Numerical Linear Algebra Exam 1 Study Guide

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Chapter one: Gaussian Elimination and variants

1.1 Concepts

- **Matrix multiplication**: Know how to do matrix-vector, and matrix-matrix multiplication.
- **Block decompositions**
- **Systems of linear equations**
- **Singularity, properties of nonsingular matrices**
- **Triangular systems**
- **Properties of triangular matrices / systems**
- **Forward and backward substitution, flop counts**
- **Positive definite matrices / systems**
- **Properties of positive definite matrices**
- **Principal submatrices, leading principal submatrices, principal minors**
- **Cholesky decomposition and the cholesky factor**
- **Algorithms to compute the Cholesky factor and flop counts**
- **Cholesky factor in a diagonal matrix**
- **Banded matrix**
- **Column envelope**
- **Envelope of Choleksy factor**
- **Elementary operations on a system**
- LU **decomposition without row interchanges (and criterion for this to be possible)**
- **Algorithms to find** LU **decomposition without row interchanges and flop counts**
- **Partial pivoting**
- **Permutation matrix**
- **Gaussian elimination with partial pivoting**
- LU **decomposition with partial pivoting**

1.2 Definitions

- **Positive definite matrix**: A matrix A is **positive definite** provided that the following two conditions are satisfied
  1. A is symmetric. That is, $A = A^\top$
  2. $x^\top Ax > 0$ for all $x \neq 0$
- **Banded matrix, lower and upper bandwidths**: A banded matrix is a sparse matrix whose nonzero entries are confined to a diagonal band, consisting of the main diagonal and a fixed number of diagonals on either side of it.

Let $A \in \mathbb{R}^{m \times n}$. Then A is called a **banded matrix** if there exist nonnegative integers p, q (called the *lower* and *upper bandwidths*) such that

$$a_{ij} = 0 \quad \text{whenever } i - j > p \text{ or } j - i > q.$$

- The *lower bandwidth* p is the number of subdiagonals (below the main diagonal) that may contain nonzero entries.
- The *upper bandwidth* q is the number of superdiagonals (above the main diagonal) that may contain nonzero entries.

The *total bandwidth* is sometimes defined as $p + q + 1$, counting the main diagonal as well.

- **Column envelope**: The column envelope of A is the set of indices (i, j) in the upper triangular part of A (including the main diagonal). Define

$$\text{colenv}\{A\} = \{(i, j)\} : \; i \leqslant j \text{ and } a_{kj} \neq 0 \text{ for } k \leqslant i\}$$

1.3 Properties

- **Singularity**: A **singular matrix** is a square matrix that does not have an inverse.

  A **nonsingular** matrix is a square matrix that does have an inverse.

  The following are equivalent, if any one holds, they all hold

  - $Ax = b$ has a unique solution
  - $\det(A) \neq 0$
  - A^{-1} exists
  - There is no nonzero vector $y \in \mathbb{R}^m$ such that $Ay = 0$
  - The columns of A are linearly independent
  - The rows of A are linearly independent
  - Given any vector b, there is exactly one vector x such that $Ax = b$

  If any one of the following are true, they all are true, and A is nonsingular

- **Properties of positive definite (p.d) matrices**:

  1. If A is p.d then A is *nonsingular*

     Note: Since A is nonsingular there is no $y \in \mathbb{R}^n$, $y \neq 0$ such that $Ay = 0$
  2. If $A = M^\top M$ for some M nonsingular than A is p.d
  3. If A is p.d than $\det(A) > 0$
  4. If A is p.d then all principal submatrices are p.d
  5. If A is p.d then $a_{ii} > 0$ for $i = 1, 2, ..., n$. So, if any $a_{ii} \leqslant 0$, A is not p.d.
  6. A is p.d if and only if all leading principal minors are positive
  7. A is p.d if and only if there exists a unique upper triangular matrix R such that $A = R^\top R$ (Cholesky factorization described below)
  8. A is p.d if and only if all eigenvalues of A are positive

Recall that λ is an eigenvalue of A if there exists $x_\lambda \neq 0$ such that $Ax_\lambda = \lambda x_\lambda$

- **Properties of a permutation matrix**:
  1. **Orthogonal:** $P^T = P^{-1}$
  2. **Determinant:**: $\det(P) = \pm 1$, depending on whether the permutation is even or odd.
  3. **Action on vectors:**: Px permutes the coordinates of x
  4. **Action on matrices:** Left multiplication permutes rows; right multiplication permutes columns.

  Note: Property two is a key property.

1.4 Theorems / Propositions

- **Theorem 1.4.7 (*Cholesky Decomposition Theorem*)**: Let A be positive definite. Then A can be decomposed in exactly one way into a product

$$A = R^\top R$$

  such that R is upper triangular and has all main diagonal entries r_{ii} positive. R is called the Cholesky factor of A.

- **Theorem**: Let $A \in \mathbb{R}^{n\times n}$ be nonsingular. Then, we can solve the system $Ax = b$, $b \in \mathbb{R}^n$ using Gaussian Elimination without row interchanges if and only if all landing principal sub-matrices of A are nonsingular.

- **Theorem 1.7.19** ***(LU Decomposition Theorem)***: Let A be an $n \times n$ matrix whose leading principal submatrices are all nonsingular. Then, A can be decomposed in exactly one way into a product $A = LU$ such that L is unit lower triangular and U is upper triangular.

1.5 Algorithms / Flop counts

Chapter 2: Sensitivity of linear systems

2.1 Concepts

- **Norms**
- **Vector norms**
- **Properties of vector norms**
- **Cauchy-schwarz inequality**
- **Entrywise matrix norms**
- **Properties of matrix norms**
- **Induced matrix norms**
- **Properties of induced matrix norms**
- **Induced matrix-norms special cases**
- **Numerical error when solving systems**
- **Residual vector**
- **Relative error**
- **Pertubation**
- **Condition number**
- **Properties of the condition number**
- **Relative error bound I**
- **Relative error bound II**
- **Condition for singularity in $\hat{A}$**
- κ**, well-conditioned, ill-conditioned**

2.2 Definitions

- **Norm:** A norm is an operation $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}_+ : x \to \|x\| \geqslant 0$ that satisfies
  1. $\|x\| = 0 \iff x = 0$
  2. $\|\alpha x\| = |\alpha| \|x\|$
  3. $\|x + y\| \leqslant \|x\| + \|y\|$ (triangle inequality)
- **Euclidean norm (2-norm)**: The standard Euclidean distance. For $x \in \mathbb{R}^n$,

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}.$$

- **Manhattan norm (1-norm)**: Denoted L^1, and also called **Taxicab norm**. For $x \in \mathbb{R}^n$,

$$\|x\|_1 = |x_1| + |x_2| + ... + |x_n|.$$

- **L-Infinity (max) norm (∞-norm)**: Denoted L^∞. for $x \in \mathbb{R}^n$,

$$\|x\|_\infty = \max_{1\leqslant i\leqslant n} |x_i| = \max\{|x_1|, |x_2|, ..., |x_n|\}.$$

- **p-norm**: In $\mathbb{R}^n$, A more general norm is

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} = (|x_1^p| + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}}$$

for $1 \leqslant p < \infty$. The general p-norm satisfies all three properties of a norm only when $p \geqslant 1$. For smaller p, the triangle inequality does not hold.

- **Induced (operator) matrix norms**: For all $A \in \mathbb{R}^{n\times m}$, we define

$$\|A\|_p := \max_{x\in\mathbb{R}^n\setminus\{0\}} \frac{\|Ax\|_p}{\|x\|_p}.$$

- **Induced matrix norms special cases**:

p	**Name**	**Explicit formula**
1	Maximum column sum	$\|A\|_1 = \max_{1\leqslant j\leqslant n} \sum_{i=1}^m \lvert a_{ij}\rvert$
2	Spectral norm	$\|A\|_2 = \sqrt{\lambda_{\max}(A^TA)}$
∞	Maximum row sum	$\|A\|_\infty = \max_{1\leqslant i\leqslant m} \sum_{j=1}^n \lvert a_{ij}\rvert$

- **Cauchy Schwarz inequality for 2-norm (vector norm)**: states

$$\left|x^Ty\right| \leqslant \|x\|_2 \|y\|_2.$$

- **Residual vector**: Suppose $Ax = b$ yields $\hat{x}$ via numerical methods, then define the residual vector as

$$\hat{r} = b - A\hat{x}.$$

Which, by the way, implies that

$$b = \hat{r} + A\hat{x}.$$

- **Intro to measuring solutions**: Consider a problem (P), where

$$(P): \; Ax = b.$$

Numerical techniques yields a solution $\hat{x}$, which may or may not be the true solution to (P). Let x be the true solution to the system. So, x solves $Ax = b$.

We want to measure the distance between the numerical solution $\hat{x}$ and the true solution x, we hope that the numerical solution $\hat{x}$ is close to x. If the distance is small, then $\hat{x}$ is a good solution.

- **Relative error**: The relative error in $\hat{x}$ is given by

$$\frac{\|\hat{x} - x\|}{\|x\|} = \frac{\|\delta x\|}{\|x\|}$$

where $\hat{x} = x + \delta x$, which implies $x = \hat{x} - \delta x$.

- **Perturbation**: If numerical methods to solve a linear system $Ax = b$ yields $\hat{x}$, then $\hat{x}$ solves $\hat{A}\hat{x} = \hat{b}$. Note that it is possible for $\hat{A} = A$ or $\hat{b} = b$. If both $\hat{A} = A$ and $\hat{b} = b$, then $\hat{x} = x$.

$\hat{A}$ and $\hat{b}$ are called perturbed if they are modified versions of the original. If $\hat{A}$ is a perturbed matrix A, and $\hat{b}$ is a perturbed vector b, then

$$\hat{A} = A + \delta A,$$
$$\hat{b} = b + \delta b.$$

- **Condition number**: We define the condition number

$$\kappa(A) = \left\|A^{-1}\right\| \|A\|,$$

which measures how sensitive the system is to perturbations in A or b, and how close A is to being singular.

2.3 Properties

- **Properties of vector norms**:
  1. $\|x\| \geqslant 0$
  2. $\|x\| = 0 \iff x = 0$
  3. $\|\alpha x\| = |\alpha| \|x\|$
  4. $\|x + y\| \leqslant \|x\| + \|y\|$ (Triangle inequality)
- **Properties of matrix norms**: Matrix norms satisfy the three required properties of norms.
  1. $\|A\| = 0 \iff A = 0$
  2. $\|\alpha A\| = |\alpha| \|A\|$
  3. $\|A + B\| \leqslant \|A\| + \|B\|$ (Triangle inequality)
- **Properties of induced matrix norms**
  - **Sub-multiplicativity**: $\|AB\|_p \leqslant \|A\|_p \|B\|_p$
  - **Consistency**: $\|Ax\|_p \leqslant \|A\|_p \|x\|_p$
  - **Normalization**: $\|I\|_p = 1$

  These are what entrywise (”flattened”) norms lack.
- **Properties of the condition number**: Let A be a matrix, and $\kappa(A)$ be the condition number that measures the system $Ax = b$. The following two properties hold
  1. $\kappa(A) \geqslant 1$
  2. $\kappa(I) = 1$
  3. $\kappa(A) = \kappa(A^{-1})$

2.4 Theorems / Propositions

- **Theorem *(Relative Error Bound I)***: Let A be nonsingular, $b \neq 0$, and $Ax = b$. If $A(x + \delta x) = b + \delta b$, then

$$\frac{\|\delta x\|}{\|x\|} \leqslant \kappa(A) \frac{\|\delta b\|}{\|b\|}$$

- **Theorem (*Singularity of perturbed* A)**: If

$$\frac{\|\delta A\|}{\|A\|} < \frac{1}{\kappa(A)}$$

then $A + \delta A$ is nonsingular.

- **Theorem *(Relative error bound II)***: Let A be nonsingular, $b \neq 0$, and $Ax = b$. If $(A + \delta A)(x + \delta x) = b$, and

$$\frac{\|\delta A\|}{\|A\|} < \frac{1}{\kappa(A)},$$

then

$$\frac{\|\delta x\|}{\|x\|} \leqslant \frac{\kappa(A)\frac{\|\delta A\|}{\|A\|}}{1 - \kappa(A)\frac{\|\delta A\|}{\|A\|}}.$$

- **Theorem *(Relative error bound III)***: Let A be nonsingular, $b \neq 0$, and $Ax = b$. If $(A + \delta A)(x + \delta x) = b + \delta b$, and

$$\frac{\|\delta A\|}{\|A\|} < \frac{1}{\kappa(A)},$$

then

$$\frac{\|\delta x\|}{\|x\|} \leqslant \frac{\kappa(A)\left(\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|}\right)}{1 - \kappa(A)\frac{\|\delta A\|}{\|A\|}}.$$

2.5 Algorithms

- **Iterative approach to improve** $\hat{x}$: Suppose for a system $Ax = b$ numerical techniques yields an approximation $\hat{x}_1$. Then, the residual vector $\hat{r}_1 = b - A\hat{x}_1$, which implies that $b = \hat{r}_1 + A\hat{x}_1$. If $\hat{x}_2$ is a different approximation, where $\hat{x}_2 = \hat{x}_1 + \delta\hat{x}_1$, then

$$\begin{aligned} A\hat{x}_2 &= b = \hat{r}_1 + A\hat{x}_1 \\ \implies A(\hat{x}_1 + \delta\hat{x}_1) &= \hat{r}_1 + A\hat{x}_1 \\ \implies A\hat{x}_1 + A\delta\hat{x}_1 &= \hat{r}_1 + A\hat{x}_1 \\ \implies A\delta\hat{x}_1 &= \hat{r}_1. \end{aligned}$$

So, we solve the system for $\delta\hat{x}_1$. Then, since $\hat{x}_2 = \hat{x}_1 + \delta\hat{x}_1$, we see that we need to update the first solution by adding the computed $\delta\hat{x}_1$.

In general, if $\hat{x}_i$ is the i^{th} numerical solution to $Ax = b$, and $\hat{r}_i$ is the residual vector to the i^{th} solution, then

$$\hat{x}_{i+1} = \hat{x}_i + A^{-1}\hat{r}_i.$$

In practice, we don't compute $A^{-1}\hat{r}_i$, as we know that this is an expensive task. Instead, we solve the system correction system

$$A\delta\hat{x}_i = \hat{r}_i.$$

In exact arithmetic,

$$\begin{aligned} A\delta\hat{x} &= b - A\hat{x} \\ \implies \delta\hat{x} &= A^{-1}(b - A\hat{x}) \\ &= A^{-1}b - A^{-1}A\hat{x} \\ &= x - \hat{x}. \end{aligned}$$

So,

$$\hat{x}_{\text{new}} = \hat{x} + \delta\hat{x} = \hat{x} + x - \hat{x} = x.$$

Thus, in exact arithmetic, we converge to the true solution in one step.

In practice, computations are done in floating-point arithmetic, so both the residual and the correction are computed approximately. Let

$$A\,\delta\hat{x} = r + \delta r,$$

where δr represents rounding or truncation errors. When we update

$$\hat{x}_{\text{new}} = \hat{x} + \delta\hat{x},$$

we hope that the new residual

$$r_{\text{new}} = b - A\hat{x}_{\text{new}}$$

is smaller than the previous residual. Each iteration ideally improves the approximation because

$$\delta\hat{x} \approx A^{-1}(b - A\hat{x}) = x - \hat{x}.$$

When floating-point errors are small enough relative to the conditioning of A, this correction moves $\hat{x}$ closer to x. But, if A is ill-conditioned, the corrections may no longer reduce the error — in fact, they can make it worse.