

Homework/Worksheet 4 - Due: Wednesday, October 4

1.) Find the length of the functions below over the given interval. If you cannot evaluate the integral exactly, use technology to approximate it.

(a) $y = x^{\frac{3}{2}}$ from $(1, 1)$ to $(8, 4)$

(b) $y = \frac{1}{3}(x^2 - 2)^{\frac{3}{2}}$ from $x = 2$ to $x = 4$

(c) $y = \frac{x}{3} + \frac{1}{4x}$ from $x = 1$ to $x = 4$

1.a

$$\frac{d}{dx}x^{\frac{3}{2}} = \frac{3}{2}x^{\frac{1}{2}}.$$

Thus:

$$\begin{aligned}s &= \int_1^8 \sqrt{1 + \left(\frac{3}{2}x^{\frac{1}{2}}\right)^2} dx \\&= \int_1^8 \sqrt{1 + 3x} dx\end{aligned}$$

$$\text{Let } u = 1 + 3x$$

$$du = 3dx$$

$$\frac{1}{3}du = dx$$

$$u(a) = 4$$

$$u(b) = 25$$

$$\frac{1}{3} \int_4^{25} \sqrt{u} du$$

$$= \frac{1}{3} \left[\frac{2}{3} u^{\frac{3}{2}} \right]_4^{25}$$

$$= \frac{2}{9} \left[u^{\frac{3}{2}} \right]_4^{25}$$

$$= \frac{2}{9} \left[25^{\frac{3}{2}} - 4^{\frac{3}{2}} \right]$$

$$= \frac{2}{9} [125 - 8]$$

$$= 26.$$

1.b

$$\begin{aligned} & \frac{1}{3} \left[\frac{d}{dx} (x^2 - 2)^{\frac{3}{2}} \right] \\ & \frac{1}{3} \left[\frac{3}{2} (x^2 - 2)^{\frac{1}{2}} \right] \cdot 2x \\ & = x(x^2 - 2)^{\frac{1}{2}}. \end{aligned}$$

Thus:

$$\begin{aligned} s &= \int_2^4 \sqrt{1 + (x(x^2 - 2)^{\frac{1}{2}})^2} \, dx \\ &= \int_2^4 \sqrt{1 + x^2(x^2 - 2)} \, dx \\ &= \int_2^4 \sqrt{1 + x^4 - 2x^2} \, dx \\ &= \int_2^4 \sqrt{(x^2 - 1)^2} \, dx. \end{aligned}$$

Since we know the domain is nonnegative, we can rewrite as:

$$\begin{aligned} & \int_2^4 x^2 - 1 \, dx \\ &= \left. \frac{1}{3}x^3 - x \right|_2^4 \\ &= \left(\frac{1}{3}(4)^3 - 4 \right) - \left(\frac{1}{3}(2)^3 - 2 \right) \\ &= \frac{64}{3} - 4 - \left(\frac{8}{3} - 2 \right) \\ &= \frac{50}{3}. \end{aligned}$$

1.c

$$\begin{aligned} & \frac{d}{dx} \frac{1}{3} x^3 + \frac{1}{4} x^{-1} \\ &= x^2 - \frac{1}{4} x^{-2}. \end{aligned}$$

Thus:

$$\begin{aligned} s &= \int_1^4 \sqrt{1 + \left(x^2 - \frac{1}{4x^2}\right)^2} dx \\ &= \int_1^4 \sqrt{1 + x^4 - 2\left(\frac{1}{4}\right) + \frac{1}{16x^4}} dx \\ &= \int_1^4 \sqrt{1 + x^4 - \frac{1}{2} + \frac{1}{16x^4}} dx \\ &= \int_1^4 \sqrt{x^4 + \frac{1}{16x^4} + \frac{1}{2}} dx \\ &= \int_1^4 \sqrt{(x^2)^2 + \frac{1}{(4x^2)^2} + \frac{1}{2}} dx \\ &= \int_1^4 \sqrt{\left(x^2 + \frac{1}{4x^2}\right)^2} dx \\ &= \int_1^4 \sqrt{\left(\frac{4x^4 + 1}{4x^2}\right)^2} dx. \end{aligned}$$

Since the domain is *nonnegative*, we can rewrite as:

$$\begin{aligned} & \int_1^4 \frac{4x^4 + 1}{4x^2} dx \\ &= \int_1^4 x^2 + \frac{1}{4x^2} dx \\ &= \int_1^4 x^2 + \frac{1}{4} x^{-2} dx \\ &= \left. \frac{1}{3} x^3 - \frac{1}{4} x^{-1} \right|_1^4 \\ &= \left(\frac{1}{3} (4)^3 - \frac{1}{4} (4)^{-1} \right) - \left(\frac{1}{3} (1)^3 - \frac{1}{4} (1)^{-1} \right) \\ &= \frac{339}{16}. \end{aligned}$$

2.) Find the surface area of the volume generated by revolving the curve $y = x^3$, $0 \leq x \leq 1$, around the x -axis.

$$\frac{d}{dx}x^3 = 3x^2.$$

Thus:

$$\begin{aligned} sa &= \int_0^1 2\pi x^3 \sqrt{1 + (3x^2)^2} \, dx \\ &= 2\pi \int_0^1 x^3 \sqrt{1 + 9x^4} \, dx. \end{aligned}$$

Thus:

$$\text{Let } u = 1 + 9x^4$$

$$du = 36x^3 dx$$

$$\frac{1}{36} du = x^3 dx$$

$$u(a) = 1$$

$$u(b) = 10.$$

$$\begin{aligned} sa &= \frac{2\pi}{36} \int_1^{10} u^{\frac{1}{2}} \, du \\ &= \frac{\pi}{18} \left[\frac{2}{3} u^{\frac{3}{2}} \right]_1^{10} \\ &= \frac{\pi}{27} \left[\left((10)^{\frac{3}{2}} - 1 \right) \right] \\ &= \frac{10^{\frac{3}{2}} \pi}{27} - \frac{\pi}{27} \\ &= \frac{10^{\frac{3}{2}} \pi - \pi}{27}. \end{aligned}$$

3.) Find the surface area of the volume generated by revolving the curve $y = 3x^4$, $0 \leq x \leq 1$, around the y -axis.

Thus:

Derivative:

$$\frac{d}{dx}3x^4 = 12x^3.$$

$$\begin{aligned} sa &= \int_0^1 2\pi x \sqrt{1 + (12x^3)^2} \, dx \\ &= 2\pi \int_0^1 x \sqrt{1 + 144x^6} \, dx \\ &\approx 15.8264. \end{aligned}$$

4.) Evaluate the following integrals

(a) $\int \frac{(\ln(x))^2}{x} dx$

(b) $\int_0^{\frac{\pi}{4}} \tan x dx$

4.a)

$$\int \frac{(\ln x)^2}{x} dx$$

Let $u = \ln x$

$$du = \frac{1}{x} dx$$

$$\implies \int u^2 du$$

$$= \frac{1}{3} u^3 + C$$

$$= \frac{1}{3} (\ln x)^3 + C$$

$$= \frac{1}{3} \ln^3 x + C.$$

4.b

Integrate:

$$\int \tan x dx$$

$$= \int \frac{\sin x}{\cos x} dx$$

Let $u = \cos x$

$$du = -\sin x dx$$

$$-du = \sin x dx$$

$$\implies \int u^{-1} du$$

$$= -\ln |u| + C$$

$$= -\ln |\cos x| + C$$

$$= \ln \left| \frac{1}{\cos x} \right| + C$$

$$= \ln |\sec x| + C.$$

Thus:

$$\begin{aligned} & \ln |\sec x| \bigg|_0^{\frac{\pi}{3}} \\ &= \ln \left| \sec \frac{\pi}{3} \right| - \ln |\sec 0| \\ &= \ln 2 - \ln 1 \\ &= \ln 2. \end{aligned}$$

5.) Compute the derivative, $\frac{dy}{dx}$ of the following functions.

(a) $y = e^{\sin x}$

(b) $y = xe^x$

(c) $y = \frac{x^{-1}}{\ln x}$

5.a)

$$\begin{aligned}\frac{d}{dx}e^{\sin x} \\ &= e^{\sin x} \cdot \frac{d}{dx} \sin x \\ &= \cos x e^{\sin x}.\end{aligned}$$

5.b)

By the product rule:

$$\begin{aligned}\frac{d}{dx}xe^x \\ &= xe^x + e^x \\ &= e^x(x+1).\end{aligned}$$

5.c

$$\begin{aligned}\frac{d}{dx}x^{-\frac{1}{\ln x}} \\ &= \frac{d}{dx}e^{\ln\left(x^{-\frac{1}{\ln x}}\right)} \\ &= \frac{d}{dx}e^{-\frac{1}{\ln x} \ln x} \\ &= \frac{d}{dx}e^{-1} \\ &= 0.\end{aligned}$$