Nate Warner MATH 232 Spring 2024

Homework/Worksheet 11 - Due: Saturday, May 2

1. Evaluate $\int_C xy^4\, ds$, where C is the right half of the circle $x^2 + y^2 = 16$.

Remark. Let f be a continuous function with a domain that includes the smooth curve C with parameterization $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Then

$$\int_C f\, ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\, dt.$$

First, we find $\mathbf{r}(t)$, if C is the right half the circle with radius 4 centered at the origin, then $\mathbf{r}(t)$i is given by

$$r(t) = \langle 4\cos(\theta), 4\sin(\theta)\rangle \quad \text{for } -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}.$$

This implies

$$f(\mathbf{r}(t)) = 4\cos(\theta)4^4\sin^4(\theta)$$
$$\text{and } \|\mathbf{r}(t)\| = \sqrt{16\sin^2(\theta) + 16\cos^2(\theta)} = 4.$$

Thus, we have the integral

$$\begin{aligned}\int_C f ds &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4^6\cos(\theta)\sin(\theta)\, d\theta \\ &= \frac{4^6}{5}\left[\sin^5(\theta)\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \frac{4^6 \cdot 2}{5}.\end{aligned}$$

2. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle -1, 0\rangle$, and C is the part of the graph $y = \frac{1}{2}x^3 - x$ from $(2, 2)$ to $(-2, -2)$.

We can define $\mathbf{r}(t)$ is this case to be

$$\mathbf{r}(t) = \left\langle t, \frac{1}{2}t^3 - t\right\rangle$$
$$\implies \mathbf{r}'(t) = \left\langle 1, \frac{3}{2}t^2 - 1\right\rangle \implies d\mathbf{r} = \left\langle 1, \frac{3}{2}t^2 - 1\right\rangle dt.$$

With t ranging from 2 to -2. Thus, we have

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_2^{-2} \langle -1, 0\rangle \cdot \left\langle 1, \frac{3}{2}t^2 - 1\right\rangle dt \\ &= \int_2^{-2} -1\, dt \\ &= -1(-2-2) \\ &= 4.\end{aligned}$$

3. Evaluate the integral $\int_C (2x - y)\, dx + (x + 3y)\, dy$, where C lies along the x-axis from $x = 0$ to $x = 5$.

We define the parameterization of the curve C to be

$$\begin{aligned} x = 5t \quad y = 0 \quad \text{for } 0 \leqslant t \leqslant 1 \\ \implies \mathbf{r}(t) = \langle 5t, 0 \rangle. \end{aligned}$$

We then find the differentials dx, dy. They are given by

$$\begin{aligned} dx &= x'(t)dt = 5dt \\ dy &= y'(t)dt = 0. \end{aligned}$$

Thus, we have the integral

$$\begin{aligned} &\int_0^1 (2(5t) - 0)5dt + \int_0^1 (5t + 3(0))0\, dt \\ &= \int_0^1 50t\, dt \\ &= 25 \left[t^2 \Big|_0 1 \right. \\ &= 25(1 - 0) = 25. \end{aligned}$$

4. Determine whether the vector field is conservative and, if it is, find the potential function.

(a) $\mathbf{F}(x, y) = \langle -y + e^x \sin x, (x + 2)e^x \cos y \rangle$

(b) $\mathbf{F}(x, y) = \langle 2x \cos y - y \cos x, -x^2 \sin y - \sin x \rangle$

(c) $\mathbf{F}(x, y) = \langle 2xye^{x^2 y}, 6x^2 e^{x^2 y} \rangle$

Problem 4a. If we call $\mathbf{F} = \langle P(x, y), Q(x, y) \rangle$, then we can identify if the vector field is conservative if $P_y = Q_x$

$$\begin{aligned} P_y &= -1 \\ Q_x &= xe^x + e^x + 2e^x \cos(y). \end{aligned}$$

Since $P_y \neq Q_x$, the given vector field is **not** conservative.

Problem 4b. Again, we check the partials

$$\begin{aligned} P_y &= -2x \sin(y) - \cos(x) \\ Q_x &= -2x \sin(y) - \cos(x). \end{aligned}$$

Since $P_y = Q_x$, the given vector field is conservative. We the find the potential function f

$$\begin{aligned}
g(x,y) &= \int P(x,y)dx = x^2\cos(y) + h(y)\\
g_y &= -x^2\sin((y)) + h'(y)\\
g_y &= Q(x,y) \implies -x^2\sin(y) + h'(y) = -x^2\sin((y)) - \sin(x)\\
&\implies h'(y) = -\sin(x)\\
&\implies \int h'(y)dy = \int -\sin(x)dy\\
&\implies h(y) = -y\sin(x) + C\\
\therefore f(x,y) &= -x^2\cos(y) - y\sin(x) + C.
\end{aligned}$$

Problem 4c.

$$\begin{aligned}
P_y &= 2xe^{x^2}\\
Q_x &= e^y(6x^2(2xe^{x^2}) + 12xe^{x^2}).
\end{aligned}$$

Since $P_y \neq Q_x$, the given vector field is not conservative

5. Evaluate the integral $\int_C \nabla f \cdot d\mathbf{r}$, where $f(x,y) = x^2y - x$ and C is any path in the plane from $(1,2)$ to $(3,2)$.

Remark. Let C be a piecewise smooth curve with parameterization $\mathbf{r}(t), a \leqslant t \leqslant b$.

Let f be a function of two or three variables with first-order partial derivatives that exist and are continuous on C. Then,

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Since $\mathbf{r}(b) = (3,2)$, and $\mathbf{r}(a) = (1,2)$, finding the parameterization is unnecessary. To evaluate the given integral we find $f(3,2) - f(1,2)$

$$\begin{aligned}
f(3,2) - f(1,2) &= 3^2(2) - 3 - (1^2(2) - 1)\\
&= 14.
\end{aligned}$$

6. Evaluate the following line integrals by applying Green's theorem:

(a) $\int_C xy\,dx + (x+y)\,dy$, where C is the boundary of the region lying between the graphs of $x^2+y^2=1$ and $x^2+y^2=9$ oriented in the counterclockwise direction.

(b) $\int_C (1-y^3)\,dx + (x^3+e^{y^2})\,dy$, where C is the circle $x^2+y^2=4$ oriented in the counterclockwise direction.

Remark. Let D be an open, simply connected region with a boundary curve C that is a piecewise smooth, simple closed curve oriented counterclockwise. Let $\mathbf{F} = \langle P, Q\rangle$ be a vector field with component functions that have continuous partial derivatives on D. Then,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA.$$

Note: Greens thoerem can only be used for a two-dimensional vector field

Problem 6a.

We know that an integral of the form $\int_C \mathbf{F}\cdot\mathbf{T}ds$ can be written as $\int_C P(x,y)dx+Q(x,y)dy$, where $\mathbf{F}=\langle P(x,y),Q(x,y)\rangle$. By this fact, we identify the following

$$\begin{aligned} &P(x,y)=xy \\ &Q(x,y)=x+y \\ &\implies Q_x=1 \\ &\implies P_y=x. \end{aligned}$$

Using greens theorem for circulation, we have

$$\iint_R Q_x - P_y dA.$$

We must next identify our region D, since we are dealing with two circles, we shall represent the region in polar form.

$$D=\{(r,\theta):\ 1\leqslant r\leqslant 3,\ 0\leqslant\theta\leqslant 2\pi\}.$$

Thus, we have the integral

$$\begin{aligned} &\iint_D {}_{xy}(Q_x-P_y)dA = \iint_D {}_{xy}(1-x)dA \\ &= \iint_{D_{r\theta}} (1-r\cos(\theta))dA \\ &= \int_0^{2\pi}\int_1^3 (1-r\cos(\theta))r\,drd\theta \\ &= \int_0^{2\pi} \frac{1}{2}r^2-\frac{1}{3}r^3\cos(\theta)\Big|_1^3 d\theta \\ &= \int_0^{2\pi} \frac{9}{2}-9\cos(\theta)\,d\theta \\ &= \frac{9}{2}\theta-9\sin(\theta)\Big|_0^{2\pi} \\ &= 9\pi. \end{aligned}$$

Problem 6b. Again, we start by identifying $P(x,y)$ and $Q(x,y)$ from the given integral

$$\begin{aligned} &P(x,y)=1-y^3 \\ &Q(x,y)=x^3+e^{y^2} \\ &\implies Q_x=3x^2 \\ &\implies P_y=-3y^2. \end{aligned}$$

We identify the region D as the polar region

$$D=\{(r,\theta):\ r\leqslant 2\,0\leqslant\theta\leqslant 2\pi\}.$$

Thus, we have the integral

$$\begin{aligned}
&\iint_{D_{xy}} (Q_x - P_y) dA = \iint_{D_{xy}} (3x^2 + 3y^2) dA \\
&= \iint_{D_{r\theta}} 3r^2 dA_{r\theta} \\
&= \int_0^{2\pi} \int_0^2 3r^3 \, dr d\theta \\
&= \int_0^{2\pi} \frac{3}{4} \left[r^4 \right|_0^2 d\theta \\
&= \int_0^{2\pi} 12 \, d\theta \\
&= 12(2\pi - 0) = 24\pi.
\end{aligned}$$