EXERCSIES 5.3 The Divergence and Integral Tests

Written assignment

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Contents

3. Use the Divergene Test to determine the whether the series converges or diverges.

$$\sum_{n=1}^{\infty}\left(1+\frac{9}{n}\right)^{n}.$$

Given the fact that Euler's number has a definition of the form:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e.$$

With a generalization of

$$e^{a}=\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{n}.$$

Using the divergence test for the series $\sum\limits_{n=1}^{\infty}\left(1+\frac{9}{n}\right)^{n}$, we get the $\lim\limits_{n\to\infty}\left(1+\frac{9}{n}\right)^{n}$. Which will trivially yield e^{9}. However, this can be shown...

$$\begin{aligned}
&\lim_{n\to\infty}\left(1+\frac{9}{n}\right)^{n}\\
&=\lim_{n\to\infty}e^{\ln\left(1+\frac{9}{n}\right)^{n}}\\
&=\lim_{n\to\infty}e^{n\ln\left(1+\frac{9}{n}\right)}.
\end{aligned}$$

Focusing on $n\ln\left(1+\frac{9}{n}\right)$...

$$\begin{aligned}
&\lim_{n\to\infty}n\ln\left(1+\frac{9}{n}\right)\quad(\text{Indeterminate }\infty\cdot 0)\\
&=\lim_{n\to\infty}\frac{\ln\left(1+\frac{9}{n}\right)}{n^{-1}}\quad\left(\frac{0}{0}\right)\\
&\stackrel{H}{=}\lim_{n\to\infty}\frac{\frac{1}{1+\frac{9}{n}}\cdot\left(-\frac{9}{n^{2}}\right)}{-\frac{1}{n^{2}}}\\
&=\lim_{n\to\infty}\frac{-\frac{9}{n^{2}+2n}}{-\frac{1}{n^{2}}}\\
&=\lim_{n\to\infty}\frac{9n^{2}}{n^{2}+2n}\\
&\lim_{n\to\infty}\frac{9n}{n+2}\quad\left(\text{Still indeterminate... }\frac{\infty}{\infty}\right)\\
&\stackrel{H}{=}\lim_{n\to\infty}\frac{9}{1}\\
&=9.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\lim_{n\to\infty}\left(1+\frac{9}{n}\right)^{n}\\
&=\lim_{n\to\infty}e^{n\ln\left(1+\frac{9}{n}\right)}\\
&=e^{9}.
\end{aligned}$$