Comprehensive Compendium:
Calculus II

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August 28,2023
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Contents

1 Calc II

1.1 Chapter 1 Key Equations

- **Mean Value Theorem For Integrals**: If $f(x)$ is continuous over an interval [a,b], then there is at least one point $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int f(x)\ dx.$$

- **Integrals resulting in inverse trig functions**

1. $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{|a|} + C.$$

2. $$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C.$$

3. $$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{|a|} \sec^{-1} \frac{|x|}{a} + C.$$

1.2 Chapter 2 Key Equations

- **Area between two curves, integrating on the x-axis**

$$A = \int_a^b [f(x) - g(x)]\, dx \tag{1}$$

Where $f(x) \geqslant g(x)$

$$A = \int_a^b \ [g(x) - f(x)] \ dx.$$

for $g(x) \geqslant f(x)$

- **Area between two curves, integrating on the y-axis**

$$A = \int_c^d [u(y) - v(y)]\, dy \tag{2}$$

- **Areas of compound regions**

$$\int_a^b \ |f(x) - g(x)| \ dx.$$

- **Area of complex regions**

$$\int_a^b \ f(x) \ dx + \int_b^c \ g(x) \ dx.$$

- **Slicing Method**

$$V(s) = \sum_{i=1}^{n} \ A(x_i^*) \ \Delta x = \int_a^b \ A(x) \ dx.$$

- **Disk Method along the x-axis**

$$V = \int_a^b \pi[f(x)]^2\, dx \tag{3}$$

- **Disk Method along the y-axis**

$$V = \int_c^d \pi[g(y)]^2\, dy \tag{4}$$

- **Washer Method along the x-axis**

$$V = \int_a^b \pi[(f(x))^2 - (g(x))^2]\, dx \tag{5}$$

- **Washer Method along the y-axis**

$$V = \int_c^d \pi[(u(y))^2 - (v(y))^2]\, dy \tag{6}$$

- **Radius if revolved around other line (Washer Method)**

$$\begin{aligned} If:\ & x = -k \\ Then:\ & r = Function + k. \end{aligned}$$

$$\begin{aligned} If:\ & x = k \\ Then:\ & r = k - Function. \end{aligned}$$

- **Method of Cylindrical Shells (x-axis)**

$$V = \int_a^b 2\pi x f(x)\, dx \tag{7}$$

- **Method of Cylindrical Shells (y-axis)**

$$V = \int_c^d 2\pi y g(y)\, dy \tag{8}$$

- **Region revolved around other line (method of cylindrical shells):**

$$If:\ x = -k$$
$$Then:\ V = \int_a^b\ 2\pi(x+k)(f(x))\ dx.$$

$$If:\ x = k$$
$$Then:\ V = \int_a^b\ 2\pi(k-x)(f(x))\ dx.$$

- **A Region of Revolution Bounded by the Graphs of Two Functions (method cylindrical shells)**

$$V = \int_a^b\ 2\pi x\left[f(x) - g(x)\right]\ dx.$$

- **Arc Length of a Function of x**

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \tag{9}$$

- **Arc Length of a Function of y**

$$\text{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy \tag{10}$$

- **Surface Area of a Function of x**

$$\text{Surface Area} = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx \tag{11}$$

- **Natural logarithm function**

$$\ln x = \int_1^x \frac{1}{t}\, dt\ Z \tag{12}$$

- **Exponential function**

$$y = e^x, \quad \ln y = \ln(e^x) = x\ Z \tag{13}$$

- **Logarithm Differentiation**

$$f'(x) = f(x) \cdot \frac{d}{dx} \ln\left(f'(x)\right).$$

Note: Use properties of logs before you differentiate whats inside the logarithm

1.3 Chapter 3 Key Equations

- **Integration by parts formula**

$$\int u\,dv = uv - \int v\,du.$$

- **Integration by parts for definite integral**

$$\int_a^b u\,dv = uv\big|_a^b - \int_a^b v\,du$$

- **To integrate products involving sin(ax), sin(bx), cos(ax), and cos(bx), use the substitutions:**
  - **Sine Products**

$$\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$$

  - **Sine and Cosine Products**

$$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$$

  - **Cosine Products**

$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$$

  - **Power Reduction Formula (sine)**

$$\int \sin^n x\ dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\ dx$$
$$\int_0^{\frac{\pi}{2}} \sin^n x\ dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x\ dx.$$

  - **Power Reduction Formula (cosine)**

$$\int \cos^n x\ dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\ dx$$
$$\int_0^{\frac{\pi}{2}} \cos^n x\ dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \cos^{n-2} x\ dx.$$

  - **Power Reduction Formula (secant)**

$$\int \sec^n x\ dx = \frac{1}{n-1}\sec^{n-1} x\sin x + \frac{n-2}{n-1}\int \sec^{n-2} x\ dx$$
$$\int \sec^n x\ dx = \frac{1}{n-1}\sec^{n-2} x\tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\ dx$$

  - **Power Reduction Formula (tangent)**

$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$$

- **Trigonometric Substitution**
  - $\sqrt{a^2-x^2}$ use $x = a\sin\theta$ with domain restriction $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
  - $\sqrt{a^2+x^2}$ use $x = a\tan\theta$ with domain restriction $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

  - $\sqrt{x^2 - a^2}$ use $x = a\sec\theta$ with domain restriction $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$

- **Steps for fraction decomposition**
  1. Ensure $deg(Q) < deg(P)$, if not, long divide
  2. Factor denominator
  3. Split up fraction into factors
  4. Multiply through to clear denominator
  5. Group terms and equalize
  6. Solve for constants
  7. Plug constants into split up fraction
  8. Compute integral

- **Solving for constants** Either:
  - Plug in values (often the roots)
  - Equalize

- **Cases for partial fractions**
  - Non repeated linear factors
  - Repeated linear factors
  - Nonfactorable quadratic factors

- **Midpoint rule**

$$M_n = \sum_{i=1}^{n} f(m_i)\ \Delta x.$$

- **Absolute error**

$$err = \Big|\text{Actual} - \text{Estimated}\Big|.$$

- **Relative error**

$$err = \left|\frac{\text{Actual} - \text{Estimated}}{\text{Actual}}\right| \cdot 100\%.$$

- **Error upper bound for midpoint rule**

$$E_M \leqslant \frac{M(b-a)^3}{24n^2}$$

  Where M is the maximum value of the second derivative

- **Trapezoidal rule**

$$T_n \frac{1}{2}\Delta x\left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right)$$

- **Error upper bound for trapezoidal rule**

$$E_T \leqslant \frac{M(b-a)^3}{12n^2}$$

  Where M is the maximum value of the second derivative

- **Simpson's rule**

$$S_n = \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)$$

- **Error upper bound for Simpson's rule**

$$E_S \leqslant \frac{M(b-a)^5}{180n^4}$$

  Where M is the maximum value of the fourth derivative

- **Finding n with error bound functions**
  1. Find $f''(x)$
  2. Find maximum values of $f''(x)$ in the interval
  3. Plug into error bound function
  4. Set value $\leqslant$ desired accuracy (ex: 0.01)
  5. Solve:
  6. If we were to truncate, we would use the ceil function $\lceil n \rceil$ DO NOT FLOOR

- **Improper integrals (Infinite interval)**
  - $\int_a^{+\infty} f(x)\ dx = \lim\limits_{t\to+\infty} \int_a^t f(x)\ dx$
  - $\int_{-\infty}^b f(x)\ dx = \lim\limits_{t\to-\infty} \int_t^b f(x)\ dx$
  - $\int_{-\infty}^{+\infty} f(x)\ dx = \int_{-\infty}^0 f(x)\ dx + \int_0^{+\infty} f(x)\ dx$

- **Improper integral (discontinuous)**
  - Let $f(x)$ be continuous on $[a, b)$, then;

$$\int_a^b f(x)\ dx = \lim_{t\to b^-} \int_a^t f(x)\ dx\ .$$

  - Let $f(x)$ be continuous on $(a, b]$, then;

$$\int_a^b f(x)\ dx = \lim_{t\to b^+} \int_t^b f(x)\ dx\ .$$

    In each case, if the limit exists, then the improper integral is said to converge. If the limit does not exist, then the improper integral is said to diverge.
  - Let $f(x)$ be continuous on $[a, b]$ except at a point $c \in (a, b)$, then;

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx.$$

    If either integral diverges, then $\int_a^b f(x)\ dx$ diverges

- **Comparison theorem** Let $f(x)$ and $g(x)$ be continuous over $[a, +\infty)$. Assume that $0 \leqslant f(x) \leqslant g(x)$ for $x \geqslant a$.
  - If $\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx = +\infty$,
    then $\int_a^{+\infty} g(x)\,dx = \lim_{t\to+\infty} \int_a^t g(x)\,dx = +\infty$.
  - If $\int_a^{+\infty} g(x)\,dx = \lim_{t\to+\infty} \int_a^t g(x)\,dx = L$, where L is a real number,
    then $\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx = M$ for some real number $M \leqslant L$.

- **P-integrals**
  - $\int_0^{+\infty} \frac{1}{x^p}\ dx = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ +\infty & \text{if } p \leqslant 1 \end{cases}$
  - $\int_0^{1} \frac{1}{x^p}\ dx = \begin{cases} \frac{1}{1-p} & \text{if } p < 1 \\ +\infty & \text{if } p \geqslant 1 \end{cases}$
  - $\int_a^{+\infty} \frac{1}{x^p}\ dx = \begin{cases} \frac{a^{1-p}}{p-1} & \text{if } p > 1 \\ +\infty & \text{if } p \leqslant 1 \end{cases}$
  - $\int_0^{a} \frac{1}{x^p}\ dx = \begin{cases} \frac{a^{1-p}}{1-p} & \text{if } p < 1 \\ +\infty & \text{if } p \geqslant 1 \end{cases}$
- **Bypass L'Hospital's Rule**

$$\ln(\ln(x)),\ \ln(x),\ \cdots\ x^{\frac{1}{100}},\ x^{\frac{1}{3}},\ \sqrt{x},\ 1,\ x^2,\ x^3,\ \cdots\ e^x,\ e^{2x},\ e^{3x},\ \cdots,\ e^{x^2},\ \cdots\ e^{e^x}.$$

  Essentially what it means is things on the right grow faster than things on the left. Thus, if we have say:

$$\lim_{x\to\infty} \frac{x^2}{e^{2x}}.$$

  We can be sure that it is zero. Because this is $x^2 \cdot e^{-2x}$. If we take $\lim\limits_{x\to\infty} x^2 e^{-2x}$, we get $\infty \cdot 0$. As we see by the sequence e^{-2x} overrules x^2 and we can say the limit is zero.
- **Consideration for Limits**: Let $f : A \to B$ be a function defined by $x \mapsto f(x)$. If a point c lies outside the domain A, then the expression $\lim\limits_{x\to c} f(x)$ is not meaningful, and we classify this limit as undefined. For instance, the function arcsine has a domain of $[-1, 1]$. Therefore, limits like $\lim\limits_{x\to a} \sin^{-1}(x)$ where $a \notin [-1, 1]$ are undefined.
- **Why does**

$$\lim_{x\to 2} \tan^{-1} \frac{1}{x-2}.$$

$$\begin{aligned} &= \lim_{x\to 2^-} \tan^{-1} \frac{1}{x-2} \\ &= \lim_{x\to -\infty} \tan^{-1} x \\ &= -\pi/2. \end{aligned}$$

$$\begin{aligned} &= \lim_{x\to 2^+} \tan^{-1} \frac{1}{x-2} \\ &= \lim_{x\to +\infty} \tan^{-1} x \\ &= \frac{\pi}{2}. \end{aligned}$$

1.4 Chapter 5 Key Equations

- **Sequence notation**

$$\{a_n\}_{n=1}^{\infty}, \text{ or simply } \{a_n\}.$$

- **Sequence notation (ordered list)**

$$a_1,\ a_2,\ a_3,\ \cdots,\ a_n,\ \cdots.$$

- **Arithemetic Sequence Difference**

$$d = a_n - a_{n-1}.$$

- **Arithmetic sequence (common difference between subsequent terms) general form**

$$\begin{aligned} &\text{Index starting at } 0: \ a_n = a + nd \\ &\text{Index starting at } 1: \ a_n = a + (n-1)d \end{aligned}$$

.

- **Arithmetic sequence (common difference between subsequent terms) recursive form**

$$a_n = a_{n-1} + d.$$

- **Sum of arithmetic sequence**

$$\begin{aligned} S_n &= \frac{n}{2}\left[a + a_n\right] \\ S_n &= \frac{n}{2}\left[2a + (n-1)d\right]. \end{aligned}$$

- **Geometric sequence form common ratio**

$$r = \frac{a_n}{a_{n-1}}.$$

- **Geometric sequence general form**

$$\begin{aligned} a_n &= ar^n \text{ (Index starting at 0)} \\ a_n &= a^{n+1} \text{(index starting at 0 and a=r)} \\ a_n &= ar^{n-1} \text{ (Index starting at 1)} \\ a_n &= a^n \text{(index starting at 1 and a=r)}. \end{aligned}$$

- **Geometric sequence recursive form**

$$a_n = ra_{n-1}.$$

- **Sum of geometric sequence (finite terms)**

$$S_n = \frac{a(1-r^n)}{1-r} \qquad r \neq 1.$$

- **Convergence / Divergence**: If

$$\lim_{n \to +\infty} a_n = L.$$

We say that the sequence converges, else it diverges

- **Formal definition of limit of sequence**

$$\lim_{n\to+\infty} a_n = L \iff \forall\varepsilon > 0, \exists N \in \mathbb{Z} \mid |a_n - L| < \varepsilon, \text{ if } n \geqslant n.$$

  Then we can say

$$\lim_{n\to+\infty} a_n = L \text{ or } a_n \to L.$$

- **Limit of a sequence defined by a function**: Consider a sequence $\{a_n\}$ such that $a_n = f(n)$ for all $n \geqslant 1$. If there exists a real number L such that

$$\lim_{x\to\infty} f(x) = L,$$

  then $\{a_n\}$ converges and

$$\lim_{n\to\infty} a_n = L.$$

- **Algebraic limit laws**: Given sequences $\{a_n\}$ and $\{b_n\}$ and any real number c, if there exist constants A and B such that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then
  - $\lim_{n\to\infty} c = c$
  - $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n = cA$
  - $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = A \pm B$
  - $\lim_{n\to\infty}(a_n \cdot b_n) = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n) = A \cdot B$
  - $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{A}{B}$, provided $B \neq 0$ and each $b_n \neq 0$.

- **Continuous Functions Defined on Convergent Sequences**: Consider a sequence $\{a_n\}$ and suppose there exists a real number L such that the sequence $\{a_n\}$ converges to L. Suppose f is a continuous function at L. Then there exists an integer N such that f is defined at all values a_n for $n \geqslant N$, and the sequence $\{f(a_n)\}$ converges to $f(L)$.

- **Squeeze Theorem for Sequences**: Consider sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$. Suppose there exists an integer N such that

$$a_n \leqslant b_n \leqslant c_n \text{ for all } n \geqslant N.$$

  If there exists a real number L such that

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n,$$

  then $\{b_n\}$ converges and $\lim_{n\to\infty} b_n = L$

- **Bounded above**: A sequence $\{a_n\}$ is bounded above if there exists a real number M such that

$$a_n \leqslant M$$

  for all positive integers n.

- **Bounded below**: A sequence $\{a_n\}$ is bounded below if there exists a real number M such that

$$M \leqslant a_n$$

  for all positive integers n.

- **Bounded**: A sequence $\{a_n\}$ is a bounded sequence if it is bounded above and bounded below.
- **Unbounded**: If a sequence is not bounded, it is an unbounded sequence.
- **If a sequence $\{a_n\}$ converges, then it is bounded.**
- **Increasing sequence**: A sequence $\{a_n\}$ is increasing for all $n \geqslant n_0$ if

$$a_n \leqslant a_{n+1} \text{ for all } n \geqslant n_0.$$

- **Decreasing sequence**: A sequence $\{a_n\}$ is decreasing for all $n \geqslant n_0$ if

$$a_n \geqslant a_{n+1} \text{ for all } n \geqslant n_0.$$

- **Monotone sequence**: A sequence $\{a_n\}$ is a **monotone sequence** for all $n \geqslant n_0$ if it is increasing for all $n \geqslant n_0$ or decreasing for all $n \geqslant n_0$

- **Monotone Convergence Theorem**: If $\{a_n\}$ is a bounded sequence and there exists a positive integer n_0 such that $\{a_n\}$ is monotone for all $n \geqslant n_0$, then $\{a_n\}$ converges.

- **Infinite Series form:**

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots ..$$

- **Partial sum (k^{th} partial sum)**

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k.$$

- **Convergence of infinity series notation**

  For a series, say...

$$\sum_{n=1}^{\infty} a_n \ .$$

  its convergence is determined by the limit of its sequence of partial sums. Specifically, if

$$\lim_{n \to +\infty} S_n = S \to \sum_{n=1}^{\infty} a_n \ = S.$$

- **Harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \ .$$

  Which diverges to $+\infty$

- **Algebraic Properties of Convergent Series** Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series. Then the following algebraic properties hold:

  1. The series $\sum_{n=1}^{\infty}(a_n + b_n)$ converges and

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n. \quad \text{(Sum Rule)}.$$

  2. The series $\sum_{n=1}^{\infty}(a_n - b_n)$ converges and

$$\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n. \quad \text{(Difference Rule)}.$$

  3. For any real number c, the series $\sum_{n=1}^{\infty} ca_n$ converges and

$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n. \quad \text{(Constant Multiple Rule)}.$$

- **Geometric series convergence or divergence:**

$$\sum_{n=1}^{\infty} ar^{n-1} \quad = \quad \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ diverges & \text{if } |r| \geqslant 1 \end{cases}.$$

- **Divergence test**: In the context of sequences, if $\lim_{n\to\infty} a_n = c \neq 0$ or the limit does not exist, then the series $\sum_{n=1}^{\infty} a_n$ is said to diverge. The converse is not true.

Because:

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} (S_k - S_{k-1}) = \lim_{k\to\infty} S_k - \lim_{k\to\infty} S_{k-1} = S - S = 0..$$

- **Integral Test Prelude**: for any integer k, the kth partial sum S_k satisfies

$$S_k = a_1 + a_2 + a_3 + \cdots + a_k < a_1 + \int_1^k f(x)\,dx < a_1 + \int_1^\infty f(x)\,dx..$$

and

$$S_k = a_1 + a_2 + a_3 + \cdots + a_k > \int_1^{k+1} f(x)\,dx..$$

- **Intgeral test** Suppose $\sum\limits_{n=1}^{\infty} a_n$ is a series with positive terms a_n Suppose there exists a function f and a positive integer N such that the following three conditions are satisfied:

  1. f positive, continuous, and decreasing on $[N, \infty)$
  2. $f(n) = a_n$ for all integers $n \geqslant N$, $N \in \mathbb{Z}^+$

  Then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ and the improper integral $\displaystyle\int_N^\infty f(x)\,dx$ either both converge or both diverge..

- **P-series** $\forall p \in \mathbb{R}$, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^P}.$$

Is called a **p-series**. Furthermore,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leqslant 1. \end{cases}.$$

- **P-series extended**

$$\sum_{n=2}^{\infty} \frac{1}{n \ln{(n)}^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leqslant 1. \end{cases}.$$

- **Remainder estimate for the integral test** Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series with positive terms. Suppose there exists a function f satisfying the following three conditions:

  1. f is continuous,
  2. f is decreasing, and
  3. $f(n) = a_n$ for all integers $n \geqslant 1$.

  Let S_N be the Nth partial sum of $\sum_{n=1}^{\infty} a_n$. For all positive integers N,

$$S_N + \int_{N+1}^\infty f(x)\,dx < \sum_{n=1}^{\infty} a_n < S_N + \int_N^\infty f(x)\,dx.$$

  In other words, the remainder $R_N = \sum_{n=1}^{\infty} a_n - S_N = \sum_{n=N+1}^{\infty} a_n$ satisfies the following estimate:

$$\int_{N+1}^\infty f(x)\,dx < R_N < \int_N^\infty f(x)\,dx.$$

  This is known as the remainder estimate

To find a value of N such that we are withing a desired margin of error, Since we know $R_n < \int_N^\infty f(x)\,dx$. Simply compute the improper integral and set the result < the desired error to solve for N

- **Find a_n given the expression for the partial sum**

$$a_n = S_n - S_{n-1}.$$

- **telescoping series**: Telescoping series are a type of series where each term cancels out a part of another term, leaving only a few terms that do not cancel. When you sum the series, most of the terms collapse or "telescope," which simplifies the calculation of the sum. Here are some key points and generalizations you can note about telescoping series:
  - Partial Fraction Decomposition
  - Cancellation Pattern: In a telescoping series, look for a pattern where a term in one fraction will cancel out with a term in another fraction.
  - Write out Terms
  - What is left is S_n, thus the sum of the series is the $\lim\limits_{n\to\infty} S_n$

  Try:

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}\ .$$

  Hint, its not only the first and last terms cancel, we also have a $\frac{\frac{1}{2}}{n}$, when a_{n-1}: Answer is $\frac{3}{4}$

- **Comparison test for series**
  1. Suppose there exists an integer N such that $0 \leqslant a_n \leqslant b_n$ for all $n \geqslant N$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
  2. Suppose there exists an integer N such that $a_n \geqslant b_n \geqslant 0$ for all $n \geqslant N$. If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

- **Limit Comparison Test** Let $a_n, b_n \geqslant 0$ for all $n \geqslant 1$.
  - If $\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.
  - If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
  - If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

  Note: Note that if $\frac{a_n}{b_n} \to 0$ and $\sum_{n=1}^{\infty} b_n$ diverges, the limit comparison test gives no information. Similarly, if $\frac{a_n}{b_n} \to \infty$ and $\sum_{n=1}^{\infty} b_n$ converges, the test also provides no information.

  Consider the series

$$\sum_{n=1}^{\infty} \frac{n^4+6}{n^5+4}\ .$$

  To find our b_n we can only focus on the leading coefficients. Thus:

$$b_n = \frac{n^4}{n^5} = \frac{1}{n}.$$

  So our test...

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\frac{n^4+6}{n^5+4}}{\frac{1}{n}}$$
$$= \lim_{n\to\infty}\frac{n(n^4+6)}{n^5+4}$$
$$= \lim_{n\to\infty}\frac{n^5+6n}{n^5+4}$$
$$= 1.$$

Since $\lim\limits_{n\to\infty}\frac{a_n}{b_n} \neq 0 \vee +\infty$. And $\frac{1}{n}$ diverges, we can conclude that a_n will also diverge.

- **Determine which series (or function) is greater**
  - **Subtraction**: Given two functions $f(x) = \frac{1}{x}$ and $g(x) = \frac{x^4+6}{x^5+4}$, we want to compare them by considering the function $h(x) = f(x) - g(x)$:

    $$h(x) = f(x) - g(x) = \frac{1}{x} - \frac{x^4+6}{x^5+4}$$

    To compare these directly, it would be helpful to have a common denominator:

    $$h(x) = \frac{x^4+4-(x^4+6)}{x(x^5+4)} = \frac{-2}{x(x^5+4)}$$

    Now, we can see that the sign of $h(x)$ depends on the sign of x because the denominator $x(x^5+4)$ is always positive for $x \neq 0$. So:
    * For $x > 0$, $h(x) < 0$, which means $f(x) < g(x)$.
    * For $x < 0$, $h(x) > 0$, which means $f(x) > g(x)$.

- **Alternating Series** Any series whose terms alternate between positive and negative values is called an alternating series. An alternating series can be written in the form

  $$\sum_{n=1}^{\infty}(-1)^{n+1}b_n = b_1 - b_2 + b_3 - b_4 + \cdots.$$

  or

  $$\sum_{n=1}^{\infty}(-1)^{n}b_n = -b_1 + b_2 - b_3 + b_4 - \cdots.$$

  Where $b_n > 0$ for all positive integers n.

- **alternating series test (Leibniz criterion)** An alternating series of the form

  $$\sum_{n=1}^{\infty}(-1)^{n+1}b_n \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^{n}b_n$$

  converges if
  - $0 < b_{n+1} \leqslant b_n \ \forall\ n \geqslant 1$
  - $\lim_{n\to\infty} b_n = 0$.

  Note: We remark that this theorem is true more generally as long as there exists some integer N such that $0 < b_{n+1} \leqslant b_n$ for all $n \geqslant N$.

Additional note: The AST allows us to consider just the positive terms to check for these two conditions because if a series of decreasing positive terms that approach zero is alternated in sign, the alternating series will converge. This is a special property of alternating series that does not generally hold for non-alternating series.

- **Show decreasing (For the AST)**: Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \ .$$

So you see we have $b_n = \frac{1}{n^2}$. For the AST, we must show that this is decreasing. If $b_{n+1} = \frac{1}{(n+1)^2}$. Then we see

$$\frac{1}{(n+1)^2} < \frac{1}{n^2}.$$

Thus it is decreasing for $n \geqslant 1$ $(b_{n+1} < b_n)$ $\blacksquare$

- **Remainders in alternating series** Consider an alternating series of the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^n b_n,$$

that satisfies the hypotheses of the alternating series test. Let S denote the sum of the series and S_N denote the N-th partial sum. For any integer $N \geqslant 1$, the remainder $R_N = S - S_N$ satisfies

$$|R_N| \leqslant b_{N+1}.$$

This tells us that if we stop at the N^{th} term, the error we are making is at most the size of the next term

- **Absolute and conditional convergence**
  - A series $\sum_{n=1}^{\infty} a_n$ exhibits absolute convergence if $\sum_{n=1}^{\infty} |a_n|$ converges.
  - A series $\sum_{n=1}^{\infty} a_n$ exhibits conditional convergence if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.
  - If $\sum\limits_{n=1}^{\infty} |a_n|$ converges then $\sum\limits_{n=1}^{\infty} a_n$ converges

Note: if $|a_n|$ diverges, we cannot have absolute convergence, thus we must examine to see if normal a_n converges, in which case we would have conditional convergence

Big Note: If a series not strictly decreasing, we can still check for absolute/conditional convergence. Take $\sum\limits_{n=1}^{\infty} \frac{\sin n}{3^n+4}$ for example.

- **Ratio test** Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms. Let

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|..$$

Then:

i. If $0 \leqslant \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
ii. If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
iii. If $\rho = 1$, the test does not provide any information.

Note: The ratio test is useful for series whose terms involve factorials

- **Root test** Consider the series $\sum_{n=1}^{\infty} a_n$. Let

$$\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}..$$

  i. If $0 \leqslant \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
  ii. If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
  iii. If $\rho = 1$, the test does not provide any information.

  Note: The root test is useful for series whose terms involve exponentials

- **Which tests require positive terms**
  - The **Integral Test:** This test applies to series where the terms come from a function that is positive, continuous, and decreasing on a certain interval. The convergence or divergence of the series is determined by the convergence or divergence of the corresponding improper integral of the function.
  - The **Remainder estimate for the integral test**
  - The **Comparison Test:** This test compares the terms of a series to those of another series with known convergence behavior. It requires that the terms of both series be positive or non-negative.
  - The **Limit Comparison Test:** Similar to the Comparison Test, this test involves comparing the terms of two series by taking the limit of the ratio of their terms. It requires that the terms of both series be positive.
  - In **alternating series**, b_n must have only positive terms

1.5 Chapter 6 Key equations

- **Euler definition for e**

$$
\begin{aligned}
e^a &= \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n \\
\frac{1}{e^a} &= \lim_{n\to\infty}\left(1+\frac{-a}{n}\right)^n \\
\frac{1}{e^a} &= \lim_{n\to\infty}\left(\frac{n}{n+a}\right)^n .
\end{aligned}
$$

- **Other definition for e**

$$
\begin{aligned}
e &= \sum_{n=0}^{\infty}\frac{1}{n!} \\
e-1 &= \sum_{n=1}^{\infty}\frac{1}{n!} \\
e^x &= \sum_{n=0}^{\infty}\frac{x^n}{n!} .
\end{aligned}
$$

- **Power series**: A series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots .$$

  is a power series centered at $x = 0$.

  A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots .$$

  is a power series centered at $x = a$.

- **Convergence of a Power Series**

  Consider the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$. The series satisfies exactly one of the following properties:

  (i) The series converges at $x = a$ and diverges for all $x \neq a$.
  (ii) The series converges for all real numbers x.
  (iii) There exists a real number $R > 0$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$. At the values x where $|x-a| = R$, the series may converge or diverge.

- **A power series always converges at its center**

- **Radius of convergence**: Consider the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$. The set of real numbers x where the series converges is the interval of convergence. If there exists a real number $R > 0$ such that the series converges for $|x-a| < R$ and diverges for $|x-a| > R$, then R is the radius of convergence. If the series converges only at $x = a$, we say the radius of convergence is $R = 0$. If the series converges for all real numbers x, we say the radius of convergence is $R = \infty$ (Figure 6.2).

- **Finding interval of convergence and radius of convergence**
  - Fact: power series is always convergent on its center
  - Use ratio test (values of ρ)
  - Use $\rho < 1$ to find Radius of convergence
  - Test end points of interval by plugging into original series and seeing whether the series is convergent or divergent

- **If $\rho = 0$, the power series converges for all x**

- **If** $\rho = \infty$, the series diverges for all $x \neq a$

- **Combining Power Series:** Suppose that the two power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to the functions f and g, respectively, on a common interval I.
  (i) The power series $\sum_{n=0}^{\infty}(c_n x^n \pm d_n x^n)$ converges to $f \pm g$ on I.
  (ii) For any integer $m \geqslant 0$ and any real number b, the power series $\sum_{n=0}^{\infty} bx^m c_n x^n$ converges to $bx^m f(x)$ on I.
  Eg: If we know $\sum\limits_{n=0}^{\infty} a_n x_n$ has $I = (-1, 1)$. Then
  $$\begin{aligned} &\sum_{n=0}^{\infty} a_n 3^n x^n \\ =&\sum_{n=0}^{\infty} a_n (3x)^n \\ &I = (-3, 3). \end{aligned}$$
  (iii) For any integer $m \geqslant 0$ and any real number b, the series $\sum_{n=0}^{\infty} c_n (bx^m)^n$ converges to $f(bx^m)$ for all x such that bx^m is in I.
- **For part I, II, and III, the interval of the combined series is the smaller interval**
- **Problem to remember:** Combining power series
- **Cauchy product (Multiplying power series)**: Suppose that the power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to f and g, respectively, on a common interval I. Let
  $$\begin{aligned} e_n &= c_0 d_n + c_1 d_{n-1} + c_2 d_{n-2} + \cdots + c_{n-1} d_1 + c_n d_0 \\ &= \sum_{k=0}^{n} c_k d_{n-k}.. \end{aligned}$$
  Then
  $$\left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=0}^{\infty} d_n x^n\right) = \sum_{n=0}^{\infty} e_n x^n$$
  and
  $$\sum_{n=0}^{\infty} e_n x^n \text{ converges to } f(x) \cdot g(x) \text{ on } I.$$
  The series $\sum_{n=0}^{\infty} e_n x^n$ is known as the Cauchy product of the series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$.
- **Sterling's Approximation**
  $$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$
- **Gamma function (extension of the factorial function)**
  $$\begin{aligned} \Gamma(z) &= \int_0^{\infty} e^{-t} t^{z-1}\, dt \\ \text{Thus, } n! &= \Gamma(n+1). \end{aligned}$$
- **Cool definition for e^x**
  $$\begin{aligned} &f'(x) = rf(x) \\ &\implies f(x) = ce^{rx}. \end{aligned}$$

- **Term-by-Term Differentiation and Integration for Power Series.** Suppose that the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges on the interval $(a-R, a+R)$ for some $R > 0$. Let f be the function defined by the series
$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$
for $|x-a| < R$. Then f is differentiable on the interval $(a-R, a+R)$ and we can find f' by differentiating the series term-by-term:
$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$
for $|x-a| < R$. Also, to find $\int f(x)\,dx$, we can integrate the series term-by-term. The resulting series converges on $(a-R, a+R)$, and we have
$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1} = C + c_0(x-a) + \frac{c_1(x-a)^2}{2} + \frac{c_2(x-a)^3}{3} + \cdots$$
for $|x-a| < R$.

  NOTE! when a power series is differentiated or integrated term-by-term, it says nothing about what happens at the endpoints.

- **Uniqueness of Power Series**: Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ and $\sum_{n=0}^{\infty} d_n(x-a)^n$ be two convergent power series such that
$$\sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} d_n(x-a)^n$$
for all x in an open interval containing a. Then $c_n = d_n$ for all $n \geqslant 0$.

- **When finding the Cauchy product of two power series, we include the zero term when finding the new power series general term. For integrating and differentiating, we do not**

- **Taylor and Maclaurin series**: If f has derivatives of all orders at $x = a$, then the Taylor series for the function f at a is
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots . \tag{14}$$
The Taylor series for f at 0 is known as the Maclaurin series for f.

- **Uniqueness of Taylor series**: If a function f has a power series at a that converges to f on some open interval containing a, then that power series is the Taylor series for f at a.

- **Taylor-Macluarin Polynomials**: If f has n derivatives at $x = a$, then the nth Taylor polynomial for f at a is
$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{15}$$
The nth Taylor polynomial for f at 0 is known as the nth Maclaurin polynomial for f.

- **Taylor's Theorem with Remainder**: Let f be a function that can be differentiated $n+1$ times on an interval I containing the real number a. Let p_n be the nth Taylor polynomial of f at a and let
$$R_n(x) = f(x) - p_n(x)$$

be the nth remainder. Then for each x in the interval I, there exists a real number c between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

If there exists a real number M such that $\left|f^{(n+1)}(x)\right| \leqslant M$ for all $x \in I$, then

$$|R_n(x)| \leqslant \frac{M}{(n+1)!}\left|x-a\right|^{n+1}$$

$\forall\ x \in I$

- **Maclaurin Series/Polynomials for sine**: The Taylor series for the sine function is

$$\sin(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} \ = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \quad \text{For } x \in \mathbb{R}.$$

Where p_n obeys

$$\begin{aligned} p_{2m+1} &= p_{2m+2} \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... + \frac{(-1)^m x^{2m+1}}{(2m+1)!}. \end{aligned}$$

Note: When discussing specific polynomials, say P_5 for example, we arnt talking about the first 5 terms in the series above, we are talking about the polynomial `up to` degree 5. Thus it would have 3 terms

- **Maclaurin Series/Polynomials for cosine**: Similar to the sine function, the Maclaurin series for the cosine function is

$$\cos(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \ = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \quad \text{For } x \in \mathbb{R}.$$

Where p_n obeys

$$\begin{aligned} p_{2m} &= p_{2m+1} \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... + (-1)^n\frac{x^{2m}}{(2m)!}. \end{aligned}$$

- **Maclaurin Series/Polynomials for** e^x: We find the Maclaurin series for the exponential function to be

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \ = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \quad \text{For } x \in \mathbb{R}.$$

Note: this definition is described above but now we have a way of showing its truthiness

- **Convergence of Taylor Series**: Suppose that f has derivatives of all orders on an interval I containing a. Then the Taylor series

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

converges to $f(x)$ for all x in I if and only if

$$\lim_{n\to\infty} R_n(x) = 0$$

for all x in I.

Note: With this theorem, we can prove that a Taylor series for f at a converges to f if we can prove that the remainder $R_n(x) \to 0$. To prove that $R_n(x) \to 0$, we typically use the bound

$$|R_n(x)| \leqslant \frac{M}{(n+1)!}|x-a|^{n+1}$$

from Taylor's theorem with remainder.

- **Using taylor series to find limits**: Consider the limit $\lim_{x \to 0+} \frac{\cos\sqrt{x}-1}{2x}$. We know we have a problem if we attempt to use the `direct substitution property`. Thus, we can substitute $\cos(\sqrt{x})$ for its `Maclaurin series` and see what happens. We know the maclaurin series for $\cos(x)$ is

$$\begin{aligned}
\cos x &\sim \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} &&= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+... \\
\implies \cos\sqrt{x} &\sim \sum_{n=0}^{\infty}(-1)^n\frac{(x^{\frac{1}{2}})^{2n}}{(2n)!} &&= 1-\frac{(x^{\frac{1}{2}})^2}{2!}+\frac{(x^{\frac{1}{2}})^4}{4!}-\frac{(x^{\frac{1}{2}})^6}{6!}+... \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{(2n)!} &&= 1-\frac{x}{2!}+\frac{x^2}{4!}-\frac{x^3}{6!}+....
\end{aligned}$$

So we have

$$\begin{aligned}
&\lim_{x\to 0^+}\frac{\left(1-\frac{x}{2!}+\frac{x^2}{4!}-\frac{x^3}{6!}+...\right)-1}{2x} \\
&= \lim_{x\to 0^+}\frac{\left(-\frac{x}{2!}+\frac{x^2}{4!}-\frac{x^3}{6!}+...\right)}{2x} \\
&= \lim_{x\to 0^+}\left(-\frac{x}{2!}+\frac{x^2}{4!}-\frac{x^3}{6!}+...\right)\cdot\frac{1}{2x} \\
&= \lim_{x\to 0^+} -\frac{1}{4} \\
&= -\frac{1}{4}.
\end{aligned}$$

- **Multiplying a known Taylor series by some other function:** Consider $f(x) = x\cos x$. Since we know that the taylor series for $\cos(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$, which converges $\forall\ x \in \mathbb{R}$. We can easily just multiply this by x to get the taylor series for $f(x) = x\cos(x)$. Since the Taylor series for $\cos(x)$ converges for all real x, multiplying it by x won't affect its convergence properties. The resulting series will still converge for all x.

  Note: The product of the Taylor series and the function will be valid only where both the series converges and the function is well-defined. Probably analyze the convergence of the product series.

- **Analytic function:**
  - An analytic function is infinitely differentiable within its domain.
  - An analytic function can be represented by a convergent power series (like a Taylor series) around any point in its domain.
  - The power series representing an analytic function not only exists but also converges to the function within a certain radius around the point of expansion

- **Maclaurin series for $\frac{1}{1-x}$:**

$$\frac{1}{1-x} \sim \sum_{n=0}^{\infty}{}^{n} \qquad \text{for } |x| < 1.$$

- **Maclaurin series for** $\ln(1+x)$:

$$\ln(1+x) \sim \sum_{n=1}^{\infty} -1)^{n+1} \frac{x^n}{n} \qquad \text{for } -1 < x \leqslant 1.$$

- **Maclaurin series for** $\tan^{-1} x$:

$$\tan^{-1} x \sim \sum_{n=0}^{\infty} -1)^{n} \frac{x^{2n+1}}{2n+1} \qquad \text{for } |x| \leqslant 1.$$

Binomial expansion for $(1+x)^r$ for $r \in \mathbb{Z}^+$

$$(1+x)^r = \sum_{n=0}^{r} \binom{r}{n} x^n \qquad r \in \mathbb{Z}^+.$$

Binomial expansion for $(1+x)^r$ for $r \in \mathbb{R}$

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!}x^n + \cdots .$$

Where

$$\binom{r}{n} = \frac{f^{(n)}(0)}{n!} = \frac{r(r-1)(r-2)\cdot \ldots \cdot (r-n+1)}{n!}.$$

Note: When $n=0$, $\binom{r}{0}=1$, when $n=1$, $\binom{r}{1}=r$, when $n=2$, $\binom{r}{2}=\frac{r(r-1)}{2!}$, etc...

The binomial theorem: For any real number r, the Maclaurin series for $f(x)=(1+x)^r$ is the binomial series. It converges to f for $|x|<1$, and we write

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!}x^n + \cdots$$

for $|x|<1$.

1.6 Chapter 6 Problems to Remember

- **Problem to remember (Properties of power series):** Evaluate the infinite series by identifying it as the value of an integral of a geometric series.

  $$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \, .$$

  Remark. If we can find which geometric power series's integral (with some bounds) gives us the given series, we can then integrate the function representation to get the value of the original series. Consider the geometric power series

  $$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n \; = \sum_{n=0}^{\infty} (-1)^n x^{2n} \, .$$

  Suppose we then integrate the power series

  $$\begin{aligned} &\int \sum_{n=0}^{\infty} (-1)^n x^{2n} \; dx \\ &= \sum_{n=0}^{\infty} (-1)^n \int x^{2n} \; dx \\ &= \frac{1}{2n+1} x^{2n+1}. \end{aligned}$$

  Now we must deduce for which bounds will the FTC give us the original series $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$. We come to the conclusion

  $$\sum_{n=0}^{\infty} (-1)^n \int_0^1 x^{2n} \; dx = \sum_{n}^{\infty} \frac{(-1)^n}{2n+1} \, .$$

  This implies we can integrate the function representation of the geometric power series we just integrated to get the value of the infinite series $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$. Thus,

  $$\int_0^1 \frac{1}{1+x^2} \; dx = \frac{\pi}{4}.$$

- **Problem to remember (Properties of power series):** Find the power series for $f(x) = \ln x$ centered at $x = 9$ by using term-by-term integration or differentiation.

  Solution. The goal is to find a function that resembles one we know (sum of geometric series $\frac{a}{1-x}$) such that if we integrate or differentiate we can get $\ln x$. Since we know the integral of $\frac{1}{x}$ is $\ln x$, and we can easily manipulate $\frac{1}{x}$ to be in the form $\frac{a}{1-x}$, we choose $\frac{1}{x}$ to be the function to examine. Thus,

  $$\frac{1}{x} = \frac{1}{9+x-9} = \frac{1}{9-(-(x-9))} = \frac{1/9}{1-\left(\frac{-(x-9)}{9}\right)}$$

  $$\text{If } f(x) = \frac{a}{1-x} \sim \sum_{n=0}^{\infty} a(x^n) \; = a + ax + ax^2 + ax^3 + ...$$

  $$\implies f(x) = \frac{1/9}{1-\left(\frac{-(x-9)}{9}\right)} \sim \sum_{n=0}^{\infty} \frac{1}{9} \left(\frac{-(x-9)}{9}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n (x-9)^n}{9^{n+1}} \, .$$

Then we can throw in some integrals

$$
\begin{aligned}
\int f(x)\ dx &= \int \frac{1/9}{1-\left(\frac{-(x-9)}{9}\right)}\ dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n (x-9)^n}{9^{n+1}}\ dx \\
\ln x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{9^{n+1}} \int (x-9)^n\ dx \\
\ln x &= \sum_{n=0}^{\infty} \frac{(-1)^n (x-9)^{n+1}}{(n+1)9^{n+1}} + C.
\end{aligned}
$$

If we let $x = 9$

$$
\begin{aligned}
\ln 9 &= \sum_{n=0}^{\infty} \frac{(-1)^n (9-9)^{n+1}}{(n+1)9^{n+1}} + C \\
\ln 9 &= C.
\end{aligned}
$$

Thus, we have

$$
f(x) = \ln 9 + \sum_{n=0}^{\infty} \frac{(-1)^n (9-9)^{n+1}}{(n+1)9^{n+1}}
$$

.

Note: the "$+C$" is initially omitted from $\ln(x)$ because we're considering a specific antiderivative. When you integrate the power series, you include "$+C$" to account for the general form of the antiderivative. The value of C is then determined using a specific condition to match the specific antiderivative you're interested in.

- **Problem to remember**: Say we want to find the power series for $7x \ln 1 + x$. We can first find the power series for $\ln(1+x)$

$$
\frac{d}{dx} \ln(1+x) = \frac{1}{1+x} = \frac{1}{1-(-x)}.
$$

We know the power series for $\frac{1}{1-(-x)}$ is

$$
\begin{aligned}
&\sum_{n=0}^{\infty} (-1)^n x^n \\
&\implies \int \frac{1}{1+x} = \int \sum_{n=0}^{\infty} (-1) x^n \\
&\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C \\
&\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}\ .
\end{aligned}
$$

Now that we have found the power series for $\ln(1+x)$, to find the power series for $7x \ln(1+x)$...

$$
\begin{aligned}
&7x \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \\
&\sum_{n=0}^{\infty} 7x \left((-1)^n \frac{x^{n+1}}{n+1} \right) \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{7x^{n+2}}{n+1}\ .
\end{aligned}
$$

- **Problem to remember (Cumbersome taylor polynomial)**: Suppose we have some function f, and we would like to find the Taylor polynomial up to degree 3. Say $f(x) = e^{2x}\cos(x)$. We could find each derivative up to degree 3, however, given that we know the taylor series for both e^{2x} and $\cos(x)$.

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + ...$$
$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ... \,.$$

We can find $P_3(x)$ by multiplying these Taylor series. Thus,

$$(1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + ...)(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ...).$$

And we can find P_3 to be

$$P_3 = 1 + 2x + \frac{3}{2}x^2 + \frac{1}{3}x^3.$$

- **Problem to Remember (Using known Taylor series to find sum of series):** Consider the series $\sum\limits_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{25}\right)^{n-3}}{2n+1}$

We notice this resembles the Taylor series for the arctangent function $\tan^{-1} x = \sum\limits_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ for $|x| \leqslant 1$. Thus, we maipulate the series to better conform to the Taylor series for $\tan^{-1} x$.

$$\begin{aligned}
&\sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{25}\right)^{n-3}}{2n+1} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{5}\right)^{2n-6}}{2n+1} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{5}\right)^{2n} 5^6}{2n+1} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{5}\right)^{2n} 5^6 \left(\frac{1}{5}\right) 5}{2n+1} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{5}\right)^{2n+1} 5^7}{2n+1}
\end{aligned}$$
.

Thus the sum will be

$$5^7 \tan^{-1} \frac{1}{5}.$$

2 Vectors

2.1 Vector vocab

- A **vector** is two pieces of information.
  1. Length
  2. Direction (Magnitude)

2.2 Vector notation

- **Defining a vector**

$$\vec{v} = [x, y] \text{ or } \begin{bmatrix} x \\ y \end{bmatrix}.$$

- **Length of a vector**

$$||\vec{v}|| \in \mathbb{R}^{\ltimes} = \sqrt{v_1^2 + v_2^2 + ... + v_n^2}.$$

- **Vector addition** Suppose we have two vectors $\vec{v} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$. Then

$$\vec{v} + \vec{u} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} = \vec{c}.$$

  Note: we call this new vector the **resultant**

- **Multiplying by a scalar** Suppose we have the vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$. Then

$$2\vec{v} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}.$$

- **Vector subtraction**

$$\vec{v} - \vec{u} = \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \end{bmatrix}.$$

- **Vector in three dimensions**

$$\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

- **Definition for $\mathbb{R}^n$**: $\mathbb{R}^n$ is the set of all $n - tuples$ of real numbers

$$\begin{aligned} &\vec{v} = [v_1, v_2] \ \vec{v} \in \mathbb{R}^2 \\ &\vec{u} = [u_1, u_2, u_3] \ \vec{u} \in \mathbb{R}^3 \\ &\vec{w} = [w_1, w_2, w_3, w_n] \ \vec{w} \in \mathbb{R}^n \\ &. \end{aligned}$$