Nate Warner MATH 230 September 06, 2023

Homework/Worksheet 2 - Due: Wednesday, September 13

1. Evaluate the Integrals

1.a $\int \frac{x^3}{\sqrt{1-x^2}} dx$

$$\begin{aligned} \text{Let } u &= 1-x^2 \\ du &= -2x\ dx \\ -\frac{1}{2}du &= x\ dx \\ x^2 &= -(u-1). \end{aligned}$$

$$\begin{aligned}
&\int \frac{x^3}{\sqrt{1-x^2}}\ dx \\
&= \int \frac{x(x^2)}{\sqrt{1-x^2}}\ dx \\
&= -\frac{1}{2}\int \frac{-(u-1)}{u^{\frac{1}{2}}}\ du \\
&= \frac{1}{2}\int \frac{u-1}{u^{\frac{1}{2}}}\ du \\
&= \frac{1}{2}\int (u-1)u^{-\frac{1}{2}}\ du \\
&= \frac{1}{2}\int u^{\frac{1}{2}} - u^{-\frac{1}{2}}\ du \\
&= \frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}} - 2^{\frac{1}{2}}\right] + C \\
&= \frac{1}{3}u^{\frac{3}{2}} + u^{\frac{1}{2}} + C \\
&= \frac{1}{3}(1-x^2)^{\frac{3}{2}} - (1-x^2)^{\frac{1}{2}} + C \\
&= \frac{(1-x^2)^{\frac{3}{2}}}{3} - (1-x^2)^{\frac{1}{2}} + C \\
&= \frac{(1-x^2)^{\frac{3}{2}}}{3} - \frac{3(1-x^2)^{\frac{1}{2}}}{3} + C \\
&= \frac{(1-x^2)^{\frac{3}{2}} - 3(1-x^2)^{\frac{1}{2}}}{3} + C \\
&= \frac{(1-x^2)^{\frac{1}{2}}\left[(1-x^2)-3\right]}{3} + C \\
&= \frac{(1-x^2)^{\frac{1}{2}}(-2-x^2)}{3} + C \\
&= -\frac{1}{3}\sqrt{1-x^2}(2+x^2) + C.
\end{aligned}$$

1.b $\int \cos x(1-\cos x)^{99} \sin x \; dx$

$$
\begin{aligned}
\text{Let } u &= 1-\cos x \\
du &= \sin x \; dx \\
\cos x &= -(u-1).
\end{aligned}
$$

$$
\begin{aligned}
&= \int \cos x(1-\cos x)^{99} \sin x \; dx \\
&= \int -(u-1)u^{99} \; du \\
&= -\int (u-1)u^{99} \; du \\
&= -\int u^{100} - u^{99} \; du \\
&= -\left[\frac{1}{101}u^{101} - \frac{1}{100}u^{100}\right] + C \\
&= -\frac{1}{101}(1-\cos x)^{101} + \frac{1}{100}(1-\cos x)^{100} + C.
\end{aligned}
$$

1.c $\int \frac{x^5}{(1-x^3)^{\frac{3}{2}}}\ dx$

$$\begin{aligned}
\text{Let } u &= 1 - x^3 \\
du &= -3^{x2}\ dx \\
-\frac{1}{3}du &= x^2\ dx \\
x^3 &= -(u-1).
\end{aligned}$$

$$\begin{aligned}
&\int \frac{x^5}{(1-x^3)^{\frac{3}{2}}}\ dx \\
&= \int \frac{x^2(x^3)}{(1-x^3)^{\frac{3}{2}}}\ dx \\
&= -\frac{1}{3}\int \frac{-(u-1)}{u^{3/2}}\ du \\
&= \frac{1}{3}\int (u-1)u^{-\frac{3}{2}}\ du \\
&= \frac{1}{3}\int u^{-\frac{1}{2}} - u^{-\frac{3}{2}}\ du \\
&= \frac{1}{3}\left[2u^{\frac{1}{2}} + 2u^{-\frac{1}{2}}\right] + C \\
&= \frac{2}{3}u^{\frac{1}{2}} + \frac{2}{3}u^{-\frac{1}{2}} + C \\
&= \frac{2}{3}(1-x^3)^{\frac{1}{2}} + \frac{2}{3}(1-x^3)^{-\frac{1}{2}} + C \\
&= \frac{2}{3}(1-x^3)^{\frac{1}{2}} + \frac{2}{3(1-x^3)^{\frac{1}{2}}} + C \\
&= \frac{2(1-x^3)^{\frac{1}{2}}}{3} + \frac{2}{3(1-x^3)^{\frac{1}{2}}} + C \\
&= \frac{2(1-x^3)^{\frac{1}{2}}(1-x^3)^{\frac{1}{2}} + 2}{3(1-x^3)^{\frac{1}{2}}} + C \\
&= \frac{2(1-x^3) + 2}{3(1-x^3)^{\frac{1}{2}}} + C \\
&= \frac{4 - 2x^3}{3(1-x^3)^{\frac{1}{2}}} + C \\
&= -\frac{2(x^3-2)}{3(1-x^3)^{\frac{1}{2}}} + C.
\end{aligned}$$

1.d $\int_0^{\frac{\pi}{4}} \sec^2 x \tan x \; dx$

$$\begin{aligned} &\text{Let } u = \sec x \\ &du = \sec x \tan x \\ &u(a) = \sec 0 = 1 \\ &u(b) = \sec \frac{\pi}{4} = \sqrt{2}. \end{aligned}$$

$$\begin{aligned} &\int_0^{\frac{\pi}{4}} \sec^2 x \tan x \; dx \\ &= \int_0^{\frac{\pi}{4}} \sec x (\sec x \tan x) \; dx \\ &= \int_1^{\sqrt{2}} u \; du \\ &= \frac{1}{2} u^2 \Big]_1^{\sqrt{2}} \\ &= \frac{1}{2}\left((\sqrt{2})^2 - (1)^2\right) \\ &= \frac{1}{2}\left(2 - 1\right) \\ &= \frac{1}{2}. \end{aligned}$$

2. Evaluate the Integrals

2.a $\int e^{\sin x} \cos x \; dx$

$$\begin{aligned} &\text{Let } u = \sin x \\ &du = \cos x. \end{aligned}$$

$$\begin{aligned} &\int e^{\sin x} \cos x \; dx \\ &= \int e^u \; du \\ &= e^{\sin x} + C. \end{aligned}$$

2.b $\int \frac{1}{x(\ln x)} \; dx$

$$\begin{aligned} &\text{Let } u = \ln x \\ &du = \frac{1}{x} \; dx. \end{aligned}$$

$$\begin{aligned} &\int \frac{1}{x(\ln x)} \; dx \\ &= \int u^{-1} \; du \\ &= \ln |u| + C \\ &= \ln (\ln |x|) + C. \end{aligned}$$

2.c $\int xe^{-x^2}\ dx$

$$\begin{aligned}&\text{Let } u = -x^2\\&du = -2x\\&-\frac{1}{2}\ du = x\ dx.\end{aligned}$$

$$\begin{aligned}&\int xe^{-x^2}\ dx\\&= -\frac{1}{2}\int e^u\ du\\&= -\frac{1}{2}e^{-x^2} + C.\end{aligned}$$

2.d $\int \ln(\cos(x))\tan x\ dx$

$$\begin{aligned}&\text{Let } u = \ln(\cos(x))\\&du = \frac{1}{\cos x} - \sin x\ dx\\&du = -\tan x\ dx\\&-du = \tan x\ dx.\end{aligned}$$

$$\begin{aligned}&\int \ln(\cos(x))\tan x\ dx\\&= -\int u\ du\\&= -\frac{1}{2}u^2 + C\\&= -\frac{1}{2}(\ln(\cos(x))^2) + C\\&= -\frac{1}{2}\ln^2(\cos(x)) + C.\end{aligned}$$

3. Evaluate the Integrals

3.a $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\ dx$

$$\begin{aligned}&\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\ dx\\&= \sin^{-1} x\Big]_{-\frac{1}{2}}^{\frac{1}{2}}\\&= \left(\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right)\\&= \frac{2\pi}{6}\\&= \frac{\pi}{3}.\end{aligned}$$

3.b $\int \frac{1}{9+x^2}\ dx$

Remark. $\int \frac{1}{x^2+a^2}\ dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

$$\begin{aligned}&\int \frac{1}{9+x^2}\ dx\\ &= \frac{1}{3}\tan^{-1}\frac{x}{3} + C.\end{aligned}$$

3.c $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\ dx$

$$\begin{aligned}&\text{Let } u = \sin^{-1} x\\ &du = \frac{1}{\sqrt{1-x^2}}\ dx.\end{aligned}$$

$$\begin{aligned}&\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\ dx\\ &= \int u\ du\\ &= \frac{1}{2}u^2 + C\\ &= \frac{1}{2}(\sin^{-1}(x))^2 + C.\end{aligned}$$

3.d $\int \frac{x\tan^{-1}(x^2)}{1+x^4}\ dx$

$$\begin{aligned}&\text{Let } u = \tan^{-1} x^2\\ &du = \frac{1}{1+(x^2)^2}\cdot 2x\ dx\\ &du = \frac{2x}{1+x^4}\ dx\\ &\frac{1}{2}du = \frac{x}{1+x^4}\ dx.\end{aligned}$$

$$\begin{aligned}&\int \frac{x\tan^{-1}(x^2)}{1+x^4}\ dx\\ &= \frac{1}{2}\int u\ du\\ &= \frac{1}{2}\left[\frac{1}{2}u^2\right] + C\\ &= \frac{1}{4}\left(\tan^{-1}(x^2)\right)^2 + C.\end{aligned}$$