Nate Warner MATH 232 Spring 2024

Homework/Worksheet 6 - Due: Wednesday, March 6

1. Let $f(x, y) = e^{xy} \cos x \sin y$. Find $f_x(x, y)$ and $f_y(x, y)$.

By the product rule, treating y as a constant, we find

$$\begin{aligned} f_x(x, y) &= \sin(y) \frac{\delta f}{\delta x} e^{xy} \cos(x) \\ &= \sin(y)(-e^{xy} \sin(x) + ye^{xy} \cos(x)). \end{aligned}$$

Treating x as a constant, we find

$$\begin{aligned} f_y(x, y) &= \cos(x) \frac{\delta f}{\delta y} e^{xy} \sin(y) \\ &= \cos(x)(e^{xy} \cos(y) + xe^{xy} \sin(y)). \end{aligned}$$

2. Let $f(x, y) = \frac{xy}{x-y}$. Find $f_x(2, -2)$ and $f_y(2, -2)$. Interpret these results as slopes.

First, we find $\frac{\delta}{\delta x}$ and $\frac{\delta}{\delta y}$. To do this, we use the chain rule.

$$\begin{aligned} \frac{\delta}{\delta x} &= \frac{xy - (y(x - y))}{(x - y)^2} = \frac{-y^2}{(x - y)^2} \\ \frac{\delta}{\delta y} &= \frac{x(x - y) - (-xy)}{(x - y)^2} = \frac{x^2}{(x - y)^2}. \end{aligned}$$

Next, we evaluate at the point $P(2, -2)$

$$\begin{aligned} f_x(2, -2) &= \frac{-(-2)^2}{(2 + 2)^2} = -\frac{1}{4} \\ f_y(2, -2) &= \frac{2^2}{(2 + 2)^2} = \frac{1}{4}. \end{aligned}$$

This implies that the slope in the x-direction is $-\frac{1}{4}$, while the slope in the y-direction is $\frac{1}{4}$.

3. Let $f(x, y) = \ln(x - y)$. Find $f_{xx}(x, y), f_{yy}(x, y)$, and $f_{xy}(x, y)$.

Using properties of differentation we find

$$\begin{aligned} f_x(x, y) &= \frac{1}{x + y} \\ f_{xx}(x, y) &= -\frac{1}{(x + y)^2} \\ f_y(x, y) &= \frac{1}{x + y} \\ f_{yy} &= -\frac{1}{(x + y)^2} \\ f_{xy} &= \frac{-1}{(x + y)^2}. \end{aligned}$$

4. Show that $f(x, y) = \ln\left(x^2 + y^2\right)$ solves Laplace's equation $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

First, we find the partial derivatives

$$\begin{aligned}
f_x(x, y) &= \frac{2x}{x^2 + y^2} \\
f_{xx}(x, y) &= \frac{2(x^2 + y^2) - 4x^2}{(x^2 + y^2)^2} = \frac{-2x^2 + 2y^2}{(x^2 + y^2)^2} \\
f_y(x, y) &= \frac{2y}{x^2 + y^2} \\
f_{yy}(x, y) &= \frac{2(x^2 + y^2 - 4y^2)}{(x^2 + y^2)^2} = \frac{-2y^2 + 2x^2}{(x^2 + y^2)^2}.
\end{aligned}$$

With these results, we can compute $f_{xx} + f_{yy}$ and confirm that it equates to zero

$$\begin{aligned}
f_{xx} + f_{yy} &= \frac{-2x^2 + 2y^2}{(x^2 + y^2)^2} + \frac{-2y^2 + 2x^2}{(x^2 + y^2)^2} \\
&= \frac{-2x^2 + 2y^2 - 2y^2 + 2x^2}{(x^2 + y^2)^2} \\
&= \frac{0}{(x^2 + y^2)^2} \\
&= 0.
\end{aligned}$$

5. Find an equation of the tangent plane to the surface $f(x, y) = \ln\left(10x^2 + 2y^2 + 1\right)$ at $P(0, 0, 0)$.

Remark. Let $P_0 = (x_0, y_0, z_0)$ be a point on a surface S, and let C be any curve passing through P_0 and lying entirely in S. If the tangent lines to all such curves C at P_0 lie in the same plane, then this plane is called the tangent plane to S at P_0.

The equation for a tangent plane at a point is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

With this, we start by computing $f(0, 0)$

$$\begin{aligned}
f(0, 0) &= \ln 10(0)^2 + 2(0)^2 + 1 \\
&= \ln 1 = 0.
\end{aligned}$$

Next, we find the partial derivatives

$$\begin{aligned}
f_x(x, y) &= \frac{20x}{10x^2 + 2y^2 + 1} \\
f_y(x, y) &= \frac{4y}{10x^2 + 2y^2 + 1}.
\end{aligned}$$

We now evaluate at our point

$$\begin{aligned}
f_x(0, 0) &= 0 \\
f_y(0, 0) &= 0.
\end{aligned}$$

This gives the tangent plane at the point $P(0,0,0)$ as $z = 0$. Thus, the plane is flat and parallel to the xy-plane

6. Let $f(x, y) = \ln\left(\sqrt{x^2 + y^2}\right)$.

   (a) Find an equation of the tangent plane to the surface $f(x, y)$ at $(3, 4, \ln 5)$.

   (b) Find the linearization $L(x, y)$ of the function $f(x, y)$ at $(3, 4)$.

   (c) Use the linear approximation of $f(x, y)$ at $(3, 4)$ to approximate $f(2.99, 4.01)$.

To find the equation of the tangent plane at the point $(3, 4, \ln(5))$, we first need to find the partial derivatives

$$\begin{aligned} f_x &= \frac{1}{(x^2+y^2)^{\frac{1}{2}}} \cdot \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2x = \frac{2x}{2(x^2+y^2)} \\ f_y &= \frac{2y}{2(x^2+y^2)}. \end{aligned}$$

Next, we evaluate at the point $(3, 4)$

$$\begin{aligned} f_x(3,4) = \frac{2(3)}{2(3^2+4^2)} &= \frac{6}{50} = \frac{3}{25} \\ f_y(3,4) &= \frac{2(4)}{50} = \frac{4}{25}. \end{aligned}$$

Now, by the equation of a tangent plane, we have

$$z = \ln(5) + \frac{3}{25}(x-3) + \frac{4}{25}(y-4).$$

The equation above is also the linearization $L(x, y)$ at $(3, 4)$. We can now use it to approximate $f(2.99, 4.01)$, we have

$$\begin{aligned} L(2.99, 4.01) &= \ln(5) + \frac{3}{25}(2.99-3) + \frac{4}{25}(4.01-4) \\ &\approx 1.60983. \end{aligned}$$

7. Find the linear approximation of the function $f(x, y) = e^x \cos y$ at $P(0, 0)$ and use it to approximate $f(0.01, -0.02)$.

First, we find the partial derivatives and evaluate them at the point $P(0, 0)$

$$\begin{aligned} p_x &= e^x \cos(y) \\ p_x(0,0) = e^0 \cos(0) = 1 & \\ p_y &= -e^x \sin(y) p_y(0,0) = -e^0 \sin(0) = 0. \end{aligned}$$

Now, the linearization is given by

$$\begin{aligned} L(x,y) &= 1 + 1(x-0) + 0(y-0) \\ &= 1 + x. \end{aligned}$$

We then use $L(x,y)$ to approximate $f(0.01,-0.02)$

$$\begin{aligned}L(0.01,-0.02) &= 1+0.01\\ &= 1.01.\end{aligned}$$

8. Let $f(x,y)=x^4, x=t, y=t$. Use the chain rule to find df/dt.

Remark. Suppose that $x=g(t)$ and $y=h(t)$ are differentiable functions of t and $z=f(x,y)$ is a differentiable function of x and y. Then $z=f(x(t),y(t))$ is a differentiable function of t and
$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt},$$
where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x,y).

By this, we find

$$\begin{aligned}\frac{df}{dt} &= 4x^3\\ &= 4t^3.\end{aligned}$$

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$$\int_0^k x^2+2x\ dx=0\notin k\neq 0.$$

9. Let $z=e^{x^2y}$, where $x=\sqrt{uv}$ and $y=1/v$. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

After constructing a tree diagram, we find

$$\begin{aligned}\frac{\delta z}{\delta u} &= \frac{\delta z}{\delta x}\cdot\frac{\delta x}{\delta u}\\ \frac{\delta z}{\delta v} &= \frac{\delta z}{\delta x}\cdot\frac{\delta x}{\delta v}+\frac{\delta z}{\delta y}\cdot\frac{\delta y}{\delta v}.\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{\delta z}{\delta u}=2xye^{x^2y}\cdot\frac{1}{2}(uv)^{-\frac{1}{2}}(v)=\frac{2xye^{x^2y}v}{2(uv)^{\frac{1}{2}}}=\frac{2(uv)^{\frac{1}{2}}v^{-1}e^{(uv)^{\frac{1}{2}2}v^{-1}}v}{2(uv)^{\frac{1}{2}}}=e^u\\ &\frac{\delta z}{\delta v}=2xye^{x^2y}\cdot\frac{1}{2}(uv)^{-\frac{1}{2}}(u)+x^2e^{x^2y}\cdot-\frac{1}{v^2}=\frac{2xye^{x^2y}u}{2(uv)^{\frac{1}{2}}}-\frac{x^2e^{x^2y}}{v^2}\\ &=\frac{(uv)^{1/2}\cdot\frac{1}{v}e^{uv\cdot\frac{1}{v}}u}{(uv)^{\frac{1}{2}}}-\frac{uve^{uv\cdot\frac{1}{v}}}{v^2}\\ &=\frac{ue^u}{v}-\frac{ue^v}{v}\\ &=0.\end{aligned}$$