Problem set 13 - Due: Monday, April 21

5. Given a proper angle $\underline{hk}$, h' the ray opposite h, that ray r bisects $\underline{hk}$ and ray s bisects $\underline{h'k}$, show that $rs = 90$ (Hint: use insertion to show that r-k-s)

Proof. Assume a proper angle $\underline{hk}$, with a ray r that bisects $\underline{hk}$, and a ray s that bisects $\underline{h'k}$. Since r bisects $\underline{hk}$, h-r-k, with $hr = rk = \frac{1}{2}hk$. Also, s bisects $\underline{kh'}$, k-s-h' and $ks = sh' = \frac{1}{2}kh'$

Consider the fan $\overrightarrow{hk}$, by the dual of Proposition 9.3, $\overrightarrow{hk} = \overline{hk} \cup \overline{kh'}$. So, since k-s-h', $s \in \overline{kh'}$. Thus, $s \in \overrightarrow{hk}$, specifically h-k-s.

So, h-r-k with h-k-s and the ROI implies h-r-k-s, which implies r-k-s. We have

$$\begin{aligned} rk + ks &= rs \\ \implies \frac{1}{2}hk + \frac{1}{2}kh' &= rs \\ \implies kh + kh' &= 2rs. \end{aligned}$$

By Theorem 14.1, kh and kh' are supplementary, so $hk + kh' = 180$. Thus,

$$\begin{aligned} kh + kh' &= 2rs \\ \implies 180 &= 2rs \\ \implies rs &= 90. \end{aligned}$$

Therefore, $rs = 90$ ■

12. Prove Theorem 14.9

Remark. *(Theorem 14.9)*: Every point of the perpendicular bisector of a segment is equidistant from the endpoints of the segment: $AX = BX$ for all X on the perpendicular bisector

Proof. Let $\overleftrightarrow{AB}$ be a line, at m be the perpendicular bisector of $\overleftrightarrow{AB}$ at the midpoint M of $\overline{AB}$. Let $X \in m$.

If $X \in \overleftrightarrow{AB}$, then $X = M$ or M^*, if $X = M$ $AX = BX$ by definition of the midpoint M of $\overline{AB}$. If $X = M^*$, then M-A-M^* and M-B-M^* by Thm 9.1, which implies M-A-X and M-B-X, which implies

$$\begin{aligned} MA + AX = MX = \omega, \\ MB + BX = MX = \omega. \end{aligned}$$

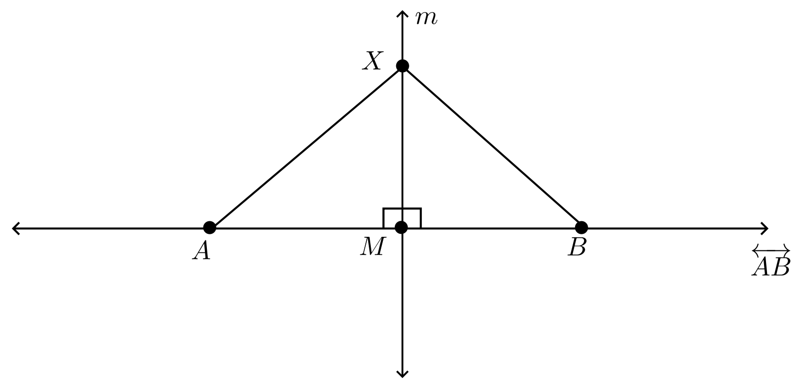
Thus,

$$\begin{aligned} AX = \omega - AM, \\ BX = \omega - MB. \end{aligned}$$

But, $AM = MB$ by definition of the midpoint M of $\overline{AB}$, so $AX = BX$

So, assume that $X \notin \overleftrightarrow{AB}$, so $X \neq M$ or M^*.

Since $X \in m$, $X \notin \overleftrightarrow{AB}$, A, M, X, B are noncollinear, so we have $\triangle AXM$ and $\triangle BXM$. By definition of perpendicular, $\angle AMX = \angle BMX = 90$.



Consider the correspondence $AXM \leftrightarrow BXM$ between the vertices of triangles $\triangle AXM$ and $\triangle BXM$. We have $\overline{MX} \cong \overline{MX}$, $\underline{\angle AMX} \cong \underline{\angle BMX}$, and $\overline{AM} \cong \overline{MB}$ (by definition of of the midpoint M of segment $\overline{AB}$), so $\triangle AXM \cong \triangle BXM$ by AX.SAS, which gives

$$\overline{AX} \cong \overline{BX}.$$

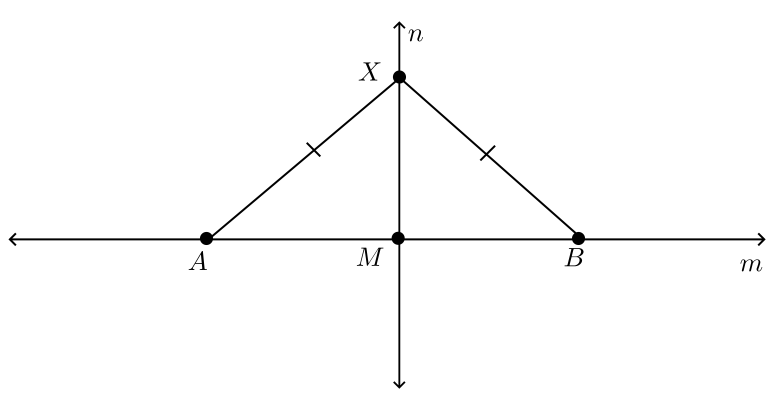
Therefore, $AX = BX$ ■

13. Prove Theorem 14.10

Remark. *Theorem 14.10 (converse of 14.9)*: Let $m = \overleftrightarrow{AB}$, suppose that line $n \neq m$ meets m at the midpoint M of $\overline{AB}$. Suppose that there is some point X on n, not on m, so that $AX = BX$. Then, $n \perp n$ at M

Proof. Assume that $m = \overleftrightarrow{AB}$, and that $n \neq m$ meets m at the midpoint M of $\overline{AB}$. Further assume that there is a point X on n, but not on m such that $AX = BX$

Since $X \in n, X \notin m$, so $X \neq M$ or M^*, thus A, M, B, X are noncollinear, and we have triangles $\triangle AMX$, and $\triangle BMX$.



We consider the correspondence $AMX \leftrightarrow BMX$ between the vertices of $\triangle AMX$ and $\triangle BMX$.

By Pons Asinorum, $\angle MAX = \angle MBX$, so $\underline{\angle MAX} \cong \underline{\angle MBX}$. Also, since $AX = BX$, we have $\overline{AX} \cong \overline{BX}$, and $\overline{AM} \cong \overline{MB}$ by definition of the midpoint M of $\overline{AB}$. So, by AX.SAS, $\triangle AMX \cong \triangle BMX$, and thus

$$\angle AMX = \angle BMX.$$

By definition of the midpoint M of segment $\overline{AB}$, A-M-B, so $\overrightarrow{MA}$ opposite to $\overrightarrow{MB}$, and $\overrightarrow{MA}\overrightarrow{MB} = \angle AMB =$ 180

Consider the ray $\overrightarrow{MX}$. By Theorem 11.8, $\overrightarrow{MA}$-$\overrightarrow{MX}$-$\overrightarrow{MB}$, so $\overrightarrow{MA}\overrightarrow{MX} + \overrightarrow{MX}\overrightarrow{MB} = \overrightarrow{MA}\overrightarrow{MB} = 180$, or equivalently, $\angle AMX + \angle BMX = 180$. Thus,

$$\begin{aligned} 2\angle AMX &= 180 \\ \implies \angle AMX &= \angle BMX = 90. \end{aligned}$$

Next, we consider the ray opposite to $\overrightarrow{MX}$, ray $\overrightarrow{MX}'$. By Ax.RR, there exists a point $C \in \overrightarrow{MX}'$ such that $MC < \omega$. So, by theorem 8.4, $\overrightarrow{MX}' = \overrightarrow{MC}$. By Theorem 11.8, we have that

$$\overrightarrow{MX}\text{-}\overrightarrow{MB}\text{-}\overrightarrow{MC}$$

and

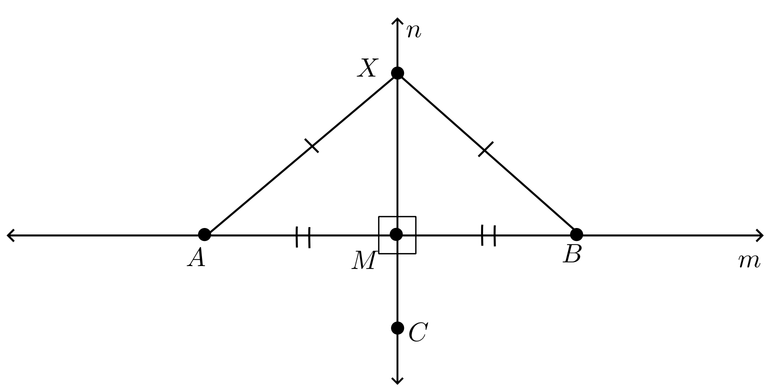
$$\overrightarrow{MX}\text{-}\overrightarrow{MA}\text{-}\overrightarrow{MC}.$$

So, $\overrightarrow{MX}\overrightarrow{MB} + \overrightarrow{MB}\overrightarrow{MC} = 180 \implies \angle BMX + \angle BMC = 180$, and $\overrightarrow{MX}\overrightarrow{MA} + \overrightarrow{MA}\overrightarrow{MC} = 180 \implies \angle AMX + \angle AMC = 180$. Since $\angle BMX = \angle AMX = 90$, we have

$$\begin{aligned} 90 + \angle BMC &= 180 \\ \implies \angle BMC &= 90, \end{aligned}$$

and

$$\begin{aligned} 90 + \angle AMC &= 180 \\ \implies \angle AMC &= 90. \end{aligned}$$

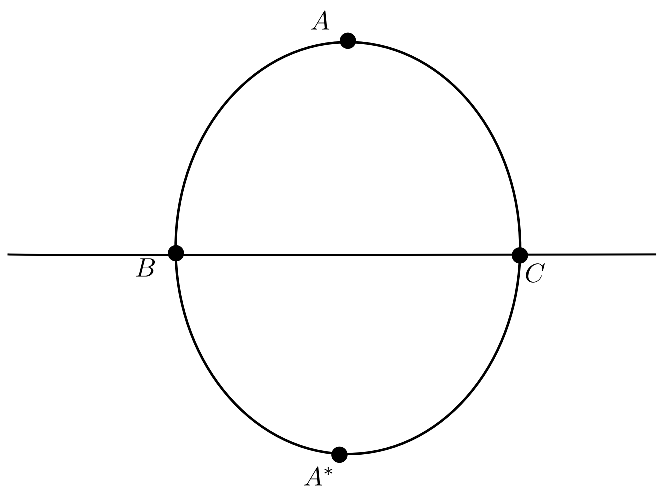


So, the four angles determined by the intersection of n with m are 90, thus $n \perp m$ at M by definition of perpendicular. ■

16. Suppose that A, B and C are noncollinear points such that $AB = AC = \frac{\omega}{2}$ $(\omega < \infty)$. Prove that A is a pole for $\overleftrightarrow{BC}$ (Hint: Consider $\triangle ABC$ and $\triangle A^*BC$)

Proof. Assume that A, B, C are noncollinear points such that $AB = AC = \frac{\omega}{2}$.

We first note that A^*, B, C are also noncollinear points, so we have $\triangle ABC$, $\triangle A^*BC$. Consider the correspondence $ABC \leftrightarrow A^*BC$ between the vertices of triangles $\triangle ABC$ and $\triangle A^*BC$



We have that $\overline{BC} \cong \overline{BC}$. Since $AB = \frac{\omega}{2}$, and A-B-A^* by Theorem 9.1, we have that

$$\begin{aligned} AB + BA^* &= AA^* = \omega \\ \implies BA^* &= \omega - AB \\ \implies BA^* &= \omega - \frac{\omega}{2} = \frac{\omega}{2}. \end{aligned}$$

Similarly, A-C-A^*, and $AC = \frac{\omega}{2}$, so

$$\begin{aligned} AC + CA^* &= AA^* = \omega \\ \implies CA^* &= \omega - AC \\ \implies CA^* &= \omega - \frac{\omega}{2} = \frac{\omega}{2}. \end{aligned}$$

So, $\overline{AB} \cong \overline{A^*B}$, and $\overline{AC} \cong \overline{A^*C}$. By Theorem 13.4 (SSS), we have that $\triangle ABC \cong \triangle A^*BC$. Thus, $\underline{\angle ABC} \cong \underline{\angle A^*BC}$, and $\underline{\angle ACB} \cong \underline{\angle A^*CB}$, which implies $\angle ABC = \angle A^*BC$, and $\angle ACB = \angle A^*CB$.

Consider the rays $\overrightarrow{BA}, \overrightarrow{BA^*}, \overrightarrow{BC}$. By Theorem 9.6, rays $\overrightarrow{BA}$ and $\overrightarrow{BA^*}$ are opposite, and Theorem 11.8 implies

$$\overrightarrow{BA}\text{-}\overrightarrow{BC}\text{-}\overrightarrow{BA^*}.$$

Thus,

$$\begin{aligned} \overrightarrow{BA}\overrightarrow{BC} + \overrightarrow{BC}\overrightarrow{BA^*} = \overrightarrow{BA}\overrightarrow{BA^*} &= 180 \\ \implies \angle ABC + \angle A^*BC &= 180 \\ \implies 2\angle ABC &= 180 \\ \implies \angle ABC = \angle A^*BC &= 90. \end{aligned}$$

Also, it can be easily shown by that the angle supplementary to $\angle ABC$ is 90, and the angle supplementary to $\angle A^*BC$ is 90. Thus, the four angles determined by the intersection of $\overleftrightarrow{AB}$ with $\overleftrightarrow{BC}$ are all 90, so $\overleftrightarrow{AB} \perp \overleftrightarrow{BC}$ at B, and $\overleftrightarrow{AB}$ meets $\overleftrightarrow{BC}$ at a point (B) distance $\frac{\omega}{2}$ from A, so by the definition of a pole, A is a pole for $\overleftrightarrow{BC}$ $\blacksquare$.