Comprehensive Compendium:
Calculus II

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Contents

Calculus II

1.1 Chapter 1 Definitions and Theorems

- **Mean Value Theorem For Integrals**: If $f(x)$ is continuous over an interval [a,b], then there is at least one point $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int f(x)\ dx.$$

- **Integrals resulting in inverse trig functions**

1.

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{|a|} + C.$$

2.

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C.$$

3.

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{|a|} \sec^{-1} \frac{|x|}{a} + C.$$

1.2 Chapter 2 Definitions and Theorems

- **Area between two curves, integrating on the x-axis**

$$A = \int_a^b [f(x) - g(x)]\, dx \tag{1}$$

Where $f(x) \geqslant g(x)$

$$A = \int_a^b \; [g(x) - f(x)] \; dx.$$

for $g(x) \geqslant f(x)$

- **Area between two curves, integrating on the y-axis**

$$A = \int_c^d [u(y) - v(y)]\, dy \tag{2}$$

- **Areas of compound regions**

$$\int_a^b \; |f(x) - g(x)| \; dx.$$

- **Area of complex regions**

$$\int_a^b \; f(x) \; dx + \int_b^c \; g(x) \; dx.$$

- **Slicing Method**

$$V(s) = \sum_{i=1}^{n} \; A(x_i^*) \; \Delta x = \int_a^b \; A(x) \; dx.$$

- **Disk Method along the x-axis**

$$V = \int_a^b \pi [f(x)]^2 \, dx \tag{3}$$

- **Disk Method along the y-axis**

$$V = \int_c^d \pi [g(y)]^2 \, dy \tag{4}$$

- **Washer Method along the x-axis**

$$V = \int_a^b \pi [(f(x))^2 - (g(x))^2] \, dx \tag{5}$$

- **Washer Method along the y-axis**

$$V = \int_c^d \pi [(u(y))^2 - (v(y))^2] \, dy \tag{6}$$

- **Radius if revolved around other line (Washer Method)**

$$If:\ x = -k$$
$$Then:\ r = Function + k.$$

$$If:\ x = k$$
$$Then:\ r = k - Function.$$

- **Method of Cylindrical Shells (x-axis)**

$$V = \int_a^b 2\pi x f(x)\, dx \tag{7}$$

- **Method of Cylindrical Shells (y-axis)**

$$V = \int_c^d 2\pi y g(y)\, dy \tag{8}$$

- **Region revolved around other line (method of cylindrical shells):**

$$If:\ x = -k$$
$$Then:\ V = \int_a^b\ 2\pi(x+k)(f(x))\ dx.$$

$$If:\ x = k$$
$$Then:\ V = \int_a^b\ 2\pi(k-x)(f(x))\ dx.$$

- **A Region of Revolution Bounded by the Graphs of Two Functions (method cylindrical shells)**

$$V = \int_a^b\ 2\pi x\left[f(x) - g(x)\right]\ dx.$$

- **Arc Length of a Function of x**

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \tag{9}$$

- **Arc Length of a Function of y**

$$\text{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy \tag{10}$$

- **Surface Area of a Function of x (Around x)**

$$\text{Surface Area} = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx \tag{11}$$

- **Surface Area of a Function of x (Around y)**

$$\text{Surface Area} = \int_a^b 2\pi\sqrt{1 + [f'(x)]^2}\, dx \tag{12}$$

$$\text{Or: } \int_a^b\ 2\pi u(y)\sqrt{1 + (u'(y))^2}\ dy \tag{13}$$

- **Natural logarithm function**

$$\ln x = \int_1^x \frac{1}{t}\,dt \tag{14}$$

- **Exponential function**

$$y = e^x, \quad \ln y = \ln(e^x) = x \tag{15}$$

- **Logarithm Differentiation**

$$f'(x) = f(x) \cdot \frac{d}{dx} \ln\left(f'(x)\right).$$

Note: Use properties of logs before you differentiate whats inside the logarithm

1.3 Chapter 3 Definitions and Theorems

- **Integration by parts formula**

$$\int u\,dv = uv - \int v\,du.$$

- **Integration by parts for definite integral**

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

- **To integrate products involving sin(ax), sin(bx), cos(ax), and cos(bx), use the substitutions:**

  - **Sine Products**

$$\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$$

  - **Sine and Cosine Products**

$$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$$

  - **Cosine Products**

$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$$

  - **Power Reduction Formula (sine)**

$$\int \sin^n x\ dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\ dx$$
$$\int_0^{\frac{\pi}{2}} \sin^n x\ dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x\ dx.$$

  - **Power Reduction Formula (cosine)**

$$\int \cos^n x\ dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\ dx$$
$$\int_0^{\frac{\pi}{2}} \cos^n x\ dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \cos^{n-2} x\ dx.$$

  - **Power Reduction Formula (secant)**

$$\int \sec^n x\ dx = \frac{1}{n-1}\sec^{n-1} x\sin x + \frac{n-2}{n-1}\int \sec^{n-2} x\ dx$$
$$\int \sec^n x\ dx = \frac{1}{n-1}\sec^{n-2} x\tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\ dx$$

  - **Power Reduction Formula (tangent)**

$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$$

- **Trigonometric Substitution**

- $\sqrt{a^2-x^2}$ use $x = a\sin\theta$ with domain restriction $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- $\sqrt{a^2+x^2}$ use $x = a\tan\theta$ with domain restriction $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
- $\sqrt{x^2-a^2}$ use $x = a\sec\theta$ with domain restriction $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$

- **Steps for fraction decomposition**
  1. Ensure $deg(Q) < deg(P)$, if not, long divide
  2. Factor denominator
  3. Split up fraction into factors
  4. Multiply through to clear denominator
  5. Group terms and equalize
  6. Solve for constants
  7. Plug constants into split up fraction
  8. Compute integral
- **Solving for constants** Either:
  - Plug in values (often the roots)
  - Equalize
- **Cases for partial fractions**
  - Non repeated linear factors
  - Repeated linear factors
  - Nonfactorable quadratic factors
- **Midpoint rule**

$$M_n = \sum_{i=1}^{n} f(m_i)\ \Delta x.$$

- **Absolute error**

$$err = \Big|\text{Actual} - \text{Estimated}\Big|.$$

- **Relative error**

$$err = \left|\frac{\text{Actual} - \text{Estimated}}{\text{Actual}}\right| \cdot 100\%.$$

- **Error upper bound for midpoint rule**

$$E_M \leqslant \frac{M(b-a)^3}{24n^2}$$

  Where M is the maximum value of the second derivative
- **Trapezoidal rule**

$$T_n \frac{1}{2}\Delta x \left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right)$$

- **Error upper bound for trapezoidal rule**

$$E_T \leqslant \frac{M(b-a)^3}{12n^2}$$

Where M is the maximum value of the second derivative

- **Simpson's rule**

$$S_n = \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)$$

- **Error upper bound for Simpson's rule**

$$E_S \leqslant \frac{M(b-a)^5}{180n^4}$$

Where M is the maximum value of the fourth derivative

- **Finding n with error bound functions**
  1. Find $f''(x)$
  2. Find maximum values of $f''(x)$ in the interval
  3. Plug into error bound function
  4. Set value $\leqslant$ desired accuracy (ex: 0.01)
  5. Solve:
  6. If we were to truncate, we would use the ceil function $\lceil n \rceil$ DO NOT FLOOR

- **Improper integrals (Infinite interval)**
  - $\int_a^{+\infty} f(x)\ dx = \lim_{t\to+\infty} \int_a^t f(x)\ dx$
  - $\int_{-\infty}^b f(x)\ dx = \lim_{t\to-\infty} \int_t^b f(x)\ dx$
  - $\int_{-\infty}^{+\infty} f(x)\ dx = \int_{-\infty}^0 f(x)\ dx + \int_0^{+\infty} f(x)\ dx$

- **Improper integral (discontinuous)**
  - Let $f(x)$ be continuous on $[a, b)$, then;

$$\int_a^b f(x)\ dx = \lim_{t\to b^-} \int_a^t f(x)\ dx\ .$$

  - Let $f(x)$ be continuous on $(a, b]$, then;

$$\int_a^b f(x)\ dx = \lim_{t\to b^+} \int_t^b f(x)\ dx\ .$$

  In each case, if the limit exists, then the improper integral is said to converge. If the limit does not exist, then the improper integral is said to diverge.

  - Let $f(x)$ be continuous on $[a, b]$ except at a point $c \in (a, b)$, then;

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx.$$

  If either integral diverges, then $\int_a^b f(x)\ dx$ diverges

- **Comparison theorem** Let $f(x)$ and $g(x)$ be continuous over $[a, +\infty)$. Assume that $0 \leqslant f(x) \leqslant g(x)$ for $x \geqslant a$.
  - If $\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx = +\infty$,
    then $\int_a^{+\infty} g(x)\,dx = \lim_{t\to+\infty} \int_a^t g(x)\,dx = +\infty$.
  - If $\int_a^{+\infty} g(x)\,dx = \lim_{t\to+\infty} \int_a^t g(x)\,dx = L$, where L is a real number,
    then $\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx = M$ for some real number $M \leqslant L$.

- **P-integrals**
  - $\int_0^{+\infty} \frac{1}{x^p}\ dx = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ +\infty & \text{if } p \leqslant 1 \end{cases}$
  - $\int_0^{1} \frac{1}{x^p}\ dx = \begin{cases} \frac{1}{1-p} & \text{if } p < 1 \\ +\infty & \text{if } p \geqslant 1 \end{cases}$
  - $\int_a^{+\infty} \frac{1}{x^p}\ dx = \begin{cases} \frac{a^{1-p}}{p-1} & \text{if } p > 1 \\ +\infty & \text{if } p \leqslant 1 \end{cases}$
  - $\int_0^{a} \frac{1}{x^p}\ dx = \begin{cases} \frac{a^{1-p}}{1-p} & \text{if } p < 1 \\ +\infty & \text{if } p \geqslant 1 \end{cases}$

- **Bypass L'Hospital's Rule**

  $$\ln(\ln(x)),\ \ln(x),\ \cdots\ x^{\frac{1}{100}},\ x^{\frac{1}{3}},\ \sqrt{x},\ 1,\ x^2,\ x^3,\ \cdots\ e^x,\ e^{2x},\ e^{3x},\ \cdots,\ e^{x^2},\ \cdots\ e^{e^x}.$$

  Essentially what it means is things on the right grow faster than things on the left. Thus, if we have say:

  $$\lim_{x\to\infty} \frac{x^2}{e^{2x}}.$$

  We can be sure that it is zero. Because this is $x^2 \cdot e^{-2x}$. If we take $\lim\limits_{x\to\infty} x^2 e^{-2x}$, we get $\infty \cdot 0$. As we see by the sequence e^{-2x} overrules x^2 and we can say the limit is zero.

- **Consideration for Limits**: Let $f : A \to B$ be a function defined by $x \mapsto f(x)$. If a point c lies outside the domain A, then the expression $\lim\limits_{x\to c} f(x)$ is not meaningful, and we classify this limit as undefined. For instance, the function arcsine has a domain of $[-1, 1]$. Therefore, limits like $\lim\limits_{x\to a} \sin^{-1}(x)$ where $a \notin [-1, 1]$ are undefined.

- **Why does**

  $$\lim_{x\to 2} \tan^{-1} \frac{1}{x-2}.$$

$$\begin{aligned} &= \lim_{x\to 2^-} \tan^{-1} \frac{1}{x-2} \\ &= \lim_{x\to -\infty} \tan^{-1} x \\ &= -\pi/2. \end{aligned}$$

$$\begin{aligned} &= \lim_{x\to 2^+} \tan^{-1} \frac{1}{x-2} \\ &= \lim_{x\to +\infty} \tan^{-1} x \\ &= \frac{\pi}{2}. \end{aligned}$$

1.4 Chapter 5 Definitions and Theorems

- **Sequence notation**

$$\{a_n\}_{n=1}^{\infty}, \text{ or simply } \{a_n\}.$$

- **Sequence notation (ordered list)**

$$a_1,\ a_2,\ a_3,\ \cdots,\ a_n,\ \cdots.$$

- **Arithemetic Sequence Difference**

$$d = a_n - a_{n-1}.$$

- **Arithmetic sequence (common difference between subsequent terms) general form**

$$\begin{aligned}&\text{Index starting at } 0: \ a_n = a + nd\\ &\text{Index starting at } 1: \ a_n = a + (n-1)d\end{aligned}$$

.

- **Arithmetic sequence (common difference between subsequent terms) recursive form**

$$a_n = a_{n-1} + d.$$

- **Sum of arithmetic sequence**

$$\begin{aligned}S_n &= \frac{n}{2}\left[a + a_n\right]\\ S_n &= \frac{n}{2}\left[2a + (n-1)d\right].\end{aligned}$$

- **Geometric sequence form common ratio**

$$r = \frac{a_n}{a_{n-1}}.$$

- **Geometric sequence general form**

$$\begin{aligned}a_n &= ar^n \text{ (Index starting at 0)}\\ a_n &= a^{n+1}\text{(index starting at 0 and a=r)}\\ a_n &= ar^{n-1} \text{ (Index starting at 1)}\\ a_n &= a^n\text{(index starting at 1 and a=r)}.\end{aligned}$$

- **Geometric sequence recursive form**

$$a_n = ra_{n-1}.$$

- **Sum of geometric sequence (finite terms)**

$$S_n = \frac{a(1-r^n)}{1-r} \quad r \neq 1.$$

- **Convergence / Divergence**: If
$$\lim_{n\to+\infty} a_n = L.$$
We say that the sequence converges, else it diverges

- **Formal definition of limit of sequence**
$$\lim_{n\to+\infty} a_n = L \iff \forall\varepsilon > 0, \exists N \in \mathbb{Z} \mid |a_n - L| < \varepsilon, \text{ if } n \geqslant n.$$
Then we can say
$$\lim_{n\to+\infty} a_n = L \text{ or } a_n \to L.$$

- **Limit of a sequence defined by a function**: Consider a sequence $\{a_n\}$ such that $a_n = f(n)$ for all $n \geqslant 1$. If there exists a real number L such that
$$\lim_{x\to\infty} f(x) = L,$$
then $\{a_n\}$ converges and
$$\lim_{n\to\infty} a_n = L.$$

- **Algebraic limit laws**: Given sequences $\{a_n\}$ and $\{b_n\}$ and any real number c, if there exist constants A and B such that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then
  - $\lim_{n\to\infty} c = c$
  - $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n = cA$
  - $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = A \pm B$
  - $\lim_{n\to\infty}(a_n \cdot b_n) = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n) = A \cdot B$
  - $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{A}{B}$, provided $B \neq 0$ and each $b_n \neq 0$.

- **Continuous Functions Defined on Convergent Sequences**: Consider a sequence $\{a_n\}$ and suppose there exists a real number L such that the sequence $\{a_n\}$ converges to L. Suppose f is a continuous function at L. Then there exists an integer N such that f is defined at all values a_n for $n \geqslant N$, and the sequence $\{f(a_n)\}$ converges to $f(L)$.

- **Squeeze Theorem for Sequences**: Consider sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$. Suppose there exists an integer N such that
$$a_n \leqslant b_n \leqslant c_n \text{ for all } n \geqslant N.$$
If there exists a real number L such that
$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n,$$
then $\{b_n\}$ converges and $\lim_{n\to\infty} b_n = L$

- **Bounded above**: A sequence $\{a_n\}$ is bounded above if there exists a real number M such that
$$a_n \leqslant M$$
for all positive integers n.

- **Bounded below**: A sequence $\{a_n\}$ is bounded below if there exists a real number M such that
$$M \leqslant a_n$$
for all positive integers n.

- **Bounded**: A sequence $\{a_n\}$ is a bounded sequence if it is bounded above and bounded below.
- **Unbounded**: If a sequence is not bounded, it is an unbounded sequence.
- **If a sequence $\{a_n\}$ converges, then it is bounded.**
- **Increasing sequence**: A sequence $\{a_n\}$ is increasing for all $n \geqslant n_0$ if

$$a_n \leqslant a_{n+1} \text{ for all } n \geqslant n_0.$$

- **Decreasing sequence**: A sequence $\{a_n\}$ is decreasing for all $n \geqslant n_0$ if

$$a_n \geqslant a_{n+1} \text{ for all } n \geqslant n_0.$$

- **Monotone sequence**: A sequence $\{a_n\}$ is a **monotone sequence** for all $n \geqslant n_0$ if it is increasing for all $n \geqslant n_0$ or decreasing for all $n \geqslant n_0$
- **Monotone Convergence Theorem**: If $\{a_n\}$ is a bounded sequence and there exists a positive integer n_0 such that $\{a_n\}$ is monotone for all $n \geqslant n_0$, then $\{a_n\}$ converges.
- **Infinite Series form:**

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots ..$$

- **Partial sum (k^{th} partial sum)**

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k.$$

- **Convergence of infinity series notation**

  For a series, say...

$$\sum_{n=1}^{\infty} a_n \ .$$

  its convergence is determined by the limit of its sequence of partial sums. Specifically, if

$$\lim_{n\to+\infty} S_n = S \to \sum_{n=1}^{\infty} a_n \ = S.$$

- **Harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \ .$$

  Which diverges to $+\infty$

- **Algebraic Properties of Convergent Series** Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series. Then the following algebraic properties hold:

  1. The series $\sum_{n=1}^{\infty}(a_n + b_n)$ converges and

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n. \quad \text{(Sum Rule).}$$

2. The series $\sum_{n=1}^{\infty}(a_n - b_n)$ converges and

$$\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n. \quad \text{(Difference Rule).}$$

3. For any real number c, the series $\sum_{n=1}^{\infty} ca_n$ converges and

$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n. \quad \text{(Constant Multiple Rule).}$$

- **Geometric series convergence or divergence:**

$$\sum_{n=1}^{\infty} ar^{n-1} \quad = \quad \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ \mathit{diverges} & \text{if } |r| \geqslant 1 \end{cases}.$$

- **Divergence test**: In the context of sequences, if $\lim_{n\to\infty} a_n = c \neq 0$ or the limit does not exist, then the series $\sum_{n=1}^{\infty} a_n$ is said to diverge. The converse is not true.

  Because:

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty}(S_k - S_{k-1}) = \lim_{k\to\infty} S_k - \lim_{k\to\infty} S_{k-1} = S - S = 0..$$

- **Integral Test Prelude**: for any integer k, the kth partial sum S_k satisfies

$$S_k = a_1 + a_2 + a_3 + \cdots + a_k < a_1 + \int_1^k f(x)\,dx < a_1 + \int_1^{\infty} f(x)\,dx..$$

  and

$$S_k = a_1 + a_2 + a_3 + \cdots + a_k > \int_1^{k+1} f(x)\,dx..$$

- **Intgeral test** Suppose $\sum\limits_{n=1}^{\infty} a_n$ is a series with positive terms a_n Suppose there exists a function f and a positive integer N such that the following three conditions are satisfied:

  1. f positive, continuous, and decreasing on $[N, \infty)$
  2. $f(n) = a_n$ for all integers $n \geqslant N$, $N \in \mathbb{Z}^+$

  Then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ and the improper integral $\displaystyle\int_N^{\infty} f(x)\,dx$ either both converge or both diverge..

- **P-series** $\forall p \in \mathbb{R}$, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^P}.$$

  Is called a **p-series**. Furthermore,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leqslant 1. \end{cases}.$$

- **P-series extended**

$$\sum_{n=2}^{\infty} \frac{1}{n \ln{(n)}^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leqslant 1. \end{cases} .$$

- **Remainder estimate for the integral test** Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series with positive terms. Suppose there exists a function f and a positive integer M satisfying the following three conditions:

  1. f is positive, decreasing, and continuous on $[M, \infty)$
  2. $f(n) = a_n$ for all integers $n \geqslant M$.

  Let S_N be the Nth partial sum of $\sum_{n=1}^{\infty} a_n$. For all positive integers N,

$$S_N + \int_{N+1}^{\infty} f(x)\,dx < \sum_{n=1}^{\infty} a_n < S_N + \int_{N}^{\infty} f(x)\,dx.$$

  In other words, the remainder $R_N = \sum_{n=1}^{\infty} a_n - S_N = \sum_{n=N+1}^{\infty} a_n$ satisfies the following estimate:

$$\int_{N+1}^{\infty} f(x)\,dx < R_N < \int_{N}^{\infty} f(x)\,dx.$$

  This is known as the remainder estimate

  To find a value of N such that we are withing a desired margin of error, Since we know $R_n < \int_N^{\infty} f(x)\ dx$. Simply compute the improper integral and set the result < the desired error to solve for N

- **Find a_n given the expression for the partial sum**

$$a_n = S_n - S_{n-1}.$$

- **telescoping series**: Telescoping series are a type of series where each term cancels out a part of another term, leaving only a few terms that do not cancel. When you sum the series, most of the terms collapse or "telescope," which simplifies the calculation of the sum. Here are some key points and generalizations you can note about telescoping series:

  – Partial Fraction Decomposition
  – Cancellation Pattern: In a telescoping series, look for a pattern where a term in one fraction will cancel out with a term in another fraction.
  – Write out Terms
  – What is left is S_n, thus the sum of the series is the $\lim\limits_{n\to\infty} S_n$

  Try:

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1} .$$

  Hint, its not only the first and last terms cancel, we also have a $\frac{\frac{1}{2}}{n}$, when a_{n-1}: Answer is $\frac{3}{4}$

- **Comparison test for series**

  1. Suppose there exists an integer N such that $0 \leqslant a_n \leqslant b_n$ for all $n \geqslant N$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
  2. Suppose there exists an integer N such that $a_n \geqslant b_n \geqslant 0$ for all $n \geqslant N$. If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

- **Limit Comparison Test** Let $a_n, b_n \geqslant 0$ for all $n \geqslant 1$.

  – If $\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.

  – If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

  – If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

  Note: Note that if $\frac{a_n}{b_n} \to 0$ and $\sum_{n=1}^{\infty} b_n$ diverges, the limit comparison test gives no information. Similarly, if $\frac{a_n}{b_n} \to \infty$ and $\sum_{n=1}^{\infty} b_n$ converges, the test also provides no information.

  Consider the series

  $$\sum_{n=1}^{\infty} \frac{n^4+6}{n^5+4} \, .$$

  To find our b_n we can only focus on the leading coefficients. Thus:

  $$b_n = \frac{n^4}{n^5} = \frac{1}{n}.$$

  So our test...

  $$\begin{aligned} &\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\frac{n^4+6}{n^5+4}}{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{n(n^4+6)}{n^5+4} \\ &= \lim_{n\to\infty} \frac{n^5+6n}{n^5+4} \\ &= 1. \end{aligned}$$

  Since $\lim\limits_{n\to\infty} \frac{a_n}{b_n} \neq 0 \vee +\infty$. And $\frac{1}{n}$ diverges, we can conclude that a_n will also diverge.

- **Determine which series (or function) is greater**

  – **Subtraction**: Given two functions $f(x) = \frac{1}{x}$ and $g(x) = \frac{x^4+6}{x^5+4}$, we want to compare them by considering the function $h(x) = f(x) - g(x)$:

  $$h(x) = f(x) - g(x) = \frac{1}{x} - \frac{x^4+6}{x^5+4}$$

  To compare these directly, it would be helpful to have a common denominator:

  $$h(x) = \frac{x^4+4-(x^4+6)}{x(x^5+4)} = \frac{-2}{x(x^5+4)}$$

  Now, we can see that the sign of $h(x)$ depends on the sign of x because the denominator $x(x^5+4)$ is always positive for $x \neq 0$. So:

  * For $x > 0$, $h(x) < 0$, which means $f(x) < g(x)$.
  * For $x < 0$, $h(x) > 0$, which means $f(x) > g(x)$.

- **Alternating Series** Any series whose terms alternate between positive and negative values is called an alternating series. An alternating series can be written in the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n = b_1 - b_2 + b_3 - b_4 + \cdots .$$

or

$$\sum_{n=1}^{\infty}(-1)^{n}b_n = -b_1 + b_2 - b_3 + b_4 - \cdots .$$

Where $b_n > 0$ for all positive integers n.

- **alternating series test (Leibniz criterion)** An alternating series of the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^{n}b_n$$

converges if

  - $0 < b_{n+1} \leqslant b_n \ \forall \ n \geqslant 1$
  - $\lim_{n\to\infty} b_n = 0$.

Note: We remark that this theorem is true more generally as long as there exists some integer N such that $0 < b_{n+1} \leqslant b_n$ for all $n \geqslant N$.

Additional note: The AST allows us to consider just the positive terms to check for these two conditions because if a series of decreasing positive terms that approach zero is alternated in sign, the alternating series will converge. This is a special property of alternating series that does not generally hold for non-alternating series.

- **Show decreasing (For the AST)**: Consider the series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} .$$

So you see we have $b_n = \frac{1}{n^2}$. For the AST, we must show that this is decreasing. If $b_{n+1} = \frac{1}{(n+1)^2}$. Then we see

$$\frac{1}{(n+1)^2} < \frac{1}{n^2}.$$

Thus it is decreasing for $n \geqslant 1$ $(b_{n+1} < b_n)$ ■

- **Remainders in alternating series** Consider an alternating series of the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^{n}b_n,$$

that satisfies the hypotheses of the alternating series test. Let S denote the sum of the series and S_N denote the N-th partial sum. For any integer $N \geqslant 1$, the remainder $R_N = S - S_N$ satisfies

$$|R_N| \leqslant b_{N+1}.$$

This tells us that if we stop at the N^{th} term, the error we are making is at most the size of the next term

- **Absolute and conditional convergence**
  - A series $\sum_{n=1}^{\infty} a_n$ exhibits absolute convergence if $\sum_{n=1}^{\infty} |a_n|$ converges.
  - A series $\sum_{n=1}^{\infty} a_n$ exhibits conditional convergence if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.
  - If $\sum\limits_{n=1}^{\infty} |a_n|$ converges then $\sum\limits_{n=1}^{\infty} a_n$ converges

  Note: if $|a_n|$ diverges, we cannot have absolute convergence, thus we must examine to see if normal a_n converges, in which case we would have conditional convergence

  Big Note: If a series not strictly decreasing, we can still check for absolute/conditional convergence. Take $\sum\limits_{n=1}^{\infty} \frac{\sin n}{3^n+4}$ for example.

- **Ratio test** Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms. Let

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|..$$

  Then:

  i. If $0 \leqslant \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
  ii. If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
  iii. If $\rho = 1$, the test does not provide any information.

  Note: The ratio test is useful for series whose terms involve factorials

- **Root test** Consider the series $\sum_{n=1}^{\infty} a_n$. Let

$$\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}..$$

  i. If $0 \leqslant \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
  ii. If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
  iii. If $\rho = 1$, the test does not provide any information.

  Note: The root test is useful for series whose terms involve exponentials

- **Which tests require positive terms**
  - The **Integral Test:** This test applies to series where the terms come from a function that is positive, continuous, and decreasing on a certain interval. The convergence or divergence of the series is determined by the convergence or divergence of the corresponding improper integral of the function.
  - The **Remainder estimate for the integral test**
  - The **Comparison Test:** This test compares the terms of a series to those of another series with known convergence behavior. It requires that the terms of both series be positive or non-negative.
  - The **Limit Comparison Test:** Similar to the Comparison Test, this test involves comparing the terms of two series by taking the limit of the ratio of their terms. It requires that the terms of both series be positive.
  - In **alternating series**, b_n must have only positive terms

1.5 Chapter 6 Definitions and Theorems

- **Euler definition for e**

$$e^a = \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n$$
$$\frac{1}{e^a} = \lim_{n\to\infty}\left(1+\frac{-a}{n}\right)^n$$
$$\frac{1}{e^a} = \lim_{n\to\infty}\left(\frac{n}{n+a}\right)^n.$$

- **Other definition for e**

$$e = \sum_{n=0}^{\infty}\frac{1}{n!}$$
$$e-1 = \sum_{n=1}^{\infty}\frac{1}{n!}$$
$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}\ .$$

- **Power series**: A series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots .$$

  is a power series centered at $x = 0$.

  A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots .$$

  is a power series centered at $x = a$.

- **Convergence of a Power Series**

  Consider the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$. The series satisfies exactly one of the following properties:

  (i) The series converges at $x = a$ and diverges for all $x \neq a$.
  (ii) The series converges for all real numbers x.
  (iii) There exists a real number $R > 0$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$. At the values x where $|x-a| = R$, the series may converge or diverge.

- **A power series always converges at its center**

- **Radius of convergence**: Consider the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$. The set of real numbers x where the series converges is the interval of convergence. If there exists a real number $R > 0$ such that the series converges for $|x-a| < R$ and diverges for $|x-a| > R$, then R is the radius of convergence. If the series converges only at $x = a$, we say the radius of convergence is $R = 0$. If the series converges for all real numbers x, we say the radius of convergence is $R = \infty$ (Figure 6.2).

- **Finding interval of convergence and radius of convergence**
  - Fact: power series is always convergent on its center
  - Use ratio test (values of ρ)
  - Use $\rho < 1$ to find Radius of convergence
  - Test end points of interval by plugging into original series and seeing whether the series is convergent or divergent
- **If $\rho = 0$, the power series converges for all x**
- **If** $\rho = \infty$, the series diverges for all $x \neq a$
- **Combining Power Series:** Suppose that the two power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to the functions f and g, respectively, on a common interval I.

  (i) The power series $\sum_{n=0}^{\infty}(c_n x^n \pm d_n x^n)$ converges to $f \pm g$ on I.

  (ii) For any integer $m \geqslant 0$ and any real number b, the power series $\sum_{n=0}^{\infty} bx^m c_n x^n$ converges to $bx^m f(x)$ on I.

  Eg: If we know $\sum\limits_{n=0}^{\infty} \ a_n x_n$ has $I = (-1, 1)$. Then

$$\begin{aligned} &\sum_{n=0}^{\infty} \ a_n 3^n x^n \\ =&\sum_{n=0}^{\infty} \ a_n (3x)^n \\ &I = (-3, 3). \end{aligned}$$

  (iii) For any integer $m \geqslant 0$ and any real number b, the series $\sum_{n=0}^{\infty} c_n (bx^m)^n$ converges to $f(bx^m)$ for all x such that bx^m is in I.

- **For part I, II, and III, the interval of the combined series is the smaller interval**
- **Cauchy product (Multiplying power series)**: Suppose that the power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to f and g, respectively, on a common interval I. Let

$$\begin{aligned} e_n &= c_0 d_n + c_1 d_{n-1} + c_2 d_{n-2} + \cdots + c_{n-1} d_1 + c_n d_0 \\ &= \sum_{k=0}^{n} c_k d_{n-k}.. \end{aligned}$$

  Then

$$\left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=0}^{\infty} d_n x^n\right) = \sum_{n=0}^{\infty} e_n x^n$$

  and

$$\sum_{n=0}^{\infty} e_n x^n \text{ converges to } f(x) \cdot g(x) \text{ on } I.$$

  The series $\sum_{n=0}^{\infty} e_n x^n$ is known as the Cauchy product of the series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$.

- **Sterling's Approximation**

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

- **Gamma function (extension of the factorial function)**

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,dt$$

Thus, $n! = \Gamma(n+1)$.

- **Cool definition for e^x**

$$f'(x) = rf(x)$$
$$\implies f(x) = ce^{rx}.$$

- **Term-by-Term Differentiation and Integration for Power Series.** Suppose that the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges on the interval $(a-R, a+R)$ for some $R > 0$. Let f be the function defined by the series

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

for $|x-a| < R$. Then f is differentiable on the interval $(a-R, a+R)$ and we can find f' by differentiating the series term-by-term:

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

for $|x-a| < R$. Also, to find $\int f(x)\,dx$, we can integrate the series term-by-term. The resulting series converges on $(a-R, a+R)$, and we have

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1} = C + c_0(x-a) + \frac{c_1(x-a)^2}{2} + \frac{c_2(x-a)^3}{3} + \cdots$$

for $|x-a| < R$.

NOTE! when a power series is differentiated or integrated term-by-term, it says nothing about what happens at the endpoints.

- **Uniqueness of Power Series**: Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ and $\sum_{n=0}^{\infty} d_n(x-a)^n$ be two convergent power series such that

$$\sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} d_n(x-a)^n$$

for all x in an open interval containing a. Then $c_n = d_n$ for all $n \geqslant 0$.

- **When finding the Cauchy product of two power series, we include the zero term when finding the new power series general term. For integrating and differentiating, we do not**

- **Taylor and Maclaurin series**: If f has derivatives of all orders at $x = a$, then the Taylor series for the function f at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots. \quad (16)$$

The Taylor series for f at 0 is known as the Maclaurin series for f.

- **Uniqueness of Taylor series**: If a function f has a power series at a that converges to f on some open interval containing a, then that power series is the Taylor series for f at a.

- **Taylor-Macluarin Polynomials**: If f has n derivatives at $x = a$, then the nth Taylor polynomial for f at a is

$$p_n(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n. \quad (17)$$

The nth Taylor polynomial for f at 0 is known as the nth Maclaurin polynomial for f.

- **Taylor's Theorem with Remainder**: Let f be a function that can be differentiated $n+1$ times on an interval I containing the real number a. Let p_n be the nth Taylor polynomial of f at a and let

$$R_n(x) = f(x) - p_n(x)$$

be the nth remainder. Then for each x in the interval I, there exists a real number c between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

If there exists a real number M such that $\left|f^{(n+1)}(x)\right| \leqslant M$ for all $x \in I$, then

$$|R_n(x)| \leqslant \frac{M}{(n+1)!}\left|x-a\right|^{n+1}$$

$\forall\ x \in I$

- **Maclaurin Series/Polynomials for sine**: The Taylor series for the sine function is

$$\sin(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} \ = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \quad \text{For } x \in \mathbb{R}.$$

Where p_n obeys

$$\begin{aligned} p_{2m+1} &= p_{2m+2} \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... + \frac{(-1)^m x^{2m+1}}{(2m+1)!}. \end{aligned}$$

Note: When discussing specific polynomials, say P_5 for example, we arnt talking about the first 5 terms in the series above, we are talking about the polynomial `up to` degree 5. Thus it would have 3 terms

- **Maclaurin Series/Polynomials for cosine**: Similar to the sine function, the Maclaurin series for the cosine function is

$$\cos(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \ = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \quad \text{For } x \in \mathbb{R}.$$

Where p_n obeys

$$\begin{aligned} p_{2m} &= p_{2m+1} \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... + (-1)^n\frac{x^{2m}}{(2m)!}. \end{aligned}$$

- **Maclaurin Series/Polynomials for e^x**: We find the Maclaurin series for the exponential function to be

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \ = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \quad \text{For } x \in \mathbb{R}.$$

Note: this definition is described above but now we have a way of showing its truthiness

- **Convergence of Taylor Series**: Suppose that f has derivatives of all orders on an interval I containing a. Then the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

converges to $f(x)$ for all x in I if and only if

$$\lim_{n\to\infty} R_n(x) = 0$$

for all x in I.

Note: With this theorem, we can prove that a Taylor series for f at a converges to f if we can prove that the remainder $R_n(x) \to 0$. To prove that $R_n(x) \to 0$, we typically use the bound

$$|R_n(x)| \leqslant \frac{M}{(n+1)!}|x-a|^{n+1}$$

from Taylor's theorem with remainder.

- **Using taylor series to find limits**: Consider the limit $\lim_{x\to 0+} \frac{\cos\sqrt{x}-1}{2x}$. We know we have a problem if we attempt to use the `direct substitution property`. Thus, we can substitute $\cos(\sqrt{x})$ for its `Maclaurin series` and see what happens. We know the maclaurin series for $\cos(x)$ is

$$\begin{aligned}
\cos x &\sim \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \;= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \\
\implies \cos\sqrt{x} &\sim \sum_{n=0}^{\infty} (-1)^n \frac{(x^{\frac{1}{2}})^{2n}}{(2n)!} \;= 1 - \frac{(x^{\frac{1}{2}})^2}{2!} + \frac{(x^{\frac{1}{2}})^4}{4!} - \frac{(x^{\frac{1}{2}})^6}{6!} + ... \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n}}{(2n)!} \;= 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + ...
\end{aligned}$$

So we have

$$\begin{aligned}
&\lim_{x\to 0^+} \frac{\left(1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + ...\right) - 1}{2x} \\
&= \lim_{x\to 0^+} \frac{\left(-\frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + ...\right)}{2x} \\
&= \lim_{x\to 0^+} \left(-\frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + ...\right) \cdot \frac{1}{2x} \\
&= \lim_{x\to 0^+} -\frac{1}{4} \\
&= -\frac{1}{4}.
\end{aligned}$$

- **Multiplying a known Taylor series by some other function:** Consider $f(x) = x\cos x$. Since we know that the taylor series for $\cos(x) = \sum\limits_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, which converges $\forall\ x \in \mathbb{R}$. We can easily just multiply this by x to get the taylor series for $f(x) = x\cos(x)$. Since the Taylor series for $\cos(x)$ converges for all real x, multiplying it by x won't affect its convergence properties. The resulting series will still converge for all x.

Note: The product of the Taylor series and the function will be valid only where both the series converges and the function is well-defined. Probably analyze the convergence of the product series.

- **Multiplying a known Taylor series by some other function where convergence is affected:** Consider the example above. Although this time suppose we multiply $\cos(x)$ by $\frac{1}{x}$ instead of x. We know the resulting taylor series must not be convergent at $x = 0$ because $\frac{1}{x}$ is not defined at zero. It is important to understand that we will not get this conclusion from the ratio test alone. The Ratio Test alone does not account for points where the series or its terms are not defined.

TLDR: Be mindful about the domain of the function you are multiplying the Taylor series by. Do not only rely on the ratio test to find points of convergence.

- **Analytic function:**
  - An analytic function is infinitely differentiable within its domain.
  - An analytic function can be represented by a convergent power series (like a Taylor series) around any point in its domain.
  - The power series representing an analytic function not only exists but also converges to the function within a certain radius around the point of expansion

- **Maclaurin series for $\frac{1}{1-x}$:**

$$\frac{1}{1-x} \sim \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1.$$

- **Maclaurin series for $\ln(1+x)$:**

$$\ln(1+x) \sim \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \quad \text{for } |x| < 1.$$

- **Maclaurin series for $\tan^{-1} x$:**

$$\tan^{-1} x \sim \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{for } |x| \leqslant 1.$$

- **Binomial expansion for $(1+x)^r$ for $r \in \mathbb{Z}^+$**

$$(1+x)^r = \sum_{n=0}^{r} \binom{r}{n} x^n \quad r \in \mathbb{Z}^+.$$

- **Binomial expansion for $(1+x)^r$ for $r \in \mathbb{R}$**

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!}x^n + \cdots .$$

Where

$$\binom{r}{n} = \frac{f^{(n)}(0)}{n!} = \frac{r(r-1)(r-2)\cdot \ldots \cdot (r-n+1)}{n!}.$$

Note: When $n = 0$, $\binom{r}{0} = 1$, when $n = 1$, $\binom{r}{1} = r$, when $n = 2$, $\binom{r}{2} = \frac{r(r-1)}{2!}$, etc...

- **The binomial theorem:** For any real number r, the Maclaurin series for $f(x) = (1+x)^r$ is the binomial series. It converges to f for $|x| < 1$, and we write

  $$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!}x^n + \cdots$$

  for $|x| < 1$.

1.6 Chapter 6 Problems to Remember

- **Problem to remember (Properties of power series):** Evaluate the infinite series by identifying it as the value of an integral of a geometric series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \ .$$

Remark. If we can find which geometric power series's integral (with some bounds) gives us the given series, we can then integrate the function representation to get the value of the original series. Consider the geometric power series

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n \ = \sum_{n=0}^{\infty} (-1)^n x^{2n} \ .$$

Suppose we then integrate the power series

$$\begin{aligned} &\int \sum_{n=0}^{\infty} (-1)^n x^{2n} \ dx \\ &= \sum_{n=0}^{\infty} (-1)^n \int x^{2n} \ dx \\ &= \frac{1}{2n+1} x^{2n+1}. \end{aligned}$$

Now we must deduce for which bounds will the FTC give us the original series $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$. We come to the conclusion

$$\sum_{n=0}^{\infty} (-1)^n \int_0^1 x^{2n} \ dx = \sum_{n}^{\infty} \frac{(-1)^n}{2n+1} \ .$$

This implies we can integrate the function representation of the geometric power series we just integrated to get the value of the infinite series $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$. Thus,

$$\int_0^1 \frac{1}{1+x^2} \ dx = \frac{\pi}{4}.$$

- **Problem to remember (Properties of power series):** Find the power series for $f(x) = \ln x$ centered at $x = 9$ by using term-by-term integration or differentiation.

Solution. The goal is to find a function that resembles one we know (sum of geometric series $\frac{a}{1-x}$) such that if we integrate or differentiate we can get $\ln x$. Since we know the integral of $\frac{1}{x}$ is $\ln x$, and we can easily manipulate $\frac{1}{x}$ to be in the form $\frac{a}{1-x}$, we choose $\frac{1}{x}$ to be the function to examine. Thus,

$$\frac{1}{x} = \frac{1}{9+x-9} = \frac{1}{9-(-(x-9))} = \frac{1/9}{1-\left(\frac{-(x-9)}{9}\right)}$$

$$\text{If } f(x) = \frac{a}{1-x} \sim \sum_{n=0}^{\infty} a(x^n) \ = a + ax + ax^2 + ax^3 + ...$$

$$\implies f(x) = \frac{1/9}{1-\left(\frac{-(x-9)}{9}\right)} \sim \sum_{n=0}^{\infty} \frac{1}{9} \left(\frac{-(x-9)}{9}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n (x-9)^n}{9^{n+1}} \ .$$

Then we can throw in some integrals

$$\int f(x)\ dx = \int \frac{1/9}{1-\left(\frac{-(x-9)}{9}\right)}\ dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n (x-9)^n}{9^{n+1}}\ dx$$

$$\ln x = \sum_{n=0}^{\infty} \frac{(-1)^n}{9^{n+1}} \int (x-9)^n\ dx$$

$$\ln x = \sum_{n=0}^{\infty} \frac{(-1)^n (x-9)^{n+1}}{(n+1)9^{n+1}} + C.$$

If we let $x = 9$

$$\ln 9 = \sum_{n=0}^{\infty} \frac{(-1)^n (9-9)^{n+1}}{(n+1)9^{n+1}} + C$$

$$\ln 9 = C.$$

Thus, we have

$$f(x) = \ln 9 + \sum_{n=0}^{\infty} \frac{(-1)^n (9-9)^{n+1}}{(n+1)9^{n+1}}$$

.

Note: the "$+C$" is initially omitted from $\ln(x)$ because we're considering a specific antiderivative. When you integrate the power series, you include "$+C$" to account for the general form of the antiderivative. The value of C is then determined using a specific condition to match the specific antiderivative you're interested in.

- **Problem to remember**: Say we want to find the power series for $7x \ln 1 + x$. We can first find the power series for $\ln(1+x)$

$$\frac{d}{dx} \ln(1+x) = \frac{1}{1+x} = \frac{1}{1-(-x)}.$$

We know the power series for $\frac{1}{1-(-x)}$ is

$$\sum_{n=0}^{\infty} (-1)^n x^n$$

$$\implies \int \frac{1}{1+x} = \int \sum_{n=0}^{\infty} (-1) x^n$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}\ .$$

Now that we have found the power series for $\ln(1+x)$, to find the power series for $7x \ln(1+x)$...

$$7x \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$$\sum_{n=0}^{\infty} 7x \left((-1)^n \frac{x^{n+1}}{n+1} \right)$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{7x^{n+2}}{n+1}\ .$$

- **Problem to remember (Cumbersome taylor polynomial)**: Suppose we have some function f, and we would like to find the Taylor polynomial up to degree 3. Say $f(x) = e^{2x}\cos(x)$. We could find each derivative up to degree 3, however, given that we know the taylor series for both e^{2x} and $\cos(x)$.

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + ...$$
$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ...$$

We can find $P_3(x)$ by multiplying these Taylor series. Thus,

$$(1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + ...)(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ...).$$

And we can find P_3 to be

$$P_3 = 1 + 2x + \frac{3}{2}x^2 + \frac{1}{3}x^3.$$

- **Problem to Remember (Using known Taylor series to find sum of series):** Consider the series $\sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{25}\right)^{n-3}}{2n+1}$

We notice this resembles the Taylor series for the arctangent function $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ for $|x| \leqslant 1$. Thus, we maipulate the series to better conform to the Taylor series for $\tan^{-1} x$.

$$\begin{aligned}
&\sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{25}\right)^{n-3}}{2n+1} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{5}\right)^{2n-6}}{2n+1} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{5}\right)^{2n} 5^6}{2n+1} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{5}\right)^{2n} 5^6 \left(\frac{1}{5}\right) 5}{2n+1} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{5}\right)^{2n+1} 5^7}{2n+1}
\end{aligned}$$
.

Thus the sum will be

$$5^7 \tan^{-1} \frac{1}{5}.$$

Vectors

2.1 The Vocabulary of Vectors

- A **vector** is two pieces of information.
  1. Length
  2. Direction (Magnitude)

2.2 Vector Notation

- **Defining a vector**

$$\vec{v} = [x, y] \text{ or } \begin{bmatrix} x \\ y \end{bmatrix}.$$

- **Length of a vector**

$$||\vec{v}|| \in \mathbb{R}^{\ltimes} = \sqrt{v_1^2 + v_2^2 + ... + v_n^2}.$$

- **Vector addition** Suppose we have two vectors $\vec{v} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$. Then

$$\vec{v} + \vec{u} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} = \vec{c}.$$

  Note: we call this new vector the **resultant**

- **Multiplying by a scalar** Suppose we have the vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$. Then

$$2\vec{v} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}.$$

- **Vector subtraction**

$$\vec{v} - \vec{u} = \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \end{bmatrix}.$$

- **Vector in three dimensions**

$$\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

- **Definition for $\mathbb{R}^n$:** $\mathbb{R}^n$ is the set of all $n - tuples$ of real numbers

$$\begin{aligned} &\vec{v} = [v_1, v_2] \ \vec{v} \in \mathbb{R}^2 \\ &\vec{u} = [u_1, u_2, u_3] \ \vec{u} \in \mathbb{R}^3 \\ &\vec{w} = [w_1, w_2, w_3, w_n] \ \vec{w} \in \mathbb{R}^n \\ &. \end{aligned}$$

Fundemental Physics: Classical Mechanics

3.1 Chapter 1: Units and Measurement

3.1.1 Key Terms

- **Physics**, which comes from the Greek *phúsis*, meaning "nature," is concerned with describing the interactions of energy, matter, space, and time to uncover the fundamental mechanisms that underlie every phenomenon.
- A **model** is a representation of something that is often too difficult (or impossible) to display directly. Although
- a **theory** is a testable explanation for patterns in nature supported by scientific evidence and verified multiple times by various groups of researchers.
- A **law** uses concise language to describe a generalized pattern in nature supported by scientific evidence and repeated experiments. Often, a law can be expressed in the form of a single mathematical equation.
- **SI units** (for the French Système International d'Unités), also known as the **metric system**.
- The Metric system may also be referred to as the **centimeter–gram–second (cgs)** system.
- **English units** (also known as the customary or imperial system). English units were historically used in nations once ruled by the British Empire and are still widely used in the United States.
- The English system may also be referred to as the **the foot–pound–second (fps)** system.
- In any system of units, the units for some physical quantities must be defined through a measurement process. These are called the **base quantities** for that system
- The base quantities units are the system's **base units**
- All other physical quantities can then be expressed as algebraic combinations of the base quantities. Each of these physical quantities is then known as a **derived quantity** and each unit is called a **derived unit.**
- A **conversion factor** is a ratio that expresses how many of one unit are equal to another unit.
- The **dimension** of any physical quantity expresses its dependence on the base quantities as a product of symbols (or powers of symbols) representing the base quantities.
- Any quantity with a dimension that can be written so that all seven powers are zero (that is, its dimension is $L^0M^0T^0I^0\Theta^0N^0J^0$) is called **dimensionless** (or sometimes "of dimension 1," because anything raised to the zero power is one).
- Physicists often call dimensionless quantities **pure numbers**.

- The importance of the concept of dimension arises from the fact that any mathematical equation relating physical quantities must be **dimensionally consistent**
  - Every term in an expression must have the same dimensions; it does not make sense to add or subtract quantities of differing dimension (think of the old saying: “You can’t add apples and oranges”). In particular, the expressions on each side of the equality in an equation must have the same dimensions.
  - The arguments of any of the standard mathematical functions such as trigonometric functions (such as sine and cosine), logarithms, or exponential functions that appear in the equation must be dimensionless. These functions require pure numbers as inputs and give pure numbers as outputs.
- **Estimation** means using prior experience and sound physical reasoning to arrive at a rough idea of a quantity’s value.
- **Accuracy** is how close a measurement is to the accepted reference value for that measurement.
- The **precision of measurements** refers to how close the agreement is between repeated independent measurements (which are repeated under the same conditions).
- The precision of a measuring system is related to the **uncertainty** in the measurements whereas the accuracy is related to the **discrepancy** from the accepted reference value.
- **Discrepancy** (or “measurement error”) is the difference between the measured value and a given standard or expected value.
- Another method of expressing uncertainty is as a percent of the measured value.

3.1.2 Defintions and Theorems (and important things)

- **Order of Magnitude (Method one)**

  $$m = \log_{10} l,\ \ l \in \mathbb{R}.$$

  For example

  $$\log 450 \approx 3.$$

  Thus the order of magnitude is 10^3

  Note: Always round s.t $m \in \mathbb{R}$

- **Order of Magnitude (Method two)**: We begin by writing the number in scientific notation. For example, suppose we want to find the order of magnitude for 800, we first write

  $$8 \cdot 10^2.$$

  We then check to see if the first factor is greater or less than $\sqrt{10} \approx 3$

  - If it is less than $\sqrt{10} \approx 3$, we round it down to one
  - If it is greater than $\sqrt{10} \approx 3$, we round it up to ten

  Since 8 is greater than $\sqrt{10} \approx 3$, our number becomes $10 \cdot 10^2 = 10^{2+1} = 10^3$. Thus, we have order 10^3

- The SI unit for time is the **second**

- The SI unit for length is the **meter**

- The SI unit for mass is the **kilogram**

- The SI unit for weight is the **Newton**

- 10^3 **(Megagram) is also called a *metric ton*, abbreviated** t

- **Average speed**

  $$\frac{\text{distance}}{\text{time}} = \frac{d}{t}.$$

- **Dimension notation**: Square brackets

  $$[r] = L.$$

- **Checking for dimensional consistency**: check that each term in a given equation has the same dimensions as the other terms in that equation and that the arguments of any standard mathematical functions are dimensionless.

- **Dimension for velocity**

  $$[v] = LT^{-1}.$$

- **Dimension for acceleration**

  $$[a] = LT^{-2}.$$

- **Dimensions for derivatives**
$$\left[\frac{dv}{dt}\right] = \frac{[v]}{[t]}.$$
- **Dimensions for integrals**
$$\left[\int v\, dt\right] = [v] \cdot [t]..$$
- **Volume**
$$V = AD.$$
Where A is the area and D is the depth
- **Mass**
$$M = \rho V.$$
Where ρ is the density and V is the volume
- **Density (Volume density)**
$$\rho = \frac{M}{V}.$$
Where M is the mass and V is the volume
- **Density (Area density)**
$$\rho = \frac{M}{A}.$$
Where M is the mass and A is the surface area
- **Geometric mean of bounds**
$$(order_1 \times order_2)^{0.5}.$$
- **Significant figures**:
  - All non-zero numbers are Significant.
  $$563 \to 3.$$
  - Zeros are significant if they reside between two significant figures.
  $$5002 \to 4.$$
  - Leading zeros are never significant.
  $$0.0056 \to 2.$$
  - Trailing zeros without a decimal point are not significant
  $$500 \to 1.$$
  - Trailing zeros with a decimal point are significant. The decimal point indicates that the zeros are measured and are significant.
  $$500.0 \to 4.$$
- **Percent uncertainty**: If a measurement A is expressed with uncertainty δA, the percent uncertainty is defined as
$$\text{Percent uncertainty} = \frac{\delta A}{A} \times 100\%.$$
where A is the average, and δA is the margin of error

Note: The value of the PU will take the place of the margin of error, so something like 5.1 $lbs \pm 0.3$ lbs will become $5.1 \pm 6\%$

- **We can find the margin of error (δA) by taking half of the range**
- **If the measurements going into the calculation have small uncertainties (a few percent or less), then the method of adding percents can be used for multiplication or division. This method states the percent uncertainty in a quantity calculated by multiplication or division is the sum of the percent uncertainties in the items used to make the calculation. For example, if a floor has a length of $4.00\,\mathbf{m}$ and a width of $3.00\,\mathbf{m}$, with uncertainties of 2% and 1%, respectively, then the area of the floor is $12.0\,\mathbf{m}^2$ and has an uncertainty of 3%. (Expressed as an area, this is $0.36\,\mathbf{m}^2$ $[12.0\,\mathbf{m}^2 \times 0.03]$, which we round to $0.4\,\mathbf{m}^2$ since the area of the floor is given to a tenth of a square meter.)**
- **When combining measurements with different degrees of precision with the mathematical operations of addition, subtraction, multiplication, or division, then the number of significant digits in the final answer can be no greater than the number of significant digits in the least-precise measured value. There are two different rules, one for multiplication and division and the other for addition and subtraction. There are two different rules**
  - **For multiplication and division**, the result should have the same number of significant figures as the quantity with the least number of significant figures entering into the calculation
  - **For addition and subtraction**, the answer can contain no more decimal places than the least-precise measurement.
- **If a quantity increases n%, that is the same as saying that it is multiplied by a factor of**
$$1 + \left(\frac{n}{100}\right).$$
- **If a quantity decreases n%, that is the same as saying that it is multiplied by a factor of**
$$1 - \left(\frac{n}{100}\right).$$
- **Proportional notation**: Suppose A is proportional to B, then we say
$$A\alpha B.$$
This means if B increases by some factor, then A must increase by the same factor.

  In other words, the ratio of two values of B is equal to the ratio of the corresponding values of A:
$$\frac{B_2}{B_1} = \frac{A_2}{A_1}.$$
Ex: Given the circumference formula for a circle
$$c = 2\pi r.$$
We can say
$$C\alpha r.$$
If the radius doubles, the circumference also doubles

- **Numbers that are exact (defined) have an infinite number of significant figures** because they are not measurements with any uncertainty, but rather are defined values or counts of discrete objects. For example, considering a conversion of 93.4 beats/min to beats/hour, we would compute

$$\frac{93.4\ beats}{1\ m} \cdot \frac{60\ m}{1\ h}$$
$$= 5604\ b/h.$$

  Since the number 60 is an exact number (it is a defined conversion factor), we should report the answer with three significant figures (since 93.4 has three). Thus, we round to 5600. To explicitly express 5600 with three sig figs, we write as $5.60 \cdot 10^3$

- **Explicity express sig figs with scientific notation**

$$5600 \to 2$$
$$5.60 \cdot 10^3 \to 3.$$

- **Forms of scientific notation**

$$10^{10} = 1.0 \cdot 10^{10} = 1.0e + 10.$$

- **Decimal places in additon and subtraction**: consider the example

$$501.258313 + 54.5235 + 350.257 = 906.038813.$$

  We would report the value as

$$906.039.$$

  Because the least precise measurement has three decimal points

- unless you are working on lab-related calculations (ie in your lab reports) or the question explicitly asks you to pay attention to sig figs, **you can err on the side of including extra ones**

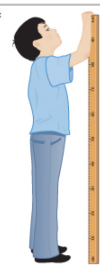
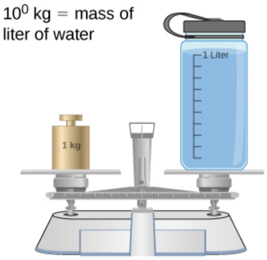
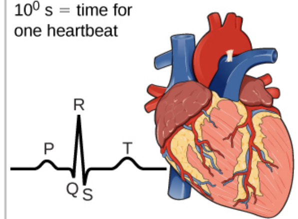
- **Weight is the measure of the force exerted on an object due to gravity. It is calculated as the mass of the object multiplied by the acceleration due to gravity.**

- **Newton**

$$N = kg \cdot \frac{m}{s^2}.$$

3.1.3 Fundemental Tables and figures

- **Known ranges of length, mass, and time**

Length in Meters (m)	Masses in Kilograms (kg)	Time in Seconds (s)
10^{-15} m = diameter of proton	10^{-30} kg = mass of electron	10^{-22} s = mean lifetime of very unstable nucleus
10^{-14} m = diameter of large nucleus	10^{-27} kg = mass of proton	10^{-17} s = time for single floating-point operation in a supercomputer
10^{-10} m = diameter of hydrogen atom	10^{-15} kg = mass of bacterium	10^{-15} s = time for one oscillation of visible light
10^{-7} m = diameter of typical virus	10^{-5} kg = mass of mosquito	10^{-13} s = time for one vibration of an atom in a solid
10^{-2} m = pinky fingernail width	10^{-2} kg = mass of hummingbird	10^{-3} s = duration of a nerve impulse
10^{0} m = height of 4 year old child 	10^{0} kg = mass of liter of water 	10^{0} s = time for one heartbeat 
10^{2} m = length of football field	10^{2} kg = mass of person	10^{5} s = one day
10^{7} m = diameter of Earth	10^{19} kg = mass of atmosphere	10^{7} s = one year
10^{13} m = diameter of solar system	10^{22} kg = mass of Moon	10^{9} s = human lifetime
10^{16} m = distance light travels in a year (one light-year)	10^{25} kg = mass of Earth	10^{11} s = recorded human history
10^{21} m = Milky Way diameter	10^{30} kg = mass of Sun	10^{17} s = age of Earth
10^{26} m = distance to edge of observable universe	10^{53} kg = upper limit on mass of known universe	10^{18} s = age of the universe

- **SI Units: Base and Derived Units**

ISQ Base Quantity	SI Base Unit
Length	meter (m)
Mass	kilogram (kg)
Time	second (s)
Electrical current	ampere (A)
Thermodynamic temperature	kelvin (K)
Amount of substance	mole (mol)
Luminous intensity	candela (cd)

- **Metric Prefixes**

Prefix	Symbol	Meaning	Prefix	Symbol	Meaning
yotta-	Y	10^{24}	yocto-	y	10^{-24}
zetta-	Z	10^{21}	zepto-	z	10^{-21}
exa-	E	10^{18}	atto-	a	10^{-18}
peta-	P	10^{15}	femto-	f	10^{-15}
tera-	T	10^{12}	pico-	p	10^{-12}
giga-	G	10^{9}	nano-	n	10^{-9}
mega-	M	10^{6}	micro-	μ	10^{-6}
kilo-	k	10^{3}	milli-	m	10^{-3}
hecto-	h	10^{2}	centi-	c	10^{-2}
deka-	da	10^{1}	deci-	d	10^{-1}

- **Base Quantity & Symbol for Dimension**

Base Quantity	Symbol for Dimension
Length	L
Mass	M
Time	T
Current	I
Thermodynamic temperature	Θ
Amount of substance	N
Luminous intensity	J

3.1.4 Memorize conversions

- **Newton to Pounds**:

$$1N = 0.225lbs.$$

- **Pounds to newtons**

$$1lb = 4.448\ N.$$

- **Length/Distance**
  - 1 inch (in) = 2.54 centimeters (cm)
  - 1 centimeter (cm) = 0.393701 inches (in)
  - 1 foot (ft) = 0.3048 meters (m)
  - 1 meter (m) = 3.28 feet (ft)
  - 1 mile (mi) = 1.6 kilometers (km)
  - 1 kilometer (km) = 0.621371 miles (mi)
- **Weight to mass or mass to weight**
  - 1 pound = 0.453592 kilograms.
  - 1 kilogram = 2.2 pounds.

3.1.5 Problems to remember

- **Restating mass**: Restate the mass 1.93×10^{13} using a metric prefix such that the resulting numerical value is bigger than one but less than 1000.

  First, we must restate in terms of grams. Since $1kg = 10^3 g$, we write

  $$1.93 \times 10^{13} \times 10^3 g$$
  $$1.93 \times 10^{16} g.$$

  Since $1Pg = 10^{15} g$, we can write

  $$1.93 \times 10^1 Pg.$$

  Since $16 - 15 = 1$

- **Unit conversion**: The distance from the university to home is 10 mi and it usually takes 20 min to drive this distance. Calculate the average speed in meters per second (m/s). (Note: Average speed is distance traveled divided by time of travel.)

  Note: There are 1609 meters in 1 mile

  First, we can compute the average speed with the units given $\left(\frac{miles}{minute}\right)$

  $$\text{Average Speed} = \frac{\text{miles}}{\text{minute}} = \frac{10}{20} = 0.5\ mi/min.$$

  Now we simply convert to m/s

  $$\frac{0.5\ \cancel{mi}}{1\ \cancel{min}} \times \frac{1\ \cancel{min}}{60\ sec} \times \frac{1609\ m}{1\ \cancel{mi}}$$
  $$\approx 13 m/s.$$

- **Unit conversion**: The density of iron is $7.86 g/cm^3$ under standard conditions. Convert this to kg/m^3.

  $$\frac{7.86\ g}{1\ cm^3} \times \left(\frac{100\ cm}{1\ m}\right)^3 \times \frac{1\ kg}{1000\ g}$$
  $$= \frac{7.86(100^3)(1\ kg)}{1000(1\ m)}$$
  $$= 7.86 \cdot 10^3 kg/m^3.$$

- **Proportional**: I found that if I drive my car 110 miles, I use 4 gallons ofgas. If I assume that the relationship between gas guzzled and distance driven is linearly proportional, how many gallons of gas do I use if I drive 275 miles?

  To answer this lets find the linear equation

  $$4\ gal = k(100\ mi).$$

  Where k is some arbitrary factor, Since we know the relationship is proportional, we can write

  $$\frac{X}{4gal} = \frac{k(275\ mi)}{k(110\ mi)}$$
  $$X = 10\ gal.$$

3.2 Chapter 2: Vectors

3.2.1 Vocabulary

- Many familiar physical quantities can be specified completely by giving a single number and the appropriate unit. For example, "a class period lasts 50 min" or "the gas tank in my car holds 65 L" or "the distance between two posts is 100 m." A physical quantity that can be specified completely in this manner is called a **scalar quantity**. Scalar is a synonym of "number." Time, mass, distance, length, volume, temperature, and energy are examples of **scalar quantities**.

- possible. Physical quantities specified completely by giving a number of units (magnitude) and a direction are called **vector quantities**. Examples of vector quantities include displacement, velocity, position, force, and torque.

3.2.2 Definitions and theorems (and important things)

- Vectors (vector quantities) have a **number of units (magnitude)**, and a **direction**
- **Algebraic Operations with vectors**
  - We can **add or subtract** two vectors
  - we can **multiply** a vector by a scalar or by another vector
  - We **cannot** divide by a vector. The operation of division by a vector is not defined.
- **Vector Notation**: We denote a vector with a bold face letter with an arrow above it. For example

$$\vec{\mathbf{V}}.$$

- **Displacement**: General term used to describe change in position.

3.2.3 Problems to remember

Calculus III

4.1 Chapter 1

4.1.1 Definitions and Theorems

- **Parametric equations, parameter, and parametric curve (*plane curve*):** If x and y are continuous functions of t on an interval I, then the equations

$$x = x(t) \quad \text{and} \quad y = y(t)$$

  are called **parametric equations** and t is called the **parameter**. The set of points (x, y) obtained as t varies over the interval I is called the graph of the parametric equations. The graph of parametric equations is called a **parametric curve** or plane curve, and is denoted by C.

- **Eliminating the parameter:** This allows us to rewrite the two equations as a single equation relating the variables x and y. Then we can apply any previous knowledge of equations of curves in the plane to identify the curve.
  - Solve one of the equations for t
  - Plug the equation for t into the equation still in terms of t
  - Sketch the curve on the interval I

- **Domain consideration when graphing by eliminating the parameter**: Suppose we have the equations

$$\begin{aligned} x &= 2t^2 \\ y &= t^4 + 1. \end{aligned}$$

  If we solve x for t, we get

$$t = \pm\sqrt{\frac{1}{2}x}.$$

  We see that the domain of t is restricted to $t \geqslant 0$, thus, the graph of this equation will only have the positive side.

- **Parameterizing a curve** is when we start with the equation of a curve and determine a pair of parametric equations for that curve.
  - To find the first Parameterization, Define $x(t) = t$ and $y(t)$ as the function given, with t instead of x
  - Verify there is no restriction on the domain of the original graph. Thus there is no restriction on the values of t
  - To find the second parameterization, choose some function for $x(t)$. Ensure that the domain is the set of all real numbers
  - Plug $x(t)$ into the original function and solve to get $y(t)$

- **Parametric equations for a cylcloid**

$$x(t) = a(t - \sin t), \quad y(t) = a(1 - \cos t).$$

- **The general parametric equations for a hypocycloid are:**

$$x(t) = (a-b)\ \cos t + b\cos\left(\frac{a-b}{b}\right)t$$
$$y(t) = (a-b)\ \sin t + b\sin\left(\frac{a-b}{b}\right)t$$

.

- **Derivatives of Parametric Equations**: Consider the plane curve defined by the parametric equations $x = x(t)$ and $y = y(t)$. Suppose that $x'(t)$ and $y'(t)$ exist, and assume that $x'(t) \neq 0$. Then the derivative $\frac{dy}{dx}$ is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}.$$

- **Second order derivatives of parametric functions**

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\left(\frac{d}{dt}\right)\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

- **Integral involving parametric equations** Consider the non-self-intersecting plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t), \quad a \leqslant t \leqslant b$$

and assume that $x(t)$ is differentiable. The area under this curve is given by

$$A = \int_a^b y(t)x'(t)\,dt.$$

- **Arc Length of a Parametric Curve**: Consider the plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t), \quad t_1 \leqslant t \leqslant t_2$$

and assume that $x(t)$ and $y(t)$ are differentiable functions of t. Then the arc length of this curve is given by

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

Or simply

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

- **Surface area for a parametric curve**: The analogous formula for a parametrically defined curve is

$$S = 2\pi\int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

provided that $y(t)$ is not negative on $[a, b]$.

- **Parametric equations for a circle (with $(h, k) = (0, 0)$)**

$$x(t) = r\cos(t)$$
$$y(t) = r\sin(t).$$

For $0 \leqslant t \leqslant 2\pi$

- **Parametric equations for the upper half of a semi-circle (with $(h,k) = (0,0)$)**

$$x(t) = r\cos(t)$$
$$y(t) = r\sin(t).$$

  For $0 \leqslant t \leqslant \pi$

- **Parametric equations for the lower half of a semi-circle (with $(h,k) = (0,0)$)**

$$x(t) = r\cos(t)$$
$$y(t) = r\sin(t).$$

  For $\pi \leqslant t \leqslant 2\pi$

- **Note: A graph has a horizontal tangent line when the derivative equals zero.**

- **Note: A graph has a vertical tangent line when the derivative does not exist**

- **Converting Points between Coordinate Systems**: Given a point P in the plane with Cartesian coordinates (x, y) and polar coordinates (r, θ), the following conversion formulas hold true:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta, .$$

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x}..$$

  These formulas can be used to convert from rectangular to polar or from polar to rectangular coordinates.

- **Sometimes finding θ with the formula is not possible.** Consider the point $P(0,3)$, using the formula $\tan\theta = \frac{3}{0}$ is undefined. Instead, we graph the point $(0,3)$ and observe the angle between the x and y axis is $\frac{\pi}{2}$

- **The polar representation of a point is not unique.** Every point in the plane has an infinite number of representations in polar coordinates. However, each point in the plane has only one representation in the rectangular coordinate system.

- The line segment starting from the center of the graph going to the right (called the positive x-axis in the Cartesian system) is the **polar axis**.

- The center point is the **pole**, or origin, of the coordinate system, and corresponds to $r = 0$.

- **If the value of r is positive**, move that distance along the terminal ray of the angle. **If it is negative**, move along the ray that is opposite the terminal ray of the given angle.

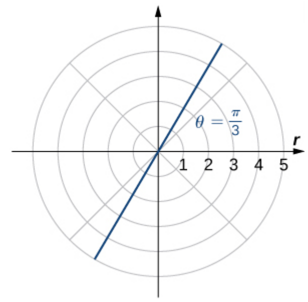
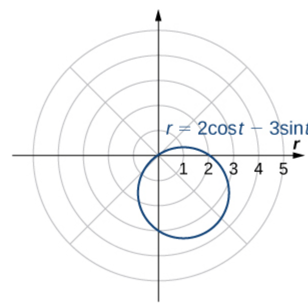
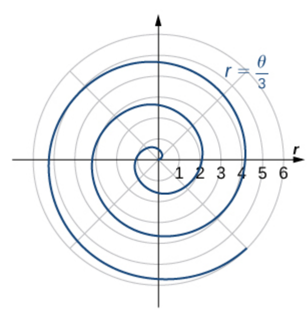
- A **Polar equation** has the form

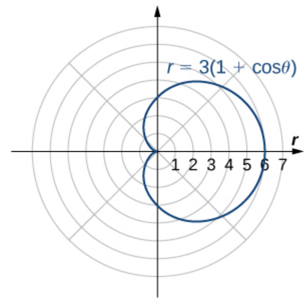
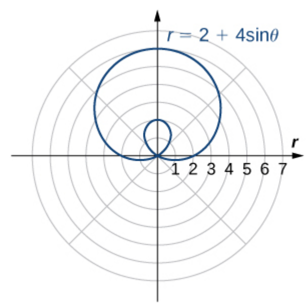
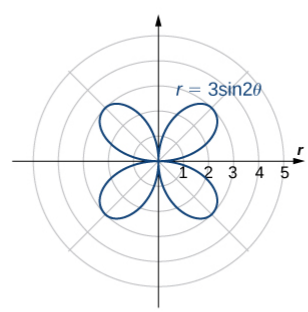
$$r = f(\theta).$$

- **Plotting a Curve in Polar Coordinates**

  1. Create a table with two columns. The first column is for θ, and the second column is for r.
  2. Create a list of values for θ.
  3. Calculate the corresponding r values for each θ.
  4. Plot each ordered pair (r, θ) on the coordinate axes.
  5. Connect the points and look for a pattern.

- **Types of polar curves**

Name	Equation	Example
Line passing through the pole with slope $\tan K$	$\theta = K$	 *Figure 1*
Circle	$r = a\cos\theta + b\sin\theta$	 *Figure 2*
Spiral	$r = a + b\theta$	 *Figure 3*

Name	Equation	Example
Cardioid	$r = a(1 + \cos\theta)$ $r = a(1 - \cos\theta)$ $r = a(1 + \sin\theta)$ $r = a(1 - \sin\theta)$ *Figure 4*	 *Figure 5*
Limacon	$r = a\cos\theta + b$ $r = a\sin\theta + b$ *Figure 6*	 *Figure 7*
Rose	$r = a\cos\theta + b$ $r = a\sin\theta + b$ *Figure 8*	 *Figure 9*

- For the rose, **If the coefficient of θ is even**, the graph has twice as many petals as the coefficient. **If the coefficient of θ is odd, then the number of petals equals the coefficient.**

- **Graphing a polar curve.**
  - Make graph in rectangular system (use θ as horizontal axis and r as vertical)
  - Translate over to polar
  - The circles represent values of r

- **Area of a Region Bounded by a Polar Curve**: Suppose f is continuous and nonnegative on the interval $\alpha \leqslant \theta \leqslant \beta$ with $0 < \beta - \alpha \leqslant 2\pi$. The area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2 \, d\theta = \frac{1}{2}\int_\alpha^\beta r^2 \, d\theta.$$

- **Area between two polar curves**:

$$A = \frac{1}{2}\int_\alpha^\beta \; f(\theta)^2 - g(\theta)^2 \; d\theta.$$

  Where $f(\theta) \geqslant g(\theta) \forall \; \alpha \leqslant \theta \leqslant \beta$

- **Bounds of integration for area outside some curve and inside some curve**: We find the bounds of integration the same way we found them for regularo functions, we find the points of intersection by setting the two functions equal to each other

- **Arc Length of a Curve Defined by a Polar Function**: Let f be a function whose derivative is continuous on an interval $\alpha \leqslant \theta \leqslant \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$\begin{aligned} L &= \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} \, d\theta \\ &= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta.. \end{aligned}$$

- **Absolute value bars**: Consider the integral

$$20\int_0^{2\pi} \; \sqrt{\cos^2\left(\frac{\theta}{2}\right)} \; d\theta.$$

  If we cancel out the sqrt and square, we must include absulute value bars. This is because $\cos\left(\frac{\theta}{2}\right)$ can be negative on the interval $[0, 2\pi]$. The integral becomes

$$20\int_0^{2\pi} \; \left|\cos\left(\frac{\theta}{2}\right)\right| \; d\theta.$$

  To solve this, we use symmetry to change the bounds. Because the graph of the polar curve $(10 + 10\cos\theta)$ is symmetric, we can integrate instead over the range $0, \pi$, and multiply the result by 2.

4.1.2 Problems to remember

- **Eliminating the parameter:** Sometimes it is necessary to be a bit creative in eliminating the parameter. The parametric equations for this example are

$$x(t) = 4\cos t, \quad y(t) = 3\sin t.$$

Solving either equation for t directly is not advisable because sine and cosine are not one-to-one functions. However, dividing the first equation by 4 and the second equation by 3 (and suppressing the t) gives us

$$\cos t = \frac{x}{4}, \quad \sin t = \frac{y}{3}.$$

Now use the Pythagorean identity $\cos^2 t + \sin^2 t = 1$ and replace the expressions for $\sin t$ and $\cos t$ with the equivalent expressions in terms of x and y. This gives

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$
$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

This is the equation of a horizontal ellipse centered at the origin, with semimajor axis 4 and semiminor axis 3 as shown in the following graph.

- **Convert from cartesian to polar**: Consider the point $P(1, 1)$. First we find r and θ

$$\begin{aligned} r^2 &= x^2 + y^2 \\ r &= \sqrt{1^2 + 1^2} \\ r &= \sqrt{2} \\ \tan\theta &= \frac{y}{x} = \frac{1}{1} \\ \theta &= \tan^{-1} 1 \\ \theta &= \frac{\pi}{4}. \end{aligned}$$

Thus we have the point $\left(\sqrt{2}, \frac{\pi}{4}\right)$

- **Graphing a polar equation**: Graph the curve defined by the function $r = 4\sin\theta$. Identify the curve and rewrite the equation in rectangular coordinates.

Because the function is a multiple of a sine function, it is periodic with period 2π, so use values for θ between 0 and 2π

θ	$r = 4\sin\theta$	θ	$r = 4\sin\theta$
0	0	π	0
$\frac{\pi}{6}$	2	$\frac{7\pi}{6}$	-2
$\frac{\pi}{4}$	$2\sqrt{2} \approx 2.8$	$\frac{5\pi}{4}$	$-2\sqrt{2} \approx -2.8$
$\frac{\pi}{3}$	$2\sqrt{3} \approx 3.4$	$\frac{4\pi}{3}$	$-2\sqrt{3} \approx -3.4$
$\frac{\pi}{2}$	4	$\frac{3\pi}{2}$	-4
$\frac{2\pi}{3}$	$2\sqrt{3} \approx 3.4$	$\frac{5\pi}{3}$	$-2\sqrt{3} \approx -3.4$
$\frac{3\pi}{4}$	$2\sqrt{2} \approx 2.8$	$\frac{7\pi}{4}$	$-2\sqrt{2} \approx -2.8$
$\frac{5\pi}{6}$	2	$\frac{11\pi}{6}$	-2
2π	0		

Plotting these points gives the graph of a circle. The equation $r = 4\sin\theta$ can be converted into rectangular coordinates by first multiplying both sides by r. This gives the equation $r^2 = 4r\sin\theta$. Next use the facts that $r^2 = x^2 + y^2$ and $y = r\sin\theta$. This gives $x^2 + y^2 = 4y$. To put this equation into standard form, subtract $4y$ from both sides of the equation and complete the square:

$$x^2 + (y^2 - 4y) = x^2 + (y^2 - 4y + 4) = x^2 + (y-2)^2 = 0 + 4.$$

This is the equation of a circle with radius 2 and center $(0, 2)$ in the rectangular coordinate system.

- **Finding the Arc Length of a Polar Curve**:

$$Find the arc length of the polar curve r = 2 + 2\cos\theta.$$

When $\theta = 0$, $r = 2 + 2\cos(0) = 4$. Furthermore, as θ goes from 0 to 2π, the cardioid is traced out exactly once. Therefore, these are the limits of integration. Using $f(\theta) = 2 + 2\cos(\theta)$, $\alpha = 0$, and $\beta = 2\pi$. Thus we have

$$\begin{aligned}
L &= \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{[2 + 2\cos(\theta)]^2 + [-2\sin(\theta)]^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{4 + 8\cos(\theta) + 4\cos^2(\theta) + 4\sin^2(\theta)}\, d\theta \\
&= \int_0^{2\pi} \sqrt{4 + 8\cos(\theta) + 4(\cos^2(\theta) + \sin^2(\theta))}\, d\theta \\
&= \int_0^{2\pi} \sqrt{8 + 8\cos(\theta)}\, d\theta \\
&= 2\int_0^{2\pi} \sqrt{2 + 2\cos(\theta)}\, d\theta..
\end{aligned}$$

Next, using the identity $\cos(2\alpha) = 2\cos^2(\alpha) - 1$, add 1 to both sides and multiply by 2. This gives $2 + 2\cos(2\alpha) = 4\cos^2(\alpha)$. Substituting $\alpha = \frac{\theta}{2}$ gives $2 + 2\cos(\theta) = 4\cos^2\left(\frac{\theta}{2}\right)$, so the integral becomes

$$\begin{aligned}
L &= 2\int_0^{2\pi} \sqrt{2 + 2\cos(\theta)}\, d\theta \\
&= 2\int_0^{2\pi} \sqrt{4\cos^2\left(\frac{\theta}{2}\right)}\, d\theta \\
&= 2\int_0^{2\pi} 2\left|\cos\left(\frac{\theta}{2}\right)\right|\, d\theta..
\end{aligned}$$

The absolute value is necessary because the cosine is negative for some values in its domain. To resolve this issue, change the limits from 0 to π and double the answer. This strategy works because cosine is positive between 0 and $\frac{\pi}{2}$. Thus,

$$\begin{aligned}
L &= 4\int_0^{2\pi} \left|\cos\left(\frac{\theta}{2}\right)\right|\, d\theta \\
&= 8\int_0^{\pi} \cos\left(\frac{\theta}{2}\right)\, d\theta \\
&= 8\left[2\sin\left(\frac{\theta}{2}\right)\right]_0^\pi \\
&= 16..
\end{aligned}$$