Calculus 2
Chapter 5

Nathan Warner

Computer Science
Northern Illinois University
October 27, 2023
United States

Contents

Sequences and Series

5.1 Sequences

Terminology of Sequences

To work with this new topic, we need some new terms and definitions. First, an infinite sequence is an ordered list of numbers of the form

$$a_1,\ a_2,\ a_3,\ ...\ a_n,\ .....$$

Each of the numbers in the sequence is called a term. The symbol n is called the index variable for the sequence. We use the notation

$$\{a_n\}_{n=1}^{\infty},\ \text{or simply}\ \{a_n\}.$$

to denote this sequence. A similar notation is used for sets, but a sequence is an ordered list, whereas a set is not ordered. Because a particular number a_n exists for each positive integer n, we can also define a sequence as a function whose domain is the set of positive integers.

Let's consider the infinite, ordered list

$$2,\ 4,\ 8,\ 16,\ 32,\ ....$$

This is a sequence in which the first, second, and third terms are given by $a_1 = 2$, $a_2 = 4$, and $a_3 = 8$. You can probably see that the terms in this sequence have the following pattern:

$$a_1 = 2^1,\ a_2 = 2^2,\ a_3 = 2^3,\ a_4 = 2^4,\ \text{and}\ a_5 = 2^5.$$

Assuming this pattern continues, we can write the n^{th} term in the sequence by the explicit formula $a_n = 2^n$. Using this notation, we can write this sequence as

$$\{2n\}_{n=1}^{\infty} \quad \text{or} \quad \{2n\}.$$

Alternatively, we can describe this sequence in a different way. Since each term is twice the previous term, this sequence can be defined recursively by expressing the n^{th} term a_n in terms of the previous term a_{n-1}. In particular, we can define this sequence as the sequence $\{a_n\}$ where $a_1 = 2$ and for all $n \geqslant 2$, each term a_n is defined by the **recurrence relation** $a_n = 2a_{n-1}$.

Definition 1:

An infinite sequence $\{a_n\}$ is an ordered list of numbers of the form

$$a_1, a_2, \ldots, a_n, \ldots$$

The subscript n is called the index variable of the sequence. Each number a_n is a term of the sequence. Sometimes sequences are defined by explicit formulas, in which case $a_n = f(n)$ for some function $f(n)$ defined over the positive integers. In other cases, sequences are defined by using a recurrence relation. In a recurrence relation, one term (or more) of the sequence is given explicitly, and subsequent terms are defined in terms of earlier terms in the sequence.

Note:-

Note that the index does not have to start at $n = 1$ but could start with other integers. For example, a sequence given by the explicit formula $a_n = f(n)$ could start at $n = 0$,in which case the sequence would be

$$a_0,\ a_1,\ a_2,\ \ldots.$$

Similarly, for a sequence defined by a recurrence relation, the term a_0 may be given explicitly, and the terms a_n for $n \geqslant 1$ may be defined in terms of a_{n-1}. Since a sequence $\{a_n\}$ has exactly one value for each positive integer n, it can be described as a function whose domain is the set of positive integers. As a result, it makes sense to discuss the graph of a sequence. The graph of a sequence $\{a_n\}$ consists of all points (n, a_n) for all positive integers n. Figure 5.2 shows the graph of $\{2n\}$.

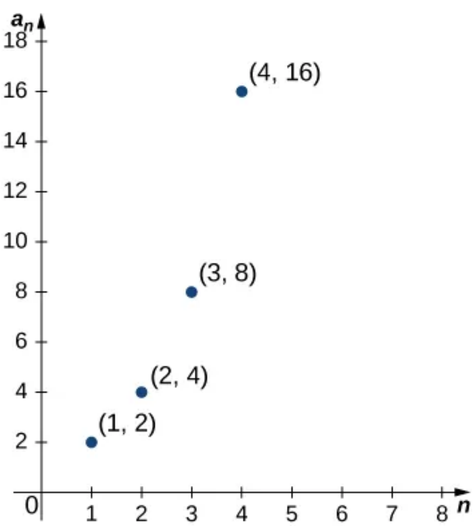

Figure 5.2