Chapter I

Parametric Equations and Polar Coordinates

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Chapter 1
Parametric Equations and Polor Coordinates

1.1 Parametric Equations

In the two-dimensional coordinate system, parametric equations are useful for describing curves that are not necessarily functions. The parameter is an independent variable that both x and y depend on, and as the parameter increases, the values of x and y trace out a path along a plane curve. For example, if the parameter is t (a common choice), then t might represent time. Then x and y are defined as functions of time, and $(x(t), y(t))$ can describe the position in the plane of a given object as it moves along a curved path.

Consider the orbit of Earth around the Sun. Our year lasts approximately 365.25 days, but for this discussion we will use 365 days. On January 1 of each year, the physical location of Earth with respect to the Sun is nearly the same, except for leap years, when the lag introduced by the extra $\frac{1}{4}$ day of orbiting time is built into the calendar. We call January 1 "day 1" of the year. Then, for example, day 31 is January 31, day 59 is February 28, and so on.

The number of the day in a year can be considered a variable that determines Earth's position in its orbit. As Earth revolves around the Sun, its physical location changes relative to the Sun. After one full year, we are back where we started, and a new year begins. According to Kepler's laws of planetary motion, the shape of the orbit is elliptical, with the Sun at one focus of the ellipse1.

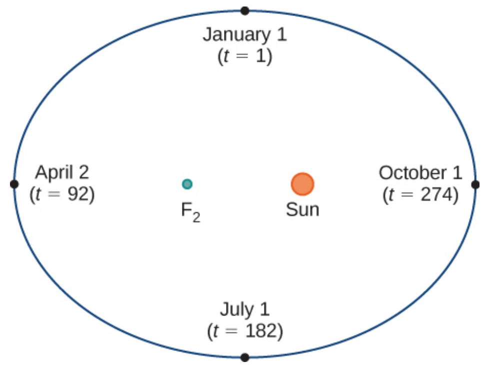

Figure 1

Figure 1 depicts Earth's orbit around the Sun during one year. The point labeled F_2 is one of the foci of the ellipse; the other focus is occupied by the Sun. If we superimpose coordinate axes over this graph, then we can assign ordered pairs to each point on the ellipse (Figure 2). Then each x value on the graph is a value of position as a function of time, and each y value is also a value of position as a function of time. Therefore, each point on the graph corresponds to a value of Earth's position as a function of time.

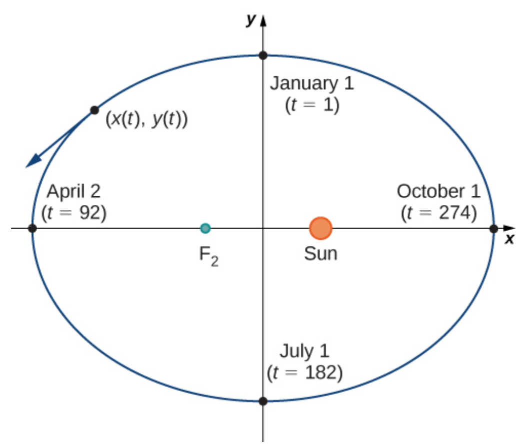

Figure 2

Definition 1.1. If x and y are continuous functions of t on an interval I, then the equations

$$x = x(t) \quad \text{and} \quad y = y(t)$$

are called **parametric equations** and t is called the **parameter**. The set of points (x, y) obtained as t varies over the interval I is called the graph of the parametric equations. The graph of parametric equations is called a **parametric curve** or plane curve, and is denoted by C.

Note:-

Notice in this definition that x and y are used in two ways. The first is as functions of the independent variable t. As t varies over the interval I, the functions $x(t)$ and $y(t)$ generate a set of ordered pairs (x, y). This set of ordered pairs generates the graph of the parametric equations. In this second usage, to designate the ordered pairs, x and y are variables. It is important to distinguish the variables x and y from the functions $x(t)$ and $y(t)$.

Example 1.1 (Graphing a Parametrically Defined Curve)**.** Sketch the curves described by the following parametric equations:

$$\begin{aligned}
&\text{(a)} \quad x:\ \mathbb{R}\to\mathbb{R}:\ t\mapsto t-1, \quad y:\ \mathbb{R}\to\mathbb{R}:\ t\mapsto 2t+4, \quad -3\leqslant t\leqslant 2\\
&\text{(b)} \quad x:\ \mathbb{R}\to\mathbb{R}:\ t\mapsto t^2-3, \quad y:\ \mathbb{R}\to\mathbb{R}:\ t\mapsto 2t+1, \quad -2\leqslant t\leqslant 3\\
&\text{(c)} \quad x:\ \mathbb{R}\to\mathbb{R}:\ t\mapsto 4\cos t, \quad y:\ \mathbb{R}\to\mathbb{R}:\ t\mapsto 4\sin t, \quad 0\leqslant t\leqslant 2\pi
\end{aligned} \tag{1}$$

Solution a. To create a graph of this curve, first set up a table of values. Since the independent variable in both $x(t)$ and $y(t)$ is t, let t appear in the first column. Then $x(t)$ and $y(t)$ will appear in the second and third columns of the table.

t	$x(t)$	$y(t)$
-3	-4	-2
-2	-3	0
-1	-2	2
0	-1	4
1	0	6
2	1	8

The second and third columns in this table provide a set of points to be plotted. The graph of these points appears in Figure 3. The arrows on the graph indicate the orientation of the graph, that is, the direction that a point moves on the graph as t varies from −3 to 2.

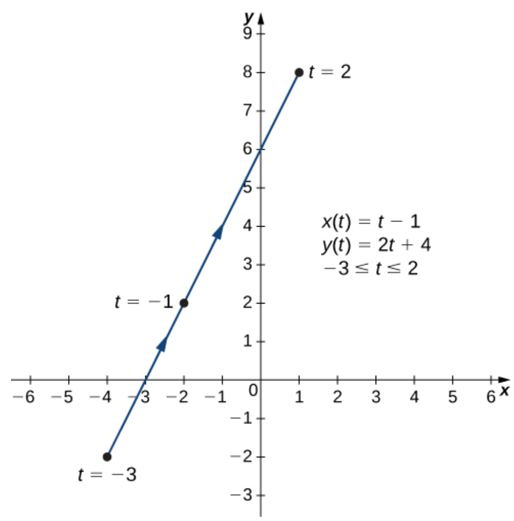

Figure 3

Solution b. To create a graph of this curve, again set up a table of values.

t	$x(t)$	$y(t)$
-2	1	-3
-1	-2	-1
0	-3	1
1	-2	3
2	1	5
3	6	7

As t progresses from -2 to 3, the point on the curve travels along a parabola. The direction the point moves is again called the orientation and is indicated on the graph.

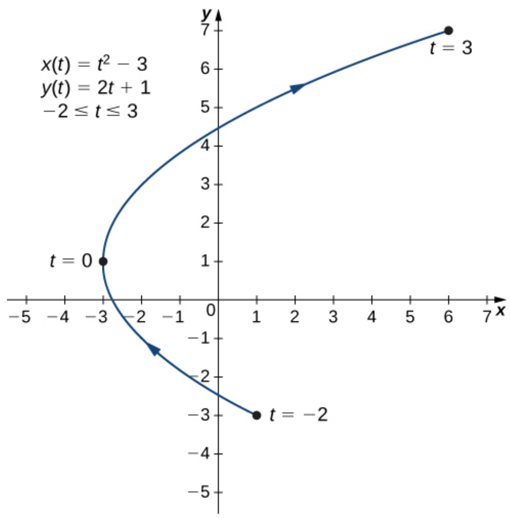

Figure 4

Solution c. In this case, use multiples of $\frac{\pi}{6}$ for t and create another table of values

t	$x(t)$	y(t)
0	4	0
$\frac{\pi}{6}$	$2\sqrt{3} \approx 3.5$	2
$\frac{\pi}{3}$	2	$2\sqrt{3}$
$\frac{\pi}{2}$	0	4
$\frac{2\pi}{3}$	-2	$2\sqrt{3}$
$\frac{5\pi}{6}$	$-2\sqrt{3}$	2
π	-4	0
$\frac{7\pi}{6}$	$-2\sqrt{3}$	2
$\frac{4\pi}{3}$	-2	$-2\sqrt{3}$
$\frac{3\pi}{2}$	0	4
$\frac{5\pi}{3}$	2	$-2\sqrt{3}$
$\frac{11\pi}{6}$	$2\sqrt{3}$	2
2π	4	0

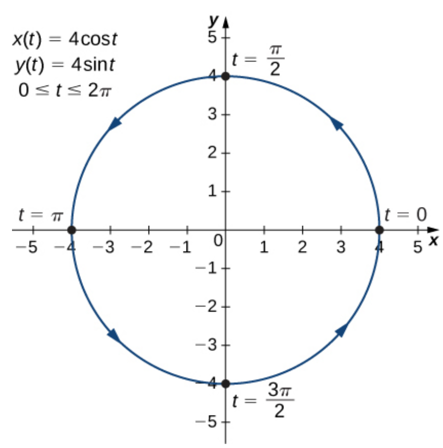

Figure 5

1.1 Eliminating the Parameter

To better understand the graph of a curve represented parametrically, it is useful to rewrite the two equations as a single equation relating the variables x and y. Then we can apply any previous knowledge of equations of curves in the plane to identify the curve. For example, the equations describing the plane curve in Example 1.1b. are

$$x(t) = t^2 - 3, \quad y(t) = 2t + 1, \quad -2 \leqslant t \leqslant 3.$$

Solving the second equation for t gives

$$t = \frac{y-1}{2}.$$

This can be substituted into the first equation:

$$\begin{aligned} x &= \left(\frac{y-1}{2}\right)^2 - 3 \\ x &= \frac{(y-1)^2}{4} - 3 \\ x &= \frac{y^2 - 2y + 1}{4} - 3 \\ x &= \frac{y^2 - 2y - 11}{4}. \end{aligned}$$

This equation describes x as a function of y. These steps give an example of *eliminating the parameter*. The graph of this function is a parabola opening to the right. Recall that the plane curve started at $(1, -3)$ and ended at $(6, 7)$. These terminations were due to the restriction on the parameter t.

Sometimes it is necessary to be a bit creative in eliminating the parameter. The parametric equations for this example are

$$x(t) = 4\cos t, \quad y(t) = 3\sin t.$$

Solving either equation for t directly is not advisable because sine and cosine are not one-to-one functions. However, dividing the first equation by 4 and the second equation by 3 (and suppressing the t) gives us

$$\cos t = \frac{x}{4}, \quad \sin t = \frac{y}{3}.$$

Now use the Pythagorean identity $\cos^2 t + \sin^2 t = 1$ and replace the expressions for $\sin t$ and $\cos t$ with the equivalent expressions in terms of x and y. This gives

$$\begin{aligned} \left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 &= 1 \\ \frac{x^2}{16} + \frac{y^2}{9} &= 1. \end{aligned}$$

This is the equation of a horizontal ellipse centered at the origin, with semimajor axis 4 and semiminor axis 3 as shown in the following graph.

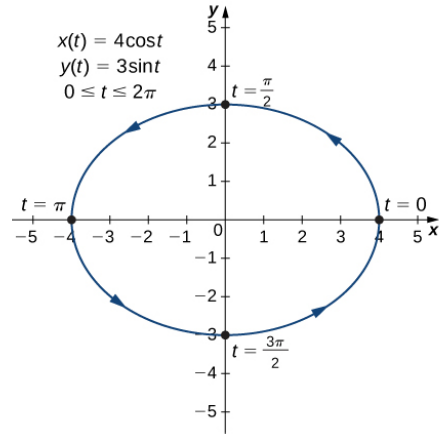

Figure 6

So far we have seen the method of eliminating the parameter, assuming we know a set of parametric equations that describe a plane curve. What if we would like to start with the equation of a curve and determine a pair of parametric equations for that curve? This is certainly possible, and in fact it is possible to do so in many different ways for a given curve. The process is known as **parameterization of a curve**.

Example 1.2 (Parameterizing a Curve)**.** Find two different pairs of parametric equations to represent the graph of $y = 2x^2 - 3$.

Solution. First, it is always possible to parameterize a curve by defining $x(t) = t$, then replacing x with t in the equation for $y(t)$. This gives the parameterization

$$x(t) = t, \quad y(t) = 2t^2 - 3.$$

We have complete freedom in the choice for the second parameterization. For example, we can choose $x(t) = 3t - 2$. The only thing we need to check is that there are no restrictions imposed on x; that is, the range of $x(t)$ is all real numbers. This is the case for $x(t) = 3t-2$. Now since $y = 2x^2 - 3$, we can substitute $x(t) = 3t - 2$ for x. This gives

$$\begin{aligned} y(t) &= 2(3t-2)^2 - 3 \\ &= 2(9t^2 - 12t + 4) - 3 \\ &= 18t^2 - 24t + 8 - 3 \\ &= 18t^2 - 24t + 5. \end{aligned}$$

Therefore, a second parameterization of the curve can be written as

$$x(t) = 3t - 2 \quad \text{and} \quad y(t) = 18t^2 - 24t + 5.$$

1.2 Cycloids and Other Parametric Curves

Imagine going on a bicycle ride through the country. The tires stay in contact with the road and rotate in a predictable pattern. Now suppose a very determined ant is tired after a long day and wants to get home. So he hangs onto the side of the tire and gets a free ride. The path that this ant travels down a straight road is called a **cycloid** (Figure 7). A cycloid generated by a circle (or bicycle wheel) of radius a is given by the parametric equations

$$x(t) = a(t - \sin t), \quad y(t) = a(1 - \cos t).$$

To see why this is true, consider the path that the center of the wheel takes. The center moves along the x-axis at a constant height equal to the radius of the wheel. If the radius is a, then the coordinates of the center can be given by the equations

$$x(t) = at, \quad y(t) = a$$

for any value of t. Next, consider the ant, which rotates around the center along a circular path. If the bicycle is moving from left to right then the wheels are rotating in a clockwise direction. A possible parameterization of the circular motion of the ant (relative to the center of the wheel) is given by

$$x(t) = -a \sin t, \quad y(t) = -a \cos t.$$

(The negative sign is needed to reverse the orientation of the curve. If the negative sign were not there, we would have to imagine the wheel rotating counterclockwise.) Adding these equations together gives the equations for the cycloid.

$$x(t) = a(t - \sin t), \quad y(t) = a(1 - \cos t).$$

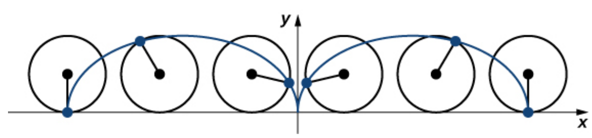

Figure 7

Now suppose that the bicycle wheel doesn't travel along a straight road but instead moves along the inside of a larger wheel, as in Figure 8. In this graph, the green circle is traveling around the blue circle in a counterclockwise direction. A point on the edge of the green circle traces out the red graph, which is called a hypocycloid.

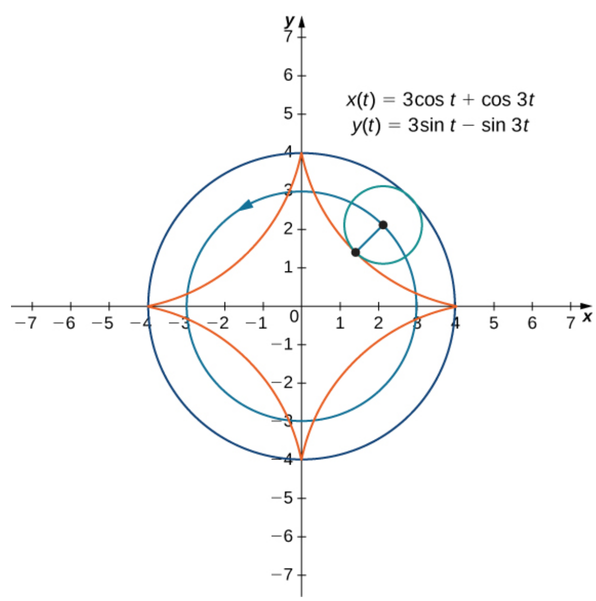

Figure 8

The general parametric equations for a hypocycloid are

$$x(t) = (a-b)\ \cos t + b\cos\left(\frac{a-b}{b}\right)t$$
$$y(t) = (a-b)\ \sin t + b\sin\left(\frac{a-b}{b}\right)t$$

.

These equations are a bit more complicated, but the derivation is somewhat similar to the equations for the cycloid. In this case we assume the radius of the larger circle is a and the radius of the smaller circle is b. Then the center of the wheel travels along a circle of radius $a-b$. This fact explains the first term in each equation above. The period of the second trigonometric function in both $x(t)$ and $y(t)$ is equal to $\frac{2\pi b}{a-b}$.

The ratio $\frac{a}{b}$ is related to the number of cusps on the graph (cusps are the corners or pointed ends of the graph), as illustrated in Figure 9. This ratio can lead to some very interesting graphs, depending on whether or not the ratio is rational. Figure 1.10 corresponds to $a = 4$ and $b = 1$. The result is a hypocycloid with four cusps. Figure 1.11 shows some other possibilities. The last two hypocycloids have irrational values for $\frac{a}{b}$. In these cases the hypocycloids have an infinite number of cusps, so they never return to their starting point. These are examples of what are known as space-filling curves.

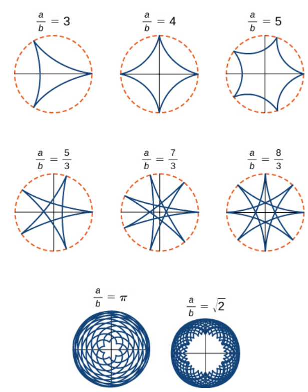

Figure 9

1.2 Calculus of Parametric Curves

Now that we have introduced the concept of a parameterized curve, our next step is to learn how to work with this concept in the context of calculus.

2.1 Derivatives of Parametric Equations

Theorem 2.1 (Derivative of Parametric Equations)**.** Consider the plane curve defined by the parametric equations $x = x(t)$ and $y = y(t)$. Suppose that $x'(t)$ and $y'(t)$ exist, and assume that $x'(t) \neq 0$. Then the derivative $\frac{dy}{dx}$ is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}.$$

Proof: This theorem can be proven using the Chain Rule. In particular, assume that the parameter t can be eliminated, yielding a differentiable function $y = F(x)$. Then $y(t) = F(x(t))$. Differentiating both sides of this equation using the Chain Rule yields

$$y'(t) = F'(x(t)) \cdot x'(t),$$

so

$$F'(x(t)) = \frac{y'(t)}{x'(t)}.$$

But $F'(x(t)) = \frac{dy}{dx}$, which proves the theorem. ■ ☺

Note:-

This theorem can be used to calculate derivatives of plane curves, as well as critical points. Recall that a critical point of a differentiable function $y = f(x)$ is any point $x = x_0$ such that either $f'(x_0) = 0$ or $f'(x_0)$ does not exist. Equation 1.1 gives a formula for the slope of a tangent line to a curve defined parametrically regardless of whether the curve can be described by a function $y = f(x)$ or not.

2.2 Second-Order Derivatives

Our next goal is to see how to take the second derivative of a function defined parametrically. The second derivative of a function $y = f(x)$ is defined to be the derivative of the first derivative; that is,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right].$$

Since $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, we can replace the y on both sides of this equation with $\frac{dy}{dx}$. This gives us

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \left(\frac{d}{dt}\right)\left(\frac{dy}{dx}\right)\frac{dx}{dt}.$$

If we know $\frac{dy}{dx}$ as a function of t, then this formula is straightforward to apply.

Example 2.1 (second order derivative)**.** Calculate the second derivative $\frac{d^2y}{dx^2}$ for the plane curve defined by the parametric equations $x(t) = t^2 - 3$, $y(t) = 2t - 1$, for $-3 \leqslant t \leqslant 4$.

Solution.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{y'(t)}{x'(t)} = \frac{2}{2t} = \frac{1}{t} \\
\implies \frac{d^2y}{dx^2} &= \frac{\left(\frac{d}{dt}\right)\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{1}{t}\right)}{2t} \\
&= -\frac{\frac{1}{t^2}}{2t} \\
&= -\frac{1}{2t^3}.
\end{aligned}$$

2.3 Integrals Involving Parametric Equations

Suppose we have a parametric curve defined by the set of equations

$$\begin{aligned} x(t) &= f(t) \\ y(t) &= g(t). \end{aligned}$$

For $c \leqslant t \leqslant d$. Assuming elimination of the parameter is possible, we have

$$y = f(x).$$

Given what we know about the riemann integral $\int_a^b f(x)\ dx$, we can use our assumption to get $\int_a^b y\ dx$. Solving for dx we get

$$\begin{aligned} \frac{dx}{dt} &= f'(t) \\ dx &= f'(t)dt. \end{aligned}$$

Thus, now we have the integral

$$\int_a^b yf'(t)\ dt.$$

Since we know $y = g(t)$, and adjusting the bounds for values of t, we arrive at

$$\int_c^d g(t)f'(t)\ dt$$

$$\text{Or simply } \int_c^d y(t)x'(t)\ dt.$$

This leads to the following theorem

Theorem 2.2 (Integral Involving Parametric Equations)**.** Consider the non-self-intersecting plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t), \quad a \leqslant t \leqslant b$$

and assume that $x(t)$ is differentiable. The area under this curve is given by

$$A = \int_a^b y(t)x'(t)\,dt.$$

2.4 Arc Length of a Parametric Curve

In addition to finding the area under a parametric curve, we sometimes need to find the arc length of a parametric curve. In the case of a line segment, arc length is the same as the distance between the endpoints. If a particle travels from point A to point B along a curve, then the distance that particle travels is the arc length. To develop a formula for arc length, we start with an approximation by line segments as shown in the following graph.

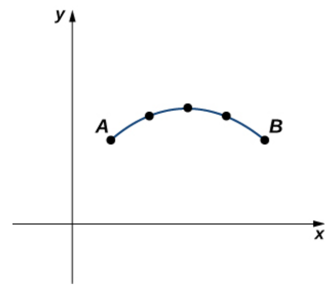

Figure 10

Given a plane curve defined by the functions $x = x(t)$, $y = y(t)$, $a \leqslant t \leqslant b$, we start by partitioning the interval $[a, b]$ into n equal subintervals: $t_0 = a < t_1 < t_2 < \cdots < t_n = b$. The width of each subinterval is given by $\Delta t = \frac{b-a}{n}$. We can calculate the length of each line segment:

$$d_1 = \sqrt{(x(t_1) - x(t_0))^2 + (y(t_1) - y(t_0))^2}$$
$$d_2 = \sqrt{(x(t_2) - x(t_1))^2 + (y(t_2) - y(t_1))^2}$$

Then add these up. We let s denote the exact arc length and s_n denote the approximation by n line segments:

$$s \approx \sum_{k=1}^{n} s_k = \sum_{k=1}^{n} \sqrt{(x(t_k) - x(t_{k-1}))^2 + (y(t_k) - y(t_{k-1}))^2}.$$

If we assume that $x(t)$ and $y(t)$ are differentiable functions of t, then the Mean Value Theorem (Introduction to the Applications of Derivatives) applies, so in each subinterval $[t_{k-1}, t_k]$ there exist $\hat{t}_k$ and $\tilde{t}_k$ such that

$$x(t_k) - x(t_{k-1}) = x'(\hat{t}_k)(t_k - t_{k-1}) = x'(\hat{t}_k)\Delta t$$
$$y(t_k) - y(t_{k-1}) = y'(\tilde{t}_k)(t_k - t_{k-1}) = y'(\tilde{t}_k)\Delta t.$$

Therefore the equation becomes

$$\begin{aligned} s &\approx \sum_{k=1}^{n} s_k \\ &= \sum_{k=1}^{n} \sqrt{(x'(\hat{t}_k)\Delta t)^2 + (y'(\tilde{t}_k)\Delta t)^2} \\ &= \sum_{k=1}^{n} \sqrt{(x'(\hat{t}_k))^2(\Delta t)^2 + (y'(\tilde{t}_k))^2(\Delta t)^2} \\ &= \left(\sum_{k=1}^{n} \sqrt{(x'(\hat{t}_k))^2 + (y'(\tilde{t}_k))^2} \right) \Delta t. \end{aligned}$$

This is a Riemann sum that approximates the arc length over a partition of the interval [a,b]. If we further assume that the derivatives are continuous and let the number of points in the partition increase without bound, the approximation approaches the exact arc length. This gives

$$\begin{aligned} s &= \lim_{n\to\infty} \sum_{k=1}^{n} s_k \\ &= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \sqrt{(x'(\hat{t}_k))^2 + (y'(\tilde{t}_k))^2} \right) \Delta t \\ &= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt. \end{aligned}$$

Note:-

When taking the limit, the values of $\hat{t}_k$ and $\tilde{t}_k$ are both contained within the same ever-shrinking interval of width Δt, so they must converge to the same value.

Theorem 2.3 (Arc Length of a Parametric Curve)**.** Consider the plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t), \quad t_1 \leqslant t \leqslant t_2$$

and assume that $x(t)$ and $y(t)$ are differentiable functions of t. Then the arc length of this curve is given by

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

A side derivation can be made. Suppose the parameter can be eliminated, leading to a function $y = F(x)$. Then $y(t) = F(x(t))$ and the Chain Rule gives $y'(t) = F'(x(t)) \cdot x'(t)$. Substituting into the theorem above we get

$$\begin{aligned} s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(F'(x)\frac{dx}{dt}\right)^2}\, dt \\ &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 (1 + (F'(x))^2)\, dt} \\ &= \int_{t_1}^{t_2} x'(t) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dt.. \end{aligned}$$

Here we have assumed that $x'(t) > 0$, which is a reasonable assumption. The Chain Rule gives $dx = x'(t)\, dt$, and letting $a = x(t_1)$ and $b = x(t_2)$ we obtain the formula

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

2.5 Surface Area Generated by a Parametric Curve

Theorem 2.4 (Surface area for a parametric curve)**.** The analogous formula for a parametrically defined curve is

$$S = 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

provided that $y(t)$ is not negative on $[a, b]$.

$$\frac{-1 - \sqrt{1 + 4x}}{2} < 0.$$