Methods Of Integration

Pre Calculus 2

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$$\int_a^b d\,x$$

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1 U-Substitution

If $u = g(x)$ is differentiable and its range $\in I$ and f is continuous on I, then

$$\int f(g(x))g'(x)\ dx = \int f(u)\ du.$$

Process:

1. Make a decent choice of what to let u equal
2. Change our integral from being in terms of x, to in terms of u
3. Integrate $\int f(u)\ du$
4. Change back to x

Note:-

Also include the constant in your u sub, if it is attached to the function you let u equal, Also note for rational trig functions, you can move trig functions from upstairs or downstairs based on their reciprical function, for example, a $\cos^2 x$ in the denominator can be moved upstairs as $\sec^2 x$

Notes:

- Our goal with u-sub is to let u equal some function in our composition of functions, such that if we derive that function, we get back something that is also in our integrand
- Say we have something like:

$$\int \sec^2\theta\ d\theta$$
$$Let\ u = 8\theta$$
$$du = 8d\theta$$
$$\frac{1}{8}du = d\theta.$$

- Know that you can rewrite equations like:

$$\int \frac{(\ln x)^{36}}{x}\ dx$$
$$as: \int \frac{1}{x}(\ln x)^{36}\ dx.$$

- Say we have:

$$-\frac{1}{2}\int_0^{-1} e^u\ du.$$

  - We can flip the limits of integration to remove the negative sign. So we will have:

$$\frac{1}{2}\int_{-1}^{0} e^u\ du.$$

- Look out for being able to turn antiderivative into inverse trig functions.

- Say we have:

$$\int \frac{x^7}{1+x^{16}}\ dx.$$

- Don't let $u = 1 + x^8$, we could do this by turning x^{16} *into* $(x^8)^2$, but instead, just let $u = x^8$. We do this because it ends up like:

$$\frac{1}{8}du = x^7 dx.$$

- So when we sub we get that nice arctan antiderivative, just something to look out for.

$$\frac{1}{8}\int \frac{1}{1+u^2}\ du.$$

- Note that we can write:

$$e^{2x}.$$

  - as

$$(e^x)^2.$$

Definite Integrals with u-sub

1. Find what u is going to equal
2. Find u(a) and u(b)
3. make u-sub and use u(a) and u(b) as limits of integration
4. find antiderivative and evaluate at new limits

Note:-

For definite integrals, don't sub back in for u, just evalute integral with u still subbed

2 Integrals of Symmetric functions.

Sometimes you will run into integrals that are either impossible, or too difficult with u Substitution. For these cases we will look at Integrals of Symmetric Functions

Even:

$$f(-x) = f(x) \; then \; \int_{-a}^{a} f(x)\; dx = 2\int_{0}^{a} f(x)\; dx.$$

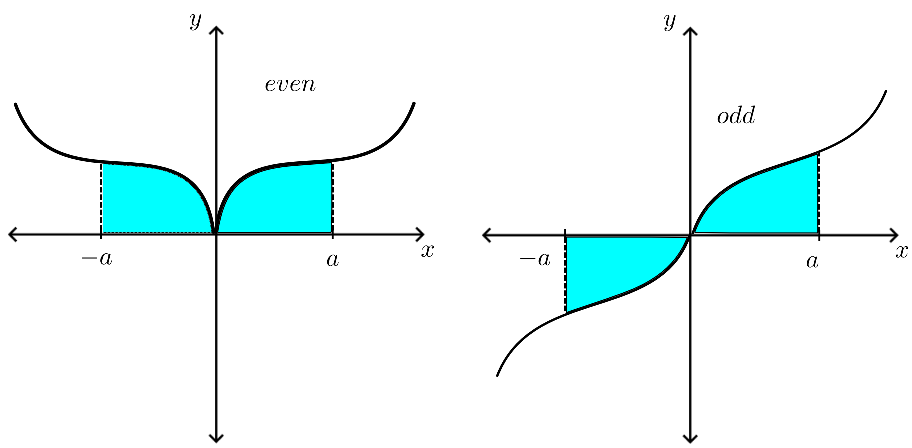
Odd:

$$f(-x) = -f(x) \; then \; \int_{-a}^{a} f(x)\; dx = 0.$$

Notes:

- If a function is even, you can replace your lower limit with zero and multiply the integral by 2
- if a function is odd, then the integral equals zero.

Figures:



3 Integration By Long Division.

Long division is typically used when evaluating integrals that involve rational functions, where the numerator's degree is equal to or greater than the denominator's degree. By performing long division, you can rewrite the rational function as a sum of a polynomial and a proper rational function, making it easier to integrate.

Consider the integral

$$\int \frac{x-5}{-2x+2}\,dx.$$

So with long division, we have:

$$-2x+2\overline{)x-5}.$$

We get $-\frac{1}{2}$, with a remainder of -4, which would be $\frac{4}{-2x+2}$. So now our integral would be:

$$\begin{aligned}
&\int \left(-\frac{1}{2}-\frac{4}{-2x+2}\right)dx\\
&=\int \left(-\frac{1}{2}-\frac{2}{-x+1}\right)dx\\
&=\int \left(-\frac{1}{2}-\left(-\frac{2}{x-1}\right)\right)dx\\
&=\int \left(-\frac{1}{2}+\frac{2}{x-1}\right)dx.
\end{aligned}$$

Now we can split up our integral:

$$\int -\frac{1}{2}\,dx+\int \frac{2}{x-1}\,dx.$$

We can see that our first integral will evaluate to $-\frac{1}{2}x+C$, and we can use U-Sub for the second integral:

$$\begin{aligned}
&\int \frac{2}{x-1}\,dx\\
&=\int \frac{1}{x-1}2\,dx\\
&=2\int \frac{1}{x-1}\,dx\\
&\text{Let } u=x-1\\
&du=dx\\
&=2\int \frac{1}{u}\,du=2\int u^{-1}\,dx\\
&=2\ln|u|+C\\
&=2\ln|x-1|+C.
\end{aligned}$$

So our full solution is:

$$-\frac{1}{2}x+2\ln|x-1|+C.$$

4 Integration By Completing The Square.

Completing the square is a technique commonly used in algebra to rewrite quadratic expressions. While it is not typically used directly for evaluating integrals, completing the square can be helpful in certain situations when dealing with quadratic forms within integrals. Here are two scenarios where completing the square can be useful for evaluating integrals:

- Quadratic Denominators: When you encounter an integral with a quadratic denominator, completing the square can help in simplifying the expression. By completing the square, you can rewrite the quadratic expression in a form that allows for a straightforward integration.

Consider the integral:

$$\int \frac{1}{5x^2 - 30x + 65}\, dx.$$

We can first start by factoring out a 5 from the denominator and moving it outside the integral.

$$\begin{aligned}&\int \frac{1}{5(x^2 - 6x + 13)}\, dx\\ &= \frac{1}{5}\int \frac{1}{x^2 - 6x + 13}\, dx.\end{aligned}$$

Now we can complete the square.

$$\begin{aligned}&\frac{1}{5}\int \frac{1}{x^2 - 6x + \left(\frac{-6}{2}\right)^2 - \left(\frac{-6}{2}\right)^2 + 13}\, dx\\ &= \frac{1}{5}\int \frac{1}{x^2 - 6x + 9 - 9 + 13}\, dx\\ &= \frac{1}{5}\int \frac{1}{(x-3)^2 + 2^2}\, dx.\end{aligned}$$

Note:-

We also subtracted 9 from the denominator as to not break any rules.

Now our goal is to make this arctan , so we are going to divide everything by 2^2 to get that $+1$ in the denominator.

So we get:

$$\begin{aligned}&\frac{\frac{1}{5}}{4} \cdot \int \frac{\frac{1}{2^2}}{\frac{(x-3)^2}{2^2} + \frac{2^2}{2^2}}\, dx\\ &\frac{1}{20}\int \frac{1}{\left(\frac{(x-3)}{2}\right)^2 + 1}\, dx.\end{aligned}$$

Now we can use U-Sub to evalute this integral:

$$\begin{aligned}\text{Let } u &= \frac{1}{2}x - \frac{3}{2}\\ du &= \frac{1}{2}dx\\ 2du &= dx.\end{aligned}$$

So we have:

$$2 \cdot \frac{1}{20} \int \arctan u \; du$$
$$= \frac{1}{10} \arctan u + C$$
$$= \frac{1}{10} \arctan \left(\frac{x-3}{2} \right) + C.$$

Example 4.1 (Evalute the integral by completing the square)

$$\int \frac{1}{\sqrt{-x^2 - 6x + 40}} \; dx.$$

By the looks of it, our goal will be to turn this into arcsine. So our first step will be to write the radical as:

$$\int \frac{1}{\sqrt{40 - x^2 - 6x)}} \; dx.$$

Because our goal is for this to resemble:

$$\frac{d}{dx} \sin^{-1} x$$
$$= \frac{1}{\sqrt{1 - x^2}}.$$

Now if we complete square:

$$\int \frac{1}{\sqrt{40 - x^2 - 6x + \left(\frac{-6}{2}\right)^2 - \left(-\frac{6}{2}\right)^2}} \; dx$$
$$= \int \frac{1}{\sqrt{40 - x^2 - 6x + 9 - 9}} \; dx$$
$$= \int \frac{1}{\sqrt{40 + 9 - x^2 - 6x - 9}} \; dx$$
$$= \int \frac{1}{\sqrt{-(-40 - 9 + x^2 + 6x + 9)}} \; dx$$
$$= \int \frac{1}{\sqrt{-(-49 + (x+3)^2)}} \; dx$$
$$= \int \frac{1}{\sqrt{49 - (x+3)^2)}} \; dx$$
$$= \int \frac{1}{\sqrt{7^2 - (x+3)^2)}} \; dx$$
$$.$$

Now if we let $u = x + 3$, then $du = dx$, so we have:

$$\begin{aligned} &\int \frac{1}{\sqrt{7^2 - u^2}}\ du \\ &= \int \frac{1}{\sqrt{\frac{7^2}{7^2} - \frac{u^2}{7^2}}}\ du \\ &= \int \frac{1}{\sqrt{1 - \left(\frac{u}{7}\right)^2}}\ du. \end{aligned}$$

Which we can evaluate as:

$$\begin{aligned} &\arcsin\left(\frac{u}{7}\right) + C \\ &\boxed{= \arcsin\left(\frac{x+3}{7}\right) + C}. \end{aligned}$$

Example 4.2

Problem:

$$\int \frac{1}{(x-7)^2 + 3^2}\ dx.$$

Accepted Answer:

$$\frac{1}{3}\arctan\left(\frac{x-7}{3}\right) + C.$$

My Work

$$\begin{aligned} &1 \cdot \int \frac{1}{(x-7)^2 + 3^2}\ dx \\ &= \frac{1}{3^2}\int \frac{1}{\left(\frac{(x-7)}{3}\right)^2 + \frac{3^2}{3^2}}\ dx \\ &= \frac{1}{9}\int \frac{1}{\left(\frac{(x-7)}{3}\right)^2 + 1}\ dx \\ &. \end{aligned}$$

$$\begin{aligned} \textit{Let } u &= \frac{1}{3}x - \frac{7}{3} \\ du &= \frac{1}{3}dx \\ 3du &= dx. \end{aligned}$$

$$\begin{aligned} &3 \cdot \frac{1}{9}\int \frac{1}{u^2 + 1}\ dx \\ &= \frac{1}{3}\tan^{-1} u + C \\ &= \frac{1}{3}\tan^{-1}\left(\frac{x-7}{3}\right) + C. \end{aligned}$$