Calculus II
Chapter 2: Applications of Integration

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Applications of Integration
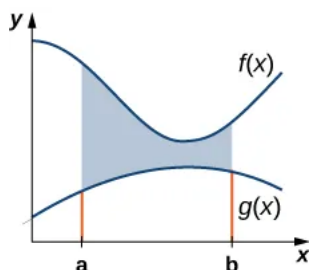
2.1: Areas between Curves

Area of a Region between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$ such that $f(x) \geqslant g(x)$ on $[a, b]$. We want to find the area between the graphs of the functions, as shown in the following figure.

As we did before, we are going to partition the interval on the x-axis and approximate the area between the graphs of the functions with rectangles. So, for $i = 0, 1, 2, \ldots, n$, let $P = \{x_i\}$ be a regular partition of $[a, b]$. Then, for $i = 1, 2, \ldots, n$, choose a point $x_i^* \in [x_{i-1}, x_i]$, and on each interval $[x_{i-1}, x_i]$ construct a rectangle that extends vertically from $g(x_i^*)$ to $f(x_i^*)$. Figure 2.3(a) shows the rectangles when x_i^* is selected to be the left endpoint of the interval and $n = 10$.

(a)   (b)
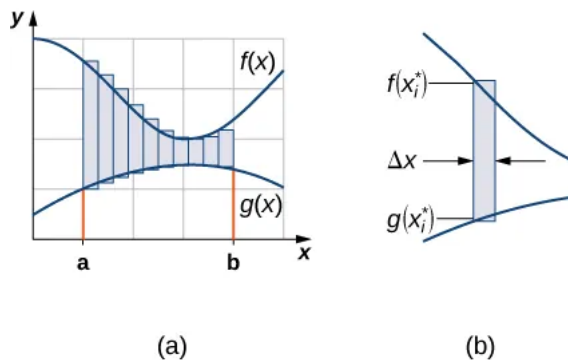
The height of each individual rectangle is $f(x_i^*) - g(x_i^*)$ and the width of each rectangle is Δx. Adding the areas of all the rectangles, we see that the area between the curves is approximated by

$$A \approx \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \, \Delta x.$$

This is a Riemann sum, so we take the limit as $n \to \infty$ and we get

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \, \Delta x = \int_a^b [f(x) - g(x)] \, dx.$$

These findings are summarized in the following theorem.

Theorem 1. Let $f(x)$ and $g(x)$ be continuous functions such that $f(x) \geqslant g(x)$ over an interval $[a, b]$. Let R denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of R is given by

$$A = \int_a^b [f(x) - g(x)] \, dx.$$

Quite often, though, we want to define our interval of interest based on where the graphs of the two functions intersect. This is illustrated in the following example.

Example 1. If R is the region bounded above by the graph of the function $f(x) = 9 - \left(\frac{x}{2}\right)^2$ and below by the graph of the function $g(x) = 6 - x$, find the area of region R.
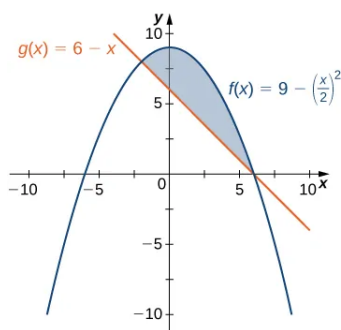
To find our limits of integration, we first need to find where the two functions intersect.

$$= 9 - \left(\frac{x}{2}\right)^2 = 6 - x$$

$$= 9 - \frac{x^2}{4} = 6 - x$$

$$= \frac{36 - x^2}{4} = 6 - x$$

$$= 36 - x^2 = 24 - 4x$$

$$= -x^2 + 4x + 12$$

$$= -(x^2 - 4x - 12) = 0$$

$$= -(x - 6)(x + 2)$$

$$= x = 6, -2.$$

Thus, we need to integrate from x=-2 to 6

By Theorem 1: $A = \int_a^b [f(x) - g(x)] \, dx$, we find that:

$$A = \frac{64}{3} \ Units^2.$$

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Areas of Compound Regions

So far, we have required $f(x) \geqslant g(x)$ over the entire interval of interest, but what if we want to look at regions bounded by the graphs of functions that cross one another? In that case, we modify the process we just developed by using the absolute value function.

Theorem 2. Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$. Let R denote the region between the graphs of $f(x)$ and $g(x)$, and be bounded on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of R is given by

$$\int_a^b |f(x) - g(x)| \ dx.$$

In practice, applying this theorem requires us to break up the interval [a,b] and evaluate several integrals, depending on which of the function values is greater over a given part of the interval. We study this process in the following example.

Example 2. If R is the region between the graphs of the functions $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[0, \pi]$, find the area of region R.

By theorem 2, we can find the area A, in the region R with the following integral:

$$\int_a^b |f(x) - g(x)| \ dx$$

$$= \int_0^\pi |\sin x - \cos x| \ dx$$

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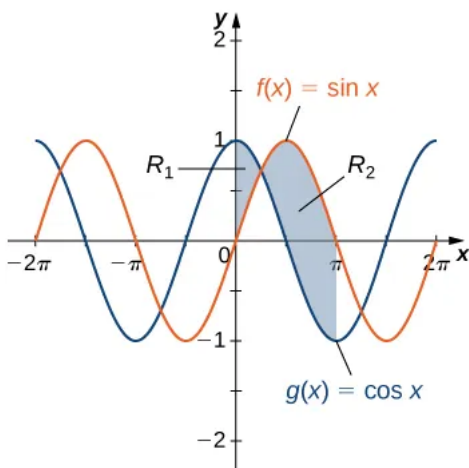
First, we need to find where the sign of the expression changes:

$$= \sin x - \cos x = 0$$

$$= \frac{\sin x}{\cos x} = \frac{\cos x}{\cos x}$$

$$= \tan x = 1$$

$$x = \tan^{-1} 1, \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$x = \frac{\pi}{4}.$$

Thus:

$$\sin x - \cos x = \begin{cases} \sin x - \cos x & \text{if } x \geqslant \frac{\pi}{4} \\ -(\sin x - \cos x) & \text{if } x < \frac{\pi}{4} \end{cases} \quad (1)$$

So our integral is now:

$$\int_0^{\frac{\pi}{4}} -(\sin x - \cos x)dx + \int_{\frac{\pi}{4}}^{\pi} \sin x - \cos x \ dx$$

$$= \int_0^{\frac{\pi}{4}} \cos x - \sin x \ dx + \int_{\frac{\pi}{4}}^{\pi} \sin x - \cos x \ dx$$

$$Where: \ I_1 = \int_0^{\frac{\pi}{4}} \cos x - \sin x \ dx$$

$$I_2 = \int_{\frac{\pi}{4}}^{\pi} \sin x - \cos x \ dx$$

$$A = I_1 + I_2.$$

Computing both integrals we get:

$$I_1 = \int_0^{\frac{\pi}{4}} \cos x - \sin x \ dx$$

$$= \sin x + \cos x \Big]_{\frac{\pi}{4}}^{0}$$

$$= \left(\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) \right) - \left(\sin(0) + \cos 0 \right)$$

$$= \sqrt{2} - 1.$$

$$= I_2 = \int_{\frac{\pi}{4}}^{\pi} \sin x - \cos x \ dx$$

$$= -\cos x - \sin x \Big]_{\pi}^{\frac{\pi}{4}}$$

$$= -\Big[\cos x + \sin x \Big]_{\frac{\pi}{4}}^{\pi}$$

$$= -\left[\left(\cos \pi + \sin \pi \right) - \left(\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right) \right) \right]$$

$$= -\left[-1 - \sqrt{2} \right]$$

$$= 1 + \sqrt{2}.$$

Thus:

$$A = I_1 + I_2$$

$$= \sqrt{2} - 1 + 1 + \sqrt{2}$$

$$= 2\sqrt{2} \quad Units^2.$$

Finding the Area of a Complex Region

Definition 1. a "complex region" between curves usually refers to an area that is not easily described by a single, continuous function over the interval of interest.

Consider the following example:

Example 3. If R is the region between the graphs of the functions $f(x) = x^2$ and $g(x) = 2 - x$ over the interval $[0, 2]$, find the area of region R.
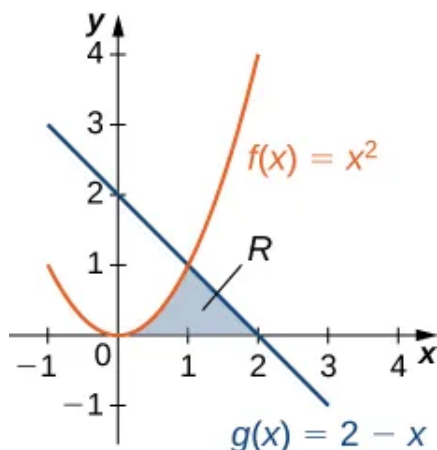
As with Example 2.3, we need to divide the interval into two pieces. The graphs of the functions intersect at $x = 1$ (set $f(x) = g(x)$ and solve for x), so we evaluate two separate integrals: one over the interval $[0, 1]$ and one over the interval $[1, 2]$.

Over the interval $[0, 1]$, the region is bounded above by $f(x) = x^2$ and below by the x-axis, so we have

$$A_1 = \int_0^1 x^2 \, dx = \left. \frac{x^3}{3} \right|_0^1 = \frac{1}{3}.$$

Over the interval $[1, 2]$, the region is bounded above by $g(x) = 2 - x$ and below by the x-axis, so we have

$$A_2 = \int_2^1 (2 - x) \, dx = \left[2x - \frac{x^2}{2} \right]_2^1 = \frac{1}{2}.$$

Adding these areas together, we obtain

$$A = A_1 + A_2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

The area of the region is $\frac{5}{6}$ units^2.