Nate Warner MATH 434 Fall 2025

Problem set 1 - Due: Sunday, September 28

1.2.4. Prove that if A^{-1} exists, then there can be no nonzero y for which $Ay = 0$

Proof. Assume that $A \in \mathbb{R}^{n \times n}$, and A^{-1} exists. Assume for the sake of contradiction that there exists $y \in \mathbb{R}^n$, $y \neq 0$ such that $Ay = 0$. So,

$$\begin{aligned} Ay &= 0, \\ \implies A^{-1}Ay &= A^{-1}0 \\ \implies Iy &= 0 \\ \implies y &= 0 \end{aligned}$$

But, $y \neq 0$, a contradiction. Therefore, if A^{-1} exists, then there can be no nonzero y for which $Ay = 0$. ■

1.2.5. Prove that if A^{-1} exists, then $\det(A) \neq 0$.

Proof. Assume that $A \in \mathbb{R}^{n \times n}$, and A^{-1} exists.

Suppose for the sake of contradiction that $\det(A) = 0$. We know that $AA^{-1} = I$, and that $\det(AB) = \det(A)\det(B)$. So,

$$\begin{aligned} AA^{-1} &= I \\ \implies \det(AA^{-1}) &= \det(I) \\ \implies \det(A)\det(A^{-1}) &= 1 \\ \implies 0\det(A^{-1}) &= 1 \\ \implies 0 &= 1 \end{aligned}$$

A contradiction. Therefore, if A^{-1} exists, then $\det(A) \neq 0$ ■

1.2.11. Check that the equations in Example 1.2.10 are correct. Check that the coefficient matrix of the system is nonsingular

The stiffness matrix A is given as

$$A = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}.$$

The system $Ax = b$ is given as

$$\begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

So, the system of linear equations is

$$\begin{aligned} 8x_1 - 4x_2 &= 1 \\ -4x_1 + 8x_2 - 4x_3 &= 2 \\ -4x_2 + 8x_3 &= 3 \end{aligned}$$

Since the second equation is derived, we only need to check the first and third equations.

For cart one, the spring exerts a leftward force of $-4\frac{N}{m}$, and is stretched by the amount that cart one moves, which is given by $x_1\ m$. So, the leftward force is $-4x_1\ N$. On the right, there is a force given by the spring of $4\frac{N}{m}$, and is stretched $(x_2 - x_1)\ m$ meters. So, the rightward force is $4(x_2 - x_1)\ N$. Finally, an external force of $1N$ is applied to the cart, so the equilibrium equation for cart one is

$$\begin{aligned} -4x_1 + 4(x_2 - x_1) + 1 &= 0, \\ \implies -4x_1 + 4x_2 - 4x_1 + 1 &= 0, \\ \implies -8x_1 + 4x_2 + 1 &= 0, \\ \implies 8x_1 - 4x_2 &= 1. \end{aligned}$$

Which is precisely the first equation in the given system.

Regarding the third cart, there is a leftward force from the spring $-4\ \frac{N}{m}$, and the spring is stretched $(x_3 - x_2)\ m$, so the leftward force is $-4(x_3 - x_2)\ N$. On the right, there is a force from the spring $4\ \frac{N}{m}$, and the spring is compressed $-x_3\ m$, so the rightward force is $-4x_3\ N$. Thus, the equilibrium equation for the third cart is

$$\begin{aligned} -4(x_3 - x_2) - 4x_3 + 3 &= 0, \\ \implies -4x_3 + 4x_2 - 4x_3 + 3 &= 0, \\ \implies -8x_3 + 4x_2 + 3 &= 0, \\ \implies 8x_3 - 4x_2 &= 3. \end{aligned}$$

Which is precisely the third equation in the given system.

Therefore, the equations are verified.

We can check that the coefficient matrix of the system is nonsingular by computing the determinant and verifying that it is nonzero.

$$\begin{aligned} \det\left(\begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}\right) &= 8(8(8) - (-4)(-4)) + 4(-4(8) - (-4)(0)) + 0(-4(-4) - 8(0)) \\ &= 256 \neq 0. \end{aligned}$$

So, A is nonsingular.

1.3.4. Use pencil and paper to solve the system

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 3 & 1 & -1 & 0 \\ 4 & 1 & -3 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 9 \end{bmatrix}$$

by forward substitution

We solve the system by forward substitution

$$\begin{aligned} 2y_1 = 2 &\implies y_1 = 1, \\ -y_1 + 2y_2 = 3 &\implies y_2 = \frac{3 + y_1}{2} = \frac{3+1}{2} = 2, \\ 3y_1 + y_2 - y_3 = 2 &\implies y_3 = \frac{2 - 3y_1 - y_2}{-1} = \frac{2 - 3(1) - 2}{-1} = 3, \\ 4y_1 + y_2 - 3y_3 + 3y_4 = 9 &\implies y_4 = \frac{9 - 4y_1 - y_2 + 3y_3}{3} = \frac{9 - 4(1) - 2 + 3(3)}{3} = 4. \end{aligned}$$

Thus, $y = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$. We verify multiplying Ay. We have,

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 3 & 1 & -1 & 0 \\ 4 & 1 & -3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2(1) \\ -1(1) + 2(2) \\ 3(1) + 1(2) - 1(3) \\ 4(1) + 1(2) - 3(3) + 3(4) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 9 \end{bmatrix}.$$

1.3.11. Use column-oriented forward substitution to solve the system from Exercise 1.3.4.

Remark. *(Column oriented forward substitution).* Suppose we have $Lx = b$ when L is lower triangular, we split the matrix into the following blocks

$$\begin{bmatrix} \ell_{11} & 0 \\ \hat{\ell} & \hat{L} \end{bmatrix} \begin{bmatrix} x_1 \\ \hat{x} \end{bmatrix} = \begin{bmatrix} b_1 \\ \hat{b} \end{bmatrix}.$$

With $\hat{\ell} \in \mathbb{R}^{n-1}$, $\hat{L} \in \mathbb{R}^{n-1 \times n-1}$, $\hat{x} \in \mathbb{R}^{n-1}$, $\ell_{11}, x_1, b_1 \in \mathbb{R}$. Note that $\hat{L}$ is also lower triangular.

1. Compute $x_1 = \frac{b_1}{\ell_{11}}$
2. Compute $\hat{b} - \hat{\ell} x_1 = \tilde{b} \in \mathbb{R}^{n-1}$
3. Find $\hat{L} x = \tilde{b}$
4. Run the algorithm on $\hat{L}$, $\tilde{b}$. That is, $\text{Alg}(\hat{L}, \tilde{b})$

The recursive column oriented forward substitution algorithm requires $\mathcal{O}(n^2)$ flops.

$\text{Alg}(L, b)$: The system is

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 3 & 1 & -1 & 0 \\ 4 & 1 & -3 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 9 \end{bmatrix}$$

We have $\ell_{11} = 2$, $b_1 = 2$, and

$$\hat{L} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -3 & 3 \end{bmatrix} \in \mathbb{R}^{3 \times 3},$$

$$\hat{\ell} = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} \in \mathbb{R}^3,$$

$$\hat{y} = \begin{bmatrix} y_2 \\ y_3 \\ y_4 \end{bmatrix}, \in \mathbb{R}^3$$

$$\hat{b} = \begin{bmatrix} 3 \\ 2 \\ 9 \end{bmatrix} \in \mathbb{R}^3.$$

By step one, we have

$$y_1 = \frac{2}{2} = 1.$$

By step two, we have

$$\tilde{b} = \begin{bmatrix} 3 \\ 2 \\ 9 \end{bmatrix} - \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} \cdot 1 = \begin{bmatrix} 3+1 \\ 2-3(1) \\ 9-4(1) \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}.$$

$\text{Alg}(\hat{L}, \tilde{b})$: The system is

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -3 & 3 \end{bmatrix} \begin{bmatrix} y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}$$

We have $\ell_{11} = 2$, $b_1 = 3$, and

$$\hat{L} = \begin{bmatrix} -1 & 0 \\ -3 & 3 \end{bmatrix} \in \mathbb{R}^{2\times 2},$$
$$\hat{\ell} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathbb{R}^2,$$
$$\hat{y} = \begin{bmatrix} y_3 \\ y_4 \end{bmatrix} \in \mathbb{R}^2,$$
$$\hat{b} = \begin{bmatrix} 2 \\ 9 \end{bmatrix} \in \mathbb{R}^2.$$

By step one,

$$y_2 = \frac{4}{2} = 2$$

By step two we have

$$\tilde{b} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot 2 = \begin{bmatrix} -1-2 \\ 5-2 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

$\mathrm{Alg}(\hat{L}, \tilde{b})$: The system is

$$\begin{bmatrix} -1 & 0 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

We have $\ell_{11} = -1$, $b_1 = -3$, and

$$\hat{L} = \begin{bmatrix} 3 \end{bmatrix} \in \mathbb{R}^1,$$
$$\hat{\ell} = \begin{bmatrix} -3 \end{bmatrix} \in \mathbb{R}^1,$$
$$\hat{y} = \begin{bmatrix} y_4 \end{bmatrix} \in \mathbb{R}^1,$$
$$\hat{b} = \begin{bmatrix} 3 \end{bmatrix}.$$

By step one, we have that

$$y_3 = \frac{-3}{-1} = 3.$$

By step two, we have that

$$\tilde{b} = \begin{bmatrix} 3 \end{bmatrix} - \begin{bmatrix} -3 \end{bmatrix} \cdot 3 = \begin{bmatrix} 3 - -3(3) \end{bmatrix} = \begin{bmatrix} 12 \end{bmatrix}$$

Alg $(\hat{L}, \tilde{b})$: The system is

$$\begin{bmatrix} 3 \end{bmatrix} \begin{bmatrix} y_4 \end{bmatrix} = \begin{bmatrix} 12 \end{bmatrix}$$

From this final call we have that

$$y_4 = \frac{12}{3} = 4$$

So, $y = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, the same result as 1.3.4.

1.3.15. Develop the row-oriented version of back substitution. Write pseudocode in the spirit of (1.3.5) and (1.3.13).

Let $A \in \mathbb{R}^{n\times n},\ x \in \mathbb{R}^n,\ b \in \mathbb{R}^n$, with

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Then,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

implies

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{nn}x_n &= b_n. \end{aligned}$$

So,

$$\begin{aligned} x_1 &= \frac{b_1 - (a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n)}{a_{11}}, \\ x_2 &= \frac{b_2 - (a_{23}x_3 + a_{24}x_4 + \cdots + a_{2n}x_n)}{a_{22}}, \\ x_n &= \frac{b_n}{a_{nn}}. \end{aligned}$$

In general, we have that

$$x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{ij}x_j}{a_{ii}}, \quad i = n, n-1, ..., 1$$

The pseudocode for the above algorithm is

```
for i = n, ..., 1
    for j = i + 1, ..., n
        b[i] = b[i] − A[i, j] · b[j]
    end
    if A[i, i] = 0, set error flag, exit
    b[i] = b[i]/A[i, i]
end
```

> 1.3.16. Develop the column-oriented version of back substitution Write pseudocode in the spirit of (1.3.5) and (1.3.13).

Let $U \in \mathbb{R}^{n\times n}$ be upper triangular, $x \in \mathbb{R}^n$, and $b \in \mathbb{R}^n$ which gives the system

$$\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Split the system into the following block decomposition

$$\begin{bmatrix} \hat{U} & u \\ 0^\top & u_{nn} \end{bmatrix} \begin{bmatrix} \hat{x} \\ x_n \end{bmatrix} = \begin{bmatrix} \hat{b} \\ b_n \end{bmatrix}$$

Then,

$$\hat{U}\hat{x} + ux_n = \hat{b} \implies \hat{U}\hat{x} = \hat{b} - ux_n = \tilde{b},$$
$$u_{nn}x_n = b_n \implies x_n = \frac{b_n}{u_{nn}}$$

Thus, the column-oriented backward substitution algorithm is defined by the following steps

1. Compute $x_n = \frac{b_n}{u_{nn}}$
2. Compute $\tilde{b} = \hat{b} - ux_n$
3. Run the algorithm on $\hat{U}, \tilde{b}$. That is, $\mathrm{Alg}(\hat{U}, \tilde{b})$

The non-recursive pseudocode in the spirit of 1.3.5 and 1.3.13 is

```
for i = n, ..., 1
    if U[i,i] = 0, set error flag, exit

    b[i] = b[i]/U[i,i]

    for j = i - 1, ..., 1
        b[j] = b[j] - U[j,i] · b[i]
    end
end
```

1.3.17. Solve the upper-triangular system

$$\begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -10 \\ 10 \\ 1 \\ 12 \end{bmatrix}$$

(a) by row-oriented back substitution, (b) by column-oriented back substitution

By row-oriented back substitution, we have

$$\begin{aligned} 4x_4 = 12 &\implies x_4 = 3, \\ -2x_3 + x_4 = 1 &\implies x_3 = \frac{1-3}{-2} = 1, \\ x_2 + 2x_3 + 3x_4 = 10 &\implies x_2 = 10 - 3(3) - 2(1) = -1, \\ 3x_1 + 2x_2 + x_3 = -10 &\implies x_1 = \frac{-10 - 1 - 2(-1)}{3} = -3. \end{aligned}$$

So, $x = \begin{pmatrix} -3 \\ -1 \\ 1 \\ 3 \end{pmatrix}$. We can verify the result with matrix multiplication Ux, we have

$$\begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3(-3) + 2(-1) + 1(1) \\ 1(-1) + 2(1) + 3(3) \\ -2(1) + 1(3) \\ 4(3) \end{bmatrix} = \begin{bmatrix} -10 \\ 10 \\ 1 \\ 12 \end{bmatrix}.$$

Which matches the given b. With column-oriented back substitution, we have

Alg(U, b): We have the system

$$\begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -10 \\ 10 \\ 1 \\ 12 \end{bmatrix}$$

Notice that $\hat{U} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$, $\hat{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\hat{b} = \begin{bmatrix} -10 \\ 10 \\ 1 \end{bmatrix}$, and $u = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$

By step one, we have that $x_4 = \frac{12}{4} = 3$.

By step two, we have that

$$\tilde{b} = \begin{bmatrix} -10 \\ 10 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \cdot 3 = \begin{bmatrix} -10 - 0 \\ 10 - 3(3) \\ 1 - 3 \end{bmatrix} = \begin{bmatrix} -10 \\ 1 \\ -2 \end{bmatrix}$$

Alg$(\hat{U}, \tilde{b})$: We have the system

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -10 \\ 1 \\ -2 \end{bmatrix}$$

Notice that $\hat{U} = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$, $\hat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\hat{b} = \begin{bmatrix} -10 \\ 1 \end{bmatrix}$, and $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

By step one, we have that $x_3 = \frac{-2}{-2} = 1$

By step two, we have that

$$\tilde{b} = \begin{bmatrix} -10 \\ 10 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot 1 = \begin{bmatrix} -10-1 \\ 1-2 \end{bmatrix} = \begin{bmatrix} -11 \\ -1 \end{bmatrix}$$

$\text{Alg}(\hat{U}, \tilde{b})$: We have the system

$$\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -11 \\ -1 \end{bmatrix}$$

Notice that $\hat{U} = \begin{bmatrix} 3 \end{bmatrix}$, $\hat{x} = \begin{bmatrix} x_1 \end{bmatrix}$, $\hat{b} = \begin{bmatrix} -11 \end{bmatrix}$, $u = \begin{bmatrix} 2 \end{bmatrix}$.

By step one, we have that $x_2 = \frac{-1}{1} = -1$

By step two, we have that

$$\tilde{b} = \begin{bmatrix} -11 \end{bmatrix} - \begin{bmatrix} 2 \end{bmatrix} \cdot (-1) = \begin{bmatrix} -11+2 \end{bmatrix} = \begin{bmatrix} -9 \end{bmatrix}$$

$\text{Alg}(\hat{U}, \tilde{b})$: In the final call, we have the system

$$\begin{bmatrix} 3 \end{bmatrix} \begin{bmatrix} x_1 \end{bmatrix} = \begin{bmatrix} -9 \end{bmatrix}$$

So, we have that $x_1 = -\frac{9}{3} = -3$. Thus,

$$x = \begin{pmatrix} -3 \\ -1 \\ 1 \\ 3 \end{pmatrix}$$

which is precisely the vector x found in the row-oriented backward substitution.

1.4.21. Let

$$A = \begin{bmatrix} 16 & 4 & 8 & 4 \\ 4 & 10 & 8 & 4 \\ 8 & 8 & 12 & 10 \\ 4 & 4 & 10 & 12 \end{bmatrix}, \qquad b = \begin{bmatrix} 32 \\ 26 \\ 38 \\ 30 \end{bmatrix}.$$

Notice that A is symmetric, (a) Use the inner-product formulation of Cholesky's method to show that A is positive definite and compute its Cholesky factor, (b) Use forward and back substitution to solve the linear system $Ax = b$.

Remark. *(Inner product formulas to compute R (Cholesky factor))*: We have the formulas

$$r_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} r_{ki}^2} \quad i = 1, 2, ..., n$$

$$r_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} r_{ki} r_{kj}}{r_{ii}} \quad j = i+1, ..., n$$

The inner product formulas to compute R requires $\mathcal{O}(n^3)$ flops.

For the first row, we have

$$r_{11} = \sqrt{a_{11} - \sum_{k=1}^{0} r_{ki}^2} = \sqrt{a_{11}} = \sqrt{16} = 4,$$

$$r_{12} = \frac{a_{12} - \sum_{k=1}^{0} r_{ki} r_{kj}}{r_{11}} = \frac{4}{4} = 1,$$

$$r_{13} = \frac{a_{13}}{r_{11}} = \frac{8}{4} = 2,$$

$$r_{14} = \frac{a_{14}}{r_{11}} = \frac{4}{4} = 1.$$

For the second row, we have

$$r_{22} = \sqrt{a_{22} - \sum_{k=1}^{1} r_{k2}^2} = \sqrt{10 - 1^2} = \sqrt{9} = 3,$$

$$r_{23} = \frac{a_{23} - \sum_{k=1}^{1} r_{k2} r_{k3}}{r_{22}} = \frac{8 - 1(2)}{3} = 2,$$

$$r_{24} = \frac{a_{24} - \sum_{k=1}^{1} r_{k2} r_{k4}}{r_{22}} = \frac{4 - 1(1)}{3} = 1.$$

For the third row, we have

$$r_{33} = \sqrt{a_{33} - \sum_{k=1}^{2} r_{k3}^2} = \sqrt{12 - (2^2 + 2^2)} = \sqrt{4} = 2,$$

$$r_{34} = \frac{a_{34} - \sum_{k=1}^{2} r_{k3} r_{k4}}{r_{33}} = \frac{10 - (2(1) + 2(1))}{2} = 3.$$

For the last row, we have

$$r_{44} = \sqrt{a_{44} - \sum_{k=1}^{3} r_{k4}^2} = \sqrt{12 - (1^2 + 1^2 + 3^2)} = \sqrt{1} = 1.$$

So, the Cholesky factor R is

$$\begin{bmatrix} 4 & 1 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, the system $Ax = b$ becomes $R^\top Rx = b$, which is given by

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & 2 & 2 & 0 \\ 1 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 32 \\ 26 \\ 38 \\ 30 \end{bmatrix}$$

First, we solve the upper triangular system $R^\top y = b$ with forward substitution. We have the system

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & 2 & 2 & 0 \\ 1 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 32 \\ 26 \\ 38 \\ 30 \end{bmatrix}.$$

Using forward substitution, we find

$$\begin{aligned} 4y_1 = 32 &\implies y_1 = 8, \\ y_1 + 3y_2 = 26 &\implies y_2 = \frac{26 - 8}{3} = 6, \\ 2y_1 + 2y_2 + 2y_3 = 38 &\implies y_3 = \frac{38 - 2(6) - 2(8)}{2} = 5, \\ y_1 + y_2 + 3y_3 + y_4 = 30 &\implies y_4 = 30 - 3(5) - 6 - 8 = 1. \end{aligned}$$

So,

$$y = \begin{pmatrix} 8 \\ 6 \\ 5 \\ 1 \end{pmatrix}$$

Now, we solve $Rx = y$. We have

$$\begin{bmatrix} 4 & 1 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 5 \\ 1 \end{bmatrix}$$

Using backward substitution, we find

$$\begin{aligned} x_4 &= 1, \\ 2x_3 + 3x_4 = 5 &\implies x_3 = \frac{5 - 3(1)}{2} = 1, \\ 3x_2 + 2x_3 + x_4 = 6 &\implies x_2 = \frac{6 - 1 - 2(1)}{3} = 1, \\ 4x_1 + x_2 + 2x_3 + x_4 = 8 &\implies x_1 = \frac{8 - 1 - 2(1) - 1}{4} = 1. \end{aligned}$$

So,

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Let's verify this solution by multiplying Ax to see that it equals the given b

$$\begin{bmatrix} 16 & 4 & 8 & 4 \\ 4 & 10 & 8 & 4 \\ 8 & 8 & 12 & 10 \\ 4 & 4 & 10 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 16+4+8+4 \\ 4+10+8+4 \\ 8+8+12+10 \\ 4+4+10+12 \end{bmatrix} = \begin{bmatrix} 32 \\ 26 \\ 38 \\ 30 \end{bmatrix}.$$

The solution is verified.

1.4.31. Use the outer-product form to work part (a) of Exercise 1.4.21.

Remark. *(Recursive column oriented method to find the Cholesky factor R (Outer product method)).* Let $A \in \mathbb{R}^{n\times n}$. Assume that A is positive definite, so $A = A^\top$, and $A = R^\top R$ for a unique upper triangular matrix R. We have,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} r_{11} & 0 & \cdots & 0 \\ r_{12} & r_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ r_{1n} & r_{2n} & \cdots & r_{nn} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & r_{nn} \end{bmatrix}$$

We then perform a matrix decomposition

$$\begin{bmatrix} a_{11} & a^\top \\ a & \hat{A} \end{bmatrix} = \begin{bmatrix} r_{11} & 0^\top \\ r & \hat{R}^\top \end{bmatrix} \begin{bmatrix} r_{11} & r^\top \\ 0 & \hat{R} \end{bmatrix}.$$

Where $\hat{A} = \hat{A}^\top \in \mathbb{R}^{n-1\times n-1}$, $a \in \mathbb{R}^{n-1}$, $\hat{R}^\top \in \mathbb{R}^{n-1\times n-1}$ lower triangular, and $\hat{R} \in \mathbb{R}^{n-1\times n-1}$ upper triangular. Further,

The recursive column oriented algorithm to compute the Cholesky factor R is given by the following steps

1. $r_{11} = \sqrt{a_{11}}$
2. $r = \frac{a}{r_{11}}$
3. $\tilde{A} = \hat{A} - rr^\top$
4. $\mathrm{Alg}(\tilde{A}) = \hat{R}$

The recursive column oriented algorithm to compute the Cholesky factor R requires $\mathcal{O}(n^3)$ flops.

Recall the given system

$$\begin{bmatrix} 16 & 4 & 8 & 4 \\ 4 & 10 & 8 & 4 \\ 8 & 8 & 12 & 10 \\ 4 & 4 & 10 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 32 \\ 26 \\ 38 \\ 30 \end{bmatrix}.$$

$\mathrm{Alg}(A)$: Notice that

$$\hat{A} = \begin{bmatrix} 10 & 8 & 4 \\ 8 & 12 & 10 \\ 4 & 10 & 12 \end{bmatrix},$$
$$a = \begin{bmatrix} 4 \\ 8 \\ 4 \end{bmatrix}.$$

By the first step, $r_{11} = \sqrt{16} = 4$.

By the second step,

$$r = \frac{1}{4}\begin{bmatrix}4\\8\\4\end{bmatrix} = \begin{bmatrix}1\\2\\1\end{bmatrix}$$

By the third step,

$$\begin{aligned}\tilde{A} &= \begin{bmatrix}10 & 8 & 4\\8 & 12 & 10\\4 & 10 & 12\end{bmatrix} - \begin{bmatrix}1\\2\\1\end{bmatrix}\begin{bmatrix}1 & 2 & 1\end{bmatrix}\\ &= \begin{bmatrix}10 & 8 & 4\\8 & 12 & 10\\4 & 10 & 12\end{bmatrix} - \begin{bmatrix}1 & 2 & 1\\2 & 4 & 2\\1 & 2 & 1\end{bmatrix}\\ &= \begin{bmatrix}9 & 6 & 3\\6 & 8 & 8\\3 & 8 & 11\end{bmatrix}\end{aligned}$$

Alg($\hat{A}$): We have

$$A = \begin{bmatrix}9 & 6 & 3\\6 & 8 & 8\\3 & 8 & 11\end{bmatrix}.$$

So,

$$\hat{A} = \begin{bmatrix}8 & 8\\8 & 11\end{bmatrix},$$
$$a = \begin{bmatrix}6\\3\end{bmatrix}.$$

By the first step, $r_{11} = \sqrt{9} = 3$

By the second step,

$$r = \frac{1}{3}\begin{bmatrix}6\\3\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

By the third step,

$$\begin{aligned}\tilde{A} &= \begin{bmatrix}9 & 6 & 3\\6 & 8 & 8\\3 & 8 & 11\end{bmatrix} - \begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}2 & 1\end{bmatrix}\\ &= \begin{bmatrix}8 & 8\\8 & 11\end{bmatrix} - \begin{bmatrix}4 & 2\\2 & 1\end{bmatrix} = \begin{bmatrix}4 & 6\\6 & 10\end{bmatrix}\end{aligned}$$

Alg($\tilde{A}$): We have

$$A = \begin{bmatrix}4 & 6\\6 & 10\end{bmatrix}.$$

So,

$$\hat{A} = \begin{bmatrix}10\end{bmatrix},$$
$$a = \begin{bmatrix}6\end{bmatrix}.$$

By the first step,

$$r_{11} = \sqrt{4} = 2$$

By the second step,

$$r = \frac{1}{2}\begin{bmatrix}6\end{bmatrix} = \begin{bmatrix}3\end{bmatrix}.$$

By the third step,

$$\begin{aligned}\tilde{A} &= \begin{bmatrix}10\end{bmatrix} - \begin{bmatrix}3\end{bmatrix}\begin{bmatrix}3\end{bmatrix}^\top \\ &= \begin{bmatrix}10\end{bmatrix} - \begin{bmatrix}9\end{bmatrix} = \begin{bmatrix}1\end{bmatrix}\end{aligned}$$

Alg($\tilde{A}$): We have

$$A = \begin{bmatrix}1\end{bmatrix}$$

So, $r_{11} = \sqrt{1} = 1$. Thus, the Cholesky factor R is

$$R = \begin{bmatrix}4 & 1 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1\end{bmatrix}$$

Which is precisely the factor that was computer in the row-oriented approach.

1.4.33. Write a nonrecursive algorithm that implements the outer-product formulation of Cholesky's algorithm (1.4.28). Your algorithm should exploit the symmetry of A by referencing only the main diagonal and upper part of A, and it should store R over A. Be sure to put in the necessary check before taking the square root.

```
for i = 1, ..., n
    if a_ii ⩽ 0, set error flag, exit
    a_ii = sqrt(a_ii)
    for j = i+1, ..., n
        a_ij = a_ij / a_ii
    end

    for k = i+1, ..., n
        for ℓ = k, ..., n
            a_kℓ = a_kℓ - a_ik · a_iℓ
        end
    end
end
```

At the end of the above algorithm, R is stored in the upper triangular part of A. We could zero out the lower triangular part of A so that A stores only the matrix R.

```
for i = 1, ..., n
    for j = 1, ..., i-1
        a_ij = 0
    end
end
```

1.4.40. Use the bordered form to work part (a) of Exercise 1.4.21.

Remark. *(Bordered form of Choleskys method)*. Suppose $A \in \mathbb{R}^{n\times n}$ is positive definite. Then, A admits a decomposition $A = R^\top R$, for a unique upper triangular matrix R called the Cholesky factor, with $r_{ii} > 0$ for $i = 1, 2, ..., n$. So,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} r_{11} & 0 & \cdots & 0 \\ r_{12} & r_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ r_{1n} & r_{2n} & \cdots & r_{nn} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & r_{nn} \end{bmatrix}$$

We then perform a matrix decomposition

$$\begin{bmatrix} \hat{A} & a \\ a^\top & a_{nn} \end{bmatrix} = \begin{bmatrix} \hat{R}^\top & 0 \\ r^\top & r_{nn} \end{bmatrix} \begin{bmatrix} \hat{R} & r \\ 0 & r_{nn} \end{bmatrix}.$$

So,

$$\begin{aligned} \hat{A} &= \hat{R}^\top \hat{R}, \\ a &= \hat{R}^\top r, \\ a_{nn} &= r^\top r + r_{nn}^2 \implies r_{nn} = \sqrt{a_{nn} - r^\top r}. \end{aligned}$$

So, the steps for the algorithm are

1. Recurse $\hat{A}$ until $A \in \mathbb{R}^{1\times 1}$
2. Solve the lower triangular system $\hat{R}^\top r = a$ by forward substitution
3. Compute $r_{nn} = \sqrt{a_{nn} - r^\top r}$
4. Return the step two on the previous call

The calls to the algorithm are

1. Alg $\left(\begin{bmatrix} 16 & 4 & 8 & 4 \\ 4 & 10 & 8 & 4 \\ 8 & 8 & 12 & 10 \\ 4 & 4 & 10 & 12 \end{bmatrix} \right)$
2. Alg $\left(\begin{bmatrix} 16 & 4 & 8 \\ 4 & 10 & 8 \\ 8 & 8 & 12 \end{bmatrix} \right)$
3. Alg $\left(\begin{bmatrix} 16 & 4 \\ 4 & 10 \end{bmatrix} \right)$
4. Alg $\left(\begin{bmatrix} 16 \end{bmatrix} \right)$

On the fourth and final call, we have that $A = \begin{bmatrix} 16 \end{bmatrix}$, and so $r_{11} = \sqrt{16} = 4$.

We return to the third call with

$$
\begin{aligned}
R &= \begin{bmatrix} 4 & r_{12} & r_{13} & r_{14} \\ 0 & r_{22} & r_{23} & r_{24} \\ 0 & 0 & r_{33} & r_{34} \\ 0 & 0 & 0 & r_{44} \end{bmatrix}, \\
\hat{R} &= \begin{bmatrix} 4 \end{bmatrix}, \\
a &= \begin{bmatrix} 4 \end{bmatrix}.
\end{aligned}
$$

So, we solve the system $\hat{R}^\top r = a$

$$
\begin{aligned}
&\begin{bmatrix} 4 \end{bmatrix} \begin{bmatrix} r_{12} \end{bmatrix} = \begin{bmatrix} 4 \end{bmatrix} \\
&\implies r_{12} = \begin{bmatrix} 1 \end{bmatrix}.
\end{aligned}
$$

Next, we find r_{22}.

$$
\begin{aligned}
r_{22} &= \sqrt{10 - \begin{bmatrix} 1 \end{bmatrix}^\top \begin{bmatrix} 1 \end{bmatrix}} \\
&= \sqrt{10 - 1} = \sqrt{9} = 3
\end{aligned}
$$

So, we return to the second call with

$$
\begin{aligned}
R &= \begin{bmatrix} 4 & 1 & r_{13} & r_{14} \\ 0 & 3 & r_{23} & r_{24} \\ 0 & 0 & r_{33} & r_{34} \\ 0 & 0 & 0 & r_{44} \end{bmatrix}, \\
\hat{R} &= \begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix}, \\
a &= \begin{bmatrix} 8 \\ 8 \end{bmatrix}.
\end{aligned}
$$

We solve the system $\hat{R}^\top r = a$ with forward substitution.

$$
\begin{bmatrix} 4 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} r_{13} \\ r_{23} \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \end{bmatrix}
$$

So,

$$
\begin{aligned}
4r_{13} = 8 &\implies r_{13} = 2, \\
r_{13} + 3r_{23} = 8 &\implies r_{23} = \frac{8-2}{3} = 2.
\end{aligned}
$$

Thus, $r = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. With this, we find r_{33}.

$$
r_{33} = \sqrt{a_{33} - r^\top r} = \sqrt{12 - 8} = \sqrt{4} = 2.
$$

We return to the first call with

$$
\begin{aligned}
R &= \begin{bmatrix} 4 & 1 & 2 & r_{14} \\ 0 & 3 & 2 & r_{24} \\ 0 & 0 & 2 & r_{34} \\ 0 & 0 & 0 & r_{44} \end{bmatrix}, \\
\hat{R} &= \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}, \\
a &= \begin{bmatrix} 4 \\ 4 \\ 10 \end{bmatrix}.
\end{aligned}
$$

We again use forward substitution to solve $\hat{R}^\top r = a$.

$$\begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & 0 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} r_{14} \\ r_{24} \\ r_{24} \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 10 \end{bmatrix}.$$

We have

$$\begin{aligned} 4r_{14} = 4 &\implies r_{14} = 1, \\ r_{14} + 3r_{24} = 4 &\implies r_{24} = \frac{4-1}{3} = 1, \\ 2r_{14} + 2r_{24} + 2r_{24} = 10 &\implies r_{24} = \frac{10 - 2(1) - 2(1)}{2} = 3. \end{aligned}$$

Lastly, we find r_{44}.

$$r_{44} = \sqrt{a_{44} - r^\top r} = \sqrt{12 - (1^2 + 1^2 + 3^2)} = \sqrt{1} = 1$$

Thus, the Cholesky factor R is

$$R = \begin{bmatrix} 4 & 1 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Which is exactly the factor found previously.

1.4.54. Prove Proposition 1.4.53.

As in the previous exercise, do not use the Cholesky decomposition in your proof; use the fact that $x^\top Ax > 0$ for all nonzero x.

Proposition 1.4.53. Let A be positive definite, and consider a partition

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

in which A_{11} and A_{22} are square. Then A_{11} and A_{22} are positive definite

Proof. Assume $A \in \mathbb{R}^{n\times n}$ is positive definite, with the block decomposition

$$A = \begin{array}{cc} & \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{array}{c} m_1 \\ m_2 \end{array} & \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \end{array}.$$

where A_{11}, A_{22} are square, $n_1 + n_2 = n$, and $m_1 + m_2 = n$. Since A is positive definite, the following properties hold.

1. $A = A^\top$
2. $x^\top Ax > 0$ for all $x \in \mathbb{R}^n$, $x \neq 0$

Since $A = A^\top$,

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11}^\top & A_{12}^\top \\ A_{21}^\top & A_{22}^\top \end{bmatrix}.$$

So, $A_{11} = A_{11}^\top$, and $A_{22} = A_{22}^\top$.

Next, consider $x = \begin{pmatrix} \bar{x} \\ 0 \end{pmatrix} \in \mathbb{R}^n$, where $\bar{x} \neq 0 \in \mathbb{R}^{n_1}$, and $y = \begin{pmatrix} 0 \\ \bar{y} \end{pmatrix}$, where $\bar{y} \neq 0 \in \mathbb{R}^{n_2}$. Note that since A is positive definite,

$$x^\top Ax > 0,$$
$$y^\top Ay > 0.$$

From $x^\top Ax$, we see

$$\begin{aligned} &\begin{pmatrix} \bar{x}^\top & 0 \end{pmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} \bar{x} \\ 0 \end{pmatrix} > 0, \\ \implies &\begin{pmatrix} \bar{x}^\top & 0 \end{pmatrix} \begin{bmatrix} A_{11}\bar{x} \\ A_{21}\bar{x} \end{bmatrix} > 0, \\ \implies &\bar{x}^\top A_{11}\bar{x} > 0. \end{aligned}$$

Since $\bar{x} \in \mathbb{R}^{n_1}$ is an arbitrary vector different from zero and A_{11} is symmetric, A_{11} is positive definite.

Next, we see

$$\begin{aligned} &\begin{pmatrix} 0 & \bar{y}^\top \end{pmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} 0 \\ \bar{y} \end{pmatrix} > 0, \\ \implies &\begin{pmatrix} 0 & \bar{y}^\top \end{pmatrix} \begin{bmatrix} A_{12}\bar{y} \\ A_{22}\bar{y} \end{bmatrix} > 0, \\ \implies &\bar{y}^\top A_{22}\bar{y} > 0 \end{aligned}$$

Again, since $\bar{y} \in \mathbb{R}^{n_2}$ is an arbitrary vector different from zero and A_{22} is symmetric, A_{22} is positive definite.

Therefore, both A_{11} and A_{22} are positive definite. ■

1.4.56. Prove Proposition 1.4.55.

Proposition 1.4.55: If A and X are $n \times n$, A is positive definite, and X is nonsingular, then the matrix $B = X^\top AX$ is also positive definite.

Considering the special case $A = I$ (which is clearly positive definite), we see that this proposition is a generalization of Theorem 1.4.4.

Proof. Assume that $A, X \in \mathbb{R}^{n\times n}$, with A positive definite, and X non-singular. First, we look at $B^\top$.

$$\begin{aligned} B^\top &= (X^\top AX)^\top = X^\top A^\top \left(X^\top\right)^\top \\ &= X^\top A^\top X. \end{aligned}$$

But, A is positive definite, so $A = A^\top$. Thus,

$$B^\top = X^\top A^\top X = X^\top AX = B.$$

So, B is symmetric. Next, we look at $x^\top Bx$. Let $x \in \mathbb{R}^n$, $x \neq 0$, we have

$$x^\top Bx = x^\top X^\top AXx = (Xx)^\top A(Xx).$$

Let $y = Xx$. Since X non-singular and $x \neq 0$, $Xx \neq 0$, so $y \neq 0$. Thus, $y^\top Ay \neq 0$. But, since A positive definite,

$$x^\top Bx = y^\top Ay > 0$$

Therefore, B is positive definite. ■

1.4.58*. Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

be positive definite, and suppose A_{11} is $j \times j$ and A_{22} is $k \times k$. By Proposition 1.4.53, A_{11} is positive definite. Let R_{11} be the Cholesky factor of A_{11}, let $R_{12} = R_{11}^{-T} A_{12}$, and let $\tilde{A}_{22} = A_{22} - R_{12}^T R_{12}$. The matrix $\tilde{A}_{22}$ is called the *Schur complement* of A_{11} in A.

1. Show that
$$\tilde{A}_{22} = A_{22} - A_{21} A_{11}^{-1} A_{12}.$$

2. Establish a decomposition of A that is similar to (1.4.57) and involves $\tilde{A}_{22}$.

3. Prove that $\tilde{A}_{22}$ is positive definite.

Proof. Assume that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

is positive definite, with $A_{11} \in \mathbb{R}^{j \times j}$, $A_{22} \in \mathbb{R}^{k \times k}$. By proposition 1.4.53, A_{11} is positive definite. Let R_{11} be the Cholesky factor of A_{11}, let $R_{12} = R_{11}^{-\top} A_{12}$, and let

$$\tilde{A}_{22} = A_{22} - R_{12}^\top R_{12}.$$

We have

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} R_{11}^\top & 0 \\ R_{21}^\top & R_{22}^\top \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}.$$

By matrix multiplication, we have

$$\begin{aligned} A_{21} &= R_{12}^\top R_{11}, \\ \implies R_{12}^\top &= A_{21} R_{11}^{-1} \end{aligned}$$

since R_{11} is non-singular. So,

$$\begin{aligned} \tilde{A}_{22} &= A_{22} - R_{12}^\top R_{12}, \\ &= A_{22} - A_{21} R_{11}^{-1} R_{12} \\ &= A_{22} - A_{21} R_{11}^{-1} R_{11}^{-\top} A_{12} \end{aligned}$$

Since R_{11} is the Cholesky factor for A_{11},

$$\begin{aligned} A_{11} &= R_{11}^\top R_{11}, \\ \implies (A_{11})^{-1} &= (R_{11}^\top R_{11})^{-1} \\ &= R_{11}^{-1} R_{11}^{-\top}. \end{aligned}$$

So,

$$\begin{aligned} \tilde{A}_{22} &= A_{22} - A_{21} R_{11}^{-1} R_{11}^{-\top} A_{12} \\ &= A_{22} - A_{21} A_{11}^{-1} A_{12}. \end{aligned}$$

As desired.

The decomposition of A similar to (1.4.57) that involves $\tilde{A}_{22}$ is

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} R_{11}^\top & 0 \\ R_{12}^\top & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ 0 & I \end{bmatrix}$$

We can show that $\tilde{A}_{22}$ is positive definite by showing the following hold

1. $\tilde{A}_{22} = \tilde{A}_{22}^\top$
2. $x^\top \tilde{A} x > 0$ for all $x \in \mathbb{R}^k$, $x \neq 0$

First, we show that $\tilde{A}_{22} = \tilde{A}_{22}^\top$.

$$\begin{aligned} \tilde{A}_{22} &= A_{22} - A_{21}A_{11}^{-1}A_{12} \\ \implies \tilde{A}_{22}^\top &= (A_{22} - A_{21}A_{11}^{-1}A_{12})^\top \\ &= A_{22}^\top - (A_{21}A_{11}^{-1}A_{12})^\top \\ &= A_{22}^\top - A_{12}^\top A_{11}^{-\top} A_{21}^\top \end{aligned}$$

But, A is symmetric, so

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11}^\top & A_{21}^\top \\ A_{12}^\top & A_{22}^\top \end{bmatrix}.$$

Which, implies that $A_{12} = A_{21}^\top$, $A_{21} = A_{12}^\top$, $A_{22}^\top = A_{22}$, and $A_{11} = A_{11}^\top$. Thus,

$$\begin{aligned} \tilde{A}^\top &= A_{22}^\top - A_{12}^\top A_{11}^{-\top} A_{21}^\top \\ &= A_{22} - A_{21}A_{11}^{-\top}A_{12}. \end{aligned}$$

But, since $A_{11} = A_{11}^\top$, we see

$$\begin{aligned} A_{11} &= A_{11}^\top, \\ \implies A_{11}^{-1} &= A_{11}^{-\top}. \end{aligned}$$

So,

$$\tilde{A}_{22}^\top = A_{22} - A_{21}A_{11}^{-1}A_{12} = \tilde{A}_{22}.$$

Thus $\tilde{A}_{22}$ is symmetric.

Let $x = \begin{pmatrix} z \\ y \end{pmatrix} \in \mathbb{R}^n$, $x \neq 0$ with $z \in \mathbb{R}^j$, and $y \in \mathbb{R}^k$. Since A is positive definite,

$$\left(z^\top y^\top\right) \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} z \\ y \end{pmatrix} > 0.$$

We aim to show that $y^\top(A_{22} - A_{21}A_{11}^{-1}A_{12})y > 0$ for all $y \in \mathbb{R}^k,\ y \neq 0$ (since $A_{22} - A_{21}A_{11}^{-1}A_{12} \in \mathbb{R}^{k\times k}$).

We have

$$\begin{pmatrix} z^\top & y^\top \end{pmatrix} \begin{bmatrix} A_{11}z + A_{12}y \\ A_{21}z + A_{22}y \end{bmatrix} > 0.$$

We need

$$A_{21}z + A_{22}y = (A_{22} - A_{21}A_{11}^{-1}A_{12})y,$$

so,

$$
\begin{aligned}
A_{21}z + A_{22}y &= A_{22}y - A_{21}A_{11}^{-1}A_{12}y, \\
\implies A_{21}z &= -A_{21}A_{11}^{-1}A_{12}y, \\
\implies z &= -A_{21}^{-1}A_{21}A_{11}^{-1}A_{12}y \\
&= -A_{11}^{-1}A_{12}y.
\end{aligned}
$$

So, let $x = \begin{pmatrix} -A_{11}^{-1}A_{12}y \\ y \end{pmatrix}$. Observe that $x \neq 0$ when $y \neq 0$. Thus, since $x \neq 0$, $y \neq 0$. Recall that $y \in \mathbb{R}^k$.

We have

$$
\begin{aligned}
\left((-A_{11}^{-1}A_{12}y)^\top y^\top\right) \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} -A_{11}^{-1}A_{12}y \\ y \end{pmatrix} &> 0, \\
\implies \left((-A_{11}A_{12}y)^\top y^\top\right) \begin{bmatrix} A_{11}(-A_{11}^{-1}A_{12}y) + A_{12}y \\ (A_{22} - A_{21}A_{11}^{-1}A_{12})y \end{bmatrix} &> 0, \\
\implies \left((-A_{11}A_{12}y)^\top y^\top\right) \begin{bmatrix} -A_{11}A_{11}^{-1}A_{12}y + A_{12}y \\ (A_{22} - A_{21}A_{11}^{-1}A_{12})y \end{bmatrix} &> 0 \\
\implies \left((-A_{11}A_{12}y)^\top y^\top\right) \begin{bmatrix} -A_{12}y + A_{12}y \\ (A_{22} - A_{21}A_{11}^{-1}A_{12})y \end{bmatrix} &> 0 \\
\implies \left((-A_{11}A_{12}y)^\top y^\top\right) \begin{bmatrix} 0 \\ (A_{22} - A_{21}A_{11}^{-1}A_{12})y \end{bmatrix} &> 0, \\
\implies y^\top (A_{22} - A_{21}A_{11}^{-1}A_{12})y &> 0 \\
\implies y^\top \tilde{A}_{22} y &> 0 \\
&.
\end{aligned}
$$

So, $\tilde{A}_{22}$ is positive definite. ■

1.4.62. Prove that if A is positive definite, then $\det(A) > 0$

Proof. Assume that A is a positive definite matrix. Since A is positive definite, then A admits a Cholesky decomposition $A = R^\top R$, for a unique upper triangular matrix R, with $r_{ii} > 0$. So,

$$\begin{aligned} & A = R^\top R \\ \Longrightarrow\ & \det(A) = \det(R^\top R) = \det(R^\top)\det(R) \end{aligned}$$

But, since the determinant of a triangular matrix is the product of the main diagonal, and $r_{ii} > 0$ for $i = 1, 2, ..., n$, both $\det(R^\top)$ and $\det(R)$ are positive, so their product is positive. Thus,

$$\det(A) = \det(R^\top)\det(R) > 0$$

■