Elementary Linear Algebra Reference

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Contents

Vectors

- **Magnitude**: For a vector $x \in \mathbb{R}^n,\ x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, the norm (magnitude) of x is

$$\|x\| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$$

- **Triangle inequality**: For vectors $x, y \in \mathbb{R}^n$, we have the inequality

$$\|x + y\| \leqslant \|x\| + \|y\|$$

- **Properties of Vector Operations**: Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in a plane. Let r and s be scalars.

  1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (Commutative property)
  2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (Associative property)
  3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$ (Additive identity property)
  4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ (Additive inverse property)
  5. $r(s\mathbf{u}) = (rs)\mathbf{u}$ (Associativity of scalar multiplication)
  6. $(r + s)\mathbf{u} = r\mathbf{u} + s\mathbf{u}$ (Distributive property)
  7. $r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$ (Distributive property)
  8. $1\mathbf{u} = \mathbf{u}, \quad 0\mathbf{u} = \mathbf{0}$ (Identity and zero properties)

- **Finding components of a vector given the magnitude and the angle θ**: If $v \in \mathbb{R}^2,\ v = \begin{pmatrix} x \\ y \end{pmatrix}$, then

$$x = \|\vec{v}\| \cos\theta$$
$$y = \|\vec{v}\| \sin\theta.$$

- **Unit vector**: A unit vector is a vector with magnitude 1. For any nonzero vector $\vec{v}$, we can use scalar multiplication to find a unit vector $\vec{u}$ that has the same direction as $\vec{v}$:. To do this, we multiply the vector by the reciprocal of its magnitude:

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}.$$

- **Properties of the dot product**: Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors, and let c be a scalar.

  1. Commutative property: $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
  2. Distributive property: $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
  3. Associative property of scalar multiplication: $(c\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$
  4. Property of magnitude: $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

- **Evaluating a dot product**: The dot product of two vectors is the product of the magnitude of each vector and the cosine of the angle between them:

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos\theta.$$

- **Find the measure of the angle between two nonzero vectors**:

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}.$$

Note: We are considering $0 \leqslant \theta \leqslant \pi$

- **Vector Projection**: The vector projection of **v** onto **u** has the same initial point as **u** and **v** and the same direction as **u**, and represents the component of **v** that acts in the direction of **u** :.

$$\mathrm{proj}_{\vec{u}}\vec{v} = \frac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u}.$$

We say "The vector projection of $\vec{v}$ onto $\vec{u}$"

- **Scalar projection notation**: This is the length of the vector projection and is denoted

$$\|\mathrm{proj}_{\vec{u}}\vec{v}\| = \mathrm{comp}_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}.$$

- **Decompose some vector $\vec{v}$ into orthogonal components such that one of the component vectors has the same direction as $\vec{u}$**:
  - First, we compute $\vec{p} = \mathrm{proj}_{\vec{u}}\vec{v}$
  - Then, we define $\vec{q} = \vec{v} - \vec{p}$
  - Check that $\vec{q}$ and $\vec{p}$ are orthogonal by finding $\vec{q}\cdot\vec{p}$

- **Two vectors are orthogonal if**:

$$\vec{u}\cdot\vec{v} = 0.$$

- **Two vectors are parallel if**: Two vectors v, u are parallel if there exists some scalar $\alpha \in \mathbb{R}$ such that $\alpha u = v$

  - If $\alpha > 0$, then v points in the same direction as u
  - If $\alpha < 0$, then v points in the opposite direction of u

- **Scalar projection componets of a vector**:

$$\vec{v} = \langle \mathrm{comp}_{\hat{i}}\vec{v}, \mathrm{comp}_{\hat{j}}\vec{v}, \mathrm{comp}_{\hat{k}}\vec{v}\rangle.$$

- **The Cross Product**: produces a vector perpendicular to both vectors involved in the multiplication

Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$. Then, the cross product $\mathbf{u}\times\mathbf{v}$ is vector

$$\begin{aligned}\mathbf{u}\times\mathbf{v} &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} \\ &= \langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1\rangle..\end{aligned}$$

Note: The cross product only works in $\mathbb{R}^3$, additionally, we measure the angle between $\vec{u}$ and $\vec{v}$ in $\vec{u}\times\vec{v}$ from $\vec{u}$ to $\vec{v}$

- **Cross product using matrix and determinant**, suppose we have vectors $\vec{u}$ und $\vec{v}$:. Then we can express them in matrix form as

$$\vec{u}\times\vec{v} = \begin{bmatrix}\hat{i} & \hat{j} & \hat{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z\end{bmatrix}.$$

Then we can find the determinant of this matrix to compute the cross product

$$\vec{u}\times\vec{v} = (u_yv_z - u_zv_y)\hat{i} - (u_xv_z - u_zv_x)\hat{k} + (u_xv_y - u_yv_x)\hat{j}.$$

- **Properties of the Cross Product**: Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in space, and let c be a scalar.
  1. Anticommutative property: $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
  2. Distributive property: $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
  3. Multiplication by a constant: $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$
  4. Cross product of the zero vector: $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
  5. Cross product of a vector with itself: $\mathbf{v} \times \mathbf{v} = \mathbf{0}$
  6. Scalar triple product: $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$
- **Magnitude of the Cross Product**: Let $\mathbf{u}$ and $\mathbf{v}$ be vectors, and let θ be the angle between them. Then, $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cdot \sin\theta$.
- **Triple Scalar Product**:

  The triple scalar product of vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

  The triple scalar product is the determinant of the 3×3 matrix formed by the components of the vectors
- **triple scalar product identities**:
  (a) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = -\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v})$
  (b) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u} = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}))$
- **The zero vector is considered to be parallel to all vectors**:
- **vector equation of a line**:

  $$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}.$$

  Where $\mathbf{v}$ is the direction vector (vector parallel to the line), t is some scalar, and $\mathbf{r}$, $\mathbf{r}_0$ are position vectors
- **Parametric and Symmetric Equations of a Line**: A line L parallel to vector $\mathbf{v} = \langle a, b, c\rangle$ and passing through point $P(x_0, y_0, z_0)$ can be described by the following parametric equations:

  $$x = x_0 + ta, \quad y = y_0 + tb, \quad \text{and} \quad z = z_0 + tc.$$

  If the constants a, b, and c are all nonzero, then L can be described by the symmetric equation of the line:

  $$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

  Note: The parametric equations of a line are not unique. Using a different parallel vector or a different point on the line leads to a different, equivalent representation. Each set of parametric equations leads to a related set of symmetric equations, so it follows that a symmetric equation of a line is not unique either.
- **Vector equation of a line reworked**: Suppose we have some line, with points $P(x_0, y_0, z_0)$, $Q(x_1, y_1, z_1)$. Where $\mathbf{p} = \langle x_0, y_0, z_0\rangle$ and $\mathbf{Q} = \langle x_1, y_1, z_1\rangle$ are the correponding position vectors. Suppose we also have $\mathbf{r} := \langle x, y, z\rangle$. Then our vector equation for a line becomes

  $$\mathbf{r} = \mathbf{p} + t\left(\vec{PQ}\right).$$

  By properties of vectors, we get the vector equation of a line passing through points P and Q to be

  $$\mathbf{r} = (1 - t)\mathbf{p} + t\mathbf{q}.$$

- **Distance from a Point to a Line**: Let L be a line in space passing through point P with direction vector $\mathbf{v}$. If M is any point not on L, then the distance from M to L is
$$d = \frac{\left\|\overrightarrow{PM} \times \mathbf{v}\right\|}{\|\mathbf{v}\|}$$

- **Vector equation of a plane**: Given a point P and vector $\mathbf{n}$, the set of all points Q satisfying the equation $\mathbf{n} \cdot \overrightarrow{PQ} = 0$ forms a plane. The equation
$$\mathbf{n} \cdot \overrightarrow{PQ} = 0$$
is known as the vector equation of a plane.

- **Scalar equation of a plane**: The scalar equation of a plane containing point $P = (x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

- **General form of the equation of a plane**: This equation (the one above) can be expressed as $ax + by + cz + d = 0$, where $d = -ax_0 - by_0 - cz_0$. This form of the equation is sometimes called the general form of the equation of a plane.

Solutions to linear systems

- **Possible solutions to a linear system of two unknowns**: The linear system can have a **unique solution, no solution, or infinitely many solutions**.

- **Does the solution set form a line, plane, hyperplane, or something else?**: The formation of the solution set depends on the number of free variables,

  - **No free variables (one unique solution)**: Intersects at a point
  - **One free variable (Uncountable solutions)**: Solution set is a line (1-dimensional subspace)
  - **Two free variable (Uncountable solutions)**: Solution set forms a plane (2-dimensional subspace)
  - **Three free variable (Uncountable solutions)**: Solution set is a three dimensional subspace (In $\mathbb{R}^3$ it would be the whole space)
  - k **free variables**: Solution set is a k-dimensional subspace in $\mathbb{R}^n$

  Note: A k-dimensional subspace in $\mathbb{R}^n$ means that the solution set spans a k-dimensional space within the n-dimensional ambient space $\mathbb{R}^n$.

- **Determine if three planes intersect at a unique point**: For this, we find all three normal vectors $\vec{\mathbf{n}}_1, \vec{\mathbf{n}}_2$, and $\vec{\mathbf{n}}_3$. Then we find the triple scalar product, that is

$$\vec{\mathbf{n}}_1 \cdot (\vec{\mathbf{n}}_2 \times \vec{\mathbf{n}}_3).$$

  If this value is non-zero, we have intersection at a unique point. If the value is zero, we either have no intersection, or intersection at a line.

Linearity

- **The properties of linear equations**: A function $f : \mathbb{R}^n \to \mathbb{R}$ representing a linear equation is linear, meaning it satisfies the following properties for all vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and all scalars $c \in \mathbb{R}$:
  - **Additivity:** $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$
  - **Homogeneity of Degree 1:** $f(c\mathbf{x}) = cf(\mathbf{x})$

    It follows from this that $f(c\mathbf{x})$, when $c = 0$ implies $f(0\mathbf{x}) = 0f(\mathbf{x}) = 0$. Thus, we add the property
  - **Scale by zero**: $f(0) = 0$

  These properties define a linear function and imply that the graph of a linear equation is a straight line (in 2D) or a plane (in 3D).

Matrix algebra

- **Laws of matrix addition**:
  - **Addition with the zero matrix**: $0 + A = A$
  - **Communitive law for matrix addition**: $A + B = B + A$
  - **Associativity of matrix addition**: $(A + B) + C = A + (B + C)$
- **Laws of matrix subtraction**:
  - $A - 0 = A$
  - $A - A = 0$
  - $B - A = (-1)(A - B)$
- **Matrix difference (subtraction)**: We can give a definition to the subtraction operator by just defining it as using matrix addition and multiplication by a scalar $A - B = A + (-1B)$
- **Note on matrix multiplication**: Matrix multiplication is general **not** communitive, it can be, but it isn't always. Also, in the real numbers, we know for

  $$ab = 0.$$

  Then either a is zero, b is zero, or they are both zero. This is not always the case with matrix multiplication, it is possible to multiply two non-zero matrices and get the zero matrix as a result.
- **Properties of matrix multiplication**:
  1. If A, B, and C are matrices of the appropriate sizes, then

     $$A(BC) = (AB)C.$$

  2. If A, B, and C are matrices of the appropriate sizes, then

     $$(A + B)C = AC + BC.$$

  3. If A, B, and C are matrices of the appropriate sizes, then

     $$C(A + B) = CA + CB.$$

- **Properties of Scalar Multiplication**: If r and s are real numbers and A and B are matrices of the appropriate sizes, then
  1. $r(sA) = (rs)A$
  2. $(r + s)A = rA + sA$
  3. $r(A + B) = rA + rB$
  4. $A(rB) = r(AB) = (rA)B$
- **Note on cancellation**: If a, b, and c are real numbers for which $ab = ac$ and $a \neq 0$, it follows that $b = c$. That is, we can cancel out the nonzero factor a. However, the cancellation law does not hold for matrices.
- **Differences between matrix multiplication and multiplication of real numbers**: We summarize some of the differences between matrix multiplication and the multiplication of real numbers as follows: For matrices A, B, and C of the appropriate sizes,
  1. AB need not equal BA.
  2. AB may be the zero matrix with $A \neq 0$ and $B \neq 0$.
  3. AB may equal AC with $B \neq C$.

Transpose

- **Squared magnitude of a vector**:

$$\|x\|^2 = x^\top x$$

- **Transpose of product of matrices**:

$$(AB)^\top = B^\top A^\top$$

  Consequence:

$$(ABC)^\top = C^\top (AB)^\top = C^\top B^\top A^\top$$

- **Properties of Transpose**: If r is a scalar and A and B are matrices of the appropriate sizes, then

  1. $(A^T)^T = A$
  2. $(A + B)^T = A^T + B^T$
  3. $(AB)^T = B^T A^T$
  4. $(rA)^T = rA^T$

Linear maps

- **Composition**: Let $L: \mathbb{R}^n \to \mathbb{R}^m, \quad v \to L(v)$, and $K: \mathbb{R}^m \to \mathbb{R}^p, \quad L(v) \to K(L(v))$. We see that $L \in \mathbb{R}^{m \times n}$, and $K \in \mathbb{R}^{p \times m}$.

  The composition is

  $$K(L(v)) = (K \circ L)(v) = (KL)(v)$$

- **2D rotation map**:

  $$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

- **3D rotation, but keeping one variable constant, ie rotating about one of the coordinate axis**.

  All there cases below will require a 3×3 matrix

  - **Rotation about the x-axis (rotation in the yz-plane)**:

    $$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin() & \cos(\theta) \end{bmatrix}.$$

  - **Rotation about the y-axis (rotation in the xz-plane)**

    $$R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}.$$

  - **Rotation about the z-axis (rotation in the xy-plane)**

    $$R_z(\theta) \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

  Notes on consecutive rotations: Two consecutive rotations about different axises is **not** communitive, however if you rotation about the same axis it is.

Injective, surjective, bijective