Make A Sequence Walkthrough

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0 Introduction

Make The Sequence is a maths puzzle game I created using Pygame. The goal of this game is to make the sequence given in each level by inputting an appropriate formula with a limited number of characters. It is preferred that the solution uses as few characters as possible.

1 Levels

Sequence 1. 1, 2, 3, 4, 5, ...

Solution 1. a(n) = $\boxed{n}$

Sequence 2. 2, 4, 6, 8, 10, ...

Solution 2. a(n) = $\boxed{2n}$

Sequence 3. 4, 5, 6, 7, 8, ...

Solution 3. a(n) = $\boxed{n+3}$

Sequence 4. 1, 1, 1, 1, 1, ...

Solution 4. a(n) = $\boxed{1}$

Sequence 5. 1, 3, 5, 7, 9, ...

Solution 5. a(n) = $\boxed{2n-1}$

Sequence 6. 5, 9, 13, 17, 21, ...

Solution 6. a(n) = $\boxed{4n+1}$

Sequence 7. 13, 23, 33, 43, 53, ...

Solution 7. a(n) = $\boxed{10n+3}$

Sequence 8. 38, 31, 24, 17, 10, ...

Solution 8. Note that this is an arithmetic sequence with common difference -7. So a(n)= $38-7(n-1)$= $\boxed{45-7n}$.

Sequence 9. 1, 4, 9, 16, 25, ...

Solution 9. a(n) = $n**2 = \boxed{nn}$.

Sequence 10. 1, 3, 6, 10, 15, ...

Solution 10. Note that this is the sequence of **triangular numbers**.
So a(n) = $\boxed{n(n+1)/2}$.

Sequence 11. 1, 8, 27, 64, 125, ...

Solution 11. a(n) = $n**3 = \boxed{nnn}$.

Sequence 12. 1, 2, 4, 8, 16, ...

Solution 12. a(n) = $\boxed{2**n/2}$.

Sequence 13. 1, 2, 4, 7, 11, ...

Solution 13. Note that the first difference of the sequence is 1, 2, 3, 4 , ... and the second difference is 1, 1, 1, ... Let $\Delta a(n)$ denote the first difference sequence and $\Delta^2 a(n)$ denote the second difference sequence.

We have $\Delta a(n+1) - \Delta a(n) = \Delta^2 a(n)$.

Suppose we want to find $\Delta a(n)$ given $\Delta^2 a(n) = 1$. Putting $\Delta^2 a(n) = 1$ into the above equation:

$$\Delta a(n+1) - \Delta a(n) = 1$$

Since $\Delta^2 a(n)$ is a constant sequence, we guess that $\Delta a(n)$ is a linear sequence, so $\Delta a(n) = cn + d$ for some c and d. Putting $\Delta a(n) = cn + d$:

$$c(n+1) + d - (cn + d) = 1$$
$$c = 1$$

From our observations above, we know that $\Delta a(1) = c(1) + d = 1$, so $d = 0$, and $\Delta a(n) = n$.

Now we want to find $a(n)$ given $\Delta a(n) = n$. We have $a(n+1) - a(n) = \Delta a(n) = n$. Since $\Delta a(n)$ is a linear sequence, we guess that $a(n)$ is a quadratic sequence, so we let $a(n) = An^2 + Bn + C$. Putting $a(n) = An^2 + Bn + C$:

$$A(n+1)^2 + B(n+1) + C - (An^2 + Bn + C) = n$$
$$A((n+1)^2 - n^2) + B((n+1) - n) + C - C = n$$
$$A(2n+1) + B = n \qquad \ldots(*)$$

Since the difference equation holds true for all positive integer n , putting any positive integer value of n will work. Putting $n = 1$ and $n = 2$ into (*):

$$\begin{cases} A(2(1)+1) + B = 1 \\ A(2(2)+1) + B = 2 \end{cases}$$

Solving, $A = \dfrac{1}{2}$ and $B = -\dfrac{1}{2}$.

So $a(n) = \dfrac{1}{2}n^2 - \dfrac{1}{2}n + C$. From the sequence given at the beginning, we know that $a(1) = 1$. Putting $a(1) = 1$:

$$a(1) = \frac{1}{2}(1)^2 - \frac{1}{2}(1) + C = 1$$
$$C = 1$$

Thus, a(n) $= \dfrac{1}{2}n^2 - \dfrac{1}{2}n + 1 = \boxed{n(n-1)/2 + 1}$.

Sequence 14. 3, 9, 27, 81, 243, ...

Solution 14. Note that this is a geometric sequence with first term = 3 and common ratio = 3 . More succinctly, this is a sequence of powers of 3.

a(n) = $\boxed{3 * *n}$.

Sequence 15. 48, 72, 108, 162, 243, ...

Solution 15. Note that this is a geometric sequence with first term = 48 and common ratio = 3/2 .

a(n) $= 48(3/2) **(n-1) = \boxed{32(3/2) **n}$.

Sequence 16. 1, 11, 111, 1111, 11111, ...

Solution 16. Note that
$a(1) = 1$,
$a(2) = 10^2 + 1$,
$a(3) = 10^3 + 10^2 + 1$,
$a(4) = 10^4 + 10^3 + 10^2 + 10 + 1$,
$a(5) = 10^5 + 10^4 + 10^3 + 10^2 + 10 + 1$.
Each term is the sum of geometric sequence with first term = 1 and common ration = 10 . Using the sum of geometric sequence formula $S(n) = \dfrac{A(r^n-1)}{r-1}$ where first term = A , common ratio = r and number of terms = n ,
We get a(n) $= \dfrac{10^n - 1}{10-1} = \boxed{(10 **n - 1)/9}$.

Sequence 17. 3, -6, 9, -12, 15, ...

Solution 17. Note that the absolute sequence 3, 6, 9, 12, 15, ... is given by $b(n) = 3n$. The corresponding alternating sequence can be obtained by adding $(-1)^n$ to the formula of the absolute sequence. However, in this alternating sequence, the odd-n th term is positive but the even-n th term is negative, so we add one more negative sign to the formula.
a(n) $= \boxed{-3n(-1) **n}$.

Sequence 18. 5, 13, 25, 41, 61, ...

Solution 18. Note that the sequence of $\Delta a(n)$ is 8, 12, 16, 20, ... and $\Delta^2 a(n) = 4$. By inspection, we see that $\Delta a(n) = 4n + 4$. Let $a(n) = An^2 + Bn + C$. Then

$$A(2n+1) + B = 4n + 4 \qquad \ldots (*)$$

Putting $n = 1$ and $n = 2$ into (*):

$$\begin{cases} A(2(1)+1) + B = 4(1) + 4 \\ A(2(2)+1) + B = 4(2) + 4 \end{cases}$$

Solving, $A = 2$ and $B = 2$. Thus $a(n) = 2n^2 + 2n + C$. Since $a(1) = 2(1)^2 + 2(1)^2 + C = 5$, we have $C = 1$.
a(n) $= \boxed{2nn + 2n + 1}$.

Discussion 18. Note that $a(n) = n^2 + (n+1)^2$.

Sequence 19. 1, 5, 12, 22, 35, ...

Solution 19. Note that the sequence of $\Delta a(n)$ is 4, 7, 10, 13, ... and $\Delta^2 a(n) = 3$. By inspection, we see that $\Delta a(n) = 3n + 1$. Let $a(n) = An^2 + Bn + C$. Then

$$A(2n+1) + B = 3n + 1 \qquad \ldots (*)$$

Putting $n = 1$ and $n = 2$ into (*):

$$\begin{cases} A(2(1)+1) + B = 3(1) + 1 \\ A(2(2)+1) + B = 3(2) + 1 \end{cases}$$

Solving, $A = \dfrac{3}{2}$ and $B = -\dfrac{1}{2}$. Thus $a(n) = \dfrac{3}{2}n^2 - \dfrac{1}{2}n + C$. Since $a(1) = \dfrac{3}{2}(1)^2 - \dfrac{1}{2}(1) + C = 1$, we have $C = 0$.

a(n) = $\boxed{n(3n-1)/2}$.

Discussion 19. This sequence is the sequence of **pentagonal numbers** .

Sequence 20. 1, 5, 14, 30, 55, ...

Solution 20. Note that the sequence of $\Delta a(n)$ is 4, 9, 16, 25, ... , so $\Delta a(n) = (n+1)^2$, which is quadratic. So we guess that $a(n)$ is a cubic sequence. Let $a(n) = An^3 + Bn^2 + Cn + D$. Then

$$a(n+1) - a(n) = \Delta a(n)$$
$$A(n+1)^3 + B(n+1)^2 + C(n+1) + D - (An^3 + Bn^2 + Cn + D) = (n+1)^2$$
$$A(3n^2 + 3n + 1) + B(2n+1) + C = (n+1)^2 \qquad \ldots (*)$$

Putting $n = 1$, $n = 2$ and $n = 3$ into (*):

$$\begin{cases} 7A + 3B + C = 4 \\ 19A + 5B + C = 9 \\ 37A + 7B + C = 16 \end{cases}$$

Solving, $A = \dfrac{1}{3}$, $B = \dfrac{1}{2}$ and $C = \dfrac{1}{6}$. Thus $a(n) = \dfrac{1}{3}n^3 + \dfrac{1}{2}n^2 + \dfrac{1}{6}n + D$. Since $a(1) = \dfrac{1}{3}(1)^3 + \dfrac{1}{2}(1)^2 + \dfrac{1}{6}(1) + D = 1$, we have $D = 0$.

a(n) = $\dfrac{1}{3}n^3 - \dfrac{1}{2}n^2 + \dfrac{1}{6}n = \boxed{n(n+1)(2n+1)/6}$.

Discussion 20. This formula is the sum of the first n square number, i.e. $n(n+1)(2n+1)/6 = \sum_{k=1}^{n} k^2$

Sequence 21. 1, 0, 1, 0, 1, ...

Solution 21. Start with the powers of (-1). The formula is $b(n) = (-1)^n$ and the sequence is: -1, 1, -1, 1, -1, ... Let $c(n) = 1 - b(n)$. The sequence is 2, 0, 2, 0, 2, ... Now divide by 2 to get $a(n)$.

a(n) = $\boxed{(1 - (-1) * *n)/2}$

Sequence 22. 23, 45, 89, 177, 353, ...

Solution 22. [1] Note that the sequence satisfies the recurrence relation $a(n + 1) = 2 * a(n) - 1$ with initial condition $a(1) = 23$.

Let b_n be a sequence that also satisfies the recurrence relation of $a(n)$ (but with different initial conditions) , so that $b_{n+1} = 2b_n - 1$. As the recurrence relation is in the form $a(n + 1) = ka(n) + f(n)$ where f is a constant function, we guess that the general formula of b_n is also a constant function, say $b_n = d$. Putting $b_n = d$ and $b_{n+1} = d$ into the recurrence relation:

$$d = 2d - 1$$

$$d = 1$$

So $b_n = 1$ for all positive integers n . We want to somehow transform b_n into $a(n)$, so let's find some help.

Let h_n be a sequence with homogeneous recurrence relation $h_{n+1} = 2h_n$. This is a geometric sequence with common ration = 2. Let A be the first term (initial condition) . Then the general formula is $h_n = A(2^{n-1})$.

Let's sum these two sequences to make a new sequence a_n and see what happens. Define $a_n = h_n + b_n$ for all positive integers n. We start with the recurrence relations :

$$h_{n+1} = 2h_n \tag{1}$$

$$b_{n+1} = 2b_n - 1 \tag{2}$$

(1) + (2):

$$h_{n+1} + b_{n+1} = 2h_n + 2b_n - 1$$

$$a_{n+1} = 2a_n - 1$$

We get the recurrence relation of $a(n)$, so we are in the right direction.

Let $a(n) = a_n$ and $a(n) = h_n + b_n = A(2^{n-1}) + 1$. We can find the constant A by considering $a(1)$:

$$a(1) = A(2^{1-1}) + 1 = 23$$

$$A = 22$$

a(n) = $22(2^{n-1}) + 1 = \boxed{2 * *n11 + 1}$

Sequence 23. 1, 4, 9, 18, 35, ...

Solution 23. Note that the sequence of $\Delta a(n)$ is 3, 5, 9, 17, ... , and the sequence of $\Delta^2 a(n)$ is 2, 4, 8, ... This suggests that the general formula of $\Delta a(n)$ has the component of 2^n . By inspection, we find that $\Delta a(n) = 2^n + 1$. This suggests that $a(n)$ is in a similar form. Let $a(n) = A(2^n) + Bn + C$. Then

$$a(n+1) - a(n) = \Delta a(n)$$

$$A(2^{n+1}) + B(n+1) + C - (A(2^n) + Bn + C) = 2^n + 1$$
$$A(2^n)(2-1) + B = 2^n + 1$$

Comparing both sides, we get $A = 1$ and $B = 1$. So $a(n) = (2^n) + n + C$. Since $a(1) = (2^1) + 1 + C = 1$, we have $C = -2$.
a(n) = $\boxed{2**n + n - 2}$

Sequence 24. 1, 9, 36, 100, 225, ...

Solution 24. Note that the sequence is $1^2, 3^2, 6^2, 10^2, 15^2$, ..., which is the square of triangular numbers. The formula of triangular number is $\dfrac{n(n+1)}{2}$, so
a(n) = $\boxed{(n(n+1)/2)**2}$

Discussion 24. $\left(\dfrac{n(n+1)}{2}\right)^2 = \displaystyle\sum_{k=1}^{n} k^3$

Sequence 25. 17, 69, 17, 69, 17, ...

Solution 25. Note that the sequence of $1 + (-1)^n$ is 0, 2, 0, 2, 0, ... Multiply by 26 to get 0, 52, 0, 52, 0 , ..., then add 17 to get 17, 69, 17, 69, 17, ...
a(n) = $26(1 + (-1)^n) + 17 = \boxed{26(-1)**n + 43}$

Sequence 26. 4, 16, 64, 256, 1024, ...

Solution 26. Note that the sequence is $2^2, 2^4, 2^6, 2^8, 2^{10}, \ldots$.
a(n) = $\boxed{2**(2n)}$

Sequence 27. 2, 16, 512, 65536, 33554432, ...

Solution 27. Note that the sequence is $2^1, 2^4, 2^9, 2^{16}, 2^{25}, \ldots$.
a(n) = $\boxed{2**n**2}$

Sequence 28. 1, 4, 27, 256, 3125, ...

Solution 28. Note that the sequence is $1^1, 2^2, 3^3, 4^4, 5^5, \ldots$.
a(n) = $\boxed{n**n}$

Sequence 29. 2, 4, 16, 256, 65536, ...

Solution 29. Note that the sequence is $2^1, 2^2, 2^4, 2^8, 2^{16}, \ldots$.
a(n) = $\boxed{2**2**(n-1)}$

Sequence 30. 1, 12, 108, 864, 6480, ...

Solution 30. Let $\delta\, a(n) = a(n+1)/a(n)$ denote the first ratio of $a(n)$ (the ratio between two successive terms) . Note that the first ratio is:

$$\begin{aligned}
\delta\, a(1) &= a(2)/a(1) = 12 \\
\delta\, a(2) &= a(3)/a(2) = 9 \\
\delta\, a(3) &= a(4)/a(3) = 8 \\
\delta\, a(4) &= a(5)/a(4) = 7.5
\end{aligned}$$

Let $\delta^2 a(n) = \delta\, a(n+1)/\delta\, a(n)$ denote the second ratio of $a(n)$ (the ratio between two successive first ratios). The second ratio is:

$$\begin{aligned}
\delta^2\, a(1) &= \frac{\delta\, a(2)}{\delta\, a(1)} = \frac{3}{4} \\
\delta^2\, a(2) &= \frac{\delta\, a(3)}{\delta\, a(2)} = \frac{8}{9} \\
\delta^2\, a(3) &= \frac{\delta\, a(4)}{\delta\, a(3)} = \frac{15}{16} \\
\delta^2\, a(n) &= \frac{\delta\, a(n+1)}{\delta\, a(n)} = 1 - \frac{1}{(n+1)^2}
\end{aligned}$$

Multiply both sides by $\delta\, a(n)$:

$$\delta\, a(n+1) = \delta\, a(n) \left(1 - \frac{1}{(n+1)^2}\right)$$

We know that $\delta\, a(1) = 12$. Writing out the following terms:

$$\begin{aligned}
&\delta\, a(2) = 12\,(1 - \frac{1}{2^2}) \\
&\delta\, a(3) = \delta\, a(2)\,(1 - \frac{1}{3^2}) = 12(1 - \frac{1}{2^2})(1 - \frac{1}{3^2})
\end{aligned}$$

$$\delta\, a(4) = \delta\, a(3)\,(1-\frac{1}{4^2}) = 12(1-\frac{1}{2^2})(1-\frac{1}{3^2})(1-\frac{1}{4^2})$$

$$\vdots$$

We are interested in finding the general formula of this product for $n \geq 2$: (LHS is a just shorthand for the product, not a general formula.)

$$\prod_{k=2}^{n}(1-\frac{1}{k^2}) = (1-\frac{1}{2^2})(1-\frac{1}{3^2})\cdots(1-\frac{1}{n^2})$$

We can write: [2]

$$\begin{aligned}
\prod_{k=2}^{n}(1-\frac{1}{k^2}) = \prod_{k=2}^{n}(\frac{k^2-1}{k^2}) &= \prod_{k=2}^{n}(\frac{(k+1)(k-1)}{k^2}) \\
&= \left(\prod_{k=2}^{n}(k+1)\right)\left(\prod_{k=2}^{n}(k-1)\right)\left(\prod_{k=2}^{n}\frac{1}{k^2}\right) \\
&= \left(\prod_{k=3}^{n+1}k\right)\left(\prod_{k=1}^{n-1}k\right)\left(\prod_{k=2}^{n}\frac{1}{k}\right)^2 \\
&= (\frac{(n+1)!}{2!})(n-1)!(\frac{1}{n!})^2 \\
&= \frac{n+1}{2n}
\end{aligned}$$

Thus, $\delta\, a(n) = 12(\frac{n+1}{2n}) = 6\,(\frac{n+1}{n})$. Now we find the formula for $a(n)$. Recall that

$$\delta\, a(n) = \frac{a(n+1)}{a(n)} = 6\,(\frac{n+1}{n})$$

$$a(n+1) = 6\,a(n)(\frac{n+1}{n})$$

We know that $a(1) = 1$. Writing out the following terms:

$$\begin{aligned}
a(2) &= 6(1)(\frac{1+1}{1}) \\
a(3) &= 6\,a(2)\cdot(\frac{2+1}{2}) = 6\left(6(\frac{1+1}{1})\right)\cdot(\frac{2+1}{2}) \\
a(4) &= 6\,a(3)\cdot(\frac{3+1}{3}) = 6\left(6\left(6(\frac{1+1}{1})\right)\cdot(\frac{2+1}{2})\right)\cdot(\frac{3+1}{3}) \\
&\vdots \\
a(n) &= 6^{n-1}\frac{n!}{(n-1)!} \\
a(n) &= n(6^{n-1})
\end{aligned}$$

a(n) = $\boxed{n6 * *n/6}$

Sequence 31. 1, 1, 2, 2, 3, ...

Solution 31. By inspection, we see that the formula of the sequence is $\left\lfloor \frac{n+1}{2} \right\rfloor$. Note that the definition of the floor division operator // is:

$$a \,//\, b = \left\lfloor \frac{a}{b} \right\rfloor$$

a(n) = $\boxed{(n+1)//2}$

Sequence 32. 1, 3, 4, 6, 7, ...

Solution 32. We start with the sequence 1, 2, 3, 4, 5 , ..., and add another sequence 0, 1, 1, 2, 2 , ..., which can be created by the formula $n//2$.

a(n) = $\boxed{n + n//2}$

Sequence 33. 1, 2, 3, 4, 4, ...

Solution 33. Consider the sequence 0, 0, 0, 0, -1 . This can be created by the formula $-n//5$. Now add this to the sequence 1, 2, 3, 4, 5 to get the desired sequence.

a(n) = $\boxed{n - n//5}$

Sequence 34. 1, 2, 0, 1, 2, ...

Solution 34. We start with the sequence 1, 2, 3, 4, 5 , ... We need to add another sequence 0, 0, -3, -3, -3, ..., which can be created by the formula $-n//3 * 3$. Adding the sequences to together, we get

a(n) = $\boxed{n - n//3 * 3}$

Sequence 35. 1, 2, 3, 2, 1, ...

Solution 35. We start with the sequence 1, 2, 3, 4, 5 , ... We need to add another sequence that has 0 for the first 3 terms and -2 for the 4th term, which can be created by the formula $-n//4 * 2$. So the sequence becomes 1, 2, 3, 2, 3, ... We need to add another sequence that has 0 for the first 4 terms and -2 for the 5th term, which can be created by the formula $-n//5 * 2$.

Adding the three sequences to together, we get a(n) = $\boxed{n - n//4 * 2 - n//5 * 2}$

Sequence 36. 1, 0, 0, 0, 0, ...

Solution 36. a(n) = $\boxed{1//n}$

Sequence 37. 1, 2, 3, 5, 9, ...

Solution 37. We start with the sequence of powers of 2 with shifted index: 1/2, 1, 2, 4, 8 , ..., which is 2^{n-2} . Taking the floor of each term, the sequence becomes 0, 1, 2, 4, 8 and the formula becomes $2^{n-2}//1$. Adding 1 to each term, we get the desired sequence 1, 2, 3, 5, 9, ...

a(n) = $\boxed{2 * *n/4//1 + 1}$

Sequence 38. 6, 14, 36, 98, 276, ...

Solution 38. Note that

$$\begin{aligned} a(2) &= 14 = 6 \times 2 + 2 \\ a(3) &= 36 = 14 \times 2 + 8 \\ a(4) &= 98 = 36 \times 2 + 26 \\ a(5) &= 276 = 98 \times 2 + 80 \end{aligned}$$

By inspection, we find that the number added at the end of each equation is $3^n - 1$. So we have the recurrence relation:

$$a(n+1) = 2\,a(n) + 3^n - 1$$

with $a(1) = 6$. We want to find the general formula of this recurrence relation.

Let b_n be a sequence that also satisfies the recurrence relation of $a(n)$ (but with different initial conditions) , so that $b_{n+1} = 2b_n + 3^n - 1$. As the recurrence relation is in the form $a(n+1) = ka(n) + f(n)$ where $f(n) = m(3^n) + l$ for some m and l , we guess that the general formula of b_n is also in the form of $f(n)$, say $b_n = A(3^n) + B$. Putting $b_n = A(3^n) + B$ and $b_{n+1} = A(3^{n+1}) + B$ into the recurrence relation:

$$A(3^{n+1}) + B = 2(A(3^n) + B) + 3^n - 1$$

$$0 = A(2(3^n) - 3^{n+1}) + 3^n + (B - 1)$$

Since this is an identity over all positive integers values of n , we have

$$\begin{aligned} A(2(3^n) - 3^{n+1}) + 3^n &= 0 & \qquad B - 1 &= 0 \\ A(3^n)(2-3) + 3^n &= 0 & B &= 1 \end{aligned}$$

$$(3^n)(-A+1) = 0$$
$$A = 1$$

Thus, $b_n = 3^n + 1$.

We want to somehow transform b_n into $a(n)$, so let's find some help.

Let h_n be a sequence with homogeneous recurrence relation $h_{n+1} = 2h_n$. This is a geometric sequence with common ration = 2. Let A be the first term (initial condition) . Then the general formula is $h_n = A(2^{n-1})$.

Let's sum these two sequences to make a new sequence a_n and see what happens. Define $a_n = h_n + b_n$ for all positive integers n. We start with the recurrence relations :

$$h_{n+1} = 2h_n \tag{1}$$
$$b_{n+1} = 2b_n + 3^n - 1 \tag{2}$$

(1) + (2):

$$h_{n+1} + b_{n+1} = 2h_n + 2b_n + 3^n - 1$$
$$a_{n+1} = 2a_n + 3^n - 1$$

We get the recurrence relation of $a(n)$, so we are in the right direction.

Let $a(n) = a_n$ and $a(n) = h_n + b_n = A(2^{n-1}) + 3^n + 1$. We can find the constant A by considering $a(1)$:

$$a(1) = A(2^{1-1}) + 3^1 + 1 = 6$$
$$A = 2$$

Thus, a(n) $= 2(2^{n-1}) + 3^n + 1 = \boxed{1 + 2**n + 3**n}$

Sequence 39. 5, 32, 176, 896, 4352, ...

Solution 39. The first ratio of the sequence is (approximately) 6.4, 5.5, 5.091, 4.857, ..., which is approaching 4 . So let's try to find out if the recurrence relation is in the form of $a(n+1) = 4\,a(n) + \text{something}$.

Note that

$$a(2) = 32 = 5 \times 4 + 12$$
$$a(3) = 176 = 32 \times 4 + 48$$
$$a(4) = 896 = 176 \times 4 + 192$$
$$a(5) = 4352 = 896 \times 4 + 768$$

By inspection, we find that the number added at the end of each equation is $3(4^n)$. So we have the recurrence relation:

$$a(n+1) = 4\,a(n) + 3(4^n)$$

with $a(1) = 5$. We want to find the general formula of this recurrence relation.

Let b_n be a sequence that also satisfies the recurrence relation of $a(n)$ (but with different initial conditions) , so that $b_{n+1} = 4\,b_n + 3(4^n)$. Let's guess that the general formula of b_n is $b_n = A(4^n)$. Putting $b_n = A(4^n)$ and $b_{n+1} = A(4^{n+1})$ into the recurrence relation:

$$A(4^{n+1}) = 4(A(4^n)) + 3(4^n)$$

$$0 = 3(4^n)$$

Clearly our guess is wrong, so we need another guess. Let's guess that $b_n = q(n)(4^{n-1})$ for some expression $q(n)$. Putting $b_n = q(n)(4^{n-1})$ and $b_{n+1} = q(n+1)(4^n)$ into the recurrence relation:

$$\begin{aligned} q(n+1)(4^n) &= 4\,(q(n)(4^{n-1})) + 3(4^n) \\ q(n+1)(4^n) - q(n)(4^n) &= 3(4^n) \\ (q(n+1) - q(n))(4^n) &= 3(4^n) \end{aligned}$$

Since this is an identity over all positive integers values of n , we have

$$q(n+1) - q(n) = 3$$

We need to find an expression $q(n)$ that satisfies this recurrence relation. An obvious choice is the formula of multiples of 3:

$$q(n) = 3n$$

$$q(n+1) - q(n) = 3(n+1) - 3n = 3$$

It works. So we have $b_n = 3n(4^{n-1})$.

Let h_n be a sequence with the corresponding homogeneous recurrence relation $h_{n+1} = 4h_n$. The general formula is $h_n = A(4^{n-1})$ for some A.

Let $a(n) = h_n + b_n = A(4^{n-1}) + 3n(4^{n-1})$. We can find the constant A by considering $a(1)$:

$$a(1) = A(4^{1-1}) + 3(1)(4^{1-1}) = 5$$

$$A = 2$$

Thus, a(n) $= 2(4^{n-1}) + 3n(4^{n-1}) = \boxed{(2+3n)4**n/4}$

Sequence 40. -1, 6, 49, 244, 1023 . . .

Solution 40. The first ratio of the sequence is (approximately) -6, 8.167, 4.980, 4.193, ... It seems that it is approaching 4 too quickly . May it is actually approaching 3. So let's guess the recurrence relation is in the form of $a(n+1) = 3\,a(n) + \text{something}$.

Note that

$$
\begin{aligned}
a(2) &= 6 = -1 \times 3 + 9 \\
a(3) &= 49 = 6 \times 3 + 31 \\
a(4) &= 244 = 49 \times 3 + 97 \\
a(5) &= 1023 = 244 \times 3 + 291
\end{aligned}
$$

The number added at the end of each equation doesn't seem to follow any obvious pattern, but they seem a bit close to the powers of 3. Let's do another recurrence analysis on these numbers using the ratio 3 again. Let $b(n)$ be the number added at the end of the equation of $a(n+1)$. We have:

$$
\begin{aligned}
b(1) &= 9 \\
b(2) &= 31 = 9 \times 3 + 4 \\
b(3) &= 97 = 31 \times 3 + 4 \\
b(4) &= 291 = 97 \times 3
\end{aligned}
$$

So frustrating. It almost follows a nice recurrence relation. Let's do a general formula analysis instead, by using the form $3^n +$ [something] :

$$
\begin{aligned}
b(1) &= 9 = 3^2 + 0 \\
b(2) &= 31 = 3^3 + 4 \\
b(3) &= 97 = 3^4 + 16 \\
b(4) &= 291 = 3^5 + 48
\end{aligned}
$$

The numbers at the end (0, 4, 16, 48) almost follows the powers of 4. Let's examine their prime factors:

$$
\begin{aligned}
0 &= 0 = 0(2^1) \\
4 &= 2^2 = 1(2^2) \\
16 &= 2^4 = 2(2^3) \\
48 &= 2^4 \cdot 3 = 3(2^4)
\end{aligned}
$$

We have somehow artificial created a pattern using these 4 terms, but let's roll with it. We have $b(n) = 3^{n+1} + (n-1)(2^n)$. So

$$
\begin{aligned}
a(n+1) &= 3\,a(n) + b(n) \\
a(n+1) &= 3\,a(n) + 3^{n+1} + (n-1)(2^n)
\end{aligned}
$$

Let b_n be a sequence that also satisfies the recurrence relation of $a(n)$, so that $b_{n+1} = 3\,b_n + 3^{n+1} + (n-1)(2^n)$. Let's guess that the general formula of b_n is $b_n = A(3^n) + (Cn+D)(2^n)$. Putting $b_n = A(3^n) + (Cn+D)(2^n)$ and $b_{n+1} = A(3^{n+1}) + (C(n+1)+D)(2^{n+1})$ into the recurrence relation:

$$
A(3^{n+1}) + (C(n+1)+D)(2^{n+1}) = 3(A(3^n) + (Cn+D)(2^n)) + 3^{n+1} + (n-1)(2^n)
$$

We quickly see a problem when the two terms $A(3^{n+1})$ and $3A(3^n)$ cancel out, leaving the 3^{n+1} on RHS alone. This is bad, because there is no values of n such that $3^n = 0$. So our guess in the part $A(3^{n+1})$ is wrong. Let's instead guess that $b_n = (An + B)(3^n) + (Cn + D)(2^n)$. Putting it into the recurrence relation again:

$$(A(n+1)+B)(3^{n+1})+(C(n+1)+D)(2^{n+1}) = 3((An+B)(3^n)+(Cn+D)(2^n))+3^{n+1}+(n-1)(2^n)$$

Grouping terms together:

$$(A(n+1) + B - (An + B) - 1)(3^{n+1}) + (2C(n + 1) + 2D - 3(Cn + D) - (n - 1))(2^{n+1}) = 0$$

$$\begin{aligned}(A-1)(3^{n+1}) + (-Cn + 2C - D - n + 1)(2^{n+1}) &= 0\\ (A-1)(3^{n+1}) + ((-C-1)n + 2C - D + 1)(2^{n+1}) &= 0\end{aligned}$$

$$\begin{array}{ccccc} A - 1 = 0 & \text{and} & -C - 1 = 0 & \text{and} & 2C - D + 1 = 0\\ A = 1 & \text{and} & C = -1 & \text{and} & D = 2(-1) + 1 = -1\end{array}$$

Since B has been cancelled out, its value doesn't matter, so we can just set $B = 0$.

Thus, $b_n = n(3^n) - (n + 1)(2^n)$. If we type this formula in the input box (tactically to minimize character used), we will see that the sequence it generates is -1, 6, 49, 244, 1023, . . . , which is exactly what we want. So we don't even need to add the homogeneous linear recurrence general formula to b_n .

a(n) = $b_n = \boxed{n3 * *n - (n + 1)2 * *n}$

Sequence 41. 1701, 7168, 21875, 54432, 117649, . . .

Solution 41. The first ratio is $\frac{1024}{243}, \frac{3125}{1024}, \frac{7776}{3125}, \frac{16807}{7776}$, ... Note that

$$\begin{aligned}\delta a(1) &= \frac{1024}{243} = \frac{4^5}{3^5}\\ \delta a(2) &= \frac{3125}{1024} = \frac{5^5}{4^5}\\ \delta a(3) &= \frac{7776}{3125} = \frac{6^5}{5^5}\\ \delta a(4) &= \frac{16807}{7776} = \frac{7^5}{6^5}\\ \delta a(n) &= \frac{(n+3)^5}{(n+2)^5}\end{aligned}$$

Also note that

$$a(1) = 7(3^5)$$
$$a(2) = 7(3^5) \cdot \frac{4^5}{3^5} = 7(4^5)$$
$$a(3) = 7(4^5) \cdot \frac{5^5}{4^5} = 7(5^5)$$
$$a(4) = 7(5^5) \cdot \frac{6^5}{5^5} = 7(6^5)$$
$$a(5) = 7(6^5) \cdot \frac{7^5}{6^5} = 7(7^5)$$

Thus, a(n) = $\boxed{7(n+2)**5}$

Sequence 42. 1, 121, 12321, 1234321, 123454321, ...

Solution 42. Note that

$$a(1) = 1$$
$$a(2) = 121 = 110 + 11$$
$$a(3) = 12321 = 11100 + 1110 + 111$$
$$a(4) = 1234321 = 1111000 + 111100 + 11110 + 1111$$
$$a(5) = 123454321 = 111110000 + 11111000 + 1111100 + 111110 + 11111$$

Let $b(n)$ be the sequence 1, 11, 111, 1111, 11111, ... This is sequence 16 , which has the general formula of $\frac{10^n - 1}{9}$. Note that each term of $a(n)$ is the sum of an n-term geometric sequence with common ratio 10 and the corresponding $b(n)$ term being the first term in the sequence. So

$$a(n) = b(n)\frac{10^n - 1}{10 - 1} = (\frac{10^n - 1}{9})(\frac{10^n - 1}{9}) = \frac{(10^n - 1)^2}{81}$$

a(n) = $\boxed{(10**n-1)**2/81}$

Sequence 43. 648, 2160, 5688, 13920, 33128, ...

Solution 43. The first ratio is 3.333 , 2.633 , 2.447 , 2.380 , ... Looks like it is approaching 2. Let's use recurrence analysis with ratio 2.

$$
\begin{aligned}
a(2) &= 2160 = 648 \times 2 + 864 \\
a(3) &= 5688 = 2160 \times 2 + 1368 \\
a(4) &= 13920 = 5688 \times 2 + 2544 \\
a(5) &= 33128 = 13920 \times 2 + 5288
\end{aligned}
$$

The numbers at the end seems growing too slow to have $\times 2$ recurrence again. Let's try to take the first difference instead. Let $b(n)$ be the sequence of numbers at the end and $\Delta b(n)$ be the first difference of $b(n)$. The sequence of $\Delta b(n)$ is 504, 1176, 2744, ... Note that the first ratio of $\Delta b(n)$ is:

$$
\frac{1176}{504} = \frac{7}{3}
$$
$$
\frac{2744}{1176} = \frac{7}{3}
$$

Thus, $\Delta b(n) = 504(\frac{7}{3})^{n-1} = 216(\frac{7}{3})^{n}$. This suggests that the formula of $b(n)$ has a similar form. Let's guess that $b(n) = A(\frac{7}{3})^n + C$. Then

$$
A(\frac{7}{3})^{n+1} - A(\frac{7}{3})^n = 216(\frac{7}{3})^n
$$
$$
A = 162
$$
$$
C = 864 - 162(\frac{7}{3})^1 = 486
$$

Thus, $b(n) = 162(\frac{7}{3})^n + 486$, and

$$
a(n+1) = 2\,a(n) + b(n) = 2\,a(n) + 162(\frac{7}{3})^n + 486
$$

Let b_n be a sequence that satisfies the recurrence relation of $a(n)$. Let's guess that the general formula of b_n is $b_n = A(\frac{7}{3})^n + C$. Putting $b_n = A(\frac{7}{3})^n + C$ and $b_{n+1} = A(\frac{7}{3})^{n+1} + C$ into the recurrence relation:

$$
A(\frac{7}{3})^{n+1} + C = 2(A(\frac{7}{3})^n + C) + 162(\frac{7}{3})^n + 486
$$
$$
(\frac{7}{3})^n(\frac{7}{3}A - 2A - 162) - C - 486 = 0
$$

$$\frac{7}{3}A - 2A - 162 = 0 \quad \text{and} \quad -C - 486 = 0$$
$$A = 486 \quad \text{and} \quad C = -486$$

Thus, $b_n = 486(\frac{7}{3})^n - 486$. Let $h_n = A(2^n)$, and let $a(n) = b_n + h_n = 486(\frac{7}{3})^n - 486 + A(2^n)$. Putting $n = 1$:

$$a(1) = 486(\frac{7}{3})^1 - 486 + A(2^1) = 648$$
$$A = 0$$

a(n) = $486(\frac{7}{3})^n - 486 = \boxed{486((7/3) **n - 1)}$

Sequence 44. 1, 2, 3, 5, 8, ...

Solution 44. This looks like the **Fibonacci sequence**, but 15 characters are not enough to type the general formula for it. So we create a fake sequence instead. Start with the n sequence: 1, 2, 3, 4, 5, ... Adding $n//4$, the sequence becomes 1, 2, 3, 5, 6, ... Adding $n//5 * 2$, the sequence becomes 1, 2, 3, 5, 8, ...
a(n) = $\boxed{n + n//4 + n//5 * 2}$

Sequence 45. 50, 67, 75, 80, 83, ...

Solution 45. Let $[n]$ denote the nearest integer of n . Note that

$$\begin{aligned}
50 &= 100 \cdot \frac{1}{2} \\
67 &= \left[100 \cdot \frac{2}{3}\right] \\
75 &= 100 \cdot \frac{3}{4} \\
80 &= 100 \cdot \frac{4}{5} \\
83 &= \left[100 \cdot \frac{5}{6}\right] \\
a(n) &= \left[100 \cdot \frac{n}{n+1}\right]
\end{aligned}$$

Note that the nearest integer function can be expressed in terms of the floor function:

$$[n] = \left\lfloor n + \frac{1}{2} \right\rfloor$$

Thus, a(n) $= \left\lfloor 100 \cdot \frac{n}{n+1} + \frac{1}{2} \right\rfloor = \boxed{(100n/(n+1)+1/2)//1}$

Sequence 46. 2, 3, 5, 7, 11, ...

Solution 46. We start with the $2n-1$ sequence: 1, 3, 5, 7, 9 , ... We need to add the sequence 1, 0, 0, 0, 0, ..., which is $1//n$. Then we add the sequence 0, 0, 0, 0, 2, ..., which is $n//5*2$.

a(n) $= \boxed{2n-1+1//n+n//5*2}$

Sequence 47. 4, 7, 15, 29, 59, ...

Solution 47. Note that

$$\begin{aligned} a(2) &= 7 = 4 \cdot 2 - 1 \\ a(3) &= 15 = 7 \cdot 2 + 1 \\ a(4) &= 29 = 15 \cdot 2 - 1 \\ a(5) &= 59 = 29 \cdot 2 + 1 \\ a(n+1) &= 2\,a(n) + (-1)^n \end{aligned}$$

Let b_n be a sequence that satisfies the recurrence relation of $a(n)$. Let's guess that $b_n = A(-1)^n$. Putting it into the recurrence relation:

$$\begin{aligned} A(-1)^{n+1} &= 2A(-1)^n + (-1)^n \\ -A &= 2A + 1 \\ A &= -\frac{1}{3} \end{aligned}$$

So $b_n = -\frac{1}{3}(-1)^n$. Let $h_n = A(2^n)$. Then $a(n) = b_n + h_n = -\frac{1}{3}(-1)^n + A(2^n)$. Putting $n = 1$:

$$-\frac{1}{3}(-1)^1 + A(2^1) = 4$$

$$A = \frac{11}{6}$$

a(n) $= -\frac{1}{3}(-1)^n + \frac{11}{6}(2^n) = \boxed{(2**n11-2(-1)**n)/6}$

Sequence 48. 20, 24, 29, 33, 38, ...

Solution 48. Note that the first difference is 4, 5, 4, 5, ..., which is 4, 4, 4, 4, ... added by 0, 1, 0, 1, ... So we have

$$a(n+1) - a(n) = 4 + \frac{1 + (-1)^n}{2} = \frac{9 + (-1)^n}{2} a(n+1) \quad = a(n) + \frac{9 + (-1)^n}{2}$$

Let b_n be a sequence that satisfies the recurrence relation of $a(n)$. Let's guess that $b_n = A(-1)^n + B$. Putting it into the recurrence relation:

$$A(-1)^{n+1} + B = A(-1)^n + B + \frac{9 + (-1)^n}{2}$$
$$A(-1)^{n+1} = A(-1)^n + \frac{9 + (-1)^n}{2}$$

There is no corresponding constant for the term $\frac{9}{2}$. That's not good. Let's try another guess: $b_n = A(-1)^n + Bn$. Putting it into the recurrence relation:

$$A(-1)^{n+1} + B(n+1) = A(-1)^n + Bn + \frac{9 + (-1)^n}{2}$$
$$(-1)^n(-A - A - \frac{1}{2}) + B - \frac{9}{2} = 0$$

$$-A - A - \frac{1}{2} = 0 \quad \text{and} \quad B - \frac{9}{2} = 0$$
$$A = -\frac{1}{4} \quad \text{and} \quad B = \frac{9}{2}$$

So $b_n = -\frac{1}{4}(-1)^n + \frac{9}{2}n$. Let $h_n = C$ and $a(n) = b_n + h_n = -\frac{1}{4}(-1)^n + \frac{9}{2}n + C$. Putting $n = 1$:

$$-\frac{1}{4}(-1)^1 + \frac{9}{2}(1) + C = 20$$
$$C = \frac{61}{4}$$

$$\text{a(n)} = -\frac{1}{4}(-1)^n + \frac{9}{2}n + \frac{61}{4} = \boxed{(18n - (-1) * *n + 61)/4}$$

Sequence 49. 4, 2, 6, 4, 8, ...

Solution 49. The first difference is -2, 4, -2, 4, ..., which is generated by the formula $-2 + 3(1 + (-1)^n) \quad = 3(-1)^n + 1$. So we have

$$a(n+1) = a(n) + 3(-1)^n + 1$$

Let b_n be a sequence that satisfies the recurrence relation of $a(n)$. Let's guess that $b_n = A(-1)^n$. Putting it into the recurrence relation:

$$\begin{aligned} A(-1)^{n+1} &= 2A(-1)^n + (-1)^n \\ -A &= 2A + 1 \\ A &= -\frac{1}{3} \end{aligned}$$

So $b_n = -\frac{1}{3}(-1)^n$. Let $h_n = A(2^n)$. Then $a(n) = b_n + h_n = -\frac{1}{3}(-1)^n + A(2^n)$. Putting $n = 1$:

$$-\frac{1}{3}(-1)^1 + A(2^1) = 4$$

$$A = \frac{11}{6}$$

$$\text{a(n)} = -\frac{1}{3}(-1)^n + \frac{11}{6}(2^n) = \boxed{n - 3/2((-1) * *n - 1)}$$

Sequence 50. 62, 363, 1364, 3905, 9330, ...

Solution 50. When we do ratio analysis, recurrence analysis and difference analysis, there doesn't seem to be any particular patterns at all. At this point, you may just give up and complain that this game is so stupid and all it involves is guessing and checking. I mean, you're not wrong. This game really sucks at creating fun and interesting gameplay.

If we are insanely lucky, we may think of listing out the prime factors of each terms in the sequence and discover something:

$$\begin{aligned} 62 &= 2 \cdot 31 & &= 2 \cdot 31 \\ 363 &= 3 \cdot 11 \cdot 11 & &= 3 \cdot 121 \\ 1364 &= 2 \cdot 2 \cdot 11 \cdot 31 & &= 4 \cdot 341 \\ 3905 &= 5 \cdot 11 \cdot 71 & &= 5 \cdot 781 \\ 9330 &= 2 \cdot 3 \cdot 5 \cdot 311 & &= 6 \cdot 1555 \end{aligned}$$

Notice that in four of the terms, the larger factor ends in '1'. If we subtract 1 from the larger factors of each term and find their prime factors:

$$\begin{aligned} 30 &= 2 \cdot 3 \cdot 5 & &= 2 \cdot 15 \\ 120 &= 2^3 \cdot 3 \cdot 5 & &= 3 \cdot 40 \\ 340 &= 2^2 \cdot 5 \cdot 17 & &= 4 \cdot 85 \\ 780 &= 2^2 \cdot 3 \cdot 5 \cdot 13 & &= 5 \cdot 156 \\ 1554 &= 2 \cdot 3 \cdot 7 \cdot 37 & &= 6 \cdot 259 \end{aligned}$$

Now do it again:

$$\begin{aligned} 14 &= 2 \cdot 7 & &= 2 \cdot 7 \\ 39 &= 3 \cdot 13 & &= 3 \cdot 13 \\ 84 &= 2^2 \cdot 3 \cdot 7 & &= 4 \cdot 21 \\ 155 &= 5 \cdot 31 & &= 5 \cdot 31 \\ 258 &= 2 \cdot 3 \cdot 43 & &= 6 \cdot 43 \end{aligned}$$

And again:

$$\begin{aligned} 6 &= 2 \cdot 3 & &= 2 \cdot 3 \\ 12 &= 2^2 \cdot 3 & &= 3 \cdot 4 \\ 20 &= 2^2 \cdot 5 & &= 4 \cdot 5 \\ 30 &= 2 \cdot 3 \cdot 5 & &= 5 \cdot 6 \\ 42 &= 2 \cdot 3 \cdot 7 & &= 6 \cdot 7 \end{aligned}$$

Thus,

$$a(n) = (n+1)(1+(n+1)(1+(n+1)(1+(n+1)(n+2))))$$

Replace $n+1$ with u to make it cleaner:

$$\begin{aligned} a(n) &= u(1+u(1+u(1+u(u+1)))) \\ &= u + u^2 + u^3 + u^4 + u^5 \\ &= \frac{u(u^5-1)}{u-1} \\ &= \frac{(n+1)((n+1)^5-1)}{n} \\ &= \frac{(n+1)^6 - n - 1}{n} \end{aligned}$$

a(n) = $\boxed{((n+1)**6-n-1)/n}$

Discussion 50. Or we can just cheat by using the OEIS (The On-Line Encyclopedia of Integer Sequences) `http://oeis.org/` and typing the sequence in the search bar. This yields sequence A152031 (with shifted index): `http://oeis.org/A152031` .

Sequence 51. 7, 5, 8, 4, 9, ...

Solution 51. The first difference is -2, 3, -4, 5, ..., which is $(n+1)(-1)^n$. So we have

$$a(n+1) = a(n) + (n+1)(-1)^n$$

Let b_n be a sequence that satisfies the recurrence relation of $a(n)$. Let's guess that $b_n = (An+B)(-1)^n$. Putting it into the recurrence relation:

$$\begin{aligned}(A(n+1)+B)(-1)^{n+1} &= (An+B)(-1)^n + (n+1)(-1)^n \\ (An+A+B)(-1) &= (An+B)+(n+1) \\ n(-2A-1)-A-2B-1=0\end{aligned}$$

$$\begin{aligned}-2A-1=0 \quad &\text{and} \quad -A-2B-1=0 \\ A=-\frac{1}{2} \quad &\text{and} \quad B=\frac{-(-\frac{1}{2})-1}{2}=-\frac{1}{4}\end{aligned}$$

So $b_n = (-\dfrac{1}{2}n - \dfrac{1}{4})(-1)^n$. Let $h_n = C$. Then $a(n) = b_n + h_n$ $= (-\dfrac{1}{2}n - \dfrac{1}{4})(-1)^n + C$. Putting $n = 1$:

$$(-\frac{1}{2}(1)+\frac{1}{4})(-1)^1 + C = 7$$

$$C = \frac{25}{4}$$

a(n) = $(-\dfrac{1}{2}n - \dfrac{1}{4})(-1)^n + \dfrac{25}{4} = \boxed{25/4-(2n+1)(-1)**n/4}$

Sequence 52. 5, 8, 13, 21, 34, ...

Solution 52. Note that this is part of the **Fibonacci sequence** , with

$$a(n+2) = a(n+1) + a(n)$$

. The first ratio is 1.6, 1.625, 1.615, 1.619, ..., which is approaching the **golden ratio** (with a value of $\frac{1+\sqrt{5}}{2}$) . Let's approximate this sequence with a geometric sequence with common ratio of $\frac{34}{21}$ and see what happens. Let $b(n) = 5(\frac{34}{21})^{n-1}$. The sequence of $b(n)$ is 5, 8.095, 13.107, 21.220, 34.356, ..., which is just a little bit larger than the desired sequence. Taking the floor, we get 5, 8, 13, 21, 34 , which is exactly the desired sequence. So $a(n) = 5(34/21)**(n-1)//1$. But there is a shorter formula. The term that precedes 5 in the Fibonacci sequence is 3, so we get $3(34/21)**n//1$, but that produces 4, 7, 12, 20, 33. Every term is 1 less than the desired terms. So we add 1 at the end.

a(n) = $\boxed{3(34/21)**n//1+1}$

Sequence 53. 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, ...

Solution 53. This is the first 15 terms of the Fibonacci sequence. We can't use floor division // on this level (because it is OP), but we have 50 characters to use, so we want to find the general formula of the Fibonacci sequence. We have

$$a(n+2) = a(n+1) + a(n)$$

with initial condition $a(1) = 1$ and $a(2) = 1$.

Let b_n be a sequence that satisfies the recurrence relation of $a(n)$. Let's guess that the general formula of b_n is $b_n = r^n$ for some r , which means we assume that there is some number r that satisfies

$$r^{n+2} = r^{n+1} + r^n$$

for all $n > 0$. Divide by r^n on both sides:

$$\begin{aligned} r^2 &= r + 1 \\ r^2 - r - 1 &= 0 \\ r &= \frac{1 \pm \sqrt{1+4}}{2} \\ r &= \frac{1+\sqrt{5}}{2} \qquad \text{or} \qquad \frac{1-\sqrt{5}}{2} \end{aligned}$$

There are two choices for the value of r. Let b_n be the sequence that takes $r = \frac{1+\sqrt{5}}{2}$ and h_n be the sequence that takes the other value of r. So $b_n = (\frac{1+\sqrt{5}}{2})^n$ and $h_n = (\frac{1-\sqrt{5}}{2})^n$. Note that both b_n and h_n satisfies the recurrence relation of a_n . Now we define a_n to be the linear combination of b_n and h_n , namely $a_n = Ab_n + Bh_n$ for some constant A and B . Starting with the recurrence relation of b_n and h_n:

$$b_{n+2} = b_{n+1} + b_n \tag{1}$$

$$h_{n+2} = h_{n+1} + h_n \tag{2}$$

$(1) \times A + (2) \times B$:

$$Ab_{n+2} + Bh_{n+2} = Ab_{n+1} + Bh_{n+1} + Ab_n + Bh_n \tag{1}$$

$$a_{n+2} = a_{n+1} + a_n \tag{2}$$

Thus, any linear combination of b_n and h_n also satisfies the recurrence relation of $a(n)$. Let $a(n) = a_n = Ab_n + Bh_n = A(\frac{1+\sqrt{5}}{2})^n + B(\frac{1-\sqrt{5}}{2})^n$. The initial conditions of $a(n)$ will determine the value of the constants. Consider $a(0)$ and $a(1)$: ($a(0)$ can be found by $a(0) = a(2) - a(1) = 1 - 1 = 0$.)

$$\begin{cases} a(0) = A(\frac{1+\sqrt{5}}{2})^0 + B(\frac{1-\sqrt{5}}{2})^0 = 0 \\ a(1) = A(\frac{1+\sqrt{5}}{2})^1 + B(\frac{1-\sqrt{5}}{2})^1 = 1 \end{cases}$$

$$\begin{cases} A+B &= 0 \\ A(1+\sqrt{5})+B(1-\sqrt{5}) &= 2 \end{cases}$$

Solving, $A = \dfrac{1}{\sqrt{5}}$ and $B = -\dfrac{1}{\sqrt{5}}$.

$$\begin{aligned} \text{a(n)} &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \\ &= \frac{(1+\sqrt{5})^n-(1-\sqrt{5})^n}{2^n\sqrt{5}} \\ &= \boxed{((1+5**(1/2))**n-(1-5**(1/2))**n)/2**n/5**(1/2)} \end{aligned}$$

Sequence 54. 2, 6, 24, 120, 720, ...

Solution 54. Since there is only 5 terms and we can use 40 characters, we can guess that $a(n)$ is a degree-4 (quartic) polynomial. Let $a(n) = An^4 + Bn^3 + Cn^2 + Dn + E$. Putting from $n = 1$ to $n = 5$, we get a system of equations:

$$\begin{cases} a(1) &= A(1)^4+B(1)^3+C(1)^2+D(1)^1+E = 2 \\ a(2) &= A(2)^4+B(2)^3+C(2)^2+D(2)^1+E = 6 \\ a(3) &= A(3)^4+B(3)^3+C(3)^2+D(3)^1+E = 24 \\ a(4) &= A(4)^4+B(4)^3+C(4)^2+D(4)^1+E = 120 \\ a(5) &= A(5)^4+B(5)^3+C(5)^2+D(5)^1+E = 720 \end{cases}$$

$$\begin{cases} A+B+C+D+E &= 2 \\ 16A+8B+4C+2D+E &= 6 \\ 81A+27B+9C+3D+E &= 24 \\ 256+64B+16C+4D+E &= 120 \\ 625A+125B+25C+5D+E &= 720 \end{cases}$$

Solving, we get $A = \dfrac{181}{12}$, $B = -\dfrac{841}{6}$, $C = \dfrac{5651}{12}$, $D = -\dfrac{3923}{6}$, $E = 310$. So

$$\begin{aligned} \text{a(n)} &= \frac{181}{12}n^4 - \frac{841}{6} + \frac{5651}{12} - \frac{3923}{6} + 310 \\ &= \frac{181n^4-1682n^3+5651n^2-7846n+3720}{12} \\ &= \boxed{(nn(181nn-1682n+5651)-7846n+3720)/12} \end{aligned}$$

Sequence 55. 3, 6, 18, 72, 360, ...

Solution 55. The first ratio is 2, 3, 4, 5, ..., so we have

$$a(n+1) = (n+1)\cdot a(n)$$

. We see that $a(n) = A(n!)$. Putting $n = 1$, we have $a(1) = A(1!) = 3$, so $A = 3$.

a(n) = $\boxed{3fact(n)}$

Sequence 56. 36, 54, 108, 270, 810, ...

Solution 56. The first ratio is 1.5, 2, 2.5, 3, ..., which is $1 + \frac{n}{2}$, so we have

$$a(n+1) = \frac{n+2}{2} \cdot a(n)$$

. Listing out the terms:

$$\begin{aligned}
a(1) &= 36 \\
a(2) &= 36(\frac{3}{2}) \\
a(3) &= 36(\frac{3}{2})(\frac{4}{2}) \\
a(4) &= 36(\frac{3}{2})(\frac{4}{2})(\frac{5}{2}) \\
a(5) &= 36(\frac{3}{2})(\frac{4}{2})(\frac{5}{2})(\frac{6}{2}) \\
a(n) &= 36 \cdot \frac{(n+1)!}{2!(2^{n-1})}
\end{aligned}$$

a(n) = $\boxed{36fact(n+1)/2**n}$

Sequence 57. 7, 28, 252, 4032, 100800, ...

Solution 57. The first ratio is 4, 9, 16, 25, ..., which is $(n+1)^2$, so we have

$$a(n+1) = (n+1)^2 \cdot a(n)$$

. We suspect that that $a(n) = a(1) \cdot (n!)^2$. Let's prove it using mathematical induction:

When $n = 1$, LHS $= a(1) = a(1) \cdot (1)^2 =$ RHS .

Assume that $a(k) = a(1) \cdot (k!)^2$ for some k .

When $n = k + 1$,

$$\begin{aligned}
\text{LHS} &= a(k+1) \\
&= (k+1)^2 \cdot a(k) && \text{(recurrence relation)} \\
&= (k+1)^2 \cdot a(1) \cdot (k!)^2 && \text{(induction hypothesis)} \\
&= a(1) \cdot ((k+1)!)^2 \\
&= \text{RHS}
\end{aligned}$$

The formula holds for $n = k + 1$.

By mathematical induction, $a(n) = a(1) \cdot (n!)^2$ for all positive integers n .

a(n) = $\boxed{7fact(n) * *2}$

Sequence 58. 23, 46, 138, 276, 828, 1656, 4968, 9936, 29808, 59616, . . .

Solution 58. The first ratio is 2, 3, 2, 3, 2, 3, 2, 3, 2, . . . , which is $2 + \dfrac{(1 + (-1)^n)}{2} = \dfrac{(5 + (-1)^n)}{2}$, so we have

$$a(n + 1) = \frac{(5 + (-1)^n)}{2} \cdot a(n)$$

. Listing out the terms:

$$\begin{aligned}
a(1) &= 23 \\
a(2) &= 23(2) \\
a(3) &= 23(2)(3) \\
a(4) &= 23(2)^2(3) \\
a(5) &= 23(2)^2(3)^2 \\
a(6) &= 23(2)^3(3)^2 \\
a(7) &= 23(2)^3(3)^3 \\
a(8) &= 23(2)^4(3)^3 \\
a(9) &= 23(2)^4(3)^4 \\
a(10) &= 23(2)^5(3)^4
\end{aligned}$$

Note that the powers of 2 follows the sequence 0, 1, 1, 2, 2, 3, 3, . . . , which is $n//2$. And the powers of 3 follows the sequence 0, 0, 1, 1, 2, 2, 3, 3, . . . , which is $(n - 1)//2$. So

a(n) = $\boxed{2 * *(n//2)3 * *((n - 1)//2)23}$

Sequence 59. 1, 19, 171, 969, 3876, 11628, 27132, 50388, 75582, 92378, . . .

Solution 59. The first ratio is $19, 9, \frac{17}{3}, 4, 3, \frac{7}{3}, \frac{13}{7}, \frac{3}{2}, \frac{11}{9}, \ldots$. We don't see any obvious patterns. Let's take the first difference of the first ratio: $-10, -\frac{10}{3}, -\frac{5}{3}, -1, -\frac{2}{3}, -\frac{10}{21}, -\frac{5}{14}, -\frac{5}{18}, \ldots$.

Note that the numerator of each term is a factor of 10. Let's make 10 the common numerator of all the terms: $-\frac{10}{1}, -\frac{10}{3}, -\frac{10}{6}, -\frac{10}{10}, -\frac{10}{15}, -\frac{10}{21}, -\frac{10}{28}, -\frac{10}{36}, \ldots$.

Note that the denominators of the terms are the triangular numbers, which is $\dfrac{n(n+1)}{2}$, so we have $\Delta\delta a(n) = -\dfrac{10}{\frac{n(n+1)}{2}} = -\dfrac{20}{n(n+1)}$. Taking the partial fraction, we have

$$\Delta\delta a(n) = -\frac{20}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

for some constant A and B. Multiply both sides by $n(n+1)$:

$$-20 = A(n+1) + Bn$$

Since this is an identity, both sides are equal no matter what value of n is substituted. Let's substitute some easy values for n , such as $n = 0$:

$$-20 = A$$

Then substitute $n = -1$:

$$-20 = -B$$
$$B = 20$$

Thus, $\Delta\delta a(n) = -\dfrac{20}{n} + \dfrac{20}{n+1} = \dfrac{20}{n+1} - \dfrac{20}{n}$.

We have $\delta a(n+1) - \delta a(n) = \dfrac{20}{n+1} - \dfrac{20}{n}$. By inspection, it is likely that $\delta a(n) = \dfrac{20}{n} + C$ for some constant C. Since we know that $\delta a(1) = 19$, we have $C = 19 - \dfrac{20}{1} = -1$. So

$$\delta a(n) = \frac{a(n+1)}{a(n)} = \frac{20}{n} - 1$$

$$a(n+1) = (\frac{20-n}{n})\, a(n)$$

Listing out the terms:

$$a(1) = 1$$
$$a(2) = (\frac{19}{1})$$
$$a(3) = (\frac{19}{1})(\frac{18}{2})$$
$$a(4) = (\frac{19}{1})(\frac{18}{2})(\frac{17}{3})$$
$$a(5) = (\frac{19}{1})(\frac{18}{2})(\frac{17}{3})(\frac{16}{4})$$
$$a(n) = \frac{\frac{19!}{(20-n)!}}{(n-1)!} = \frac{19!}{(n-1)!(20-n)!}$$

a(n) = $\boxed{fact(19)/fact(n-1)/fact(20-n)}$

Sequence 60. 0, 1, 2, 9, 44, 265, 1854, 14833, 133496, 1334961, ...

Solution 60. If we are familiar with some basic combinatorics stuff, we will recognize this sequence, which is the sequence of **derangement** numbers. One formula of the derangement numbers is:

$$a(n) = \left[\frac{n!}{e}\right]$$

where $e \approx 2.71828$ is the **Euler's number** and the square bracket means rounding the stuff inside to the nearest integer.

We can't type an irrational number, but we can type a rational approximation of e. The continued fraction of e is:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \cfrac{1}{1 + ...}}}}}}}}}}$$

Expressed in continued fraction notation, it is

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, \ldots]$$

(The proof is in [3].)

When we evaluate the continued fractions and stop after a certain point, the fraction obtained is called the **convergent** of the continued fractions, and it is a close rational approximation of e. We find that

$$e \approx 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \cfrac{1}{1}}}}}}}}}} = \frac{2721}{1001}$$

Coupled with the fact that $[n] = \lfloor n + \frac{1}{2} \rfloor$, we have

$$a(n) = \left\lfloor \frac{n!}{\frac{2721}{1001}} + \frac{1}{2} \right\rfloor = \left\lfloor \frac{1001n!}{2721} + \frac{1}{2} \right\rfloor$$

a(n) = $\boxed{(1001fact(n)/2721 + 1/2)//1}$

Congrats! You beat the game!

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