Nate Warner MATH 353 Spring 2025

Problem set 4 - Due: Friday, Feb 7

1. For each set, either find the supremum (l.u.b) or explain why none exists
   (a) $\{-2, -\frac{1}{2}, 0, \frac{4}{5}, \frac{3}{2}\}$
   (b) $\{x : x \in \mathbb{R} \text{ and } 5x^2 < 45\}$
   (c) $\{.6, .66, .666, .6666, ...\}$
   (d) $\{x^2 : x \in \mathbb{R} \text{ and } x < 2\}$
   (e) $\{x^3 : x \in \mathbb{R} \text{ and } x < 2\}$
   (f) $\{\frac{x}{3+x} : x \in \mathbb{R} \text{ and } x > 0\}$
   (g) $\{x : x = d_{\mathbb{S}}(PQ) \text{ for some points } P, Q \text{ in } \mathbb{S}(\text{radius } 1)\ \}$
   (h) $\{x : x = d_{\mathbb{M}}(PQ) \text{ for some points } P, Q \text{ in } \mathbb{M}\}$
   (i) $\{x : x = d_{\mathbb{G}}(PQ) \text{ for some points } P, Q \text{ in } \mathbb{G}\}$

a.) The supremum is $\frac{3}{2}$

b.) We have

$$\begin{aligned} 5x^2 &< 45 \\ x^2 &< 9 \\ |x| &< 3 \\ -3 < x &< 3 \end{aligned}$$

Thus, $\{x : x \in \mathbb{R} \text{ and } 5x^2 < 45\}$ is precisely the open interval $(-3, 3)$ and the supremem is therefore 3

c.) The supremum is $\frac{2}{3} = 0.6666666667$

d.) The set $\{x^2 : x \in \mathbb{R} \text{ and } x < 2\}$ is the open interval $[0, 4)$. Thus, the supremum is 4

e.) The set $\{x^3 : x \in \mathbb{R} \text{ and } x < 2\}$ is the open interval $(-\infty, 8)$. Thus, the supremum is 8

f.) We find the limit as $x \to \infty$

$$\begin{aligned} \lim_{x\to\infty} \frac{x}{3+x} &= \lim_{x\to\infty} \frac{\frac{x}{x}}{\frac{3}{x} + \frac{x}{x}} \\ &= \lim_{x\to\infty} \frac{1}{\frac{3}{x} + 1} = \frac{1}{0+1} = 1 \end{aligned}$$

Therefore, the set has supremum one.

g.) The set of distances $\mathbb{D}$ for the spherical plane with radius r is bounded above by πr. Thus, the supremum is $\pi(1) = \pi$

h.) The set of distances $\mathbb{D}$ on the Minkowski plane is unbounded. Distance is given by

$$|x_1 - x_2| + |y_1 - y_2|$$

For $P(x_1, y_1), Q(x_2, y_2)$ which grows arbitrarily large as Q gets further from P. Therefoer, there is no supremum

i.) The set of distances $\mathbb{D}$ in the gap plane is also unbounded. Therefore there is no supremum

2. Prove proposition 4.1

Proposition. Let S be a nonempty set of real numbers that has a least upper bound $b \in \mathbb{R}$. Let $t \in \mathbb{R}$ such that $t < b$. Then, there exists some $s \in S$ such that $t < s \leqslant b$.

Proof. Assume S is a nonempty subset of the real numbers with a least upper bound b. Let $t \in \mathbb{R}$ such that $t < b$. Since b is a least upper bound of S, we have

$$\forall\, s \in S,\ \ s \leqslant b$$

Since $t < b$, t cannot be an upper bound for S. If it were, then that would contradict b being the least upper bound. Since t is not an upper bound of S, then this implies the existence of some $s \in S$ such that $t < s$. If this were not the case, then the negation which states, for all $s \in S$, $t \geqslant s$ would be true. Since the negation implies that t is an upper bound, which we know can't be the case, there must exist some $s \in S$ such that $t < s$.

Since $s \leqslant b$ for all $s \in S$, and we know that there exists some $s \in S$ such that $t < s$, there must be at least one s that satisfies

$$t < s \leqslant b$$

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3. Show that in the $\mathbb{H}$ model, $\mathbb{D} = [0, \infty)$. (Hints: Compute $d_{\mathbb{H}}(AB)$ (in terms of x) for $A = (0,0)$ and $B = (x, 0)$, $0 < x < 1$. Then use the fact that ln sends the interval $(1, \infty)$ onto $(0, \infty)$)

Fix A at $(0,0)$, let $B = (x, 0)$ for $0 < x < 1$. Thus, $M = (-1, 0)$, and $N = (1, 0)$. If the hyperbolic distance is given by

$$\ln\left(\frac{e(AN)e(BM)}{e(AM)e(BN)}\right)$$

Where $e(PQ)$ is the Euclidean distance $e(PQ) = |x_1 - x_2|\sqrt{1 + m^2}$ for all points $P(x_1, y_1), Q(x_2, y_2)$ on the line $y = mx + b$, then we have $e(PQ) = |x_1 - x_2|\sqrt{1 + 0^2} = |x_1 - x_2|$, which implies $e(AN) = |0 - 1| = 1$, and $e(AM) = |0 - (-1)| = 1$.

Also,

$$E(BM) = |x - (-1)| = |x + 1|$$
$$E(BN) = |x - 1|$$

Therefore,

$$d_{\mathbb{H}} = \ln\left(\frac{|x+1|}{|x-1|}\right)$$

Analyzing the input function of the natural log, we see the domain is $(-\infty, 1) \cup (1, \infty)$. We have

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{(x+1)^2}}{\sqrt{(x-1)^2}} &= \sqrt{\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^2} \\
&= \sqrt{\lim_{x\to\infty} \frac{x^2+2x+1}{x^2-2x+1}} \\
&= \sqrt{\lim_{x\to\infty} \frac{1+\frac{2}{x}+\frac{1}{x^2}}{1-\frac{2}{x}+\frac{1}{x^2}}} \\
&= \sqrt{\frac{1+0+0}{1-0+0}} = 1
\end{aligned}$$

Similarly, $\lim\limits_{x\to-\infty} \frac{|x+1|}{|x-1|} = 1$. Further, $\lim\limits_{x\to 1} |x-1| = 0$. Since the denominator tends to zero, we have $\lim\limits_{x\to 1} \frac{|x+1|}{|x-1|} = \infty$. Thus, the range of $f(x) = \frac{|x+1|}{|x-1|}$ is $(1, \infty)$. Therefore, the domain of $\ln\left(\frac{|x+1|}{|x-1|}\right)$ is $(1, \infty)$. Since we know $\ln : (1, \infty) \to [0, \infty)$, the set $\mathbb{D}$ is therefore $\{x : x \geqslant 0\} = [0, \infty)$

4. Let $\mathbb{P} = \{1, 2, 3\}, \mathbb{L} = \{\{1\}, \{1, 2\}, \{2, 3\}\}$. Define distance by

$$d(PQ) = P - Q$$

for all P, Q in $\mathbb{P}$ (equal or not).

(a) Tell which of the seven axioms fail to hold in this example, and explain why

(b) Find $\mathbb{D}$, the set of all distances (ie the image of d), and find ω, the supremum of $\mathbb{D}$

Remark. Distance axioms: For all points P, Q

1. $PQ \geqslant 0$
2. $PQ = 0 \iff P = Q$
3. $PQ = QP$

Incidence axioms:

1. At least two lines
2. Each line contains at least two different points
3. Each pair of points are together in at least one line
4. Each pair of points with $PQ < \omega$ are together in at most one line

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We have the distances

$$\begin{aligned}
d(1,1) &= 0 \\
d(1,2) &= -1 \\
d(1,3) &= -2 \\
d(2,1) &= 1 \\
d(2,2) &= 0 \\
d(2,3) &= -1 \\
d(3,3) &= 0 \\
d(3,2) &= 1 \\
d(3,1) &= 2
\end{aligned}$$

Thus, the distance function $d: \mathbb{P} \times \mathbb{P} \to \mathbb{R}$ is given by

	1	2	3
1	0	-1	-2
2	1	0	-1
3	2	1	0

a.) The first and third distance axioms do not hold. Observe that $d(12) = -1$ and $d(12) = -1$ while $d(21) = 1$

Moreover, the second and third incidence axioms do not hold. Observe that the line $\{1\}$ contains only one point, and the pair of points $\{1, 3\}$ are in no line.

b.) The set of distances is $\mathbb{D} = \{-2, -1, 0, 1, 2\}$, which has $\omega = \sup \mathbb{D} = 2$

5. Give an example of a plane (which satisfies the first seven axioms) in which all the points are collinear

The following plane satisfies the first seven axioms. Let $\mathbb{P} = \{A, B, C, D\}, \mathbb{L} = \{\{A, B, C, D\}, \{C, D\}\}$, with distance function

	A	B	C	D
A	0	1	4	3
B	1	0	3	4
C	4	3	0	1
D	3	4	1	0

Satisfies all three distance axioms and the four incidence axioms. Note that line $\{B, C\}$ is contained within $\{A, B, C, D\}$