1. Solve the initial-value problem

$$\dot{x}_1 = x_2 + e^{-t}, \qquad x_1(0) = 1,$$
$$\dot{x}_2 = 6\left(t+1\right)^{-2} x_1 + \sqrt{t}, \quad x_2(0) = 2.$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6(t+1)^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad f(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$X_1(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad X_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$X_2(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad X_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

case 1 when $x_1 = x_2$ $\quad x_1 = (t+1)^3$ and $(t+1)^{-2}$

$x_1 = c_1(t+1)^3 + c_2(t+1)^{-2}$

$x_1(0) = 1 = c_1 + c_2$

$x_2(0) = x_1'(0) = 0 = 3c_1 - 3c_2$

$\Rightarrow \quad x_1(t) = \frac{2}{5}(t+1)^3 + \frac{3}{5}(t+1)^{-2}$

Case 2

$$x_1 = c_1(t+1)^3 + c_2(t+1)^{-2}$$

$$x_1(0) = 0 = c_1 + c_2$$

$$x_2(0) = x_1'(0) = 1 = 3c_1 - 2c_2$$

$$\left.\begin{aligned} c_1 + c_2 &= 0 \\ 3c_1 - 2c_2 &= 1 \end{aligned}\right\} \Rightarrow c_1 = \frac{1}{5} \quad c_2 = -\frac{1}{5}$$

$$\Rightarrow x_1(t) = \frac{1}{5}(t+1)^3 - \frac{1}{5}(t+1)^{-2}$$

then:

$$x_h(t) = \begin{bmatrix} \frac{4}{5}(t+1)^3 + \frac{1}{5}(t+1)^{-2} \\ \frac{12}{5}(t+1)^2 - \frac{2}{5}(t+1)^{-3} \end{bmatrix}$$

$$x(t) = \Phi(t)(x_0 + \Phi^{-1}(s) f(s)\, ds$$

2. Solve the initial-value problem

$$\frac{dy(t)}{dt} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix} y(t), \quad y(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}.$$

lee $y(t) = [y_1(t)\ \ y_2(t)\ \ y_3(t)\ \ y_4(t)\ \ y_5(t)]^T$

Hence $\begin{bmatrix} y_1 t \\ y_2 t \\ y_3 t \\ y_4 t \\ y_5 t \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \\ y_5(t) \end{bmatrix}$

A $\begin{cases} \dot y_1(t) = 2y_1(t) \\ \dot y_2(t) = y_1(t) + y_2(t) \times 2 \end{cases}$

B $\begin{cases} \dot y_3(t) = 3y_3(t) \\ \dot y_4(t) = y_3(t) + 3y_4(t) \\ \dot y_5(t) = y_4(t) + 3y_5(t) \end{cases}$

$y_1(t) = e^{2t}$

$y_2(t) = (2+t)\, e^{-t}$

From 3) $\begin{cases} y_3(t) = 3e^{t} \\ y_4(t) = (4+3t)e^{3t} \\ y_5(t) = (5+4t+\frac{3}{2}t^2)\, e^{3t} \end{cases}$

$\Rightarrow \begin{cases} y_1(t) = e^{2t} \\ y_2(t) = (2+t)\, e^{2t} \\ y_3(t) = 3e^{3t} \\ y_5(t) = (5+4t+\frac{3}{2}t^2)\, e^{3t} \end{cases}$