Elementary Linear Algebra Reference

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Contents

Solutions to linear systems

- **Possible solutions to a linear system of two unknowns**: The linear system can have a **unique solution, no solution, or infinitely many solutions**.
- **Does the solution set form a line, plane, hyperplane, or something else?**: The formation of the solution set depends on the number of free variables,
  - **No free variables (one unique solution)**: Intersects at a point
  - **One free variable (Uncountable solutions)**: Solution set is a line (1-dimensional subspace)
  - **Two free variable (Uncountable solutions)**: Solution set forms a plane (2-dimensional subspace)
  - **Three free variable (Uncountable solutions)**: Solution set is a three dimensional subspace (In $\mathbb{R}^3$ it would be the whole space)
  - k **free variables**: Solution set is a k-dimensional subspace in $\mathbb{R}^n$

    Note: A k-dimensional subspace in $\mathbb{R}^n$ means that the solution set spans a k-dimensional space within the n-dimensional ambient space $\mathbb{R}^n$.
- **Determine if three planes intersect at a unique point**: For this, we find all three normal vectors $\vec{\mathbf{n}}_1, \vec{\mathbf{n}}_2$, and $\vec{\mathbf{n}}_3$. Then we find the triple scalar product, that is

$$\vec{\mathbf{n}}_1 \cdot (\vec{\mathbf{n}}_2 \times \vec{\mathbf{n}}_3).$$

  If this value is non-zero, we have intersection at a unique point. If the value is zero, we either have no intersection, or intersection at a line.

Linearity

- **The properties of linear equations**: A function $f : \mathbb{R}^n \to \mathbb{R}$ representing a linear equation is linear, meaning it satisfies the following properties for all vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and all scalars $c \in \mathbb{R}$:
  - **Additivity:** $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$
  - **Homogeneity of Degree 1:** $f(c\mathbf{x}) = cf(\mathbf{x})$

    It follows from this that $f(c\mathbf{x})$, when $c = 0$ implies $f(0\mathbf{x}) = 0f(\mathbf{x}) = 0$. Thus, we add the property
  - **Scale by zero**: $f(0) = 0$

  These properties define a linear function and imply that the graph of a linear equation is a straight line (in 2D) or a plane (in 3D).

Matrix algebra

Transpose