Calculus 2
Chapter 5

Nathan Warner

Northern Illinois University

Computer Science
Northern Illinois University
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United States

Contents

Sequences and Series

5.1 Sequences

Terminology of Sequences

To work with this new topic, we need some new terms and definitions. First, an infinite sequence is an ordered list of numbers of the form

$$a_1,\ a_2,\ a_3,\ \ldots\ a_n,\ \ldots..$$

Each of the numbers in the sequence is called a term. The symbol n is called the index variable for the sequence. We use the notation

$$\{a_n\}_{n=1}^{\infty},\ \text{or simply}\ \{a_n\}.$$

to denote this sequence. A similar notation is used for sets, but a sequence is an ordered list, whereas a set is not ordered. Because a particular number a_n exists for each positive integer n, we can also define a sequence as a function whose domain is the set of positive integers.

Let's consider the infinite, ordered list

$$2,\ 4,\ 8,\ 16,\ 32,\ \ldots.$$

This is a sequence in which the first, second, and third terms are given by $a_1 = 2$, $a_2 = 4$, and $a_3 = 8$. You can probably see that the terms in this sequence have the following pattern:

$$a_1 = 2^1,\ a_2 = 2^2,\ a_3 = 2^3,\ a_4 = 2^4,\ \text{and}\ a_5 = 2^5.$$

Assuming this pattern continues, we can write the n^{th} term in the sequence by the explicit formula $a_n = 2^n$. Using this notation, we can write this sequence as

$$\{2n\}_{n=1}^{\infty} \quad \text{or} \quad \{2n\}.$$

Alternatively, we can describe this sequence in a different way. Since each term is twice the previous term, this sequence can be defined recursively by expressing the n^{th} term a_n in terms of the previous term a_{n-1}. In particular, we can define this sequence as the sequence $\{a_n\}$ where $a_1 = 2$ and for all $n \geqslant 2$, each term a_n is defined by the **recurrence relation** $a_n = 2a_{n-1}$.

Definition 1:

An infinite sequence $\{a_n\}$ is an ordered list of numbers of the form

$$a_1, a_2, \ldots, a_n, \ldots$$

The subscript n is called the index variable of the sequence. Each number a_n is a term of the sequence. Sometimes sequences are defined by explicit formulas, in which case $a_n = f(n)$ for some function $f(n)$ defined over the positive integers. In other cases, sequences are defined by using a recurrence relation. In a recurrence relation, one term (or more) of the sequence is given explicitly, and subsequent terms are defined in terms of earlier terms in the sequence.

Note:-

Note that the index does not have to start at $n = 1$ but could start with other integers. For example, a sequence given by the explicit formula $a_n = f(n)$ could start at $n = 0$,in which case the sequence would be

$$a_0,\ a_1,\ a_2,\ \ldots.$$

Similarly, for a sequence defined by a recurrence relation, the term a_0 may be given explicitly, and the terms a_n for $n \geqslant 1$ may be defined in terms of a_{n-1}. Since a sequence $\{a_n\}$ has exactly one value for each positive integer n, it can be described as a function whose domain is the set of positive integers. As a result, it makes sense to discuss the graph of a sequence. The graph of a sequence $\{a_n\}$ consists of all points (n, a_n) for all positive integers n. Figure 5.2 shows the graph of $\{2n\}$.

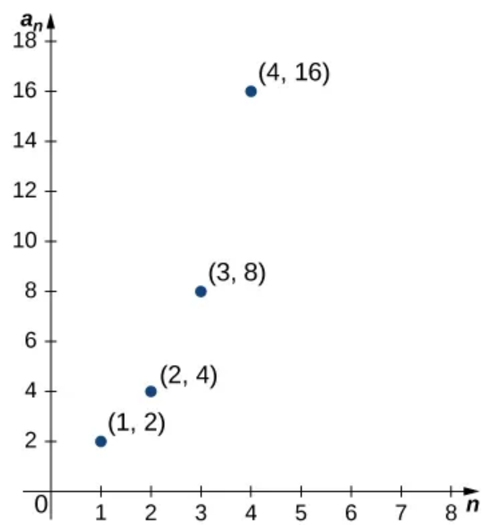


Figure 5.2

Two types of sequences occur often and are given special names: arithmetic sequences and geometric sequences. In an arithmetic sequence, the difference between every pair of consecutive terms is the same. For example, consider the sequence

$$3,\ 7,\ 11,\ 15,\ 19,\ \cdots.$$

You can see that the difference between every consecutive pair of terms is 4. Assuming that this pattern continues, this sequence is an arithmetic sequence. It can be described by using the recurrence relation

$$\begin{cases} a_1 &= 3 \\ a_n &= a_{n-1} + 4 \text{ for } n \geqslant 2 \end{cases} \tag{1}$$

Note that:

$$\begin{aligned} a_2 &= 3 + 4 \\ a_3 &= 3 + 4 + 4 = 3 + 2 \cdot 4 \\ a_4 &= 3 + 4 + 4 + 4 = 3 + 3 \cdot 4. \end{aligned}$$

Thus the sequence can also be described using the explicit formula

$$a_n = a + (n-1)d$$
$$\text{Thus } a_n = 3 + (n-1)4.$$

Where d is the difference $d = a_n - a_{n-1}$. In general, an arithmetic sequence is any sequence of the form $a_n = a + dn$

In a **geometric sequence**, the ratio of every pair of consecutive terms is the same. For example, consider the sequence

$$2,\ -\frac{2}{3},\ \frac{2}{9},\ -\frac{2}{27},\ \frac{2}{81},\ \cdots.$$

We see that the ratio of any term to the preceding term is $-\frac{1}{3}$. Assuming this pattern continues, this sequence is a geometric sequence. It can be defined recursively as

$$\begin{aligned} a_1 &= 2 \\ &= a_n = -\frac{1}{3} \cdot a_{n-1} \text{ for } n \geqslant 2. \end{aligned}$$

Alternatively, since

$$\begin{aligned} a_2 &= -\frac{1}{3} \cdot 2 \\ a_3 &= \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)(2) = \left(-\frac{1}{3}\right)^2 \cdot 2 \\ a_4 &= \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)(2) = \left(-\frac{1}{3}\right)^3 \cdot 2. \end{aligned}$$

we see that the sequence can be described by using the explicit formula

$$a_n = 2\left(-\frac{1}{3}\right)^{n-1}.$$

The sequence {2n} that we discussed earlier is a geometric sequence, where the ratio of any term to the previous term is 2. In general, a geometric sequence is any sequence of the form $a_n = ar^n$.

Finding explicit formulas

Example 1

Find an explicit formula for the sequence

$$-\frac{1}{2},\ \frac{2}{3},\ -\frac{3}{4},\ \frac{4}{5},\ -\frac{5}{6}.$$

Solution. First, note that the sequence is alternating from negative to positive. The odd terms in the sequence are negative, and the even terms are positive. Therefore, the n^{th} term includes a factor of $(-1)^n$. Next, consider the sequence of numerators $\{1, 2, 3, \ldots\}$ and the sequence of denominators $\{2, 3, 4, \ldots\}$. We can see that both of these sequences are arithmetic sequences. The n^{th} term in the sequence of numerators is n, and the n^{th} term in the sequence of denominators is $n+1$. Therefore, the sequence can be described by the explicit formula.

$$a_n = \frac{(-1)^n n}{n+1}.$$

Example 2

Find an explicit formula for the sequence

$$\frac{3}{4},\ \frac{9}{7},\ \frac{27}{10},\ \frac{81}{13},\ \frac{243}{16}.$$

Solution. The sequence of numerators $3, 9, 27, 81, 243, \ldots$ is a geometric sequence. The numerator of the n^{th} term is 3^n. The sequence of denominators $4, 7, 10, 13, 16, \ldots$ is an arithmetic sequence. The denominator of the n^{th} term is $4 + 3(n-1) = 3n + 1$. Therefore, we can describe the sequence by the explicit formula $a_n = \frac{3^n}{3n+1}$.

Example 3: Defined by Recurrence Relations

For each of the following recursively defined sequences, find an explicit formula for the sequence.

(a) $a_1 = 2,\ a_n = -3a_{n-1}$ for $n \geqslant 2$

(b) $a_1 = \frac{1}{2},\ a_n = a_{n-1} + \left(\frac{1}{2}\right)^n$ for $n \geqslant 2$

A: Solution.

$$a = 2,\ r = -3$$
$$\text{Thus: } a_n = 2(-3)^{n-1}.$$

B: Solution.

If we write out the first few terms we get:

$$a_1 = \frac{1}{2},\ a_2 = \frac{3}{4},\ a_3 = \frac{7}{8},\ a_4 = \frac{15}{16}.$$

We see the denominator is the geometric sequence:

$$2,\ 4,\ 8,\ 16$$
$$\text{Thus: } a_n = 2^n.$$

And for the numerator, examining the difference between the terms, we get $2, 4, 8$, thus we can see that the pattern is $a_n = 2^n - 1$

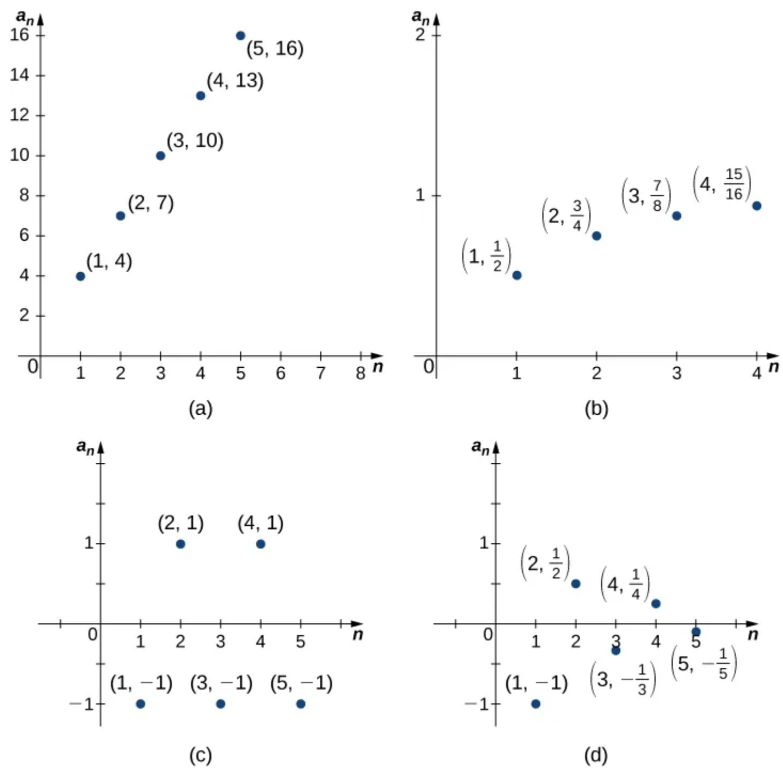
Thus, we have the explicit formula

$$a_n = \frac{2^n - 1}{2^n}.$$

Limit of a Sequence

A fundamental question that arises regarding infinite sequences is the behavior of the terms as n gets larger. Since a sequence is a function defined on the positive integers, it makes sense to discuss the limit of the terms as $n \to \infty$. For example, consider the following four sequences and their different behaviors as $n \to \infty$ (see Figure 5.3):

1. $\{1+3n\} = \{4, 7, 10, 13, \ldots\}$. The terms $1+3n$ become arbitrarily large as $n \to \infty$. In this case, we say that $1+3n \to \infty$ as $n \to \infty$.
2. $\{1-\left(\frac{1}{2}\right)^n\} = \{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots\}$. The terms $\left(1-\frac{1}{2}\right)^n \to 1$ as $n \to \infty$.
3. $\{(-1)^n\} = \{-1, 1, -1, 1, \ldots\}$. The terms alternate but do not approach one single value as $n \to \infty$.
4. $\{\frac{(-1)^n}{n}\} = \{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots\}$ The terms alternate for this sequence as well, but $\frac{(-1)^n}{n} \to 0$ as $n \to \infty$.



From these examples, we see several possibilities for the behavior of the terms of a sequence as $n \to \infty$. In two of the sequences, the terms approach a finite number as $n \to \infty$. In the other two sequences, the terms do not. If the terms of a sequence approach a finite number L as $n \to \infty$, we say that the sequence is a convergent sequence and the real number L is the limit of the sequence. We can give an informal definition here.

Definition 2:

Given a sequence $\{a_n\}$,
if the terms a_n become arbitrarily close to a finite number L as n becomes sufficiently large, we say $\{a_n\}$ is a convergent sequence and L is the limit of the sequence. In this case, we write

$$\lim_{n\to\infty} a_n = L.$$

If a sequence $\{a_n\}$ is not convergent, we say it is a divergent sequence.

From Figure 5.3, we see that the terms in the sequence $\{1-\left(\frac{1}{2}\right)^n\}$ are becoming arbitrarily close to 1 as n becomes very large. We conclude that $\{1-\left(\frac{1}{2}\right)^n\}$ is a convergent sequence and its limit is 1. In contrast, from Figure 5.3, we see that the terms in the sequence $1+3n$ are not approaching a finite number as n becomes larger. We say that $\{1+3n\}$ is a divergent sequence.

In the informal definition for the limit of a sequence, we used the terms "arbitrarily close" and "sufficiently large." Although these phrases help illustrate the meaning of a converging sequence, they are somewhat vague. To be more precise, we now present the more formal definition of limit for a sequence and show these ideas graphically in Figure 5.4.

Definition 3:

A sequence $\{a_n\}$ converges to a real number L if for all $\varepsilon > 0$, there exists an integer N such that $|a_n - L| < \varepsilon$ if $n \geqslant N$. The number L is the limit of the sequence and we write

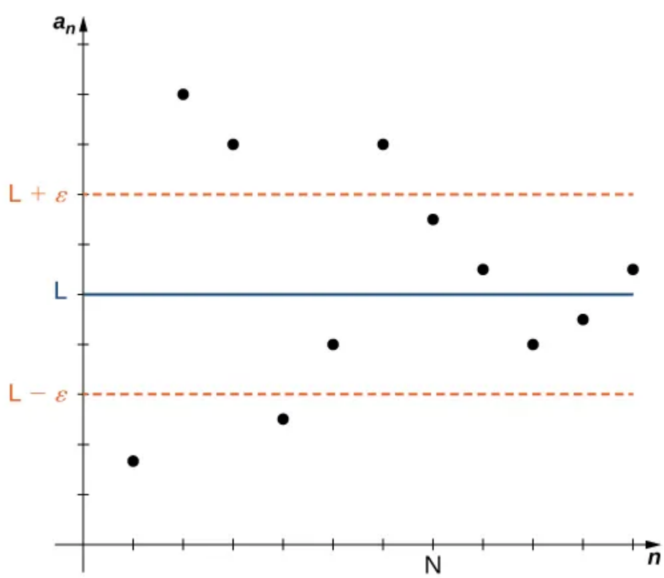
$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L.$$

In this case, we say the sequence $\{a_n\}$ is a convergent sequence. If a sequence does not converge, it is a divergent sequence, and we say the limit does not exist.

We remark that the convergence or divergence of a sequence $\{a_n\}$ depends only on what happens to the terms a_n as $n \to \infty$. Therefore, if a finite number of terms $b_1, b_2, \ldots, b_N$ are placed before a_1 to create a new sequence

$$b_1, b_2, \ldots, b_N, a_1, a_2, \ldots$$

This new sequence will converge if $\{a_n\}$ converges and diverge if $\{a_n\}$ diverges. Further, if the sequence $\{a_n\}$ converges to L, this new sequence will also converge to L.



As n increases, the terms a_n become closer to L. For values of $n \geqslant N$, the distance between each point (n, a_n) and the line $y = L$ is less than ε.

As defined above, if a sequence does not converge, it is said to be a divergent sequence. For example, the sequences $\{1+3n\}$ and $\{(-1)^n\}$ shown in Figure 5.4 diverge. However, different sequences can diverge in different ways. The sequence $\{(-1)^n\}$ diverges because the terms alternate between 1 and -1, but do not approach one value as $n \to \infty$. On the other hand, the sequence $\{1+3n\}$ diverges because the terms $1+3n \to \infty$ as $n \to \infty$. We say the sequence $\{1+3n\}$ diverges to infinity and write

$$\lim_{n\to\infty}(1+3n) = \infty.$$

It is important to recognize that this notation does not imply the limit of the sequence $\{1+3n\}$ exists. The sequence is, in fact, divergent. Writing that the limit is infinity is intended only to provide more information about why the sequence is divergent. A sequence can also diverge to negative infinity. For example, the sequence $\{-5n+2\}$ diverges to negative infinity because $-5n+2 \to -\infty$ as $n \to -\infty$. We write this as

$$\lim_{n\to\infty}(-5n+2) = -\infty.$$

Because a sequence is a function whose domain is the set of positive integers, we can use properties of limits of functions to determine whether a sequence converges. For example, consider a sequence $\{a_n\}$ and a related function f defined on all positive real numbers such that $f(n) = a_n$ for all integers $n \geqslant 1$. Since the domain of the sequence is a subset of the domain of f, if $\lim_{x\to\infty} f(x)$ exists, then the sequence converges and has the same limit. For example, consider the sequence $\{\frac{1}{n}\}$ and the related function $f(x) = \frac{1}{x}$. Since the function f defined on all real numbers $x > 0$ satisfies $f(x) = \frac{1}{x} \to 0$ as $x \to \infty$, the sequence $\{\frac{1}{n}\}$ must satisfy $\frac{1}{n} \to 0$ as $n \to \infty$.

Theorem 1: Limit of a Sequence Defined by a Function

Consider a sequence $\{a_n\}$ such that $a_n = f(n)$ for all $n \geqslant 1$. If there exists a real number L such that

$$\lim_{x\to\infty} f(x) = L,$$

then $\{a_n\}$ converges and

$$\lim_{n\to\infty} a_n = L.$$

We now consider slightly more complicated sequences. For example, consider the sequence $\left\{\left(\frac{2}{3}\right)^n + \left(\frac{1}{4}\right)^n\right\}$. The terms in this sequence are more complicated than other sequences we have discussed, but luckily the limit of this sequence is determined by the limits of the two sequences $\left\{\left(\frac{2}{3}\right)^n\right\}$ and $\left\{\left(\frac{1}{4}\right)^n\right\}$. As we describe in the following algebraic limit laws, since $\left\{\left(\frac{2}{3}\right)^n\right\}$ and $\left\{\left(\frac{1}{4}\right)^n\right\}$ both converge to 0, the sequence $\left\{\left(\frac{2}{3}\right)^n + \left(\frac{1}{4}\right)^n\right\}$ converges to $0+0=0$. Just as we were able to evaluate a limit involving an algebraic combination of functions f and g by looking at the limits of f and g (see Introduction to Limits), we are able to evaluate the limit of a sequence whose terms are algebraic combinations of a_n and b_n by evaluating the limits of $\{a_n\}$ and $\{b_n\}$.

Theorem 2: Algebraic Limit Laws

Given sequences $\{a_n\}$ and $\{b_n\}$ and any real number c, if there exist constants A and B such that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then

- $\lim_{n\to\infty} c = c$
- $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n = cA$
- $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = A \pm B$
- $\lim_{n\to\infty}(a_n \cdot b_n) = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n) = A \cdot B$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{A}{B}$, provided $B \neq 0$ and each $b_n \neq 0$.

Recall that if f is a continuous function at a value L, then $f(x) \to f(L)$ as $x \to L$. This idea applies to sequences as well. Suppose a sequence $a_n \to L$, and a function f is continuous at L. Then $f(a_n) \to f(L)$. This property often enables us to find limits for complicated sequences. For example, consider the sequence $\sqrt{5 - \frac{3}{n^2}}$. From Example 5.3a, we know the sequence $5 - \frac{3}{n^2} \to 5$. Since $\sqrt{x}$ is a continuous function at $x = 5$,

$$\lim_{n\to\infty} \sqrt{5 - \frac{3}{n^2}} = \sqrt{\lim_{n\to\infty} 5 - \frac{3}{n^2}} = \sqrt{5}$$

.

Theorem 3: Continuous Functions Defined on Convergent Sequences

Consider a sequence $\{a_n\}$ and suppose there exists a real number L such that the sequence $\{a_n\}$ converges to L. Suppose f is a continuous function at L. Then there exists an integer N such that f is defined at all values a_n for $n \geqslant N$, and the sequence $\{f(a_n)\}$ converges to $f(L)$.

Example 4: Limits Involving Continuous Functions Defined on Convergent Sequences

Determine whether the sequence $\{\cos\left(\frac{3}{n}\right)^2\}$ converges. If it converges, find its limit.

Solution. Since the sequence $\left(\frac{3}{n^2}\right)$ converges to 0 and $\cos x$ is continuous at 0, we can conclude that the sequence $\left(\frac{3}{n^2}\right)$ converges and

$$\lim_{n\to\infty} \cos\left(\frac{3}{n^2}\right) = \cos 0 = 1.$$

Another theorem involving limits of sequences is an extension of the Squeeze Theorem for limits

Theorem 4: Squeeze Theorem for Sequences

Consider sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$. Suppose there exists an integer N such that

$$a_n \leqslant b_n \leqslant c_n \text{ for all } n \geqslant N.$$

If there exists a real number L such that

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n,$$

then $\{b_n\}$ converges and $\lim_{n\to\infty} b_n = L$

Bounded Sequences

We now turn our attention to one of the most important theorems involving sequences: the Monotone Convergence Theorem. Before stating the theorem, we need to introduce some terminology and motivation. We begin by defining what it means for a sequence to be bounded.

Definition 4:

A sequence $\{a_n\}$ is bounded above if there exists a real number M such that

$$a_n \leqslant M$$

for all positive integers n.

A sequence $\{a_n\}$ is bounded below if there exists a real number M such that

$$M \leqslant a_n$$

for all positive integers n.

A sequence $\{a_n\}$ is a bounded sequence if it is bounded above and bounded below.

If a sequence is not bounded, it is an unbounded sequence.

For example, the sequence $\left\{\frac{1}{n}\right\}$ is bounded above because $\frac{1}{n} \leqslant 1$ for all positive integers n. It is also bounded below because $\frac{1}{n} \geqslant 0$ for all positive integers n. Therefore, $\left\{\frac{1}{n}\right\}$ is a bounded sequence. On the other hand, consider the sequence $\{2n\}$. Because $2n \geqslant 2$ for all $n \geqslant 1$, the sequence is bounded below. However, the sequence is not bounded above. Therefore, $\{2n\}$ is an unbounded sequence.

We now discuss the relationship between boundedness and convergence. Suppose a sequence $\{a_n\}$ is unbounded. Then it is not bounded above, or not bounded below, or both. In either case, there are terms a_n that are arbitrarily large in magnitude as n gets larger. As a result, the sequence $\{a_n\}$ cannot converge. Therefore, being bounded is a necessary condition for a sequence to converge.

Theorem 5

If a sequence $\{a_n\}$ converges, then it is bounded.

Note that a sequence being bounded is not a sufficient condition for a sequence to converge. For example, the sequence $\{(-1)^n\}$ is bounded, but the sequence diverges because the sequence oscillates between 1 and -1 and never approaches a finite number. We now discuss a sufficient (but not necessary) condition for a bounded sequence to converge.

Consider a bounded sequence $\{a_n\}$. Suppose the sequence $\{a_n\}$ is increasing. That is, $a_1 \leqslant a_2 \leqslant a_3 \ldots$. Since the sequence is increasing, the terms are not oscillating. Therefore, there are two possibilities. The sequence could diverge to infinity, or it could converge. However, since the sequence is bounded, it is bounded above and the sequence cannot diverge to infinity. We conclude that $\{a_n\}$ converges. For example, consider the sequence

$$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}.$$

Since this sequence is increasing and bounded above, it converges. Next, consider the sequence

$$\{2, 0, 3, 0, 4, 0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}..$$

Definition 5:

A sequence $\{a_n\}$ is increasing for all $n \geqslant n_0$ if

$$a_n \leqslant a_{n+1} \text{ for all } n \geqslant n_0.$$

A sequence $\{a_n\}$ is decreasing for all $n \geqslant n_0$ if

$$a_n \geqslant a_{n+1} \text{ for all } n \geqslant n_0.$$

A sequence $\{a_n\}$ is a **monotone sequence** for all $n \geqslant n_0$ if it is increasing for all $n \geqslant n_0$ or decreasing for all $n \geqslant n_0$.

We now have the necessary definitions to state the Monotone Convergence Theorem, which gives a sufficient condition for convergence of a sequence.

Theorem 6: Monotone Convergence Theorem

If $\{a_n\}$ is a bounded sequence and there exists a positive integer n_0 such that $\{a_n\}$ is monotone for all $n \geqslant n_0$, then $\{a_n\}$ converges.

Example 5: Monotone Convergence Theorem

Determine whether the sequence is convergent or divergent. If it is convergent, find its limit.

$$a_n = \frac{1000^n}{n!}.$$

Solution. 7.f

Remark. A sequence $\{a_n\}$ is a monotone sequence $\forall\ n \geqslant n_0$ if it is increasing $\forall\ n \geqslant n_0$ or decreasing $\forall\ n \geqslant n_0$. If $\{a_n\}$ is a bounded sequence and there exists a positive integer n_0 s.t $\{a_n\}$ is monotone for all $n \geqslant n_0$, then $\{a_n\}$ converges

The first thing to notice about this sequence, is that it begins by increasing, but eventually must become a decreasing sequence as $n!$ grows much faster than 1000^n, to find the value of n for which this switch occurs...

$$a_{n+1} = \frac{1000^{n+1}}{(n+1)!} = \frac{1000}{n+1} \cdot \frac{1000^n}{n!} = \frac{1000}{n+1} \cdot a_n.$$

Now that we have an equation for the $n+1$ term, we can deduce for which value of n the sequence will start decreasing

$$\begin{aligned}
&a_{n+1} < a_n \\
&\frac{1000}{n+1} \cdot a_n < a_n \\
&\frac{1000}{n+1} < 1 \\
&1000 < n+1 \\
&n > 999.
\end{aligned}$$

By induction, we can show that this is true

Proposition. $\forall n \geqslant 1000$, $a_n > a_{n+1}$

Proof:

Base case: $a_{1000} > a_{1001}$

$$\frac{1000^{1000}}{1000!} > \frac{1000^{1001}}{1001!}$$
$$1000^{1000}(1001)! > 1000^{1001}(1000)!$$
$$1000^{1000}(1001)(1000)! > 1000^{1001}(1000)!$$
$$1001 > \frac{1000^{1001}}{1000^{1000}}$$
$$10001 > 1000.$$

Inductive step: $a_n > a_{n+1}$ if we divide $\frac{a_{n+1}}{a_{n+2}}$ ...

$$\frac{\frac{1000^n}{n!}}{\frac{1000^{n+1}}{(n+1)!}}$$
$$= \frac{1000^n(n+1)!}{1000^{n+1}n!}$$
$$= \frac{1000^n(n+1)n!}{1000^{n+1}n!}$$
$$= \frac{1000^n(n+1)}{1000^{n+1}}$$
$$= \frac{1}{1000}(n+1).$$

for $n \geqslant 1000$, $\frac{1}{1000}(n+1) > 1$. $\therefore \frac{a_n}{a_{n+1}} > 1 \implies a_n > a_{n+1}$

Induction: $a_{n+1} > a_{n+2}$, we can divide $\frac{a_{n+1}}{a_{n+2}}$

$$\frac{\frac{1000^{n+1}}{(n+1)!}}{\frac{1000^{n+2}}{(n+2)!}}$$
$$= \frac{1000^{n+1}(n+2)!}{1000^{n+2}(n+1)!}$$
$$= \frac{1000^{n+1}(n+2)(n+1)!}{1000^{n+2}(n+1)!}$$
$$= \frac{1000^{n+1}(n+2)}{1000^{n+2}}$$
$$(n+2)\left(\frac{1}{1000}\right).$$

For $n \geqslant 1000$, $(n+2)\left(\frac{1}{1000}\right) > 1$. $\therefore \frac{a_{n+1}}{a_{n+2}} > 1 \implies a_{n+1} > a_{n+2}$

■ ☺

Thus, this sequence is decreasing for $n \geqslant 1000$. Furthermore, this sequence is bounded below by 0 because $\frac{(1000)^n}{n!} \geqslant 0$, $\forall\ n \in \mathbb{Z}^+$. Therefore, the conditions for the monotone convergence theorem are met and this sequence must converge.

Using the fact that this sequence converges, and a finite number of terms does not affect the convergence of a sequence, we can propose

$$\lim_{n\to+\infty} a_{n+1} = \lim_{n\to+\infty} a_n = L$$

.

Since we know...

$$a_{n+1} = \frac{1000}{n+1} \cdot a_n.$$

We can take the limit of both sides,

$$\begin{aligned}
&\lim_{n\to+\infty} a_{n+1} = \lim_{n\to+\infty} \frac{1000}{n+1} a_n \\
&L = \frac{1000}{\lim\limits_{n\to+\infty} n+1} \cdot \lim_{n\to+\infty} a_n \\
&L = 0 \cdot \lim_{n\to+\infty} a_n \\
&L = 0.
\end{aligned}$$

5.2 Infinite Series

Definition 6:

An infinite series is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots .$$

For each positive integer k, the sum

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$$

is called the k^{th} partial sum of the infinite series. The partial sums form a sequence $\{S_k\}$. If the sequence of partial sums converges to a real number S, the infinite series converges. If we can describe the convergence of a series to S, we call S the sum of the series, and we write

$$\sum_{n=1}^{\infty} a_n = S.$$

If the sequence of partial sums diverges, we have the divergence of a series.

Note that the index for a series need not begin with $n = 1$ but can begin with any value. For example, the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} .$$

Can also be written as:

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} \quad \text{or: } \sum_{n=5}^{\infty} \left(\frac{1}{2}\right)^{n-5} .$$

Often it is convenient for the index to begin at 1, so if for some reason it begins at a different value, we can reindex by making a change of variables. For example, consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2} .$$

By introducing the variable $m = n - 1$ so that $n = m + 1$, we can rewrite the series as

$$\sum_{m}^{\infty} \frac{1}{(m+1)^2} .$$

Example 6: Evaluating Limits of Sequences of Partial Sums

use the sequence of partial sums to determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n}{n+1} \ .$$

Solution. The sequence of partial sums S_k satisfies

$$\begin{aligned} S_1 &= \frac{1}{2} \\ S_2 &= \frac{1}{2} + \frac{2}{3} \\ S_3 &= \frac{1}{2} + \frac{2}{3} + \frac{3}{4} \\ S_4 &= \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} \end{aligned}$$

Notice that each term added is greater than $\frac{1}{2}$ As a result, we see that

$$\begin{aligned} S_1 &= \frac{1}{2} \\ S_2 &= \frac{1}{2} + \frac{2}{3} > \frac{1}{2} + \frac{1}{2} = 2\left(\frac{1}{2}\right) \\ S_3 &= \frac{1}{2} + \frac{2}{3} + \frac{3}{4} > \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 3\left(\frac{1}{2}\right) \\ S_4 &= \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} > \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 4\left(\frac{1}{2}\right). \end{aligned}$$

From this pattern, we can see that $S_k > k\left(\frac{1}{2}\right)$ for every integer k. Therefore, $\{S_k\}$ is unbounded and consequently, diverges. Therefore, the infinite series

$$\sum_{n=1}^{\infty} \frac{n}{n+1}$$

diverges.

Example 7

determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} (-1)^n \ .$$

Solution. The sequence of partial sums S_k satisfies

$$\begin{aligned} S_1 &= -1 \\ S_2 &= -1 + 1 = 0 \\ S_3 &= -1 + 1 - 1 = -1 \\ S_4 &= -1 + 1 - 1 + 1 = 0. \end{aligned}$$

From this pattern, we can see the sequence of partial sums is

$$\{S_k\} = \{-1, 0, -1, 0, \ldots\}.$$

Since this sequence diverges, the infinite series

$$\sum_{n=1}^{\infty} (-1)^n$$

diverges.

Example 8

Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \ .$$

Solution. The sequence of partial sums S_k satisfies

$$\begin{aligned}
S_1 &= \frac{1}{1\cdot 2} = \frac{1}{2} \\
S_2 &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3} \\
S_3 &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4} \\
S_4 &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} = \frac{4}{5} \\
S_5 &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} = \frac{5}{6}.
\end{aligned}$$

From this pattern, we can see that the k^{th} partial sum is given by the explicit formula

$$S_k = \frac{k}{k+1}.$$

And since

$$\lim_{k\to+\infty} \frac{k}{k+1} = 1.$$

We can conclude that the series converges, Thus:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \ = 1.$$

The Harmonic Series

A useful series to know about is the harmonic series. The harmonic series is defined as

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{+}\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \ .$$

This series is interesting because it diverges, but it diverges very slowly. By this, we mean that the terms in the sequence of partial sums S_k approach infinity, but do so very slowly. We will show that the series diverges, but first, we illustrate the slow growth of the terms in the sequence S_k in the following table.

k	10	100	$1,000$	$10,000$	$100,000$	$1,000,000$
S_k	2.92897	5.18738	7.48547	9.78761	12.09015	14.39273

Even after 1,000,000 terms, the partial sum is still relatively small. From this table, it is not clear that this series actually diverges. However, we can show analytically that the sequence of partial sums diverges, and therefore the series diverges.

To show that the sequence of partial sums diverges, we show that the sequence of partial sums is unbounded. We begin by writing the first several partial sums:

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 1 + \frac{1}{2} \\ S_3 &= 1 + \frac{1}{2} + \frac{1}{3} \\ S_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}. \end{aligned}$$

Notice that for the last two terms in S_4,

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4}.$$

Therefore, we conclude that

$$S_4 > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2\left(\frac{1}{2}\right).$$

Using the same idea for S_8, we see that

$$\begin{aligned} S_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + 3\left(\frac{1}{2}\right). \end{aligned}$$

From this pattern, we see that

$$\begin{aligned} S_1 &= 1, \\ S_2 &= 1 + \frac{1}{2}, \\ S_4 &> 1 + 2\left(\frac{1}{2}\right), \\ S_8 &> 1 + 3\left(\frac{1}{2}\right). \end{aligned}$$

More generally, it can be shown that

$$S_{2^j} > 1 + j\left(\frac{1}{2}\right)$$

for all $j > 1$. Since $1 + j\left(\frac{1}{2}\right) \to \infty$, we conclude that the sequence $\{S_k\}$ is unbounded and therefore diverges. In the previous section, we stated that convergent sequences are bounded. Consequently, since $\{S_k\}$ is unbounded, it diverges. Thus, the harmonic series diverges.

Algebraic Properties of Convergent Series

Since the sum of a convergent infinite series is defined as a limit of a sequence, the algebraic properties for series listed below follow directly from the algebraic properties for sequences.

Theorem 7: Algebraic Properties of Convergent Series

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series. Then the following algebraic properties hold:

1. The series $\sum_{n=1}^{\infty}(a_n + b_n)$ converges and
$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n. \quad \text{(Sum Rule)}.$$
2. The series $\sum_{n=1}^{\infty}(a_n - b_n)$ converges and
$$\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n. \quad \text{(Difference Rule)}.$$
3. For any real number c, the series $\sum_{n=1}^{\infty} ca_n$ converges and
$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n. \quad \text{(Constant Multiple Rule)}.$$

Geometric Series

A geometric series is of the form:

$$\sum_{n=1}^{\infty} a(r)^{n-1} \quad = a + ar + ar^2 + ar^3 + \ldots.$$

So when does it converge? When $a > 0$, its sequence of partial sums is given by:

$$\sum_{n=1}^{\infty} ar^{n-1} \quad = a + ar + ar^3 + \ldots + ar^{k-1}.$$

When $r = 1$, we have:

$$S_k = a + a(1) + a(1)^2 + a(1)^3 + \ldots + a(1)^{k-1} \quad = ak.$$

Since $a > 0$, we know that $ak \to \infty$ as $k \to \infty$, Thus, the sequence of partial sums is unbounded and diverges, which means when $r = 1$ the series diverges. When $r \neq 1$, we have:

$$S_k = a + ar + ar^2 + ar^3 + \ldots + ar^{k-1}.$$

We can then multiply by $(1 - r)$ to get some nice cancellations:

$$\begin{aligned}
(1-r)S_k &= (1-r)(a + ar + ar^2 + ar^3 + \ldots + ar^{k-1}) \\
(1-r)S_k &= a(1-r)(1 + r + r^2 + r^3 + \ldots + r^{k-1}) \\
(1-r)S_k &= a(1 + r - r + r^2 - r^2 + r^3 - r^3 - r^k) \\
(1-r)S_k &= a(1 - r^k) \\
S_k &= \frac{a(1-r^k)}{1-r} \quad \text{for } r \neq 1.
\end{aligned}$$

Since we know for any $r \in \mathbb{R}$, s.t $0 \leqslant r < 1$, $r^k \to 0$. Whereas $r^k \to \infty$ for $r > 1$. Thus, $r^k \to 0$ if $|r| < 1$ and $r \to \infty$ if $|r| > 1$. This means we can write:

Definition 7:

Geometric series convergence or divergence:

$$\sum_{n=1}^{\infty} ar^{n-1} \quad = \quad \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ diverges & \text{if } |r| \geqslant 1 \end{cases}.$$

Example 9

Determine if the series converges or diverges

$$\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{4^{n-1}} \; .$$

Solution. We can rewrite this as:

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{(-3)^{n-1}(-3)^2}{4^{n-1}} \\ &= \sum_{n=1}^{\infty} 9\left(-\frac{3}{4}\right)^{n-1} \; . \end{aligned}$$

Which means we have $a = 9$, $r = -\frac{3}{4}$. Since $|r| < 1$, this sequence converges to...

$$\begin{aligned} \frac{a}{1-r} &= \frac{9}{1+\frac{3}{4}} \\ &= 9 + \frac{7}{4} \\ &= \frac{36}{7}. \end{aligned}$$

Example 10

Determine if the series converges or diverges

$$\sum_{n=1}^{\infty} e^{2n} \; .$$

Solution. We can rewrite this as

$$\begin{aligned} &\sum_{n=1}^{\infty} (e^n)^2 \\ &= \sum_{n=1}^{\infty} (e^{n-1})^2 e^2 \\ &= \sum_{n=1}^{\infty} (e^2)^{n-1} e^2 \; . \end{aligned}$$

Which means we have $r = e^2$. Since $|r| > 1$, this series diverges

We now turn our attention to a nice application of geometric series. We show how they can be used to write repeating decimals as fractions of integers.

Example 11: Writing Repeating Decimals as Fractions of Integers

Use a geometric series to write $3.\bar{26}$ as a fraction of integers.

Solution. Since $3.\bar{26} = 3.262626...$, first we write

$$\begin{aligned} 3.262626 &= 3 + \frac{26}{100} + \frac{26}{10,000} + \frac{26}{1,000,000} + ... \\ &= 3 + \frac{26}{10^2} + \frac{26}{10^4} + \frac{26}{10^6} + ... \end{aligned}$$

Ignoring the first term 3, the rest of this expression is a geometric series with initial term $a = \frac{26}{10^2}$ and $r = \frac{1}{10^2}$. Thus, the sum of the series is

$$\begin{aligned} &\frac{\frac{26}{10^2}}{1 - \left(\frac{1}{10^2}\right)} \\ &= \frac{\frac{26}{10^2}}{\frac{99}{10^2}} \\ &= \frac{26}{99}. \end{aligned}$$

Thus $3.\bar{26} =$

$$\begin{aligned} &3 + \frac{26}{99} \\ &= \frac{323}{99}. \end{aligned}$$

5.3 The Divergence and Integral Tests

Divergence Test

For a series $\sum_{n=1}^{\infty} a_n$ to converge, the nth term a_n must satisfy $a_n \to 0$ as $n \to \infty$.

Therefore, from the algebraic limit properties of sequences,

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty}(S_k - S_{k-1}) = \lim_{k\to\infty} S_k - \lim_{k\to\infty} S_{k-1} = S - S = 0.$$

Therefore, if $\sum_{n=1}^{\infty} a_n$ converges, the nth term $a_n \to 0$ as $n \to \infty$. An important consequence of this fact is the following statement:

$$\text{If } a_n \nrightarrow 0 \text{ as } n \to \infty, \sum_{n=1}^{\infty} a_n \text{ diverges..}$$

This test is known as the divergence test because it provides a way of proving that a series diverges.

Theorem 8: Divergence Test

In the context of sequences, if $\lim_{n\to\infty} a_n = c \neq 0$ or the limit does not exist, then the series $\sum_{n=1}^{\infty} a_n$ is said to diverge.

It is important to note that the converse of this theorem is not true. That is, if $\lim_{n\to\infty} a_n = 0$, we cannot make any conclusion about the convergence of $\sum_{n=1}^{\infty} a_n$. For example, $\lim_{n\to\infty} \frac{1}{n} = 0$, but the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. In this section and the remaining sections of this chapter, we show many more examples of such series. Consequently, although we can use the divergence test to show that a series diverges, we cannot use it to prove that a series converges. Specifically, if $a_n \to 0$, the divergence test is inconclusive.

Integral Test

In the previous section, we proved that the harmonic series diverges by looking at the sequence of partial sums $\{S_k\}$ and showing that $S_{2^k} > 1 + \frac{k}{2}$ for all positive integers k. In this section, we use a different technique to prove the divergence of the harmonic series. This technique is important because it is used to prove the divergence or convergence of many other series. This test, called the integral test, compares an infinite sum to an improper integral. It is important to note that this test can only be applied when we are considering a series whose terms are all positive.

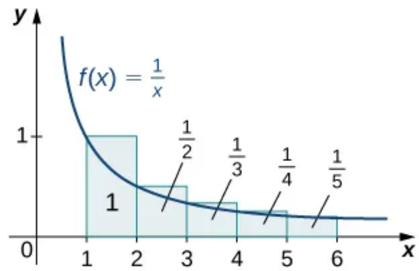
To illustrate how the integral test works, use the harmonic series as an example. In Figure 5.12, we depict the harmonic series by sketching a sequence of rectangles with areas $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ along with the function $f(x) = \frac{1}{x}$. From the graph, we see that

$$\sum_{n=1}^{k} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} > \int_1^{k+1} \frac{1}{x}\,dx..$$

Therefore, for each k, the k^{th} partial sum S_k satisfies

$$S_k = \sum_{n=1}^{k} \frac{1}{n} > \int_1^{k+1} \frac{1}{x}\,dx = \ln(x)\Big|_1^{k+1} = \ln(k+1) - \ln(1) = \ln(k+1)..$$

Since $\lim_{k\to\infty} \ln(k+1) = \infty$, we see that the sequence of partial sums $\{S_k\}$ is unbounded. Therefore, $\{S_k\}$ diverges, and, consequently, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ also diverges.



Now consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. We show how an integral can be used to prove that this series converges. In Figure 5.13, we sketch a sequence of rectangles with areas $1, \frac{1}{2^2}, \frac{1}{3^2}, \ldots$ along with the function $f(x) = \frac{1}{x^2}$. From the graph, we see that

$$\sum_{n=1}^{k} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < 1 + \int_1^k \frac{1}{x^2}\,dx..$$

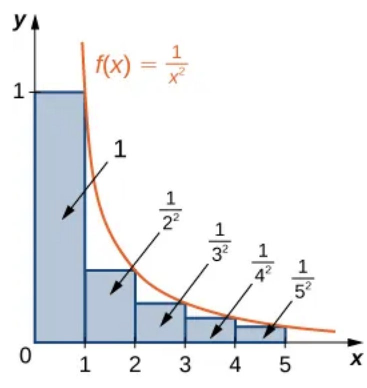
Therefore, for each k, the kth partial sum S_k satisfies

$$S_k = \sum_{n=1}^{k} \frac{1}{n^2} < 1 + \int_1^k \frac{1}{x^2}\,dx = 1 - \frac{1}{x}\bigg|_1^k = 1 - \frac{1}{k} + 1 = 2 - \frac{1}{k} < 2.$$

We conclude that the sequence of partial sums $\{S_k\}$ is bounded. We also see that $\{S_k\}$ is an increasing sequence:

$$S_k = S_{k-1} + \frac{1}{k^2} \quad \text{for} \quad k \geqslant 2.$$

Since $\{S_k\}$ is increasing and bounded, by the Monotone Convergence Theorem, it converges. Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.



We can extend this idea to prove convergence or divergence for many different series. Suppose $\sum_{n=1}^{\infty} a_n$ is a series with positive terms a_n such that there exists a continuous, positive, decreasing function f where $f(n) = a_n$ for all positive integers. Then, as in Figure 5.14(a), for any integer k, the kth partial sum S_k satisfies

$$S_k = a_1 + a_2 + a_3 + \cdots + a_k < a_1 + \int_1^k f(x)\,dx < a_1 + \int_1^{\infty} f(x)\,dx.$$

Therefore, if $\int_1^{\infty} f(x)\,dx$ converges, then the sequence of partial sums $\{S_k\}$ is bounded. Since $\{S_k\}$ is an increasing sequence, if it is also a bounded sequence, then by the Monotone Convergence Theorem, it converges. We conclude that if $\int_1^\infty f(x)\,dx$ converges, then the series $\sum_{n=1}^\infty a_n$ also converges. On the other hand, from Figure 5.14(b), for any integer k, the kth partial sum S_k satisfies

$$S_k = a_1 + a_2 + a_3 + \cdots + a_k > \int_1^{k+1} f(x)\,dx.$$

If $\lim_{k\to\infty} \int_1^{k+1} f(x)\,dx = \infty$, then $\{S_k\}$ is an unbounded sequence and therefore diverges. As a result, the series $\sum_{n=1}^\infty a_n$ also diverges. We conclude that if $\int_1^\infty f(x)\,dx$ diverges, then $\sum_{n=1}^\infty a_n$ diverges.

Theorem 9: Integral Test

Suppose $\sum\limits_{n=1}^{\infty} a_n$ is a series with positive terms a_n Suppose there exists a function f and a positive integer N such that the following three conditions are satisfied:

1. f continuous
2. f positive and decreasing
3. $f(n) = a_n$ for all integers $n \geqslant N$,

Then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ and the improper integral $\displaystyle\int_N^\infty f(x)\,dx$ either both converge or both diverge..

Although convergence of $\int_N^\infty f(x)\,dx$ implies convergence of the related series $\sum_{n=1}^\infty a_n$, it does not imply that the value of the integral and the series are the same. They may be different, and often are. For example,

$$\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n = \frac{1}{e} + \left(\frac{1}{e}\right)^2 + \left(\frac{1}{e}\right)^3 + \cdots$$

is a geometric series with initial term $a = \frac{1}{e}$ and ratio $r = \frac{1}{e}$, which converges to

$$\frac{\frac{1}{e}}{1 - \left(\frac{1}{e}\right)} = \frac{1}{e} \cdot \frac{e-1}{e} = \frac{1}{e-1}.$$

However, the related integral $\int_1^\infty \left(\frac{1}{e}\right)^x\,dx$ satisfies

$$\int_1^\infty \left(\frac{1}{e}\right)^x dx = \int_1^\infty e^{-x}\,dx = \lim_{b\to\infty} \int_1^b e^{-x}\,dx = \lim_{b\to\infty} \left[-e^{-x}\right]_1^b = \lim_{b\to\infty} \left[-e^{-b} + e^{-1}\right] = \frac{1}{e}.$$

The p-series

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ are both examples of a type of series called a p-series.

Definition 8:

$\forall p \in \mathbb{R}$, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^P}.$$

Is called a **p-series**

We know the p-series converges if $p = 2$ and diverges if $p = 1$. What about other values of p? In general, it is difficult, if not impossible, to compute the exact value of most p-series. However, we can use the tests presented thus far to prove whether a p-series converges or diverges.

If $p < 0$, then $\frac{1}{n^p} \to \infty$, and if $p = 0$, then $\frac{1}{n^p} \to 1$. Therefore, by the divergence test,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ diverges if } p \leqslant 0. \tag{2}$$

If $p > 0$, then $f(x) = \frac{1}{x^p}$ is a positive, continuous, decreasing function. Therefore, for $p > 0$, we use the integral test, comparing

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{and} \quad \int_1^{\infty} \frac{1}{x^p}\, dx. \tag{3}$$

We have already considered the case when $p = 1$. Here we consider the case when $p > 0, p \neq 1$. For this case,

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \lim_{b \to \infty} \int_1^{b} \frac{1}{x^p}\, dx \tag{4}$$

$$= \lim_{b \to \infty} \left[\frac{1}{1-p} x^{1-p} \right]_1^b \tag{5}$$

$$= \lim_{b \to \infty} \left[\frac{1}{1-p} \left(b^{1-p} - 1 \right) \right]. \tag{6}$$

Because

$$b^{1-p} \to 0 \text{ if } p > 1 \text{ and} \tag{7}$$

$$b^{1-p} \to \infty \text{ if } p < 1, \tag{8}$$

we conclude that

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \infty & \text{if } p \leqslant 1. \end{cases} \tag{9}$$

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $0 < p < 1$.

In summary,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leqslant 1. \end{cases} \tag{10}$$

Estimating the Value of a Series

Suppose we know that a series $\sum_{n=1}^{\infty} a_n$ converges and we want to estimate the sum of that series. Certainly, we can approximate that sum using any finite sum $\sum_{n=1}^{N} a_n$ where N is any positive integer. The question we address here is, for a convergent series $\sum_{n=1}^{\infty} a_n$, how good is the approximation $\sum_{n=1}^{N} a_n$? More specifically, if we let

$$R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n$$

be the remainder when the sum of an infinite series is approximated by the Nth partial sum, how large is R_N? For some types of series, we are able to use the ideas from the integral test to estimate R_N.

Theorem 10: Remainder Estimate from the Integral Test

Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series with positive terms. Suppose there exists a function f satisfying the following three conditions:

1. f is continuous,
2. f is decreasing, and
3. $f(n) = a_n$ for all integers $n \geqslant 1$.

Let S_N be the Nth partial sum of $\sum_{n=1}^{\infty} a_n$. For all positive integers N,

$$S_N + \int_{N+1}^{\infty} f(x)\,dx < \sum_{n=1}^{\infty} a_n < S_N + \int_{N}^{\infty} f(x)\,dx.$$

In other words, the remainder $R_N = \sum_{n=1}^{\infty} a_n - S_N = \sum_{n=N+1}^{\infty} a_n$ satisfies the following estimate:

$$\int_{N+1}^{\infty} f(x)\,dx < R_N < \int_{N}^{\infty} f(x)\,dx.$$

This is known as the remainder estimate

We illustrate the Remainder Estimate from the Integral Test in Figure 5.15. In particular, by representing the remainder $R_N = a_{N+1} + a_{N+2} + a_{N+3} + \cdots$ as the sum of areas of rectangles, we see that the area of those rectangles is bounded above by $\int_N^{\infty} f(x)\,dx$ and bounded below by $\int_{N+1}^{\infty} f(x)\,dx$. In other words,

$$R_N = a_{N+1} + a_{N+2} + a_{N+3} + \cdots > \int_{N+1}^{\infty} f(x)\,dx$$

and

$$R_N = a_{N+1} + a_{N+2} + a_{N+3} + \cdots < \int_{N}^{\infty} f(x)\,dx.$$

We conclude that

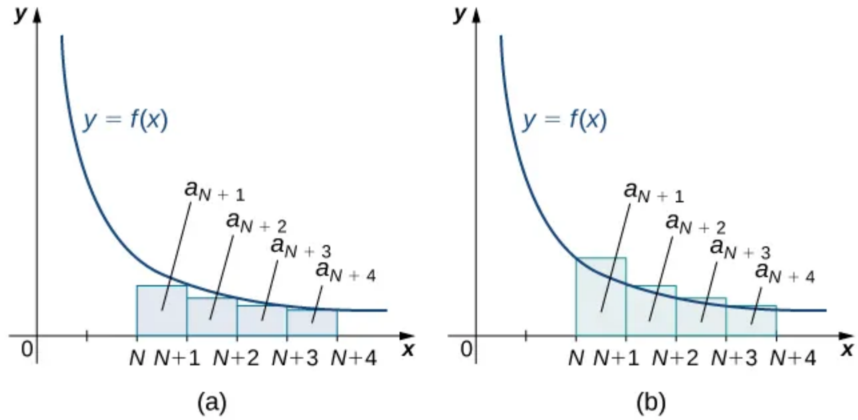
$$\int_{N+1}^{\infty} f(x)\,dx < R_N < \int_{N}^{\infty} f(x)\,dx.$$

Since

$$\sum_{n=1}^{\infty} a_n = S_N + R_N,$$

where S_N is the Nth partial sum, we conclude that

$$S_N + \int_{N+1}^{\infty} f(x)\,dx < \sum_{n=1}^{\infty} a_n < S_N + \int_{N}^{\infty} f(x)\,dx.$$



Example 12: Estimating the Value of a Series

Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$

(a) Calculate $S_{10} = \sum_{n=1}^{10} \frac{1}{n^3}$

(b) Determine the least value of N necessary such that S_N will estimate $\sum_{n=1}^{\infty} \frac{1}{n^3}$ to within 0.001

Solution. Computing S_{10}, we get:

$$S_{10} \approx 1.19753.$$

By the remainder estimate, we know

$$R_N < \int_N^{\infty} \frac{1}{x^3}\,dx.$$

We have:

$$\int_{10}^{\infty} \frac{1}{x^3}\,dx = \lim_{b\to\infty} \int_{10}^{b} \frac{1}{x^3}\,dx = \lim_{b\to\infty} \left[-\frac{1}{2x^2}\right]_{10}^{b} = \lim_{b\to\infty} \left[-\frac{1}{2b^2} + \frac{1}{2N^2}\right] = \frac{1}{2N^2}.$$

Thus, the error is $R_{10} < \frac{1}{2(10)^2} = 0.005$

Since we have showed that $R_N < \frac{1}{2N^2}$, the remainder $R_N < 0.001$ as long as $\frac{1}{2N^2} < 0.001$. That is, we need $2N^2 > 1000$. Solving for N we get 22.36. Therefore, we should use $N = 23$

5.4 Comparison Tests

In the preceding two sections, we discussed two large classes of series: geometric series and p-series. We know exactly when these series converge and when they diverge. Here we show how to use the convergence or divergence of these series to prove convergence or divergence for other series, using a method called the comparison test. For example, consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}.$$

This series looks similar to the convergent series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Since the terms in each of the series are positive, the sequence of partial sums for each series is monotone increasing. Furthermore, since

$$0 < \frac{1}{n^2+1} < \frac{1}{n^2}$$

for all positive integers n, the kth partial sum S_k of $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ satisfies

$$S_k = \sum_{n=1}^{k} \frac{1}{n^2+1} < \sum_{n=1}^{k} \frac{1}{n^2} < \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Since the series on the right converges, the sequence $\{S_k\}$ is bounded above. We conclude that $\{S_k\}$ is a monotone increasing sequence that is bounded above. Therefore, by the Monotone Convergence Theorem, $\{S_k\}$ converges, and thus

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} \tag{11}$$

converges.

Theorem 11: Comparison Test for series

1. Suppose there exists an integer N such that $0 \leqslant a_n \leqslant b_n$ for all $n \geqslant N$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. Suppose there exists an integer N such that $a_n \geqslant b_n \geqslant 0$ for all $n \geqslant N$. If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example 13: Using the comparison test for series

Use theorem 11 to show the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3 + 3n + 1} .$$

Solution. We can compare this series to $\sum\limits_{n=1}^{\infty} \frac{1}{n^3}$. Since this new series is a p-series with $p > 1$, we know it will converge. Furthermore, since

$$\frac{1}{n^3 + 3n + 1} < \frac{1}{n^3}.$$

$\forall\ n \in \mathbb{Z}^+$, we can canclude that the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^3+3n+1}$ converges

Example 14: Using the comparison test for series

Use theorem 11 to show the convergence or divergence of the series

$$\sum_{n=2}^{\infty} \frac{1}{\ln n} .$$

Solution. We can compare this series to $\sum\limits_{n=2}^{\infty} \frac{1}{n}$. Since

$$\frac{1}{\ln n} > \frac{1}{n}.$$

$\forall\ n \geqslant 2, n \in \mathbb{Z}^+$, and since $\sum\limits_{n=2}^{\infty} \frac{1}{n}$ diverges, we can conclude that the series $\sum\limits_{n=2}^{\infty} \frac{1}{\ln n}$ also diverges

Limit Comparison Test

The comparison test works nicely if we can find a comparable series satisfying the hypothesis of the test. However, sometimes finding an appropriate series can be difficult. Consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}.$$

It is natural to compare this series with the convergent series

$$\sum_{n=2}^{\infty} \frac{1}{n^2}.$$

However, this series does not satisfy the hypothesis necessary to use the comparison test because

$$\frac{1}{n^2 - 1} > \frac{1}{n^2}$$

for all integers $n \geqslant 2$. Although we could look for a different series with which to compare $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$, instead we show how we can use the limit comparison test to compare

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1} \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{1}{n^2}.$$

Let us examine the idea behind the limit comparison test. Consider two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ with positive terms a_n and b_n and evaluate

$$\lim_{n\to\infty} \frac{a_n}{b_n}.$$

If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0,$$

then, for n sufficiently large, $a_n \approx Lb_n$. Therefore, either both series converge or both series diverge. For the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ and $\sum_{n=2}^{\infty} \frac{1}{n^2}$, we see that

$$\lim_{n\to\infty} \frac{1/(n^2-1)}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = 1.$$

Since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges, we conclude that

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

converges.

The limit comparison test can be used in two other cases. Suppose

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

In this case, $\{a_n/b_n\}$ is a bounded sequence. As a result, there exists a constant M such that $a_n \leqslant Mb_n$. Therefore, if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. On the other hand, suppose

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty.$$

In this case, $\{a_n/b_n\}$ is an unbounded sequence. Therefore, for every constant M there exists an integer N such that $a_n \geqslant Mb_n$ for all $n \geqslant N$. Therefore, if $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges as well.

Theorem 12: Limit Comparison Test

Let $a_n, b_n \geqslant 0$ for all $n \geqslant 1$.

- If $\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Note that if $\frac{a_n}{b_n} \to 0$ and $\sum_{n=1}^{\infty} b_n$ diverges, the limit comparison test gives no information. Similarly, if $\frac{a_n}{b_n} \to \infty$ and $\sum_{n=1}^{\infty} b_n$ converges, the test also provides no information. For example, consider the two series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$. These series are both p-series with $p = \frac{1}{2}$ and $p = 2$, respectively. Since $p = \frac{1}{2} < 1$, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges. On the other hand, since $p = 2 > 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. However, suppose we attempted to apply the limit comparison test, using the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ as our comparison series. First, we see that

$$\frac{\frac{1}{\sqrt{n}}}{\frac{1}{n^3}} = \frac{n^3}{\sqrt{n}} = n^{\frac{5}{2}} \to \infty \text{ as } n \to \infty.$$

Similarly, we see that

$$\frac{\frac{1}{n^2}}{\frac{1}{n^3}} = n \to \infty \text{ as } n \to \infty.$$

Therefore, if $\frac{a_n}{b_n} \to \infty$ when $\sum_{n=1}^{\infty} b_n$ converges, we do not gain any information on the convergence or divergence of $\sum_{n=1}^{\infty} a_n$.

Example 15

use the limit comparison test to determine whether the series converges or diverges. If the test does not apply, say so.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}\,.$$

Solution. We compare to $\frac{1}{\sqrt{n}}$, by the p-series we know that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ Diverges, since $\frac{1}{\sqrt{n}+1} < \frac{1}{\sqrt{n}}$. We must use the limit comparison test. Thus,

$$\begin{aligned} &\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} \\ &= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} \\ &= 1. \end{aligned}$$

Since $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$, then since b_n diverges, so does a_n