$$\binom{n}{k} = \begin{cases} \binom{n-1}{k-1} + \binom{n-1}{k} & 0<k<n \\ 1 & k=0 \ \text{ or } \ k=n. \end{cases} \qquad (3.1)$$

1. $\binom{n}{k} = 1$ for $k=0$ or $k=n$.

⇒ i) $k=0$

$$\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{1\cdot n!} = \frac{n!}{n!} = 1.$$

ii) $k=n$

$$\binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n!\cdot 1!} = \frac{n!}{n!} = 1. \qquad \square.$$

2. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ for $0<k<n$

⇒ Base Case: $n=2 \Rightarrow k=1 \because 0<k<n$

$$\text{LHS} = \binom{2}{1} = 2$$

$$\text{RHS} = \binom{2-1}{1-1} + \binom{2-1}{1} = \binom{1}{0} + \binom{1}{1} = 1+1 = 2$$

Assume: $\binom{m}{k} = \binom{m-1}{k-1} + \binom{m-1}{k}$

Prove: $\binom{m+1}{k} = \binom{m}{k-1} + \binom{m}{k}$

$$\text{RHS} = \binom{m}{k-1} = \frac{m!}{(k-1)!(m-(k-1))!} = \frac{m!}{(k-1)!(m-k+1)!}$$

$$= \frac{km!}{k\cdot(k-1)!(m-k+1)!} = \frac{km!}{k!(m-k+1)!}$$

$$\binom{m}{k} = \frac{m!}{k!(m-k)!} = \frac{(m-k+1)m!}{(m-k+1)\cdot k!(m-k)!} = \frac{(m-k+1)m!}{k!(m-k+1)!}$$

$$\therefore \frac{km!}{k!(m-k+1)!} + \frac{(m-k+1)m!}{k!(m-k+1)!} = \frac{km! + (m-k+1)m!}{k!(m-k+1)!}$$

$$= \frac{m!(k+(m-k+1))}{k!(m-k+1)!}$$

$$= \frac{m!(m+1)}{k!(m-k+1)!}$$

$$= \frac{(m+1)!}{k!(m-k+1)!}$$

$$\text{LHS} = \binom{m+1}{k} = \frac{(m+1)!}{k!(m+1-k)!} = \frac{(m+1)!}{k!(m-k+1)!} \qquad \square$$