Nate Warner MATH 232 Spring 2024

Homework/Worksheet 2 - Due: Wednesday, January 31

1. Convert the rectangular equation $y^2 = 4x$ to polar form and sketch its graph.

Remark. Given a point P with cartesian coordinates (x, y), and polar coordinates (r, θ) the following conversion formulas are true.

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ x^2 + y^2 &= r^2 \\ \tan\theta &= \frac{y}{x}. \end{aligned}$$

With the formulas mentioned above, we can convert $y^2 = 4x$ to polar form.

$$\begin{aligned} y^2 &= 4x \\ y &= 4 \cdot \frac{x}{y} \\ r\sin\theta &= 4\cot\theta \\ r &= 4\cot\theta\csc\theta. \end{aligned}$$

To graph this equation, we first make a table of points

θ	r
0	undefined
$\frac{\pi}{2}$	0
π	undefined
$\frac{3\pi}{2}$	0
2π	undefined

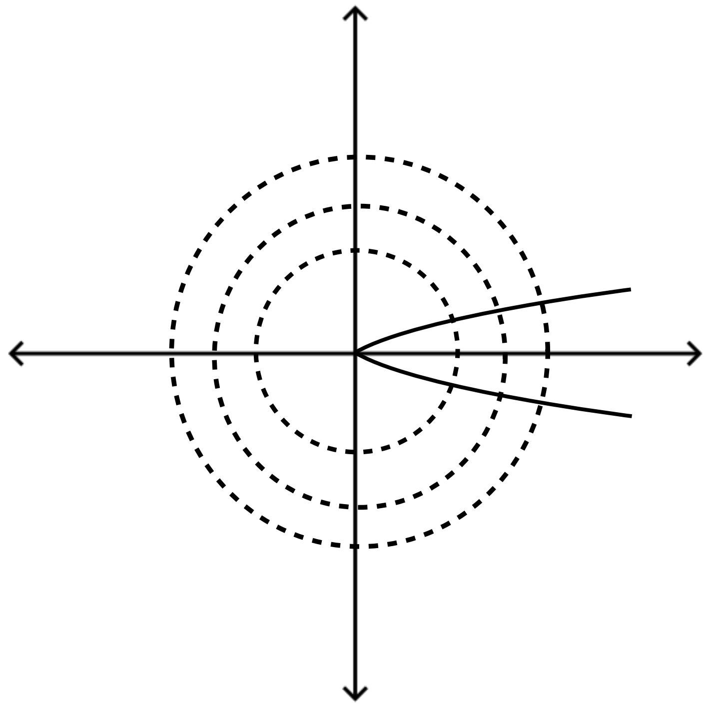
Additionally, we know $\csc\theta$ has period 2π, and $\cot\theta$ has period π. Thus, $4\cot\theta\csc\theta$ will have period π. Since both functions have vertical asymptotes at $x = k\pi, k \in \mathbb{R}$, we know the graph of $4\cot\theta\csc\theta$ will also have these asymptotes. Moreover, we can find the zeros by setting the equation equal to zero and solving for θ

$$\begin{aligned} 4\cot\theta\csc\theta &= 0 \\ \frac{\cos\theta}{\sin^2\theta} &= 0 \\ \cos\theta &= 0 \\ \theta &= \cos^{-1} 0 \\ \theta &= \frac{\pi}{2} + k\pi,\ k \in \mathbb{R}. \end{aligned}$$

Now we need to determine the behavior of the graph as θ approaches 0 and π

$$\begin{aligned} \lim_{\theta\to 0} 4\frac{\cos\theta}{\sin^2 0} &= \infty \\ \lim_{\theta\to \pi} 4\frac{\cos\theta}{\sin^2 0} &= -\infty. \end{aligned}$$

From this information, the polar curve can be sketched

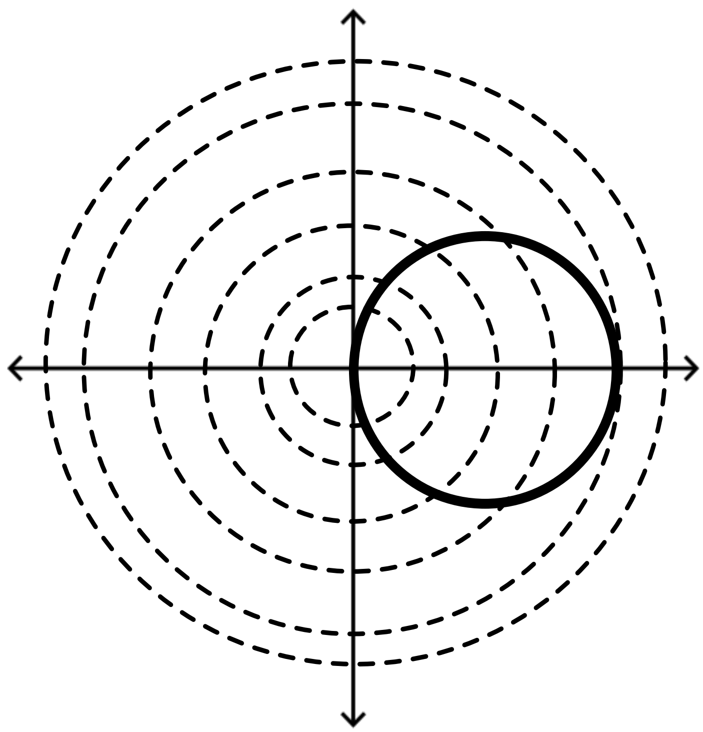


2. Convert the polar equation $r = 6\cos\theta$ to rectangular form and sketch its graph.

Converting to rectangular form we get.

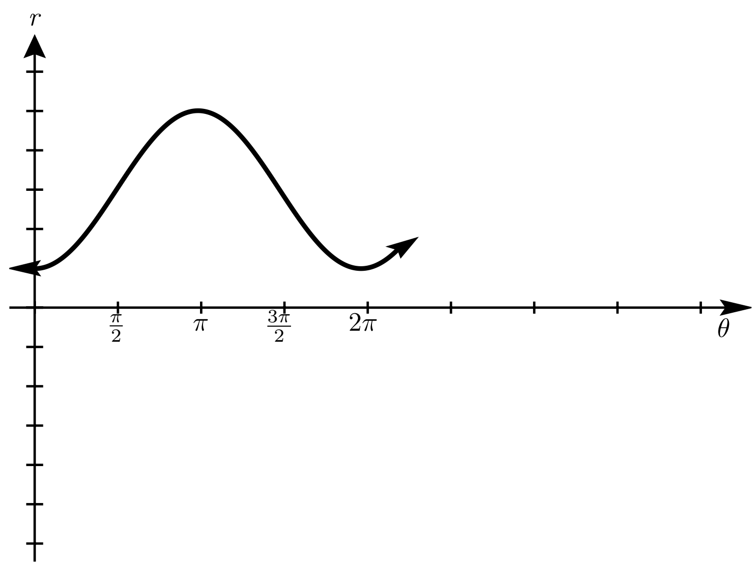
$$
\begin{aligned}
r^2 &= 6r\cos\theta \\
x^2 + y^2 &= 6x \\
x^2 - 6x + y^2 &= 0 \\
(x-3)^2 + y^2 &= 3^2.
\end{aligned}
$$

We see that this is the equation of a circle, with center (3,0) and radius $r = 3$. Thus we have

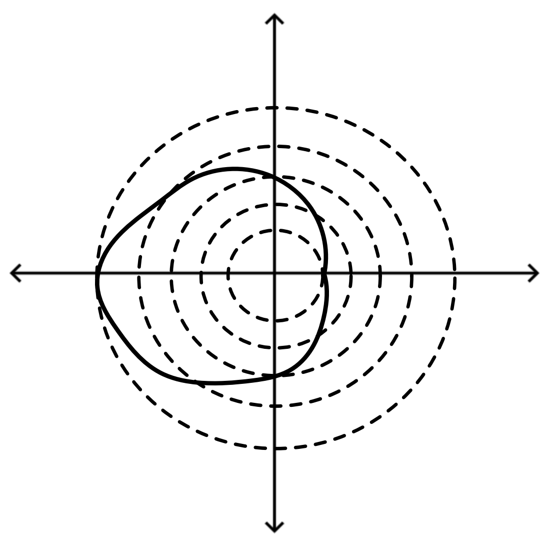


3. Sketch the curve $r = 3 - 2\cos\theta$ by first sketching the graph of r as a function of θ in Cartesian coordinates.

Sketching this curve in the rectangular system,



From this we can sketch the polar curve (Very Rough sketch)



4. Determine a definite integral that represents the area of the region in the first quadrant enclosed by $r = 2 - \cos\theta$

Remark. Suppose f is continuous and nonnegative on the interval $\alpha \leqslant \theta \leqslant \beta$ with $0 < \alpha - \beta \leqslant 2\pi$. The area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by

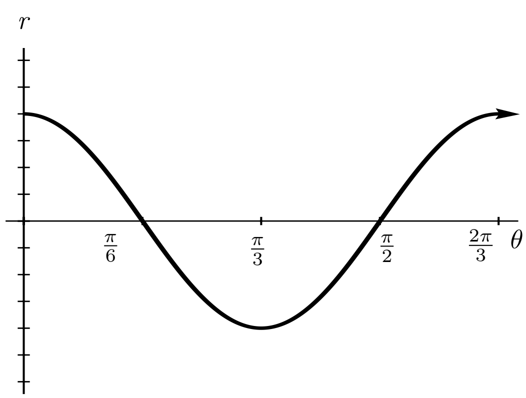
$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \ d\theta.$$

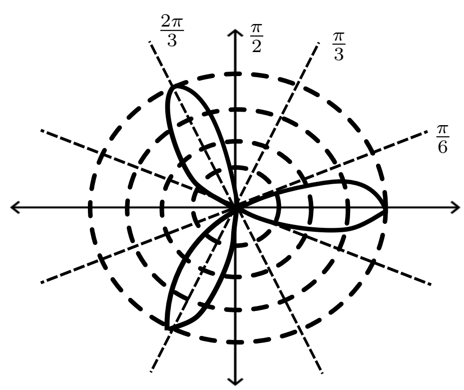
The first quadrant of this polar curve lies between the radial lines $\theta = 0$ and $\theta = \frac{\pi}{2}$. Thus, we have the integral

$$\frac{1}{2}\int_{0}^{\frac{\pi}{2}} (2 - \cos\theta)^2 \ d\theta.$$

5. Sketch the curve $r = 4\cos 3\theta$ and find the area enclosed by one petal.

We first sketch the graph in the rectangular system, and then we tranlate over to polar.





Because this curve has exactly three petals, where each petal is symmetric covering a total range of 2π, we divide 2π by 3 to get the range of just one petal. Thus, each petal covers an angle of $\frac{2\pi}{3}$. To find the area of one of the petals, we can compute the integral

$$\begin{aligned}
&\frac{1}{2}\int_{0}^{\frac{2\pi}{3}} r^2 \ d\theta \\
&= \frac{1}{2}\int_{0}^{\frac{2\pi}{3}} (4\cos 3\theta)^2 \ d\theta \\
&= \frac{1}{2}\int_{0}^{\frac{2\pi}{3}} 16\cos^2 3\theta \ d\theta \\
&= 8\int_{0}^{\frac{2\pi}{3}} \cos^2 3\theta \ d\theta.
\end{aligned}$$

From this, we can use the double angle formula $\cos 2\alpha = 2\cos^2\alpha - 1$ to solve for $\cos^2 3\theta$

$$\begin{aligned}\cos 2\alpha &= 2\cos^2\alpha - 1\\ \cos^2\alpha &= \frac{1}{2} + \frac{1}{2}\cos 2\alpha.\end{aligned}$$

Now if we let $\alpha = 3\theta$, we can get a form that perfectly matches our integrand.

$$\cos^2 3\theta = \frac{1}{2} + \frac{1}{2}\cos 6\theta.$$

Thus, we have the integral

$$\begin{aligned}&8\int_0^{\frac{2\pi}{3}} \frac{1}{2} + \frac{1}{2}\cos 6\theta \; d\theta\\ &= 8\left[\frac{1}{2}\theta + \frac{1}{12}\sin 6\theta\right|_0^{\frac{2\pi}{3}}\\ &= 8\left[\left(\frac{\pi}{3} - 0\right) - \left(\frac{1}{12}\sin(4\pi) - \sin 0\right)\right]\\ &= \frac{8\pi}{3}.\end{aligned}$$

6. Find the length of the curve $r = 2\cos\theta$ on the interval $0 \leqslant \theta \leqslant 2\pi$

Remark. Let f be a function whose derivative is continous on the interval $\alpha \leqslant \theta \leqslant \beta$. The length of the graph $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$\int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta.$$

With this, we can find the length of the curve

$$\begin{aligned}
&\int_0^{2\pi} \sqrt{4\cos^2\theta + 4\sin^2\theta}\ d\theta \\
&= \int_0^{2\pi} \sqrt{4(\cos^2\theta + \sin^2\theta)}\ d\theta \\
&= \int_0^{2\pi} \sqrt{4}\ d\theta \\
&= 2\left[\theta\Big|_0^{2\pi}\right. \\
&= 4\pi.
\end{aligned}$$

7. Find the slope of the tangent line to the polar curve $r = 3\cos\theta$ at the point $\theta = \frac{\pi}{3}$.

First, we find r

$$\begin{aligned}
r &= 3\cos\left(\frac{\pi}{3}\right) \\
&= \frac{3}{2}.
\end{aligned}$$

From here we must find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{3}$ to get the slope

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \\
\frac{dy}{dx} &= \frac{-3\sin^2\theta + \frac{3}{2}\cos\theta}{-3\sin\theta\cos\theta - \frac{3}{2}\sin\theta} \\
\frac{dy}{dx} &= \frac{-3\sin^2\left(\frac{\pi}{3}\right) + \frac{3}{2}\cos\left(\frac{\pi}{3}\right)}{-3\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right) - \frac{3}{2}\sin\left(\frac{\pi}{3}\right)} \\
&= \frac{-3\left(\frac{\sqrt{3}}{2}\right)^2 + \frac{3}{2}\left(\frac{1}{2}\right)}{-3\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) - \frac{3}{2}\left(\frac{\sqrt{3}}{2}\right)} \\
&= \frac{\sqrt{3}}{3}.
\end{aligned}$$

8. Find the area of the region that lies inside $r = 1 + \cos\theta$ and outside $r = \cos\theta$

First, we find the area of $1 + \cos\theta$

$$
\begin{aligned}
&\frac{1}{2}\int_0^{2\pi} (1+\cos\theta)^2 \ d\theta \\
&\frac{1}{2}\int_0^{2\pi} 1 + 2\cos\theta + \cos^2\theta \ d\theta \\
&\frac{1}{2}\int_0^{2\pi} d\theta + \frac{1}{2}\int_0^{2\pi} 2\cos\theta \ d\theta + \frac{1}{2}\int_0^{2\pi} \frac{1}{2} + \frac{1}{2}\cos 2\theta \ d\theta \\
&= \frac{1}{2}\left[\theta\Big|_0^{2\pi} + \left[\sin\theta\Big|_0^{2\pi} + \frac{1}{4}\left[\theta + \frac{1}{2}\sin 2\theta\Big|_0^{2\pi}\right.\right.\right. \\
&= \pi + (0-0) + \left(\frac{1}{4}\left[2\pi + \frac{1}{2}\sin 4\pi - \left(0 + \frac{1}{2}\sin(0)\right)\right]\right) \\
&= \pi + \left(\frac{\pi}{2} + 0 - 0\right) \\
&= \pi + \frac{\pi}{2} \\
&= \frac{3\pi}{2}.
\end{aligned}
$$

Then we can find the area of $r = \cos\theta$

$$
\begin{aligned}
&\frac{1}{2}\int_0^{\pi} \cos^2\theta \ d\theta \\
&\frac{1}{2}\int_0^{2\pi} \frac{1}{2} + \frac{1}{2}\cos 2\theta \ d\theta \\
&= \frac{1}{2}\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\Big|_0^{2\pi}\right. \\
&= \frac{1}{2}\left[\pi + 0 - 0 + 0\right] \\
&= \frac{\pi}{2}.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&A_1 - A_2 \\
&\frac{3\pi}{2} - \frac{\pi}{2} \\
&= \pi.
\end{aligned}
$$