Calc 1 Chapters 2-5 All Theorems

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Secant Lines

$$m_{pq} = \frac{y_2 - y_1}{x_2 - x_1}\ rather,\ \frac{P_y - Q_y}{P_x - Q_x}.$$

And, later we learned that the slope of the secant line is defined by:

$$m_{PQ} = \frac{f(x+h) - f(x)}{h}$$
$$And: m_{PQ} = \frac{f(x) - f(a)}{x - a}.$$

Tangent lines

Approximation of slope of tangent line

$$\lim_{Q \to P} m_{PQ} = m.$$

And, later we learned that the slope of the secant line is defined by:

$$m_{tan} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$And: m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Point-slope form

$$y - y_1 = m(x - x_1).$$

Limits

$$\lim_{x \to a} f(x) = L$$
$$\lim_{x \to a} f(x) = l \iff \lim_{x \to a-} f(x) = l \wedge \lim_{x \to a+} f(x) = l.$$

Asymptotes with limits

Vertical if:

$$\lim_{x \to a-} f(x) = \infty\ or\ -\infty$$
$$\lim_{x \to a+} f(x) = \infty\ or\ -\infty$$
$$\lim_{x \to a} f(x) = \infty\ or\ -\infty.$$

Horizontal if:

$$\lim_{x \to \infty} f(x) = L$$
$$Or \lim_{x \to -\infty} f(x) = l.$$

Continuity

For a function to have continuity ata, 3 things must be true:

1. f(x) is defined at a
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

If no. 3 is true, the function is automatically continous at a

One-sided continuity

- Continuity from the right:

$$\lim_{x \to a+} f(x) = f(a).$$

- Continuity from the left:

$$\lim_{x \to a-} f(x) = f(a).$$

If f and g are continuous at a, then:

- $f + g$
- $f - g$
- fg
- $\frac{f}{g}$
- cf

Are all continuous at a

Also:

- Any polynomial is continous on its domain ($\mathbb{R}$)
- Any rational function is continous on its domain

Note:-

If $\lim_{x \to a} f(x)$ exists, Then you don't need to worry about which side the continuity is coming from

Intermediate value theorem

Suppose f is continuous on [a,b], Let N be any number between f(a) and f(b), where $f(a) \neq f(b)$, then:

$$\exists\ c \in (a,b) \mid f(c) = N.$$

Derivatives and rates of change

- Slope of Secant Line:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$
$$or : \frac{f(a+h) - f(a)}{h}.$$

- Slope of Tangent Line:

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
$$or : \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

- Average Velocity:

$$v_{ave} = \frac{f(x) - f(x)}{x - a}.$$

- Instantaneous Velocity:

$$v_{inst} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
$$or : \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

- Speed:

$$Speed\ = |Velocity|.$$

- Derivatives Definition:

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
$$Or : f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

- Know that:

$$s(t) = postition\ function$$
$$v(t) = s'(t) = velocity\ function$$
$$a(t) = v'(t) = acceleration\ function.$$

Note:-

If f(x) is differentiable at a, then it is continous at a, the converse is not true

Derivatives of common functions

Exponential Functions:

- $\frac{d}{dx}e^x = e^x \cdot \frac{d}{dx}x$
- $\frac{d}{dx}a^x = a^x \cdot \ln a \cdot \frac{d}{dx}x$
- $\frac{d}{dx}\ln x = \frac{1}{x} \cdot \frac{d}{dx}x$
- $\frac{d}{dx}\log_a x = \frac{1}{x \cdot \ln a} \cdot \frac{d}{dx}x$

Trig Functions:

- $\frac{d}{dx}\sin x = \cos x$
- $\frac{d}{dx}\cos x = -\sin x$
- $\frac{d}{dx}\tan x = \sec^2 x$
- $\frac{d}{dx}\csc x = -\csc x \cot x$
- $\frac{d}{dx}\sec x = \sec x \tan x$
- $\frac{d}{dx}\cot x = -\csc^2 x$

Inverse Trig:

- $\frac{d}{dx}$arcsin(x) $= \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}$arccos(x) = -$\frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}$arctan(x) $= \frac{1}{x^2+1}$
- $\frac{d}{dx}$arccsc(x) $= -\frac{1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}$arcsec(x) $= -\frac{1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}$arccot(x) $= -\frac{1}{x^2+1}$

Hyperbolic Trig

- $\frac{d}{dx}\sinh x = \cosh x$
- $\frac{d}{dx}\cosh x = \sinh x$
- $\frac{d}{dx}\tan x = sech^2 x$
- $\frac{d}{dx} cschx = -cschxcothx$
- $\frac{d}{dx} sechx = -sechxtanhx$
- $\frac{d}{dx} cothx = -csc^2 x$

Normal Line

$$m_{tan} \cdot m_{normal} = -1.$$

Product and quotient rule

Product rule:

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)].$$

Quotient Rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

Chain rule

- If:

$$F(x) = f(g(x)).$$

- Then:

$$F'(x) = f'(g(x)) \cdot g'(x).$$

Exponential growth and decay

$$y = Ce^{kt}.$$

Newton's law of cooling

$$\begin{aligned} T(t) &= t_s + Ce^{kt} \\ C &= t_0 - t_s. \end{aligned}$$

Linear Approximation

$$f(x) \approx L(x) = f(a) - f'(a)(x - a).$$

Differentials

$$dy = f'(x)dx$$
$$\Delta x = dx$$
$$\Delta y = (f(x + \Delta x) - f(x)).$$

Note:-

Δy, can sometimes be difficult to find so we can use $dy \approx \Delta y$

Extreme value theorem

- If f is continuous on a closed interval [a,b], then f attains an obsulute maximum value $f(c)$ and an absolute minimum value $f(d)$ where $c, d \in [a, b]$

Fermats theorem

- If f has a local minimum or maximum at c, and if $f'(c)$ exists, then $f'(c) = 0$

Critical number theorem

- c in the domain of $f(x)$ is a critical number if $f'(c) = 0$ or if $f'(c)$ does not exist.
  Note: If f has a local max or min at c, then c is a critical number of f
- Critical number has to obey restriction

Rolles Theorem

If f(x) satisfies the following:

1. continuous on [a,b]
2. differentiable on (a,b)
3. f(a) = f(b)

Then there is a $c \in (a, b)$ such that $f'(c) = 0$

Notes:

- If rolle's theorem can be applied, just set $f'(x) = 0$ to find c, remember you are finding all c in the ***open interval***, so if c does not obey this interval, it is not a solution

The mean value theorem

if f(x) satisfies the following:

1. continuous on [a,b]
2. differentiable on (a,b)

then there is a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

In other words,

$$m_{tan} = m_{sec}.$$

Notes:

- If rational function, find where function is undefined, if that number is not an element of the interval, then it is continous on the closed interval
- If $f'(x)$ is defined on the open interval, then it is differentiable on the open interval
- use the theorem, then set $f'(c) = c$

L'Hospital's Rule

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}.$$

Newton's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Common Antiderivatives

- Exponential
  - $x^n = \frac{x^{n+1}}{n+1} + C$
  - $\frac{1}{x} = \ln|x| + C$
  - $a^x = \frac{a^x}{\ln a} + C$
  - $\ln x = x\ln x - x + C$
  - $e^x = e^x + C$

- Trig:
  - $\sin x = -\cos x + C$
  - $\cos x = \sin x + C$
  - $\tan x = \ln|\sec x| + C$
  - $\csc x = \ln|\csc x - \cot x| + C$
  - $\sec x = \ln|\sec x + \tan x| + C$
  - $\cot x = \ln|\sin x| + C$
  - $\sin^2 x = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$
  - $\cos^2 x = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$
  - $\tan^2 x = -x + \tan x + C$
  - $\csc^2 x = -\cot x + C$
  - $\sec^2 x = \tan x + C$
  - $\cot^2 x = -x - \cot x + C$

- Hyperbolic Trig
  - $\sinh x = \cosh x + C$
  - $\cosh x = \sinh x + C$
  - $\tanh x = \ln|\cos x| + C$
  - $csch x = \ln|\tan h(\frac{1}{2}x)| + C$
  - $sech x = \tan^{-1}(\sinh(x)) + C$
  - $coth x = \ln|\sinh x| + C$
  - $csch^2 x = -\coth x + C$
  - $sech^2 x = \tanh x + C$

Riemann sum

$$\sum_{i=1}^{n} \Delta x(f(x_i)) \ .$$

Definition of definite integrals

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(x_i^*)$$
$$\Delta x = \frac{b-a}{n}$$
$$x_i = a + i\Delta x.$$

The fundemental theorem of calculus

$$Part\ 1: \ \frac{d}{dx}\int_a^x f(x)\ dt = f(x), a \leqslant x \leqslant b$$
$$Part\ 2: \int_a^b f(x)\ dx = F(b) - F(a)\ where\ F' = f.$$
$$and$$
$$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx.$$

The Substitution Rule (u-sub)

If $u = g(x)$ is differentiable and its range $\in I$ and f is continuous on I, then

$$\int f(g(x))g'(x)\ dx = \int f(u)\ du.$$

Laws of Limits, Derivatives, Summations

Limits:

- $\lim\limits_{x\to a} f(x)+g(x) = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$
- $\lim\limits_{x\to a} f(x)-g(x) = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x)$
- $\lim\limits_{x\to a} cf(x) = c\cdot \lim\limits_{x\to a} f(x)$
- $\lim\limits_{x\to a} f(x)\cdot g(x) = \lim\limits_{x\to a} f(x) \cdot \lim\limits_{x\to a} g(x)$
- $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, if g(x) $\neq 0$
- $\lim\limits_{x\to a} \left(f(x)\right)^n = \left[\lim\limits_{x\to a} f(x)\right]^n$, where n is a positive integer
- $\lim\limits_{x\to a} c = c$
- $\lim\limits_{x\to a} x = a$
- $\lim\limits_{x\to a} x^n = a^n$, where n is a positive integer
- $\lim\limits_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$, where n is a positive integer
- $\lim\limits_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)}$, where n is a positive integer

Derivatives:

- $\frac{d}{dx}c = 0$
- $\frac{d}{dx}x = 1$
- $\frac{d}{dx}(x^n) = n\cdot x^{n-1} \rightarrow$ ***Power Rule***
- $\frac{d}{dx}[c\cdot f(x)] = c\cdot \frac{d}{dx}[f(x)]$
- $\frac{d}{dx}[f(x)\pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

Summation:

- $\sum\limits_{i=m}^{n} c\cdot a_i = c\sum\limits_{i=m}^{n} a_i$, where c is a constant
- $\sum\limits_{i=m}^{n} (a_i + b_i) = \sum\limits_{i=m}^{n} a_i + \sum\limits_{i=m}^{n} b_i$
- $\sum\limits_{i=m}^{n} (a_i - b_i) = \sum\limits_{i=m}^{n} a_i - \sum\limits_{i=m}^{n} b_i$
- $\sum\limits_{i=1}^{n} 1 = n$
- $\sum\limits_{i=1}^{n} c = c\cdot n$, where c is a constant
- $\sum\limits_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\sum\limits_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum\limits_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

Properties of Integrals:

- $\int_a^b c dx = c(b-a)$
- $\int_a^b cf(x)dx = c \cdot \int_a^b f(x)dx$
- $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
- $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$
- $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$
- if $f(x) \geqslant 0$ for all $a \leqslant x \leqslant b$, then $\int_a^b f(x)dx \geqslant 0$
- if $f(x) \geqslant g(x)$ for all $a \leqslant x \leqslant b$, then $\int_a^b f(x)dx \geqslant \int_a^b g(x)dx$
- if $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then $m(b-a) \leqslant \int_a^b f(x)dx \leqslant M(b-a)$