Nate Warner MATH 434 Fall 2025

Problem set 3 - Due: Tuesday, December 2

3.1.5. Consider the following data.

t_i	1.0	1.5	2.0	2.5	3.0
y_i	1.1	1.2	1.3	1.3	1.4

(a) Set up an overdetermined system of the form (3.1.3) for a straight line passing through the data points. Use the standard basis polynomials $\phi_1(t) = 1$, $\phi_2(t) = t$.

(b) Use MATLAB to calculate the least-squares solution of the system from part (a). This is a simple matter. Given an overdetermined system $Ax = b$, the MATLAB command `x = A\b` computes the least squares solution. Recall that this is exactly the same command as would be used to tell MATLAB to solve a square system $Ax = b$ by Gaussian elimination. Some useful MATLAB commands:

```
t = 1:.5:3; t = t'; s = ones(5,1); A = [s t];
```

We already know that MATLAB uses Gaussian elimination with partial pivoting in the square case. In the next two sections you will find out what MATLAB does in the overdetermined case.

(c) Use the MATLAB `plot` command to plot the five data points and your least squares straight line. Type `help plot` for information about using the plot command.

(d) Use MATLAB to compute $\|r\|_2$, the norm of the residual.

a.) The overdetermined system for a straight line passing through the given data points using the standard basis polynomials $\phi_1(t) = 1,\ \phi_2(t) = t$ is

$$\begin{bmatrix} 1 & 1.0 \\ 1 & 1.5 \\ 1 & 2.0 \\ 1 & 2.5 \\ 1 & 3.0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1.1 \\ 1.2 \\ 1.3 \\ 1.3 \\ 1.4 \end{pmatrix}.$$

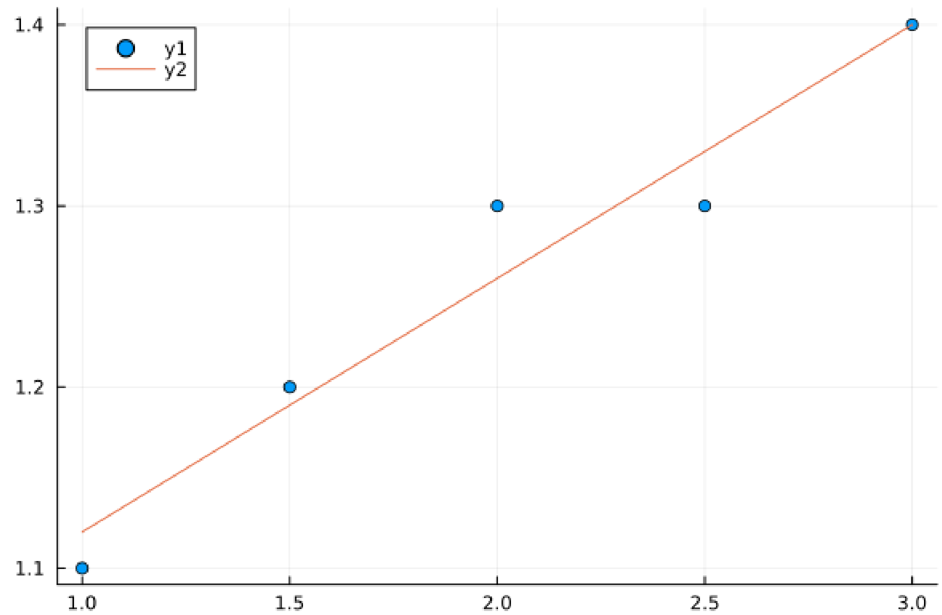
b.)

```
ts = [1.0, 1.5, 2.0, 2.5, 3.0]
ys = [1.1,1.2,1.3,1.3,1.4]
A = [1 1.0; 1 1.5; 1 2.0; 1 2.5; 1 3.0]
x = A\ys

# Out
# 2-element Vector{Float64}:
#0.9799999999999994
#0.14000000000000004
```

c.)

```
scatter(ts, ys)
plot!(t -> x[1] + x[2] * t)
```



d.)

```
r = ys - A *x
err = norm(r)

# Out
# 0.05477225575051658
```

3.1.6. Repeat Exercise 3.1.5, but this time compute the best least squares polynomial of degree ≤ 2. Notice that in this case the norm of the residual is smaller. This is to be expected; the space of quadratic polynomials contains the space of linear polynomials, so the quadratic fit should be better or in any case no worse than the linear polynomial fit.

In this case, we use standard basis vectors $\phi_1(t) = 1,\ \phi_2(t) = t,\ \phi_3(t) = t^2$. Thus, the system is

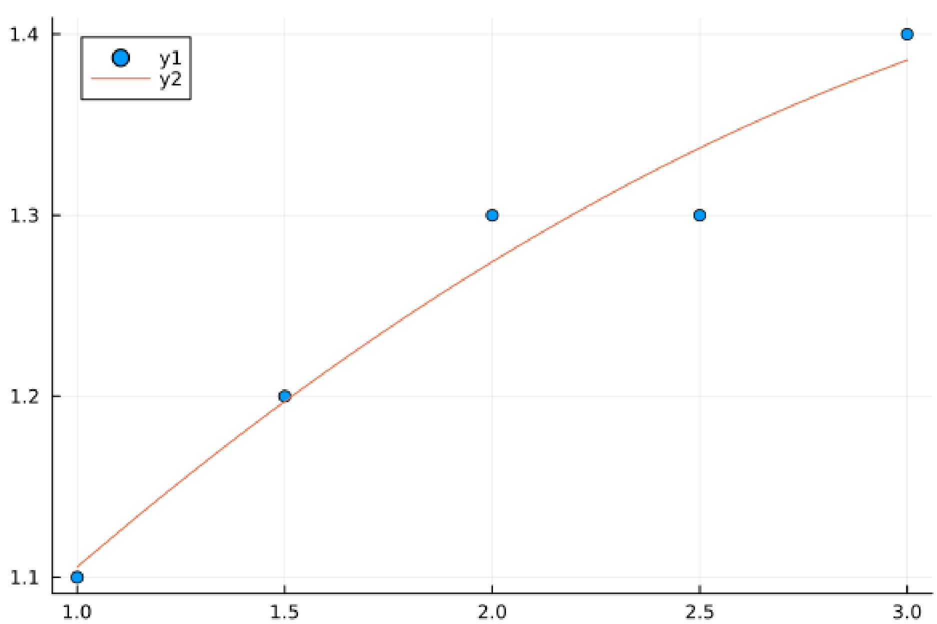
$$\begin{bmatrix} 1 & 1.0 & 1.0^2 \\ 1 & 1.5 & 1.5^2 \\ 1 & 2.0 & 2.0^2 \\ 1 & 2.5 & 2.5^2 \\ 1 & 3.0 & 3.0^2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1.1 \\ 1.2 \\ 1.3 \\ 1.3 \\ 1.4 \end{pmatrix}.$$

Solving this system in Julia gives

```
ts = [1.0, 1.5, 2.0, 2.5, 3.0]
ys = [1.1,1.2,1.3,1.3,1.4]
A = [ t^(j-1) for t in ts, j in 1:3 ]
x = A\ys

# Out
# 3-element Vector{Float64}:
# 0.8799999999999981
# 0.25428571428571595
# -0.028571428571428966
```

With plot



The norm of the residual $r = y - Ax$, where x is the solution of the least squares problem, and y is the vector of $y_i s$ is

```
r = ys - A *x
err = norm(r)

# Out
# 0.04780914437337568
```

3.1.7. Repeat Exercise 3.1.5, but this time compute the best least squares polynomial of degree ≤ 4. The space of quartic polynomials has dimension 5. Thus we have $n = m = 5$; the system is not overdetermined. The solution interpolates the data exactly (except for roundoff errors). Plot the data points and the solution on the same set of axes. Make the spacing between points small enough that the curve appears smooth. Sample code.

```
tt = 1:.01:3;
p4 = x(1) + x(2)*tt + x(3)*tt.^2 + x(4)*tt.^3 + x(5)*tt.^4;
plot(...,tt,p4,'k-',...)
```

In the same plot include your least squares linear and quadratic polynomials from the previous two exercises. Notice that the latter are much less oscillatory than the fourth-degree interpolant is. They seem to represent the trend of the data better.

This time, we have basis vectors $\phi_1(t) = 1,\ \phi_2(t) = t,\ \phi_3(t) = t^2,\ \phi_4(t) = t^3,\ \phi_5(t) = t^4$. Thus, our system is

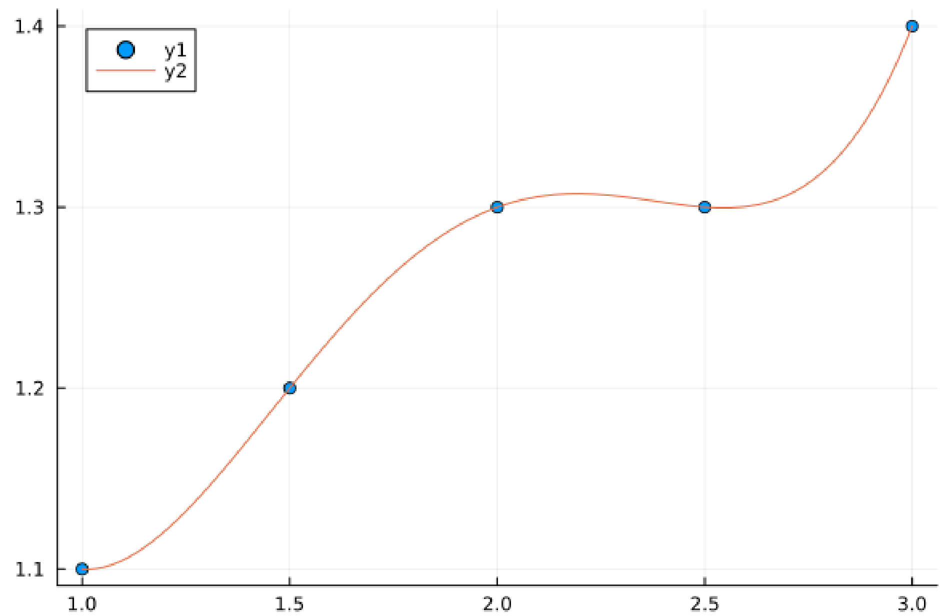
$$\begin{bmatrix} 1 & 1.0 & 1.0^2 & 1.0^3 & 1.0^4 \\ 1 & 1.5 & 1.5^2 & 1.5^3 & 1.5^4 \\ 1 & 2.0 & 2.0^2 & 2.0^3 & 2.0^4 \\ 1 & 2.5 & 2.5^2 & 2.5^3 & 2.5^4 \\ 1 & 3.0 & 3.0^2 & 3.0^3 & 3.0^4 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1.1 \\ 1.2 \\ 1.3 \\ 1.3 \\ 1.4 \end{pmatrix}.$$

The solution to this system is

```
ts = [1.0, 1.5, 2.0, 2.5, 3.0]
ys = [1.1,1.2,1.3,1.3,1.4]
A = [t^(j-1) for t in ts, j in 1:5]
x = A\ys

# Out
# 5-element Vector{Float64}:
# 2.800000000000004
# -4.516666666666675
# 4.150000000000006
# -1.533333333333335
# 0.20000000000000018
```

With plot

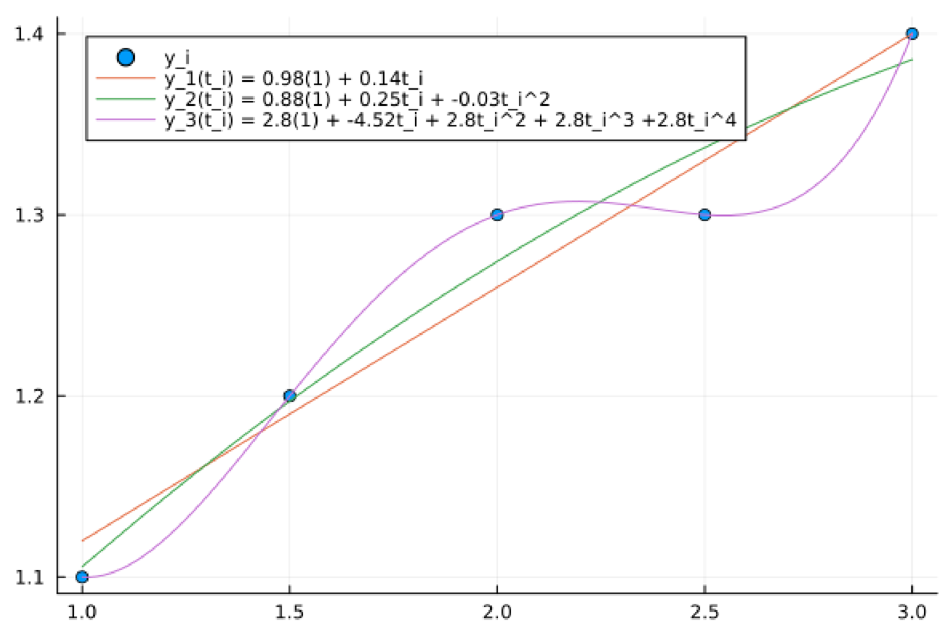


And residual norm

```
r = ys - A *x
err = norm(r)

# Out
# 6.564413678897588e-15
```

In total, we have the plots



3.2.4. Show that if Q is orthogonal, then $\det(Q) = \pm 1$.

Proof. Assume Q is an orthogonal matrix. Since Q is orthogonal, $Q^T Q = QQ^T = I$. Thus,

$$\begin{aligned} \det(Q^T Q) &= \det(I) = 1 \\ \implies \det(Q^T)\det(Q) &= (\det(Q))^2 = 1 \\ \therefore \det(Q) &= \pm 1. \end{aligned}$$

3.2.8. Show that if Q is orthogonal, then $\|Q\|_2 = 1$, $\|Q^{-1}\|_2 = 1$, and $\kappa_2(Q) = 1$. Thus Q is perfectly conditioned with respect to the 2-condition number. This suggests that orthogonal matrices will have good computational properties. □

Remark. The matrix 2-norm is the spectral norm

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}.$$

Consider the identity matrix I. The eigenvalues are the roots of the characteristic polynomial $\det(I - \lambda I)$. Observe that the face of $I - \lambda I$ is

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 0 & 0 & \cdots & 0 \\ 0 & 1-\lambda & 0 & \cdots & 0 \\ 0 & 0 & 1-\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1-\lambda \end{bmatrix}.$$

Since this matrix is diagonal, the determinant is the product of the main diagonal. That is,

$$\det\left(\begin{bmatrix} 1-\lambda & 0 & 0 & \cdots & 0 \\ 0 & 1-\lambda & 0 & \cdots & 0 \\ 0 & 0 & 1-\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1-\lambda \end{bmatrix}\right) = \prod_{i=1}^{n}(1-\lambda) = (1-\lambda)^n.$$

Thus, the eigenvalue is $\lambda = 1$, with multiplicity n.

So, the 2-norm of the identity matrix is

$$\|I\|_2 = \sqrt{\lambda_{\max}(I^T I)} = \sqrt{\lambda_{\max}(I)} = \sqrt{1} = 1.$$

Regarding the 2-norm of Q, we have

$$\|Q\|_2 = \sqrt{\lambda_{\max}(Q^T Q)} = \sqrt{\lambda_{\max}(I)} = \sqrt{1} = 1.$$

Next, we remark that since $QQ^T = I$,

$$(QQ^T)^{-1} = Q^{-T}Q^{-1} = I^{-1} = I.$$

With this, the 2-norm of Q^{-1} is

$$\left\|Q^{-1}\right\|_2 = \sqrt{\lambda_{\max}(Q^{-T}Q^{-1})} = \sqrt{\lambda_{\max}(I)} = \sqrt{1} = 1.$$

Since $\|Q\|_2 = \left\|Q^{-1}\right\|_2 = 1$, the condition number of Q using the 2-norm $\kappa_2(Q)$ is

$$\kappa_2(Q) = \|Q\|_2 \left\|Q^{-1}\right\|_2 = 1(1) = 1.$$

3.2.14. Use a QR decomposition to solve the linear system

$$\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 12 \\ 29 \end{pmatrix}.$$

We use Givens rotations to find the QR decomposition. We start by mapping

$$Q_1^T c_1(A) = Q_1^T \begin{pmatrix} 2 \\ 5 \end{pmatrix} \mapsto \begin{pmatrix} y_1 \\ 0 \end{pmatrix}.$$

Where $y_1 = \sqrt{x_1^2 + x_2^2} = \sqrt{2^2 + 5^2} = \sqrt{29}$. The rotation matrix is

$$Q_1^T = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$$

with $c = \frac{x_1}{y_1} = \frac{2}{\sqrt{29}}$, and $s = \frac{x_2}{y_1} = \frac{5}{\sqrt{29}}$. So,

$$Q_1^T = \frac{1}{\sqrt{29}} \begin{pmatrix} 2 & 5 \\ -5 & 2 \end{pmatrix}.$$

With Q_1, we transform the remaining column of A, $Q_1^T c_2(A)$. We have

$$Q_1^T c_2(A) = \frac{1}{\sqrt{29}} \begin{pmatrix} 2 & 5 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 7 \end{pmatrix} = \frac{1}{\sqrt{29}} \begin{pmatrix} 6+35 \\ -15+14 \end{pmatrix} = \begin{pmatrix} \frac{41}{\sqrt{29}} \\ -\frac{1}{\sqrt{29}} \end{pmatrix}.$$

Thus,

$$Q_1 A = \begin{pmatrix} \sqrt{29} & \frac{41}{\sqrt{29}} \\ 0 & -\frac{1}{\sqrt{29}} \end{pmatrix}.$$

Notice that $Q_1 A$ is upper triangular, so $Q_1 A = R = \hat{R}$. With this, we can solve the system using $\hat{R}x = Rx = Q^T b = c$. First, we find c

$$c = Q^T b = \frac{1}{\sqrt{29}} \begin{pmatrix} 2 & 5 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} 12 \\ 29 \end{pmatrix} = \frac{1}{\sqrt{29}} \begin{pmatrix} 24+145 \\ -60+58 \end{pmatrix} = \begin{pmatrix} \frac{169}{\sqrt{29}} \\ -\frac{2}{\sqrt{29}} \end{pmatrix}.$$

Now,

$$Rx = c \implies \begin{pmatrix} \sqrt{29} & \frac{41}{\sqrt{29}} \\ 0 & -\frac{1}{\sqrt{29}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{169}{\sqrt{29}} \\ -\frac{2}{\sqrt{29}} \end{pmatrix}.$$

We can solve this system with backward substitution.

$$-\frac{1}{\sqrt{29}} x_2 = -\frac{2}{\sqrt{29}} \implies x_2 = -\frac{2}{\sqrt{29}} \left(-\sqrt{29}\right) = 2,$$

$$\sqrt{29} x_1 + \frac{41}{\sqrt{29}} x_2 = \frac{169}{\sqrt{29}} \implies x_1 = \frac{\frac{169}{\sqrt{29}} - \frac{41(2)}{\sqrt{29}}}{\sqrt{29}} = \frac{169 - 82}{\sqrt{29}^2} = 3.$$

So, $x = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. We can verify this solution by showing that $Ax = b$. We have

$$\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 6+6 \\ 15+14 \end{pmatrix} = \begin{pmatrix} 12 \\ 29 \end{pmatrix}.$$

The solution is verified.

3.2.17. Let Q be the plane rotator (3.2.15). Show that the transformations $x \to Qx$ and $x \to Q^T x$ alter only the ith and jth entries of x and that the effect on these entries is the same as that of the 2-by-2 rotators

$$\hat{Q} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \qquad \text{and} \qquad \hat{Q}^T = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

on the vector $\begin{bmatrix} x_i \\ x_j \end{bmatrix}$.

Let $x \in \mathbb{R}^n$, where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix}$. The plane rotator Q is $Q = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & c & & -s & & \\ & & & 1 & & & \\ & & s & & c & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$, where

$$q_{ii} = c = \cos(\theta),\ q_{ij} = -s = -\sin(\theta),$$
$$q_{ji} = s = \sin(\theta),\ q_{jj} = c = \cos(\theta)..$$

Notice that in Qx,

$$Qx = x_1 e_1 + x_2 e_2 + \ldots + x_{i-1} e_{i-1} + x_i(\cos(\theta) e_i + \sin(\theta) e_j) + x_{i+1} e_{i+1}$$
$$+ \ldots + x_{j-1} e_{j-1} + x_j(-\sin(\theta)\, e_i + \cos(\theta) e_j) + x_{j+1} e_{j+1} + \ldots + x_n e_n.$$

Recall that e_i is the i^{th} standard basis vector in $\mathbb{R}^n$. Observe that only x_i, x_j are altered, all other entries are left unchanged. We have

$$Qx = \begin{cases} x_k = x_k & \text{if } k \neq i \text{ and } k \neq j \\ x_k = x_i \cos(\theta) - x_j \sin(\theta) & \text{if } k = i \\ x_k = x_i \sin(\theta) + x_j \cos(\theta) & \text{if } k = j \end{cases}.$$

So,

$$Q \begin{pmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_i \cos(\theta) - x_j \sin(\theta) \\ \vdots \\ x_i \sin(\theta) + x_j \cos(\theta) \\ \vdots \\ x_n \end{pmatrix}.$$

If we transpose Q, the rotation matrix is

$$Q^T = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & c & & s & & \\ & & & 1 & & & \\ & & -s & & c & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix},$$

and $Q^T x$ is

$$\begin{aligned} Q^T x = x_1 e_1 + x_2 e_2 + \ldots + x_{i-1} e_{i-1} + x_i(\cos(\theta)\, e_i - \sin(\theta) e_j) + x_{i+1} e_{i+1} \\ + \ldots + x_{j-1} e_{j-1} + x_j(\sin(\theta)\, e_i + \cos(\theta) e_j) + x_{j+1} e_{j+1} + \ldots + x_n e_n. \end{aligned}$$

Thus,

$$Q^T x = \begin{cases} x_k = x_k & \text{if } k \neq i \text{ and } k \neq j \\ x_k = x_i \cos(\theta) + x_j \sin(\theta) & \text{if } k = i \\ x_k = -x_i \sin(\theta) + x_j \cos(\theta) & \text{if } k = j \end{cases}.$$

And

$$Q^T \begin{pmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_i \cos(\theta) + x_j \sin(\theta) \\ \vdots \\ -x_i \sin(\theta) + x_j \cos(\theta) \\ \vdots \\ x_n \end{pmatrix}.$$

If we set $\hat{x} = \begin{bmatrix} x_i \\ x_j \end{bmatrix}$, then the rotation matrices are

$$\hat{Q} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \qquad \text{and} \qquad \hat{Q}^T = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}.$$

We have

$$\begin{aligned} \hat{Q}\hat{x} &= \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} x_i \\ x_j \end{bmatrix} = \begin{bmatrix} x_i c - x_j s \\ x_i s + x_j c \end{bmatrix} = \begin{bmatrix} x_i \cos(\theta) - x_j \sin(\theta) \\ x_i \sin(\theta) + x_j \cos(\theta) \end{bmatrix}, \\ \hat{Q}^T \hat{x} &= \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} x_i \\ x_j \end{bmatrix} = \begin{bmatrix} x_i c + x_j s \\ -x_i s + x_j c \end{bmatrix} = \begin{bmatrix} x_i \cos(\theta) + x_j \sin(\theta) \\ -x_i \sin(\theta) + x_j \cos(\theta) \end{bmatrix}. \end{aligned}$$

So, the effect is the same.

3.2.26. Prove Theorem 3.2.24

Theorem 3.2.24. Let $u \in \mathbb{R}^n$ with $\|u\|_2 = 1$, and define $P \in \mathbb{R}^{n \times n}$ be $P = uu^T$. Then,

(a) $Pu = u$

(b) $Pv = 0$ if $\langle u, v \rangle = 0$

(c) $P^2 = P$

(d) $P^T = P$

Proof. Assume $u \in \mathbb{R}^n$, $\|u\|_2 = 1$. Define $P \in \mathbb{R}^{n \times n}$, $P = uu^T$.

(a)

$$Pu = uu^T u = u(u^T u) = u \|u\|_2^2 = u(1) = u.$$

(b)

$$Pv = uu^T v = u(u^T v) = u(0) = 0.$$

(c)

$$P^2 = uu^T uu^T = u(u^T u)u^T = u \left\| u \right\|_2^2 u^T = u(1)u^T = uu^T = P.$$

(d)

$$P^T = \left(uu^T\right)^T = \left(u^T\right)^T u^T = uu^T = P.$$

3.2.39(a). Find a reflector Q that maps the vector $$\begin{bmatrix} 3 & 4 & 1 & 3 & 1 \end{bmatrix}^T.$$ to a vector of the form $\begin{bmatrix} -\tau & 0 & 0 & 0 & 0 \end{bmatrix}^T$. (You need not concern yourself with precautions to avoid over/underflow.) Write Q two ways: (i) in the form $I - \gamma uu^T$ and (ii) as a completely assembled matrix.

We have

$$\begin{aligned} \tau &= \operatorname{sgn}(x_1) \left\| x \right\|_2 = \operatorname{sgn}(3)\sqrt{3^2 + 4^2 + 1^2 + 3^2 + 1^2} = 6, \\ \gamma &= \frac{\tau + x_1}{\tau} = \frac{6+3}{6} = \frac{3}{2}, \\ \tau + x_1 &= 6 + 3 = 9, \\ u &= \begin{pmatrix} 1 \\ \frac{x_2}{\tau + x_1} \\ \frac{x_3}{\tau + x_1} \\ \frac{x_4}{\tau + x_1} \\ \frac{x_5}{\tau + x_1} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{4}{9} \\ \frac{1}{9} \\ \frac{1}{3} \\ \frac{1}{9} \end{pmatrix}. \end{aligned}$$

Thus (i),

$$Q = I - \gamma uu^T = I - \frac{3}{2} \begin{pmatrix} 1 \\ \frac{4}{9} \\ \frac{1}{9} \\ \frac{1}{3} \\ \frac{1}{9} \end{pmatrix} \begin{pmatrix} 1 & \frac{4}{9} & \frac{1}{9} & \frac{1}{3} & \frac{1}{9} \end{pmatrix}$$

.

Which is

$$\begin{aligned} Q &= I - \frac{3}{2} \begin{pmatrix} 1 \\ \frac{4}{9} \\ \frac{1}{9} \\ \frac{1}{3} \\ \frac{1}{9} \end{pmatrix} \begin{pmatrix} 1 & \frac{4}{9} & \frac{1}{9} & \frac{1}{3} & \frac{1}{9} \end{pmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{2}{3} & -\frac{1}{6} & -\frac{1}{2} & -\frac{1}{6} \\ -\frac{2}{3} & \frac{19}{27} & -\frac{2}{27} & -\frac{2}{9} & -\frac{2}{27} \\ -\frac{1}{6} & -\frac{2}{27} & \frac{53}{54} & -\frac{1}{18} & -\frac{1}{54} \\ -\frac{1}{2} & -\frac{2}{9} & -\frac{1}{18} & \frac{5}{6} & -\frac{1}{18} \\ -\frac{1}{6} & -\frac{2}{27} & -\frac{1}{54} & -\frac{1}{18} & \frac{53}{54} \end{bmatrix}. \end{aligned}$$

3.2.47.

(a) Let
$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}.$$
Find a reflector $\hat{Q}$ and an upper triangular $\hat{R}$ such that $A = \hat{Q}\hat{R}$. Assemble $\hat{Q}$ and simplify it.
$$\left(\text{Solution: } \hat{Q} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \hat{R} = \frac{1}{\sqrt{2}} \begin{bmatrix} -2 & -5 \\ 0 & 1 \end{bmatrix} \right).$$

(b) Compare your solution from part (a) with the QR decomposition of the same matrix (obtained using a rotator) in Example 3.2.13. Find a diagonal matrix D with main diagonal entries ± 1 such that $Q = \hat{Q}D$ and $\hat{R} = DR$.

(a) We have
$$\begin{aligned}
\tau &= \operatorname{sgn}(x_1)\, \|x\|_2 = \operatorname{sgn}(1)\sqrt{1^2 + 1^2} = \sqrt{2}, \\
\gamma &= \frac{\tau + x_1}{\tau} = \frac{\sqrt{2}+1}{\sqrt{2}}, \\
u &= \begin{pmatrix} 1 \\ x_2/(\tau + x_1) \end{pmatrix} = \begin{pmatrix} 1 \\ 1/(\sqrt{2}+1) \end{pmatrix} = \begin{pmatrix} 1 \\ (\sqrt{2}-1)/(\sqrt{2}^2 - 1^2) \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix} \\
\hat{Q} &= I - \gamma u u^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{\sqrt{2}+1}{\sqrt{2}} \begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix} \begin{pmatrix} 1 & \sqrt{2}-1 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{\sqrt{2}+1}{\sqrt{2}} \begin{pmatrix} 1 & \sqrt{2}-1 \\ \sqrt{2}-1 & 3-2\sqrt{2} \end{pmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{pmatrix} \frac{\sqrt{2}+1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{\sqrt{2}-1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 - \frac{\sqrt{2}+1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 1 - \frac{\sqrt{2}-1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.
\end{aligned}$$
So,
$$\begin{aligned}
\hat{Q}\begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} -\sqrt{2} \\ 0 \end{pmatrix}, \\
\hat{Q}\begin{pmatrix} 2 \\ 3 \end{pmatrix} &= \begin{pmatrix} -\frac{2}{\sqrt{2}} - \frac{3}{\sqrt{2}} \\ -\frac{2}{\sqrt{2}} + \frac{3}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{5}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, .
\end{aligned}$$
Therefore,
$$\hat{Q} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix},\ \hat{Q}A = \begin{pmatrix} -\sqrt{2} & -\frac{5}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -2 & -5 \\ 0 & 1 \end{pmatrix} = \hat{R}.$$
As desired.

(b) The decomposition in example 3.2.13, which was obtained using a rotator, is
$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},\ Q^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},\ R = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}.$$
Which has the same numbers, but with different signs.

We need $Q = \hat{Q}D$, so $\hat{Q}Q = D$. Thus,
$$\begin{aligned}
\hat{Q}Q &= \left(\frac{1}{\sqrt{2}}\right)^2 \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \\
\implies \frac{1}{2} \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}.
\end{aligned}$$

So,

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We can verify that $\hat{R} = DR$

$$DR = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -2 & -5 \\ 0 & 1 \end{pmatrix} = \hat{R}.$$

3.3.7. Work this problem by hand. Consider the overdetermined system

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} [x] = \begin{bmatrix} 9 \\ 5 \end{bmatrix},$$

whose coefficient matrix obviously has full rank.

(a) Before you do anything else, guess the least squares solution of the system.

(b) Calculate a QR decomposition of the coefficient matrix, where Q is a 2×2 rotator, and R is 2×1. Use the QR decomposition to calculate the least squares solution. Also deduce the norm of the residual (without calculating the residual directly).

(a) We need a vector $x \in \mathbb{R}^1$, when used to scale $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ produces a vector in $\mathbb{R}^2$ closest to $\begin{pmatrix} 9 \\ 5 \end{pmatrix}$ out of all other vectors in the span of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. My guess is that the x that produces this scaled vector is the average of the entries in $\begin{pmatrix} 9 \\ 5 \end{pmatrix}$. So,

$$[x] = \left[\frac{9+5}{2}\right] = [7].$$

(b) We have $y_1 = \sqrt{1^2 + 1^2} = \sqrt{2}$, and

$$Q^T = \begin{pmatrix} c & s \\ -s & c \end{pmatrix},$$

where

$$c = \frac{x_1}{y_1} = \frac{1}{\sqrt{2}}, \quad s = \frac{x_2}{y_1} = \frac{1}{\sqrt{2}}.$$

So,

$$Q^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

With $Q^T \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}$. So,

$$R = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}, \quad Q^T b = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 9 \\ 5 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 14 \\ -4 \end{pmatrix} = \begin{pmatrix} 7\sqrt{2} \\ -2\sqrt{2} \end{pmatrix}.$$

Thus, $\hat{R} = \sqrt{2}$, and $\hat{c} = \left(7\sqrt{2}\right)$. With this, we can solve the system $\hat{R}x = \hat{c}$ to get the least squares solution. We have

$$\left(\sqrt{2}\right)(x) = \left(7\sqrt{2}\right).$$

So,

$$\sqrt{2}x = 7\sqrt{2} \implies x = 7.$$

Thus $x = 7$, As expected. The norm of the residual is given by the norm of the $\bar{c}$ component in $Q^T b = c$. So, the norm of the residual is $(\sqrt{(-2\sqrt{2})^2} = 2\sqrt{2}$.