

Problem set 2 - Due: Wednesday, October 15

1.7.10. Let

$$A = \begin{bmatrix} 2 & 1 & -1 & 3 \\ -2 & 0 & 0 & 0 \\ 4 & 1 & -2 & 6 \\ -6 & -1 & 2 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} 13 \\ -2 \\ 24 \\ -14 \end{bmatrix}.$$

- (a) Calculate the appropriate (four) determinants to show that A can be transformed to (nonsingular) upper-triangular form by operations of type 1 only. (By the way, this is strictly an academic exercise. In practice one never calculates these determinants in advance.)
- (b) Carry out the row operations of type 1 to transform the system $Ax = b$ to an equivalent system $Ux = y$, where U is upper triangular. Save the multipliers for use in Exercise 1.7.18.
- (c) Carry out the back substitution on the system $Ux = y$ to obtain the solution of $Ax = b$. Don't forget to check your work.

Remark. Let $A \in \mathbb{R}^{n \times n}$. A admits an LU factorization $A = LU$ where L is unit lower triangular and U is upper triangular if and only if all leading principal submatrices are nonsingular. ■

a.) So, we check that

$$\det([2]), \det\left(\begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix}\right), \det\left(\begin{bmatrix} 2 & 1 & -1 \\ -2 & 0 & 0 \\ 4 & 1 & -2 \end{bmatrix}\right), \det\left(\begin{bmatrix} 2 & 1 & -1 & 3 \\ -2 & 0 & 0 & 0 \\ 4 & 1 & -2 & 6 \\ -6 & -1 & 2 & -3 \end{bmatrix}\right)$$

are all nonzero. We see that

$$\begin{aligned} \det([2]) &= 2 \neq 0, \\ \det\left(\begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix}\right) &= 2(0) - (1)(-2) = 2 \neq 0, \\ \det\left(\begin{bmatrix} 2 & 1 & -1 \\ -2 & 0 & 0 \\ 4 & 1 & -2 \end{bmatrix}\right) &= -1 \cdot -2(1(-2) - (-1)(1)) = -1 \cdot 2(-2 + 1) = -2 \neq 0, \\ \det\left(\begin{bmatrix} 2 & 1 & -1 & 3 \\ -2 & 0 & 0 & 0 \\ 4 & 1 & -2 & 6 \\ -6 & -1 & 2 & -3 \end{bmatrix}\right) &= -1 \cdot -2 \det\left(\begin{bmatrix} 1 & -1 & 3 \\ 1 & -2 & 6 \\ -1 & 2 & -3 \end{bmatrix}\right) \\ &= 2(1(-2(-3) - 6(2)) - (-1)(1(-3) - 6(-1)) + 3(1(2) - (-2)(-1))) \\ &= 2(-3) = -6 \neq 0. \end{aligned}$$

Thus, all leading principal submatrices are nonsingular and A can be transformed to nonsingular upper-triangular form by operations of type 1 only.

b.) We use Gaussian Elimination on the augmented system $[A|b] \rightarrow [U|y]$. We have

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 3 & 13 \\ -2 & 0 & 0 & 0 & -2 \\ 4 & 1 & -2 & 6 & 24 \\ -6 & -1 & 2 & -3 & -14 \end{array} \right].$$

The operations to get $a_{21} = a_{31} = a_{41} = 0$ are

$$\begin{aligned} -(-1)r_1 + r_2 &\rightarrow r'_2, \\ -2r_1 + r_3 &\rightarrow r'_3, \\ -(-3)r_1 + r_4 &\rightarrow r'_4. \end{aligned}$$

Thus, $m_{21} = -1$, $m_{31} = 2$, $m_{41} = -3$ and the system becomes

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 3 & 13 \\ 0 & 1 & -1 & 3 & 11 \\ 0 & -1 & 0 & 0 & -2 \\ 0 & 2 & -1 & 6 & 25 \end{array} \right].$$

Next, we set a_{22} as the pivot element, r_2 as the pivot row, and perform the following operations to get $a_{23} = a_{24} = 0$. The operations are

$$\begin{aligned} r'_3 &\leftarrow r_3 - (-1)r_2 \implies m_{32} = -1, \\ r'_4 &\leftarrow r_4 - 2r_2 \implies m_{42} = 2. \end{aligned}$$

After these operations, the system becomes

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 3 & 13 \\ 0 & 1 & -1 & 3 & 11 \\ 0 & 0 & -1 & 3 & 9 \\ 0 & 0 & 1 & 0 & 3 \end{array} \right].$$

Next, we set a_{33} as the pivot element, and r_3 as the pivot row, and perform the operation

$$r'_4 \leftarrow r_4 - (-1)r_3 \implies m_{43} = -1.$$

After this operation, the system becomes

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 3 & 13 \\ 0 & 1 & -1 & 3 & 11 \\ 0 & 0 & -1 & 3 & 9 \\ 0 & 0 & 0 & 3 & 12 \end{array} \right].$$

Thus, the system $Ux = y$ is

$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 13 \\ 11 \\ 9 \\ 12 \end{bmatrix}.$$

(c) Using back substitution, we can solve the above system.

$$\begin{aligned} 3x_4 &= 12 \implies x_4 = 4, \\ -x_3 + 3x_4 &= 9 \implies x_3 = -1(9 - 3(4)) = 3, \\ x_2 - x_3 + 3x_4 &= 11 \implies x_2 = 11 - 3(4) + 3 = 2, \\ 2x_1 + x_2 - x_3 + 3x_4 &= 13 \implies x_1 = \frac{13 - 3(4) + 3 - 2}{2} = 1. \end{aligned}$$

So, the solution to $Ax = b$ is

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

We can verify this solution by computing Ax , and observing that it equals the given b . We see that

$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ -2 & 0 & 0 & 0 \\ 4 & 1 & -2 & 6 \\ -6 & -1 & 2 & -3 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2+2-3+12 \\ -2+0+0+0 \\ 4+2-6+24 \\ -6-2+6-12 \end{pmatrix} = \begin{pmatrix} 13 \\ -2 \\ 24 \\ -14 \end{pmatrix}.$$

Thus, the solution is verified.

Note: We can assemble our multipliers to form L , we have

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ -3 & 2 & -1 & 1 \end{bmatrix}.$$

1.7.18. Solve the linear system $Ax = \hat{b}$, where A is as in Exercise 1.7.10 and

$$\hat{b} = [12 \quad -8 \quad 21 \quad -26]^T.$$

Use the L and U that you calculated in Exercise 1.7.10.

From the previous exercise, we have that

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ -3 & 2 & -1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

So, the system $Ax = \hat{b}$ is solved using our LU decomposition for A . We have

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ -3 & 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 12 \\ -8 \\ 21 \\ -26 \end{pmatrix}.$$

Let $Ux = y$, and $Ly = b$. First, we solve $Ly = b$ for y using forward substitution.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ -3 & 2 & -1 & 1 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 12 \\ -8 \\ 21 \\ -26 \end{pmatrix}$$

implies

$$\begin{aligned} y_1 &= 12, \\ -y_1 + y_2 &= -8 \implies y_2 = -8 + 12 = 4, \\ 2y_1 - y_2 + y_3 &= 21 \implies y_3 = 21 + 4 - 2(12) = 1, \\ -3y_1 + 2y_2 - y_3 + y_4 &= -26 \implies y_4 = -26 + 1 - 2(4) + 3(12) = 3. \end{aligned}$$

So,

$$y = \begin{pmatrix} 12 \\ 4 \\ 1 \\ 3 \end{pmatrix}.$$

Now, we solve $Ux = y$ with backward substitution. We have

$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 12 \\ 4 \\ 1 \\ 3 \end{pmatrix}.$$

Which, implies that

$$\begin{aligned} 3x_4 &= 3 \implies x_4 = 1, \\ -x_3 + 3x_4 &= 1 \implies x_3 = -1(1 - 3(1)) = 2, \\ x_2 - x_3 + 3x_4 &= 4 \implies x_2 = 4 - 3(1) + 2 = 3, \\ 2x_1 + x_2 - x_3 + 3x_4 &= 12 \implies x_1 = \frac{12 - 3(1) + 2 - 3}{2} = 4. \end{aligned}$$

So,

$$x = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}.$$

We verify the result by computing Ax , and comparing it against \hat{b} . We have

$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ -2 & 0 & 0 & 0 \\ 4 & 1 & -2 & 6 \\ -6 & -1 & 2 & -3 \end{bmatrix} \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2(4) + 1(3) - 1(2) + 3(1) \\ -2(4) + 0 + 0 + 0 \\ 4(4) + 1(3) - 2(2) + 6(1) \\ -6(4) - 1(3) + 2(2) - 3(1) \end{pmatrix} = \begin{pmatrix} 12 \\ -8 \\ 21 \\ -26 \end{pmatrix}.$$

The result is verified.

1.7.26. Use the inner-product formulation to calculate the LU decomposition of the matrix A in Exercise 1.7.10

Remark. The inner-product formulas to compute the LU decomposition are

$$u_{ij} = a_{ij} - \sum_{k=1}^{i-1} \ell_{ik} u_{kj} \quad j = i, i+1, \dots, n, \quad (1)$$

$$\ell_{ij} = \frac{a_{ij} - \sum_{k=1}^{j-1} \ell_{ik} u_{kj}}{u_{jj}} \quad i = j+1, j+2, \dots, n. \quad (2)$$

To use these formulas to find each u_{ij} we first need to plug $i = 1$ into (1), then after we get the first row of U , we can plug in $j = 1$ into (2) to get the first column of L , and so on. ■

Recall that the matrix A is given as

$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ -2 & 0 & 0 & 0 \\ 4 & 1 & -2 & 6 \\ -6 & -1 & 2 & -3 \end{bmatrix}.$$

So, we first find the first row of U (set $i = 1$), we have

$$\begin{aligned}u_{11} &= a_{11} = 2, \\u_{12} &= a_{12} = 1, \\u_{13} &= a_{13} = -1, \\u_{14} &= a_{14} = 3.\end{aligned}$$

Next, we find the first column of L (set $j = 1$),

$$\begin{aligned}\ell_{11} &= 1, \\\ell_{21} &= \frac{a_{21}}{u_{11}} = \frac{-2}{2} = -1, \\\ell_{31} &= \frac{a_{31}}{u_{11}} = \frac{4}{2} = 2, \\\ell_{41} &= \frac{a_{41}}{u_{11}} = -\frac{6}{2} = -3.\end{aligned}$$

For the second row of U ($i = 2$),

$$\begin{aligned}u_{22} &= a_{22} - \sum_{k=1}^1 \ell_{2k} u_{k2} = 0 - (-1)(1) = 1, \\u_{23} &= a_{23} - \sum_{k=1}^1 \ell_{2k} u_{k3} = 0 - (-1)(-1) = -1, \\u_{24} &= a_{24} - \sum_{k=1}^1 \ell_{2k} u_{k4} = 0 - (-1)(3) = 3.\end{aligned}$$

For the second column of L ($j = 2$),

$$\begin{aligned}\ell_{22} &= 1, \\\ell_{32} &= \frac{a_{32} - \sum_{k=1}^1 \ell_{3k} u_{k2}}{u_{22}} = \frac{1 - 2(1)}{1} = -1, \\\ell_{42} &= \frac{a_{42} - \sum_{k=1}^1 \ell_{4k} u_{k2}}{u_{22}} = \frac{-1 - (-3)(1)}{1} = 2.\end{aligned}$$

For the third row of U ($i = 3$),

$$\begin{aligned}u_{33} &= a_{33} - \sum_{k=1}^2 \ell_{3k} u_{k3} = a_{33} - (\ell_{31} u_{13} + \ell_{32} u_{23}) = -2 - (2(-1) + (-1)(-1)) = -1, \\u_{34} &= a_{34} - \sum_{k=1}^2 \ell_{3k} u_{k4} = a_{34} - (\ell_{31} u_{14} + \ell_{32} u_{24}) = 6 - (2(3) + (-1)(3)) = 3.\end{aligned}$$

For the third column of L ($j = 3$),

$$\begin{aligned}\ell_{33} &= 1, \\\ell_{43} &= \frac{a_{43} - \sum_{k=1}^2 \ell_{4k} u_{k3}}{u_{33}} = \frac{a_{43} - (\ell_{41} u_{13} + \ell_{42} u_{23})}{u_{33}} = \frac{2 - ((-3)(-1) + (2)(-1))}{-1} = -1.\end{aligned}$$

For the fourth row of U ($i = 4$),

$$u_{44} = a_{44} - \sum_{k=1}^3 \ell_{4k} u_{k4} = a_{44} - (\ell_{41} u_{14} + \ell_{42} u_{24} + \ell_{43} u_{34}) = -3 - ((-3)(3) + 2(3) + (-1)(3)) = -3.$$

For the fourth column of L ($j = 4$),

$$\ell_{44} = 1.$$

So, the LU decomposition according to the inner-product formulas is

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ -3 & 2 & -1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Which is exactly the same decomposition that we got with Gaussian Elimination.

1.7.34. In this exercise you will show that performing an elementary row operation of type 1 is equivalent to left multiplication by a matrix of a special type. Suppose \tilde{A} is obtained from A by adding m times the j th row to the i th row.

- (a) Show that $\tilde{A} = MA$, where M is the triangular matrix obtained from the identity matrix by replacing the zero by an m in the (i, j) position. For example, when $i > j$, M has the form

$$M = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & m & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{bmatrix}.$$

Notice that this is the matrix obtained by applying the type 1 row operation directly to the identity matrix. We call M an *elementary matrix of type 1*.

- (b) Show that $\det(M) = 1$ and $\det(\tilde{A}) = \det(A)$. Thus we see (again) that \tilde{A} is nonsingular if and only if A is.
- (c) Show that M^{-1} differs from M only in that it has $-m$ instead of m in the (i, j) position. M^{-1} is also an elementary matrix of type 1. To which elementary operation does it correspond?

1.7.36. Suppose \tilde{A} is obtained from A by multiplying the i th row by the nonzero constant c .

- Find the form of the matrix M (an *elementary matrix of type 3*) such that $\tilde{A} = MA$.
- Find M^{-1} and state its function as an elementary matrix.
- Find $\det(M)$ and determine the relationship between $\det(\tilde{A})$ and $\det(A)$. Deduce that \tilde{A} is nonsingular if and only if A is.

1.8.4. Let

$$A = \begin{bmatrix} 2 & 2 & -4 \\ 1 & 1 & 5 \\ 1 & 3 & 6 \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ -2 \\ -5 \end{bmatrix}.$$

Use Gaussian elimination with partial pivoting (by hand) to find matrices L and U such that U is upper

triangular, L is unit lower triangular with $|l_{ij}| \leq 1$ for all $i > j$, and $LU = \tilde{A}$, where \tilde{A} can be obtained from A by making row interchanges. Use your LU decomposition to solve the system $Ax = b$.

1.8.9. Let A be the matrix in Exercise 1.8.4. Determine matrices P , L , and U with the properties stated in Theorem 1.8.8, such that $A = P^T LU$.

1.8.12. Write an algorithm that implements Gaussian elimination with partial pivoting. Store L and U over A , and save a record of the row interchanges.

2.1.10. Prove that the 1-norm is a norm.

2.1.13. Prove that the ∞ -norm is a norm.

2.1.17.

1. Let A be a positive definite matrix, and let R be its Cholesky factor, so that $A = R^T R$. Verify that for all $x \in \mathbb{R}^n$,

$$\|x\|_A = \|Rx\|_2.$$

2. Using the fact that the 2-norm is indeed a norm on \mathbb{R}^n , prove that the A -norm is a norm on \mathbb{R}^n .

2.2.6.

- (a) Show that $\kappa(A) = \kappa(A^{-1})$
- (b) Show that for any nonzero scalar c , $\kappa(cA) = \kappa(A)$

2.2.15. Let us take another look at the ill-conditioned matrices

$$A = \begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} -998 & 999 \\ 999 & -1000 \end{bmatrix}$$

from Example 2.2.8. Notice that

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1999 \\ 1997 \end{bmatrix}. \quad (2.2.16)$$

If we use the ∞ -norm to measure lengths, the magnification factor

$$\frac{\|Ax\|_\infty}{\|x\|_\infty}$$

is 1999, which equals $\|A\|_\infty$. Thus $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a vector that is magnified maximally by A .

Since the amount by which a vector is magnified depends only on its direction and not on its length, we say that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in a *direction of maximum magnification* by A .

Equivalently we can say that $\begin{bmatrix} 1999 \\ 1997 \end{bmatrix}$ lies in a *direction of minimum magnification*.

Looking now at A^{-1} , we note that

$$A^{-1} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1997 \\ -1999 \end{bmatrix}.$$

The magnification factor $\frac{\|A^{-1}x\|_{\infty}}{\|x\|_{\infty}}$ is 1999, which equals $\|A^{-1}\|_{\infty}$, so $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is in a direction of maximum magnification by A^{-1} . Equivalently

$$A \begin{bmatrix} 1997 \\ -1999 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad (2.2.17)$$

and $\begin{bmatrix} 1997 \\ -1999 \end{bmatrix}$ is in a direction of minimum magnification by A .

We will use the vectors in (2.2.16) and (2.2.17) to construct a spectacular example. Suppose we wish to solve the system

$$\begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1999 \\ 1997 \end{bmatrix}. \quad (2.2.18)$$

That is, $Ax = b$, where $b = \begin{bmatrix} 1999 \\ 1997 \end{bmatrix}$. Then by (2.2.16) the unique solution is

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Now suppose that we solve instead the slightly perturbed system

$$\begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 1998.99 \\ 1997.01 \end{bmatrix}. \quad (2.2.19)$$

This is $\hat{A}x = b + \delta b$, where $\delta b = \begin{bmatrix} -0.01 \\ 0.01 \end{bmatrix} = 0.01 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, which is in a direction of maximum magnification by A^{-1} . By (2.2.17), $A\delta x = \delta b$, where $\delta x = \begin{bmatrix} 19.97 \\ -19.99 \end{bmatrix}$. Therefore $\hat{x} = x + \delta x = \begin{bmatrix} 20.97 \\ -18.99 \end{bmatrix}$. Thus the nearly identical problems (2.2.18) and (2.2.19) have very different solutions.

Repeat the proof of Theorem 2.3.3.

Remark. Theorem 2.3.3. Let A be nonsingular, let $b \neq 0$, and let x and $\hat{x} = x + \delta x$ be solutions of $Ax = b$ and $(A + \delta A)\hat{x} = b$, respectively. Then,

$$\frac{\|\delta x\|}{\|\hat{x}\|} \leq \kappa(A) \frac{\|\delta A\|}{\|A\|}. \quad (2.3.4)$$

Proof. Rewriting the equation $(A + \delta A)\hat{x} = b$ as $Ax + A\delta x + \delta A\hat{x} = b$, using the equation $Ax = b$, and reorganizing the resulting equation, we obtain

$$\delta x = -A^{-1}\delta A\hat{x}.$$

Thus

$$\|\delta x\| \leq \|A^{-1}\| \|\delta A\| \|\hat{x}\|. \quad (2.3.5)$$

Dividing through by $\|\hat{x}\|$ and using the definition $\kappa(A) = \|A\| \|A^{-1}\|$, we obtain the desired result.