PSET 4 - Due: Wednesday, July 3

1. An individual who has automobile insurance from a certain company is randomly selected. Let X = the number of moving violations for which the individual was cited during the last 3 years. The probability mass function of X is given below.

x	0	1	2	3	4
$p(x)$	0.50	0.20	0.15	0.10	0.05

(a) Calculate the probability of each of the following events.

(i) Exactly one moving violation

(ii) At most one moving violation

(iii) More than two moving violations

(iv) Between 1 and 3 (inclusive of the endpoints) moving violations

(b) Find the cumulative distribution function $F(x)$. Be sure to write your answer in the appropriate way.

The probability of one moving violation is

$$p(1) = 0.2.$$

The probability of at most one moving violation is the sum of $p(0)$ and $p(1)$

$$p(0) + p(1) = 0.5 + 0.2 = 0.7.$$

The probability of more than two moving violations is the sum of the following probabilities

$$p(3) + p(4) = 0.1 + 0.05 = 0.15.$$

The sum of between one and three (inclusive) moving violations is

$$p(1) + p(2) + p(3) = 0.2 + 0.15 + 0.1 = 0.45.$$

Remark. The **cumulative distribution function (cdf)** $F(x)$ of a discrete rv variable X with pmf $p(x)$ is defined for every number x by

$$F(x) = P(X \leqslant x) = \sum_{y:y \leqslant x} p(y) \tag{3.3}$$

For any number x, $F(x)$ is the probability that the observed value of X will be at most x.

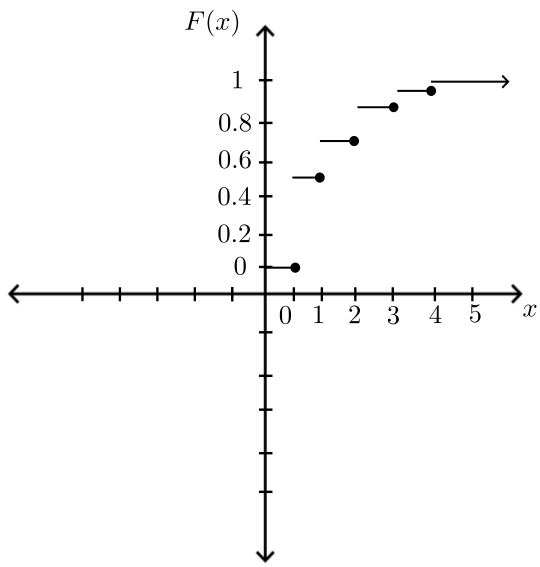
To find the cdf, we first find $F(x)$ for each value of x in the above table.

$$\begin{aligned}
&F(0) = P(X \leqslant 0) = p(0) = 0.5 \\
&F(1) = P(X \leqslant 1) = p(0) + p(1) = 0.5 + 0.2 = 0.7 \\
&F(2) = P(X \leqslant 2) = p(0) + p(1) + p(2) = 0.5 + 0.2 + 0.15 = 0.85 \\
&F(3) = P(X \leqslant 3) = p(0) + p(1) + p(2) + p(3) = 0.5 + 0.2 + 0.15 + 0.1 = 0.95 \\
&F(4) = P(X \leqslant 4) = 1.
\end{aligned}$$

Thus, the cdf is given by

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0.5 & \text{if } 0 \leqslant x < 1 \\ 0.7 & \text{if } 1 \leqslant x < 2 \\ 0.85 & \text{if } 2 \leqslant x < 3 \\ 0.95 & \text{if } 3 \leqslant x < 4 \\ 1 & \text{if } x \geqslant 4 \end{cases}.$$

A graph of the cdf is shown below



2. Suppose that two fair, six-sided dice are rolled independently of each other. For each possible roll define X = the smaller of the two dice. Find the probability mass function of X. Write your answer using a table similar to that given in Problem 1.

First, we need to find the number of outcomes for each possible value of x. For this, we consider orderded pairs. For $x = 1$, we need at least one of the die to show a one. We have

$$(1, \lambda)$$
$$(\lambda, 1).$$

In the first case, λ can be any number from 1 to 6. Thus we have $6P1 = \frac{6!}{5!} = 6$ possibilities. For the second case, λ can range from 1 to 5 (so we dont double count (1,1)). Thus we have $5P1 = 5$ possibilities. In total, we have 11 favorable outcomes for $x = 1$. Similarly, when

$$\begin{aligned}
x = 2 &\implies 5P1 + 4P1 = 9 \quad \text{favorable outcomes} \\
x = 3 &\implies 4P1 + 3P1 = 7 \quad \text{favorable outcomes} \\
x = 4 &\implies 3P1 + 2P1 = 5 \quad \text{favorable outcomes} \\
x = 5 &\implies 2P1 + 1P1 = 3 \quad \text{favorable outcomes} \\
x = 6 &\implies 1P1 = 1 \quad \text{favorable outcomes.}
\end{aligned}$$

If the total number of possible outcomes after rolling both dice is $n^k = 6^2 = 36$, then we can find the probabilities for each value of x. For example when $x = 1$ we have $p(x = 1) = \frac{11}{36} = 0.3056$. With these computations we build the following pmf

x	1	2	3	4	5	6
$p(x)$	0.3056	0.25	0.1944	0.1389	0.0833	0.0278

3. Let X be a discrete random variable having the following cumulative distribution function (cdf).

$$F(x) = \begin{cases} 0.00 & \text{if } x < 1 \\ 0.05 & \text{if } 1 \leqslant x < 3 \\ 0.10 & \text{if } 3 \leqslant x < 5 \\ 0.25 & \text{if } 5 \leqslant x < 6 \\ 0.65 & \text{if } 6 \leqslant x < 8 \\ 0.90 & \text{if } 8 \leqslant x < 9 \\ 1.00 & \text{if } 9 \leqslant x \end{cases}$$

(a) Graph the cdf. It should be neat, accurate and well-labeled.

(b) Using just the cdf calculate the following probabilities. (Your work should clearly show how you are using $F(x)$ to find these.)

(i) $P(X \leqslant 3)$

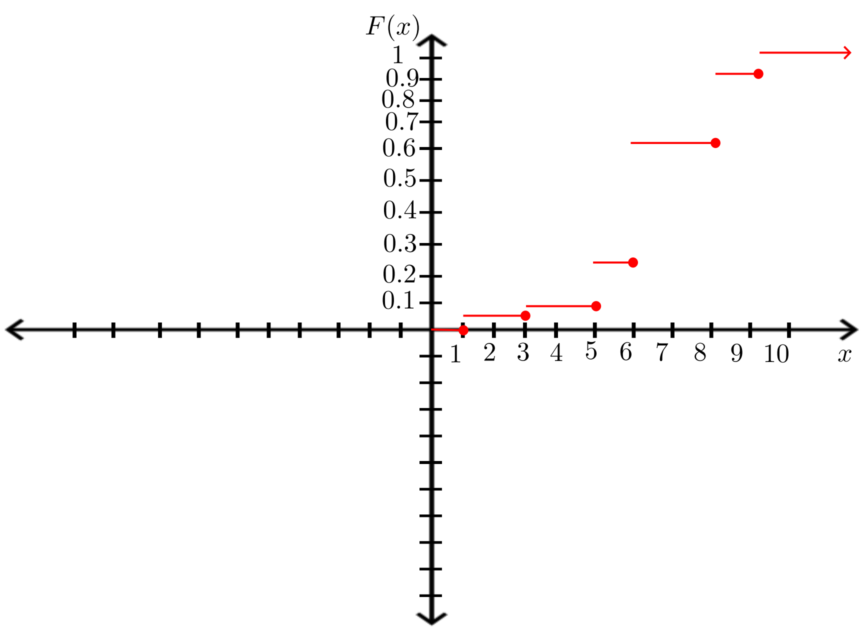
(ii) $P(X \geqslant 6)$

(iii) $P(X = 5)$

(iv) $P(3 \leqslant X \leqslant 8)$

(c) Find the probability mass function. Write your answer using a table similar to that given in Problem 1.

a.) The graph of the cdf is shown below



The possible x values are $1, 3, 5, 6, 8, 9$. Using the cdf, we see

$$\begin{aligned}
F(1) &= 0.05 \\
F(3) &= 0.1 \\
F(5) &= 0.25 \\
F(6) &= 0.65 \\
F(8) &= 0.9 \\
F(9) &= 1.
\end{aligned}$$

b.)

(i) To find $P(X \leqslant 3)$, we simply use $F(3) = 0.1$.

(ii) To find $P(X \geqslant 6)$. We need $p(6) + p(8) + p(9)$. To obtain this from the cdf, we use

$$\begin{aligned}
F(9) - F(5) &= p(1) + p(3) + p(5) + p(6) + p(8) + p(9) - (p(1) + p(3) + p(5)) \\
&= p(6) + p(8) + p(9).
\end{aligned}$$

Thus, $F(9) - F(5) = 1 - 0.25 = 0.75$. We could also use $1 - F(5) = 1 - 0.25 = 0.75$

(iii) Similarly, to find $P(x = 5)$, we use $F(5) - F(3) = 0.25 - 0.1 = 0.15$

(iv) Finally, to find $P(3 \leqslant X \leqslant 8)$, we use $F(8) - F(1) = 0.85$

c.) To obtain the pmf from the cdf, we remark

Remark. For any two numbers a and b with $a \leqslant b$,

$$P(a \leqslant X \leqslant b) = F(b) - F(a-)$$

where "$a-$" represents the largest possible X value that is strictly less than a. In particular, if the only possible values are integers and if a and b are integers, then

$$P(a \leqslant X \leqslant b) = P(X = a \text{ or } a + 1 \text{ or } \dots \text{ or } b) = F(b) - F(a - 1)$$

Taking $a = b$ yields $P(X = a) = F(a) - F(a - 1)$ in this case.

Thus, to find each $p(x)$, we simply take $F(x) - F(x-)$ for each value of x. We find

$$\begin{aligned}
p(1) &= F(1) - F(0) = 0.05 - 0 = 0.05 \\
p(3) &= F(3) - F(1) = 0.1 - 0.05 = 0.05 \\
p(5) &= F(5) - F(3) = 0.25 - 0.1 = 0.15 \\
p(6) &= F(6) - F(8) = 0.65 - 0.25 = 0.4 \\
p(8) &= F(8) - F(6) = 0.9 - 0.65 = 0.25 \\
p(9) &= F(9) - F(8) = 1 - 0.9 = 0.1.
\end{aligned}$$

Thus, we have the pmf given by the following table.

x	1	3	5	6	8	9
p(x)	0.05	0.05	0.15	0.4	0.25	0.1