Toddler Geometry

Jes Modian

April 2, 2023

Abstract

Geometry is an ancient branch of Mathematics, dating as far back as 4000 years ago. Humanity has been fascinated and puzzled by these 'simple' lines and shapes for millennia, so it is only natural for a maths person like me to want to study Geometry and uncover its mysteries. But unlike other branches of mathematics such as Calculus and Linear Algebra, why are all the geometry theorems so useless and unapplicable in real life? I have no idea. After studying some circle theorems in high school, we don't even touch them again in University, which is doing Geometry a disservice in my opinion. So here I am, fully embracing the uselessness of Geometry and just studying for the fun of it, because it is the purest form of art.

Contents

0 Introduction

Hi

1 Lines, angles and shapes

1.0 Symbols and abbreviations

(The hyphen and the bullet point aren't a part of symbol.)

- $\angle$ - angle
- $\triangle$ - triangle
- ◺ - right-angled triangle
- $\perp$ - perpendicular to
- // - parallel to
- $\cong$ - congruent to
- $\because$ - since
- $\therefore$ - therefore
- adj. - adjacent
- opp. - opposite
- pt. - point
- st. - straight
- vert. - vertical
- prop. - property
- corr. - corresponding
- isos. - isosceles
- equil. - equilateral
- //gram - parallelogram
- inc. - included

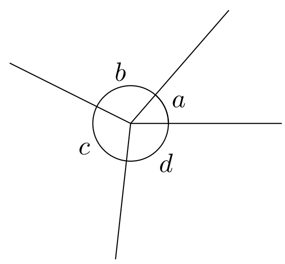
1.1 Basic properties

[1] [2]

Preposition 1. Two lines can intersect at one point at most. (property of line intersection)

Proof. Suppose that two lines intersect at two distinct points called P and Q. We have two lines passing through P and Q , which contradicts Euclid's postulate 1 (which states that there is only one line that passes through two points). So the two lines can also never intersect at three distinct points or more because they would have to intersect at two of the points, which we have just shown to be impossible. So two lines can intersect at one point at most. □

Preposition 2. The sum of all angles sharing the same vertex is 360° . ($\angle$s at a pt.) * [1]
Example

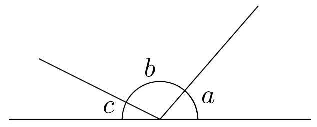


[1]Reasons marked with * are used in HK secondary school maths.

$$a + b + c + d = 360° \qquad (\angle\text{s at a pt.})$$

Proof. By definition, a whole revolution is 360° . By angle addition postulate, when a whole revolution is split into several smaller angles, the sum of these angles must be a whole revolution, which is 360°. ☐

Preposition 3. The sum of adjacent angles on a straight line is 180° (adj. ∠s on st. line) *

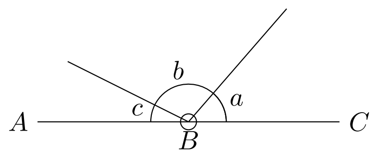


$$a + b + c = 180° \qquad (\text{adj. } \angle\text{s on st. line})$$

Proof. By definition, a straight angle (which is 180°) is half a revolution or two right angles, so two straight angles sharing a vertex makes up the whole revolution which is 360° (∠s at a pt.). Since all right angles are equal by Euclid's 4th postulate, all straight angles are also equal. So one straight angle is 180° . By angle addition postulate, the straight angle can be split into several smaller angles whose sum is the straight angle, which is 180° . ☐

Preposition 4. If the sum of some adjacent angles is 180° , then these angles make a straight line. (adj. ∠s supp.) *

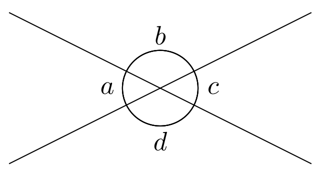
(The tiny circle at B indicates that we are not sure if the 'line' passing through B is actually a straight line.)



$$\text{Observation: } a + b + c = 180°$$
$$\therefore\ ABC \text{ is a straight line.} \qquad (\text{converse of adj. } \angle\text{s on st. line})$$

Proof. By protractor postulate, given ray BC, there is a unique ray BA such that $\angle ABC$ is 180° (a straight angle), and a straight angle is a straight line. ☐

Preposition 5. Vertically opposite angles are equal. (vert. opp. ∠s) *



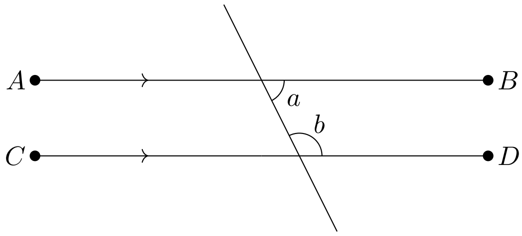
$$a = c \text{ and } b = d \qquad (\text{vert. opp. } \angle\text{s})$$

Proof.

$$\begin{aligned} a + b &= 180° && (\text{adj. } \angle\text{s on st. line}) \\ b + c &= 180° && (\text{adj. } \angle\text{s on st. line}) \\ \therefore\quad a + b &= b + c \\ a &= c \end{aligned}$$

By similar reasoning, we have $b = d$. ☐

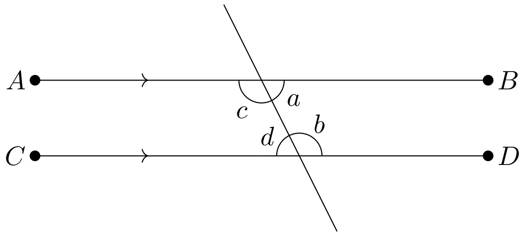
Preposition 6. For a pair of parallel lines, the interior angles formed by a transversal line are **supplementary** [2]. (int. $\angle$s, $AB//CD$) *



$$a + b = 180° \qquad \text{(int. } \angle\text{s, } AB//CD)$$

Proof. By the contrapositive of parallel postulate, if two lines never intersect each other (meaning they are parallel), then the two lines are **not** drawn in such a way that intersect a third line (the transversal line) such that the sum of the interior angles on one side is less than two right angles. This means that $a + b \geq 180°$.

However, if $a + b > 180°$, then we can focus on the interior angles of the other side: c and d .

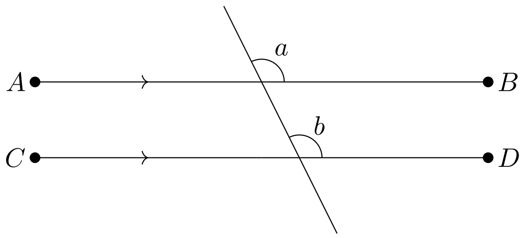


Note that we have $c = 180° - a$ and $d = 180° - b$ (adj. $\angle$s on st. line). Thus, starting from the inequality:

$$\begin{aligned} a + b &> 180° \\ 360° - (a + b) &< 360° - 180° \\ 180° - a + 180° - b &< 180° \\ c + d &< 180° \end{aligned}$$

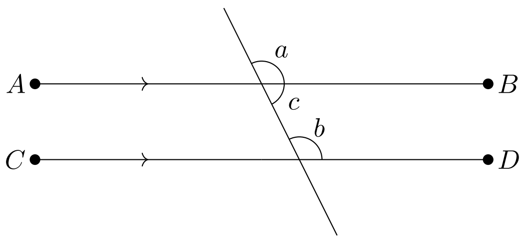
By the parallel postulate, line AB and CD must meet at the left of the transversal line, but this contradicts the initial assumption that the two lines never intersect each other. Thus, it must be the case that $a + b = 180°$. □

Preposition 7. For a pair of parallel lines, the corresponding angles formed by a transversal line are equal. (corr. $\angle$s, $AB//CD$) *



$$a = b \qquad \text{(corr. } \angle\text{s, } AB//CD)$$

Proof. .

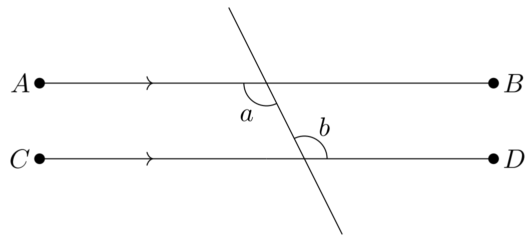


[2]Two angles are supplementary if they add up to 180°.

$$
\begin{aligned}
c + b &= 180^\circ && \text{(int. } \angle\text{s, } AB//CD) \\
a + c &= 180^\circ && \text{(adj. } \angle\text{s on st. line)} \\
\therefore\ a + c &= c + b \\
a &= b
\end{aligned}
$$

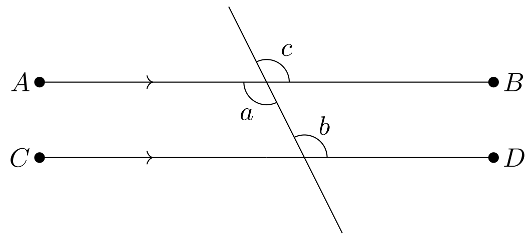
□

Preposition 8. For a pair of parallel lines, the alternate angles formed by a transversal line are equal. (alt. $\angle$s, $AB//CD$) *



$$a = b \qquad \text{(alt. } \angle\text{s, } AB//CD)$$

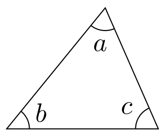
Proof. .



$$
\begin{aligned}
a &= c && \text{(vert. opp. } \angle\text{s)} \\
b &= c && \text{(corr. } \angle\text{s, } AB//CD) \\
\therefore\ a &= b
\end{aligned}
$$

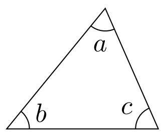
□

Preposition 9. The sum of interior angles of a triangle is 180° . ($\angle$ sum of $\triangle$)



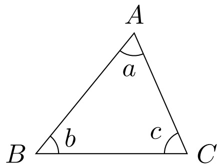
$$a + b + c = 180^\circ \qquad (\angle \text{ sum of } \triangle)$$

Preposition 10. The sum of any two interior angles of a triangle is less than 180° . (2 $\angle$ sum of $\triangle$)



$$
\begin{aligned}
a + b &< 180^\circ && (2\ \angle \text{ sum of } \triangle) \\
b + c &< 180^\circ && (2\ \angle \text{ sum of } \triangle) \\
a + c &< 180^\circ && (2\ \angle \text{ sum of } \triangle)
\end{aligned}
$$

Proof. Note that the three vertices of the triangle must be non-collinear (otherwise it will be just a line segment), so any interior angle is larger than zero.

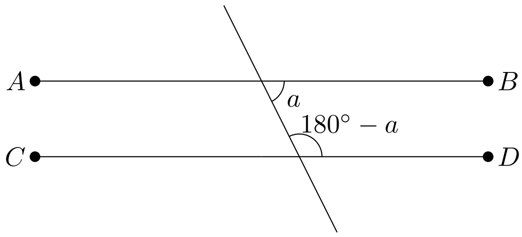


Refer to the figure, we have $a+b+c = 180°$ ($\angle$ sum of $\triangle$), with $a > 0°$, $b > 0°$, $c > 0°$. Thus $a+b < 180°$, $b+c < 180°$, $a+c < 180°$.

$\square$

An (obvious) implication of this preposition is that any interior angle of a triangle is less than $180°$.

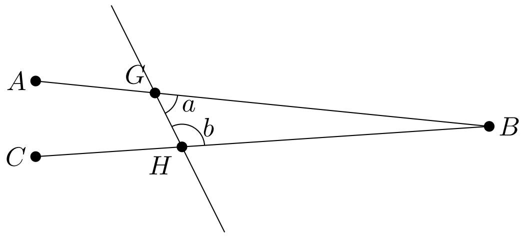
Preposition 11. For two lines, if the interior angles formed by another transversal line are supplementary , then the two lines are parallel. (int. $\angle$s supp.) *



$AB \,//\, CD$ (int. $\angle$s, supp.)

(// means ‘is parallel to’)

Proof. If the two lines are not parallel, then they intersect at some point. These two lines and the transversal line form a triangle (assuming the three lines are not concurrent):



Refer to the figure, a and b are the interior angles formed by the transversal line that are on the same side as B. Note that $a+b+\angle GBH = 180°$ ($\angle$ sum of $\triangle$). Since $\angle GBH > 0°$ (as G is not on HB), we have $a+b < 180°$.

If we want to consider the interior angles that are on the opposite side from B , then two interior angles are $\angle AGH$ and $\angle CHG$ instead. We have $\angle AGH = 180° - a$ and $\angle CHG = 180° - b$ (adj. $\angle$s on st. line). So

$$\begin{aligned} a+b &< 180° \\ 360° - (a+b) &> 360° - 180° \\ 180° - a + 180° - b &> 180° \\ \angle AGH + \angle CHG &> 180° \end{aligned}$$

No matter which side we look at, the two interior angles formed by the transversal line are not equal to $180°$. So we have proved the statement:

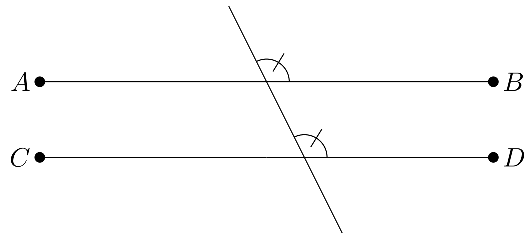
If the two lines are not parallel, then the interior angles formed by another transversal line are not supplementary.

Thus, the contrapositive of this statement is also true:

If the interior angles formed by another transversal line are supplementary , then the two lines are parallel.

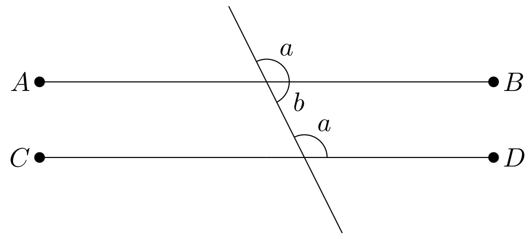
$\square$

Preposition 12. For two lines, if the corresponding angles formed by another transversal line are equal, then the two lines are parallel. (corr. ∠s equal.) *



$$AB\,//\,CD \qquad \text{(corr. } \angle\text{s equal.)}$$

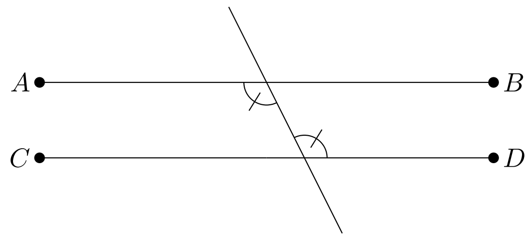
Proof. .



$$a + b = 180^\circ \qquad \text{(adj. } \angle\text{s on st. line)}$$
$$AB\,//\,CD \qquad \text{(int. } \angle\text{s, supp.)}$$

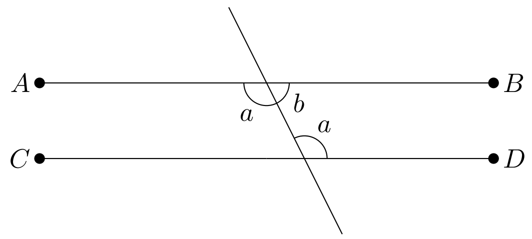
□

Preposition 13. For two lines, if the alternate angles formed by another transversal line are equal, then the two lines are parallel. (alt. ∠s equal.) *



$$AB\,//\,CD \qquad \text{(alt. } \angle\text{s equal.)}$$

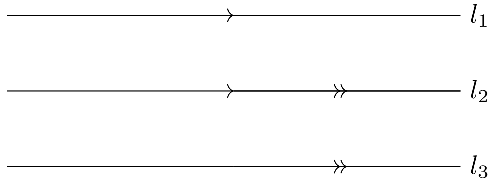
Proof. .



$$a + b = 180^\circ \qquad \text{(adj. } \angle\text{s on st. line)}$$
$$AB\,//\,CD \qquad \text{(int. } \angle\text{s, supp.)}$$

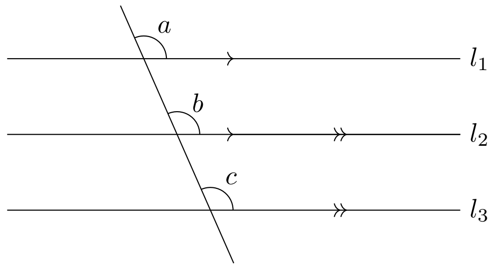
□

Preposition 14. If there are three lines, in which the first line is parallel to the second line, and the second line is parallel to the third line, then the first line is parallel to the third line. (transitivity of parallel lines)



$$\begin{aligned} \because\ & l_1 // l_2 \text{ and } l_2 // l_3 \\ \therefore\ & l_1 // l_3 \qquad \text{(transitivity of parallel lines)} \end{aligned}$$

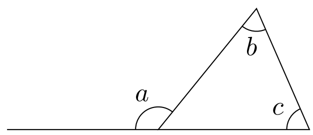
Proof. Draw a transversal line.



$$\begin{aligned} a &= b && \text{(corr. } \angle\text{s , } l_1 // l_2\text{)} \\ b &= c && \text{(corr. } \angle\text{s , } l_2 // l_3\text{)} \\ \therefore\ a &= c && \text{(transitivity of equality)} \\ \therefore\ l_1 &// \, l_3 && \text{(corr. } \angle\text{s equal)} \end{aligned}$$

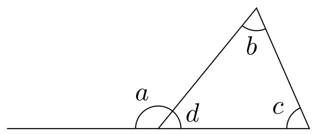
□

Preposition 15. An exterior angle of a triangle is the sum of the two opposite interior angles. (ext. $\angle$ of $\triangle$)



$$a = b + c \qquad \text{(ext. } \angle \text{ of } \triangle\text{)}$$

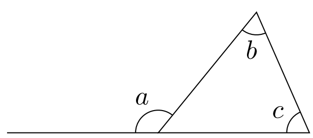
Proof. .



$$\begin{aligned} d + b + c &= 180^\circ && (\angle \text{ sum of } \triangle) \\ a + d &= 180^\circ && \text{(adj. } \angle\text{s on st. line)} \\ \therefore\ d + b + c &= a + d \\ a &= b + c \end{aligned}$$

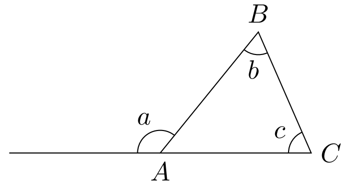
□

Preposition 16. An exterior angle of a triangle is greater than any of its opposite interior angle. (ext. $\angle$ of $\triangle$ < int. opp. $\angle$)



$$a > b \text{ and } a > c \qquad (\text{ext. } \angle \text{ of } \triangle < \text{int. opp. } \angle)$$

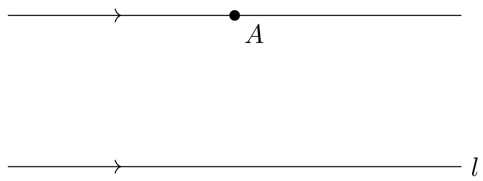
Proof. .



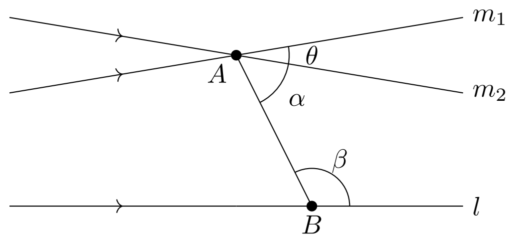
Since $\triangle ABC$ is a valid triangle, A , B , C are non-collinear, so $b > 0$ and $c > 0$.

Since $a = b + c$ (ext. $\angle$ of $\triangle$), we have $a > b$ and $a > c$. ☐

Preposition 17. Given a line and a point not on it, there is exactly one line passing through the point that is parallel to the given line. (Playfair's theorem / property of parallel line)



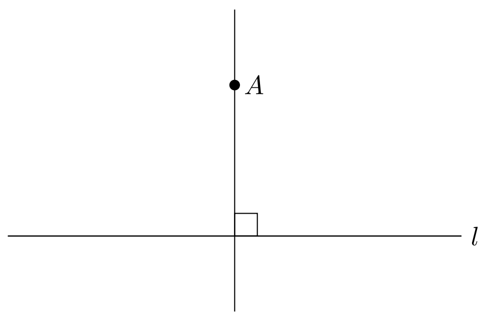
Proof. Label the given line as l and the given point as A . If there are two distinct lines m_1 and m_2 both passing through A , then A must be the only point of intersection (property of line intersection) . Let θ be the angle formed (facing right) between m_1 and m_2 . Let B be an arbitrary point on l and connect AB.



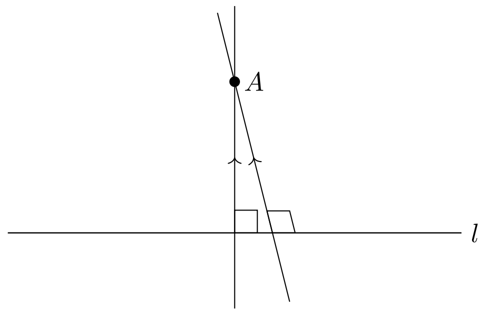
Refer to the figure. If m_1 and m_2 are both parallel to l , then we have $\alpha + \beta = 180°$ (int. $\angle$s, $m_2 // l$) , and $\alpha + \beta + \theta = 180°$ (int. $\angle$s, $m_1 // l$) . This means $\theta = 0°$. But this would mean that m_1 and m_2 are actually the same line, which is a contradiction.

Thus, there can only be one unique line passing through A that is parallel to l . ☐

Preposition 18. Given a line and a point, there is exactly one line passing through the point that is perpendicular to the given line. (property of perpendicular line)

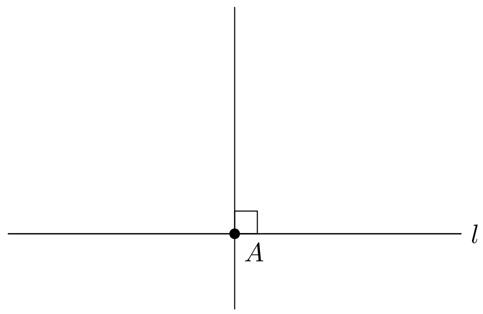


Proof. Label the given line as l and the given point as A . First, consider the case that A is not on l . Suppose there are two distinct lines passing through A that are perpendicular to l . Then they must meet l at two distinct points, or otherwise they are actually the same line (Euclid's first postulate). Then they must be parallel since the interior angles formed by the lines sum up to $180°$ (int. $\angle$s supp.) . But parallel lines never intersect, which contradicts the assumption that the two lines intersect at A .



Thus, there is a unique line passing A that is perpendicular to l .

Now consider the case that A is on line l . Then by protractor postulate, there is a unique line that intersects l at A at $90°$, so there is a unique line passing A that is perpendicular to l .

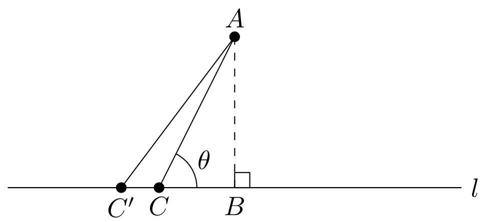


□

Preposition 19. Suppose B is a point on line l and A is a point vertically above B (meaning $AB \perp l$) . If C is a point on l that is not B, then the longer BC is, the smaller the angle $\angle ACB$ is. (property of hypotenuse inclination)

In other words, if length BC is a variable over the domain $(0, \infty)$, then $\angle ACB$ is a **strictly decreasing** [3] function of BC .

Proof. Assume that C is at the left of B. Let C' be a point on l to the left of C. So $C'B > CB$.



In $\triangle ACC'$, we have $\angle AC'B + \angle C'AC = \angle ACB$ (ext. $\angle$ of $\triangle$), so $\angle AC'B < ACB$.
If C is at the right of B , then we can let C' be a point to the right of C, and the (ext. $\angle$ of $\triangle$)reason will still hold.

□

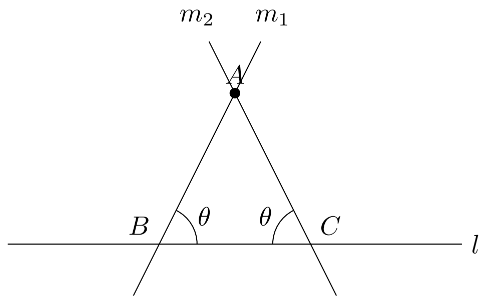
An implication of this preposition is that for any given acute angle θ , there is exactly one point C that is at the left of B such that $\angle ACB = \theta$. (And also exactly one point at the right of B for that.) Otherwise, say, if there are two distinct points C and C' at the left of B for that, with C' at the leftmost, then we have $\theta + \angle C'AC = \theta$ with $\angle C'AC > 0$, which violates the law of non-contradiction.

Note 1: If C is not on B, then $\angle ACB$ must be smaller than $90°$ because $\angle CAB > 0$, and $\angle ACB + \angle CAB = 90°$ (ext. $\angle$ of $\triangle$), which means $\angle ACB < 90°$.
Thus, we also have the alternative statement: if length BC is a variable over the domain $[0, \infty)$, then the smaller angle formed by AC and l is a strictly decreasing function of BC .

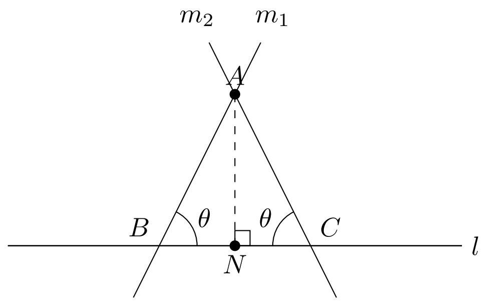
Note 2: If point A is below line l , then the preposition still holds because of symmetry (reflection postulate). I just state point A is above line l to simplify the statement.

[3] A function $f(x)$ is said to be strictly decreasing on an interval I if $f(b) < f(a)$ for all $b > a$, where $a, b \in I$.

Preposition 20. If there is a point A above line l , then for a given acute angle θ , there is exactly two lines m_1 and m_2 passing through A such that θ is the smaller angle formed between m_1 and l , and also m_2 and l . (property of falling lines)



Proof. Let N be the projection of A on l .(Alternatively, we can say 'draw $AN \perp l$ '.) Let m_1 , m_2 intersect l at B , C respectively.



By property of hypotenuse inclination, given an angle θ , for each side of line l divided by N , there is exactly one point B or C on l that satisfies $\angle ABN = \theta$ or $\angle ACN = \theta$. Since there is exactly one A, B and C , there is exactly one line AB and exactly one line AC .

An implication of this preposition is that for a given obtuse angle θ , there is exactly two lines m_1 and m_2 passing through A such that θ is the larger angle formed between m_1 and l , and also m_2 and l. This is because for every obtuse angle θ , there is a unique corresponding acute angle $180° - \theta$, which is the smaller angle formed by m and l. (m can be m_1 or m_2.) , which means that the property of falling lines can be applied and there are two unique lines m_1 and m_2 that satisfy the requirement.

Note: The previous preposition's note also applies here.

□

1.2 Congruent triangles

Two triangles are called **congruent** if one triangle can be translated, rotated, and reflected in any way to perfectly overlap with another triangle. In real life analogy, if there are two triangles made of hard paper, and we can stack them another perfectly (flipping is allowed), then the triangles are congruent.

A pair of congruent triangles have the corresponding sides and corresponding angles that are equal:

To denote that the two triangles are congruent, we say that $\triangle ABC \cong \triangle DEF$. Note that the order of the corresponding vertices must be the same. We cannot say that $\triangle ABC \cong \triangle FED$.

Note that congruence ($\cong$) is an **equivalence relation** , meaning that it satisfies the three properties shared by equality:

1. $x \cong x$ (reflexive property)
2. If $x \cong y$, then $y \cong x$ (symmetric property)
3. If $x \cong y$ and $y \cong z$, then $x \cong z$ (transitive property)

Conditions for determining congruence

In practice, we don't need to know that all of the sides and angles are corresponding in order to determine that two triangles are congruent, and there are a couple of minimum conditions that are sufficient to determine congruence.

1.2.1 SAS (Side-Angle-Side)

For side-angle-side condition, in the same triangle, the corresponded angle must be between the two corresponded sides:

This is also allowed:

In the figure, we have $AB = DE$, $\angle ABC = \angle DEF$, $BC = EF$. Thus $\triangle ABC \cong \triangle DEF$ (SAS) *.

Proof. (Proof of congruence) If $\triangle DEF$ is the flipped version of $\triangle ABC$, then we can just reflect $\triangle DEF$ once since reflection is allowed for congruence. And by reflection postulate, reflection preserves side lengths and angle sizes. So we only need to look at the case that a triangle is not the flipped version of another.

If we translate and rotate $\triangle ABC$ such that vertex B coincide with vertex E and vertex C lies on line EF, then C coincides with E because $BC = EF$. By polar coordinate postulate, given an angle $\angle ABC$ and a length AB , there is a unique position of A above BC. And since $AB = DE$ and $\angle ABC = \angle DEF$, it must be the case that A is in the same position as D . Since all the vertices coincide, all the sides must also coincide. Thus, $\triangle ABC \cong \triangle DEF$.

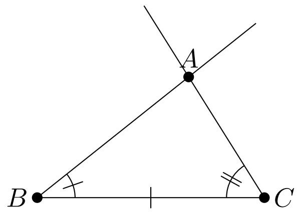
□

1.2.2 ASA (Angle-Side-Angle)

For angle-side-angle, in the same triangle, the corresponded side is the common side shared by the two corresponded angles

In the figure, we have $BC = EF$, $\angle ABC = \angle DEF$, $\angle ACB = \angle DFE$. Thus $\triangle ABC \cong \triangle DEF$ (ASA) *.

Proof. (We neglect the situation where a triangle is the flipped version of another since if so, we just need to flip a triangle back.) By protractor postulate, for a given line segment BC and a given value of $\angle ABC$, there is a unique ray such that the clockwise angle between BC and the ray is equal to $\angle ABC$. Same goes for $\angle ACB$ (but the clockwise angle is the reflex angle). The two rays BA and CA must intersect at one point if $\triangle ABC$ is a valid triangle:



The position (relative to BC) of this point is necessarily unique when given $\angle ABC, \angle ACB$ and segment BC , as we observe that placing A in any other position will cause at least one corresponded angle to change (since that will make A not lie on at least one of the original rays). Thus, if we overlap segment BC with EF , point A and point D must also coincide. Since all three vertices coincide, it must be the case that $\triangle ABC \cong \triangle DEF$. □

1.2.3 AAS (Angle-Angle-Side)

For angle-angle-side, in the same triangle, the corresponded side is not the common side shared by the two corresponded angles, and can be any one of the non-common sides.

In the figure, we have $AB = DE$, $\angle ABC = \angle DEF$, $\angle ACB = \angle DFE$. Thus $\triangle ABC \cong \triangle DEF$ (AAS) *.

Proof. Suppose we have two triangles, $\triangle ABC$ and $\triangle DEF$, in which $AB = DE$, $\angle ABC = \angle DEF$, $\angle ACB = \angle DFE$. Note that

$$\begin{aligned}\angle BAC &= 180^\circ - \angle ABC - \angle ACB \qquad (\angle \text{ sum of } \triangle)\\ &= 180^\circ - \angle DEF - \angle DFE\\ &= \angle EDF \qquad (\angle \text{ sum of } \triangle)\end{aligned}$$

So when there are two corresponding angles in two triangles, the third angle is also corresponding.

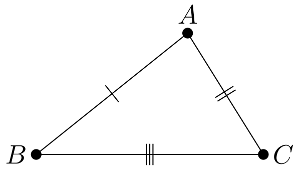
Note that we now have an angle-side-angle situation:

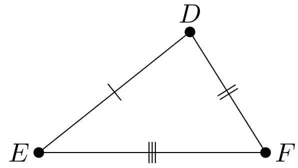
$$\begin{aligned}\angle ABC &= \angle DEF \qquad \text{(given)}\\ AB &= DE \qquad \text{(given)}\\ \angle BAC &= \angle EDF \qquad (\angle \text{ sum of } \triangle)\\ \therefore\ \triangle ABC &\cong \triangle DEF \qquad \text{(ASA)}\end{aligned}$$

□

1.2.4 SSS (Side-Side-Side)

For side-side-side, three sides are corresponding sides.

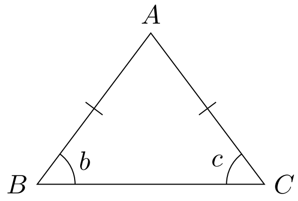




In the figure, we have $AB = DE$, $AC = DF$, $BC = EF$. Thus $\triangle ABC \cong \triangle DEF$ (SSS) *.

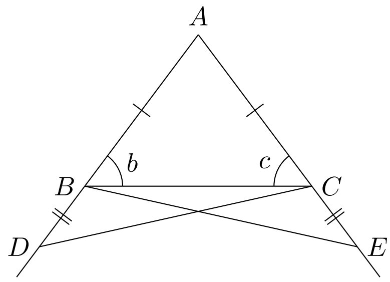
Proof. First we prove the preposition of 'base $\angle$s, isos. $\triangle$'

Preposition. The base angles of an isosceles triangle are equal. (base $\angle$s, isos. $\triangle$)



$$b = c \qquad \text{(base } \angle\text{s, isos. } \triangle)$$

<u>Proof.</u> [3] Let there be $\triangle ABC$ where $AB = AC$. Extend AB and AC into rays. Pick an arbitrary point on ray AB below B called D. Let E be on ray AC below C such that $BD = CE$.



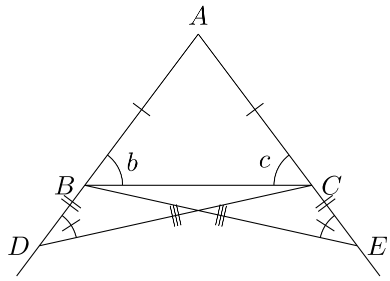
In $\triangle ADC$ and $\triangle AEB$,

$AC = AB$ (given)
$AD = AB + BD = AC + CE = AE$ (segment addition postulate + substitution of equals)
$\angle CAD = \angle BAE$ (common $\angle$)
$\therefore\ \triangle ABC \cong \triangle ACB$ (SAS)

Since the two triangles are congruent, all the corresponding angles of the triangles are equal.

We have $CD = BE$ (corr. sides, $\cong \triangle$s) ,
$\angle ADC = \angle AEB$ (corr. $\angle$s, $\cong \triangle$s)

Updated figure:



In $\triangle BDC$ and $\triangle CEB$,

$$\begin{aligned} BD &= CE && \text{(constructed)} \\ \angle BDC &= \angle CEB && \text{(proven)} \\ DC &= EB && \text{(proven)} \\ \therefore\ \triangle BDC &\cong \triangle CEB && \text{(SAS)} \end{aligned}$$

$\therefore\ \angle BCD = \angle CBE$ (corr. $\angle$s, $\triangle BDC \cong \triangle CEB$)

Note that we also have $\angle ACD = \angle ABE$ (corr. $\angle$s, $\triangle ADC \cong \triangle AEB$) .

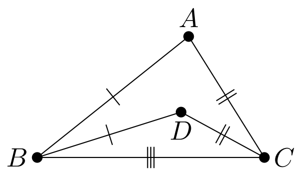
Thus,

$$\begin{aligned}\angle ABC &= \angle ABE - \angle CBE \qquad \text{(angle addition postulate)}\\ &= \angle ACD - \angle BCD \qquad \text{(substitution of equals)}\\ &= \angle ACB \qquad \text{(angle addition postulate)}\end{aligned}$$

Thus $b = c$, and the preposition 'the base angles of an isosceles triangle are equal' is proven.

Now back to our side-side-side business. Suppose we have two side-side-side corresponding triangle $\triangle ABC$ and $\triangle DEF$. Move (meaning translate and rotate) $\triangle DEF$ such that side EF coincides with side BC , and both A and D are above BC. Suppose that vertex D does not coincide with vertex A . There are 4 possibilities:

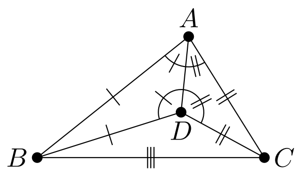
1. D lies inside $\triangle ABC$



Connect AD .

Note that $\angle BAC < 180°$ and $\angle BDC < 180°$ since they are interior angles of a triangle. Thus reflex$\angle BDC = 360° - \angle BDC > 180°$ ($\angle$s at a pt.).

Note that $\angle BAD = \angle BDA$, and $\angle CAD = \angle CDA$ (base $\angle$s, isos. $\triangle$).

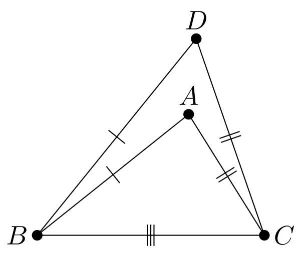


Also note that $\angle BAC = \angle BAD + \angle CAD$, and reflex$\angle BDC = \angle BDA + \angle CDA$. By substitution of equals,

$$\text{reflex}\angle BDC = \angle BDA + \angle CDA = \angle BAD + \angle CAD = \angle BAC$$

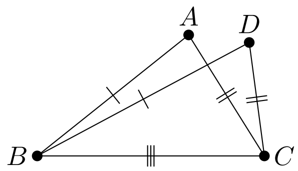
But $\angle BAC < 180°$ while reflex$\angle BDC > 180°$, which means both $\angle BAC < 180°$ and $\angle BAC > 180°$ are true, which violates the law of trichotomy. Thus, it cannot be the case that D lies inside $\triangle ABC$.

2. A lies inside $\triangle DBC$

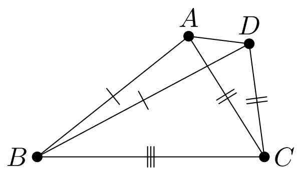


We can use arguments similar to case 1 to arrive at the conclusion that $\angle BDC < 180°$ and reflex$\angle BAC > 180°$ with $\angle BDC =$ reflex$\angle BAC$, violating the law of trichotomy. Thus, it cannot be the case that A lies inside $\triangle ADC$.

3. D lies to the right of line AB [4]

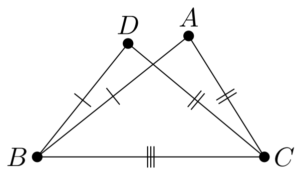


Connect AD . Since $AB = DB$, we have $\angle BAD = \angle BDA$ (base $\angle$s, isos. $\triangle$). Since $AC = DC$, we have $\angle CAD = \angle CDA$ (base $\angle$s, isos. $\triangle$).



We have $\angle BAD > \angle CAD$ since AC is between the angle $\angle BAD$. Similarly, $\angle CDA > \angle BDA$. Substituting $\angle CDA = \angle CAD$ and $\angle BDA = \angle BAD$, we have $\angle CAD > \angle BAD$. But this is impossible since we have both $\angle BAD > \angle CAD$ and $\angle CAD > \angle BAD$, which violates the law of trichotomy. Thus, it cannot be the case that D lies to the right of line AB .

4. D lies to the left of line AB



We can use arguments similar to case 3 to arrive at the conclusion that $\angle BDA > \angle CDA$ and $\angle CDA > \angle BDA$, which violates the law of trichotomy. Thus, it cannot be the case that D lies to the left of line AB .

Therefore, the only possible position of D is the same position as A , so A and D coincide. Since the three vertices of the triangles coincide, we have $\angle ABC \cong \angle DEF$.

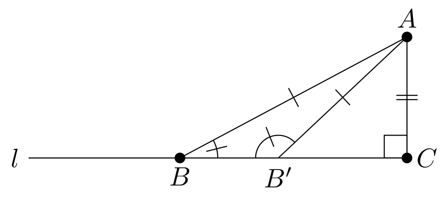
□

1.2.5 RHS (Right Angle-Hypotenuse-Side)

For right angle-hypotenuse-side, the corresponding angle is a right angle, and there are (any) two corresponding sides.

In the figure, we have $AB = DE$, $AC = DF$, $\angle ACB = \angle DFE = 90°$. Thus $\triangle ABC \cong \triangle DEF$ (RHS) *.

Proof. Note that for a given line segment AC and a given angle $90°$, there is a unique ray l such that the clockwise angle between AC and l is $90°$. Let B be a point on l . We want to show that for a given length AB , there is a unique position of B on l.



Suppose B' is a point between B and C such that $AB = AB'$. Then $\angle ABB' = \angle AB'B$ (base $\angle$s, isos. $\triangle$), and $\angle AB'C = 180° - \angle AB'B$ (adj. $\angle$s on st. line) $= 180° - \angle ABB'$.

Note that $\angle AB'C < 90°$ (property of hypotenuse inclination), so we have $180° - \angle ABB' < 90°$, which means $\angle ABB' > 90°$. But that cannot be true because $\angle ABB' < 90°$ (property of hypotenuse inclination). Law of trichotomy is violated. Thus it cannot be the case that B' is between B and C.

Now suppose B' is a point at the left of B such that $AB = AB'$. By similar argument to above, we can arrive at the conclusion that $\angle AB'B < 90°$ and $AB'B > 90°$, which violates the law of trichotomy. Thus it cannot be the case that B' is at the left of B.

Therefore there is a unique position of B when given length AB , line segment AC and clockwise angle $\angle ACB = 90°$, and two triangles with RHS correspondence must coincide, making them congruent triangles. □

1.2.6 Special case

ASS (Angle-Side-Side) with special conditions

For angle-side-side, there are a corresponding angle and two corresponding sides. Generally, this is not enough to determine congruence since there are two possibilities for a triangle given two sides and an angle.

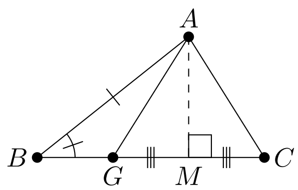
Suppose that we are given two triangles $\triangle ABC$ and $\triangle DEF$ in which $\angle ABC = \angle DEF$, $AB = DE$, $AC = DF$. There are several cases to consider:

1. $\angle ACB > \angle ABC$ where $\angle ACB \neq 90°$

Type 1 Type 2

In the figure, we have $\angle ABC = \angle DEF$, $AB = DE$, $AC = DF$. If $\angle ACB > \angle ABC$ where $\angle ACB \neq 90°$, then $\triangle ABC$ and $\triangle DEF$ may or may not be congruent, as there exists two unique type of triangles when given an angle $\angle ABC$, a side AB and a side AC . Suppose $\triangle ABC$ and $\triangle DEF$ are different types of triangle. Then $\angle DFE$ must be $180° - \angle ACB$. (ASS case 1)

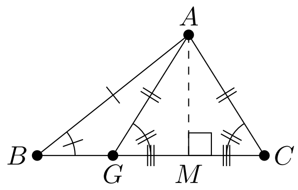
Proof. First consider the case that $\angle ACB$ is an acute angle.



Draw $AM \perp BC$. Let G be a point distinct from C on line BM such that $GM = MC$.
In $\triangle GMA$ and $\triangle CMA$,

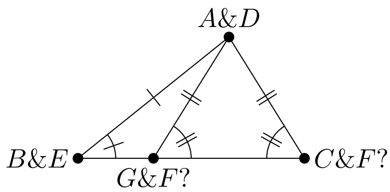
$$\begin{aligned} GM &= MC \qquad \text{(constructed)} \\ \angle GMA &= \angle CMA = 90° \qquad \text{(constructed)} \\ AM &= AM \qquad \text{(common side)} \\ \therefore\ \triangle GMA &\cong \triangle CMA \qquad (SAS) \end{aligned}$$

Thus, $AG = AC$ (corr. sides, $\cong \triangle$s) and $\angle AGM = \angle ACM$ (corr. $\angle$s, $\cong \triangle$s) .



Note that $\angle ACB > \angle ABC$ by initial assumption, so we have $\angle AGC > \angle ABC$. This means that G must lie between B and M (property of hypotenuse inclination), making $\triangle ABG$ a valid triangle.

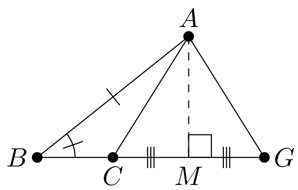
Beside point G and C , there must be no other distinct points N on line BC such that $AN = AC$ (by RHS proof).



If we move side EF of $\triangle DEF$ to coincide with line BC of $\triangle ABC$, A and D must also coincide by polar coordinate postulate. Since vertice F of $\triangle DEF$ must either lie on point G or point C in the figure above, $\triangle DEF$ must be either a 'type 1' triangle or a 'type 2' triangle mentioned above.

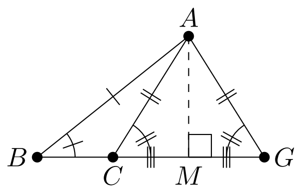
Suppose $\triangle DEF$ is a 'type 2' triangle (thus different type from $\triangle ABC$). Then $\angle DFE = 180° - \angle AGC$ (adj. $\angle$s on st. line) $= 180° - \angle ACB$, as desired.

Let's now consider the case that $\angle ACB$ is an obtuse angle.



Draw $AM \perp$ line BC . Let G be a point distinct from C on line BM such that $GM = MC$. We have $\triangle CMA \cong \triangle GMA$ (SAS).

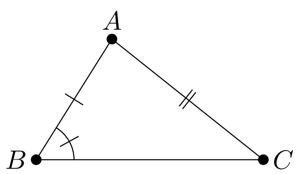
Thus, $AG = AC$ (corr. sides, $\cong \triangle$s) and $\angle AGM = \angle ACM$ (corr. $\angle$s, $\cong \triangle$s) .



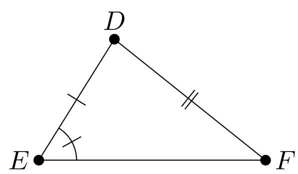
The argument proceeds similar to the acute angle case. Let the two triangles overlap at one side. The vertex F of $\triangle DEF$ can lie on either C or G . Suppose vertex F lies on G . Then $\angle DFE = \angle DCF = 180° - \angle ACB$ (adj. $\angle$s on st. line).

□

2. $\angle ACB < \angle ABC$



Type 1

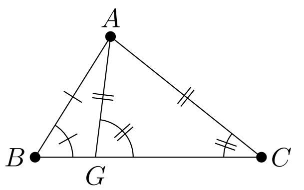


Also type 1

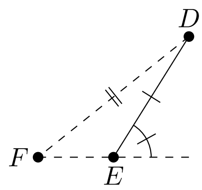
In the figure, we have $\angle ABC = \angle DEF$, $AB = DE$, $AC = DF$. If $\angle ACB < \angle ABC$, then $\triangle ABC \cong \triangle DEF$. (ASS case 2)

Proof. Note that $\angle ACB < 90°$, or otherwise we will also have $\angle ABC \geq 90°$ and $\angle ACB + \angle ABC \geq 180°$, which violates (2 $\angle$ sum of $\triangle$) .

Suppose given ASS correspondence where $\angle ACB < \angle ABC$, we have two unique type of triangles. Then similar to the proof of ASS case 1 acute situation, we can uniquely make a point G between B and C such that $AG = AC$. Then $\angle AGC = \angle ACB$ (base $\angle$s, isos. $\triangle$).



Imagination



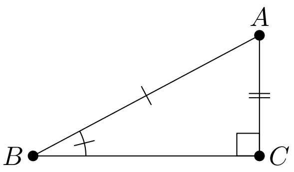
Reality

In $\triangle ABG$, we have $\angle ABG + \angle BAG = \angle AGC$, so $\angle AGC > \angle ABC$, and thus $\angle ACB > \angle ABC$. But we have assumed that $\angle ACB < \angle ABC$, so this contradicts the law of trichotomy. Thus, there must not be more than one unique type of triangle.

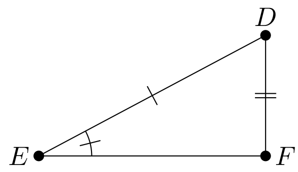
Since there is only one unique type of triangle, when we try to overlap $\triangle ABC$ and $\triangle DEF$, they must completely coincide, making them congruent triangles.

□

3. $\angle ACB = 90°$



Type 1



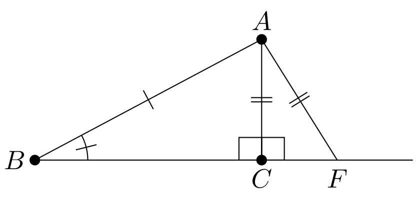
Also type 1

In the figure, we have $\angle ABC = \angle DEF$, $AB = DE$, $AC = DF$. If $\angle ACB = 90°$, then $\triangle ABC \cong \triangle DEF$. (ASS case 3)

Proof. By polar coordinate postulate, given angle $\angle ABC$ and length of line segment AB , there is a unique position of A above line BC . Since there is only one unique line segment with endpoint A that is perpendicular to BC (property of perpendicular line), the position of C and the length of AC can also be uniquely determined.

If we move vertex E to coincide with B such that F is on line BC, then D must coincide with A (polar coordinate postulate). Since $DF = AC$ and F is on BC, F must also coincide with C.

Otherwise, suppose F does not coincide with C . Let's say it is at the right of C.



Then $\angle ACF = \angle AFC = 90°$ (base $\angle$s, isos. $\triangle$). But this means in $\triangle ACF$, the sum of two interior angles $\angle ACF + \angle AFC = 180°$, which violates (2 $\angle$ sum of $\triangle$) . If F is at the left of C instead, the same thing will happen. Thus, it must be the case that F coincides with C . □

1.3 Triangle properties

Let’s summarize the conditions for congruent triangles in a preposition:

Preposition 21. Two triangles are congruent if one of the conditions holds:
SAS, ASA, AAS, SSS, RHS .

1. SAS (Side-Angle-Side)

If $AB = DE$, $\angle ABC = \angle DEF$, $BC = EF$, then $\triangle ABC \cong \triangle DEF$ (SAS) *.

2. ASA (Angle-Side-Angle)

If $BC = EF$, $\angle ABC = \angle DEF$, $\angle ACB = \angle DFE$, then $\triangle ABC \cong \triangle DEF$ (ASA) *.

3. AAS (Angle-Angle-Side)

If $AB = DE$, $\angle ABC = \angle DEF$, $\angle ACB = \angle DFE$, then $\triangle ABC \cong \triangle DEF$ (AAS) *.

4. SSS (Side-Side-Side)

If $AB = DE$, $AC = DF$, $BC = EF$, then $\triangle ABC \cong \triangle DEF$ (SSS) *.

5. RHS (Right Angle-Hypotenuse-Side)

If $AB = DE$, $AC = DF$, $\angle ACB = \angle DFE = 90^\circ$, then $\triangle ABC \cong \triangle DEF$ (RHS) *.

Preposition 22. Two triangles are congruent if they have ASS correspondence with one of the additional conditions:

1. Non-corresponded & non-included side is smaller than corresponded side

   If $\angle ABC = \angle DEF$, $AB = DE$, $AC = DF$, $\angle ACB < \angle ABC$, then $\triangle ABC \cong \triangle DEF$. (ASS case 2)

2. Non-corresponded & non-included side is right angle

   If $\angle ABC = \angle DEF$, $AB = DE$, $AC = DF$, $\angle ACB = 90°$, then $\triangle ABC \cong \triangle DEF$. (ASS case 3)

   If they have the following condition, the triangles may or may not be congruent.

3. Non-corresponded & non-included side is larger than corresponded side

   Type 1 Type 2

   If $\angle ABC = \angle DEF$, $AB = DE$, $AC = DF$, $\angle ACB > \angle ABC$, $\angle ACB \neq 90°$, then $\triangle ABC$ and $\triangle DEF$ may or may not be congruent.

   Suppose $\triangle ABC$ and $\triangle DEF$ are not congruent. Then $\angle DFE = 180° - \angle ACB$. (ASS case 1)

Preposition 23. If two triangles are congruent, then:

- Their corresponding sides are equal. (corr. sides, $\cong \triangle$s)*
- Their corresponding angles are equal. (corr. $\angle$s, $\cong \triangle$s)*

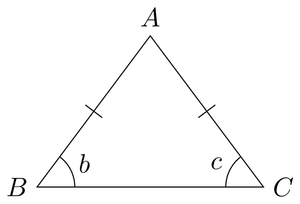
Observation: $\triangle ABC \cong \triangle DEF$

$\therefore AB = DE$, $BC = EF$, $AC = DF$ (corr. sides, $\cong \triangle$s),

$\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$ (corr. $\angle$s, $\cong \triangle$s)

Proof. If two triangles are congruent, then they can be moved (and flipped) to completely coincide. Thus all the corresponding line segments and angles coincide. By the common notion of 'coincidable line segments and angles are equal' , the corresponding line segments and angles are equal. ☐

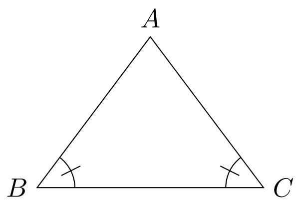
Preposition 24. The base angles of an isosceles triangle are equal. (base $\angle$s, isos. $\triangle$)



$$b = c \qquad \text{(base } \angle\text{s, isos. } \triangle\text{)}$$

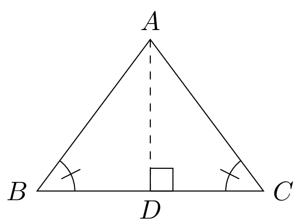
For the proof, refer to Section 1.2.4 (it's long).

Preposition 25. If two angles of a triangle are equal, then the triangle is an isosceles triangle. (sides opp. equal $\angle$s)



$$AB = AC \qquad \text{(sides opp. equal } \angle\text{s)}$$

Proof. Draw $AD \perp BC$.



In $\triangle ABD$ and $\triangle ACD$,

$$\begin{aligned}
\angle ABD &= \angle ACD \qquad \text{(given)} \\
\angle BDA &= \angle CDA = 90^\circ \qquad \text{(constructed)} \\
AD &= AD \qquad \text{(common side)} \\
\therefore\ \triangle ABD &\cong \triangle ACD \qquad \text{(AAS)} \\
\therefore\ AB &= AC \qquad \text{(corr. sides, } \cong \triangle\text{s)}
\end{aligned}$$

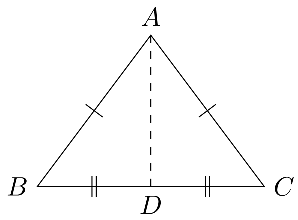
☐

Preposition 26. For an isosceles triangle $\triangle ABC$ with $AB = AC$ and D on side BC, if one of the following conditions is true, then the other two conditions are also true:

1. $BD = DC$
2. $\angle BAD = \angle CAD$
3. $AD \perp BC$

(prop. of isos. $\triangle$)

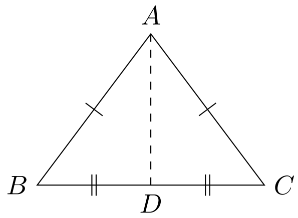
<u>Example</u>



$$\text{Observation: } AB = AC \text{ and } BD = DC$$
$$\therefore\ \angle BAD = \angle CAD \text{ and } AD \perp BC \qquad \text{(prop. of isos. } \triangle)$$

Proof. Let there be an isosceles triangle $\triangle ABC$ with $AB = AC$ and D on side BC . Let's look at what happens for each condition being true.

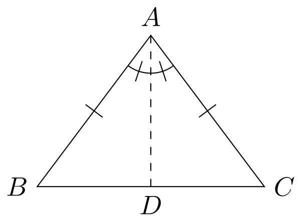
1. $BD = DC$



In $\triangle ABD$ and $\angle ACD$,

$$\begin{aligned}
AB &= AC && \text{(given)}\\
BD &= DC && \text{(given)}\\
AD &= AD && \text{(commomn side)}\\
\therefore\ \triangle ABD &\cong \triangle ACD && \text{(SSS)}\\
\therefore\ \text{(condition 2)}\quad \angle BAD &= \angle CAD && \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}\\
\angle ADB &= \angle ADC && \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}\\
\angle ADB &= \angle ADC = 180^\circ/2 = 90^\circ && \text{(adj. } \angle\text{s on st. line)}\\
\therefore\ \text{(condition 3)}\quad AD &\perp BC
\end{aligned}$$

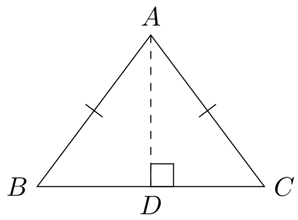
2. $\angle BAD = \angle CAD$



In $\triangle ABD$ and $\triangle ACD$,

$$\begin{aligned}
AB &= AC && \text{(given)}\\
\angle BAD &= \angle CAD && \text{(given)}\\
AD &= AD && \text{(common side)}\\
\therefore\ \triangle ABD &\cong \triangle ACD && \text{(SAS)}\\
\therefore\ \text{(condition 1)}\quad BD &= DC && \text{(corr. sides, } \cong \triangle\text{s)}\\
\angle ADB &= \angle ADC && \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}\\
\therefore\ \text{(condition 3)}\quad AD &\perp BC
\end{aligned}$$

3. $AD \perp BC$

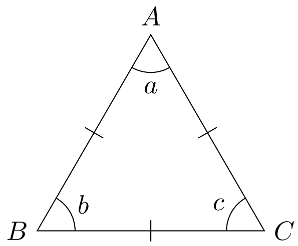


In $\triangle ABD$ and $\triangle ACD$,

$$\begin{aligned}
\angle ADB = \angle ADC = 90° \qquad & (AD \perp BC) \\
AB = AC \qquad & \text{(given)} \\
AD = AD \qquad & \text{(common side)} \\
\therefore\ \triangle ABD \cong \triangle ACD \qquad & \text{(RHS)} \\
\therefore\ \text{(condition 1)} \quad BD = DC \qquad & \text{(corr. sides, } \cong \triangle\text{s)} \\
\text{(condition 2)} \quad \angle BAD = \angle CAD \qquad & \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}
\end{aligned}$$

□

Preposition 27. Each interior angle of an equilateral triangle is 60°. (prop. of equil. △) *



$$a = b = c = 60° \qquad \text{(prop. of equil. } \triangle)$$

Proof.

$$\begin{aligned}
AB = AC \qquad & \text{(given)} \\
\therefore\ b = c \qquad & \text{(base } \angle\text{s, isos. } \triangle) \\
BC = BA \qquad & \text{(given)} \\
\therefore\ c = a \qquad & \text{(base } \angle\text{s, isos. } \triangle) \\
\therefore\ a = b = c & \\
a + b + c = 180° \qquad & (\angle \text{ sum of } \triangle) \\
\therefore\ a = b = c = 180°/3 = 60° &
\end{aligned}$$

□

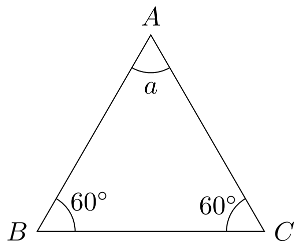
Preposition 28. A triangle is an equilateral triangle if it satisfies one of the following conditions:

1. Two angles are 60° .
2. The triangle is isosceles with one 60° angle.
3. Two angles are equal and one angle is 60° (the 60° angle may or may not be in the equal pair).
4. Three angles are equal.

(con. of equil. △)

Proof. Let there be $\triangle ABC$. Let's consider the conditions.

1. $\angle B = \angle C = 60°$.



$\angle B = \angle C = 60°$ (given)

$AB = AC$ (sides opp. equal $\angle$s)

$\angle A = 180° - 60° - 60° = 60°$ ($\angle$ sum of $\triangle$)

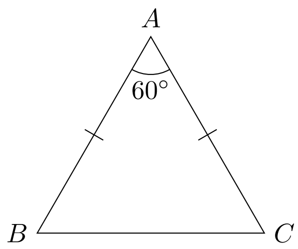
$\therefore\ \angle C = \angle A = 60°$

$\therefore\ BC = BA$ (sides opp. equal $\angle$s)

$\therefore\ AB = AC = BC$

$\therefore\ \triangle ABC$ is an equil. $\triangle$.

2a. $AB = AC$ with $\angle A = 60°$.



$AB = AC$ (given)

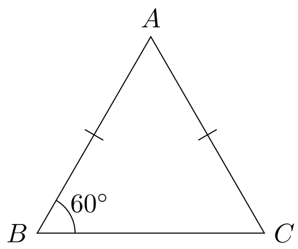
$\angle B = \angle C$ (base $\angle$s, isos. $\triangle$)

$\angle B + \angle C + 60° = 180°$ ($\angle$ sum of $\triangle$)

$\angle B = \angle C = (180° - 60°)/2 = 60°$

$\therefore\ \triangle ABC$ is an equil. $\triangle$ (condition 1 of this preposition)

2b. $AB = AC$ with $\angle B = 60°$.

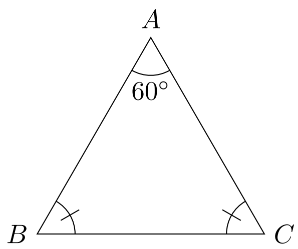


$AC = AB$ (given)

$\angle C = \angle B = 60°$ (base $\angle$s, isos. $\triangle$)

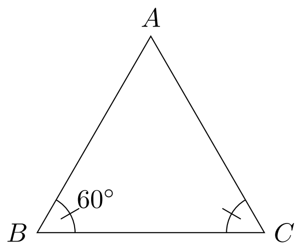
$\therefore\ \triangle ABC$ is an equil. $\triangle$ (condition 1 of this preposition)

3a. $\angle B = \angle C$ with $\angle A = 60°$.



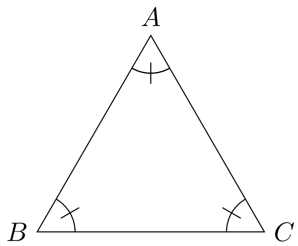
$$\angle B = \angle C \qquad \text{(given)}$$
$$\angle B + \angle C + 60^\circ = 180^\circ \qquad (\angle \text{ sum of } \triangle)$$
$$\angle B = \angle C = (180^\circ - 60^\circ)/2 = 60^\circ$$
$$\therefore\ \triangle ABC \text{ is an equil. } \triangle \qquad \text{(condition 1 of this preposition)}$$

3b. $\angle B = \angle C$ with $\angle B = 60^\circ$.



$$\angle C = \angle B = 60^\circ$$
$$\therefore\ \triangle ABC \text{ is an equil. } \triangle \qquad \text{(condition 1 of this preposition)}$$

4. $\angle A = \angle B = \angle C$

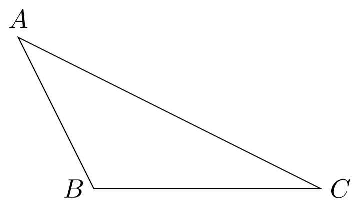


$$\angle A = \angle B = \angle C \qquad \text{(given)}$$
$$\angle A + \angle B + \angle C = 180^\circ \qquad (\angle \text{ sum of } \triangle)$$
$$\angle A = \angle B = \angle C = 180^\circ/3 = 60^\circ$$
$$\therefore\ \triangle ABC \text{ is an equil. } \triangle \qquad \text{(condition 1 of this preposition)}$$

□

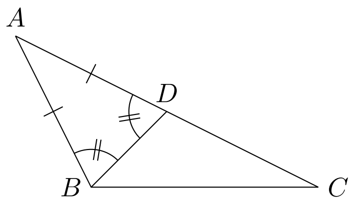
Preposition 29. In a triangle, the longer side subtends the larger angle. (longer side, larger $\angle$)

In other words, in a triangle, the greater side has larger opposite angle.



Observation: $AC > AB$
$\therefore\ \angle B > \angle C$ (longer side, larger $\angle$)

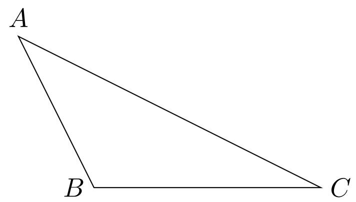
Proof. Let $\triangle ABC$ be a triangle where $AC > AB$. Let D be a point on side AC such that $AB = AD$. Connect BD .



Then $\angle ADB$ is an exterior angle of $\triangle BCD$. Thus $\angle ADB > \angle ACB$ (ext. $\angle$ of $\triangle$ > int. opp. $\angle$) .

Note that $\angle ABD = \angle ADB$ (base $\angle$s, isos. $\triangle$). So $\angle ABD > \angle ACB$.
Since $\angle ABC > \angle ABD$, we have $\angle ABC > \angle ACB$ (transitive property of inequality). □

Preposition 30. In a triangle, the larger angle subtends the longer side. (larger $\angle$, longer side)
In other words, in a triangle, the larger the angle, the longer the opposite side.



Observation: $\angle B > \angle C$
$\therefore$ $AC > AB$ (longer $\angle$, larger side)

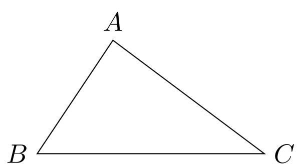
Proof. Let $\triangle ABC$ be a triangle where $\angle B > \angle C$.

Suppose that AC is not longer AB . If $AC = AB$, then $\angle B = \angle C$ (base $\angle$s, isos. $\triangle$), which contradicts the initial assumption $\angle B > \angle C$. So it cannot be the case that $AB = AC$.

If $AC < AB$, then by 'longer side, larger $\angle$', we have $\angle C > \angle B$, which contradicts the initial assumption $\angle B > \angle C$. So it cannot be the case that $AB = AC$.

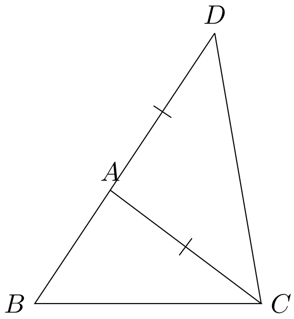
Thus it can only be the case that $AC > AB$. □

Preposition 31. In a triangle, the sum of lengths of any two sides is greater than the length of the remaining side. (triangle inequality)



$$AB + AC > BC$$
$$AB + BC > AC$$
$$AC + CB > AB$$
(triangle inequality)

Proof. Extend BA past A. Make a point D on line BA above A such that $AD = AC$.

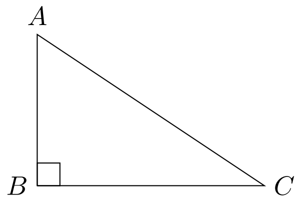


Note that $\angle ACD = \angle ADC$ (base $\angle$s, isos. $\triangle$). Note that $\angle BCD > \angle ACD$, so $\angle BCD > \angle ADC$.

Since $\angle BCD > \angle BDC$, we have $BD > BC$ (larger $\angle$, longer side) .
But $BD = BA + AD$ and $AD = AC$. Thus, $BD = BA + AC$.
Replace BD with $BA + AC$ in the inequality $BD > BC$, we get $BA + AC > BC$.

We can use similar argument with the other two sides to get the rest of the inequalities. □

Preposition 32. In a right triangle, the hypotenuse is the longest side. (hypotenuse is longest side of ◺)



$$AC > AB \text{ and } AC > BC \qquad \text{(hypotenuse is longest side of ◺)}$$

Proof. Note that in a right triangle, the right angle must be the largest angle. Otherwise, say, $\angle ABC = 90°$ and $\angle BAC \geq 90°$, then $\angle ABC + \angle BAC \geq 180°$, which violates '2 $\angle$ sum of $\triangle$'.

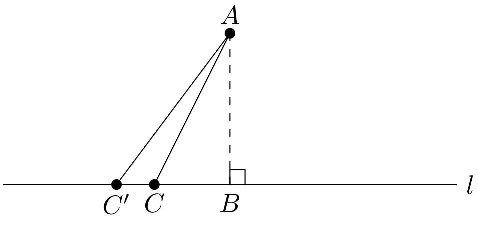
By 'larger $\angle$, longer side' , in a triangle, the largest angle must have the longest opposite side. In a right triangle, the opposite side of the right angle is the hypotenuse, so the hypotenuse must be the longest side.

□

Preposition 33. Suppose B is a point on line l and A is a point vertically above B (meaning $AB \perp l$) . If C is a point on l that is not B, then the longer BC is, the longer AC is. (property of hypotenuse length)

In other words, if length BC is a variable over the domain $(0, \infty)$, then AB is a strictly increasing function of BC .

Proof. Assume that C is at the left of B. Let C' be a point on l to the left of C. So $C'B > CB$.



In $\triangle ACC'$, since ACC' is an obtuse angle while $AC'C$ is an acute angle, we have $ACC' > AC'C$. By 'larger $\angle$, longer side' , we have $AC' > AC$.

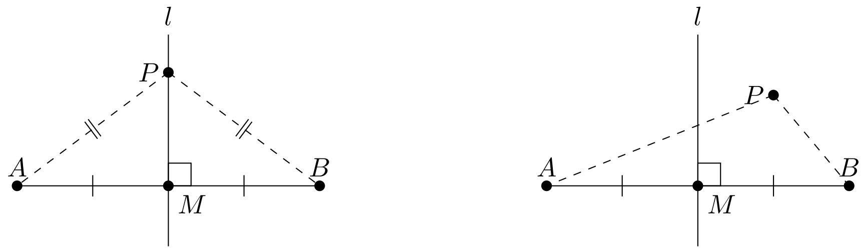
If C is at the right of B , then we can let C' be at the right of C and use similar reasoning to show that $AC' > AC$.

□

An implication of this preposition is that if there are a point not on a line, then the shortest distance between the point and the line is length of the line segment perpendicular to that line with that point as the endpoint.

Preposition 34. For a line segment AB and a point P on the same plane, $PA = PB$ if and only if P lies on the perpendicular bisector of AB .

If and only if P lies on the same side of the perpendicular bisector as an endpoint, then the distance between P and this endpoint and shorter than the distance from the other endpoint. (prop. of $\perp$ bisector)



Case 1 Case 2

(Let l be the perpendicular bisector of AB .)
Case 1a:

$$\because\ P \text{ is on line } l\ .$$
$$\therefore\ PA = PB \qquad \text{(prop. of } \perp \text{ bisector)}$$

Case 1b:

$$\because PA = PB$$
$$\therefore\ P \text{ is on line } l\ . \qquad \text{(prop. of } \perp \text{ bisector)}$$

Case 2a:

$$\because P \text{ is at the right of } l \text{ (same side as } B).$$
$$\therefore\ PB < PA \qquad \text{(prop. of } \perp \text{ bisector)}$$

Case 2b:

$$\because PB < PA$$
$$\therefore\ P \text{ is at the right of } l \qquad \text{(prop. of } \perp \text{ bisector)}$$

Proof. Let's consider each case:

Case 1a: P is on line l .

In $\triangle PMA$ and $\triangle PMB$,

$$AM = BM \qquad \text{(given)}$$
$$\angle PMA = \angle PMB = 90^\circ \qquad \text{(PM}\perp\text{ AB)}$$
$$PM = PM \qquad \text{(common side)}$$
$$\therefore\ \triangle PMA \cong \triangle PMB \qquad \text{(SAS)}$$
$$\therefore\ PA = PB \qquad \text{(corr. sides, } \cong \triangle\text{s)}$$

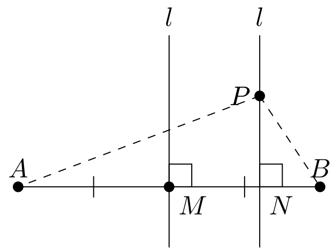
Case 1b: $PA = PB$

If P is on line segment AB , then P is actually the midpoint of AB , and since the perpendicular bisector of AB passes through midpoint of AB , P must lie on the perpendicular bisector (line l) .

If P is not on line segment AB , then $\triangle PAB$ is an isos. $\triangle$. By 'prop. of isos. $\triangle$' , if there is a point M on AB such that $AM = MB$, then $PM \perp AB$, making PM a perpendicular bisector of AB . Since for any given segment, there is a unique perpendicular bisector (since there is a unique midpoint and a unique perpendicular line passing through a given midpoint) , thus the perpendicular bisector of AB must pass through P .

Case 2a: P is at the right of l (same side as B).

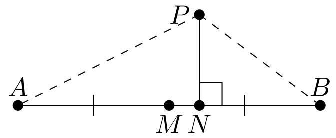
Draw $PN \perp AB$.



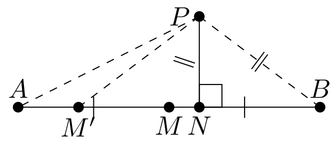
Note that $BN > AN$, so by property of hypotenuse length, we have $PB < PA$.

Case 2b: $PB < PA$

Draw $PN \perp AB$.



Since $PB < PA$, we have $\angle PAB < \angle PBA$ by (longer side, larger $\angle$). Since $\angle APN = 90° - \angle PAB$ and $\angle BPN = 90° - \angle PBA$ ($\angle$ sum of $\triangle$), we have $\angle APN > \angle BPN$ (subtractive property of inequality).



Make a point M' on line AB such that $NB = NM'$. Since $\angle PM'B = \angle PBA$ ($\triangle PNM' \cong \triangle PNB$), we have $\angle PAB > \angle PM'B$. Thus $PM'B$ is the exterior angle of $\triangle PAM'$ and M' lies between AB. Since $M'B = 2NB < AB$, we have $AN > NB$. Thus the perpendicular line passing through N and P must be at the right of the perpendicular bisector l , meaning P must be at the right of l .

□

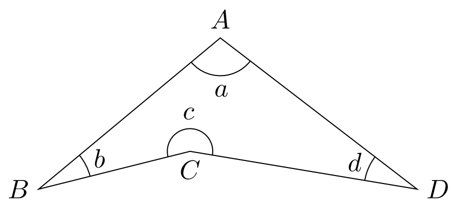
1.3.1 Problems

Time for some problems. (Cut due to runtime error)

1.4 Quadrilateral properties

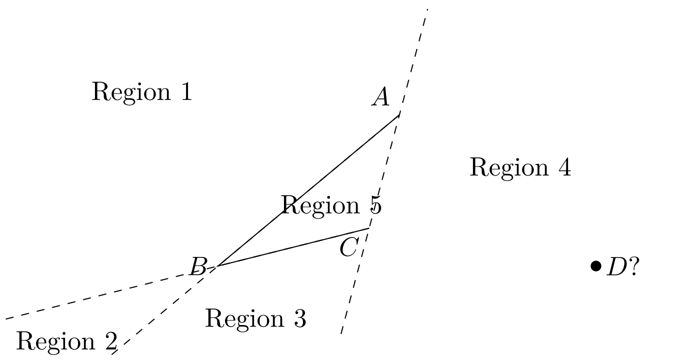
1.4.1 General properties

Preposition 35. The sum of interior angles of a quadrilateral is $360°$. ($\angle$ sum of quad.)



$$a + b + c + d = 360° \qquad (\angle \text{ sum of quad.})$$

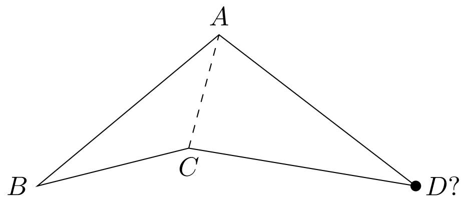
Proof. Note that every quadrilateral can be split into two triangles. To see why, arbitrarily choose three vertices of the quadrilateral and label them A, B, C . Suppose these vertices make side AB and BC . For the quadrilateral to be valid, the fourth vertex must be placed in a position such that any two sides will not intersect each other at a point other than the vertices.



Refer to the figure, the plane is split into 5 regions by the (dotted) lines (/rays). We see that D must either be placed in region 2, 4 or 5. Otherwise, say, D is in region 1, then side CD will intersect with AB at a point between A and B , which invalidates the quadrilateral.

If D is in region 4, then we can draw diagonal AC to split $ABCD$ into two triangles.

If D is in region 2 or 5 instead, then we can draw diagonal BD to split $ABCD$ into two triangles.

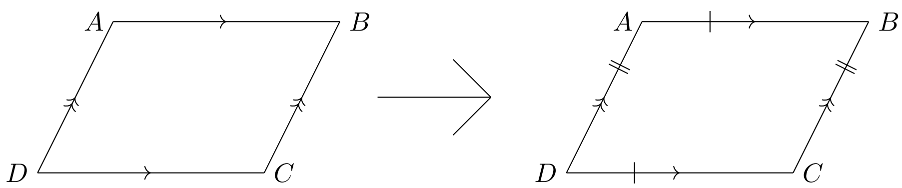


After splitting, the two triangles must share a common side. Note that the sum of interior angles of a triangle is $180°$, so the sum of interior angles of a triangle of two triangles is $360°$. But the interior angles of triangles combined are also the interior angles of the quadrilateral. So the sum of interior angles of a quadrilateral is $360°$. ☐

1.4.2 Parallelograms

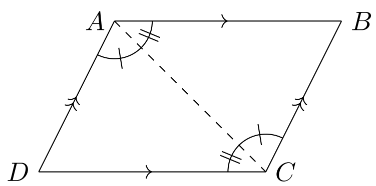
Properties of parallelogram:

Preposition 36. The opposite sides of a parallelogram are equal. (opp. sides of //gram) *



$$\because\ AB//DC \text{ and } AD//BC$$
$$\therefore\ AB = CD \text{ and } AD = BC \qquad \text{(opp. sides of //gram)}$$

Proof. Let there be parallelogram $ABCD$.

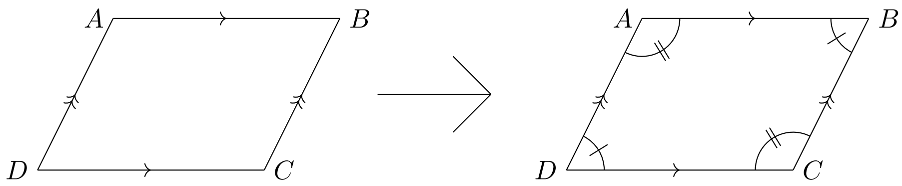


Join AC. In $\triangle ADC$ and $\triangle CBA$,

$$\begin{aligned} \angle ACD &= \angle CAB \qquad \text{(alt. } \angle \text{ s , } AB//DC) \\ AC &= AC \qquad \text{(common side)} \\ \angle CAD &= \angle ACB \qquad \text{(alt. } \angle \text{ s , } AD//BC) \\ \therefore\ \triangle ADC &\cong \triangle CBA \qquad \text{(ASA)} \\ \therefore\ AB = DC \ \text{ and } \ AD &= BC \qquad \text{(corr. sides, } \cong \triangle\text{s)} \end{aligned}$$

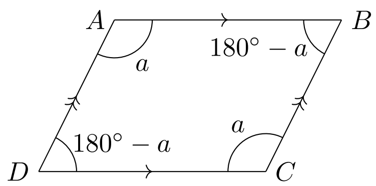
□

Preposition 37. The opposite angles of a parallelogram are equal. (opp. ∠s of //gram) *



$$\because\ AB//DC \text{ and } AD//BC$$
$$\therefore\ \angle ADC = \angle ABC \text{ and } \angle BAD = \angle BCD \qquad \text{(opp. } \angle\text{s of //gram)}$$

Proof. Let there be parallelogram $ABCD$.



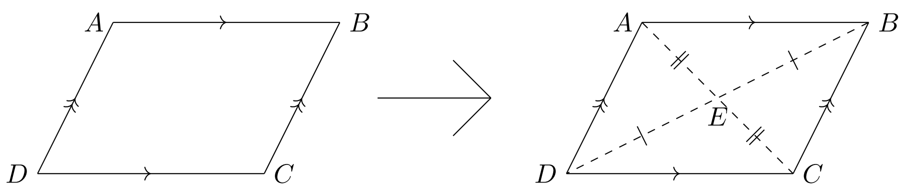
$$\begin{aligned} \angle A + \angle B &= 180^\circ \qquad \text{(int. } \angle\text{s , } AD//BC) \\ \angle B + \angle C &= 180^\circ \qquad \text{(int. } \angle\text{s , } AB//DC) \\ \therefore\ \angle A + \angle B &= \angle B + \angle C \\ \angle A &= \angle C \end{aligned}$$

Similarly,

$$\begin{aligned} \angle B + \angle C &= 180^\circ \qquad \text{(int. } \angle\text{s , } AB//DC) \\ \angle C + \angle D &= 180^\circ \qquad \text{(int. } \angle\text{s , } AD//BC) \\ \therefore\ \angle B + \angle C &= \angle C + \angle D \\ \angle B &= \angle D \end{aligned}$$

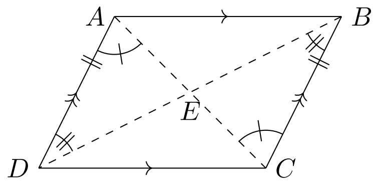
□

Preposition 38. The diagonals of a parallelogram bisect each other. (diags of //gram) *



$$\because\ AB//DC \text{ and } AD//BC$$
$$\therefore\ AE = EC \text{ and } DE = EB \qquad \text{(diags of //gram)}$$

Proof. Let there be parallelogram $ABCD$. Let diagonals AC and BD intersect at E .



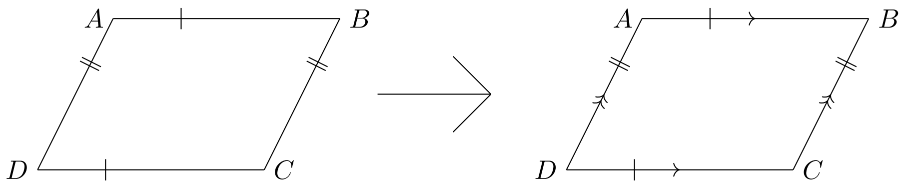
In $\triangle ADE$ and $\triangle CBE$,

$$\begin{aligned}
\angle DAE &= \angle BCE && \text{(alt. } \angle\text{s , } AD//BC\text{)} \\
AD &= BC && \text{(opp. sides of //gram)} \\
\angle ADE &= \angle CBE && \text{(alt. } \angle\text{s , } AD//BC\text{)} \\
\therefore\ \triangle ADE &\cong \triangle CBE && \text{(ASA)} \\
\therefore\ AE &= EC && \text{(corr. sides, } \cong \triangle\text{s)} \\
DE &= EB && \text{(corr. sides, } \cong \triangle\text{s)}
\end{aligned}$$

□

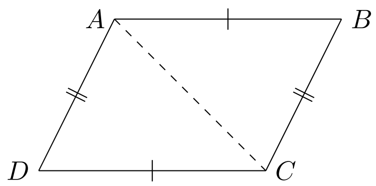
Conditions for determining a quadrilateral is a parallelogram:

Preposition 39. If there are two equal pairs of opposite sides in a quadrilateral, then the quadrilateral is a parallelogram. (opp. sides equal)



$$\because\ AB = DC \text{ and } AD = BC$$
$$\therefore\ ABCD \text{ is a //gram.} \qquad \text{(opp. sides equal)}$$

Proof. Let there be a quadrilateral $ABCD$ where $AB = DC$ and $AD = BC$.



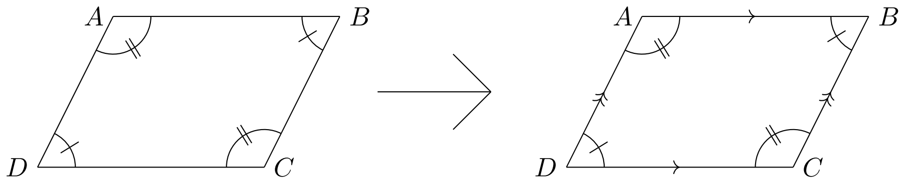
Join AC. In $\triangle ADC$ and $\triangle CBA$,

$$\begin{aligned}
DC &= AB && \text{(given)} \\
AD &= BC && \text{(given)} \\
AC &= AC && \text{(common side)} \\
\therefore\ \triangle ADC &\cong \triangle CBA && \text{(SSS)} \\
\therefore\ \angle DCA &= \angle BAC && \text{(corr. } \angle\text{s, } \cong \triangle\text{s)} \\
\therefore\ AB &// DC && \text{(alt. } \angle\text{s equal)} \\
\therefore\ \angle DAC &= \angle BCA && \text{(corr. } \angle\text{s, } \cong \triangle\text{s)} \\
\therefore\ AD &// BC && \text{(alt. } \angle\text{s equal)}
\end{aligned}$$

$$\therefore\ ABCD\text{is a //gram.}$$

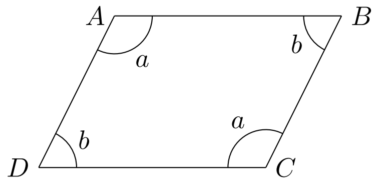
□

Preposition 40. If there are two equal pairs of opposite angles in a quadrilateral, then the quadrilateral is a parallelogram. (opp. $\angle$s equal) *



$$\because \therefore \angle ADC = \angle ABC \text{ and } \angle BAD = \angle BCD$$
$$AB//DC \text{ and } AD//BC \qquad \text{(opp. } \angle\text{s of //gram)}$$

Proof. Let there be a quadrilateral $ABCD$ where $\angle A = \angle C$ and $\angle B = \angle D$.



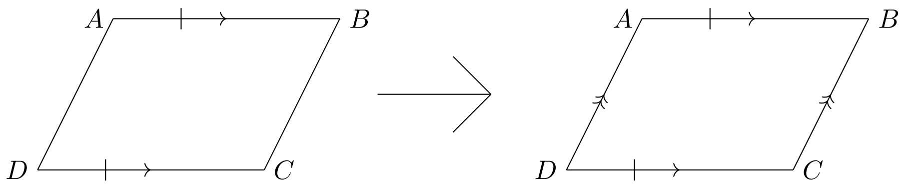
$$\angle A + \angle B + \angle C + \angle D = 360^\circ \qquad (\angle \text{ sum of quad.})$$
$$\angle A + \angle B + \angle A + \angle B = 360^\circ$$
$$\angle A + \angle B = 180^\circ$$
$$\therefore AD//BC \qquad \text{(int. } \angle\text{s supp.)}$$

$\angle A = \angle C$ and $\angle B = \angle D$, we also have

$$\angle C + \angle D = 180^\circ$$
$$\therefore AB//DC \qquad \text{(int. } \angle\text{s supp.)}$$
$$\therefore ABCD \text{ is a //gram.}$$

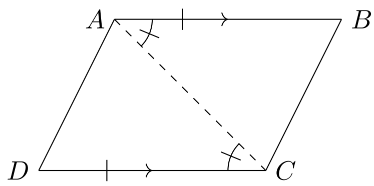
□

Preposition 41. If there is one equal and parallel pair of opposite sides in a quadrilateral, then the quadrilateral is a parallelogram. (opp. sides equal and //) *



$$\because AB = DC \text{ and } AB//DC$$
$$\therefore ABCD \text{ is a //gram.} \qquad \text{(opp. sides equal and //)}$$

Proof. Let there be a quadrilateral $ABCD$ where $AB = DC$ and $AB//DC$.

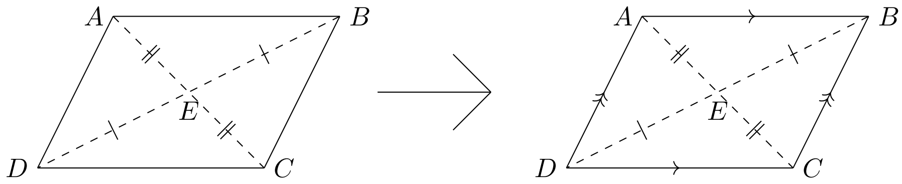


Join AC. In $\triangle ADC$ and $\triangle CBA$,

$$DC = AB \qquad \text{(given)}$$
$$\angle DCA = \angle BAC \qquad (\text{alt. } \angle\text{s , } AB//DC)$$
$$AC = AC \qquad \text{(common side)}$$
$$\therefore \triangle ADC \cong \triangle CBA \qquad \text{(SAS)}$$
$$\therefore \angle DAC = \angle BCA \qquad (\text{corr. } \angle\text{s, } \cong \triangle\text{s})$$
$$\therefore AD//BC \qquad \text{(alt. } \angle\text{s equal)}$$
$$\therefore ABCD \text{ is a //gram.}$$

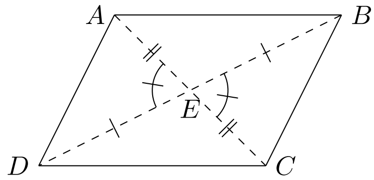
$\square$

Preposition 42. If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram. (diags bisect each other) *



$$\because \ AE = EC \text{ and } DE = EB$$
$$\therefore \ ABCD \text{ is a //gram.} \qquad \text{(diags of //gram)}$$

Proof. Let there be quadrilateral $ABCD$ with bisecting diagonals.



In $\triangle ADE$ and $\triangle CBE$,

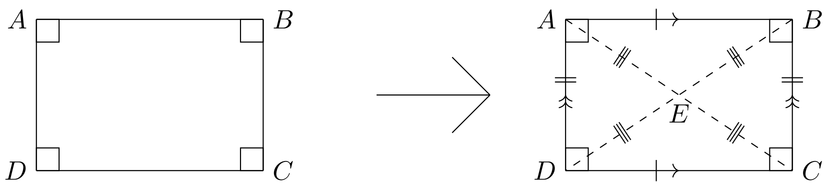
$$
\begin{aligned}
AE &= EC \qquad \text{(given)}\\
\angle AED &= \angle CEB \qquad \text{(vert. opp. } \angle\text{s)}\\
DE &= EB \qquad \text{(given)}\\
\therefore \ \triangle ADE &\cong \triangle CBE \qquad \text{(SAS)}\\
\therefore \ \angle DCA &= \angle BAC \qquad \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}\\
\therefore \ \angle DAE &= \angle BCE \qquad \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}\\
\therefore \ AD &// BC \qquad \text{(alt. } \angle\text{s equal)}\\
\text{Also, } AD &= BC \qquad \text{(corr. sides, } \cong \triangle\text{s)}\\
\therefore \ ABCD & \text{ is a //gram.} \qquad \text{(opp. sides equal and //)}
\end{aligned}
$$

$\square$

1.4.3 Rectangles

Properties of rectangles:

Preposition 43. A rectangle has four right angles, two equal and parallel pairs of opposite sides, and two equal diagonals that bisect each other. (prop. of rectangle) *



$$\because \angle A = \angle B = \angle C = \angle D = 90^\circ \qquad \text{(definition of a rectangle)}$$
$$\therefore \ AB = DC \ , \ AB//DC \ , \ AD = BC \ , \ AD//BC$$
$$\text{Also, } AE = BE = CE = DE \qquad \text{(prop. of rectangle)}$$

Proof. By definition, a rectangle has 4 right angles. Let there be rectangle $ABCD$.

$$
\begin{aligned}
\angle A = \angle C = 90^\circ \ \text{ and } \ \angle B = \angle D = 90^\circ & \qquad \text{(definition of rectangle)}\\
\therefore \ ABCD \text{ is a parallogram.} & \qquad \text{(opp. } \angle\text{s equal)}\\
\therefore \ AB//DC \ \text{ and } \ AD//BC & \qquad \text{(definition of //gram)}\\
\therefore \ AB = DC \ \text{ and } \ AD = BC & \qquad \text{(opp. sides of //gram)}
\end{aligned}
$$

To show that the diagonal AC is equal to diagonal BD , consider $\triangle ADC$ and $\triangle BCD$:

$$\begin{aligned} AD &= BC && \text{(opp. sides of //gram)} \\ \angle D &= \angle C && \text{(definition of rectangle)} \\ DC &= CD && \text{(common side)} \\ \therefore\ \triangle ADC &\cong \triangle BCD && \text{(SAS)} \\ \therefore\ AC &= BD && \text{(corr. sides, } \cong \triangle\text{s)} \end{aligned}$$

By 'diags of //gram' , we have $AE = EC$ and $DE = EB$. Since $AC = AE + EC$ and $BD = DE + EB$, we have $AE + EC = DE + EB \Rightarrow AE + AE = DE + DE \Rightarrow AE = DE \Rightarrow AE = BE = CE = DE$. □

Conditions of rectangle

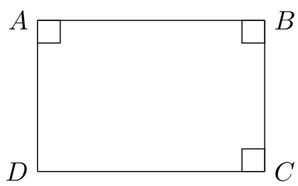
Preposition 44. A quadrilateral is a rectangle if it satisfies one of the following conditions:

1. Three angles are $90°$. (3 right $\angle$s)
2. Four angles are equal. (4 $\angle$s equal)
3. It is a parallelogram with a $90°$ angle. (//gram with right $\angle$)
4. One pair of opposite sides are parallel, with two $90°$ angles not sharing the same uncertain side. (1 // pair, 2 right $\angle$s)
5. Two angles are $90°$, with one equal pair of opposite sides. (1 equal pair, 2 right $\angle$s)
6. One pair of opposite sides are parallel, another pair of opposite sides are equal, with one right angle. (1 equal pair, 1 // pair, 1 right $\angle$s)
7. Diagonals are equal and bisect each other. (diags equal and bisect each other)
8. It is a parallelogram with diagonals equal. (//gram with diags equal)
9. Two angles are $90°$, and the diagonals are equal. (2 right $\angle$s , diags equal)
10. It is a parallelogram with opposite angles supplementary. (//gram with opp. $\angle$s supp.)

Note: If a specific reason is not to be named, use the general reason (con. of rectangle) .

Proof. Let there be quadrilateral $ABCD$. Let's consider the conditions.

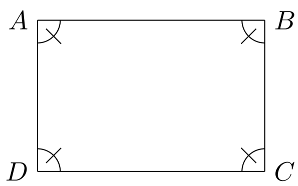
1. $\angle A = \angle B = \angle C = 90°$



$$\begin{aligned} \angle D &= 360° - 90° - 90° - 90° && (\angle \text{ sum of quad.}) \\ &= 90° \end{aligned}$$

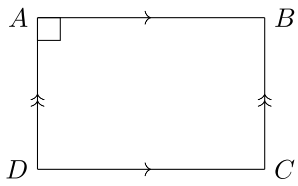
$\therefore$ $ABCD$ is a rectangle. (definition of rectangle)

2. $\angle A = \angle B = \angle C = \angle D$



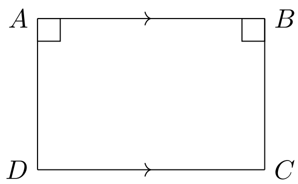
$$\angle A = \angle B = \angle C = \angle D$$
$$\angle A + \angle B + \angle C + \angle D = 360°$$
$$\therefore\ \angle A = \angle B = \angle C = \angle D = 360°/4 = 90°$$
$\therefore$ $ABCD$ is a rectangle. (definition of rectangle)

3. $AB//DC$, $AD//BC$ and $\angle A = 90°$



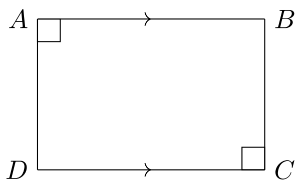
$\angle D = 180° - 90° = 90°$ (int. $\angle$s , $AB//DC$)
$\angle B = 180° - 90° = 90°$ (int. $\angle$s , $AD//BC$)
$\therefore\ \angle A = \angle D = \angle B = 90°$
$\therefore$ $ABCD$ is a rectangle. (3 right $\angle$s)

4a. $AB//DC$, $\angle A = \angle B = 90°$



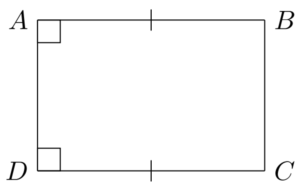
$\angle D = 180° - 90° = 90°$ (int. $\angle$s , $AB//DC$)
$\therefore\ \angle A = \angle D = \angle B = 90°$
$\therefore$ $ABCD$ is a rectangle. (3 right $\angle$s)

4b. $AB//DC$, $\angle A = \angle C = 90°$



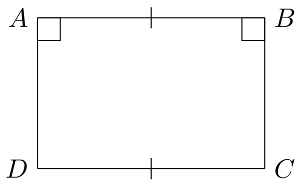
$\angle B = 180° - 90° = 90°$ (int. $\angle$s , $AD//BC$)
$\therefore\ \angle A = \angle B = \angle C = 90°$
$\therefore$ $ABCD$ is a rectangle. (3 right $\angle$s)

5a. $AB = DC$, $\angle A = \angle D = 90°$



$$\because \angle A + \angle D = 90^\circ + 90^\circ = 180 \qquad \text{(given)}$$
$$\therefore \angle AB // \angle DC \qquad \text{(int. } \angle\text{s supp.)}$$
$$\therefore ABCD \text{ is a parallelogram.} \qquad \text{(opp. sides equal and //)}$$
$$\because ABCD \text{ is a parallelogram with } \angle A = 90^\circ,$$
$$\therefore ABCD \text{ is a rectangle.} \qquad \text{(//gram with right } \angle\text{)}$$

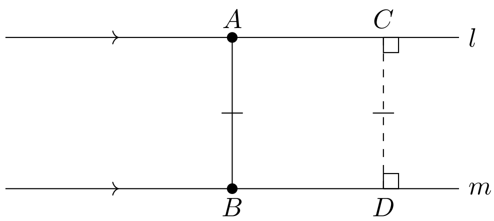
5b. $AB = DC$, $\angle A = \angle B = 90^\circ$



$$\because \angle A + \angle B = 90^\circ + 90^\circ = 180 \qquad \text{(given)}$$
$$\therefore AD // BC \qquad \text{(int. } \angle\text{s supp.)}$$

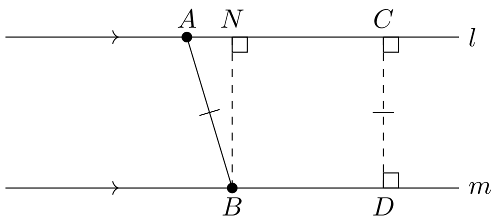
We will use a preposition not shown before.

Preposition. If there are a pair of parallel lines m and l , for which A is a point on line l and B is a point on line m such that the length of AB is equal to the perpendicular distance of l and m, then AB is perpendicular to both l and m . (property of parallel line distance)



$$\because AB = CD \text{ and } CD \perp l \text{ and } CD \perp m$$
$$\therefore AB \perp m \text{ and } AB \perp l \qquad \text{(property of parallel line distance)}$$

Proof. Let there be a pair of parallel lines m and l , for which A is a point on line l and B is a point on line m such that the length of AB is equal to the perpendicular distance of l and m .
Suppose (for the sake of contradiction) that AB is not perpendicular to line l and line m . Refer to the figure, let's say $\angle ABD > 90^\circ$. Then we can make a point N on line l such that $BN \perp l$:



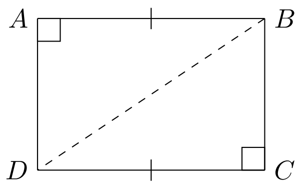
In quadrilateral $NCDB$, $\angle BNC = \angle NCD = \angle CDB = 90^\circ$. Thus $NBCD$ is a rectangle . (3 right $\angle$s) , and $AB = CD$ (prop. of rectangle) . So we have $AB = NB$. But $AB = NB$ can't be true since in right triangle $\triangle ANB$, the hypotenuse AB must be the longest side, so we have $AB > NB$, which is a contradiction.
If we suppose that $\angle ABD > 90^\circ$ instead, then we can draw $AN \perp m$ and arrive at the contradiction similarly.
Thus, the only possible angle of $\angle ABD$ is 90° , so $AB \perp m$, and thus, we also have $AB \perp l$ (int. $\angle$s, $l // m$) . $\square$

Return to the 5b condition of rectangle.

Since $AD//BC$, $AB \perp AD$, $AB \perp BC$, and $DC = AB$, by property of parallel line distance, we have $DC \perp AD$ and $DC \perp BC$.

Thus, $\angle A = \angle B = \angle C = \angle D = 90°$, which means $ABCD$ is a rectangle (definition of rectangle) .

5c. $AB = DC$, $\angle A = \angle C = 90°$

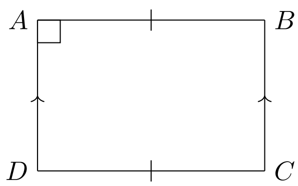


Join BD . In $\triangle ABD$ and $\triangle CDB$,

$$\angle A = \angle C = 90° \qquad \text{(given)}$$
$$BD = DB \qquad \text{(common side)}$$
$$AB = DC \qquad \text{(given)}$$
$$\therefore\ \triangle ABD \cong \triangle CDB \qquad \text{(RHS)}$$
$$\therefore\ AD = BC \qquad \text{(corr. sides, } \cong \triangle\text{s)}$$
$$\therefore\ ABCD \text{ is a parallelogram.} \qquad \text{(opp. sides equal)}$$
$$\therefore\ ABCD \text{ is a rectangle.} \qquad \text{(//gram with right } \angle\text{)}$$

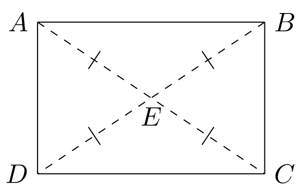
□

6. $AB = DC$, $AD//BC$, $\angle A = 90°$



$$\angle B = 180° - 90° = 90° \qquad \text{(int. } \angle\text{s , } AD//BC\text{)}$$
$$\therefore\ ABCD \text{ is a rectangle.} \qquad \text{(1 equal pair, 2 right } \angle\text{s)}$$

7. $AE = BE = CE = DE$



In $\triangle AED$ and $\triangle CEB$,

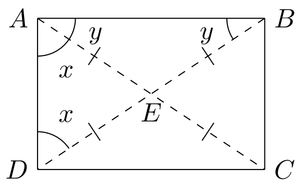
$$AE = CE \qquad \text{(given)}$$
$$\angle AED = \angle CEB \qquad \text{(vert. opp. } \angle\text{s)}$$
$$DE = BE \qquad \text{(given)}$$
$$\therefore\ \triangle AED \cong \triangle CEB \qquad \text{(SAS)}$$
$$\therefore\ AD = BC \qquad \text{(corr. sides, } \cong \triangle\text{s)}$$

In $\triangle AEB$ and $\triangle CED$,

$$\begin{aligned} AE &= CE \qquad \text{(given)} \\ \angle AEB &= \angle CED \qquad \text{(vert. opp. } \angle\text{s)} \\ BE &= DE \qquad \text{(given)} \\ \therefore\ \triangle AEB &\cong \triangle CED \qquad \text{(SAS)} \\ \therefore\ AB &= DC \qquad \text{(corr. sides, } \cong \triangle\text{s)} \end{aligned}$$

Since $AD = BC$ and $AB = DC$, $ABCD$ is a //gram (opp. sides equal) .

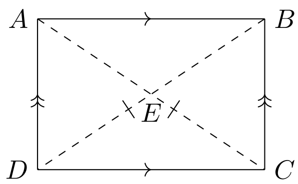
To show that $\angle A$ is a right angle, let's focus on $\triangle AED$ and $\triangle AEB$. Note that $\angle EAD = \angle EDA$ and $\angle EAB = \angle EBA$ (base $\angle$s, isos. $\triangle$).



Let $\angle EAD = \angle EDA = x$, and $\angle EAB = \angle EBA = y$. Note that $\angle A = x + y$. In $\triangle ABD$,

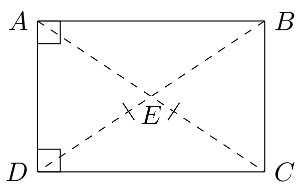
$$\begin{aligned} \angle A + \angle ABD + \angle ADB &= 180^\circ \qquad (\angle \text{ sum of } \triangle) \\ (x+y) + y + x &= 180^\circ \\ x + y &= 90^\circ \\ \angle A &= 90^\circ \\ \therefore\ ABCD \text{ is a rectangle.} & \qquad \text{(//gram with right } \angle) \end{aligned}$$

8. $AB//DC$, $AD//BC$, $AC = BD$



$$\begin{aligned} AE = CE \text{ and } BE &= DE \qquad \text{(diags of //gram)} \\ \text{Also, } AE + CE &= BE + DE \qquad \text{(given)} \\ \therefore\ AE + AE &= BE + BE \\ AE &= BE \\ \therefore\ AE = BE &= CE = DE \\ \therefore\ ABCD \text{ is a rectangle.} & \qquad \text{(diags equal and bisect each other)} \end{aligned}$$

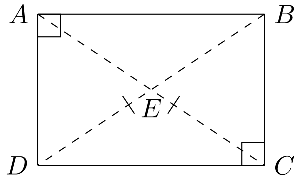
9a. $\angle A = \angle D = 90^\circ$, $AC = BD$



In $\triangle ABD$ and $\triangle DCA$,

$$\begin{aligned} \angle A = \angle D &= 90^\circ \qquad \text{(given)} \\ AC &= BD \qquad \text{(given)} \\ AD &= DA \qquad \text{(common side)} \\ \therefore\ \triangle ABD &\cong \triangle DCA \qquad \text{(RHS)} \\ \therefore\ AB &= DC \qquad \text{(corr. sides, } \cong \triangle\text{s)} \\ \therefore\ ABCD \text{ is a rectangle.} & \qquad \text{(1 equal pair, 2 right } \angle\text{s)} \end{aligned}$$

9b. $\angle A = \angle C = 90°$, $AC = BD$

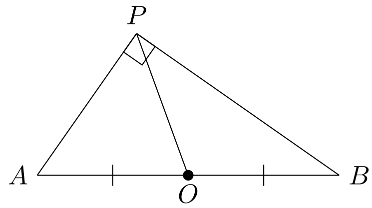


Let's use a preposition not shown before.

Preposition. In a right triangle, if a line segment joins the right angle vertex and the mid-point of the hypotenuse, then the line segment has half the length of the hypotenuse. (prop. of hypotenuse mid-pt.)

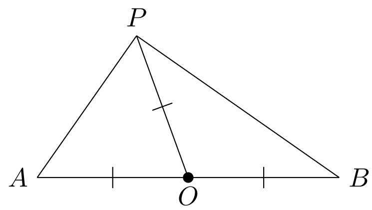
Conversely, in a triangle, if a line segment joins a vertex and the mid-point of the opposite side, and has half the length of this opposite side, then the angle of the vertex is a right angle. (converse of prop. of hypotenuse mid-pt.)

Case 1a:



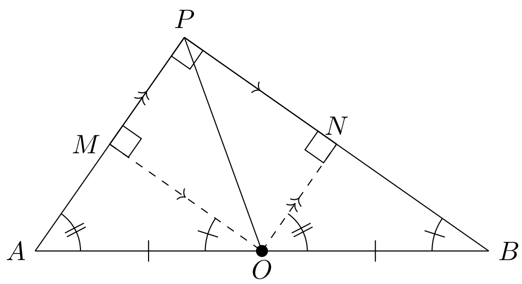
$$\because \angle APB = 90°,\ AO = OB$$
$$\therefore\ OP = AO = OB \qquad \text{(prop. of hypotenuse mid-pt.)}$$

Case 1b:



$$\because OA = OB = OP$$
$$\therefore\ \angle APB = 90° \qquad \text{(converse of prop. of hypotenuse mid-pt.)}$$

Proof. Case 1a: Let $OM \perp AP$ and $ON \perp PB$.



Note that $PNOM$ is a rectangle (3 right $\angle$s) . Thus, $MO//PB$ and $AP//ON$ (prop. of rectangle).

In $\triangle AMO$ and $\triangle ONB$,

$$\begin{aligned} \angle MAO &= \angle NOB \qquad \text{(corr. } \angle\text{s , } PA//NO) \\ AO &= OB \qquad \text{(given)} \\ \angle MOA &= \angle NBO \qquad \text{(corr. } \angle\text{s , } MO//PB) \\ \therefore\ \triangle AMO &\cong \triangle ONB \qquad \text{(ASA)} \\ \therefore\ AM &= ON \qquad \text{(corr. sides, } \cong \triangle\text{s)} \end{aligned}$$

Note that $ON = MP$ (opp. sides of rectangle). Thus, $AM = MP$ by transitivity of equality.

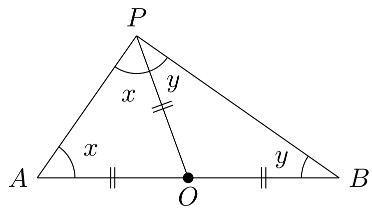
In $\triangle OPM$ and $\triangle OAM$,

$$\begin{aligned} MP &= AM \\ \angle OMP &= \angle OMA = 90^\circ \\ OM &= OM \qquad \text{(common side)} \\ \therefore\ \triangle OPM & \triangle OAM \qquad \text{(SAS)} \\ \therefore\ OP &= OA \qquad \text{(corr. sides, } \cong \triangle\text{s)} \end{aligned}$$

Therefore, $OA = OB = OP$.

<u>Case 1b:</u>

Note that $\angle OPA = \angle OAP$ and $\angle OPB = \angle OBP$ (base $\angle$s, isos. $\triangle$). Let $\angle OPA = \angle OAP = x$ and $\angle OPB = \angle OBP = y$.



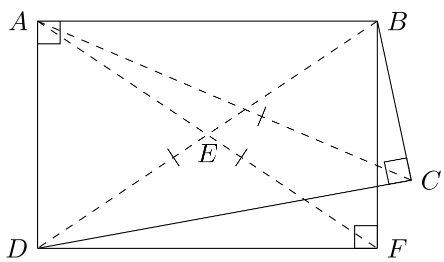
In $\triangle APB$,

$$\begin{aligned} \angle A + \angle APB + \angle B &= 180^\circ \\ x + (x+y) + y &= 180^\circ \\ x + y &= 90^\circ \\ \therefore\ \angle APB &= 90^\circ \end{aligned}$$

□

Return to condition 9b.

Suppose that $ABCD$ is not a rectangle. Then make a point F such that $ABFD$ is a rectangle.

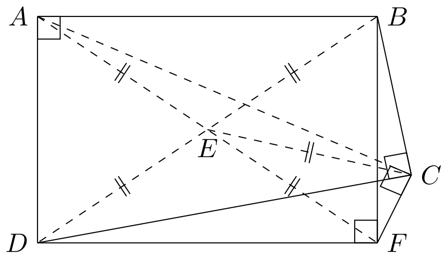


Note that $AF = BD$ (diags of rectangle). Since $BD = AC$ (given), we have $AF = AC$.

Join CF and CE .

Since $\triangle BCD$ is a right triangle and $DE = EB$ (diags of rectangle), we have $EC = DE = EB$ (prop. of hypotenuse mid-pt.) .

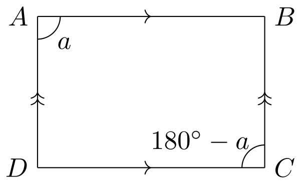
Note that $DE = EB = AE = EF$ (diags of rectangle), so we have $EC = AE = EF$. Thus, $\angle ACF = 90^\circ$ (converse of prop. of hypotenuse mid-pt.) .



But $AF = AC$. So $\angle AFC = \angle ACF = 90°$ (base $\angle$s, isos. $\triangle$). But this means $AF//AC$ (int. $\angle$s supp.) , which cannot be true since $\triangle AFC$ is a triangle. There is a contradiction.

Thus, we conclude that it is impossible for C to not lie on F . This means $ABCD$ must be a rectangle.

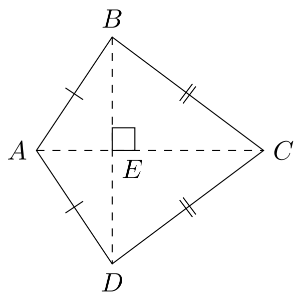
10. $AB//DC$, $AD//BC$, $\angle A + \angle C = 180°$.



$$\begin{aligned} \angle A + \angle C &= 180° && \text{(given)} \\ \angle A &= \angle C && \text{(opp. } \angle\text{s of //gram)} \\ \therefore\ \angle A = \angle C &= 180°/2 = 90° \\ \therefore\ ABCD &\text{ is a rectangle.} && \text{(//gram with right } \angle\text{).} \end{aligned}$$

1.4.4 Kites

Preposition 45. The diagonals of a kite are perpendicular to each other. (prop. of kite diags)



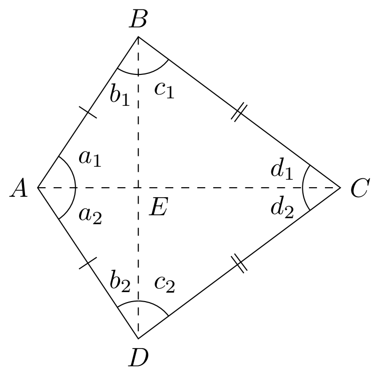
$$\because\ AB = AD \text{ and } CB = CD$$
$$\therefore\ BD \perp AC \qquad \text{(prop. of kite diags)}$$

Proof. In $\triangle ABC$ and $\triangle ADC$,

$$\begin{aligned} AB &= AD && \text{(given)} \\ CB &= CD && \text{(given)} \\ AC &= AC && \text{(common side)} \\ \therefore\ \triangle ABC &\cong \triangle ADC && \text{(SSS)} \\ \therefore\ \angle BCE &= \angle DCE && \text{(corr. } \angle\text{s, } \cong \triangle\text{s)} \\ \therefore\ BD &\perp CE && \text{(prop. of isos. } \triangle) \end{aligned}$$

Since AEC is a straight line, we also have $BD \perp AC$. □

Preposition 46. In a kite, the angles formed by a side and a diagonal form equal pairs. (prop. of kite $\angle$s)



$$AB = AD \text{ and } CB = CD$$
$$\therefore\ a_1 = a_2\ ,\ b_1 = b_2\ ,\ c_1 = c_2\ ,\ d_1 = d_2$$
(prop. of kite diags)

Proof. In $\triangle ABC$ and $\triangle ADC$,

$$\begin{aligned} AB &= AD && \text{(given)} \\ CB &= CD && \text{(given)} \\ AC &= AC && \text{(common side)} \\ \therefore\ \triangle ABC &\cong \triangle ADC && \text{(SSS)} \end{aligned}$$

$$\therefore\ \angle BAC = \angle DAC \text{ and } \angle BCA = \angle DCA \qquad (\text{corr. } \angle\text{s}, \cong \triangle\text{s})$$

Also, $\angle ABD = \angle ADB$ and $\angle CBD = \angle CDB$ (base $\angle$s, isos. $\triangle$). □

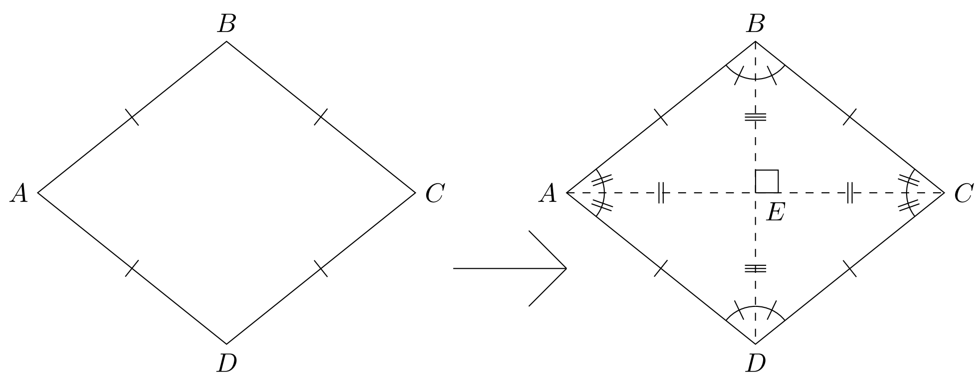
1.4.5 Rhombuses

Properties of rhombus

Preposition 47. A rhombus has the following properties:

1. Four sides are equal. (definition of rhombus)
2. Opposite sides are parallel (which means it is a parallelogram).
3. Opposite angles are equal and bisected by diagonals.
4. Diagonals are perpendicular and bisect each other.

(prop. of rhombus) *



$$\because\ AB = BC = CD = DA \qquad \text{(definition of rhombus)}$$
$$\therefore\ AB//CD\ ,\ BC//AD$$
$$\angle BAC = \angle DAC = \angle BCA = \angle DCA$$
$$\angle ABD = \angle ADB = \angle CBD = \angle CDB$$
$$BD \perp AC$$
$$BE = ED\ ,\ AE = EC$$
$$\text{(prop. of rhombus)}$$

Proof.

$$\because\ AB = CD \ \text{ and } \ BC = AD \qquad \text{(given)}$$
$$\therefore \qquad \text{(prop. 2) } AB//CD \ \text{ and } \ BC//AD \qquad \text{(opp. sides equal)}$$
$$\therefore \qquad \text{(prop. 4) } AE = EC \ \text{ and } \ BE = ED \qquad \text{(diags of //gram)}$$

Now we prove that the four triangles formed by the rhombus' diagonals are congruent.

In $\triangle EAB$, $\triangle ECB$, $\triangle EAD$ and $\triangle ECD$,

$$\text{(prop. 1) } AB = BC = AD = CD \qquad \text{(given)}$$
$$BE = BE = ED = ED \qquad \text{(common side \& diags of //gram)}$$
$$AE = EC = AE = EC \qquad \text{(common side \& diags of //gram)}$$
$$\therefore\ \triangle EAB \cong \triangle ECB \cong \triangle EAD \cong \triangle ECD \qquad \text{(SSS)}$$
$$\therefore \qquad \text{(prop. 3) } \angle BAC = \angle BCA = \angle DAC = \angle DCA \qquad \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}$$
$$\text{(prop. 3) } \angle ABD = \angle CBD = \angle ADB = \angle CDB \qquad \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}$$

Also note that a rhombus is a special type of kite since $AB = AD$ and $CB = CD$. By property of kite diags, we have $BD \perp AC$ (prop. 4). □

<u>Conditions of rhombus</u>

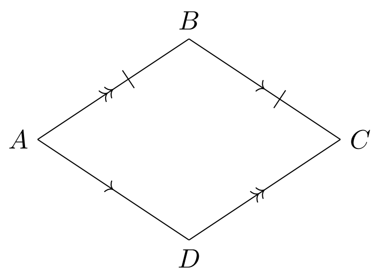
Preposition 48. A quadrilateral is a rhombus if it satisfies one of the following conditions:

1. It is a parallelogram with a pair of adjacent side equal. (//gram with equal adj. side)
2. The diagonal bisects an equal pair of opposite angles. (diag bisects equal opp. $\angle$s)
3. Diagonals are perpendicular and bisect each other. (diags $\perp$ and bisect each other)

Non-specific reason: (con. of rhombus)

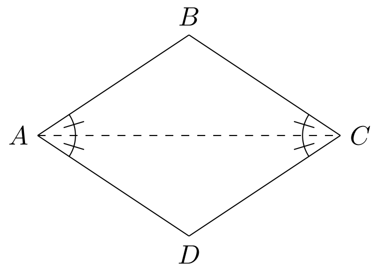
Proof. Let there be quadrilateral $ABCD$. Let's consider the conditions.

1. $AB//CD$, $BC//AD$, $AB = BC$



$$AB = CD \ \text{ and } \ BC = AD \qquad \text{(opp. sides of //gram)}$$
$$\therefore \qquad \text{(condition 1) } AB = BC = CD = AD$$
$$\therefore\ ABCD \text{ is a rhombus.} \qquad \text{(ddefinition of rhombus)}$$

2. $\angle BAC = \angle BCA = \angle DAC = \angle DCA$



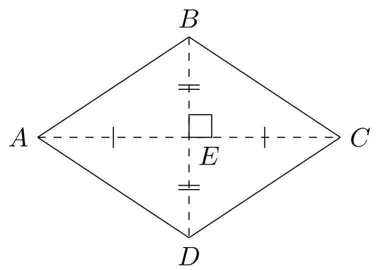
$$\angle BAC = \angle BCA \qquad \text{(given)}$$
$$\therefore\ BA = BC \qquad \text{(sides opp. equal } \angle\text{s)}$$
$$\angle DAC = \angle DCA \qquad \text{(given)}$$
$$\therefore\ DA = DC \qquad \text{(sides opp. equal } \angle\text{s)}$$

In $\triangle BAC$ and $\triangle DAC$,

$$\angle BAC = \angle DAC \qquad \text{(given)}$$
$$AC = AC \qquad \text{(common side)}$$
$$\angle BCA = \angle DCA \qquad \text{(given)}$$
$$\therefore\ \triangle BAC \cong \triangle DAC \qquad \text{(ASA)}$$
$$\therefore\ AB = AD \ \text{ and } \ CB = CD \qquad \text{(corr. sides, } \cong \triangle\text{s)}$$

$\therefore\ AB = BC = CD = AD$, and $ABCD$ is a rhombus.

3. $BD \perp AC$, $AE = EC$, $BE = ED$



In $\triangle EAB$, $\triangle ECB$, $\triangle EAD$ and $\triangle ECD$,

$$AE = EC = AE = EC \qquad \text{(given \& common side)}$$
$$\angle AEB = \angle CEB = \angle AED = \angle CED = 90^\circ \qquad (BD \perp AC)$$
$$BE = BE = ED = ED \qquad \text{(given \& common side)}$$
$$\therefore\ \triangle EAB \cong \triangle ECB \cong \triangle EAD \cong \triangle ECD \qquad \text{(SAS)}$$
$$\therefore\ AB = BC = CD = AD \qquad \text{(corr. sides, } \cong \triangle\text{s)}$$

$\therefore\ ABCD$ is a rhombus. □

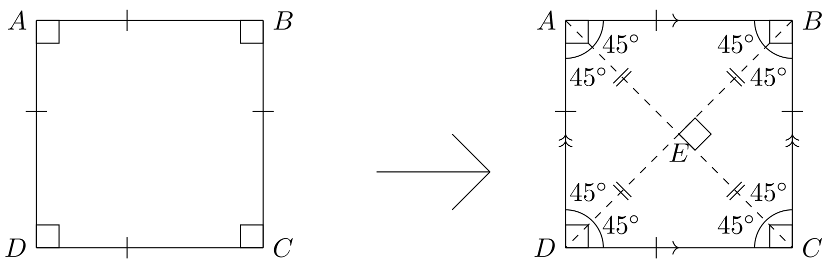
1.4.6 Squares

Properties of square

Preposition 49. A square has the following properties:

1. Four equal sides. (definition of square part I)
2. Four right angles. (definition of square part II)
3. Opposite sides are parallel.
4. Diagonals are perpendicular, equal and bisect each other.
5. The angles formed between a side and a diagonal is 45° .

(prop. of square) *



$$\because\ \angle A = \angle B = \angle C = \angle D = 90^\circ\ ,\ AB = BC = CD = DA \qquad \text{(definition of square)}$$
$$\therefore\ AB//DC\ ,\ AD//BC$$
$$AE = BE = CE = DE$$
$$BD \perp AC$$
$$\angle EAD = \angle EDA = \angle EDC = \angle ECD = \angle ECB = \angle EBC = \angle EBA = \angle EAB = 45^\circ$$
(prop. of square)

Proof.

$$AB = BC = CD = DA \qquad \text{(definition of square part I)}$$
$$\therefore\ ABCD \text{ is a rhombus.} \qquad \text{(definition of rhombus)}$$
$$\therefore \qquad \text{(condition 3) } AB//DC\ ,\ AD//BC \qquad \text{(prop. of rhombus)}$$
$$\text{(condition 4) } BD \perp AC \qquad \text{(prop. of rhombus)}$$
$$\text{(condition 4) } AE = EC \ \text{ and } \ BE = ED \qquad \text{(prop. of rhombus)}$$

In $\triangle ADC$ and $\triangle DAB$,

$$AD = AD \qquad \text{(common side)}$$
$$\angle D = \angle A \qquad \text{(given)}$$
$$DC = AB \qquad \text{(given)}$$
$$\therefore\ \triangle ADC \cong \triangle DAB \qquad \text{(SAS)}$$
$$\therefore\ AC = BD \qquad \text{(corr. sides, } \cong \triangle\text{s)}$$

Since we also have $AE = EC$ and $BE = ED$, we have $AE = BE = CE = DE$ (condition 4).

Note that since $ABCD$ is a rhombus, the four triangles formed by diagonals are congruent (proven in prop. of rhombus).

Focus on one of the triangles, say $\triangle AED$. Since $AE = DE$, we have $\angle EAD = \angle EDA$ (base $\angle$s, isos. $\triangle$).

$$\angle EAD + \angle EDA + \angle AED = 180^\circ \qquad (\angle \text{ sum of } \triangle)$$
$$2 \times \angle EAD + 90^\circ = 180^\circ$$
$$\angle EAD = \angle EDA = 45^\circ$$

By congruent triangles, we have $\angle EAB = \angle ECB = \angle ECD = \angle EAD = 45^\circ$, and $\angle EBA = \angle EBC = \angle EDC = \angle EDA = 45^\circ$ (condition 5). □

Preposition 50. A quadrilateral is a square if it satisfies one of the following conditions:

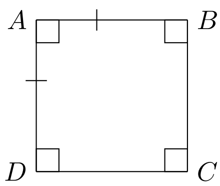
1. It is a rectangle with two adjacent sides are equal. (rectangle with equal adj. pair)
2. It is a rhombus with a 90° angle. (rhombus with right $\angle$)
3. It is a rhombus with an angle between side and diagonal being 45° . (rhombus with 45 ° inclination)
4. Three sides are equal, with two 90° angles. (3 sides equal, 2 right $\angle$s)

5. Diagonals are perpendicular, equal and bisect each other. (diags $\perp$, equal and bisect each other)

Non-specific reason: (con. of square)

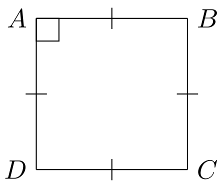
Proof. Let there be quadrilateral $ABCD$. Let's consider the conditions.

1. $\angle A = \angle B = \angle C = \angle D = 90°$, $AB = AD$



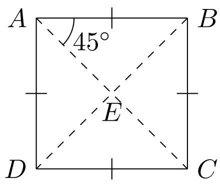
$$AD = BC \text{ and } AB = DC \qquad \text{(opp. sides of rectangle)}$$
$$\therefore AB = BC = CD = AD$$
$$\therefore ABCD \text{ is a square.} \qquad \text{(definition of square)}$$

2. $AB = BC = CD = AD$, $\angle A = 90°$



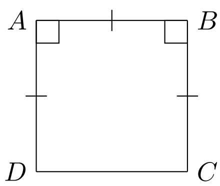
$$AB//DC \text{ and } AD//BC \qquad \text{(prop. of rhombus)}$$
$$\angle D = 180° - 90° = 90° \qquad (\text{int. } \angle\text{s} , AB//DC)$$
$$\angle B = 180° - 90° = 90° \qquad (\text{int. } \angle\text{s} , AD//BC)$$
$$\angle C = 180° - 90° = 90° \qquad (\text{int. } \angle\text{s} , AB//DC)$$
$$\therefore ABCD \text{ is a square.} \qquad \text{(definition of square)}$$

3. $AB = BC = CD = AD$, $\angle BAC = 45°$



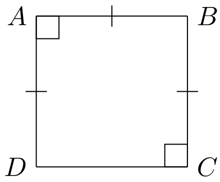
$$ABCD \text{ is a rhombus.} \qquad \text{(definition of rhombus)}$$
$$\therefore \angle EAD = \angle EAB = 45° \qquad \text{(prop. of rhombus)}$$
$$\angle A = 45° + 45° = 90°$$
$$\therefore ABCD \text{ is a square.} \qquad (\text{rhombus with right } \angle)$$

4a. $AB = BC = AD$, $\angle A = \angle B = 90°$



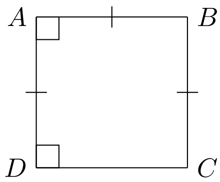
$$\because AD = BC \ \text{ and } \ \angle A = \angle B = 90^\circ \qquad \text{(given)}$$
$$\therefore \ ABCD \text{ is a rectangle.} \qquad \text{(1 equal pair, 2 right } \angle\text{s)}$$
$$\therefore \ \angle D = \angle C = 90^\circ \ \text{ and } \ DC = AB \qquad \text{(prop. of rectangle)}$$
$$\therefore \ ABCD \text{ is a square.} \qquad \text{(definition of square)}$$

4b. $AB = BC = AD$, $\angle A = \angle C = 90^\circ$



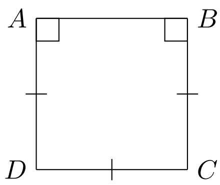
$$\because AD = BC \ \text{ and } \ \angle A = \angle C = 90^\circ \qquad \text{(given)}$$
$$\therefore \ ABCD \text{ is a rectangle.} \qquad \text{(1 equal pair, 2 right } \angle\text{s)}$$
$$\therefore \ \angle B = \angle D = 90^\circ \ \text{ and } \ DC = AB \qquad \text{(prop. of rectangle)}$$
$$\therefore \ ABCD \text{ is a square.} \qquad \text{(definition of square)}$$

4c. $AB = BC = AD$, $\angle A = \angle D = 90^\circ$



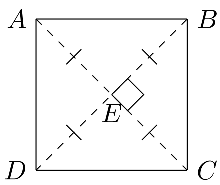
$$\because AD = BC \ \text{ and } \ \angle A = \angle D = 90^\circ \qquad \text{(given)}$$
$$\therefore \ ABCD \text{ is a rectangle.} \qquad \text{(1 equal pair, 2 right } \angle\text{s)}$$
$$\therefore \ \angle B = \angle C = 90^\circ \ \text{ and } \ DC = AB \qquad \text{(prop. of rectangle)}$$
$$\therefore \ ABCD \text{ is a square.} \qquad \text{(definition of square)}$$

4d. $AD = BC = DC$, $\angle A = \angle B = 90^\circ$



$$\because AD = BC \ \text{ and } \ \angle A = \angle B = 90^\circ \qquad \text{(given)}$$
$$\therefore \ ABCD \text{ is a rectangle.} \qquad \text{(1 equal pair, 2 right } \angle\text{s)}$$
$$\therefore \ \angle D = \angle C = 90^\circ \ \text{ and } \ DC = AB \qquad \text{(prop. of rectangle)}$$
$$\therefore \ ABCD \text{ is a square.} \qquad \text{(definition of square)}$$

5. $AE = BE = CE = DE$, $BD \perp AC$



In $\triangle EAB$, $\triangle ECB$, $\triangle EAD$ and $\triangle ECD$,

$$\begin{gathered} AE = CE = AE = CE \qquad \text{(given \& common side)} \\ \angle AEB = \angle CEB = \angle AED = \angle CED = 90^\circ \qquad (BD \perp AC) \\ BE = BE = ED = ED \qquad \text{(given \& common side)} \\ \therefore\ \triangle EAB \cong ECB \cong EAD \cong ECD \qquad \text{(SAS)} \\ \therefore\ AB = BC = CD = AD \qquad (\text{corr. sides}, \cong \triangle\text{s}) \end{gathered}$$

Note that $\angle EAB = \angle EBA$ (base $\angle$s, isos. $\triangle$). Thus $\angle EAB = \angle EBA = (180^\circ - 90^\circ)/2 = 45^\circ$ ($\angle$ sum of $\triangle$).

By congruent triangles, $\angle EAB = \angle ECB = \angle EAD = \angle ECD = 45^\circ$, and $\angle EBA = \angle EBC = \angle EDA = \angle EDC = 45^\circ$.

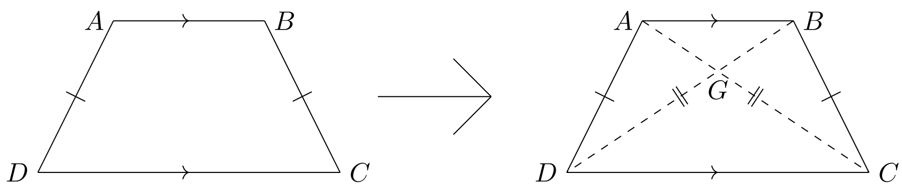
Thus $\angle A = \angle B = \angle C = \angle D = 45^\circ + 45^\circ = 90^\circ$, and $ABCD$ is a square. (definition of square). $\square$

1.4.7 Isosceles trapeziums

Preposition 51. An isosceles trapezium has the following properties:

1. It has exactly one pair of parallel opposite sides. (definition of isos. trapezium part I)
2. The pair of non-parallel sides are equal. (definition of isos. trapezium part II)
3. Angles sharing the same parallel side are equal.
4. Diagonals are equal.
5. Angles formed between a side and a diagonal form equal pairs.

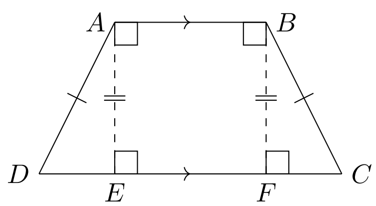
   (prop. of isos. trapezium)



$$\begin{gathered} \because\ AB//DC\,,\ AD\neg//BC\,,\ AC = BC \qquad \text{(definition of isos. trapezium)} \\ \therefore\ \angle A = \angle B \ \text{ and } \ \angle D = \angle C \\ AC = BD \\ \angle GAB = \angle GBA\ ,\ \angle DAC = \angle CBD\ ,\ \angle ADB = \angle BCA\ ,\ \angle GDC = \angle GCD \\ \text{(prop. of isos. trapezium)} \end{gathered}$$

Proof. Let there be isos. trapezium $ABCD$ where $AB//DC$, $AD = BC$, and $AB < DC$.

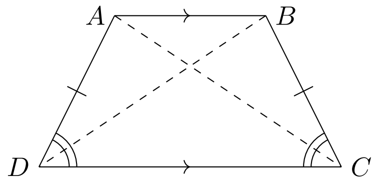
Draw $AE \perp DC$ and $BF \perp DC$. Note that $AE \perp AB$ and $BF \perp AB$ (int. $\angle$s , $AB//DC$)



Note that $ABFE$ is a rectangle, so $AE = BF$ (opp. sides of rectangle).

In $\triangle AED$ and $\triangle BFC$,

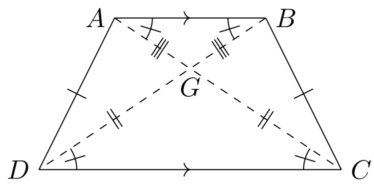
$$\begin{aligned}
\angle AED = \angle BFC = 90^\circ \qquad & (AE \perp DC, BF \perp DC) \\
AD = BC \qquad & \text{(given)} \\
AE = BF \qquad & \text{(opp. sides of rectangle)} \\
\therefore\ \angle AED \cong BFC \qquad & \text{(RHS)} \\
\therefore \qquad \text{(prop. 3) } \angle D = \angle C \qquad & \text{(corr. } \angle\text{s, } \cong \triangle\text{s)} \\
\angle DAE = \angle CBF \qquad & \text{(corr. } \angle\text{s, } \cong \triangle\text{s)} \\
\text{(prop. 3) } \angle A &= \angle DAE + 90^\circ \\
&= \angle CBF + 90^\circ \\
&= \angle B
\end{aligned}$$



In $\triangle ADC$ and $\triangle BCD$,

$$\begin{aligned}
AD = BC \qquad & \text{(given)} \\
\angle D = \angle C \qquad & \text{(prop. 3 of this preposition)} \\
DC = CD \qquad & \text{(common side)} \\
\therefore\ \triangle ADC \cong BCD \qquad & \text{(SAS)} \\
\therefore \qquad \text{(prop. 4) } AC = BD \qquad & \text{(corr. sides, } \cong \triangle\text{s)} \\
\text{(prop. 5a) } \angle ACD = \angle BDC \qquad & \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}
\end{aligned}$$

Let G denote the intersection of AC and BD . Since $\angle ACD = \angle BDC$, we have $GD = GC$ (sides opp. equal $\angle$s). Since $AC = BD$, we also have $GA = GB$, so $\angle GAB = \angle GBA$ (base $\angle$s, isos. $\triangle$) (prop. 5b).



Since $AB//DC$, we have $\angle ABD = \angle BDC$ (alt. $\angle$s , $AB//DC$) , which means $\angle GAB = \angle GBA = \angle GDC = \angle GCD$.

Finally, in $\triangle GAD$ and $\triangle GBC$,

$$\begin{aligned}
GA = GB \qquad & \text{(proven above)} \\
AD = BC \qquad & \text{(given)} \\
GD = GC \qquad & \text{(proven above)} \\
\therefore\ \triangle GAD \cong \triangle GBC \qquad & \text{(SSS)} \\
\therefore \qquad \text{(prop. 5c) } \angle DAG = \angle CBG \qquad & \text{(corr. } \angle\text{s, } \cong \triangle\text{s)} \\
\text{(prop. 5d) } \angle ADG = \angle BCG \qquad & \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}
\end{aligned}$$

□

Preposition 52. A quadrilateral is an isosceles trapezium or a rectangle if it satisfies one of the following conditions:

1. A pair of opposite sides is parallel, and a pair of angles sharing the same parallel side is equal. (opp. sides // and adj. $\angle$s equal)

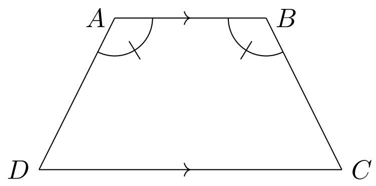
2. A pair of opposite sides is equal, and a pair of angles sharing the side joining the equal sides is equal. (opp. sides and adj. $\angle$s equal)
3. A pair of opposite sides is parallel, and the diagonals are equal. (opp. sides // and diags equal)
4. A pair of opposite sides is equal, and the diagonals are equal. (opp. sides and diags equal)
5. A pair of adjacent angles is equal and not acute, and the diagonals are equal. (non-acute adj. $\angle$s and diags equal)

(con. of isos. trapezium or rectangle)

Note: Since my definition of isosceles trapezium does not include rectangles, but it is too annoying to distinguish between the two, we consider both in our conditions.

Proof. Let there be quadrilateral $ABCD$. Let's consider the conditions.

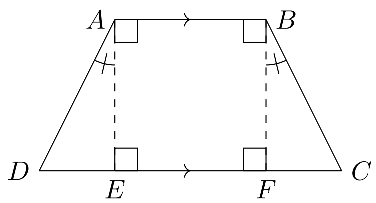
<u>1.</u> $AB//DC$, $\angle A = \angle B$



If $\angle A = \angle B = 90°$, then $ABCD$ is a rectangle by '1 // pair, 2 right $\angle$s' .

Assume that $\angle A$ and $\angle B$ are not right angles. Then AD is not parallel to BC since $\angle A + \angle B \neq 180°$.

Assume that $\angle A, \angle B > 90°$. Draw $AE \perp DC$ and $BF \perp DC$.

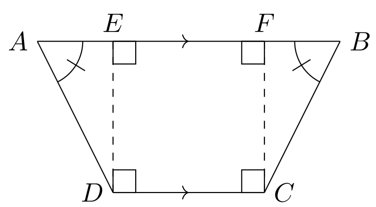


Note that $ABFE$ is a rectangle by '1 // pair, 2 right $\angle$s' . Thus $\angle BAE = \angle ABF = 90°$ (definition of rectangle) .
Thus, $\angle DAE = \angle A - 90° = \angle B - 90° = \angle CBF$.

In $\triangle DAE$ and $\triangle CBF$,

$$\begin{aligned} \angle DAE &= \angle CBF \\ AE &= BF \qquad \text{(opp. sides of rectangle)} \\ \angle AED &= \angle BFC = 90° \qquad (AE \perp DC \text{ and } BF \perp DC) \\ \therefore \triangle DAE &\cong \triangle CBF \qquad \text{(ASA)} \\ \therefore AD &= BC \qquad \text{(corr. sides, } \cong \triangle\text{s)} \\ \therefore ABCD & \text{ is a isosceles trapezium by definition.} \end{aligned}$$

If $\angle A \angle B < 90°$, then draw $DE \perp AB$ and $CF \perp AB$ instead.



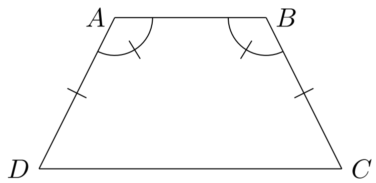
Note that $EFCD$ is a rectangle by '1 // pair, 2 right $\angle$s' . Thus $ED = FC$ (opp. sides of rectangle). .

In $\triangle DAE$ and $\triangle CBF$,

$$\begin{aligned}
\angle DAE &= \angle CBF \qquad \text{(given)} \\
\angle AED &= \angle BFC = 90^\circ \qquad (AB \perp ED \text{ and } AB \perp FC) \\
ED &= FC \qquad \text{(opp. sides of rectangle)} \\
\therefore\ \triangle DAE &\cong \triangle CBF \qquad \text{(AAS)} \\
\therefore\ AD &= BC \qquad \text{(corr. sides, } \cong \triangle\text{s)}
\end{aligned}$$

$\therefore$ $ABCD$ is an isosceles trapezium by definition.

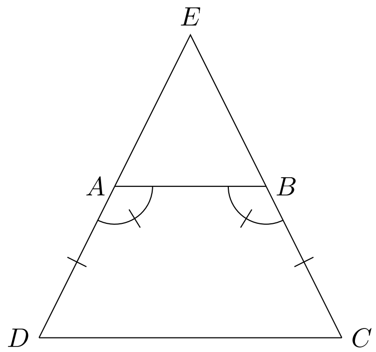
2. $AD = BC$, $\angle A = \angle B$



If $\angle A = \angle B = 90^\circ$, then $ABCD$ is a rectangle by '1 equal pair, 2 right $\angle$s' .

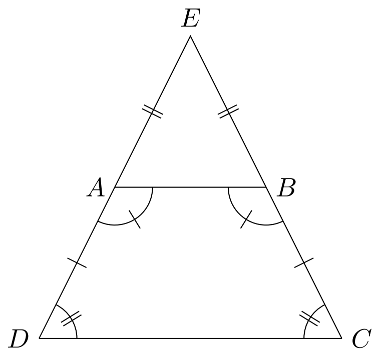
Assume that $\angle A$ and $\angle B$ are not right angles. Then AD is not parallel to BC since $\angle A + \angle B \neq 180^\circ$.

Assume that $\angle A, \angle B > 90^\circ$. Extend DA and CB to intersect at E . Note that E is above AB since $\angle EAB + \angle EBA < 180^\circ$.



Note that $\angle EAB = 180^\circ - \angle BAD = 180^\circ - \angle ABC = \angle EBA$ (adj. $\angle$s on st. line).
Since $\angle EAB = \angle EBA$, we have $EA = EB$ (sides opp. equal $\angle$s).

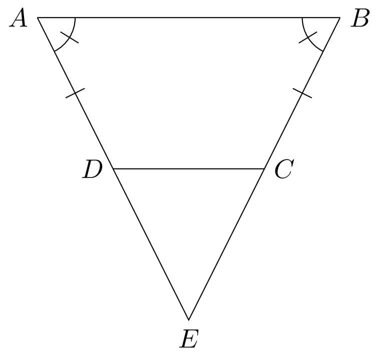
Note that $ED = EA + AD = EB + BC = EC$. Thus, $\angle EDC = \angle ECD$ (base $\angle$s, isos. $\triangle$).



In quadrilateral $ABCD$,

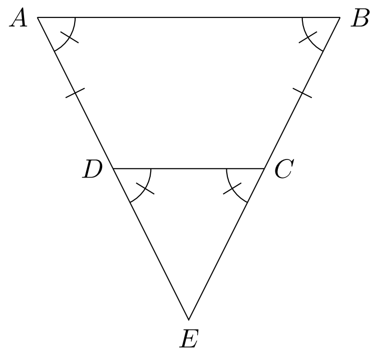
$$\begin{aligned} \angle A+\angle B+\angle C+\angle D &= 360^\circ \\ \angle A+\angle A+\angle D+\angle D &= 360^\circ \\ \angle A+\angle D &= 180^\circ \\ \therefore\ AB//DC & \qquad \text{(int. } \angle\text{s supp.)} \\ \therefore\ ABCD & \text{ is an isosceles trapezium by definition.} \end{aligned}$$

If $\angle A, \angle B < 90^\circ$ instead, then E is below DC .



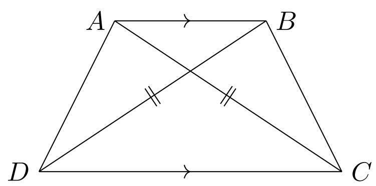
In $\triangle EAB$, we have $EA = EB$ (sides opp. equal $\angle$s), so $DE = EA - AD = EB - BC = CE$.
Since $DE = CE$, we have $\angle EDC = \angle ECD$ (base $\angle$s, isos. $\triangle$).

Note that $\angle EDC = \angle ECD = \dfrac{180 - \angle E}{2} = \angle A = \angle B$ ($\angle$ sum of $\triangle$).



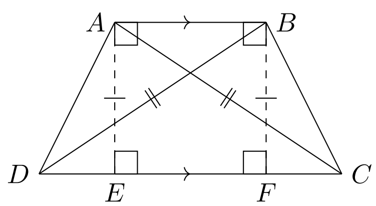
Since $\angle EDC = \angle EAB$, we have $AB//DC$ (corr. $\angle$s equal) .

<u>3.</u> $AB//DC$, $AC = BD$



If $AB = DC$, then $ABCD$ is a parallelogram by ‘opp. sides equal and //’, and since $AC = BD$, $ABCD$ is also a rectangle (//gram with equal diags).

Assume that $AB < DC$. Draw $AE \perp DC$ and $BF \perp DC$.
‘



Note that $EFCD$ is a rectangle by '1 // pair, 2 right $\angle$s' . Thus $AE = BF$ (opp. sides of rectangle). .

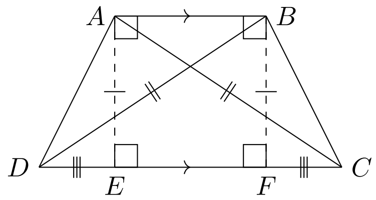
In $\triangle AEC$ and $\triangle BFD$,

$$\begin{aligned}
\angle AEC &= \angle BFD \qquad (AE \perp DC \text{ and } BF \perp DC) \\
AE &= BF \qquad \text{(shown above)} \\
AC &= BD \qquad \text{(given)} \\
\therefore\ \triangle AEC &\cong \triangle BFD \qquad \text{(RHS)} \\
\therefore\ EC &= DF \qquad \text{(corr. sides, } \cong \triangle\text{s)}
\end{aligned}$$

Since $EF < DC$, note that E , F must lie between D, C .

Otherwise, say, F is at the right of C or lies on C . Then $EC < DF$, but this contradicts $EC = DF$, which is what we have just shown.

So E , F must lie between D, C , and $DE = DF - EF = CE - EF = FC$.



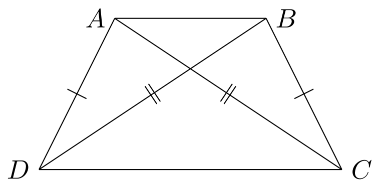
In $\triangle AED$ and $\triangle BFC$,

$$\begin{aligned}
AE &= BF \\
\angle AED &= \angle BFC = 90^\circ \\
DE &= FC \\
\therefore\ \triangle AED &\cong \triangle BFC \qquad \text{(SAS)} \\
\therefore\ AD &= BC \qquad \text{(corr. sides, } \cong \triangle\text{s)}
\end{aligned}$$

$\therefore\ ABCD$ is an isosceles trapezium by definition.

If $AB > DC$ instead, we can use similar argument to show that $ABCD$ is an isosceles trapezium.

4. $AD = BC$, $AC = BD$



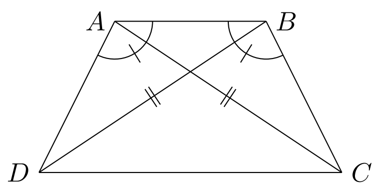
In $\triangle ADC$ and $\triangle BCD$,

$$\begin{aligned}
AD &= BC \qquad \text{(given)} \\
AC &= BD \qquad \text{(given)} \\
DC &= CD \qquad \text{(common side)} \\
\therefore\ \triangle ADC &\cong \triangle BCD \qquad \text{(SSS)} \\
\therefore\ \angle ADC &= \angle BCD \qquad \text{(corr. } \angle\text{s, } \cong \triangle\text{s)}
\end{aligned}$$

$\therefore\ ABCD$ is an isosceles trapezium or a rectangle.

(opp. sides and adj. $\angle$s equal)

5. $\angle A = \angle B \geq 90^\circ$, $AC = BD$



If $\angle A = \angle B = 90^\circ$, then $ABCD$ is a rectangle (2 right $\angle$s, diags equal) .

If $\angle A, \angle B > 90^\circ$, then $\angle ADB < 90^\circ$ and $\angle BCA < 90^\circ$ since the sum of two interior angles in a triangle must be less than 180° . This means $\angle A > \angle ADB$ and $\angle B > \angle BCA$.

In $\triangle ABD$ and $\triangle BAC$,

$$\begin{aligned}
\angle A &= \angle B && \text{(given)} \\
AB &= AB && \text{(common side)} \\
BD &= AC && \text{(given)} \\
\angle A &> \angle ADB && \\
\therefore\ \triangle ABD &\cong \triangle BAC && \text{(ASS case 2)} \\
\therefore\ AD &= BC && \text{(corr. sides, } \cong \triangle\text{s)} \\
\therefore\ ABCD &\text{ is an isosceles trapezium.} && \\
& && \text{(opp. sides diags equal)}
\end{aligned}$$

Note: If $\angle A, \angle B < 90^\circ$, then it is possible that $ABCD$ is not an isosceles trapezium, because the ASS case 2 condition no longer holds.

$\square$

References

[1] The University of Hong Kong, “Deductive geometry.” [Online]. Available: https://www.studocu.com/hk/document/the-university-of-hong-kong/university-mathematics-i/chapter-4-basic-geometry/39019206

[2] Queen’s College, “Euclidean geometry.” [Online]. Available: https://qcweb.qc.edu.hk/math/Teaching_Learning/Euclean%20geometry%20theorem.pdf

[3] Proof Wiki, “Isosceles triangle has two equal angles.” [Online]. Available: https://proofwiki.org/wiki/Isosceles_Triangle_has_Two_Equal_Angles

[4] ——, “Two lines meet at unique point.” [Online]. Available: https://proofwiki.org/wiki/Two_Lines_Meet_at_Unique_Point