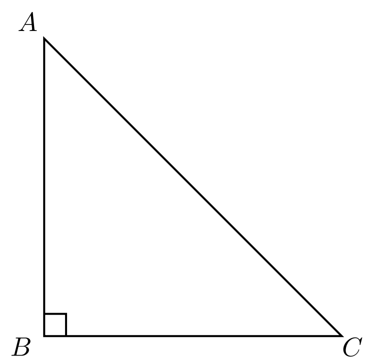
Nate Warner MATH 353 Spring 2025

Problem set 14 - Due: Monday, April 28

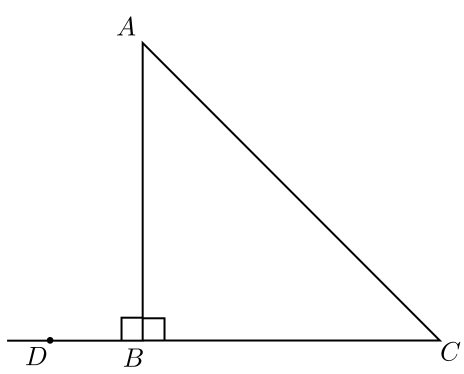
15.3. Prove Corollary 15.4

Remark. *Corollary 15.4.* The nonright angles of a small right triangle are accute

***Proof*.** Assume a small triangle $\triangle ABC$, with $\angle ABC = 90$.



Extend the segment $\overline{BC}$ to form exterior angle $\underline{\angle DBA}$. Since $\underline{\angle ABC}$ and $\underline{\angle DBA}$ are supplementary, $\angle DBA = 180 - \angle ABC = 180 - 90 = 90$.



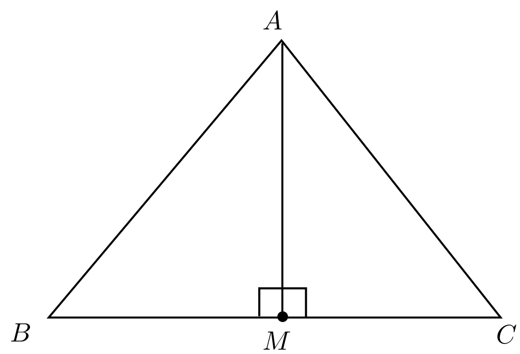
By theorem 15.3, $\angle DBA > \angle ACB$ and $\angle BAC$, which implies $\angle BAC, \angle ACB < 90$. Since $\underline{\angle ACB}$ and $\underline{\angle BAC}$ are the nonright angles and they have angle measure less than 90, the nonright angles of a small right triangle are therefore acute. ■

15.4. Prove Corollary 15.5 (Hint: Show that if M is the midpoint of the base $\overline{BC}$ of isosceles triangle $\triangle ABC$, then $\triangle ABM$ and $\triangle ACM$ are both small right triangles)

Remark. *Corollary 15.5.* The base angles of an isosceles triangle whose congruent sides are $< \frac{\omega}{2}$ are acute.

Proof. Assume an isosceles triangle $\triangle ABC$, with congruent sides $< \frac{\omega}{2}$, let AB, AC be the congruent sides, so $\overline{AB} \cong \overline{AC} \implies AB = AC$.

Let M be the midpoint of $\overline{BC}$, call the line that contains A, M $\overleftrightarrow{AM}$. Since $A \in \overleftrightarrow{AM}$, $A \notin \overleftrightarrow{BC}$, and $BA = BC$, $\overleftrightarrow{AM} \perp \overleftrightarrow{BC}$ at M, so $\angle AMB = \angle AMC = 90$.



Since $AB, AC < \frac{\omega}{2}$, and B-M-C, Theorem 15.1 implies $AM < \frac{\omega}{2}$. Since B-M-C, we have $BM + MC = BC$. Since A, B, C noncollinear, $BC < \omega$. By definition of the midpoint M of the segment $\overline{BC}$, $BM = MC = \frac{1}{2}BC$, so $BC = 2BM = 2BC$. So,

$$\begin{aligned} & BC < \omega \\ \implies & 2BM < \omega \\ \implies & BM < \frac{\omega}{2}. \end{aligned}$$

And,

$$\begin{aligned} & BC < \omega \\ \implies & 2MC < \omega \\ \implies & MC < \frac{\omega}{2}. \end{aligned}$$

So, $\triangle AMB$ and $\triangle AMC$ are both small.

Therefore, by Corollary 15.4, $\angle ABM < 90$, and $\angle ACM < 90$. ■

15.7. Show that if $\omega < \infty$, then for any triangle $\triangle ABC$, $$AB + BC + CA < 2\omega.$$ Hint: Apply the Triangle Inequality to $\triangle BCA^*$

Proof. Assume $\omega < \infty$, and the existence of triangle $\triangle ABC$.

Consider the triangle $\triangle BCA^*$, by the Triangle Inequality, we have

$$BA^* + CA^* > BC.$$

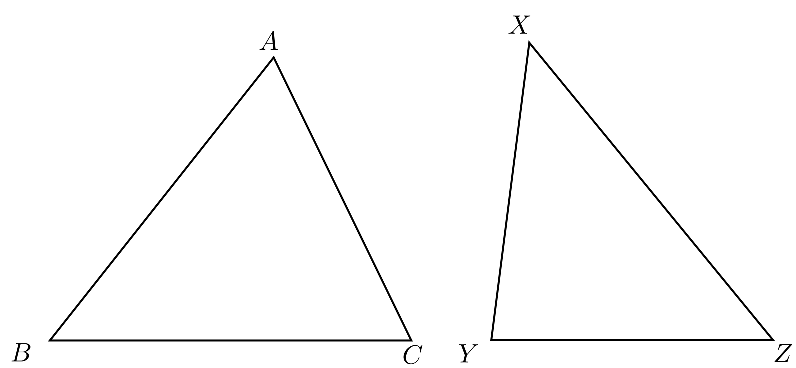
Note that by Theorem 9.1 A-B-$A^* \implies AB + BA^* = AA^* = \omega \implies BA^* = \omega - AB$, and similarly $CA^* = \omega - CA$. So,

$$\begin{aligned} BA^* + CA^* &> BC \\ \implies \omega - BA + \omega - CA &> BC \\ \therefore AB + BC + CA &< 2\omega. \end{aligned}$$

As desired. ■

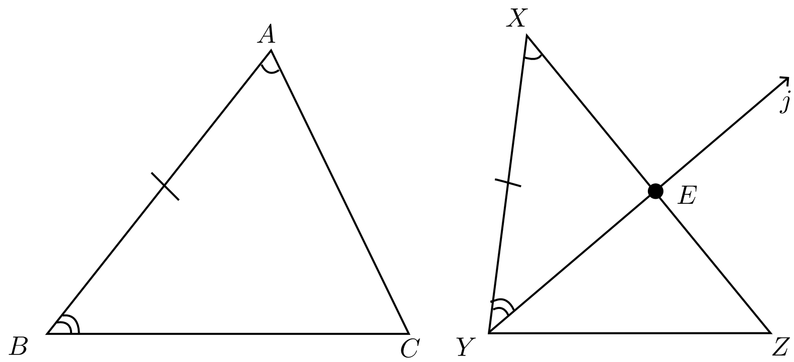
16.2. Suppose that $\triangle ABC$ and $\triangle XYZ$ are two small triangles with $\angle A = \angle X$, $AB = XY$, and $\angle B < \angle Y$. Prove that $\angle C > \angle Z$

Proof. Assume that $\triangle ABC$ and $\triangle XYZ$ are two small triangles with $\angle A = \angle X$, $AB = XY$, and $\angle B < \angle Y$.



Consider the rays, $\overrightarrow{YZ}, \overrightarrow{YX}$ and the fan $\overrightarrow{\overrightarrow{YZ}\overrightarrow{YX}}$ (which exists since X, Y, Z noncollinear). By Theorem 11.6, there exists a ray $j \in \overrightarrow{\overrightarrow{YZ}\overrightarrow{YX}}$ such that $\overrightarrow{YX}j = \angle B$, $\overrightarrow{YX}j$ must be in the wedge $\overrightarrow{\overrightarrow{YZ}\overrightarrow{YX}}$ since $\angle B < \angle Y = \overrightarrow{YZ}\overrightarrow{YX}$.

By the Crossbar Theorem, there exists a point $E \in j^0$ such that X-E-Z.



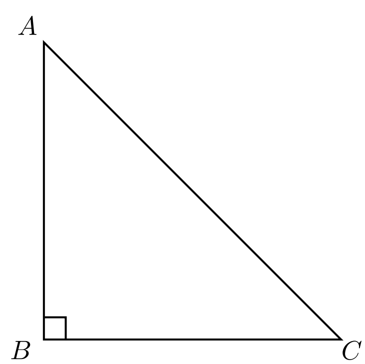
Notice that by Theorem 13.1 (ASA), we have congruence of triangles, specifically $\triangle ABC \cong \triangle XYE$ under the correspondence $ABC \leftrightarrow XYE$

Thus, $\angle C = \angle XEY$. Observe that $\angle XEY = \angle C$ is exterior to $\triangle EYZ$, thus $\angle XEY = \angle C > \angle EZY = \angle Z$. Thus, $\angle C > \angle Z$. ■

16.5. Prove Corollary 16.2

Remark. *Corollary 16.2.* The hypotenuse of a small right triangle is its longest side

Proof. Assume a small right triangle, call this triangle $\triangle ABC$, where $\angle B = 90$.



By Theorem 15.4, $\angle A$ and $\angle C < 90$, so $\angle B > \angle A$, and $\angle B > \angle C$. Thus, by Theorem 16.1 (comparison), $CB < CA$ and $AB < AC$. Therefore, the hypotenuse $\overline{AC}$ is the longest side. ■