Discrete Structures

Notes

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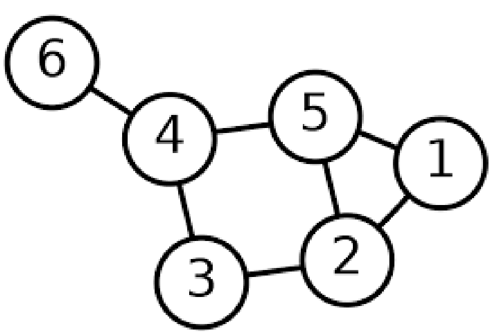

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1 Set Theory

1.1 Definition of a set

Definition: A **set** is a collection of elements
We denote sets with the following syntax:

$$A = \{1, 2, 3, 4\}.$$

Where in this case A is the identifier and it's elements are delimited by commas and encapsulated among braces.
Note: The identifier for sets are commonly represented with capital letters

We can also indicate *infinitely many* elements in a set by use of the **ellipsis**, which would look like:

$$A = \{1, 2, 3, ...\}$$
$$Generally: \ A = \{A_1, A_2, A_3, ..., A_n\}.$$

More Notation: We can indicate that an object is an **element** of a set with the following syntax:

$$A = \{1, 2, 3, 4\}$$
$$3 \in A.$$

1.2 Number Sets

The set of **Natural Numbers** (whole numbers) is denoted by $\mathbb{N}$:

$$\mathbb{N} : 1, 2, 3, ...$$

The set of **Integers** is denoted by $\mathbb{Z}$:

$$\mathbb{Z} : -5, -4, -3-, 2, -1, 0, 1, 2, 3, 4, 5, ...$$

So you can see the set of all integers is similar to that of the natural numbers, however this set includes *negative numbers*

The set of **Rational numbers**, (ratio of two integers), is denoted by $\mathbb{Q}$:

$$\mathbb{Q} : \frac{1}{6}, \frac{1}{4}, \frac{1}{2}, ...$$

The set of **Irrational numbers**, is denoted by $\bar{\mathbb{Q}}$:

$$\bar{\mathbb{Q}} : \pi, e, \sqrt{2}, \ etc.$$

Note:-

for a number to be considered irrational, they cannot be exactly represented as fractions of integers and have non-repeating, non-terminating decimal representations. Thus, the following condition must hold:

$$x \text{ is irrational} \iff \frac{a}{b}, \quad \text{where } a \wedge b \notin \mathbb{Z} \text{ and } \gcd(a, b) = 1..$$

The set of all **Real numbers** is denoted by $\mathbb{R}$:

$$\mathbb{R}: \text{ Both rational and irrational numbers.}$$

The set of all **imaginary numbers** is denoted by $\mathbb{I}$:

$$\mathbb{I}: \; i^2 = -1, \; i = \sqrt{-1}$$
$$Ex: \; \sqrt{-9} = \sqrt{9} \cdot \sqrt{-1} = 3i.$$

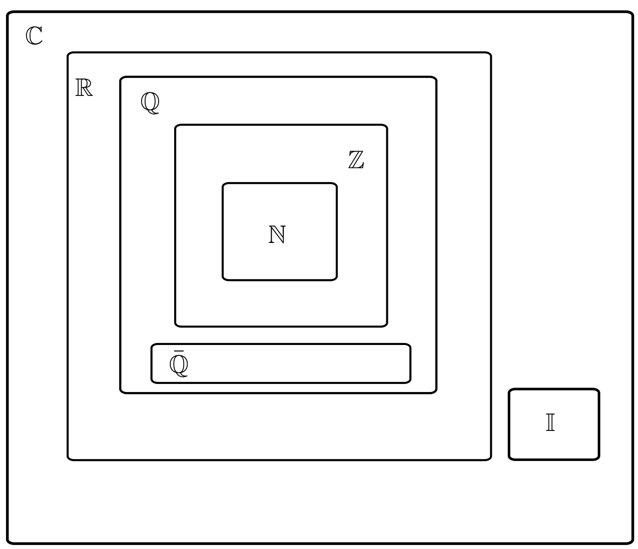
The set of **Complex numbers**, which describes numbers that are comprised of two components, one real and one imaginary, and is denoted by:

$$\mathbb{C}: 2 + 3i.$$

In summary:

- $\mathbb{N}$: Denotes the set of all **Natural Numbers**
- $\mathbb{Z}$: Denotes the set of all **Integers**
- $\mathbb{Q}$: Denotes the set of all **Rational Numbers**
- $\bar{\mathbb{Q}}$: Denotes the set of all **Irrational Numbers**
- $\mathbb{I}$: Denotes the set of all **Imaginary Numbers**
- $\mathbb{C}$: Denotes the set of all **Complex Numbers**

Figure:



1.3 Set Equality

> **Definition:** An **axiom** is a rule or statement that is generally accepted to be true without proof.
> An **Axiom of Extension** is a set determined by what its elements are - not the order in which they might be listed or the fact that some elements might be listed more than once.

Consider the sets:

$$A = \{1, 3, 5, 1, 5, 5, 3\}$$
$$B = \{1, 3, 5\}.$$

Because of the **Axiom of Extension**, which states that a set is not determined by the order or possible repetitions, we can conclude that $A = B$.

Furthermore, we can conclude that we only have 3 elements amongst set A, although it may seem like we have 7.

1.4 Set-Builder Notation

Set-Builder is a convention we can use when dealing with sets to imply the elements of a set without listing all of its values.

Suppose we have:

$$x = -5, 4, 3, -10, -5, 2, 0.$$

Then:

$$\{x|x < 0\} \text{ Reads: "The set of all x's such that (pipe) x is less than zero"}$$
$$= \{-10, -5\}.$$

So naturally you can infer that this set would be all x's from are defined pool of x values that are negative.

Additionally, we can utilize *Number Sets*:

$$\{x \in \mathbb{R}| - 2 < x < 5\}.$$

1.5 Types of Sets

- **Universal Set**: Denoted $\mathbb{U}$, represents the collection of all possible elements or objects that are under consideration for a particular context or problem.
- **Empty Set (Null set)**: Denoted $\emptyset$ (phi), represents a set that contains no elements
- **Singleton Set**: Represents a set that only has one element
- **Finite Set**: Represents a set that has a countable number of elements
- **Infinite Set**: Represents a set that has an infinite amount of elements
- **Subset**: A set in which all elements are part of a larger set

Definition: Cardinal Number of a Set: is the number of elements in a set, denoted $n(A)$. Where, in this case, A represents the name of the set.

Consider the set:

$$A = \{1, 2, 3\}$$
$$Then: \; n(A) = 3.$$

Where $n(A) = 3$ represents the cardinal number of the set

Definition: Equivalent Set: Represents sets that have the same *Cardinal Number*. To show that two sets are equivalent, we can use the notation:

$$A \sim B.$$

Which shows that the cardinality of A equals the cardinality of B

Consider the sets:

$$A = \{1, 4, 5\}$$
$$B = \{6, 8, 10\}.$$

Then we can say:

$$A \sim B.$$

1.6 Subsets

Definition. If **A** and **B** are sets, then **A** is called a **subset** of **B**, written $A \subseteq B$, if and only if every element of **A** is also an element of **B**
If **A** is a **subset** of **B**, and **B** has at least one additional element that is not in **A**, then **A**a is called a **proper subset** of **B**

1.7 Power Sets

Definition. The **Power set** of A, denoted $P(A)$, is the set of all subsets of A

Consider the set:

$$A = \{1, 2, 3\}.$$

Then by the power set of A, $P(A)$, would be:

$$P(A) = \{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

To calculate how many subsets are possible within a set, we can compute:

$$2^n$$

Where na is the number of elements in the set.

1.8 Cartesian Product

Definition: Given sets **A** and **B**, the **Cartesian product** of **A** and **B**, denoted $A \times B$, and read "**A** cross **B**", is the set of all ordered pairs (a, b), where a is in **A**, and b is in **B**.

$$A \times B = \{(a, b) \mid a \in A \textit{ and } b \in B\}.$$

Consider the sets:

$$A = \{1, 2\}$$
$$B = \{c, d\}.$$

Then:

$$A \times B = \{(1, c), (1, d), (2, c), (2, d)\}.$$

Consider the sets:

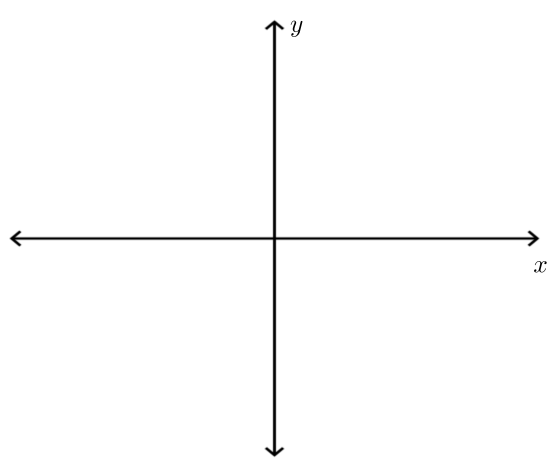
$$A = \{1, 2\}$$
$$B = \{\$, !\}$$
$$C = \{x, y\}.$$

Then:

$$A \times B = \{(1, \$), (1, !), (2, \$), (2, !)\}$$
$$(A \times B) \times C = \{((1, \$), x), ((1, \$), y), ..., so\ forth\}.$$

1.9 Cartesian Plane

Figure:



The way in which we denote all the possible points on the Cartesian plane is by denoting a cartesian product

$$\mathbb{R} \times \mathbb{R}$$
$$Or: \ \{(a, b) \mid a \in \mathbb{R},\ b \in \mathbb{R}\}$$
$$Or: \ \{(a, b) \mid (a, b) \in \mathbb{R}^2\}.$$

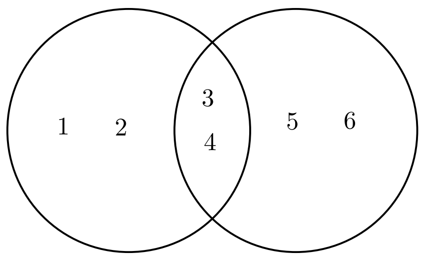
1.10 Venn Diagram

Definition: We use **Venn Diagram** to show relationships between sets

Consider the sets:

$$A = \{1, 2, 3, 4\}$$
$$B = \{3, 4, 5, 6\}.$$

With these two sets, we can construct the following Venn Diagram:



1.11 Set Operations (Union and Intersection)

We can use **set operations** on sets to create new sets

Set operators:

- $\cup$: **Denotes Union**, to find the union of two sets, we combine the elements of both sets into a new set
- $\cap$: **Denotes Intersection**, to find the intersection of two sets, we find the elements that are common in both sets.

Union:

Consider the sets:

$$A = \{1, 2, 3\}$$
$$B = \{4, 5, 6\}.$$

Then:

$$A \cup B = \{1, 2, 3, 4, 5, 6\}.$$

Intersection:

Consider the sets:

$$A = \{1, 2, 3\}$$
$$B = \{3, 5, 6\}.$$

Then:

$$A \cap B = \{3\}.$$

1.12 Properties of Union and Intersection

- $A \cup B = B \cup A, \quad A \cap B = B \cap A$ (Commutative Law)
- $(A \cup B) \cup C = A \cup (B \cup C), \quad (A \cap B) \cap C = A \cap (B \cap C)$ (Associative Law)
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive Law)
- $A \cup \emptyset = A, \quad A \cap \emptyset = \emptyset$
- $A \cup U = U$
- $A \cup A = A, \quad A \cap A = A$ (Idempotent Law)

1.13 Set Operations (Difference and Complement)

Difference:

Consider the sets:

$$A = \{1, 2, 3, 4\}$$
$$B = \{4, 5, 5\}.$$

Then $A - B$, read "A Difference B", would be:

$$A - B = \{1, 2, 3\}.$$

Complement

Consider the sets:

$$U = \{1, 2, 3, 4, 5\}$$
$$A = \{1, 2\}.$$

Then the complement of A would be:

$$A^c = \{3, 4, 5\}.$$

1.14 Properties of Difference and Complement

- $A \cup A^c = U$
- $(A^c)^c = A$
- $U^c = \emptyset$, $\emptyset^c = U$
- $A - B = A \cap B^c$

1.15 De Morgan's Laws

- $(A \cup B)^c = A^c \cap B^c$
- $(A \cap B)^c = A^c \cup B^c$

1.16 Partition of Sets

Definition: Two sets are called **disjoint**, if and only if they have no elements in common.
A finite or infinite collection of non empty sets $\{A_1, A_2, A_3, ...,\}$ is a Partition of a set A, if and only if

1. A is the union of all of the sets
2. The sets are mutually disjoint

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