Nate Warner MATH 230 December 6, 2023

Homework/Worksheet 11 - Due: Wednesday, November 15

1. Use term-by-term differentiation or integration to find a power series representation for each function centered at the given point.

   (a) $f(x) = \ln(1-x)$ centered at $x = 0$

   (b) $f(x) = \frac{2x}{(1-x^2)^2}$ centered at $x = 0$

   (c) $f(x) = \tan^{-1} x^2$ centered at $x = 0$

   (d) $f(x) = \ln(1+x^2)$ centered at $x = 0$

Problem 1a. Using the fact that $\frac{d}{dx}\ln(1-x) = -\frac{1}{1-x}$ and $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$

$$\begin{aligned} -\int \frac{1}{1-x}\,dx &= -\int \sum_{n=0}^{\infty} x^n\,dx = -\int (1+x+x^2+x^3+...)\,dx && \text{for } |x|<1 \\ \ln(1-x) &= \sum_{n=0}^{\infty} -\frac{x^{n+1}}{n+1} + C = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - ... && \text{for } |x|<1 \end{aligned} \tag{1}$$

Conclusion. When $x = 0$, we find $C = 0$. Thus,

$$\ln(1-x) = \sum_{n=0}^{\infty} -\frac{x^{n+1}}{n+1} = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + ... \qquad \text{for } |x| < 1.$$

Problem 1b. Using the fact that $\int \frac{2x}{(1-x^2)^2}\,dx = \frac{1}{1-x^2}$, and $\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$ for $|x| < 1$

$$\begin{aligned} \frac{d}{dx}\frac{1}{1-x^2} &= \frac{d}{dx}\sum_{n=0}^{\infty} x^{2n} = \frac{d}{dx}(1+x^2+x^4+x^6+...) && \text{for } |x|<1 \\ \frac{2x}{(1-x^2)^2} &= \sum_{n=0}^{\infty} 2nx^{2n-1} = 0 + 2x + 4x^3 + 6x^5 + ... && \text{for } |x|<1 \end{aligned} \tag{2}$$

Conclusion. Thus we have

$$\frac{2x}{(1-x^2)^2} = \sum_{n=1}^{\infty} 2nx^{2n-1} \qquad \text{for } |x| < 1.$$

Problem 1c. Using the fact that $\frac{d}{dx}\tan^{-1}x^2 = \frac{2x}{1-(-x^4)}$ and $\frac{2x}{1-(-x^4)} = \sum\limits_{n=0}^{\infty}(-1)^n 2x^{4n+1}$ for $|x|<1$

Remark. The series $\sum\limits_{n=0}^{\infty} 2x(-x^4)^n$ is in the form $\sum\limits_{n=0}^{\infty} bx^m c_n x^n$. By properties of combining power series, we know that this series must converge to $bx^m f(x)$ on the same interval of convergence as the simpler series, Since we know that $\sum\limits_{n=0}^{\infty}(-x^4)^n$ converges for $|x^4|<1$, or $-1<x<1$. We can conclude that $2x(-x^4)^n$ must do the same.

Thus we have:

$$\begin{aligned}
\int \frac{2x}{1+x^4}\,dx &= \int \sum_{n=0}^{\infty}(-1)^n 2x^{4n+1}\,dx = \int (2x-2x^5+2x^9-2x^{13}+...)\,dx && \text{for } |x|<1\\
\tan^{-1}x^2 &= \sum_{n=0}^{\infty}(-1)^n\frac{2x^{4n+2}}{4n+2}+C \;= C+x^2-\frac{1}{3}x^6+\frac{1}{5}x^{10}-\frac{1}{7}x^{14}+... && \text{for } |x|<1\\
\tan^{-1}x^2 &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{2n+1}+C \;= C+x^2-\frac{1}{3}x^6+\frac{1}{5}x^{10}-\frac{1}{7}x^{14}+... && \text{for } |x|<1
\end{aligned} \tag{3}$$

Conclusion. When $x=0$, $C=0$. Thus the power series for $\tan^{-1}x^2 = \sum\limits_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{2n+1}$ for $|x|<1$

Problem 1d. Using the fact that $\frac{d}{dx}\ln(1+x^2) = \frac{2x}{1-(-x^2)}$ and $\frac{2x}{1-(-x^2)} = \sum\limits_{n=0}^{\infty}(-1)^n 2x^{2n+1}$ for $|x|<1$. Similar to the last problem, we know that this series converges for $|x|<1$ by properties of combining power series. Thus

$$\begin{aligned}
\int \frac{2x}{1+x^2}\,dx &= \int \sum_{n=0}^{\infty}(-1)^n 2x^{2n+1}\,dx = \int (2x-2x^3+2x^5-2x^7+...)\,dx && \text{for } |x|<1\\
\ln(1+x^2) &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{n+1} \;+C = C+x^2-\frac{1}{2}x^4+\frac{1}{3}x^6-\frac{1}{4}x^8+... && \text{for } |x|<1
\end{aligned} \tag{4}$$

Conclusion. When $x=0$, $C=0$. Thus, the power series for $\ln(1+x^2) = \sum\limits_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{n+1}$ for $|x|<1$

2. Find the Taylor polynomial of degree two approximating the function $f(x) = \cos(2x)$ at $a = \pi$.

$$\begin{aligned} f(x) &= \cos(2x) & f(\pi) &= \cos(2\pi) = 1 \\ f'(x) &= -2\sin(2x) & f'(\pi) &= -2\sin(2\pi) = 0 \\ f''(x) &= -4\cos(2x) & f''(\pi) &= -4\cos(2\pi) = -4 \end{aligned} \tag{5}$$

We know the taylor series for a function f conforms to the form

$$f(x) \sim \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Conclusion. Thus $P_2(x)$ conforms to

$$\begin{aligned} &P_2(x) = 1 + 0(x-\pi) + \frac{-4}{2!}(x-\pi)^2 \\ &= 1 - 2(x-\pi)^2 \ . \end{aligned}$$

3. Find the Taylor series of the functions $f(x)$ centered at the given value of a

(a) $f(x) = \sin(x), \quad a = \pi$

(b) $f(x) = e^x, \quad a = -1$

(c) $f(x) = \ln(x), \quad a = 1$

(d) $f(x) = \frac{1}{2x-x^2}, \quad a = 1$

Problem 3a.

$$\begin{aligned} f(x) &= \sin(x) & f(\pi) &= \sin(\pi) = 0 \\ f'(x) &= \cos(x) & f'(\pi) &= \cos(\pi) = -1 \\ f''(x) &= -\sin(x) & f''(\pi) &= -\sin(\pi) = 0 \\ f'''(x) &= -\cos(x) & f'''(\pi) &= -\cos(\pi) = 1 \\ f^{(4)}(x) &= \sin(x) & f^{(4)}(\pi) &= \sin(\pi) = 0 \end{aligned} \tag{6}$$

Again, we use the Taylor series form

$$f(x) \sim \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ...$$

To get

$$\sin(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi)}{n!}(x-\pi)^n \ = 0 + (-1)(x-\pi) + \frac{0}{2!}(x-\pi)^2 + \frac{1}{3!}(x-\pi)^3 + \frac{0}{4!}(x-\pi)^4 + \frac{-1}{5!}(x-\pi)^5 + \frac{0}{6!}(x-\pi)^6 + \frac{1}{7!}(x$$
$$= -(x-\pi) + \frac{1}{3!}(x-\pi^3) - \frac{1}{5!}(x-\pi)^5 + \frac{1}{7!}(x-\pi)^7 + ...$$

Conclusion. Thus the Taylor series has the form

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-\pi)^{2n+1}}{(2n+1)!} \ .$$

Problem 3b.

$$\begin{aligned}
f(x) &= e^x & f(-1) &= \frac{1}{e} \\
f'(x) &= e^x & f(-1) &= \frac{1}{e} \\
f''(x) &= e^x & f(-1) &= \frac{1}{e} \\
&\vdots & &\vdots
\end{aligned} \tag{7}$$

By the definition of a Taylor series, we have

$$e^x = \frac{1}{e} + \frac{1}{e}(x+1) + \frac{2!}{e}(x+1)^2 + \frac{3!}{e}(x+1)^3 + ...$$

Thus

$$e^x = \sum_{n=0}^{\infty} \frac{n!(x+1)^n}{e} .$$

3c.

$$\begin{aligned}
f(x) &= \ln(x) & f(1) &= 0 \\
f'(x) &= \frac{1}{x} & f'(1) &= 1 \\
f''(x) &= -\frac{1}{x^2} & f''(1) &= -1 \\
f'''(x) &= \frac{2}{x^3} & f(1) &= 2 \\
f'''(x) &= -\frac{6}{x^4} & f(1) &= -6
\end{aligned} \tag{8}$$

By the definition of a Taylor series, we have

$$\begin{aligned}
&\ln(x) = 0 + 1(x-1) + -\frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \frac{-6}{4!}(x-1)^4 + ... \\
&= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4.
\end{aligned}$$

Conclusion. Thus the Taylor series has the form

$$\ln(x) \sim \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1} .$$

Problem 3d.