Problem 1. Consider the series

$$\sum_{n=1}^{\infty} \frac{3n^3}{2n^3+4}\ .$$

Based on the Divergence Test, does this series Diverge?

By the divergence test

$$\lim_{n\to\infty} \frac{3n^3}{2n^3+4}$$
$$= \frac{3}{2}.$$

Thus, since the limit is not zero. This series will diverge

Problem 2. What does the divergence test tell you about each of the series below?

(a) $\sum_{n=1}^{\infty} 3^n$

(b) $\sum_{n=1}^{\infty} 7^{-n}$

(c) $\sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n$

(d) $\sum_{n=0}^{\infty} \left(\frac{7}{3}\right)^n$

Part A.

$$\lim_{n\to\infty} 3^n$$
$$= +\infty.$$

Thus, since the limit is not zero. The divergence test tells us that this series will diverge

Part B.

$$\lim_{n\to\infty} 7^{-n}$$
$$= 0.$$

Since the limit is zero, the divergence test is inconclusive

Part C.

$$\lim_{n\to\infty} \left(\frac{1}{e}\right)^n$$
$$= 0.$$

Since the limit is zero, the divergence test is inconclusive

Part D.

$$\lim_{n\to\infty} \left(\frac{7}{3}\right)^n$$
$$= +\infty.$$

Thus, since the limit is not zero. The divergence test tells us that this series will diverge

Problem 3. Use the Divergene Test to determine the whether the series converges or diverges.

$$\sum_{n=1}^{\infty}\left(1+\frac{9}{n}\right)^{n}.$$

Given the fact that Euler's number has a definition of the form:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e.$$

With a generalization of

$$e^{a}=\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{n}.$$

Using the divergence test for the series $\sum_{n=1}^{\infty}\left(1+\frac{9}{n}\right)^{n}$, we get the $\lim_{n\to\infty}\left(1+\frac{9}{n}\right)^{n}$. Which will trivially yield e^{9}. However, this can be shown...

$$\begin{aligned}&\lim_{n\to\infty}\left(1+\frac{9}{n}\right)^{n}\\&=\lim_{n\to\infty}e^{\ln\left(1+\frac{9}{n}\right)^{n}}\\&=\lim_{n\to\infty}e^{n\ln\left(1+\frac{9}{n}\right)}.\end{aligned}$$

Focusing on $n\ln\left(1+\frac{9}{n}\right)$...

$$\begin{aligned}&\lim_{n\to\infty}n\ln\left(1+\frac{9}{n}\right)\quad(\text{Indeterminate }\infty\cdot 0)\\&=\lim_{n\to\infty}\frac{\ln\left(1+\frac{9}{n}\right)}{n^{-1}}\quad\left(\frac{0}{0}\right)\\&\overset{H}{=}\lim_{n\to\infty}\frac{\frac{1}{1+\frac{9}{n}}\cdot\left(-\frac{9}{n^{2}}\right)}{-\frac{1}{n^{2}}}\\&=\lim_{n\to\infty}\frac{-\frac{9}{n^{2}+2n}}{-\frac{1}{n^{2}}}\\&=\lim_{n\to\infty}\frac{9n^{2}}{n^{2}+2n}\\&\lim_{n\to\infty}\frac{9n}{n+2}\quad\left(\text{Still indeterminate... }\frac{\infty}{\infty}\right)\\&\overset{H}{=}\lim_{n\to\infty}\frac{9}{1}\\&=9.\end{aligned}$$

Thus,

$$\begin{aligned}&\lim_{n\to\infty}\left(1+\frac{9}{n}\right)^{n}\\&=\lim_{n\to\infty}e^{n\ln\left(1+\frac{9}{n}\right)}\\&=e^{9}.\end{aligned}$$

Problem 4. To test the series $\sum_{n=1}^{\infty} e^{-3n}$ for convergence, you can use the Integral Test. (This is also a geometric series, so we could also investigate convergence using other methods.) What does this value tell you about the convergence of the series

$$\sum_{n=1}^{\infty} e^{-3n} .$$

Since a_n has positive terms, and $a_n = f(n)$, for $f(x) = e^{-3x}$ on $[1, +\infty)$, satisfying

- Continuous
- Positive, decreasing

Then by the integral test

$$\begin{aligned}
&\int_1^{\infty} e^{-3x}\ dx \\
&= \lim_{t\to\infty} \int_1^t e^{-3x}\ dx \\
&= \lim_{t\to\infty} -\frac{1}{3}e^{-3x}\ \Big|_1^t \\
&= \lim_{t\to\infty} -\frac{1}{3}e^{-3t} - \left(-\frac{1}{3}e^{-3}\right) \\
&= \frac{1}{3e^3}.
\end{aligned}$$

Since the improper integral converges, so does the series

Problem 5. Compute the value of the following improper integral

$$\int_1^\infty \frac{2\ln(x)}{x^6}\, dx.$$

What does the value of the improper integral tell use about the convergence of the series

$$\sum_{n=1}^\infty \frac{2\ln(n)}{n^6}\ .$$

$$\begin{aligned}
&\int_1^\infty \frac{2\ln(x)}{x^6}\, dx \\
&= \lim_{t\to\infty} \int_1^t \frac{2\ln(x)}{x^6}\, dx \\
&= 2\lim_{t\to\infty} \int_1^\infty x^{-6}\ln(x)\, dx \\
&= 2\lim_{t\to\infty} -\frac{1}{5x^5}\ln x\,\Big|_1^t - \int_1^t -\frac{1}{5}x^{-6}\, dx \\
&= 2\lim_{t\to\infty} -\frac{1}{5t^5}\ln t + \int_1^t \frac{1}{5}x^{-6}\, dx \\
&= 2\lim_{t\to\infty} -\frac{1}{5t^5}\ln t + \left(-\frac{1}{25x^5}\,\Big|_1^t\right) \\
&= 2\lim_{t\to\infty} -\frac{1}{5t^5}\ln t - \frac{1}{25t^5} + \frac{1}{25} \\
&= 2\lim_{t\to\infty} -\cancelto{0}{\frac{1}{5t^5}\ln t} - \cancelto{0}{\frac{1}{25t^5}} + \frac{1}{25} \\
&= \frac{2}{25}.
\end{aligned}$$

$$\begin{aligned}
u &= \ln(x) \quad & dv &= x^{-6}\, dx \\
du &= \frac{1}{x}\, dx \quad & v &= -\frac{1}{5}x^{-5}.
\end{aligned}$$

Since the improper integral converges, so does the series

Problem 6. Compute the value of the improper integral

$$\int_2^\infty \frac{dx}{(2x+3)^7}\ dx.$$

Use your answer to help determine whether the series

$$\sum_{n=2}^\infty \frac{1}{(2n+3)^7}\ .$$

converges or diverges

$$\begin{aligned}
&\int_2^\infty \frac{dx}{(2x+3)^7}\ dx \\
&= \frac{1}{2}\lim_{t\to\infty}\int_2^t \frac{dx}{(2x+3)^7}\ dx \\
&= \frac{1}{2}\lim_{t\to\infty} -\frac{1}{6}(2x+3)^{-6}\ \Big|_2^t \\
&= \frac{1}{2}\lim_{t\to\infty} -\frac{1}{6}\left[(2t+3)^{-6} - (7)^{-6}\right] \\
&= \frac{1}{2}\lim_{t\to\infty} -\frac{1}{6}\left[\cancel{(2t+3)^{-6}}^{\,0} - (7)^{-6}\right] \\
&= -\frac{1}{12}\left(-\frac{1}{7^6}\right) \\
&= \frac{1}{1411788}.
\end{aligned}$$

Since the improper integral converges, so does the series

Problem 7. Compute the value of the improper integral

$$\int_1^\infty \frac{3}{1+x^2}\ dx.$$

Use the value of the improper integral to determine whether or not the series

$$\sum_{n=1}^{\infty} \frac{3}{1+n^2}\ .$$

converges or diverges

$$\int_1^\infty \frac{3}{1+x^2}\ dx$$
$$= 3 \lim_{t\to\infty} \frac{1}{1+x^2}$$
$$= 3 \lim_{t\to\infty} \tan^{-1} x \Big|_1^t$$
$$3 \lim_{t\to\infty} \tan^{-1} t - \tan^{-1} 1$$
$$= 3\left[\frac{pi}{2} - \frac{\pi}{4}\right]$$
$$= \frac{3\pi}{4}.$$

Since the improper integral converges, so does the series

Problem 8. To test the series

$$\sum_{n=1}^{\infty} \frac{1}{k^3}\ .$$

For convergence, you can use the P-test. Then compute S_3, the partial sum consisting of the first 3 terms of $\sum\limits_{k=1}^{\infty} \frac{1}{k^3}$

Since $P = 3 > 1$, this series will converge. For S_3...

$$S_3 = 1 + \frac{1}{8} + \frac{1}{27}$$
$$= \frac{251}{216}$$
$$\approx 1.16.$$

Problem 9. To test the series

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[5]{k^4}}\ .$$

for convergence, you can use the P-test. Then compute S_3, the partial sum consisting of the first 3 terms of $\sum\limits_{k=1}^{\infty} \frac{1}{\sqrt[5]{k^4}}$

Since $P = \frac{4}{5} \leqslant 1$. This series will diverge. For S_3...

$$\begin{aligned} S_3 &= 1 + \frac{1}{2^{\frac{4}{5}}} + \frac{1}{3^{\frac{4}{5}}} \\ &\approx 1.9896. \end{aligned}$$