Group pset 2 - Due: Wednesday, March 19

1. Let $\overrightarrow{AB}$ be a ray with carrier m, and C a point in $\overrightarrow{AB}^0$. Prove that if $\omega < \infty$, then $C_m^* \notin \overrightarrow{AB}$

Proof. Assume ray $\overrightarrow{AB}$ with carrier m. Let C be a point in $\text{Int}\overrightarrow{AB}$, and $\omega < \infty$.

Since $C \in \text{Int}\overrightarrow{AB}$, $C \neq A$ by the definition of the interior of a ray. Further, by the definition of $\overrightarrow{AB}$, one of A-C-B or A-B-C are true.

Suppose for the sake of contradiction that $C_m^* \in \overrightarrow{AB}$. Then, one of A-C_m^*-B, A-B-C_m^*. We consider four cases.

1. A-C-B and A-C_m^*-B
2. A-C-B and A-B-C_m^*
3. A-B-C and A-C_m^*-B
4. A-B-C and A-B-C_m^*

We first remark that since $\overrightarrow{AB}$ defined, $AB < \omega$. Also, $C \in \text{Int}\overrightarrow{AB}$ implies $C \neq A$

Assume (1) is true. Thus, we have A-C-B and A-C_m^*-B. Since $AB < \omega$ and $AC_m^* + C_m^*B = AB$, $AC_m^* < AB < \omega$, and by theorem 8.4, $\overrightarrow{AB} = \overrightarrow{AC_m^*}$. Next, observe that since $C \in \overrightarrow{AB} = \overrightarrow{AC_m^*}$, one of

$$A\text{-}C\text{-}C_m^* \quad A\text{-}C_m^*\text{-}C$$

Assume A-C-C_m^*. In this case, $AC + CC_m^* = AC_m^*$, which implies $CC_m^* < AC_m^*$. But, with $A \neq C$ and Theorem 9.1, $CC_m^* = \omega$, and $AC_m^* < \omega$. Thus, $CC_m^* < AC_m^* \implies \omega < \omega$, a contradiction.

Next, assume A-C_m^*-C, which implies $AC_m^* + C_m^*C = AC$, and $C_m^*C < AC$. But, since A-C-B, and $AB < \omega$, we have $AC < AB$. Thus, $C_m^*C < AC < AB < \omega$ is a contradiction, since $CC_m^* = \omega$. Thus, not (A-C-B and A-C_m^*-B)

Assume (2) is true, then A-C-B and A-B-C_m^*. In this case, ROI yields A-C-B-C_m^*, which yields A-C-C_m^*. This new relation gives $AC + CC_m^* = AC_m^*$, which again implies $CC_m^* < AC_m^* < \omega$, a contradiction by theorem 9.1. Thus, not (A-C-B and A-B-C_m^*)

Assume (3) is true, in a similar fashion to the previous case, from A-B-C, A-C_m^*-B and the ROI, we get A-C_m^*-B-C, which gives A-C_m^*-C. From this, $AC_m^* + CC_m^* = AC$. Which means we have $CC_m^* < AC < \omega$, which is a contradiction by theorem 9.1 ($CC_m^* = \omega$). Thus, not (A-B-C and A-C_m^*-B)

Lastly, assume (4). Thus, A-B-C and A-B-C_m^*. In this case, A-B-C_m^* gives $AB + BC = AC_m^*$, but A-B-C tells us that $A \neq C$, and by theorem 9.1, $AC_m^* < \omega$. Thus, by theorem 8.4, $\overrightarrow{AB} = \overrightarrow{AC_m^*}$. This means one of

$$A\text{-}C\text{-}C_m^* \quad A\text{-}C_m^*\text{-}C$$

Which we saw in case (1) both give contradictions. So, not (A-B-C and A-B-C_m^*)

Therefore, $C \notin \overrightarrow{AB}$ ■

6. Prove Theorem 9.10

Proof. Let A, B be points on line m with $0 < AB < \omega < \infty$. Let $C \neq A, B, A_m^*, B_m^*$ be another point on m.

First, assume $C \in \overrightarrow{AB} \cup \overrightarrow{BA}$, which equals $\overline{AB} \cup \overline{BA_m^*} \cup \overline{AB_m^*}$ By proposition 9.3. If $C \in \overrightarrow{AB} \cup \overrightarrow{BA}$, then one of

$$A\text{-}C\text{-}B \quad A\text{-}B\text{-}C \quad B\text{-}C\text{-}A \quad B\text{-}A\text{-}C$$

By definition of a ray. Observe that in any case, there is a betweenness relation among A, B, C.

By corollary 9.9, the only segment left to examine is $\overline{A_m^* B_m^*}$. Thus, assume $C \in \text{Int}\overline{A_m^* B_m^*}$ (since $C \neq A_m^*$ or B_m^*), which implies A_m^*-C-B_m^*

Assume for the sake of contradiction that there does exist a betweenness relation among A, B, C. Then, one of

$$A\text{-}B\text{-}C \quad A\text{-}C\text{-}B \quad B\text{-}A\text{-}C$$

Assume A-B-C, then $C \in \overrightarrow{AB}$ by the definition of a ray. But, by proposition 9.3, $\overline{A_m^* B_m^*}^0$ is not included in $\overrightarrow{AB}$. Thus, a contradiction. Similarly, B-A-C implies $C \in \overrightarrow{BA}$, another contradiction.

lastly, assume A-C-B, then $C \in \overline{AB}^0$. But, by prop 9.9, $\overline{AB}^0 \cap \overline{A_m^* B_m^*}^0 = \varnothing$. Thus, a contradiction.

Therefore, there is no betweenness relation among A, B, C if and only if $C \in \overline{A_m^* B_m^*}^0$ ■