Nate Warner MATH 353 Spring 2025

Problem set 6 - Due: Wednesday, March 5

1. Prove that if $0 < AB < \omega$, $X \neq Y$, A-X-B and A-Y-B, then either A-X-Y-B or A-Y-X-B. Does the same conclusion follow if $AB = \omega$?

Proof. Assume $0 < AB < \omega$, $X \neq Y$, A-X-B and A-Y-B. Since $0 < AB < \omega$, $\overrightarrow{AB}$ is defined.

A-X-B and A-Y-B together with $X \neq Y$ implies A, B, X, Y are distinct, collinear. Thus, we can apply theorem 8.3, which says one of the following relations must hold.

$$A\text{-}X\text{-}Y \quad \text{or} \quad A\text{-}Y\text{-}X$$

Assume A-X-Y, then A-Y-B with the ROI yields A-X-Y-B

Assume A-Y-X, then A-X-B with the ROI yields A-Y-X-B ■

Note: If $AB = \omega$, then ray $\overrightarrow{AB}$ would not be defined, and we would not be able to invoke theorem 8.3. Thus, the same conclusion would not follow.

2. Prove proposition 8.11

Proposition 8.11 Let A, B be any two points on line m, with $0 < AB < \omega$. Then, there exists a point C on m with C-A-B and $CB < \omega$.

Proof. Assume A, B are any two points on line m, with $0 < AB < \omega$. Thus, $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are defined. We let our choice of C be on the ray $\overrightarrow{BA}$ but not in $\overline{BA}$. We note that there are infinitely many choices for C. By Ax.RR, there exists a choice of C on the ray $\overrightarrow{BA}$ such that $BC = CB < \omega$.

Since C on $\overrightarrow{BA}$ but not in $\overline{BA}$, C-A-B by definition of the ray $\overrightarrow{BA}$. ■

3. Show via the following steps that $\mathbb{H}$ satisfies axiom RR.

Let $M = (r, mr+b)$ and $N = (t, mt+b)$ be the points of intersection of the line $y = mx+b$ with the unit circle, $r < t$. So $l = \{(x, mx+b) : \ r < x < t\}$ is a (typical nonvertical) line in $\mathbb{H}$. Let $A = (a, ma+b), C = (c, mc+b)$ be two points on l. We will assume $a < c$, so that $r < a < c < t$

(a) Show that $X = (x, y)$ is on $\overrightarrow{AC}$ (in $\mathbb{H}$) if and only if $a \leqslant x < t$ and $y = mx+b$

(b) For $X = (x, y) \neq A$ on $\overrightarrow{AC}$, show

$$AX = \ln\left(\frac{(t-a)(x-r)}{(a-r)(t-x)}\right)$$

(c) For any real number $s > 0$, show that there exists an x with $a < x < t$ such that

$$\ln\left(\frac{(t-a)(x-r)}{(a-r)(t-x)}\right) = s$$

and hence $AX = s$ for $X = (x, mx+b) \in \overrightarrow{AC}$

a.) We show part (a) in two parts. First (1) that $X = (x,y) \in \overrightarrow{AC} \implies a \leqslant x < t$, and $y = mx+b$. Then (2), $a \leqslant x < t$ and $y = mx+b \implies X = (x,y) \in \overrightarrow{AC}$

(1) assume $X = (x, y)$ exists on the ray $\overrightarrow{AC}$. $X = A$, $X = C$, or one of A-X-C, A-C-X. If $X = A$, then $(x, y) = (a, ma+b)$ implies $x = a$, and $y = ma+b = mx+b$. Thus, $a \leqslant x < t$ and $y = mx+b$ are satisfied. If $X = C$, then $(x,y) = (c, mc+b)$ implies $x = c$, and $y = mc+b = mx+b$. Since $t > c > a$, $a \leqslant x < t$ and $y = mx+b$ are satisfied.

Assume $X \neq A$ or C. Then, $X \in \overrightarrow{AC}$ implies one of A-X-C or A-C-X. Assume A-X-C. Since $t > c > a$ x must live somewhere between a and c for A-X-C to be satisfied on the hyperbolic plane. Thus, $a < x < c$ satisfies $a \leqslant x < t$. Next, since A-X-C, X is collinear with A and C, which implies $y = mx+b$ is satisfied.

Assume A-C-X. Again, since $t > c > a$, A-C-X implies $c < x < t$, which satisfies $a \leqslant x < t$. Also, X is collinear with A and C, thus $y = mx+b$ is also satisfied.

(2) Assume $a \leqslant x < t$ and $y = mx+b$. Since $y = mx+b$, X is collinear with A and C. Thus, X exists somewhere on the line $\overleftrightarrow{AC}$. But, since $a \leqslant x < t$, X must be somewhere between A and N. Notice that this is precisely the definition of the ray $\overrightarrow{AC} = \{A, C\} \cup \{X : A\text{-}X\text{-}C\} \cup \{X : A\text{-}C\text{-}X\}$ on the hyperbolic line $\overleftrightarrow{AC}$. Thus, for $a \leqslant x < t$ to be satisfied, one of A-X-C or A-C-X must be true.

Note that $x = a$ or $x = c$ implies $X = A$ or $X = C$, both of which make X be on the ray $\overrightarrow{AC}$, as desired.

b.) Assume $X = (x, y) \neq A$ on $\overrightarrow{AC}$, then

$$AX = \ln\left(\frac{e(AN)e(XM)}{e(AM)e(XN)}\right) = \ln\left(\frac{|a-t| \cdot |x-r|}{|a-r| \cdot |x-t|}\right)$$

But, since $r < a \leqslant x < c < t$, we have

$$AX = \ln\left(\frac{(t-a)(x-r)}{(a-r)(t-x)}\right)$$

As desired.

c.) We first note that as $x \to t$, $(t-x) \to 0 \implies \frac{(t-a)(x-r)}{(a-r)(t-x)} \to +\infty \implies \ln\left(\frac{(t-a)(x-r)}{(a-r)(t-x)}\right) \to +\infty$. Next, we note that as $x \to a$, $\frac{(t-a)(x-r)}{(a-r)(t-x)} \to 1$, since $\frac{(t-a)(x-r)}{(a-r)(t-x)} = \frac{(t-a)(a-r)}{(a-r)(t-a)} = 1$ when $x = a$. Therefore the domain of $AX = \ln\left(\frac{(t-a)(x-r)}{(a-r)(t-x)}\right)$ is $(1, \infty)$

Since the natural log function is continuous over $[1, \infty)$ and maps $[1, \infty) \to (0, \infty)$, any $s > 0$ has an x satisfying $a < x < t$ such that

$$AX = \ln\left(\frac{(t-a)(x-r)}{(a-r)(t-x)}\right) = s$$

$\blacksquare$

4. Let $A = (0, 0)$ and $B = (.8, 0)$ in $\mathbb{H}$, and compute the midpoint of $\overline{AB}$ (It's not (.4,0))

We require a point K such that $AK = KB = \frac{1}{2}AB$. First, we compute $d_{\mathbb{H}}(AB)$. If $A = (0, 0)$, and $B = (0.8, 0)$, then $M = (-1, 0)$ and $N = (1, 0)$. Further, distance is given by $e(PQ) = |x_1 - x_2|$ for all points on the line $y = 0$ collinear with A and B

$$\begin{aligned} d_{\mathbb{H}}(AB) &= \ln\left(\frac{e(AN)e(BM)}{e(AM)e(BN)}\right) = \ln\left(\frac{1\,|0.8-(-1)|}{1\,|0.8-1|}\right) \\ &= \ln\left(\frac{1.8}{0.2}\right) = \ln(9) \end{aligned}$$

Thus, we require K such that $AK = KB = \frac{1}{2}AB = \frac{1}{2}\ln(9) = \ln\left(9^{\frac{1}{2}}\right) = \ln(3)$. That is,

$$\begin{aligned} d_{\mathbb{H}}(AK) &= \ln\left(\frac{e(AN)e(KM)}{e(AM)e(KN)}\right) = \ln(3) \\ \implies \frac{e(AN)e(KM)}{e(AM)e(KN)} &= 3 \end{aligned}$$

Note that $e(AN) = e(AM) = 1$. Thus, if $K = (x, 0)$ for $0 < x < 0.8$

$$\begin{aligned} \frac{e(KM)}{e(KN)} &= 3 \\ \implies \frac{|x+1|}{|x-1|} &= 3 \\ \implies \frac{\sqrt{(x+1)^2}}{\sqrt{x-1}^2} &= 3 \\ \implies \sqrt{(x+1)^2} &= 3\sqrt{(x-1)^2} \\ \implies (x+1)^2 &= 9(x-1)^2 \\ \implies x^2 + 2x + 1 &= 9x^2 - 18x + 9 \\ \implies 8x^2 - 20x + 8 &= 0 \end{aligned}$$

By the quadratic formula

$$\begin{aligned} x &= \frac{20 \pm \sqrt{20^2 - 4(8)(8)}}{2(8)} \\ &= \frac{20 \pm 12}{16} \end{aligned}$$

Thus, $x = \frac{1}{2}, 2$. Observe that since $2 > 1 > 0.8$, it cannot be a solution. Thus, $x = \frac{1}{2}, d_{\mathbb{H}}(AK) = \ln(3)$ and the midpoint is therefore $K = (0.5, 0)$. We quickly verify that $d_{\mathbb{H}}(KB) = \ln(3)$

$$d_{\mathbb{H}}(KB) = \ln\left(\frac{e(KN)e(BM)}{e(KM)e(BN)}\right) = \ln\left(\frac{|0.5-1| \cdot |0.8+1|}{|0.5+1| \cdot |0.8-1|}\right) = \ln(3)$$

Thus, $d_{\mathbb{H}}(AK) = d_{\mathbb{H}}(KB) = \frac{1}{2}AB = \ln(3)$, and $K = (0.5, 0)$ is the midpoint of $\overline{AB}$