Nate Warner MATH 230 October 27, 2023

Homework/Worksheet 7 - Due: Wednesday, November 1

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1. Use the trapezoidal rule and Simpson's rule to approximate the integral

$$\int_0^2 \frac{e^x}{1+x^2} \, dx.$$

With $n = 10$

Trapezoidal approximation:

Remark. $T_n = \frac{h}{2} \left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n) \right]$ with $h = \frac{b-a}{n}$

We have,

$$x_0 = 0, \; x_1 = \frac{1}{5}, \; x_2 = \frac{2}{5}, \; x_3 = \frac{3}{5}, \; x_4 = \frac{4}{5}, \; x_5 = 1, \; x_6 = \frac{6}{5}, \; x_7 = \frac{7}{5}, \; x_8 = \frac{8}{5}, \; x_9 = \frac{9}{5}, \; x_{10} = 2$$

$$f(x_0) = 1, \; f(x_1) = \frac{e^{\frac{1}{5}}}{1+\frac{1}{25}}, \; f(x_2) = \frac{e^{\frac{2}{5}}}{1+\frac{4}{25}}, \; f(x_3) = \frac{e^{\frac{3}{5}}}{1+\frac{9}{25}}, \; f(x_4) = \frac{e^{\frac{4}{5}}}{1+\frac{16}{25}}, \; f(x_5) = \frac{e}{2}$$

$$f(x_6) = \frac{e^{\frac{6}{5}}}{1+\frac{36}{25}}, \; f(x_7) = \frac{e^{\frac{7}{5}}}{1+\frac{49}{25}}, \; f(x_8) = \frac{e^{\frac{8}{5}}}{1+\frac{64}{25}}, \; f(x_9) = \frac{e^{\frac{9}{5}}}{1+\frac{81}{25}}, \; f(x_{10}) = \frac{e^2}{5}.$$

Thus:

$$T_{10} = \frac{1}{10} \left[1 + \frac{2e^{\frac{1}{5}}}{1+\frac{1}{25}} + \frac{2e^{\frac{2}{5}}}{1+\frac{4}{25}} + \frac{2e^{\frac{3}{5}}}{1+\frac{9}{25}} + \frac{2e^{\frac{4}{5}}}{1+\frac{16}{25}} + e + \frac{2e^{\frac{6}{5}}}{1+\frac{36}{25}} + \frac{2e^{\frac{7}{5}}}{1+\frac{49}{25}} + \frac{2e^{\frac{8}{5}}}{1+\frac{64}{25}} + \frac{2e^{\frac{9}{5}}}{1+\frac{81}{25}} + \frac{e^2}{5} \right]$$
$$\approx 2.6608.$$

Simpson's rule approximation

Remark. $S_n = \frac{h}{3} \left[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right]$ where $h = \frac{b-a}{n}$ and $n \in 2k$ for some integer k

Thus,

$$S_{10} = \frac{1}{10} \left[1 + \frac{4e^{\frac{1}{5}}}{1+\frac{1}{25}} + \frac{2e^{\frac{2}{5}}}{1+\frac{4}{25}} + \frac{4e^{\frac{3}{5}}}{1+\frac{9}{25}} + \frac{2e^{\frac{4}{5}}}{1+\frac{16}{25}} + 2e + \frac{2e^{\frac{6}{5}}}{1+\frac{36}{25}} + \frac{4e^{\frac{7}{5}}}{1+\frac{49}{25}} + \frac{2e^{\frac{8}{5}}}{1+\frac{64}{25}} + \frac{4e^{\frac{9}{5}}}{1+\frac{81}{25}} + \frac{e^2}{5} \right]$$
$$\approx 2.6632.$$

2. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

(a) $\int_0^{+\infty} \frac{1}{4+x^2} \, dx$

(b) $\int_e^{+\infty} \frac{1}{x \ln^2 x}$

(c) $\int_{-\infty}^{+\infty} \frac{e^x}{1+e^{2x}} \, dx$

(d) $\int_1^{+\infty} \frac{5}{x^3} \, dx$

2.a

$$\int_0^{+\infty} \frac{1}{4+x^2} \, dx$$

$$= \lim_{t \to +\infty} \int_0^t \frac{1}{4+x^2} \, dx$$

$$= \lim_{t \to +\infty} \frac{1}{2} \tan^{-1} \frac{x}{2} \Big|_0^t$$

$$= \lim_{t \to +\infty} \frac{1}{2} \tan^{-1} \frac{t}{2} - \left(\frac{1}{2} \tan^{-1} 0 \right)$$

$$= \frac{1}{2} \lim_{t \to +\infty} \tan^{-1} \frac{t}{2}$$

$$= \frac{\pi}{4}.$$

2.b

Let $u = \ln x$

$du = \frac{1}{x} \, dx$

When $x = t$, $u = \ln t$

When $x = e$, $u = \ln e = 1$.

$$\int_e^{+\infty} \frac{1}{x \ln^2 x} \, dx$$

$$= \lim_{t \to +\infty} \int_e^t \frac{1}{x \ln^2 x} \, dx$$

$$= \lim_{t \to +\infty} \int_1^{\ln t} \frac{1}{u^2} \, du$$

$$= \lim_{t \to +\infty} -u^{-1} \Big|_1^{\ln t}$$

$$= \lim_{t \to +\infty} -\frac{1}{\ln(t)} + 1$$

$$= 1.$$

2.c

$$\int_{-\infty}^{+\infty} \frac{e^x}{1 + (e^x)^2} \, dx$$

$$\int_{-\infty}^{0} \frac{e^x}{1 + (e^x)^2} \, dx + \int_{0}^{+\infty} \frac{e^x}{1 + (e^x)^2} \, dx$$

$$\lim_{t \to -\infty} \int_{t}^{0} \frac{e^x}{1 + (e^x)^2} \, dx + \lim_{t \to +\infty} \int_{0}^{t} \frac{e^x}{1 + (e^x)^2} \, dx.$$

Let $u = e^x$

$du = e^x \, dx.$

I_1 : When $x = t$, $u = e^t$ I_2 : When $x = 0$, $u = 1$

When $x = 0$, $u = 1$.   When $x = t$, $u = e^t$.

I_1 :

$$\lim_{t \to -\infty} \int_{t}^{0} \frac{e^x}{1 + (e^x)^2} \, dx$$

$$= \lim_{t \to -\infty} \int_{e^t}^{1} \frac{du}{1 + u^2}$$

$$= \lim_{t \to -\infty} \tan^{-1} u \, \Big|_{e^t}^{1}$$

$$= \lim_{t \to -\infty} \tan^{-1} 1 - \tan^{-1} e^t$$

$$= \frac{\pi}{4} - \lim_{t \to -\infty} \tan^{-1} e^t$$

$$= \frac{\pi}{4} - \lim_{t \to 0} \tan^{-1} t \quad (\text{Since } \lim_{t \to -\infty} e^t = 0)$$

$$= \frac{\pi}{4} - 0$$

$$= \frac{\pi}{4}.$$

Thus, I_1 converges to $\frac{\pi}{4}$

I_2:

$$\lim_{t \to +\infty} \int_{0}^{t} \frac{e^x}{1 + (e^x)^2} \, dx$$

$$= \lim_{t \to +\infty} \int_{1}^{e^t} \frac{1}{1 + u^2} \, du$$

$$= \lim_{t \to +\infty} \tan^{-1} u \, \Big|_{1}^{e^t}$$

$$= \lim_{t \to +\infty} \tan^{-1} e^t - \tan^{-1} 0$$

$$= \lim_{t \to +\infty} \tan^{-1} e^t - \frac{\pi}{4}$$

$$= \lim_{t \to +\infty} \tan^{-1} t - \frac{\pi}{4} \quad (\text{Since } \lim_{t \to +\infty} e^t = +\infty)$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}.$$

Thus, I_2 also converges to $\frac{\pi}{4}$, Which means we have:

$$I = \frac{\pi}{4} + \frac{\pi}{4}$$

$$= \frac{\pi}{2}.$$

2.d

$$\int_1^{+\infty} \frac{5}{x^3}\,dx$$

$$= \lim_{t\to+\infty} \int_1^t \frac{5}{x^3}\,dx$$

$$= \lim_{t\to+\infty} \left. -\frac{5}{2x^2}\right|_1^t$$

$$= \lim_{t\to+\infty} \overset{0}{\cancel{-\frac{5}{2t^2}}} + \frac{5}{2}$$

$$= \frac{5}{2}.$$

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3. Use the Comparison Theorem to determine whether the integral

$$\int_1^{+\infty} \frac{dx}{1+\sqrt{x}}\,dx.$$

is convergent or divergent

Remark. Let $f(x)$ and $g(x)$ be continuous over $[a, +\infty)$, assume $0 \leqslant f(x) \leqslant g(x)$

if $\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx = +\infty$

then $\int_a^{+\infty} g(x)\,dx = \lim_{t\to+\infty} \int_a^t g(x)\,dx = +\infty$

Alternatively,

if $\int_a^{+\infty} g(x)\,dx = \lim_{t\to+\infty} \int_a^t g(x)\,dx = L$ for $L \in \mathbb{R}$

then $\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx = M$ for $M \leqslant L$ where $M \in \mathbb{R}$

Let $f(x)$ be $\frac{1}{1+\sqrt{x}}$, choose $g(x) = \frac{1}{\sqrt{x}}$. If $g(x)$ diverges to $+\infty$, then $f(x)$ diverges to $+\infty$

$$\int_1^{+\infty} \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{t\to+\infty} \int_1^t \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{t\to+\infty} \left. 2x^{\frac{1}{2}}\right|_1^t$$

$$= \lim_{t\to+\infty} \overset{+\infty}{\cancel{2t^{\frac{1}{2}}}} - 2.$$

Thus, $f(x) = \frac{1}{1+\sqrt{x}}$ diverges to $+\infty$ since $g(x) = \frac{1}{\sqrt{x}}$ diverges to $+\infty$

> **4.** Find a formula for the general term of the arithmetic sequence whose first term is $a_1 = 1$ such that $a_{n-1} - a_n = 17$ for $n \geqslant 1$.

Remark. The general form of an arithmetic sequence is of the type $a_1 + (n-1)d$ for $n \geqslant 1$. S.t d is the common difference defined $a_n - a_{n-1}$, and a_1 is the first term in the sequence

With this, we can deduce $a_{n-1} - a_n = 17 \to d = -17$. With a_1 of course defined as 1. Consequently, the general form of this sequence would be

$$a_n = 1 + (n-1)(-17)$$
$$a_n = 1 + -17n + 17$$
$$a_n = -17n + 18.$$

> **5.** Find a formula for the general term of the geometric sequence whose first term is $a_1 = 1$ such that $\frac{a_{n+1}}{a_n} = 10$ for $n \geqslant 1$.

Remark. The general form of a geometric sequence is of the type $a_n = ar^{n-1}$ for $n \geqslant 1$, where r is the common ratio defined $\frac{a_n}{a_{n-1}}$

With this, we can see that the common ratio r, is defined as 10, which makes the general form:

$$a_n = 10^{n-1}.$$

> **6.** Find a formula for the general term of the sequence $\{4, -1, \frac{1}{4}, -\frac{1}{16}, \frac{1}{64}, \ldots\}$.

For $n \geqslant 1$ this sequence has the general form

$$a_n = 4 \left(-\frac{1}{4} \right)^{n-1}.$$

Where the common ratio $r = -\frac{1}{4}$

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7. Determine whether the sequence is convergent or divergent. If it is convergent, find its limit.

(a) $a_n = \frac{4+5n^2}{1+n}$

(b) $a_n = \tan^{-1}\left(n^2\right)$

(c) $a_n = \ln\left(\frac{n+2}{n^2-3}\right)$

(d) $a_n = n\sin\left(\frac{1}{n}\right)$

(e) $a_n = \left(1 - \frac{2}{n}\right)^n$

(f) $a_n = \frac{1000^n}{n!}$

7.a

$$\lim_{n\to+\infty} \frac{4+5n^2}{1+n}$$
$$= \lim_{n\to+\infty} \frac{\frac{4}{n} + \frac{5n^2}{n}}{\frac{1}{n} + \frac{n}{n}}$$
$$= \lim_{n\to+\infty} \frac{\frac{4}{n} + 5n}{\frac{1}{n} + 1}$$
$$= \frac{\lim_{n\to+\infty} \frac{4}{n} + \lim_{n\to+\infty} 5n}{\lim_{n\to+\infty} \frac{1}{n} + \lim_{n\to+\infty} 1}$$
$$= \frac{0 + \infty}{0 + 1}$$
$$= +\infty.$$

Thus, this sequence is divergent

7.b

$$\lim_{n\to+\infty} \tan^{-1}\left(n^2\right)$$
$$= \frac{\pi}{2}.$$

Thus, this sequence converges

7.c

$$\lim_{n\to+\infty} \ln\left(\frac{n+2}{n^2-3}\right).$$

Considering the rational function:

$$\lim_{n \to +\infty} \left(\frac{n+2}{n^2 - 3} \right)$$

$$= \lim_{n \to +\infty} \left(\frac{\frac{n}{n^2} + \frac{2}{n^2}}{\frac{n^2}{n^2} - \frac{3}{n^2}} \right)$$

$$= \lim_{n \to +\infty} \left(\frac{\frac{1}{n} + \frac{2}{n^2}}{1 - \frac{3}{n^2}} \right)$$

$$= \frac{\lim_{n \to +\infty} \frac{1}{n} + \lim_{n \to +\infty} \frac{2}{n^2}}{\lim_{n \to +\infty} 1 - \lim_{n \to +\infty} \frac{3}{n^2}}$$

$$= \frac{0 + 0}{1 - 0}$$

$$= 0.$$

Consequently:

$$\lim_{n \to 0} \ln (n)$$

$$= -\infty.$$

Thus, this sequence diverges

7.d

$$\lim_{n\to+\infty} n\sin\left(\frac{1}{n}\right)$$

$$= \lim_{n\to+\infty} \frac{\sin\left(\frac{1}{n}\right)}{n^{-1}}$$

$$= \frac{\lim_{n\to+\infty} \sin\left(\frac{1}{n}\right)}{\lim_{n\to+\infty} n^{-1}} \quad \text{(Indeterminate...)}$$

$$= \lim_{n\to+\infty} \frac{\cos\left(\frac{1}{n}\right)\cdot\left(-\frac{1}{n^2}\right)}{-\frac{1}{n^2}}$$

$$= \lim_{n\to+\infty} \cos\left(\frac{1}{n}\right)$$

$$= \lim_{n\to 0} \cos(n) \quad \left(\text{Since } \lim_{n\to+\infty}\left(\frac{1}{n}\right) = 0\right)$$

$$= 1.$$

7.e

Thus:

$$\lim_{n\to+\infty} \left(1-\frac{2}{n}\right)^n$$

$$= \lim_{n\to+\infty} e^{\ln\left(1-\frac{2}{n}\right)^n}$$

$$= \lim_{n\to+\infty} e^{n\ln\left(1-\frac{2}{n}\right)}$$

$$\lim_{n\to+\infty} n\ln\left(1-\frac{2}{n}\right)$$

$$= \lim_{n\to+\infty} \frac{\ln\left(1-\frac{2}{n}\right)}{n^{-1}} \quad \text{(Indeterminate...)}$$

$$\implies \lim_{n\to+\infty} \frac{\frac{1}{1-\frac{2}{n}}\cdot\frac{2}{n^2}}{-\frac{1}{n^2}}$$

$$= \lim_{n\to+\infty} \frac{\frac{2}{\left(1-\frac{2}{n}\right)n^2}}{-\frac{1}{n^2}}$$

$$= \lim_{n\to+\infty} \frac{\frac{2}{n(n-2)}}{-\frac{1}{n^2}}$$

$$= \lim_{n\to+\infty} -\frac{2n^2}{n(n-2)}$$

$$= \lim_{n\to+\infty} -\frac{2n}{n-2}$$

$$= -2.$$

$$\lim_{n\to+\infty} \left(1-\frac{2}{n}\right)^n$$

$$= \lim_{n\to+\infty} e^{\ln\left(1-\frac{2}{n}\right)^n}$$

$$= \lim_{n\to+\infty} e^{n\ln\left(1-\frac{2}{n}\right)}$$

$$= \lim_{n\to+\infty} e^{-2}$$

$$= e^{-2}$$

$$= \frac{1}{e^2}.$$

7.f

$$\lim_{n \to +\infty} \frac{1000^n}{n!}.$$

Proposition. Since factorial growth is much faster than exponential growth, as n increases without bound, the denominator $n!$ will become much larger than the numerator 1000^n. Thus, this limit must be zero and we can conclude that this sequence converges.

So we can say:

$$\lim_{n \to +\infty} \frac{1000^n}{n!} = 0.$$