Nate Warner MATH 230 October 27, 2023

Homework/Worksheet 7 - Due: Wednesday, November 1

1. Use the trapezoidal rule and Simpson's rule to approximate the integral

$$\int_0^2 \frac{e^x}{1+x^2}\ dx.$$

With $n = 10$

Trapezoidal approximation:

Remark. $T_n = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$ with $h = \frac{b-a}{n}$

We have,

$$x_0 = 0,\ x_1 = \frac{1}{5},\ x_2 = \frac{2}{5},\ x_3 = \frac{3}{5},\ x_4 = \frac{4}{5},\ x_5 = 1,\ x_6 = \frac{6}{5},\ x_7 = \frac{7}{5},\ x_8 = \frac{8}{5},\ x_9 = \frac{9}{5},\ x_{10} = 2$$
$$f(x_0) = 1,\ f(x_1) = \frac{e^{\frac{1}{5}}}{1+\frac{1}{25}},\ f(x_2) = \frac{e^{\frac{2}{5}}}{1+\frac{4}{25}},\ f(x_3) = \frac{e^{\frac{3}{5}}}{1+\frac{9}{25}},\ f(x_4) = \frac{e^{\frac{4}{5}}}{1+\frac{16}{25}},\ f(x_5) = \frac{e}{2}$$
$$f(x_6) = \frac{e^{\frac{6}{5}}}{1+\frac{36}{25}},\ f(x_7) = \frac{e^{\frac{7}{5}}}{1+\frac{49}{25}},\ f(x_8) = \frac{e^{\frac{8}{5}}}{1+\frac{64}{25}},\ f(x_9) = \frac{e^{\frac{9}{5}}}{1+\frac{81}{25}},\ f(x_{10}) = \frac{e^2}{5}.$$

Thus:

$$\begin{aligned} T_{10} &= \frac{1}{10}\left[1 + \frac{2e^{\frac{1}{5}}}{1+\frac{1}{25}} + \frac{2e^{\frac{2}{5}}}{1+\frac{4}{25}} + \frac{2e^{\frac{3}{5}}}{1+\frac{9}{25}} + \frac{2e^{\frac{4}{5}}}{1+\frac{16}{25}} + e + \frac{2e^{\frac{6}{5}}}{1+\frac{36}{25}} + \frac{2e^{\frac{7}{5}}}{1+\frac{49}{25}} + \frac{2e^{\frac{8}{5}}}{1+\frac{64}{25}} + \frac{2e^{\frac{9}{5}}}{1+\frac{81}{25}} + \frac{e^2}{5}\right] \\ &\approx 2.6608. \end{aligned}$$

Simpson's rule approximation

Remark. $S_n = \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$ where $h = \frac{b-a}{n}$ and $n \in 2k$ for some integer k

Thus,

$$\begin{aligned} S_{10} &= \frac{1}{10}\left[1 + \frac{4e^{\frac{1}{5}}}{1+\frac{1}{25}} + \frac{2e^{\frac{2}{5}}}{1+\frac{4}{25}} + \frac{4e^{\frac{3}{5}}}{1+\frac{9}{25}} + \frac{2e^{\frac{4}{5}}}{1+\frac{16}{25}} + 2e + \frac{2e^{\frac{6}{5}}}{1+\frac{36}{25}} + \frac{4e^{\frac{7}{5}}}{1+\frac{49}{25}} + \frac{2e^{\frac{8}{5}}}{1+\frac{64}{25}} + \frac{4e^{\frac{9}{5}}}{1+\frac{81}{25}} + \frac{e^2}{5}\right] \\ &\approx 2.6632. \end{aligned}$$

2. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

(a) $\int_0^{+\infty} \frac{1}{4+x^2}\ dx$

(b) $\int_e^{+\infty} \frac{1}{x\ln^2 x}$

(c) $\int_{-\infty}^{+\infty} \frac{e^x}{1+e^{2x}}\ dx$

(d) $\int_1^{+\infty} \frac{5}{x^3}\ dx$

2.a

$$
\begin{aligned}
&\int_0^{+\infty} \frac{1}{4+x^2}\ dx\\
&= \lim_{t\to+\infty} \int_0^t \frac{1}{4+x^2}\ dx\\
&= \lim_{t\to+\infty} \frac{1}{2}\tan^{-1}\frac{x}{2}\Bigg|_0^t\\
&= \lim_{t\to+\infty} \frac{1}{2}\tan^{-1}\frac{t}{2} - \cancelto{0}{\left(\frac{1}{2}\tan^{-1} 0\right)}\\
&= \frac{1}{2}\lim_{t\to+\infty} \cancelto{\frac{\pi}{2}}{\tan^{-1}\frac{t}{2}}\\
&= \frac{\pi}{4}.
\end{aligned}
$$

2.b

Let $u = \ln x$

$du = \dfrac{1}{x}\ dx$

When $x = t,\ u = \ln t$

When $x = e,\ u = \ln e = 1.$

$$
\begin{aligned}
&\int_e^{+\infty} \frac{1}{x\ln^2 x}\ dx\\
&= \lim_{t\to+\infty} \int_e^t \frac{1}{x\ln^2 x}\ dx\\
&= \lim_{t\to+\infty} \int_1^{\ln t} \frac{1}{u^2}\ du\\
&= \lim_{t\to+\infty} -u^{-1}\Bigg|_1^{\ln t}\\
&= \lim_{t\to+\infty} \cancelto{0}{-\frac{1}{\ln(t)}} + 1\\
&= 1.
\end{aligned}
$$

2.c

$$\int_{-\infty}^{+\infty} \frac{e^x}{1+(e^x)^2}\ dx$$
$$\int_{-\infty}^{0} \frac{e^x}{1+(e^x)^2}\ dx + \int_{0}^{+\infty} \frac{e^x}{1+(e^x)^2}\ dx$$
$$\lim_{t\to-\infty}\int_{t}^{0} \frac{e^x}{1+(e^x)^2}\ dx + \lim_{t\to+\infty}\int_{0}^{t} \frac{e^x}{1+(e^x)^2}\ dx.$$

Let $u = e^x$

$du = e^x\ dx.$

$I_1:$ When $x = t,\ u = e^t$
When $x = 0,\ u = 1.$

$I_2:$ When $x = 0,\ u = 1$
When $x = t,\ u = e^t.$

$I_1:$

$$\begin{aligned}
&\lim_{t\to-\infty}\int_{t}^{0} \frac{e^x}{1+(e^x)^2}\ dx \\
&= \lim_{t\to-\infty}\int_{e^t}^{1} \frac{du}{1+u^2} \\
&= \lim_{t\to-\infty} \tan^{-1} u \Big|_{e^t}^{1} \\
&= \lim_{t\to-\infty} \tan^{-1} 1 - \tan^{-1} e^t \\
&= \frac{\pi}{4} - \lim_{t\to-\infty} \tan^{-1} e^t \\
&= \frac{\pi}{4} - \lim_{t\to 0} \tan^{-1} t \quad (\text{Since } \lim_{t\to-\infty} e^t = 0) \\
&= \frac{\pi}{4} - 0 \\
&= \frac{\pi}{4}.
\end{aligned}$$

Thus, I_1 converges to $\frac{\pi}{4}$

I_2:

$$\begin{aligned}
&\lim_{t\to+\infty}\int_{0}^{t} \frac{e^x}{1+(e^x)^2}\ dx \\
&= \lim_{t\to+\infty}\int_{1}^{e^t} \frac{1}{1+u^2}\ du \\
&= \lim_{t\to+\infty} \tan^{-1} u \Big|_{1}^{e^t} \\
&= \lim_{t\to+\infty} \tan^{-1} e^t - \tan^{-1} 0 \\
&= \lim_{t\to+\infty} \tan^{-1} e^t - \frac{\pi}{4} \\
&= \lim_{t\to+\infty} \tan^{-1} t - \frac{\pi}{4} \quad (\text{Since } \lim_{t\to+\infty} e^t = +\infty) \\
&= \frac{\pi}{2} - \frac{\pi}{4} \\
&= \frac{\pi}{4}.
\end{aligned}$$

Thus, I_2 also converges to $\frac{\pi}{4}$, Which means we have:

$$\begin{aligned}
I &= \frac{\pi}{4} + \frac{\pi}{4} \\
&= \frac{\pi}{2}.
\end{aligned}$$

2.d

$$
\begin{aligned}
&\int_1^{+\infty} \frac{5}{x^3}\ dx \\
&= \lim_{t\to+\infty} \int_1^t \frac{5}{x^3}\ dx \\
&= \lim_{t\to+\infty} -\frac{5}{2x^2}\ \Bigg|_1^t \\
&= \overset{\nearrow 0}{\lim_{t\to+\infty} -\frac{5}{2t^2}} + \frac{5}{2} \\
&= \frac{5}{2}.
\end{aligned}
$$

3. Use the Comparison Theorem to determine whether the integral

$$\int_1^{+\infty} \frac{dx}{1+\sqrt{x}}\ dx.$$

is convergent or divergent

Remark. Let $f(x)$ and $g(x)$ be continuous over $[a, +\infty)$, assume $0 \leqslant f(x) \leqslant g(x)$
if $\int_a^{+\infty}\ f(x)\ dx = \lim\limits_{t\to+\infty} \int_a^t\ f(x)\ dx = +\infty$
then $\int_a^{+\infty}\ g(x)\ dx = \lim\limits_{t\to+\infty} \int_a^t\ g(x)\ dx = +\infty$

Alternatively,
if $\int_a^{+\infty}\ g(x)\ dx = \lim\limits_{t\to+\infty} \int_a^t\ g(x)\ dx = L$ for $L \in \mathbb{R}$
then $\int_a^{+\infty}\ f(x)\ dx = \lim\limits_{t\to+\infty} \int_a^t\ f(x)\ dx = M$ for $M \leqslant L$ where $M \in \mathbb{R}$

Let $f(x)$ be $\frac{1}{1+\sqrt{x}}$, choose $g(x) = \frac{1}{\sqrt{x}}$. If $g(x)$ diverges to $+\infty$, then $f(x)$ diverges to $+\infty$

$$
\begin{aligned}
&\int_1^{+\infty} \frac{1}{\sqrt{x}}\ dx \\
&= \lim_{t\to+\infty} \int_1^t \frac{1}{\sqrt{x}}\ dx \\
&= \lim_{t\to+\infty} 2x^{\frac{1}{2}}\ \Bigg|_1^t \\
&= \overset{\nearrow +\infty}{\lim_{t\to+\infty} 2t^{\frac{1}{2}}} - 2.
\end{aligned}
$$

Thus, $f(x) = \frac{1}{1+\sqrt{x}}$ diverges to $+\infty$ since $g(x) = \frac{1}{\sqrt{x}}$ diverges to $+\infty$

4. Find a formula for the general term of the arithmetic sequence whose first term is $a_1 = 1$ such that $a_{n-1} - a_n = 17$ for $n \geqslant 1$.

Remark. The general form of an arithmetic sequence is of the type $a_1 + (n-1)d$ for $n \geqslant 1$. S.t d is the common difference defined $a_n - a_{n-1}$, and a_1 is the first term in the sequence

With this, we can deduce $a_{n-1} - a_n = 17 \to d = -17$. With a_1 of course defined as 1. Consequently, the general form of this sequence would be

$$\begin{aligned} a_n &= 1 + (n-1)(-17) \\ a_n &= 1 + -17n + 17 \\ a_n &= -17n + 18. \end{aligned}$$

5. Find a formula for the general term of the geometric sequence whose first term is $a_1 = 1$ such that $\frac{a_{n+1}}{a_n} = 10$ for $n \geqslant 1$.

Remark. The general form of a geometric sequence is of the type $a_n = ar^{n-1}$ for $n \geqslant 1$, where r is the common ratio defined $\frac{a_n}{a_{n-1}}$

With this, we can see that the common ratio r, is defined as 10, which makes the general form:

$$a_n = 10^{n-1}.$$

6. Find a formula for the general term of the sequence $\{4, -1, \frac{1}{4}, -\frac{1}{16}, \frac{1}{64}, \ldots\}$.

For $n \geqslant 1$ this sequence has the general form

$$a_n = 4\left(-\frac{1}{4}\right)^{n-1}.$$

Where the common ratio $r = -\frac{1}{4}$

7. Determine whether the sequence is convergent or divergent. If it is convergent, find its limit.

   (a) $a_n = \frac{4+5n^2}{1+n}$

   (b) $a_n = \tan^{-1}(n^2)$

   (c) $a_n = \ln\left(\frac{n+2}{n^2-3}\right)$

   (d) $a_n = n\sin\left(\frac{1}{n}\right)$

   (e) $a_n = \left(1-\frac{2}{n}\right)^n$

   (f) $a_n = \frac{1000^n}{n!}$

7.a

$$\begin{aligned}
&\lim_{n\to+\infty} \frac{4+5n^2}{1+n} \\
&= \lim_{n\to+\infty} \frac{\frac{4}{n}+\frac{5n^2}{n}}{\frac{1}{n}+\frac{n}{n}} \\
&= \lim_{n\to+\infty} \frac{\frac{4}{n}+5n}{\frac{1}{n}+1} \\
&= \frac{\lim\limits_{n\to+\infty}\frac{4}{n}+\lim\limits_{n\to+\infty} 5n}{\lim\limits_{n\to+\infty}\frac{1}{n}+\lim\limits_{n\to+\infty} 1} \\
&= \frac{0+\infty}{0+1} \\
&= +\infty.
\end{aligned}$$

Thus, this sequence is divergent

7.b

$$\begin{aligned}
&\lim_{n\to+\infty} \tan^{-1}(n^2) \\
&= \frac{\pi}{2}.
\end{aligned}$$

Thus, this sequence converges

7.c

$$\lim_{n\to+\infty} \ln\left(\frac{n+2}{n^2-3}\right).$$

Considering the rational function:

$$\lim_{n\to+\infty}\left(\frac{n+2}{n^2-3}\right)$$
$$=\lim_{n\to+\infty}\left(\frac{\frac{n}{n^2}+\frac{2}{n^2}}{\frac{n^2}{n^2}-\frac{3}{n^2}}\right)$$
$$=\lim_{n\to+\infty}\left(\frac{\frac{1}{n}+\frac{2}{n^2}}{1-\frac{3}{n^2}}\right)$$
$$=\frac{\lim_{n\to+\infty}\frac{1}{n}+\lim_{n\to+\infty}\frac{2}{n^2}}{\lim_{n\to+\infty}1-\lim_{n\to+\infty}\frac{3}{n^2}}$$
$$=\frac{0+0}{1-0}$$
$$=0.$$

Consequently:

$$\lim_{n\to0}\ln(n)$$
$$=-\infty.$$

Thus, this sequence diverges

7.d

$$
\begin{aligned}
&\lim_{n\to+\infty} n\sin\left(\frac{1}{n}\right) \\
&= \lim_{n\to+\infty} \frac{\sin\left(\frac{1}{n}\right)}{n^{-1}} \\
&= \frac{\lim_{n\to+\infty} \sin\left(\frac{1}{n}\right)}{\lim_{n\to+\infty} n^{-1}} \quad \text{(Indeterminate...)} \\
&= \lim_{n\to+\infty} \frac{\cos\left(\frac{1}{n}\right)\cdot\left(-\frac{1}{n^2}\right)}{-\frac{1}{n^2}} \\
&= \lim_{n\to+\infty} \cos\left(\frac{1}{n}\right) \\
&= \lim_{n\to 0} \cos(n) \quad \left(\text{Since } \lim_{n\to+\infty}\left(\frac{1}{n}\right) = 0\right) \\
&= 1.
\end{aligned}
$$

7.e

Thus:

$$
\begin{aligned}
&\lim_{n\to+\infty} \left(1-\frac{2}{n}\right)^n \\
&= \lim_{n\to+\infty} e^{\ln\left(1-\frac{2}{n}\right)^n} \\
&= \lim_{n\to+\infty} e^{n\ln\left(1-\frac{2}{n}\right)}
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{n\to+\infty} n\ln\left(1-\frac{2}{n}\right) \\
&= \lim_{n\to+\infty} \frac{\ln\left(1-\frac{2}{n}\right)}{n^{-1}} \quad \text{(Indeterminate...)} \\
&\implies \lim_{n\to+\infty} \frac{\frac{1}{1-\frac{2}{n}}\cdot\frac{2}{n^2}}{-\frac{1}{n^2}} \\
&= \lim_{n\to+\infty} \frac{\frac{2}{\left(1-\frac{2}{n}\right)n^2}}{-\frac{1}{n^2}} \\
&= \lim_{n\to+\infty} \frac{\frac{2}{n(n-2)}}{-\frac{1}{n^2}} \\
&= \lim_{n\to+\infty} -\frac{2n^2}{n(n-2)} \\
&= \lim_{n\to+\infty} -\frac{2n}{n-2} \\
&= -2.
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{n\to+\infty} \left(1-\frac{2}{n}\right)^n \\
&= \lim_{n\to+\infty} e^{\ln\left(1-\frac{2}{n}\right)^n} \\
&= \lim_{n\to+\infty} e^{n\ln\left(1-\frac{2}{n}\right)} \\
&= \lim_{n\to+\infty} e^{-2} \\
&= e^{-2} \\
&= \frac{1}{e^2}.
\end{aligned}
$$

7.f

Remark. A sequence $\{a_n\}$ is a monotone sequence $\forall\ n \geqslant n_0$ if it is increasing $\forall\ n \geqslant n_0$ or decreasing $\forall\ n \geqslant n_0$. If $\{a_n\}$ is a bounded sequence and there exists a positive integer n_0 s.t $\{a_n\}$ is monotone for all $n \geqslant n_0$, then $\{a_n\}$ converges

The first thing to notice about this sequence, is that it begins by increasing, but eventually must become a decreasing sequence as $n!$ grows much faster than 1000^n, to find the value of n for which this switch occurs...

$$a_{n+1} = \frac{1000^{n+1}}{(n+1)!} = \frac{1000}{n+1} \cdot \frac{1000^n}{n!} = \frac{1000}{n+1} \cdot a_n.$$

Now that we have an equation for the $n+1$ term, we can deduce for which value of n the sequence will start decreasing

$$\begin{aligned}
&a_{n+1} < a_n \\
&\frac{1000}{n+1} \cdot a_n < a_n \\
&\frac{1000}{n+1} < 1 \\
&1000 < n+1 \\
&n > 999.
\end{aligned}$$

By induction, we can show that this is true

Proposition. $\forall n \geqslant 1000$, $a_n > a_{n+1}$

Proof:

Base case: $a_{1000} > a_{1001}$

$$\begin{aligned}
&\frac{1000^{1000}}{1000!} > \frac{1000^{1001}}{1001!} \\
&1000^{1000}(1001)! > 1000^{1001}(1000)! \\
&1000^{1000}(1001)(1000)! > 1000^{1001}(1000)! \\
&1001 > \frac{1000^{1001}}{1000^{1000}} \\
&1001 > 1000.
\end{aligned}$$

Inductive step: $a_n > a_{n+1}$ if we divide $\frac{a_{n+1}}{a_{n+2}}$ ...

$$\begin{aligned}
&\frac{\frac{1000^n}{n!}}{\frac{1000^{n+1}}{(n+1)!}} \\
&= \frac{1000^n(n+1)!}{1000^{n+1}n!} \\
&= \frac{1000^n(n+1)n!}{1000^{n+1}n!} \\
&= \frac{1000^n(n+1)}{1000^{n+1}} \\
&= \frac{1}{1000}(n+1).
\end{aligned}$$

for $n \geqslant 1000$, $\frac{1}{1000}(n+1) > 1$. $\therefore \frac{a_n}{a_{n+1}} > 1 \implies a_n > a_{n+1}$

Induction: $a_{n+1} > a_{n+2}$, we can divide $\frac{a_{n+1}}{a_{n+2}}$

$$\begin{aligned}
&\frac{\frac{1000^{n+1}}{(n+1)!}}{\frac{1000^{n+2}}{(n+2)!}} \\
&= \frac{1000^{n+1}(n+2)!}{1000^{n+2}(n+1)!} \\
&= \frac{1000^{n+1}(n+2)(n+1)!}{1000^{n+2}(n+1)!} \\
&= \frac{1000^{n+1}(n+2)}{1000^{n+2}} \\
&(n+2)\left(\frac{1}{1000}\right).
\end{aligned}$$

For $n \geqslant 1000$, $(n+2)\left(\frac{1}{1000}\right) > 1$. $\therefore \frac{a_{n+1}}{a_{n+2}} > 1 \implies a_{n+1} > a_{n+2}$

■ ☺

Thus, this sequence is decreasing for $n \geqslant 1000$. Furthermore, this sequence is bounded below by 0 because $\frac{(1000)^n}{n!} \geqslant 0$, $\forall\ n \in \mathbb{Z}^+$. Therefore, the conditions for the monotone convergence theorem are met and this sequence must converge.

Using the fact that this sequence converges, and a finite number of terms does not affect the convergence of a sequence, we can propose

$$\lim_{n\to+\infty} a_{n+1} = \lim_{n\to+\infty} a_n = L$$

.

Since we know...

$$a_{n+1} = \frac{1000}{n+1} \cdot a_n.$$

We can take the limit of both sides,

$$\begin{aligned}
&\lim_{n\to+\infty} a_{n+1} = \lim_{n\to+\infty} \frac{1000}{n+1} a_n \\
&L = \frac{1000}{\lim\limits_{n\to+\infty} n+1} \cdot \lim_{n\to+\infty} a_n \\
&L = 0 \cdot \lim_{n\to+\infty} a_n \\
&L = 0.
\end{aligned}$$