   Proof. $x < y \Rightarrow x + d = y$ for some d. (definition of <) $\Rightarrow x + d = z$ (transitive property) $\Rightarrow x < z$ (definition of <) □

[2] Generally, an if-then statement is a statement in the form of 'if p, then q' . The contrapositive of the statement is 'if not q, then not p' .

[3] A binary relation associates elements of one set, called the domain, with elements of another set, called the codomain. A binary relation over sets X and Y is a new set of ordered pairs (x, y) consisting of elements x in X and y in Y.

[4] A binary operation is a rule for combining two elements (called operands) to produce another element. More precisely, a binary operation on a set S is a mapping of the elements of the Cartesian product $S \times S$ to S ,written as $f : S \times S \to S$.

10. If $x = y$ and $y < z$, then $x < z$ (substitution of equals [for inequality] (another version))

    Proof. $y < z \;\Rightarrow\; y + d = z$ for some d. (definition of $<$) $\Rightarrow\; x + d = z$ (substitution of equals) $\Rightarrow\; x < z$ (definition of $<$) □

11. If $x < y$ and $y < z$, then $x < z$ (transitive property of inequality)

    Proof. $x < y$ and $y < z \;\Rightarrow\; x + d = y$ and $y + k = z$ for some d, k

    $\Rightarrow\; (x + d) + k = z$ (substitution of equals) $\Rightarrow\; x + (d + k) = z$ (associative property)

    $\Rightarrow\; x < z$ (definition of $<$) □

12. If $x < y$, then $x + z < y + z$, and $z + x < z + y$ (additive property of inequality)

    Proof. $x < y \;\Rightarrow\; x + d = y$ for some $d \;\Rightarrow\; x + z + d = y + z$ (additive property)

    $\Rightarrow\; x+z < y+z$ (definition of $<$) $\Leftrightarrow\; z+x < z+y$ (commutative property and substitution of equals for inequality) □

Next, assume an axiom for cancellation:

13. If $x + z = y + z$, then $x = y$ (cancellation property)

With this axiom, subtraction ($-$) can be defined. Subtraction is the inverse operation of addition,and is characterized by the property that

$$x + z = y \text{ if and only if } z = y - x$$

The cancellation happens as follows: let $x + z = y$. Then since $x < y$, we have $y = x + k$ for some k. By cancellation property, we have $k = z$. Thus, given two numbers x, y where $x < y$, there is a unique number z such that $x + z = y$. We can write this z in terms of subtraction operator: $z = y - x$.

Note that in the following properties, whenever a difference is indicated, such as $x - y$, it is implicitly assumed that $x > y$.

Some properties involving subtraction:

14. If $x = y$, then $x - z = y - z$, and $w - x = w - y$ (subtractive property)

    Proof. Assume that $x = y$. Let $m = x - z$ and $n = y - z$. Then $z + m = x$ and $z + n = y$ (definition of subtraction). By transitive property, $z + m = z + n$. By cancellation property, $m = n$, and thus $x - z = y - z$.

    (Forget variable m, n). For the latter then-statement, assume $x = y$ again. Let $m = w - x$ and $n = w - y$. Then $x + m = w$ and $y + n = w$. By transitive property, $x + m = y + n$. By substitution of equals, $x + m = x + n$. By cancellation property, $m = n$, and thus $w - x = w - y$. □

15. If $x = y$ and $x - d = z$, then $y - d = z$ (substitution of equals (another version))

    Proof. $x = y$ and $x - d = z \;\Rightarrow\; x - d = y - d$ and $x - d = z$ (subtractive property)

    $\Rightarrow\; y - d = z$ (transitive property) □

16. If $x = y$ and $w = z$, then $x - w = y - z$ (subtractive property (another version))

    Proof. Assume $x = y$ and $w = z$. Then $x - w = y - w$. By subtractive property (the latter one), $y - w = y - z$. Thus by transitive property, $x - w = y - z$. □

17. If $x - z = y - z$ or $z - x = z - y$, then $x = y$ (cancellation property (another version))

    Proof. Assume that $x - z = y - z$. Let $m = x - z = y - z$. Then $z + m = x$ and $z + m = y$. By transitive property, $x = y$.

    Assume that $z - x = z - y$. Let $m = z - x = z - y$. Then $x + m = z$ and $y + m = z$. By transitive property, $x + m = y + m$. By cancellation property, $x = y$. □

18. $(x + y) - y = x$ (property of additive inverse)

Proof. Let $z = (x + y) - y$. Then $z + y = x + y$. By cancellation property, $z = x$, and thus $(x + y) - y = x$. $\square$

19. $(x - y) + y = x$ (property of additive inverse (another version))

Proof. Let $z = (x - y) + y$. Then $z - y = x - y$. By cancellation property, $z = x$, and thus $(x + y) - y = x$. $\square$

20. $(x + y) - z = (x - z) + y$ (commutative property of operations)

Proof. Let $m = (x + y) - z$. Then $m + z = x + y \Rightarrow (m + z) - y = x$.

Let $n = (x - z) + y$. Then $n - y = x - z \Rightarrow (n - y) + z = x$.

By transitive property, $(m + z) - y = (n - y) + z \Rightarrow m + z = (n - y) + z + y$

$\Rightarrow m + z = (n - y) + y + z$ (commutative property)

$\Rightarrow m = (n - y) + y$ (cancellation property)

$\Rightarrow m = n$ (property of additive inverse)

$\Rightarrow (x + y) - z = (x - z) + y$ $\square$

21. $(x + y) - z = x + (y - z)$ (property of added difference)

Proof. Let $m = (x+y)-z$ and $n = x+(y-z)$. Then $n = (y-z)+x$ (commutative property) $\Rightarrow n = (y + x) - z$ (commutative property of operations)

$\Rightarrow n = (x + y) - z$ (commutative property and substitution of equals)

$\Rightarrow m = n = (x + y) - z$ (transitive property)

$\Rightarrow (x + y) - z = x + (y - z)$ $\square$

22. $(x - y) - z = x - (y + z)$ (property of subtracted sum)

Proof. Let $m = (x - y) - z$. Then $m + z = x - y \Rightarrow m + z + y = x$.

Let $n = x - (y + z)$. Then $n + y + z = x$. By transitive property, $m + z + y = n + y + z$.

By commutative property and property of cancellation, $m = n \Rightarrow (x-y)-z = x-(y+z)$ $\square$

23. $(x - y) + z = x - (y - z)$ (property of subtracted difference)

Proof. Let $m = (x - y) + z$. Then $(m - z) + y = x$.

Let $n = x - (y - z)$. Then $n + (y - z) = x$. By transitive property, $(m - z) + y = n + (y - z)$.

Note that $n+(y-z) = (n+y)-z = (n-z)+y$ (property of added difference and commutative property of operations) .

Thus $(m-z)+y = (n-z)+y \Rightarrow m = n$ (cancellation property) $\Rightarrow (x-y)+z = x-(y-z)$ $\square$

24. If $x < y$, then $x - z < y - z$ and $w - x > w - y$ (subtractive property of inequality)

Proof. $x < y \Rightarrow x + d = y$ for some d . $\Rightarrow x + d - z = y - z$ (subtractive property) $\Rightarrow$ $x - z < y - z$ (definition of $<$)

For the latter then-statement:

$x < y \Rightarrow x + d = y$ for some d . $\Rightarrow w - (x + d) = w - y$ (subtractive property)

$\Rightarrow (w - x) - d = w - y$ (property of subtracted sum) $\Rightarrow w - x = w - y + d$

$\Rightarrow w - y < w - x \Rightarrow w - x > w - y$ $\square$

25. If $x < y$ and $w = z$, then $x - w < y - z$ (subtractive property of inequality + substitution of equals)

Proof. Assume that $x < y$ and $w = z$. Then $x - w < y - w$ (subtractive property of inequality)

Since $y - w = y - z$ (subtractive property) , we have $x - w < y - z$ (substitution property for inequality) □

26. If $x = y$ and $w < z$, then $x - w > y - z$ (subtractive property of inequality + substitution of equals (another version))

    Proof. Assume that $x = y$ and $w < z$. Then $x - w > x - z$ (subtractive property of inequality) . $\Rightarrow$ $x - z < x - w$

    Since $x - z = y - z$ (subtractive property) , we have $y - z < x - w$ (substitution property for inequality) $\Rightarrow$ $x - w > y - z$ □

27. If $x < y$ and $w > z$, then $x - w < y - z$ (uneven subtractive property)

    Proof. $x < y$ and $w > z$ $\Rightarrow x + d = y$ for some d and $w = z + k$ for some k .

    $\Rightarrow (x + d) - w = y - (z + k)$ (subtractive property)

    $\Rightarrow (x - w) + d = (y - z) - k$ (commutative property of operations and property of subtracted sum)

    $\Rightarrow (x - w) + (d + k) = (y - z)$ $\Rightarrow x - w < y - z$ (definition of $<$) □

    We have overlooked that the equations $x + y = x$ and $x < x$ haven't been falsified by these axioms and properties. Thus we need a new axiom:

28. It is not the case that $x = x + y$ (law of non-contradiction)

    In other words, $x \neq x + y$. This implies that $x \not< x$ and $x \not> x$.

We still need one more axiom:

29. For each x and y, either $x = y$, or there is some z such that $x + z = y$, or there is some z such that $x = y + z$. (law of trichotomy)

    In other words, only one of the following can be true: $x = y$ or $x < y$ or $x > y$.

Geometry

1. Objects that completely coincide in position are the same. Similarly, the same object can only appear at one position at a time. (position uniqueness of objects)

   If there are 'two' points A , B on the plane that are at the exact same position on the plane, then $A = B$, and there are actually only one distinct points.

   Similarly, if there are 'two' lines l , m that completely coincide with each other, then they are in fact the same line ($m = l$), and there is actually only one distinct line.

   As for the latter statement, a point cannot simultaneously appear at two distinct positions. If there are two objects at two distinct positions, then the two objects must be distinct.

   However, if we allow the passage of time, then we may allow one object to travel to another position at another time. For example, it is possible that point A is at the origin at $t = 0$, and point A moved to coordinate (1,0) at $t = 2$.

2. Objects on the plane will stay at the same position indefinitely unless stated otherwise. (Object permanence)

   If we have constructed a circle on the plane, then the circle will still exist at the same position even when we look away.

3. A point cannot lie between itself and another point.

   This means that a point cannot lie to the left or right of itself, as that would imply that the same point exists in two distinct positions.

4. If all points of a straight line are divided into two non-empty sets, such that every point of the first set lies to the left of any point of the second set, then either there exists a point that is the rightmost point of the first set, or there exists a point that is that the leftmost point of the second set. (Dedekind's axiom)

(Filled dot is in 1st set. Unfilled dot is in 2nd set.)

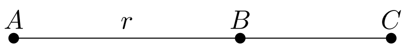
This axiom makes the points on the line behave like real numbers. For real numbers, if we cut the real number line into two intervals at real number k , then we have either the intervals [5] $(-\infty, k]$ and $(k, \infty]$, or the intervals $(-\infty, k)$ and $[k, \infty)$. Note that for each case, only one of the two intervals (the interval that has square brackets) can have a the maximum / minimum number (which is k) . For an interval like $(-\infty, k)$, there doesn't exist a maximum or minimum number.

Similarly, when a line is divided into two sets stated above, only one set can have a leftmost / rightmost point. The other set simply doesn't have a leftmost / rightmost point.

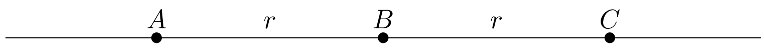
5. The points on a line can be matched one to one with the **real numbers** . (ruler postulate)

   The real number that corresponds to a point is the **coordinate** of the point. The **distance** between points A , B (written as AB), is the absolute value of the difference of the coordinates of A and B. This means distance and length can only be non-negative numbers.

   An implication of ruler postulate is that when we have a line segment AC and B is a point on AC , then for a given length of AB , there is a unique position of point B on the line segment.



6. If B is a point on a line, then for a given length r , there are exactly two points A , C on the line such that $AB = r$ and $AC = r$. (left-right property)



   This is because point B divides the line into two parts: the left part and the right part. In each part, there is exactly one point that has a distance r away from B (because of ruler postulate).

7. There exists a point between any two distinct points on a line.

   This is because points on the line are like real numbers, and real numbers also have the property that for any a, b where $a < b$, there exists a number c such that $a < c < b$.

8. If one line segment / angle can be moved (translated and rotated) to completely coincide with another, then they are equal. (coincidable line segments and angles are equal)

   Consider the two line segments:

[5] An interval of the real number line / real number set is a continuous subset (/connected space) of the real number set. A closed (/ inclusive) interval with lower bound a and upper bound b is denoted $[a, b]$, and if x is in the interval $[a, b]$, then $a \leq x \leq b$. An open (/ exclusive) interval with lower bound a and upper bound b is denoted (a, b) , and if x is in the interval (a, b) , then $a < x < b$.

Since AB and CD have the same length, AB can be moved to coincide with CD completely, so we say that $AB = CD$.

So for convenience, we simply write AB to represent the length of line segment AB , and CD for length of line segment CD .

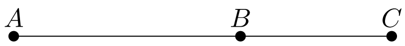
Similarly, if we have two angles:

First we extend the line segments of the angles into a ray (in the direction from vertex to endpoint). Since these angles have the same size, we can move $\angle ABC$ to overlap with $\angle DEF$ perfectly. Thus $\angle ABC = \angle DEF$.

For convenience, we simply write $\angle ABC$ to represent the size of $\angle ABC$ (usually in degrees), and $\angle DEF$ for size of $\angle DEF$.

Note that saying that two line segments / angles are equal is just the shorthand way of saying that they are equal in **magnitude**. It doesn't actually mean that the two line segments / angles are the same, which would require them to completely coincide with each other.

9. A line segment can be split into the sum of two shorter line segments, and two shorter line segments on the same line can be combined to be a longer line segment. (segment addition postulate)



As shown in the figure, if there is a point B **between** A and C (/on the line segment AC), then we have $AC = AB + BC$.

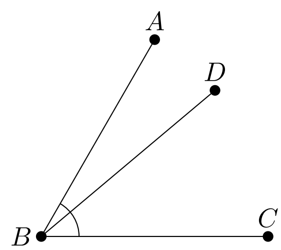
Similarly, if we have a point B such that $AC = AB + BC$, then B lies between A and C.

10. The rays of any angle can be matched one to one with the real numbers from 0 to 360. (protractor postulate)

This means that for a given angle θ and a given initial ray, there is, uniquely, another ray such that the clockwise angle formed by the two rays is θ. (Clockwise angle means that the angle is measured in clockwise direction.)

This postulate also makes angle behave like real numbers, so it has the properties that for every two angles, there exists an angle between them, and any angle is infinitely divisible.

11. An angle can be split into the sum of two smaller angles, and two smaller angles sharing the same vertex and a common side can be combined to be a larger angle. (angle addition postulate)

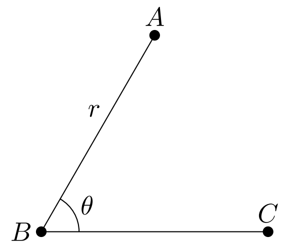


In the figure, there is a line (segment) between $\angle ABC$, so we have $\angle ABC = \angle ABD + \angle DBC$.

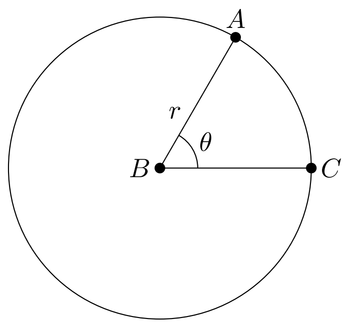
$\angle ABD$ and $\angle DBC$ are called vertically adjacent angles (or simply adjacent angles).

Note that we can only add angles and angles together, or line segments and line segments together. We cannot add an angle and a line segment together.

12. If there is a line segment BC, and A is a point above line BC, then for a given angle θ and a given length d , there is a unique position of point A such that $\angle ABC = \theta$ and $AB = r$. (polar coordinate postulate)



This is because by protractor postulate, every distinct point A on a circle corresponds to a distinct angle θ:

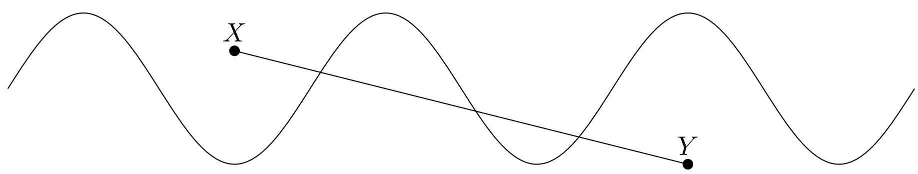


And to make the length of AB to be r , the circle's radius will also be r . Since there exists only one distinct circle with the same centre and same radius, the position of point A can be uniquely specified when given a radius and an angle.

Note that $\angle ABC = 0°$ if and only if point A is on line BC .

13. A plane contains at least three non-collinear points.
14. The length of a straight line segment connecting points A and B is shorter than any curve segment connecting A and B . (property of straight lines)

   In other words, a straight line segment is the shortest distance between two points.
15. When a line / curve l separates the plane into two disconnected parts A and B , and X is a point in A while Y is a point in B , then the line / curve (segment) connecting A and B must intersect l at least once. (property of continuous path)



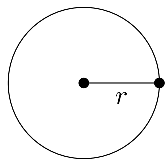
   If there is a path from X to Y without crossing line/curve l , then A and B are connected by definition.
16. The position of the point of intersection of any two lines / curves can be determined with infinite precision . We can also determine whether two lines / curves / points intersect or coincide or not.

   This isn't like when we draw two lines on paper and have to eyeball the intersection point. We have the power of mathematics to help us.

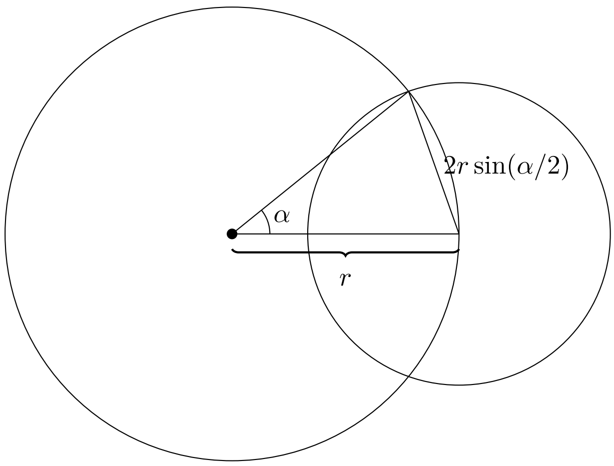
17. The position of any objects on the plane, distance between two points / lines, the length of line segments, and sizes of angles can be measured, specified and drawn infinite precision.

    Imagine that we have a measuring tape that can measure any distance on the plane with infinite precision, and a protractor that can measure any angle with infinite precision. This means we can even construct the so-called uncontructible length such as the cube root of 2 ($\sqrt[3]{2}$), and trisect an arbitrary angle (which we cannot do with only an unmarked straight edge and compass).

    This ability seems overpowered, but it is given by Euclid's axiom 3, which states that a circle can be drawn with *any* centre and *any* radius. To get a line segment with length r , we can just draw a circle with radius r and mark a point on the circumference of the circle:

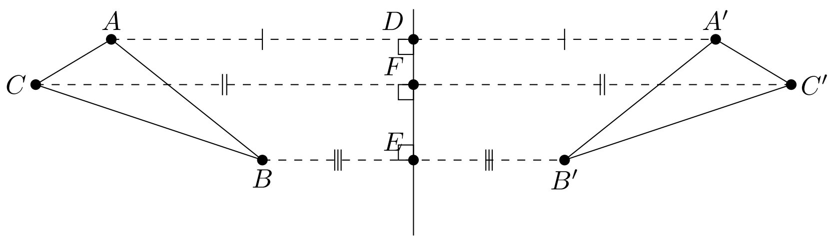


    To get a specific angle α , we can draw a circle with its centre on the initial point of the angle and its radius being something like $2r\sin(\alpha/2)$ where r is the distance between the initial point and vertex of the angle.



18. Reflection preserves segment length and angle size. (reflection postulate)

    This postulate allows us to prove stuff using reflection symmetry.



    In the figure, $\triangle A'B'C'$ is a reflection of $\triangle ABC$ about the line DE , so we have $AD = A'D$, $CE = C'E$, $BE = B'E$, with $AA' \perp DE$, $CC' \perp DE$, $BB' \perp DE$. By reflection postulate, we have $AB = A'B'$, $BC = B'C'$, $AC = A'C'$, and $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$.

1 Lines, angles and shapes

1.0 Symbols and abbreviations

(The hyphen and the bullet point aren't a part of symbol.)

- $\angle$ - angle
- $\triangle$ - triangle
- ◺ - right-angled triangle
- $\perp$ - perpendicular to
- // - parallel to
- $\cong$ - congruent to
- $\because$ - since
- $\therefore$ - therefore
- adj. - adjacent
- opp. - opposite
- pt. - point
- st. - straight
- vert. - vertical
- prop. - property
- corr. - corresponding
- isos. - isosceles
- equil. - equilateral
- //gram - parallelogram
- inc. - included

1.1 Basic properties

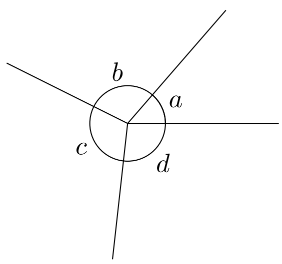
[4] [5]

Preposition 1. Two lines can intersect at one point at most. (property of line intersection)

Proof. Suppose that two lines intersect at two distinct points called P and Q. We have two lines passing through P and Q , which contradicts Euclid's postulate 1 (which states that there is only one line that passes through two points). So the two lines can also never intersect at three distinct points or more because they would have to intersect at two of the points, which we have just shown to be impossible. So two lines can intersect at one point at most. □

Preposition 2. The sum of all angles sharing the same vertex is 360° . ($\angle$s at a pt.) * [6]

Example

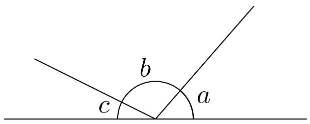


[6]Reasons marked with * are used in HK secondary school maths.

$$a + b + c + d = 360° \qquad (\angle\text{s at a pt.})$$

Proof. By definition, a whole revolution is 360° . By angle addition postulate, when a whole revolution is split into several smaller angles, the sum of these angles must be a whole revolution, which is 360°. ☐

Preposition 3. The sum of adjacent angles on a straight line is 180° (adj. ∠s on st. line) *

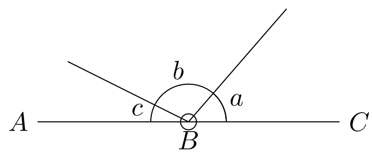


$$a + b + c = 180° \qquad (\text{adj. } \angle\text{s on st. line})$$

Proof. By definition, a straight angle (which is 180°) is half a revolution or two right angles, so two straight angles sharing a vertex makes up the whole revolution which is 360° (∠s at a pt.). Since all right angles are equal by Euclid's 4th postulate, all straight angles are also equal. So one straight angle is 180° . By angle addition postulate, the straight angle can be split into several smaller angles whose sum is the straight angle, which is 180° . ☐

Preposition 4. If the sum of some adjacent angles is 180° , then these angles make a straight line. (adj. ∠s supp.) *

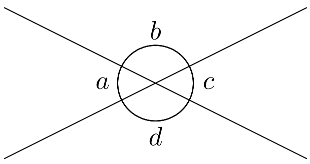
(The tiny circle at B indicates that we are not sure if the 'line' passing through B is actually a straight line.)



$$\text{Observation: } a + b + c = 180°$$
$$\therefore\ ABC \text{ is a straight line.} \qquad (\text{converse of adj. } \angle\text{s on st. line})$$

Proof. By protractor postulate, given ray BC, there is a unique ray BA such that $\angle ABC$ is 180° (a straight angle), and a straight angle is a straight line. ☐

Preposition 5. Vertically opposite angles are equal. (vert. opp. ∠s) *



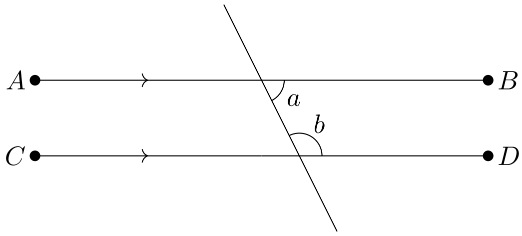
$$a = c \text{ and } b = d \qquad (\text{vert. opp. } \angle\text{s})$$

Proof.

$$\begin{aligned} a + b &= 180° && (\text{adj. } \angle\text{s on st. line}) \\ b + c &= 180° && (\text{adj. } \angle\text{s on st. line}) \\ \therefore\quad a + b &= b + c \\ a &= c \end{aligned}$$

By similar reasoning, we have $b = d$. ☐

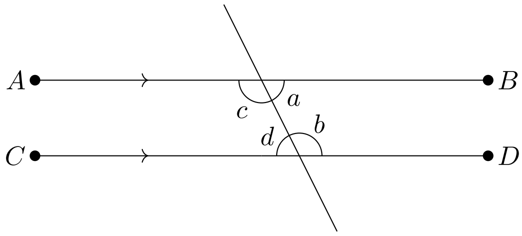
Preposition 6. For a pair of parallel lines, the interior angles formed by a transversal line are **supplementary** [7]. (int. $\angle$s, $AB//CD$) *



$$a + b = 180^\circ \qquad \text{(int. } \angle\text{s, } AB//CD)$$

Proof. By the contrapositive of parallel postulate, if two lines never intersect each other (meaning they are parallel), then the two lines are **not** drawn in such a way that intersect a third line (the transversal line) such that the sum of the interior angles on one side is less than two right angles. This means that $a + b \geq 180^\circ$.

However, if $a + b > 180^\circ$, then we can focus on the interior angles of the other side: c and d .

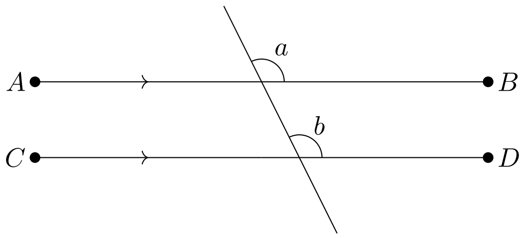


Note that we have $c = 180^\circ - a$ and $d = 180^\circ - b$ (adj. $\angle$s on st. line). Thus, starting from the inequality:

$$\begin{aligned} a + b &> 180^\circ \\ 360^\circ - (a + b) &< 360^\circ - 180^\circ \\ 180^\circ - a + 180^\circ - b &< 180^\circ \\ c + d &< 180^\circ \end{aligned}$$

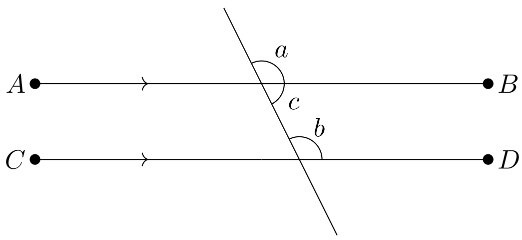
By the parallel postulate, line AB and CD must meet at the left of the transversal line, but this contradicts the initial assumption that the two lines never intersect each other. Thus, it must be the case that $a + b = 180^\circ$. □

Preposition 7. For a pair of parallel lines, the corresponding angles formed by a transversal line are equal. (corr. $\angle$s, $AB//CD$) *



$$a = b \qquad \text{(corr. } \angle\text{s, } AB//CD)$$

Proof. .

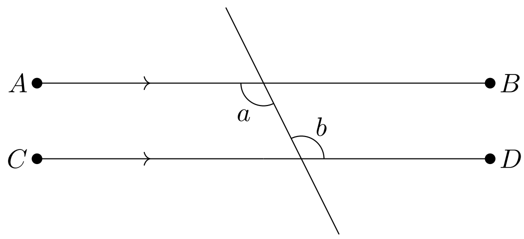


[7] Two angles are supplementary if they add up to 180°.

$$
\begin{aligned}
c + b &= 180^\circ && \text{(int. } \angle\text{s, } AB//CD) \\
a + c &= 180^\circ && \text{(adj. } \angle\text{s on st. line)} \\
\therefore\ a + c &= c + b \\
a &= b
\end{aligned}
$$

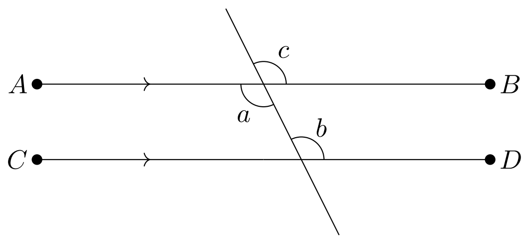
□

Preposition 8. For a pair of parallel lines, the alternate angles formed by a transversal line are equal. (alt. $\angle$s, $AB//CD$) *



$$a = b \qquad \text{(alt. } \angle\text{s, } AB//CD)$$

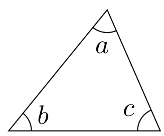
Proof. .



$$
\begin{aligned}
a &= c && \text{(vert. opp. } \angle\text{s)} \\
b &= c && \text{(corr. } \angle\text{s, } AB//CD) \\
\therefore\ a &= b
\end{aligned}
$$

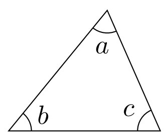
□

Preposition 9. The sum of interior angles of a triangle is 180° . ($\angle$ sum of $\triangle$)



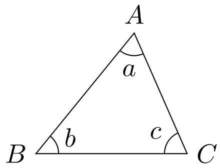
$$a + b + c = 180^\circ \qquad (\angle \text{ sum of } \triangle)$$

Preposition 10. The sum of any two interior angles of a triangle is less than 180° . (2 $\angle$ sum of $\triangle$)



$$
\begin{aligned}
a + b &< 180^\circ && (2\ \angle \text{ sum of } \triangle) \\
b + c &< 180^\circ && (2\ \angle \text{ sum of } \triangle) \\
a + c &< 180^\circ && (2\ \angle \text{ sum of } \triangle)
\end{aligned}
$$

Proof. Note that the three vertices of the triangle must be non-collinear (otherwise it will be just a line segment), so any interior angle is larger than zero.

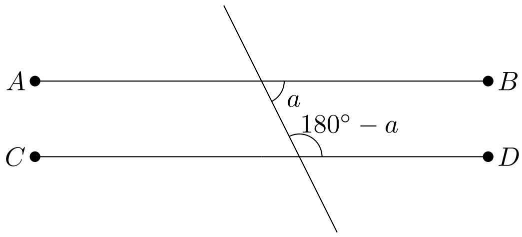


Refer to the figure, we have $a+b+c=180^\circ$ ($\angle$ sum of $\triangle$), with $a>0^\circ$, $b>0^\circ$, $c>0^\circ$. Thus $a+b<180^\circ$, $b+c<180^\circ$, $a+c<180^\circ$.

□

An (obvious) implication of this preposition is that any interior angle of a triangle is less than 180° .

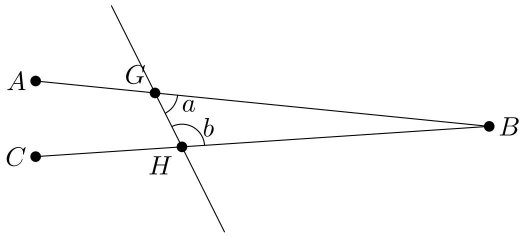
Preposition 11. For two lines, if the interior angles formed by another transversal line are supplementary , then the two lines are parallel. (int. $\angle$s supp.) *



$AB \,//\, CD$ (int. $\angle$s, supp.)

(// means 'is parallel to')

Proof. If the two lines are not parallel, then they intersect at some point. These two lines and the transversal line form a triangle (assuming the three lines are not concurrent):



Refer to the figure, a and b are the interior angles formed by the transversal line that are on the same side as B. Note that $a+b+\angle GBH=180^\circ$ ($\angle$ sum of $\triangle$). Since $\angle GBH>0^\circ$ (as G is not on HB), we have $a+b<180^\circ$.

If we want to consider the interior angles that are on the opposite side from B , then two interior angles are $\angle AGH$ and $\angle CHG$ instead. We have $\angle AGH=180^\circ-a$ and $\angle CHG=180^\circ-b$ (adj. $\angle$s on st. line). So

$$\begin{aligned} a+b &< 180^\circ \\ 360^\circ-(a+b) &> 360^\circ-180^\circ \\ 180^\circ-a+180^\circ-b &> 180^\circ \\ \angle AGH+\angle CHG &> 180^\circ \end{aligned}$$

No matter which side we look at, the two interior angles formed by the transversal line are not equal to 180° . So we have proved the statement:

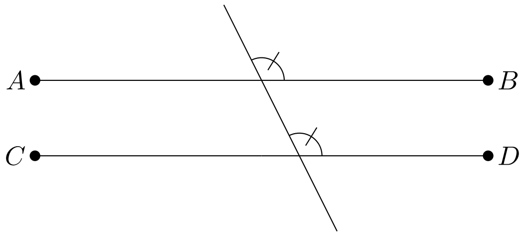
If the two lines are not parallel, then the interior angles formed by another transversal line are not supplementary.

Thus, the contrapositive of this statement is also true:

If the interior angles formed by another transversal line are supplementary , then the two lines are parallel.

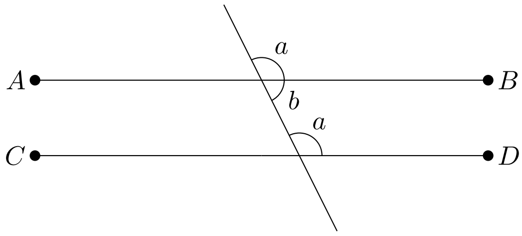
□

Preposition 12. For two lines, if the corresponding angles formed by another transversal line are equal, then the two lines are parallel. (corr. ∠s equal.) *



$$AB \,//\, CD \qquad \text{(corr. } \angle\text{s equal.)}$$

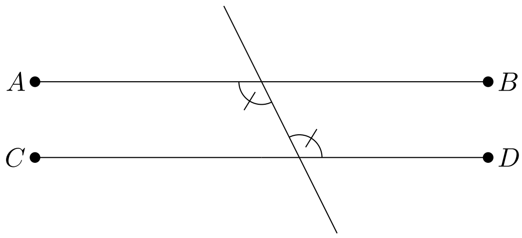
Proof. .



$$a + b = 180^\circ \qquad \text{(adj. } \angle\text{s on st. line)}$$
$$AB \,//\, CD \qquad \text{(int. } \angle\text{s, supp.)}$$

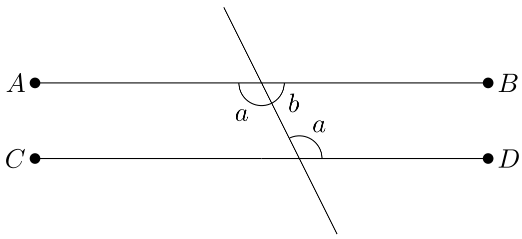
□

Preposition 13. For two lines, if the alternate angles formed by another transversal line are equal, then the two lines are parallel. (alt. ∠s equal.) *



$$AB \,//\, CD \qquad \text{(alt. } \angle\text{s equal.)}$$

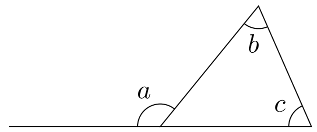
Proof. .



$$a + b = 180^\circ \qquad \text{(adj. } \angle\text{s on st. line)}$$
$$AB \,//\, CD \qquad \text{(int. } \angle\text{s, supp.)}$$

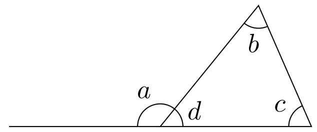
□

Preposition 14. An exterior angle of a triangle is the sum of the two opposite interior angles. (ext. ∠ of △)



$$a = b + c \qquad \text{(ext. } \angle \text{ of } \triangle)$$

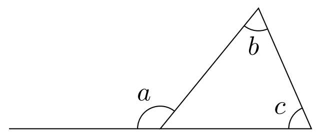
Proof. .



$$\begin{aligned} d + b + c &= 180^\circ && (\angle \text{ sum of } \triangle) \\ a + d &= 180^\circ && (\text{adj. } \angle\text{s on st. line}) \\ \therefore\ d + b + c &= a + d \\ a &= b + c \end{aligned}$$

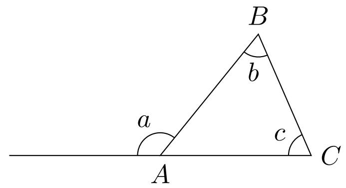
□

Preposition 15. An exterior angle of a triangle is greater than any of its opposite interior angle. (ext. $\angle$ of $\triangle$ < int. opp. $\angle$)



$$a > b \text{ and } a > c \qquad (\text{ext. } \angle \text{ of } \triangle < \text{int. opp. } \angle)$$

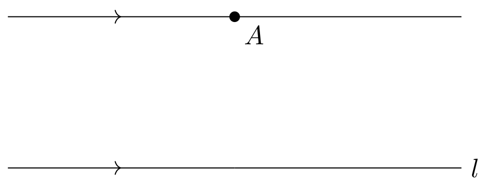
Proof. .



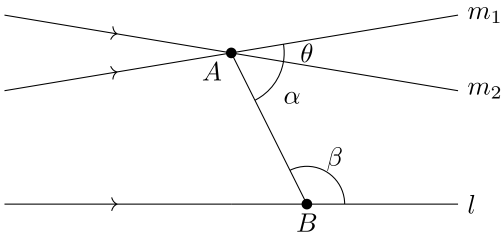
Since $\triangle ABC$ is a valid triangle, A , B , C are non-collinear, so $b > 0$ and $c > 0$.

Since $a = b + c$ (ext. $\angle$ of $\triangle$), we have $a > b$ and $a > c$. □

Preposition 16. Given a line and a point not on it, there is exactly one line passing through the point that is parallel to the given line. (Playfair's theorem / property of parallel line)



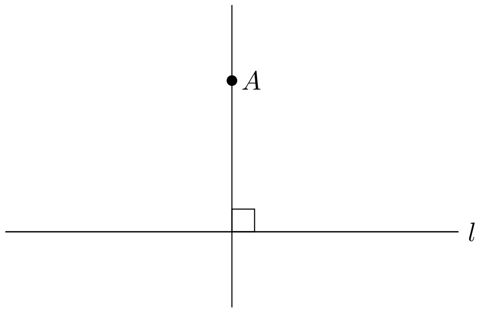
Proof. Label the given line as l and the given point as A . If there are two distinct lines m_1 and m_2 both passing through A , then A must be the only point of intersection (property of line intersection) . Let θ be the angle formed (facing right) between m_1 and m_2 . Let B be an arbitrary point on l and connect AB.



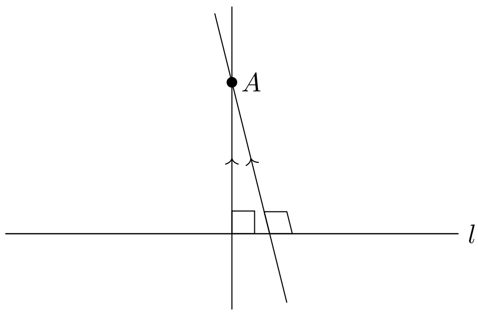
Refer to the figure. If m_1 and m_2 are both parallel to l , then we have $\alpha + \beta = 180°$ (int. $\angle$s, $m_2//l$) , and $\alpha + \beta + \theta = 180°$ (int. $\angle$s, $m_1//l$) . This means $\theta = 0°$. But this would mean that m_1 and m_2 are actually the same line, which is a contradiction.

Thus, there can only be one unique line passing through A that is parallel to l . $\square$

Preposition 17. Given a line and a point, there is exactly one line passing through the point that is perpendicular to the given line. (property of perpendicular line)

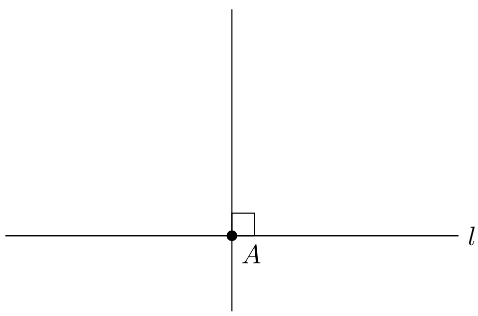


Proof. Label the given line as l and the given point as A . First, consider the case that A is not on l . Suppose there are two distinct lines passing through A that are perpendicular to l . Then they must meet l at two distinct points, or otherwise they are actually the same line (Euclid's first postulate). Then they must be parallel since the interior angles formed by the lines sum up to $180°$ (int. $\angle$s supp.) . But parallel lines never intersect, which contradicts the assumption that the two lines intersect at A .



Thus, there is a unique line passing A that is perpendicular to l .

Now consider the case that A is on line l . Then by protractor postulate, there is a unique line that intersects l at A at $90°$, so there is a unique line passing A that is perpendicular to l .

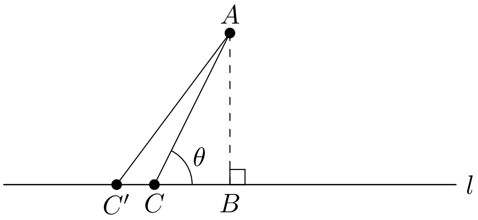


$\square$

Preposition 18. Suppose B is a point on line l and A is a point vertically above B (meaning $AB \perp l$) . If C is a point on l that is not B, then the longer BC is, the smaller the angle $\angle ACB$ is. (property of hypotenuse inclination)

In other words, if length BC is a variable over the domain $(0, \infty)$, then $\angle ACB$ is a **strictly decreasing** [8] function of BC .

Proof. Assume that C is at the left of B. Let C' be a point on l to the left of C. So $C'B > CB$.



In $\triangle ACC'$, we have $\angle AC'B + \angle C'AC = \angle ACB$ (ext. $\angle$ of $\triangle$), so $\angle AC'B < ACB$.

If C is at the right of B , then we can let C' be a point to the right of C, and the (ext. $\angle$ of $\triangle$)reason will still hold.

□

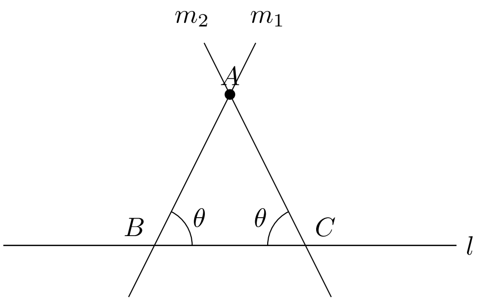
An implication of this preposition is that for any given acute angle θ , there is exactly one point C that is at the left of B such that $\angle ACB = \theta$. (And also exactly one point at the right of B for that.) Otherwise, say, if there are two distinct points C and C' at the left of B for that, with C' at the leftmost, then we have $\theta + \angle C'AC = \theta$ with $\angle C'AC > 0$, which violates the law of non-contradiction.

Note 1: If C is not on B, then $\angle ACB$ must be smaller than $90°$ because $\angle CAB > 0$, and $\angle ACB + \angle CAB = 90°$ (ext. $\angle$ of $\triangle$), which means $\angle ACB < 90°$.

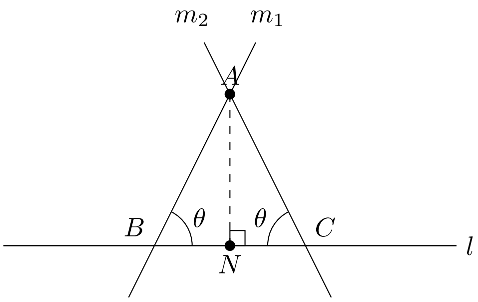
Thus, we also have the alternative statement: if length BC is a variable over the domain $[0, \infty)$, then the smaller angle formed by AC and l is a strictly decreasing function of BC .

Note 2: If point A is below line l , then the preposition still holds because of symmetry (reflection postulate). I just state point A is above line l to simplify the statement.

Preposition 19. If there is a point A above line l , then for a given acute angle θ , there is exactly two lines m_1 and m_2 passing through A such that θ is the smaller angle formed between m_1 and l , and also m_2 and l . (property of falling lines)



Proof. Let N be the projection of A on l .(Alternatively, we can say 'draw $AN \perp l$ '.) Let m_1 , m_2 intersect l at B , C respectively.



[8]A function $f(x)$ is said to be strictly decreasing on an interval I if $f(b) < f(a)$ for all $b > a$, where $a, b \in I$.

By property of hypotenuse inclination, given an angle θ , for each side of line l divided by N , there is exactly one point B or C on l that satisfies $\angle ABN = \theta$ or $\angle ACN = \theta$. Since there is exactly one A, B and C , there is exactly one line AB and exactly one line AC .

An implication of this preposition is that for a given obtuse angle θ , there is exactly two lines m_1 and m_2 passing through A such that θ is the larger angle formed between m_1 and l , and also m_2 and l. This is because for every obtuse angle θ , there is a unique corresponding acute angle $180° - \theta$, which is the smaller angle formed by m and l. (m can be m_1 or m_2.) , which means that the property of falling lines can be applied and there are two unique lines m_1 and m_2 that satisfy the requirement.

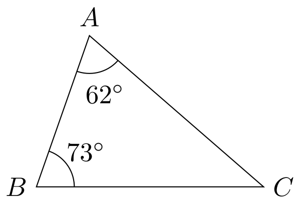
Note: The previous preposition's note also applies here.

□

1.1.1 Problems

After all the preposition stating, let's try some practical problems. (The diagrams in the problems are not necessarily to scale.)

Problem 1. In $\triangle ABC$, $\angle A = 62°$ and $\angle B = 73°$. What is $\angle C$?

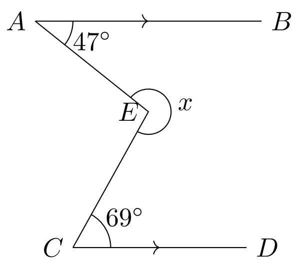


(Difficulty: 1 [Beginner])

Solution 1.

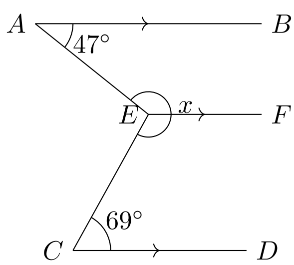
$$\begin{aligned} \angle C &= 180° - \angle A - \angle B \qquad (\angle \text{ sum of } \triangle) \\ &= 180° - 62° - 73° \\ &= \boxed{45°} \end{aligned}$$

Problem 2. In the figure, $AB//CD$, and E is a point between line AB and line CD . $\angle BAE = 47°$ and $\angle DCE = 69°$. What is x ?



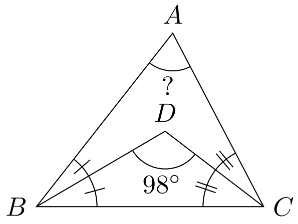
(Difficulty: 3 [Easy])

Solution 2. Draw $EF//AB//CD$.



$$
\begin{aligned}
\angle AEF + 47^\circ &= 180^\circ \qquad \text{(alt. } \angle\text{s , } AB//EF\text{)} \\
\angle AEF &= 133^\circ \\
\angle CEF + 69^\circ &= 180^\circ \qquad \text{(alt. } \angle\text{s , } EF//CD\text{)} \\
\angle CEF &= 111^\circ \\
x &= \angle AEF + \angle CEF \\
&= 133^\circ + 111^\circ \\
&= \boxed{244^\circ}
\end{aligned}
$$

Problem 3. D is a point inside $\triangle ABC$ such that $\angle ABD = \angle DBC$ and $\angle ACD = \angle DCB$, $\angle BDC = 98^\circ$. What is $\angle BAC$?



(Difficulty: 3)

Solution 3. Let $\angle ABD = \angle DBC = x$ and $\angle ACD = \angle DCB = y$. In $\triangle DBC$,

$$
\begin{aligned}
x + y + 98^\circ &= 180^\circ \qquad (\angle \text{ sum of } \triangle) \\
x + y &= 82^\circ
\end{aligned}
$$

In $\triangle ABC$,

$$
\begin{aligned}
\angle BAC + 2x + 2y &= 180^\circ \qquad (\angle \text{ sum of } \triangle) \\
\angle BAC &= 180^\circ - 2(x+y) \\
&= 180^\circ - 2(82^\circ) \\
&= \boxed{16^\circ}
\end{aligned}
$$

1.2 Congruent triangles

Two triangles are called **congruent** if one triangle can be translated, rotated, and reflected in any way to perfectly overlap with another triangle. In real life analogy, if there are two triangles made of hard paper, and we can stack them another perfectly (flipping is allowed), then the triangles are congruent.

A pair of congruent triangles have the corresponding sides and corresponding angles that are equal:

To denote that the two triangles are congruent, we say that $\triangle ABC \cong \triangle DEF$. Note that the order of the corresponding vertices must be the same. We cannot say that $\triangle ABC \cong \triangle FED$.

Note that congruence ($\cong$) is an **equivalence relation** , meaning that it satisfies the three properties shared by equality:

1. $x \cong x$ (reflexive property)
2. If $x \cong y$, then $y \cong x$ (symmetric property)
3. If $x \cong y$ and $y \cong z$, then $x \cong z$ (transitive property)

Conditions for determining congruence

In practice, we don't need to know that all of the sides and angles are corresponding in order to determine that two triangles are congruent, and there are a couple of minimum conditions that are sufficient to determine congruence.

1.2.1 SAS (Side-Angle-Side)

For side-angle-side condition, in the same triangle, the corresponded angle must be between the two corresponded sides:

This is also allowed:

In the figure, we have $AB = DE$, $\angle ABC = \angle DEF$, $BC = EF$. Thus $\triangle ABC \cong \triangle DEF$ (SAS) *.

Proof. (Proof of congruence) If $\triangle DEF$ is the flipped version of $\triangle ABC$, then we can just reflect $\triangle DEF$ once since reflection is allowed for congruence. And by reflection postulate, reflection preserves side lengths and angle sizes. So we only need to look at the case that a triangle is not the flipped version of another.

If we translate and rotate $\triangle ABC$ such that vertex B coincide with vertex E and vertex C lies on line EF, then C coincides with E because $BC = EF$. By polar coordinate postulate, given an angle $\angle ABC$ and a length AB , there is a unique position of A above BC. And since $AB = DE$ and $\angle ABC = \angle DEF$, it must be the case that A is in the same position as D . Since all the vertices coincide, all the sides must also coincide. Thus, $\triangle ABC \cong \triangle DEF$.

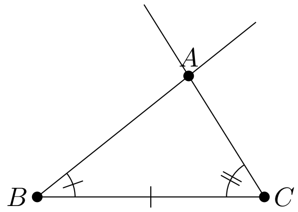
□

1.2.2 ASA (Angle-Side-Angle)

For angle-side-angle, in the same triangle, the corresponded side is the common side shared by the two corresponded angles

In the figure, we have $BC = EF$, $\angle ABC = \angle DEF$, $\angle ACB = \angle DFE$. Thus $\triangle ABC \cong \triangle DEF$ (ASA) *.

Proof. (We neglect the situation where a triangle is the flipped version of another since if so, we just need to flip a triangle back.) By protractor postulate, for a given line segment BC and a given value of $\angle ABC$, there is a unique ray such that the clockwise angle between BC and the ray is equal to $\angle ABC$. Same goes for $\angle ACB$ (but the clockwise angle is the reflex angle). The two rays BA and CA must intersect at one point if $\triangle ABC$ is a valid triangle:



The position (relative to BC) of this point is necessarily unique when given $\angle ABC, \angle ACB$ and segment BC , as we observe that placing A in any other position will cause at least one corresponded angle to change (since that will make A not lie on at least one of the original rays). Thus, if we overlap segment BC with EF , point A and point D must also coincide. Since all three vertices coincide, it must be the case that $\triangle ABC \cong \triangle DEF$. □

1.2.3 AAS (Angle-Angle-Side)

For angle-angle-side, in the same triangle, the corresponded side is not the common side shared by the two corresponded angles, and can be any one of the non-common sides.

In the figure, we have $AB = DE$, $\angle ABC = \angle DEF$, $\angle ACB = \angle DFE$. Thus $\triangle ABC \cong \triangle DEF$ (AAS) *.

Proof. Suppose we have two triangles, $\triangle ABC$ and $\triangle DEF$, in which $AB = DE$, $\angle ABC = \angle DEF$, $\angle ACB = \angle DFE$. Note that

$$\begin{aligned}\angle BAC &= 180^\circ - \angle ABC - \angle ACB \qquad (\angle \text{ sum of } \triangle)\\ &= 180^\circ - \angle DEF - \angle DFE\\ &= \angle EDF \qquad (\angle \text{ sum of } \triangle)\end{aligned}$$

So when there are two corresponding angles in two triangles, the third angle is also corresponding.

Note that we now have an angle-side-angle situation:

$$\begin{aligned}\angle ABC &= \angle DEF \qquad \text{(given)}\\ AB &= DE \qquad \text{(given)}\\ \angle BAC &= \angle EDF \qquad (\angle \text{ sum of } \triangle)\\ \therefore\ \triangle ABC &\cong \triangle DEF \qquad \text{(ASA)}\end{aligned}$$

□

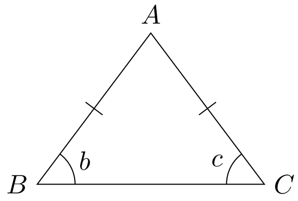
1.2.4 SSS (Side-Side-Side)

For side-side-side, three sides are corresponding sides.

In the figure, we have $AB = DE$, $AC = DF$, $BC = EF$. Thus $\triangle ABC \cong \triangle DEF$ (SSS) *.

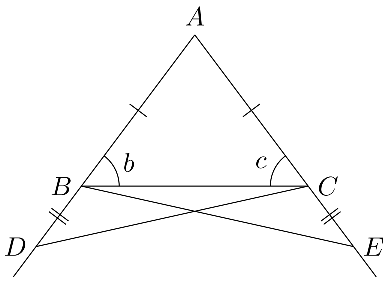
Proof. First we prove the preposition of 'base $\angle$s, isos. $\triangle$'

Preposition. The base angles of an isosceles triangle are equal. (base $\angle$s, isos. $\triangle$)



$$b = c \qquad \text{(base } \angle\text{s, isos. } \triangle\text{)}$$

<u>Proof.</u> [6] Let there be $\triangle ABC$ where $AB = AC$. Extend AB and AC into rays. Pick an arbitrary point on ray AB below B called D. Let E be on ray AC below C such that $BD = CE$.



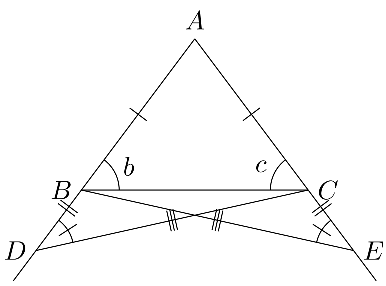
In $\triangle ADC$ and $\triangle AEB$,

$$\begin{aligned} AC &= AB && \text{(given)} \\ AD &= AB + BD = AC + CE = AE && \text{(segment addition postulate + substitution of equals)} \\ \angle CAD &= \angle BAE && \text{(common } \angle\text{)} \\ \therefore\ \triangle ABC &\cong \triangle ACB && \text{(SAS)} \end{aligned}$$

Since the two triangles are congruent, all the corresponding angles of the triangles are equal.

We have $CD = BE$ (corr. sides, $\cong \triangle$s) ,
$\angle ADC = \angle AEB$ (corr. $\angle$s, $\cong \triangle$s)

Updated figure:



In $\triangle BDC$ and $\triangle CEB$,

$$\begin{aligned} BD &= CE && \text{(constructed)} \\ \angle BDC &= \angle CEB && \text{(proven)} \\ DC &= EB && \text{(proven)} \\ \therefore\ \triangle BDC &\cong \triangle CEB && \text{(SAS)} \end{aligned}$$

$\therefore\ \angle BCD = \angle CBE$ (corr. $\angle$s, $\triangle BDC \cong \triangle CEB$)

Note that we also have $\angle ACD = \angle ABE$ (corr. $\angle$s, $\triangle ADC \cong \triangle AEB$) .

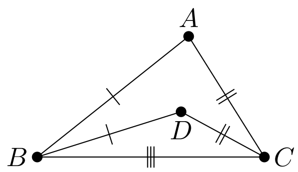
Thus,

$$\begin{aligned}\angle ABC &= \angle ABE - \angle CBE \qquad \text{(angle addition postulate)}\\ &= \angle ACD - \angle BCD \qquad \text{(substitution of equals)}\\ &= \angle ACB \qquad \text{(angle addition postulate)}\end{aligned}$$

Thus $b = c$, and the preposition 'the base angles of an isosceles triangle are equal' is proven.

Now back to our side-side-side business. Suppose we have two side-side-side corresponding triangle $\triangle ABC$ and $\triangle DEF$. Move (meaning translate and rotate) $\triangle DEF$ such that side EF coincides with side BC , and both A and D are above BC. Suppose that vertex D does not coincide with vertex A . There are 4 possibilities:

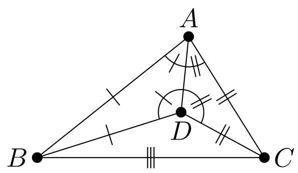
1. D lies inside $\triangle ABC$



Connect AD .

Note that $\angle BAC < 180°$ and $\angle BDC < 180°$ since they are interior angles of a triangle. Thus reflex$\angle BDC = 360° - \angle BDC > 180°$ ($\angle$s at a pt.).

Note that $\angle BAD = \angle BDA$, and $\angle CAD = \angle CDA$ (base $\angle$s, isos. $\triangle$).

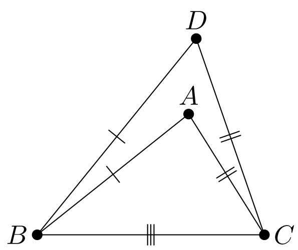


Also note that $\angle BAC = \angle BAD + \angle CAD$, and reflex$\angle BDC = \angle BDA + \angle CDA$. By substitution of equals,

$$\text{reflex}\angle BDC = \angle BDA + \angle CDA = \angle BAD + \angle CAD = \angle BAC$$

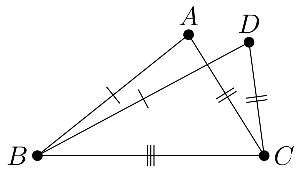
But $\angle BAC < 180°$ while reflex$\angle BDC > 180°$, which means both $\angle BAC < 180°$ and $\angle BAC > 180°$ are true, which violates the law of trichotomy. Thus, it cannot be the case that D lies inside $\triangle ABC$.

2. A lies inside $\triangle DBC$

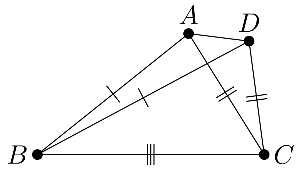


We can use arguments similar to case 1 to arrive at the conclusion that $\angle BDC < 180°$ and reflex$\angle BAC > 180°$ with $\angle BDC =$ reflex$\angle BAC$, violating the law of trichotomy. Thus, it cannot be the case that A lies inside $\triangle ADC$.

3. D lies to the right of line AB [7]

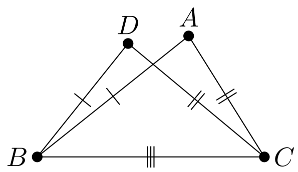


Connect AD . Since $AB = DB$, we have $\angle BAD = \angle BDA$ (base $\angle$s, isos. $\triangle$). Since $AC = DC$, we have $\angle CAD = \angle CDA$ (base $\angle$s, isos. $\triangle$).



We have $\angle BAD > \angle CAD$ since AC is between the angle $\angle BAD$. Similarly, $\angle CDA > \angle BDA$. Substituting $\angle CDA = \angle CAD$ and $\angle BDA = \angle BAD$, we have $\angle CAD > \angle BAD$. But this is impossible since we have both $\angle BAD > \angle CAD$ and $\angle CAD > \angle BAD$, which violates the law of trichotomy. Thus, it cannot be the case that D lies to the right of line AB .

4. D lies to the left of line AB



We can use arguments similar to case 3 to arrive at the conclusion that $\angle BDA > \angle CDA$ and $\angle CDA > \angle BDA$, which violates the law of trichotomy. Thus, it cannot be the case that D lies to the left of line AB .

Therefore, the only possible position of D is the same position as A , so A and D coincide. Since the three vertices of the triangles coincide, we have $\angle ABC \cong \angle DEF$.

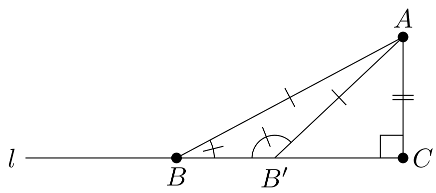
□

1.2.5 RHS (Right Angle-Hypotenuse-Side)

For right angle-hypotenuse-side, the corresponding angle is a right angle, and there are (any) two corresponding sides.

In the figure, we have $AB = DE$, $AC = DF$, $\angle ACB = \angle DFE = 90°$. Thus $\triangle ABC \cong \triangle DEF$ (RHS) *.

Proof. Note that for a given line segment AC and a given angle $90°$, there is a unique ray l such that the clockwise angle between AC and l is $90°$. Let B be a point on l . We want to show that for a given length AB , there is a unique position of B on l.



Suppose B' is a point between B and C such that $AB = AB'$. Then $\angle ABB' = \angle AB'B$ (base $\angle$s, isos. $\triangle$), and $\angle AB'C = 180° - \angle AB'B$ (adj. $\angle$s on st. line) $= 180° - \angle ABB'$.

Note that $\angle AB'C < 90°$ (property of hypotenuse inclination), so we have $180° - \angle ABB' < 90°$, which means $\angle ABB' > 90°$. But that cannot be true because $\angle ABB' < 90°$ (property of hypotenuse inclination). Law of trichotomy is violated. Thus it cannot be the case that B' is between B and C.

Now suppose B' is a point at the left of B such that $AB = AB'$. By similar argument to above, we can arrive at the conclusion that $\angle AB'B < 90°$ and $AB'B > 90°$, which violates the law of trichotomy. Thus it cannot be the case that B' is at the left of B.

Therefore there is a unique position of B when given length AB , line segment AC and clockwise angle $\angle ACB = 90°$, and two triangles with RHS correspondence must coincide, making them congruent triangles. □

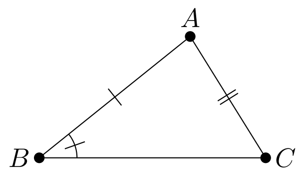
1.2.6 Special case

ASS (Angle-Side-Side) with special conditions

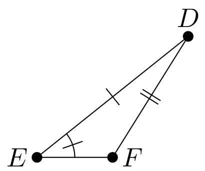
For angle-side-side, there are a corresponding angle and two corresponding sides. Generally, this is not enough to determine congruence since there are two possibilities for a triangle given two sides and an angle.

Suppose that we are given two triangles $\triangle ABC$ and $\triangle DEF$ in which $\angle ABC = \angle DEF$, $AB = DE$, $AC = DF$. There are several cases to consider:

1. $\angle ACB > \angle ABC$ where $\angle ACB \neq 90°$



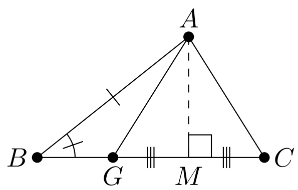
Type 1



Type 2

In the figure, we have $\angle ABC = \angle DEF$, $AB = DE$, $AC = DF$. If $\angle ACB > \angle ABC$ where $\angle ACB \neq 90°$, then $\triangle ABC$ and $\triangle DEF$ may or may not be congruent, as there exists two unique type of triangles when given an angle $\angle ABC$, a side AB and a side AC . Suppose $\triangle ABC$ and $\triangle DEF$ are different types of triangle. Then $\angle DFE$ must be $180° - \angle ACB$. (ASS case 1)

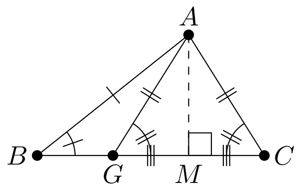
Proof. First consider the case that $\angle ACB$ is an acute angle.



Draw $AM \perp BC$. Let G be a point distinct from C on line BM such that $GM = MC$.
In $\triangle GMA$ and $\triangle CMA$,

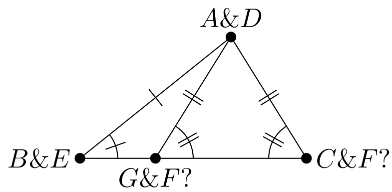
$$\begin{aligned} GM &= MC \qquad \text{(constructed)} \\ \angle GMA &= \angle CMA = 90° \qquad \text{(constructed)} \\ AM &= AM \qquad \text{(common side)} \\ \therefore\ \triangle GMA &\cong \triangle CMA \qquad (SAS) \end{aligned}$$

Thus, $AG = AC$ (corr. sides, $\cong \triangle$s) and $\angle AGM = \angle ACM$ (corr. $\angle$s, $\cong \triangle$s) .



Note that $\angle ACB > \angle ABC$ by initial assumption, so we have $\angle AGC > \angle ABC$. This means that G must lie between B and M (property of hypotenuse inclination), making $\triangle ABG$ a valid triangle.

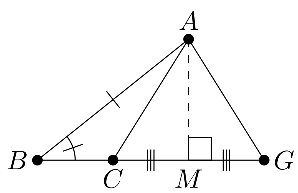
Beside point G and C , there must be no other distinct points N on line BC such that $AN = AC$ (by RHS proof).



If we move side EF of $\triangle DEF$ to coincide with line BC of $\triangle ABC$, A and D must also coincide by polar coordinate postulate. Since vertice F of $\triangle DEF$ must either lie on point G or point C in the figure above, $\triangle DEF$ must be either a 'type 1' triangle or a 'type 2' triangle mentioned above.

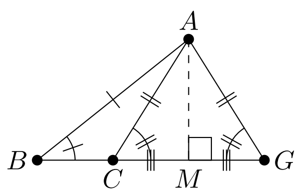
Suppose $\triangle DEF$ is a 'type 2' triangle (thus different type from $\triangle ABC$). Then $\angle DFE = 180° - \angle AGC$ (adj. $\angle$s on st. line) $= 180° - \angle ACB$, as desired.

Let's now consider the case that $\angle ACB$ is an obtuse angle.



Draw $AM \perp$ line BC . Let G be a point distinct from C on line BM such that $GM = MC$. We have $\triangle CMA \cong \triangle GMA$ (SAS).

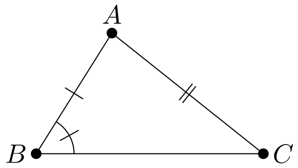
Thus, $AG = AC$ (corr. sides, $\cong \triangle$s) and $\angle AGM = \angle ACM$ (corr. $\angle$s, $\cong \triangle$s) .



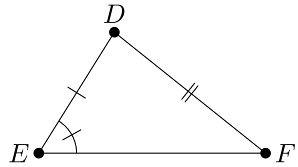
The argument proceeds similar to the acute angle case. Let the two triangles overlap at one side. The vertex F of $\triangle DEF$ can lie on either C or G . Suppose vertex F lies on G . Then $\angle DFE = \angle DCF = 180° - \angle ACB$ (adj. $\angle$s on st. line).

$\square$

2. $\angle ACB < \angle ABC$



Type 1

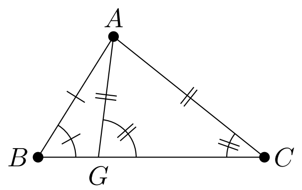


Also type 1

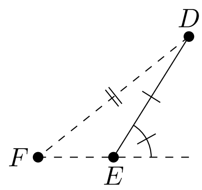
In the figure, we have $\angle ABC = \angle DEF$, $AB = DE$, $AC = DF$. If $\angle ACB < \angle ABC$, then $\triangle ABC \cong \triangle DEF$. (ASS case 2)

Proof. Note that $\angle ACB < 90°$, or otherwise we will also have $\angle ABC \geq 90°$ and $\angle ACB + \angle ABC \geq 180°$, which violates (2 $\angle$ sum of $\triangle$) .

Suppose given ASS correspondence where $\angle ACB < \angle ABC$, we have two unique type of triangles. Then similar to the proof of ASS case 1 acute situation, we can uniquely make a point G between B and C such that $AG = AC$. Then $\angle AGC = \angle ACB$ (base $\angle$s, isos. $\triangle$).



Imagination



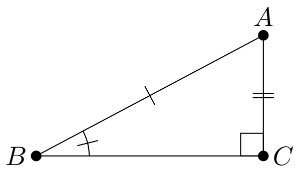
Reality

In $\triangle ABG$, we have $\angle ABG + \angle BAG = \angle AGC$, so $\angle AGC > \angle ABC$, and thus $\angle ACB > \angle ABC$. But we have assumed that $\angle ACB < \angle ABC$, so this contradicts the law of trichotomy. Thus, there must not be more than one unique type of triangle.

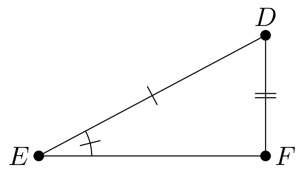
Since there is only one unique type of triangle, when we try to overlap $\triangle ABC$ and $\triangle DEF$, they must completely coincide, making them congruent triangles.

□

3. $\angle ACB = 90°$



Type 1



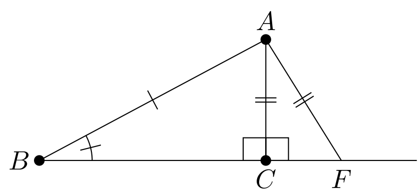
Also type 1

In the figure, we have $\angle ABC = \angle DEF$, $AB = DE$, $AC = DF$. If $\angle ACB = 90°$, then $\triangle ABC \cong \triangle DEF$. (ASS case 3)

Proof. By polar coordinate postulate, given angle $\angle ABC$ and length of line segment AB , there is a unique position of A above line BC . Since there is only one unique line segment with endpoint A that is perpendicular to BC (property of perpendicular line), the position of C and the length of AC can also be uniquely determined.

If we move vertex E to coincide with B such that F is on line BC, then D must coincide with A (polar coordinate postulate). Since $DF = AC$ and F is on BC, F must also coincide with C.

Otherwise, suppose F does not coincide with C . Let's say it is at the right of C.



Then $\angle ACF = \angle AFC = 90°$ (base $\angle$s, isos. $\triangle$). But this means in $\triangle ACF$, the sum of two interior angles $\angle ACF + \angle AFC = 180°$, which violates (2 $\angle$ sum of $\triangle$) . If F is at the left of C instead, the same thing will happen. Thus, it must be the case that F coincides with C . □

1.3 Triangle properties

Let's summarize the conditions for congruent triangles in a preposition:

Preposition 20. Two triangles are congruent if one of the conditions holds:
SAS, ASA, AAS, SSS, RHS .

1. <u>SAS (Side-Angle-Side)</u>

If $AB = DE$, $\angle ABC = \angle DEF$, $BC = EF$, then $\triangle ABC \cong \triangle DEF$ (SAS) *.

2. <u>ASA (Angle-Side-Angle)</u>

If $BC = EF$, $\angle ABC = \angle DEF$, $\angle ACB = \angle DFE$, then $\triangle ABC \cong \triangle DEF$ (ASA) *.

3. <u>AAS (Angle-Angle-Side)</u>

If $AB = DE$, $\angle ABC = \angle DEF$, $\angle ACB = \angle DFE$, then $\triangle ABC \cong \triangle DEF$ (AAS) *.

4. <u>SSS (Side-Side-Side)</u>

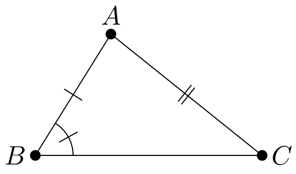
If $AB = DE$, $AC = DF$, $BC = EF$, then $\triangle ABC \cong \triangle DEF$ (SSS) *.

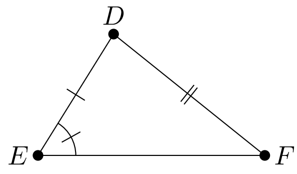
5. <u>RHS (Right Angle-Hypotenuse-Side)</u>

If $AB = DE$, $AC = DF$, $\angle ACB = \angle DFE = 90°$, then $\triangle ABC \cong \triangle DEF$ (RHS) *.

Preposition 21. Two triangles are congruent if they have ASS correspondence with one of the additional conditions:

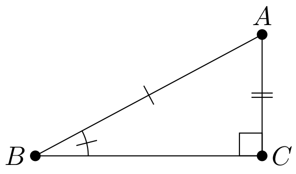
1. <u>Non-corresponded & non-included side is smaller than corresponded side</u>

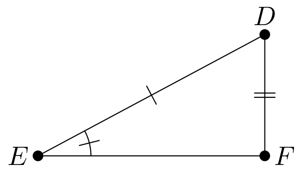




   If $\angle ABC = \angle DEF$, $AB = DE$, $AC = DF$, $\angle ACB < \angle ABC$, then $\triangle ABC \cong \triangle DEF$. (ASS case 2)

2. <u>Non-corresponded & non-included side is right angle</u>

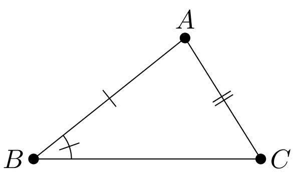




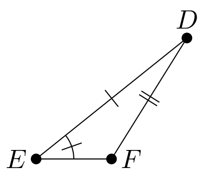
   If $\angle ABC = \angle DEF$, $AB = DE$, $AC = DF$, $\angle ACB = 90°$, then $\triangle ABC \cong \triangle DEF$. (ASS case 3)

   If they have the following condition, the triangles may or may not be congruent.

3. <u>Non-corresponded & non-included side is larger than corresponded side</u>



Type 1



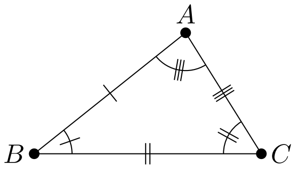
Type 2

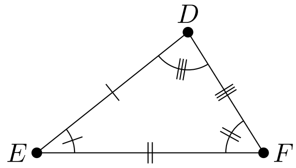
   If $\angle ABC = \angle DEF$, $AB = DE$, $AC = DF$, $\angle ACB > \angle ABC$, $\angle ACB \neq 90°$, then $\triangle ABC$ and $\triangle DEF$ may or may not be congruent.

   Suppose $\triangle ABC$ and $\triangle DEF$ are not congruent. Then $\angle DFE = 180° - \angle ACB$. (ASS case 1)

Preposition 22. If two triangles are congruent, then:

- Their corresponding sides are equal. (corr. sides, $\cong \triangle$s)*
- Their corresponding angles are equal. (corr. $\angle$s, $\cong \triangle$s)*





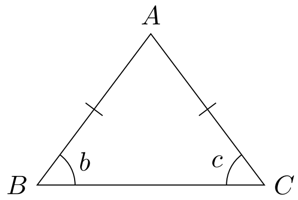
Observation: $\triangle ABC \cong \triangle DEF$

$$\therefore\ AB = DE\ ,\ BC = EF\ ,\ AC = DF \qquad \text{(corr. sides, } \cong \triangle\text{s),}$$

$$\angle A = \angle D\ ,\ \angle B = \angle E\ ,\ \angle C = \angle F \qquad \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}$$

Proof. If two triangles are congruent, then they can be moved (and flipped) to completely coincide. Thus all the corresponding line segments and angles coincide. By the common notion of 'coincidable line segments and angles are equal' , the corresponding line segments and angles are equal. ☐

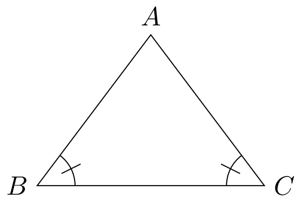
Preposition 23. The base angles of an isosceles triangle are equal. (base $\angle$s, isos. $\triangle$)



$$b = c \qquad \text{(base } \angle\text{s, isos. } \triangle\text{)}$$

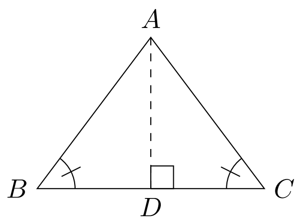
For the proof, refer to Section 1.2.4 (it's long).

Preposition 24. If two angles of a triangle are equal, then the triangle is an isosceles triangle. (sides opp. equal $\angle$s)



$$AB = AC \qquad \text{(sides opp. equal } \angle\text{s)}$$

Proof. Draw $AD \perp BC$.



In $\triangle ABD$ and $\triangle ACD$,

$$\begin{aligned} \angle ABD &= \angle ACD \qquad \text{(given)} \\ \angle BDA &= \angle CDA = 90^\circ \qquad \text{(constructed)} \\ AD &= AD \qquad \text{(common side)} \\ \therefore\ \triangle ABD &\cong \triangle ACD \qquad \text{(AAS)} \\ \therefore\ AB &= AC \qquad \text{(corr. sides, } \cong \triangle\text{s)} \end{aligned}$$

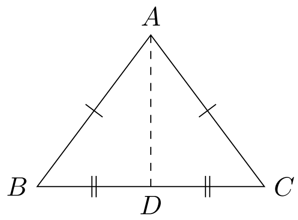
☐

Preposition 25. For an isosceles triangle $\triangle ABC$ with $AB = AC$ and D on side BC, if one of the following conditions is true, then the other two conditions are also true:

1. $BD = DC$
2. $\angle BAD = \angle CAD$
3. $AD \perp BC$

(prop. of isos. $\triangle$)

Example

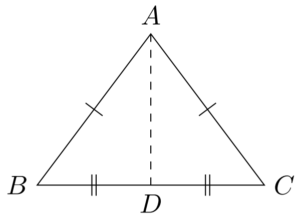


Observation: $AB = AC$ and $BD = DC$

$\therefore\ \angle BAD = \angle CAD$ and $AD \perp BC$ (prop. of isos. $\triangle$)

Proof. Let there be an isosceles triangle $\triangle ABC$ with $AB = AC$ and D on side BC . Let's look at what happens for each condition being true.

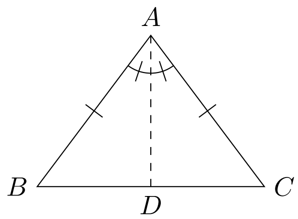
1. $BD = DC$



In $\triangle ABD$ and $\angle ACD$,

$$\begin{aligned}
AB &= AC && \text{(given)}\\
BD &= DC && \text{(given)}\\
AD &= AD && \text{(commomn side)}\\
\therefore\ \triangle ABD &\cong \triangle ACD && \text{(SSS)}\\
\therefore\ \text{(condition 2)}\quad \angle BAD &= \angle CAD && \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}\\
\angle ADB &= \angle ADC && \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}\\
\angle ADB &= \angle ADC = 180^\circ/2 = 90^\circ && \text{(adj. } \angle\text{s on st. line)}\\
\therefore\ \text{(condition 3)}\quad & AD \perp BC
\end{aligned}$$

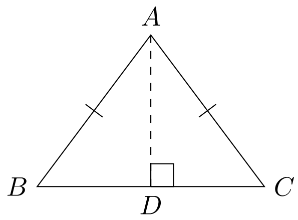
2. $\angle BAD = \angle CAD$



In $\triangle ABD$ and $\triangle ACD$,

$$\begin{aligned}
AB &= AC && \text{(given)}\\
\angle BAD &= \angle CAD && \text{(given)}\\
AD &= AD && \text{(common side)}\\
\therefore\ \triangle ABD &\cong \triangle ACD && \text{(SAS)}\\
\therefore\ \text{(condition 1)}\quad BD &= DC && \text{(corr. sides, } \cong \triangle\text{s)}\\
\angle ADB &= \angle ADC && \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}\\
\therefore\ \text{(condition 3)}\quad & AD \perp BC
\end{aligned}$$

3. $AD \perp BC$

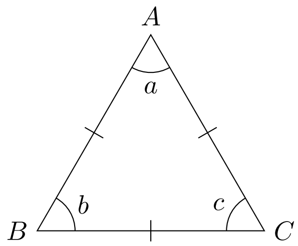


In $\triangle ABD$ and $\triangle ACD$,

$$\begin{aligned} \angle ADB = \angle ADC = 90^\circ \qquad & (AD \perp BC) \\ AB = AC \qquad & (\text{given}) \\ AD = AD \qquad & (\text{common side}) \\ \therefore\ \triangle ABD \cong \triangle ACD \qquad & (\text{RHS}) \\ \therefore\ (\text{condition 1}) \quad BD = DC \qquad & (\text{corr. sides, } \cong \triangle\text{s}) \\ (\text{condition 2}) \quad \angle BAD = \angle CAD \qquad & (\text{corr. } \angle\text{s, } \cong \triangle\text{s}) \end{aligned}$$

□

Preposition 26. Each interior angle of an equilateral triangle is 60°. (prop. of equil. △) *



$$a = b = c = 60^\circ \qquad (\text{prop. of equil. } \triangle)$$

Proof.

$$\begin{aligned} AB = AC \qquad & (\text{given}) \\ \therefore\ b = c \qquad & (\text{base } \angle\text{s, isos. } \triangle) \\ BC = BA \qquad & (\text{given}) \\ \therefore\ c = a \qquad & (\text{base } \angle\text{s, isos. } \triangle) \\ \therefore\ a = b = c & \\ a + b + c = 180^\circ \qquad & (\angle \text{ sum of } \triangle) \\ \therefore\ a = b = c = 180^\circ/3 = 60^\circ & \end{aligned}$$

□

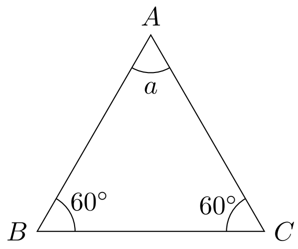
Preposition 27. A triangle is an equilateral triangle if it satisfies one of the following conditions:

1. Two angles are 60° .
2. The triangle is isosceles with one 60° angle.
3. Two angles are equal and one angle is 60° (the 60° angle may or may not be in the equal pair).
4. Three angles are equal.

(con. of equil. △)

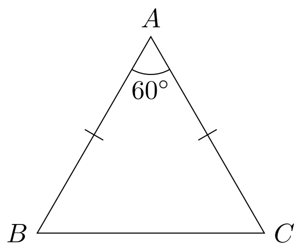
Proof. Let there be $\triangle ABC$. Let's consider the conditions.

1. $\angle B = \angle C = 60^\circ$.



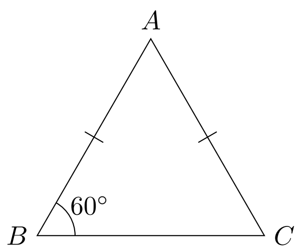
$$\begin{aligned}
&\angle B = \angle C = 60^\circ \qquad \text{(given)}\\
&AB = AC \qquad \text{(sides opp. equal } \angle\text{s)}\\
&\angle A = 180^\circ - 60^\circ - 60^\circ = 60^\circ \qquad (\angle \text{ sum of } \triangle)\\
\therefore\ &\angle C = \angle A = 60^\circ\\
\therefore\ &BC = BA \qquad \text{(sides opp. equal } \angle\text{s)}\\
\therefore\ &AB = AC = BC\\
\therefore\ &\triangle ABC \text{ is an equil. } \triangle\ .
\end{aligned}$$

2a. $AB = AC$ with $\angle A = 60^\circ$.



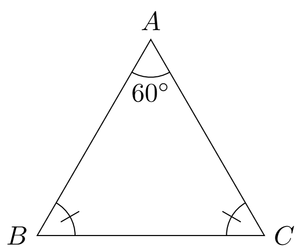
$$\begin{aligned}
AB &= AC \qquad \text{(given)}\\
\angle B &= \angle C \qquad \text{(base } \angle\text{s, isos. } \triangle)\\
\angle B + \angle C + 60^\circ &= 180^\circ \qquad (\angle \text{ sum of } \triangle)\\
\angle B = \angle C &= (180^\circ - 60^\circ)/2 = 60^\circ\\
\therefore\ \triangle ABC &\text{ is an equil. } \triangle \qquad \text{(condition 1 of this preposition)}
\end{aligned}$$

2b. $AB = AC$ with $\angle B = 60^\circ$.



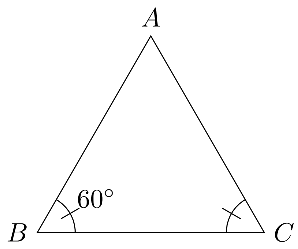
$$\begin{aligned}
AC &= AB \qquad \text{(given)}\\
\angle C &= \angle B = 60^\circ \qquad \text{(base } \angle\text{s, isos. } \triangle)\\
\therefore\ \triangle ABC &\text{ is an equil. } \triangle \qquad \text{(condition 1 of this preposition)}
\end{aligned}$$

3a. $\angle B = \angle C$ with $\angle A = 60^\circ$.



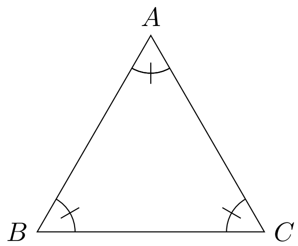
$$\angle B = \angle C \qquad \text{(given)}$$
$$\angle B + \angle C + 60^\circ = 180^\circ \qquad (\angle \text{ sum of } \triangle)$$
$$\angle B = \angle C = (180^\circ - 60^\circ)/2 = 60^\circ$$
$$\therefore\ \triangle ABC \text{ is an equil. } \triangle \qquad \text{(condition 1 of this preposition)}$$

3b. $\angle B = \angle C$ with $\angle B = 60^\circ$.



$$\angle C = \angle B = 60^\circ$$
$$\therefore\ \triangle ABC \text{ is an equil. } \triangle \qquad \text{(condition 1 of this preposition)}$$

4. $\angle A = \angle B = \angle C$

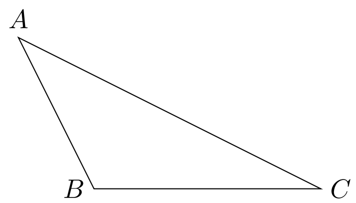


$$\angle A = \angle B = \angle C \qquad \text{(given)}$$
$$\angle A + \angle B + \angle C = 180^\circ \qquad (\angle \text{ sum of } \triangle)$$
$$\angle A = \angle B = \angle C = 180^\circ/3 = 60^\circ$$
$$\therefore\ \triangle ABC \text{ is an equil. } \triangle \qquad \text{(condition 1 of this preposition)}$$

□

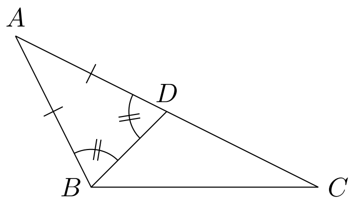
Preposition 28. In a triangle, the longer side subtends the larger angle. (longer side, larger $\angle$)

In other words, in a triangle, the greater side has larger opposite angle.



$$\text{Observation: } AC > AB$$
$$\therefore\ \angle B > \angle C \qquad \text{(longer side, larger } \angle)$$

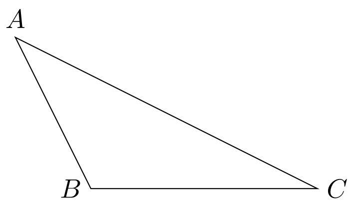
Proof. Let $\triangle ABC$ be a triangle where $AC > AB$. Let D be a point on side AC such that $AB = AD$. Connect BD .



Then $\angle ADB$ is an exterior angle of $\triangle BCD$. Thus $\angle ADB > \angle ACB$ (ext. $\angle$ of $\triangle$ > int. opp. $\angle$)
.

Note that $\angle ABD = \angle ADB$ (base $\angle$s, isos. $\triangle$). So $\angle ABD > \angle ACB$.
Since $\angle ABC > \angle ABD$, we have $\angle ABC > \angle ACB$ (transitive property of inequality). ☐

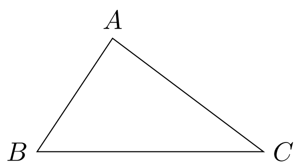
Preposition 29. In a triangle, the larger angle subtends the longer side. (larger $\angle$, longer side)
In other words, in a triangle, the larger the angle, the longer the opposite side.



Observation: $\angle B > \angle C$
$\therefore\ AC > AB$ (longer $\angle$, larger side)

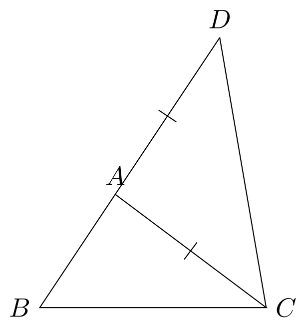
Proof. Let $\triangle ABC$ be a triangle where $\angle B > \angle C$.
Suppose that AC is not longer AB . If $AC = AB$, then $\angle B = \angle C$ (base $\angle$s, isos. $\triangle$), which contradicts the initial assumption $\angle B > \angle C$. So it cannot be the case that $AB = AC$.
If $AC < AB$, then by 'longer side, larger $\angle$', we have $\angle C > \angle B$, which contradicts the initial assumption $\angle B > \angle C$. So it cannot be the case that $AB = AC$.
Thus it can only be the case that $AC > AB$. ☐

Preposition 30. In a triangle, the sum of lengths of any two sides is greater than the length of the remaining side. (triangle inequality)



$$AB + AC > BC$$
$$AB + BC > AC$$
$$AC + CB > AB$$
(triangle inequality)

Proof. Extend BA past A. Make a point D on line BA above A such that $AD = AC$.

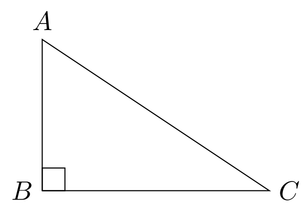


Note that $\angle ACD = \angle ADC$ (base $\angle$s, isos. $\triangle$). Note that $\angle BCD > \angle ACD$, so $\angle BCD > \angle ADC$.

Since $\angle BCD > \angle BDC$, we have $BD > BC$ (larger $\angle$, longer side) .
But $BD = BA + AD$ and $AD = AC$. Thus, $BD = BA + AC$.
Replace BD with $BA + AC$ in the inequality $BD > BC$, we get $BA + AC > BC$.

We can use similar argument with the other two sides to get the rest of the inequalities. □

Preposition 31. In a right triangle, the hypotenuse is the longest side. (hypotenuse is longest side of ◺)



$AC > AB$ and $AC > BC$ (hypotenuse is longest side of ◺)

Proof. Note that in a right triangle, the right angle must be the largest angle. Otherwise, say, $\angle ABC = 90°$ and $\angle BAC \geq 90°$, then $\angle ABC + \angle BAC \geq 180°$, which violates '2 $\angle$ sum of $\triangle$'.

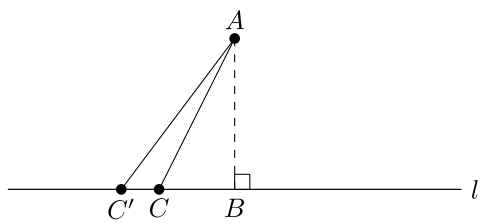
By 'larger $\angle$, longer side' , in a triangle, the largest angle must have the longest opposite side. In a right triangle, the opposite side of the right angle is the hypotenuse, so the hypotenuse must be the longest side.

□

Preposition 32. Suppose B is a point on line l and A is a point vertically above B (meaning $AB \perp l$) . If C is a point on l that is not B, then the longer BC is, the longer AC is. (property of hypotenuse length)

In other words, if length BC is a variable over the domain $(0, \infty)$, then AB is a strictly increasing function of BC .

Proof. Assume that C is at the left of B. Let C' be a point on l to the left of C. So $C'B > CB$.



In $\triangle ACC'$, since ACC' is an obtuse angle while $AC'C$ is an acute angle, we have $ACC' > AC'C$. By 'larger $\angle$, longer side' , we have $AC' > AC$.

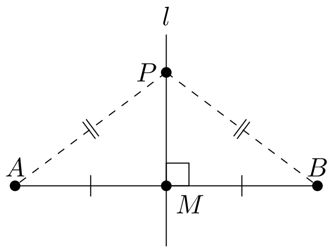
If C is at the right of B , then we can let C' be at the right of C and use similar reasoning to show that $AC' > AC$.

□

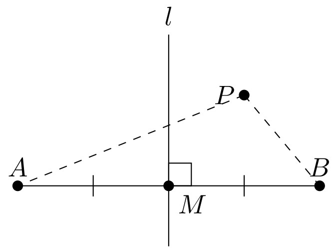
An implication of this preposition is that if there are a point not on a line, then the shortest distance between the point and the line is length of the line segment perpendicular to that line with that point as the endpoint.

Preposition 33. For a line segment AB and a point P on the same plane, $PA = PB$ if and only if P lies on the perpendicular bisector of AB .

If and only if P lies on the same side of the perpendicular bisector as an endpoint, then the distance between P and this endpoint and shorter than the distance from the other endpoint. (prop. of $\perp$ bisector)



Case 1



Case 2

(Let l be the perpendicular bisector of AB .)

Case 1a:

$$\because\ P \text{ is on line } l\ .$$
$$\therefore\ PA = PB \qquad \text{(prop. of } \perp \text{ bisector)}$$

Case 1b:

$$\because PA = PB$$
$$\therefore\ P \text{ is on line } l\ . \qquad \text{(prop. of } \perp \text{ bisector)}$$

Case 2a:

$$\because P \text{ is at the right of } l \text{ (same side as } B).$$
$$\therefore\ PB < PA \qquad \text{(prop. of } \perp \text{ bisector)}$$

Case 2b:

$$\because PB < PA$$
$$\therefore\ P \text{ is at the right of } l \qquad \text{(prop. of } \perp \text{ bisector)}$$

Proof. Let's consider each case:

Case 1a: P is on line l .

In $\triangle PMA$ and $\triangle PMB$,

$$AM = BM \qquad \text{(given)}$$
$$\angle PMA = \angle PMB = 90^\circ \qquad \text{(PM}\perp \text{ AB)}$$
$$PM = PM \qquad \text{(common side)}$$
$$\therefore\ \triangle PMA \cong \triangle PMB \qquad \text{(SAS)}$$
$$\therefore\ PA = PB \qquad \text{(corr. sides, } \cong \triangle\text{s)}$$

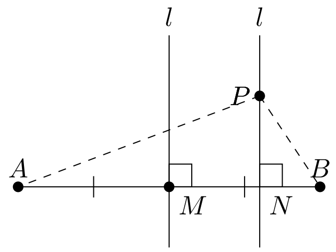
Case 1b: $PA = PB$

If P is on line segment AB , then P is actually the midpoint of AB , and since the perpendicular bisector of AB passes through midpoint of AB , P must lie on the perpendicular bisector (line l) .

If P is not on line segment AB , then $\triangle PAB$ is an isos. $\triangle$. By 'prop. of isos. $\triangle$' , if there is a point M on AB such that $AM = MB$, then $PM \perp AB$, making PM a perpendicular bisector of AB . Since for any given segment, there is a unique perpendicular bisector (since there is a unique midpoint and a unique perpendicular line passing through a given midpoint) , thus the perpendicular bisector of AB must pass through P .

Case 2a: P is at the right of l (same side as B).

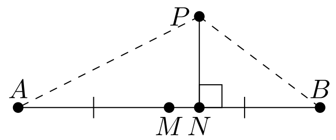
Draw $PN \perp AB$.



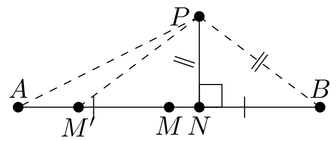
Note that $BN > AN$, so by property of hypotenuse length, we have $PB < PA$.

Case 2b: $PB < PA$

Draw $PN \perp AB$.



Since $PB < PA$, we have $\angle PAB < \angle PBA$ by (longer side, larger $\angle$). Since $\angle APN = 90° - \angle PAB$ and $\angle BPN = 90° - \angle PBA$ ($\angle$ sum of $\triangle$), we have $\angle APN > \angle BPN$ (subtractive property of inequality).



Make a point M' on line AB such that $NB = NM'$. Since $\angle PM'B = \angle PBA$ ($\triangle PNM' \cong \triangle PNB$), we have $\angle PAB > \angle PM'B$. Thus $PM'B$ is the exterior angle of $\triangle PAM'$ and M' lies between AB. Since $M'B = 2NB < AB$, we have $AN > NB$. Thus the perpendicular line passing through N and P must be at the right of the perpendicular bisector l , meaning P must be at the right of l .

□

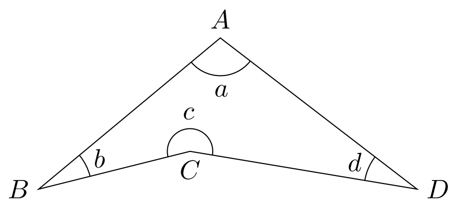
1.3.1 Problems

Time for some problems. (Cut due to runtime error)

1.4 Quadrilateral properties

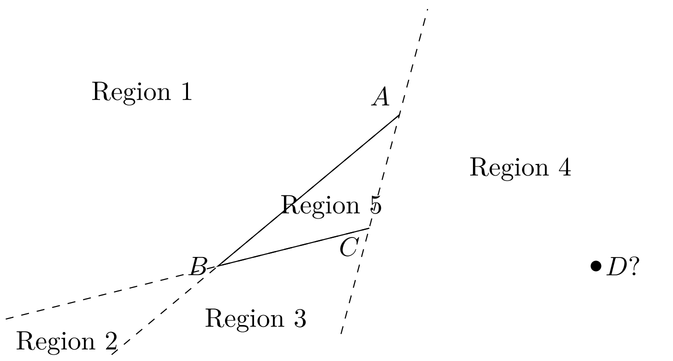
1.4.1 General properties

Preposition 34. The sum of interior angles of a quadrilateral is $360°$. ($\angle$ sum of quad.)



$$a + b + c + d = 360° \qquad (\angle \text{ sum of quad.})$$

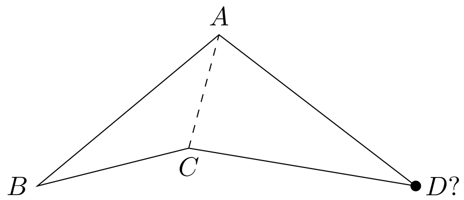
Proof. Note that every quadrilateral can be split into two triangles. To see why, arbitrarily choose three vertices of the quadrilateral and label them A, B, C . Suppose these vertices make side AB and BC . For the quadrilateral to be valid, the fourth vertex must be placed in a position such that any two sides will not intersect each other at a point other than the vertices.



Refer to the figure, the plane is split into 5 regions by the (dotted) lines (/rays). We see that D must either be placed in region 2, 4 or 5. Otherwise, say, D is in region 1, then side CD will intersect with AB at a point between A and B , which invalidates the quadrilateral.

If D is in region 4, then we can draw diagonal AC to split $ABCD$ into two triangles.

If D is in region 2 or 5 instead, then we can draw diagonal BD to split $ABCD$ into two triangles.

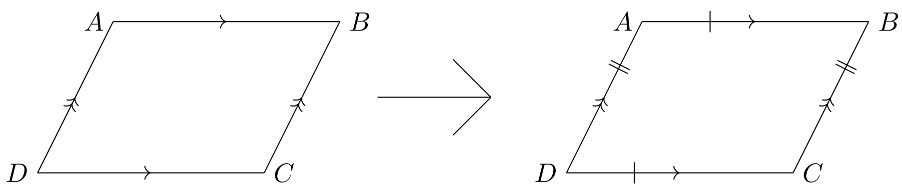


After splitting, the two triangles must share a common side. Note that the sum of interior angles of a triangle is $180°$, so the sum of interior angles of a triangle of two triangles is $360°$. But the interior angles of triangles combined are also the interior angles of the quadrilateral. So the sum of interior angles of a quadrilateral is $360°$. □

1.4.2 Parallelograms

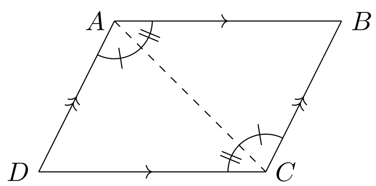
Properties of parallelogram:

Preposition 35. The opposite sides of a parallelogram are equal. (opp. sides of //gram) *



$$\because\ AB//DC \text{ and } AD//BC$$
$$\therefore\ AB = CD \text{ and } AD = BC \qquad \text{(opp. sides of //gram)}$$

Proof. Let there be parallelogram $ABCD$.

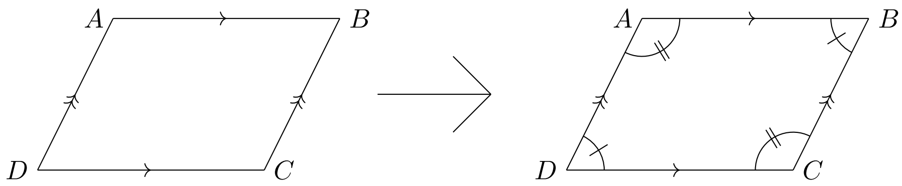


Join AC. In $\triangle ADC$ and $\triangle CBA$,

$$\begin{aligned} \angle ACD &= \angle CAB \qquad (\text{alt. } \angle \text{ s , } AB//DC) \\ AC &= AC \qquad (\text{common side}) \\ \angle CAD &= \angle ACB \qquad (\text{alt. } \angle \text{ s , } AD//BC) \\ \therefore\ \triangle ADC &\cong \triangle CBA \qquad (\text{ASA}) \\ \therefore\ AB = DC \ \text{ and } \ AD &= BC \qquad (\text{corr. sides, } \cong \triangle\text{s}) \end{aligned}$$

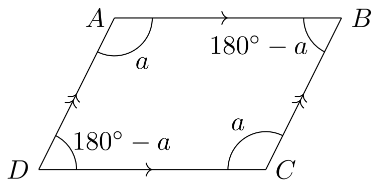
□

Preposition 36. The opposite angles of a parallelogram are equal. (opp. $\angle$s of //gram) *



$$\because\ AB//DC \text{ and } AD//BC$$
$$\therefore\ \angle ADC = \angle ABC \text{ and } \angle BAD = \angle BCD \qquad (\text{opp. } \angle\text{s of //gram})$$

Proof. Let there be parallelogram $ABCD$.



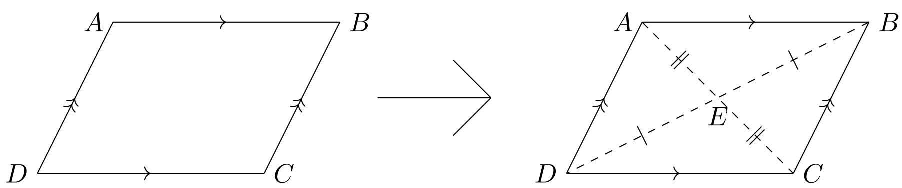
$$\begin{aligned} \angle A + \angle B &= 180^\circ \qquad (\text{int. } \angle\text{s , } AD//BC) \\ \angle B + \angle C &= 180^\circ \qquad (\text{int. } \angle\text{s , } AB//DC) \\ \therefore\ \angle A + \angle B &= \angle B + \angle C \\ \angle A &= \angle C \end{aligned}$$

Similarly,

$$\begin{aligned} \angle B + \angle C &= 180^\circ \qquad (\text{int. } \angle\text{s , } AB//DC) \\ \angle C + \angle D &= 180^\circ \qquad (\text{int. } \angle\text{s , } AD//BC) \\ \therefore\ \angle B + \angle C &= \angle C + \angle D \\ \angle B &= \angle D \end{aligned}$$

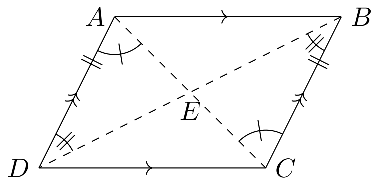
□

Preposition 37. The diagonals of a parallelogram bisect each other. (diags of //gram) *



$$\because\ AB//DC \text{ and } AD//BC$$
$$\therefore\ AE = EC \text{ and } DE = EB \qquad (\text{diags of //gram})$$

Proof. Let there be parallelogram $ABCD$. Let diagonals AC and BD intersect at E .



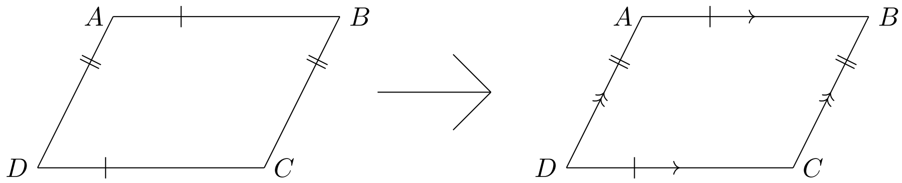
In $\triangle ADE$ and $\triangle CBE$,

$$\begin{aligned}
\angle DAE &= \angle BCE \qquad \text{(alt. } \angle\text{s , } AD//BC\text{)} \\
AD &= BC \qquad \text{(opp. sides of //gram)} \\
\angle ADE &= \angle CBE \qquad \text{(alt. } \angle\text{s , } AD//BC\text{)} \\
\therefore\ \triangle ADE &\cong \triangle CBE \qquad \text{(ASA)} \\
\therefore\ AE &= EC \qquad \text{(corr. sides, } \cong \triangle\text{s)} \\
DE &= EB \qquad \text{(corr. sides, } \cong \triangle\text{s)}
\end{aligned}$$

□

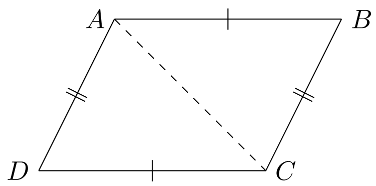
Conditions for determining a quadrilateral is a parallelogram:

Preposition 38. If there are two equal pairs of opposite sides in a quadrilateral, then the quadrilateral is a parallelogram. (opp. sides equal)



$\because\ AB = DC$ and $AD = BC$
$\therefore\ ABCD$ is a //gram. (opp. sides equal)

Proof. Let there be a quadrilateral $ABCD$ where $AB = DC$ and $AD = BC$.



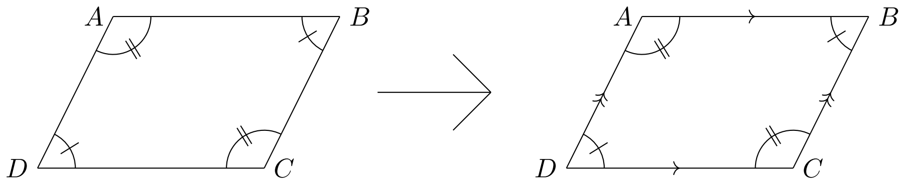
Join AC. In $\triangle ADC$ and $\triangle CBA$,

$$\begin{aligned}
DC &= AB \qquad \text{(given)} \\
AD &= BC \qquad \text{(given)} \\
AC &= AC \qquad \text{(common side)} \\
\therefore\ \triangle ADC &\cong \triangle CBA \qquad \text{(SSS)} \\
\therefore\ \angle DCA &= \angle BAC \qquad \text{(corr. } \angle\text{s, } \cong \triangle\text{s)} \\
\therefore\ AB&//DC \qquad \text{(alt. } \angle\text{s equal)} \\
\therefore\ \angle DAC &= \angle BCA \qquad \text{(corr. } \angle\text{s, } \cong \triangle\text{s)} \\
\therefore\ AD&//BC \qquad \text{(alt. } \angle\text{s equal)}
\end{aligned}$$

$\therefore\ ABCD$is a //gram.

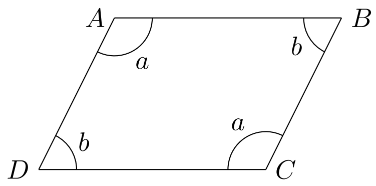
□

Preposition 39. If there are two equal pairs of opposite angles in a quadrilateral, then the quadrilateral is a parallelogram. (opp. $\angle$s equal) *



$$\because \therefore \ \angle ADC = \angle ABC \text{ and } \angle BAD = \angle BCD$$
$$AB//DC \text{ and } AD//BC \qquad (\text{opp. } \angle\text{s of //gram})$$

Proof. Let there be a quadrilateral $ABCD$ where $\angle A = \angle C$ and $\angle B = \angle D$.



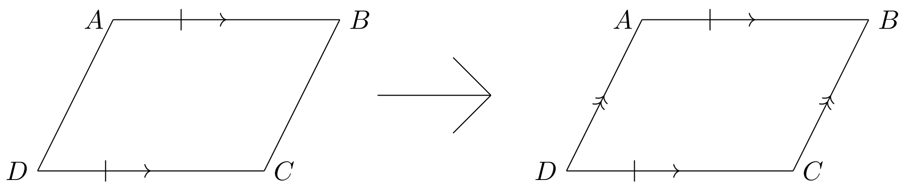
$$\angle A + \angle B + \angle C + \angle D = 360^\circ \qquad (\angle \text{ sum of quad.})$$
$$\angle A + \angle B + \angle A + \angle B = 360^\circ$$
$$\angle A + \angle B = 180^\circ$$
$$\therefore \ AD//BC \qquad (\text{int. } \angle\text{s supp.})$$

$\angle A = \angle C$ and $\angle B = \angle D$, we also have

$$\angle C + \angle D = 180^\circ$$
$$\therefore \ AB//DC \qquad (\text{int. } \angle\text{s supp.})$$
$$\therefore \ ABCD \text{ is a //gram.}$$

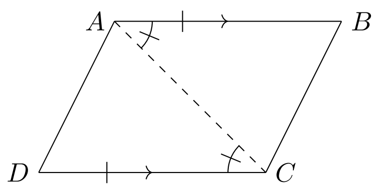
□

Preposition 40. If there is one equal and parallel pair of opposite sides in a quadrilateral, then the quadrilateral is a parallelogram. (opp. sides equal and //) *



$$\because \ AB = DC \text{ and } AB//DC$$
$$\therefore \ ABCD \text{ is a //gram.} \qquad (\text{opp. sides equal and //})$$

Proof. Let there be a quadrilateral $ABCD$ where $AB = DC$ and $AB//DC$.

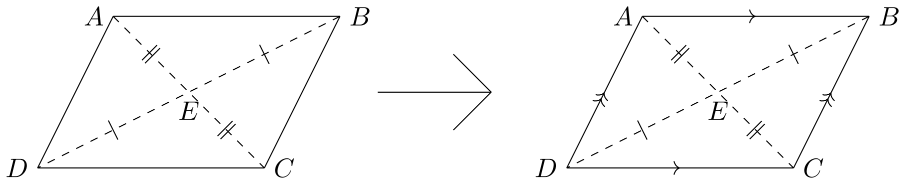


Join AC. In $\triangle ADC$ and $\triangle CBA$,

$$DC = AB \qquad (\text{given})$$
$$\angle DCA = \angle BAC \qquad (\text{alt. } \angle\text{s , } AB//DC)$$
$$AC = AC \qquad (\text{common side})$$
$$\therefore \ \triangle ADC \cong \triangle CBA \qquad (\text{SAS})$$
$$\therefore \ \angle DAC = \angle BCA \qquad (\text{corr. } \angle\text{s, } \cong \triangle\text{s})$$
$$\therefore \ AD//BC \qquad (\text{alt. } \angle\text{s equal})$$
$$\therefore \ ABCD \text{ is a //gram.}$$

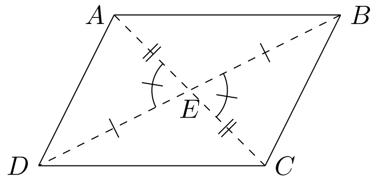
$\square$

Preposition 41. If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram. (diags bisect each other) *



$$\because\ AE = EC \text{ and } DE = EB$$
$$\therefore\ ABCD \text{ is a //gram.} \qquad \text{(diags of //gram)}$$

Proof. Let there be quadrilateral $ABCD$ with bisecting diagonals.



In $\triangle ADE$ and $\triangle CBE$,

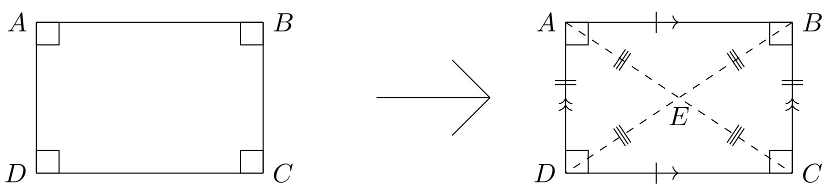
$$\begin{aligned}
AE &= EC \qquad \text{(given)}\\
\angle AED &= \angle CEB \qquad \text{(vert. opp. } \angle\text{s)}\\
DE &= EB \qquad \text{(given)}\\
\therefore\ \triangle ADE &\cong \triangle CBE \qquad \text{(SAS)}\\
\therefore\ \angle DCA &= \angle BAC \qquad \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}\\
\therefore\ \angle DAE &= \angle BCE \qquad \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}\\
\therefore\ AD&//BC \qquad \text{(alt. } \angle\text{s equal)}\\
\text{Also, } AD &= BC \qquad \text{(corr. sides, } \cong \triangle\text{s)}\\
\therefore\ ABCD &\text{ is a //gram.} \qquad \text{(opp. sides equal and //)}
\end{aligned}$$

$\square$

1.4.3 Rectangles

Properties of rectangles:

Preposition 42. A rectangle has four right angles, two equal and parallel pairs of opposite sides, and two equal diagonals that bisect each other. (prop. of rectangle) *



$$\because \angle A = \angle B = \angle C = \angle D = 90^\circ \qquad \text{(definition of a rectangle)}$$
$$\therefore\ AB = DC\ ,\ AB//DC\ ,\ AD = BC\ ,\ AD//BC$$
$$\text{Also, } AE = BE = CE = DE \qquad \text{(prop. of rectangle)}$$

Proof. By definition, a rectangle has 4 right angles. Let there be rectangle $ABCD$.

$$\begin{aligned}
\angle A = \angle C = 90^\circ \ \text{ and } \ \angle B = \angle D = 90^\circ &\qquad \text{(definition of rectangle)}\\
\therefore\ ABCD \text{ is a parallogram.} &\qquad \text{(opp. } \angle\text{s equal)}\\
\therefore\ AB//DC \ \text{ and } \ AD//BC &\qquad \text{(definition of //gram)}\\
\therefore\ AB = DC \ \text{ and } \ AD = BC &\qquad \text{(opp. sides of //gram)}
\end{aligned}$$

To show that the diagonal AC is equal to diagonal BD , consider $\triangle ADC$ and $\triangle BCD$:

$$\begin{aligned} AD &= BC && \text{(opp. sides of //gram)} \\ \angle D &= \angle C && \text{(definition of rectangle)} \\ DC &= CD && \text{(common side)} \\ \therefore\ \triangle ADC &\cong \triangle BCD && \text{(SAS)} \\ \therefore\ AC &= BD && \text{(corr. sides, } \cong \triangle\text{s)} \end{aligned}$$

By 'diags of //gram' , we have $AE = EC$ and $DE = EB$. Since $AC = AE + EC$ and $BD = DE + EB$, we have $AE + EC = DE + EB \Rightarrow AE + AE = DE + DE \Rightarrow AE = DE \Rightarrow AE = BE = CE = DE$. □

Conditions of rectangle

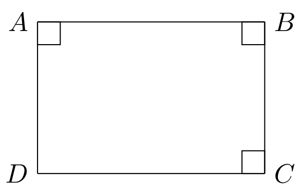
Preposition 43. A quadrilateral is a rectangle if it satisfies one of the following conditions:

1. Three angles are $90°$. (3 right ∠s)
2. Four angles are equal. (4 ∠s equal)
3. It is a parallelogram with a $90°$ angle. (//gram with right ∠)
4. One pair of opposite sides are parallel, with two $90°$ angles not sharing the same uncertain side. (1 // pair, 2 right ∠s)
5. Two angles are $90°$, with one equal pair of opposite sides. (1 equal pair, 2 right ∠s)
6. One pair of opposite sides are parallel, another pair of opposite sides are equal, with one right angle. (1 equal pair, 1 // pair, 1 right ∠s)
7. Diagonals are equal and bisect each other. (diags equal and bisect each other)
8. It is a parallelogram with diagonals equal. (//gram with equal diags)

Note: If a specific reason is not to be named, use the general reason (con. of rectangle) .

Proof. Let there be quadrilateral $ABCD$. Let's consider the conditions.

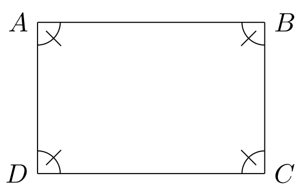
1. $\angle A = \angle B = \angle C = 90°$



$$\begin{aligned} \angle D &= 360° - 90° - 90° - 90° && (\angle \text{ sum of quad.}) \\ &= 90° \end{aligned}$$

$\therefore\ ABCD$ is a rectangle. (definition of rectangle)

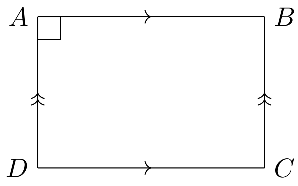
2. $\angle A = \angle B = \angle C = \angle D$



$$\begin{aligned} \angle A = \angle B = \angle C &= \angle D \\ \angle A + \angle B + \angle C + \angle D &= 360° \\ \therefore\ \angle A = \angle B = \angle C = \angle D &= 360°/4 = 90° \end{aligned}$$

$\therefore\ ABCD$ is a rectangle. (definition of rectangle)

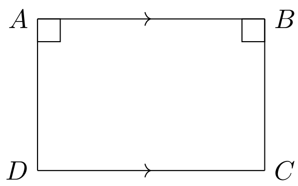
3. $AB//DC$, $AD//BC$ and $\angle A = 90°$



$$\angle D = 180^\circ - 90^\circ = 90^\circ \qquad (\text{int. } \angle\text{s , } AB//DC)$$
$$\angle B = 180^\circ - 90^\circ = 90^\circ \qquad (\text{int. } \angle\text{s , } AD//BC)$$
$$\therefore\ \angle A = \angle D = \angle B = 90^\circ$$

$\therefore$ $ABCD$ is a rectangle. (3 right $\angle$s)

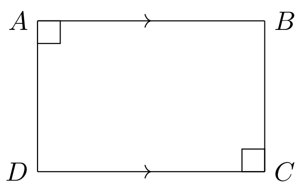
4a. $AB//DC$, $\angle A = \angle B = 90^\circ$



$$\angle D = 180^\circ - 90^\circ = 90^\circ \qquad (\text{int. } \angle\text{s , } AB//DC)$$
$$\therefore\ \angle A = \angle D = \angle B = 90^\circ$$

$\therefore$ $ABCD$ is a rectangle. (3 right $\angle$s)

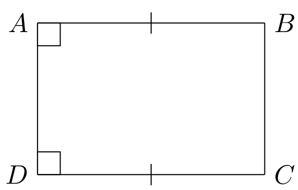
4b. $AB//DC$, $\angle A = \angle C = 90^\circ$



$$\angle B = 180^\circ - 90^\circ = 90^\circ \qquad (\text{int. } \angle\text{s , } AD//BC)$$
$$\therefore\ \angle A = \angle B = \angle C = 90^\circ$$

$\therefore$ $ABCD$ is a rectangle. (3 right $\angle$s)

5a. $AB = DC$, $\angle A = \angle D = 90^\circ$



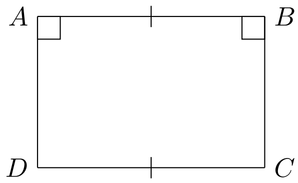
$$\because \angle A + \angle D = 90^\circ + 90^\circ = 180 \qquad (\text{given})$$
$$\therefore\ \angle AB//\angle DC \qquad (\text{int. } \angle\text{s supp.})$$

$\therefore$ $ABCD$ is a parallelogram. (opp. sides equal and //)

$\because$ $ABCD$ is a parallelogram with $\angle A = 90^\circ$,

$\therefore$ $ABCD$ is a rectangle. (//gram with right $\angle$)

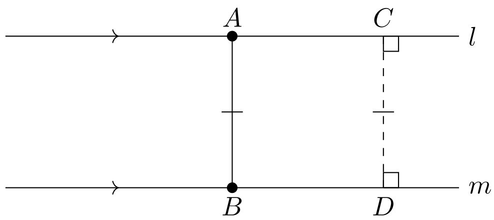
5b. $AB = DC$, $\angle A = \angle B = 90^\circ$



$$\because \angle A + \angle B = 90^\circ + 90^\circ = 180 \qquad \text{(given)}$$
$$\therefore\ AD//BC \qquad \text{(int. } \angle\text{s supp.)}$$

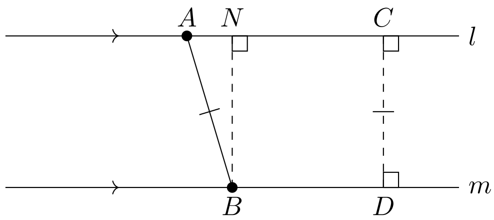
We will use a preposition not shown before.

Preposition. If there are a pair of parallel lines m and l , for which A is a point on line l and B is a point on line m such that the length of AB is equal to the perpendicular distance of l and m, then AB is perpendicular to both l and m . (property of parallel line distance)



$$\because AB = CD \text{ and } CD \perp l \text{ and } CD \perp m$$
$$\therefore\ AB \perp m \text{ and } AB \perp l \qquad \text{(property of parallel line distance)}$$

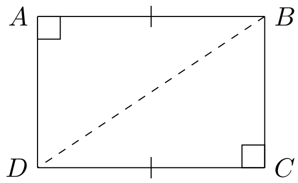
Proof. Let there be a pair of parallel lines m and l , for which A is a point on line l and B is a point on line m such that the length of AB is equal to the perpendicular distance of l and m .
Suppose (for the sake of contradiction) that AB is not perpendicular to line l and line m . Refer to the figure, let's say $\angle ABD > 90^\circ$. Then we can make a point N on line l such that $BN \perp l$:



In quadrilateral $NCDB$, $\angle BNC = \angle NCD = \angle CDB = 90^\circ$. Thus $NBCD$ is a rectangle . (3 right $\angle$s) , and $AB = CD$ (prop. of rectangle) . So we have $AB = NB$. But $AB = NB$ can't be true since in right triangle $\triangle ANB$, the hypotenuse AB must be the longest side, so we have $AB > NB$, which is a contradiction.
If we suppose that $\angle ABD > 90^\circ$ instead, then we can draw $AN \perp m$ and arrive at the contradiction similarly.
Thus, the only possible angle of $\angle ABD$ is 90° , so $AB \perp m$, and thus, we also have $AB \perp l$ (int. $\angle$s, $l//m$) . □

Return to the 5b condition of rectangle.
Since $AD//BC$, $AB \perp AD$, $AB \perp BC$, and $DC = AB$, by property of parallel line distance, we have $DC \perp AD$ and $DC \perp BC$.
Thus, $\angle A = \angle B = \angle C = \angle D = 90^\circ$, which means $ABCD$ is a rectangle (definition of rectangle) .

5c. $AB = DC$, $\angle A = \angle C = 90^\circ$

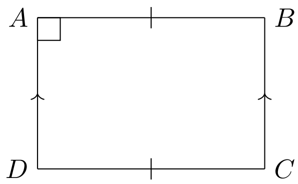


Join BD . In $\triangle ABD$ and $\triangle CDB$,

$$\angle A = \angle C = 90^\circ \qquad \text{(given)}$$
$$BD = DB \qquad \text{(common side)}$$
$$AB = DC \qquad \text{(given)}$$
$$\therefore\ \triangle ABD \cong \triangle CDB \qquad \text{(RHS)}$$
$$\therefore\ AD = BC \qquad \text{(corr. sides, } \cong \triangle\text{s)}$$
$$\therefore\ ABCD \text{ is a parallelogram.} \qquad \text{(opp. sides equal)}$$
$$\therefore\ ABCD \text{ is a rectangle.} \qquad \text{(//gram with right } \angle\text{)}$$

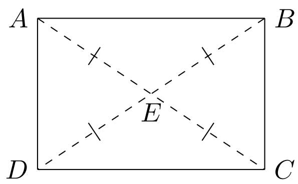
□

6. $AB = DC$, $AD//BC$, $\angle A = 90^\circ$



$$\angle B = 180^\circ - 90^\circ = 90^\circ \qquad \text{(int. } \angle\text{s , } AD//BC\text{)}$$
$$\therefore\ ABCD \text{ is a rectangle.} \qquad \text{(1 equal pair, 2 right } \angle\text{s)}$$

7. $AE = BE = CE = DE$



In $\triangle AED$ and $\triangle CEB$,

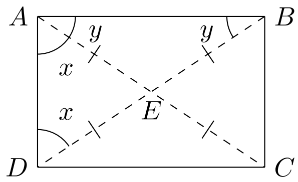
$$AE = CE \qquad \text{(given)}$$
$$\angle AED = \angle CEB \qquad \text{(vert. opp. } \angle\text{s)}$$
$$DE = BE \qquad \text{(given)}$$
$$\therefore\ \triangle AED \cong \triangle CEB \qquad \text{(SAS)}$$
$$\therefore\ AD = BC \qquad \text{(corr. sides, } \cong \triangle\text{s)}$$

In $\triangle AEB$ and $\triangle CED$,

$$AE = CE \qquad \text{(given)}$$
$$\angle AEB = \angle CED \qquad \text{(vert. opp. } \angle\text{s)}$$
$$BE = DE \qquad \text{(given)}$$
$$\therefore\ \triangle AEB \cong \triangle CED \qquad \text{(SAS)}$$
$$\therefore\ AB = DC \qquad \text{(corr. sides, } \cong \triangle\text{s)}$$

Since $AD = BC$ and $AB = DC$, $ABCD$ is a //gram (opp. sides equal) .

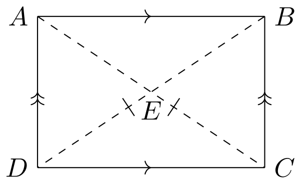
To show that $\angle A$ is a right angle, let's focus on $\triangle AED$ and $\triangle AEB$. Note that $\angle EAD = \angle EDA$ and $\angle EAB = \angle EBA$ (base $\angle$s, isos. $\triangle$).



Let $\angle EAD = \angle EDA = x$, and $\angle EAB = \angle EBA = y$. Note that $\angle A = x + y$. In $\triangle ABD$,

$$\begin{aligned}
\angle A + \angle ABD + \angle ADB &= 180^\circ \qquad (\angle \text{ sum of } \triangle)\\
(x+y) + y + x &= 180^\circ\\
x + y &= 90^\circ\\
\angle A &= 90^\circ\\
\therefore\ ABCD \text{ is a rectangle.} &\qquad (//\text{gram with right } \angle)
\end{aligned}$$

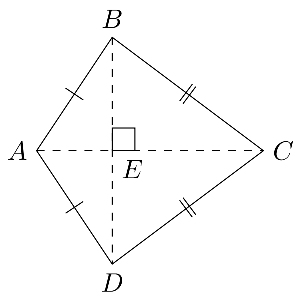
8. $AB//DC$, $AD//BC$, $AC = BD$



$$\begin{aligned}
AE = CE \text{ and } BE = DE &\qquad (\text{diags of } //\text{gram})\\
\text{Also, } AE + CE = BE + DE &\qquad (\text{given})\\
\therefore\ AE + AE = BE + BE &\\
AE = BE &\\
\therefore\ AE = BE = CE = DE &\\
\therefore\ ABCD \text{ is a rectangle.} &\qquad (\text{diags equal and bisect each other})
\end{aligned}$$

1.4.4 Kites

Preposition 44. The diagonals of a kite are perpendicular to each other. (prop. of kite diags)



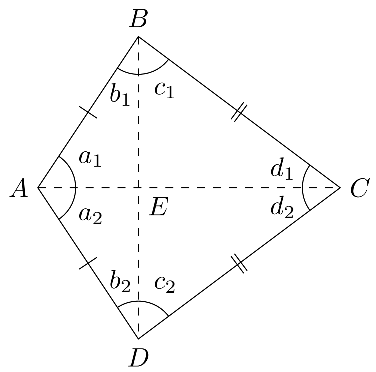
$$\begin{aligned}
\because\ AB = AD \text{ and } CB = CD &\\
\therefore\ BD \perp AC &\qquad (\text{prop. of kite diags})
\end{aligned}$$

Proof. In $\triangle ABC$ and $\triangle ADC$,

$$\begin{aligned}
AB &= AD \qquad (\text{given})\\
CB &= CD \qquad (\text{given})\\
AC &= AC \qquad (\text{common side})\\
\therefore\ \triangle ABC &\cong \triangle ADC \qquad (\text{SSS})\\
\therefore\ \angle BCE &= \angle DCE \qquad (\text{corr. } \angle\text{s}, \cong \triangle\text{s})\\
\therefore\ BD &\perp CE \qquad (\text{prop. of isos. } \triangle)
\end{aligned}$$

Since AEC is a straight line, we also have $BD \perp AC$. □

Preposition 45. In a kite, the angles formed by a side and a diagonal form equal pairs. (prop. of kite $\angle$s)



$$AB = AD \text{ and } CB = CD$$
$$\therefore\ a_1 = a_2\ ,\ b_1 = b_2\ ,\ c_1 = c_2\ ,\ d_1 = d_2$$
(prop. of kite diags)

Proof. In $\triangle ABC$ and $\triangle ADC$,

$$\begin{aligned} AB &= AD && \text{(given)} \\ CB &= CD && \text{(given)} \\ AC &= AC && \text{(common side)} \end{aligned}$$
$$\therefore\ \triangle ABC \cong \triangle ADC \qquad \text{(SSS)}$$
$$\therefore\ \angle BAC = \angle DAC \ \text{ and } \ \angle BCA = \angle DCA \qquad (\text{corr. } \angle\text{s}, \cong \triangle\text{s})$$

Also, $\angle ABD = \angle ADB$ and $\angle CBD = \angle CDB$ (base $\angle$s, isos. $\triangle$). $\square$

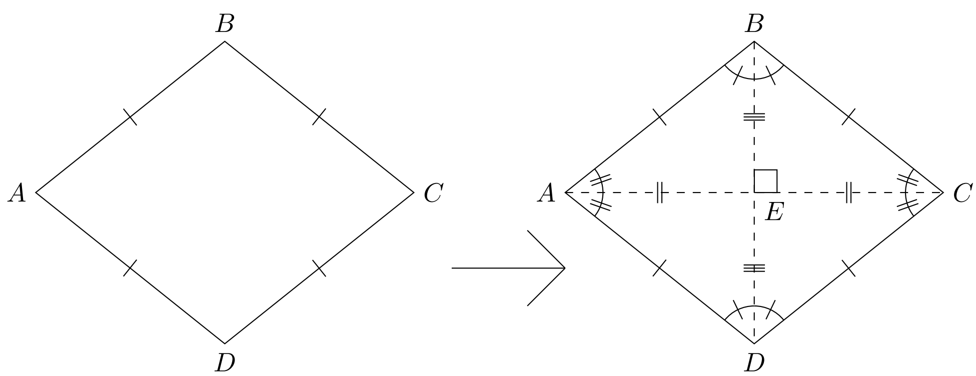
1.4.5 Rhombuses

Properties of rhombus

Preposition 46. A rhombus has the following properties:

1. Four sides are equal. (definition of rhombus)
2. Opposite sides are parallel (which means it is a parallelogram).
3. Opposite angles are equal and bisected by diagonals.
4. Diagonals are perpendicular and bisect each other.

(prop. of rhombus) *



$$\because\ AB = BC = CD = DA \qquad \text{(definition of rhombus)}$$
$$\therefore\ AB//CD\ ,\ BC//AD$$
$$\angle BAC = \angle DAC = \angle BCA = \angle DCA$$
$$\angle ABD = \angle ADB = \angle CBD = \angle CDB$$
$$BD \perp AC$$
$$BE = ED\ ,\ AE = EC$$
$$\text{(prop. of rhombus)}$$

Proof.

$$\because\ AB = CD \text{ and } BC = AD \qquad \text{(given)}$$
$$\therefore \qquad \text{(prop. 2) } AB//CD \text{ and } BC//AD \qquad \text{(opp. sides equal)}$$
$$\therefore \qquad \text{(prop. 4) } AE = EC \text{ and } BE = ED \qquad \text{(diags of //gram)}$$

Now we prove that the four triangles formed by the rhombus' diagonals are congruent.
In $\triangle EAB$, $\triangle ECB$, $\triangle EAD$ and $\triangle ECD$,

$$\text{(prop. 1) } AB = BC = AD = CD \qquad \text{(given)}$$
$$BE = BE = ED = ED \qquad \text{(common side \& diags of //gram)}$$
$$AE = EC = AE = EC \qquad \text{(common side \& diags of //gram)}$$
$$\therefore\ \triangle EAB \cong \triangle ECB \cong \triangle EAD \cong \triangle ECD \qquad \text{(SSS)}$$
$$\therefore \qquad \text{(prop. 3) } \angle BAC = \angle BCA = \angle DAC = \angle DCA \qquad \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}$$
$$\text{(prop. 3) } \angle ABD = \angle CBD = \angle ADB = \angle CDB \qquad \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}$$

Also note that a rhombus is a special type of kite since $AB = AD$ and $CB = CD$. By property of kite diags, we have $BD \perp AC$ (prop. 4). □

<u>Conditions of rhombus</u>

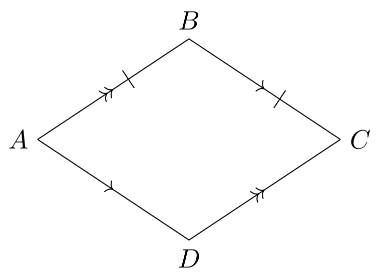
Preposition 47. A quadrilateral is a rhombus if it satisfies one of the following conditions:

1. It is a parallelogram with a pair of adjacent side equal. (//gram with equal adj. side)
2. The diagonal bisects an equal pair of opposite angles. (diag bisects equal opp. $\angle$s)
3. Diagonals are perpendicular and bisect each other. (diags $\perp$ and bisect each other)

Non-specific reason: (con. of rhombus)

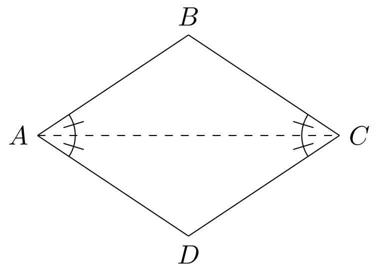
Proof. Let there be quadrilateral $ABCD$. Let's consider the conditions.

1. $AB//CD$, $BC//AD$, $AB = BC$



$$AB = CD \text{ and } BC = AD \qquad \text{(opp. sides of //gram)}$$
$$\therefore \qquad \text{(condition 1) } AB = BC = CD = AD$$
$$\therefore\ ABCD \text{ is a rhombus.} \qquad \text{(ddefinition of rhombus)}$$

2. $\angle BAC = \angle BCA = \angle DAC = \angle DCA$



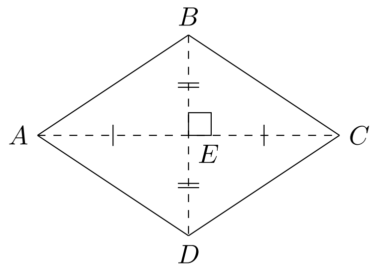
$$\angle BAC = \angle BCA \qquad \text{(given)}$$
$$\therefore\ BA = BC \qquad \text{(sides opp. equal } \angle\text{s)}$$
$$\angle DAC = \angle DCA \qquad \text{(given)}$$
$$\therefore\ DA = DC \qquad \text{(sides opp. equal } \angle\text{s)}$$

In $\triangle BAC$ and $\triangle DAC$,

$$\angle BAC = \angle DAC \qquad \text{(given)}$$
$$AC = AC \qquad \text{(common side)}$$
$$\angle BCA = \angle DCA \qquad \text{(given)}$$
$$\therefore\ \triangle BAC \cong \triangle DAC \qquad \text{(ASA)}$$
$$\therefore\ AB = AD \text{ and } CB = CD \qquad \text{(corr. sides, } \cong \triangle\text{s)}$$

$\therefore\ AB = BC = CD = AD$, and $ABCD$ is a rhombus.

3. $BD \perp AC$, $AE = EC$, $BE = ED$



In $\triangle EAB$, $\triangle ECB$, $\triangle EAD$ and $\triangle ECD$,

$$AE = EC = AE = EC \qquad \text{(given \& common side)}$$
$$\angle AEB = \angle CEB = \angle AED = \angle CED = 90^\circ \qquad (BD \perp AC)$$
$$BE = BE = ED = ED \qquad \text{(given \& common side)}$$
$$\therefore\ \triangle EAB \cong \triangle ECB \cong \triangle EAD \cong \triangle ECD \qquad \text{(SAS)}$$
$$\therefore\ AB = BC = CD = AD \qquad \text{(corr. sides, } \cong \triangle\text{s)}$$

$\therefore\ ABCD$ is a rhombus. □

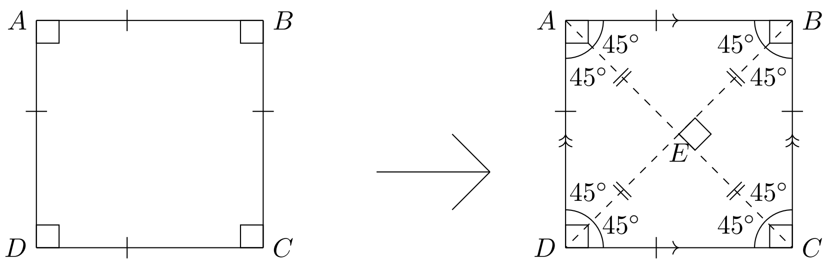
1.4.6 Squares

Properties of square

Preposition 48. A square has the following properties:

1. Four equal sides. (definition of square part I)
2. Four right angles. (definition of square part II)
3. Opposite sides are parallel.
4. Diagonals are perpendicular, equal and bisect each other.
5. The angles formed between a side and a diagonal is 45° .

(prop. of square) *



$$\because\ \angle A = \angle B = \angle C = \angle D = 90° \ , AB = BC = CD = DA \qquad \text{(definition of square)}$$
$$\therefore\ AB//DC \ , AD//BC$$
$$AE = BE = CE = DE$$
$$BD \perp AC$$
$$\angle EAD = \angle EDA = \angle EDC = \angle ECD = \angle ECB = \angle EBC = \angle EBA = \angle EAB = 45°$$
(prop. of square)

Proof.

$$AB = BC = CD = DA \qquad \text{(definition of square part I)}$$
$$\therefore\ ABCD \text{ is a rhombus.} \qquad \text{(definition of rhombus)}$$
$$\therefore \qquad \text{(condition 3) } AB//DC\,,\ AD//BC \qquad \text{(prop. of rhombus)}$$
$$\text{(condition 4) } BD \perp AC \qquad \text{(prop. of rhombus)}$$
$$\text{(condition 4) } AE = EC \ \text{ and } \ BE = ED \qquad \text{(prop. of rhombus)}$$

In $\triangle ADC$ and $\triangle DAB$,

$$AD = AD \qquad \text{(common side)}$$
$$\angle D = \angle A \qquad \text{(given)}$$
$$DC = AB \qquad \text{(given)}$$
$$\therefore\ \triangle ADC \cong \triangle DAB \qquad \text{(SAS)}$$
$$\therefore\ AC = BD \qquad \text{(corr. sides, } \cong \triangle\text{s)}$$

Since we also have $AE = EC$ and $BE = ED$, we have $AE = BE = CE = DE$ (condition 4).

Note that since $ABCD$ is a rhombus, the four triangles formed by diagonals are congruent (proven in prop. of rhombus).

Focus on one of the triangles, say $\triangle AED$. Since $AE = DE$, we have $\angle EAD = \angle EDA$ (base $\angle$s, isos. $\triangle$).

$$\angle EAD + \angle EDA + \angle AED = 180° \qquad (\angle \text{ sum of } \triangle)$$
$$2 \times \angle EAD + 90° = 180°$$
$$\angle EAD = \angle EDA = 45°$$

By congruent triangles, we have $\angle EAB = \angle ECB = \angle ECD = \angle EAD = 45°$, and $\angle EBA = \angle EBC = \angle EDC = \angle EDA = 45°$ (condition 5). □

Preposition 49. A quadrilateral is a square if it satisfies one of the following conditions:

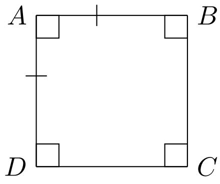
1. It is a rectangle with two adjacent sides are equal. (rectangle with equal adj. pair)
2. It is a rhombus with a 90° angle. (rhombus with right $\angle$)
3. It is a rhombus with an angle between side and diagonal being 45° . (rhombus with 45 ° inclination)
4. Three sides are equal, with two 90° angles. (3 sides equal, 2 right $\angle$s)

5. Diagonals are perpendicular, equal and bisect each other. (diags $\perp$, equal and bisect each other)

Non-specific reason: (con. of square)

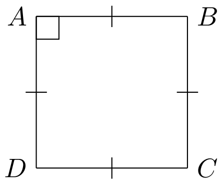
Proof. Let there be quadrilateral $ABCD$. Let's consider the conditions.

1. $\angle A = \angle B = \angle C = \angle D = 90°$, $AB = AD$



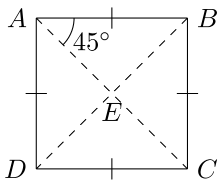
$$AD = BC \text{ and } AB = DC \qquad \text{(opp. sides of rectangle)}$$
$$\therefore\ AB = BC = CD = AD$$
$\therefore$ $ABCD$ is a square. (definitoin of square)

2. $AB = BC = CD = AD$, $\angle A = 90°$



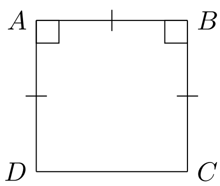
$$AB//DC \text{ and } AD//BC \qquad \text{(prop. of rhombus)}$$
$$\angle D = 180° - 90° = 90° \qquad \text{(int. } \angle\text{s , } AB//DC)$$
$$\angle B = 180° - 90° = 90° \qquad \text{(int. } \angle\text{s , } AD//BC)$$
$$\angle C = 180° - 90° = 90° \qquad \text{(int. } \angle\text{s , } AB//DC)$$
$\therefore$ $ABCD$ is a square. (definition of square)

3. $AB = BC = CD = AD$, $\angle BAC = 45°$



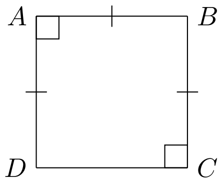
$ABCD$ is a rhombus. (definition of rhombus)
$$\therefore\ \angle EAD = \angle EAB = 45° \qquad \text{(prop. of rhombus)}$$
$$\angle A = 45° + 45° = 90°$$
$\therefore$ $ABCD$ is a square. (rhombus with right $\angle$)

4a. $AB = BC = AD$, $\angle A = \angle B = 90°$



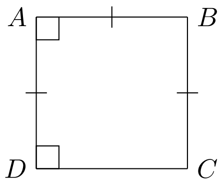
$$\begin{aligned}
&\because AD = BC \ \text{ and } \ \angle A = \angle B = 90^\circ \qquad \text{(given)}\\
&\therefore\ ABCD \text{ is a rectangle.} \qquad \text{(1 equal pair, 2 right } \angle\text{s)}\\
&\therefore\ \angle D = \angle C = 90^\circ \ \text{ and } \ DC = AB \qquad \text{(prop. of rectangle)}\\
&\therefore\ ABCD \text{ is a square.} \qquad \text{(definition of square)}
\end{aligned}$$

4b. $AB = BC = AD$, $\angle A = \angle C = 90^\circ$



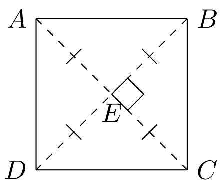
$$\begin{aligned}
&\because AD = BC \ \text{ and } \ \angle A = \angle C = 90^\circ \qquad \text{(given)}\\
&\therefore\ ABCD \text{ is a rectangle.} \qquad \text{(1 equal pair, 2 right } \angle\text{s)}\\
&\therefore\ \angle B = \angle D = 90^\circ \ \text{ and } \ DC = AB \qquad \text{(prop. of rectangle)}\\
&\therefore\ ABCD \text{ is a square.} \qquad \text{(definition of square)}
\end{aligned}$$

4c. $AB = BC = AD$, $\angle A = \angle D = 90^\circ$



$$\begin{aligned}
&\because AD = BC \ \text{ and } \ \angle A = \angle D = 90^\circ \qquad \text{(given)}\\
&\therefore\ ABCD \text{ is a rectangle.} \qquad \text{(1 equal pair, 2 right } \angle\text{s)}\\
&\therefore\ \angle B = \angle C = 90^\circ \ \text{ and } \ DC = AB \qquad \text{(prop. of rectangle)}\\
&\therefore\ ABCD \text{ is a square.} \qquad \text{(definition of square)}
\end{aligned}$$

5. $AE = BE = CE = DE$, $BD \perp AC$



In $\triangle EAB$, $\triangle ECB$, $\triangle EAD$ and $\triangle ECD$,

$$\begin{aligned}
&AE = CE = AE = CE \qquad \text{(given \& common side)}\\
&\angle AEB = \angle CEB = \angle AED = \angle CED = 90^\circ \qquad (BD \perp AC)\\
&BE = BE = ED = ED \qquad \text{(given \& common side)}\\
&\therefore\ \triangle EAB \cong ECB \cong EAD \cong ECD \qquad \text{(SAS)}\\
&\therefore\ AB = BC = CD = AD \qquad \text{(corr. sides, } \cong \triangle\text{s)}
\end{aligned}$$

Note that $\angle EAB = \angle EBA$ (base $\angle$s, isos. $\triangle$). Thus $\angle EAB = \angle EBA = (180^\circ - 90^\circ)/2 = 45^\circ$ ($\angle$ sum of $\triangle$).

By congruent triangles, $\angle EAB = \angle ECB = \angle EAD = \angle ECD = 45^\circ$, and $\angle EBA = \angle EBC = \angle EDA = \angle EDC = 45^\circ$.

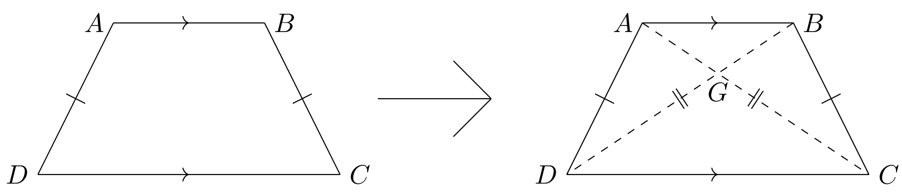
Thus $\angle A = \angle B = \angle C = \angle D = 45^\circ + 45^\circ = 90^\circ$, and $ABCD$ is a square. (definition of square). □

1.4.7 Proper trapeziums

Preposition 50. An isosceles trapezium has the following properties:

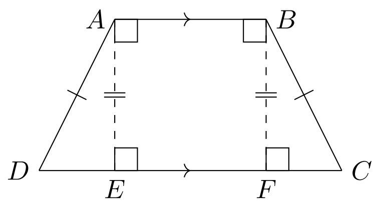
1. It has exactly one pair of parallel opposite sides. (definition of isos. trapezium part I)
2. The pair of non-parallel sides are equal. (definition of isos. trapezium part II)
3. Angles sharing the same parallel side are equal.
4. Diagonals are equal.
5. Angles formed between a side and a diagonal form equal pairs.

   (prop. of isos. trapezium)



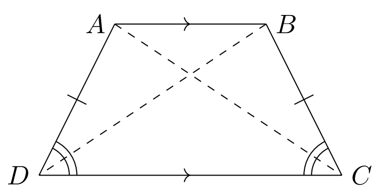
$$\because AB//DC \,,\, AD\neg//BC \,,\, AC = BC \qquad \text{(definition of isos. trapezium)}$$
$$\therefore\ \angle A = \angle B \text{ and } \angle D = \angle C$$
$$AC = BD$$
$$\angle GAB = \angle GBA \,,\, \angle DAC = \angle CBD \,,\, \angle ADB = \angle BCA \,,\, \angle GDC = \angle GCD$$
$$\text{(prop. of isos. trapezium)}$$

Proof. Let there be isos. trapezium $ABCD$ where $AB//DC$, $AD = BC$, and $AB < DC$.
Draw $AE \perp DC$ and $BF \perp DC$. Note that $AE \perp AB$ and $BF \perp AB$ (int. $\angle$s , $AB//DC$)



Note that $ABFE$ is a rectangle, so $AE = BF$ (opp. sides of rectangle).
In $\triangle AED$ and $\triangle BFC$,

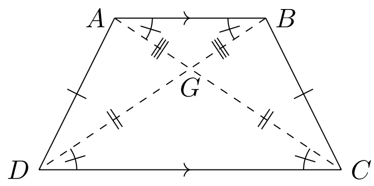
$$\begin{aligned}
\angle AED = \angle BFC &= 90^\circ \qquad (AE \perp DC, BF \perp DC)\\
AD &= BC \qquad \text{(given)}\\
AE &= BF \qquad \text{(opp. sides of rectangle)}\\
\therefore\ \angle AED &\cong BFC \qquad \text{(RHS)}\\
\therefore \qquad \text{(prop. 3) } \angle D &= \angle C \qquad (\text{corr. } \angle\text{s}, \cong \triangle\text{s})\\
\angle DAE &= \angle CBF \qquad (\text{corr. } \angle\text{s}, \cong \triangle\text{s})\\
\text{(prop. 3) } \angle A &= \angle DAE + 90^\circ\\
&= \angle CBF + 90^\circ\\
&= \angle B
\end{aligned}$$



In $\triangle ADC$ and $\triangle BCD$,

$$\begin{aligned}
AD &= BC && \text{(given)}\\
\angle D &= \angle C && \text{(prop. 3 of this preposition)}\\
DC &= CD && \text{(common side)}\\
\therefore\ \triangle ADC &\cong BCD && \text{(SAS)}\\
\therefore\quad \text{(prop. 4) } AC &= BD && \text{(corr. sides, } \cong \triangle\text{s)}\\
\text{(prop. 5a) } \angle ACD &= \angle BDC && \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}
\end{aligned}$$

Let G denote the intersection of AC and BD . Since $\angle ACD = \angle BDC$, we have $GD = GC$ (sides opp. equal $\angle$s). Since $AC = BD$, we also have $GA = GB$, so $\angle GAB = \angle GBA$ (base $\angle$s, isos. $\triangle$) (prop. 5b).



Since $AB//DC$, we have $\angle ABD = \angle BDC$ (alt. $\angle$s , $AB//DC$) , which means $\angle GAB = \angle GBA = \angle GDC = \angle GCD$.

Finally, in $\triangle GAD$ and $\triangle GBC$,

$$\begin{aligned}
GA &= GB && \text{(proven above)}\\
AD &= BC && \text{(given)}\\
GD &= GC && \text{(proven above)}\\
\therefore\ \triangle GAD &\cong \triangle GBC && \text{(SSS)}\\
\therefore\quad \text{(prop. 5c) } \angle DAG &= \angle CBG && \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}\\
\text{(prop. 5d) } \angle ADG &= \angle BCG && \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}
\end{aligned}$$

□

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