Vectors in Mathematics

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Contents

1 What is a vector

A **vector** is two pieces of information.

1. Length
2. Direction (Magnitude)

1.1 Vector notation

The notation for vectors is simply a variable name, with an arrow over top.

$$\vec{v}.$$

We can also specify the **components** of a vector

$$v = [x, y] \text{ or } \begin{bmatrix} x \\ y \end{bmatrix}.$$

1.2 Length of a vector

Furthermore, the length of the vector would be denoted

$$||\vec{v}||.$$

We can find the length by observation, if we have the x and y denominations, then we can use Pythagorean's theorem to find the length, or hypotenuse. Thus, the length of a vector would be

$$||\vec{v}|| = \sqrt{x^2 + y^2}.$$

Note:-

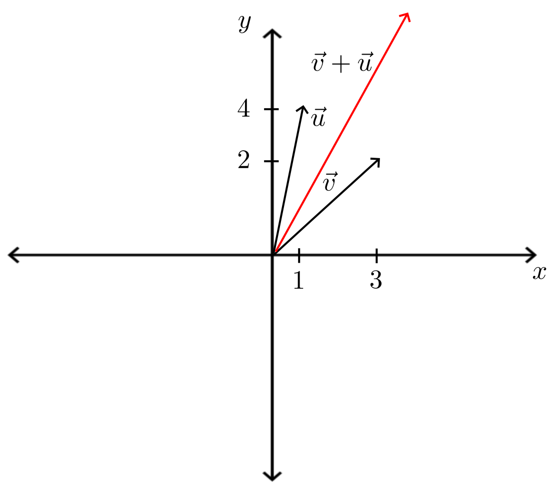
The name of a specific vector does **not** have to be v

1.3 Vector addition

Suppose we have two vectors $\vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$. Then

$$\begin{aligned} \vec{v} + \vec{u} &= \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 3+1 \\ 2+4 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 6 \end{bmatrix}. \end{aligned}$$

Let's take a look at this graphically...



1.4 Multiplying a vector by a scalar

Suppose we have the vector $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then

$$\begin{aligned} 2\vec{v} &= 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot 1 \\ 2 \cdot 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 4 \end{bmatrix}. \end{aligned}$$

So you can imagine we just double the length of the vector

1.5 Vector Subtraction

Suppose we have the vectors $\vec{v} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$. Then how might we compute $\vec{v} - \vec{u}$?

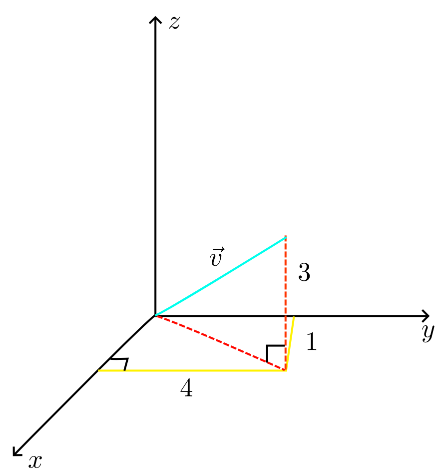
$$\begin{aligned} &\vec{v} - \vec{u} \\ =&\vec{v} + (-\vec{u}). \end{aligned}$$

Or just simply

$$\vec{v} - \vec{u} = \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \end{bmatrix}.$$

1.6 Vectors in 3 dimensions

With a three dimension vector, instead of having $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, we will have $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Suppose we have the vector $\vec{v} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$. Then graphically we would have



1.7 Length of a vector in three dimensions

By examining the above figure, we notice that to find $||\vec{v}||$, we need to find the hypotenuse of two separate triangles. Thus, we can generalize the length of a three dimensional vector with

$$\begin{aligned} ||\vec{v}|| &= \sqrt{(\sqrt{x^2 + y^2})^2 + z^2} \\ &= \sqrt{x^2 + y^2 + z^2}. \end{aligned}$$

1.8 Definition of $\mathbb{R}^n$

Definition 1:

$\mathbb{R}^n$ is the set of all $n - tuples$ of real numbers

For example, a vector $\vec{v}$ in two dimensions has two components $\begin{bmatrix} x \\ y \end{bmatrix}$. Thus, we say that $\vec{v}$ is a $2 - tuple$. Similarly, for any vector $\vec{v} = [v_1, v_2, v_3, ..., v_n]$, we say it is a $n - tuple$. Thus we can declare:

$$\begin{aligned}
&\vec{v} = [v_1, v_2] \ \vec{v} \in \mathbb{R}^2 \\
&\vec{u} = [u_1, u_2, u_3] \ \vec{u} \in \mathbb{R}^3 \\
&\vec{w} = [w_1, w_2, w_3, w_n] \ \vec{w} \in \mathbb{R}^n \\
&.
\end{aligned}$$

This may seem intuitive, if we recall the definition for the set of all (x, y) pairs on the Cartesian plane, we have

$$\begin{aligned}
&\mathbb{R} \times \mathbb{R} = \{(x, y) \mid x \in \mathbb{R}^2\} \\
&\text{or } \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}.
\end{aligned}$$

So for a three dimensional plane

$$\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid (x, y, z) \in \mathbb{R}^3\}.$$

1.9 Algebraic Properties of Vectors

(i) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(ii)