Nate Warner MATH 336 Spring 2026

Problem set 4 - Due: Monday, February 9

1. Solve the initial value problem

$$xy' + 2y = 3x, \quad y(1) = 5.$$

This is a first order linear differential equation, with standard form

$$y' + \frac{2}{x}y = 3.$$

So, let $P(x) = \frac{2}{x}$, $Q(x) = 3$, and define

$$\mu(x) = e^{\int P(x)\ dx} = e^{2\int \frac{1}{x}\ dx} = e^{2\ln(x)} = x^2.$$

So,

$$y' + \frac{2}{x}y = 3 \implies x^2y' + 2xy = 3x^2$$
$$\implies (x^2y)' = 3x^2 \implies x^2y = \int 3x^2\ dx$$
$$\implies x^2y = x^3 + C \implies y = x + \frac{C}{x^2}.$$

With the initial condition $y(1) = 5$,

$$5 = 1 + \frac{c}{1^2} \implies C = 4.$$

Thus,

$$y(x) = x + \frac{4}{x^2}.$$

Where the initial of definition of this solution is $(0, \infty)$.

2. Solve the initial value problem

$$xy' - y = x, \quad y(1) = 7.$$

This is a linear first order differential equation with standard form

$$y' - \frac{1}{x}y = 1.$$

Let $P(x) = -\frac{1}{x}$, $Q(x) = 1$, and define

$$\mu(x) = e^{\int P(x)\ dx} = e^{-\int \frac{1}{x}\ dx} = e^{-\ln(x)} = \frac{1}{x}.$$

So,

$$\begin{aligned}
y' - \frac{1}{x}y = 1 &\implies \frac{1}{x}y' - \frac{1}{x^2}y = \frac{1}{x} \\
\implies \left(\frac{1}{x}y\right)' = \frac{1}{x} &\implies \frac{1}{x}y = \int \frac{1}{x}\,dx \\
\implies \frac{1}{x}y = \ln|x| + C &\implies y = x\ln|x| + Cx.
\end{aligned}$$

With the initial condition $y(1) = 7$,

$$7 = (1)\ln(1) + C(1) \implies C = 7.$$

Thus,

$$y(x) = x\ln|x| + 7x.$$

Notice that the initial of definition is $(0, \infty)$, so

$$y(x) = x\ln(x) + 7x$$

since $x > 0$ for all x in the interval.

3. Solve the initial value problem

$$y' + 2xy = x, \quad y(0) = -2.$$

This is a first order linear differential equation already in standard form. Let $P(x) = 2x$, $Q(x) = x$, and define

$$\mu(x) = e^{\int P(x)\,dx} = e^{2\int x\,dx} = e^{x^2}.$$

Then,

$$\begin{aligned}
y' + 2xy = x &\implies e^{x^2}y' + 2xe^{x^2}y = xe^{x^2} \\
\implies (e^{x^2}y)' = xe^{x^2} &\implies e^{x^2}y = \int xe^{x^2}\,dx.
\end{aligned}$$

Regarding the integral $\int xe^{x^2}\,dx$, let $u = x^2$, then $du = 2x\,dx$, so $\frac{1}{2}\,du = x\,dx$. Thus,

$$\int xe^{x^2}\,dx = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u = \frac{1}{2}e^{x^2} + C.$$

So,

$$\begin{aligned}
e^{x^2}y = \int xe^{x^2}\,dx &\implies e^{x^2}y = \frac{1}{2}e^{x^2} \\
\implies y = \frac{1}{e^{x^2}}\left(\frac{1}{2}e^{x^2} + C\right) &= \frac{1}{2} + \frac{C}{e^{x^2}}.
\end{aligned}$$

With the initial condition $y(0) = -2$,

$$-2 = \frac{1}{2} + \frac{C}{e^0} \implies C = -2 - \frac{1}{2} = -\frac{5}{2}.$$

Thus,

$$y(x) = \frac{1}{2} - \frac{5}{2e^{x^2}} = \frac{1}{2}\left(1 - \frac{5}{e^{x^2}}\right)$$

with interval of definition $(-\infty, \infty)$.

4. Solve the initial value problem

$$(1+x)y' + y = \cos(x), \quad y(0) = 1.$$

This is a first order linear differential equation with standard form

$$y' + \frac{1}{1+x}y = \frac{\cos(x)}{1+x}.$$

Let $P(x) = \frac{1}{1+x}$, $Q(x) = \frac{\cos(x)}{1+x}$, and define

$$\mu(x) = e^{\int P(x)\,dx} = e^{\int \frac{1}{1+x}\,dx} = e^{\ln(1+x)} = 1+x.$$

Then,

$$\begin{aligned} y' + \frac{1}{1+x}y = \frac{\cos(x)}{1+x} &\implies (1+x)y' + y = \cos(x) \\ \implies ((1+x)y)' = \cos(x) &\implies (1+x)y = \int \cos(x)\,dx \\ \implies (1+x)y = \sin(x) + C &\implies y = \frac{1}{1+x}\left(\sin(x) + C\right). \end{aligned}$$

With the initial condition $y(0) = 1$,

$$1 = \frac{1}{1+0}\left(\sin(0) + C\right) \implies C = 1.$$

Thus,

$$y(x) = \frac{1}{1+x}\left(\sin(x) + 1\right)$$

with interval of definition $(-1, \infty)$.

5. A tank contains 1000 liters of a solution consisting of 100 kg of salt dissolved in water. Pure water is pumped into the tank at the rate of 5 L/s, and the mixture, kept uniform by stirring, is pumped out at the same rate. How long will it be until only 10 kg of salt remains in the tank?

Let $x(t)$ be the amount of salt (in kg) in the tank at time t, and $v(t)$ be the volume of liquid in the tank at time t, we have

$$v(t) = 1000 + (5-5)t = 1000 \text{ L}.$$

Now, the inflow of salt into the tank is given by

$$0\ \frac{\text{kg}}{\text{L}}\left(5\ \frac{\text{L}}{\text{s}}\right) = 0\ \frac{\text{kg}}{\text{s}}.$$

The outflow of salt water from the tank is given by

$$\frac{x(t)}{v(t)}\ \frac{\text{kg}}{\text{L}}\left(5\ \frac{\text{L}}{\text{s}}\right) = \frac{5x(t)}{1000}\ \frac{\text{kg}}{\text{s}}.$$

Thus, the differential equation is

$$\frac{dx}{dt} = 0 - \frac{5}{1000}x(t) = \frac{1}{200}x(t).$$

Observe that the standard from is

$$\frac{dx}{dt} + \frac{1}{200}x(t) = 0.$$

Notice that this is a first order linear differential equation. Let $P(t) = \frac{1}{200}$, $Q(t) = 0$, and define

$$\mu(t) = e^{\int P(t)\, dt} = e^{\int \frac{1}{200}\, dt} = e^{\frac{1}{200}t}.$$

Then,

$$e^{\frac{1}{200}t}\frac{dx}{dt} + \frac{e^{\frac{1}{200}t}}{200} = 0 \implies \left(e^{\frac{1}{200}t}x(t)\right)' = 0$$
$$\implies e^{\frac{1}{200}t}x(t) = \int 0\, dt \implies x(t) = \frac{C}{e^{\frac{1}{200}t}}.$$

With the initial condition $x(0) = 100$,

$$100 = \frac{C}{e^0} \implies C = 100.$$

Thus,

$$x(t) = \frac{100}{e^{\frac{1}{200}t}}.$$

Now we can find $x(10)$,

$$10 = \frac{100}{e^{\frac{1}{200}t}} \implies e^{\frac{1}{200}t} = 10$$
$$\implies \frac{t}{200} = \ln(10).$$

Thus, $t = 200\ln(10)$.

6. A tank initially contains 60 gal of pure water. Brine containing 1 lb of salt per gallon enters the tank at 2 gal/min, and the (perfectly mixed) solution leaves the tank at 3 gal/min; thus the tank is empty after exactly 1 h.

(a) Find the amount of salt in the tank after t minutes.

(b) What is the maximum amount of salt ever in the tank?

Let $x(t)$ be the amount of salt in the tank at time t, and $v(t)$ be the volume of liquid in the tank at time t, we have

$$v(t) = 60 + (2-3)t = 60 - t \text{ gal}.$$

The inflow of salt into the tank is given by

$$1\ \frac{\text{lb}}{\text{gal}}\left(2\ \frac{\text{gal}}{\text{min}}\right) = 2\ \frac{\text{lbs}}{\text{min}}.$$

The outflow of solution is given by

$$\frac{x(t)}{v(t)}\ \frac{\text{lbs}}{\text{gal}}\left(3\ \frac{\text{gal}}{\text{min}}\right) = \frac{3x(t)}{v(t)}\ \frac{\text{lbs}}{\text{min}}.$$

Thus, the differential equation is

$$\frac{dx}{dt} = 2 - \frac{3}{60-t}x(t) \implies \frac{dx}{dt} + \frac{3}{60-t}x(t) = 2.$$

Let $P(t) = \frac{3}{60-t}$, $Q(t) = 2$, and define

$$\mu(t) = e^{\int P(t)\ dt} = e^{3\int \frac{1}{60-t}\ dt} = e^{-3\ln(60-t)} = \frac{1}{(60-t)^3}.$$

So,

$$\begin{aligned} \frac{dx}{dt} + \frac{3}{60-t}x(t) = 2 = \frac{1}{(60-t)^3}\frac{dx}{dt} + \frac{3}{(60-t)^4}x(t) = \frac{2}{(60-t)^3} \\ \implies \left(\frac{1}{(60-t)^3}x(t)\right)' = \frac{2}{(60-t)^3} \implies \frac{1}{(60-t)^3}x(t) = 2\int \frac{1}{(60-t)^3}\ dt. \end{aligned}$$

Regarding $2\int \frac{1}{(60-t)^3}\ dt$, let $u = 60-t$, then $-du = dt$. So,

$$2\int \frac{1}{(60-t)^3}\ dt = -2\int u^{-3}\ du = u^{-2} = \frac{1}{(60-t)^2} + C.$$

Thus,

$$x(t) = (60-t)^3\left(\frac{1}{(60-t)^2} + C\right) = (60-t) + C(60-t)^3.$$

With the initial condition $y(0) = 0$,

$$0 = 60 + 60^3 C \implies C = -\frac{60}{60^3} = -\frac{1}{60^2}.$$

Thus,

$$x(t) = (60-t) - \frac{1}{60^2}(60-t)^3.$$

If we differentiate $x(t)$ with respect to t, we find

$$\frac{dx}{dt} = -1 - \frac{1}{60^2}(3(60-t)^2(-1)) = \frac{3}{60^2}(60-t)^2 - 1.$$

Setting equal to zero gives

$$\begin{aligned} \frac{3}{60^2}(60-t)^2 - 1 = 0 \implies 60 - t = \frac{60}{\sqrt{3}} \\ \implies t = 60 - \frac{60}{\sqrt{3}} = 60 - 20\sqrt{3} \approx 25.4. \end{aligned}$$

Thus, the maximum amount of salt in the tank occurs at $t = 25.4$ minutes.

7. A 400-gal tank initially contains 100 gal of brine containing 50 lb of salt. Brine containing 1 lb of salt per gallon enters the tank at the rate of 5 gal/s, and the well mixed brine in the tank flows out at the rate of 3 gal/s. How much salt will the tank contain when it is full of brine?

Let $x(t)$ be the amount of salt at time t, and $v(t)$ be the amount of liquid at time t, we have

$$v(t) = 100 + (5 - 3)t = 100 + 2t = 2(50 + t).$$

The inflow of salt into the tank is given by

$$1(5) = 5\ \frac{\text{lbs}}{\text{s}}.$$

The outflow is given by

$$\frac{x(t)}{v(t)}(3) = \frac{3x(t)}{100 + 2t}\ \frac{\text{lbs}}{\text{s}}.$$

Thus, the differential equation is given by

$$\frac{dx}{dt} = 5 - \frac{3}{100 + 2t}x(t) \implies \frac{dx}{dt} + \frac{3}{100 + 2t}x(t) = 5.$$

Let $P(t) = \frac{3}{100+2t}$, $Q(t) = 5$, and define

$$\mu(t) = e^{\int P(t)\ dt} = e^{3\int \frac{1}{100+2t}\ dt} = e^{\frac{3}{2}\int \frac{1}{50+t}\ dt} = e^{\frac{3}{2}\ln(50+t)} = (50 + t)^{\frac{3}{2}}.$$

Then,

$$\begin{aligned}
&\frac{dx}{dt} + \frac{3}{100 + 2t}x(t) = 5 \implies (50 + t)^{\frac{3}{2}}\frac{dx}{dt} + \frac{3(50 + t)^{\frac{3}{2}}}{2(50 + t)} = 5(50 + t)^{\frac{3}{2}} \\
&\implies \left((50 + t)^{\frac{3}{2}}x(t)\right)' = 5(50 + t)^{\frac{3}{2}} \implies (50 + t)^{\frac{3}{2}}x(t) = \int 5(50 + t)^{\frac{3}{2}}\ dt \\
&\implies x(t) = \frac{1}{(50 + t)^{\frac{3}{2}}}\left(2(50 + t)^{\frac{5}{2}} + C\right) = 2(50 + t) + \frac{C}{(50 + t)^{\frac{3}{2}}}.
\end{aligned}$$

With the initial condition $x(0) = 50$,

$$50 = 2(50) + \frac{C}{50^{\frac{3}{2}}} \implies C = -50(50^{\frac{3}{2}}) = -50^{\frac{5}{2}}.$$

Thus,

$$x(t) = 2(50 + t) + \frac{-50^{\frac{5}{2}}}{(50 + t)^{\frac{3}{2}}}.$$

To find how much salt will be in the tank when the tank is full of brine, we observe that the tank is full of brine when $v(t) = 400$, so

$$2(50 + t) = 400 \implies 50 + t = 200 \implies t = 150.$$

Thus,

$$x(150) = 2(50 + 150) - \frac{50^{\frac{5}{2}}}{(50 + 150)^{\frac{3}{2}}} = 400 - \frac{50^{\frac{5}{2}}}{200^{\frac{3}{2}}} \approx 393.75 \text{ lbs}.$$