Remark. For a series of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n \qquad \text{or} \quad \sum_{n=1}^{\infty} (-1)^{n} b_n \ .$$

If the following conditions hold

i. $b_{n+1} \leqslant b_n \ \forall \ n \geqslant 1$

ii. $\lim\limits_{n \to \infty} b_n = 0$

Then it follows that the series is convergent. This is known as the Leibniz criterion (or alternating series test)

Problem 1. We want to use the Alternating Series Test to determine if the series:

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^4}{\sqrt{k^3+6}} \ .$$

converges or diverges.

First we verify that b_k is monotone decreasing $\forall \ k \geqslant 1$. If $b_{k+1} = \frac{(k+1)^4}{\sqrt{(k+1)^3+6}}$. Then we can quite clearly see that

$$b_{k+1} \leqslant b_k.$$

Furthermore...

$$\begin{aligned}
\lim_{k \to \infty} b_k &= \lim_{k \to \infty} \frac{k^4}{\sqrt{k^3+6}} \\
&= \lim_{k \to \infty} \frac{\frac{k^4}{k^3}}{\sqrt{\frac{k^3}{k^3} + \frac{6}{k^3}}} \\
&= \lim_{k \to \infty} \frac{k}{\sqrt{1 + \frac{6}{k^3}}} \\
&= \lim_{k \to \infty} \frac{\cancelto{\infty}{k}}{\sqrt{1 + \cancelto{0}{\frac{6}{k^3}}}} \\
&= \infty.
\end{aligned}$$

Conclusion. The Alternating Series Test does not apply because the absolute value of the terms do not approach 0, and the series diverges for the same reason.

Problem 2. We want to use the Alternating Series Test to determine if the series:

$$\sum_{k=4}^{\infty} (-1)^{k+2} \frac{k^3}{\sqrt{k^8+12}} \ .$$

converges or diverges.

First we verify that b_k is monotone decreasing $\forall\ x \geqslant 1$. If $b_{k+1} = \frac{(k+1)^3}{\sqrt{(k+1)^8+12}}$. Then we can see that

$$\frac{(k+1)^3}{\sqrt{(k+1)^8+12}} \leqslant \frac{k^3}{\sqrt{k^8+12}}$$

Thus $b_{k+1} \leqslant b_k$.

Now we examine

$$\begin{aligned}
&\lim_{k\to\infty} b_k = \lim_{k\to\infty} \frac{k^3}{\sqrt{k^8+12}} \\
&= \lim_{k\to\infty} \frac{\frac{k^3}{k^8}}{\sqrt{\frac{k^8}{k^8}+\frac{12}{k^8}}} \\
&= \lim_{k\to\infty} \frac{\frac{1}{k^5}}{\sqrt{1+\frac{12}{k^8}}} \\
&= \lim_{k\to\infty} \frac{\cancelto{0}{\frac{1}{k^5}}}{\sqrt{1+\cancelto{0}{\frac{12}{k^8}}}} \\
&= 0.
\end{aligned}$$

Conclusion. Since b_k is monotone decreasing $\forall\ k \geqslant 1$ and $\lim\limits_{k\to\infty} b_k = 0$. We conclude that the series must converge.

Problem 3. We want to use the Alternating Series Test to determine if the series:

$$\sum_{k=1}^{\infty} \left(\frac{\sin^2\left(\frac{k\pi}{2}\right)}{k} - \frac{\cos^2\left(\frac{k\pi}{2}\right)}{2^k} \right) .$$

Converges or diverges

This is an interesting series so let's give it a closer look. The $\frac{\pi}{2}k$ in the sine and cosine functions arguments insinuate that these functions will oscillate with a period $T = \frac{2\pi}{w} = \frac{2\pi}{\frac{\pi}{2}} = 4$. Thus, we check cases k $\in [1, 4]$

$$\begin{aligned}
k = 1: &\ \sin^2\left(\frac{\pi}{2}\right) = 1, \quad \cos^2\left(\frac{\pi}{2}\right) = 0 \\
k = 2: &\ \sin^2(\pi) = 0 \quad \cos^2\left(\frac{\pi}{2}\right) = 1 \\
k = 3: &\ \sin^2\left(\frac{3\pi}{2}\right) = 1 \quad \cos^2\left(\frac{\pi}{2}\right) = 0 \\
k = 4: &\ \sin^2(2\pi) = 0 \quad \cos^2\left(\frac{\pi}{2}\right) = 1 \\
.
\end{aligned}$$

Which means we have...

$$\begin{aligned}
k = 1: &\ 1 - \frac{0}{2} = 1 \\
k = 2: &\ 0 - \frac{1}{4} = -\frac{1}{4} \\
k = 3: &\ \frac{1}{3} - 0 = \frac{1}{3} \\
k = 4: &\ 0 - \frac{1}{16} = -\frac{1}{16}.
\end{aligned}$$

So we see that this series is indeed alternating, however, when we examine the absolute value of these terms

$$1, \frac{1}{4}, \frac{1}{3}, \frac{1}{16}.$$

We notice that they are **not** monotone decreasing $\forall\, k \geqslant 1$

Conclusion. The Alternating Series Test does not apply because the absolute value of the terms are not decreasing.

Problem 4. Does the series

$$\sum_{k=1}^{\infty} \frac{k^2}{\sqrt{k^{10}+4}} \ .$$

converge absolutely, converge conditionally or diverge?

Does the series

$$\sum_{k=1}^{\infty} \frac{(-1)^k k^2}{\sqrt{k^{10}+4}} \ .$$

converge absolutely, converge conditionally or diverge?

converge absolutely, converge conditionally or diverge?

The most efficient way to determine the end behavior of these series is to first look at the series $\sum_{k=1}^{\infty} \frac{k^2}{\sqrt{k^{10}+4}}$. We notice that this series is the absolute value of the series $\sum_{k=1}^{\infty} \frac{(-1)^k k^2}{\sqrt{k^{10}+4}}$. Thus, if we find $\sum_{k=1}^{\infty} \frac{k^2}{\sqrt{k^{10}+4}}$ to converge, we know that $\sum_{k=1}^{\infty} \frac{(-1)^k k^2}{\sqrt{k^{10}+4}}$ converges absolutely

For the series $\sum_{k=1}^{\infty} \frac{k^2}{\sqrt{k^{10}+4}}$, we use the comparison test. Let $b_k = \frac{1}{k^3}$. Which, by the p-series, we know will converge. Since

$$\frac{k^2}{\sqrt{k^{10}+4}} < \frac{1}{k^3} \ .$$

Then, we can conclude that by the simple comparison test, the series will converge.

Conclusion. Since the absolute value series converges, we can conclude that both series converge absolutely

Problem 5. Does the series

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[5]{k^8+7}} \ .$$

Converge absolutely, converge conditionally or diverge?

Does the series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[5]{k^8+7}} \ .$$

Converge absolutely, converge conditionally or diverge?

We make a similar claim as we did in problem 4. Comparing the absolute value series to $b_k = \frac{1}{k^{\frac{8}{5}}}$. Which, by the p-series, we know will converge. We see

$$\frac{1}{\sqrt{k^8+7}} < \frac{1}{k^{\frac{8}{5}}}.$$

Conclusion. Thus, the absolute value series also converges and we conclude that both series converge absolutely

Problem 6. Does the series

$$\sum_{k=2}^{\infty} (-1)^k \frac{\ln(k^6)}{k+7}.$$

converge absolutely, converge conditionally or diverge?

Examining the absolute value of this series $\sum_{k=2}^{\infty} \frac{\ln k^6}{k}$, we can again use a comparison test to see whether it diverges or converges. If we choose b_n to be $\frac{1}{k}$. Then we can see that

$$\frac{\ln k^6}{k} > \frac{1}{k}.$$

And since we know $\frac{1}{k}$ diverges, we can conclude $\sum_{k=1}^{\infty} \frac{\ln k^6}{k}$ will also diverge. Thus, we can have no absolute convergence.

We know must look at the alternating series. Since

$$\frac{\ln(k+1)^6}{k+1} < \frac{\ln k^6}{k} \implies b_{k+1} \leqslant b_k.$$

We conclude that the series b_k is monotone decreasing. Now if we look at the limit

$$\lim_{k\to\infty} b_k = \lim_{k\to\infty} \frac{\ln k^6}{k}$$

.

Since $\ln k^6$ grows slower than $\frac{1}{k}$, the end behavior of $\frac{1}{k}$ will determine what this limit is. Thus, the limit is zero and we can conclude that the alternating series will converge

Conclusion. Since $|a_k|$ diverges while a_k converges, we say that the series $\sum_{k=2}^{\infty} \frac{\ln k^6}{k}$ converges conditionally.