

Valid Generators via Parsing*

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Abstract

When generating random data structures, it is often important that the generated values satisfy some validity condition. This is especially important in property-based testing, where test inputs are only useful if they satisfy the precondition of a functional specification. We tackle this “valid generation problem” by observing that, perhaps surprisingly, random data *generators* are closely related to *parsers*. In particular, both structures make a sequence of choices that are used to build a data structure.

This insight leads us to a rich theory of *free generators* that makes the connection between generators and parsers explicit; free generators can be interpreted as either generators or parsers as needed. Free generators admit an efficiently computable notion of *Brzowski derivative* that can be used to preview the effect of a particular choice in the generator. We use this derivative operation in a novel algorithm that aims to solve the problem of generating valid data structures in certain domains.

Keywords: Property-based testing, Formal languages, Random generation

1 Introduction

Various uses for random data are subject to the *valid generation problem*—they need a large number of the random values that satisfy some validity condition. For example, we may want to generate random binary trees such that they are valid binary search trees (BSTs). The challenge is to randomly generate a variety of valid data as quickly as possible.

This problem is especially common when using tools like QUICKCHECK [4] for *property-based testing*. Property-based testing checks that a program obeys a functional specification using hundreds or thousands of random inputs. Testers run into the valid generation problem when the random inputs are expected to satisfy some precondition. For example, in order to test that an insert procedure preserves the BST invariants, we need to generate lots of inputs that are already BSTs.

The valid generation problem is computationally hard in general, but various approaches have shown promise [3, 6, 10]. We choose to focus on *generator optimization* methods,

that start with an existing generator (which does not produce exclusively valid inputs) and tune or modify the generator to produce valid results with higher-than-normal probability [13, 14]. In particular, we explore new avenues for generator optimization that hinge on a simple observation: generators operate like *parsers* of random choices.

A random generator is program that makes a sequence of random choices—essentially coin-flips—to decide how some data structure should be constructed. As the program runs, each choice is made in real-time by querying some kind of random number generator. Of course, there is nothing stopping us from pre-computing choices for the generator to use; we could provide a list of choices ahead of time and tell the generator to make those choices one after the other. This subtle shift in perspective turns our generator into a parser of sequences of choices!

We make this intuition concrete with structures that we call *free generators*, which re-imagine standard random generators as data structures that can be interpreted and manipulated at will. Free generators can be viewed as normal generators, which make real-time choices, but they can also be interpreted as deterministic parsers of sequences of choices in the way we observed above. This gives a concrete understanding of the close connection between generation and parsing, clarifying existing folklore into a useful formalism.

Free generators are inspired by *Free Applicative Functors* [2], and accordingly, they are incredibly flexible. They are not tied to a particular output distribution, so we can modify the average size and shape of output values without modifying the actual generator. But changing the distribution does not yet give us a way to solve the valid generation problem. For that, we rely on the fact that free generators are represented as data structures that can be manipulated by programs. In particular, we can define a *derivative* operation on free generators that computes a new generator that is identical to the first, except with the first choice already fixed. In the case of our binary search tree example, this might fix a particular value for a node, or fix a stopping point for a particular sub-tree. We can use derivatives to evaluate how likely a given choice is to produce valid results, which yields a new approach to the valid generation problem that has done remarkably well in our experiments.

In the next section, we describe the high-level motivation behind our free generator formalism and give an overview of potential applications that justify the theory in the remainder of the paper (§2). After that, we make the following contributions:

*Placeholder title

- We present a novel theory of *free generators* that gives a formal account of the connections between random generators and parsers (§3).
- We define a *derivative* operation on free generators that computes a view of a generator after making a particular choice and prove this operation consistent with related notions in formal languages (§4).
- We give an algorithm for tackling the *valid generation problem* that relies on free generator derivatives (§5).

We conclude with related work (§6) and ideas for future research (§7).

2 The High-Level Story

This section gives a bird's-eye view of our motivation, theory, and experimental results.

2.1 Generators and Parsers

The goal of this work is to optimize an existing generator by observing that generators are like parsers. Let us start by looking at a specific generator and understanding what it has in common with a specific parser. Consider `genTree`, defined in Figure 1. The program `genTree` is a generator that produces random binary trees of Booleans like

Node True Leaf Leaf and
Node True Leaf (Node False Leaf Leaf)

by making a series of random choices (represented here as uniform coin flips). The choices lead the program to generate trees up to a given height via a simple recursive procedure. Note that the generator does not always make the same number of choices: sometimes it chooses to return a Leaf and terminate, and other times it chooses to build a Node, which requires more choices.

Now, consider `parseTree` (also in Figure 1), which parses a tree from a string containing the characters `n`, `l`, `t`, and `f`. The parser turns

`ntl` into Node True Leaf Leaf and
`ntlnfl` into Node True Leaf (Node False Leaf Leaf).

The parser operates character by character, sometimes changing the way it handles the next character based on the previous character.

At this point the superficial similarities between `genTree` and `parseTree` should be clear, but what is really going on? It all comes down to *choices*. In `genTree`, choices are made randomly during the execution of the program, while in `parseTree` the choices were made ahead of time and manifest as the characters in the input string. The programs have the same structure, and only differ in their expectation of when choices should be made. Ideally, we should always be able to re-interpret a generator as a parser of sequences of choices, or a parser as a generator over its output set.

```

genTree n =
  if n == 0 then
    return Leaf
  else
    c ← flip()
    if c == Head then return Leaf
    if c == Tail then
      c ← flip()
      if c == Head then x ← True
      if c == Tail then x ← False
      l ← genTree (n - 1)
      r ← genTree (n - 1)
      return Node x l r

parseTree n =
  if n == 0 then
    return Leaf
  else
    c ← consume()
    if c == l then return Leaf
    if c == n then
      c ← consume()
      if c == t then x ← True
      if c == f then x ← False
      else fail
      l ← parseTree (n - 1)
      r ← parseTree (n - 1)
      return Node x l r
    else fail

```

Figure 1. A generator and a parser for Boolean binary trees.

2.2 Free Generators

We can unify random generation with parsing by abstracting both ideas into a single data structure. For this, we introduce *free generators*.¹ Free generators are not traditional programs like `genTree` or `parseTree`; taking inspiration from *Free Applicative Functors* [2] they are more like abstract syntax trees. Free generators are data structures that can be *interpreted* as we see fit.

In §3 we give a domain-specific language for constructing free generators that is almost identical to the normal way users would construct generators with `QUICKCHECK`. But we give an example here that is explicitly built from data constructors order to make it clear that these are data structures, not normal programs.

This makes everything a bit harder to interpret, but the details are not critical; instead, look at Figure 2 and notice the structural similarities between that free generator and the examples in Figure 1.

¹This document uses the knowledge package in \LaTeX to make definitions interactive. Readers viewing the PDF electronically can click on technical terms and symbols to see where they are defined in the document.

```

fgenTree n =
  if n = 0 then
    Pure Leaf
  else
    Select
      [ (1, Pure Leaf),
        (n, MapR
          (Pair (Select
            [ (t, Pure True),
              (f, Pure False) ])
            (Pair (fgenTree (n - 1))
                  (fgenTree (n - 1))))
          )
        (λ (x, (l, r)) → Node x l r) ]

```

Figure 2. A free generator for Boolean binary trees.

We can use `fgenTree` to construct a free generator for Boolean binary trees of a given height. When reading the data structure, think of `Pure` in almost the same way as **return** from the previous examples, returning a pure value without making any choices or parsing any characters. `MapR` takes two arguments, a free generator and a function that can be applied to the result that is generated/parsed. (In §3 we define `Map` instead, which is the same, but the arguments are reversed.) The `Pair` constructor does a sort of sequencing: it generates/parses using its first argument, then it does the same with its second argument, and finally it pairs the results together. Finally, the real magic is in the way we interpret the `Select` structure. When we want a generator, we treat it as making a uniform random choice, and when we want a parser we treat it as consuming a character and checking `c` against the first elements of the pairs.

The almost line-to-line correspondence between `fgenTree` and `genTree` (or `parseTree`) is no accident! In §3 we give formal definitions of free generators, along with a number of interpretation functions. We use $\mathcal{G}[\![\cdot]\!]$ to mean the **generator interpretation** of a free generator and $\mathcal{P}[\![\cdot]\!]$ to mean the **parser interpretation** of a free generator. In other words,

$$\mathcal{G}[\![\text{fgenTree } 5]\!] \approx \text{genTree } 5 \quad \text{and} \quad \mathcal{P}[\![\text{fgenTree } 5]\!] \approx \text{parseTree } 5.$$

These functions mean that we can write one free generator that can be used in different ways depending on what we need at the moment.

We also define $\mathcal{C}[\![g]\!]$ to be the **choice distribution** of a free generator. Intuitively, the choice distribution should produce the sequences of choices that the generator interpretation can make, or equivalently the sequences that the parser interpretation can parse. To make this relationship formal, we prove Theorem 3.4, which says that for any free generator g ,

$$\mathcal{P}[\![g]\!] \langle \$ \rangle \mathcal{C}[\![g]\!] \approx \mathcal{G}[\![g]\!]$$

(where $\langle \$ \rangle$ is defined as a kind of “mapping” operation over distributions). Another way to read this theorem is that generators can be factored into two pieces: a distribution over choice sequences (given by $\mathcal{C}[\![\cdot]\!]$), and a parser of those sequences (given by $\mathcal{P}[\![\cdot]\!]$).

We can work with those pieces independently if we wish. For example, we can replace $\mathcal{C}[\![g]\!]$ with some other distribution over choice sequences to obtain a generator with a totally different output distribution but the same overall structure.

Besides swapping out a generator’s distribution, there are other reasons to prefer a free generator to a normal generator. In the next section, we see how free generators can be syntactically manipulated.

2.3 Derivatives of Free Generators

Every parser defines a formal language of the strings that it can successfully parse. Similarly, we define the **language interpretation** of a free generator $\mathcal{L}[\![g]\!]$ to be the set of choice sequences parsed (or made) by the generator. Viewing free generators this way suggests some interesting ways that free generators might be manipulated.

Formal languages have a notion of *derivative* that is attributed to Brzozowski [1]. For a formal language L , the Brzozowski derivative is defined as:

$$\delta_c L = \{s \mid cs \in L\}$$

In other words, the derivative of L with respect to c is all strings in L that start with c , with the first c removed.

Some models of languages, including regular expressions and context-free grammars, allow syntactic transformations that correspond to derivatives. While parser programs like `parseTree` cannot literally be modified in this way, we can still imagine what such a transformation would look like. Conceptually, the derivative of a parser with respect to a character c is whatever parser remains assuming c has just been parsed.

For example, if we fix `height = 5` for simplicity, we can think of the derivative of `parseTree 5` with respect to `n` as:

```

δn(parseTree 5) ≈
  c ← consume()
  if c == t then x ← True
  if c == f then x ← False
  else fail
  l ← parseTree 4
  r ← parseTree 4
  return Node x l r

```

This parser is “one step” simpler than `parseTree`. After parsing the character `n`, the next step is to parse either `t` or `f` and then construct a `Node`, so the derivative does just that.

We can imagine another step and look at the derivative of $\delta_n(\text{parseTree } 5)$ with respect to `t`:

```

331  $\delta_t \delta_n(\text{parseTree } 5) \approx$ 
332    $l \leftarrow \text{parseTree } 4$ 
333    $r \leftarrow \text{parseTree } 4$ 
334   return Node True  $l$   $r$ 
335

```

Now we have fixed the value True for x , and we can continue by making the recursive calls and constructing the final tree.

Our free generators also have a closely related notion of derivative! We can take derivatives of the free generator produced by `fgenTree` that look almost identical to the ones that we saw for `parseTree`:

```

343  $\delta_n(\text{fgenTree } 5) \approx$ 
344   MapR
345     (Pair (Select
346           [ (t, Pure True),
347             (f, Pure False) ]))
348     (Pair (fgenTree 4)
349           (fgenTree 4)))
350     ( $\lambda (x, (l, r)) \rightarrow \text{Node } x \mid l \mid r$ )
351

```

and

```

355  $\delta_t \delta_n(\text{fgenTree } 5) \approx$ 
356   MapR
357     (Pair (fgenTree 4)
358           (fgenTree 4))
359     ( $\lambda (l, r) \rightarrow \text{Node True } l \mid r$ )
360

```

These derivatives are also easy to compute. In §4 we define a simple procedure for computing the derivative of a free generator and show that the definition behaves the way we want. Formally, we prove Theorem 4.2 which says:

$$\delta_c \mathcal{L}[\![g]\!] = \mathcal{L}[\![\delta_c g]\!].$$

Intuitively, the derivative of a free generator can be thought of as the generator that remains after parsing *or making* a choice. This means that we can essentially preview the result of making a choice without actually interpreting the generator. In the next section, we show the practical applications of this technique.

2.4 Experimental Results

Since we can preview a single choice via a derivative, we can preview all possible choices by taking a *gradient*. If we repeatedly take gradients and make choices based on how good each choice is, we get an algorithm for generating valid data relative to a precondition φ .

We compare this approach to simple rejection sampling, which is the default way `QUICKCHECK` handles properties with preconditions, in two small benchmarks. The first benchmark (BST) uses a free generator similar the ones produced

by `fgenTree`, and a validity condition that ensures the tree is a valid binary search tree. The second benchmark (STLC) uses a generator for simply-typed lambda calculus terms, where valid terms are those that are well-typed. We ran both algorithms (REJECTION doing normal rejection sampling, and GRADIENT doing our gradient search) for two minutes, and tracked the number of unique valid values that each produced.

	REJECTION	GRADIENT
BST	12, 344	26, 527
STLC	147, 712	427, 005

Table 1. Unique valid values generated in 120 seconds.

Our results in Table 1 are clearly promising! Our algorithm consistently beat rejection sampling by more than a factor of two. Looking at plots of these values over time (Figure 4) is also instructive: for STLC especially, neither plot seems to be leveling off, and the GRADIENT line is increasing quite quickly.

Finally, we compared the relative sizes of the generated results. For BST, the GRADIENT algorithm increased the average term size by around 2, and for STLC the average size almost doubled from 5 to 10. The results are in Figure 5.

The algorithm is shown in more detail in Figure 3. Starting from g we take the *gradient* of g by taking the derivative with respect to each possible character, in this case a , b , and c . Then we evaluate each of the derivatives by (1) interpreting the generator with $\mathcal{G}[\![\cdot]\!]$, (2) sampling values from the resulting

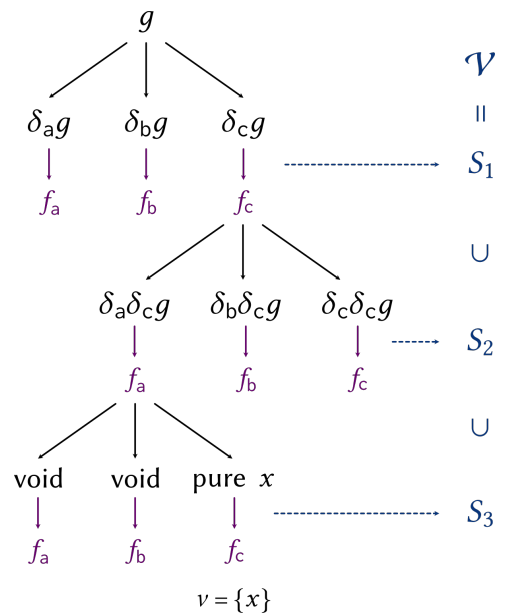


Figure 3. Generating valid values, \mathcal{V} , with free generator derivatives.

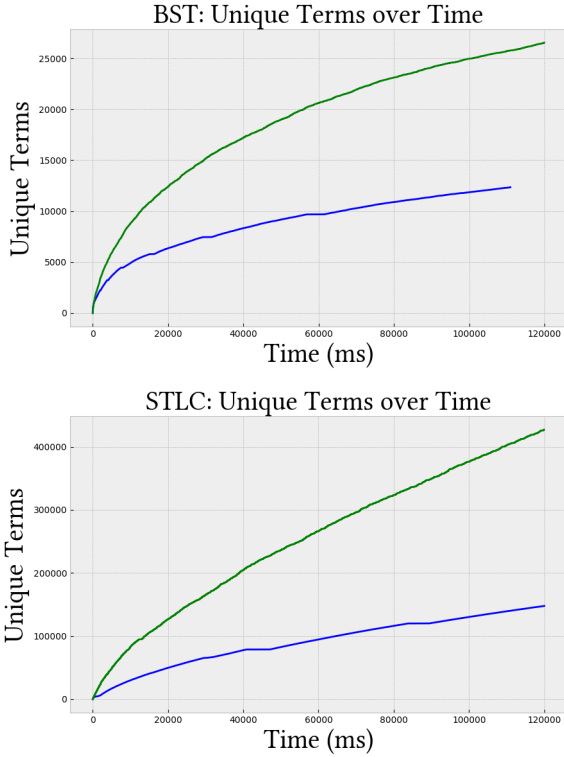


Figure 4. Unique valid values generated over 120 seconds. (REJECTION in blue, GRADIENT in green.)

generator, and (3) counting how many of those results are valid with respect to φ . The result of (3) is values f_a , f_b , and f_c , which we can think of as relative weights for how likely each choice is to lead us to a valid result. We then pick a choice randomly, weighted based on the values for f , and continue until our choices produce an output. There are more algorithmic details that we elide until §5.

Overall we are encouraged by these early results. They represent a strong proof of concept that indicates that free generators and their derivatives are practically useful. For more details on our gradient algorithm and our experimental design, see §5.

3 Free Generators in Detail

In this section, we develop the theory of *free generators*. We start with some background information on applicative abstractions for parsing and random generation and then present our new framing of generators that makes choices explicit and enables interesting syntactic manipulations.

3.1 Background: Applicative Parsers and Generators

In §2 we represent generators and parsers with pseudo-code, but moving forward we should to make our abstractions more concrete.

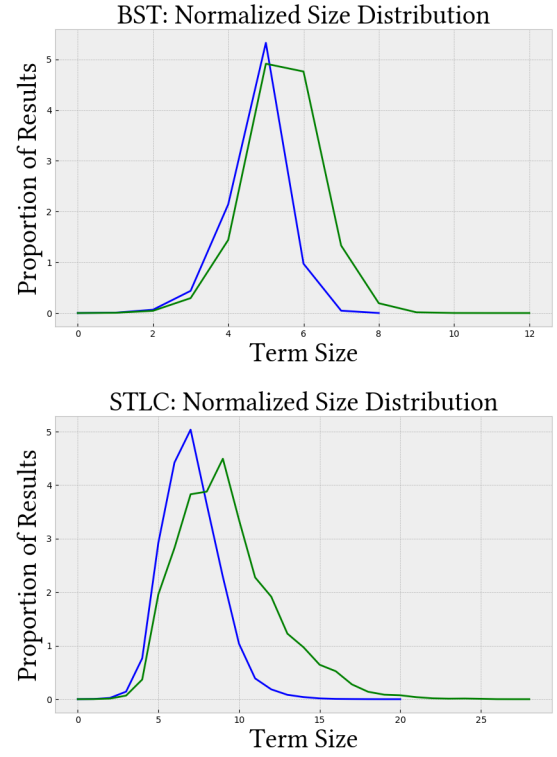


Figure 5. Normalized size distributions of valid values generated in 120 seconds. (REJECTION in blue, GRADIENT in green.)

We represent both generators and parsers using *applicative functors* [12]. At a high level, an applicative functor as a type constructor f with an operation,

$$(\langle \$ \rangle) :: (a \rightarrow b) \rightarrow f a \rightarrow f b$$

also known as “fmap” and operations,

$$\text{pure} :: a \rightarrow f a$$

$$(\langle * \rangle) :: f (a \rightarrow b) \rightarrow f a \rightarrow f b$$

the latter of which is often called “TIE”. At first they may seem fairly esoteric, but these operations really just define a convenient way to apply functions over some data structure. For example, the idiom

$$f \langle \$ \rangle x \langle * \rangle y \langle * \rangle z$$

applies a pure function f “over” three structures x , y , and z .

We can use these operations to define `genTree` like we would in QUICKCHECK [4]:

```
genTree 0 = pure Leaf
genTree n =
  oneof [ pure Leaf,
          Node ($) genInt
            (*) genTree (n - 1)
            (*) genTree (n - 1) ]
```

In this context, `pure` is the trivial generator that always generates the same value, and `Node <$> g1 <*> g2 <*> g3` means apply the constructor `Node` “over” three sub-generators to produce a new generator. (Operationally, this means sampling `x1` from `g1`, `x2` from `g2`, and `x3` from `g3`, and then constructing `Node x1 x2 x3`.) Notice that we need one extra function beyond the applicative interface: `oneof` makes a uniform choice between generators, just as we saw in the pseudo-code.

We can do the same thing for `parseTree`, using combinators inspired by *Parsec* [11]:

```
parseTree 0 = pure Leaf
parseTree n =
  choice [ (1, pure Leaf),
           (n, Node <$> parseInt
                        <*> parseTree (n - 1)
                        <*> parseTree (n - 1)) ]
```

Parsers of this form represent functions of type

`String → Maybe (a, String)`,

where `a` is the result type of the parser. The `String` in the input and output is the state of the parser, which changes as characters are consumed. The `Maybe` is there to handle possible failures. In this context, `pure` consumes no characters and never fails, it just produces the value passed to it. We can interpret `Node <$> p1 <*> p2 <*> p3` as parsing each sub-parser in sequence (failing if any of them fail) and then wrapping the results in the `Node` constructor. Finally, we have replaced `oneof` with `choice`, but the idea behavior is the same: choose between sub-parsers.² The only difference is that now the choice is based on an input string instead of a pseudo-random number generator, and we might fail if the consumed character is not in the list.

3.2 Representing Generators

With the applicative interface in mind, we can now give the formal definition of a *free generator*. We represent free generators as an inductive data type:

3.2.1 Type Definition. The type `FGen` is defined as follows:

```
data FGen a where
  Pure  :: a → FGen a
  Pair  :: FGen a → FGen b → FGen (a, b)
  Map   :: (a → b) → FGen a → FGen b
  Select :: [(Char, FGen a)] → FGen a
```

These constructors form a sort of abstract syntax tree with constructors that each correspond to some function from the interfaces in the previous section. Clearly `Pure` represents `pure`. `Pair` is a slightly different form of `<*>`—one is definable from the other, but this version makes more sense as a data constructor. `Map` corresponds to `<$>`. Finally, `Select`

²For clarity, we have slightly modified the interface for applicative parsers; *Parsec* has different way of expressing choice.

subsumes both `oneof` and `choice`—it might mean either, depending on the interpretation.

Recall that free generators are inspired by free applicative functors. In essence, all this means is that, since this generator is represented syntactically, we can write a transformation from `FGen a → f a` for any `f` with similar structure. This fact motivates the rest of this section.

3.2.2 Language of a Free Generator. We say that the *language of a free generator* is the set of choice sequences that it might make or parse. We define a generator’s language recursively, by cases:

$$\begin{aligned} \mathcal{L}[\text{Void}] &= \emptyset \\ \mathcal{L}[\text{Pure } a] &= \epsilon \\ \mathcal{L}[\text{Map } f \ x] &= \mathcal{L}[x] \\ \mathcal{L}[\text{Pair } x \ y] &= \mathcal{L}[x] \cdot \mathcal{L}[y] \\ \mathcal{L}[\text{Select } xs] &= \{cs \mid (c, x) \in xs \wedge s \in \mathcal{L}[x]\} \end{aligned}$$

(Recall that $S \cdot T = \{st \mid s \in S \wedge t \in T\}$, the standard definition of language concatenation.)

3.2.3 Smart Constructors and Simplified Forms. Free generators admit a useful *simplified form*. We ensure this property by requiring that free generators are built with *smart constructors*. Instead of `Pair`, users should use `⊗`:

$$\begin{aligned} (\otimes) :: \text{FGen } a \rightarrow \text{FGen } b \rightarrow \text{FGen } (a, b) \\ \text{Void} \otimes _ &= \text{Void} \\ _ \otimes \text{Void} &= \text{Void} \\ \text{Pure } a \otimes y &= (\lambda b \rightarrow (a, b)) \ \langle \$ \rangle \ y \\ x \otimes \text{Pure } b &= (\lambda a \rightarrow (a, b)) \ \langle \$ \rangle \ x \\ x \otimes y &= \text{Pair } x \ y \end{aligned}$$

which makes sure that `Void` and `Pure` are simplified as much as possible with respect to `Pair`. Next, `<$>` is a version of `Map` that does similar simplifications:

$$\begin{aligned} (\langle \$ \rangle) :: (a \rightarrow b) \rightarrow \text{FGen } a \rightarrow \text{FGen } b \\ f \ \langle \$ \rangle \ \text{Void} &= \text{Void} \\ f \ \langle \$ \rangle \ \text{Pure } a &= \text{Pure } (f \ a) \\ f \ \langle \$ \rangle \ x &= \text{Map } f \ x \end{aligned}$$

We define `pure` and `<*>` to make `FGen` an applicative functor:

```
pure :: a → FGen a
pure = Pure
(<*>) :: FGen (a → b) → FGen a → FGen b
f <*> x = (λ (f, x) → f x) <$> (f ⊗ x)
```

Finally, we define a smart constructor for `Select`:

```
select :: [(Char, FGen a)] → FGen a
select xs =
  case filter (λ (_, p) → p ≠ Void) xs of
    [] → ⊥
    xs | duplicates (map (λ (c, _) → c) xs) → ⊥
    xs → Select xs
```

Unlike the other smart constructors, `select` can fail. This helps us ensure that generators make sense later on—for example, our interpretation of choices means that it is not clear what it would mean the same choice label to appear twice in a given call to `Select`. When generators are built with only smart constructors, we say they are in simplified form.

3.2.4 Examples. Before we continue, here are a couple of examples of free generators that illustrate how they should be used; these examples are used later in our evaluation. First, we can generalize our definitions from earlier in this section to get a single free generator `fgenTree` that subsumes `genTree` and `parseTree`.

```
fgenTree :: Depth → FGen Tree
fgenTree 0 = pure Leaf
fgenTree n =
  select [ (1, pure Leaf),
           (n, Node ⟨$⟩ fgenInt
                  ⟨*⟩ fgenTree (n - 1)
                  ⟨*⟩ fgenTree (n - 1)) ]
```

This free generator looks almost exactly like `genTree`, but remember that whereas `genTree` is a program that makes choices as it runs, `fgenTree` is just a data structure.

Another example of a generator that we use later produces random terms of a simply-typed lambda-calculus. We discuss the details of the language in Appendix A. For now, we can just look at the generator:

```
data Type = TInt | TFun Type Type
data Expr
  = Lit Int | Plus Expr Expr
  | Lam Type Expr | Var Int | App Expr Expr

fgenExpr :: Depth → FGen Expr
fgenExpr 0 =
  select [ (i, Lit ⟨$⟩ fgenInt),
           (v, Var ⟨$⟩ fgenVar) ]
fgenExpr n =
  select [ (i, Lit ⟨$⟩ fgenInt),
           (p, Plus ⟨$⟩ fgenExpr (n - 1)
                  ⟨*⟩ fgenExpr (n - 1)),
           (l, Lam ⟨$⟩ fgenType
                  ⟨*⟩ fgenExpr (n - 1)),
           (a, App ⟨$⟩ fgenExpr (n - 1)
                  ⟨*⟩ fgenExpr (n - 1)),
           (v, Var ⟨$⟩ fgenVar) ]
```

Structurally this is very similar to the previous generator, it just has more cases and more choices.

3.3 Interpreting Free Generators

A **free generator** does not *do* anything on its own—it is simply a data structure. In this section, we see the formal definitions of the interpretation functions that we discussed in §2 and prove a theorem that links those interpretations together.

3.3.1 As a Generator of Values. The most natural way to interpret a free generator is as a `QUICKCHECK` generator—that is, as a distribution over some data structure. We define the *generator interpretation* of a generator to be:

$$\begin{aligned} \mathcal{G}[\text{Void}] &= \perp \\ \mathcal{G}[\text{Pure } v] &= \text{pure } v \\ \mathcal{G}[\text{Map } f \ x] &= f \ \langle \$ \rangle \ \mathcal{G}[x] \\ \mathcal{G}[\text{Pair } x \ y] &= (\lambda x \ y \rightarrow (x, y)) \ \langle \$ \rangle \ \mathcal{G}[x] \ \langle * \rangle \ \mathcal{G}[y] \\ \mathcal{G}[\text{Select } xs] &= \text{oneof} \ (\text{map} \ (\lambda _, x) \rightarrow \mathcal{G}[x]) \ xs \end{aligned}$$

One detail worth noting is that the interpretation behaves poorly on `Void`, but we can prove a lemma to show that this does not cause problems in practice:

Lemma 3.1. *If a generator g is **simplified**,*

$$g = \text{Void} \iff g \text{ contains Void.}$$

Proof. By induction on the structure of g and inspection of the smart constructors. \square

Thus we can conclude that as long as g is in simplified form and not `Void`, $\mathcal{G}[g]$ is defined.

Example 3.2. As desired, $\mathcal{G}[\text{fgenTree } 5]$ is equivalent to `genTree 5`.

3.3.2 As Parser of Choice Sequences. Of course, there would be no point in defining free generators if we were only going to interpret them as `QUICKCHECK` generators. We can make use of the choice labels using the free generator’s *parser interpretation*—in other words, viewing it as a parser of choices as we originally wanted. The translation looks like:

$$\begin{aligned} \mathcal{P}[\text{Void}] &= \lambda s \rightarrow \text{Nothing} \\ \mathcal{P}[\text{Pure } a] &= \text{pure } a \\ \mathcal{P}[\text{Map } f \ x] &= f \ \langle \$ \rangle \ \mathcal{P}[x] \\ \mathcal{P}[\text{Pair } x \ y] &= (\lambda x \ y \rightarrow (x, y)) \ \langle \$ \rangle \ \mathcal{P}[x] \ \langle * \rangle \ \mathcal{P}[y] \\ \mathcal{P}[\text{Select } xs] &= \text{choice} \ (\text{map} \ (\lambda (c, x) \rightarrow (c, \mathcal{P}[x])) \ xs) \end{aligned}$$

This definition uses the representation of parsers as functions of type `String → Maybe (a, String)` that we saw earlier. Error handling happens under the hood for operations like `Pair`.

Example 3.3. As desired, $\mathcal{P}[\text{fgenTree } 5]$ is equivalent to `parseTree 5`.

3.3.3 As a Generator of Choice Sequences. Our final interpretation of free generators is a sort of dual to the previous one. Instead of throwing away randomness in favor of deterministic parsing, we can extract only the random distribution and ignore everything about how result values are constructed. We define the *choice distribution* of a generator to be:

$$\begin{aligned} C[\text{Void}] &= \perp \\ C[\text{Pure } a] &= \text{Gen.pure } \varepsilon \\ C[\text{Map } f \ x] &= C[x] \\ C[\text{Pair } x \ y] &= \\ &(\lambda \ s \ t \rightarrow st) \text{ Gen.}\langle \$ \rangle C[x] \text{ Gen.}\langle * \rangle C[y] \\ C[\text{Select } xs] &= \\ &\text{Gen.oneof} \\ &(\text{map} \\ &(\lambda \ (c, x) \rightarrow (\lambda \ s \rightarrow cs) \text{ Gen.}\langle \$ \rangle C[x]) \ xs) \end{aligned}$$

Note that we can equivalently think of this as a distribution over $\mathcal{L}[g]$.

3.3.4 Coherence. We would like for our three interpretations of free generators to make sense together. We call this property coherence, and prove it in this section.

We say two generators are *equivalent*, written $g_1 \equiv g_2$, if and only if the generators represent the same distribution over values. Any equal generators are trivially equivalent, but for technical reasons properties concerning $\langle \$ \rangle$, $\langle * \rangle$, etc. are generally only valid up to this notion of equivalence. With this in mind, we can state our coherence theorem and sketch the proof:

Theorem 3.4 (Factoring Coherence). *Every generator can be factored coherently into a parser and a distribution over choice sequences. In other words, for all simplified generators $g \neq \text{Void}$,*

$$\mathcal{P}[g] \langle \$ \rangle C[g] \equiv (\lambda x \rightarrow \text{Just } (x, \varepsilon)) \langle \$ \rangle \mathcal{G}[g].$$

Proof sketch. We proceed by induction on the structure of g .

Case $g = \text{Pure } a$. Straightforward.

Case $g = \text{Map } f \ x$. Straightforward.

Case $g = \text{Pair } x \ y$. This case is the most interesting one. The difficulty is that it is not immediately obvious why

$$\mathcal{P}[\text{Pair } x \ y] \langle \$ \rangle C[\text{Pair } x \ y]$$

should be a function of

$$\mathcal{P}[x] \langle \$ \rangle C[x] \quad \text{and} \quad \mathcal{P}[y] \langle \$ \rangle C[y].$$

Showing the correct relationship between these requires a lemma that says that for any sequence s of choices generated by $C[x]$ and an arbitrary sequence t , $\mathcal{P}[x] \ st$ returns $\text{Just } (a, t)$ for some a .

Case $g = \text{Select } xs$. The reasoning in this case is a bit subtle, since it requires certain operations to commute with Select , but the details are not particularly instructive.

See Appendix B for the full proof. \square

3.4 Replacing a Generator's Distribution

Since a generator g can be factored using $C[\cdot]$ and $\mathcal{P}[\cdot]$, we can explore what it would look like to modify a generator's distribution (i.e., change or replace $C[\cdot]$) without having to modify the entire generator.

Suppose we have some other distribution that we want our choices to follow, represented as a function from a history to a generator of next characters:

type DistF = String \rightarrow Gen (Maybe Char)

We interpret these functions as specifying the distribution over the a next choice, given a history of choices. (If the next choice is `Nothing`, then generation stops.) We can be a bit more general and pair a distribution function with a “current” history, to get our formal definition of a custom choice distribution:

type Dist = (String, DistF)

A Dist may be arbitrarily complex—in particular, it might contain information obtained from a machine learning model, example-based tuning, or some other automated tuning process. How would we use such a distribution in place of the standard distribution given by $C[g]$?

The solution is actually quite elegant: we just replace $C[g]$ with a distribution to yield the *definition*:

$$\begin{aligned} \overline{\mathcal{G}}[(h, d), g] &= \mathcal{P}[g] \langle \$ \rangle \text{genDist } h \\ \text{where} \\ \text{genDist } h &= d \ h \gg \lambda x \rightarrow \text{case } x \ \text{of} \\ &\quad \text{Nothing} \rightarrow \text{Gen.pure } h \\ &\quad \text{Just } c \rightarrow \text{genDist } hc \end{aligned}$$

Whereas before we proved an equivalence between $\mathcal{G}[g]$ and $\mathcal{P}[g] \langle \$ \rangle C[g]$, we can now use that relationship as a definition of what it means to interpret a generator under a new distribution.

Since replacing a free generator's distribution does not actually change the structure of the generator, we can theoretically have a different distribution for each use-case of the free generator. In a property-based testing scenario, this might mean a finely-tuned distribution for each property, carefully optimized to find bugs as quickly as possible.

4 Brzowski Derivatives

So far we have shown how to construct a free generator that factors into a distribution over choice sequences and a parser. This is useful on its own, but it would be a shame to stop there. In particular, free generators, like many other representations of languages, can be manipulated in interesting and useful ways. In this section, we recall the idea of a *Brzowski derivative* for languages and regular expressions, and we show that a similar notion can be applied to free generators.

4.1 Background: Derivatives of Languages

The *Brzowski derivative* [1] of a formal language L with respect to some character c is defined as

$$\delta_c L = \{s \mid cs \in L\}.$$

In other words, it is the set of strings in L with c removed from the front. For example,

$$\delta_a\{abc, aaa, bba\} = \{bc, aa\}.$$

Certain language representations have syntactic transformations that correspond to Brzowski derivatives. For example, we can take the derivative of a regular expression in the following way:

$$\begin{aligned} \delta_c \emptyset &= \emptyset \\ \delta_c \{\varepsilon\} &= \emptyset \\ \delta_c \{c\} &= \{\varepsilon\} \\ \delta_c \{a\} &= \emptyset & (a \neq c) \\ \delta_c(r_1 + r_2) &= \delta_c r_1 + \delta_c r_2 \\ \delta_c(r_1 \cdot r_2) &= \delta_c r_1 \cdot r_2 + vr_1 \cdot \delta_c r_2 \\ \delta_c(r^*) &= \delta_c r \cdot r^* \end{aligned}$$

where *nullability* (the v operator in the “ \cdot ” rule) is defined as:

$$\begin{aligned} v\emptyset &= \emptyset \\ v\varepsilon &= \varepsilon \\ vc &= \emptyset \\ v(r_1 + r_2) &= vr_1 + vr_2 \\ v(r_1 \cdot r_2) &= vr_1 \cdot vr_2 \\ v(r^*) &= \{\varepsilon\} \end{aligned}$$

As one would hope, if r has language L , it is always the case that $\delta_c r$ has language $\delta_c L$.

It is worth noting that the derivative operation is called a “derivative” because of the “ \cdot ” rule, which is closely related to the product rule for derivatives in other contexts. The only difference is the nullability check, which enforces that that a character can only be removed from r_2 if it is possible for r_1 to accept ε .

4.2 Derivatives of Free Generators

Since *free generators* define a language (given by $\mathcal{L}[\![\cdot]\!]$), it makes sense to ask: can we take their derivatives? Yes!

4.2.1 Definitions. We define the *derivative of a free generator* in the following way:

$$\begin{aligned} \delta_c \text{Void} &= \text{Void} \\ \delta_c (\text{Pure } v) &= \text{Void} \\ \delta_c (\text{Map } f \ x) &= f \ \langle \$ \rangle \ \delta_c x \\ \delta_c (\text{Pair } x \ y) &= \delta_c x \otimes y \\ \delta_c (\text{Select } xs) &= \text{if } (c, x) \in xs \text{ then } x \text{ else Void} \end{aligned}$$

These definitions should be mostly intuitive. The derivative of a generator that does not make a choice (i.e., *Void* and *Pure*) is *Void*, since the corresponding language would be empty. *Map* does not affect the derivative, since it is concerned with choices, not the final result. The derivative of *Select* is just the argument generator corresponding to the appropriate choice.

The one potentially confusing case is the one for *Pair*. We have defined the derivative of a pair of generators by taking a derivative to the first generator in the pair, but what about when the first generator’s language is nullable? One might imagine that the rule for *Pair* should look more like the one for “ \cdot ” in the definition for the derivative of a regular expression. Luckily, our *simplified form* clears up the confusion. We can prove a lemma:

Lemma 4.1. *If g is in *simplified form*, then either $g = \text{Pure } a$ or $v\mathcal{L}[\![g]\!] = \emptyset$.*

Proof. We proceed by induction on the structure of g .

Case $g = \text{Void}$. Trivial.

Case $g = \text{Pure } a$. Trivial.

Case $g = \text{Pair } x \ y$. By our inductive hypothesis, $x = \text{Pure } a$ or $v\mathcal{L}[\![x]\!] = \emptyset$.

Since the smart constructor \otimes never constructs a *Pair* with *Pure* on the left, it must be that $v\mathcal{L}[\![x]\!] = \emptyset$.

Therefore, it must be the case that $v\mathcal{L}[\![\text{Pair } x \ y]\!] = \emptyset$.

Case $g = \text{Map } f \ x$.

Similarly to the previous case, our inductive hypothesis and simplification assumptions imply that $v\mathcal{L}[\![x]\!] = \emptyset$.

Therefore, $v\mathcal{L}[\![\text{Map } f \ y]\!] = \emptyset$.

Case $g = \text{Select } xs$.

It is always the case that $v\mathcal{L}[\![\text{Select } xs]\!] = \emptyset$.

Thus, we have shown that every simplified free generator is either *Pure* a or has an empty nullable set. \square

A simple corollary of this lemma is that if $\text{Pair } x \ y$ is in simplified form, $\mathcal{L}[\![x]\!]$ is not nullable. In addition, the lemma means we can define *nullability* for free generators simply as:

$$\begin{aligned} v(\text{Pure } v) &= \{v\} \\ vg &= \emptyset \end{aligned}$$

4.2.2 Consistency. Before we prove our main theorem, one quick aside.

Remark. The derivative of a simplified generator is simplified. This follows simply from the definition, since we only use smart constructors to build the derivative generators. By induction, this also means that repeated derivatives preserve simplification.

This is mostly a technical detail, but it means that anything we know about simplified generators also applies to derivatives of simplified generators.

With that in mind, we can prove a concrete theorem that says our definition of derivative acts the way we expect:

Theorem 4.2 (Generator Derivative Consistency). *For all simplified free generators g and characters c ,*

$$\delta_c \mathcal{L}[\![g]\!] = \mathcal{L}[\![\delta_c g]\!].$$

Proof. We again proceed by induction on the structure of g .

Case $g = \text{Void}$. $\emptyset = \emptyset$.

Case $g = \text{Pure } a$. $\emptyset = \emptyset$.

Case $g = \text{Pair } x \ y$.

$$\begin{aligned} \delta_c \mathcal{L}[\![\text{Pair } x \ y]\!] &= \delta_c(\mathcal{L}[\![x]\!] \cdot \mathcal{L}[\![y]\!]) && \text{(by defn)} \\ &= \delta_c(\mathcal{L}[\![x]\!] \cdot \mathcal{L}[\![y]\!]) && \text{(by defn)} \\ &\quad + \nu \mathcal{L}[\![x]\!] \cdot \delta_c \mathcal{L}[\![y]\!] \\ &= \delta_c(\mathcal{L}[\![x]\!] \cdot \mathcal{L}[\![y]\!]) && \text{(by Lemma 4.1)} \\ &= \mathcal{L}[\![\delta_c x]\!] \cdot \mathcal{L}[\![y]\!] && \text{(by IH)} \\ &= \mathcal{L}[\![\text{Pair } (\delta_c x) \ y]\!] && \text{(by defn)} \\ &= \mathcal{L}[\![\delta_c \text{Pair } x \ y]\!] && \text{(by defn)} \end{aligned}$$

Case $g = \text{Map } f \ x$.

$$\begin{aligned} \delta_c \mathcal{L}[\![\text{Map } f \ x]\!] &= \delta_c \mathcal{L}[\![x]\!] && \text{(by defn)} \\ &= \mathcal{L}[\![\delta_c x]\!] && \text{(by IH)} \\ &= \mathcal{L}[\![\text{Map } f \ (\delta_c x)]\!] && \text{(by defn)} \\ &= \mathcal{L}[\![\delta_c(\text{Map } f \ x)]\!] && \text{(by defn*)} \end{aligned}$$

*Note that the last step follows because $\text{Map } f \ x$ is assumed to be simplified, so $x \neq \text{Pure } a$. This means that $f \ \langle \$ \rangle \ x$ is equivalent to $\text{Map } f \ x$.

Case $g = \text{Select } xs$. If there is no pair $(c, \ x)$ in xs , then $\emptyset = \emptyset$. Otherwise,

$$\begin{aligned} \delta_c \mathcal{L}[\![\text{Select } xs]\!] &= \delta_c \{cs \mid s \in \mathcal{L}[\![x]\!]\} && \text{(by defn)} \\ &= \mathcal{L}[\![x]\!] && \text{(by defn)} \\ &= \mathcal{L}[\![\delta_c(\text{Select } xs)]\!] && \text{(by defn)} \end{aligned}$$

Thus we have shown that the symbolic derivative of free generators is compatible with the derivative of the generator's language. \square

5 Generating Valid Results with Gradients

So far we have shown that factoring generators through the sequences of choices that they make leads to the natural abstraction of **free generators**. We have also established that free generators admit **derivatives**. This gives a symbolic way to compute a view of a generator after making a particular choice. We now put these theoretical ideas into practice—we present an algorithm that helps to solve the *valid generation problem* by guiding a generator to outputs that satisfy a computable precondition.

The basic intuition behind our algorithm is that a generator's derivative with respect to a choice c contains information about how “good” or “bad” the choice c would be—in other

words, how likely that choice is to lead the generator to a valid result. In order to access this information, we can sample from $\mathcal{G}[\![\delta_c g]\!]$ a number of times and count up how many of the samples are valid. The more valid samples we get, the more likely c will lead us to a valid result at the end of generation.

5.1 The Algorithm

With this intuition in mind, we present an algorithm that searches for valid results using repeated free generator *gradients* (i.e., derivatives with respect to every available choice). Given a free generator G in **simplified form** and a validity predicate φ , the following algorithm produces a set of outputs \mathcal{O} such that $\forall x \in \mathcal{O}. \varphi(x)$.

Let $N \in \mathbb{N}$ be the *sample rate constant*. The algorithm proceeds as follows:

```

1:  $g \leftarrow G$ 
2:  $\mathcal{V} \leftarrow \emptyset$ 
3: while true do
4:   if  $\nu g \neq \emptyset$  then return  $\nu g \cup \mathcal{V}$ 
5:   if  $g = \text{Void}$  then  $g \leftarrow G$ 
6:    $\nabla g \leftarrow \{\delta_c g \mid c \in C\}$   $\triangleright \nabla g$  is the gradient of  $g$ 
7:   for  $\delta_c g \in \nabla g$  do
8:     if  $\delta_c g = \text{Void}$  then
9:        $V \leftarrow \emptyset$ 
10:    else
11:       $x_1, \dots, x_N \leftarrow \mathcal{G}[\![\delta_c g]\!]$   $\triangleright$  Sample  $\mathcal{G}[\![\delta_c g]\!]$ 
12:       $V \leftarrow \{x_j \mid \varphi(x_j)\}$ 
13:       $f_c \leftarrow |V|$   $\triangleright f_c$  is the fitness of  $c$ 
14:       $\mathcal{V} \leftarrow \mathcal{V} \cup V$ 
15:   if  $\max_{c \in C} f_c = 0$  then
16:     for  $c \in C$  do  $f_c \leftarrow 1$ 
17:    $g \leftarrow \text{weightedChoice } \{(f_c, \delta_c g) \mid c \in C\}$ 

```

Figure 6. An algorithm for generating valid random values, using free generator gradients.

The intuition from earlier plays out in lines 7–14. We take the derivative of our current generator with respect to each choice, sample values from each derivative, and then choose one of the derivatives with weights proportional to the number of valid samples from each. Since each choice is made based on the “fitness” of the associated derivative, we are likely to keep making choices that lead us to valid results.

One critical thing that makes this process work is the set \mathcal{V} which keeps track of all of the valid samples that we generate while evaluating derivatives. Storing the valid samples that we find along the way ensures that no effort is wasted.

5.2 Experiments

In §2 we showed the results of our experiments using the gradient search algorithm. This section goes into more detail on the experimental setup.

Overall our goal was not to prove that this algorithm should be implemented as-is in any concrete tools, though we do believe a similar algorithm could be used in practice. Instead, our goal was to see how our implementation does in comparison to the baseline, and gain confidence that free generators and their derivatives are useful.

We ran one main experiment that uses two different benchmarks. The benchmarks are based on the example generators given in §3.2.4: the first benchmark, “BST,” uses `fgenTree` to generate valid binary search trees; and the second, “STLC,” uses `fgenExpr` to generate well-typed terms in a simply-typed lambda calculus. We assume readers are familiar with binary search trees and their invariants, but we provide a brief explanation of our STLC example in Appendix A.

In our experiment, we compared our gradient algorithm to simple rejection sampling. Concretely, we took our free generator g , interpreted it as $\mathcal{G}[\![g]\!]$, sampled from that generator, and ignored any results that were invalid. Admittedly, comparing against rejection sampling is not a terribly high bar, but our goals with these experiments were modest: we aimed to understand the pros and possible cons of our approach in a controlled environment.

We ran each of the two algorithms, REJECTION (rejection sampling) and GRADIENT (our algorithm), for two minutes on a MacBook Pro with an M1 processor. We felt two minutes was appropriate for a small-scale experiment like this, as it is within the amount of time that a tester might reasonably spend on a particular property. For BST we used $N = 50$, and for STLC we used $N = 400$ —in general, higher sample rates yield better performance more complex data structures, although this must be balanced by the added cost of sampling. As the algorithms ran, we kept track of every valid value that each generated and counted the number of *unique* values. We cared about unique values specifically, since there is no value in testing the same input more than once in a property-based testing scenario.

We do not repeat our charts here (Figures 4 and 5), but we can briefly review the results from Table 1:

	REJECTION	GRADIENT
BST	12, 344	26, 527
STLC	147, 712	427, 005

It is clear that the GRADIENT algorithm produces significantly more unique, valid values than REJECTION in the same amount of time, which is very promising. We expected to see this effect: the gradient algorithm is able to rule out choices that are unlikely to result in valid results and make choices that

tend to work out well. The high incidence of *unique* values is a natural side-effect.

5.3 Modified Distributions

In §3.4, we show that replacing a generator’s distribution is as simple as pairing it with a distribution over choices (represented by the type `Dist`). It turns out that there is a straightforward way to adapt the above algorithm to work for distribution-modified generators!

The algorithm requires that our generator structure admit three basic operations: δ_c so we can compute the gradients, $\mathcal{G}[\![\cdot]\!]$ so we can sample from the gradients, and ν so we know when to stop. We already showed how to define $\mathcal{G}[\![\cdot]\!]$ for generators with modified distributions, so we just need definitions of δ_c and ν .

As a starting point, we can observe that `Dist` admits a simple kind of derivative:

$$\delta_c(h, d) = (hc, d).$$

Intuitively, the derivative just internalizes `c` into the history, so future queries of the distribution take that character into account. Given that, we can further define the derivative of a pair (d, g) of a `Dist` and a free generator to be:

$$\delta_c(d, g) = (\delta_c d, \delta_c g)$$

Finally, to complete the construction, we can say that a modified generator’s nullable set is the same as the nullable set of the underlying generator:

$$\nu(d, g) = \nu g$$

With these definitions in hand, we can adapt our algorithm with essentially no changes!

6 Related Work

The valid generation problem need not be solved automatically. Libraries like QUICKCHECK [8] provide domain specific languages that make it easier to write manual generators that produce valid inputs by construction.

Languages like LUCK [9] provide a sort of middle-ground solution; users are still required to put in some effort, but they are able to define generators and validity predicates at the same time. We ultimately still want a more automated solution.

When validity predicates are expressed as inductive relations, approaches like the one in GENERATING GOOD GENERATORS FOR INDUCTIVE RELATIONS [10] are extremely powerful. This is a sufficiently automatic solution to the valid generation problem—we simply choose to tackle situations with less structured validity predicates.

Some approaches have tried to use machine learning to generate valid inputs. LEARN@FUZZ [6] generates valid PDF documents using a recurrent neural network. While the results are promising, this solution seems to work best when a large corpus of inputs is already available and the validity

condition is more structural than semantic. In the same vein, `RLCHECK` [14] uses reinforcement learning to guide a generator to valid inputs. We hope to incorporate ideas from `RLCHECK` into this work in the future, but we felt that getting the theory of free generators right was a critical first step.

Claessen et al. present a structure that is superficially similar to our free generator structure, but with a different underlying approach. They primarily use the syntactic structure of their generators (they call them “Spaces”) to control the size distribution of the outputs; in particular, Spaces do not make choice information explicit. Claessen et al.’s generation approach uses `HASKELL`’s laziness, rather than derivatives and sampling, to prune unhelpful paths in the generation process. It seems plausible that we could incorporate some of these ideas into our work to improve performance and give finer control over size distributions.

Finally, while it does not explicitly aim to solve the valid generation problem, `CLOTHO` [5] is an interesting point in the generator design space.

7 Future Directions

There are a number of exciting paths forward from this work: some continue the theoretical exploration, while others look towards algorithmic improvements that would be incredibly useful in practice.

7.1 Distribution Optimization

We believe that we have only scratched the surface of what is possible with free generators. Making choices explicit opens the door for other transformations and interpretations that take advantage of the extra information. One concrete idea would be to expand merge the theory of free generators with the emerging theory of *ungenerators* [7]. That work expresses generators that can be run both forward (to generate values as usual) and *backward*. In the backward direction, the program takes a value that the generator might have generated and “un-generates” it to give a sequence of choices that the generator might have made when generating that value.

The free generator formalism is quite compatible with these ideas—since free generators make choices explicit, turning one into a bidirectional generator that can both generate and ungenerate should be fairly straightforward. From there, we can build on the ideas in the ungenerators work and use the backward direction of the generator to learn a distribution of choices that approximates some user-provided samples of “desirable” values. Used in conjunction with the extended algorithm from §5.3, this would give a better starting point for generation with little extra work from the user.

7.2 Making Choices with Neural Networks

As our gradient algorithm makes choices, it generates a lot of useful data about the frequencies with which choices should be made. Specifically, every iteration of the algorithm produces a pair of a history and a distribution over next choices that looks something like:

$$abcca \mapsto \{a : 0.3, b : 0.7, c : 0.0\}$$

In the course of the gradient algorithm, this information is used once (to make the next choice) and then forgotten—what if there was a way to learn from it?

In theory, it should be possible to use pairs like this to train a recurrent neural network (RNN) to make choices that are similar to the ones made by the gradient algorithm. If this went well, we could run the gradient algorithm for a while, then train the model, and after that point only use the model to generate valid data. Staging things this way could potentially save a lot of time that is currently wasted by the constant sampling of derivative generators.

Of course, there are a number of hurdles that one would need to overcome to make this work properly. For a start, the right network architecture and hyper-parameters would need to be chosen, which might need to be different from one context to the next. If this is the case, the RNN approach would be significantly less useful from the perspective of the valid generation problem. In addition, the training process will be lossy, so extra care will need to be taken to make sure the network is still able to produce mostly valid data.

Still, if we can overcome these challenges, this approach could be extremely useful. One could imagine a user writing a definition of a type and a predicate for that type, and then setting the model to train while they work on their algorithm. By the time the algorithm is finished and ready to test, the RNN model would be trained and ready to produce valid test inputs. A workflow like this could significantly increase adoption of property-based testing in industry and give developers more control over the quality of their software.

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Appendix

A STLC Details

$$\tau ::= \text{int} \mid \tau_1 \rightarrow \tau_2$$

$$e ::= n \mid e_1 + e_2 \mid \lambda \tau. e \mid e_1 e_2 \mid v$$

```

typeOf :: Expr → Maybe Type
typeOf expr = runReaderT (aux expr) []
  where
    aux (Int _) = return TInt
    aux (Plus e1 e2) = do
      TInt ← aux e1
      TInt ← aux e2
      return TInt
    aux (Lam t e) = do
      u ← local (t :) (aux e)
      return (Fun t u)
    aux (App e1 e2) = do
      (Fun t1 t2) ← aux e1
      u1 ← aux e2
      guard (t1 == u1)
      return t2
    aux (Var n) = do
      ctx ← ask
      if length ctx <= n then lift Nothing else return (ctx !! n)

hasType :: Expr → Bool
hasType = isJust . typeOf

```

B Proof of Theorem 3.4

Lemma B.1. *Pairing two parsers and mapping over the concatenation of the associated choice distributions is equal to a function of the two parsers mapped over the distributions individually. Specifically, for all simplified free generators x and y ,*

$$\begin{aligned}
 ((\lambda x y \rightarrow (x, y)) \langle \$ \rangle \mathcal{P} \llbracket x \rrbracket \langle * \rangle \mathcal{P} \llbracket y \rrbracket) \langle \$ \rangle ((\cdot) \langle \$ \rangle \mathcal{C} \llbracket x \rrbracket \langle * \rangle \mathcal{C} \llbracket y \rrbracket) &\equiv (\lambda a_{\perp} b_{\perp} \rightarrow \text{case } (a_{\perp}, b_{\perp}) \text{ of} \\
 &\quad (Just (a, _), Just (b, _)) \rightarrow Just ((a, b), \varepsilon) \\
 &\quad _ \rightarrow \text{Nothing}) \\
 \langle \$ \rangle (\mathcal{P} \llbracket x \rrbracket \langle \$ \rangle \mathcal{C} \llbracket x \rrbracket) \langle * \rangle (\mathcal{P} \llbracket y \rrbracket \langle \$ \rangle \mathcal{C} \llbracket y \rrbracket) &
 \end{aligned}$$

Proof. First, note that for any simplified generator, g , if $\mathcal{C} \llbracket g \rrbracket$ generates a string s , for any other string t $\mathcal{P} \llbracket g \rrbracket st = \text{Just } (a, t)$ for some value a . This can be shown by induction on the structure of g .

Now, assume $\mathcal{C} \llbracket x \rrbracket$ generates a string s , and $\mathcal{C} \llbracket y \rrbracket$ generates t . This means that $(\cdot) \langle \$ \rangle \mathcal{C} \llbracket x \rrbracket \langle * \rangle \mathcal{C} \llbracket y \rrbracket$ generates st .

By the above fact, it is simple to show that both sides of the above equation simplify to $\text{Just } ((a, b), \varepsilon)$ for some values a and b that depend on the particular interpretations of x and y .

Since this is true for any s and t that the choice distributions generate, the desired fact holds. \square

Theorem 3.4. *Every generator can be factored coherently into a parser and a distribution over choice sequences. In other words, for all simplified generators $g \neq \text{Void}$,*

$$\mathcal{P} \llbracket g \rrbracket \langle \$ \rangle \mathcal{C} \llbracket g \rrbracket \equiv (\lambda x \rightarrow \text{Just } (x, \varepsilon)) \langle \$ \rangle \mathcal{G} \llbracket g \rrbracket.$$

Proof. We proceed by induction on the structure of g .

Case $g = \text{Pure } a$.

$$\begin{aligned} \mathcal{P}[\![\text{Pure } a]\!] \langle \$ \rangle \mathcal{C}[\![\text{Pure } a]\!] &\equiv \text{pure } (\text{Just } (a, \varepsilon)) && \text{(by defn)} \\ &\equiv (\lambda x \rightarrow \text{Just } (x, \varepsilon)) \langle \$ \rangle \mathcal{G}[\![\text{Pure } a]\!] && \text{(by defn)} \end{aligned}$$

Case $g = \text{Pair } x \ y$.

$$\begin{aligned} \mathcal{P}[\![\text{Pair } x \ y]\!] \langle \$ \rangle \mathcal{C}[\![\text{Pair } x \ y]\!] &\equiv ((\lambda x \ y \rightarrow (x, y)) \langle \$ \rangle \mathcal{P}[\![x]\!] \langle * \rangle \mathcal{P}[\![y]\!] \langle \$ \rangle ((\cdot) \langle \$ \rangle \mathcal{C}[\![x]\!] \langle * \rangle \mathcal{C}[\![y]\!] \langle \$ \rangle)) && \text{(by defn)} \\ &\equiv (\lambda a_{\perp} \ b_{\perp} \rightarrow \text{case } (a_{\perp}, b_{\perp}) \text{ of} && \\ &\quad (\text{Just } (a, _), \text{Just } (b, _)) \rightarrow \text{Just } ((a, b), \varepsilon) && \\ &\quad _ \rightarrow \text{Nothing}) && \\ &\quad \langle \$ \rangle (\mathcal{P}[\![x]\!] \langle \$ \rangle \mathcal{C}[\![x]\!] \langle * \rangle (\mathcal{P}[\![y]\!] \langle \$ \rangle \mathcal{C}[\![y]\!] \langle \$ \rangle)) && \text{(by Lemma B.1)} \\ &\equiv (\lambda a_{\perp} \ b_{\perp} \rightarrow \text{case } (a_{\perp}, b_{\perp}) \text{ of} && \\ &\quad (\text{Just } (a, _), \text{Just } (b, _)) \rightarrow \text{Just } ((a, b), \varepsilon) && \\ &\quad _ \rightarrow \text{Nothing}) && \\ &\quad \langle \$ \rangle ((\lambda x \rightarrow \text{Just } (x, \varepsilon)) \langle \$ \rangle \mathcal{G}[\![x]\!] \langle * \rangle ((\lambda x \rightarrow \text{Just } (x, \varepsilon)) \langle \$ \rangle \mathcal{G}[\![y]\!] \langle \$ \rangle)) && \text{(by IH)} \\ &\equiv (\lambda x \rightarrow \text{Just } (x, \varepsilon)) \langle \$ \rangle ((\lambda x \ y \rightarrow (x, y)) \langle \$ \rangle \mathcal{G}[\![x]\!] \langle * \rangle \mathcal{G}[\![y]\!] \langle \$ \rangle)) && \text{(by app. properties)} \\ &\equiv (\lambda x \rightarrow \text{Just } (x, \varepsilon)) \langle \$ \rangle \mathcal{G}[\![\text{Pair } x \ y]\!] && \text{(by defn)} \end{aligned}$$

Case $g = \text{Map } f \ x$.

$$\begin{aligned} \mathcal{P}[\![\text{Map } f \ x]\!] \langle \$ \rangle \mathcal{C}[\![\text{Map } f \ x]\!] &\equiv (f \langle \$ \rangle \mathcal{P}[\![x]\!] \langle \$ \rangle \mathcal{C}[\![x]\!] \langle \$ \rangle) && \text{(by defn)} \\ &\equiv f \langle \$ \rangle (\mathcal{P}[\![x]\!] \langle \$ \rangle \mathcal{C}[\![x]\!] \langle \$ \rangle) && \text{(by functor properties)} \\ &\equiv f \langle \$ \rangle ((\lambda x \rightarrow \text{Just } (x, \varepsilon)) \langle \$ \rangle \mathcal{G}[\![x]\!] \langle \$ \rangle) && \text{(by IH)} \\ &\equiv (\lambda x \rightarrow \text{Just } (x, \varepsilon)) \langle \$ \rangle (f \langle \$ \rangle \mathcal{G}[\![x]\!] \langle \$ \rangle) && \text{(by functor properties)} \\ &\equiv (\lambda x \rightarrow \text{Just } (x, \varepsilon)) \langle \$ \rangle \mathcal{G}[\![\text{Map } f \ x]\!] && \text{(by defn)} \end{aligned}$$

Case $g = \text{Select } xs$.

$$\begin{aligned} \mathcal{P}[\![\text{Select } xs]\!] \langle \$ \rangle \mathcal{C}[\![\text{Select } xs]\!] &\equiv (\text{choice } (\text{map } (\lambda (c, x) \rightarrow (c, \mathcal{P}[\![x]\!] \langle \$ \rangle)) xs)) && \\ &\quad \langle \$ \rangle \text{oneof } (\text{map } (\lambda (c, x) \rightarrow (c, \cdot) \langle \$ \rangle \mathcal{C}[\![x]\!] \langle \$ \rangle) xs) && \text{(by defn)} \\ &\equiv \text{oneof } (\text{map } (\lambda (c, x) \rightarrow && \\ &\quad (\text{choice } (\text{map } (\lambda (c, x) \rightarrow (c, \mathcal{P}[\![x]\!] \langle \$ \rangle)) xs)) \circ (\cdot) \langle \$ \rangle \mathcal{C}[\![x]\!] \langle \$ \rangle) && \\ &\quad) xs) && \text{(by generator properties)} \\ &\equiv \text{oneof } (\text{map } (\lambda (_, x) \rightarrow \mathcal{P}[\![x]\!] \langle \$ \rangle \mathcal{C}[\![x]\!] \langle \$ \rangle) xs) && \text{(by parser properties)} \\ &\equiv \text{oneof } (\text{map } (\lambda (_, x) \rightarrow (\lambda x \rightarrow \text{Just } (x, \varepsilon)) \langle \$ \rangle \mathcal{G}[\![x]\!] \langle \$ \rangle) xs) && \text{(by IH)} \\ &\equiv (\lambda x \rightarrow \text{Just } (x, \varepsilon)) \langle \$ \rangle \text{oneof } (\text{map } (\lambda (_, x) \rightarrow \mathcal{G}[\![x]\!] \langle \$ \rangle) xs) && \text{(by generator properties)} \\ &\equiv (\lambda x \rightarrow \text{Just } (x, \varepsilon)) \langle \$ \rangle \mathcal{G}[\![\text{Select } xs]\!] && \text{(by defn)} \end{aligned}$$

Thus, generators can be coherently factored into a parser and a distribution. \square