Nate Warner | MATH 232 | Spring 2024

Homework/Worksheet 9 - Due: Saturday, April 13

1. For the double integrals below, convert the integrals to polar coordinates and evaluate them.

(a) $\int_0^3 \int_0^{\sqrt{9-y^2}} (x^2+y^2)\,dxdy$

(b) $\int_0^1 \int_0^{\sqrt{1-x^2}} (x+y)\,dydx$

Problem 1a. We identify the given region as

$$D = \{(x,y) : \ 0 \leqslant y \leqslant 3, \ 0 \leqslant x \leqslant \sqrt{9-y^2}\}.$$

To transform this region to one of the form $S = \{(r,\theta) : \ \alpha \leqslant \theta \leqslant \beta, \ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$, we first notice that the region D describes a quarter circle in the first quadrant of the xy-plane. Thus, our new region S is given by

$$S = \{(r,\theta) : \ 0 \leqslant \theta \leqslant \frac{\pi}{2}, \ 0 \leqslant r \leqslant 3\}.$$

So we have the integral

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \int_0^3 (r^2) r \, dr d\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{4} \Big[r^4 \Big]_0^3 d\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{81}{4} \, d\theta \\
&= \frac{81}{4} (\frac{\pi}{2} - 0) \\
&= \frac{81\pi}{8}.
\end{aligned}$$

Problem 1b. Again, we identify the given region and then transform it to polar

$$\begin{aligned}
&D = \{(x,y) : \ 0 \leqslant x \leqslant 1, \ 0 \leqslant y \leqslant \sqrt{1-x^2}\} \\
\implies &S = \{(r,\theta) : \ 0 \leqslant r \leqslant 1, \ 0 \leqslant \theta \leqslant \frac{\pi}{2}\}.
\end{aligned}$$

Thus, we have the integral

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \int_0^1 (r\cos(\theta) + r\sin(\theta))\, r dr d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 r^2 \cos(\theta) + r^2 \sin(\theta) \, dr d\theta \\
&= \int_0^{\frac{\pi}{2}} \left[\cos(\theta) \int_0^1 r^2 dr + \sin(\theta) \int_0^1 r^2 \, dr \right] d\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{3}\cos(\theta) + \frac{1}{3}\sin(\theta) \, d\theta \\
&= \frac{1}{3} \Big[\sin(\theta) - \cos(\theta) \Big]_0^{\frac{\pi}{2}} \\
&= \frac{1}{3}(1 - 0 - (0-1)) = \frac{2}{3}.
\end{aligned}$$

2. Find the volume of the solid bounded by the paraboloid $z = 2 - 9x^2 - 9y^2$ and the plane $z = 1$.

We first define this region as

$$E = \{(x, y, z) : (x, y) \in D,\ 1 \leqslant z \leqslant 2 - 9x^2 - 9y^2\}.$$

We must next deduce D, which is the projection of E onto the xy-plane. To accomplish this, we can set $z = 1$, We set $z = 1$ instead of $z = 0$ because the floor of our region is defined by the plane $z = 1$.

$$\begin{aligned} 2 - 9x^2 - 9y^2 &= 1 \\ \implies x^2 + y^2 &= \frac{1}{9}. \end{aligned}$$

Thus, the projection of E onto the xy-plane is a disk centered at the origin of radius $\frac{1}{3}$, with this information we determine that $D = \{(x, y) :\ -\frac{1}{3} \leqslant x \leqslant \frac{1}{3},\ -\sqrt{\frac{1}{9} - x^2} \leqslant y \leqslant \sqrt{\frac{1}{9} - x^2}\}$

Thus, region E is given by

$$E = \{(x, y, z) :\ -\frac{1}{3} \leqslant x \leqslant \frac{1}{3},\ -\sqrt{\frac{1}{9} - x^2} \leqslant y \leqslant \sqrt{\frac{1}{9} - x^2},\ 1 \leqslant z \leqslant 9 - x^2 - y^2\}.$$

We then move this region to cylindrical coordinates, of the form $G = \{(r, \theta, z) :\ g_1(\theta) \leqslant r \leqslant g_2(\theta)\alpha \leqslant \theta \leqslant \beta,\ u_1(r, \theta) \leqslant z \leqslant u_2(r, \theta)\}$, thus region G in cylindrical coordinates is given by

$$G = \{(r, \theta, z) :\ 0 \leqslant r \leqslant \frac{1}{3},\ 0 \leqslant \theta \leqslant 2\pi,\ 1 \leqslant z \leqslant 2 - 9r^2\}.$$

Then the volume is given by the triple integral over the cylindrical region

$$\begin{aligned} &\int_0^{2\pi} \int_0^{\frac{1}{3}} \int_1^{2-9r^2} r dz dr d\theta \\ &= \int_0^{2\pi} r \int_0^{\frac{1}{3}} 2 - 9r^2 - 1 \, dr d\theta \\ &= \int_0^{2\pi} \int_0^{\frac{1}{3}} r - 9r^3 \, dr d\theta \\ &= \int_0^{2\pi} \frac{1}{2} r^2 - \frac{9}{4} r^4 \, d\theta \\ &= \int_0^{2\pi} \frac{1}{2} \left(\frac{1}{3}\right)^2 - \frac{9}{4} \left(\frac{1}{3}\right)^4 \, d\theta \\ &= \int_0^{2\pi} \frac{1}{36} \, d\theta \\ &= \frac{1}{36}(2\pi - 0) = \frac{\pi}{18}. \end{aligned}$$

3. The solid E is bounded by $y^2 + z^2 = 9$, $z = 0$, $x = 0$, and $x = 5$ (see picture in problem 5.4.211). Evaluate the triple integral $\iiint_E z\,dV$ by integrating first with respect to z, then y, and then x.

We see that the solid E describes a cylinder with radius 3 centered along the x-axis. Thus, we have the integral

$$\begin{aligned}
&\int_0^5 \int_{-3}^3 \int_0^3 dzdydx \\
&= \int_0^5 \int_{-3}^3 3\,dydx \\
&= \int_0^5 3y\Big|_{-3}^3 dx \\
&= \int_0^5 9 + 9\,dx \\
&= \int_0^5 18\,dx \\
&= 18(5) \\
&= 90.
\end{aligned}$$

4. The solid E is bounded by $y = \sqrt{x}$, $x = 4$, $y = 0$, and $z = 1$ (see picture in problem 5.4.212). Evaluate the triple integral $\iiint_E z\,dV$ by integrating first with respect to x, then y, and then z.

We describe the given region as

$$E = \{(x, y, z) : \ (x, y) \in D,\ 0 \leqslant z \leqslant 1\}$$
$$\text{With } D_{\text{t}2} = \{0 \leqslant x \leqslant 4,\ 0 \leqslant y \leqslant \sqrt{x}\}.$$

However, since the problem states to integrate first with respect to x, then y, our integral would be

$$\int_0^1 \int_0^{\sqrt{x}} \int_0^4 z\,dxdydz.$$

This is a problem because the upper bound for the middle intgeral involve an x variable. Thus, we must change the region D from type 1 to type 2.

$$D_{t1} = \{(x, y) : \ y^2 \leqslant x \leqslant 4,\ 0 \leqslant y \leqslant 2\}.$$

The intgeral then becomes

$$\begin{aligned}
&\int_0^1 \int_0^2 \int_{y^2}^4 z\,dxdydz \\
&= \int_0^1 z \int_0^2 4 - y^2\,dydz = \int_0^1 z\left[4y - \frac{1}{3}y^3\right]_0^2 dz \\
&= \int_0^1 z\left(8 - \frac{8}{3}\right) dz = \int_0^1 \frac{16}{3} z\,dz \\
&= \frac{8}{3}(1 - 0) = \frac{8}{3}.
\end{aligned}$$

5. Find the volume of the solid E bounded by $z = 10 - 2x - y$ and situated in the first octant (see picture in problem 5.4.231).

Since we know we are only focused on the first octant, we have $x = 0$, $y = 0$, and $z = 0$. This deduction gives the lower bounds for each integral. Furthermore, we can find the upper bounds for the x and y integral by first setting $z = 0$, solving for y, then setting $y = 0$ and solving for x

$$\begin{aligned} & 0 = 10 - 2x - y \\ \implies & y = 10 - 2x \\ \implies & x = 5. \end{aligned}$$

Thus we have the region

$$E = \{(x, y, z) : \ 0 \leqslant x \leqslant 5, \ 0 \leqslant y \leqslant 10 - 2x, \ 0 \leqslant z \leqslant 10 - 2x - y\}.$$

And the integral is given by

$$\begin{aligned} & \int_0^5 \int_0^{10-2x} \int_0^{10-2x-y} dzdydx \\ &= \int_0^5 \int_0^{10-2x} 10 - 2x - y \, dydx \\ &= \int_0^5 10y - 2xy - \frac{1}{2}y^2 \Big|_0^{10-2x} dx \\ &= \int_0^5 10(10 - 2x) - 2x(10 - 2x) - \frac{1}{2}(10 - 2x)^2 \, dx \\ &= \int_0^5 2x^2 - 20x + 50 \, dx \\ &= \frac{2}{3}x^3 - 10x^2 + 50x \Big|_0^5 \\ &= \frac{250}{3}. \end{aligned}$$

6. Let $f(x, y, z) = x^2 + y^2$ and $E = \{(x, y, z) : 0 \leqslant x^2 + y^2 \leqslant 4, y \geqslant 0, 0 \leqslant z \leqslant 3 - x\}$. Convert the integral $\iiint_E f(x, y, z) \, dV$ into cylindrical coordinates and evaluate it.

Since the region D on the xy-plane is a circle, we can easily change the given region to cylindrical coordinates. We note that because $y \geqslant 0$, we only consider the top half of the circle, this gives the new region as

$$E = \{(r, \theta, z) : \ 0 \leqslant r \leqslant 2, \ 0 \leqslant \theta \leqslant \pi, \ 0 \leqslant z \leqslant 3 - r\cos(\theta)\}.$$

Hence, the volume is given by the integral

$$\begin{aligned} & \int_0^\pi \int_0^2 \int_0^{3 - r\cos(\theta)} r^2 r \, dzdrd\theta \\ &= \int_0^\pi \int_0^2 r^3(3 - r\cos(\theta)) \, drd\theta = \int_0^\pi \int_0^2 3r^3 - r^4 \cos(\theta) \, drd\theta \\ &= \int_0^\pi \frac{3}{4}r^4 - \frac{1}{5}r^5 \cos(\theta) \Big|_0^2 d\theta = \int_0^\pi 12 - \frac{32}{5}\cos(\theta) \, d\theta \\ &= 12(\pi - 0) - \frac{32}{5}(\sin(\pi) - \sin(0) = 12\pi. \end{aligned}$$

7. Convert the integral

$$\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{\sqrt{x^2+y^2}}^{x^2+y^2} xz\, dz\, dx\, dy$$

into an integral in cylindrical coordinates.

We express the given integral as the region

$$E = \{(x, y, z) :\ 0 \leqslant y \leqslant 1,\ -\sqrt{1-y^2} \leqslant x \leqslant \sqrt{1-y^2},\ x^2 + y^2 \leqslant z \leqslant \sqrt{x^2+y^2}\}.$$

We notice that the bounds for x and y define a circle on the xy-plane centered at the origin with radius 1. Using this and converting the bounds of z and the integrand to polar form, we get the cylindrical region

$$E = \{(r, \theta, z) :\ 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant r \leqslant 1,\ r^2 \leqslant z \leqslant r\}.$$

Thus, the integral becomes

$$\begin{aligned}
&\int_0^{2\pi} \int_0^1 \int_{r^2}^{r} r^2 \cos(\theta) z\, dz dr d\theta \\
&= \int_0^{2\pi} \int_0^1 \frac{1}{2} r^2 \cos(\theta) \left[z^2\right]_{r^2}^{r},\ dr d\theta \\
&= \int_0^{2\pi} \int_0^1 \frac{1}{2} r^5 \cos(\theta) - \frac{1}{2} r^6 \cos(\theta)\, dr d\theta \\
&= \int_0^{2\pi} \frac{1}{10} r^5 \cos(\theta) - \frac{1}{14} r^7 \cos(\theta) \Big|_0^1\, d\theta \\
&= \frac{1}{35} \int_0^{2\pi} \cos(\theta)\, d\theta \\
&= -\frac{1}{35} \left[\sin(\theta)\right]_0^{2\pi} \\
&= 0.
\end{aligned}$$