1、$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

we can set $(\lambda-1)^2-1=0$

$\lambda^2-2\lambda=0$

$\lambda(\lambda-2)=0$

$\lambda=0 \quad \lambda=2$

when $\lambda=0$

$\begin{bmatrix} \lambda-1 & -1 \\ -1 & \lambda-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda=2$

$\begin{bmatrix} \lambda-1 & -1 \\ -1 & \lambda-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1-x_2=0$

$-x_1+x_2=0$

$\Rightarrow \begin{bmatrix} x_2 \\ x_{22} \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Then, we can know the matrix A has two independent eigenvectors.

2、$\det(\lambda I - A) = \det[(\lambda I - A)^T] = \det[(\lambda I)^T - A^T]$

$= \det(\lambda I - A^T)$

The character polynomial to the matrix A and the transposing to A is same, A and A^T have the same eigenvalues

3、$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B=\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

$AB=\begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{bmatrix}$

$BA=\begin{bmatrix} a_{11}b_{11}+a_{21}b_{12} & a_{12}b_{11}+a_{22}b_{12} \\ a_{11}b_{21}+a_{21}b_{22} & a_{12}b_{21}+a_{22}b_{22} \end{bmatrix}$

① $\det(\lambda I - AB) = \lambda^2 - (a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22})$

$\lambda + a_{12}a_{21}(b_{12}b_{21} - b_{11}b_{22})$

$+ a_{11}a_{22}(b_{11}b_{22} - b_{12}b_{21})$

② $\det(\lambda I - BA) = \lambda^2 - (a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22})\lambda$

$+ a_{12}a_{21}(b_{12}b_{21} - b_{11}b_{22}) + a_{11}a_{22}$

$(b_{11}b_{22} - b_{12}b_{21})$

Both of them have the same characteristic equation

4、 1. let $B = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$ $\quad \lambda I - B = 0$

$(\lambda-2)(\lambda-1) = 0$

$\lambda = 2, 1$

$\lambda = 2$ corresponding $p_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 1$ corresponding $p_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$P = [p_1\ p_2] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\quad P^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$

$P^{-1}BP = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

let $D = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

$\lambda I - D = 0 \Rightarrow (\lambda-1)(\lambda-2) = 0 \quad \lambda = 1, 2$

$P = [p_1 \ p_2] = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$

$P^{-1} b P = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

we can turn $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$