Nate Warner MATH 230 December 6, 2023

Homework/Worksheet 11 - Due: Wednesday, November 15

1. Use term-by-term differentiation or integration to find a power series representation for each function centered at the given point.
   (a) $f(x) = \ln(1-x)$ centered at $x = 0$
   (b) $f(x) = \frac{2x}{(1-x^2)^2}$ centered at $x = 0$
   (c) $f(x) = \tan^{-1} x^2$ centered at $x = 0$
   (d) $f(x) = \ln(1+x^2)$ centered at $x = 0$

Problem 1a. Using the fact that $\frac{d}{dx} \ln(1-x) = -\frac{1}{1-x}$ and $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$

$$\begin{aligned}
-\int \frac{1}{1-x}\, dx &= -\int \sum_{n=0}^{\infty} x^n\, dx = -\int (1 + x + x^2 + x^3 + ...)\, dx && \text{for } |x| < 1 \\
\ln(1-x) &= \sum_{n=0}^{\infty} -\frac{x^{n+1}}{n+1} + C = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - ... && \text{for } |x| < 1
\end{aligned} \tag{1}$$

Conclusion. When $x = 0$, we find $C = 0$. Thus,

$$\ln(1-x) = \sum_{n=0}^{\infty} -\frac{x^{n+1}}{n+1} = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + ... \qquad \text{for } |x| < 1.$$

Problem 1b. Using the fact that $\int \frac{2x}{(1-x^2)^2}\, dx = \frac{1}{1-x^2}$, and $\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$ for $|x| < 1$

$$\begin{aligned}
\frac{d}{dx}\frac{1}{1-x^2} &= \frac{d}{dx}\sum_{n=0}^{\infty} x^{2n} = \frac{d}{dx}(1 + x^2 + x^4 + x^6 + ...) && \text{for } |x| < 1 \\
\frac{2x}{(1-x^2)^2} &= \sum_{n=0}^{\infty} 2nx^{2n-1} = 0 + 2x + 4x^3 + 6x^5 + ... && \text{for } |x| < 1
\end{aligned} \tag{2}$$

Conclusion. Thus we have

$$\frac{2x}{(1-x^2)^2} = \sum_{n=1}^{\infty} 2nx^{2n-1} \qquad \text{for } |x| < 1.$$

Problem 1c. Using the fact that $\frac{d}{dx}\tan^{-1}x^2 = \frac{2x}{1-(-x^4)}$ and $\frac{2x}{1-(-x^4)} = \sum_{n=0}^{\infty} (-1)^n 2x^{4n+1}$ for $|x| < 1$

Remark. The series $\sum_{n=0}^{\infty} 2x(-x^4)^n$ is in the form $\sum_{n=0}^{\infty} bx^m c_n x^n$. By properties of combining power series, we know that this series must converge to $bx^m f(x)$ on the same interval of convergence as the simpler series, Since we know that $\sum_{n=0}^{\infty} (-x^4)^n$ converges for $|x^4| < 1$, or $-1 < x < 1$. We can conclude that $2x(-x^4)^n$ must do the same.

Thus we have:

$$\begin{aligned}
\int \frac{2x}{1+x^4}\,dx &= \int \sum_{n=0}^{\infty} (-1)^n 2x^{4n+1}\,dx = \int (2x - 2x^5 + 2x^9 - 2x^{13} + ...)\,dx && \text{for } |x| < 1 \\
\tan^{-1} x^2 &= \sum_{n=0}^{\infty} (-1)^n \frac{2x^{4n+2}}{4n+2} + C \ = C + x^2 - \frac{1}{3}x^6 + \frac{1}{5}x^{10} - \frac{1}{7}x^{14} + ... && \text{for } |x| < 1 \qquad (3) \\
\tan^{-1} x^2 &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1} + C \ = C + x^2 - \frac{1}{3}x^6 + \frac{1}{5}x^{10} - \frac{1}{7}x^{14} + ... && \text{for } |x| < 1
\end{aligned}$$

Conclusion. When $x = 0$, $C = 0$. Thus the power series for $\tan^{-1} x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$ for $|x| < 1$

Problem 1d. Using the fact that $\frac{d}{dx}\ln(1+x^2) = \frac{2x}{1-(-x^2)}$ and $\frac{2x}{1-(-x^2)} = \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1}$ for $|x| < 1$. Similar to the last problem, we know that this series converges for $|x| < 1$ by properties of combining power series. Thus

$$\begin{aligned}
\int \frac{2x}{1+x^2}\,dx &= \int \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1}\,dx = \int (2x - 2x^3 + 2x^5 - 2x^7 + ...)\,dx && \text{for } |x| < 1 \\
\ln(1+x^2) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n+1} + C = C + x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 + ... && \text{for } |x| < 1
\end{aligned} \qquad (4)$$

Conclusion. When $x = 0$, $C = 0$. Thus, the power series for $\ln(1+x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n+1}$ for $|x| < 1$

2. Find the Taylor polynomial of degree two approximating the function $f(x) = \cos(2x)$ at $a = \pi$.

$$\begin{aligned} f(x) &= \cos(2x) & f(\pi) &= \cos(2\pi) = 1 \\ f'(x) &= -2\sin(2x) & f'(\pi) &= -2\sin(2\pi) = 0 \\ f''(x) &= -4\cos(2x) & f''(\pi) &= -4\cos(2\pi) = -4 \end{aligned} \tag{5}$$

We know the taylor series for a function f conforms to the form

$$f(x) \sim \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Conclusion. Thus $P_2(x)$ conforms to

$$\begin{aligned} P_2(x) &= 1 + 0(x-\pi) + \frac{-4}{2!}(x-\pi)^2 \\ &= 1 - 2(x-\pi)^2 \ . \end{aligned}$$

3. Find the Taylor series of the functions $f(x)$ centered at the given value of a

(a) $f(x) = \sin(x), \quad a = \pi$

(b) $f(x) = e^x, \quad a = -1$

(c) $f(x) = \ln(x), \quad a = 1$

(d) $f(x) = \frac{1}{2x-x^2}, \quad a = 1$

Problem 3a.

$$\begin{aligned} f(x) &= \sin(x) & f(\pi) &= \sin(\pi) = 0 \\ f'(x) &= \cos(x) & f'(\pi) &= \cos(\pi) = -1 \\ f''(x) &= -\sin(x) & f''(\pi) &= -\sin(\pi) = 0 \\ f'''(x) &= -\cos(x) & f'''(\pi) &= -\cos(\pi) = 1 \\ f^{(4)}(x) &= \sin(x) & f^{(4)}(\pi) &= \sin(\pi) = 0 \end{aligned} \tag{6}$$

Again, we use the Taylor series form

$$f(x) \sim \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ...$$

To get

$$\begin{aligned} \sin(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi)}{n!}(x-\pi)^n \ &= 0 + (-1)(x-\pi) + \frac{0}{2!}(x-\pi)^2 + \frac{1}{3!}(x-\pi)^3 + \frac{0}{4!}(x-\pi)^4 + \frac{-1}{5!}(x-\pi)^5 \\ &+ \frac{0}{6!}(x-\pi)^6 + \frac{1}{7!}(x-\pi)^7 + ... \\ &= -(x-\pi) + \frac{1}{3!}(x-\pi^3) - \frac{1}{5!}(x-\pi)^5 + \frac{1}{7!}(x-\pi)^7 + ... \end{aligned}$$

Thus we have

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-\pi)^{2n+1}}{(2n+1)!} \ .$$

Using the ratio test we see

$$\begin{aligned}
&\lim_{n\to\infty}\left|\frac{(x-\pi)^{2n}(x-\pi)^3}{(2n+3)(2n+2)(2n+1)!}\cdot\frac{(2n+1)!}{(x-\pi)^{2n}(x-\pi)}\right|\\
&(x-\pi)^2\lim_{n\to\infty}\frac{1}{(2n+3)(2n+2)}\\
&=0.
\end{aligned}$$

Conclusion. Thus the Taylor series has the form

$$\sin(x)=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-\pi)^{2n+1}}{(2n+1)!}\qquad \forall x\in\mathbb{R}.$$

Problem 3b.

$$\begin{aligned}
f(x)&=e^x & f(-1)&=\frac{1}{e}\\
f'(x)&=e^x & f(-1)&=\frac{1}{e}\\
f''(x)&=e^x & f(-1)&=\frac{1}{e}\\
&\vdots & &\vdots
\end{aligned}\tag{7}$$

By the definition of a Taylor series, we have

$$e^x=\frac{1}{e}+\frac{1}{e}(x+1)+\frac{1}{2!e}(x+1)^2+\frac{1}{3!e}(x+1)^3+....$$

Thus

$$e^x=\sum_{n=0}^{\infty}\frac{(x+1)^n}{n!e}\ .$$

Using the ratio test we see

$$\begin{aligned}
&\lim_{n\to\infty}\left|\frac{(x+1)^n(x+1)}{(n+1)n!e}\cdot\frac{n!e}{(x+1)^n}\right|\\
&|x+1|\lim_{n\to\infty}\frac{1}{n+1}\\
&=0.
\end{aligned}$$

Conclusion. Thus the Taylor series for e^x at center $a=-1$ is

$$e^x=\sum_{n=0}^{\infty}\frac{(x+1)^n}{n!e}\qquad \forall x\in\mathbb{R}.$$

Problem 3c.

$$\begin{aligned}
f(x)&=\ln(x) & f(1)&=0\\
f'(x)&=\frac{1}{x} & f'(1)&=1\\
f''(x)&=-\frac{1}{x^2} & f''(1)&=-1\\
f'''(x)&=\frac{2}{x^3} & f(1)&=2\\
f'''(x)&=-\frac{6}{x^4} & f(1)&=-6
\end{aligned}\tag{8}$$

By the definition of a Taylor series, we have

$$\begin{aligned}\ln(x) &= 0 + 1(x-1) + -\frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \frac{-6}{4!}(x-1)^4 + ... \\ &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4.\end{aligned}$$

Using the ratio test

$$\begin{aligned}&\lim_{n\to\infty}\left|\frac{(x-1)^n(x-1)^2}{n+2}\cdot\frac{n+1}{(x-1)^n(x-1)}\right| \\ &= |x-1|\lim_{n\to\infty}\frac{n+1}{n+2} \\ &\implies |x-1| < 1 \text{ or } 0 < x < 2 \\ &\therefore R = 2.\end{aligned}$$

When $x = 0$ we have

$$\begin{aligned}&\sum_{n=0}^{\infty}(-1)^n\frac{(-1)^{n+1}}{n+1} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^{2n+1}}{n+1} \\ &= \sum_{n=0}^{\infty}\frac{-1}{n+1} \\ &= -\sum_{n=0}^{\infty}\frac{1}{n+1}\end{aligned}$$

Which we know is divergent by the harmonic series. When $x = 2$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}.$$

Which converges by the alternating series test.

Conclusion. Thus the Taylor series has the form

$$\ln(x) \sim \sum_{n=0}^{\infty}(-1)^n\frac{(x-1)^{n+1}}{n+1} \qquad \forall x \in (0, 2].$$

Problem 3d.

Remark. By Uniqueness of Taylor series, if a function f has a power series at a that converges to f on some open interval containing a, then that power series is the Taylor series for f at a

We find the power series.

$$\frac{1}{2x - x^2} = \frac{1}{x(2-x)}.$$

By partial fraction decomposition

$$\begin{aligned} \frac{1}{x(2-x)} &= \frac{A}{x} + \frac{B}{2-x} \\ 1 &= A(2-x) + Bx. \end{aligned}$$

When $x = 0$, $A = \frac{1}{2}$. When $x = 2$, $B = \frac{1}{2}$. Thus we have

$$\begin{aligned} &\frac{1}{2x} + \frac{1}{2(2-x)} \\ &= \frac{\frac{1}{2}}{1-(-(x-1))} + \frac{\frac{1}{2}}{1-(x-1)}. \end{aligned}$$

With

$$\begin{aligned} \frac{\frac{1}{2}}{1-(-(x-1))} &= \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{2} \qquad \text{for } |x-1| < 1 \\ \frac{\frac{1}{2}}{1-(x-1)} &= \sum_{n=0}^{\infty} \frac{(x-1)^n}{2} \qquad \text{for } |x-1| < 1 \end{aligned} \tag{9}$$

Combining these series we have

$$\sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{2} + \sum_{n=0}^{\infty} \frac{(x-1)^n}{2} \quad \text{for } |x-1| < 1 \ .$$

Writing out the first few terms we see.

$$\begin{aligned} &\left(\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{x-1}{2} + \frac{x-1}{2}\right) + \left(\frac{(x-1)^2}{2} + \frac{(x-1)^2}{2}\right) + \left(-\frac{(x-1)^3}{2} + \frac{(x-1)^3}{2}\right) + \left(\frac{(x-1)^4}{2} + \frac{(x-1)^4}{2}\right) + ... \\ &= 1 + (x-1)^2 + (x-1)^4 + ... \end{aligned}$$

Which can be represented as

$$\sum_{n=0}^{\infty} (x-1)^{2n} \quad \text{for } (x-1)^2 < 1 \ .$$

4. Find the Maclaurin series for $f(x) = x\cos(x)$ using the definition of a Maclaurin series. Also find the associated radius of convergence.

$$\begin{aligned} f(x) &= x\cos(x) & f(0) &= 0 \\ f'(x) &= -x\sin(x) + \cos(x) & f'(0) &= 1 \\ f''(x) &= -x\cos(x) - 2\sin(x) & f''(0) &= 0 \\ f'''(x) &= x\sin(x) - 3\cos(x) & f'''(0) &= -3 \\ f^{(4)}(x) &= x\cos(x) + 4\sin(x) & f^{(4)}(0) &= 0 \\ f^{(5)}(x) &= -x\sin(x) + 5\cos(x) & f^{(5)}(0) &= 5 \end{aligned} \tag{10}$$

Thus,

$$\begin{aligned} x\cos(x) &= 0 + 1(x-0) + \frac{0}{2!}(x-0)^2 + \frac{-3}{3!}(x-0)^3 + \frac{0}{4!}(x-0)^4 + \frac{5}{5!}(x-0)^5 \\ &= x - \frac{1}{2!}x^3 + \frac{1}{4!}x^5 + ... \end{aligned}$$

So we see

$$x\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!} \ .$$

To find the interval of convergence we use the ratio test

$$\begin{aligned} \rho = &\lim_{n\to\infty} \left| \frac{x^{2n}x^2}{(2n+1)(2n)!} \cdot \frac{(2n)!}{x^{2n}x} \right| \\ &|x| \lim_{n\to\infty} \frac{1}{2n+1} \\ &= 0. \end{aligned}$$

Thus, this series must converge $\forall x \in \mathbb{R}$, which implies the radius of convergence is $R = \infty$

5. Use a known Maclaurin series to obtain the Maclaurin series for the given functions.

   (a) $f(x) = x\cos(2x)$

   (b) $f(x) = e^{3x} - e^{2x}$

Problem 5a.

$$\begin{aligned}
\cos(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\
\implies \cos(2x) &= \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{4^n x^{2n}}{(2n)!} \\
\therefore x\cos(2x) &= x \cdot \sum_{n=0}^{\infty} (-1)^n \frac{4^n x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{4^n x^{2n+1}}{(2n)!} .
\end{aligned}$$

Problem 5b.

$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
\implies e^{3x} &= \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} \\
\implies e^{2x} &= \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} \\
\therefore e^{3x} - e^{2x} &= \sum_{n=0}^{\infty} \frac{3^n x^n}{n!} - \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{x^n (3^n - 2^n)}{n!} .
\end{aligned}$$