Nate Warner MATH 336 Spring 2026

Problem set 1 - Due: Friday, January 23

1. Solve the following ODE

$$\frac{dy}{dx} = \frac{\cos\left(\frac{1}{x}\right)}{x^3}.$$

First, we see that

$$\frac{dy}{dx} = \frac{\cos\left(\frac{1}{x}\right)}{x^3}$$

implies that

$$dy = \frac{\cos\left(\frac{1}{x}\right)}{x^3}dx.$$

So, we integrate both sides,

$$\int dy = \int \frac{\cos\left(\frac{1}{x}\right)}{x^3}\ dx.$$

Thus,

$$y + C_1 = \int \frac{\cos\left(\frac{1}{x}\right)}{x^3}\ dx.$$

To integrate the right hand side, we first let $u = \frac{1}{x}$. Then,

$$\frac{du}{dx} = \frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}.$$

So,

$$du = -\frac{1}{x^2}\ dx \implies -du = \frac{1}{x^2}\ dx.$$

With this, we see that

$$\frac{1}{x^3}\ dx = \frac{1}{x}\cdot\frac{1}{x^2}\ dx = \frac{1}{x}\cdot(-du) = u(-du) = -u\ du.$$

So,

$$\int \frac{\cos\left(\frac{1}{x}\right)}{x^3}\ dx = -\int u\cos(u)\ du.$$

Now, we can integrate by parts. Let $\hat{u} = u$, and $dv = \cos(u)du$. Then,

$$\frac{d\hat{u}}{du} = 1 \implies d\hat{u} = du,$$

and

$$\int dv = v = \int \cos(u)\ du = \sin(u)$$
$$\implies v = \sin(u).$$

Now, since

$$\int \hat{u}\ dv = \hat{u}v - \int v\ d\hat{u} = \hat{u}v - \int v\ du,$$

we have

$$\begin{aligned}
-\int u\cos(u)\ du &= -\left(u\sin(u) - \int \sin(u)\ du\right) \\
&= -\left(u\sin(u) + \cos(u)\right) \\
&= -u\sin(u) - \cos(u).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&y + C_1 = \int \frac{\cos\left(\frac{1}{x}\right)}{x^3}\ dx = -\frac{1}{x}\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \\
&\implies y = -\frac{1}{x}\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) + C.
\end{aligned}$$

2. Solve the following ODE

$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2-7}}.$$

We have

$$\begin{aligned}
dy &= \frac{x}{\sqrt{x^2-7}}\ dx \\
\implies \int\ dy = y &= \int \frac{x}{\sqrt{x^2-7}}\ dx.
\end{aligned}$$

If we let $u = x^2 - 7$, then

$$\frac{du}{dx} = 2x \implies du = 2x\ dx \implies x\ dx = \frac{1}{2}\ du.$$

Thus,

$$y = \frac{1}{2}\int \frac{1}{u^{\frac{1}{2}}}\ du.$$

Now, since

$$\int u^{-\frac{1}{2}}\ du = \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = 2u^{\frac{1}{2}} = 2\sqrt{u},$$

we have

$$y = \frac{1}{2}\left(2\sqrt{u}\right) = \sqrt{u} = \sqrt{x^2-7} + C.$$

3. Solve the following ODE

$$\frac{dy}{dx} = \frac{x}{\sqrt{1+x^2}}.$$

Similar to the last problem,

$$dy = \frac{x}{\sqrt{1+x^2}}\ dx,$$

so

$$y = \int \frac{x}{\sqrt{1+x^2}}\ dx.$$

Let $u = 1 + x^2$, so

$$du = 2x\ dx \implies x\ dx = \frac{1}{2}\ du.$$

Thus,

$$y = \frac{1}{2}\int \frac{1}{\sqrt{u}}\ du.$$

Which is exactly the same as the last problem. So, we know that

$$y = \sqrt{u} = \sqrt{1+x^2} + C.$$

1. Solve the following Initial value problem

$$\begin{cases} \frac{dy}{dx} = x^2 e^x \\ y(0) = 2 \end{cases}.$$

We have

$$dy = x^2 e^x\ dx,$$

so

$$y = \int x^2 e^x\ dx.$$

If we let $u = x^2$, and $dv = e^x\ dx$, then

$$du = 2x\ dx,$$

and

$$v = e^x.$$

Thus, by integration by parts, we have

$$y = x^2 e^x - \int 2x e^x\ dx.$$

Now, we preform a second integration by parts for the integral $\int 2xe^x\ dx$. Let $u = 2x$, $dv = e^x\ dx$. Then,

$$du = 2\ dx,$$

and

$$v = e^x.$$

Thus,

$$\begin{aligned} y &= x^2e^x - \left(2xe^x - 2\int e^x\ dx\right) = x^2e^x - (2xe^x - 2e^x) \\ &= x^2e^x - 2xe^x + 2e^x + C. \end{aligned}$$

With the initial value $y(0) = 2$,

$$2 = 0^2e^0 - 2(0)e^0 + 2e^0 + C \implies 2 = 2 + C.$$

Thus, $C = 0$, and

$$y = x^2e^x - 2xe^x + 2e^x = e^x\left(x^2 - 2x + 2\right).$$