Discrete Structures
Logic

A Document By:
Nathan Warner

Northern Illinois University

Computer Science
Northern Illinois University
August 11, 2023
United States

Contents

1 Statements

Definition: A statement (or proposition) is a sentence that is either true or false (but not both)

Example: for the following, state whether it is a statement, or not a statement

A.) "I think it will rain tomorrow" is a statement.

B.) 3 - x = 12

C.) 2 + 2 =3

Solutions:

A.) The sentence is not a statement because there is a chance it will rain, or not.

B.) Though the bellow sentence is a mathematical expression, however it is not a statement because it is not either true or false. Depending on what x is, the sentence is either true or false, but right now it is neither.

C.) Even though the bellow expression is false, but it is a statement because it is either true or false, but not both, and in this case it is false. Therefore "2 + 2 = 3" is a statement.

2 Compound Statements

Components of a statement

- p: Represents predicate
- q: Represents conclusion

Logical Connectives

- $\wedge$: Represents **and**
- $\wedge$: Represents **or**
- $\neg$ **or** $\sim$: Represents **negation**

By utilizing logical connectives, we can create compound statements

Example: For each sentence, choose the correct compound statement

A.) The sentence "It is not hot or it is sunny" in symbols is:

B.) The expression, "$3 \leqslant a$" in writing is:

C.) If p: a week has seven days, q: there are 20 hours in a day, and r: there are 60 minutes in an hour, then ∼p ∧ ∼r is:

Solutions:

A.) $\sim p \vee q$

B.) $3 > a$ **or** $3 = a$

C.) A week doesn't have 7 days and there are not 60 minutes in an hour.

3 Truth Tables

Here is a simple example of a truth table for logical and:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Here is a simple example of a truth table for logical or (inclusive):

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Here is a simple example of a truth table for logical or (exclusive):

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

Here is a simple example of a truth table for logical not (negation):

P	$\neg P$
T	F
F	T

Example: Construct a truth table for $(p \wedge q) \vee \neg r$

Figure:

p	q	r	$(p \wedge q)$	$\neg r$	$(p \wedge q) \vee \neg r$
T	T	T	T	F	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	F	T	T
F	T	T	F	F	F
F	T	F	F	T	T
F	F	T	F	T	F
F	F	F	F	F	T

4 Logical Equivalence

Definition: Statements p and q are said to be logically equivalent if they have the same truth value in every model.
Notation: the notation for logical equivalence is $\equiv$

Say we have statements p and q, and we want to show that $p \wedge q$, and $q \wedge p$ are logically equivalent, to do this, we must first construct a truth table:

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

So we can see that the columns $p \wedge q$, $q \wedge p$ have the same truth values, therefore they are said to be **logically equivalent**

5 Tautologies and Contradictions

Definition: A **Tautology** is a Statement that is always true, a assertion that is true in every possible interpretation
A **Contradiction** is a statement that is always false.

Consider the following compound statement

$$p \vee \neg p.$$

Because this statement can never be false, we say it is a **Tautology**

Contradiction

Consider the statement:

$$p \wedge \neg p.$$

Because this statement can never be true, we say it is a **Contradiction**

6 De Morgan's Laws

De Morgan's Laws are:

- $\neg(p \wedge q) = \neg p \vee \neg q$
- $\neg(p \vee q) = \neg p \wedge \neg q$

Consider the statement:

$$0 < x \leqslant 3.$$

To use De Morgan's Law, which states:

$$\neg(p \wedge q) = \neg p \vee \neg q.$$

We can rewrite the statement as:

$$0 \geqslant x \ or \ x > 3.$$

7 Logical Equivalence Laws

Commutative laws:	$\mathrm{p} \wedge \mathrm{q} \equiv \mathrm{q} \wedge \mathrm{p}$	$p \vee q \equiv q \vee p$
Associative laws:	$(\mathrm{p} \wedge \mathrm{q}) \wedge \mathrm{r} \equiv \mathrm{p} \wedge (\mathrm{q} \wedge \mathrm{r})$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive laws:	$\mathrm{p} \wedge (\mathrm{q} \vee \mathrm{r}) \equiv (\mathrm{p} \wedge \mathrm{q}) \vee (\mathrm{p} \wedge \mathrm{r})$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Identity laws:	$\mathrm{p} \wedge \mathbf{t} \equiv \mathrm{p}$	$p \vee \mathbf{c} \equiv p$
Negation laws:	$\mathrm{p} \vee \sim \mathrm{p} \equiv \mathbf{t}$	$p \wedge \sim p \equiv \mathbf{c}$
Double negative law:	$\sim (\sim \mathrm{p}) \equiv \mathrm{p}$	
Idempotent laws:	$\mathrm{p} \wedge \mathrm{p} \equiv \mathrm{p}$	$p \vee p \equiv p$
Universal bound laws:	$\mathrm{p} \vee \mathrm{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
DeMorgan's laws:	$\sim (\mathrm{p} \wedge \mathrm{q}) \equiv \sim \mathrm{p} \vee \sim \mathrm{q}$	$\sim (p \vee q) \equiv \sim p \wedge \sim q$
Absorption laws:	$\mathrm{p} \vee (\mathrm{p} \wedge \mathrm{q}) \equiv \mathrm{p}$	$p \wedge (p \vee q) \equiv p$
Negation of t and c	$\neg \mathrm{t} = \mathrm{c}$	$\neg \mathrm{c} = \mathrm{t}$

8 Conditional Statements

Definition: A **conditional statement** is a statement that can be written in the form "If P then Q," where P and Q are sentences.
Syntax: if *statement* then *statement*

Consider the statment

$$p \rightarrow q.$$

This statement, read "if p then q", can be described with the following truth table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Note:-

To get a truth value of "true" in $p \rightarrow q$, either p and q both need to be true, or both need to be false, or q needs to be true

9 Negation of Conditional Statements

Definition: The **Negation of conditional statement** is logically equivalent to a conjunction of the antecedent and the negation of the consequent.

$$Negation: \ p \rightarrow q \equiv p \wedge \neg q.$$

The **Contrapositive** of a conditional statement is a combination of the converse and the inverse

$$p \rightarrow q \equiv \neg q \rightarrow \neg p.$$

Consider the following condition

$$p \rightarrow q.$$

Which we know is logically equivalent to:

$$p \rightarrow q \equiv \neg p \vee q.$$

By use of De Morgan's Law, which states that:

$$\neg(p \vee q) \equiv \neg p \wedge \neg q.$$

We can negate $p \rightarrow q$, so:

$$\neg(p \vee q) \equiv \neg(\neg p) \wedge \neg q.$$

Consider the following Conditional Statement

If my dad is at home then he cant pick me up
p →q

Then the negation would be:

$$\begin{aligned} &p \wedge \neg q \\ &\equiv \text{my dad is at home and he can pick me up.} \end{aligned}$$

10 Converse and Inverse

Definition: The **Converse** of a conditional statement is created when the hypothesis and conclusion are reversed
The **Inverse** of a conditional statement is when both the hypothesis and conclusion are negated

Consider the statement

$$p \to q.$$

Then the **Converse** would be:

$$q \to p.$$

And the **Inverse** would be:

$$\neg p \to \neg q.$$

11 Biconditional Statements

Definition: A **Biconditional Statement** is a true statement that combines a hypothesis and conclusion with the words 'if and only if' instead of the words 'if' and 'then'

Say we have the following statement

$$q \iff p.$$

Then the truth table would be:

p	q	q $\iff p$
T	T	T
F	T	F
T	F	F
F	F	T

Note:-

Similar to conditional statements, in the truth table, $p \iff q$ is true if both p and q have the same value. So false false will be true