Comprehensive Compendium:
Calculus II

Nathan Warner

Computer Science
Northern Illinois University
August 28,2023
United States

Contents

1 Calc II

1.1 Chapter 1 Key Equations

- **Mean Value Theorem For Integrals**: If $f(x)$ is continuous over an interval [a,b], then there is at least one point $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int f(x)\ dx.$$

- **Integrals resulting in inverse trig functions**

  1. $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{|a|} + C.$$

  2. $$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C.$$

  3. $$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{|a|} \sec^{-1} \frac{|x|}{a} + C.$$

1.2 Chapter 2 Key Terms / Ideas

- **Finding limits of integration for region between two functions**: Usually, we want our limits of integration to be the points where the functions intersect
- A **"complex region"** between curves usually refers to an area that is not easily described by a single, continuous function over the interval of interest.
- **compound regions** are regions bounded by the graphs of functions that cross one another
- **Cross-section:** The intersection of a plane and a solid object.
- a **cylinder** is a three-dimensional shape that has two parallel, congruent bases connected by a curved surface. The bases are usually circles, but they can be other shapes as well
- The line segment connecting the centers of the two bases is called the **"axis" of the cylinder.**
- **Slicing method:** A method of calculating the volume of a solid that involves cutting the solid into pieces, estimating the volume of each piece, then adding these estimates to arrive at an estimate of the total volume; as the number of slices goes to infinity, this estimate becomes an integral that gives the exact value of the volume.
    1. Examine the solid and determine the shape of a cross-section of the solid. It is often helpful to draw a picture if one is not provided.
    2. Determine a formula for the area of the cross-section.
    3. Integrate the area formula over the appropriate interval to get the volume.
- **Solid of revolution:** A solid generated by revolving a region in a plane around a line in that plane.
- **Disk method:** A special case of the slicing method used with solids of revolution when the slices are disks.
- A **Washer (Annuli)** is a disk with holes in the center.
- **Washer method:** A special case of the slicing method used with solids of revolution when the slices are washers.
- **Method of cylindrical shells:** A method of calculating the volume of a solid of revolution by dividing the solid into nested cylindrical shells; this method is different from the methods of disks or washers in that we integrate with respect to the opposite variable.
- **Arc length:** The arc length of a curve can be thought of as the distance a person would travel along the path of the curve.
- **Surface area:** The surface area of a solid is the total area of the outer layer of the object; for objects such as cubes or bricks, the surface area of the object is the sum of the areas of all of its faces.

1.3 Chapter 2 Key Equations

- **Area between two curves, integrating on the x-axis**

$$A = \int_a^b [f(x) - g(x)]\, dx \tag{1}$$

Where $f(x) \geqslant g(x)$

$$A = \int_a^b \ [g(x) - f(x)] \ dx.$$

for $g(x) \geqslant f(x)$

- **Area between two curves, integrating on the y-axis**

$$A = \int_c^d [u(y) - v(y)]\, dy \tag{2}$$

- **Areas of compound regions**

$$\int_a^b \ |f(x) - g(x)| \ dx.$$

- **Area of complex regions**

$$\int_a^b \ f(x) \ dx + \int_b^c \ g(x) \ dx.$$

- **Slicing Method**

$$V(s) = \sum_{i=1}^{n} \ A(x_i^*) \ \Delta x = \int_a^b \ A(x) \ dx.$$

- **Disk Method along the x-axis**

$$V = \int_a^b \pi[f(x)]^2\, dx \tag{3}$$

- **Disk Method along the y-axis**

$$V = \int_c^d \pi[g(y)]^2\, dy \tag{4}$$

- **Washer Method along the x-axis**

$$V = \int_a^b \pi[(f(x))^2 - (g(x))^2]\, dx \tag{5}$$

- **Washer Method along the y-axis**

$$V = \int_c^d \pi[(u(y))^2 - (v(y))^2]\, dy \tag{6}$$

- **Radius if revolved around other line (Washer Method)**

$$\begin{aligned} If: \ x &= -k \\ Then: \ r &= Function + k. \end{aligned}$$

$$\begin{aligned} If: \ x &= k \\ Then: \ r &= k - Function. \end{aligned}$$

- **Method of Cylindrical Shells (x-axis)**

$$V = \int_a^b 2\pi x f(x)\, dx \tag{7}$$

- **Method of Cylindrical Shells (y-axis)**

$$V = \int_c^d 2\pi y g(y)\, dy \tag{8}$$

- **Region revolved around other line (method of cylindrical shells):**

$$If:\ x = -k$$
$$Then:\ V = \int_a^b\ 2\pi(x+k)(f(x))\ dx.$$

$$If:\ x = k$$
$$Then:\ V = \int_a^b\ 2\pi(k-x)(f(x))\ dx.$$

- **A Region of Revolution Bounded by the Graphs of Two Functions (method cylindrical shells)**

$$V = \int_a^b\ 2\pi x\left[f(x) - g(x)\right]\ dx.$$

- **Arc Length of a Function of x**

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \tag{9}$$

- **Arc Length of a Function of y**

$$\text{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy \tag{10}$$

- **Surface Area of a Function of x**

$$\text{Surface Area} = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx \tag{11}$$

- **Natural logarithm function**

$$\ln x = \int_1^x \frac{1}{t}\, dt\ Z \tag{12}$$

- **Exponential function**

$$y = e^x, \quad \ln y = \ln(e^x) = x\ Z \tag{13}$$

- **Logarithm Differentiation**

$$f'(x) = f(x) \cdot \frac{d}{dx} \ln\left(f'(x)\right).$$

Note: Use properties of logs before you differentiate whats inside the logarithm

1.4 Chapter 3 Key Terms

- **integration by parts**: a technique of integration that allows the exchange of one integral for another using the formula
- **integration table**: a table that lists integration formulas.
- **power reduction formula**: a rule that allows an integral of a power of a trigonometric function to be exchanged for an integral involving a lower power.
- **trigonometric integral**: an integral involving powers and products of trigonometric functions.
- **trigonometric substitution**: an integration technique that converts an algebraic integral containing expressions of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$ into a trigonometric integral.
- **partial fraction decomposition**: a technique used to break down a rational function into the sum of simple rational functions.
- **improper integral**: an integral over an infinite interval or an integral of a function containing an infinite discontinuity on the interval; an improper integral is defined in terms of a limit. The improper integral converges if this limit is a finite real number; otherwise, the improper integral diverges.

1.5 Chapter 3 Key Equations

- **Integration by parts formula**

$$\int u\,dv = uv - \int v\,du.$$

- **Integration by parts for definite integral**

$$\int_a^b u\,dv = uv\big|_a^b - \int_a^b v\,du$$

- **To integrate products involving sin(ax), sin(bx), cos(ax), and cos(bx), use the substitutions:**
  - **Sine Products**

$$\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$$

  - **Sine and Cosine Products**

$$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$$

  - **Cosine Products**

$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$$

  - **Power Reduction Formula (sine)**

$$\int \sin^n x\ dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\ dx$$
$$\int_0^{\frac{\pi}{2}} \sin^n x\ dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x\ dx.$$

  - **Power Reduction Formula (cosine)**

$$\int \cos^n x\ dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\ dx$$
$$\int_0^{\frac{\pi}{2}} \cos^n x\ dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \cos^{n-2} x\ dx.$$

  - **Power Reduction Formula (secant)**

$$\int \sec^n x\ dx = \frac{1}{n-1}\sec^{n-1} x\sin x + \frac{n-2}{n-1}\int \sec^{n-2} x\ dx$$
$$\int \sec^n x\ dx = \frac{1}{n-1}\sec^{n-2} x\tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\ dx$$

  - **Power Reduction Formula (tangent)**

$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$$

- **Trigonometric Substitution**
  - $\sqrt{a^2-x^2}$ use $x = a\sin\theta$ with domain restriction $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
  - $\sqrt{a^2+x^2}$ use $x = a\tan\theta$ with domain restriction $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

    - $\sqrt{x^2 - a^2}$ use $x = a \sec\theta$ with domain restriction $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$

- **Steps for fraction decomposition**
    1. Ensure $deg(Q) < deg(P)$, if not, long divide
    2. Factor denominator
    3. Split up fraction into factors
    4. Multiply through to clear denominator
    5. Group terms and equalize
    6. Solve for constants
    7. Plug constants into split up fraction
    8. Compute integral
- **Solving for constants** Either:
    - Plug in values (often the roots)
    - Equalize
- **Cases for partial fractions**
    - Non repeated linear factors
    - Repeated linear factors
    - Nonfactorable quadratic factors
- **Midpoint rule**

$$M_n = \sum_{i=1}^{n} f(m_i)\ \Delta x.$$

- **Absolute error**

$$err = \Big|\text{Actual} - \text{Estimated}\Big|.$$

- **Relative error**

$$err = \left|\frac{\text{Actual} - \text{Estimated}}{\text{Actual}}\right| \cdot 100\%.$$

- **Error upper bound for midpoint rule**

$$E_M \leqslant \frac{M(b-a)^3}{24n^2}$$

  Where M is the maximum value of the second derivative

- **Trapezoidal rule**

$$T_n \frac{1}{2} \Delta x \left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right)$$

- **Error upper bound for trapezoidal rule**

$$E_T \leqslant \frac{M(b-a)^3}{12n^2}$$

  Where M is the maximum value of the second derivative

- **Simpson's rule**

$$S_n = \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)$$

- **Error upper bound for Simpson's rule**

$$E_S \leqslant \frac{M(b-a)^5}{180n^4}$$

  Where M is the maximum value of the fourth derivative

- **Finding n with error bound functions**
  1. Find $f''(x)$
  2. Find maximum values of $f''(x)$ in the interval
  3. Plug into error bound function
  4. Set value $\leqslant$ desired accuracy (ex: 0.01)
  5. Solve:
  6. If we were to truncate, we would use the ceil function $\lceil n \rceil$ DO NOT FLOOR

- **Improper integrals (Infinite interval)**
  - $\int_a^{+\infty} f(x)\ dx = \lim_{t\to+\infty} \int_a^t f(x)\ dx$
  - $\int_{-\infty}^b f(x)\ dx = \lim_{t\to-\infty} \int_t^b f(x)\ dx$
  - $\int_{-\infty}^{+\infty} f(x)\ dx = \int_{-\infty}^0 f(x)\ dx + \int_0^{+\infty} f(x)\ dx$

- **Improper integral (discontinuous)**
  - Let $f(x)$ be continuous on $[a, b)$, then;

$$\int_a^b f(x)\ dx = \lim_{t\to b^-} \int_a^t f(x)\ dx\ .$$

  - Let $f(x)$ be continuous on $(a, b]$, then;

$$\int_a^b f(x)\ dx = \lim_{t\to b^+} \int_t^b f(x)\ dx\ .$$

    In each case, if the limit exists, then the improper integral is said to converge. If the limit does not exist, then the improper integral is said to diverge.
  - Let $f(x)$ be continuous on $[a, b]$ except at a point $c \in (a, b)$, then;

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx.$$

    If either integral diverges, then $\int_a^b f(x)\ dx$ diverges

- **Comparison theorem** Let $f(x)$ and $g(x)$ be continuous over $[a, +\infty)$. Assume that $0 \leqslant f(x) \leqslant g(x)$ for $x \geqslant a$.
  - If $\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx = +\infty$,
    then $\int_a^{+\infty} g(x)\,dx = \lim_{t\to+\infty} \int_a^t g(x)\,dx = +\infty$.
  - If $\int_a^{+\infty} g(x)\,dx = \lim_{t\to+\infty} \int_a^t g(x)\,dx = L$, where L is a real number,
    then $\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx = M$ for some real number $M \leqslant L$.

- **P-integrals**
  - $\int_0^{+\infty} \frac{1}{x^p}\ dx = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ +\infty & \text{if } p \leqslant 1 \end{cases}$
  - $\int_0^{1} \frac{1}{x^p}\ dx = \begin{cases} \frac{1}{1-p} & \text{if } p < 1 \\ +\infty & \text{if } p \geqslant 1 \end{cases}$
  - $\int_a^{+\infty} \frac{1}{x^p}\ dx = \begin{cases} \frac{a^{1-p}}{p-1} & \text{if } p > 1 \\ +\infty & \text{if } p \leqslant 1 \end{cases}$
  - $\int_0^{a} \frac{1}{x^p}\ dx = \begin{cases} \frac{a^{1-p}}{1-p} & \text{if } p < 1 \\ +\infty & \text{if } p \geqslant 1 \end{cases}$
- **Bypass L'Hospital's Rule**

$$\ln(\ln(x)),\ \ln(x),\ \cdots\ x^{\frac{1}{100}},\ x^{\frac{1}{3}},\ \sqrt{x},\ 1,\ x^2,\ x^3,\ \cdots\ e^x,\ e^{2x},\ e^{3x},\ \cdots,\ e^{x^2},\ \cdots\ e^{e^x}.$$

Essentially what it means is things on the right grow faster than things on the left. Thus, if we have say:

$$\lim_{x\to\infty} \frac{x^2}{e^{2x}}.$$

We can be sure that it is zero. Because this is $x^2 \cdot e^{-2x}$. If we take $\lim_{x\to\infty} x^2 e^{-2x}$, we get $\infty \cdot 0$. As we see by the sequence e^{-2x} overrules x^2 and we can say the limit is zero.

- **Consideration for Limits**: Let $f : A \to B$ be a function defined by $x \mapsto f(x)$. If a point c lies outside the domain A, then the expression $\lim_{x\to c} f(x)$ is not meaningful, and we classify this limit as undefined. For instance, the function arcsine has a domain of $[-1, 1]$. Therefore, limits like $\lim_{x\to a} \sin^{-1}(x)$ where $a \notin [-1, 1]$ are undefined.
- **Why does**

$$\lim_{x\to 2} \tan^{-1} \frac{1}{x-2}.$$

$$\begin{aligned} &= \lim_{x\to 2^-} \tan^{-1} \frac{1}{x-2} \\ &= \lim_{x\to -\infty} \tan^{-1} x \\ &= -\pi/2. \end{aligned}$$

$$\begin{aligned} &= \lim_{x\to 2^+} \tan^{-1} \frac{1}{x-2} \\ &= \lim_{x\to +\infty} \tan^{-1} x \\ &= \frac{\pi}{2}. \end{aligned}$$

1.6 Chapter 5 Key Terms

- Alternating series:

  A series of the form $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ or $\sum_{n=1}^{\infty}(-1)^{n}b_n,$ where $b_n \geqslant 0,$ is called an alternating series.

- Alternating series test:

  For an alternating series of either form, if $b_{n+1} \leqslant b_n$ for all integers $n \geqslant 1$ and $b_n \to 0,$ then an alternating series converg

- Arithmetic sequence:

  A sequence in which the difference between every pair of consecutive terms is the same is called an arithmetic sequence.

- Bounded above:

  A sequence $\{a_n\}$ is bounded above if there exists a constant M such that $a_n \leqslant M$ for all positive integers n.

- Bounded below:

  A sequence $\{a_n\}$ is bounded below if there exists a constant M such that $M \leqslant a_n$ for all positive integers n.

- Bounded sequence:

  A sequence $\{a_n\}$ is bounded if there exists a constant M such that $|a_n| \leqslant M$ for all positive integers n.

- Convergence of a series:

  A series converges if the sequence of partial sums for that series converges.

- Convergent sequence:

  A convergent sequence is a sequence $\{a_n\}$ for which there exists a real number L such that a_n is arbitrarily close to L as

- Divergence of a series:

  A series diverges if the sequence of partial sums for that series diverges.

- Divergence test:

  If $\lim_{n\to\infty} a_n \neq 0,$ then the series $\sum_{n=1}^{\infty} a_n$ diverges.

- Divergent sequence:

  A sequence that is not convergent is divergent.

- Explicit formula:

  A sequence may be defined by an explicit formula such that $a_n = f(n)$.

- Geometric sequence:

  A sequence $\{a_n\}$ in which the ratio $\frac{a_{n+1}}{a_n}$ is the same for all positive integers n is called a geometric sequence.

- Geometric series:

  A geometric series is a series that can be written in the form $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots.$

- Harmonic series:

  The harmonic series takes the form $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$.

- Index variable:

  The subscript used to define the terms in a sequence is called the index.

- Infinite series:

  An infinite series is an expression of the form $a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$.

- Integral test:

  For a series $\sum_{n=1}^{\infty} a_n$ with positive terms a_n, if there exists a continuous, decreasing function f such that $f(n) = a_n$ for all

- Limit comparison test:

  Suppose $a_n, b_n \geqslant 0$ for all $n \geqslant 1$. If

1.7 Chapter 5 Key Equations

- **Sequence notation**

$$\{a_n\}_{n=1}^{\infty}, \text{ or simply } \{a_n\}.$$

- **Sequence notation (ordered list)**

$$a_1,\ a_2,\ a_3,\ \cdots,\ a_n,\ \cdots.$$

- **Arithemetic Sequence Difference**

$$d = a_n - a_{n-1}.$$

- **Arithmetic sequence (common difference between subsequent terms) general form**

$$\begin{aligned}&\text{Index starting at } 0: \ a_n = a + nd\\ &\text{Index starting at } 1: \ a_n = a + (n-1)d\end{aligned}$$
.

- **Arithmetic sequence (common difference between subsequent terms) recursive form**

$$a_n = a_{n-1} + d.$$

- **Sum of arithmetic sequence**

$$\begin{aligned}S_n &= \frac{n}{2}\left[a + a_n\right]\\ S_n &= \frac{n}{2}\left[2a + (n-1)d\right].\end{aligned}$$

- **Geometric sequence form common ratio**

$$r = \frac{a_n}{a_{n-1}}.$$

- **Geometric sequence general form**

$$\begin{aligned}a_n &= ar^n \text{ (Index starting at 0)}\\ a_n &= a^{n+1}\text{(index starting at 0 and a=r)}\\ a_n &= ar^{n-1} \text{ (Index starting at 1)}\\ a_n &= a^n\text{(index starting at 1 and a=r)}.\end{aligned}$$

- **Geometric sequence recursive form**

$$a_n = ra_{n-1}.$$

- **Sum of geometric sequence (finite terms)**

$$S_n = \frac{a(1-r^n)}{1-r} \qquad r \neq 1.$$

- **Convergence / Divergence**: If

$$\lim_{n\to+\infty} a_n = L.$$

We say that the sequence converges, else it diverges

- **Formal definition of limit of sequence**

$$\lim_{n\to+\infty} a_n = L \iff \forall\varepsilon > 0, \exists N \in \mathbb{Z} \mid |a_n - L| < \varepsilon,\ \text{if } n \geqslant n.$$

  Then we can say

$$\lim_{n\to+\infty} a_n = L \text{ or } a_n \to L.$$

- **Limit of a sequence defined by a function**: Consider a sequence $\{a_n\}$ such that $a_n = f(n)$ for all $n \geqslant 1$. If there exists a real number L such that

$$\lim_{x\to\infty} f(x) = L,$$

  then $\{a_n\}$ converges and

$$\lim_{n\to\infty} a_n = L.$$

- **Algebraic limit laws**: Given sequences $\{a_n\}$ and $\{b_n\}$ and any real number c, if there exist constants A and B such that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then
  - $\lim_{n\to\infty} c = c$
  - $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n = cA$
  - $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = A \pm B$
  - $\lim_{n\to\infty}(a_n \cdot b_n) = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n) = A \cdot B$
  - $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{A}{B}$, provided $B \neq 0$ and each $b_n \neq 0$.

- **Continuous Functions Defined on Convergent Sequences**: Consider a sequence $\{a_n\}$ and suppose there exists a real number L such that the sequence $\{a_n\}$ converges to L. Suppose f is a continuous function at L. Then there exists an integer N such that f is defined at all values a_n for $n \geqslant N$, and the sequence $\{f(a_n)\}$ converges to $f(L)$.

- **Squeeze Theorem for Sequences**: Consider sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$. Suppose there exists an integer N such that

$$a_n \leqslant b_n \leqslant c_n \text{ for all } n \geqslant N.$$

  If there exists a real number L such that

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n,$$

  then $\{b_n\}$ converges and $\lim_{n\to\infty} b_n = L$

- **Bounded above**: A sequence $\{a_n\}$ is bounded above if there exists a real number M such that

$$a_n \leqslant M$$

  for all positive integers n.

- **Bounded below**: A sequence $\{a_n\}$ is bounded below if there exists a real number M such that

$$M \leqslant a_n$$

  for all positive integers n.

- **Bounded**: A sequence $\{a_n\}$ is a bounded sequence if it is bounded above and bounded below.

- **Unbounded**: If a sequence is not bounded, it is an unbounded sequence.

- **If a sequence $\{a_n\}$ converges, then it is bounded.**

- **Increasing sequence**: A sequence $\{a_n\}$ is increasing for all $n \geqslant n_0$ if

$$a_n \leqslant a_{n+1} \text{ for all } n \geqslant n_0.$$

- **Decreasing sequence**: A sequence $\{a_n\}$ is decreasing for all $n \geqslant n_0$ if
$$a_n \geqslant a_{n+1} \text{ for all } n \geqslant n_0.$$
- **Monotone sequence**: A sequence $\{a_n\}$ is a **monotone sequence** for all $n \geqslant n_0$ if it is increasing for all $n \geqslant n_0$ or decreasing for all $n \geqslant n_0$
- **Monotone Convergence Theorem**: If $\{a_n\}$ is a bounded sequence and there exists a positive integer n_0 such that $\{a_n\}$ is monotone for all $n \geqslant n_0$, then $\{a_n\}$ converges.
- **Infinite Series form:**
$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots ..$$
- **Partial sum (k^{th} partial sum)**
$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k.$$
- **Convergence of infinity series notation**

  For a series, say...
$$\sum_{n=1}^{\infty} a_n \ .$$
  its convergence is determined by the limit of its sequence of partial sums. Specifically, if
$$\lim_{n\to+\infty} S_n = S \to \sum_{n=1}^{\infty} a_n \ = S.$$
- **Harmonic series**
$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \ .$$
  Which diverges to $+\infty$
- **Algebraic Properties of Convergent Series** Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series. Then the following algebraic properties hold:
  1. The series $\sum_{n=1}^{\infty}(a_n + b_n)$ converges and
$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n. \quad \text{(Sum Rule)}.$$
  2. The series $\sum_{n=1}^{\infty}(a_n - b_n)$ converges and
$$\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n. \quad \text{(Difference Rule)}.$$
  3. For any real number c, the series $\sum_{n=1}^{\infty} ca_n$ converges and
$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n. \quad \text{(Constant Multiple Rule)}.$$
- **Geometric series convergence or divergence:**
$$\sum_{n=1}^{\infty} ar^{n-1} \quad = \quad \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ diverges & \text{if } |r| \geqslant 1 \end{cases}.$$
- **Divergence test**: In the context of sequences, if $\lim_{n\to\infty} a_n = c \neq 0$ or the limit does not exist, then the series $\sum_{n=1}^{\infty} a_n$ is said to diverge. The converse is not true.

Because:

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} (S_k - S_{k-1}) = \lim_{k\to\infty} S_k - \lim_{k\to\infty} S_{k-1} = S - S = 0..$$

- **Integral Test Prelude**: for any integer k, the kth partial sum S_k satisfies

$$S_k = a_1 + a_2 + a_3 + \cdots + a_k < a_1 + \int_1^k f(x)\,dx < a_1 + \int_1^\infty f(x)\,dx..$$

and

$$S_k = a_1 + a_2 + a_3 + \cdots + a_k > \int_1^{k+1} f(x)\,dx..$$

- **Intgeral test** Suppose $\sum\limits_{n=1}^{\infty} a_n$ is a series with positive terms a_n Suppose there exists a function f and a positive integer N such that the following three conditions are satisfied:

  1. f positive, continuous, and decreasing on $[N, \infty)$
  2. $f(n) = a_n$ for all integers $n \geqslant N$, $N \in \mathbb{Z}^+$

  Then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ and the improper integral $\displaystyle\int_N^\infty f(x)\,dx$ either both converge or both diverge..

- **P-series** $\forall p \in \mathbb{R}$, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^P}.$$

Is called a **p-series**. Furthermore,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leqslant 1. \end{cases}.$$

- **P-series extended**

$$\sum_{n=2}^{\infty} \frac{1}{n \ln(n)^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leqslant 1. \end{cases}.$$

- **Remainder estimate for the integral test** Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series with positive terms. Suppose there exists a function f satisfying the following three conditions:

  1. f is continuous,
  2. f is decreasing, and
  3. $f(n) = a_n$ for all integers $n \geqslant 1$.

  Let S_N be the Nth partial sum of $\sum_{n=1}^{\infty} a_n$. For all positive integers N,

$$S_N + \int_{N+1}^{\infty} f(x)\,dx < \sum_{n=1}^{\infty} a_n < S_N + \int_N^\infty f(x)\,dx.$$

  In other words, the remainder $R_N = \sum_{n=1}^{\infty} a_n - S_N = \sum_{n=N+1}^{\infty} a_n$ satisfies the following estimate:

$$\int_{N+1}^{\infty} f(x)\,dx < R_N < \int_N^\infty f(x)\,dx.$$

  This is known as the remainder estimate

To find a value of N such that we are withing a desired margin of error, Since we know $R_n < \int_N^\infty f(x)\,dx$. Simply compute the improper integral and set the result < the desired error to solve for N

- **Find a_n given the expression for the partial sum**

$$a_n = S_n - S_{n-1}.$$

- **telescoping series**: Telescoping series are a type of series where each term cancels out a part of another term, leaving only a few terms that do not cancel. When you sum the series, most of the terms collapse or "telescope," which simplifies the calculation of the sum. Here are some key points and generalizations you can note about telescoping series:
  - Partial Fraction Decomposition
  - Cancellation Pattern: In a telescoping series, look for a pattern where a term in one fraction will cancel out with a term in another fraction.
  - Write out Terms
  - What is left is S_n, thus the sum of the series is the $\lim\limits_{n\to\infty} S_n$

  Try:

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}\ .$$

  Hint, its not only the first and last terms cancel, we also have a $\frac{\frac{1}{2}}{n}$, when a_{n-1}: Answer is $\frac{3}{4}$

- **Comparison test for series**
  1. Suppose there exists an integer N such that $0 \leqslant a_n \leqslant b_n$ for all $n \geqslant N$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
  2. Suppose there exists an integer N such that $a_n \geqslant b_n \geqslant 0$ for all $n \geqslant N$. If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

- **Limit Comparison Test** Let $a_n, b_n \geqslant 0$ for all $n \geqslant 1$.
  - If $\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.
  - If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
  - If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

  Note: Note that if $\frac{a_n}{b_n} \to 0$ and $\sum_{n=1}^{\infty} b_n$ diverges, the limit comparison test gives no information. Similarly, if $\frac{a_n}{b_n} \to \infty$ and $\sum_{n=1}^{\infty} b_n$ converges, the test also provides no information.

  Consider the series

$$\sum_{n=1}^{\infty} \frac{n^4+6}{n^5+4}\ .$$

  To find our b_n we can only focus on the leading coefficients. Thus:

$$b_n = \frac{n^4}{n^5} = \frac{1}{n}.$$

  So our test...

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\frac{n^4+6}{n^5+4}}{\frac{1}{n}}$$
$$= \lim_{n\to\infty} \frac{n(n^4+6)}{n^5+4}$$
$$= \lim_{n\to\infty} \frac{n^5+6n}{n^5+4}$$
$$= 1.$$

Since $\lim_{n\to\infty} \frac{a_n}{b_n} \neq 0 \vee +\infty$. And $\frac{1}{n}$ diverges, we can conclude that a_n will also diverge.

- **Determine which series (or function) is greater**
  - **Subtraction**: Given two functions $f(x) = \frac{1}{x}$ and $g(x) = \frac{x^4+6}{x^5+4}$, we want to compare them by considering the function $h(x) = f(x) - g(x)$:

$$h(x) = f(x) - g(x) = \frac{1}{x} - \frac{x^4+6}{x^5+4}$$

    To compare these directly, it would be helpful to have a common denominator:

$$h(x) = \frac{x^4+4-(x^4+6)}{x(x^5+4)} = \frac{-2}{x(x^5+4)}$$

    Now, we can see that the sign of $h(x)$ depends on the sign of x because the denominator $x(x^5+4)$ is always positive for $x \neq 0$. So:
    * For $x > 0$, $h(x) < 0$, which means $f(x) < g(x)$.
    * For $x < 0$, $h(x) > 0$, which means $f(x) > g(x)$.

- **Alternating Series** Any series whose terms alternate between positive and negative values is called an alternating series. An alternating series can be written in the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots.$$

  or

$$\sum_{n=1}^{\infty} (-1)^{n} b_n = -b_1 + b_2 - b_3 + b_4 - \cdots.$$

  Where $b_n > 0$ for all positive integers n.

- **alternating series test (Leibniz criterion)** An alternating series of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^{n} b_n$$

  converges if
  - $0 < b_{n+1} \leqslant b_n \ \forall \ n \geqslant 1$
  - $\lim_{n\to\infty} b_n = 0$.

  Note: We remark that this theorem is true more generally as long as there exists some integer N such that $0 < b_{n+1} \leqslant b_n$ for all $n \geqslant N$.

Additional note: The AST allows us to consider just the positive terms to check for these two conditions because if a series of decreasing positive terms that approach zero is alternated in sign, the alternating series will converge. This is a special property of alternating series that does not generally hold for non-alternating series.

- **Show decreasing (For the AST)**: Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \ .$$

So you see we have $b_n = \frac{1}{n^2}$. For the AST, we must show that this is decreasing. If $b_{n+1} = \frac{1}{(n+1)^2}$. Then we see

$$\frac{1}{(n+1)^2} < \frac{1}{n^2}.$$

Thus it is decreasing for $n \geqslant 1$ $(b_{n+1} < b_n)$ ■

- **Remainders in alternating series** Consider an alternating series of the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^{n}b_n,$$

that satisfies the hypotheses of the alternating series test. Let S denote the sum of the series and S_N denote the N-th partial sum. For any integer $N \geqslant 1$, the remainder $R_N = S - S_N$ satisfies

$$|R_N| \leqslant b_{N+1}.$$

This tells us that if we stop at the N^{th} term, the error we are making is at most the size of the next term

- **Absolute and conditional convergence**
  - A series $\sum_{n=1}^{\infty} a_n$ exhibits absolute convergence if $\sum_{n=1}^{\infty} |a_n|$ converges.
  - A series $\sum_{n=1}^{\infty} a_n$ exhibits conditional convergence if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.
  - If $\sum\limits_{n=1}^{\infty} |a_n|$ converges then $\sum\limits_{n=1}^{\infty} a_n$ converges

Note: if $|a_n|$ diverges, we cannot have absolute convergence, thus we must examine to see if normal a_n converges, in which case we would have conditional convergence

- **Ratio test** Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms. Let

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|..$$

Then:

i. If $0 \leqslant \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
ii. If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
iii. If $\rho = 1$, the test does not provide any information.

Note: The ratio test is useful for series whose terms involve factorials

- **Root test** Consider the series $\sum_{n=1}^{\infty} a_n$. Let

$$\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}..$$

i. If $0 \leqslant \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
ii. If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
iii. If $\rho = 1$, the test does not provide any information.

Note: The root test is useful for series whose terms involve exponentials

2 Vectors

2.1 Vector vocab

- A **vector** is two pieces of information.
  1. Length
  2. Direction (Magnitude)

2.2 Vector notation

- **Defining a vector**

$$\vec{v} = [x, y] \text{ or } \begin{bmatrix} x \\ y \end{bmatrix}.$$

- **Length of a vector**

$$||\vec{v}|| = \sqrt{x^2 + y^2}.$$

- **Vector addition** Suppose we have two vectors $\vec{v} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$. Then

$$\vec{v} + \vec{u} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} = \vec{c}.$$

  Note: we call this new vector the **resultant**

- **Multiplying by a scalar** Suppose we have the vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$. Then

$$2\vec{v} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}.$$

- **Vector subtraction**

$$\vec{v} - \vec{u} = \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \end{bmatrix}.$$

- **Vector in three dimensions**

$$\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

- **Length of a vector in three dimensions**

$$= \sqrt{x^2 + y^2 + z^2}.$$

  Note: The length of a vector in n dimensions is the square root of the squares of all the components

- **Definition for $\mathbb{R}^n$**: $\mathbb{R}^n$ is the set of all $n - tuples$ of real numbers

$$\begin{aligned} &\vec{v} = [v_1, v_2] \ \vec{v} \in \mathbb{R}^2 \\ &\vec{u} = [u_1, u_2, u_3] \ \vec{u} \in \mathbb{R}^3 \\ &\vec{w} = [w_1, w_2, w_3, w_n] \ \vec{w} \in \mathbb{R}^n \end{aligned}$$

.