Calculus 1 Chapter 5 Notes

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Chapter 5

Contents

Review

Sigma (Summation) Notation

Sigma Notation is written like:

$$\sum_{i=m}^{n} a^i.$$

Where n is the number to stop the sum, and i=m tells us where to start our sum. a_i is the general term.

Expanded sum:

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + ... + a_n.$$

Note:-

i is called the ***index of summation***

Properties:

- $\sum_{i=m}^{n} c \cdot a_i = c \sum_{i=m}^{n} a_i$, where c is a constant
- $\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$
- $\sum_{i=m}^{n} (a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$
- $\sum_{i=1}^{n} 1 = n$
- $\sum_{i=1}^{n} c = c \cdot n$, where c is a constant
- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

Example: Write in expanded form:

$$\sum_{i=1}^{5} \sqrt{i}.$$

So:

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5}$$

Example: Write in sigma notation:

$$1 + 2 + 4 + 8 + 16 + 32.$$

So:

$$\sum_{i=0}^{6} 2^i \text{ or } \sum_{i=1}^{6} 2^{i-1}$$

Example: Write in expanded form:

$$\sum_{k=0}^{4} \frac{2k-1}{2k+1}.$$

So:

$$\begin{aligned}\frac{0-1}{0+1} + \frac{2-1}{2+1} + \frac{4-1}{4+1} + \frac{6-1}{6+1} + \frac{8-1}{8+1} \\ = -1 + \frac{1}{3} + \frac{3}{5} + \frac{5}{7} + \frac{7}{9}\end{aligned}$$

Example: Find the value of the sum:

$$\sum_{i=1}^{n} (i^3 - i - 2).$$

Using properties of summations we can rewrite as:

$$
\begin{aligned}
&\sum_{i=1}^{n} i^3 - \sum_{i=1}^{n} i - \sum_{i=1}^{n} 2 \\
&= \left[\frac{n(n+1)}{2}\right]^2 - \frac{n(n+1)}{2} - 2n \\
&= \frac{n^2(n+1)^2}{4} - \frac{n(n+1)}{2} - 2n \\
&= \frac{n^2(n+1)^2}{4} - \frac{2n(n+1)}{4} - \frac{8n}{4} \\
&= \frac{n^2(n^2+2n+1) - 2n^2 - 2n - 8n}{4} \\
&= \frac{n^4 + 2n^3 + n^2 - 2n^2 - 10n}{4} \\
&= \frac{n^4 + 2n^3 - n^2 - 10n}{4} \\
&\boxed{= \frac{n(n^3 + 2n^2 - n - 10)}{4}}.
\end{aligned}
$$

Example: Evaluate the telescoping sum

$$\sum_{i=1}^{n} i^4 - (i-1)^4.$$

Start by expanding the sum and see what pattern emerges

So:

$$(i^4 - 0^4) + (2^4 - 1^4) + (3^4 - 2^4) + ... \ + [(n-2)^4 - (n-3)^4] + [(n-1)^4 - (n-2)^4] + [n^4 - (n-1)^4]$$

You can see that we have Evaluated the sum for the first 3 terms, as well as for the last 3, and if you look at the terms, you can see that most of them will end up being canceled except for 0^4 *in the beginning and* n^4 *at the end*

So we are left with:

$$0 + n^4$$
$$\boxed{n^4}.$$

Note:-

Most of the terms ended up canceling out or *collapsing*, which is why it's called a ***telescoping sum***, it's important to note that hoever many terms end up surviving in the beginning of the sum, will be matched at the end of the sum.

Example: Evaluate the telescoping sum:

$$\sum_{i=3}^{99} \left(\frac{1}{i} - \frac{1}{i+1}\right).$$

So:

$$\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+...+\left(\frac{1}{97}-\frac{1}{98}\right)+\left(\frac{1}{98}-\frac{1}{99}\right)+\left(\frac{1}{99}-\frac{1}{100}\right)$$

Again you can see that most of the terms will end up canceling out except:

$$\begin{aligned} &\frac{1}{3}-\frac{1}{100} \\ &= \frac{100}{300}-\frac{3}{300} \\ &\boxed{= \frac{97}{300}} \end{aligned}$$

.

Example: Find the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2.$$

We will start by pulling out constants:

So:

$$\lim_{n\to\infty} \frac{1}{n} \cdot \frac{1}{n^2} \sum_{i=1}^{n} i^2$$

From here we can clean up the expression and replace i^2 with the property of summations

So:

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} \\ &= \lim_{n\to\infty} \frac{1}{n^2} \frac{(n+1)(2n+1)}{6} \\ &= \lim_{n\to\infty} \frac{2n^2+3n+1}{6n^2} \end{aligned}$$

Since the degree of the numerator is equal to the degree of the denominator the limit is going to be the ratio of the leading coefficients.

So:

$$\begin{aligned} &\frac{2}{6} \\ &\boxed{= \frac{1}{3}} \end{aligned}$$

Example: Find the limit:

$$\lim_{n\to\infty}\sum_{i=1}^{n} \frac{2}{n}\left[\left(\frac{2i}{n}\right)^3+5\left(\frac{2i}{n}\right)\right].$$

So again we will start by pulling out constants:

So:

$$\lim_{n\to\infty}\frac{2}{n}\cdot\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^3+5\left(\frac{2i}{n}\right)\right]$$
$$=\lim_{n\to\infty}\frac{2}{n}\cdot\sum_{i=1}^{n}\left[\left(\frac{8i^3}{n^3}\right)+5\left(\frac{2i}{n}\right)\right]$$

From here we can factor out an $\frac{2}{n}$

$$=\lim_{n\to\infty}\frac{2}{n}\cdot\frac{2}{n}\cdot\sum_{i=1}^{n}\left[\frac{4i^3}{n^2}+5i\right]$$
$$=\lim_{n\to\infty}\frac{4}{n^2}\left[\frac{4}{n^2}\sum_{i=1}^{n} i^3+5\sum_{i=1}^{n} i\right]$$
$$=\lim_{n\to\infty}\frac{4}{n^2}\left[\frac{4}{n^2}\left(\left(\frac{n(n+1)}{2}\right)\right)^2+5\left(\frac{n(n+1)}{2}\right)\right]$$
$$=\lim_{n\to\infty}\frac{4}{n^2}\left[\frac{n(n+1)}{2}\left(\frac{4n(n+1)}{2n^2}\right)+5\right]$$
$$=\lim_{h\to\infty}\frac{4}{n^2}\left(\frac{n(n+1)}{2}\right)\left(\frac{4n^2+4n+10n^2}{2n^2}\right)$$
$$=\lim_{n\to\infty}\frac{(n+1)(14n^2+4n)}{n^3}$$
$$=\lim_{n\to\infty}\frac{14n^3+18n^2+4n}{n^3}.$$

Again, since the degree of the numerator is equal to the degree of the denominator, we take the ration of the leading coefficients to evaluate the limit.

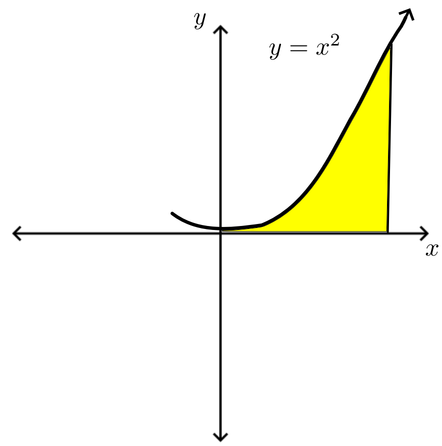
So:

$$\frac{14}{1}$$
$$\boxed{=14}$$

5.1

Areas and Distances

The Area Problem

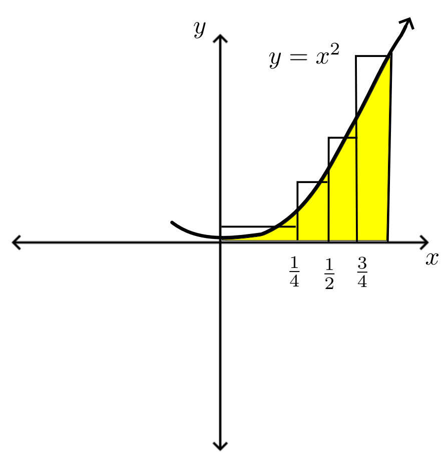


Find the shaded area.

We dan't have an exact formula to compute, so we'll use approximation.

Process.

We'll use rectangles to estimate the area.



Using right endpoints, we draw 4 rectangles:

- $A_1 = (\frac{1}{4})(\frac{1}{4})^2 = \frac{1}{64}$
- $A_2 = (\frac{1}{4})(\frac{1}{2})^2 = \frac{1}{16}$
- $A_3 = (\frac{1}{4})(\frac{3}{4})^2 = \frac{9}{64}$
- $A_4 = (\frac{1}{4})(1)^2 = \frac{1}{4}$

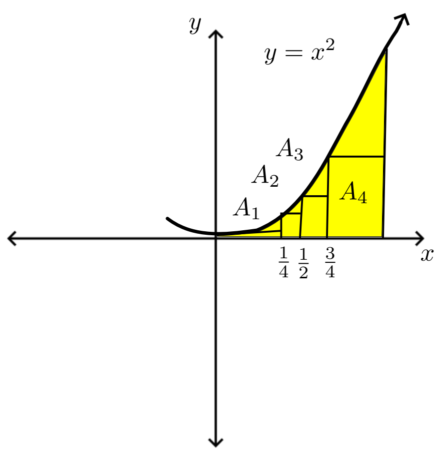
Note:-

We find height by pulling x into the formula for the graph, (x^2)

Let $r_4 = A_1 + A_2 + A_3 + A_4 = \frac{15}{32}$

So we can say the shaded region is $< R_4$

Similarly, we form 4 rectangles using left endpoints.



- $A_1 = (\frac{1}{4})(0)^2 = 0$
- $A_2 = (\frac{1}{4})(\frac{1}{4})^2 = \frac{1}{64}$
- $A_3 = (\frac{1}{4})(\frac{1}{2})^2 = \frac{1}{16}$
- $A_4 = (\frac{1}{4})(\frac{3}{4})^2 = \frac{9}{64}$

Let $L_4 = A_1 + A_2 + A_3 + A_4 = \frac{7}{32}$

So we can say the shaded region $> L_4$

Now we have:

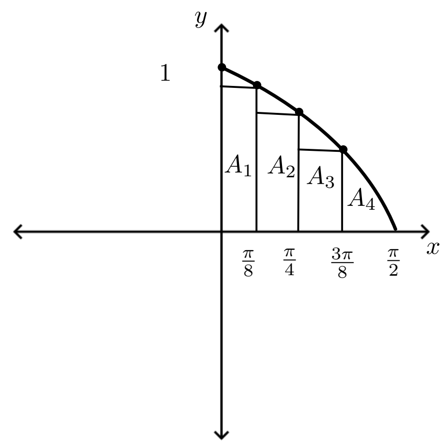
$$L_4 < A < R_4.$$

We can improve the estimation by increasing the nmuber of rectangles. Therefore:

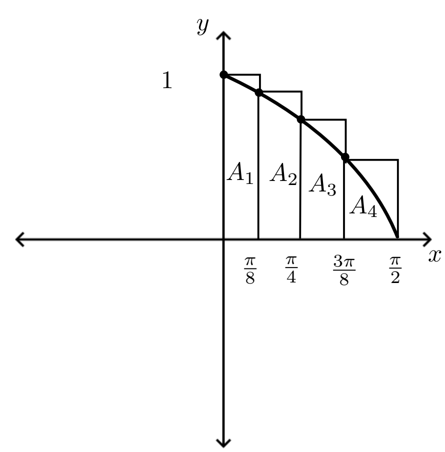
$$A \approx 0.33.$$

Example: Estimate the area under the graph of $f(x) = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$ using 4 rectangles

- a.) Right endpoints
- b.) Left endpoints
- c.) Which is an understimate/overestimate

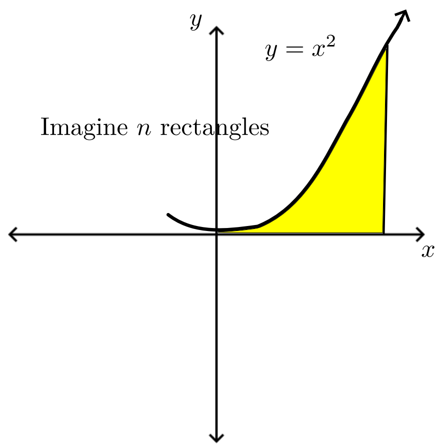


$$R_4 = \frac{\pi}{8}(\cos\frac{\pi}{8} + \cos\frac{\pi}{4} + \cos\frac{3\pi}{8} + \cos\frac{\pi}{2}) \approx 0.7908 \quad \boxed{underestimate}.$$



$$\begin{aligned} L_4 &= \frac{\pi}{8}(\cos 0 + \cos\frac{\pi}{8} + \cos\frac{3\pi}{8} + \cos\frac{\pi}{2}) \\ &\approx 1.1835 \\ &\boxed{overestimate}. \end{aligned}$$

Now we'll use a formula for n number of rectangles:



In the previous x^2 example, we found the length of the base given the interval $[0, 1]$ by:

$$\frac{1-0}{4} = \frac{1}{4}.$$

Now that we have n rectangles, each rectangle has widith $\frac{1-0}{n}$ and height $\left(\frac{1}{n}\right)^2, \left(\frac{2}{n}\right)^2$...

So:

$$\begin{aligned} R_n &= \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \frac{1}{n}\left(\frac{3}{n}\right)^2 + ... + \frac{1}{n}\left(\frac{n}{n}\right)^2 \\ &= \frac{1}{n}\left(\frac{1}{n^2} + \frac{2^2}{n^2} + \frac{3^3}{n^2} + ... + \frac{n^2}{n^2}\right) \\ &= \frac{1}{n} \cdot \frac{1}{n^2}(1 + 2^2 + 3^2 + ... + n^2) \\ &= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\ &= \frac{n(n+1)(2n+1)}{6n^2} \end{aligned}$$

And:

$$\begin{aligned} \lim_{n\to\infty} R_n &= \frac{2n^2+3n+1}{6n^2} \\ &= \frac{2}{6} \\ &= \frac{1}{3}. \end{aligned}$$

Similarly, we can show that:

$$\lim_{n\to\infty} L_n = \frac{1}{3}.$$

Therefore:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \frac{1}{3}.$$

Definition:

The area under the graph of a continuous function $f(x)$ is:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [\Delta x f(x_1) + \Delta x f(x_2) + ... + \Delta x f(x_n)].$$

or

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [\Delta x f(x_0) + \Delta x f(x_1) + ... + \Delta x f(x_{n-1})].$$

or

$$A = \lim_{n\to\infty} [\Delta x f(x_1^*) + ... + \Delta x f(x_n^*)].$$

Where x_i^* is any number in the ith interval.

Note:-

$\Delta x =$ Base of each rectangle

Find Δx with $\frac{b-a}{n}$, on [a,b]

Using $\sum$ ”Sigma” notation, we have:

- $A = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(x_i)$ (Right endpoints)
- $A = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(x_{i-1})$ (Left endpoints)
- $A = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(x_i^*)$ (Arbitrary partiton)

Note:-

The one with the star denotes not using left or right endpoints

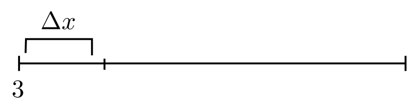
Example: Find an expression for the area under the graph of f as a limit.

$$f(x) = x^2 + \sqrt{1+2x},\ 3 \leqslant x \leqslant 10.$$

We need:

$$\begin{aligned}\Delta x &= \frac{b-a}{n}\\ &= \frac{10-3}{n}\\ &= \frac{7}{n}.\end{aligned}$$

Now we need a formula for x_i, if we draw a number line:



We can see that:

$$x_1 = 3 + \Delta x.$$

And:

$$x_2 = 3 + 2\Delta x.$$

So we can see that the general formula for x_i would be:

$$x_i = a + i\Delta x.$$

So our x_i is:

$$\begin{aligned}x_i &= 3 + i \cdot \frac{7}{n}\\ &= 3 + \frac{7i}{n}.\end{aligned}$$

Now substitute x_i into the function

So:

$$f(x_i) = \left(3 + \frac{7i}{n}\right)^2 + \sqrt{1 + 2\left(3 + \frac{7i}{n}\right)}$$

Put everything together:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{7}{n}\right)\left[\left(3 + \frac{7i}{n}\right)^2 + \sqrt{7 + \frac{14i}{n}}\,\right].$$

Where:

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \underbrace{\left(\frac{7}{n}\right)}_{Base} \underbrace{\left[\left(3 + \frac{7i}{n}\right)^2 + \sqrt{7 + \frac{14i}{n}}\,\right]}_{Height}.$$

5.2

The Definite Integral

From 5.1, Definition 2 gave the following limit for computing exact areas under the graph of $f(x)$ on [a,b]:

$$A = \lim_{n\to\infty} R_n = \lim_{x\to\infty} \sum_{i=1}^{n} \Delta x \cdot f(x_i).$$

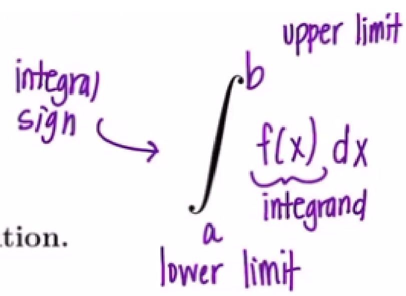
In 5.2, we give the above setup a name: **Definite Integral:**

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(x_i^*).$$

Provided the limit exists and x_i^* is any point in each subinterval

This technique is called **integration**.

Terms:



ation.

Note:-

From definition 2:

$$A = \lim_{x\to\infty} \sum_{i=1}^{n} \Delta x \cdot f(x_i).$$

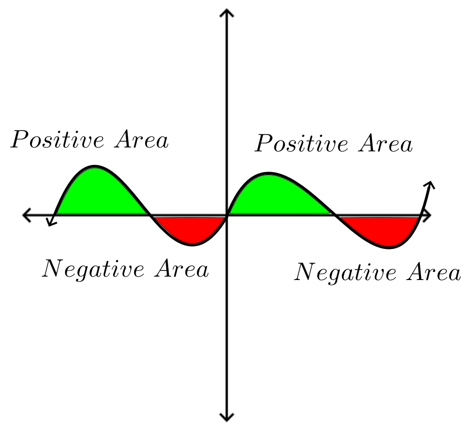
Just:

$$\sum_{i=1}^{n} \Delta x f(x_i).$$

is called the **Rieman sum**

Also, if the regoin lies below the x-axis, its area will have a - sign

Example:



Properties of integrals:

- $\int_a^b c dx = c(b-a)$
- $\int_a^b cf(x)dx = c \cdot \int_a^b f(x)dx$
- $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
- $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$
- $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$
- if $f(x) \geqslant 0$ for all $a \leqslant x \leqslant b$, then $\int_a^b f(x)dx \geqslant 0$
- if $f(x) \geqslant g(x)$ for all $a \leqslant x \leqslant b$, then $\int_a^b f(x)dx \geqslant \int_a^b g(x)dx$
- if $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then $m(b-a) \leqslant \int_a^b f(x)dx \leqslant M(b-a)$

Example: $f(x) = \sin x,\ 0 \leqslant x \leqslant \frac{3\pi}{2}$

Find the riemann sum with 6 terms, taking the sample points to be the right endpoints.

First find Δx

If:

$$\Delta x = \frac{b-a}{n}.$$

Then:

$$\Delta x = \frac{\frac{3\pi}{2} - 0}{6} = \frac{\pi}{4}.$$

Next, find your 6 x values by adding $\frac{\pi}{4}$ *to* Δx

$$\left\{ \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2} \right\}.$$

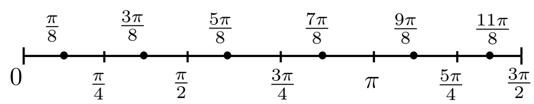
Now compute:

$$\Delta x \cdot (f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6)).$$

So:

$$\begin{aligned}\frac{\pi}{4}\left(f\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{2}\right) + f\left(\frac{3\pi}{4}\right) + f\left(\pi\right) + f\left(\frac{5\pi}{4}\right) + f\left(\frac{3\pi}{2}\right)\right)\\ = \frac{\pi}{4}\left(\frac{\sqrt{2}}{2}\right)\\ = \frac{\pi\sqrt{2}}{4}\\ \boxed{\approx 0.5554}\end{aligned}$$

Repeat with midpoints as sample points:



Here we divided $\frac{\pi}{4}$ by 2 and added $\frac{\pi}{4}$ to each new subinterval to aquire our midpoints

Note:-

we can see that our Δx is still $\frac{\pi}{4}$

So:

$$\begin{aligned}\frac{\pi}{4}\left(\sin\frac{\pi}{8} + \sin\frac{3\pi}{8} + \sin\frac{5\pi}{8} + \sin\frac{7\pi}{8} + \sin\frac{9\pi}{8} + \sin\frac{11\pi}{8}\right)\\ \boxed{\approx 1.0262}\end{aligned}$$

Example: Use midpoints with the given value of n to approximate the integral

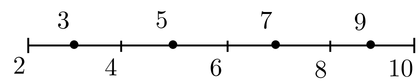
$$\int_2^{10} \sqrt{x^3+1}dx, \ n = 4.$$

So:

$$a = 2, \ b = 10, \ f(x) = \sqrt{x^3+1}, \ n = 4$$

Find Δx

$$\begin{aligned}\Delta x &= \frac{10-2}{4}\\ &= 2.\end{aligned}$$



Find midpoints:

So:

$$A \approx \sum_{i=1}^{4} \Delta x f(x_i) =$$

Which is:

$$\begin{aligned} A &\approx 2(f(3) + f(5) + f(7) + f(9)) \\ &\boxed{\approx 124.16}. \end{aligned}$$

Theorem 4:

$$\int_a^b f(x)dx = \lim_{x \to \infty} \Delta x \cdot f(x_i).$$

Example: Evaluate using theorem 4:

$$\int_1^4 (x^2 + 2x - 5)dx.$$

Steps:

1. Find Δx
2. Find x_i $(a + i \cdot \Delta x)$
3. Find $f(x_i)$

1.)

$$\begin{aligned} \Delta x &= \frac{4-1}{n} \\ &= \frac{3}{n}. \end{aligned}$$

2.)

$$x_i = 1 + \frac{3i}{n}$$
.

3.)

$$f(x_i) = \left(1 + \frac{3i}{n}\right)^2 + 2\left(1 + \frac{3i}{n}\right) - 5.$$

Now use the fact that:

$$\int_a^b f(x)dx = \lim_{x\to\infty} \Delta x \cdot f(x_i).$$

So:

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=1}^{n} \left(\frac{3}{n}\right)\left[\left(1+\frac{3i}{n}\right)^2 + 2\left(1+\frac{3i}{n}\right) - 5\right] \\
&= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n} \left(1+\frac{6i}{n}+\frac{9i^2}{n^2}+2+\frac{6i}{n}-5\right) \\
&= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n} \left(\frac{9i^2}{n^2}+\frac{12i}{n}-2\right)
\end{aligned}$$

From here use properties of summation:

$$\begin{aligned}
&\lim_{n\to\infty}\frac{3}{n}\left[\sum_{i=1}^{n} \frac{9}{n}i^2 + \sum_{i=1}^{n} \frac{12}{n}i - \sum_{i=1}^{n} 2\right] \\
&= \lim_{n\to\infty}\frac{3}{n}\left[\frac{9}{n^2} \sum_{i=1}^{n} i^2 + \frac{12}{n}\sum_{i=1}^{n} i - 2n\right] \\
&= \lim_{n\to\infty}\frac{3}{n}\left[\frac{9}{n^2}\frac{n(n+1)(2n+1)}{6} + \frac{12}{n}\cdot\frac{n(n+1)}{2} - 2n\right].
\end{aligned}$$

Now using limit rules we have:

$$\begin{aligned}
&\lim_{n\to\infty}\frac{9}{2}\frac{(n+1)(2n+1)}{n^2} + \lim_{n\to\infty}\frac{18(n+1)}{n} - \lim_{n\to\infty} 6 \\
&= \lim_{n\to\infty}\frac{9}{2}\frac{(2n^2+3n+1)}{n^2} + \lim_{n\to\infty}\frac{18n+18}{n} - \lim_{n\to\infty} 6 \\
&= \frac{9}{2}\cdot 2 + 18 - 6 \\
&\boxed{21}.
\end{aligned}$$

Note:-

Recall if the degree of the numerator is equal to the degree of the denominator we take the ratio of the leading terms

Example: Evaluate in terms of areos. (Hint: Use Geometry!)

$$\int_{-2}^{2} \sqrt{4-x^2}dx.$$

So:

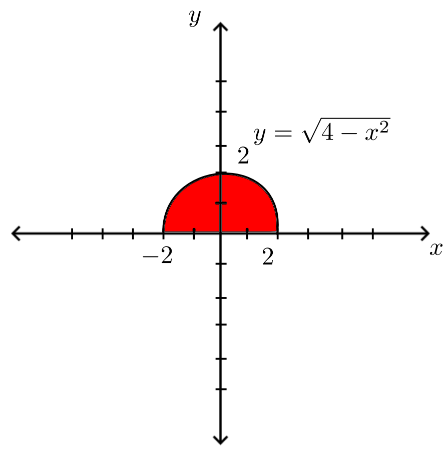
$$f(x) = \sqrt{4-x^2}$$

Note:-

This is an equation for a semicircle, because:

$$y = \sqrt{4-x^2}$$
$$y^2 = 4-x^2$$
$$x^2+y^2 = 4 \ r = 2.$$

We know it's only the upper half of the semicircle because we only have the positive version



So:

$$A = \frac{\pi r^2}{2}$$
$$= \frac{\pi(2)^2}{2}$$
$$\boxed{= 2\pi}$$

Example: Evaluate in terms of areas. (Hint: Use Geometry)

$$\int_{-1}^{3}(3-2x)dx.$$

So:

$$f(x) = 3-2x$$

Note:-

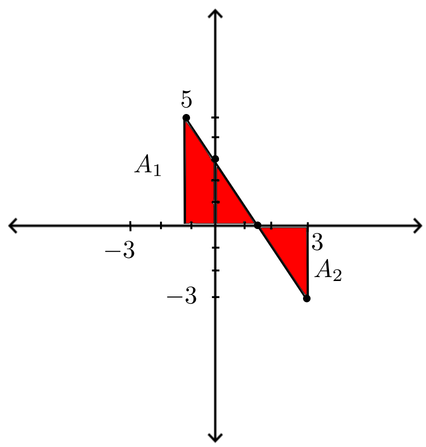
This is an equation for a line ($y = mx + b$)

$$x - int: \ 0 = -2x+3$$
$$x = \frac{3}{2}, \ so \ (\frac{3}{2}, 0).$$

$$y - int: \ (0, 3).$$

Evaluate at endpoints:

$$f(-1) = 5$$
$$f(3) = -3.$$



Notice we have 2 triangles, so find the area for each, $(A = \frac{1}{2}bh)$

$$\begin{aligned} A_1 &= \frac{1}{2}(2.5)(5) \\ &= 6.25 \\ A_2 &= \frac{1}{2}(1.5)(3) \\ &= 2.25 \\ Net\ Area &= A_1 - A_2 \\ &\boxed{6.25 - 2.25 = 3.25}. \end{aligned}$$

Note:-

We subtracted A_2 because A_2 was below the x axis therefore it is ***negative area***

Comparison Properties of the integral inequalities with integrals

Properties:

1. If $f(x) \geqslant 0$ for all $a \leqslant x \leqslant b$, then $\int_a^b f(x)dx \geqslant 0$
2. If $f(x) \geqslant g(x)$ for all $a \leqslant x \leqslant b$, then $\int_a^b f(x)dx \geqslant \int_a^b g(x)dx$
3. if $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then

$$m(b-a) \leqslant \int_a^b f(x)dx \leqslant M(b-a).$$

Example: Use the properties of integrals to verify the inequality without evaluating the integrals.

$$\int_0^1 \sqrt{1+x^2}\ dx \leqslant \int_0^1 \sqrt{1+x}\ dx.$$

With **Property 2.**

$$If\ f(x) \geqslant g(x)\ for\ all\ a \leqslant x \leqslant b,\ then\ \int_a^b f(x)\ dx \geqslant \int_a^b g(x)\ dx.$$

On, the interval [0,1] We know

$$x^2 \leqslant x.$$

Therefore

$$\begin{aligned} 1+x^2 &\leqslant 1+x \\ &so \\ \sqrt{1+x^2} &\leqslant \sqrt{1+x}\ on\ [0,1]. \end{aligned}$$

Hence we have showed that

$$\int_0^1 \sqrt{1+x^2}\ dx \leqslant \int_0^1 \sqrt{1+x}\ dx.$$

Example:

$$2 \leqslant \int_{-1}^{1} \sqrt{1+x^2}\ dx \leqslant 2\sqrt{2}.$$

With property 3, which states:

- if $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then

$$m(b-a) \leqslant \int_a^b f(x)dx \leqslant M(b-a).$$

We know x has to obey

$$\begin{aligned} x &\in [-1,1] \\ &or \\ -1 \leqslant x &\leqslant 1. \end{aligned}$$

Next we want to try to make the x in the inequality above become our function $\sqrt{1+x^2}$

Step 1.) Square everything in the equality to get x^2

$$0 \leqslant x^2 \leqslant 1.$$

Note:-

Since we squared the inequality, all the negatives are gone, but zero remains. Therefore our inequality is now bounded between 0 and 1.

Step 2.) Add one to the inequality.

$$1 \leqslant 1 + x^2 \leqslant 2.$$

Step 3.) Take the square root of the inequality.

$$1 \leqslant \sqrt{1+x^2} \leqslant \sqrt{2}.$$

Now we can apply the third comparison property.

$$\begin{aligned} 1(1-(-1)) \leqslant \int_{-1}^{1} \sqrt{1+x^2}\ dx &\leqslant \sqrt{2}(1-(-1)) \\ 2 \leqslant \int_{-1}^{1} \sqrt{1+x^2}\ dx &\leqslant 2\sqrt{2}. \end{aligned}$$

Example: Use the comparison properties of integras to prove the inequality.

$$\int_1^3 \sqrt{x^4+1}\ dx \geqslant \frac{26}{3}.$$

Since our function is not easily integratable, we will show that it is greater than something we can easily integrate.

$$\sqrt{x^4+1} \geqslant \sqrt{x^4} = x^2.$$

So:

$$\int_1^3 \sqrt{x^4+1}\ dx \geqslant \int_1^3 x^2\ dx$$

Evaluate $\int_1^3 x^2\ dx$ by finding the antiderivitive

$$\begin{aligned} &\frac{1}{3}x^3\Big]_1^3 \\ &= \frac{1}{3}(3^3 - 1^3) \\ &\boxed{= \frac{26}{3}}. \end{aligned}$$

Therefore we have shown that

$$\int_1^3 \sqrt{x^4+1}\ dx \geqslant \frac{26}{3}.$$

Example

$$\int_0^{\frac{\pi}{2}} x \sin x\ dx \leqslant \frac{\pi^2}{8}.$$

Let's try and bound our $f(x)$ above something we can easily integrate. We know $\sin x$ is bounded between -1 and 1, but in this case we are only looking at quadrant I. In this scenario $\sin x$ is bounded between 0 and 1.

So.

$$\sin x \leqslant 1\ if\ 0 \leqslant x \leqslant \frac{\pi}{2}.$$

Like the previous example, we want our inequality to involve our $f(x)$, so we multiply everything by x

$$x \sin x \leqslant x.$$

Now we can integrate

$$\int_0^{\frac{\pi}{2}} x \sin x \ dx \leqslant \int_0^{\frac{\pi}{2}} x \ dx.$$

So:

$$\begin{aligned} &\frac{1}{2}x^2\Big]_0^{\frac{\pi}{2}} \\ &= \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - 0^2\right] \\ &\boxed{= \frac{\pi^2}{8}} \end{aligned}$$

Example:

$$\int_0^1 \sqrt{1+e^{2x}} \ dx \geqslant e - 1.$$

$$\begin{aligned} \sqrt{1+e^{2x}} > \sqrt{e^{2x}} &= e^x \\ \sqrt{1+e^x} &\geqslant e^x \\ &. \end{aligned}$$

$$\begin{aligned} \int_0^1 \sqrt{1+e^x} \ dx \geqslant \int_0^1 e^x \ dx \\ = e^x\Big]_0^1 \\ = e^1 - e^0 \\ \boxed{= e - 1}. \end{aligned}$$

Example:

$$\int_0^1 e^x \cos x \ dx \leqslant e - 1.$$

We know:

$$\begin{aligned} \cos x &\leqslant 1 \\ &\mathit{and} \\ e^x &> 0. \end{aligned}$$

Therefore:

$$e^x \cos x \leqslant e^x.$$

So:

5.3

The Fundamental Theorem of Calculus

Theorem:

If f is continuous on [a,b], then:

$$Part\ 1:\ \frac{d}{dx}\int_a^x f(x)\ dt = f(x), a \leqslant x \leqslant b$$
$$Part\ 2:\ \int_a^b f(x)\ dx = F(b) - F(a)\ where\ F' = f..$$

Example: Use Part 1 of the Fundamental Theorem of calculus to find the derivative of the function

$$g(x) = \int_1^x e^{t^2-t}\ dt.$$

We will make use of the property:

$$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx.$$

So:

$$g'(x) = \frac{d}{dx}\int_1^x e^{t^2-t}\ dt.$$

Replace instances of t with x, because x is our upper limit.

$$g'(t) = \frac{d}{dx}\int_1^x = \boxed{e^{x^2-x}}.$$

Example:

$$g(t) = \int_0^t \sqrt{r^2+4}\ dx.$$

Replace instances of r with t

$$g'(t) = \frac{d}{dt}\int_0^t \sqrt{r^2+4}\ dr = \boxed{\sqrt{t^2+4}}.$$

Example:

$$G(x) = \int_x^1 \cos\sqrt{t}\ dt.$$

Notice we cannot apply the Fundamental thoerem of calculus because our upper limit is not a constant, therefore we must utilize the property:

$$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx.$$

So:

$$-\int_1^x \cos\sqrt{t}\ dt$$

Which means:

$$\boxed{G'(x) = -\cos\sqrt{t}\ dt}.$$

Example:

$$y = \int_1^{\cos r} (a + x^2)^{10}\ dx.$$

Replace x with upper limit:

$$\begin{aligned} y' &= (1 + (\cos r)^2)^{10}\ \frac{d}{dr}(\cos r) \\ &= (1 + \cos^2 r)^{10}(-\sin r) \\ &\boxed{= -\sin r(1 + \cos^2 r)^{10}}. \end{aligned}$$

Now let's look at part 2 of the Fundamental theorem of calculus.

Which States:

$$\int_a^b f(x)\ dx = F(b) - F(a)\ where\ F' = f.$$

Example:

$$\int_1^8 \sqrt[3]{x}\ dx.$$

Find the antiderivitive

$$\frac{x^{\frac{1}{3}+1}}{\frac{4}{3}} = \frac{3}{4}x^{\frac{4}{3}}$$

Now

$$\frac{3}{4}x^{\frac{4}{3}}\Big]_1^8.$$

To evaluate, use

$$F(b) - f(a).$$

So:

$$\begin{aligned} &\frac{3}{4}(8^{\frac{4}{3}} - 1^{\frac{4}{3}}) \\ &= \frac{3}{4}(16-1) \\ &\boxed{= \frac{45}{4}} \end{aligned}$$

Example:

$$\int_0^2 (y-1)(2y+1)\ dy.$$

So:

$$\int_0^2 (2y^2 - y - 1)\ dy$$

Find the antiderivitive and evaluate

$$\begin{aligned} &\frac{2}{3}y^3 - \frac{1}{2}y^2 - y\Big]_0^2 \\ &= \left(\frac{2}{3}\cdot 2^3 - \frac{1}{2}\cdot 2^2 - 2\right) - (0) \\ &\boxed{\frac{4}{3}}. \end{aligned}$$

Example:

$$\int_0^1 10^x \ dx.$$

So:

$$\frac{10x}{\ln 10}\Big]_0^1$$

$$= \frac{1}{\ln 10}\left[10^1 - 10^0\right]$$

$$\boxed{\frac{9}{\ln 10}}$$

Example:

$$\int_1^2 \frac{4+u^2}{u^3} \ du.$$

Rewrite

$$\int_1^2 \left(\frac{4}{u^3} + \frac{u^2}{u^3}\right) du$$

$$= \int_1^2 (4u^{-3} + u^{-1}) \ du.$$

Find the antiderivitive

$$-2u^2 + \ln|u|\Big]_1^2$$

$$= (-2(2)^{-2} + \ln 2) - (-2(1)^{-2} + \ln 1)$$

$$\boxed{\frac{3}{2} + \ln 2}.$$

Example: Find $g'(x)$ if:

$$\int_{\tan x}^{x^2} \frac{1}{\sqrt{2+t^4}} \ dt.$$

rewrite:

$$g'(x) = \frac{d}{dx}\int_{\tan x}^{x^2} \frac{1}{\sqrt{2+t^4}} \ dt.$$

Now we are going to split up the integral

$$\frac{d}{dx}\left(\int_{\tan x}^{0} \frac{1}{\sqrt{2+t^4}} \ dt + \int_0^{x^2} \frac{1}{\sqrt{2+t^4}} \ dt\right).$$

Since the first integrals upper limit is not a function of x, we will flip the limits of integration and add a negative sign.

$$\frac{d}{dx}\left(-\int_0^{\tan x} \frac{1}{\sqrt{2+t^4}}\,dt + \int_0^{x^2} \frac{1}{\sqrt{2+t^4}}\,dt\right).$$

Substitute upper limit for x

$$\frac{-1}{\sqrt{2+\tan^4 x}} \cdot \sec^2 x + \frac{1}{\sqrt{2+x^8}} \cdot 2x$$
$$\boxed{= \frac{2x}{\sqrt{2+x^8}} - \frac{\sec^2 x}{\sqrt{2+\tan^4 x}}}.$$

Note:-

Note that you have to multiply by the derivative of the upper limit when you substitute

5.4

Indefinite Integrals and the Net Change Theorem.

Indefinite Integrals

$$\int f(x)\ dx = F(x).$$

Means:

$$F'(x) = f(x).$$

Note:-

When you have an indefinite integral, instead of computing the area under the curve like the definite integral, indefinite integrals is essentially just finding the antiderivative. So when you see $\int$ just think antiderivative

Properties of Indefinite Integrals:

- $\int cf(x)\ dx = c\int f(x)\ dx$
- $\int [f(x) + g(x)]\ dx = \int f(X)dx + \int g(x)dx$
- $\int kdx = kx + C$
- $\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ (n \neq -1)$
- $\int \sin x dx = -\cos x + C$
- $\int \cos x dx = \sin x + C$
- $\int \sec^2 x dx = \tan x + C$
- $\int \csc^2 x dx = -\cot x + C$
- $\int \sec x \tan x\ dx = \sec x + C$
- $\int \csc x \cot x dx = -\csc x + C$

Example: Find the general indefinit integral

$$\int (\sqrt{x^3} + \sqrt[3]{x^2})\ dx.$$

Rewrite:

$$\int (x^{\frac{3}{2}} + x^{\frac{2}{3}})\ dx.$$

Antidifferentiate:

$$\boxed{\frac{2}{5}x^{\frac{5}{2}} + \frac{3}{5}x^{\frac{5}{3}} + C}.$$

Example:

$$\int v(v^2+2)^2 \ dx.$$

So:

$$\int v(v^4+4v^2+4) \ dv$$
$$= \int v^5+4v^3+4v \ dv$$
$$\boxed{= \frac{1}{6}v^6+v^4+2v^2+C}$$

Example:

$$\int \sec t(\sec t+\tan t) \ dt$$

.

So:

$$\int (\sec^2 t+\sec t\tan t) \ dt$$
$$\boxed{= \tan t+\sec t+C}$$

Example:

$$\int \frac{\sin 2x}{\sin x} \ dx.$$

So:

$$\int \frac{2\sin x\cos x}{\sin x} \ dx$$
$$= \int 2\cos x \ dx$$
$$\boxed{= 2\sin x+C}$$

Example: Definite Integrals

$$\int_0^4 (2v+5)(3v-1) \ dv.$$

So:

$$\int_0^4 (6v^2+13v-5) \ dv$$
$$= 2v^3+\frac{13}{2}v^2-5v\Big]_0^4$$
$$= (2\cdot 4^3+\frac{13}{2}\cdot 4^2-5\cdot 4)-(0)$$
$$\boxed{= 212}$$

Example: Definite Integral

$$\int_0^9 \sqrt{2t}\ dt.$$

Rewrite:

$$\begin{aligned}
&\int_0^9 2^{\frac{1}{2}}\ t^{\frac{1}{2}}\ dt \\
&= 2^{\frac{1}{2}} \cdot \frac{2}{3} t^{\frac{3}{2}} \Big]_0^9 \\
&= \sqrt{2}\left[\frac{2}{3}\cdot 9^{3/2} - \frac{2}{3}\cdot 0\right] \\
&\boxed{= 18\sqrt{2}}.
\end{aligned}$$

Example: Definite Integral

$$\int_0^{\frac{\pi}{3}} \frac{\sin\theta + \sin\theta(\tan^2\theta)}{\sec^2\theta}\ d\theta$$

.

So:

$$\begin{aligned}
&= \int_0^{\frac{\pi}{3}} \left(\sin\theta\cos^2\theta + \sin\theta\cdot\cos^2\theta\cdot\frac{\sin^2\theta}{\cos^2\theta}\right) d\theta \\
&= \int_0^{\frac{\pi}{3}} (\sin\theta\cos^2\theta + \sin^3\theta)\ d\theta \\
&= \int_0^{\frac{\pi}{3}} \sin\theta(\cos^2\theta + \sin^2\theta)\ d\theta \\
&= \int_0^{\frac{\pi}{3}} \sin\theta\ d\theta \\
&= -\cos\theta\Big]_0^{\frac{\pi}{3}} \\
&= -\frac{1}{2} - (-1) \\
&\boxed{= \frac{1}{2}}.
\end{aligned}$$

Example: Definite Integral

$$\int_0^2 |2x-1| \; dx.$$

Rewrite as picewise:

$$\begin{aligned} 2x-1 &= 0 \\ x &= \frac{1}{2}. \end{aligned}$$

So:

$$f(x) = |2x+1| = \begin{cases} 2x-1 & \text{if } x \geqslant \frac{1}{2} \\ -(2x-1) & \text{if } x < \frac{1}{2} \end{cases} \tag{1}$$

Split integral:

$$\begin{aligned} \int_0^{\frac{1}{2}} -(2x+1) \; dx + \int_{\frac{1}{2}}^2 (2x-1) \; dx \\ = -\left[x^2-x\right]_{\frac{1}{2}}^0 + \left[x^2-x\right]_{\frac{1}{2}}^2 \\ = -\left(\left(\frac{1}{4}-\frac{1}{2}\right)-0\right) + \left(\left(4-2\right)-\left(\frac{1}{4}-\frac{1}{2}\right)\right) \\ = 2+\frac{1}{2} \\ \boxed{=\frac{5}{2}}. \end{aligned}$$

The Net Change Theorem

$$\int_a^b f(x)\ dx = \int_a^b F'(x)\ dx = F(b) - f(a).$$

the integral of a rate of change is the net change.

Example: The current in a wire is defined as the derivative of the charge:

$$I(t) = Q'(t).$$

What does $\int_a^b I(t)\ dt$ represent?

$$\int_a^b I(t)\ dt = \int_a^b Q'(t)\ dt$$

$= Q(b) - Q(a) \rightarrow$ Net change of the charge, Q(t), over [a,b].

Example: A honeybee population starts with 100 bees and increases at a rate of $n'(t)$ bees per week. What does $100 + \int_0^{15} n'(t)\ dt$ represent?

100: Initial number of bees.

$\int_0^{15} n'(t)\ dt$: Net change in the number of bees after 15 weeks.

The total quantity represents the nember of bees after 15 weeks.

Example: If $f(x)$ is the slope of a trail at a distance of x miles from the start of the trail, what does $\int_3^5 f(x)\ dx$ represent?

Antiderivitive of $f(x)$ = elevation

$\int_3^5 f(x)\ dx$ overall change in elevation from 3 to 5 miles.

Particle Motion:

Displacement of an object moving along a straight line:

$$\int_{t_1}^{t_2} v(t)\ dt = s(t_2) - s(t_1).$$

Distance traveled by an object:

$$\int_{t_1}^{t_2} |v(t)|\ dt.$$

Example:

$$v(t) = t^2 - 2t - 8$$
$$1 \leqslant t \leqslant 6.$$

Find the displacement and the distnace traveled by a particle with the given velocity function.

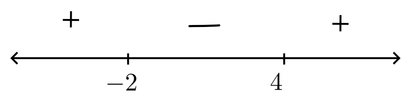
a.) Displacement:

$$\begin{aligned} Displacement &= \int_1^6 v(t)\ dt \\ &= \int_1^6 (t^2 - 2t - 8)\ dt \\ &= \frac{1}{3}t^3 - t^2 - 8t\Big]_1^6 \\ &= \left(\frac{1}{3}(216) - 36 - 48\right) - \left(\frac{1}{3} - 1 - 8\right) \\ &= -12 + 9 - \frac{1}{3} \\ &\boxed{= -\frac{10}{3}}. \end{aligned}$$

b.) Distance Traveled:

$$\begin{aligned} Distance\ Traveled: &\int_1^6 |v(t)|\ dt \\ &= \int_1^6 |t^2 - 2t - 8|\ dt \\ |v(t)| &= |(t-4)(t+2)|. \end{aligned}$$

Test with number line:



So:

$$|v(t)| = |(t-4)(t+2)| = \begin{cases} t^2 - 2t - 8 & \text{if } t \leqslant -2,\ t \geqslant 4 \\ -(t^2 - 2t - 8) & \text{if } -2 \leqslant t \leqslant 4 \end{cases} \tag{2}$$

$$\int_1^4 -(t^2 - 2t - 8)\ dt + \int_4^6 (t^2 - 2t - 8)\ dt$$

$$\boxed{= \frac{98}{3}}.$$

Neat Trick

If we have:

$$\int_1^6 |v(t)|\ dt.$$

Which in our previous example would be:

$$\int_1^6 |t^2 - 2t - 8|\ dt.$$

Then find the zeros:

$$(t-4)(t+2).$$

Just find where you need to split the integral, and take the absolute value of both:

$$\left| \int_1^4 (t^2 - 2t - 8)\ dt \right| + \left| \int_4^6 (t^2 - 2t - 8)\ dt \right|.$$

5.5

Substitution Rule

Definition:

If $u = g(x)$ is differentiable and its range $\in I$ and f is continuous on I, then

$$\int f(g(x))g'(x)\ dx = \int f(u)\ du.$$

Process:

1. Make a decent choice of what to let u equal
2. Change our integral from being in terms of x, to in terms of u
3. Integrate $\int f(u)\ du$
4. Change back to x

Note:-

Also include the constant in your u sub, if it is attached to the function you let u equal, Also note for rational trig functions, you can move trig functions from upstairs or downstairs based on their reciprical function, for example, a $\cos^2 x$ in the denominator can be moved upstairs as $\sec^2 x$

Example: Evaluate the following:

$$\int x^2(x^3+5)^9\ dx.$$

Let $u = x^3 + 5$, We know this is a good choice because the derivative of u should be sitting somewhere in our original intergal. You might notice that $\frac{du}{dx} = 3x^2$, but we will fix this shortly.

To make:

$$\frac{du}{dx} = 3x^2.$$

Match our original intergal, we do the following:

$$\frac{du}{dx} = 3x^2$$
$$du = 3x^2 dx$$
$$\frac{1}{3}du = x^2 dx.$$

From here we can substitute in u, so:

$$\begin{aligned} &\int u^9 \cdot \frac{1}{3}\ du \\ &= \frac{1}{3}\int u^9\ du \\ &= \frac{1}{3} \cdot \frac{1}{10} u^{10} + C. \end{aligned}$$

Lastly, replace swap back in our function that we let u equal:

$$\boxed{\frac{1}{30}(x^3+5)^{10} + C}.$$

Example:

$$\int e^x (\sin e^x)\ dx.$$

Let $u = e^x$, so:

$$\begin{aligned} \frac{du}{dx} &= e^x \\ du &= e^x dx. \end{aligned}$$

Rewrite integral:

$$\begin{aligned} &\int \sin u\ du \\ &= -\cos u + C. \end{aligned}$$

Replace u:

$$\boxed{-\cos e^x + C}.$$

Example:

$$\int \frac{dt}{\cos^2 t\sqrt{1+\tan t}}\ .$$

Let $u = 1 + \tan t$

$$\begin{aligned} \frac{du}{dt} &= 1 + \tan t \\ \frac{du}{dt} &= \sec^2 t \\ du &= \sec^2 t\ dt. \end{aligned}$$

Let's rewrite our integral such that $\cos^2 t$ is moved upstairs as $\sec^2 t$

So we have:

$$\int \frac{\sec^2 t dt}{\sqrt{1+\tan t}} \ .$$

So with our u sub:

$$\begin{aligned}\int \frac{du}{\sqrt{u}} &= \int \frac{1}{\sqrt{u}}\ du \\ &= \int u^{-\frac{1}{2}}\ du.\end{aligned}$$

Now if we Antidifferentiate and un sub:

$$\begin{aligned}&2u^{\frac{1}{2}} + C \\ &\boxed{= 2\sqrt{1+\tan t} + C}.\end{aligned}$$

Example:

$$\int \frac{\tan^{-1} x}{1+x^2}\ dx.$$

Let $u = \arctan x$

$$\begin{aligned}\frac{du}{dx} &= \frac{1}{1+x^2} \\ du &= \frac{1}{1+x^2}dx.\end{aligned}$$

Substitute:

$$\begin{aligned}&\int \frac{1}{1+x^2}\tan^{-1} x\ dx \\ &= u\ du.\end{aligned}$$

Antidifferentiate:

$$\frac{1}{2}u^2 + C.$$

Sub back in:

$$\frac{1}{2}(\arctan x)^2 + C.$$

Example:

$$\int \tan\theta \ d\theta.$$

Rewrite in terms of sine and cosine

$$\int \frac{\sin\theta}{\cos\theta} \ dx.$$

let $u = \cos\theta$

$$du = -\sin\theta d\theta$$
$$-du = \sin\theta d\theta.$$

Substitute:

$$\int \frac{1}{\cos\theta} \sin\theta \ d\theta$$
$$= -\frac{1}{u} du$$
$$= -u^{-1}$$
$$.$$

Antidifferentiate and sub back in:

$$-\ln|u| + C$$
$$-\ln|\cos\theta| + C$$
$$= \ln|\cos\theta|^{-1} + C$$
$$= \ln\left|\frac{1}{\cos\theta}\right| + C$$
$$\boxed{= \ln|\sec\theta| + C}.$$

Example:

$$\int \frac{x}{1+x^4} \ dx.$$

Let $u = x^2$, this works because we can write x^4 as $(x^2)^2$, and if we attempt to let $u = 1 + x^4$ we can quickly see that it will not work.

$$du = 2x \ dx.$$

Now if we substitute:

$$\int \frac{1}{1+x^2} x \; dx$$
$$\int \frac{1}{1+u^2} \frac{1}{2} du$$
$$\frac{1}{2} \int \frac{1}{1+u^2} du$$
$$.$$

and Antidifferentiate:

$$\frac{1}{2} \tan^{-1} u + C$$
$$\boxed{= \frac{1}{2} \tan^{-1} x^2 + C}.$$

Know This:

$$\int x\sqrt{x+1} \; dx.$$

Let $u = x + 1$

Then solve for x:

$$x = u - 1.$$

So:

$$\int (u-1)\sqrt{u} \; du$$
$$= \int (u-1)u^{\frac{1}{2}} \; du$$
$$= \int (u^{\frac{3}{2}} - u^{\frac{1}{2}}) \; du$$

Antidifferentiate:

$$\frac{2}{5} u^{\frac{5}{2}} - \frac{2}{3} u^{\frac{4}{2}} + C$$
$$= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C.$$

Example: Use the information from the "Know This" box

$$\int \frac{x^2}{\sqrt{1-x}}\ dx.$$

Let $u = \sqrt{1-x}$

$$\begin{aligned} u^2 &= 1-x \\ x &= 1-u^2 \\ dx &= -2u\ du. \end{aligned}$$

U-sub:

$$\begin{aligned} &\int \frac{(1-u^2)^2}{u}\ (-2u\ du) \\ &-2\int (1-u^2)^2\ du \\ &-2\left[\left(1-2u^2+u^4\right)\right]du \\ &-2\left[u-\frac{2}{3}u^3+\frac{1}{5}u^5\right]+C \\ &= -2u+\frac{4}{3}u^3-\frac{2}{5}u^5+C. \end{aligned}$$

Replace U:

$$\boxed{= -2(1-x)^{\frac{1}{2}}+\frac{4}{3}(1-x)^{\frac{3}{2}}-\frac{2}{5}(1-x)^{\frac{5}{2}}+C}.$$

The Substitution Rule for Definite Integrals

If g' is continous on [a,b] and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du.$$

Note:-

You can think of that last part as $\int_{u(a)}^{u(b)}$

Example: Evaluate.

$$\int_0^1 xe^{-x^2}\ dx.$$

Let $u = -x^2$

$$du = -2x\ dx$$
$$-\frac{1}{2}du = x\ dx.$$

Find $u(a)$ and $u(b)$

$$u(0) = -0^2 = 0$$
$$u(1) = -1^2 = -1.$$

So we have:

$$-\frac{1}{2}\int_0^{-1} e^u\ du.$$

If we flip the limits of integration, we can remove the negative infront

$$\frac{1}{2}\int_{-1}^{0} e^u\ du.$$

Now Antidifferentiate and evaluate

$$\begin{aligned}
&\frac{1}{2}e^u\Big]_{-1}^{0} \\
&= \frac{1}{2}[e^0 - e^1] \\
&= \frac{1}{2}\left(1 - \frac{1}{e}\right) \\
&= \frac{1}{2}\left(\frac{e-1}{e}\right) \\
&\boxed{= \frac{e-1}{2e}}.
\end{aligned}$$

Example:

$$\int_0^{\frac{1}{2}} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\ dx.$$

Let $u = \arcsin x$

$$du = \frac{1}{\sqrt{1-x^2}}dx.$$

So:

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} \sin^{-1} x\ dx \\
&= \int_0^{\frac{1}{2}} u\ du
\end{aligned}$$

Find $u(a)$ and $u(b)$

$$\begin{aligned}
u(a) &= \sin^{-1} 0 = 0 \\
a(b) &= \sin^{-1} \frac{1}{2} = \frac{\pi}{6}.
\end{aligned}$$

Which Means we have:

$$\int_0^{\frac{\pi}{6}} u\ du.$$

Antidifferentiate and evaluate

$$\begin{aligned}
&\frac{1}{2}u^2\Big]_0^{\frac{\pi}{6}} \\
&= \frac{1}{2}\left[\left(\frac{\pi}{6}\right)^2 - (0)^2\right] \\
&= \frac{1}{2}\left[\frac{\pi^2}{36}\right] \\
&\boxed{= \frac{\pi^2}{72}}.
\end{aligned}$$

Example:

$$\int_{\frac{1}{6}}^{\frac{1}{2}} \csc \pi t \cot \pi t \ dt.$$

Let $u = \pi t$

$$du = \pi \ dt$$
$$\frac{1}{\pi} \ du = dt.$$

Find limits:

$$u\left(\frac{1}{6}\right) = \frac{\pi}{6}$$
$$u\left(\frac{1}{2}\right) = \frac{\pi}{2}.$$

So we have:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \csc u \cot u \ \frac{1}{\pi} du$$
$$= \frac{1}{\pi} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \csc u \cot u \ du.$$

And if we Antidifferentiate and evaluate:

$$-\frac{1}{\pi} \csc u \Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$
$$= -\frac{1}{\pi}\left[\csc \frac{\pi}{2} - \csc \frac{\pi}{2}\right]$$
$$= -\frac{1}{\pi}\left[1 - 2\right]$$
$$\boxed{= \frac{1}{\pi}}.$$

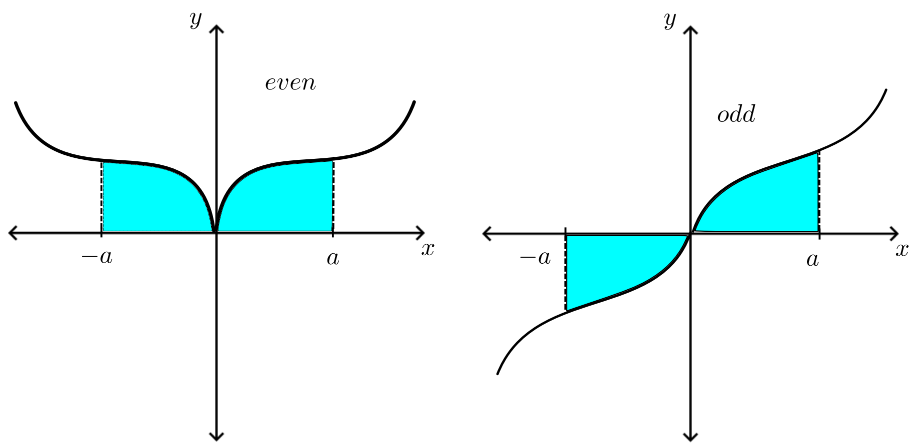
Sometimes you will run into integrals that are either impossible, or too difficult with u Substitution. For these cases we will look at Integrals of Symmetric Functions

Even:

$$f(-x) = f(x) \text{ then } \int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx.$$

Odd:

$$f(-x) = -f(x) \text{ then } \int_{-a}^{a} f(x)\, dx = 0.$$



Example:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 \sin x}{1+x^6}\, dx.$$

So since our limits of integration are in the form $-a$ *and* a, we can look for symmetry

Lets let's check $f(-x)$

$$\begin{aligned} f(-x) &= \frac{(-x)^2 \sin -x}{1+(-x)^6} \\ &= \frac{x^2(-\sin x)}{1+x^6} \\ &= -\frac{x^2 \sin x}{1+x^6}. \end{aligned}$$

So you can see that our function is odd, therefore:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 \sin x}{1+x^6}\ dx$$
$$\boxed{=0}.$$