Nate Warner MATH 230 October 14, 2023

Homework/Worksheet 6 - Due: Wednesday, October 18

1. Evaluate the following integrals using trigonometric substitution
   (a) $\int \frac{x^2}{\sqrt{1-x^2}}\ dx$
   (b) $\int \sqrt{x^2+9}\ dx$
   (c) $\int \frac{\sqrt{x^2-25}}{x}\ dx$
   (d) $\int \frac{1}{(x^2-9)^{\frac{3}{2}}}\ dx$
   (e) $\int \frac{x^2}{\sqrt{x^2+4}}\ dx$
   (f) $\int_{-1}^{1} (1-x^2)^{\frac{3}{2}}\ dx$

1.a

Trig sub:

$$\begin{aligned} x &= \sin\theta \\ dx &= \cos\theta\ d\theta. \end{aligned}$$

Thus we have:

$$\begin{aligned} &\int \frac{\sin^2\theta\cos\theta}{\sqrt{1-\sin^2\theta}}\ d\theta \\ &= \int \sin^2\theta\ d\theta. \end{aligned}$$

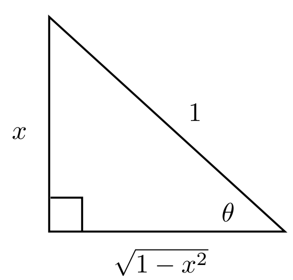
Interlude. By double angle formulas, we can solve for $\sin^2\theta$ to get an easier integrand

$$\begin{aligned} \cos 2\theta &= 1-2\sin^2\theta \\ \sin^2\theta &= \frac{1}{2}-\frac{1}{2}\cos 2\theta. \end{aligned}$$

Following this, we have:

$$\begin{aligned} &\int \frac{1}{2}-\frac{1}{2}\cos 2\theta\ d\theta \\ &= \int \frac{1}{2}d\theta - \frac{1}{2}\int \cos 2\theta\ d\theta \\ &= \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C. \end{aligned}$$

Reference triangle:



By the reference triangle we have:

$$\int \frac{x^2}{\sqrt{1-x^2}} = \frac{1}{2}\sin^{-1}(x) - \frac{1}{4}\sin\left(2\sin^{-1}(x)\right) + C.$$

1.b

$$x = 3\tan\theta$$
$$dx = 3\sec^2\theta \ d\theta.$$

Thus, we have:

$$\begin{aligned}
&\int \sqrt{x^2+9} \ dx \\
&= \int \sqrt{9\tan^2\theta + 9} \ 3\sec^2\theta \ d\theta \\
&= 9\int \sqrt{\tan^2\theta + 1}\sec^2\theta \ d\theta \\
&= 9\int \sec^3\theta \ d\theta \\
&= 9\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta \ d\theta\right] \\
&= 9\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right] + C \\
&= \frac{9}{2}\sec\theta\tan\theta + \frac{9}{2}\ln|\sec\theta + \tan\theta| + C
\end{aligned}$$

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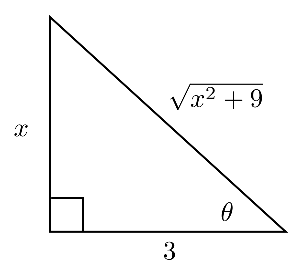
Reference Triangle:



$$\begin{aligned}
&\therefore \int \sqrt{x^2+9} \ dx = \frac{9}{2}\cdot\frac{1}{3}\sqrt{x^2+9}\frac{1}{3}x + \frac{9}{2}\ln|\frac{1}{3}\sqrt{x^2+9} + \frac{1}{3}x| + C \\
&= \frac{3}{2}\sqrt{x^2+9} \ \frac{1}{3}x + \frac{9}{2}\ln|\frac{1}{3}\sqrt{x^2+9} + \frac{1}{3}x| + C
\end{aligned}$$

.

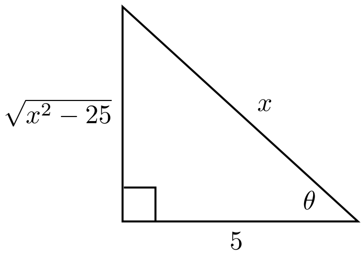
1.c

$$x = 5\sec\theta$$
$$dx = 5\sec\theta\tan\theta \ d\theta.$$

Thus we have:

$$\begin{aligned}
&\int \frac{\sqrt{x^2-25}}{x} \ dx = \int \frac{\sqrt{25\sec^2\theta - 25}}{5\sec\theta} \ 5\sec\theta\tan\theta \ d\theta \\
&= 5\int \frac{\tan\theta}{\sec\theta}\sec\theta\tan\theta \ d\theta \\
&= 5\int \tan^2\theta \ d\theta \\
&= 5\int \sec^2\theta - 1 \ d\theta \\
&= 5\tan\theta - 5\theta + C.
\end{aligned}$$

Reference Triangle



$$\therefore \int \frac{\sqrt{x^2-25}}{x}\ dx = 5\left(\frac{\sqrt{x^2-25}}{5}\right) - 5\sec^{-1}\left(\frac{1}{5}\right)x + C$$
$$= \sqrt{(x-5)(x+5)} - 5\sec^{-1}\left(\frac{1}{5}x\right) + C.$$

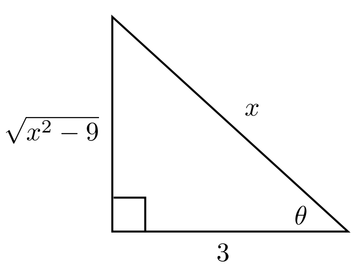
1.d

$$x = 3\sec\theta$$
$$dx = 3\sec\theta\tan\theta\ d\theta.$$

Thus we have:

$$\int \frac{1}{(x^2-9)^{\frac{3}{2}}}\ dx = \int \frac{3\sec\theta\tan\theta}{\sqrt{(9\sec^2\theta-9)^3}}$$
$$= \int \frac{3\sec\theta\tan\theta}{\sqrt{729(\sec^2\theta-1)^3}}\ d\theta$$
$$= \int \frac{3\sec\theta\tan\theta}{\sqrt{729\tan^6\theta}}\ d\theta$$
$$= \int \frac{3\sec\theta\tan\theta}{27\tan^3\theta}$$
$$= \frac{1}{9}\int \frac{\sec\theta}{\tan^2\theta}\ d\theta$$
$$= \frac{1}{9}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\ d\theta$$
$$= \frac{1}{9}\int \frac{\cos\theta}{\sin^2\theta}\ d\theta$$
$$= \frac{1}{9}\int u^{-2}\ du$$
$$= -\frac{1}{9u} + C$$
$$= -\frac{1}{9\sin\theta} + C$$
$$= -\frac{1}{9}\csc\theta + C.$$

Reference Triangle:



$$\therefore \int \frac{1}{(x^2-9)^{\frac{3}{2}}}\ dx = -\frac{x}{9\sqrt{x^2-9}} + C$$
$$= -\frac{x\sqrt{(x-3)(x+3)}}{9(x-3)(x+3)} + C.$$

1.e

$$x = 2\tan\theta$$
$$dx = 2\sec^2\theta \; d\theta.$$

Thus we have:

$$\begin{aligned}
&\int \frac{x^2}{\sqrt{x^2+4}}\; dx = \int \frac{4\tan^2\theta}{2\sec\theta}\cdot 2\sec^2\theta \; d\theta \\
&= 4\int \tan^2\theta\sec\theta \; d\theta \\
&= 4\int (\sec^2\theta - 1)\sec\theta \; d\theta \\
&= 4\int \sec^3\theta - \sec\theta \; d\theta \\
&= 4\Bigg[-\int \sec\theta \; d\theta + \int \sec^3\theta \; d\theta\Bigg] \\
&= 4\Bigg[-\ln|\sec\theta+\tan\theta| + \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta\Bigg] \\
&= 4\Bigg[-\ln|\sec\theta+\tan\theta| + \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|\Bigg] + C \\
&= 4\Bigg[-\frac{1}{2}\ln|\sec\theta+\tan\theta| + \frac{1}{2}\sec\theta\tan\theta\Bigg] + C \\
&= -2\ln|\sec\theta+\tan\theta| + 2\sec\theta\tan\theta + C
\end{aligned}$$

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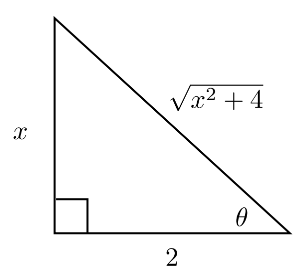
Reference Triangle:

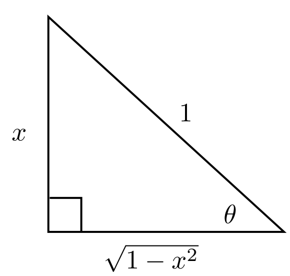


$$\begin{aligned}
&-2\ln|\frac{1}{2}\sqrt{x^2+4} + \frac{1}{2}x| + \frac{1}{2}\sqrt{x^2+4}\; x + C \\
&= -2\ln\left(\frac{1}{2}|\sqrt{x^2+4}+x|\right) + \frac{1}{2}\sqrt{x^2+4}\; x + C \\
&= \frac{1}{2}\left(-4\ln\left(\frac{1}{2}|\sqrt{x^2+4}+x|\right) + \sqrt{x^2+4}\; x\right) + C.
\end{aligned}$$

1.f

$$x = \sin\theta$$
$$dx = \cos\theta \ d\theta.$$

Reference Triangle:



Thus:

$$\int_{-1}^{1} (1-x^2)^{\frac{3}{2}} \ dx = \int_{-1}^{1} \sqrt{(1-\sin^2\theta)^3} \ \cos\theta \ d\theta$$
$$\int_{-1}^{1} \sqrt{\cos^6\theta} \ \cos\theta \ d\theta$$
$$= \int_{-1}^{1} \cos^4\theta \ d\theta$$
$$= \frac{1}{4}\cos^3\theta\sin\theta + \frac{3}{4}\int \cos^2\theta \ d\theta$$
$$= \frac{1}{4}\cos^3\theta\sin\theta + \frac{3}{4}\left[\frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}\int d\theta\right]$$
$$= \frac{1}{4}\cos^3\theta\sin\theta + \frac{3}{8}\cos\theta\sin\theta + \frac{3}{8}\theta + C.$$

Consequently...

$$2\int_{0}^{1} (1-x^2)^{\frac{3}{2}} \ dx = 2\left(\frac{1}{4}(1-x^2)^{\frac{3}{2}}x + \frac{3}{8}(1-x^2)^{\frac{1}{2}}x + \frac{3}{8}\sin^{-1}x\right)\Bigg|_0^1$$
$$= 2\left(\frac{3}{8}\sin^{-1}1\right)$$
$$= 2\left(\frac{3\pi}{16}\right)$$
$$= \frac{3\pi}{8}.$$

Evaluate the following integrals using partial fractions

(a) $\int \frac{dx}{x^2-5x+6}$

(b) $\int \frac{dx}{x(x-1)(x-2)(x-3)}$

(c) $\int \frac{2}{(x+2)^2(2-x)}\ dx$

(d) $\int \frac{x^3+6x^2+3x+6}{x^3+2x^2}\ dx$

(e) $\int \frac{2}{(x-4)(x^2+2x+6)}\ dx$

(f) $\int \frac{\sin x}{1-\cos^2 x}\ dx$

2.a

$$\begin{aligned}
&\int \frac{dx}{x^2-5x+6} \\
&\frac{1}{(x-2)(x-3)} = \frac{A}{(x-2)} + \frac{B}{(x-3)} \\
&1 = A(x-3) + B(x-2) \\
&A = -1 \quad B = 1.
\end{aligned}$$

Thus we have:

$$\begin{aligned}
&\int \frac{-1}{(x-2)} + \int \frac{1}{(x-3)} \\
&= -\ln|x-2| + \ln|x-3| + C.
\end{aligned}$$

2.b

$$\begin{aligned}
&\int \frac{dx}{x(x-1)(x-2)(x-3)} \\
&\frac{1}{x(x-1)(x-2)(x-3)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2} + \frac{D}{x-3} \\
&1 = A(x-1)(x-2)(x-3) + Bx(x-2)(x-3) + Cx(x-1)(x-3) + Dx(x-1)(x-2) \\
&A = -\frac{1}{6},\ B = \frac{1}{2},\ C = -\frac{1}{2},\ D = \frac{1}{6} \quad \text{(By plugging in zeros).}
\end{aligned}$$

So we have:

$$\begin{aligned}
&\int -\frac{1}{6x} + \frac{1}{2(x-1)} - \frac{1}{2(x-2)} + \frac{1}{6(x-3)}\ dx \\
&= -\frac{1}{6}\int \frac{1}{x}\ dx + \frac{1}{2}\int \frac{1}{(x-1)}\ dx - \frac{1}{2}\int \frac{1}{(x-2)}\ dx + \frac{1}{6}\int \frac{1}{(x-3)}\ dx \\
&= -\frac{1}{6}\ln|x| + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x-2| + \frac{1}{6}\ln|x-3| + C.
\end{aligned}$$

2.c

$$
\begin{aligned}
&\int \frac{2}{(x+2)^2(2-x)}\ dx\\
&\frac{2}{(x+2)^2(2-x)} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2} + \frac{C}{(2-x)}\\
&2 = A(x+2)(2-x) + B(2-x) + C(x+2)^2\\
&B = \frac{1}{2},\ C = \frac{1}{8} \quad \text{(Plugging in zeros)}\\
&2 = -Ax^2 + 4A + 2B - Bx + Cx^2 + 4Cx + 4C\\
&2 = (-A+C)x^2 + (C-B)x + (4A+B+4C)\\
&-A + C = 0\\
&A = \frac{1}{8}\\
&Thus:\ A = \frac{1}{8},\ B = \frac{1}{2},\ C = \frac{1}{8}.
\end{aligned}
$$

So we have the integral:

$$
\begin{aligned}
&\int -\frac{1}{8(x+2)} + \frac{1}{2(x+2)^2} + \frac{1}{8(2-x)}\ dx\\
&\frac{1}{8}\int \frac{1}{(x+2)}\ dx + \frac{1}{2}\int \frac{1}{(x+2)^2}\ dx + \frac{1}{8}\int \frac{1}{(2-x)}\ dx\\
&= \frac{1}{8}\ln|x+2| - \frac{1}{2(x-2)} + \frac{1}{8}\ln|2-x| + C.
\end{aligned}
$$

1.d

$$
\begin{aligned}
&\int \frac{x^3+6x^2+3x+6}{x^3+2x^2}\ dx\\
&\int\ dx + \int \frac{4x^2+3x+6}{x^3+2x^2}\ dx \quad \text{(After long division)}\\
&= x + \int \frac{4x^2+3x+6}{x^3+2x^2}\ dx\\
&= x + \int \frac{4x^2+3x+6}{x^2(x+2)}\ dx\\
&\frac{4x^2+3x+6}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x+2)} \quad \text{(Omitting evaluation of first integral for now...)}\\
&4x^2+3x+6 = Ax(x+2) + B(x+2) + Cx^2\\
&4x^2+3x+6 = Ax^2 + 2Ax + Bx + 2B + Cx^2\\
&(A+C)x^2 + (2A+B)x + 2B\\
&A + C = 4 \quad \text{(Begin system...)}\\
&2A + B = 3\\
&2B = 6 \quad \text{(End system...)}\\
&A = 0,\ B = 3,\ C = 4.
\end{aligned}
$$

Thus we have the integral:

$$\int \frac{3}{x^2}\ dx + \frac{4}{(x+2)}\ dx$$
$$= -\frac{3}{x} + 4\ln|x+2|.$$

Bringing back the first integral that we evaluated we get the full solution of:

$$x - \frac{3}{x} + 4\ln|x+2| + C.$$

1.e

$$\int \frac{2}{(x-4)(x^2+2x+6)}\ dx$$
$$\frac{2}{(x-4)(x^2+2x+6)} = \frac{A}{(x-4)} + \frac{Bx+C}{(x^2+2x+6)}$$
$$2 = A(x^2+2x+6) + (Bx+C)(x-4)$$
$$A = \frac{1}{15} \quad \text{(Plugging in 4)}$$
$$2 = Ax^2 + 2Ax + 6A + Bx^2 - 4Bx - 4C$$
$$2 = (A+B)x^2 + (2A-4B+C)x + (6A-4C)$$
$$A + B = 0 \quad \text{(Begin system...)}$$
$$2A - 4B + C = 0$$
$$6A - 4C = 2 \quad \text{(End system...)}$$
$$B = -\frac{1}{15}$$
$$C = -\frac{1}{5}$$
$$Thus:\ A = \frac{1}{15},\ B = -\frac{1}{15},\ C = -\frac{1}{5}.$$

By this we have the integral:

$$\begin{aligned}
&\int \frac{1}{15(x-4)} + \frac{\left(-\frac{1}{15}\right)x + \left(-\frac{1}{5}\right)}{x^2+2x+6}\ dx \\
&= \int \frac{1}{15(x-4)}\ dx + \int \frac{-x-3}{15(x^2+2x+6)}\ dx \\
&= \int \frac{1}{15(x-4)}\ dx - \int \frac{x+3}{15(x^2+2x+6)}\ dx \\
&I_1 = \frac{1}{15}\int \frac{1}{x-4}\ dx \\
&= \frac{1}{15}\ln|x-4| \\
&I_2 = -\frac{1}{15}\int \frac{x}{x^2+2x+6} + \frac{3}{x^2+2x+6}\ dx \\
&= -\frac{1}{15}\int \frac{x}{(x+1)^2+5} + \frac{3}{(x+1)^2+5} \quad \text{(By completing the square)} \\
&= -\frac{1}{15}\int \frac{u-1}{u^2+5} + \frac{3}{u^2+5}\ du \quad \text{(Where } u = x+1\text{)} \\
&= -\frac{1}{15}\int \frac{u}{u^2+5} - \frac{1}{u^2+(\sqrt{5})^2} + \frac{3}{u^2+(\sqrt{5})^2}\ du \\
&= -\frac{1}{15}\left[\frac{1}{2}\ln|u^2+5| - \frac{1}{\sqrt{5}}\tan^{-1}\frac{u}{\sqrt{5}} + \frac{3}{\sqrt{5}}\tan^{-1}\frac{u}{\sqrt{5}}\right] \\
&= -\frac{1}{15}\left[\frac{1}{2}\ln|(x+1)^2+5| - \frac{1}{\sqrt{5}}\tan^{-1}\frac{x+1}{\sqrt{5}} + \frac{3}{\sqrt{5}}\tan^{-1}\frac{x+1}{\sqrt{5}}\right] + C \\
&= -\frac{1}{15}\left[\frac{1}{2}\ln|(x+1)^2+5| + \frac{2}{\sqrt{5}}\tan^{-1}\frac{x+1}{\sqrt{5}}\right] + C \\
&\therefore \int \frac{2}{(x-4)(x^2+2x+6)}\ dx \\
&= \frac{1}{15}\ln|x-4| - \frac{1}{15}\left[\frac{1}{2}\ln|(x+1)^2+5| + \frac{2}{\sqrt{5}}\tan^{-1}\frac{x+1}{\sqrt{5}}\right] + C \\
&= \frac{1}{15}\ln|x-4| - \frac{1}{30}\ln|(x+1)^2+5| + \frac{2}{15\sqrt{5}}\tan^{-1}\frac{x+1}{\sqrt{5}} + C.
\end{aligned}$$

2.f

$$\begin{aligned}
&\int \frac{\sin x}{1-\cos^2 x}\ dx \\
&= -\int \frac{du}{1-u^2} \quad \text{(Let } u = \cos x\text{)} \\
&= -\int \frac{du}{(1-u)(1+u)}\ du \\
&\frac{1}{(1-u)(1+u)} = \frac{A}{(1-u)} + \frac{B}{(1+u)} \\
&1 = A(1+u) + B(1-u) \\
&A = \frac{1}{2},\ B = \frac{1}{2} \quad \text{(By zeros).}
\end{aligned}$$

Thus:

$$
\begin{aligned}
&-\int \frac{1}{2(1-u)} + \frac{1}{2(1+u)}\ du \\
&-\left[-\frac{1}{2}\ln|1-u| + \frac{1}{2}\ln|1+u|\right] + C \\
&= \frac{1}{2}\ln|1-\cos x| - \frac{1}{2}\ln|1+\cos x| + C.
\end{aligned}
$$