Nate Warner MATH 230 November 3, 2023

Homework/Worksheet 8 - Due: Wednesday, November 8

1. Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.
   (a) $1 + \frac{e}{\pi} + \frac{e^2}{\pi^2} + \frac{e^3}{\pi^3} + \cdots$
   (b) $a_1 = 2$ and $an/a_{n+1} = \frac{1}{2}$ for $n \geqslant 1$
   (c) $\sum\limits_{n=2}^{\infty} \frac{1}{n^2-1}$
   (d) $\sum\limits_{n=1}^{\infty} (\sin n - \sin(n+1))$

Remark. Regarding a geometric series, we know:

$$\sum_{n=1}^{\infty} ar^{n-1} \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{Diverges} & \text{if } |r| \geqslant 1 \end{cases}.$$

Problem 1.a: We can see this series conforms to

$$\sum_{n=1}^{\infty} \left(\frac{\pi}{e}\right)^{n-1}.$$

Thus we have $a = 1$, $r = \frac{\pi}{e}$, and we can assert

$$\begin{aligned} S &= \frac{a}{1-r} \\ &= \frac{1}{1-\frac{e}{\pi}} \\ &= \frac{1}{\frac{\pi-e}{\pi}} \\ &= \frac{\pi}{\pi-e}. \end{aligned}$$

Problem 1.b: We can see that $r = \frac{1}{2}$, $a = 2$. Thus we have the series

$$\sum_{n=1}^{\infty} 2\left(\frac{1}{2}\right)^{n-1}.$$

Where

$$\begin{aligned} S &= \frac{2}{1-\frac{1}{2}} \\ &= \frac{2}{\frac{1}{2}} \\ &= 4. \end{aligned}$$

Problem 1.c

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$
$$= \sum_{n=2}^{\infty} \frac{1}{(n-1)(n+1)}.$$

By a partial fraction decomposition, we have

$$\frac{1}{(n-1)(n+1)} = \frac{A}{(n-1)} + \frac{B}{(n+1)}$$
$$1 = A(n+1) + B(n-1)$$
$$.$$

Thus, $A = \frac{1}{2}$, $B = -\frac{1}{2}$

$$1 = \frac{1/2}{n-1} - \frac{1/2}{(n+1)}.$$

Writing out the first few terms we get

$$\left(\frac{\frac{1}{2}}{1} - \frac{\frac{1}{2}}{3}\right) + \left(\frac{\frac{1}{2}}{2} - \frac{\frac{1}{2}}{4}\right) + \left(\frac{\frac{1}{2}}{3} - \frac{\frac{1}{2}}{5}\right) + \left(\frac{\frac{1}{2}}{4} - \frac{\frac{1}{2}}{6}\right) + \left(\frac{\frac{1}{2}}{5} - \frac{\frac{1}{2}}{7}\right) + ... + \left(\frac{\frac{1}{2}}{n-1} - \frac{\frac{1}{2}}{n+1}\right).$$

Where most of these terms cancel

$$\left(\frac{\frac{1}{2}}{1} - \cancel{\frac{\frac{1}{2}}{3}}\right) + \left(\frac{\frac{1}{2}}{2} - \cancel{\frac{\frac{1}{2}}{4}}\right) + \left(\cancel{\frac{\frac{1}{2}}{3}} - \cancel{\frac{\frac{1}{2}}{5}}\right) + \left(\cancel{\frac{\frac{1}{2}}{4}} - \cancel{\frac{\frac{1}{2}}{6}}\right) + \left(\cancel{\frac{\frac{1}{2}}{5}} - \cancel{\frac{\frac{1}{2}}{7}}\right) + ... + \left(\cancel{\frac{\frac{1}{2}}{n-1}} - \frac{\frac{1}{2}}{n+1}\right).$$

(We also have the right side of the a_{n-1} term not having a cancellation), leaving:

$$S_n = \frac{1}{2} - \frac{1}{4} - \frac{\frac{1}{2}}{n} - \frac{\frac{1}{2}}{n+1}$$
$$\therefore \lim_{n\to\infty} S_n \implies \lim_{n\to\infty} \frac{1}{2} + \frac{1}{4} - \cancelto{0}{\frac{\frac{1}{2}}{n}} - \cancelto{0}{\frac{\frac{1}{2}}{n+1}}$$
$$= \frac{3}{4}.$$

Problem 1.d

$$\sum_{n=1}^{\infty} (\sin n - \sin n + 1)\ .$$

Writing out the first few terms we get:

$$(\sin 1 - \sin 2) + (\sin(2) - \sin(3)) + (\sin(3) - \sin(4)) + ... + (\sin n - \sin n + 1).$$

Where all terms cancel except

$$\sin 1 - \sin n + 1.$$

Thus,

$$S_n = \sin 1 - \sin(n+1)$$
$$\therefore \lim_{n\to\infty} S_n \implies \lim_{n\to\infty} \sin 1 - \sin n + 1$$
Diverges.

2. Determine whether the series is convergent or divergent.

   (a) $\sum\limits_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

   (b) $\sum\limits_{n=1}^{\infty} \frac{n^e}{n^\pi}$

   (c) $\sum\limits_{n=1}^{\infty} \cos n$

   (d) $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n+5}}$

   (e) $\sum\limits_{n=1}^{\infty} \frac{2n}{1+n^4}$

Remark. Divergence test: For a series a_n, if $\lim\limits_{n\to\infty} a_n \neq 0$ or DNE, the series is said to diverge
Integral test: For a series a_n with positive terms, if there exists a function f and a positive integer N s.t

1. f is positive, continuous, and decreasing on $[N, \infty)$
2. $a_n = f(n)\ \forall n \geqslant N, N \in \mathbb{Z}^+$

Then:

$$\sum_{n=N}^{\infty} a_n \text{ and } \int_N^{\infty} f(x)\, dx.$$

Either both converge or both diverge
We also have the p-series, which states

$$\sum_{n=1}^{\infty} \frac{1}{n^P} = \begin{cases} \text{Converges} & \text{if } p > 1 \\ \text{Diverges} & \text{if } p \leqslant 1 \end{cases}.$$

Which can be extended to

$$\sum_{n=2}^{\infty} \frac{1}{n \ln^P n}.$$

Problem 2.a

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$$
$$= \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}.$$

By the p-series, this series will converge. $P > 1$

Problem 2.b

$$\sum_{n=1}^{\infty} \frac{n^e}{n^\pi}$$
$$= \sum_{n=1}^{\infty} n^{e-\pi}$$
$$= \sum_{n=1}^{\infty} \frac{1}{n^{\pi-e}} \ .$$

By the p-series, this series will diverge. $P \leqslant 1$

Problem 2.c

$$\sum_{n=1}^{\infty} \cos n \ .$$

By the divergence test, we can conclude that this series diverges, as the $\lim\limits_{n\to\infty} \cos n$ DNE

Problem 2.d

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+5}}$$

Since $\lim_{n\to\infty} \frac{1}{\sqrt{n+5}} = 0$, the divergence test does not yield conclusive results. Furthermore, since this series has positive terms, we can compare the series to an integral of a function $f(x)$ where $a_n = f(n)$.
Let $f(x) = \frac{1}{\sqrt{x+5}}$, which is positive, continuous, and decreasing for all $x \geqslant 1$. We can then examine the improper integral $\int_1^\infty \frac{1}{\sqrt{x+5}} \ dx$:

$$\int_1^\infty \frac{1}{\sqrt{x+5}} \ dx$$
$$= \lim_{t\to\infty} \int_1^t \frac{1}{\sqrt{x+5}} \ dx$$
$$= \lim_{t\to\infty} \left[2\sqrt{x+5}\right]_1^t$$
$$= \lim_{t\to\infty} \left(2\sqrt{t+5} - 2\sqrt{6}\right)$$
$$= +\infty$$

Since the improper integral diverges, by the integral test, the series also diverges.

Problem 2.e

$$\sum_{n=1}^{\infty} \frac{2n}{1+n^4} \ .$$

First, we check the divergence test

$$\lim_{n\to\infty} \frac{2n}{1+n^4}$$
$$= \lim_{n\to\infty} \frac{\frac{2n}{n^4}}{\frac{1}{n^4} + \frac{n^4}{n^4}}$$
$$= \lim_{n\to\infty} \frac{\frac{2}{n^3}}{\frac{1}{n^4} + 1}$$
$$= 0.$$

Since the limit is zero, the divergence test does not yield conclusive results. For the integral test:

$$
\begin{aligned}
&\int_1^{\infty} \frac{2x}{1+x^4}\,dx \\
&= \lim_{t\to\infty} \int_1^{t} \frac{2x}{1+x^4}\,dx \\
&= \lim_{t\to\infty} \int_1^{t^2} \frac{1}{1+u^2}\,du \\
&\lim_{t\to\infty} \tan^{-1} u \,\Big|_1^{t^2} \\
&= \lim_{t\to\infty} \tan^{-1} t^2 - \tan^{-1} 1 \\
&= \frac{\pi}{2} - \frac{\pi}{4} \\
&= \frac{\pi}{4}.
\end{aligned}
$$

Let $u = x^2$
$du = 2x\ dx$
when $x = 1,\ u = 1$
when $x = t,\ u = t^2$.

Therefore, Since the improper integral converges, by the integral test, the series also converges.