Chapters 9-12

Stat 128: Elementary Statistics

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Contents

Chapter 9 3

0.1 9.1 Estimating a Population Proportion 3

0.2 9.2: Estimating a Population Mean 14

0.3 9.3: Putting It Together: Which Procedure Do I Use? 21

Chapter 9

0.1 9.1 Estimating a Population Proportion

Learning Objectives For This Section:

1. **Obtain a Point Estimate for the Population Proportion**
2. **Construct and Interpret a Confidence Interval for the Population Proportion**
3. **Determine the Sample Size Necessary for Estimating a Population Proportion within a Specified Margin of Error**

Vocab:

- A **point estimate** is the value of a statistic that estimates the value of a parameter.
- **A Confidence Interval** for an unknown parameter consists of an interval of numbers based on a point estimate.
- **The level of confidence** represents the expected proportion of intervals that will contain the parameter if a large number of different samples is obtained.
- **Critical Value** represents the number of standard deviations the sample statistic can be from the parameter and still result in an interval that includes the parameter.

Formulas/Notation:

- **Point Estimate = Sample Proportion**

$$\hat{p} = \frac{x}{n}.$$

- The **Margin of Error** is denoted:

$$E.$$

- **Confidence Intervals for a Proportion** are of the form:

  point estimate ± margin of error

  That is:

$$\hat{p} \pm E.$$

  We have 95% confidence that the sampling proportion will lie between:

$$\hat{p} \pm 1.96\sigma_{\hat{p}}.$$

- **The level of confidence** is denoted:

$$(1-\alpha)\cdot 100\%.$$

- **Standard Error:**

$$\sigma_{\hat{p}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

- **Constructing any $(1-\alpha)\cdot 100\%$ Confidence Interval**

$$\hat{p} - z_{\alpha/2}\cdot\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} < p < \hat{p} + z_{\alpha/2}\cdot\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

- **Critical Value**

$$z_{\frac{\alpha}{2}}.$$

- **Constructing a (1—α)$\cdot$ 100% Confidence Interval for a Population Proportion:** Suppose that a simple random sample of size n is taken from a population or the data are the result of a randomized experiment. A (1—α)$\cdot$100% confidence interval for p is given by the following quantities:

  $$Lower\ bound:\ \hat{p} - z_{\frac{\alpha}{2}} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$
  $$Upper\ bound:\ \hat{p} + z_{\frac{\alpha}{2}} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

  Note: It must be the case that $n\hat{p}(1-\hat{p}) \geqslant 10$ and $n \leqslant 0.05N$ to construct this interval. Use $\hat{p}$ in place of p in the standard deviation. This is because p is unknown, and $\hat{p}$ is the best point estimate of p.

- The **margin of error**, E, in a (1—α)$\cdot$100% confidence interval for a population proportion is given by

  $$E = Z_{\frac{\alpha}{2}} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$
  $$or:\ \frac{Upper\ Limit\ - Lower\ Limit}{2}.$$

- **Sample Size Needed for Estimating the Population Proportion**

  $$n = \hat{p} \cdot (1-\hat{p}) \cdot \left(\frac{z_{\alpha/2}}{E}\right)^2.$$

  (rounded up to the next integer) where $\hat{p}$ is a prior estimate of p.

  If a prior estimate of p is unavailable, the sample size required is

  $$n = 0.25 \cdot \left(\frac{z_{\alpha/2}}{E}\right)^2.$$

  rounded up to the next integer. The margin of error should always be expressed as a decimal when using these formulas.

- **Width**

  $$2(E).$$

Introduction

Two Types of Inferential Statistics:

1. Estimation
2. Hypothesis Testing

Obtain a Point Estimate for the Population Proportion

Suppose we want to estimate the proportion of adult Americans who believe that the amount they pay in federal income taxes is fair. It is unreasonable to expect that we could survey every adult American. Instead, we use a sample of adult Americans to arrive at an estimate of the proportion. We call this estimate a point estimate.

Example:

Problem:

The Gallup Organization conducted a poll in April 2017 in which a simple random sample of 1019 Americans aged 18 and older were asked, “Do you regard the income tax that you will have to pay this year as fair?” Of the 1019 adult Americans surveyed, 620 said yes. Obtain a point estimate for the proportion of Americans aged 18 and older who believe that the amount of income tax they pay is fair.

Approach: The point estimate of the population proportion is $\hat{p} = \frac{x}{n}$, where $x = 620$ and $n = 1019$.

Solution:

Substituting into the formula, we get $\hat{p} = \frac{x}{n} = \frac{620}{1019} = 0.608 = 60.8\%$.

We estimate that 60.8% of Americans aged 18 and older believe that the amount of income tax they pay is fair.

Construct and Interpret a Confidence Interval for the Population Proportion

What if we conducted a different random sample of 1019 Americans aged 18 years or older? Would we get the same sample proportion who believe the amount of income tax they pay is fair? Probably not. Why? Because statistics such as $\hat{p}$ vary from sample to sample. So a different random sample of adult Americans might result in a different point estimate of the population proportion, such as $\hat{p} = 0.593$.

If the method used to select the adult Americans was done appropriately, both point estimates would be good guesses of the population proportion. Due to variability in the sample proportion, we report a range (or interval) of values, including a measure of the likelihood that the interval includes the unknown population proportion.

Consider the following:

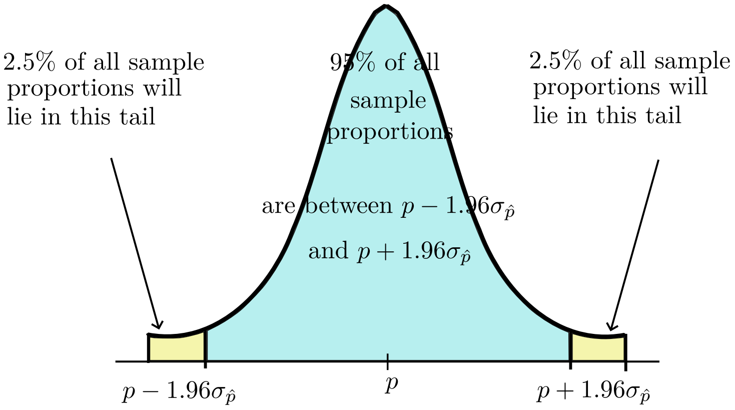
$$\hat{p} = 0.54 \quad E = 0.034.$$

Then:

$$0.54 - 0.034 = 0.506$$
$$0.54 + 0.034 = 0.574.$$

With this we can say ”We are 95% confident that the proportion is between 0.506 and 0.574”

Figure:


2.5% of all sample proportions will lie in this tail
95% of all sample proportions
are between $p - 1.96\sigma_{\hat{p}}$
and $p + 1.96\sigma_{\hat{p}}$
2.5% of all sample proportions will lie in this tail
$p - 1.96\sigma_{\hat{p}}$
p
$p + 1.96\sigma_{\hat{p}}$

Where does the 1.96 comes from?

$$z_\alpha$$
$$z_{0.025} = 1.96.$$

This graphic also infers that 95% of all sample proportions will lie between:

$$p - 1.96\sigma_{\hat{p}} < \hat{p} < p + 1.96\sigma_{\hat{p}}.$$

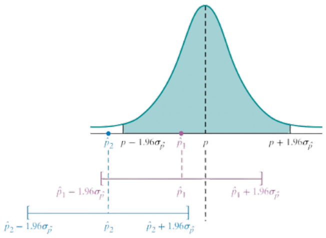
By solving this inequality such that p is in the center:

$$\hat{p} - 1.96\sigma_{\hat{p}} < p < \hat{p} + 1.96\sigma_{\hat{p}}.$$

And we can write this in shorthand form:

$$\hat{p} \pm 1.96\sigma_{\hat{p}}$$
$$or: \ \hat{p} \pm E.$$

figure:


$\hat{p}_2$
$p - 1.96\sigma_{\hat{p}}$
$\hat{p}_1$
p
$p + 1.96\sigma_{\hat{p}}$
$\hat{p}_1 - 1.96\sigma_{\hat{p}}$
$\hat{p}_1$
$\hat{p}_1 + 1.96\sigma_{\hat{p}}$
$\hat{p}_2 - 1.96\sigma_{\hat{p}}$
$\hat{p}_2$
$\hat{p}_2 + 1.96\sigma_{\hat{p}}$

Summary:

- for a 95% confidence interval, any sample proportion that lies withing 1.96 standard errors of the population proportion will result in a confidence interval that includes p. This will happen in 95% of all possible samples.
- Any sample proportion that is more than 1.96 standard errors from the population proportion will result in a confidence interval that does not contain p. This will happen in 5% of all possible samples (those sample proportions in the tails of the distribution).
- A confidence interval for an unknown parameter consists of an interval of numbers based on a point estimate.
- The level of confidence represents the expected proportion of intervals that will contain the parameter if a large number of different samples is obtained. The level of confidence is denoted $(1-\alpha)\cdot$ 100%.
- Whether a confidence interval contains the population parameter depends solely on the value of the sample statistic. Any sample statistic that is in the tails of the sampling distribution will result in a confidence interval that does not include the population parameter. See Figure 1. Notice that $\hat{p}_1$ results in a confidence interval that includes the population proportion, p. However, $\hat{p}_2$ results in a confidence interval that does not include the population proportion, p, because $\hat{p}_2$ is in the tails of the distribution.

Question 1

The horizontal axis in the sampling distribution of represents all possible sample proportions from a simple random sample of size n.

a.) What percent of sample proportions results in a 75% confidence interval that includes the population proportion?

b.) What percent of sample proportions results in a 75% confidence interval that does not include the population proportion?

Solution:: ☺

a.) 75%

b.) 25%

Caution:

95% confident does not mean 95% probability. Probability describes the likelihood of undetermined event.

It does not make sense to talk about the probability that the interval contains the parameter, because the parameter is an unknown, but fixed value. Thus, it is either in the interval, or not.

Consider a coin flip, if a coin has been flipped but its outcome has not been revealed, the flip resulting in a head is not $\frac{1}{2}$, because the outcome has already been determined. Instead the probability is either 0, or 1

Therefore, the probability that any confidence interval contains a parameter is either 0, or 1.

We do say ”we are 95% confident the interval contains the parameter” because the method ”works” in 95% of all samples.

Constructing any $(1-\alpha)\cdot 100\%$ **Confidence interval**

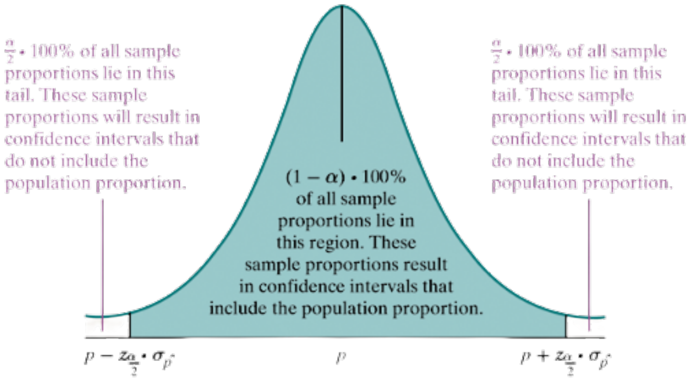
We need a method for constructing any $(1-\alpha)\cdot 100\%$ confidence interval. When $\alpha = 0.05$, we are constructing a 95% confidence interval.

We generalize

$$\begin{array}{ccccccccc} p & - & 1.96\sigma_{\hat{p}} & < & \hat{p} & < & p & + & 1.96\sigma_{\hat{p}} \\ \text{parameter} & - & \text{1.96 standard error} & < & \text{point estimate} & < & \text{parameter} & + & \text{1.96 standard error} \end{array}$$

by first noting that $(1-\alpha)\cdot 100\%$ of all sample proportions are in the interval as shown in Figure 2.

Figure 2


$\frac{\alpha}{2}\cdot 100\%$ of all sample proportions lie in this tail. These sample proportions will result in confidence intervals that do not include the population proportion.
$\frac{\alpha}{2}\cdot 100\%$ of all sample proportions lie in this tail. These sample proportions will result in confidence intervals that do not include the population proportion.
$(1-\alpha)\cdot 100\%$ of all sample proportions lie in this region. These sample proportions result in confidence intervals that include the population proportion.
$p - z_{\frac{\alpha}{2}}\cdot\sigma_{\hat{p}}$
p
$p + z_{\frac{\alpha}{2}}\cdot\sigma_{\hat{p}}$

Rewrite this inequality with p in the middle and obtain

$$\hat{p} - z_{\frac{\alpha}{2}}\cdot\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} < p < \hat{p} + z_{\frac{\alpha}{2}}\cdot\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

So $(1-\alpha)$ 100% of all sample proportions will result in confidence intervals that contain the population proportion. The sample proportions that are in the tails of the distribution in Figure 2 will not result in confidence intervals that contain the population proportion.

Table 1 shows some of the common critical values used in the construction of confidence intervals. Notice that higher levels of confidence correspond to higher critical values. After all, if your level of confidence that the interval includes the unknown parameter increases, the width of your interval (through the margin of error) should increase.

Level of confidence $(1-\alpha)\cdot 100\%$	Area in each tail $\frac{\alpha}{2}$	Critical Value $z_{\frac{\alpha}{2}}$
90%	0.05	1.645
95%	0.025	1.96
99%	0.005	2.575

Interpretation of a Confidence Interval

A $(1-\alpha)\cdot 100\%$ confidence interval indicates that $(1-\alpha)\cdot 100\%$ of all simple random samples of size n from the population whose parameter is unknown will result in an interval that contains the parameter.

For example, a 90% confidence interval for a parameter suggests that 90% of all possible samples will result in an interval that includes the unknown parameter and 10% of the samples will result in an interval that does not capture the parameter.

Example: Interpreting a Confidence Interval

Problem: The Gallup Organization conducted a poll in April 2017 in which a simple random sample of 1019 Americans aged 18 and older were asked, “Do you regard the income tax that you will have to pay this year as fair?” We learned from Example 1 that the proportion of those surveyed who responded yes was 0.608. Gallup reported its “survey methodology” as follows:

Approach: Confidence intervals for a proportion are of the form point estimate ± margin of error. So add and subtract the margin of error from the point estimate to obtain the confidence interval. Interpret the confidence interval, ”We are 95% confident that the proportion of Americans aged 18 and older who believe that the income tax they will have to pay this year is fair is between lower bound and upper bound.”

Solution:

The point estimate is 0.608, and the margin of error is 0.04. The confidence interval is 0.608 ± 0.04. Therefore, the lower bound of the confidence interval is $0.608-0.04=0.568$ and the upper bound of the confidence interval is $0.608+0.04=0.648$. We are 95% confident that the proportion of Americans aged 18 and older who believe that the income tax they will have to pay this year is between 0.568 and 0.648.

We are now prepared to present a method for constructing a confidence interval about the population proportion, p

Constructing a (1−α) 100% Confidence Interval for a Population Proportion:

Suppose that a simple random sample of size n is taken from a population or the data are the result of a randomized experiment. A (1−α) 100% confidence interval for p is given by the following quantities:

$$\textit{Lower bound}:\ \hat{p}-z_{\frac{\alpha}{2}}\cdot\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$
$$\textit{Upper bound}:\ \hat{p}+z_{\frac{\alpha}{2}}\cdot\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

Note:-

It must be the case that $n\hat{p}(1-\hat{p})\geqslant 10$ and $n\leqslant 0.05N$ to construct this interval. Use $\hat{p}$ in place of p in the standard deviation. This is because p is unknown, and $\hat{p}$ is the best point estimate of p.

Example: Constructing a Confidence Interval for a Population Proportion (Statcrunch)

Problem: In the Parent–Teen Cell Phone Survey conducted by Princeton Survey Research Associates International, 800 randomly sampled 16- to 17-year-olds living in the United States were asked whether they have ever used their cell phone to text while driving. Of the 800 teenagers surveyed, 272 indicated that they text while driving. Obtain a 95% confidence interval for the proportion of 16- to 17-year-olds who text while driving.

Approach: It is important to verify the requirements for constructing the confidence interval first. It must be the case that $n\hat{p}(1\hat{p}) \geqslant 10$ and the sample size is no more than 5% of the population size (n $\leqslant$0.05N).Then, construct the confidence interval either by hand or using technology.

Solution:

First, we obtain $\hat{p}$

$$\hat{p} = \frac{272}{800} = .34.$$

Verify Necessary Conditions:

$$\begin{aligned} n\hat{p}(1-\hat{p}) &\geqslant 10 \\ 800(.34)(.66) = 179.52 &\geqslant 10. \end{aligned}$$

Also, we know that $n \leqslant 0.05N$:

now in statcrunch:

1. Stat > Proportion Stats > One Sample > with summary
2. Input no. of successes
3. Input no. of observations
4. Input confidence level
5. Method: Standard-Wald
6. Compute

after doing the StatCrunch steps, we get a result of 0.307 and 0.373, therefore, we can conclude that we are 95% confident that the proportion of 16 to 17-year-olds who text while driving is between 0.307 and 0.373

Example: Constructing a Confidence Interval for a Population Proportion (By hand)

Problem: In the Parent–Teen Cell Phone Survey conducted by Princeton Survey Research Associates International, 800 randomly sampled 16- to 17-year-olds living in the United States were asked whether they have ever used their cell phone to text while driving. Of the 800 teenagers surveyed, 272 indicated that they text while driving. Obtain a 95% confidence interval for the proportion of 16- to 17-year-olds who text while driving.

Approach: It is important to verify the requirements for constructing the confidence interval first. It must be the case that $n\hat{p}(1\hat{p}) \geqslant 10$ and the sample size is no more than 5% of the population size (n $\leqslant$0.05N).Then, construct the confidence interval either by hand or using technology.

Solution:

First, we obtain $\hat{p}$

$$\hat{p} = \frac{272}{800} = .34.$$

Verify Necessary Conditions:

$$\begin{aligned} n\hat{p}(1-\hat{p}) &\geqslant 10 \\ 800(.34)(.66) = 179.52 &\geqslant 10. \end{aligned}$$

Also, we know that $n \leqslant 0.05N$:

Now, to use the formula:

$$\begin{aligned} &\hat{p} \pm E \\ Where\ E = z_{\frac{\alpha}{2}} \cdot &\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}. \end{aligned}$$

We will need to find alpha, given that the formula for confidence level is:

$$\begin{aligned} (a-\alpha) \cdot 100\% \\ 1-\alpha = 0.95 \\ \alpha = 0.05. \end{aligned}$$

So our critical value is:

$$\begin{aligned} z_{\frac{\alpha}{2}} &= z_{\frac{0.05}{2}} \\ &= z_{0.025}. \end{aligned}$$

Using the normal distribution table, we get:

$$\alpha = 1.96.$$

Now we can compute our bounds:

$$\begin{aligned} LB :&= \hat{p} - z_{\frac{\alpha}{2}} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \\ &= 0.34 - 19.6 \cdot \sqrt{\frac{0.34(0.66)}{800}} \\ &= 0.307 \\ UB &= \hat{p} + z_{\frac{\alpha}{2}} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \\ &= 0.34 + 1.96 \cdot \sqrt{\frac{0.34(0.66)}{800}} \\ &= 0.373. \end{aligned}$$

Therefore:

$$\begin{aligned} Upper\ bound:\ 0.307 \quad Upper\ Bound&:\ 0.373 \\ Or&: (0.307, 0.373). \end{aligned}$$

The Effect of Level of Confidence on the Margin of Error

Example: The Role of the Level of Confidence in the Margin of Error

Problem: In the Parent–Teen Cell Phone Survey conducted by Princeton Survey Research Associates International, 800 randomly sampled 16- to 17-year-olds living in the United States were asked whether they have ever used their cell phone to text while driving. Of the 800 teenagers surveyed, 272 indicated that they text while driving. From the last example, we concluded that we are 95% confident that the proportion of 16- to 17-year-olds who text while driving is between 0.307 and 0.373. Determine the effect on the margin of error by increasing the level of confidence from 95% to 99%.

Approach: We would expect the margin of error to increase with a larger level of confidence. Construct the confidence interval either by hand or using technology.

Solution:

The margin of error for the 95% confidence interval found in Example 3 is 0.033, and the margin of error for the 99% confidence interval is 0.043. So increasing the level of confidence increases the margin of error, resulting in a wider confidence interval.

The Effect of Sample Size on the Margin of Error

We know that larger sample sizes produce more precise estimates (the Law of Large Numbers). Given that the margin of error is $E = z_{\alpha/2} \cdot \hat{p} \cdot \sqrt{\frac{(1-\hat{p})}{n}}$, we can see that increasing the sample size n decreases the standard error, so the margin of error decreases. Therefore, larger sample sizes will result in narrower confidence intervals.

What If We Do Not Satisfy the Normality Condition?

When the normality condition is not satisfied, the proportion of intervals that capture the parameter is below the level of confidence.

Determine the Sample Size Necessary for Estimating a Population Proportion within a Specified Margin of Error

Example: Determining Sample Size

Problem: An economist wants to know if the proportion of the U.S. population who commutes to work via car-pooling is on the rise. What size sample should be obtained if the economist wants an estimate within 2 percentage points of the true proportion with 90% confidence?

Assume that the economist uses the estimate of 10% obtained from the American Community Survey.

Approach: Since $1 - \alpha = 0.9$, we know that $\alpha = 0.10$. Use $E = 0.02$, $z_{\alpha/2} = z_{0.12} = z_{0.05} = 1.645$, and $\hat{p} = 0.10$ (the prior estimate).

Solution: Using the formula assuming that a prior estimate is available, $n = \hat{p}(1-\hat{p})\left(\frac{z_{\alpha/2}}{E}\right)^2$, we obtain

$$n = \hat{p}(1-\hat{p})\left(\frac{z_{\alpha/2}}{E}\right)^2 = 0.10 \times (1 - 0.10) \times \left(\frac{1.645}{0.02}\right)^2 = 608.9.$$

Round this value up to 609. So the economist must survey 609 randomly selected residents of the United States.

statcrunch steps:

First:

$$\begin{aligned} E &= 0.02 \text{ (The 2 percentage points)} \\ W &= 2(E) = 0.04 \\ C.Level &= 0.9 \\ Target\ Proportion &= 0.1. \end{aligned}$$

1. Stats > Proportion Stats > One Sample > Width/Sample Size
2. Input confidence level
3. Input Target Proportion
4. Compute

And we find that the minimum sample size would be 609

Now consider that the economist does not use any prior estimates (target proportion), in this case, our target proportion will be 0.5

With this target proportion we get 1691

0.2 9.2: Estimating a Population Mean

Learning Objectives For This Section:

1. **Obtain a Point Estimate for the Population Mean**
2. **State Properties of Student's t-Distribution**
3. **Determine t-Values**
4. **Construct and Interpret a Confidence Interval for a Population Mean**
5. **Determine the Sample Size Necessary for Estimating a Population Mean within a Given Margin of Error**

Vocab:

- **The point estimate of the population mean**, μ, is the sample mean, $\overline{x}$.
- **T-Interval:** confidence interval that uses the t-distribution

Notation/Formulas:

- **Student's t-distribution**

$$t = \frac{\overline{x} - \mu}{\frac{s}{\sqrt{n}}}.$$

- **Constructing a (1−α) ·100% Confidence Interval for** μ Provided:
  - sample data come from a simple random sample or randomized experiment
  - sample size is small relative to the population size (n $\leqslant$0.05N)
  - the data come from a population that is normally distributed with no outliers, or the sample size is large

  A (1−α)·100% confidence interval for μ is given by

$$LB := \overline{x} - t_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}$$
$$UB := \overline{x} + t_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}$$
$$.$$

  where $t_{\frac{\alpha}{2}}$ is the critical value with n−1 degrees of freedom.

- **Margin of Error:**

$$E = t_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}.$$

- **Determine the Sample Size Necessary for Estimating a Population Mean within a Given Margin of Error**

$$n = \left(\frac{z_{\frac{\alpha}{2}} \cdot s}{E}\right)^2.$$

Obtain a Point Estimate for the Population Mean

To find the point estimate of the population mean, compute the sample mean, $\overline{x}$

State Properties of Student's t-Distribution

a different random sample of 16 cars would likely result in a different point estimate of μ. For this reason, we want to construct a confidence interval for the population mean, just as we did for the population proportion.

A confidence interval for the population mean is of the form

$$\text{Point estimate} \pm \text{Margin of Error}$$

(just like the confidence interval for a population proportion). To determine the margin of error, we need to know the sampling distribution of the sample mean.

Recall that the distribution of $\bar{x}$ is approximately normal if the population from which the sample is drawn is normal or the sample size is sufficiently large. In addition, the distribution of $\bar{x}$ has the same mean as the parent population, $\mu_{\bar{x}} = \mu$, and a standard deviation equal to the parent population's standard deviation divided by the square root of the sample size, $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$.

Following the same logic used in constructing a confidence interval about a population proportion, our confidence interval would be

$$\begin{aligned} point\ estimate &\pm Margin\ of\ Error \\ &= \bar{x} \pm z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}. \end{aligned}$$

This presents a problem because we need to know the population standard deviation to construct this interval. It does not seem likely that we would know the population standard deviation but not know the population mean. So what can we do? A logical option is to use the sample standard deviation, s, as an estimate of σ. Then the standard deviation of the sampling distribution of $\bar{x}$ would be estimated by $\frac{s}{\sqrt{n}}$ and our confidence interval would be

$$\overline{x} \pm z_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}.$$

Unfortunately, there is a problem with this approach. The sample standard deviation, s, is a statistic and therefore will vary from sample to sample. Using the normal model to determine the critical value, $z_{\frac{\alpha}{2}}$, in the margin of error does not take into account the additional variability introduced by using s in place of σ. This is not much of a problem for large samples because the variability in the sample standard deviation decreases as the sample size increases (Law of Large Numbers), but for small samples, we have a real problem.

Put another way, the z-score of $\bar{x}$, $\frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}$, is normally distributed with mean 0 and standard deviation 1 (provided $\bar{x}$ is normally distributed). However, $\frac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}}$ is not normally distributed with mean 0 and standard deviation 1. So a new model must be used to determine the margin of error in a confidence interval that accounts for this additional variability. This leads to the story of William Gosset.

In the early 1900s, William Gosset worked for the Guinness brewery. Gosset was in charge of conducting experiments at the brewery to identify the best barley variety. When working with beer, Gosset was limited to small data sets. At the time, the model used for constructing confidence intervals about a mean was the normal model, regardless of whether the population standard deviation was known. Gosset did not know the population standard deviation, so he substituted the sample standard deviation for the population standard deviation and used the formula $\bar{x} \pm z_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}$. While doing this, he was finding that his confidence intervals did not include the population mean at the rate expected. This led Gosset to develop a model that accounts for the additional variability introduced by using s in place of σ when determining the margin of error. Guinness would not allow Gosset to publish his results under his real name (Guinness was very secretive about its brewing practices), but did allow the results to be published under a pseudonym. Gosset chose ”Student.”

Definition: Student's T-Distribution

We know that we can create a standard normal random variable, with the formula:

$$z = \frac{x - \mu}{s}.$$

Where x is the value of the statistic, μ is the population mean, and σ is the population standard deviation, Thus, if we wanted to created a standard normal random variable with a sample mean statistic, we would have the equation:

$$z = \frac{\overline{x} - \mu}{\frac{\sigma}{\sqrt{n}}}.$$

However this poses a problem, as states previously, we may not always know the value of σ, in this case, we can use students $t - distribution$, which states:

$$t = \frac{\overline{x} - \mu}{\frac{s}{\sqrt{n}}}.$$

So with the $t - distribution$, we are assuming we do not have access to σ.

It is important to note that when using students method, there will be additional variability in the $t - distribution$ compared to the $z - distribution$

Note:-

Student's t-distribution is asymptotically normal, this means as the sample size n increases, student's t-distribution looks more and more like the standard normal distribution

Properties of the $t - distribution$

1. The t-distribution is different for different degrees of freedom
2. the t-distribution is centered at 0 and is symmetric about 0
3. The area under the curve is 1. The area around each curve to the right of 0 equals the area under the curve to the left of 0, which equals 1/2
4. As t increases or decreases without bound, the graph approaches, but never equals, zero.

$$\lim_{t \to -\infty} f(x) = 0$$
$$\lim_{t \to \infty} f(x) = 0.$$

5. The area in the tails of the t-distribution is a little greater than the area in the tails of the standard normal distribution, because we are using s as an estimate of σ, thereby introducing further variability into the t-statistic
6. As the sample size n increases, the density curve of t get closer to the standard normal density curve. This result occurs because, as the sample size increases, the values of s get closer to the value of σ, by the Law of Large Numbers.

Construct and Interpret a Confidence Interval for a Population Mean

We are now ready to construct a confidence interval for a population mean.

Constructing a (1−α)·100% Confidence Interval for μ

Provided

- sample data come from a simple random sample or randomized experiment
- sample size is small relative to the population size (n$\leqslant 0.05N$)
- the data come from a population that is normally distributed with no outliers, or the sample size is large

A (1−α)·100% confidence interval for μ is given by:

$$\begin{aligned} Lower\ Bound: &\ \overline{x} - t_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}} \\ Upper\ Bonud: &\ \overline{x} + t_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}} \\ &. \end{aligned}$$

where $t_{\frac{\alpha}{2}}$ is the critical value with n−1 degrees of freedom.

Because this confidence interval uses the *t*-distribution, it is often referred to as the *t*-interval.

Note:-

The margin of error is $E = t_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}$

A robust procedure

Notice that a confidence interval about μ can be computed for non-normal populations even though Student's t-distribution requires a normal population. This is because the procedure for constructing the confidence interval is robust—it is accurate despite minor departures from normality.

If a data set has outliers, the confidence interval is not accurate because neither the sample mean nor the sample standard deviation is resistant to outliers. Sample data should always be inspected for serious departures from normality and for outliers. This is easily done with normal probability plots and boxplots.

Example: Constructing a confidence interval about a population mean (By Hand)

Problem: The website fueleconomy.gov allows drivers to report the miles per gallon of their vehicle. The data in Table 3 show the reported miles per gallon of 2011 Ford Focus automobiles for 16 different owners. Treat the sample as a simple random sample of all 2011 Ford Focus automobiles. Construct a 95% confidence interval for the mean miles per gallon of a 2011 Ford Focus. Interpret the interval.

Before we begin, let's keep the following equation in mind:

$$\overline{x} \pm t_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}.$$

Approach: Before we begin, let's insure that the model requirements for constructing a confidence interval about a mean are satisfied by drawing a normal probability plot and boxplot.

We can confidently conclude that the data was obtained randomly, and the sample size 16 is much less than 5% of the population size

Because the sample size is small, we first need to verify that the data came from a population that is *normally distributed*, and also that the sample has not outliers. If we take a lot at the probability plot, we can see that the data is roughly linear, which insinuates that the data did indeed come from a population that is *normally distributed*, furthermore, if we examine the boxplot, it becomes obvious that there are no outliers in the data.

Using technology, we can compute the sample mean $\overline{x}$ and the sample standard deviation s

$$\overline{x} = 36.8$$
$$s = 2.92.$$

Now let's find the critical value for the $t - distribution$

$$\begin{aligned}(1-\alpha) &= 0.95\\ \alpha &= 0.05\\ \frac{\alpha}{2} &= 0.025\\ &.\end{aligned}$$

Thus: $z_{0.025} = 2.131$ By the t-distribution area in right tail table. From here, we can compute our bounds:

$$\begin{aligned}LB:\ \overline{x} - t_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}\\ = 36.8 - 2.131 \cdot \frac{2.92}{\sqrt{15}}\\ = 35.24.\end{aligned}$$

$$\begin{aligned}UB:\ \overline{x} + t_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}\\ = 36.8 + 2.131 \cdot \frac{2.92}{\sqrt{15}}\\ = 38.36.\end{aligned}$$

We are 95% confident that the mean miles per gallon of all 2011 Ford Focus cars is between 35.24 and 38.36 mpg.

Example: Constructing a confidence interval about a population mean (Using Statcrunch)

First, verify that the population is normally distributed by looking at the probability plot, and also that there are no outliers by looking at the boxplot

Statrunch Steps

1. Stat > T Stats > One Sample > with data
2. Select column containig data
3. Input confidence level
4. Compute

Note:-

In the above example (by hand), we found the critical value to be $t_{0.025} = 2.131$ for 15 degrees of freedom, whereas $z_{0.025} = 1.96$. The t-distribution gives a larger critical value, so the width of the interval is wider. This larger critical value using Student's t-distribution is necessary to account for the increased variability due to using s as an estimate of σ.

Remember, 95% confidence refers to our confidence in the method. If we obtained 100 samples of size $n = 16$ from the population of 2011 Ford Focuses, we would expect about 95 of the samples to result in confidence intervals that include μ. We do not know whether the interval in Example 3 includes μ or does not include μ.

Determine the Sample Size Necessary for Estimating a Population Mean within a Given Margin of Error

The margin of error in constructing a confidence interval about the population mean is $E = t_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}$.

Solving this for n, we obtain $n = \left(\frac{t_{\frac{\alpha}{2}} \cdot s}{E}\right)^2$.

The problem with this formula is that the critical value $t_{\frac{\alpha}{2}}$ requires that we know the sample size to determine the degrees of freedom, $n - 1$. Obviously, if we do not know n, we cannot know the degrees of freedom.

The solution to this problem lies in the fact that the t-distribution approaches the standard normal z-distribution as the sample size increases. To convince yourself of this, look at the last few rows of Table VII and compare them to the corresponding z-scores for 95

Determining the Sample Size n

The sample size required to estimate the population mean, μ, with a level of confidence $(1 - \alpha) \times 100\%$ within a specified margin of error, E, is given by

$$n = \left(\frac{z_{\frac{\alpha}{2}} \cdot s}{E}\right)^2$$

Where n is rounded up to the nearest whole number.

Example: Determining Sample Size (By Hand)

Problem: We again consider the problem of estimating the miles per gallon of a 2011 Ford Focus. How large a sample is required to estimate the mean miles per gallon within 0.5 miles per gallon with 95% confidence? Note: The sample standard deviation is s=2.92 mpg.

Approach: Use $n = \left(\frac{z_{\alpha/2} \cdot s}{E}\right)^2$ with $z_{\alpha/2} = z_{0.025} = 1.96$, $s = 2.92$, and $E = 0.5$ to find the required sample size.

Solution:

$$n = \left(\frac{z_{\alpha/2} \cdot s}{E}\right)^2 = \left(\frac{1.96 \cdot 2.92}{0.5}\right)^2 = 131.02.$$

Round 131.02 up to 132. A sample size of $n = 132$ results in an interval estimate of the population mean miles per gallon of a 2011 Ford Focus with a margin of error of 0.5 mile per gallon with 95% confidence.

Example: Determining Sample Size (With Statcrunch)

1. Find Standard Deviation
2. Stat > Z Stats > One Sample > Width/Sample Size
3. Enter Confidence Lever
4. Enter Standard Deviation
5. Enter Sample Size
6. Compute

0.3 9.3: Putting It Together: Which Procedure Do I Use?

Learning Objectives For This Section:

1. **Determine the Appropriate Confidence Interval to Construct**

Determine the Appropriate Confidence Interval to Construct

Questions to ask

1. What is the variable of interest?
2. Are the conditions for constructing the interval satisfied?

For question one. *What is the variable of interest*, there are two possible situations for our purposes.

1. Qualitative variable with two outcomes: Analyze with proportions p
2. Quantitative variable: μ

Now that we know which model we are going to construct, we need to verify that the conditions for constructing an interval are satisfied.

1. Population Proportion:
   - $n\hat{p}(1-\hat{p}) \geqslant 10$, and $n \leqslant 0.05N$
2. Mean:
   - $n \geqslant 30$
   - if $n \leqslant 30$, we need to verify that the data comes from a population that is normally distributed. Thus, we need to make a normal probability plot to verify this and make a boxplot to check for outliers.