Nate Warner MATH 230 November 11, 2023

Homework/Worksheet 9 - Due: Wednesday, November 15

1. Use the comparison test to determine whether the series is convergent or divergent

   (a) $\sum\limits_{n=1}^{\infty} \frac{1}{2n-1}$

   (b) $\sum\limits_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$

   (c) $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$

Remark. Suppose we have two series a_n, b_n and $\exists\ N \in \mathbb{Z}^+$ s.t $0 \leqslant a_n \leqslant b_n \ \ \forall\ n \geqslant N$. If b_n converges then a_n will also converge. Conversely, if $a_n \geqslant b_n \geqslant 0 \ \ \forall\ n \geqslant N$, and b_n diverges, then a_n will also diverge

Problem 1.a. If we let b_n be the series $\sum\limits_{n=1}^{\infty} \frac{1}{2n}$. We may conject that this series will diverge since it is know that the harmonic series $\sum \frac{1}{n}$ diverges, and multiplying a series by a constant factor will not affect the convergence or divergence. Furthermore,

$$\frac{1}{2n-1} > \frac{1}{2n}.$$

Conclusion. Thus, since $\sum\limits_{n=1}^{\infty} \frac{1}{2n}$ diverges, we can conclude that $\sum\limits_{n=1}^{\infty} \frac{1}{2n-1}$ will also diverge

Problem 1.b Let b_n be the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ Since we know the sine function produces outputs in the range [-1,1], the sine function squared will produce outputs withing the range [0,1]. However, since we are only considering integer values $[1, \infty)$, $\sin^2 n$ will only produce outputs $(0, 1)$. This is because the sine function has outputs of 1 at $\frac{\pi}{2} + 2k\pi,\ k \in \mathbb{Z}$, and outputs of 0 at $k\pi,\ k \in \mathbb{Z}$,

Problem 1.b: Let $b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$. We know The function $\sin^2(x)$ yields values in the range [0,1], as $\sin(x)$ varies between -1 and 1. For integer values n in the range [1, ∞), $\sin^2(n)$ will produce values in the interval (0,1). This is because $\sin(x)$ equals 1 at $\frac{\pi}{2} + 2k\pi$ and 0 at $k\pi$, where k is an integer, and these points are not integers. Thus we can conclude

$$\frac{\sin^2 n}{n^2} < \frac{1}{n^2}.$$

Conclusion. Since we know by the p-series $\frac{1}{n^2}$ will converge, $\frac{\sin^2 n}{n^2}$ will also converge

Problem 1.c Let b_n be the series $\sum\limits_{n=1}^{\infty} \frac{1}{n+1}$. We know this series will diverge because it is just the harmonic series $\frac{1}{n}$ shifted over by 1. We can deduce that $\frac{1}{\sqrt{n^2+1}} > \frac{1}{n+1}$ by looking at their reciprocals

$$\begin{aligned}
&\sqrt{n^2+1} < n+1\\
&n^2+1 < (n+1)^2\\
&n^2+1 < n^2+2n+1\\
&n^2 < n^2+2n.
\end{aligned}$$

Conclusion. Since this is clearly a true statement, then by the reciprocal identify for inequalities, which states if $0 \leqslant a \leqslant b$, then $\frac{1}{a} \geqslant \frac{1}{b}$ it holds that $\frac{1}{\sqrt{n^2+1}} > \frac{1}{n+1}$. and since we know $\sum\limits_{n=1}^{\infty} \frac{1}{n+1}$ diverges, then $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ will also diverge.

2. Use the Limit Comparison Test to determine whether the series is convergent or divergent.

   (a) $\sum\limits_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$

   (b) $\sum\limits_{n=1}^{\infty} \frac{1}{4^n-3^n}$

   (c) $\sum\limits_{n=1}^{\infty} (1-\cos\frac{1}{n})$

Remark. Suppose we have two series $a_n,\ b_n$ where $a_n,\ b_n \geqslant 0\ \forall\ n \geqslant 1$. Then if

- $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$ or $+\infty$. Then a_n and b_n either both converge or both diverge
- $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$, then if b_n converges, so does a_n
- $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = +\infty$, then if b_n diverges, so does a_n

Problem 2.a Let $b_n = \frac{1}{n^2}$, which by the p-series, converges

$$\begin{aligned}
&\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\ln\left(1+\frac{1}{n^2}\right)}{\frac{1}{n^2}}\\
&= \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n^2}\right)}{n^{-2}}\\
&\stackrel{H}{=} \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n^2}} \cdot -\frac{2}{n^3}}{-\frac{2}{n^3}}\\
&= \lim_{n\to\infty} \frac{-2n^3}{-2n^3\left(1+\frac{1}{n^2}\right)}\\
&= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n^2}\right)}\\
&= 1.
\end{aligned}$$

Conclusion. Since $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges, so does $\sum\limits_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$

Problem 2.b Choose $b_n = \frac{1}{4^n} = \left(\frac{1}{4}\right)^n$, which is a geometric series with $|r| < 1$ and thus converges

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\frac{1}{4^n-3^n}}{\frac{1}{4^n}} \\
&= \lim_{n\to\infty} \frac{4^n}{4^n - 3^n} \\
&= \lim_{n\to\infty} \frac{\frac{4^n}{4^n}}{\frac{4^n}{4^n} - \frac{3^n}{4^n}} \\
&= \lim_{n\to\infty} \frac{1}{1 - \left(\frac{3}{4}\right)^n} \\
&= 1.
\end{aligned}$$

Thus, since $\sum\limits_{n=1}^{\infty} \frac{1}{4^n}$ converges, so does $\sum\limits_{n=1}^{\infty} \frac{1}{4^n-3^n}$

Problem 2.c Choose $b_n = \frac{1}{n^2}$, which we know converges

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1-\cos\left(\frac{1}{n}\right)}{\frac{1}{n^2}} \quad (\text{Indeterminate } \frac{0}{0}) \\
&\overset{H}{=} \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)\cdot -\frac{1}{n^2}}{-\frac{2}{n^3}} \\
&= \lim_{n\to\infty} \frac{n^3\sin\left(\frac{1}{n}\right)}{2n^2} \\
&= \frac{1}{2}\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) \\
&= \frac{1}{2}\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{n^{-1}} \quad (\text{Indeterminate } \frac{0}{0}) \\
&\overset{H}{=} \frac{1}{2}\lim_{n\to\infty} \frac{\cos\left(\frac{1}{n}\right)\cdot -\frac{1}{n^2}}{-\frac{1}{n^2}} \\
&= \frac{1}{2}\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) \\
&= \frac{1}{2}\cos 0 \\
&= \frac{1}{2}.
\end{aligned}$$

Conclusion. Since $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L \neq 0 \text{ or } +\infty$, since b_n converges, so does a_n

3. Use the Alternating Series Test to determine whether the series is convergent or divergent.

   (a) $\frac{2}{3} - \frac{2}{5} + \frac{2}{7} - \frac{2}{9} + \frac{2}{11} - ...$

   (b) $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \ \frac{1}{\sqrt{n+3}}$

   (c) $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \ \cos^2 n$

   (d) $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \ \frac{n^2}{1+n^4}$

   (e) $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \ \frac{\cos n\pi}{n}$

Problem 3a. We can see that the general term for this series is given by

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{2n+1} \ .$$

Which is decreasing by

$$\frac{2}{2n+3} \leqslant \frac{2}{2n+1}.$$

And

$$\begin{aligned} &\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{2}{2n+1} \\ &= 0. \end{aligned}$$

Since the series is decreasing and $\lim\limits_{n\to\infty} b_n = 0$, by Leibniz's criterion, this series will converge

Problem 3b. This series is decreasing by

$$\frac{1}{\sqrt{n+4}} \leqslant \frac{1}{\sqrt{n+3}}.$$

And

$$\begin{aligned} &\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+3}} \\ &= 0. \end{aligned}$$

Since the series is decreasing and $\lim\limits_{n\to\infty} b_n = 0$, by Leibniz's criterion, this series will converge

Problem 3c. In this case, the AST does not apply because $b_n = \cos^2 n$ is not monotone decreasing $\ \forall\ n \geqslant 1$. In fact, because $\cos^2 n$ oscillates, there will be no such N s.t $\cos^2 n$ is monotone decreasing $\ \forall\ n \geqslant N$. Furthermore. Because

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \cos^2 n \neq 0.$$

This series is likely divergent

Problem 3d. We show $b_n = \frac{n^2}{1+n^4}$ is decreasing by

$$\frac{(n+1)^2}{1+(n+1)^4} \leqslant \frac{n^2}{1+n^4}.$$

And

$$\begin{aligned} \lim_{n\to\infty} b_n &= \lim_{n\to\infty} \frac{n^2}{1+n^4} \\ &= \lim_{n\to\infty} \frac{\frac{n^2}{n^4}}{\frac{1}{n^4}+\frac{n^4}{n^4}} \\ &= \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n^4}+1} \\ &= \frac{0}{1} \\ &= 0. \end{aligned}$$

Since the series is decreasing and $\lim\limits_{n\to\infty} b_n = 0$, by Leibniz's criterion, this series will converge

Problem 3e. Upon examination of $b_n = \cos n\pi$, we realize that this series will only ever be -1 or 1 for n in the family of integers. As $\cos x = -1$ for $(2k+1)\pi,\ k \in \mathbb{Z}$, and $\cos x = 1$ for $2k\pi,\ k \in \mathbb{Z}$, which essentially boils down to a simple deduction. For $n \in 2k+1,\ k \in \mathbb{Z}$, $\cos(n\pi) = -1$, for $n \in 2k,\ k \in \mathbb{Z}$, $\cos n\pi = 1$ and we can rewrite this series as

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^n}{n} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n} \\ &= \sum_{n=1}^{\infty} -\frac{1}{n} \\ &= -\sum_{n=1}^{\infty} \frac{1}{n}\ . \end{aligned}$$

Which we know diverges

4. Use the Root or Ratios Test to determine whether the series is convergent or divergent.

   (a) $\sum\limits_{n=1}^{\infty} \frac{2^{3n}(n!)^3}{(3n!)}$

   (b) $\sum\limits_{n=1}^{\infty} \frac{n!}{\left(\frac{n}{e}\right)^n}$

   (c) $\sum\limits_{n=1}^{\infty} \left(\frac{n-1}{2n+3}\right)^n$

   (d) $\sum\limits_{n=1}^{\infty} \frac{1}{(1+\ln n)^n}$

Problem 4a. By the ratio test

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{\frac{2^{3(n+1)}((n+1)!)^3}{3(n+1)!}}{\frac{2^{3n}(n!)^3}{3(n!)}} \right| \\
&= \lim_{n\to\infty} \left| \frac{2^{3n}2^3(n+1)^3(n!)^3}{3n!(n+1)} \cdot \frac{3n!}{2^{3n}(n!)^3} \right| \\
&= \lim_{n\to\infty} 8(n+1)^2 \\
&= +\infty.
\end{aligned}$$

Thus, by the ratio test, this series will diverge

Problem 4b. By the ratio test

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{\left(\frac{n+1}{e}\right)^{n+1}}}{\frac{n!}{\left(\frac{n}{e}\right)^n}} \right| \\
&= \lim_{n\to\infty} \left| \frac{n!(n+1)}{\left(\frac{n+1}{e}\right)^{n+1}} \cdot \frac{\left(\frac{n}{e}\right)^n}{n!} \right| \\
&= \lim_{n\to\infty} \frac{(n+1)\left(\frac{n}{e}\right)^n}{\left(\frac{n+1}{e}\right)^{n+1}} \\
&= \lim_{n\to\infty} \frac{n+1 \cdot \frac{n^n}{e^n}}{\frac{(n+1)^{n+1}}{e^{n+1}}} \\
&= \lim_{n\to\infty} \frac{(n+1)n^n e^n e}{e^n(n+1)^n(n+1)} \\
&= \lim_{n\to\infty} \frac{en^n}{(n+1)^n} \\
&= e \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n.
\end{aligned}$$

By Euler's definition $e = \lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^n \implies \frac{1}{e} = \lim\limits_{n\to\infty} \left(\frac{n}{n+1}\right)^n$. This means we have

$$e \cdot \frac{1}{e} = 1.$$

Since $\rho = 1$, the ratio test does not yield conclusive results

Problem 4c. By the root test

$$\begin{aligned}\rho &= \lim_{n\to\infty} \left| \left(\frac{n-1}{2n+3}\right)^n \right|^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{n-1}{2n+3} \\ &= \frac{1}{2}.\end{aligned}$$

Thus, by the root test, this series will converge

Problem 4d. By the root test

$$\begin{aligned}\rho &= \lim_{n\to\infty} \left| \left(\frac{1}{1+\ln n}\right)^n \right|^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{1}{1+\ln n} \\ &= 0.\end{aligned}$$

Thus, by the root test, this series will converge

5. Use an appropriate test to determine whether the series is convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{n+1}{n^3+n^2+n+1}$

(b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+1}{n^3+3n^2+3n+1}$

(c) $\sum_{n=1}^{\infty} \frac{(n-1)^n}{(n+1)^n}$

(d) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Problem 5a. Choose $b_n = \frac{1}{n^2}$, which by the p-series, converges. We can show by a simple comparison test that $\sum_{n=1}^{\infty} \frac{n+1}{n^3+n^2+n+1}$ also converges. Since

$$\frac{n+1}{n^3+n^2+n+1} \leqslant \frac{1}{n^2}.$$

By simple comparison test, the series converges

Problem 5b. Since $|a_n| = \sum_{n=1}^{\infty} \frac{n+1}{n^3+3n^2+3n+1}$, we can compare this series to $\frac{1}{n^2}$. Since $\frac{1}{n^2}$ converges by the p-series, and

$$\frac{n+1}{n^3+3n^2+3n+1} \leqslant \frac{1}{n^2}.$$

$\frac{n+1}{n^3+3n^2+3n+1}$ also converges, since $|a_n|$ converges, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+1}{n^3+3n^2+3n+1}$ converges absolutely

Problem 5c. By the divergence test

$$\begin{aligned}
&\lim_{n\to\infty} \left(\frac{n-1}{n+1}\right)^n \\
&= \lim_{n\to\infty} \left(\frac{n(1-\frac{1}{n})}{n(1+\frac{1}{n})}\right)^n && \textit{dotted} \\
&= \lim_{n\to\infty} \left(\frac{(1-\frac{1}{n})}{(1+\frac{1}{n})}\right)^n && \textit{dotted} \\
&= \lim_{n\to\infty} \frac{(1-\frac{1}{n})^n}{(1+\frac{1}{n})^n} \\
&= \lim_{n\to\infty} \frac{(1+\frac{-1}{n})^n}{(1+\frac{1}{n})^n}
\end{aligned}$$

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Knowing Euler's definition $e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$, which is generalized as $e^a = \lim_{n\to\infty} \left(1+\frac{a}{n}\right)^n$, our limit becomes

$$\begin{aligned}
&\lim_{n\to\infty} \frac{e^{-1}}{e} \\
&= \frac{1}{e^2}.
\end{aligned}$$

Since this limit is not zero, our series diverges

Problem 5.d Using the ratio test

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} \right| \\
&= \lim_{n\to\infty} \frac{n^2+2n+1}{2n^2} \\
&= \frac{1}{2}.
\end{aligned}$$

Since $0 \leqslant \rho < 1$, by the ratio test, this series will converge