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MIXED-HYBRID FINITE ELEMENT APPROXIMATIONS OF SECOND-ORDER ELLIPTIC BOUNDARY-VALUE PROBLEMS

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Mixed-hybrid finite element approximations are described for second-order elliptic boundary-value problems in which independent approximations are used for the solution and its gradient on the interior of an element and the trace of the gradients on the boundary of the element. A priori error estimates are derived with conditions for convergence. Several other finite element models are also obtained as special cases. Some numerical examples are included.

1. Introduction

During the last decade three nonconventional finite element methods have emerged that have simultaneously found wide application to a variety of technical problems and have also met with some notorious failures owing to their delicate convergence and stability properties. We refer to the so-called hybrid methods, the mixed methods, and the methods which use nonconforming elements.

1.1. Hybrid methods

In 1964 Jones [1] proposed an interesting method of finite element approximations in which reduced continuity requirements on the coordinate functions were made possible by treating element continuity conditions as constraints and carrying along Lagrange multipliers as new dependent variables. Similar methods were proposed also by Pian [2] and were subsequently expanded and extensively applied by him and Tong and ultimately dubbed "hybrid finite element methods" (see [2]-[4], and references therein). Recently Raviart [5], Raviart and Thomas [6], and Thomas [7] outlined some interesting mathematical properties of a class of hybrid elements for second order problems that appear to fall into the category of the displacement models of Yamamoto [8] and the assumed stress model of Pian [2]. Brezzi [9] and Brezzi and Marini [10] studied stress-assumed hybrid models for fourth-order problems. See also [11] for some interesting observations concerning variational formulations of hybrid models.

1.2. Mixed methods

The so-called mixed finite element methods were introduced by Herrmann [12], and their convergence properties were studied by Oden and Reddy [13, 14], Ciarlet and Raviart [15], and Johnson [16]. In mixed methods, operators in a given boundary-value problem are decomposed into two or more parts, and the corresponding dependent variables are represented by independent finite element approximations. For example, independent approximations might be used simultaneously for the solution u of Laplace's equation and for its gradient, and the result is often improved accuracy for approximations of grad u.

1.3. Nonconforming elements

A characteristic of nonconforming finite element methods for boundary-value problems of order $2m$ (in which the energy involves derivatives of order m) is that approximations to derivatives of order $m-1$ may suffer simple discontinuities across interelement boundaries. Noncomforming elements have also been investigated by several authors (see for example [17], [18], [19] and [20]).

We mention that the engineering literature contains many variants of these methods. Some quite general models have been suggested by Atluri [21] and Wolfe [22], and extensive references and computational accounts can be found in [3] or [23], among others.

In the present paper we develop a theory of mixed-hybrid approximations in connection with a model second-order elliptic boundary-value problem in which both independent approximations are used for the solution and its gradient on the interior of the element, and an independent approximation is used for the normal derivative on the boundary. Thus, our method is simultaneously a hybrid method and a mixed method, and we show that either can be extracted from our theory as a special case. But what is also interesting is that it is a nonconforming method as well.

2. Notations and preliminaries

Throughout this paper Ω shall denote an open bounded Lipschitzian domain in a two-dimensional euclidean plane with a piecewise smooth boundary $\partial\Omega$. The notation $\boldsymbol{x} = \{x_1, x_2\}$ is used to denote a point in Ω, a differential element of Ω is denoted $\mathrm{d}x = \mathrm{d}x_1\,\mathrm{d}x_2$, and an element of $\partial\Omega$ is denoted $\mathrm{d}s$. The boundary $\partial\Omega$ is taken to be a collection of N smooth arcs, and the interior angles α_j of tangents to the arcs meeting at the jth joint are in the range $0 < \alpha_j < 2\pi$, $1 \leqslant j \leqslant N$. Polygonal domains are thus an important special subclass of the domains considered here.

For an integer $m \geqslant 0$ it is well-known that the Sobolev space $H^m(\Omega)$ is a Hilbert space defined as the completion of the space of functions, infinitely differentiable on Ω, in the Sobolev norm

$$\|u\|_{m,\Omega} = \left[\int_\Omega \sum_{|\boldsymbol{\alpha}| \leqslant m} |D^{\boldsymbol{\alpha}}u|^2\,\mathrm{d}x\right]^{1/2}, \tag{2.1}$$

wherein $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2\}$, $\alpha_i \geqslant 0$ integral, $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2$, and

$$D^{\boldsymbol{\alpha}}u = \frac{\partial^{|\boldsymbol{\alpha}|}u}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}.$$

The inner product on $H^m(\Omega)$ is denoted

$$(u, v)_{m,\Omega} = \int_\Omega \sum_{|\alpha| \le m} D^\alpha u \, D^\alpha v \, dx , \qquad u, v \in H^m(\Omega) . \tag{2.2}$$

We also use interchangeably the notation $H^0(\Omega) = L_2(\Omega)$. Likewise, $H_0^m(\Omega) \subset H^m(\Omega)$ is the completion in the norm $\|\cdot\|_{m,\Omega}$ of the space of infinitely smooth functions with compact support in Ω. In addition, for $m < 0$ we define $H^m(\Omega)$ as the completion of $L_2(\Omega)$ in the norm

$$\|u\|_{m,\Omega} = \sup_{v \in H^{-m}(\Omega)} \frac{|\int_\Omega uv \, dx|}{\|v\|_{-m,\Omega}} , \qquad m < 0 . \tag{2.3}$$

In much of the subsequent analysis we also deal with functions defined on the boundary $\partial\Omega$. Obviously, the notation $L_2(\partial\Omega)$ denotes the space of functions whose squares are Lebesgue-integrable on $\partial\Omega$. It is also well-known that if $u \in H^1(\Omega)$, the trace $\gamma_0 u$ on $\partial\Omega$ is well defined. We denote by $H^{1/2}(\partial\Omega)$ the space of functions defined on $\partial\Omega$ which are traces of functions in $H^1(\Omega)$, and we furnish $H^{1/2}(\partial\Omega)$ with the norm

$$\|\phi\|_{1/2,\partial\Omega} = \inf_{u \in H^1(\Omega)} \{\|u\|_{1,\Omega} : \phi = \gamma_0 u\} . \tag{2.4}$$

Clearly,

$$\|\gamma_0 u\|_{1/2,\partial\Omega} \le \|u\|_{1,\Omega} \qquad \forall u \in H^1(\Omega) . \tag{2.5}$$

Conversely, there also exists a continuous map ν of $H^{1/2}(\partial\Omega)$ into $H^1(\Omega)$ such that

$$\|\nu\phi\|_{1,\Omega} \le \|\phi\|_{1/2,\partial\Omega} \qquad \forall \phi \in H^{1/2}(\partial\Omega) . \tag{2.6}$$

Indeed, if $v \in H^1(\Omega)$ is such that

$$\begin{aligned} -\Delta v + v &= 0 \quad \text{on } \Omega , \\ \gamma_0 v &= \phi \quad \text{on } \partial\Omega , \end{aligned} \tag{2.7}$$

then $\|v\|_{1,\Omega} = \|\phi\|_{1/2,\partial\Omega}$.

We recall that the Sobolev imbedding theorem establishes that [24]

$$\|\gamma_0 v\|_{0,\partial\Omega} \le C(\Omega) \|v\|_{1,\Omega} \qquad \forall v \in H^1(\Omega) , \tag{2.8}$$

where $C(\Omega)$ is a positive constant depending only on Ω. It follows that $H^{1/2}(\partial\Omega) \subset L_2(\partial\Omega)$. Therefore, if ψ is a given function in $L_2(\partial\Omega)$, and if $\phi \in H^{1/2}(\partial\Omega)$, the integral $\oint_{\partial\Omega} \psi\phi \, ds$ has meaning. Consequently, we can introduce a norm $\|\psi\|_{-1/2,\partial\Omega}$ on ψ defined by

$$\|\psi\|_{-1/2,\partial\Omega} = \sup_{\phi\in H^{1/2}(\partial\Omega)} \frac{|\oint_{\partial\Omega} \psi\phi \, ds|}{\|\phi\|_{1/2,\partial\Omega}} . \tag{2.9}$$

We now define the space $H^{-1/2}(\partial\Omega)$ as the completion of $L_2(\partial\Omega)$ in the norm $\|\cdot\|_{-1/2,\partial\Omega}$.

The space $H^{-1/2}(\partial\Omega)$ plays an important role in later developments, and it is fitting to analyze its structure in more detail. First, we establish a lemma similar to one given by Babuska [25]:

LEMMA 2.1. Let $\psi \in L_2(\partial\Omega)$ be given, and let $z \in H^1(\Omega)$ be the weak solution of the Neumann problem

$$\begin{aligned} -\Delta z + z &= 0 \quad \text{in } \Omega , \\ \partial z/\partial n &= \psi \quad \text{on } \partial\Omega . \end{aligned} \tag{2.10}$$

Then

$$\|\psi\|^2_{-1/2,\partial\Omega} = \oint_{\partial\Omega} \psi\gamma_0 z \, ds = \|z\|^2_{1,\Omega} . \tag{2.11}$$

Proof. This follows from the definition (2.9), together with (2.5) and (2.6), and the fact that z is uniquely characterized by

$$\int_{\Omega} [(\nabla z)\nabla v + zv] \, dx = \oint_{\partial\Omega} \psi\gamma_0 v \, ds \qquad \forall v \in H^1(\Omega) \tag{2.12}$$

and

$$\|z\|^2_{1,\Omega} = \oint \psi\gamma_0 z \, ds . \quad \blacksquare \tag{2.13}$$

We easily see from these results that

$$\|\psi\|_{-1/2,\partial\Omega} \leqslant \bar{C}(\Omega) \|\psi\|_{0,\partial\Omega} , \tag{2.14}$$

where $\bar{C}(\Omega)$ is again a positive constant depending on Ω. Hence, $L_2(\partial\Omega) \subset H^{-1/2}(\partial\Omega)$.

Some additional properties of $H^{-1/2}(\partial\Omega)$ should be noted. In particular, if $\xi(x)$ is a function defined on $\partial\Omega$ such that $\xi(x) = 1$ on some measurable subset $\Gamma_1 \subset \partial\Omega$ and $\xi(x) = 0$ on $\partial\Omega - \Gamma_1$, then $\psi\xi \in L_2(\partial\Omega)$ for every $\psi \in L_2(\partial\Omega)$ and $\|\xi\psi\|_{0,\partial\Omega} \leqslant \|\psi\|_{0,\partial\Omega}$. On the other hand, if $\psi \in H^{-1/2}(\partial\Omega)$, then, in general, $\xi\psi \notin H^{-1/2}(\partial\Omega)$.

We shall also make use of another special space of functions defined on portions of the boundaries of two-dimensional domains. Let $\partial\Omega^i$ be a smooth arc of $\partial\Omega$. Let $v \in H^m(\Omega)$, $m \geqslant 2$. Then $\partial v/\partial n \in L_2(\partial\Omega^i)$. We now denote by $H^{m-3/2}(\partial\Omega, \Omega)$ the space of all normal derivatives $\partial v/\partial n$ of functions $v \in H^m(\Omega)$ furnished with the norm

$$\|\psi\|_{H^{m-3/2}(\partial\Omega,\Omega)} = \inf\{\|v\|_{m,\Omega} : m \geqslant 2, \partial v/\partial n|_{\partial\Omega^i} = \psi\}, \qquad i = 1, 2, \dots N\} . \tag{2.15}$$

We shall also deal with spaces of vector-valued functions. For instance, if $\boldsymbol{\sigma} = \{\sigma_1, \sigma_2\}$ is such that $\sigma_i \in H^m(\Omega)$, $i = 1, 2$, we write $\boldsymbol{\sigma} \in H^m(\Omega)$ and

$$\|\boldsymbol{\sigma}\|_{m,\Omega} = [\|\sigma_1\|^2_{m,\Omega} + \|\sigma_2\|^2_{m,\Omega}]^{1/2} . \tag{2.16}$$

Analogously, $L_2(\Omega) = H^0(\Omega)$.

Finally, we conclude our preliminary remarks by recording two theorems which play a fundamental role in later developments. Proofs of these theorems can be found in [26] (see also [27] and [28]):

THEOREM 2.1. Let $\mathcal{U}$ and $\mathcal{V}$ be two real Hilbert spaces, and let B: $\mathcal{U} \times \mathcal{V} \to \Re$ be a linear functional on $\mathcal{U} \times \mathcal{V}$ such that for every $u \in \mathcal{U}$ and $v \in \mathcal{V}$ the following hold:

$$|B(u, v)| \leqslant C_1 \|u\|_{\mathcal{U}} \|v\|_{\mathcal{V}} , \tag{2.17}$$

$$\inf_{\|u\|_{\mathcal{U}}=1} \sup_{\|v\|_{\mathcal{V}} \leqslant 1} |B(u, v)| \geqslant C_2 > 0 , \tag{2.18}$$

$$\sup_{u \in \mathcal{U}} |B(u, v)| > 0 , \qquad v \neq 0 , \qquad v \in \mathcal{V} . \tag{2.19}$$

Here C_1 and C_2 are positive constants independent of u and v, and $\|\cdot\|_{\mathcal{U}}$ and $\|\cdot\|_{\mathcal{V}}$ denote the norms on $\mathcal{U}$ and $\mathcal{V}$, respectively. In addition, let $f \in \mathcal{V}'$ be given, where $\mathcal{V}'$ is the dual space of $\mathcal{V}$. Then there exists a unique element $u_0 \in \mathcal{U}$ such that

$$B(u_0, v) = f(v) \qquad \forall v \in \mathcal{V} . \tag{2.20}$$

Moreover,

$$\|u_0\|_{\mathcal{U}} \leqslant \frac{1}{C_2} \|f\|_{\mathcal{V}'} . \quad \blacksquare \tag{2.21}$$

THEOREM 2.2. Let $\mathcal{U}_h$ and $\mathcal{V}_h$ be finite-dimensional subspaces of real Hilbert spaces $\mathcal{U}$ and $\mathcal{V}$, respectively, and let the bilinear form B: $\mathcal{U} \times \mathcal{V} \to \Re$ of Theorem 2.1 be such that for $U \in \mathcal{U}_h$, $V \in \mathcal{V}_h$, the following hold:

$$\inf_{\|U\|_{\mathcal{U}}=1} \sup_{\|V\|_{\mathcal{V}} \leqslant 1} |B(U, V)| \geqslant C_2^h > 0 , \tag{2.22}$$

$$\sup_{U \in \mathcal{U}_h} |B(U, V)| > 0 \qquad V \neq 0, \qquad V \in \mathcal{V}_h . \tag{2.23}$$

In addition, let $f \in \mathcal{U}'$ be given as in theorem 2.1. Then

(i) there exists a unique $U_0 \in \mathcal{U}_h$ such that

$$B(U_0, V) = f(V) \qquad \forall V \in \mathcal{V}_h \,. \tag{2.24}$$

with

$$\|U_0\|_{\mathcal{U}} \leqslant (1/C_2^h)\|f\|_{\mathcal{V}'} \,, \tag{2.25}$$

(ii) if u_0 is the unique solution of (2.20),

$$\|u_0 - U_0\|_{\mathcal{U}} \leqslant (1 + C_1/C_2^h) \inf_{\tilde{U} \in \mathcal{U}_h} \|u_0 - \tilde{U}\|_{\mathcal{U}} \,. \quad \blacksquare \tag{2.26}$$

Hereafter, C shall denote generic constants which are not necessarily the same in different places; we shall always indicate the major dependence of such constants.

3. A model Dirichlet problem and a special variational principle

We shall concentrate on the following model Dirichlet problem:

$$\begin{aligned} -\Delta u + u &= f \quad \text{in } \Omega \,, \\ \gamma_0 u &= g \quad \text{on } \partial\Omega \,, \end{aligned} \tag{3.1}$$

where $f \in L_2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$. As usual, a function $u^* \in H^1(\Omega)$ is the weak solution to (3.1) if $\gamma_0 u^* = g$ on $\partial\Omega$ and

$$(u^*, v)_{1,\Omega} = \int_\Omega f v \,\mathrm{d}x \qquad \forall v \in H_0^1(\Omega) \,. \tag{3.2}$$

We are interested here in a special formulation of a variational boundary-value problem – equivalent to (3.2) under certain conditions – which is designed to lead naturally to a theory of hybrid finite element approximation. We begin by considering a domain $\Omega \equiv \Omega_0 \subset \Re^2$ of the type described in section 2. Let P be a partition of Ω into a collection of E subdomains Ω_e, 1,2,... ..., $E(P)$ such that the following hold:

(i) $E = E(P) < \infty$,

(ii) $\bar{\Omega} = \bigcup_{e=1}^{E} \bar{\Omega}_e \,, \qquad \Omega_e \cap \Omega_f = \emptyset \,, \quad e \neq f \,,$

(iii) the subdomains Ω_e are Lipschitzian with piecewise smooth (C^∞) boundaries $\partial\Omega_e$,

(iv) the boundary segments

$$\Gamma_{ef} = \partial\Omega_e \cap \partial\Omega_f , \qquad e > f , \quad 0 \leqslant e , f \leqslant E ,$$

are sets with a finite number $\mu(e,f)$ of components, and

$$\Gamma_{ef} = \bigcup_{g=1}^{\rho(e,f)} \Gamma_{ef}^g + S_{ef} ;$$

where Γ_{ef}^g are smooth (closed) arcs, $\rho(e,f) \leqslant \mu(e,f)$, and S_{ef} is a set of isolated points such that

$$S_{ef} \cap \bigcup_{g=1}^{\rho(e,f)} \Gamma_{ef}^g = \emptyset , \qquad 0 \leqslant e, f \leqslant E .$$

In this way, we can define unambiguously the collection of smooth boundary pieces

$$\Gamma = \Gamma(P) = \bigcup_{\substack{e,f=0 \\ e>f \\ 1 \leqslant g \leqslant \rho(e,f)}}^{E} \Gamma_{ef}^g .$$

We assume that Γ is oriented so that Ω_e is on the left side of Γ_{ef}^g, $e > f$, $E \geqslant e, f \geqslant 1$.

In what will follow we deal with integrals over $\partial\Omega_e$ which should be understood throughout in the following sense:

$$\oint_{\partial\Omega_e} \nu \, \mathrm{d}s = \sum_{i=1}^{N(e)} \int_{\Gamma_i} \nu\rho_i |\mathrm{d}s| . \tag{3.3}$$

where Γ_i, $i = 1, 2, \ldots N(e)$, are smooth pieces of $\partial\Omega_e$ which can be uniquely identified with Γ_{ef}^g, $\rho_i = +1$ if Ω_e is on the left side of Γ_i, and $\rho_i = -1$ if Ω_e is on the right side of Γ_i. The orientation of Γ_i is the same as Γ_{ef}^g, $e > f$ and the integral $\int(\cdot)|\mathrm{d}s|$ is understood as a nonoriented one.

In addition, if v is any function defined on Ω, we shall denote its restriction to Ω_e by v_e, i.e.

$$v_e = v|_{\Omega_e} , \qquad 1 \leqslant e \leqslant E .$$

We now introduce a space $H^m(P)$ of functions defined on the subdomains Ω_e, $1 \leqslant e \leqslant E$, endowed with the norm and inner product

$$|||u|||_{m,P} = \left[\sum_{e=1}^{E} \|u\|_{m,\Omega_e}^2 \right]^{1/2} .$$

$$((u,v))_{m,P} = \sum_{e=1}^{E} (u,v)_{m,\Omega_e} .$$

Clearly,

$$H^m(\Omega) \subset H^m(P) , \qquad m > 0 ,$$

and if $u \in H^0(P)$, then also $u \in H^0(\Omega)$. We continue to use the notation $L_2(P) = H^0(P)$.

In addition, we introduce the space $L_2(\Gamma)$ of square-integrable functions on Γ with the norm

$$|||\psi|||^2_{0,\Gamma} = \int_\Gamma |\psi|^2 |ds| = \sum_{\substack{e,f,g \\ e>f}} \int_{\Gamma^g_{ef}} |\psi_e|^2 |ds| \,. \qquad 1 \le e \le E, \quad 0 \le f \le E, \quad 1 \le g \le \rho(e,f)\,.$$

wherein ψ_e denotes the restriction of $\psi \in L_2(\Gamma)$ to $\partial\Omega_e$. Then, for any $\psi \in L_2(\Gamma)$, we define

$$\|\psi\|^2_{\mathcal{W}(\Gamma)} = \sum_{e=1}^{E} \|\psi_e\|^2_{-1/2,\partial\Omega_e}\,, \tag{3.4}$$

where $\mathcal{W}(\Gamma)$ is the completion of the space $L_2(\Gamma)$ in the norm $\|\cdot\|_{\mathcal{W}(\Gamma)}$.

In (2.11), $\|\psi\|_{-1/2,\partial\Omega}$ is defined with the aid of the weak solution of the Neumann problem. We now define $z_e \in H^1(\Omega_e)$ such that for a $\psi_e = \psi|_{\partial\Omega_e}$, $\psi \in L_2(\Gamma)$,

$$(z_e, v)_{1,\Omega_e} = \oint_{\partial\Omega_e} \psi_e \gamma_0 v \, ds \qquad \forall v \in H^1(\Omega_e)\,. \tag{3.5}$$

Denoting by $\boldsymbol{n}_e$ the unit outward normal to $\partial\Omega_e$, we have

$$\partial z_e/\partial n_e = \rho_i \psi_e \quad \text{on} \quad \Gamma_i = \Gamma^g_{ef} \subset \partial\Omega_e\,.$$

Then, as in Lemma 2.1,

$$\|\psi_e\|^2_{-1/2,\partial\Omega_e} = \|z_e\|^2_{1,\Omega_e} = \oint_{\partial\Omega_e} \psi_e \gamma_0 z_e \, ds\,. \tag{3.6}$$

Similarly, we use the notation (see also (2.15))

$$\|\psi_e\|_{H^{m-3/2}(\partial\Omega_e,\Omega_e)} = \inf\{\|v\|_{m,\Omega_e};\ m \ge 2,\ \partial v/\partial n_e|_{\Gamma_i} = \rho_i \psi_e\}\,.$$

We remark that a given $\psi \in \mathcal{W}(\Gamma)$ can be restricted to $\partial\Omega_e$ and that $\|\psi_e\|_{-1/2,\partial\Omega_e} \le \|\psi\|_{\mathcal{W}(\Gamma)}$; however, the set of restrictions of $\psi \in \mathcal{W}(\Gamma)$ on $\partial\Omega_e$ is not necessarily closed in $H^{-1/2}(\partial\Omega_e)$.

Now, for a given partition P of Ω of the type described above we introduce the product space

$$\mathcal{X} = H^1(P) \times L_2(P) \times \mathcal{W}(\Gamma) \tag{3.7}$$

together with the norm

$$\begin{aligned} \|\lambda\|^2 &= |||u|||^2_{1,P} + \|\sigma\|^2_{L_2(P)} + \|\psi\|^2_{\mathcal{W}(\Gamma)} \\ &= \sum_{e=1}^{E} [\|u_e\|^2_{1,\Omega_e} + \|\sigma_e\|^2_{0,\Omega_e} + \|\psi_e\|^2_{-1/2,\partial\Omega_e}]\,. \end{aligned} \tag{3.8}$$

where $\lambda = \{u, \sigma, \psi\}$, $u \in H^1(P)$, $\sigma = \{\sigma_1, \sigma_2\} \in L_2(P)$, $\psi \in \mathcal{W}(\Gamma)$. It is also convenient to denote by λ_e the triple of restrictions $\{u_e, \sigma_e, \psi_e\}$ to $\bar{\Omega}_e$ and to use the notation

$$\|\lambda_e\|^2_{\mathcal{X}_e} \equiv \|\{u_e, \sigma_e, \psi_e\}\|^2_{\mathcal{X}_e} = \|u_e\|^2_{1,\Omega_e} + \|\sigma_e\|^2_{0,\Omega_e} + \|\psi_e\|^2_{-1/2,\partial\Omega_e}, \qquad 1 \leqslant e \leqslant E, \tag{3.9}$$

so that

$$\|\lambda\|^2 = \sum_{e=1}^{E} \|\lambda_e\|^2_{\mathcal{X}_e}. \tag{3.10}$$

Let $B(\lambda, \bar{\lambda})$ denote a bilinear form on $\mathcal{X} \times \mathcal{X}$ given by

$$B(\lambda, \bar{\lambda}) = \sum_{e=1}^{E} B_e(\lambda_e, \bar{\lambda}_e), \tag{3.11}$$

where the bar denotes an arbitrary element, and

$$B_e(\lambda_e, \bar{\lambda}_e) = \int_{\Omega_e} [\sigma_e \cdot \nabla \bar{u}_e + u_e \bar{u}_e + (\nabla u_e - \sigma_e) \cdot \bar{\sigma}_e]\, dx + \oint_{\partial\Omega_e} (\psi_e \bar{u}_e + \bar{\psi}_e u_e)\, ds. \tag{3.12}$$

The fundamental properties of this form are established in the following theorem.

THEOREM 3.1. The bilinear form $B(\cdot, \cdot)$ of (3.11) satisfies all the conditions of theorem 2.1 with the choice of constants $C_1 = 2$ and $C_2 = 15^{-1/2}$ (hence, independent of the partition P) when we set $\mathcal{U} = \mathcal{V} = \mathcal{X}$, where $\mathcal{X}$ is the space defined in (3.7).

Proof: We first show that $B(\lambda, \bar{\lambda})$ is continuous on $\mathcal{X} \times \mathcal{X}$; i.e. we show that (2.17) is satisfied. Recalling (3.5) and (3.6),

$$\int_{\partial\Omega_e} \psi_e \bar{u}_e\, ds = (z_e, \bar{u}_e)_{1,\Omega_e} \leqslant \|z_e\|_{1,\Omega_e} \|\bar{u}_e\|_{1,\Omega_e} = \|\psi_e\|_{-1/2,\partial\Omega_e} \|\bar{u}_e\|_{1,\Omega_e}, \qquad 1 \leqslant e \leqslant E. \tag{3.13}$$

Similarly,

$$\oint_{\partial\Omega_e} \bar{\psi}_e u_e\, ds \leqslant \|\bar{\psi}_e\|_{-1/2,\partial\Omega_e} \|u_e\|_{1,\Omega_e}, \qquad 1 \leqslant e \leqslant E. \tag{3.14}$$

Likewise, the use of Schwarz's inequality reveals that

$$\oint_{\Omega_e} [\sigma_e \cdot \nabla \bar{u}_e + u_e u_e + (\nabla u_e - \sigma_e) \cdot \bar{\sigma}_e]\, dx$$
$$\leqslant \|\sigma_e\|_{0,\Omega_e} \|\nabla \bar{u}_e\|_{0,\Omega_e} + \|u_e\|_{0,\Omega_e} \|\bar{u}_e\|_{0,\Omega_e} + \|\nabla u_e\|_{0,\Omega_e} \|\bar{\sigma}\|_{0,\Omega_e} + \|\sigma_e\|_{0,\Omega_e} \|\bar{\sigma}_e\|_{0,\Omega_e}. \tag{3.15}$$

Thus, by combining (3.13), (3.14) and (3.15) and successively using the Schwarz inequality, we obtain

$$B(\lambda, \bar{\lambda}) = \sum_{e=1}^{E} B_e(\lambda_e, \bar{\lambda}_e) \leqslant 2\|\lambda\|_{\mathcal{X}} \|\bar{\lambda}\|_{\mathcal{X}} . \tag{3.16}$$

Thus (2.17) holds for the bilinear form in (3.11), and $C_1 = 2$ is independent of the partition P.

Next, we verify that $B(\lambda, \bar{\lambda})$ satisfies the conditions (2.18) and (2.19). Toward this end, we introduce for arbitrary $\lambda = \{u, \sigma, \psi\}$ the special triple $\hat{\lambda} = \{\hat{u}, \hat{\sigma}, \hat{\psi}\}$, where $\hat{\lambda}|_{\Omega_e} = \hat{\lambda}_e$ is given by

$$\hat{\lambda}_e = \{\hat{u}_e, \hat{\sigma}_e, \hat{\psi}_e\} \qquad \begin{Bmatrix} \hat{u}_e = 2u_e + z_e \\ \hat{\sigma}_e = \nabla u_e - \sigma_e + \nabla z_e \\ \hat{\psi}_e = -3\psi_e \end{Bmatrix} \qquad 1 \leqslant e \leqslant E ,$$

where z_e is the solution of the local auxiliary problem for a $\psi \in \mathcal{W}(\Gamma)$:

$$(z_e, v_e)_{1,\Omega_e} = \oint_{\partial\Omega_e} \psi_e v_e \, \mathrm{d}s \qquad \forall v_e \in H^1(\Omega_e) . \tag{3.17}$$

We easily verify that

$$\|\hat{\lambda}_e\|^2_{\mathcal{X}_e} \leqslant 9\|u_e\|^2_{1,\Omega_e} + 3\|\lambda_e\|^2_{0,\Omega_e} + 15\|\psi_e\|^2_{-1/2,\partial\Omega_e} \leqslant 15\|\lambda_e\|^2_{\mathcal{X}_e} . \tag{3.18}$$

Now we obtain directly from (3.12) the inequality

$$B_e(\lambda_e, \hat{\lambda}_e) = \int_{\Omega_e} [(\nabla u_e)\cdot\nabla u_e + 2u_e^2 + \sigma_e\cdot\sigma_e] \, \mathrm{d}x + \int_{\Omega_e} [(\nabla z_e)\cdot\nabla u_e + z_e u_e] \, \mathrm{d}x - \oint_{\partial\Omega_e} \psi_e u_e \, \mathrm{d}s + \oint_{\partial\Omega_e} \psi_e z_e \, \mathrm{d}s$$

$$= \|u_e\|^2_{1,\Omega_e} + \|u_e\|^2_{0,\Omega_e} + \|\sigma_e\|^2_{0,\Omega_e} + \|\psi_e\|^2_{-1/2,\partial\Omega_e} \geqslant \|\lambda_e\|^2_{\mathcal{X}_e} ,$$

where we have used (3.17), (3.5) and (3.6). Therefore,

$$B(\lambda, \hat{\lambda}) = \sum_{e=1}^{E} B_e(\lambda_e, \hat{\lambda}_e) \geqslant \sum_{e=1}^{E} \|\lambda_e\|^2_{\mathcal{X}_e} = \|\lambda\|^2_{\mathcal{X}} ,$$

which, in view of (3.18), means that

$$B(\lambda, \hat{\lambda}) \geqslant \frac{1}{\sqrt{15}} \|\lambda\|_{\mathcal{X}} \|\hat{\lambda}\|_{\mathcal{X}} .$$

Thus,

$$\inf_{\|\lambda\|_{\mathcal{X}}=1} \sup_{\|\bar{\lambda}\|_{\mathcal{X}}\leqslant 1} |B(\lambda, \bar{\lambda})| \geqslant \inf_{\|\lambda\|_{\mathcal{X}}=1} \frac{|B(\lambda, \hat{\lambda})|}{\|\hat{\lambda}\|_{\mathcal{X}}} \geqslant \inf_{\|\lambda\|_{\mathcal{X}}=1} 15^{-1/2}\|\lambda\|_{\mathcal{X}} = 15^{-1/2} . \tag{3.19}$$

In other words, condition (2.18) of theorem 2.1 holds. That condition (2.19) also holds follows by interchanging the roles of λ and $\hat{\lambda}$ and noting that $B(\lambda, \bar{\lambda})$ is symmetric. We emphasize that the constants $C_1 = 2$ and $C_2 = 15^{-1/2}$ are independent of the partition P. This completes the proof. ∎

We next introduce a linear functional F on $\mathcal{X}$ given by

$$F(\bar{\lambda}) = \sum_{e=1}^{E} \left[\int_{\Omega_e} f_e \bar{u}_e \, dx + \int_{\Gamma_{e0}^g} g_e \bar{\psi}_e |ds| \right], \tag{3.20}$$

wherein $f \in L_2(\Omega) = L_2(P)$, $g \in H^{1/2}(\partial\Omega)$, $g_e = g|_{\Gamma_{e0}}$.

THEOREM 3.2. The functional F: $\mathcal{X} \to \Re$ given by (3.20) is continuous on the space $\mathcal{X}$ defined in (3.7).

Proof: The fact that the first term is continuous on $\mathcal{X}$ follows immediately from the Schwarz inequality. The second term requires some additional consideration.

Let $\bar{\psi} \in \mathcal{W}(\Gamma) \cap L_2(\Gamma)$ be given. Then, in view of (3.5), there exists a $z \in H^1(P)$ such that

$$((z, v))_{1,P} \equiv \sum_{e=1}^{E} (z_e, v_e)_{1,\Omega_e} = \sum_{e=1}^{E} \oint_{\partial\Omega_e} \bar{\psi}_e v_e \, ds . \tag{3.21}$$

Next, we take $v = u^*$, where u^* is the solution of (3.2) (recall that $H^1(\Omega) \subset H^1(P)$). Thus,

$$\sum_{e=1}^{E} \oint_{\partial\Omega_e} \bar{\psi}_e u_e^* \, ds = \oint_{\partial\Omega} \bar{\psi} g \, ds = \sum_{e=1}^{E} \int_{\Gamma_{ef}^g} g_e \bar{\psi}_e |ds| ,$$

and, according to (3.21),

$$\oint_{\partial\Omega} \bar{\psi} g \, ds = ((z, u^*))_{1,P} \leq |||z|||_{1,P} \, |||u^*|||_{1,P} \leq \|\bar{\psi}\|_{\mathcal{W}(\Gamma)} \|u^*\|_{1,\Omega} \leq \|\bar{\lambda}\|_{\mathcal{X}} \, \|g\|_{1/2,\partial\Omega} .$$

This completes the proof ∎

We are thus led to the following special variational boundary-value problem: find $\lambda \in \mathcal{X}$ such that

$$B(\lambda, \bar{\lambda}) = F(\bar{\lambda}) , \qquad \forall \bar{\lambda} \in \mathcal{X} , \tag{3.22}$$

where $B(\lambda, \bar{\lambda})$ is given by (3.11) and (3.12), and $F(\bar{\lambda})$ is given by (3.20). By virtue of Theorem 2.1 and Theorems 3.1 and 3.2 just proved, we immediately have the following:

THEOREM 3.3. There exists a unique solution $\lambda^0 \in \mathcal{X}$ of problem (3.22). Moreover, there exists a constant $C > 0$, independent of the partition P (indeed we can take $C \geq 15^{-1/2}$), such that

$$\|\lambda^0\|_{\mathcal{X}} \leq C \|F\|_{\mathcal{X}'} = C [\|f\|_{0,\Omega}^2 + \|g\|_{1/2,\partial\Omega}^2]^{1/2} \quad ∎ \tag{3.23}$$

The overriding question now is: what is the relationship between the solution λ^0 of (3.22) and the solution of our original model problem (3.1)? We resolve this question in the following theorem.

THEOREM 3.4. Let $\lambda^0 \in \mathfrak{X}$ denote the solution of (3.22) and $u^* \in H^1(\Omega)$ the solution of the weak Dirichlet problem (3.2). Moreover, let $\lambda^* \in \mathfrak{X}$ denote the triple

$$\lambda^* = \{u^*, \nabla u^*, \psi^*\}, \tag{3.24}$$

where $\psi_e^* = -\partial u_e^* / \partial n_e$ on Γ_{ef}^g, $e > f$. Then

$$\lambda^0 = \lambda^* . \tag{3.25}$$

[Note that when $u^* \in H^1(\Omega)$, $\partial u^*/\partial n$ has no sense in general. However, we define ψ_e^* here in the weak sense such that

$$\oint_{\partial\Omega_e} \psi_e^* \gamma_0 v \, ds = (u^*, v)_{1,\Omega_e} - \int_{\Omega_e} f v \, dx \qquad \forall v \in H^1(\Omega_e) .$$

On the other hand, we know that $u^* \in H^2(\tilde{\Omega})$, where $\tilde{\Omega}$ is arbitrary but such that $\bar{\tilde{\Omega}} \subset \Omega$. Therefore, $\partial u_e^*/\partial n$ has a normal sense on $\Gamma - \partial\Omega$.]

Proof: Introducing (3.24) into the definition (3.11) and (3.12), we get

$$\begin{aligned} B(\lambda^*, \bar{\lambda}) &= \sum_{e=1}^{E} \left[\int_{\Omega_e} [(\nabla u_e^*) \cdot \nabla \bar{u}_e + u_e^* \bar{u}_e] \, dx + \oint_{\partial\Omega_e} \psi_e^* \bar{u}_e \, ds + \oint_{\partial\Omega_e} u_e^* \bar{\psi}_e \, ds \right] \\ &= \sum_{e=1}^{E} \left[\int_{\Omega_e} f_e \bar{u}_e \, dx + \oint_{\partial\Omega_e} u_e \bar{\psi}_e \, ds \right] . \end{aligned} \tag{3.26}$$

However, using the definition of $\oint_{\partial\Omega_e}(\cdot)\, ds$, we obtain

$$\sum_{e=1}^{E} \oint_{\partial\Omega_e} u_e \bar{\psi}_e \, dx = \int_{\partial\Omega} g \bar{\psi} \, ds , \tag{3.27}$$

and, therefore,

$$B(\lambda^*, \bar{\lambda}) = F(\bar{\lambda}) \qquad \forall \bar{\lambda} \in \mathfrak{X} .$$

Hence, (3.25) follows from the uniqueness of the solution λ^0 to (3.22) ∎

Remark 3.1. Theorem 2.1 is still valid when $\mathcal{W}(\Gamma)$ is restricted to any closed nonempty subspace $\mathcal{T}(\Gamma) \subset \mathcal{W}(\Gamma)$, and the constants C_i are not altered. Likewise, the functional $F(\bar{\lambda})$ of (3.20) is then also continuous, and λ^0 is again uniquely determined. Of course, $\psi^0 = \psi^*$ only when $\psi^* \in T(\Gamma)$.

Remark 3.2. The space $\mathfrak{X}$ of (3.7) can be restricted in a variety of ways, each of which will lead to a different type of variational problem. For example, we could use $\Omega_1 = \Omega_0$, $E = 1$, and $\boldsymbol{\sigma} = \nabla u$. Then we obtain the Lagrange multiplier theory of Babuska [25] (see also Babuska and Aziz [26]). When $\boldsymbol{\sigma} \neq \nabla u$, we obtain a space used in the mixed finite element formulations of Oden and Reddy [13, 14]. Still other special cases could be cited. We consider some of these in section 7.

4. Mixed-hybrid finite element methods

The stage is now set for the construction of finite element approximations of the variational boundary-value problem (3.22). As expected, we will investigate a set of partitions P of Ω into $E(P)$ subdomains which is now viewed as a finite element model of Ω; i.e. each domain $\bar{\Omega}_e$, $1 \leqslant e \leqslant E$, is now viewed as a finite element. The particular formulation that we have described in the previous section provides a direct vehicle for the construction of mixed-hybrid finite elements: over each element we introduce local approximations of u_e, $\boldsymbol{\sigma}_e$, and ψ_e, using polynomials of possibly differing degree.

As usual, we associate with every partition P a parameter h such that

$$h_e = \max_{x,y \in \Omega_e} \left[\sum_{i=1}^{2} |x_i - y_i|^2 \right]^{1/2}, \qquad 1 \leqslant e \leqslant E(P), \tag{4.1}$$

and

$$h = \max_{1 \leqslant e \leqslant E} h_e . \tag{4.2}$$

We also construct a collection of finite-dimensional subspaces over the partition which have the following properties:

$$Q_k^1(P) = \{U \in H^1(P)\colon U_e \in \mathcal{P}_{k'}(\Omega_e), \quad 1 \leqslant e \leqslant E, \quad 1 \leqslant k \leqslant k'\}, \tag{4.3}$$

where $\mathcal{P}_{k'}(\Omega_e)$ is a space of polynomials of degree $\leqslant k'$ over Ω_e.

For any function $u \in H^l(\Omega_e)$ there is a constant $C_3 > 0$, independent of h_e, and a $\widetilde{U} \in Q_k^1(P)$ such that

$$\|u - \widetilde{U}\|_{s,\Omega_e} \leqslant C_3 h_e^{\eta} \|u\|_{l,\Omega_e}, \tag{4.4}$$

where $s = 0, 1$, $l \geqslant 1$, and

$$\eta = \min(k + 1 - s,\ l - s). \tag{4.5}$$

Moreover,

$$Q_r^0(P) = \{\boldsymbol{\Sigma} \in L_2(P)\colon \boldsymbol{\Sigma}_e \in \mathcal{P}_{r'}(\Omega_e), \quad 1 \leqslant e \leqslant E, \quad 0 \leqslant r \leqslant r'\}, \tag{4.6}$$

where $\mathcal{P}_{r'}(\Omega_e)$ is a space of vector polynomials of degree $\leqslant r'$ over Ω_e.

For any $\sigma \in H^q(\Omega_e)$ there is a constant $C_4 > 0$, independent of h_e, and a $\tilde{\Sigma} \in Q_r^0(P)$ such that

$$\|\sigma - \tilde{\Sigma}\|_{0,\Omega_e} \leqslant C_4 h_e^{\omega} \|\sigma\|_{q,\Omega_e}, \tag{4.7}$$

wherein $q \geqslant 0$ and

$$\omega = \min(r+1, q). \tag{4.8}$$

Recalling that Γ_{ef}^g are smooth arcs contained in $\Gamma(P)$, we define

$$Q_t^{-1/2}(\Gamma) = \{\Psi \in \mathcal{W}(\Gamma) : \psi_{ef}^g = \Psi|_{\Gamma_{ef}^g} \in \mathcal{P}_{t'}(\Gamma_{ef}^g),\ t' \geqslant t \geqslant 0,\ E \geqslant e > f \geqslant 0\}, \tag{4.9}$$

where $\mathcal{P}_{t'}(\Gamma_{ef}^g)$ is a space of polynomials of degree $\leqslant t'$ on Γ_{ef}^g.

For any $\psi \in \mathcal{W}(\Gamma) \cap L_2(\Gamma)$ so that $\psi_e = \psi|_{\partial\Omega_e}$, and $\psi_{ef}^g = \psi|_{\Gamma_{ef}^g}$ there exists a constant $C_5 > 0$, independent of h_e, and a $\tilde{\Psi} \in Q_t^{-1/2}(\Gamma)$ such that

$$\|\psi_e - \tilde{\Psi}_e\|_{-1/2,\partial\Omega_e} \leqslant C_5 h_e^{\theta} \|\psi_e\|_{H^{m-3/2}(\partial\Omega_e,\Omega_e)}, \tag{4.10}$$

where

$$\theta = \min(t + 3/2,\ m-1), \qquad m \geqslant 2. \tag{4.11}$$

In addition, we denote by $Q_t^{-1/2}(\partial\Omega_e)$ the subspace of $Q_t^{-1/2}(\Gamma)$ consisting of functions which vanish on $\Gamma - \partial\Omega_e$.

Clearly, the product space

$$Q = Q_k^1(P) \times Q_r^0(P) \times Q_t^{-1/2}(\Gamma) \tag{4.12}$$

is a subspace of the space $\mathcal{X}$ defined in (3.7).

The inequalities (4.4), (4.7) and (4.10) are assumptions that we formulated. The question is: under what conditions are these inequalities valid? It is possible to show, for example, that if all Ω_e are quasi-uniform triangles or quadrilaterals, then our assumptions are valid. We call an element Ω_e quasi-uniform if there exists a constant $\sigma_0 > 0$ independent of P such that $h_e/\bar{h}_e \leqslant \sigma_0$, $1 \leqslant e \leqslant E(P)$, where $\bar{h}_e$ is the diameter of the largest circle that can be inscribed in Ω_e. We mention that there are many other cases where the assumptions are valid, e.g. curvilinear elements. We will assume that the partition is such that (4.4)–(4.12) hold everywhere.

The *mixed-hybrid finite element method* involves seeking an element

$$\Lambda = \{U, \Sigma, \Psi\} \in Q$$

such that

$$B(\Lambda, \bar{\Lambda}) = F(\bar{\Lambda}) \qquad \forall \bar{\Lambda} \in Q, \tag{4.13}$$

where $B(\cdot,\cdot)$ is the bilinear form defined in (3.11) and (3.12), and $F(\cdot)$ is the linear form in (3.20).

The existence of a unique solution to (4.13) depends upon whether or not the basic properties (2.17), (2.18) and (2.19) of $B(\cdot, \cdot)$, established in theorem 3.1, are carried over to the approximate problem (recall theorem 2.2). The remainder of this section is devoted to a study of these conditions for (4.13).

Let Π_e^1 and Π_e^0 denote orthogonal projections of $H^1(\Omega_e)$ and $L^2(\Omega_e)$ onto $Q_k^1(\Omega_e)$ and $Q_r^0(\Omega_e)$, respectively, and let us construct a special element $\hat{\Lambda} = \{\hat{U}, \hat{\Sigma}, \hat{\Psi}\} \in Q$ such that

$$\hat{\Lambda}_e = \{\hat{U}_e, \hat{\Sigma}_e, \hat{\Psi}_e\} \quad \begin{cases} \hat{U}_e = 2U_e + \Pi_e^1 z_e \\ \hat{\Sigma}_e = -\Sigma_e + \Pi_e^0(\nabla U_e) + \Pi_e^0(\nabla \Pi_e^1 z_e)\,, \\ \hat{\Psi}_e = -3\Psi_e \end{cases} \tag{4.14}$$

where $z_e \in H^1(\Omega_e)$ satisfies (3.5) for $\psi = \Psi_e \in Q_t^{-1/2}(\Omega_e)$. We observe that

$$(z_e, U_e)_{1,\Omega_e} = \oint_{\partial\Omega_e} \Psi_e U_e \,\mathrm{d}s = (\Pi_e^1 z_e, U_e)_{1,\Omega_e} \qquad \forall U_e \in Q_k^1(\Omega_e)\,, \tag{4.15}$$

and, owing to the continuity of Π_0^e and Π_e^1,

$$\|\hat{\Lambda}_e\|^2_{\mathfrak{X}_e} \leq 15 \|\Lambda_e\|^2_{\mathfrak{X}_e}\,, \qquad 1 \leq e \leq E\,. \tag{4.16}$$

Then, making use of the identities

$$\left.\begin{aligned} &(\Sigma_e, \nabla U_e - \Pi_e^0(\nabla U_e))_{0,\Omega_e} = 0 \\ &(\Sigma_e, \nabla \Pi_e^1 z_e - \Pi_e^0(\nabla \Pi_e^1 z_e))_{0,\Omega_e} = 0 \end{aligned}\right. \qquad \forall \Sigma_e \in Q_r^0(\Omega_e)\,,$$

$$(\nabla U_e, \Pi_e^0 \nabla U_e)_{0,\Omega_e} = \|\Pi_e^0 \nabla U_e\|^2_{0,\Omega_e} \qquad \forall U_e \in Q_k^1(\Omega_e)\,,$$

we easily establish that

$$\begin{aligned} B_e(\Lambda_e, \hat{\Lambda}_e) = {} & 2\|U_e\|^2_{0,\Omega_e} + \|\Sigma_e\|^2_{0,\Omega_e} + \|\Pi_e^1 z_e\|^2_{1,\Omega_e} \\ & + \|\Pi_e^0 \nabla U_e\|^2_{0,\Omega_e} - \int_{\Omega_e} (\nabla U_e)\cdot(\nabla \Pi_e^1 z_e - \Pi_e^0 \nabla \Pi_e^1 z_e)\,\mathrm{d}x\,. \end{aligned} \tag{4.17}$$

To obtain proper bounds on these terms, we now introduce three basic parameters:

$$\mu_e = \mu_e(Q_k^1(\Omega_e), Q_t^{-1/2}(\partial\Omega_e)) = \inf_{\Psi_e \in Q_t^{-1/2}(\partial\Omega_e)} \frac{\|\Pi_e^1 z_e\|^2_{1,\Omega_e}}{\|\Psi_e\|^2_{-1/2,\partial\Omega_e}}\,, \tag{4.18}$$

$$\nu_e = \nu_e(Q_k^1(\Omega_e), Q_r^0(\Omega_e)) = \inf_{V_e \in Q_k^1(\Omega_e)} \frac{\|\Pi_e^0 \nabla V_e\|^2_{0,\Omega_e}}{\|\nabla V_e\|^2_{0,\Omega_e}}\,, \tag{4.19}$$

$$\gamma_e = \gamma_e(Q_k^1(\Omega_e), Q_r^0(\Omega_e)) = \sup_{V_e \in Q_k^1(\Omega_e)} \frac{\| \nabla V_e - \Pi_e^0 \nabla V_e \|_{0,\Omega_e}}{\| \nabla V_e \|_{0,\Omega_e}} . \tag{4.20}$$

Then, using the elementary inequality $ab \leqslant (a^2 + b^2)/2$ and noting that $0 \leqslant \mu_e, \nu_e, \mu_e \leqslant 1$, we have

$$B_e(\Lambda_e, \hat{\Lambda}_e) \geqslant (\nu_e - \gamma_e/2) \| U_e \|_{1,\Omega_e}^2 + \| \Sigma_e \|_{0,\Omega_e}^2 + (\mu_e - \gamma_e/2) \| \Psi_e \|_{-1/2,\partial\Omega_e}^2 \geqslant \beta_e(\Omega_e) \| \Psi_e \|^2_e , \tag{4.21}$$

wherein

$$\beta_e(\Omega_e) = \min(\nu_e - \gamma_e/2,\ \mu_e - \gamma_e/2) . \tag{4.22}$$

Finally, summing (4.21) over all E elements, making arguments analogous to those we used in Theorem 3.1, and recalling Theorem 2.2, we arrive at the following approximation theorem:

THEOREM 4.1. Let $B(\Lambda, \bar{\Lambda})$ denote the bilinear form on $Q \times Q$ in (4.13). Then

$$\inf_{\|\Lambda\| = 1} \sup_{\|\bar{\Lambda}\| = 1} |B(\Lambda, \bar{\Lambda})| \geqslant 15^{-1/2} \beta(P) , \tag{4.23}$$

wherein

$$\beta(P) = \min_{1 \leqslant e \leqslant E} \beta_e(\Omega_e) , \tag{4.24}$$

and $\beta_e(\Omega_e)$ is defined in (4.22). Moreover, if $\beta(P) > 0$ for a given partition P of Ω, then there exists a unique solution Λ^0 to the mixed-hybrid finite element approximation problem (4.13), and

$$\| \lambda^0 - \Lambda^0 \|_{\mathcal{X}} \leqslant (1 + C_1/C_2^h) \inf_{\Lambda \in Q} \| \lambda^0 - \Lambda \|_{\mathcal{X}} , \tag{4.25}$$

where λ^0 is the unique solution of (3.22), $C_1 = 2$ is the constant of continuity appearing in (3.16), and $C_2^h = 15^{-1/2} \beta(P)$ ■

Remark 4.1. It is clear that in the special case that

$$Q_r^0(P) \supset \nabla(Q_k^1(P)) , \tag{4.26}$$

which occurs whenever $k' - 1 \leqslant r'$ (i.e. $\nabla U_e \in Q_r^0(\Omega_e)\ \ \forall U_e \in Q_k^1(\Omega_e)$), then

$$\gamma_e = 0 , \qquad \nu_e = 1 , \qquad 1 \leqslant e \leqslant E . \tag{4.27}$$

In this case

$$\beta(P) = \min(1, \mu) \quad \text{and} \quad \mu = \min_{1 \leqslant e \leqslant E} \mu_e , \tag{4.28}$$

and $\mu > 0$ is a sufficient condition for the existence of a unique solution.

We now give a necessary and sufficient condition for μ to be positive.

THEOREM 4.2. The parameter μ_e defined by (4.18) is positive if and only if the following condition holds for any $\Psi_e \in Q_t^{-1/2}(\partial\Omega_e)$:

$$\oint_{\partial\Omega_e} \Psi_e U_e \, \mathrm{d}s = 0 \qquad \forall U_e \in Q_k^1(\Omega_e) \qquad \text{implies that } \Psi_e = 0 \,. \tag{4.29}$$

Proof: Suppose that $\mu_e = 0$. Then there exists $\Psi_e \in Q_t^{-1/2}(\partial\Omega_e)$, $\Psi_e \neq 0$. such that

$$\|\Pi_e^1 z_e\|_{1,\Omega_e} = 0 \,.$$

Using (4.15), we have

$$\int_{\partial\Omega_e} \Psi_e U_e \, \mathrm{d}s = 0 \qquad \forall U_e \in Q_k^1(\Omega_e) \,,$$

which implies $\Psi_e = 0$ by (4.29), a contradiction.

Now let $\mu_e \neq 0$ and $\Psi_e \neq 0$. Then, by (4.18) and (4.15),

$$\oint_{\partial\Omega_e} \Psi_e \Pi_e^1 z_e \, \mathrm{d}s \neq 0 \,,$$

which is the contrapositive form of (4.29) ∎

We refer to (4.29) as the *rank condition* because it obviously depends upon the rank of a rectangular matrix produced by introducing members of the spaces $Q_k^1(\Omega_e)$ and $Q_t^{-1/2}(\partial\Omega_e)$ into the contour integral. The same condition has been imposed by Raviart and Thomas [6]. Discussions on this condition are given in section 8 for a specific case.

Remark 4.2. If the inclusion (4.26) does not hold, i.e. if $k' - 1 > r'$, then it could happen that $\nu_e = 0$. For this case, assuming that the inverse property (cf. [27] p. 89)

$$\|U_e\|_{0,\Omega_e}^2 \geqslant Ch^2 \|\nabla U_e\|_{0,\Omega_e}^2 \qquad \forall U_e \in Q_k^1(\Omega_e)$$

holds, we obtain from (4.17)

$$\begin{aligned} B_e(\Lambda_e, \Lambda_e) &\geqslant 2\|U_e\|_{0,\Omega_e}^2 + \|\Sigma_e\|_{0,\Omega_e}^2 + \mu_e \|\Psi_e\|_{-1/2,\partial\Omega_e}^2 - \gamma_e \|\nabla U_e\|_{0,\Omega_e} \|\Psi_e\|_{-1/2,\partial\Omega_e} \\ &\geqslant \|U_e\|_{0,\Omega_e}^2 + (Ch^2 - \gamma_e/2)\|\nabla U_e\|_{0,\Omega_e}^2 + \|\Sigma_e\|_{0,\Omega_e} + (\mu_e - \gamma_e/2)\|\Psi_e\|_{-1/2,\partial\Omega_e}^2 \,. \end{aligned}$$

The inequality (4.21) still holds for this case with the choice of β_e as

$$\beta_e(\Omega_e) = \min(Ch_e^2 - \gamma_e/2,\ \mu_e - \gamma_e/2) \,.$$

Then the parameter $\beta(P)$ in (4.23) may still be positive so as to guarantee a unique solution of the approximate problem. However, in view of the form of the estimate (4.25), the dependence of $\beta(P)$ on h may destroy convergence of the method.

Clearly, $\beta(P) > 0$ is only sufficient for the existence of a unique solution to (4.13). However, we furnish a necessary condition in the following theorem:

THEOREM 4.3. In order that the approximate problem (4.13) have a unique solution, it is necessary that for a $\Psi \in Q_t^{-1/2}(\Gamma)$,

$$\sum_{e=1}^{E} \oint_{\partial\Omega_e} \Psi_e V_e \, ds = 0 \qquad \forall V \in Q_k^1(P) \tag{4.30}$$

implies that $\Psi = 0$.

Proof: Let there exist a unique solution to (4.13) for arbitrary data $F \in \mathcal{X}'$ and let $\Psi^0 \in Q_t^{-1/2}(\Gamma)$ be such that $\Psi^0 \neq 0$ and (4.30) holds. Then

$$B(\{0, \boldsymbol{o}, \Psi^0\}, \Lambda) = 0\,, \qquad \forall \Lambda \in Q\,.$$

However, $B(\boldsymbol{o}, \Lambda) = 0$, $\forall \Lambda \in Q$, also. Hence, $\boldsymbol{Q}$ is not a unique solution for $F = 0$, a contradiction.

Remark 4.3. When the equation $-\Delta u = f$ in Ω is approximated instead of (3.1a), one can show that the inclusion (4.26) is also a necessary condition in addition to Theorem 4.3. For details on this see [29].

5. The dependence of μ, ν and γ on h

It is important that we establish precisely how the parameters μ_e, ν_e, and γ_e of (4.18), (4.19) and (4.20) depend upon the mesh parameter h. Toward this end, consider an open bounded Lipschitzian domain $\mathcal{G} \subset \mathcal{R}^2$ of diameter h, $0 < h \leqslant 1$, and a fixed domain $\hat{\mathcal{G}}$ defined by

$$\hat{\mathcal{G}} = \{\hat{\boldsymbol{x}} : \hat{\boldsymbol{x}} = \boldsymbol{x}/h,\ \boldsymbol{x} \in \mathcal{G}\} \tag{5.1}$$

so that

$$\operatorname{dia} \hat{\mathcal{G}} = 1\,. \tag{5.2}$$

Using obvious notation analogous to (4.3), (4.6) and (4.9), we introduce the spaces

$$\hat{Q}_k^1(\hat{\mathcal{G}}) = \{\hat{U} : \hat{U}(\hat{\boldsymbol{x}}) = U(\boldsymbol{x}) \in Q_k^1(\mathcal{G})\}\,, \tag{5.3}$$

$$\hat{Q}_r^0(\hat{\mathcal{G}}) = \{\hat{\boldsymbol{\Sigma}} : \hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{x}}) = \boldsymbol{\Sigma}(\boldsymbol{x}) \in Q_r^0(\mathcal{G})\}\,, \tag{5.4}$$

$$\hat{Q}_t^{-1/2}(\partial\hat{\mathcal{G}}) = \{\hat{\Psi} : \hat{\Psi}(\hat{\boldsymbol{x}}) = \Psi(\boldsymbol{x}) \in Q_t^{-1/2}(\partial\mathcal{G})\}\,. \tag{5.5}$$

Here, for example, $Q_k^1(\mathcal{G}) = \{U: U \in \mathcal{P}_{k'}(\mathcal{G}), 1 \leq k \leq k'\}$ etc. We shall establish a collection of lemmas pertaining to properties of functions defined on the spaces described by (5.3), (5.4) and (5.5) and then apply these to the finite element subspaces described in the previous article.

Next, we introduce a Hilbert space $H_h^1(\hat{\mathcal{G}})$ endowed with the inner product

$$[\hat{u}, \hat{v}]_{1,h} = \int_{\hat{\mathcal{G}}} [(\hat{\nabla}\hat{u})\cdot\hat{\nabla}\hat{v} + h^2 \hat{u}\hat{v}]\, \mathrm{d}\hat{x} \tag{5.6}$$

wherein $\hat{u}(\hat{x}) = u(x)$, $x \in \mathcal{G}$, and $\hat{\nabla} = h\nabla = h\{\partial/\partial x_1, \partial/\partial x_2\}$. Likewise,

$$[]\hat{u}[]_{1,h} = [\hat{u}, \hat{u}]_{1,h}^{1/2} \tag{5.7}$$

and

$$[]\hat{u}[]_{1,h}^2 = \int_{\mathcal{G}} h^2 [(\nabla u)\cdot\nabla u + u^2]\, \mathrm{d}x/h^2 = \|u\|_{1,\mathcal{G}}^2 . \tag{5.8}$$

Now, let $\hat{\Psi} \in \hat{Q}_t^{-1/2}(\partial\hat{\mathcal{G}})$ be given, and let $\hat{z}$, $\hat{p}$ and $\hat{q} \in H^1(\hat{\mathcal{G}})$ denote solutions to the following auxiliary problems:

$$[\hat{z}, w]_{1,h} = \int_{\partial\hat{\mathcal{G}}} \hat{\Psi} w\, \mathrm{d}\hat{s} \qquad \forall w \in H^1(\hat{\mathcal{G}}) , \tag{5.9}$$

$$[\hat{p}, w]_{1,h} = \int_{\partial\hat{\mathcal{G}}} \hat{\Psi} w\, \mathrm{d}\hat{s} - A(\hat{\Psi}) \int_{\hat{\mathcal{G}}} w\, \mathrm{d}\hat{x} \qquad \forall w \in H^1(\hat{\mathcal{G}}) , \tag{5.10}$$

$$|\hat{q}, w|_{1,\hat{\mathcal{G}}} = \int_{\partial\hat{\mathcal{G}}} \hat{\Psi} w\, \mathrm{d}\hat{s} - A(\hat{\Psi}) \int_{\hat{\mathcal{G}}} w\, \mathrm{d}\hat{x} \qquad \forall w \in H^1(\hat{\mathcal{G}}) , \tag{5.11}$$

wherein

$$|\cdot, \cdot|_{1,\hat{\mathcal{G}}} = \int_{\mathcal{G}} (\hat{\nabla}(\cdot))\cdot\hat{\nabla}(\cdot)\, \mathrm{d}\hat{x} \qquad \text{and} \qquad A(\hat{\Psi}) = \int_{\partial\hat{\mathcal{G}}} \hat{\Psi}\, \mathrm{d}\hat{s} \Big/ \int_{\hat{\mathcal{G}}} \mathrm{d}\hat{x} . \tag{5.12}$$

We have thus selected $A(\Psi)$ so that (5.11) has a solution, i.e.

$$\int_{\hat{\mathcal{G}}} A(\hat{\Psi})\, \mathrm{d}\hat{x} - \int_{\partial\hat{\mathcal{G}}} \hat{\Psi}\, \mathrm{d}\hat{s} = 0 . \tag{5.13}$$

Likewise, in order that the solution q of (5.11) be unique, we also require that

$$\int_{\hat{\mathcal{G}}} \hat{q}\, \mathrm{d}\hat{x} = 0 . \tag{5.14}$$

We now come to the first of two lemmas:

LEMMA 5.1. Let $\hat{z}, \hat{q} \in H^1(\hat{\mathcal{G}})$ be the solutions of (5.9) and (5.11). Then

$$[]\hat{z}[]_{1,h}^2 \leq |\hat{q}|_{1,\hat{\mathcal{G}}}^2 + \mathcal{A}(h) , \tag{5.15}$$

wherein

$$|\hat{q}|^2_{1,\hat{\mathcal{G}}} = |\hat{q}, \hat{q}|_{1,\hat{\mathcal{G}}}\,, \qquad (h) = (A^2/h^2) \int_{\hat{\mathcal{G}}} \mathrm{d}\hat{x}\,, \tag{5.16}$$

and A is defined in (5.12).

Proof: From (5.9) and (5.10) it is clear that $\hat{z} = \hat{p} + A/h^2$, and, therefore, setting $w = \hat{p}$ in (5.10) gives

$$\Box \hat{p} \Box^2_{1,h} = \int_{\partial\hat{\mathcal{G}}} \hat{\Psi}\hat{p}\,\mathrm{d}\hat{s} - A \int_{\mathcal{G}} \hat{p}\,\mathrm{d}\hat{x} = [\hat{z}, \hat{p}]_{1,h} - A \int_{\hat{\mathcal{G}}} \hat{p}\,\mathrm{d}\hat{x}$$

$$= \Box \hat{z} \Box^2_{1,h} - A \int_{\hat{\mathcal{G}}} \hat{z}\,\mathrm{d}\hat{x} - A \int_{\hat{\mathcal{G}}} \hat{p}\,\mathrm{d}\hat{x}\,.$$

By setting $w = 1$ in (5.9) and using (5.12) and the fact that $\hat{z} = \hat{p} + A/h^2$, we also find that $\int_{\hat{\mathcal{G}}} \hat{z}\,\mathrm{d}\hat{x} = \mathcal{A}(h)/A$ and $\int_{\hat{\mathcal{G}}} \hat{p}\,\mathrm{d}\hat{x} = 0$. Thus,

$$\Box \hat{z} \Box^2_{1,h} = \Box \hat{p} \Box^2_{1,h} + \mathcal{A}(h)\,. \tag{5.17}$$

Now from (5.10) and (5.11) we see that

$$\Box \hat{p} \Box^2_{1,h} \leqslant |\hat{q}|_{1,\hat{\mathcal{G}}} \Box \hat{p} \Box_{1,h}\,. \tag{5.18}$$

Thus, combining (5.18) and (5.17) gives (5.15) ∎

We denote by $\hat{\Pi}^1_h$ and T^1 the orthogonal projections of $H^1_h(\hat{\mathcal{G}})$ onto $\hat{Q}^1_k(\hat{\mathcal{G}})$ with respect to the inner products $[\cdot, \cdot]_{1,h}$ and $|\cdot, \cdot|_{1,\hat{\mathcal{G}}}$, respectively. Let

$$\hat{Z} = \hat{\Pi}^1_h \hat{z}\,, \qquad \hat{\mathfrak{P}} = \hat{\Pi}^1_h \hat{p} \qquad \text{and} \qquad \hat{Q} = T^1 \hat{q}\,. \tag{5.19}$$

Then, $\forall\, W \in \hat{Q}^1_k(\hat{\mathcal{G}})$,

$$[\hat{Z}, W]_{1,h} = \oint_{\partial\hat{\mathcal{G}}} \hat{\Psi} W\,\mathrm{d}\hat{s}\,, \tag{5.20}$$

$$[\hat{\mathfrak{P}}, W]_{1,h} = \oint_{\partial\hat{\mathcal{G}}} \hat{\Psi} W\,\mathrm{d}\hat{s} - A(\hat{\Psi}) \int_{\hat{\mathcal{G}}} W\,\mathrm{d}\hat{x}\,, \tag{5.21}$$

$$|\hat{Q}, W|_{1,} = \oint_{\partial\hat{\mathcal{G}}} \hat{\Psi} W\,\mathrm{d}\hat{s} - A(\hat{\Psi}) \int_{\hat{\mathcal{G}}} W\,\mathrm{d}\hat{x}\,, \tag{5.22}$$

with the normalization $\int_{\mathcal{G}} \hat{Q}\,\mathrm{d}\hat{x} = 0$. Again, we easily verify that

$$\hat{Z} = \hat{\mathfrak{P}} + A/h^2\,, \quad \Box \hat{Z} \Box^2_{1,h} = \Box \hat{\mathfrak{P}} \Box^2_{1,h} + \mathcal{A}(h) \tag{5.23}$$

and

$$\|\hat{Z}\|^2_{1,h} \leqslant |\hat{Q}|^2_{1,} + \mathscr{A}(h)\,. \tag{5.24}$$

LEMMA 5.2. Let $\hat{Z}$ and $\hat{Q}$ be given by (5.19). Then there exists a constant $C_6 > 0$, independent of h, such that

$$\square \hat{Z} \square^2_{1,h} \geqslant (1 - C_6 h^2)|\hat{Q}|^2_{1,\hat{\mathscr{G}}} + \mathscr{A}(h)\,. \tag{5.25}$$

Proof: In view of (5.21) and (5.22),

$$|\hat{Q}|^2_{1,\hat{\mathscr{G}}} = [\hat{Q}, \hat{P}]_{1,h} \leqslant \tfrac{1}{2}(\square \hat{Q} \square^2_{1,h} + \square \hat{\mathfrak{P}} \square^2_{1,h})\,.$$

However,

$$\square \hat{Q} \square^2_{1,h} = |\hat{Q}|^2_{1,\hat{\mathscr{G}}} + h^2\|\hat{Q}\|^2_{0,\hat{\mathscr{G}}} \leqslant (1 + C_6 h^2)|\hat{Q}|^2_{1,\hat{\mathscr{G}}}\,.$$

where we have used Poincaré's inequality ($\|\hat{Q}\|_{0,\hat{\mathscr{G}}} \leqslant C_6 |Q|_{1,\hat{\mathscr{G}}}$, C_6 dependent on $\hat{\mathscr{G}}$ but not on h, because $\int_{\hat{\mathscr{G}}} \hat{Q}\, \mathrm{d}\hat{x} = 0$). Thus,

$$\square \hat{\mathfrak{P}} \square^2_{1,h} \geqslant 2|\hat{Q}|^2_{1,\mathscr{G}} - \square \hat{Q} \square^2_{1,h} \geqslant (1 - C_6 h^2)|\hat{Q}|^2_{1,\mathscr{G}}\,,$$

from which (5.25) follows in view of (5.23) ∎

We are finally ready to apply these results to a typical finite element of the type described in the previous section:

THEOREM 5.1. Let Ω_e, $1 \leqslant e \leqslant E$, be a finite element and $\hat{\Omega}_e = \{\hat{x}_e : \hat{x}_e = x_e/h_e,\ x_e \in \Omega_e,\ h_e = \mathrm{dia}(\Omega_e)\}$. Let $\hat{q}_e \in H^1(\hat{\Omega}_e)$ be such that

$$\int_{\hat{\Omega}_e} (\hat{\nabla}\hat{q}_e)\cdot \hat{\nabla}\hat{w}_e\, \mathrm{d}\hat{x} = \oint_{\partial\hat{\Omega}_e} \hat{\Psi}_e \hat{w}_e\, \mathrm{d}\hat{s} - A_e(\hat{\Psi}) \int_{\hat{\Omega}_e} \hat{w}_e\, \mathrm{d}\hat{x} \qquad \forall \hat{w}_e \in H^1(\hat{\Omega}_e)\,, \tag{5.26}$$

wherein

$$A_e = A_e(\hat{\Psi}_e) = \oint_{\partial\hat{\Omega}_e} \hat{\Psi}_e\, \mathrm{d}\hat{x} \Big/ \int_{\hat{\Omega}_e} \mathrm{d}\hat{x}\,, \qquad \int_{\hat{\Omega}_e} \hat{q}_e\, \mathrm{d}\hat{x} = 0\,, \tag{5.27}$$

and $\hat{\Psi}_e \in Q_t^{-1/2}(\partial\hat{\Omega}_e)$, $1 \leqslant e \leqslant E$. In addition, let T^1 denote the orthogonal projection of $H^1(\Omega_e)$ onto $Q^1_k(\Omega_e)$ with respect to the inner product $|\cdot,\cdot|_{1,\hat{\Omega}_e}$. Then

$$\mu_e \geqslant (1 - C_6 h_e^2) \inf_{\Psi \in Q_t^{-1/2}(\partial\Omega_e)} \frac{|T^1\hat{q}|^2_{1,\hat{\Omega}_e}}{|\hat{q}|^2_{1,\hat{\Omega}_e}}\,, \tag{5.28}$$

$$\nu_e(Q^1_k(\Omega_e), Q^0_r(\Omega_e) = \nu_e(Q^1_k(\Omega_e), Q^0_r(\Omega_e))\,, \tag{5.29}$$

$$\gamma_e(Q_k^1(\Omega_e), Q_r^0(\Omega_e)) = \gamma_e(\hat{Q}_k^1(\hat{\Omega}_e), Q_r^0(\hat{\Omega}_e)) . \tag{5.30}$$

wherein C_6 is a constant independent of h_e, while μ_e, ν_e and γ_e are the parameters defined in (4.18), (4.19) and (4.20), respectively.

Proof: Equations (5.29) and (5.30) follow immediately from the definitions (4.19) and (4.20) and the definition of $\hat{\Omega}_e$. Thus, we concentrate on (5.28). We have, by (5.20), (5.8) and Lemma 5.2,

$$\oint_{\partial\Omega_e} \Psi_e Z_e \, ds = \|Z_e\|^2_{1,\Omega_e} = \square \hat{Z}_e \square^2_{1,h_e} \geq (1 - C_6 h_e^2)|T^1 \hat{q}|_{1,\hat{\Omega}_e} + \mathcal{A}(h_e) ,$$

On the other hand, from (2.11) and (5.15) ,

$$\|\Psi_e\|^2_{-1/2,\partial\Omega_e} = \|z_e\|^2_{1,\Omega_e} = \square \hat{z}_e \square^2_{1,h_e} \leq |\hat{q}|^2_{1,\hat{\Omega}_e} + \mathcal{A}(h_e) .$$

Thus

$$\mu_e \geq \inf_{\Psi_e \in Q_T^{-1/2}(\partial\Omega_e)} \frac{(1 - C_6 h_e^2)|T^1 q|^2_{1,\hat{\Omega}_e} + \mathcal{A}(h_e)}{|\hat{q}|^2_{1,\hat{\Omega}_e} + \mathcal{A}(h_e)} .$$

Inequality (5.28) now follows from the elementary inequality $b/c \leq (b+a)/(c+a)$ for $a, c \geq 0$ and $b \leq c$ ∎

Theorem 5.1 is important because we obtain bounds in the parameters μ_e, ν_e and γ_e by studying the master element $\hat{\Omega}_e$ only. Using Theorem 5.1 and Theorem 4.2, it is easy to show that, for example, when only quasi-uniform quadrilateral or triangular elements are used, then for all sufficiently small h, if the rank condition (4.29) is satisfied,

$$\mu = \min_{1 \leq e \leq E} \mu_e \geq \delta > 0 ,$$

where δ is independent of h.

A similar analysis can be made to study the dependence of μ on h by constructing a smooth mapping of Ω_e onto a master element $\hat{\mathcal{G}}$.

6. A priori error estimate

We now come to the question of convergence and error estimates for the approximate scheme prescribed. The principal results follow easily from theorem 2.2 and the properties of subspaces (4.1)–(4.12).

THEOREM 6.1. Let $u \in H^l(\cdot)$, $l \geq 2$, be the exact solution of (3.1). Assume that the parameter $\beta(P)$ of (4.24) is positive. And let $\Lambda = \{U, \Sigma^t, \Psi\} \in Q$ be the mixed-hybrid finite element solution. Then the following error estimate holds:

$$\|e\|_{\mathcal{X}} \leq C^* h^\alpha |||u|||_{l,P} , \tag{6.1}$$

where

$$e = \{u - U, (\nabla u - \Sigma)^t, \psi - \Psi\}, \qquad \alpha = \min\{k, r+1, t+3/2, l-1\},$$
$$C^* = A_0 \max\{C_3, C_4, C_5\}, \qquad A_0 = 1 + 2 \times 15^{-1/2}\beta(P), \tag{6.2}$$

in which $\psi = -\partial u/\partial n$ on Γ^g_{ef}, $e > f$, $\beta(P)$ is as in (4.24), and C_3, C_4, C_5 are as in (4.4), (4.7), (4.10).

Proof. By combining the results of section 3 and theorem 4.1, we have, in accordance with theorem 2.2,

$$\|e\| \leqslant A_0 \inf_{\tilde{\Lambda} \in Q} \|\lambda - \tilde{\Lambda}\| \leqslant A_0 \{|||u - \tilde{U}|||_{1,P} + \|\sigma - \tilde{\Sigma}\|_{0,P} + \|\psi - \tilde{\Psi}\|_{\mathcal{W}(\Gamma)}\}, \tag{6.3}$$

where $\tilde{U}$, $\tilde{\Sigma}$, and $\tilde{\Psi}$ are as in (4.4), (4.7) and (4.10), respectibely. Introducing (4.4), (4.7) and (4.10) into (6.2) with the aid of (3.8),

$$\|e\|_{\mathfrak{X}} \leqslant C^* \left\{ h^\eta \left(\sum_{e=1}^{E} \|u_e\|^2_{l,\Omega_e} \right)^{1/2} + h^\omega \left(\sum_{e=1}^{E} \|\nabla u_e\|^2_{q,\Omega_e} \right)^{1/2} + h^\theta \left(\sum_{e=1}^{E} \|\psi_e\|^2_{H^{m-3/2}(\partial\Omega_e, \Omega_e)} \right)^{1/2} \right\},$$

from which we deduce (6.1) in view of (2.15) due to the fact that $u \in H^l(P)$, $l \geqslant 2$ ∎

In view of the estimate (6.1) and remarks made in section 4, if subspaces satisfy

(i) the rank condition (4.29) and

(ii) the inclusion property (4.26),

then the finite element solutions will converge at the rate of α as in (6.2) if all Ω_e are quasi-uniform. However, if (i) is satisfied but not (ii), then in general $A_0 \to \infty$ as $h \to 0$ – this will destroy convergence.

If $\bar{\Omega}$ is convex and if (4.26) holds, then by applying the technique of Nitsche (see for example [27] or [28]), the following L_2 estimate can also be obtained:

$$\|e_u\|_{0,\Omega} \leqslant C' h^{\alpha+1}(\cdot), \tag{6.4}$$

where C' is independent of h, and α is as in (6.2).

7. Special cases

Several special cases can be deduced from our general theory by imposing various restrictions on the spaces $\mathfrak{X}$ and Q. A brief discussion is given below for each case.

7.1. Hybrid-displacement model 1 (primal hybrid model)

If we let $\sigma = \nabla u$ throughout the formulation, we obtain a hybrid model similar to the one proposed by Yamamoto [8] and studied by Raviart and Thomas [6]. Naturally, we set

$$\mathcal{X}^1 = H^1(P) \times \mathcal{W}(\Gamma) . \tag{7.1}$$

The bilinear form is now

$$B^1(\lambda, \bar{\lambda}) = \sum_{e=1}^{E} B_e^1(\lambda_e, \bar{\lambda}_e) = \sum_{e=1}^{E} \left\{ (u_e, \bar{u}_e)_{1,\Omega_e} + \oint_{\partial\Omega_e} (\psi_e \bar{u}_e + \bar{\psi}_e u_e)\, ds \right\} . \tag{7.2}$$

Following the analysis similar to that in section 3, it is easy to see that the bilinear form $B^1(\cdot, \cdot)$ satisfies all the hypotheses of theorem 2.1. Indeed, we can take $C_1 = 2$ and $C_2 = 6^{-1/2}$, where C_1 and C_2 are constants in (2.17) and (2.18), respectively. Therefore, there exists a unique $\lambda^1 = \{u^1, \psi^1\} \in \mathcal{X}^1$ such that

$$B^1(\lambda^1, \bar{\lambda}) = F(\bar{\lambda}) \qquad \forall \bar{\lambda} \in \mathcal{X}^1 , \tag{7.3}$$

where $F(\cdot)$ is as in (3.20).

Denoting the corresponding finite dimensional space by

$$\mathcal{Q}^1 = Q_k^1(P) \times Q_t^{-1/2}(\Gamma) \tag{7.4}$$

(with the same notations as in section 4), we have the following theorem:

THEOREM 7.1. There exists a unique finite element solution $\Lambda^1 \in \mathcal{Q}^1$ to (7.3) if the rank condition (4.29) is satisfied.

Proof: Following the same analysis as in section 4, we can take

$$C_2^h = \min_{1 \leq e \leq E} 6^{-1/2} \mu_e , \tag{7.5}$$

where C_2^h is as in (2.26), and μ_e is as in (4.18). By Theorem 4.2 we see that $C_2^h > 0$ if (4.29) holds. This proves the theorem ∎

The necessary condition is the same as in theorem 4.3. Finally, we obtain the following error estimate in view of theorem 7.1 and theorem 2.2:

THEOREM 7.2. Let the rank condition (4.29) hold. Then the following error estimates holds:

$$\|e\|_{\mathcal{X}^1} \leq A_1 h^{\alpha_1} \|u\|_{l,\Omega} , \tag{7.6}$$

where

$$e = \lambda^1 - \Lambda^1 , \qquad \alpha_1 = \max\{k, t + 3/2, l - 1\},\ l \geq 2, \qquad A_1 = (1 + 2/C_2^h) \min\{C_3, C_5\} ,$$

in which C_2^h, C_3, and C_5 are as in (7.5), (4.4) and (4.10), respectively ∎

We comment here that the estimate (7.6) slightly differs from one in [5] which reads

$$|||e_u|||_{1,P} + \|e_\psi\|_{L_2(\Gamma)} \leq C h^{\tau} \|u\|_{\tau+1,\Omega} , \tag{7.7}$$

where $\tau = \min(k, t+1)$, the obvious reason being the difference in norms on Γ.

7.2. Mixed method with a constrained boundary condition

This is a generalization of the Lagrange multiplier method studied by Babuska [25]. We begin by setting

$$\mathcal{X}^2 = H^1(\Omega) \times L_2(\Omega) \times H^{-1/2}(\partial\Omega)\,, \tag{7.8}$$

$$Q^2 = Q_k^1(\Omega) \times Q_r^0(\Omega) \times Q_t^{-1/2}(\partial\Omega)\,, \tag{7.9}$$

where

$$Q_k^1(\Omega) = \{U \in H^1(\Omega)\colon\ U_e \in \mathcal{P}_{k'}(\Omega_e)\,,\ \ 1 \leqslant e \leqslant E\}\,, \tag{7.10}$$

$$Q_r^0(\Omega) = \{\Sigma \in L_2(\Omega)\colon\ \Sigma_e \in \mathcal{P}_{r'}(\Omega_e)\,,\ \ 1 \leqslant e \leqslant E\}\,, \tag{7.11}$$

$$Q_t^{-1/2}(\partial\Omega) = \{\Psi \in H^{-1/2}(\partial\Omega)\colon\ \Psi|_{\Gamma_{e0}} \in \mathcal{P}_{t'}(\Gamma_{e0})\,,\ \ 1 \leqslant e \leqslant E\}\,. \tag{7.12}$$

Note that $Q_k^1(\Omega)$ is C^0-finite element space that does not satisfy the boundary condition. The corresponding bilinear form on $\mathcal{X}$ is

$$B^2(\lambda, \bar{\lambda}) = \int_\Omega [\sigma \cdot \nabla\bar{u} + u\bar{u} + (\nabla u - \sigma)\cdot\bar{\sigma}]\,\mathrm{d}x + \oint_{\partial\Omega} (\psi\bar{u} + \bar{\psi}u)\,\mathrm{d}s\,. \tag{7.13}$$

As clearly indicated by the choice of spaces, the partition prescribed in section 3 is no longer needed to study the variational problem associated with the bilinear form (7.13). Therefore, the dependency of variables and parameters on Ω_e will be dropped throughout the analysis. This simply means that all the analysis made in sections 3, 4 and 5 is valid when we set $E = 1$ and $\Omega_1 = \Omega_0 = \Omega$ throughout.

Finally, assuming that the rank condition is satisfied, we obtain the similar error estimate

$$\|e\|_{\mathcal{X}} \leqslant A_2 h^{\alpha_2} \|u\|_{l,\Omega}\,, \tag{7.14}$$

where

$$e = \{u - U, \sigma - \Sigma\,, -\partial u/\partial n|_{\partial\Omega} - \Psi\}\,,$$

$$A_2 = (1 + 2/C_2^h)\max\{C_3, C_4, C_5\}\,,$$

$$\alpha_2 = \min\{k, r+1, t+3/2, l-1\}\,, \qquad l \geqslant 2\,;$$

in which

$$C_2^h = 15^{-1/2}\beta(\Omega)\,, \qquad \beta(\Omega) = \max(0, \min(\nu - \gamma/2,\ \mu - \gamma/2))\,.$$

Here the parameters μ, γ and ν are defined in analogy to those in (4.18)–(4.20) with the index e dropped.

A further obvious restriction would be to take $\boldsymbol{\sigma} = \nabla u$ throughout. Then the method is exactly the one studied by Babuska [25].

7.3. Mixed method

If we assume that the boundary condition is satisfied a priori and if we set

$$\mathcal{X}^3 = H^1(\Omega) \times L_2(\Omega) , \tag{7.15}$$

$$Q^3 = Q_k^1(\Omega) \times Q_r^0(\Omega) , \tag{7.16}$$

where $Q_k^1(\Omega)$ is the usual C^0-finite element subspace which satisfies the boundary condition and $Q_r^0(\Omega)$ is as in (7.11), then the corresponding bilinear form on $\mathcal{X}^3$ will be

$$B(\omega, \bar{\omega}) = \int_\Omega [\boldsymbol{\sigma}\cdot\nabla\bar{u} + u\bar{u} + (\nabla u - \boldsymbol{\sigma})\cdot\bar{\boldsymbol{\sigma}}]\,\mathrm{d}x . \tag{7.17}$$

A slight modification of our analysis leads us to the following estimate:

$$\|e\|_{\mathcal{X}^3} \leqslant A_3 h^{\alpha_3} \|u\|_{l,\Omega} , \tag{7.18}$$

where

$$e = \{u - U, \nabla u - \boldsymbol{\Sigma}\} ,$$

$$A_3 = (1 + 2 \times 6^{1/2}/\nu) \max C_3, C_4\} ,$$

$$\alpha_3 = \min\{k, r+1, l-1\} ,$$

in which ν is as in (4.19) with e dropped. For details of this method see [9] and [10].

Obviously, by letting $\boldsymbol{\sigma} = \nabla u$, we obtain the usual conforming displacement method, and the estimate is

$$\|u - U\|_{1,\Omega} \leqslant A_4 h^{\alpha_4} \|u\|_{l,\Omega} , \tag{7.19}$$

where

$$A_4 = 2C_3 , \qquad \alpha_4 = \min\{k, l-1\}, \quad l \geqslant 1 .$$

8. The rank condition

We discuss here the rank condition mentioned in section 4 in connection with a specific example.

In particular, let the domain Ω be quasiuniformly *triangulated*, and let τ denote a master element with the boundary $\partial\tau$. Further, let $k = k'$ and $t = t'$; that is $Q_k^1(\tau) = \mathcal{P}_k(\tau)$ and $Q_t^{-1/2}(\partial\tau) = \mathcal{P}_t(\partial\tau)$ are spaces of complete polynomials of order k over τ and t over each side of $\partial\tau$, respectively. Then, since the rank condition is invariant under coordinate transformation, the condition (4.29) can be rewritten for a $\Psi \in \mathcal{P}_t(\partial\tau)$ as

$$\int_{\partial\tau} \Psi U \, ds = 0 \qquad \forall U \in \mathcal{P}_k(\tau) \qquad \text{implies} \qquad \Psi = 0 . \tag{8.1}$$

The following theorem, proved by Raviart and Thomas in [6], also applies to this particular aspect of our problem:

THEOREM 8.1. The rank condition (8.1) is satisfied if and only if

$$k \geq \begin{cases} t+1 & \text{if} \quad t \text{ is even,} \\ t+2 & \text{if} \quad t \text{ is odd} \end{cases} \quad \blacksquare$$

For further discussions on criteria such as this, consult [6].

Remark: The sufficiency condition holds for the case when $k' > k$ but not the necessary condition.

Finally, we mention that the rank condition for the case discussed in section 7.2 takes a bit different form; namely, for a given $\Psi \in Q_t^{-1/2}(\partial\Omega)$,

$$\oint_{\partial\Omega} \Psi U \, ds = 0 \qquad \forall U \in Q_k^1(\Omega) \qquad \text{implies} \qquad U = 0 . \tag{8.2}$$

And it is easy to show that this condition is satisfied if and only if $k \geq t$, provided that no more than two sides of $\partial\tau$ coincide with $\partial\Omega$. Here τ need not be a triangle.

9. Numerical experiments

As a simple example designed to verify our theoretical estimates, we consider the following one-dimensional problem:

$$\begin{cases} -u'' + u = f(x) , & x \in \Omega = (0, 1) , \\ u(0) = u(1) = 0 , \end{cases} \tag{9.1}$$

where

$$f(x) = \frac{2\alpha(1 + \alpha^2(1-\bar{x})(x-\bar{x}))}{(1+\alpha^2(x-\bar{x})^2)} + (1-x)\{\tan^{-1}\alpha(x-\bar{x}) + \tan^{-1}\alpha\bar{x}\} .$$

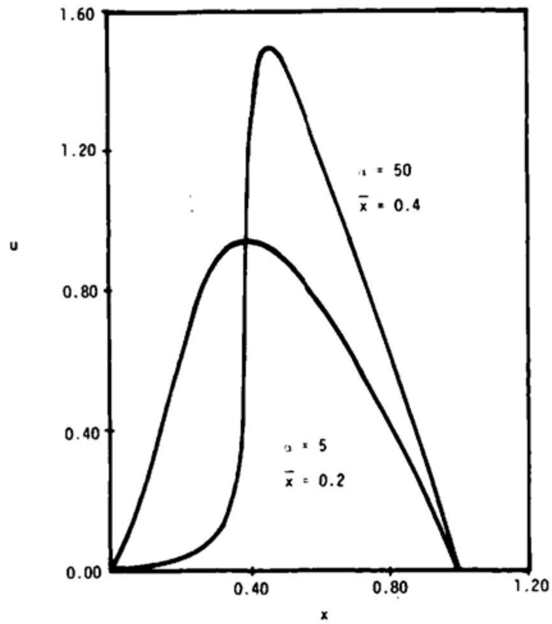

Fig. 1. Exact solutions.

in which α and $\bar{x}$ are some given constants. The solution [30] is

$$u(x) = (1 - x)\,(\tan^{-1} \alpha(x - \bar{x}) + \tan^{-1} \alpha\bar{x}) \,. \tag{9.2}$$

The solution has a sharp knee near $\bar{x}$ when α is large (see Fig. 1). We consider the following two cases:

1. Smooth solution; $\alpha = 5, \bar{x} = 0.2$,
2. Rough solution; $\alpha = 50, \bar{x} = 0.4$,

both solutions are shown in Fig. 1.

Following the notation in section 3, Γ now consists of $E + 1$ knots; hence, $\mathcal{W}(\Gamma) = \Re^{E+1}$ and $\psi \in \mathcal{W}(\Gamma)$ is an $(E + 1)$-tuple

$$\psi = \{\psi^0, \psi^1, ..., \psi^E\} \,,$$

where ψ^i are real numbers defined at each knot x_i, $0 \leqslant i \leqslant E = 1/h$. According to Lemma 2.1, (3.4) and (3.6), we have

$$\|\psi\|^2_{\mathcal{W}(\Gamma)} = \sum_{e=1}^{E} \left\{ [(\psi^e)^2 + (\psi^{e-1})^2] \, \frac{\cosh h_e}{\sinh h_e} + \frac{2\psi^e \psi^{e-1}}{\sinh h_e} \right\} , \tag{9.3}$$

and one can show that

$$|\psi|_{\max} \leqslant c \|\psi\|_{\mathcal{W}(\Gamma)} \,, \tag{9.4}$$

where c does not depend on h.

The corresponding spaces are

$$\mathcal{X} = H^1(P) \times L_2(P) \times \mathcal{W}(\Gamma)\,,$$

and

$$\mathcal{Q} = H^1(P) \times L_2(P) \times \mathcal{W}(\Gamma)\,.$$

Clearly, the rank condition (4.29) is satisfied if and only if $k \geqslant 1$. Hence, if $k \geqslant 1$, the estimates (6.1) and (6.4) can be written as

$$\{|||e_u|||_{1,P}\,,\ \|e_\sigma\|_{0,P}\,,\ \|e_\psi\|_{\mathcal{W}(\Gamma)}\} \leqslant C^* h^\alpha(\cdot) \tag{9.5}$$

and

$$\|e_u\|_{0,P} \leqslant CC^* h^{\alpha+1}(\cdot)\,, \tag{9.6}$$

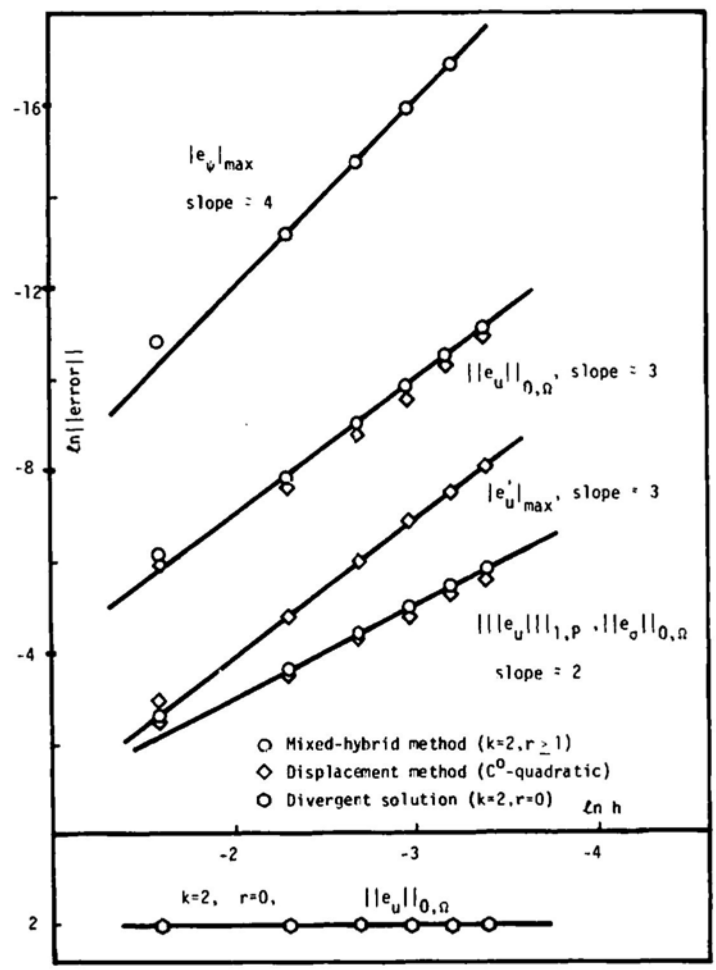

Fig. 2. Convergence curve for the smooth problem.

where $\alpha = \min\{k, r+1\}$, C is a constant independent of h, and C^* is such that

1. C^* is independent of h if $r \geqslant k-1$ (see remark 4.1).
2. $C^* = C'(1 + C''h^{-2})$ if $r < k-1$ (see remark 4.2).

Numerical results show that the actual rates of convergence are as the estimates derived with the exception of $|e_\psi|_{max}$, where the actual rate is better than what follows from (9.4). This is in complete agreement with the behavior of the usual finite element method where the rate in $|\cdot|_{max}$ is better than the theoretical estimate obtained by simple application of the imbedding theorem. This effect has been completely proven by Nitsche [17] very recently. So we can expect that the above-mentioned effect for the behavior of $|e_\psi|_{max}$ could be theoretically explained. Figs. 2 and 3 show that the approximation is convergent for the case $k = 2$, $r \geqslant 1$ and divergent when $k = 2$, $r = 0$ for smooth and rough problems, respectively. Also shown for comparisons of accuracy are solutions obtained using the mixed-hybrid method and the conventional conforming finite method using C^0-quadratic elements. As can be seen from Figs. 2 and 3, accuracies in approximating u and u' are about the same for both models in an average sense. However, as anticipated, the approximation Ψ of u' at the knots is quite accurate compared to that of the C^0-quadratic element. This could be an advantage of the mixed-hybrid method at the expense of a bit more manipulation.

For both the mixed-hybrid and displacement models, the interval is divided into E equal elements, where $E = 5, 10, 15, 20, 25, 30$. To insure sufficiently accurate numerical integrations, a 5-point Gauss quadrature rule is used, and values of $|e'_u|_{max}$ are computed for the displacement

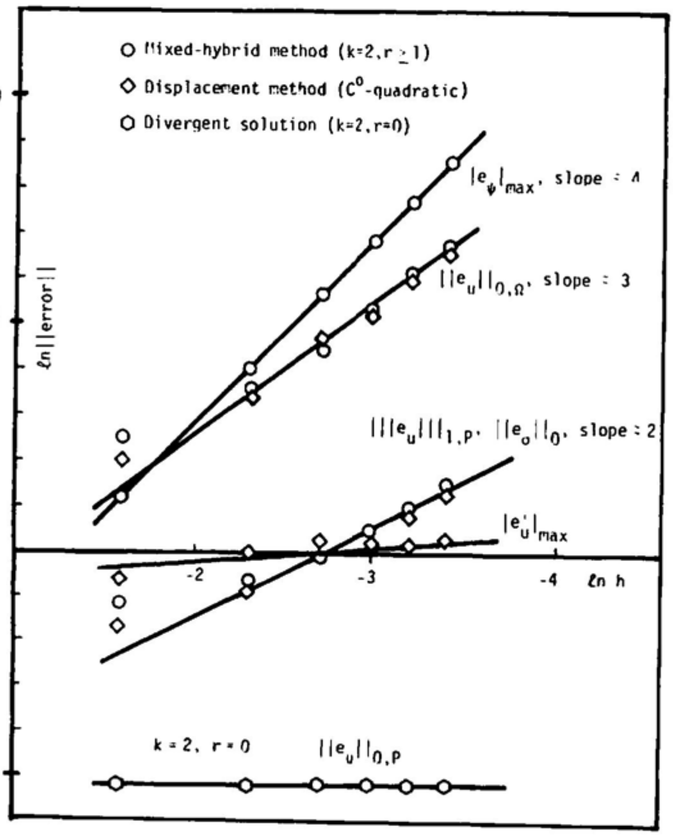

Fig. 3. Convergence curve for the rough problem.

model at 2 Gauss points. The very slow convergence of $|e'_u|_{max}$ in fig. 3 is evidently due to steep gradient of the solution and might have been improved if nodal points were placed tactically.

No meaningful comparisons of computing cost could be made because the computer program for the mixed-hybrid method is specialized in this particular problem while the other is a quite versatile one.

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References

[1] E. Jones, A generalization of the direct-stiffness method of structural analysis, AIAA J. 2 (1964) 821–826.
[2] T.H.H. Pian, Element stiffness matrices for boundary compatibility and for prescribed boundary stresses, Proceedings of the First Conference on Matrix Methods in Structural Mechanics, Wright-Patterson Air Force Base, 1965, AFDL-TR-66-80 (1966) 457–477.
[3] T.H.H. Pian and P. Tong, Basis of finite element methods for solid continua, Int. J. Numer. Meths. Eng. 1 (1969) 3–85.
[4] P. Tong, New displacement hybrid finite element models for solid continua, Int. J. Numer. Meths. Eng. 2 (1970) 73–85.
[5] P.A. Raviart, Hybrid finite element methods for solving 2nd order elliptic equations, in: J.J. Miller (ed.), Topics in Numerical Analysis II, Proc. of the Royal Irish Conf. on Num. Analysis, 1974 (Academic Press, N.Y., 1976).
[6] P.A. Raviart and J.M. Thomas, Primal hybrid finite element methods for 2nd order elliptic equations, Report 75025, Univ. Paris VI, Laboratoire Analyse Numérique (1975).
[7] J.M. Thomas, Méthodes des éléments finis hybrides duaux pour les problèmes elliptiques du second-ordre, Report 75006, Univ. Paris VI et Centre National de la Recherche Scientifique (1975).
[8] Y. Yamamoto, A formulation of matrix displacement method, Technical Report, Mass. Inst. Tech., Dept. Aeron. Astron. (Nov. 1966).
[9] F. Brezzi, Sur la méthode des éléments finis hybrides pour le problème biharmonique, Numer. Math. 24 (1975) 103–131.
[10] F. Brezzi and L.D. Marini, On the numerical solution of plate bending problems by hybrid methods, RAIRO Report (1975).
[11] B. Fraeijs de Veubeke, Variational principles and the patch test, Int. J. Numer. Meths. Eng. 8 (1974) 783–801.
[12] L. Herrmann, Finite element bending analysis for plates, J. Eng. Mech. Div. ASCE 93 (1967) 000–000.
[13] J.T. Oden and J.N. Reddy, On mixed finite element approximations, SIAM J. Numer. Anal. 13 (1976) 392–404.
[14] J.N. Reddy and J.T. Oden, Mathematical theory of mixed finite element approximations, Q. Appl. Math. 33 (1975) 255–280.
[15] P.G. Ciarlet and P.A. Raviart, A mixed finite element method for the biharmonic equation, in: C. de Boor (ed.), Mathematical aspects of finite elements in partial differential equations (Academic Press, N.Y., 1974) 125–145.
[16] C. Johnson, On the convergence of a mixed finite-element method for plate bending problems, Numer. Math. 21 (1973) 43–62.
[17] J. Nitsche, Convergence of nonconforming methods, in: C. de Boor (ed.), Mathematical aspects of finite elements in partial differential equations (Academic Press, N.Y., 1974) 15–53.
[18] I. Babuska and M. Zlamal, Nonconforming elements in the finite element method, Tech. Note BN-729, Univ. Maryland (1972).
[19] T. Miyoshi, Convergence of finite element solutions represented by a non-conforming basis, Kumamoto J. Sci. Math. 9 (1972) 11–20.
[20] G. Strang, Variational crimes in the finite element method, in: A.K. Aziz (ed.), Mathematical foundations of the finite element method with applications to partial differential equations (Academic Press, N.Y., 1972) 689–710.
[21] S. Atluri, A new assumed stress hybrid finite element model for solid continua, AIAA J. 9 (1971) 1647–1649.
[22] J.P. Wolf, Generalized hybrid stress finite element models, AIAA J. 11 (1973) 386–388.
[23] J.P. Wolf, Generalized stress models for finite element analysis, Ph.D. Thesis, Swiss Fed. Inst. Tech. (ETH), Zurich (1974).
[24] S. Agmon, Lectures on elliptic boundary value problems (Van Nostrand, 1965).
[25] I. Babuska, The finite element method with Lagrange multipliers, Numer. Math. 20 (1973) 172–192.

[26] I. Babuska, Error bounds for finite element method, Numer. Meth. 16 (1971) 322–333.
[27] I. Babuska and A.K. Aziz, Survey lectures on the mathematical theory of finite elements, in: A.K. Aziz (ed.), The mathematical foundations of the finite element method with applications to partial differential equations (Academic Press, N.Y., 1972) 3–359.
[28] J.T. Oden and J.N. Reddy, An introduction to the mathematical theory of finite elements (Wiley Interscience, New York, 1976).
[29] J.K. Lee, Convergence of mixed-hybrid finite element methods, Ph.D. Dissertation, Div. Eng. Mech., Univ. Texas at Austin (1976).
[30] H.H. Rachford and M.F. Wheeler, An H^{-1}-Galerkin procedure for the two-point boundary-value problem, in: C. de Boor (ed.), Math. aspects of finite elements in partial differential equations (Academic Press, N.Y., 1974) 353–382.