

Homework/Worksheet 6 - Due: Wednesday, October 18

1. Evaluate the following integrals using trigonometric substitution

(a) $\int \frac{x^2}{\sqrt{1-x^2}} dx$

(b) $\int \sqrt{x^2 + 9} dx$

(c) $\int \frac{\sqrt{x^2-25}}{x} dx$

(d) $\int \frac{1}{(x^2-9)^{\frac{3}{2}}} dx$

(e) $\int \frac{x^2}{\sqrt{x^2+4}} dx$

(f) $\int_{-1}^1 (1-x^2)^{\frac{3}{2}} dx$

1.a

Trig sub:

Thus we have:

$$\begin{aligned} x &= \sin \theta \\ dx &= \cos \theta d\theta. \end{aligned}$$

$$\begin{aligned} &\int \frac{\sin^2 \theta \cos \theta}{\sqrt{1 - \sin^2 \theta}} d\theta \\ &= \int \sin^2 \theta d\theta. \end{aligned}$$

Interlude. By double angle formulas, we can solve for $\sin^2 \theta$ to get an easier integrand

$$\begin{aligned} \cos 2\theta &= 1 - 2\sin^2 \theta \\ \sin^2 \theta &= \frac{1}{2} - \frac{1}{2} \cos 2\theta. \end{aligned}$$

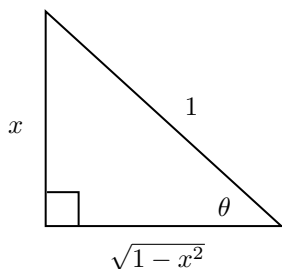
Following this, we have:

$$\begin{aligned} &\int \frac{1}{2} - \frac{1}{2} \cos 2\theta d\theta \\ &= \int \frac{1}{2} d\theta - \frac{1}{2} \int \cos 2\theta d\theta \\ &= \frac{1}{2} \theta - \frac{1}{4} \sin 2\theta + C. \end{aligned}$$

Reference triangle:

By the reference triangle we have:

$$\int \frac{x^2}{\sqrt{1-x^2}} = \frac{1}{2} \sin^{-1}(x) - \frac{1}{4} \sin(2 \sin^{-1}(x)) + C.$$



1.b

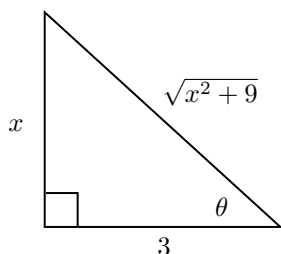
$$x = 3 \tan \theta$$

$$dx = 3 \sec^2 \theta \, d\theta.$$

Thus, we have:

$$\begin{aligned} \int \sqrt{x^2 + 9} \, dx &= \int \sqrt{9 \tan^2 \theta + 9} \, 3 \sec^2 \theta \, d\theta \\ &= 9 \int \sqrt{\tan^2 \theta + 1} \sec^2 \theta \, d\theta \\ &= 9 \int \sec^3 \theta \, d\theta \\ &= 9 \left[\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \int \sec \theta \, d\theta \right] \\ &= 9 \left[\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| \right] + C \\ &= \frac{9}{2} \sec \theta \tan \theta + \frac{9}{2} \ln |\sec \theta + \tan \theta| + C \end{aligned}$$

Reference Triangle:



$$\begin{aligned} \therefore \int \sqrt{x^2 + 9} \, dx &= \frac{9}{2} \cdot \frac{1}{3} \sqrt{x^2 + 9} \frac{1}{3} x + \frac{9}{2} \ln \left| \frac{1}{3} \sqrt{x^2 + 9} + \frac{1}{3} x \right| + C \\ &= \frac{3}{2} \sqrt{x^2 + 9} \frac{1}{3} x + \frac{9}{2} \ln \left| \frac{1}{3} \sqrt{x^2 + 9} + \frac{1}{3} x \right| + C \end{aligned}$$

1.c

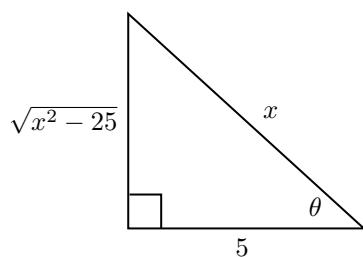
$$x = 5 \sec \theta$$

$$dx = 5 \sec \theta \tan \theta \, d\theta.$$

Thus we have:

$$\begin{aligned} \int \frac{\sqrt{x^2 - 25}}{x} \, dx &= \int \frac{\sqrt{25 \sec^2 \theta - 25}}{5 \sec \theta} \, 5 \sec \theta \tan \theta \, d\theta \\ &= 5 \int \frac{\tan \theta}{\sec \theta} \sec \theta \tan \theta \, d\theta \\ &= 5 \int \tan^2 \theta \, d\theta \\ &= 5 \int \sec^2 \theta - 1 \, d\theta \\ &= 5 \tan \theta - 5\theta + C. \end{aligned}$$

Reference Triangle



$$\begin{aligned}\therefore \int \frac{\sqrt{x^2 - 25}}{x} dx &= 5 \left(\frac{\sqrt{x^2 - 25}}{5} \right) - 5 \sec^{-1} \left(\frac{1}{5} x \right) + C \\ &= \sqrt{(x-5)(x+5)} - 5 \sec^{-1} \left(\frac{1}{5} x \right) + C.\end{aligned}$$

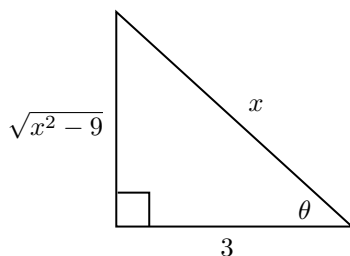
1.d

$$\begin{aligned}x &= 3 \sec \theta \\ dx &= 3 \sec \theta \tan \theta d\theta.\end{aligned}$$

Thus we have:

$$\begin{aligned}\int \frac{1}{(x^2 - 9)^{\frac{3}{2}}} dx &= \int \frac{3 \sec \theta \tan \theta}{\sqrt{(9 \sec^2 \theta - 9)^3}} \\ &= \int \frac{3 \sec \theta \tan \theta}{\sqrt{729(\sec^2 \theta - 1)^3}} d\theta \\ &= \int \frac{3 \sec \theta \tan \theta}{\sqrt{729 \tan^6 \theta}} d\theta \\ &= \int \frac{3 \sec \theta \tan \theta}{27 \tan^3 \theta} \\ &= \frac{1}{9} \int \frac{\sec \theta}{\tan^2 \theta} d\theta \\ &= \frac{1}{9} \int \frac{\frac{1}{\cos \theta}}{\frac{\sin^2 \theta}{\cos^2 \theta}} d\theta \\ &= \frac{1}{9} \int \frac{\cos \theta}{\sin^2 \theta} d\theta \\ &= \frac{1}{9} \int u^{-2} du \\ &= -\frac{1}{9u} + C \\ &= -\frac{1}{9 \sin \theta} + C \\ &= -\frac{1}{9} \csc \theta + C.\end{aligned}$$

Reference Triangle:



$$\begin{aligned}\therefore \int \frac{1}{(x^2 - 9)^{\frac{3}{2}}} dx &= -\frac{x}{9\sqrt{x^2 - 9}} + C \\ &= -\frac{x\sqrt{(x-3)(x+3)}}{9(x-3)(x+3)} + C.\end{aligned}$$

1.e

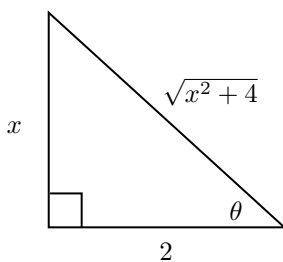
$$x = 2 \tan \theta$$

$$dx = 2 \sec^2 \theta \, d\theta.$$

Thus we have:

$$\begin{aligned} \int \frac{x^2}{\sqrt{x^2+4}} \, dx &= \int \frac{4 \tan^2 \theta}{2 \sec \theta} \cdot 2 \sec^2 \theta \, d\theta \\ &= 4 \int \tan^2 \theta \sec \theta \, d\theta \\ &= 4 \int (\sec^2 \theta - 1) \sec \theta \, d\theta \\ &= 4 \int \sec^3 \theta - \sec \theta \, d\theta \\ &= 4 \left[- \int \sec \theta \, d\theta + \int \sec^3 \theta \, d\theta \right] \\ &= 4 \left[- \ln |\sec \theta + \tan \theta| + \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \int \sec \theta \, d\theta \right] \\ &= 4 \left[- \ln |\sec \theta + \tan \theta| + \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| \right] + C \\ &= 4 \left[- \frac{1}{2} \ln |\sec \theta + \tan \theta| + \frac{1}{2} \sec \theta \tan \theta \right] + C \\ &= -2 \ln |\sec \theta + \tan \theta| + 2 \sec \theta \tan \theta + C \end{aligned}$$

Reference Triangle:



$$\begin{aligned} &-2 \ln \left| \frac{1}{2} \sqrt{x^2+4} + \frac{1}{2} x \right| + \frac{1}{2} \sqrt{x^2+4} x + C \\ &= -2 \ln \left(\frac{1}{2} |\sqrt{x^2+4} + x| \right) + \frac{1}{2} \sqrt{x^2+4} x + C \\ &= \frac{1}{2} \left(-4 \ln \left(\frac{1}{2} |\sqrt{x^2+4} + x| \right) + \sqrt{x^2+4} x \right) + C. \end{aligned}$$

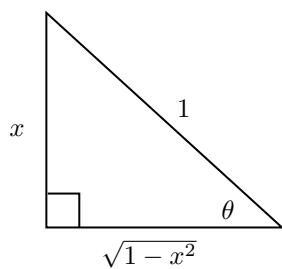
1.f

$$\begin{aligned}x &= \sin \theta \\dx &= \cos \theta \, d\theta.\end{aligned}$$

Thus:

$$\begin{aligned}\int_{-1}^1 (1-x^2)^{\frac{3}{2}} dx &= \int_{-1}^1 \sqrt{(1-\sin^2 \theta)^3} \cos \theta \, d\theta \\&= \int_{-1}^1 \sqrt{\cos^6 \theta} \cos \theta \, d\theta \\&= \int_{-1}^1 \cos^4 \theta \, d\theta \\&= \frac{1}{4} \cos^3 \theta \sin \theta + \frac{3}{4} \int \cos^2 \theta \, d\theta \\&= \frac{1}{4} \cos^3 \theta \sin \theta + \frac{3}{4} \left[\frac{1}{2} \cos \theta \sin \theta + \frac{1}{2} \int d\theta \right] \\&= \frac{1}{4} \cos^3 \theta \sin \theta + \frac{3}{8} \cos \theta \sin \theta + \frac{3}{8} \theta + C.\end{aligned}$$

Reference Triangle:



Consequently...

$$\begin{aligned}2 \int_0^1 (1-x^2)^{\frac{3}{2}} dx &= 2 \left(\frac{1}{4} (1-x^2)^{\frac{3}{2}} x + \frac{3}{8} (1-x^2)^{\frac{1}{2}} x + \frac{3}{8} \sin^{-1} x \right) \Big|_0^1 \\&= 2 \left(\frac{3}{8} \sin^{-1} 1 \right) \\&= 2 \left(\frac{3\pi}{16} \right) \\&= \frac{3\pi}{8}.\end{aligned}$$

Evaluate the following integrals using partial fractions

(a) $\int \frac{dx}{x^2 - 5x + 6}$

(b) $\int \frac{dx}{x(x-1)(x-2)(x-3)}$

(c) $\int \frac{2}{(x+2)^2(2-x)} dx$

(d) $\int \frac{x^3 + 6x^2 + 3x + 6}{x^3 + 2x^2} dx$

(e) $\int \frac{2}{(x-4)(x^2 + 2x + 6)} dx$

(f) $\int \frac{\sin x}{1 - \cos^2 x} dx$

2.a

$$\begin{aligned} \int \frac{dx}{x^2 - 5x + 6} \\ \frac{1}{(x-2)(x-3)} &= \frac{A}{(x-2)} + \frac{B}{(x-3)} \\ 1 &= A(x-3) + B(x-2) \\ A = -1 \quad B &= 1. \end{aligned}$$

Thus we have:

$$\begin{aligned} \int \frac{-1}{(x-2)} + \int \frac{1}{(x-3)} \\ = -\ln|x-2| + \ln|x-3| + C. \end{aligned}$$

2.b

$$\begin{aligned} \int \frac{dx}{x(x-1)(x-2)(x-3)} \\ \frac{1}{x(x-1)(x-2)(x-3)} &= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2} + \frac{D}{x-3} \\ 1 &= A(x-1)(x-2)(x-3) + Bx(x-2)(x-3) + Cx(x-1)(x-3) + Dx(x-1)(x-2) \\ A = -\frac{1}{6}, \quad B &= \frac{1}{2}, \quad C = -\frac{1}{2}, \quad D = \frac{1}{6} \quad (\text{By plugging in zeros}). \end{aligned}$$

So we have:

$$\begin{aligned} \int -\frac{1}{6x} + \frac{1}{2(x-1)} - \frac{1}{2(x-2)} + \frac{1}{6(x-3)} dx \\ = -\frac{1}{6} \int \frac{1}{x} dx + \frac{1}{2} \int \frac{1}{(x-1)} dx - \frac{1}{2} \int \frac{1}{(x-2)} dx + \frac{1}{6} \int \frac{1}{(x-3)} dx \\ = -\frac{1}{6} \ln|x| + \frac{1}{2} \ln|x-1| - \frac{1}{2} \ln|x-2| + \frac{1}{6} \ln|x-3| + C. \end{aligned}$$

2.c

$$\begin{aligned}
& \int \frac{2}{(x+2)^2(2-x)} dx \\
& \frac{2}{(x+2)^2(2-x)} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2} + \frac{C}{(2-x)} \\
& 2 = A(x+2)(2-x) + B(2-x) + C(x+2)^2 \\
& B = \frac{1}{2}, \quad C = \frac{1}{8} \quad (\text{Plugging in zeros}) \\
& 2 = -Ax^2 + 4A + 2B - Bx + Cx^2 + 4Cx + 4C \\
& 2 = (-A+C)x^2 + (C-B)x + (4A+B+4C) \\
& -A+C=0 \\
& A = \frac{1}{8} \\
& \text{Thus : } A = \frac{1}{8}, \quad B = \frac{1}{2}, \quad C = \frac{1}{8}.
\end{aligned}$$

So we have the integral:

$$\begin{aligned}
& \int -\frac{1}{8(x+2)} + \frac{1}{2(x+2)^2} + \frac{1}{8(2-x)} dx \\
& \frac{1}{8} \int \frac{1}{(x+2)} dx + \frac{1}{2} \int \frac{1}{(x+2)^2} dx + \frac{1}{8} \int \frac{1}{(2-x)} dx \\
& = \frac{1}{8} \ln|x+2| - \frac{1}{2(x-2)} + \frac{1}{8} \ln|2-x| + C.
\end{aligned}$$

1.d

$$\begin{aligned}
& \int \frac{x^3 + 6x^2 + 3x + 6}{x^3 + 2x^2} dx \\
& \int dx + \int \frac{4x^2 + 3x + 6}{x^3 + 2x^2} dx \quad (\text{After long division}) \\
& = x + \int \frac{4x^2 + 3x + 6}{x^3 + 2x^2} dx \\
& = x + \int \frac{4x^2 + 3x + 6}{x^2(x+2)} dx \\
& \frac{4x^2 + 3x + 6}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x+2)} \quad (\text{Omitting evaluation of first integral for now...}) \\
& 4x^2 + 3x + 6 = Ax(x+2) + B(x+2) + Cx^2 \\
& 4x^2 + 3x + 6 = Ax^2 + 2Ax + Bx + 2B + Cx^2 \\
& (A+C)x^2 + (2A+B)x + 2B \\
& A+C=4 \quad (\text{Begin system...}) \\
& 2A+B=3 \\
& 2B=6 \quad (\text{End system...}) \\
& A=0, \quad B=3, \quad C=4.
\end{aligned}$$

Thus we have the integral:

$$\begin{aligned} \int \frac{3}{x^2} dx + \frac{4}{(x+2)} dx \\ = -\frac{3}{x} + 4 \ln |x+2|. \end{aligned}$$

Bringing back the first integral that we evaluated we get the full solution of:

$$x - \frac{3}{x} + 4 \ln |x+2| + C.$$

1.e

$$\begin{aligned} \int \frac{2}{(x-4)(x^2+2x+6)} dx \\ \frac{2}{(x-4)(x^2+2x+6)} &= \frac{A}{(x-4)} + \frac{Bx+C}{(x^2+2x+6)} \\ 2 &= A(x^2+2x+6) + (Bx+C)(x-4) \\ A &= \frac{1}{15} \quad (\text{Plugging in } 4) \\ 2 &= Ax^2 + 2Ax + 6A + Bx^2 - 4Bx - 4C \\ 2 &= (A+B)x^2 + (2A-4B+C)x + (6A-4C) \\ A+B &= 0 \quad (\text{Begin system...}) \\ 2A-4B+C &= 0 \\ 6A-4C &= 2 \quad (\text{End system...}) \\ B &= -\frac{1}{15} \\ C &= -\frac{1}{5} \\ \text{Thus : } A &= \frac{1}{15}, B = -\frac{1}{15}, C = -\frac{1}{5}. \end{aligned}$$

By this we have the integral:

$$\begin{aligned}
& \int \frac{1}{15(x-4)} + \frac{\left(-\frac{1}{15}\right)x + \left(-\frac{1}{5}\right)}{x^2 + 2x + 6} dx \\
&= \int \frac{1}{15(x-4)} dx + \int \frac{-x-3}{15(x^2 + 2x + 6)} dx \\
&= \int \frac{1}{15(x-4)} dx - \int \frac{x+3}{15(x^2 + 2x + 6)} dx \\
I_1 &= \frac{1}{15} \int \frac{1}{x-4} dx \\
&= \frac{1}{15} \ln |x-4| \\
I_2 &= -\frac{1}{15} \int \frac{x}{x^2 + 2x + 6} + \frac{3}{x^2 + 2x + 6} dx \\
&= -\frac{1}{15} \int \frac{x}{(x+1)^2 + 5} + \frac{3}{(x+1)^2 + 5} \quad (\text{By completing the square}) \\
&= -\frac{1}{15} \int \frac{u-1}{u^2 + 5} + \frac{3}{u^2 + 5} du \quad (\text{Where } u = x+1) \\
&= -\frac{1}{15} \int \frac{u}{u^2 + 5} - \frac{1}{u^2 + (\sqrt{5})^2} + \frac{3}{u^2 + (\sqrt{5})^2} du \\
&= -\frac{1}{15} \left[\frac{1}{2} \ln |u^2 + 5| - \frac{1}{\sqrt{5}} \tan^{-1} \frac{u}{\sqrt{5}} + \frac{3}{\sqrt{5}} \tan^{-1} \frac{u}{\sqrt{5}} \right] \\
&= -\frac{1}{15} \left[\frac{1}{2} \ln |(x+1)^2 + 5| - \frac{1}{\sqrt{5}} \tan^{-1} \frac{x+1}{\sqrt{5}} + \frac{3}{\sqrt{5}} \tan^{-1} \frac{x+1}{\sqrt{5}} \right] + C \\
&= -\frac{1}{15} \left[\frac{1}{2} \ln |(x+1)^2 + 5| + \frac{2}{\sqrt{5}} \tan^{-1} \frac{x+1}{\sqrt{5}} \right] + C \\
&\therefore \int \frac{2}{(x-4)(x^2 + 2x + 6)} dx \\
&= \frac{1}{15} \ln |x-4| - \frac{1}{15} \left[\frac{1}{2} \ln |(x+1)^2 + 5| + \frac{2}{\sqrt{5}} \tan^{-1} \frac{x+1}{\sqrt{5}} \right] + C \\
&= \frac{1}{15} \ln |x-4| - \frac{1}{30} \ln |(x+1)^2 + 5| - \frac{2}{15\sqrt{5}} \tan^{-1} \frac{x+1}{\sqrt{5}} + C.
\end{aligned}$$

2.f

$$\begin{aligned}
& \int \frac{\sin x}{1 - \cos^2 x} dx \\
&= - \int \frac{du}{1 - u^2} \quad (\text{Let } u = \cos x) \\
&= - \int \frac{du}{(1-u)(1+u)} du \\
&\frac{1}{(1-u)(1+u)} = \frac{A}{(1-u)} + \frac{B}{(1+u)} \\
1 &= A(1+u) + B(1-u) \\
A &= \frac{1}{2}, \quad B = \frac{1}{2} \quad (\text{By zeros}).
\end{aligned}$$

Thus:

$$\begin{aligned} & - \int \frac{1}{2(1-u)} + \frac{1}{2(1+u)} du \\ & - \left[-\frac{1}{2} \ln |1-u| + \frac{1}{2} \ln |1+u| \right] + C \\ & = \frac{1}{2} \ln |1-\cos x| - \frac{1}{2} \ln |1+\cos x| + C. \end{aligned}$$