Discrete Structures
Logic

A Document By:
Nathan Warner

Northern Illinois University

Computer Science
Northern Illinois University
August 11, 2023
United States

Contents

1 Statements

> **Definition:** A statement (or proposition) is a sentence that is either true or false (but not both)

> **Example: for the following, state whether it is a statement, or not a statement**
>
> **A.) "I think it will rain tomorrow" is a statement.**
>
> **B.) 3 - x = 12**
>
> **C.) 2 + 2 =3**
>
> **Solutions:**
>
> **A.) The sentence is not a statement because there is a chance it will rain, or not.**
>
> **B.) Though the bellow sentence is a mathematical expression, however it is not a statement because it is not either true or false. Depending on what x is, the sentence is either true or false, but right now it is neither.**
>
> **C.) Even though the bellow expression is false, but it is a statement because it is either true or false, but not both, and in this case it is false. Therefore "2 + 2 = 3" is a statement.**

2 Compound Statements

Components of a statement

- p: Represents predicate
- q: Represents conclusion

Logical Connectives

- $\wedge$: Represents **and**
- $\wedge$: Represents **or**
- $\neg$ **or** $\sim$: Represents **negation**

By utilizing logical connectives, we can create compound statements

> **Example: For each sentence, choose the correct compound statement**
>
> **A.) The sentence "It is not hot or it is sunny" in symbols is:**
>
> **B.) The expression, "$3 \leqslant a$" in writing is:**
>
> **C.) If p: a week has seven days, q: there are 20 hours in a day, and r: there are 60 minutes in an hour, then $\sim$p $\wedge$ $\sim$r is:**
>
> **Solutions:**
>
> **A.)** $\sim p \vee q$
>
> **B.)** $3 > a$ **or** $3 = a$
>
> **C.) A week doesn't have 7 days and there are not 60 minutes in an hour.**

3 Truth Tables

Here is a simple example of a truth table for logical and:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Here is a simple example of a truth table for logical or (inclusive):

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Here is a simple example of a truth table for logical or (exclusive):

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

Here is a simple example of a truth table for logical not (negation):

P	$\neg P$
T	F
F	T

Example: Construct a truth table for $(p \wedge q) \vee \neg r$

Figure:

p	q	r	$(p \wedge q)$	$\neg r$	$(p \wedge q) \vee \neg r$
T	T	T	T	F	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	F	T	T
F	T	T	F	F	F
F	T	F	F	T	T
F	F	T	F	T	F
F	F	F	F	F	T

4   Logical Equivalence

Definition: Statements p and q are said to be logically equivalent if they have the same truth value in every model.
Notation: the notation for logical equivalence is $\equiv$

Say we have statements p and q, and we want to show that $p \wedge q$, and $q \wedge p$ are logically equivalent, to do this, we must first construct a truth table:

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

So we can see that the columns $p \wedge q$, $q \wedge p$ have the same truth values, therefore they are said to be **logically equivalent**

5 Tautologies and Contradictions

> **Definition:** A **Tautology** is a Statement that is always true, a assertion that is true in every possible interpretation
>
> A **Contradiction** is a statement that is always false.

Consider the following compound statement

$$p \vee \neg p.$$

Because this statement can never be false, we say it is a **Tautology**

Contradiction

Consider the statement:

$$p \wedge \neg p.$$

Because this statement can never be true, we say it is a **Contradiction**

6 De Morgan's Laws

De Morgan's Laws are:

- $\neg(p \wedge q) = \neg p \vee \neg q$
- $\neg(p \vee q) = \neg p \wedge \neg q$

Consider the statement:

$$0 < x \leqslant 3.$$

To use De Morgan's Law, which states:

$$\neg(p \wedge q) = \neg p \vee \neg q.$$

We can rewrite the statement as:

$$0 \geqslant x \text{ or } x > 3.$$

7 Logical Equivalence Laws

Commutative laws:	p ∧ q ≡ q ∧ p	$p \vee q \equiv q \vee p$
Associative laws:	(p ∧ q) ∧ r ≡ p ∧ (q ∧ r)	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive laws:	p ∧ (q ∨ r) ≡ (p ∧ q) ∨ (p ∧ r)	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Identity laws:	p ∧ **t** ≡ p	$p \vee \mathbf{c} \equiv p$
Negation laws:	p∨ ∼ p ≡ **t**	$p \wedge \sim p \equiv \mathbf{c}$
Double negative law:	∼ (∼ p) ≡ p	
Idempotent laws:	p ∧ p ≡ p	$p \vee p \equiv p$
Universal bound laws:	p ∨ t ≡ **t**	$p \wedge \mathbf{c} \equiv \mathbf{c}$
DeMorgan's laws:	∼ (p ∧ q) ≡∼ p∨ ∼ q	$\sim (p \vee q) \equiv \sim p \wedge \sim q$
Absorption laws:	p ∨ (p ∧ q) ≡ p	$p \wedge (p \vee q) \equiv p$
Negation of t and c	¬t = c	¬c = t

8 Conditional Statements

> **Definition:** A **conditional statement** is a statement that can be written in the form "If P then Q," where P and Q are sentences.
> **Syntax:** if *statement* then *statement*

Consider the statment

$$p \to q.$$

This statement, read "if p then q", can be described with the following truth table:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Note:-

To get a truth value of "true" in $p \to q$, either p and q both need to be true, or both need to be false, or q needs to be true

9   Negation of Conditional Statements

Definition: The **Negation of conditional statement** is logically equivalent to a conjunction of the antecedent and the negation of the consequent.

$$Negation: \ p \to q \equiv p \land \neg q.$$

The **Contrapositive** of a conditional statement is a combination of the converse and the inverse

$$p \to q \equiv \neg q \to \neg p.$$

Consider the following condition

$$p \to q.$$

Which we know is logically equivalent to:

$$p \to q \equiv \neg p \lor q.$$

By use of De Morgan's Law, which states that:

$$\neg(p \lor q) \equiv \neg p \land \neg q.$$

We can negate $p \to q$, so:

$$\neg(p \lor q) \equiv \neg(\neg p) \land \neg q.$$

Consider the following Conditional Statement

If my dad is at home then he cant pick me up
p →q

Then the negation would be:

$p \land \neg q$
$\equiv$ my dad is at home and he can pick me up.

10 Converse and Inverse

Definition: The **Converse** of a conditional statement is created when the hypothesis and conclusion are reversed
The **Inverse** of a conditional statement is when both the hypothesis and conclusion are negated

Consider the statement

$$p \to q.$$

Then the **Converse** would be:

$$q \to p.$$

And the **Inverse** would be:

$$\neg p \to \neg q.$$

11 Biconditional Statements

Definition: A **Biconditional Statement** is a true statement that combines a hypothesis and conclusion with the words 'if and only if' instead of the words 'if' and 'then'

Say we have the following statement

$$q \iff p.$$

Then the truth table would be:

p	q	q $\iff$ p
T	T	T
F	T	F
T	F	F
F	F	T

Note:-

Similar to conditional statements, in the truth table, $p \iff q$ is true if both p and q have the same value. So false false will be true

12   Digital Logic Circuits

Definition: A **Circuit** is a complete circular path that electricity flows through
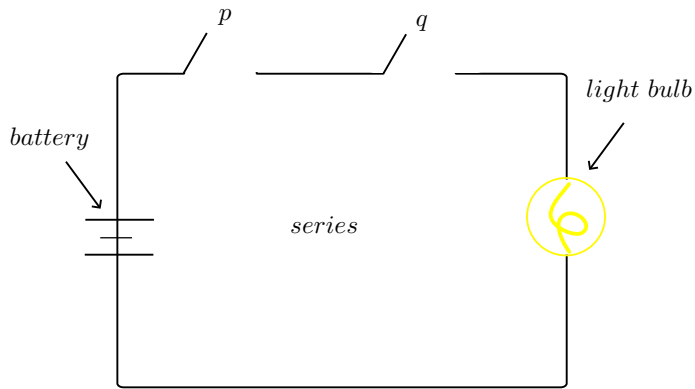
First, let's consider the following circuit diagram.

Figure 1: Circuit Diagram

p	q	Light Bulb
Closed	Closed	On
Open	Closed	Off
Closed	Open	Off
Open	Open	Off

Figure 2: Truth Table

So assuming the electricity can only flow up from the battery, then we can clearly see that both p and q need to be closed to get power to the light bulb. Thus, we can see that this case is $p \wedge q$

Next, let's consider the circuit diagram:
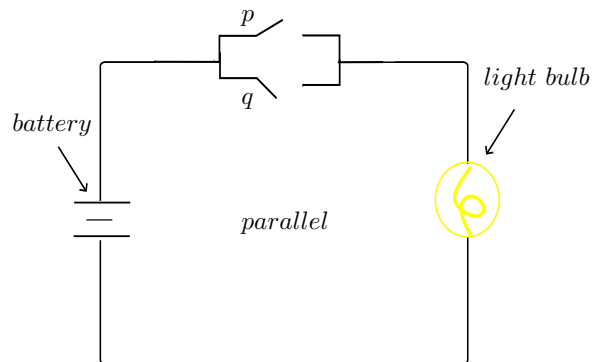

Figure 3: Circuit Diagram

p	q	Light Bulb
Closed	Closed	On
Open	Closed	On
Closed	Open	On
Open	Open	Off

Figure 4: Truth Table

Again, asumming the electricity can only flow up from the battery, we can clearly see that either gate can be closed while the other is open to result in the light bulb getting power.
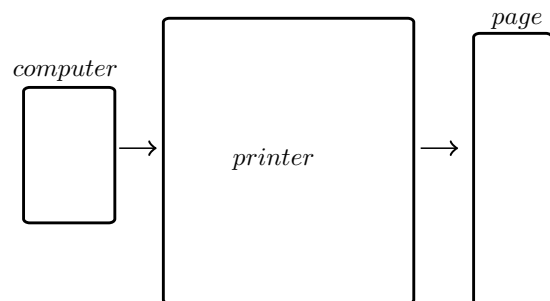
13 Black Boxes and Gates

> **Definition:** A **Black Box** is a circuit that is observed in terms of its input, output or transfer characteristics without any knowledge of its internal workings.
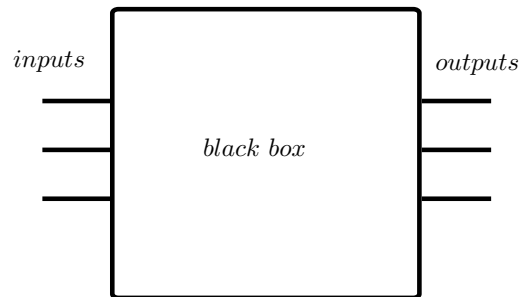
With this black box, we don't care about the inner workings, we are just concerned with the inputs and outputs

Figure 5:

Let's consider an example that's a bit more pragmatic

In this case, the printer is the black box, we aren't required to know the inner workings of the printer, we are just concerned with the inputs and outputs. If we send some input to the printer through the computer, we know that we will get some output via the page.

> **Definition:** A **Gate** is circuit with one or more inputs and exactly one output

Types of gates:

- **Not (negation)**: One input and one output, negates the input

- **And**: Two inputs and one output, requires both inputs to be true

- **Or**: Two inputs and one output, requires only one input to be true

Representation of gates:
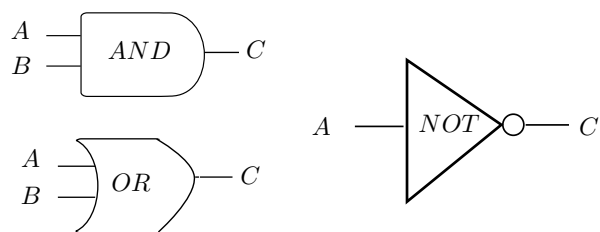
14 Boolean Expressions

Definition: A **Boolean Variable** is a variable with only two possible values, True, or False.
A **Boolean expression** is an expression that is composed of Boolean variables, connected with logic connectives
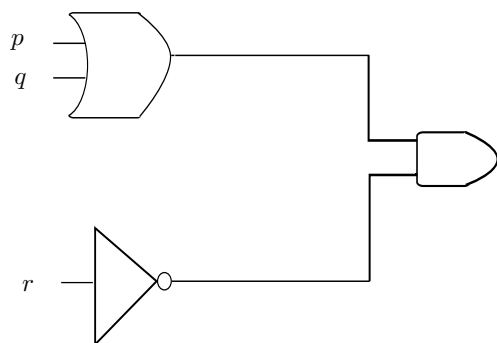

$$(p \lor q) \land (\neg r).$$

Figure 7: Boolean Expression

Figure 6: Boolean Expression

$$(p \lor q) \land (p \land q).$$

Figure 9: Boolean Expression
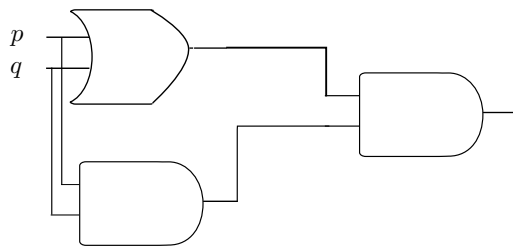
Figure 8: Boolean Expression

Now suppose we are given the following expression, how might we construct the circuit?

$$(\neg p \land q) \lor q.$$
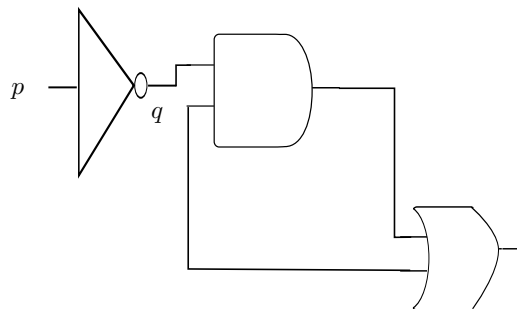
15   Truth Tables and Circuits