Nate Warner MATH 434 Fall 2025

Problem set 1 - Due: Sunday, September 28

1.2.4. Prove that if A^{-1} exists, then there can be no nonzero y for which $Ay = 0$

Proof. Assume that $A \in \mathbb{R}^{n\times n}$, and A^{-1} exists. Assume for the sake of contradiction that there exists $y \in \mathbb{R}^n$, $y \neq 0$ such that $Ay = 0$. So,

$$\begin{aligned} Ay &= 0, \\ \implies A^{-1}Ay &= A^{-1}0 \\ \implies Iy &= 0 \\ \implies y &= 0 \end{aligned}$$

But, $y \neq 0$, a contradiction. Therefore, if A^{-1} exists, then there can be no nonzero y for which $Ay = 0$. ■

1.2.5. Prove that if A^{-1} exists, then $\det(A) \neq 0$.

Proof. Assume that $A \in \mathbb{R}^{n\times n}$, and A^{-1} exists.

Suppose for the sake of contradiction that $\det(A) = 0$. We know that $AA^{-1} = I$, and that $\det(AB) = \det(A)\det(B)$. So,

$$\begin{aligned} AA^{-1} &= I \\ \implies \det(AA^{-1}) &= \det(I) \\ \implies \det(A)\det(A^{-1}) &= 1 \\ \implies 0\det(A^{-1}) &= 1 \\ \implies 0 &= 1 \end{aligned}$$

A contradiction. Therefore, if A^{-1} exists, then $\det(A) \neq 0$ ■

1.2.11. Check that the equations in Example 1.2.10 are correct. Check that the coefficient matrix of the system is nonsingular

1.3.4. Use pencil and paper to solve the system

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 3 & 1 & -1 & 0 \\ 4 & 1 & -3 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 9 \end{bmatrix}$$

by forward substitution

1.3.11. Use column-oriented forward substitution to solve the system from Exercise 1.3.4.

1.3.15. Develop the row-oriented version of back substitution. Write pseudocode in the spirit of (1.3.5) and (1.3.13).

1.3.16. Develop the column-oriented version of back substitution Write pseudocode in the spirit of (1.3.5) and (1.3.13).

1.3.17. Solve the upper-triangular system

$$\begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -10 \\ 10 \\ 1 \\ 12 \end{bmatrix}$$

(a) by row-oriented back substitution, (b) by column-oriented back substitution

1.4.21. Let

$$A = \begin{bmatrix} 16 & 4 & 8 & 4 \\ 4 & 10 & 8 & 4 \\ 8 & 8 & 12 & 10 \\ 4 & 4 & 10 & 12 \end{bmatrix}, \qquad b = \begin{bmatrix} 32 \\ 26 \\ 38 \\ 30 \end{bmatrix}.$$

Notice that A is symmetric, (a) Use the inner-product formulation of Cholesky's method to show that A is positive definite and compute its Cholesky factor, (b) Use forward and back substitution to solve the linear system $Ax = b$.

1.4.31. Use the outer-product form to work part (a) of Exercise 1.4.21.

1.4.33. Write a nonrecursive algorithm that implements the outer-product formulation of Cholesky's algorithm (1.4.28). Your algorithm should exploit the symmetry of A by referencing only the main diagonal and upper part of A, and it should store R over A. Be sure to put in the necessary check before taking the square root.

1.4.40. Use the bordered form to work part (a) of Exercise 1.4.21.

1.4.54. Prove Proposition 1.4.53.

As in the previous exercise, do not use the Cholesky decomposition in your proof; use the fact that $x^\top Ax > 0$ for all nonzero x.

Proposition 1.4.53. Let A be positive definite, and consider a partition

in which A_{11} and A_{22} are square. Then A_{11} and A_{22} are positive definite

1.4.56. Prove Proposition 1.4.55.

Proposition 1.4.55: If A and X are $n \times n$, A is positive definite, and X is nonsingular, then the matrix $B = X^\top AX$ is also positive definite.

Considering the special case $A = I$ (which is clearly positive definite), we see that this proposition is a generalization of Theorem 1.4.4.

1.4.58*. Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

be positive definite, and suppose A_{11} is $j \times j$ and A_{22} is $k \times k$. By Proposition 1.4.53, A_{11} is positive definite. Let R_{11} be the Cholesky factor of A_{11}, let $R_{12} = R_{11}^{-T} A_{12}$, and let $\tilde{A}_{22} = A_{22} - R_{12}^T R_{12}$. The matrix $\tilde{A}_{22}$ is called the *Schur complement* of A_{11} in A.

1. Show that
$$\tilde{A}_{22} = A_{22} - A_{21} A_{11}^{-1} A_{12}.$$
2. Establish a decomposition of A that is similar to (1.4.57) and involves $\tilde{A}_{22}$.
3. Prove that $\tilde{A}_{22}$ is positive definite.

1.4.62. Prove that if A is positive definite, then $\det(A) > 0$