Calculus 2
Chapter 3

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September 27, 2023
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Techniques of Integration

3.1 Integration by Parts

Definition 1:

Many students want to know whether there is a product rule for integration. There isn't, but there is a technique based on the product rule for differentiation that allows us to exchange one integral for another. We call this technique **integration by parts.**

The Integration-by-Parts Formula

If, $h(x) = f(x)g(x)$, then by using the product rule, we obtain $h'(x) = f'(x)g(x) + g'(x)f(x)$. Although at first it may seem counterproductive, let's now integrate both sides of this equation:

$$\int h'(x)\,dx = \int \left(g(x)f'(x) + f(x)g'(x)\right)\,dx.$$

This gives us

$$h(x) = f(x)g(x) = \int g(x)f'(x)\,dx + \int f(x)g'(x)\,dx.$$

Now we solve for $\int f(x)g'(x)\,dx$:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx.$$

By making the substitutions $u = f(x)$ and $v = g(x)$, which in turn make $du = f'(x)\,dx$ and $dv = g'(x)\,dx$, we have the more compact form

$$\int u\,dv = uv - \int v\,du.$$

Theorem 1: Integration by Parts

Let $u = f(x)$ and $v = g(x)$ be functions with continuous derivatives. Then, the integration-by-parts formula for the integral involving these two functions is:

$$\int u\,dv = uv - \int v\,du.$$

Example 1: Using Integration by Parts

Use integration by parts with $u = x$ and $dv = \sin x\,dx$ to evaluate

$$\int x \sin x\,dx.$$

Solution: So to use the formula:

$$\int \ u \ dv = uv - \int v \ du.$$

We need:

$$\begin{aligned} u = x &\quad du = dx \\ dv = \sin x dx &\quad v = -\cos x. \end{aligned}$$

Thus:

$$\begin{aligned} \int x \sin x \ dx &= -x \cos x - \int -\cos x \ dx \\ &= -x \cos x + \sin x + C. \end{aligned}$$

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The natural question to ask at this point is: How do we know how to choose u and dv? Sometimes it is a matter of trial and error; however, the acronym **LIATE** can often help to take some of the guesswork out of our choices. This acronym stands for

- **L**ogarithmic Functions
- **I**nverse Trigonometric Functions
- **A**lgebraic Functions
- **T**rigonometric Functions
- **E**ponential Functions

This mnemonic serves as an aid in determining an appropriate choice for u.

Applying integration by parts more than once

Example 2: Evaluate

$$\int x^2 e^{3x}\ dx.$$

Solution: By **LIATE**, we let $u = x^2$, and $dv = e^{3x}$. Thus, we get:

$$u = x^2 \quad dv = e^{3x}$$
$$du = 2xdx \quad v = \frac{1}{3}e^{3x}.$$

Then by theorem 1, we get:

$$\begin{aligned}\int udv &= uv - \int vdu \\ &= \int x^2 e^{3x}\ dx = x^2 \frac{1}{3}e^{3x} - \int \frac{1}{3}e^{3x} 2x\ dx \\ &= \int x^2 e^{3x}\ dx = x^2 \frac{1}{3}e^{3x} - \int \frac{2}{3}e^{3x} x\ dx\end{aligned}$$

.

At this point, we will notice that we still cannot evaluate the integral $\int \frac{2}{3}e^{3x}x\ dx$. Thus, we must apply the theorem once more.

$$\int \frac{2}{3}e^{3x} x\ dx$$
$$u = x \quad dv = \frac{2}{3}e^{3x}$$
$$du = dx \quad v = \frac{2}{9}e^{3x}.$$

Thus:

$$\begin{aligned}\int \frac{2}{3}e^{3x}\ dx &= \frac{2}{9}e^{3x}x - \int \frac{2}{9}e^{3x}\ dx \\ &= \frac{2}{9}xe^{3x} - \frac{2}{27}e^{3x}.\end{aligned}$$

In full we have:

$$\begin{aligned}&\int x^2 e^{3x}\ dx = \frac{1}{3}x^2 e^{3x} - \left(\frac{2}{9}xe^{3x} - \frac{2}{27}e^{3x}\right) \\ &= \frac{1}{3}e^{3x}x^2 - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C.\end{aligned}$$

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Applying Integration by Parts When LIATE Doesn't Quite Work

Example 3: Evaluate

$$\int t^3 e^{t^2}\ dt.$$

Solution: If we use a strict interpretation of the mnemonic **LIATE** to make our choice of u, we end up with $u = t^3$ and $dv = e^{t^2} dt$. Unfortunately, this choice won't work because we are unable to evaluate $\int e^{t^2}\ dt$. However, since we can evaluate $\int te^{t^2}\ dt$, we can try choosing $u = t^2$ and $dv = te^{t^2}\ dt$. With these choices we have

$$\begin{aligned} u &= t^2 \quad dv = te^{t^2} \\ du &= 2t\ dt \quad v = \frac{1}{2}e^{t^2}. \end{aligned}$$

Thus, we obtain:

$$\begin{aligned} \int t^3 e^{t^2}\ dt &= \frac{1}{2}t^2 e^{t^2} - \int \frac{1}{2}e^{t^2} 2t \\ &= \frac{1}{2}t^2 e^{t^2} - \int e^{t^2} t\ dt \\ &= \frac{1}{2}t^2 e^{t^2} - \frac{1}{2}e^{t^2} + C. \end{aligned}$$

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Example 4: Evaluate

$$\int \sin(\ln(x))\ dx.$$

Solution: Here, we let $u = \sin(\ln(x))$ and $dv = 1dx$, so we have:

$$\begin{aligned} u &= \sin(\ln(x)) \quad dv = dx \\ du &= \frac{1}{x}\cos(\ln(x))\ dx \quad v = x. \end{aligned}$$

Which gives us:

$$\int \sin(\ln(x))\ dx = x\sin(\ln(x)) - \int \cos(\ln(x))\ dx.$$

Which leaves us in no better shape than the original integral, so we apply the theorem once more:

$$\int \cos(\ln(x))$$

$$\begin{aligned} u &= \cos(\ln(x)) \quad dv = dx \\ du &= -\frac{1}{x}\sin(\ln(x))\ dx \quad v = x. \end{aligned}$$

Thus we have:

$$\int \cos(\ln(x)) = x\cos(\ln(x)) - \int -\sin(\ln(x))\ dx.$$

At this point, we have:

$$\begin{aligned}\int \sin\left(\ln\left(x\right)\right)\,dx &= x\sin\left(\ln\left(x\right)\right) - \left(x\cos\left(\ln\left(x\right)\right) - \int -\sin\left(ln(x)\right)\,dx\right)\\ &= x\sin\left(\ln\left(x\right)\right) - \left(x\cos\left(\ln\left(x\right)\right) + \int \sin\left(ln(x)\right)\,dx\right)\\ &= x\sin\left(\ln\left(x\right)\right) - x\cos\left(\ln\left(x\right)\right) - \int \sin\left(ln(x)\right)\,dx\end{aligned}$$

.

The last integral is now the same as the original. It may seem that we have simply gone in a circle, but now we can actually evaluate the integral. To see how to do this more clearly, substitute:

$$I = \int \sin\left(\ln\left(x\right)\right)\,dx.$$

Thus, the equation becomes:

$$\begin{aligned}I &= x\sin\left(\ln\left(x\right)\right) - x\cos\left(\ln\left(x\right)\right) - I\\ 2I &= x\sin\left(\ln\left(x\right)\right) - x\cos\left(\ln\left(x\right)\right)\\ I &= \frac{1}{2}x\sin\left(\ln\left(x\right)\right) - \frac{1}{2}x\cos\left(\ln\left(x\right)\right).\end{aligned}$$

Substituting back in for I we get:

$$\int \sin\left(\ln\left(x\right)\right)\,dx = \frac{1}{2}x\sin\left(\ln\left(x\right)\right) - \frac{1}{2}x\cos\left(\ln\left(x\right)\right) + C.$$

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Integration by parts for definite integrals

Now that we have used integration by parts successfully to evaluate indefinite integrals, we turn our attention to definite integrals. The integration technique is really the same, only we add a step to evaluate the integral at the upper and lower limits of integration.

Theorem 2: Integration by Parts for Definite Integrals

Let $u = f(x)$ and $v = g(x)$ be functions with continuous derivatives on $[a, b]$. Then:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$$

3.2 Trigonometric Integrals

In this section we look at how to integrate a variety of products of trigonometric functions. These integrals are called trigonometric integrals. They are an important part of the integration technique called trigonometric substitution, which is featured in Trigonometric Substitution. This technique allows us to convert algebraic expressions that we may not be able to integrate into expressions involving trigonometric functions, which we may be able to integrate using the techniques described in this section. In addition, these types of integrals appear frequently when we study polar, cylindrical, and spherical coordinate systems later. Let's begin our study with products of $\sin x$ and $\cos x$.

Integrating $\cos^j x \sin x$

In this case, we can preform a simple u-substation, where we let $u = \cos x$, and from there we can evaluate.

Example 5: Evaluate

$$\int \cos^5 x \sin x \; dx$$
$$= -\int u^5 \; du$$
$$-\frac{1}{6}u^6 + C$$
$$= -\frac{1}{6}\cos^6 x + C.$$

Integrating $\cos^j x \sin^k x$ when k is odd

In this case, we can use the trigonometric identity: $\sin^2 x = 1 - \cos^2 x$ to rewrite the expression such that using a u-substation will work. In general:

$$\int \cos^j x \sin^k x \; dx \quad \text{s.t k = 2l+1 l} \in \text{Z} \qquad \textit{dotted}$$
$$= \int \cos^j x(1-\cos^2 k)^{\frac{k-1}{2}} \sin(x).$$

Example 6: Evaluate

$$\int \cos^2 x \sin^5 x \; dx$$
$$= \int \cos^2 x(1-\cos^2 x)^{\frac{5-1}{2}} \sin x \; dx$$
$$= \int \cos^2 x(1-\cos^2 x)^2 \sin x \; dx$$
$$= etc....$$

Note:-

This fact also holds for $\int \sin^j x \cos^k x \; for \; k = 2l+1, \; l \in \mathbb{Z}$

Integrating even powers of $\sin x$

In the next example, we see the strategy that must be applied when there are only even powers of $\sin(x)$ and $\cos(x)$. For integrals of this type, the identities

$$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x) = \frac{1-\cos(2x)}{2}$$

and

$$\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x) = \frac{1+\cos(2x)}{2}$$

are invaluable. These identities are sometimes known as power-reducing identities and they may be derived from the double-angle identity $\cos(2x) = \cos^2(x) - \sin^2(x)$ and the Pythagorean identity $\cos^2(x) + \sin^2(x) = 1$

Example 7: Evaluate

$$\int \sin^2 x \; dx.$$

By the identity described above, we can derive:

$$\begin{aligned} \cos 2x &= 1 - 2\sin^2 x \\ \sin^2 x &= \frac{1}{2} - \frac{1}{2}\cos 2x. \end{aligned}$$

Thus we have:

$$\begin{aligned} &\int \frac{1}{2} - \frac{1}{2}\cos 2x \; dx \\ &= \frac{1}{2}x - \frac{1}{4}\sin 2x + C. \end{aligned}$$

Problem-Solving Strategy: Integrating Products and Powers of sin x and cos x

To integrate

$$\int \cos^j(x)\sin^k(x)\,dx$$

use the following strategies:

1. If k is odd, rewrite $\sin^k(x)$ as $\sin^{k-1}(x)\sin(x)$ and use the identity $\sin^2(x) = 1-\cos^2(x)$ to rewrite $\sin^{k-1}(x)$ in terms of $\cos(x)$. Integrate using the substitution $u = \cos(x)$. This substitution makes $du = -\sin(x)\,dx$.

2. If j is odd, rewrite $\cos^j(x)$ as $\cos^{j-1}(x)\cos(x)$ and use the identity $\cos^2(x) = 1-\sin^2(x)$ to rewrite $\cos^{j-1}(x)$ in terms of $\sin(x)$. Integrate using the substitution $u = \sin(x)$. This substitution makes $du = \cos(x)\,dx$. (Note: If both j and k are odd, either strategy 1 or strategy 2 may be used.)

3. If both j and k are even, use
$$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$$
and
$$\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x).$$
After applying these formulas, simplify and reapply strategies 1 through 3 as appropriate.