Discrete Structures

Graph Theory

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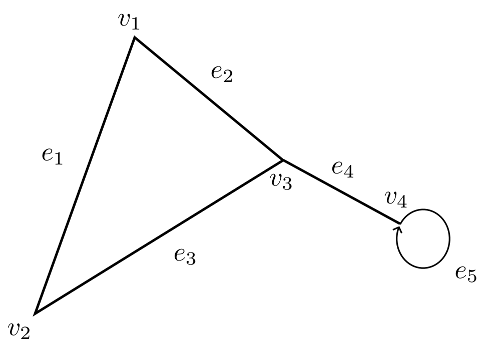
Contents

1 Graphs

Definition 1. A graph G consists of two finite sets: a nonempty set $V(G)$ of vertices and a set $E(G)$ of edges, where each edge is associated with a set consisting of either one or two vertices called its endpoints. Formally, a graph is defined as an ordered pair $G = (V, E)$, where V is the set of vertices and E is the set of edges

$$G = (V, E)$$
$$V = \{v_1, v_2, v_3, ..., v_n\}$$
$$E = \{e_1, e_2, e_3, ..., e_m\}.$$



$$V = \{v_1, v_2, v_3, v_4\}$$
$$E = \{e_1, e_2, e_3, e_4, e_5\}.$$

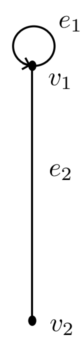
We can also represent the edges by only stating the vertices which connect the edges

Edges	Endpoints
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_3\}$
e_3	$\{v_2, v_3\}$
e_4	$\{v_3, v_4\}$
e_5	$\{v_4\}$

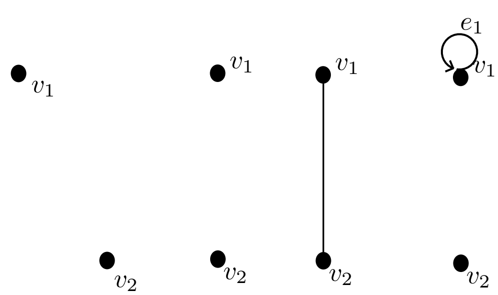
2 Subgraphs

Definition 2. Graph H is said to be a subgraph of a graph H iff every vertex in H is also a vertex in G, every edge in H is also an edge in G, and every edge in H has the same endpoints as it has in G.

Consider the graph:



Then the possible **sub graphs** could be:



Note:-

These graphs are not **all** the possibilites, just a few.

3 Degree

Definition 3. In graph theory, the **degree** of a vertex refers to the number of edges that are connected to that vertex.

Definition 4. **Parallel edges** are two or more edges that have the same pair of end vertices.

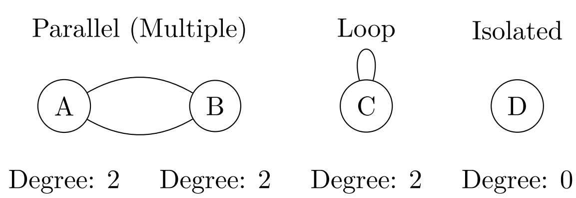
Definition 5. **Multiple Edges** is a term used interchangeably with parallel edges.

Definition 6. An **isolated vertex** is a vertex that has a degree of zero

Definition 7. A **loop** is an edge that connects a vertex to itself.

Definition 8. A **Degree Sequence** is an **n-tuple** of the degrees on vertices, in increasing order and with repetition.

Definition 9. The **overall degree** is the sum of all the degrees.



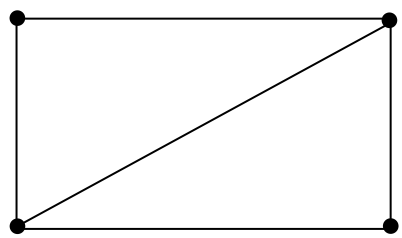
4 Sum of Degrees and Vertices Theorem

Definition 10. To denote the number of vertices in a graph, we say $||V||$, or just $|v|$. To denote the number of edges in a graph, we say $||E||$, or just $|E|$

Definition 11. The number of vertices in a graph is called the **order** of the graph

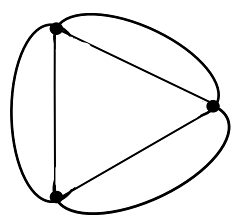
Definition 12. The number of edges in a graph is called the **size** of the graph.

Consider the graphs:



Then we have:

$$||V|| = 4$$
$$||E|| = 5$$
$$\sum deg = 10.$$



$$||V|| = 3$$
$$||E|| = 6$$
$$\sum deg = 12.$$

So you might notice from these two examples that the total degree of the graph ($\sum deg$) is exactly **twice** the number of edges. Thus, we can conclude:

Theorem 1.

$$\sum deg = 2||E||.$$

Proof. Let G be a graph, that has n vertices $v_1, v_2, v_3, v_4, ..., v_n$ and m edges, where n is a positive integer and m is a nonnegative integer.

If e_1 is an edge, then

$$v_i, v_j = \begin{cases} 1\ edge,\ 1V & \rightarrow degree = 2 \\ 1\ edge,\ 2V & \rightarrow degree = 2 \end{cases} \tag{1}$$

Thus, no matter the case, the edge always contributes 2 to the total degree. 😄

Corollary 1. The total degree of a graph is even.

Corollary 2. In any graph, there are an even number of vertices of odd degree.

5 Adjacency and Incidence

Definition 13.