Problem set 1 - Due: Fri, Jan 24

1.

(a) In E, find the distance between $A\left(-\frac{1}{3},0\right)$ and $B\left(0,\frac{1}{3}\right)$.

(b) In M, find the distance between $A\left(-\frac{1}{3},0\right)$ and $B\left(0,\frac{1}{3}\right)$.

(c) In H, find the distance between $A\left(-\frac{1}{3},0\right)$ and $B\left(0,\frac{1}{3}\right)$.

(d) In S (radius $r=1$), find the distance between $C\,(0,0,1)$ and $D\left(0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$.

(e) In S (radius $r=\frac{1}{2}$), find the distance between $P\left(\frac{1}{4},\frac{\sqrt{2}}{4},-\frac{1}{4}\right)$ and $Q\left(\frac{1}{6},-\frac{1}{3},\frac{1}{3}\right)$.

(f) In G, find the distance between $A(-2,-3)$ and $B(4,6)$.

Remark. The *Euclidean distance* $e(AB)$ between A and B satisfies

$$e(AB)=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Let $\mathbb{M}$ denote the Minkowski plane. If $A(x_1,y_1)$ and $B(x_2,y_2)$ are on the line $y=mx+b$, then the *Minkowski distance* $d_{\mathbb{M}}(AB)=|x_1-x_2|(1+|m|)$

Let $\mathbb{S}(r)$ denote the spherical plane, which describes the surface of a sphere with radius r. For any two points A,B that lie on this plane, the distance between them is the arc length given $d_{\mathbb{S}}=r\theta$. An explicit formula using the points coordinates is $d_{\mathbb{S}}=r\cos^{-1}\left(\frac{ax+by+cz}{r^2}\right)$

Let $\mathbb{G}$ denote the gap plane. For points A,B in $\mathbb{G}$, we define $d_{\mathbb{G}}(AB)$ as

$$d_{\mathbb{G}}(AB)=\begin{cases} e(AB) & \text{for } A,B \text{ on the same side of the gap} \\ e(AB)-e(CD) & \text{for } A,B \text{ on the opposite sides of the gap}\end{cases}$$

Let $\mathbb{H}$ denote the Hyperbolic plane. For two points A,B that lie on this plane, then M,N are the points where the chord AB meets the unit circle. The Hyperbolic distance $d_{\mathbb{H}}(AB)$ is given by

$$d_{\mathbb{H}}=\ln\left(\frac{e(AN)e(BM)}{e(AM)e(BN)}\right)$$

Where $e(AN),e(BM),...$ denotes the Euclidean distance.

Note: $d_H(AA)$ is defined to be one. That is, $d_{\mathbb{H}}(AA)=1$.

😊

a.) If points on the Euclidean plane A,B are given by coordinates $\left(-\frac{1}{3},0\right)$, and $\left(0,\frac{1}{3}\right)$ respectively, then the Euclidean distance $e(AB)$ is given by

$$\begin{aligned} e(AB)&=\sqrt{\left(0-\left(-\frac{1}{3}\right)\right)^2+\left(\frac{1}{3}-0\right)^2} \\ &=\sqrt{\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2}=\sqrt{\frac{2}{9}}=\frac{\sqrt{2}}{3}\end{aligned}$$

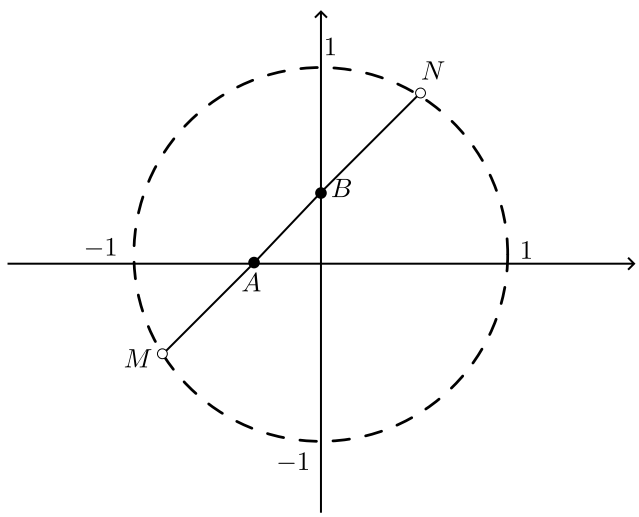
b.) If points on the Minkowski plane A, B are given by coordinates $\left(-\frac{1}{3}, 0\right)$, and $\left(0, \frac{1}{3}\right)$ respectively, then the Minkowski distance $d_{\mathbb{M}}$ is given by

$$\begin{aligned} d_{\mathbb{M}} &= \left|0 - \left(-\frac{1}{3}\right)\right| + \left|\frac{1}{3} - 0\right| \\ &= \left|\frac{1}{3}\right| + \left|\frac{1}{3}\right| = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \end{aligned}$$

c.) If points on the Hyperbolic plane A, B are given by coordinates $\left(-\frac{1}{3}, 0\right)$, and $\left(0, \frac{1}{3}\right)$ respectively, then the Hyperbolic distance $d_{\mathbb{H}}$ is given by

$$d_{\mathbb{H}} = \ln\left(\frac{e(AN)e(BM)}{e(AM)e(BN)}\right)$$

Thus, we first find points M, N.



If the line ℓ that passes through A, B has slope $m = \frac{\frac{1}{3}}{\frac{1}{3}} = 1$. Then, the equation of the line is given by

$$\begin{aligned} y - 0 &= 1\left(x - \left(-\frac{1}{3}\right)\right) \\ \implies y &= x + \frac{1}{3} \end{aligned}$$

Since the circle is given by $x^2 + y^2 = 1$, or $y = \pm\sqrt{1 - x^2}$. The line ℓ meets this circle at

$$\begin{aligned} x + \frac{1}{3} &= \sqrt{1 - x^2} \\ \implies 3x + 1 &= 3\sqrt{1 - x^2} \\ \implies (3x + 1)^2 &= 9(1 - x^2) = 9 - 9x^2 \\ \implies 9x^2 + 6x + 1 - 9 + 9x^2 &= 0 \\ \implies 18x^2 + 6x - 8 &= 0 \end{aligned}$$

Thus,

$$x = \frac{-6 \pm \sqrt{6^2 - 4(18)(-8)}}{2(18)}$$
$$= -\frac{1}{6} \pm \frac{\sqrt{17}}{6}$$

Let $\xi(x) = x + \frac{1}{3}$. Then, $M = \left(-\frac{1}{6} - \frac{\sqrt{17}}{6}, \xi\left(-\frac{1}{6} - \frac{\sqrt{17}}{6}\right)\right)$, and $N = \left(-\frac{1}{6} + \frac{\sqrt{17}}{6}, \xi\left(-\frac{1}{6} + \frac{\sqrt{17}}{6}\right)\right)$. Since

$$\xi\left(-\frac{1}{6} - \frac{\sqrt{17}}{6}\right) = \left(-\frac{1}{6} - \frac{\sqrt{17}}{6}\right) + \frac{1}{3} = \frac{1}{6} - \frac{\sqrt{17}}{6}$$
$$\xi\left(-\frac{1}{6} + \frac{\sqrt{17}}{6}\right) = \left(-\frac{1}{6} + \frac{\sqrt{17}}{6}\right) + \frac{1}{3} = \frac{1}{6} + \frac{\sqrt{17}}{6}$$

$M = \left(-\frac{1}{6} - \frac{\sqrt{17}}{6}, \frac{1}{6} - \frac{\sqrt{17}}{6}\right)$, and $N = \left(-\frac{1}{6} + \frac{\sqrt{17}}{6}, \frac{1}{6} + \frac{\sqrt{17}}{6}\right)$. We can now find the Euclidean distances required to compute $d_{\mathbb{H}}(AB)$. We use the Euclidean distance formula $e(CD) = \Big|x_1 - x_2\Big|\sqrt{1+m^2}$. Since $m = 1$, let $\lambda = \sqrt{1+m^2} = \sqrt{1+1^2} = \sqrt{2}$, then $e(CD) = \Big|x_1 - x_2\Big| \cdot \sqrt{2}$, for all $C(x_1, y_1), D(x_2, y_2)$ on the line ℓ through A, B.

$$e(AN) = \left| -\frac{1}{3} - \left(-\frac{1}{6} + \frac{\sqrt{17}}{6}\right)\right| \sqrt{2} = \frac{\sqrt{2} + \sqrt{34}}{6}$$
$$e(BM) = \left| 0 - \left(-\frac{1}{6} - \frac{\sqrt{17}}{6}\right)\right| \sqrt{2} = \frac{\sqrt{2} + \sqrt{34}}{6}$$
$$e(AM) = \left| -\frac{1}{3} - \left(-\frac{1}{6} - \frac{\sqrt{17}}{6}\right)\right| \sqrt{2} = \frac{-\sqrt{2} + \sqrt{34}}{6}$$
$$e(BN) = \left| 0 - \left(-\frac{1}{6} + \frac{\sqrt{17}}{6}\right)\right| \sqrt{2} = \frac{-\sqrt{2} + \sqrt{34}}{6}$$

Thus,

$$d_{\mathbb{H}} = \ln\left(\frac{\left(\frac{\sqrt{2}+\sqrt{34}}{6}\right)^2}{\left(\frac{-\sqrt{2}+\sqrt{34}}{6}\right)^2}\right) \approx 0.9899$$

d.) If points C, D on the spherical plane $\mathbb{S}(1)$ are given by coordinates $(0, 0, 1)$, and $\left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ respectively, then the distance $d_{\mathbb{S}}$ is given by

$$d_{\mathbb{S}} = r\cos^{-1}\left(\frac{c_1 d_1 + c_2 d_2 + c_3 d_3}{r^2}\right)$$
$$= 1\cos^{-1}\left(\frac{0(0) + 0\left(-\frac{1}{\sqrt{2}}\right) + 1\left(\frac{1}{\sqrt{2}}\right)}{1^2}\right)$$
$$= cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

e.) Next, consider $\mathbb{S}\left(\frac{1}{2}\right)$, with $P\left(\frac{1}{4}, \frac{\sqrt{2}}{4}, -\frac{1}{4}\right)$, and $Q\left(\frac{1}{6}, -\frac{1}{3}, \frac{1}{3}\right)$. Then,

$$\begin{aligned} d_{\mathbb{S}} &= \frac{1}{2}\cos^{-1}\left(\frac{\frac{1}{4}\left(\frac{1}{6}\right) + \frac{\sqrt{2}}{4}\left(-\frac{1}{3}\right) - \frac{1}{4}\left(\frac{1}{3}\right)}{\left(\frac{1}{2}\right)}\right)^2 \\ &= \frac{1}{2}\cos^{-1}\left(\frac{\frac{1}{24} - \frac{\sqrt{2}}{24} - \frac{1}{12}}{\frac{1}{4}}\right) \approx 1.1314 \end{aligned}$$

f.) Let $\mathbb{G}$ denote the gapped plane. If points $A(-2,-3), B(4,6)$ lie on $\mathbb{G}$, then their distance $d_{\mathbb{G}}$ is given by

$$\begin{cases} e(AB) & \text{if } A, B \text{ lie on the same side} \\ e(AB) - e(CD) & \text{otherwise} \end{cases}$$

If the line ℓ that passes through A, B has slope $m = \frac{6+3}{4+2} = \frac{3}{2}$, then the equation of the line is given by

$$\begin{aligned} y + 3 &= \frac{3}{2}(x+2) \\ \implies y &= \frac{3}{2}x \end{aligned}$$

When $x = 0$, $y = 0$. When $x = 1, y = \frac{3}{2}$. Thus, $C = (0,0)$, and $D = \left(1, \frac{3}{2}\right)$. Thus,

$$\begin{aligned} d_{\mathbb{G}} &= e(AB) - e(CD) \\ &= \left|4+2\right|\sqrt{1 + \left(\frac{3}{2}\right)^2} - \left|1-0\right|\sqrt{1 + \left(\frac{3}{2}\right)^2} \\ &= 6\sqrt{1 + \frac{9}{4}} - \sqrt{1 + \frac{9}{4}} \\ &= 6\sqrt{\frac{13}{4}} - \sqrt{\frac{13}{4}} \\ &= \frac{6\sqrt{13}}{2} - \frac{\sqrt{13}}{2} \approx 9.0139 \end{aligned}$$

2. Find two points A, B in H such that $d_H(AB) > 13$. Show the calculation that justifies your answer.

Let A, B lie on the x-axis. Then $M = (-1, 0)$, and $N = (1, 0)$. Thus, we require $A(\alpha, 0), B(\beta, 0)$ such that

$$d_{\mathbb{H}}(AB) = \ln\left(\frac{e(AN)e(BM)}{e(AM)e(BN)}\right) > 13$$

Let's try $A(-0.99, 0)$, $B(0.99, 0)$. Since the chord of the circle passes through $(-1, 0)$, and $(1, 0)$, it has slope $m = 0$, and equation $y = 0$. Thus, $e(PQ) = |q_1 - p_1|\sqrt{1 + 0^2} = |q_1 - p_1|$ for all points $P(p_1, p_2), Q(q_1, q_2)$

$$\begin{aligned} e(AN) &= |1 + 0.99| = 1.99 \\ e(BM) &= |-1 - 0.99| = 1.99 \\ e(AM) &= |-1 + 0.99| = 0.01 \\ e(BN) &+ |1 - 0.99| = 0.01 \end{aligned}$$

Thus,

$$d_{\mathbb{H}} = \ln\left(\frac{1.99^2}{0.01^2}\right) \approx 10.59$$

Not quite, let's instead try $A(-0.9999, 0), B(0.9999, 0)$. Which has distance

$$d_{\mathbb{H}} = \ln \frac{1.9999^2}{0.0001^2} \approx 19.81 > 13$$

Thus, $A = (-0.9999, 0)$, and $B = (0.9999, 0)$

3. Prove Proposition 2.1.

Proposition 1.1 If $A(x_1, y_1)$ and $B(x_2, y_2)$ are on the line $y = mx + b$, then $e(AB) = |x_1 - x_2|\sqrt{m^2 + 1}$

Proof. Assume $A(x_1, y_1)$ and $B(x_2, y_2)$ are on the line $y = mx + b$, and $e(AB) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. Observe that the slope m of the line is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Which implies

$$y_2 - y_1 = m(x_2 - x_1)$$

Plugging this expression for $y_2 - y_1$ into $e(AB)$ yields

$$\begin{aligned} e(AB) &= \sqrt{(x_2 - x_1)^2 + (m(x_2 - x_1))^2} \\ &= \sqrt{(x_2 - x_1)^2 + (m^2(x_2 - x_1)^2)} \\ &= \sqrt{(x_2 - x_1)^2[1 + m^2]} \\ &= \sqrt{(x_2 - x_1)^2} \cdot \sqrt{m^2 + 1} \\ &= |x_2 - x_1|\sqrt{m^2 + 1} \\ &= |-(x_1 - x_2)|\sqrt{m^2 + 1} \\ &= |x_1 - x_2|\sqrt{m^2 + 1} \end{aligned}$$

As desired ■

4. Prove Proposition 2.2.

Proposition 1.2 If $A(x_1, y_1)$ and $B(x_2, y_2)$ are on the line $y = mx + b$, then $d_{\mathbb{M}}(AB) = |x_1 - x_2|(1 + |m|)$

Proof. Assume $A(x_1, y_1)$ and $B(x_2, y_2)$ are on the line $y = mx + b$, and $d_{\mathbb{M}} = |x_2 - x_1| + |y_2 - y_1|$. Observe that the slope m of the line is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Which implies

$$y_2 - y_1 = m(x_2 - x_1)$$

Plugging this expression for $y_2 - y_1$ into $d_{\mathbb{M}}$ yields

$$\begin{aligned} d_{\mathbb{M}} &= |x_2 - x_1| + |y_2 - y_1| \\ &= |x_2 - x_1| + |m(x_2 - x_1)| \\ &= |x_2 - x_1| + |m||x_2 - x_1| \\ &= |x_2 - x_1|(1 + |m|) \\ &= |-(x_1 - x_2)|(1 + |m|) \\ &= |-1||x_1 - x_2|(1 + |m|) \\ &= |x_1 - x_2|(1 + |m|) \end{aligned}$$

As desired ■

> **5.** Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points on opposite sides of the gap in G and on the line $l : y = mx + b$. Derive a formula for $d_G(AB)$ in terms of x_1, x_2, and m.

If $A = (x_1, y_1)$, and $B = (x_2, y_2)$, then the gapped distance $d_{\mathbb{G}}$ is given by

$$d_{\mathbb{G}} = e(AB) - e(CD)$$

Where $C = (0, b)$, and $D = (1, m + b)$. Using $e(AB) = |x_1 - x_2|\sqrt{1 + m^2}$, we get

$$\begin{aligned} d_{\mathbb{G}} &= |x_1 - x_2| \sqrt{1 + m^2} - |1 - 0|\sqrt{1 + m^2} \\ &= |x_1 - x_2|\sqrt{1 + m^2} - \sqrt{1 + m^2} \\ &= \sqrt{1 + m^2}\,((|x_1 - x_2|) - 1) \end{aligned}$$