Nate Warner MATH 353 Spring 2025

**Problem set 11 - Due: Wednesday, April 2**

1. Assume that $AC < \omega$, $A$-$B$-$C$, and $D$ is a point not on $\overleftrightarrow{AC}$. Let $h$ be a ray with endpoint $C$ such that $h$ meets $\overline{BD}^0$. Prove that $h$ meets $\overline{AD}$

***Proof.*** By Ax.C, point $D \notin \overleftrightarrow{AC}$ and $A$-$B$-$C$ yields $\overrightarrow{DA}$-$\overrightarrow{DB}$-$\overrightarrow{DC}$. Let $E \in \overline{DA}^0$. By the Crossbar theorem, there exists a point $F \in \overline{DB}^0$ such that $E$-$F$-$C$, let $h$ be the ray with endpoint $C$ that contains points $E, F, C$. Note that point $F$ is where $h$ meets $\overline{BD^0}$

Thus, $h$ meets $\overline{AD}$ at point $E$ ■

2. Suppose that $A, P$ and $R$ are noncollinear, $A$-$X$-$P$, $A$-$Z$-$R$ and $P$-$Q$-$R$

(a) Prove there is a point $Y$ on $\overrightarrow{AQ}$ so that $X$-$Y$-$Z$

(b) Prove further that $A$-$Y$-$Q$

***Proof.*** Observe that since $P$-$Q$-$R$, $P, Q, R$ are collinear. Let $\overleftrightarrow{PR}$ be the line that contains these three points. Since $A, P, R$ noncollinear, $A \notin \overleftrightarrow{PR}$. By Ax.C $\overrightarrow{AP}$-$\overrightarrow{AQ}$-$\overrightarrow{AR}$. Note that $X \in \overrightarrow{AP}^0, Z \in \overrightarrow{AR}^0$. Thus, by the crossbar theorem, there exits a point $Y \in \overrightarrow{AQ}^0$ such that $X$-$Y$-$Z$

Since $Y \in \overrightarrow{AQ}$, one of $A$-$Y$-$Q$ or $A$-$Q$-$Y$ must be true. Assume for the sake of contradiction that $A$-$Q$-$Y$.

Consider the line $\overleftrightarrow{XZ}$, note that $Y \in \overleftrightarrow{XZ}$. By Ax.S, $\overleftrightarrow{XZ}$ splits the plane into a pair of opposite halfplanes with edge $\overleftrightarrow{XZ}$, call this pair $H, K$. By $A$-$Q$-$Y$ $=$ $Y$-$Q$-$A$ and theorem 10.3, $\overrightarrow{YQ}^0 \subseteq$ one of the halfplanes, let's say its $H$. Since $Q \in \overrightarrow{YQ}^0$, $Q \in H$. $Y$-$Q$-$A$ implies $A \in \overrightarrow{YQ}^0$, thus $A \in H$. So, $A, Q$ in the same halfplane ($H$).

Next, we consider $A$-$Z$-$R$, which implies by theorem 10.6 that $A, R$ in opposite halfplanes (since $Z \in \overleftrightarrow{XZ}$).

Similarly, since $X \in \overleftrightarrow{XZ}$, and $A$-$X$-$P$, $A, P$ in opposite halfplanes by Thm 10.6.

Observe that $R$-$Q$-$P$ implies $Q \in \overline{PR}$, and since $R, P$ not in the halfplane with $A$, they must be in the same halfplane. Namely, the halfplane $K$ since $A \in H$. Thus, since $K$ convex (by definition of $\frac{1}{2}$planes), $\overline{PR} \in K$, and since $Q \in \overline{PR}$, $Q \in K$, which is a contradiction, since $Q \in H$ implies $Q \notin K$. Thus, $A$-$Q$-$Y$ is not a valid assumption and must be thrown out. However, we know that $y \in \overrightarrow{AQ}$, which means the only possibility left is that $A$-$Y$-$Q$ ■

3. Let $P = (0.8, 0)$, $Q = (0.9, 0)$, $R = (0.9, 0.1)$. Compute both the $\mathbb{E}$-measure and the $\mathbb{H}$-measure of $\underline{\angle QPR}$. Repeat for $P = (0.98, 0)$, $Q = (0.99, 0)$, $R = (0.99, 0.1)$.

**Remark.** The $\mathbb{E}$-measure for $\underline{\angle QPR}$, if $\overleftrightarrow{PR}$ given by $y = mx + b$, $\overleftrightarrow{PQ}$ by $y = nx + c$ is

$$\angle QPR = \cos^{-1}\left(\frac{1 + mn}{\sqrt{1 + m^2}\sqrt{1 + n^2}}\right).$$

Let $P = (0.8, 0)$, $Q = (0.9, 0)$, $R = (0.9, 0.1)$. Then,

$$\begin{aligned} m &= \frac{0.1 - 0}{0.9 - 0.8} = 1, \\ n &= \frac{0 - 0}{0.9 - 0.8} = 0. \end{aligned}$$

Thus,

$$\angle QPR = \cos^{-1}\left(\frac{1 + 1(0)}{\sqrt{1 + 1^2}\sqrt{1 + 0^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}.$$

For $P = (0.98, 0), Q = (0.99, 0), R = (0.99, 0.1)$, we have

$$\begin{aligned} m &= \frac{0.1 - 0}{0.99 - 0.98} = \frac{0.1}{0.01} = 10, \\ n &= 0. \end{aligned}$$

Thus,

$$\underline{\angle QPR} = \cos^{-1}\left(\frac{1 + 1(0)}{\sqrt{1 + 10^2}\sqrt{1 + 0^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{101}}\right) \approx 1.47.$$

**Remark.** The $\mathbb{H}$-measure for $\underline{\angle QPR}$ is given by

$$\mu_{\mathbb{H}}(\overrightarrow{PQ}, \overrightarrow{PR}) = \cos^{-1}\left(\frac{1 + mn - bc}{\sqrt{1 + m^2 - b^2}\sqrt{1 + n^2 - c^2}}\right)$$

provided $\overleftrightarrow{PR}$ given by $y = mx + b$, $\overleftrightarrow{PQ}$ given by $y = nx + c$.

For $P = (0.8, 0), Q = (0.9, 0), R = (0.9, 0.1)$, we have

$$\begin{aligned} &\overleftrightarrow{PQ}: \; y = 0, \quad n = 0, c = 0 \\ &\overleftrightarrow{PR}: \; y - 0 = 1(x - 0.8) \implies y = x - 0.8, \quad m = 1, b = 0.8. \end{aligned}$$

Thus,

$$\mu_{\mathbb{H}}(\overrightarrow{PQ}, \overrightarrow{PR}) = \cos^{-1}\left(\frac{1 + 1(0) - 0.8(0)}{\sqrt{1 + 1^2 - 0.8^2}\sqrt{1 + 0^2 - 0^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2 - 0.8^2}}\right) \approx 0.9078.$$

For $P = (0.98, 0), Q = (0.99, 0), R = (0.99, 0.1)$, we have

$$\begin{aligned} &\overleftrightarrow{PQ}: \; y = 0, \quad n = 0, c = 0 \\ &\overleftrightarrow{PR}: \; y - 0 = 1(x - 0.98) \implies y = x - 0.98, \quad m = 1, b = 0.98. \end{aligned}$$

Thus,

$$\mu_{\mathbb{H}}(\overrightarrow{PQ}, \overrightarrow{PR}) = \cos^{-1}\left(\frac{1 + 1(0) - 0.98(0)}{\sqrt{1 + 1^2 - 0.98^2}\sqrt{1 + 0^2 - 0^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2 - 0.98^2}}\right) \approx 0.9506.$$