Nate Warner MATH 230 August 5, 2023

Homework/Worksheet 5 - Due: Wednesday, October 11

1. Evaluate the following integrals using the method of integration by parts:

(a) $\int xe^{4x}dx$

(b) $\int \ln x dx$

(c) $\int x^4 \ln x dx$

(d) $\int x \cos 3x dx$

(e) $\int e^{2x} \sin(5x) dx$

(f) $\int_0^1 e^{\sqrt{x}} dx$

1.a

$$u = x \quad dv = e^{4x}\ dx$$
$$du = dx \quad v = \frac{1}{4}e^{4x}.$$

$$\int xe^{4x}\ dx$$
$$= \frac{1}{4}xe^{4x} - \int \frac{1}{4}e^{4x}\ dx$$
$$= \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} + C.$$

1.b

$$u = \ln x \quad dv = dx$$
$$du = \frac{1}{x}\ dx \quad v = x.$$

$$\int \ln x\ dx$$
$$= x \ln x - \int x \cdot \frac{1}{x}\ dx$$
$$= x \ln x - x + C.$$

1.c

$$u = \ln x \quad dv = x^4\ dx$$
$$du = \frac{1}{x}\ dx \quad v = \frac{1}{5}x^5.$$

$$\int x^4 \ln x\ dx$$
$$= \frac{1}{5}x^5 \ln x - \int \frac{1}{5}x^5 \cdot \frac{1}{x}\ dx$$
$$= \frac{1}{5}x^5 \ln x - \int \frac{1}{5}x^4\ dx$$
$$= \frac{1}{5}x^5 \ln x - \frac{1}{25}x^5 + C.$$

1.d

$$u = x \quad dv = \cos 3x\ dx$$
$$du = dx \quad v = \frac{1}{3}\sin 3x.$$

$$\begin{aligned} &\int x\cos 3x\ dx \\ &= \frac{1}{3}x\sin 3x - \int \frac{1}{3}\sin 3x\ dx \\ &= \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + C. \end{aligned}$$

1.e

$$u = \sin 5x \quad dv = e^{2x}\ dx$$
$$du = 5\cos 5x\ dx \quad v = \frac{1}{2}e^{2x}.$$

$$\begin{aligned} &\int e^{2x}\sin 5x\ dx \\ &= \frac{1}{2}e^{2x}\sin 5x - \int \frac{5}{2}e^{2x}\cos 5x\ dx \\ &= \frac{1}{2}e^{2x}\sin 5x - \frac{5}{2}\int e^{2x}\cos 5x\ dx. \end{aligned}$$

$$u = \cos 5x \quad dv = e^{2x}\ dx$$
$$du = -5\sin 5x\ dx \quad v = \frac{1}{2}e^{2x}.$$

$$\begin{aligned} &\int e^{2x}\cos 5x\ dx \\ &= \frac{1}{2}e^{2x}\cos 5x - \int -\frac{5}{2}e^{2x}\sin 5x\ dx \\ &= \frac{1}{2}e^{2x}\cos 5x + \frac{5}{2}\int e^{2x}\sin 5x\ dx \end{aligned}$$

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Thus we have:

$$\begin{aligned} &\int e^{2x}\sin 5x\ dx = \frac{1}{2}e^{2x}\sin 5x - \frac{5}{2}\left[\frac{1}{2}e^{2x}\cos 5x + \frac{5}{2}\int e^{2x}\sin 5x\ dx\right] \\ &= \frac{1}{2}e^{2x}\sin 5x - \frac{5}{4}e^{2x}\cos 5x - \frac{25}{4}\int e^{2x}\sin 5x\ dx. \end{aligned}$$

Let $I = \int e^{2x}\sin 5x\ dx$

$$\begin{aligned} &\left(\frac{25}{4} + 1\right) I = \frac{1}{2}e^{2x}\sin 5x - \frac{5}{4}e^{2x}\cos 5x \\ &I = \frac{4}{29}\left(\frac{1}{2}e^{2x}\sin 5x - \frac{5}{4}e^{2x}\cos 5x\right) \\ &\int e^{2x}\sin 5x\ dx = \frac{4}{29}\left(\frac{1}{2}e^{2x}\sin 5x - \frac{5}{4}e^{2x}\cos 5x\right) + C \end{aligned}$$

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1.f

$$\text{Let } u = x^{\frac{1}{2}}$$
$$du = \frac{1}{2x^{\frac{1}{2}}} dx$$
$$2x^{\frac{1}{2}} du = dx$$
$$u(a) = 0$$
$$u(b) = 1.$$

$$\int_0^1 e^{x^{\frac{1}{2}}} \; dx$$
$$= 2 \int_0^1 ue^u \; du.$$

$$w = u \quad dv = e^u$$
$$dw = du \quad v = e^u.$$

$$2\left(ue^u - \int e^u \; du\right)$$
$$= 2\left(ue^u - e^u\right) + C$$
$$= 2\left(x^{\frac{1}{2}} e^{x^{\frac{1}{2}}} - e^{x^{\frac{1}{2}}}\right) + C$$
$$= 2x^{\frac{1}{2}} e^{x^{\frac{1}{2}}} - 2e^{x^{\frac{1}{2}}} + C.$$

2. Evaluate the following trigonometric integrals:

(a) $\int \sin^7(2x)\cos(2x)dx$

(b) $\int \sin^3 x \cos^3 x dx$

(c) $\int \sin^3 x dx$

(d) $\int \tan^2 x \sec x dx$

(e) $\int \cos^5 x dx$

(f) $\int_0^\pi \sin 3x \cos 5x dx$

2.a

Let $u = \sin 2x$

$du = 2\cos 2x \ dx$

$\frac{1}{2}du = \cos 2x \ dx.$

$$\int \sin^7 2x \cos 2x \ dx$$
$$= \frac{1}{2}\int u^7 \ du$$
$$= \frac{1}{2}\left(\frac{1}{8}u^8\right) + C$$
$$= \frac{1}{16}\sin^8(2x) + C.$$

2.b

$$\int \sin^3 x \cos^3 x \ dx$$
$$= \int \sin^3 x(1 - \sin^2 x)\cos x \ dx.$$

Let $u = \sin x$

$du = \cos x \ dx.$

$$\int \sin^3 x(1 - \sin^2 x)\cos x \ dx$$
$$= \int u^3(1 - u^2) \ du$$
$$= \int u^3 - u^5 \ du$$
$$= \frac{1}{4}u^4 - \frac{1}{6}u^6 + C$$
$$= \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C.$$

2.c

By the reduction formula $\int \cos^j x = \frac{1}{j}\cos^{j-1} x \sin x + \frac{j-1}{j}\int \cos^{j-2} x\ dx$:

$$\begin{aligned}
&\int \cos^5 x\ dx \\
&= \frac{1}{5}\cos^4 x \sin x + \frac{4}{5}\int \cos^3 x\ dx \\
&= \frac{1}{5}\cos^4 x \sin x + \frac{4}{5}\left[\frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\int \cos x\ dx\right] \\
&= \frac{1}{5}\cos^4 x \sin x + \frac{4}{15}\cos^2 x \sin x + \frac{8}{15}\int \cos x\ dx \\
&= \frac{1}{5}\cos^4 x \sin x + \frac{4}{15}\cos^2 x \sin x + \frac{8}{15}\sin x + C.
\end{aligned}$$

2.d

$$\begin{aligned}
&\int \sin^3 x \cos^3 x\ dx \\
&= \int \sin^3 x(1-\sin^2 x)\cos x\ dx \\
&= \int u^3(1-u^2)\ du \\
&= \int u^3 - u^5\ du \\
&= \frac{1}{4}u^4 - \frac{1}{6}u^6 + C \\
&= \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C.
\end{aligned}$$

2.e

$$\begin{aligned}
&\int \tan^2 x \sec x\ dx \\
&= \int (\sec^2 x - 1)\sec x\ dx \\
&= \int \sec^3 x - \sec x\ dx \\
&= \int \sec^3 x\ dx - \int \sec x\ dx \\
&I_1 = \frac{1}{2}\sec^2 x \sin x + \frac{1}{2}\int \sec x\ dx \quad \text{(By the reduction formula)} \\
&= \frac{1}{2}\sec^2 x \sin x + \frac{1}{2}\ln\left(|\sec x + \tan x|\right) \\
&I_2 = \ln\left(|\sec x + \tan x|\right) \\
&\therefore \int \tan^2 x \sec x\ dx = \frac{1}{2}\sec^2 x \sin x + \frac{1}{2}\ln\left(|\sec x + \tan x|\right) - \ln\left(|\sec x + \tan x|\right) + C \\
&= \int \tan^2 x \sec x\ dx = \frac{1}{2}\sec^2 x \sin x - \frac{1}{2}\ln\left(|\sec x + \tan x|\right) + C.
\end{aligned}$$

2.f

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\begin{aligned}
&\int_0^{\pi} \sin(3x)\cos(5x)\ dx \\
&= \int_0^{\pi} \frac{1}{2}\Big[\sin((3-5)x) - \sin((3+5)x)\Big]\ dx \\
&= \frac{1}{2}\Big[\int_0^{\pi} \sin(-2x) - \sin(8x)\ dx\Big] \\
&= \frac{1}{2}\Big[\int_0^{\pi} \sin(-2x)\ dx - \int_0^{\pi} \sin(8x)\ dx\Big] \\
&= \frac{1}{2}\Big[-\frac{1}{2}\int_0^{-2\pi} \sin u\ du - \frac{1}{8}\int_0^{8\pi} \sin u\ du\Big] \\
&= \frac{1}{2}\Big[\frac{1}{2}(\cos 2\pi - \cos 0) + \frac{1}{8}(\cos 2\pi - \cos 0)\Big] \\
&= 0.
\end{aligned}
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