Exam 1

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Contents

Axioms

Axiom of distance: For all points P, Q

1. $PQ \geqslant 0$
2. $PQ = 0 \iff P = Q$
3. $PQ = QP$

Axioms of incidence

1. There are at least two different lines
2. Each line contains at least two different points
3. Each pair of points are together in at least one line
4. Each pair of points P, Q, with $PQ < \omega$ are together in at most one line

Betweenness of points axiom (Ax. BP): If A, B, C are distinct, collinear points, and if $AB + BC \leqslant \omega$, then there exists a betweenness relation among A, B, C

What this is really saying is that if **any** of $AB + BC$, $BA + AC$, $AC + CB$ is $\leqslant \omega$, then there is a betweenness relation.

Note: If Ax.BP is true for a plane $\mathbb{P}$, and if $AB + BC \leqslant \omega$ for distinct collinear A, B, C, then there is a betweenness relation, but not necessarily A-B-C

When $\omega = \infty$, then for any distinct collinear A, B, C, $AB + BC < \infty = \omega$, so there will be a betweenness relation

Quadrichotomy Axiom for Points (Ax.QP): If A, B, C, X are distinct, collinear points, and if A-B-C. Then, at least one of the following must hold

$$X\text{-}A\text{-}B, \quad A\text{-}X\text{-}B, \quad B\text{-}X\text{-}C, \quad \text{or} \quad B\text{-}C\text{-}X$$

Thus, Ax.QP says that whenever A-B-C (say on line ℓ), then any other point X on line ℓ is in either $\overrightarrow{BA}$ or $\overrightarrow{BC}$. That is,

$$\ell = \overrightarrow{BA} \cup \overrightarrow{BC}$$

Nontriviality Axiom (Ax.N): For any point A on a line ℓ there exists a point B on ℓ with $0 < AB < \omega$

This axiom is true for the planes in which $\omega = \infty$ ($\mathbb{E}$, $\mathbb{M}$, $\mathbb{H}$, $\mathbb{G}$, $\mathbb{R}^3$, $\hat{\mathbb{E}}$, ws)

This axiom is also true for $\mathbb{S}$ and Fano, where $\omega < \infty$

Definitions

Theorems

- **Theorem 6.1 (Symmetry of betweenness)**. For a general plane $\mathbb{P}$ with points, lines, distance, and satisfy the seven axioms, $A - B - C \iff C - B - A$
- **Theorem 6.2 (UMT)**: If $A - B - C$ then $B - A - C$ and $A - C - B$ are false.
- **Theorem 7.6**: For any point A on a line ℓ there exists a point C not on ℓ with $0 < AC < \omega$
- **Triangle inequality for the line**: If A, B, C are any three distinct, collinear points, then

$$AB + BC \geqslant AC$$

- **Rule of insertion**:
  - If A-B-C and A-X-B, then A-X-B-C
  - If A-B-C and B-X-C, then A-B-X-C

Propositions

- **Proposition 6.3**
  (a) $\overline{AB}$ lies in one line, the line $\overleftrightarrow{AB}$
  (b) $\overline{AB} = \overline{BA}$
  (c) If $x \in \overline{AB}$, with $X \neq B$, then $AX < AB$
- **Proposition 6.4**: Let A, B, C, D be collinear points with $0 < AB < \omega$, $0 < CD < \omega$, and $\overline{AB} = \overline{CD}$, then
  (a) Either $\{A, B\} = \{C, D\}$ or $\{A, B\} \cap \{C, D\} = \varnothing$
  (b) $AB = CD$
- **Proposition 7.1**: If A-B-C and A-C-D, then A, B, C, D are distinct and collinear
- **Proposition 7.2** If A-B-C-D, then A, B, C, D are distinct and collinear, and D-C-B-A
- **Proposition 7.5**: If $X \neq Y$ are points distinct from A or ray $\overrightarrow{AB}$, then at least one of A-X-Y or A-Y-X or X, Y in $\overline{AB}$ is true.
- **Important fact**: Suppose X is a point on a ray $\overrightarrow{AB}$ in a general plane.
  1. If A-X-B then $AX < AB$
  2. If A-B-X then $AX > AB$
  3. IF $X = B$ then $AX = AB$