# Comprehensive Compendium: Geometry, Pre-Algebra, Pre-Calculus, Trigonometry, Calculus 1, and Intro Statistics

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## Contents

# 1 Geometry

## 1.1 Line

- Equation

$$y = mx + b.$$

## 1.2 Square

- Perimeter

$$4s.$$

- Area

$$s^2.$$

## 1.3 Rectangle

- Perimeter

$$2w + 2l.$$

- Area

$$l \cdot w.$$

## 1.4 Triangle

Types:

- Right (one angles measures $90°$)
- acute (no angles measure more than $90°$)
- obtuse (one of the angles measures more than $90°$)
- isosceles (Two sides have the same length)
- equilateral (all 3 sides have the same length)
- scalene (no sides are the same length)
- Perimeter

$$a + b + c.$$

- Area

$$\frac{1}{2}bh.$$

## 1.5 Circle

- Diameter

$$2r.$$

- Circumference

$$2\pi r \ or \ \pi d.$$

- Area

$$\pi r^2.$$

- Area of a sector

$$\frac{1}{2}\pi r\theta.$$

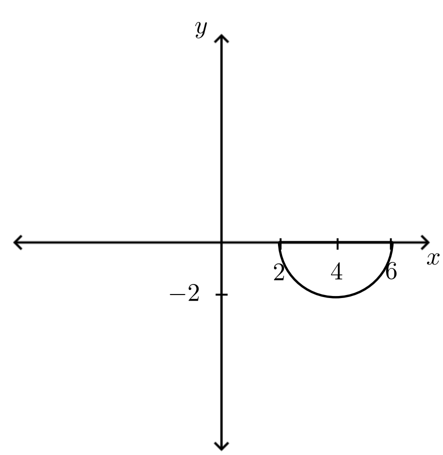
- Equation of a circle whos center is not the origin (0,0)

$$(x-h)^2+(y-k)^2=r^2, \text{ Where (h,k) is the center of the circle and r is the radius.}$$

- equation of a circle whos center is at the origin (0,0)

$$x^2+y^2=r^2.$$

- equation of a semicirle whos center is not at the origin
  - say we have




We can see we have a radius of 2, and a center at (4,0), our equation for the full circle would be:

$$(x-4)^2+(y-0)^2=2^2.$$

Solving for y we get:

$$(y-0)^2=4-(x-4)^2$$
$$y=\pm\sqrt{4-(x-4)^2}.$$

And since we only have the bottom portion of this circle we **only** use the negative sign of the $\pm$

### 1.6 Trapezoid

- Perimeter

$$a + b + c + d.$$

- Area

$$\frac{a+b}{2}h.$$

### 1.7 Parallelogram

- Perimeter

$$2l + 2w.$$

- Area

$$lh.$$

### 1.8 Cube

- Volume

$$s^3.$$

- Surface Area

$$6s^2.$$

### 1.9 Square Prism

- Volume

$$a^2h.$$

- Surface Area

$$2a^2 + 4ah.$$

### 1.10 Rectangular Prism

- Volume

$$lwh.$$

- Surface Area

$$2lh + 2wh + 2wl.$$

## 1.11 Sphere

- Volume

$$\frac{4}{3}\pi r^3.$$

- Surface Area

$$4\pi r^2.$$

## 1.12 Cylinder

- Volume

$$\pi r^2 h.$$

- Surface Area

$$2\pi rh + 2\pi r^2.$$

## 1.13 Cone

- Volume

$$\frac{1}{3}\pi r^2 h.$$

- Surface Area

$$\pi r(r + \sqrt{h^2 + r^2}).$$

## 1.14 Square Pyramid

- Volume

$$a^2\frac{h}{3}.$$

- Surface Area

$$a^2 + 2a\sqrt{\frac{a^2}{4} + h^2}.$$

## 1.15 Rectangular Pyramid

- Volume

$$\frac{1}{3}lwh.$$

- Surface Area

$$lw + l\sqrt{\left(\frac{w}{2}\right)^2 + h^2} + w\sqrt{\left(\frac{l}{2}\right)^2 + h^2}.$$

# 2 Pre-Algebra

## 2.1 Types of numbers

- Integer
$$2.$$
- Rational
$$\frac{1}{2}.$$
- Irrational
$$\sqrt{2}.$$
- Prime Number
$$7.$$
- Complex
$$3+2i.$$

## 2.2 Slope

We can calculate slope with:

$$\frac{y_2-y_1}{x_2-x_1}.$$

And we can write the equation of the line with point-slope form:

$$y-y_1=m(x-x_1),\ \text{Where } m \text{ denotes slope.}$$

Additionally, if we know the y-intercept $b$, we can write the equation with slope intercept form:

$$y=mx+b.$$

## 2.3 Solving Inequalites

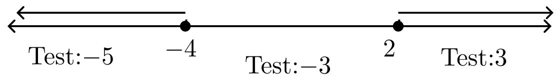
- If you divide by a negative, flip the inequality
- Quadratic Inequalites

$$\begin{aligned} x^2+2x-8 &\geqslant 0 \\ (x-2)(x+4) &\geq 0 \\ x=2 \quad x &= -4. \end{aligned}$$

$$\begin{aligned} -5:(-)(-)\to + &\geqslant 0 \\ -3:(-)(+)\to + &\leqslant 0 \\ -5:(+)(+)\to + &\geqslant 0 \\ &. \end{aligned}$$

Therefore:

$$(-\infty,-4)\cup(2,\infty).$$




## 2.4 Solving absolute value equations

***Definition:***

An absolute value equation is an equation that involves the absolute value of a variable. The absolute value of a number is its distance from zero on a number line, and it is always non-negative (or zero itself). The absolute value of a real number x is denoted as |x|.

**Example 2.1**

Consider the equation

$$|3x + 4| = 9.$$

- Solve for x with positive 9
- Solve for x with negative 9

So:

$$\begin{aligned} 3x + 4 &= 9 \\ x &= \frac{5}{3} \end{aligned}$$

And:

$$\begin{aligned} 3x + 4 &= -9 \\ x &= -\frac{13}{3}. \end{aligned}$$

**Example 2.2**

Consider the equation.

$$4|2x + 3| - 8 = 36.$$

- Get the absolute value alone
- solve for positive right side
- solve for negative right side

So:

$$\begin{aligned} |2x + 3| &= 11 \\ 2x + 3 &= 11 \\ x &= 4 \\ And: 2x + 3 &= -11 \\ x &= -7 \end{aligned}$$

## 2.5 Solving Systems of Equations.

***Definition:***

Systems of equations refer to a set of two or more equations that are considered together as a unit. These equations involve multiple variables and are interconnected, meaning the solution to the system must satisfy all the equations simultaneously. The variables in the system are typically related to each other in some way.

The two methods we will look at for solving these systems are:

- Elimination (Additon)
- Substitution

**Example 2.3**

Consider the system:

$$\begin{aligned} 2x + 3y &= 8 \\ 5x - 3y &= -1. \end{aligned}$$

To solve this system using **Elimination**, we will add the two equations together, so:

$$\begin{aligned} 7x &= 7 \\ x &= 1. \end{aligned}$$

Now that we have the value of $x$, we can pick one of the equations to plug it into to solve for y.

$$\begin{aligned} 2(1) + 3y &= 8 \\ 2 + 3y &= 8 \\ 3y &= 6 \\ y &= 2. \end{aligned}$$

**Example 2.4**

Consider the system:

$$\begin{aligned} 2x + 5y &= 19 \\ x - 2y &= -4. \end{aligned}$$

We can see that if we use the **elimination** method, none of our variable terms will cancel. In order to make it so our $x's$ will cancel with each other, we must multiply the equation by $-2$.

$$-2x + 4y = 8.$$

From here we can then use the elimination method.

**Example 2.5**

Now we will look at using the **Substitution method**, say we have the system:

$$\begin{aligned} y &= 5 - 2x \\ 4x + 3y &= 13. \end{aligned}$$

Using the Substitution method, we can plug our first equation into the $y$ term in our second equation.

$$\begin{aligned} 4x + 3(5 - 2x) &= 13 \\ 4x + 15 - 6x &= 13 \\ -2x + 15 &= 13 \\ -2x &= -2 \\ x &= 1. \end{aligned}$$

Then we can plug $x$ into our first equation.

$$\begin{aligned} y &= 5 - 2(1) \\ y &= 3. \end{aligned}$$

**Example 2.6**

Consider:

$$\begin{aligned} y &= 3x + 2 \\ y &= 7x - 6. \end{aligned}$$

With this system, we can replace the $y$ in our second equation, with the first equation. So:

$$\begin{aligned} 3x + 2 &= 7x - 6 \\ 4x &= 8 \\ x &= 2. \end{aligned}$$

Then we can plug $x$ into any of our equations to get $y$

$$\begin{aligned} y &= 3(2) + 2 \\ y &= 8. \end{aligned}$$

# 3 Algebra

## 3.1 Theorems of algebra

## 3.2 Difference Quotient

**Theorem 3.1** Difference Quotient

$$\frac{f(x+h)-f(x)}{h},\ h \neq 0.$$

The difference quotient is a mathematical concept used to calculate the average rate of change of a function over a given interval. Specifically, it is defined as the change in the function's output value divided by the change in the function's input value over the interval.

## 3.3 Distance Formula

**Theorem 3.2** Distance Formula

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

The distance formula is a mathematical formula used to calculate the distance between two points in a coordinate plane.

The distance formula can be used to calculate the distance between any two points in a two-dimensional coordinate plane, regardless of whether the points are on a straight line or a curved line. For example, if you have two points on a circle, you can use the distance formula to calculate the length of the arc between the two points. Similarly, if you have two points on a parabola, you can use the distance formula to calculate the distance between them.

## 3.4 Average Rate of Change

**Theorem 3.3** Average Rate of Change

$$m = \frac{f(b)-f(a)}{b-a}.$$

The average rate of change is a mathematical concept used to measure the average rate at which a quantity changes over a given time interval. In calculus, the average rate of change is typically used to describe the average slope of a curve over a specific interval.

## 3.5 Midpoint

**Theorem 3.4** Midpoint

$$(x_m, y_m) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

The midpoint formula can be used to find the coordinates of the center point of a line segment, which is the point that divides the line segment into two equal parts. This formula can be used in a variety of situations, such as when determining the center of a geometric shape, finding the average of two values, or calculating the half-way point between two locations.

The midpoint formula is also a useful tool in geometry for finding the center of a circle or the midpoint of an arc. By using the midpoint formula to calculate the center point of a circle or arc, it is possible to find its radius and diameter, as well as its position relative to other objects in the coordinate plane.

## 3.6 Quadratic Formula

**Theorem 3.5** Quadratic Formula

$$\frac{-b \pm \sqrt{(b)^2 - 4(a)(c)}}{2(a)}.$$

## 3.7 Sum / Difference of squares

**Theorem 3.6** Difference of Squares

Differnce of Squares:

$$(a^2 - b^2) = (a - b)(a + b).$$

Also:

$$(a^2 + b^2) = (a^2 + 2ab + b^2).$$

## 3.8 Sum / Differnce of Cubes:

**Theorem 3.7** Sum / Difference of Cubes:

Difference of Cubes:

$$(a^3 - b^3) = (a - b)(a^2 + ab + b^2).$$

Sum of Cubes:

$$(a^3 + b^3) = (a + b)(a^2 - ab + b^2).$$

## 3.9 Types of Functions

- Linear (Graph is straight line) (Standard form, Point slope form, slope intercept form)

Standard Form:

$$Ax + By = C \text{ Where A and B are integers.}$$

- Quadratic

$$f(x) = x^2.$$

- Cubic

$$f(x) = x^3.$$

- Polynomial (More than 2 terms)
- Monomial (One term)
- Binomial (2 Terms)
- Trinomial (3 Terms)
- Logarithmic

$$\log_a x = y \longrightarrow a^y = x$$
$$Where\ a > 0,\ a \neq 1,\ y \neq 0.$$

$$\ln x = y \longrightarrow x = e^y.$$

- Rational
- Radical

## 3.10 Intercepts

- X-Int (Set function equal to zero and solve for x)
- Y-Int (plug zero in for x and solve)

## 3.11 Symmetry

- Even (Symmetric with respect to y axis)

$$f(-x) = f(x).$$

- Odd (Symmetric with respect to origin)

$$f(-x) = -f(x).$$

Consider the function:

$$f(x) = x^3 - 8x.$$

If we compute $f(-x)$, we get:

$$f(-x) = -x^3 + 8x.$$

And we can show that this is the same as $-f(x)$:

$$-f(x) = -(x^3 - 8x)$$
$$-x^3 + 8x.$$

Therefore the function $f(x) = x^3 - 8x$

**Note:-**

If a function has a constant term, the function cannot be odd.

- Symmetric with respect to x-axis

$$f(x, y) = f(x, -y).$$

### 3.12 Parallel or Perpendicular

- Parallel → Two lines are parallel if their slopes are the same
- Perpendicular → Two lines are Perpendicular if the product of their slopes is -1 which means the slope of one line is the negative reciprocal of the slope of the other line.

### 3.13 Lines (Slope and Equation)

Given a line, we can find slope by:

$$\frac{\Delta y \to Rise}{\Delta x \to Run}.$$

And we can find the equation of the line with point slope form:

$$y - y_1 = x(x - x_1).$$

A line has a standard form of

$$Ax + By = C, \text{ Where A, B and C are all integers.}$$

### 3.14 Circles

- Radius $r$
- Center (h,k)
- Equation of a circle

$$(x - h)^2 + (y - k)^2 = r^2.$$

## 3.15 Determine if an equation is a function of x

- each input has precisely one output
- Vertical Line test
- $\pm$ disqualifes (will not pass the vertical line test)

## 3.16 Function Notation

1. $(f+g)(x) = f(x) + g(x)$
2. $(f-g)(x) = f(x) - g(x)$
3. $(f \cdot g)(x) = f(x) \cdot g(x)$
4. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, assuming g(x) $\neq 0$

## 3.17 Domain and Range of Functions

Domain:

- Polynomial

$$D : \mathbb{R}.$$

- Rational (Set denominator $= 0$ and solve)
- Radical (Set whats inside the radical $\geqslant 0$) (If you need to factor first and get multiple x values, make number line and test points with the factored function, domain is where it is positive)
- Logarithmic (Set inside logarithm $> 0$.) (base domain is $(0, \infty)$)

**Note:-**

If the radical is in the denominator of a radical function, set whats inside radical $> 0$

Range:

- Look for transformations
- For a rational function, we can find the range of $f(x)$ by finding the domain of $f^{-1}(x)$.

## 3.18 Finding slope of secant line

We can find the slope of the secant line by utilizing the difference quotient

## 3.19 Piecewise Defined Functions

A function is called a **Piecewise Defined Function** if it defined by two or more functions on its domain

Ex:

$$f(x) = \begin{cases} -3x & \text{if } x < -1 \\ 0 & \text{if } x = -1 \\ 2x^2 + 1 & \text{if } x > -1 \end{cases} \tag{1}$$

## 3.20 Transformations

- $-f(x)$ (reflected over x-axis)
- $f(-x)$ (reflected over y axis)
- $-f(-x)$ (reflected over origin)
- $2f(x)$ (vert stretch)
- $\frac{1}{2}f(x)$ (vert shrink)
- $f(2x)$ (Horizontal shrink)
- $f(\frac{1}{2})x$ (Hor stretch)
- $f(x-h)$ (Horizontal shift)
- $f(x)+k$ (Vertical shift)

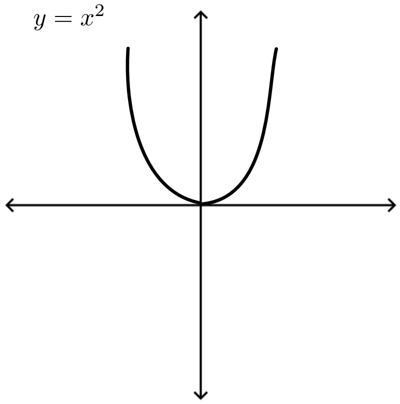
Transformations of a quadratic function

$$y = a(x-h) + k.$$

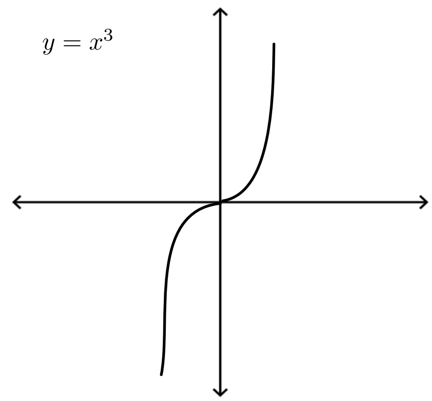
a is vertical stretch, h is horizontal shift, k is vertical shift

### 3.21 Common Functions

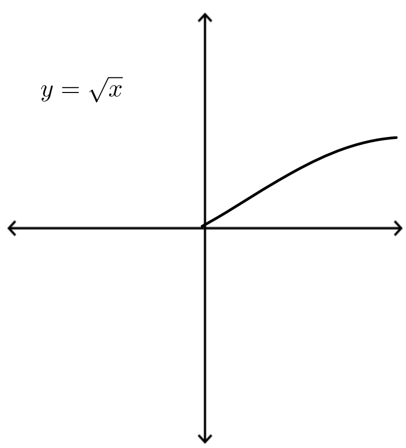
$y = x^2$



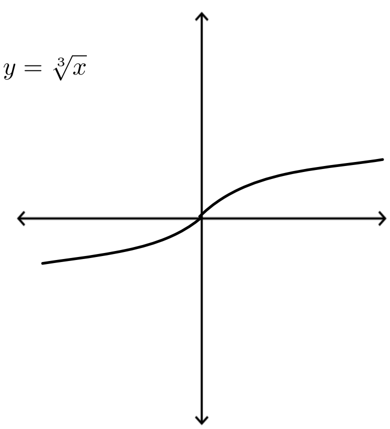
$y = x^3$

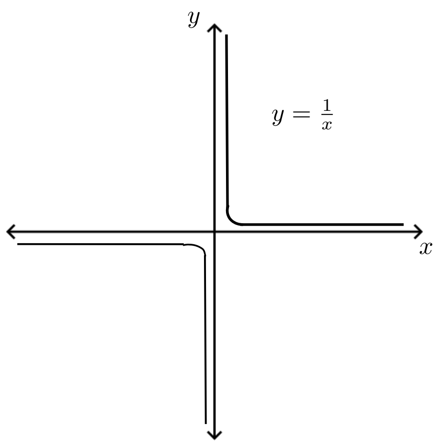


$y = \sqrt{x}$

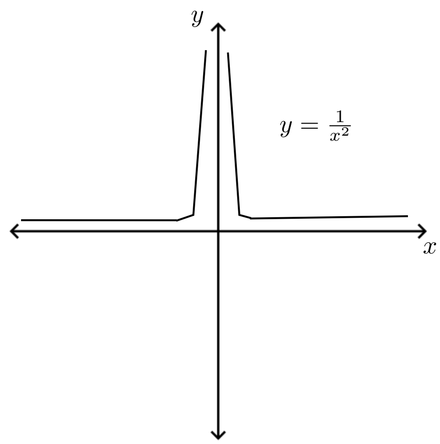


$y = \sqrt[3]{x}$

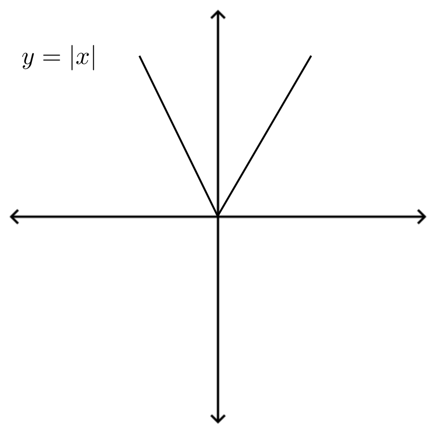




y
$y = \frac{1}{x}$
x

y
$y = \frac{1}{x^2}$
x

$y = |x|$

### 3.22 Linear Functions

A linear function is in the form:

$$f(x) = mx + b.$$

The graph of a linear function is a line with slope $m$ and a y-intercept $b$. Its domain is the set of all real numbers.

A linear function is increasing if its slope is positive, and decreasing if its slope is negative

**Note:-**

We can tell if a function is linear if $\Delta x$ and $\Delta y$ is a constant value througout all input / outputs

### 3.23 Quadratic Functions and their zeros

A zero of a function $f(x)$ is a number $r$, such that $f(r) = 0$

- Basic factoring, simply set the function equal to 0 and solve.
- Box method (when $a > 1$ and not common factors)

  *If:*

  $$5x^2 - 18x + 9.$$

  *Left as an exercise to the reader.*

- Factor by grouping (4 terms)

  *If:*

  $$5v^3 - 2v^2 + 25v - 10.$$

  *Then:*

  $$\begin{aligned} &v^2(5v-2) + 5(5v-2) \\ &= (5v-2)(v^2+5). \end{aligned}$$

- Completing the square
  - Say we have the equation:

  $$2x^2 + 7x - 4 = 0.$$

  a.) Isolate variables
  b.) Get a's coefficient 1
  c.) $\left(\frac{b}{2}\right)^2$
  d.) Add answer from c to both sides
  e.) If you factored out a constant, you also need to multiply the right side by that constant

$$2\left(x^2 + \frac{7}{2}\right) = 4$$

$$\left(\frac{\frac{7}{2}}{2}\right)^2 = \frac{49}{16}.$$

So:

$$\begin{aligned}
2\left(x^2+\frac{7}{2}+\frac{49}{16}\right) &= 4+\frac{49}{16}(2) \\
2\left(x+\frac{7}{4}\right)^2 &= \frac{81}{8} \\
\left(x+\frac{7}{4}\right)^2 &= \frac{81}{16} \\
x+\frac{7}{4} &= \pm\sqrt{\frac{81}{16}} \\
x+\frac{7}{4} &= \pm\frac{9}{4} \\
\boxed{x &= -4, \frac{1}{2}}.
\end{aligned}$$

## 3.24 Finding the zeros of a quadratic function with u Substitution

Say we have the equation:

$$(x-3)^2+5(x-3)-6=0.$$

If we let $u = x - 3$, then we have:

$$\begin{aligned}
u^2+5u-6 &= 0 \\
(u+6)(u-1) &= 0 \\
So\ u &= -6, 1.
\end{aligned}$$

Now we substitute back in for u:

$$\begin{aligned}
x-3=-6\ and\ x-3 &= 1 \\
x &= -3,\ 4.
\end{aligned}$$

## 3.25 Finding Where Two Functions Intersect

If given $f(x)$ and $g(x)$, What we do is set the functions equal to each other, and then solve such that the equation is in standard form.

With this new equation, if we solve for the zeros, these are the x values in which the two functions intersect. Then we plug this zero into $g(x)$ to get the corresponding $y$ value.

**Example 3.1**

Consider:

$$f(x)=x+6 \quad g(x)=-x.$$

So:

$$\begin{aligned} x+6 &= -x \\ 2x+6 &= 0 \\ x &= -3. \end{aligned}$$

Then we plug $-3$ into $g(x)$ to get the $y$ value.

$$g(-3) = 3.$$

Therefore our point of intersection is at $(-3, 3)$

For systems of equations, what we do is solve the system with the elimination method to get $x$ and $y$.

**Example 3.2**
Consider:

$$\begin{aligned} x-4y &= -8 \\ 2x+3y &= -5. \end{aligned}$$

So:

$$\begin{aligned} -2(x-4y) &= -2(8) \\ -2x+8y &= 16. \end{aligned}$$

Subtracting this equation from $2x+3y=-5$ we get:

$$y = 1.$$

Now if we plug $y=1$ to one of our original equations we get that:

$$x = -4.$$

Therefore our point of intersection is at $(-4, 1)$

## 3.26 Quadratic Functions and their properties

A quadratic function is a function of the form $f(x) = ax^2 + bx + c$ where $a$, $b$, and $c$ are real numbers and $a \neq 0$ The domain of a quadratic function consists of all real numbers.

A quadratic equation is an equation of the form $ax^2 + bx + c$ where $a$, $b$, and $c$ are real numbers and $a \neq 0$

The graph of the quadratic function is called the **parabola**. More on the parabola later.

The quadratic function can be written in the form $f(x) = a(x-h)^2 + k$ (standard form) either by completing the square or by using the formulas:

$$h = -\frac{b}{2a} \textit{ and } k = f(h).$$

The axis of symmetry of the parabola is $x = h$. This is the line that if we fold the parabola, it will be symmetric The vertex of the parabola is located at the point (h, k). It represents the highest point (called the maximum point) of a parabola if the parabola opens down (recall: $a < 0$). It represents the lowest point (called the minimum point) of a parabola if the parabola opens up (recall: $a > 0$).

**Example 3.3**

$$f(x) = 3x^2 + 6.$$

If:

- Vertex $= (h, k)$
- h $= \frac{-b}{2a}$
- A.O.S: x= h

Then:

$$h = \frac{0}{2(6)} = 0$$
$$k = f(0) = 6.$$

Therefore:

$$AOS: \ x = 0$$
$$Vertex: \ (0, 6) \text{ Shifted up 6 units from (0,0).}$$

## 3.27 Solving Quadratic Inequalites

Say we have the equation:

$$x^2 - x < 12$$
$$x^2 - x - 12 < 0$$
$$(x - 4)(x + 3) < 0.$$

So we have x-ints at (4, 0) and (-3,0), from this we can deduce that our porabola is below the x-axis from (-3,4)

## 3.28 Polynomial Functions and Their Models

A polynomial function is a function in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0.$$

Where $a_n, a_{n-1} ..., a_1, a_0$ are real numbers and n is a nonnegative integer.

**Note:-**

The degree of a polynomial is the highest degree of its terms

Recall: the domain of a polynomial is $\mathbb{R}$

If a function has a negative exponent then it is **not** a polynomial.

If the degree of a function is not an integer, than it is **not** a polynomial

If the multiplicity of the zero is even, the graph **touches** the x-axis at that x-intercept, if the multiplicity of the zero is odd, the graph **crosses** the x-axis at that x-intercept

Find the max number of turning points by computing: $Degree - 1$

**Example 3.4**

Label the terms of the polynomial function

$$\begin{aligned} f(x) &= 2x^5 - x^4 + 3x^2 - 7 \\ a_5 &= 2 \\ a_4 &= -1 \\ a_3 &= 0 \\ a_2 &= 3 \\ a_1 &= 0 \\ a_0 &= -7. \end{aligned}$$

**Example 3.5**

For the polynomial function:

- Find the degree of the polynomial
- List the real zeros and its multiplicity
- Find the x and y intercepts
- Determine whether the graph crosses or touches the x-axis at each x-intercept
- Determine the maximum number of turning points on the graph
- Determine the end behavior of the function
- Sketch a graph of the polynomial

$$f(x) = \left(x - \frac{1}{3}\right)^2 \left(x - 1\right)^3.$$

- Degree: 5 (Add the exponents)
- Zeros: $x = \frac{1}{3}, 1$
- multiplicity: 2 and 3 (Look at exponents)
- x-intercepts: $\left(\frac{1}{3}, 0\right)$ and $(1, 0)$

- y-intercepts: $f(0) = -\frac{1}{9}$, so $(0, -\frac{1}{9})$
- Touches: $(\frac{1}{3}, 0)$. Crosses: (1,0 )
- Max turning points: $5 - 1 = 4$
- End behavior:
  - Since the power function is of an odd degree, the graph will resemble that of $x^3$, so we can determine

$$\lim_{x \to \infty} f(x) = \infty$$
$$\lim_{x \to -\infty} f(x) = -\infty.$$

## 3.29 Power Functions

A power function of degree $n$, is a monomial of the form $f(x) = ax^n$, where $a$ is a real number, $a \neq 0$ and $n$ is a positive integer

If $n$ is **even**, then the following are true.

- The graph of the function is symmetric over the y-axis (The function is even)
- Domain: $\mathbb{R}$, range: $[0, \infty]$, if $a$ is positive and R= $[-\infty, 0]$, if $a$ is negative
- The graph resembles the graph of $x^2$

If $n$ is **odd**, then the following are true

- The graph of the function is symmetric over the origin
- Domain: $\mathbb{R}$, Range: $\mathbb{R}$
- The graph resembles the graph of $x^3$

## 3.30 Properties of Rational Functions

A Rational Function is a function of the form $\frac{g(x)}{h(x)}$, where g(x) and h(x) are polynomials and $h(x) \neq 0$

The domain of a rational function is $\mathbb{R}$, except those that make the denominator equal 0

Asymptotes:

- Vertical Asymptotes can be found by setting the denominator equal to zero and solving for x.
- Horizontal Asymptotes: If we let the highest degree of the numerator equal $n$, and the highest degree of the denominator equal $k$, then:
  1. If $n < k$, then the equation of the horizontal asymptote is $y = 0$
  2. If $n = k$, then we take the ratio of the leading coefficients
  3. If $n > k$, then the graph has no horizontal asymptote, instead the:

$$\lim_{x \to \infty \ or \ -\infty} f(x) = \infty \ or \ -\infty.$$

- Oblique (Slant) Asymptotes will occur if the degree of the numerator is precisely one higher than the degree of the denominator, to find the oblique asymptote, we must do long division

### 3.31 Sketching the graph of a rational function

To sketch the graph of a rational function, we must do the following:

- Factor the rational function and find the domain.
- Find the intercepts
- Find the asymptotes
- If there is a horizontal or oblique asymptote, determine if it intesects the graph. To do this, set unfactored function equal to the value of the asymptote

### 3.32 Polynomial and Rational Inequalties

Note: $ab < 0$ is not equivalent to $a < 0$ or $b < 0$

Procedure:

1. Get 0 on one side of the inequality and factor
2. Make a number line with the intervals from step 1 and test points between these intervals
   - If the test points lead to a true statement, that that interval is part of the solution
   - If the test points lead to a false statement, that that interval is not part of the solution
3. The solution to the inequality is the union of all the true intervals

For rational inequalities, the numbers which make the expression undefined are not part of the solution.

### 3.33 Synthetic Division

Consider the polynomial:

$$f(x) = \frac{3x^4 - 2x^2 + 5x^2 + 8}{x - 2}.$$

To preform **synthetic division**, we need to write out the coefficients like so. Note: because we dont have a $a_n x$ term, we will add a zero in its place. Also, our divisor is 2 because our factor we are dividing by is $x - 2$

$$\begin{array}{r|rrrrr} 2 & 3 & -2 & 5 & 0 & 8 \\ \hline & & & & & \\ \hline \end{array}.$$

From here, we want to drop down that first number, 3. Then what we do is multiply our divisor by that 3. Then take what you get from multipling and add to the term to the right of 3. Repaet these steps.

So

$$\begin{array}{r|rrrrr} 2 & 3 & -2 & 5 & 0 & 8 \\ \hline & & 6 & 8 & 26 & 52 \\ \hline & 3 & 4 & 13 & 26 & 60 \end{array}.$$

Which means our new polynomial is:

$$f(x) = 3x^3 + 4x^2 + 13x + 26 + \frac{60}{x - 2}.$$

Note that the last term is the **remainder** over the **divisor**

## 3.34 The real zeros of a polynomial function

If $f(x)$ and $p(x)$ are polynomials and if $p(x) \neq 0$, then there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x) = p(x) \cdot q(x) + r(x)$ where either $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $p(x)$. The polynomial $q(x)$ is the quotient, and $r(x)$ is the remainder in the division of $f(x)$ by $p(x)$.

**Remainder Theorem.**

If a polynomial $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

**Factor Theorem.**

A polynomial $f(x)$ has a factor $x - c$ if and only if $f(c) = 0$ (i.e. remainder $= 0$). That is:

- If $f(c) = 0$, then $x - c$ is a factor of $f(x)$.
- If $x - c$ is a factor of $f(x)$, then $f(c) = 0$.

**Example 3.6** (Use the remainder theorem to find the remainder when $f(x)$ is divided by $x - c$. Then use the factor theorem to determine whether $x - c$ is a factor of $f(x)$)

$$f(x) = x^4 + 3x^2 - 12, \quad x + 2 \ \ (divisor).$$

So instead of long dividing with $x + 2$, the remainder theorem states that our remainder will be $f(c)$, therefore:

$$\begin{aligned} Remainder &= f(-2) = (-2)^4 + (-2)^3 - 12 \\ &= 16. \end{aligned}$$

Since this number is not zero, 16 is not a factor (by the factor theorem)

**Theorem.**

The maximum number of zeros for a polynomial equation is less than or equal to the degree of the polynomial.

**Definition.**

- The **Constant Term** is the term that does not contain $x$
- A **Variation in sign** in $f(x)$ is when two consecutive coefficients have opposite signs.

**Descarte's rule of signs.**

Let $f(x)$ be a polynomial with real coefficients and a nonzero constant term.

1. The number of positive real zeros of $f(x)$ either is equal to the number of variations in sign of $f(x)$ or is less than that number by an even integer.
2. The number of negative real zeros of $f(x)$ either is equal to the number of variations in sign of $f(-x)$ or is less than that number by an even integer.

**Example 3.7** (Descarte's Rule of signs)

$$\begin{aligned} f(x) &= 3x^3 - 4x^2 + 3x + 7 \\ f(-x) &= -3x^3 - 4x^2 - 3x + 7. \end{aligned}$$

- Degree: 3
- Positive Solutions: 2 or 0
- Negative Solutions: 1

**Rational Zeros Theorem.**

If the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients and if $p/q$ is a rational zero of $f(x)$ such that $p$ and $q$ have no common prime factor, then

1. the numerator $p$ of the zero is a factor of the constant term $a_0$
2. the denominator $q$ of the zero is a factor of the leading coefficient $a_n$.

**Steps for finding the real zeros of a polynomial function**

1. Use the degree of the polynomial to determine the maximum number of real zeros. For example, a polynomial of degree $n$ can have at most $n$ real zeros.
2. Use Descarte's Rule of Signs to determine the possible number of positive and negative zeros. Count the number of sign changes in the coefficients of $f(x)$ and $f(-x)$. The number of positive real zeros is either equal to this number or less than it by an even integer. The number of negative real zeros is either equal to this number or less than it by an even integer.
3. Use the Rational Zero Theorem to identify rational numbers that potentially could be zeros. The possible rational zeros are of the form $\pm\frac{p}{q}$, where $p$ is a factor of the constant term and $q$ is a factor of the leading coefficient.
4. Use Factor Theorem, synthetic division, or long division to test each potential rational zero. If a potential zero is indeed a zero, it means that the polynomial can be factored by ($x -$ potential zero).
5. Each time that a zero is found, repeat step 4 on the depressed equation. The "depressed equation" is the polynomial after dividing by a factor of ($x-$found zero). This reduces the degree of the polynomial and simplifies the search for additional zeros.
6. Factor the polynomial if possible. Once all real zeros have been found, you can use the factorization to sketch the graph of the polynomial.

**Example 3.8**

Use the rational zeros theorem to find all real zeros of the polynomial. Use the zeros to factor $f$ over the real numbers

$$\begin{aligned} f(x) &= 3x^3 + 6x^2 - 15x - 30 \\ f(x) &= 3(x^3 + 2x^2 - 5x - 10) \\ f(-x) &= -3x^3 + 6x + 15x - 30. \end{aligned}$$

So:

- Degree: 3
- Max real zeros: 3
- By Decarte's rule of signs:
  - Positive real zeros: at least 1
  - Negative real zeros: either 2 or 0
- By the rational zeros theorem:

$$\pm\left\{1, 2, 5, 10\right\}.$$

  Using synthetic division (with factored $f(x)$), we can find that

$$\begin{array}{r|rrrr} -2 & 1 & 2 & -5 & -10 \\ & & -2 & 0 & -10 \\ \hline & 1 & 0 & -5 & 0 \end{array}.$$

So we see that our remainder is 0, which means $-2$ is found to be a real zero. And we have a new polynomial that has been depressed (degree lowered by 1). So we can use the depressed polynomial to find the remaining zeros.

So our depressed polynomial is:

$$x^2 - 5.$$

And we can set this equal to zero to find the remaining zeros.

$$\begin{aligned} x^2 - 5 &= 0 \\ x^2 &= 5 \\ x &= \pm\sqrt{5}. \end{aligned}$$

Therefore our 3 zeros are.

$$\boxed{\sqrt{5},\ -\sqrt{5},\ -2}.$$

## 3.35 Complex Zeros and the Fundamental theorem of algebra

**Fundamental Theorem of Algebra.**

If a polynomial $f(x)$ has positive degrees and complex coefficients, then $f(x)$ has at least one complex zero.

**Complete Factorization Theorem for Polynomials**

If $f(x)$ is a polynomial of degree $n > 0$, then there exist $n$ complex numbers $c_1, c_2, \ldots, c_n$ such that $f(x) = a(x - c_1)(x - c_2)\ldots(x - c_n)$ where $a$ is the leading coefficient of $f(x)$. Each number $c_k$ is a zero of $f(x)$.

**Conjugate Pairs Theorem**

Let $f(x)$ be a polynomial whose coefficients are real numbers. If $r = a + bi$ is a zero of $f$, then the conjugate $r = a - bi$ is also a zero of $f$.

**Note:-**

$i = \sqrt{-1}$ and $i^2 = -1$

**Example 3.9** (Find $f(x)$ given the zeros)

$$-3,\ 1 - 7i \quad \textit{degree } 3.$$

By the conjugate pairs theorem, we know that $1 + 7i$ is also a zero.

Now:

$$\begin{aligned}(x+3)(x-(1-7i))(x-(1+7i))\\ (x+3)(x-1+7i)(x-1-7i).\end{aligned}$$

And by difference of squares, which states:

$$\begin{aligned}(a+b)(a-b)\\ =(a^2-b^2).\end{aligned}$$

We have:

$$\begin{aligned}(x+3)((x-1)^2-49i^2)\\ =(x+3)(x^2-2x+1+49)\\ =(x+3)(x^2-2x+50)\\ \boxed{=x^3+x^2+44x+150}.\end{aligned}$$

**Example 3.10**

Use the given zero to find the remaining zeros of the function

$$f(x) = x^3 + 3x^2 + 25x + 75, \quad zero: \ -5i.$$

So by the conjugate pairs theorem, we know that $5i$ is a also a zero. Furthermore, because this is a degree 3 polynomial, we know we are only missing **one** zero.

**Note:-**

We can use synthetic division to divide our polynomial by the know zeros, but long division will be easier.

We will write our polynomial as:

$$f(x) = (x - 5i)(x + 5i)(x - c)$$
$$f(x) = (x^2 - 25i^2)(x - c).$$

Where $(x - c)$ is the zero we are trying to find. To solve $(x - c)$, we can divide our function $f(x)$ by the two known factors.

So if we compute

$$x^2 + 0x + 25\overline{)x^3 + 3x^2 + 25x + 75}.$$

We get $(x + 3)$, with no remainder.

Now we will set the factor $(x + 3) = 0$ and solve for the missing zero

$$x + 3 = 0$$
$$x = -3.$$

So our solution set is:

$$s = \left\{ -3, \pm 5i \right\}$$
$$\boxed{f(x) = (x - 3)(x + 5i)(x - 5i)}.$$

## 3.36 Composite Functions

***Definition.***

Given two functions $f$ and $g$, the composite function, denoted by $f \circ g$ (read as ”f composed with g”), is defined by $f \circ g = f(g(x))$.

The domain of $f \circ g$ is the set of all numbers $x$ in the domain of $g$ such that $g(x)$ is in the domain of $f$.

**Finding domain of composite function.**

Say we have the functions:

$$f(x) = \frac{4}{x+2}, \quad g(x) = \frac{1}{x}.$$

We first have to find $(f \circ g)(x)$, and then we can determine the domain.

So:

$$\begin{aligned}(f \circ g)(x) &= \frac{4}{\frac{1}{x}+2} \\ &= \frac{4x}{1+2x}.\end{aligned}$$

Now we can set the denominator equal to zero and solve.

$$\begin{aligned}1+2x &= 0 \\ 2x &= -1 \\ x &= -\frac{1}{2}.\end{aligned}$$

So for our composite function:

$$x \neq 0, \quad x \neq -\frac{1}{2}.$$

**Note:-**

Note that the domain restrictions for $g(x)$ transfer over to the composite function, this is **not** true for $f(x)$

So in interval notation we have:

$$\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 0\right) \cup \left(0, \infty\right).$$

## 3.37 One-to-One Functions

***Definition***

A function $f$ with domain $\mathcal{D}$ and range $\mathcal{R}$ is a one-to-one function if either of the following equivalent conditions is satisfied:

1. Whenever $a \neq b$ in $\mathcal{D}$, then $f(a) \neq f(b)$ in $\mathcal{R}$.
2. Whenever $f(a) = f(b)$ in $\mathcal{R}$, then $a = b$ in $\mathcal{D}$.

**Example 3.11** (Determine whether the function is one-to-one)

$$f(x) = 2x^3 - 4.$$

So:

$$\begin{aligned} f(a) &= f(b) \\ 2a^3 - 4 &= 2b^3 - 4 \\ 2a^3 &= 2b^3 \\ a^3 &= b^3 \\ \sqrt[3]{a^3} &= \sqrt[3]{b^3} \\ a &= b \end{aligned}$$

Thus, $a = b$ so the function is **one-to-one**

**Example 3.12** (Show that the function is one-to-one)

$$f(x) = \frac{4x}{x-2}.$$

So:

$$\begin{aligned} f(a) &= f(b) \\ \frac{4a}{a-2} &= \frac{4b}{b-2} \\ 4b(a-2) &= 4a(b-2) \\ 4ab - 8b &= 4ab - 8a \\ -8b &= -8a \\ b &= a \end{aligned}$$

Thus, this function is one-to-one

**Horizontal Line Test.**

A function $f$ is one-to-one if and only if every horizontal line intersects the graph of $f$ in at most one point.

## 3.38 Inverse Functions

**Steps for finding inverse functions**

1. Verify that $f(x)$ is one-to-one on its domain
2. let $y = f(x)$
3. swap x and y
4. solve for y
5. write as $f^{-1}$

**Theorem.**

Let $f$ be a one-to-one function with domain $\mathcal{D}$ and range $\mathcal{R}$. If $g$ is a function with domain $\mathcal{R}$ and range $\mathcal{D}$, then $g$ is the inverse function of $f$ if and only if both of the following conditions are true:

1. $g(f(x)) = x$ for every $x$ in $\mathcal{D}$.
2. $f(g(y)) = y$ for every $y$ in $\mathcal{R}$.

Domain of $f^{-1}$ = Range of $f$
Range of $f^{-1}$ = Domain of $f$

The graph of a function $f$ and the graph of its inverse $f^{-1}$ are symmetric with respect to the line $y = x$

**Example 3.13** (Find the inverse function)

$$f(x) = \frac{4x}{x-2}.$$

Let's first verify that this function is one-to-one

So:

$$\begin{aligned} f(a) &= f(b) \\ \frac{4a}{a-2} &= \frac{4b}{b-2} \\ 4b(a-2) &= 4a(b-2) \\ 4ab - 8b &= 4ab - 8a \\ -8b &= -8a \\ b &= a \end{aligned}$$

Thus, this function is one-to-one

Now let's let y = f(x), interchange x and y and then solve for y.

$$
\begin{aligned}
x &= \frac{4y}{y-2} \\
x(y-2) &= 4y \\
xy - 2x &= 4y \\
xy - 4y &= 2x \\
y(x-4) &= 2x \\
y &= \frac{2x}{x-4} \\
\boxed{f^{-1} = \frac{2x}{x-4}}&.
\end{aligned}
$$

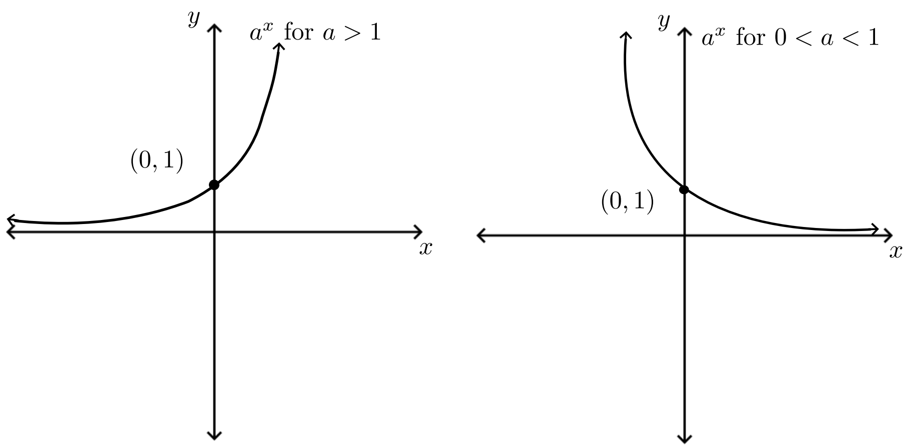
### 3.39 Exponential Functions

***Definition.***

Exponential function $f$ with base $a$ is written as $a^x$

- $\mathcal{D} : \mathbb{R}$
- $\mathcal{R} : (0, \infty)$

Figures for exponential functions.




**$a^x$ key points**

$$(0, 1)$$
$$(1, a)$$
$$\left(-1, \frac{1}{a}\right).$$

And a horizontal asymptote at $y = 0$

**Theorem.**

The exponential function $f$ given by $f(x) = a^x$ for $0 < a < 1$ or $a > 1$ is one-to-one. Thus, the following conditions are satisfied for real numbers $x_1$ and $x_2$.

1. If $x_1 \neq x_2$, then $a^{x_1} \neq a^{x_2}$.
2. If $a^{x_1} = a^{x_2}$, then $x_1 = x_2$.

**Laws of exponents**

- Product of Powers

$$x^n \cdot x^m = x^{n+m}.$$

- Quotient of Powers

$$\frac{x^n}{x^m} = x^{n-m}.$$

- Power of a Power

$$(a^n)^m = a^{n \cdot m}.$$

- Product of a Power

$$(x \cdot y)^n = x^n \cdot y^n.$$

- Product of a Quotient

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}.$$

- Negative powers

$$x^{-n} = \frac{1}{x^n}.$$

- Power of 0

$$a^0 = 1.$$

**Example 3.14**

$$\begin{aligned}
9^{2x} \cdot 27^{x^2} &= 3^{-1} \\
(3^2)^{2x} \cdot (3^3)^{x^2} &= 3^{-1} \\
3^{4x} \cdot 3^{3x^2} &= 3^{-1} \\
3^{4x+3x^2} &= 3^{-1} \\
4x + 3x^2 &= -1 \\
3x^2 + 4x + 1 &= 0 \\
(x+1)(x+3) &= 0 \\
x &= -3, -1.
\end{aligned}$$

## 3.40 Natural Exponential Function

***Definition:***

If $n$ is a positive integer, then:

$$\left(1+\frac{1}{n}\right)^n \rightarrow e \approx 2.71828 \;\; as \; n \rightarrow \infty.$$

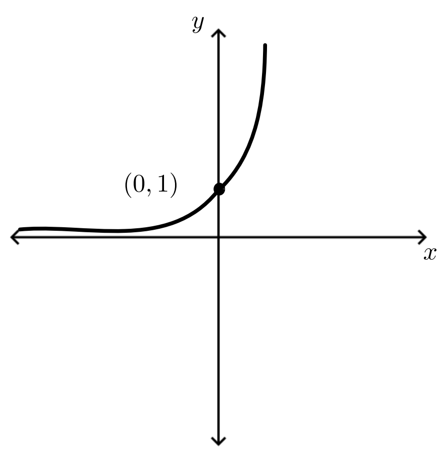
Thus:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 = e.$$

The natural exponential function $f$ is defined by

$$f(x) = e^x \text{ for every real number x..}$$

Graph of $e^x$:




$e^x$ **key points**

$$(0,1)$$
$$(1,e)$$
$$\left(-1,\frac{1}{e}\right).$$

Horizontal Asymptote at $y = 0$

## 3.41 Logarithmic Function

**Recall:**

1. Exponential functions are one-to-one with a horizontal asymptote at $y = 0$.
2. One-to-one functions have inverse functions.

The inverse of an exponential function is the logarithmic function.

***Definition:***

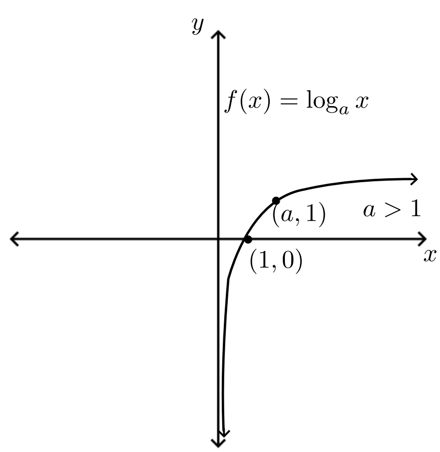
Let $a$ be a positive real number different from 1. The logarithm of $x$ with base $a$ is defined by $\log_a x = y$ if and only if $x = a^y$ for every $x > 0$ and every real number $y$.

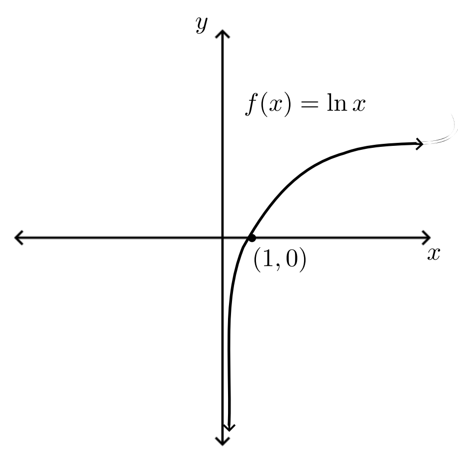
Domain of logarithm: $(0, \infty)$
Range of logarithm: $(-\infty, \infty)$

**Common Logarithm:** (Logarithm with base 10) $\log x = \log_{10} x$ for every $x > 0$

**Natural Logarithm:** (Logarithm base e) $\ln x = \log_e x$ for every $x > 0$

Graphs of logarithmic functions:




**Key Points for $\log_a x$:**

$$(1, 0)$$
$$(a, 1)$$
$$\left(\frac{1}{a}, -1\right).$$

V.A at $x = 0$

**Key Points for $\ln x$:**

$$
\begin{aligned}
&(1, 0)\\
&(e, 1)\\
&\left(\frac{1}{e}, -1\right).
\end{aligned}
$$

V.A at $x = 0$

**Example 3.15** (Changing to logarithmic form)

$$
\begin{aligned}
3^{-4} &= \frac{1}{81}\\
&= \log_3 \frac{1}{81} = -4.
\end{aligned}
$$

**Example 3.16** (Changing to exponential form)

$$
\begin{aligned}
\log_a \frac{1}{256} &= -4\\
a^{-4} &= \frac{1}{256}.
\end{aligned}
$$

**Example 3.17** (Finding the exact value)

$$\log_5 \sqrt[3]{25}.$$

1. Change to exponential form
2. Solve for x

$$
\begin{aligned}
5^x &= 25^{\frac{1}{3}}\\
5^x &= (5^2)^{\frac{1}{3}}\\
5^x &= 5^{\frac{2}{3}}\\
&\boxed{x = \frac{2}{3}}.
\end{aligned}
$$

**Finding Domain of Logarithmic Function.**

1. Set inside of logarithm strictly greater than zero.
2. Solve the inequality.
3. If you have more than one $x$, construct a number and test points, only positive outputs are included in the solution. Much like a radical function

**Theorem.**

The logarithmic function $f$ given by $f(x) = \log_a x$ for $a \neq 1$ and $a > 0$, $x > 0$ is one-to-one.

Thus, the following conditions are satisfied for real numbers $x_1$ and $x_2$.

1. If $x_1 \neq x_2$, then $\log_a x_1 \neq \log_a x_2$.
2. If $\log_a x_1 = \log_a x_2$, then $x_1 = x_2$.

**Example 3.18**

$$\begin{aligned} \log_6 36 &= 5x + 3 \\ 2 &= 5x^3 \\ x &= -\frac{1}{5}. \end{aligned}$$

**Example 3.19**

$$\begin{aligned} 4e^{x+2} &= 5 \\ e^{x+2} &= \frac{5}{4} \\ \log_e \frac{5}{4} &= x + 2 \\ \ln \frac{5}{4} &= x + 2 \\ x &= ln\frac{5}{4} - 2. \end{aligned}$$

**Properties of Logarithms.**

1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\log_a(ax) = x$
4. $a^{\log_a x} = x$
5. $\log_a(u \cdot w) = \log_a u + \log_a w$
6. $\log_a\left(\frac{u}{w}\right) = \log_a u - \log_a w$
7. $\log_a(u^c) = c \cdot \log_a u$

**Change of Base Formula.**

If $u > 0$ and if $a$ and $b$ are positive real numbers different from 1, then $\log_b u = \frac{\log_a u}{\log_a b}$.

**Notes**

- $\log_a(u + w) \neq \log_a u + \log_a w$
- $\log_a(u - w) \neq \log_a u - \log_a w$

**Example 3.20**

$$2^{\log_2 x}$$
$$= x.$$

**Example 3.21**

$$\log_3 9 \cdot \log_8 9.$$

We can use the change of base formula to rewrite this as:

$$\begin{aligned} &\left(\frac{\log_8}{\log_3}\right)\left(\frac{\log_9}{\log_8}\right) \\ &= \frac{\log_9}{\log_3} \\ &= \log_3 9 \to \text{Reverse change of base formula} \\ &= 2. \end{aligned}$$

**Example 3.22**

$$e^{\log_{e^2} 9}.$$

Using change of base formula.

$$\begin{aligned} &e^{\frac{\log_e 9}{\log_e e^2}} \\ &= e^{\frac{\log_e 9}{2}} \\ &= e^{\frac{1}{2}\log_e 9} \\ &= e^{\log_e 9^{\frac{1}{2}}} \\ &= \sqrt{9} \\ &\boxed{= 3}. \end{aligned}$$

**Shortcut.**

Say we have:

$$\log \frac{ab}{xyz}.$$

To expand this:

$$\begin{aligned} \log a + \log b - (\log x + \log y + \log z) \\ \log a + \log b - \log x - \log y - \log z \\ . \end{aligned}$$

**Example 3.23** (Expand)

$$\begin{aligned} &\log \frac{\sqrt{x}}{y^4 \sqrt[3]{z}} \\ &= \log x^{\frac{1}{2}} - \log y^4 - \log z^{\frac{1}{3}} \\ &= \frac{1}{2}\log x - 4\log y - \frac{1}{3}\log z. \end{aligned}$$

**Example 3.24** (Write the expressions as a single logarithm)

$$\begin{aligned} &5\log_a x - \frac{1}{2}\log_a(3x-4) - 3\log_a(5x+1) \\ &= \log_a x^5 - \log_a \sqrt{3x-4} - \log_a(5x+1)^3 \\ &= \log_a\left(\frac{x^5}{\sqrt{3x-4}(5x+1)^3}\right). \end{aligned}$$

**Tip.**

$$\log \frac{1}{x}$$
$$= -\log x.$$

**Example 3.25** (Solve the equation)

$$3\log_2 x = 2\log_2 3$$
$$= \log_2 x^3 = \log_2 3^2.$$

Since both the logarithms have the same base, we can set $x^3 = 3^2$

$$x^3 = 9$$
$$x = \sqrt[3]{9}.$$

**Example 3.26** (Solve the equation)

$$\begin{aligned}
\log_6(x+5) + \log_6 x &= 2\\
= \log_6(x(x+5)) &= 2\\
\log_6(x^2+5x) &= 2\\
6^2 &= x^2+5x\\
36 &= x^2+5x\\
x^2+5x-36 &= 0\\
(x-9)(x+4)&\\
x &= -4, 9.
\end{aligned}$$

Since logarithms can't be negative numbers, only 9 is a solution.

**Example 3.27** (Solve the equation)

$$\begin{aligned}
\log(57x) &= 2 + \log(x-2)\\
\log(57x) - \log(x-2) &= 2\\
\log\left(\frac{57x}{x-2}\right) &= 2\\
10^2 &= \frac{57x}{x-2}\\
100 &= \frac{57x}{x-2}\\
100(x-2) &= 57x\\
100x - 200 &= 57x\\
43x &= 200\\
x &= \frac{200}{43}.
\end{aligned}$$

If we plug $\frac{200}{43}$ into the logs in our original equation, nothing comes out negative therefore this is a **valid** solution

**Note:-**

After solving logarithmic equations, make sure you plug solutions into original equations logarithms to make sure they are valid solutions, If you get a negative output, it is not a valid solution.

**Example 3.28** (Find the exact solution, using common logarithms, and four decimal place solution, when appropriate)

$$e^{x+3} = \pi^x.$$

Since we can't get a common base on both sides, we can take the common log, or natural log of both sides.

$$\begin{aligned}
\ln e^{x+3} &= \ln \pi^x\\
x + 3 &= x \cdot \ln \pi\\
3 &= x\ln\pi - x\\
3 &= x(\ln\pi - 1)\\
x &= \frac{3}{\ln\pi - 1}\\
&\approx 20.7283.
\end{aligned}$$

**Example 3.29**

$$4^{2x+3} = 5^{x-2}.$$

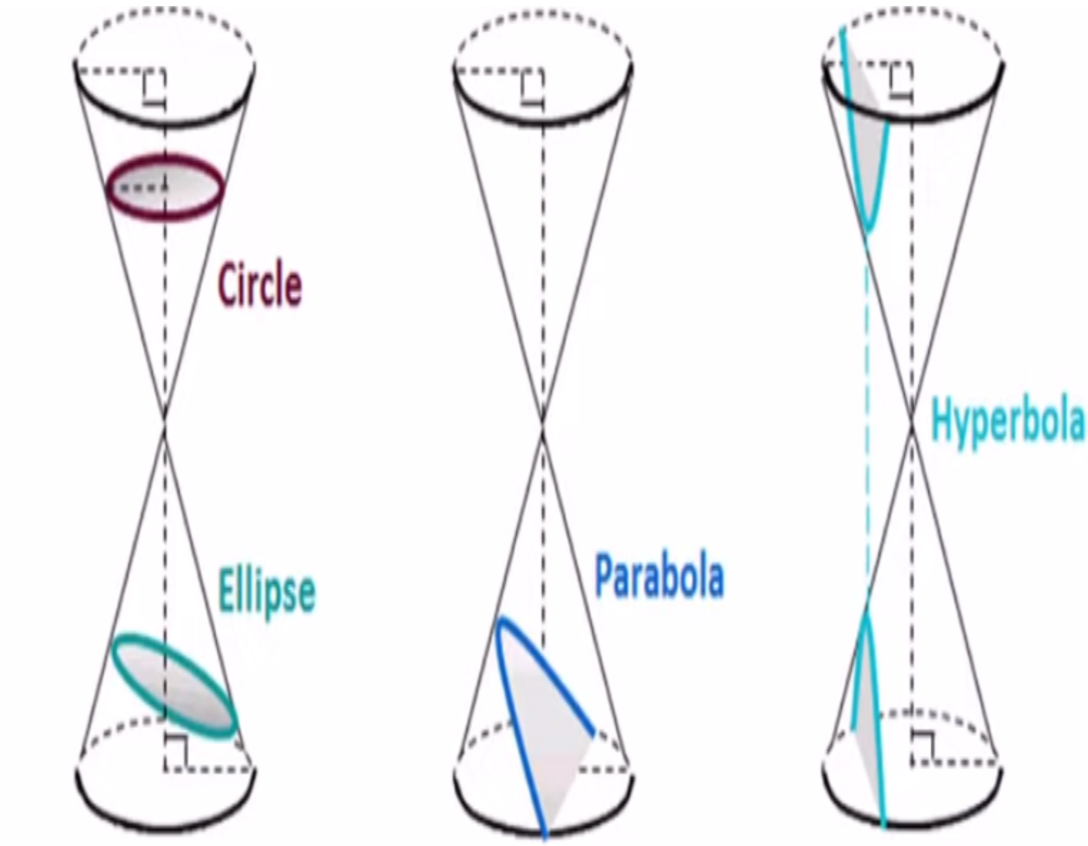
Again, take the log of both sides.

$$\begin{aligned}\log\left(4^{2x+3}\right) &= \log\left(5^{x-2}\right)\\ (2x+3)\log 4 &= (x-2)\log 5\\ 2x\log 4 + 3\log 4 &= x\log 5 - 2\log 5.\end{aligned}$$

From here we want to get all x's on the same side.

$$\begin{aligned}3\log 4 + 2\log 5 &= 2x\log 4 - x\log 5\\ 3\log 4 + 2\log 5 &= x(2\log 4 - \log 5)\\ x &= \frac{3\log 4 + 2\log 5}{2\log 4 - \log 5}.\end{aligned}$$

### 3.42 Conic Sections/The Parabola



**The Parabola.**

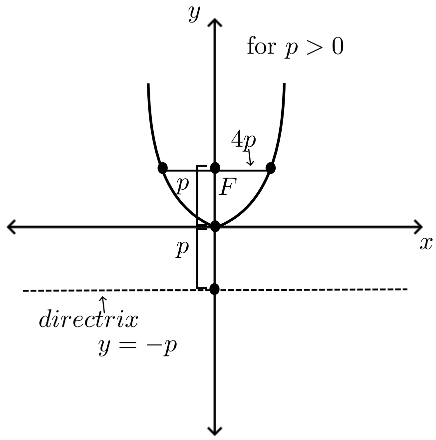
A parabola is the set of all points in a plane equidistant from a fixed point $F$ (the focus) and a fixed line $\ell$ (the directrix) that lie in the plane.

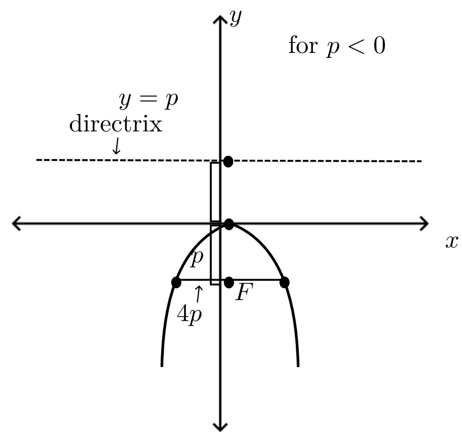
The axis of the parabola is the line through $F$ that is perpendicular to the directrix.

The vertex of the parabola is the point $V$ on the axis halfway from $F$ to $\ell$.

A Parabola with vertex $V(h, k)$ that opens up or down will have the standard form:

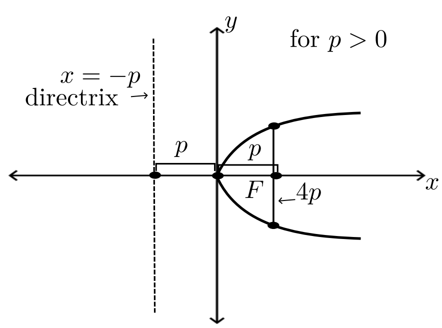
$$
\begin{aligned}
(x-h)^2 &= 4p(y-k) \\
Focus: &\ \ F(h, k+p) \\
Directrix: &\ \ y = k - p \\
\text{Length of latus rectum (focal width)}: &\ \ 4p.
\end{aligned}
$$

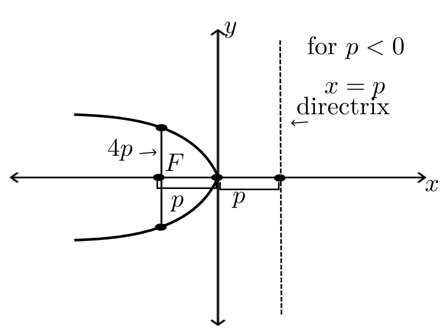



A Parabola with vertex $V(h, k)$ that left or right will have the standard form:

$$
\begin{aligned}
(y-k)^2 &= 4p(x-h) \\
Focus: &\ \ F(h+p, k) \\
Directrix: &\ \ x = h - p \\
\text{Length of latus rectum (focal width)}: &\ \ 4p.
\end{aligned}
$$



**Example 3.30**

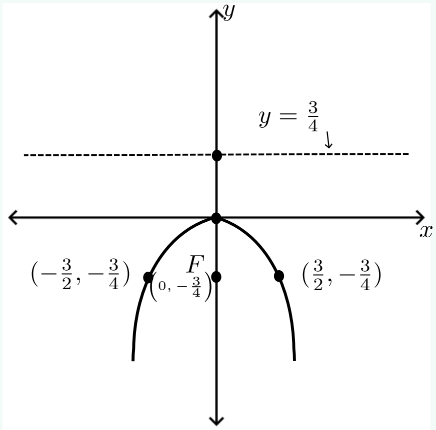
Sketch the graph and find the focus, vertex and directrix

$$x^2 = -3y.$$

So we know it is opening *down*, and:

$$\begin{aligned} 4p &= -3 \\ p &= -\frac{3}{4} \\ Vertex: &\ (0,0) \\ Focus: &\ (0, -\frac{3}{4}) \end{aligned}$$

To get the other two points, since we know that the focal width is 3, we can split that inhalf to get $\frac{3}{2}$, so our graph would look like:



**Example 3.31**

$$(y+1)^2 = -12(x+2).$$

Since $y$ is the quadratic term, we know that it is either going to open left or right.

$$\begin{aligned} Vertex:\ & (-2,-1) \\ 4p &= -12 \\ p &= -3. \end{aligned}$$

Since $p < 0$, we know that it is going to open left.

If:

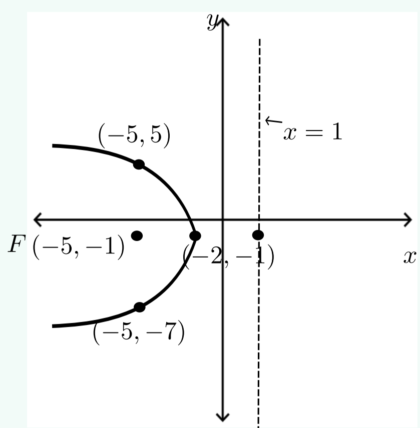
$$\begin{aligned} & F(h+4,k) \\ Then:\ & F(-5,-1). \end{aligned}$$

If:

$$\begin{aligned} Directrix:\ & x = h-p \\ Then:\ directrix\ is\ & x = -2+3 \\ & x = 1. \end{aligned}$$

To find points since $4p = -12$ and $p = -3$, we know our focal width is 12, and that split in half is 6, so:

$$\begin{aligned} p_1 &= (-5,-1+6) = (-5,5) \\ p_2 &= (-5,-1-6) = (-5,-7). \end{aligned}$$



**Example 3.32**

$$x = 2y^2 - 6y + 7.$$

For this equation, to get in standard form $(y-h)^2 = 4p(x-k)$, we must **complete the square.**

$$\begin{aligned} x - 7 &= 2y^2 - 6y \\ x - 7 &= 2(y^2 - 3y) \\ x - 7 + \left(2\left(\frac{3}{2}\right)^2\right) &= 2\left(y^2 - 3y + \left(\frac{3}{2}\right)^2\right) \\ x - 7 + \frac{9}{2} &= 2\left(y^2 - 3y + \frac{9}{4}\right) \\ x - \frac{5}{2} &= 2\left(y - \frac{3}{2}\right)^2 \\ \left(y - \frac{3}{2}\right)^2 &= \frac{1}{2}\left(x - \frac{5}{2}\right). \end{aligned}$$

From here we can procede normally.

**Example 3.33** (Find the equation of the parabola that satisfies the given conditions)

$$F(-3,-2), \quad Directrix: \; y = 1.$$

Since the directrix is a value of y, we know that this parabola is either going to open up or down. And from the fact that the directrix is **above** the focus, we can deduce that this parabola will open **Down.**

Since the Vertex is always halfway between the directrix and the focus, we can add the directrix value, 1, to the $|y|$ value of the focus to get the total distance between the two points. Then, we can divide this numebr in half to get the distance between the two points. Therefore:

$$\begin{aligned} &V\left(-3, -2 + \frac{3}{2}\right) \\ &= \left(-3, -\frac{1}{2}\right). \end{aligned}$$

Next, since the distance between the vertex and the focus is $-\frac{3}{2}$ (negative because it is opening down). We know that $p = -\frac{3}{2}$

So with all of this information, our equation would be:

$$\begin{aligned} (x+3)^2 &= 4p\left(-\frac{3}{2}\right)(y + \frac{1}{2}) \\ (x+3)^2 &= -6\left(x + \frac{1}{2}\right). \end{aligned}$$

**Example 3.34**

$$V(3,-2),\ \text{axis of symmetry parallel to the x-axis, and the y-intercept is 1.}$$

So since the A.O.S is a value of x, we know that this parabola will open left or right.

So our equation would be:

$$(y+2)^2 = 4p(x-3).$$

Now we can plug the y-intercept (0,1) into this equation to solve for $p$:

$$\begin{aligned}(1+2)^2 &= 4p(0-3)\\ 9 &= 4p(-3)\\ -3 &= 4p.\end{aligned}$$

Therefore:

$$(y+2)^2 = -3(x-3).$$

## 3.43 Ellipses

***Definition:***

An ellipse is the set of all points in a plane, the sum of whose distances from two fixed points (the foci) in the plane is a positive constant. The foci are a distance $c$ from the center, where $c^2 = a^2 - b^2$.

The major axis of the ellipse is the longest line segment passing through the center and foci. The end points of the major axis are called the vertices of the ellipse. Vertices are a distance of $a$ from the center.

The minor axis of the ellipse is the shortest line segment passing through the center. The length of the major axis is $2a$, and the length of the minor axis is $2b$.

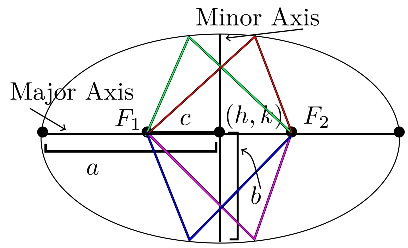
Standard Equation of an ellipse with major axis parallel to the x-axis and with center $(h,k)$ and $c^2 = a^2 - b^2$

$$\begin{aligned}\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} &= 1,\ \text{Where } a > b > 0\\ Foci:\ F(h \pm c, k)\\ Vertices:\ (h \pm a, k).\end{aligned}$$

Standard Equation of an ellipse with major axis parallel to the y-axis and with center $(h,k)$ and $c^2 = a^2 - b^2$

$$\begin{aligned}\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} &= 1,\ \text{Where } a > b > 0\\ Foci:\ F(h, k \pm c)\\ Vertices:\ (h, k \pm a).\end{aligned}$$

*Figure: Ellipse*




Consider each set of colored line segments connecting the two foci of the ellipse. If we take the sum of the line segments within a particular set, such as the green ones, we will find that their total length is equal to the combined length of all other possible line segments that can be drawn on the ellipse.

**Note:-**

The distance between the foci and the center is $c$, $c^2 = a^2 - b^2$

The endpoints of the major axis are called the **Vertices**, there are only 2 vertices on an ellipse

The distance between a vertice and the center is $a$, total length of the major axis is $2a$

The distance between the center and the end of the minor axis is $b$, total length is $2b$

**Example 3.35**

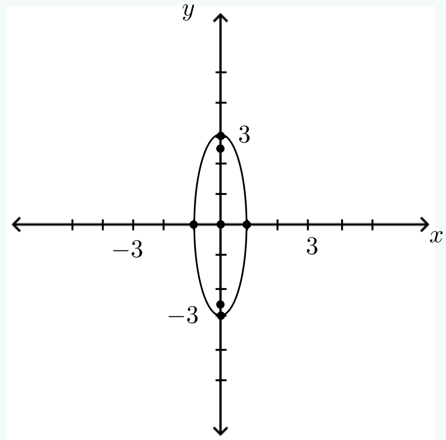
$$\begin{aligned} y^2 + 9x^2 &= 9 \\ \frac{x^2}{1} + \frac{y^2}{9} &= 1 \\ \frac{x^2}{1^2} + \frac{y^2}{3^2} &= 1 \\ x^2 + \left(\frac{y}{3}\right)^2 &= 1. \end{aligned}$$

Which means we have:

$$\begin{aligned} Center:\ (0,0) \\ a = 3 \\ b = 1 \\ c^2 = 9 - 1 \\ c = \sqrt{8} = 2\sqrt{2}. \end{aligned}$$

Since the value under $y$ is greater than the value under $x$, we know that this ellipse will have a major axis parallel to the x-axis (tall)

$$Vertices: \ (0, \pm 3)$$
$$Foci: \ (0, \pm 2\sqrt{2}).$$




**Example 3.36**

$$9x^2 + 4y^2 - 18x + 16y - 11 = 0.$$

So:

$$\begin{aligned}
9x^2 - 18x + 4y^2 + 16y &= 11 \\
9(x^2 - 2x) + 4(y^2 + 4y) &= 11 \\
9\left(x^2 - 2x + \left(-\frac{2}{2}\right)^2\right) + 4\left(y^2 + 4y + \left(\frac{4}{2}\right)^2\right) &= 11 + 9\big(1\big) + 4\big(4\big) \\
9(x^2 - 2x + 1) + 4(y^2 + 4y + 4) &= 36 \\
9(x-1)^2 + 4(y+2)^2 &= 36 \\
= \frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} &= 1 \\
= \frac{(x-1)^2}{2^2} + \frac{(y+2)^2}{3^2} &= 1
\end{aligned}$$

From here we can procede as normal.

**Example 3.37** (find the equation of the ellipse that has its center at the origin and satsifies the given conditions)

$$F(\pm 3, 0),\ \text{Minor axis length of 2.}$$

So since the foci lies across the x-axis, we know that this ellipse will have a major axis parallel to the x-axis.

So we would have:

$$Center:\ (0,0)$$
$$\frac{(x-0)^2}{a^2} + \frac{(y-0)^2}{b^2} = 1.$$

If:

$$F(h \pm c, k)$$
$$Then:\ c = 3.$$

And since:

$$\begin{aligned} Minor\ Axis :&= 2 \\ 2b &= 2 \\ b &= 1. \end{aligned}$$

Then:

$$\begin{aligned} 3^2 &= a^2 - 1^2 \\ 9 &= a^2 - 1 \\ a^2 &= 10 \\ &. \end{aligned}$$

Therefore our equation is:

$$\boxed{\frac{x^2}{10} + \frac{y^2}{1} = 1}.$$

## 3.44 Hyperbola

***Definition:***

A hyperbola is the set of all points in a plane, the difference of whose distances from two fixed points (the foci) in the plane is a positive constant. The foci are a distance $c$ from the center, where $c^2 = a^2 + b^2$.

The vertices of the hyperbola are the points obtained at the intersection of the graph and the $x$-axis or $y$-axis. Vertices are a distance $a$ from the center.

The transverse axis of the hyperbola is the line segment passing through the center and the vertices. The length of the transverse axis is $2a$.

The conjugate axis of the hyperbola is the line segment passing through the center and the points that are not on the hyperbola intersecting the other axis. The length of the conjugate axis is $2b$.

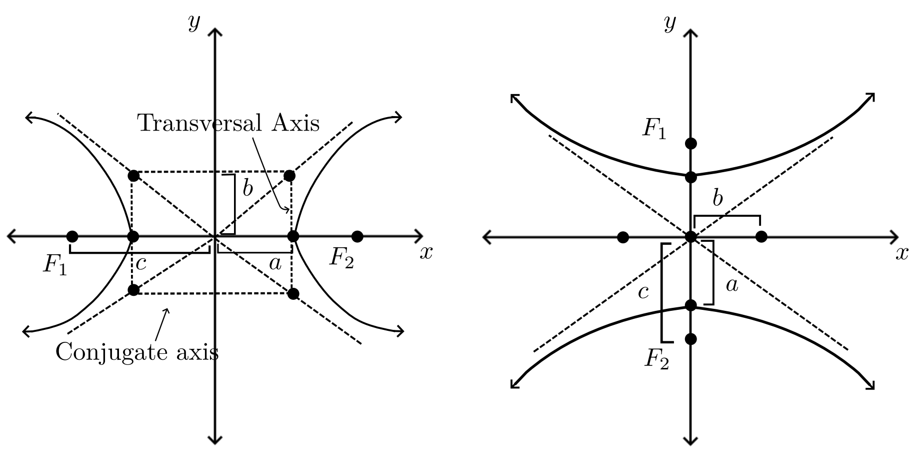
Standard Equations of a hyperbola with transversal axis parallel to the x-axis, center $(h, k)$ and $c^2 = a^2 + b^2$.

$$\begin{aligned}\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} &= 1,\\ Foci:\ F(h \pm c, k)\\ Vertices:\ (h \pm a, k)\\ Asymptotes:\ y - k = \pm\frac{b}{a}(x-h).\end{aligned}$$

Standard Equation of a hyperbola with transversal axis parallel to the y-axis and with center $(h, k)$ and $c^2 = a^2 - b^2$

$$\begin{aligned}\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} &= 1,\\ Foci:\ F(h, k \pm c)\\ Vertices:\ (h, k \pm a)\\ Asymptotes:\ y - k = \pm\frac{a}{b}(x-h).\end{aligned}$$

*Figure: Hyperbola*



**Note:-**

Whichever constant is under the positive variable is $a$

**Example 3.38** (Consider the equation)

$$y^2 - \frac{x^2}{15} = 1.$$

Since the constant under the positive variable $y^2$, is 1, we know $a = 1$ and $b = \sqrt{15}$

$$\begin{aligned} a &= 1 \\ b &= \sqrt{15} \\ c^2 &= 1 + 15 \\ c &= 4. \end{aligned}$$

If:

$$\begin{aligned} &F(h, k \pm c) \\ &F(0, \pm 4). \end{aligned}$$

If:

$$\begin{aligned} &V(h, k \pm a) \\ &V(0, \pm 1). \end{aligned}$$

If:

$$\begin{aligned} Asymptote: \ y - k &= \pm\frac{a}{b}(x - h) \\ y &= \pm\frac{1}{\sqrt{15}}x. \end{aligned}$$

**Example 3.39** (Find the equation of the hyperbola that has a center at the origin and satisfies the given conditions.)

$$Vertices: \ V(\pm 4, 0), \text{ passing through } (8, 2).$$

Since the vertices x value has the $\pm$, we know that this hyperbola is going to be opening left and right, and it has $a = 4$

So we have:

$$\frac{x^2}{4^2} - \frac{y^2}{b^2} = 1.$$

And we can plug in our given point to solve for b.

$$\frac{(8)^2}{4^2} - \frac{(2)^2}{b^2} = 1$$
$$4 - \frac{4}{b^2} = 1$$
$$-\frac{4}{b^2} = -3$$
$$\frac{4}{b^2} = 3$$
$$3b^2 = 4$$
$$b^2 = \frac{4}{3}.$$

Therefore:

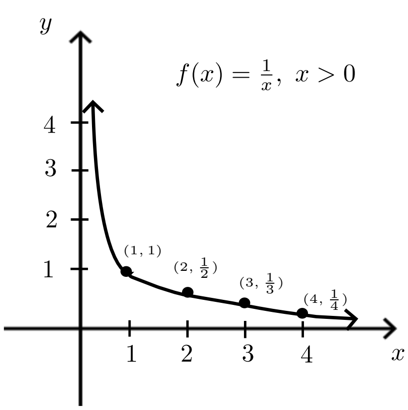
$$\boxed{\frac{x^2}{16} - \frac{y^2}{\frac{4}{3}} = 1}.$$

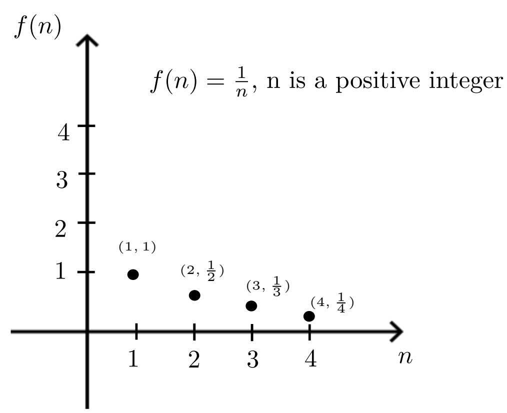
## 3.45 Sequences

***Definition:***

A sequence is a function whos domain is the set of positive integers

A sequence uses curly braces and has subscript notation with the form $a_n$




**Note:-**

The left graph is that of a **function**

The right graph is that of a **sequence**, notice it does not have a smooth curve. It only contains a series of points

**Writing the first several terms of a sequence.**

If:

$$\{a_n\} = \left\{\frac{n^2}{2n+1}\right\}.$$

Then:

$$\begin{aligned} a_1 &= \frac{1^2}{2(1)+1} = \frac{1}{3} \\ a_2 &= \frac{2^2}{2(2)+1} = \frac{4}{5} \\ a_3 &= \frac{3^2}{2(3)+1} = \frac{9}{7} \\ a_4 &= \frac{4^2}{2(4)+1} = \frac{16}{9} \end{aligned}$$

.

**Using Calculator (Texas-Instrument Graphing) to get the terms of a sequence.**

1. 2nd $\rightarrow$ stat
2. ops $\rightarrow$ seq
3. Syntax: seq(*function*, *variable*, *start*, *stop*, *step*)
   - Ex: seq($(x^2)/(2x+1)$, x,1,6,1)
4. Optionaly: math $\rightarrow$ frac

To get table:

1. mode $\rightarrow$ $\left[\text{func, par, pol, } \boxed{seq}\right]$
2. $n$Min: start value
3. $u(n)$: function
4. $u(nMin)$: value at $a_1$

**Finding function by looking at terms in a sequence.**

Consider:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots.$$

We can deduce that the function would be:

$$\{a_n\} = \left\{\frac{1}{2^n}\right\}.$$

**The Factorial Symbol.**

If $n \geq 0$, is an integer, the factorial symbol $n!$ is defined as follows

$$n! = n(n-1) \cdot ... \cdot 3 \cdot 2 \cdot 1, \qquad if\ n \geq 2$$
$$Or : n! = n(n-1)!.$$

## 3.46 Write The Terms of a Sequence Defined By Recursive Formula.

***Definition:***

A second way of defining a sequence is to assign a value to the first (or the first few) term(s) and specify the nth term by a formula that involves one or more of the terms preceding it.

**Example 3.40** (Consider)

$$s_1 = 5, \quad s_n = 2 \cdot s_{n-1}.$$

So:

$$s_2 = 2 \cdot 5 = 10$$
$$s_3 = 2 \cdot 10 = 20$$
$$s_4 = 2 \cdot 20 = 40$$

## 3.47 Sequences with summation notation

Using summation notation is a short hand way of representing a sum of a sequence of terms.

For example.

$$a_1 + a_2 + a_3 + ... + a_n = \sum_{n}^{k=1} a_k \ .$$

**Example 3.41** (Write out each sum)

$$\sum_{n}^{k=1} \frac{k}{k+1} \ .$$

So:

$$\frac{1}{1+1} + \frac{2}{2+1} + \frac{3}{3+1} + ... + \frac{n}{n+1}$$

**Example 3.42** (Writing a sum in summation notation)

$$\left(\frac{1}{1}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{5}\right)^2.$$

So:

$$\sum_{5}^{k=1} \left(\frac{1}{k}\right)^2$$

## 3.48 Properties of Summations

If $\{a_n\}$ and $\{b_n\}$ are two sequences and $c$ is a real number, then

$$\sum_{k=1}^{n} c = ca_1 + ca_2 + \cdots + ca_n = c(a_1 + a_2 + \cdots + a_n) = c\sum_{k=1}^{n} a_k$$

$$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

$$\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

$$\sum_{k=j+1}^{n} a_k = \sum_{k=1}^{n} a_k - \sum_{k=1}^{j} a_k \quad \text{where } 0 < j < n$$

Formulas for Sums of Sequences:

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{1}{2} + \frac{2^2}{2} + \frac{3^2}{2} + \cdots + \frac{n^2}{2} = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{1}{3} + \frac{2^3}{3} + \frac{3^3}{3} + \cdots + \frac{n^3}{3} = \left(\frac{n(n+1)}{2}\right)^2$$

## 3.49 Finding sum of sequence using calculator

1. 2nd → stat → math → sum
2. 2nd → stat → ops → seq
3. Syntax: sum(seq(*function, variable, start, end, step*))

## 3.50 Arithmetic Sequences.

***Definition:***

An arithmetic sequence is a sequence that is defined recursive as follows:

$$a_1 = 1, \quad a_n = a_{n-1} + d.$$

where $d$ is $a_n - a_{n-1}$. In other words, $d$ is the **common difference.**

**Example 3.43** (Show that the sequence is arithmetic.)

**a.)**

$$4, 2, 0, -2, \ \ldots$$

So:

$$d = 2 - 4 = 0 - 2 = -2 - 0 = -2$$

**b.)**

$$\{s_n\} = \{4n - 1\}.$$

So we must find the first term:

$$s_1 = 4 \cdot 1 - 1 = 3.$$

Now we want to show that two consecutive terms in the sequence is a constant. We must show this algebrically.

Since we know that:

$$d = s_n - s_{n-1}.$$

Then:

$$\begin{aligned} d &= 4n - 1 - (4(n-1) - 1) \\ &4n - 1 - (4n - 5) \\ &4n - 1 - 4n + 5 \\ &d = 4. \end{aligned}$$

**Note:-**

Since these functions are in the form $mx+b$, we can safely say that $m$ will be the common difference.

## 3.51 Find a Formula For an Arithmetic Sequence.

**nth Term of an Arithetic sequence.**

For an arithmetic sequence $\{a_n\}$ whose first term is $a$ and whose common difference is $d$, the $n$th term is determined by the formula:

$$a_n = a + (n-1)d.$$

**Example 3.44** (Find the twenty fourth term of the arithmetic sequence)

$$-3, 0, 3, 6, \ \ldots$$

Using the formula $a_n = a + (n-1)d$:

$$\begin{aligned} If:\ & a = -3,\ n = 24,\ d = 3 \\ Then:\ & a_{24} = -3 + (24-1)3 \\ & = 66. \end{aligned}$$

**Example 3.45**

The sixth term of an arithmetic sequence is 26, and the nineteenth term is 78. Find the first term and the common difference. Give a recursive formula for the sequence. What is the nth term of the sequence?

So we have:

$$\begin{aligned} 26 &= a + (6-1)d \\ 78 &= a + (19-1)d. \end{aligned}$$

$$= \begin{cases} 26 = a + 5d \\ 78 = a + 18d \end{cases} \tag{2}$$

So we can see that we have a **system of equations**, with two unknows. Thus we can solve this system with **elimination.**

So Subtracting our two equations:

$$\begin{aligned} 78 &= a + 18d \\ 26 &= a + 5d \end{aligned}$$

We get:

$$\begin{aligned} 52 &= 13d \\ d &= 4. \end{aligned}$$

Now that we have found $d$, we can plug $d$ into one of our equations to solve for $a$:

$$\begin{aligned} 26 &= a + 5(4) \\ a &= 6. \end{aligned}$$

Now that we have both $a$ and $d$, we can find the general formula by pluggion $a$ and $d$ into the general formula $a_n = a + (n-1)d$

$$\begin{aligned} a_n &= 6 + (n-1)4 \\ a_n &= 6 + 4n - 4 \\ \boxed{a_n = 4n + 2}. \end{aligned}$$

## 3.52 Finding The Sum of n Terms in an Arithmetic Sequence.

***Definition:***

Let $\{a_n\}$ be an arithmetic sequence with first term $a$ and common difference $d$. The sum $S_n$ of the first $n$ terms of $\{a_n\}$ is

$$\begin{aligned} S_n &= \frac{n}{2}[2a + (n-1)d] \\ Or: \ &\frac{n}{2}(a + a_n). \end{aligned}$$

**Example 3.46** (Find the sum of the first $n$ terms of the sequence)

$$a_n = \{4n + 2\}.$$

If:

$$s_n = \frac{n}{2}(a + a_n).$$

Then:

$$\begin{aligned} s_n &= \frac{n}{2}(6 + 4n + 2) \\ &= \frac{n}{2}(4n + 8) \\ &= \frac{n(4n+8)}{2} \\ &= \frac{4n(n+2)}{2} \\ &= 2n(n+2). \end{aligned}$$

Suppose $n = 100$, then:

$$\begin{aligned} &2(100)(100 + 2) \\ &= 20,400. \end{aligned}$$

## 3.53 Geometric Sequences.

***Definition:***

A geometric sequence is a sequence of numbers in which each term after the first is found by multiplying the preceding term by a constant factor called the common ratio, denoted by $r$.

$$a_n = a, \quad a_n = ra_{n-1}.$$

We can find the common ration $r$ with:

$$r = \frac{a_n}{a_{n-1}}.$$

**Example 3.47** (Show that the sequence is geometric)

**a.)**

$$2, 8, 32, 128, \ ...$$

So:

$$r = \frac{8}{2} = \frac{32}{8} = \frac{128}{32} = 4$$

**b.)**

$$\{s_n\} = \{3^{n+1}\}.$$

Using the formula for $r$:

$$\begin{aligned} r &= \frac{3^{n+1}}{3^{(n-1)+1}} \\ &= \frac{3^{n+1}}{3^n} \\ &= 3^{n+1-n} \\ &= 3^1 \\ r &= 3. \end{aligned}$$

And:

$$\begin{aligned} a_1 &= 3^2 \\ &= 9. \end{aligned}$$

**c.)**

$$\{t_n\} = \{3(2)^n\}.$$

Using the formula for $r$:

$$\begin{aligned} r &= \frac{3(2^n)}{3(2^{n-1})} \\ &= \frac{2^n}{2^{n-1}} \\ &= 2^{n-(n-1)} \\ &= 2^{n-n+1} \\ &= 2^1 \\ &= 2. \end{aligned}$$

## 3.54 Find a Formula For a Geometric Sequence.

***Definition:***

For a geometric sequence $\{a_n\}$ whose first term is $a$ and whose common ratio is $r$, the $n$th term is determined by the formula

$$a_n = a \cdot r^{n-1}, \quad r \neq 0$$

**Example 3.48** (Find the ninth term of the geometric sequence)

$$3, 2, \frac{4}{3}, \frac{8}{9}, \ \ldots$$

So:

$$a = 3, \quad r = \frac{2}{3}$$

Therefore:

$$a_n = 3\left(\frac{2}{3}\right)^{n-1}$$
$$a_9 = 3\left(\frac{2}{3}\right)^{8}$$
$$.$$

To find a recursive formula for this sequence, we will use the recursive formula $a_n = ra_{n-1}$

$$a_n = \left(\frac{2}{3}\right) \cdot a_{n-1}.$$

## 3.55 Sum of n Terms of a Geometric Sequence

***Definition:***

Let $\{a_n\}$ be a geometric sequence with first term $a$ and common ratio $r$, where $r \neq 0, r \neq 1$. The sum $S_n$ of the first terms of $\{a_n\}$ is

$$S_n = \frac{a \cdot (1 - r^n)}{1 - r}, \quad r \neq 0, 1$$

**Example 3.49** (Find the sum of the first $n$ terms of the sequence)

$$3^n.$$

So:

$$\begin{aligned} a_1 &= 3^1 = 3 \\ r &= \frac{3^n}{3^{n-1}} \\ &= 3^{n(n-1)} \\ &= 3^1 \\ &= 3 \end{aligned}$$

Now using the formula $S_n = \frac{a(1-r^n)}{1-r}$:

$$\begin{aligned} S_n &= \frac{3(1-3^n)}{1-3} \\ S_n &= -\frac{3}{2}(1-3^n). \end{aligned}$$

Now say we want the sum of the first 8 terms:

$$\begin{aligned} S_8 &= -\frac{3}{2}(1-3^8) \\ &= 9840. \end{aligned}$$

## 3.56 Determine Whether a Geometric Series Converges or Diverges.

***Definition:***

An infinite sum of the form

$$a + ar + ar^2 + \cdots + ar^{n-1}$$

with the first term $a$ and a common ratio $r$, is called an infinite geometric series and is denoted by

$$\sum_{k=1}^{\infty} ar^{k-1}$$

Sum of an Infinite Geometric Series If $|r| < 1$, the sum of the infinite geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ is

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$

In the context of infinite geometric sequences, convergence and divergence refer to the behavior of the sum of the terms in the sequence as more terms are added.

If an infinite geometric sequence converges, it means that the sum of its terms approaches a finite value as more terms are added. In other words, there is a well-defined sum for the infinite series. The sum of a convergent infinite geometric series can be calculated using the formula above: $\frac{a}{1-r}$

On the other hand, if an infinite geometric sequence diverges, it means that the sum of its terms does not approach a finite value as more terms are added. In this case, the sum of the infinite series is said to be infinite or undefined. The divergence can occur if the absolute value of the common ratio is greater than or equal to 1 ($|r| >= 1$).

Determining whether an infinite geometric sequence converges or diverges depends on the value of the common ratio '$r$'. If $|r|$ is less than 1, the sequence converges, and if $|r|$ is greater than or equal to 1, the sequence diverges.

**Example 3.50** (Find the sum of the infinite geometric sequence.)

$$1 + \frac{1}{3} + \frac{1}{9}.$$

So:

$$a = 1$$
$$r = \frac{1}{3}.$$

And if:

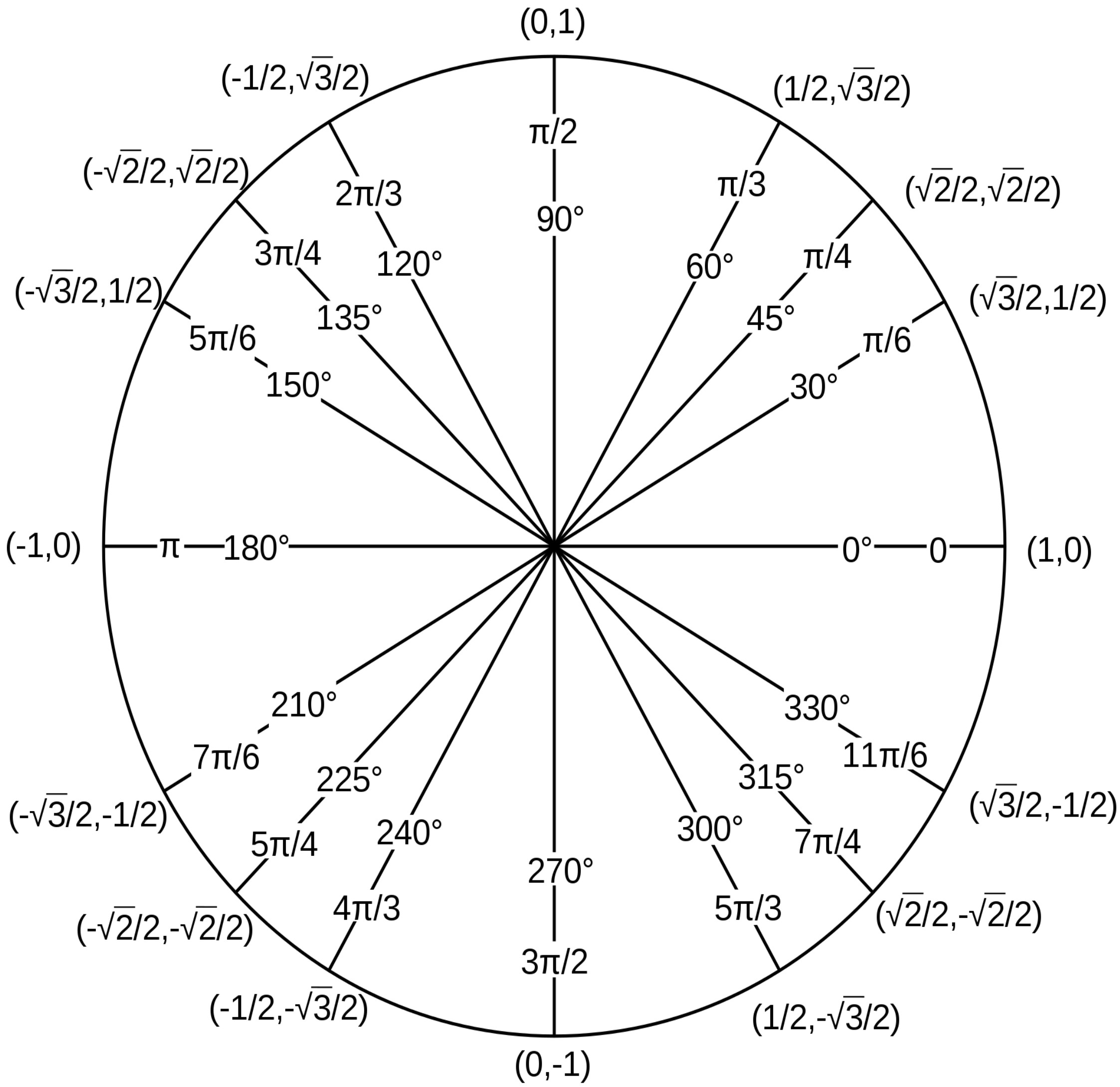
$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$

Then:

$$\frac{1}{1-\frac{1}{3}}$$
$$= \frac{2}{3}.$$

# 4 Trigonometry

## 4.1 Unit Circle Diagram



## 4.2 Standard Position / Central Angle

- Standard Position (Vertex at origin and initial side coincides with positive x-axis)
- Central Angle (Positive angle whose center is at the origin)

## 4.3 Linear / Angular Speed

- Linear Speed

$$v = \frac{s}{t}$$
$$and\ v = rw.$$

- Angular Speed

$$w = \frac{\theta}{t}.$$

## 4.4 Decimal to Degrees / Degrees to Decimal

- Convert Decimal to Degrees

$$deg + \frac{min}{60} + \frac{sec}{3600}.$$

- Convert Degrees to Decimal

*If:*

$$669.$$

*Then:*

$$29 \cdot 60 = 17$$
$$0 \cdot 60 = 24.$$

*So:*

$$66^\circ 17' 24''.$$

## 4.5 Degrees to Radians / Radians to Degrees

- Degrees to Radians

$$30^\circ \cdot \frac{\pi}{180}$$
$$= \frac{\pi}{6}.$$

- Radians to Degrees

$$\frac{\pi}{6} \cdot \frac{180}{\pi}$$
$$= 30^\circ.$$

## 4.6 Arc Length of a Circle

Arc length is computed with:

$$s = r \cdot \theta.$$

## 4.7 Trig Functions

On Unit Circle

- $\sin\theta = y$
- $\cos\theta = x$
- $\tan\theta = \frac{\sin\theta}{\cos\theta}$
- $\csc\theta = \frac{1}{\sin\theta}$
- $\sec\theta = \frac{1}{\cos\theta}$
- $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Not on Unit Circle

- $\sin\theta = \frac{y}{r}$
- $\cos\theta = \frac{x}{r}$
- $\tan\theta = \frac{y}{x}$
- $\csc\theta = \frac{r}{y}$
- $\sec\theta = \frac{r}{x}$
- $\cot\theta = \frac{x}{y}$

## 4.8 Periodicity

- sin, cos, csc, sec $= 2\pi$
- tan, cot $= \pi$

## 4.9 Even/Odd

- The cosine and Secant functions are **even**
- The sine, tangent, cosecant and cotangent functions are **odd**

## 4.10 Finding Values With Calculator

- $csc = \frac{1}{\sin\theta}$
- $sec = \frac{1}{\cos\theta}$
- $cot = \frac{1}{\tan\theta}$

## 4.11 Domain and Range of Trig Functions

- sin

$$\begin{aligned} Domain:&\ (-\infty,\infty)\\ Range:&\ [-1,1] \\ &. \end{aligned}$$

- cos

$$\begin{aligned} Domain:&\ (-\infty,\infty)\\ Range:&\ [-1,1]. \end{aligned}$$

- tan

$$\begin{aligned} Domain:&\ \ \mathbb{R}\text{ except odd integer multiples of }\frac{\pi}{2}\\ Range:&\ (-\infty,\infty). \end{aligned}$$

- csc

$$\begin{aligned} Domain:&\ \ \mathbb{R}\text{ except odd integer multiples of }\pi\\ Range:&\ (-\infty,-1]\cup[1,\infty]. \end{aligned}$$

- sec

$$\begin{aligned} Domain:&\ \ \mathbb{R}\text{ except odd integer multiples of }\frac{\pi}{2}\\ Range:&\ (-\infty,-1]\cup[1,\infty]. \end{aligned}$$

- cot

$$\begin{aligned} Domain:&\ \ \mathbb{R}\text{ except odd integer multiples of }\pi\\ Range:&\ (-\infty,\infty). \end{aligned}$$

## 4.12 Graphing

- Amplitude, Period
  - $|A|$ = Amplitude
  - T = Period

$$T = \frac{2\pi}{w}.$$

*If:*

$$y = -4\cos 3x.$$

*Then:*

$$\begin{aligned} T &= \frac{2\pi}{3}\\ A &= 4. \end{aligned}$$

- Basic Transformations

$$y = A\sin(wx - c) + d.$$

  - A = Amplitude
  - w = period
  - c = phase shift (Horizontal Shift)
  - d = vertical shift

## 4.13 Asymptotes of Trig Functions

Only Tan, Secant, cosecant and cotangent have Asymptotes, and they occur at:

- Tan: When $\cos\theta = 0$ at $\frac{\pi}{2} + \pi n$
- Cosecant: When $\sin\theta = 0$ at $\pi n$, where n is an integer
- Secant: When $\cos\theta = 0$ at $\frac{\pi}{2} + \pi n$, where n is an integer
- Cotangent When $\sin\theta = 0$ at $\pi n$, where n is an integer

## 4.14 Inverse Trig Functions

- Domain/Range (with inverse trig functions, domain and range flip, on graph, x and y flip)
- Restrictions (to make one to one)
  - $\sin^{-1}$
  $$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$
  - $\cos^{-1}$
  $$[0, \pi].$$
  - $\tan^{-1}$
  $$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$
  - $\csc^{-1}$
  $$[-\frac{\pi}{2}, \frac{\pi}{2}].$$
  - $\sec^{-1}$
  $$[0, \pi].$$
  - $\cot^{-1}$
  $$[0, \pi].$$

## 4.15 How to Find r

$$r^2 = x^2 + y^2$$
$$r = \sqrt{x^2 + y^2}.$$

## 4.16 Find Inverse of Trig Functions

1. replace f(x) with y
2. swap x and y
3. isolate siny
4. solve for y

## 4.17 Pythagorean Identites

- $\sin^2\theta + \cos^2\theta = 1$
- $\sin\theta = 1 - \cos\theta$
- $\sin^2\theta = 1 - \cos^2\theta$
- $\tan^2\theta + 1 = \sec^2\theta$
- $\cot^2\theta + 1 = \csc^2\theta$

## 4.18 Sum and Difference

- $\cos(a+b) = \cos a\cos b - \sin a\sin b$
- $\cos(a-b) = \cos a\cos b + \sin a\sin b$
- $\sin(a+b) = \sin a\cos b - \cos a\sin b$
- $\sin(a-b) = \sin a\cos b + \cos a\sin b$

## 4.19 Double Angle/Half Angle Formulas

- $\sin(2\theta) = 2\sin\theta\cos\theta$
- $\cos 2\theta = 1 - 2\sin^2\theta$
- $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$
- $\sin\frac{\theta}{2}$

$$\pm\sqrt{\frac{1-\cos\theta}{2}}.$$

- $\cos\frac{\theta}{2}$

$$\pm\sqrt{\frac{1+\cos\theta}{2}}.$$

- $\tan\frac{\theta}{2}$

$$\pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}.$$

## 4.20 Product to Sum

- $\sin a \sin b$

$$\frac{1}{2}[\cos(a-b) - \cos(a+b)].$$

- $\cos a \cos b$

$$\frac{1}{2}[\cos(a-b) + \cos(a+b)].$$

- $\sin a \cos b$

$$\frac{1}{2}[\sin(a+b) + \sin(a-b)].$$

## 4.21 Sum to Product

- $\sin a + \sin b$

$$2\sin\frac{a+b}{2}\cos\frac{a-b}{2}.$$

- $\sin a - \sin b$

$$2\sin\frac{a-b}{2}\cos\frac{a+b}{2}.$$

- $\cos a + \cos b$

$$2\cos\frac{a+b}{2}\cos\frac{a-b}{2}.$$

- $\cos a - \cos b$

$$-2\sin\frac{a+b}{2}\sin\frac{a-b}{2}.$$

## 4.22 Right Triangles

- Pythagorean Theoroem

$$a^2 + b^2 = c^2.$$

- sohcahtoa

$$sin = \frac{opp}{hyp}$$
$$cos = \frac{adj}{hyp}$$
$$tan = \frac{opp}{adj}.$$

- Complementary Angles

$$\begin{array}{cc} \sin B = \frac{b}{c} = \cos A & \csc B = \frac{c}{b} = \sec A \\ \cos B = \frac{a}{c} = \sin A & \sec B = \frac{c}{a} = \csc A \\ \tan B = \frac{b}{a} = \cot A & \cot B = \frac{a}{b} = \tan A \end{array}$$

.

## 4.23 Law of Sin / Cos

- cases
  - Law of sines

$$ASA$$
$$SAA$$
$$SSA$$
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

  - Law of Cosines

$$SAS$$
$$SSS$$
$$c^2 = a^2 + b^2 - 2ab\cos c.$$

## 4.24 Area

- S

$$s = \frac{1}{2}(a + b + c).$$

- A

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$

## 4.25 Hyerbolic Trig

- $\sinh x$

$$\frac{e^x - e^{-x}}{2}.$$

- $\cosh x$

$$\frac{e^x + e^{-x}}{2}.$$

- $\tanh x$

$$\frac{\sinh x}{\cosh x}.$$

- csch $x$

$$\frac{1}{\sinh x}.$$

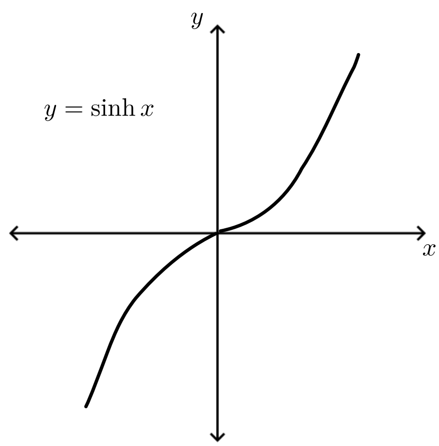
- sech $x$

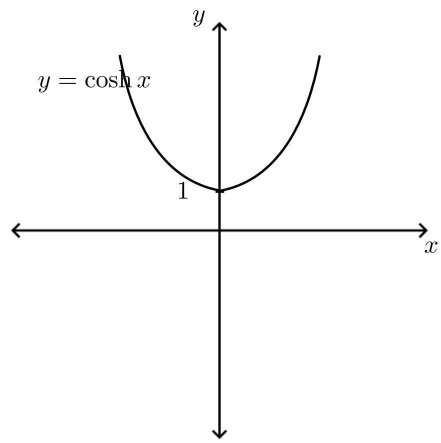
$$\frac{1}{\cosh x}.$$

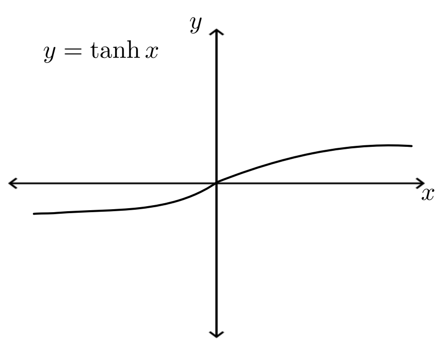
- $\coth x$

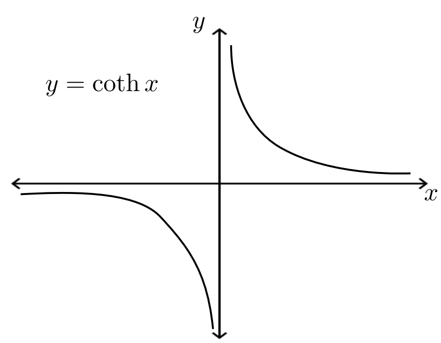
$$\frac{\cosh x}{\sinh x}.$$

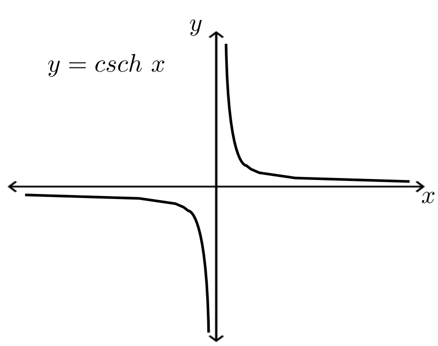
## 4.26 Graphs of Hyperbolic Trig Functions

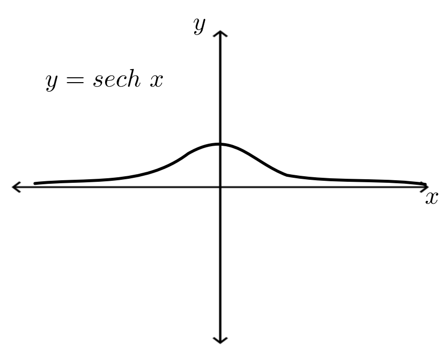


## 4.27 The Triangle

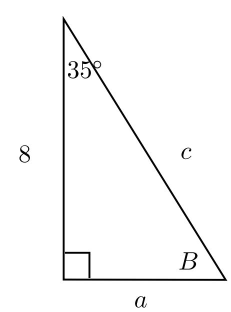
### Right Triangles

- the sum of the 2 acute angles A and B must equal 90°
- SOHCAHTOA
- complementary angles theorem

Complementary Angles:

$$\begin{aligned} \sin B &= \frac{b}{c} = \cos A & \csc B &= \frac{c}{b} = \sec A \\ \cos B &= \frac{a}{c} = \sin A & \sec B &= \frac{c}{a} = \csc A \\ \tan B &= \frac{b}{a} = \cot A & \cot B &= \frac{a}{b} = \tan A \end{aligned}.$$

Say we have the triangle:




Find B:

$$90 - 35 = 55°.$$

To find a, if we choose to use the measure of A, then the function that links opposite and adjancent is the tan function, so:

$$\begin{aligned} \tan 35 &= \frac{a}{8} \\ &= 5.6. \end{aligned}$$

To find c,

$$\begin{aligned} \sin 55 &= \frac{8}{c} \\ c &= \frac{8}{\sin 55} \\ &= 9.77. \end{aligned}$$

## Non Right Triangles

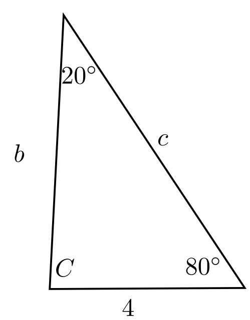
- Cases:
  - ASA
  - SAA
  - SSA

Will use the law of sines, which states:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Say we have the triangle

$$A = 20^\circ,\ b = 80^\circ,\ a = 4.$$




Use the fact that all three angles must measure to 180° to find C

$$C = 80^\circ.$$

To find b using the law of sines:

$$\frac{\sin 20}{4} = \frac{\sin 80}{b}$$
$$b = \frac{4\sin 80}{\sin 20}$$
$$b = 11.52.$$

Now find c:

$$\frac{\sin 20}{4} = \frac{\sin 80}{c}$$
$$c = \frac{4\sin 80}{\sin 20}$$
$$= 11.52.$$

## Figure out if you have more than one triangle (SSA)

Say we have:

$$c = 15^\circ, c = 6, a = 10.$$

- Find the angle of the side you are given.
- compute $(180 - A_1) + 15$
- if this equates to less than 180 degrees, you have 2 triangles.
- find $A_2$ by computing $(180 - A_1)$

## Law of cosines

The law of cosines states:

$$a^2 = b^2 + c^2 - 2bc \cdot \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cdot \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cdot \cos C.$$

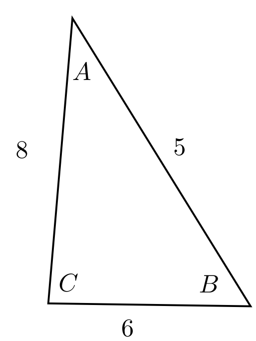
And we can rewrite to find missing angles:

$$B = \cos^{-1} \frac{b^2 - a^2 - c^2}{-2ac}.$$

## Using law of cosines to solve SSS triangles

Say we have the triangle:

$$a = 6, \; b = 8, \; c = 5.$$




We can solve this triangle by using the law of cosines

To find C:

$$\begin{aligned} 5^2 &= 6^2 + 8^2 - 2(6)(8)\cos C \\ 25 &= 36 + 64 - 96\cos C \\ -75 &= -96\cos C \\ \cos C &= \frac{75}{96} \\ C &= \cos^{-1}\frac{75}{96} \\ C &= 38.6. \end{aligned}$$

And we can do the same to find B

## Area of triangles

If we are given a triangle with 2 sides and a angle measurement, we can find area using the fact that:

$$A = \frac{1}{2} side \cdot side \cdot \sin(angle).$$

If we are given a triangle with 3 sides known, we can find area using:

$$\begin{aligned} s &= \frac{1}{2}(a+b+c) \\ A &= \sqrt{s(s-a)(s-b)(s-c)}. \end{aligned}$$

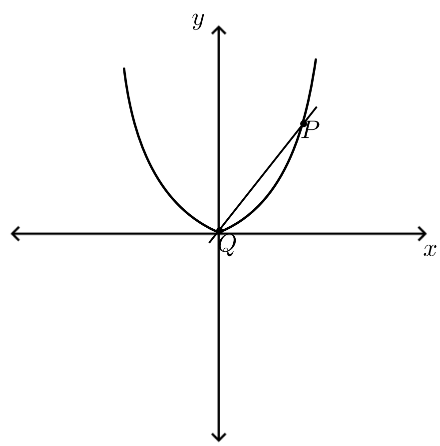
# 5 Calc 1

## 5.1 Secant Lines

- Secent lines on a graph is a line through the curve that touces at 2 places.
- To find the slope of the secent line:

$$m_{pq} = \frac{y_2 - y_1}{x_2 - x_1} \ rather, \ \frac{P_y - Q_y}{P_x - Q_x}.$$

Secant line:




## 5.2 Tangent Lines

- Tangent lines on a graph is a line that touches the curve exactly one time.
- Achieve an aproximation of the slope of the tangent line by moving Q closer to the tangent line and finding the slope of the secent line, we write

$$\lim_{Q \to P} m_{PQ} = m.$$

- To write the equation of the tangent line, use point slope form:

$$y - y_1 = x(x - x_1).$$

## 5.3 The Velocity Problem

- Average velocity is denoted by the slope of the secant line
- Instantaneous velocity is denoted by the slope of the tangent line

## 5.4 The Limit of a Function

- Consider the function:

$$f(x) = x^2 + 2.$$

  - If we want to investigate x=2, we can draw a table and examine what does the value of $f(x)$ approach as x approaches 2

- So we say:

$$\lim_{x \to a} f(x) = L.$$

## 5.5 One Sided Limits

- Approaching from the left (left hand limit):

$$\lim_{x \to a-} f(x).$$

  - Plug in a and evaluate

- Approaching from the right (right hand limit):

$$\lim_{x \to a+} f(x).$$

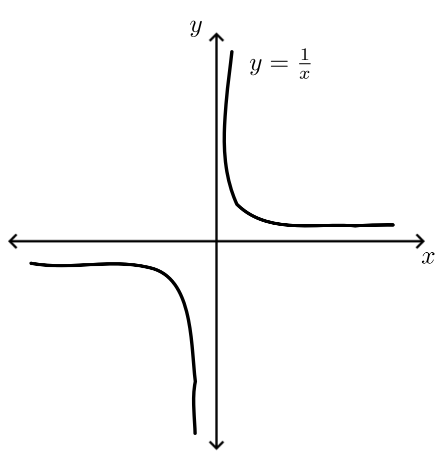
  - Plug in a and evaluate

- Know that:

$$\lim_{x \to a} f(x) = l \iff \lim_{x \to a-} f(x) = l \wedge \lim_{x \to a+} f(x) = l.$$

## 5.6 Infinite Limits

Consider:




We can see that:

$$\lim_{x \to 0+} f(x) = \infty.$$

And:

$$\lim_{x\to 0-} f(x) = -\infty.$$

Therefore we have determined that x=0 is a vertical asymptote, in general, we can safely say that f(x) has a vertical asymptote at a if any of the limits as $x \to a = \infty \ or - \infty$

What if we have a $\frac{Nonzero\ Constant}{Approaching\ Zero}$?

- say we have the equation:

$$\lim_{x\to 5-} \frac{x+1}{x-5}.$$

We can see that if we plug in 5, we get $\frac{6}{0}$, so since we have a $\frac{Nonzero\ Constant}{Approaching\ Zero}$, we know that our limit is either going to be $\infty \ or \ -\infty$, the way we determite the sign of infinity is as follows:

1. Plug in a number close to $a$ in whatever direction your limit is, the sign of the output will be the sign of infinity.

**Note:-**

If the limit does not specify a side, test with number close to $a$ in both directions, if the sign is not the same, the limit is DNE

## 5.7 Limit Laws

- $\lim_{x\to a} f(x) + g(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
- $\lim_{x\to a} f(x) - g(x) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
- $\lim_{x\to a} cf(x) = c \cdot \lim_{x\to a} f(x)$
- $\lim_{x\to a} f(x) \cdot g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
- $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, if g(x) $\neq 0$
- $\lim_{x\to a} \left(f(x)\right)^n = \left[\lim_{x\to a} f(x)\right]^n$, where n is a positive integer
- $\lim_{x\to a} c = c$
- $\lim_{x\to a} x = a$
- $\lim_{x\to a} x^n = a^n$, where n is a positive integer
- $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$, where n is a positive integer
- $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, where n is a positive integer

## 5.8 Direct Substitution Property

- You can plug in $a$ and evaluate, if a is not in the domain of $f(x)$, you can try factoring

## 5.9 Continuity

For a function to have continuity $ata$, 3 things must be true:

1. f(x) is defined at $a$
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

If no. 3 is true, the function is automatically continous at $a$

## 5.10 One-Sided Continuity

- Continuity from the right:

$$\lim_{x \to a+} f(x) = f(a).$$

- Continuity from the left:

$$\lim_{x \to a-} f(x) = f(a).$$

If $f$ and $g$ are continuous at $a$, then:

- $f + g$
- $f - g$
- $fg$
- $\frac{f}{g}$
- $cf$

Are all continuous at $a$

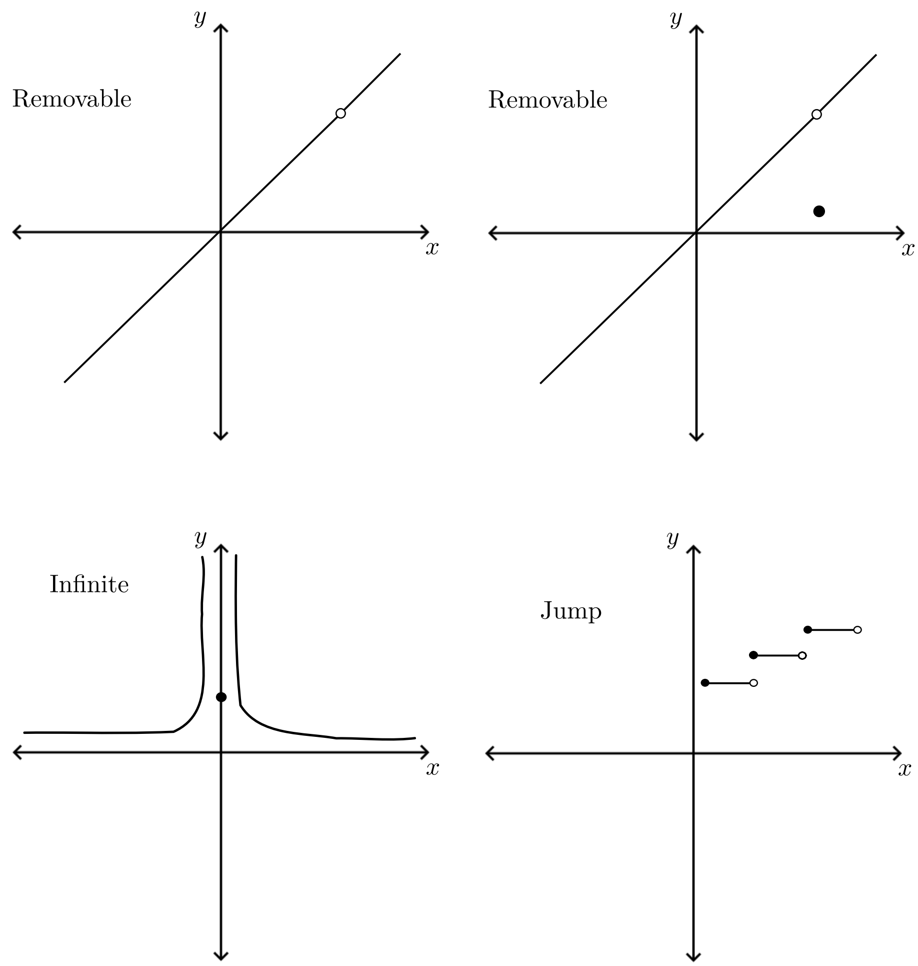
Also:

- Any polynomial is continous on its domain ($\mathbb{R}$)
- Any rational function is continous on its domain

**Note:-**

If $\lim_{x \to a} f(x)$ exists, Then you don't need to worry about which side the continuity is coming from

## 5.11 Discontinuitys




## 5.12 Finding where a function is Discontinuous without the graph of f

- Zeros of a rational function
- if $\lim_{x \to a} f(x) \neq f(a)$
- Piecewise functions, examine values of x where the domain flips to see if the limit is DNE

## 5.13 Intermediate Value Theorem

Suppose f is continuous on [a,b], Let N be any number between f(a) and f(b), where $f(a) \neq f(b)$, then:

$$\exists \; c \in (a, b) \mid f(c) = N.$$

## 5.14 Evaluating limits at infinity (horizontal asymptotes)

We have a horizontal Asymptote if:

$$\lim_{x\to\infty} f(x) = L$$
$$Or: \lim_{x\to-\infty} f(x) = L.$$

- Recall: If the degree of the numerator is higher than the degree of the denominator, the H.A is automatically y = 0
- For rational Functions: divide every term in the numerator and denominator by the highest degree in the denominator, then take the limit of each new term.
- Radical types like:

$$\lim_{x\to\infty} \sqrt{x^2+4x} - x.$$

  1. Conjugate (Multiply both numerator and denominator)
  2. pull out the 4
  3. divide numerate by x
  4. divide insider radical by $x^2$ and +x by x
  5. evaluate

- Example:

$$\lim_{x\to-\infty} \frac{\sqrt{9x^6 - x}}{x^3+1}.$$

  Since $\sqrt{x^6} = x^3$, we need to put absolute value bars and write as piecewise:

$$|x^3| = \begin{cases} x^3 & \text{if } x \geqslant 0 \\ -(x^3) & \text{if } x < 0 \end{cases} \tag{3}$$

  And since our limit is going to negative infinity, we use $-x^3$

$$\begin{aligned} &\lim_{x\to-\infty} \frac{-\sqrt{\frac{9x^6-x}{x^6}}}{\frac{x^3+1}{x^3}} \\ &= \frac{-\sqrt{9-\frac{1}{x^5}}}{1+\frac{1}{x^3}} \\ &= \frac{-\sqrt{9-0}}{1+0} \\ &= -3. \end{aligned}$$

- Limits at infinty of Polynomials
  - Say we have:

$$\lim_{x\to-\infty} 5 + 2x - x^3.$$

  - In this case, we can forget about the insignificant terms and only focus on $-x^3$, which means.

$$\begin{aligned} \lim_{x\to-\infty} -x^3 &= -(-\infty)^3 \\ &= \infty. \end{aligned}$$

- Find equation of horizontal Asymptote by looking at limtit at infinity
  - If the limit as x approaches infinity is a constant, then the equation of the H.A is y = L, where L is the constant
  - if the limit as x approches infinity of a rational function is bottom heavy then the equation of the H.A is y = 0

## 5.15 Derivatives and rates of change (Definitions)

- Slope of Secant Line:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$
$$or : \frac{f(a+h) - f(a)}{h}.$$

- Slope of Tangent Line:

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
$$or : \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

- Average Velocity:

$$v_{ave} = \frac{f(x) - f(x)}{x - a}.$$

- Instantaneous Velocity:

$$v_{inst} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
$$or : \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

- Speed:

$$Speed \; = |Velocity|.$$

- Derivatives Definition:

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
$$Or : f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

- Know that:

$$s(t) = postition\ function$$
$$v(t) = s'(t) = velocity\ function$$
$$a(t) = v'(t) = acceleration\ function.$$

**Note:-**

If f(x) is differentiable at $a$, then it is continous at $a$, the converse is not true

## 5.16 Differential Rule

Properties of Derivatives:

- $\frac{d}{dx}c = 0$
- $\frac{d}{dx}x = 1$
- $\frac{d}{dx}(x^n) = n \cdot x^{n-1} \rightarrow$ ***Power Rule***
- $\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}[f(x)]$
- $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

## 5.17 Derivatives of common functions

Exponential Functions:

- $\frac{d}{dx}e^x = e^x \cdot \frac{d}{dx}x$
- $\frac{d}{dx}a^x = a^x \cdot \ln a \cdot \frac{d}{dx}x$
- $\frac{d}{dx}\ln x = \frac{1}{x} \cdot \frac{d}{dx}x$
- $\frac{d}{dx}\log_a x = \frac{1}{x \cdot \ln a} \cdot \frac{d}{dx}x$

Trig Functions:

- $\frac{d}{dx}\sin x = \cos x$
- $\frac{d}{dx}\cos x = -\sin x$
- $\frac{d}{dx}\tan x = \sec^2 x$
- $\frac{d}{dx}\csc x = -\csc x \cot x$
- $\frac{d}{dx}\sec x = \sec x \tan x$
- $\frac{d}{dx}\cot x = -\csc^2 x$

Inverse Trig:

- $\frac{d}{dx}$arcsin(x) = $\frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}$arccos(x) = -$\frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}$arctan(x) = $\frac{1}{x^2+1}$
- $\frac{d}{dx}$arccsc(x) = $-\frac{1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}$arcsec(x) = $-\frac{1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}$arccot(x) = $-\frac{1}{x^2+1}$

Hyperbolic Trig

- $\frac{d}{dx}\sinh x = \cosh x$
- $\frac{d}{dx}\cosh x = \sinh x$
- $\frac{d}{dx}\tan x = sech^2 x$
- $\frac{d}{dx}cschx = -cschxcothx$
- $\frac{d}{dx}sechx = -sechxtanhx$
- $\frac{d}{dx}cothx = -csc^2 x$

## 5.18 Normal Line

- The normal line is Perpendicular to the tangent line at the point of tangency, which means:

$$m_{tan} \cdot m_{normal} = -1.$$

  - In other words, they are opposite reciprocals so flip and change sign

## 5.19 Find equation of tangent and normal line

Say we have the equation and point:

$$y = x^4 + 8e^x \text{ at the point (0,8)}..$$

1. Find derivative
2. Plug in x into derivative function
3. Flip $m_{tan}$ and change the sign to find $m_{normal}$
4. Use point slope form to find equations

## 5.20 Product and Quotient Rule

- Product rule:

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)].$$

- Quotient Rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

## 5.21 Chain Rule

- Know the chain rule:
  - If:

$$F(x) = f(g(x)).$$

  - Then:

$$F'(x) = f'(g(x)) \cdot g'(x).$$

## 5.22 Implicit Differentation

Procedure:

1. Differentiate
2. Leave $\frac{dy}{dx}$ next to terms of y that you differentiate
3. solve for $\frac{dy}{dx}$

Example:

$$\begin{aligned} 2x^3 + x^2y - xy^3 &= 2 \\ = 6x^2 + (x^2 \cdot 1 \cdot \frac{dy}{dx} + 2x \cdot y) - (x \cdot 3y^2 \cdot \frac{dy}{dx} + y^3 \cdot 1) &= 0 \\ = 6x^2 + x^2\frac{dy}{dx} + 2xy - 3xy^2\frac{dy}{dx} - y^3 &= 0 \\ = x^2\frac{dy}{dx} - 3xy^2\frac{dy}{dx} &= y^3 - 6x^2 - 2xy \\ = \frac{dy}{dx}(x^2 - 3xy^2) &= y^3 - 6x^2 - 2xy \\ = \frac{dy}{dx} &= \frac{y^3 - 6x^2 - 2xy}{x^2 - 3xy^2}. \end{aligned}$$

## 5.23 Logarithmic Differentation

If you have a problem that uses chain rule, product rule, quotient rule, all together. It's best to use this Logarithmic definition

Or if you have an equation with a variable in the base, and as the exponent, like:

$$y = (\cos 5x)^x.$$

Procedure:

1. Take ln of both sides
2. Differentiate implicitly with respect to x
3. solve for $y'$

Example:

$$
\begin{aligned}
y &= \sqrt[4]{\frac{x^2+1}{x^2-1}} \\
\ln y &= \ln \left(\frac{x^2+1}{x^2-1}\right)^{\frac{1}{4}} \\
\frac{1}{y}\frac{dy}{dx} &= \frac{1}{4}\ln\left(\frac{x^2+1}{x^2-1}\right) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{1}{4}\ln(x^2+1) - \frac{1}{4}\ln(x^2-1) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{1}{4}\left(\frac{1}{x^2+1}\cdot 2x\right) - \frac{1}{4}\left(\frac{1}{x^2-1}\cdot 2x\right) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{x}{2(x^2+1)} - \frac{x}{2(x^2-1)} \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{x}{2(x^2+1)}\cdot\frac{x^2-1}{x^2-1} - \frac{x}{2(x^2-1)}\cdot\frac{x^2+1}{x^2+1} \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{x(x^2-1)}{2(x^2+1)(x^2-1)} - \frac{x(x^2+1)}{2(x^2-1)(x^2+1)} \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{x^3-x}{2(x^2+1)(x^2-1)} - \frac{x^3+x}{2(x^2-1)(x^2+1)} \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{x^3-x-(x^3+x)}{2(x^2+1)(x^2-1)} \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{x^3-x-x^3-x}{2(x^4-1)} \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{-2x}{2(x^4-1)} \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{-x}{x^4-1} \\
&.
\end{aligned}
$$

Solve for $\frac{dy}{dx}$ and plug in original equation for y:

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{-x}{x^4-1}\cdot y \\
\frac{dy}{dx} &= \frac{-x}{x^4-1}\cdot\sqrt[4]{\frac{x^2+1}{x^2-1}}.
\end{aligned}
$$

## 5.24 Rates of Change in the natural and social sciences

Velocity:

- Velocity is derivative of position function
- A particle is at rest when $v(t) = 0$
- a particle reaches its maximum height when v(t) = 0
- A particle is moving in the positive direction when $v(t) > 0$
- A particle is moving in the negative direction when $v(t) < 0$

Acceleration:

- A particle is speeding up when v(t) and a(t) have the same signs
- A particle is slowing down when v(t) and a(t) have opposite signs

## 5.25 Exponential Growth and Decay

Formula:

$$\begin{aligned} y &= Ce^{kt} \\ y &= Population \\ C &= \ intitial\ Population \\ k &= \frac{Growth\ Rate}{Population} \\ t &= time. \end{aligned}$$

For example, if a bacterial culture starts with 4000 bacteria and triples every half hour.:

So:

$$y = 4000e^{kt}, \quad (\frac{1}{2}, 12000).$$

Then we can solve for k:

$$\begin{aligned} 12000 &= 4000e^{k(\frac{1}{2})} \\ 3 &= e^{\frac{k}{2}} \\ \ln 3 &= \frac{k}{2} \\ k &= 2\ln 3. \end{aligned}$$

So we have:

$$y = 4000e^{2\ln 3(t)}.$$

## 5.26 Newton's Law of Cooling

Newton's Law of cooling states:

$$\begin{aligned} T(t) &= t_s + Ce^{kt} \\ T(t) &= \text{ Temperature at time } t \\ t_s &= \text{ Temperature of surrounding area} \\ C &= t_0 - t_s. \end{aligned}$$

Say we have a freshly brewed cup of coffee has a temperature of $95°C$ in a $20°C$ room, five minutes later, its temerature is $88°C$

Then

$$\begin{aligned} T(t) &= 88 \\ t_s &= 20 \\ C = t_0 - t_s &= 95 - 20 = 75 \\ t &= 5. \end{aligned}$$

So our equation would be.

$$\begin{aligned} 88 &= 20 + 75e^{k(5)} \\ 68 &= 75e^{5k} \\ \frac{68}{75} &= e^{5k} \\ \ln\frac{68}{75} &= 5k \\ k &= \frac{\ln\frac{68}{75}}{5} \\ &\approx -0.0196. \end{aligned}$$

## 5.27 Related Rates

Many things change with item. Our goal is to find the rate at which some quantity is changing by relating it to other quantities whose rates of change are given or more easily measured

**Strategy**

1. Read the problem carefully
2. Draw a diagram whenever possible
3. Introduce notation
4. Express rates in terms of derivatives
5. Write an equation
6. Differentiate both sides with repect to $t$ using Implicit Differentiation/Chain Rule
7. Solve for the unknown rate

Say the length of a rectangle is increasing at a rate of 8 cm/s and it's width is increasing at a rate of 3 cm/s. When the length is 20cm and the width is 10cm, how fast is the area increasing?

So our givens:

$$\begin{aligned} \frac{dl}{dt} &= 8 \\ \frac{dw}{dt} &= 3 \\ l &= 20 \\ w &= 10. \end{aligned}$$

We want:

$$\frac{dA}{dt}.$$

And we know that:

$$A = l \cdot w.$$

So if we derive the area formula with Implicit differentation, and using the product rule, we get:

$$\frac{dA}{dt} = l \cdot \frac{dw}{dt} + w \cdot \frac{dl}{dt}.$$

From here we can plug in our givens:

$$\begin{aligned} \frac{dA}{dt} &= 20(3) + 10(8) \\ &= 60 + 80 \\ &= 140cm^2/s. \end{aligned}$$

## 5.28 Linear Approximation

Formula:

$$f(x) \approx L(x) = f(a) + f'(a)(x - a).$$

**Note:-**

Finding the linearization of a function is the same thing as ”find the equation of the tangent line to the curve”

Procedure,

1. find derivative of function
2. plug in a to get $m_{tan}$
3. Get point by plugging in a to $f(x)$
4. plug everything into formula $L(x) = f(a) - f'(a)(x - a)$

If asked to use linear aproximation formula to approximate values, then:

1. set original function equal to value
2. solve for x
3. plug in for x in linear approximation formula

## 5.29 Differentials

Formula:

$$\begin{aligned} dy &= f'(x)dx \\ \Delta x &= dx \\ \Delta y &= (f(x + \Delta x) - f(x)). \end{aligned}$$

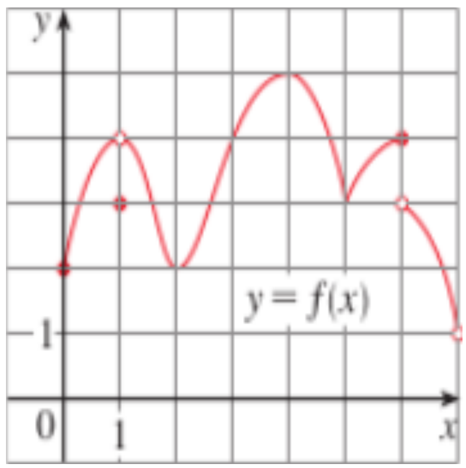
**Note:-**

$\Delta y$, can sometimes be difficult to find so we can use $dy \approx \Delta y$

## 5.30 Maximum and Minimum Values

- A function $f$ has an absolute maximum at $c$ if $f(c) \geqslant f(x)$ for all $x \in$ Domain of $f$
- A function $f$ has an absolute minimum at $k$ if $f(c) \leqslant f(x)$ for all $x \in$ Domain of $f$
- A function $f$ has a local maximum at $b$ if $f(b) \geqslant f(x)$ when $x$ is near $b$
- A function $f$ has a local minimum at $m$ if $f(m) \leqslant f(x)$ when $x$ is near $m$
- endpoints are not considered local min max

Cosider This Graph




Note that the textbook denotes $f(1) = 3$ as a local minimum.

## 5.31 Extreme Value Theorem

- If $f$ is continuous on a closed interval [a,b], then $f$ attains an obsulute maximum value $f(c)$ and an absolute minimum value $f(d)$ where $c, d \in [a, b]$

## 5.32 Fermat's Theorem

- If $f$ has a local minimum or maximum at $c$, and if $f'(c)$ exists, then $f'(c) = 0$

## 5.33 Critical Number

- $c$ in the domain of $f(x)$ is a critical number if $f'(c) = 0$ or if $f'(c)$ does not exist.
  Note: If $f$ has a local max or min at $c$, then $c$ is a critical number of $f$
- Critical number has to obey restriction

## 5.34 How to find the absolute maximum and minimum values of a continuous function f on a closed interval [a,b]

1. Find critical values of f in (a,b)
2. Find $f(a)$ and $f(b)$
3. Absolute Max: largest from 1.) and 2.)
4. Absolute Min: smallest from 1.) and 2.)

### Rolle's Theorem:

If f(x) satisfies the following:

1. continuous on [a,b]
2. differentiable on (a,b)
3. f(a) = f(b)

Then:

$$\{\exists\ c \in (a,b) \mid f'(c) = 0\}.$$

Notes:

- If rolle's theorem can be applied, just set $f'(x) = 0$ to find c, remember you are finding all c in the ***open interval***, so if c does not obey this interval, it is not a solution

## 5.35 The Mean Value Theorem

if f(x) satisfies the following:

1. continuous on [a,b]
2. differentiable on (a,b)

Then:

$$\left\{\exists\ c \in (a,b) \mid f'(c) = \frac{f(b) - f(a)}{b - a}\right\}.$$

In other words,

$$m_{tan} = m_{sec}.$$

Then after you solve for $f'(c)$, set $f'(x)$ equal to this value. This is how you compute c.

Notes:

- If rational function, find where function is undefined, if that number is not an element of the interval, then it is continous on the closed interval
- If $f'(x)$ is defined on the open interval, then it is differentiable on the open interval
- use the theorem, then set $f'(c) = c$

What if you just have a graph?

1. Draw a secant line through a and b
2. Draw a tangent line such that the slopes are the same (they are parallel)

## 5.36 First Derivative Test

- If $f'(x)$ changes from + to - at $c$, then $f(x)$ has a local maximum at c.
- If $f'(x)$ changes from - to + at $c$, then $f(x)$ has a local minimumc.

## 5.37 Second Derivative Test

- f(x) has a local minimum at c if $f'(c) = 0$ and $f''(c) > 0$
- f(x) has a local maximum at c if $f'(c) = 0$ and $f''(c) < 0$

## 5.38 Domain

- Polynomial: $\mathbb{R}$
- rational: Where denominator $= 0$
- Radical: Set denominator $\geqslant 0$, if needed, make a number line and test points to see where its positive.

## 5.39 Intercepts

- x-ints: set function equal to zero and solve, if the function is rational, only set the numerator equal to zero
- y-ints: plug in zero for x and solve

## 5.40 Asymptotes

- Horizontal Asymptote, find the limit of the function as x approaches infinity and -infinty, see limits at infinity for more info on rational functions
- vertical Asymptoteo, only for rational functions, it is the zeros of the function
- Slant (oblique), if numerater degree is one higher than denominator, use long division to find equation of slant Asymptote

$$\frac{-8x^4 + 8x^3 + 4}{2x^3 - x}.$$

$$2x^3 + 0x^2 - x\overline{)-8x^4 + 8x^3 + 4}.$$

*And:*

$$\frac{x^2 + 4}{x + 4}.$$

$$x + 4\overline{)x^2 + 0x + 4}.$$

## 5.41 Symmetric

- Even if (symmetric with respect to y axis)
  - $f(-x) = f(x)$
- Odd if (symmetric with respect to origin)
  - $f(-x) = -f(x)$

## 5.42 Increasing/Decreasing (Numberline with Domain (not as critical values but still test))

- Find derivative
- set numerator equal to 0 to find critical values (make sure they are in the domain.)
- Make number line and test with $f'(x)$, ontop of the critical values, use the values are found in domain
  - ex.

$$(-\infty, -4] \cup [0, \infty].$$

  - use -4 and 0 on number line

Note: no brackets on intervals of increase/decrease

### Local Min/Max

- First derivative test:
  - If $f'(x)$ switches from positive to negative, you have a local max at f(c)
  - If $f'(x)$ switches from negative to positive, you have a local min at f(c)
- Second Derivative test:
  - If $f''(c) < 0$, you have a local max at $f(c)$
  - If $f''(c) > 0$, you have a local min at $f(c)$

## 5.43 Abs Max and Min

1. find $f'(x)$
2. find critical values, plug into $f(x)$
3. find $f(a)$ *and* $f(b)$ from (a,b)
4. Abs max: largest form top steps
5. Abs min: smallest form top steps

## 5.44 Concave Up/Down and inflection points

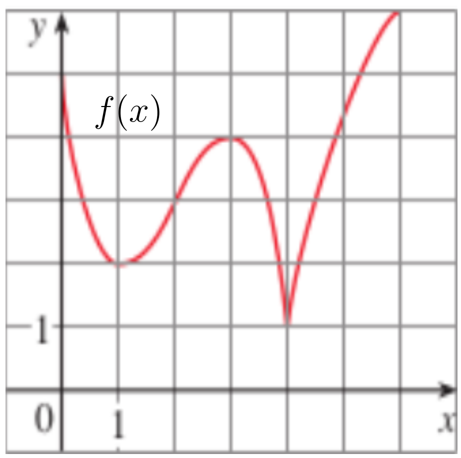
1. Find $f''(x)$
2. Find inflection points (Critical Values)
3. Test inflection points on a number line
4. If concavity changes at critical value, then plug in critical value to $f(x)$ to find inflection points

Note: If no $c \in D$, but domain is say $(-\infty, 15]$, use 15 on number line, but 15 cannot be an inflection point or numbers after used to test concavity

Find inflection points of $f(x)$, $f'(x)$, and $f''(x)$, given the graph of $f(x)$

- $f(x)$: find inflection points by locating where on the graph does it switch concavity. (x values)
- $f'(x)$: find inflection points by locating local min/maxs. (x values)
- $f''(x)$: Find inflection points by locating where it switches from positive to negative

Consider the graph of f(x):




$$\begin{aligned} CU &: \ (0, 2) \\ CD &: \ (2, 4) \cup (4, 6) \\ IP &: \ (2, 3). \end{aligned}$$

**Note:-**

Write inflection points as an ordered pair

## 5.45 L'Hospital's Rule

Main Concept:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

- Important concepts
  - Review limits at infinity
  - Any number other than zero or 1 to the power of negative infinity is 0
  - Any number other than zero or 1 to the power of infinity is infinity
  - The natural log of a negative number is undefined, therefore the limit as x approaches negative infinity of $\ln x$ is hence undefined
  - Same with log
  - Any contant over infinity approaches 0
- Types of indeterminate forms we do want
  - $\frac{0}{0}$
  - $\frac{\infty}{\infty}$
- Types of indeterminate forms we don't want:
  - $0 \cdot \infty$
  - $\infty - \infty$
  - $0^0$
  - $\infty^0$
  - $1^\infty$
- type $\infty - \infty$
  - Change to the type we want by:
    1. Common Denominators
    2. rationalizing
    3. factoring
- Types $0^0$, $\infty^0$, $1^\infty$
  - Change to the type we want by:
    1. taking natural log of function or writing as exponential

**Note:-**

If you use natural log method, you must write final answer as $e^{\lim_{x\to\infty}\ln y}$

## 5.46 Curve Sketching

- Find A-G and sketch graph
  - A: Domain
  - B: Intercepts
  - C: Asymptotes
  - D: Symmetry
  - E: Local min/max
  - F: Increasing / decreasing
  - G: Concavity / Inflection Points

## 5.47 Optimization Problems

**Strategy:**

1. Read the problem carefully
2. Draw a diagram whenever possible
3. Introducte notation
4. Expess the quantity to be optimized in terms of other variables.
5. Reduce the number of variables from Step 4 to only 1 (write a function of 1 variable)
6. find the absolute minimum/maximum

Procedure:

1. Find formula
2. Get to one variable
3. Find derivative
4. Find critical values
5. Use second derivative test to test if min / max

## 5.48 Newton's Method

Concept:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

What if you have a problem like:

$$\sqrt[4]{75}.$$

*Rewrite as:*

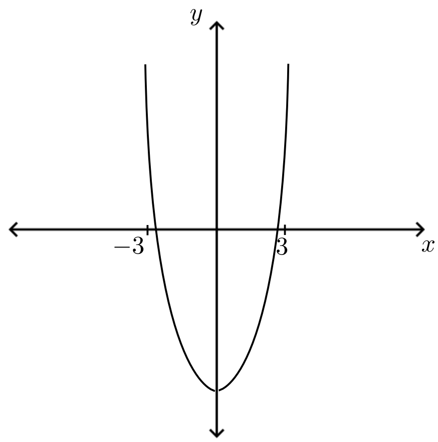
$$x^4 = 75$$
$$f(x) = x^4 - 75.$$

How do you get $x_1$?

$$x_1 = 3.$$

This is because we took $\sqrt[4]{81} = 3$, 81 is close to 75 so we will use that as our starting point, remember this is just an approximation.

What if it is a porabola, (ie more than one root), say we have the graph:




*Then we will use 2 different $x_1$, at -3 and 3*

## 5.49 Antiderivitates

- Know what it means to be an Antiderivitate and how to find them
- know the most common Antiderivitates
- understand +C
- know why

$$x^{\frac{1}{3}}$$
$$= \frac{3}{4}x^{\frac{4}{3}}.$$

- and

$$2y^2$$
$$= \frac{2}{3}y^3.$$

- Know that

$$x^{-1}$$
$$= \ln|x|.$$

Common Antiderivitates:

- Exponential
  - $x^n = \frac{x^{n+1}}{n+1} + C$
  - $\frac{1}{x} = \ln|x| + C$
  - $a^x = \frac{a^x}{\ln a} + C$
  - $\ln x = x \ln x - x + C$
  - $e^x = e^x + C$

- Trig:
  - $\sin x = -\cos x + C$
  - $\cos x = \sin x + C$
  - $\tan x = \ln|\sec x| + C$
  - $\csc x = \ln|\csc x - \cot x| + C$
  - $\sec x = \ln|\sec x + \tan x| + C$
  - $\cot x = \ln|\sin x| + C$
  - $\sin^2 x = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$
  - $\cos^2 x = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$
  - $\tan^2 x = -x + \tan x + C$
  - $\csc^2 x = -\cot x + C$
  - $\sec^2 x = \tan x + C$
  - $\cot^2 x = -x - \cot x + C$

- Hyperbolic Trig
  - $\sinh x = \cosh x + C$
  - $\cosh x = \sinh x + C$
  - $\tanh x = \ln|\cos x| + C$
  - $cschx = \ln|\tan h(\frac{1}{2}x)| + C$
  - $sechx = \tan^{-1}(\sinh(x)) + C$
  - $cothx = \ln|\sinh x| + C$
  - $csch^2x = -\coth x + C$
  - $sech^2x = \tanh x + C$

## 5.50 derivative of hyperbolic trig fuctions

- $\sinh x = \cosh x$
- $\cosh x = \sinh x$
- $\tanh x = \text{sech}^2 x$
- sech $x =$ -sech $x$ $\tanh x$
- csch $x =$ -csch $x$ $\coth x$
- $\coth x =$ -csch$^2 x$

**lnx**

- when you have f(x) like:

$$f(x) = 3x^{-2} - 7x^{-1} + 6.$$

*you can see if we tried to find F(x), by adding 1 to -1, we would get:*

$$\frac{-7^0}{0}.$$

*Which is a problem, so instead, the Antiderivitive would be* $-7\ln|x|$, this is because

$$\frac{d}{dx} - 7\ln x$$
$$= -\frac{7}{x}.$$

*And don't forget the absolute value bars*

## 5.51 Sigma notation

- Know properties of summation:
  - $\sum_{i=m}^{n} c \cdot a_i = c\sum_{i=m}^{n} a_i$, where c is a constant
  - $\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$
  - $\sum_{i=m}^{n} (a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$
  - $\sum_{i=1}^{n} 1 = n$
  - $\sum_{i=1}^{n} c = c \cdot n$, where c is a constant
  - $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
  - $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
  - $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$
- Know what is means to be a telescoping sum and how to evaluate it.
- Know how to eval limits of summation, recall that if you have the limit as n -> infinty, of a rational function whos degress are equal, evaluate the limit by taking the ration of the leading coefficients.

## 5.52 Area and Distance Problem

- Know the definition for the area under the curve.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [\Delta x f(x_1) + \Delta x f(x_2) + ... + \Delta x f(x_n)].$$

or

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [\Delta x f(x_0) + \Delta x f(x_1) + ... + \Delta x f(x_{n-1})].$$

or

$$A = \lim_{n\to\infty} [\Delta x f(x_1^*) + ... + \Delta x f(x_n^*)].$$

Where $x_i^*$ is any number in the $i$th interval.

**Note:-**

$\Delta x$ = Base of each rectangle

Find $\Delta x$ with $\frac{b-a}{n}$, on [a,b]

- And the sigma versions:
- $A = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(x_i)$ (Right endpoints)
- $A = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(x_{i-1})$ (Left endpoints)
- $A = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(x_i^*)$ (Arbitrary partiton)

**Note:-**

The one with the star denotes not using left or right endpoints

- Know how to find $\Delta x$

$$\Delta x = \frac{b-a}{n}.$$

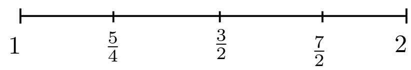
- Know how to find $x_i$ (right endpoints)

$$x_i = a + i\Delta x.$$

- Know how to find area with a fixed number of rectangles.
  - Find $\Delta x$
  - multiply $\Delta x$ by the sum of all the heights of the rectangles.

Notes for riemann sum:

- Say we are using right endpoints with $n = 4$, imagine we had the number line:
  - Since we are using right endpoints with $n = 4$, we only use $\frac{5}{4}$ onward
  - If we were using left endpoints, we would use 1 to $\frac{7}{2}$




## 5.53 Definite Integrals

- Know:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(x_i^*).$$

- Know riemann sum:

$$\sum_{i=1}^{n} \Delta x f(x_i).$$

- Know the properties of integrals:
  - $\int_a^b c dx = c(b-a)$
  - $\int_a^b cf(x)dx = c \cdot \int_a^b f(x)dx$
  - $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
  - $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$
  - $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$
  - if $f(x) \geqslant 0$ for all $a \leqslant x \leqslant b$, then $\int_a^b f(x)dx \geqslant 0$
  - if $f(x) \geqslant g(x)$ for all $a \leqslant x \leqslant b$, then $\int_a^b f(x)dx \geqslant \int_a^b g(x)dx$
  - if $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then $m(b-a) \leqslant \int_a^b f(x)dx \leqslant M(b-a)$

- Know how to find midpoints
  - Make number line with right endpoints, then find the middle value of each interval
  - If given an interval say, [1,3], and your asked to find midpoints, first plot number line with right endpoints. Then divide $\Delta x$ by 2 and add this number to each left point. Note that $\Delta x$ will remain the same when you find the riemann sum.
- Know when you have equations for geometric shapes (circle, half cirle, line)

- integral with the same bounds is **zero**
- Know how to split up integrals like

$$\int_{-1}^{2} (x - 8|x|)\ dx.$$

  - Set up piecewise

$x = 0$ *Set whats inside abs = 0 to see where sign flips.

$$= \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases} \tag{4}$$

- split integral

$$\int_{-1}^{0} x - 8(-x)\ dx + \int_{0}^{2} x - 8x\ dx.$$

- Then evaluate

## 5.54 The Fundemental theorem of calculus

- Know

$$Part\ 1: \ \frac{d}{dx}\int_{a}^{x} f(t)\ dt = f(x), a \leqslant x \leqslant b$$
$$Part\ 2: \ \int_{a}^{b} f(x)\ dx = F(b) - F(a)\ where\ F' = f.$$
$$and$$
$$\int_{a}^{b} f(x)\ dx = -\int_{b}^{a} f(x)\ dx.$$

- Know that you have to multiply by the derivative of the upper limit when you substitute for part 1
- Know how to split up integrals asked to find the derivative and if given something like:

$$F(x) = \int_{x}^{x^2} e^{t^7}\ dx.$$

  - so this would end up

$$F'(x) = \frac{d}{dx}\int_{x}^{0} e^{t^7}\ dt + \frac{d}{dx}\int_{0}^{x^2} e^{t^7}\ dt.$$

  - and we would have to flip our limits of integration such that our upper limit is a function of x

$$F'(x) = \frac{d}{dx} - \int_{0}^{x} e^{t^7}\ dt + \frac{d}{dx}\int_{0}^{x^2} e^{t^7}\ dt$$
$$F'(x) = e^{x^7} + e^{14} \cdot 2x$$
$$= e^{x^7} + 2xe^{14}$$
$$.$$

- Know the notation
  - If given:

$$g(x) = \int_{0}^{x} \sqrt{t^3 + t^5}\ dt.$$

  - Notation for $g'$ (using part 1) would be:

$$g'(x) = \frac{d}{dx}\int_{0}^{x} \sqrt{t^3 + t^5}\ dt$$
$$= \sqrt{x^3 + x^5}.$$

  - definite integrals do not get +C

## 5.55 Indefinite Integrals

- Indefinite Integrals are essentially just asking for the Antiderivitate, do not forget +C
- review common Antiderivitates
- know how to indeterminate antiderivative for inverse trig
- Know this antiderivative:

$$F(x) = 5^x$$
$$f(x) = \frac{5^x}{\ln 5}.$$

Because:

$$\begin{aligned} &\frac{d}{dx}\frac{5^x}{\ln 5} \\ &= \frac{1}{\ln 5}\left[5^x\right] \\ &= \frac{1}{\ln 5}\left[5^x \cdot \ln 5\right] \\ &= \frac{5^x \cdot \ln 5}{\ln 5} \\ &\boxed{= 5^x}. \end{aligned}$$

## 5.56 Velocity Functions with integrals

If given:

$$v(t) = 3t - 8,\ 0 \leqslant t \leqslant 3.$$

- Know that to find displacement:

$$\int_0^3 3t - 8\ dt.$$

- To find distance traveled:

$$\int_0^3 |3t - 8|\ dx.$$

  - So you must write as piecewise and split up the integral (add them)

## 5.57 The Substitution Rule (u-sub)

If $u = g(x)$ is differentiable and its range $\in I$ and $f$ is continuous on $I$, then

$$\int f(g(x))g'(x)\ dx = \int f(u)\ du.$$

***Process:***

1. Make a decent choice of what to let u equal
2. Change our integral from being in terms of $x$, to in terms of $u$
3. Integrate $\int f(u)\ du$
4. Change back to $x$

**Note:-**

Also include the constant in your u sub, if it is attached to the function you let u equal, Also note for rational trig functions, you can move trig functions from upstairs or downstairs based on their reciprical function, for example, a $\cos^2 x$ in the denominator can be moved upstairs as $\sec^2 x$

***Notes:***

- Our goal with u-sub is to let u equal some function in our composition of functions, such that if we derive that function, we get back something that is also in our integrand
- Say we have something like:

$$\begin{aligned} &\int \sec^2\theta\ d\theta \\ &Let\ u = 8\theta \\ &du = 8d\theta \\ &\frac{1}{8}du = d\theta. \end{aligned}$$

- Know that you can rewrite equations like:

$$\begin{aligned} &\int \frac{(\ln x)^{36}}{x}\ dx \\ as: &\int \frac{1}{x}(\ln x)^{36}\ dx. \end{aligned}$$

- Say we have:

$$-\frac{1}{2}\int_0^{-1} e^u\ du.$$

  - We can flip the limits of integration to remove the negative sign. So we will have:

$$\frac{1}{2}\int_{-1}^{0} e^u\ du.$$

- Look out for being able to turn antiderivative into inverse trig functions.

- Say we have:

$$\int \frac{x^7}{1+x^{16}}\ dx.$$

- Don't let $u = 1 + x^8$, we could do this by turning $x^{16}$ *into* $(x^8)^2$, but instead, just let $u = x^8$. We do this because it ends up like:

$$\frac{1}{8}du = x^7 dx.$$

- So when we sub we get that nice arctan antiderivative, just something to look out for.

$$\frac{1}{8}\int \frac{1}{1+u^2}\ du.$$

- Note that we can write:

$$e^{2x}.$$

  - as

$$(e^x)^2.$$

***Definite Integrals with u-sub***

1. Find what u is going to equal
2. Find u(a) and u(b)
3. make u-sub and use u(a) and u(b) as limits of integration
4. find antiderivative and evaluate at new limits

**Note:-**

For definite integrals, don't sub back in for u, just evalute integral with u still subbed

## 5.58 Integrals of Symmetric functions:

**Sometimes you will run into integrals that are either impossible, or too difficult with u Substitution. For these cases we will look at Integrals of Symmetric Functions**

***Even:***

$$f(-x) = f(x) \text{ then } \int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\ dx.$$

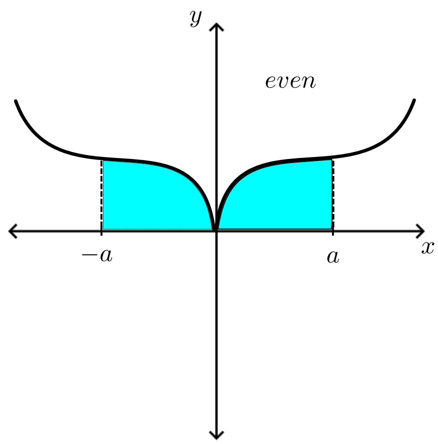
***Odd:***

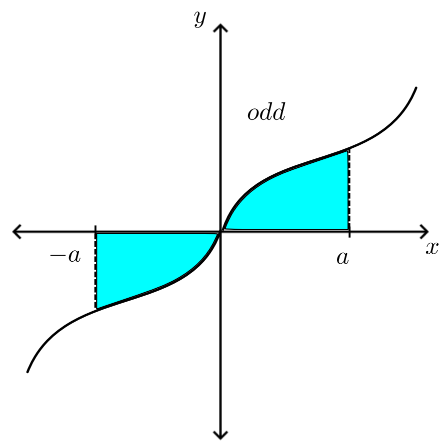
$$f(-x) = -f(x) \text{ then } \int_{-a}^{a} f(x)\ dx = 0.$$

***Notes:***

- If a function is even, you can replace your lower limit with zero and multiply the integral by 2
- if a function is odd, then the integral equals zero.

***Figures:***



# 6 Discrete Structures

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## 6.1 Vocabulary:

- A **set** is a collection of elements
- **Element** Is a member of a set
- An **axiom** Is a rule or statement that is generally accepted to be true without proof.
- An **Axiom of Extension** Is a set determined by what its elements are - not the order in which they might be listed or the fact that some elements might be listed more than once.
- **Set-Builder** Is a convention we can use when dealing with sets to imply the elements of a set without listing all of its values.
- **Universal Set**: Represents the collection of all possible elements or objects that are under consideration for a particular context or problem.
- **Empty Set (Null set)**: Represents a set that contains no elements
- **Singleton Set**: Represents a set that only has one element
- **Finite Set**: Represents a set that has a countable number of elements
- **Infinite Set**: Represents a set that has an infinite amount of elements
- **Subset**: $A$ is called a subset of $B$ if, and only if, every element of $A$ is also an element of $B$
- **Cardinal Number of a Set:** is the number of elements in a set
- **Equivalent Sets**: When two sets have the same cardinality

## 6.2 All Notation

- **Set:** $A = \{A_1, A_2, A_3, ..., A_n\}$
- **Declare element or non element**: $\in$, $\notin$
- $\mathbb{N}$: Denotes the set of all **Natural Numbers**
- $\mathbb{Z}$: Denotes the set of all **Integers**
- $\mathbb{Q}$: Denotes the set of all **Rational Numbers**
- $\bar{\mathbb{Q}}$: Denotes the set of all **Irrational Numbers**
- $\mathbb{I}$: Denotes the set of all **Imaginary Numbers**
- $\mathbb{C}$: Denotes the set of all **Complex Numbers**
- **Set-Builder**: $\{Expression \mid Condition\}$
- $\mathbb{U}$: **Universal Set**:
- $\emptyset$ (phi): **Empty Set (Null set)**
- $n(A)$: **Cardinal number of a set**
- $A \sim B$: **Denotes Equivalent Sets**

## 6.3 Definition of a set

**Definition:** A **set** is a collection of elements
We denote sets with the following syntax:

$$A = \{1, 2, 3, 4\}.$$

Where in this case $A$ is the identifier and it's elements are delimited by commas and encapsulated among braces.
**Note:** The identifier for sets are commonly represented with capital letters

We can also indicate *infinitely many* elements in a set by use of the **ellipsis**, which would look like:

$$A = \{1, 2, 3, ...\}.$$

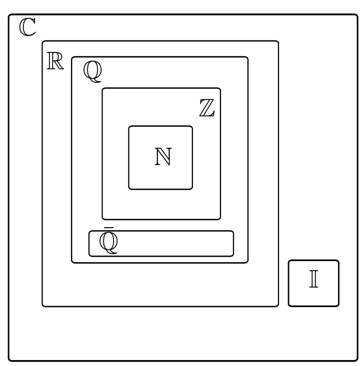
## 6.4 Notation for sets

- $\in$: ("in")
- $\notin$: ("not in")

## 6.5 Number Sets

- $\mathbb{N}$: Denotes the set of all **Natural Numbers**
- $\mathbb{Z}$: Denotes the set of all **Integers**
- $\mathbb{Q}$: Denotes the set of all **Rational Numbers**
- $\bar{\mathbb{Q}}$: Denotes the set of all **Irrational Numbers**
- $\mathbb{I}$: Denotes the set of all **Imaginary Numbers**
- $\mathbb{C}$: Denotes the set of all **Complex Numbers**

*Figure:*



## 6.6 Set Equality

Consider the sets:

$$A = \{1, 3, 5, 1, 5, 5, 3\}$$
$$B = \{1, 3, 5\}.$$

Because of the **Axiom of Extension**, which states that a set is not determined by the order or possible repetitions, we can conclude that $A = B$.

Furthermore, we can conclude that we only have 3 elements amongst set $A$, although it may seem like we have 7.

## 6.7 Set-Builder

Since the definition and syntax is already covered above, here is an example:

$$\{x \in \mathbb{R} | -2 < x < 4\}.$$

## 6.8 Types of Sets

# 7 Intro Statistics

## Chapters 1-4

### Titles

**Chapter 1:**

- 1.1: Introduction to the Practice of Statistics
- 1.2: Observational Studies versus Designed Experiments
- 1.3: Simple Random Sampling
- 1.4: Other Effective Sampling Methods
- 1.5: Bias in Sampling
- 1.6: The Design of Experiments

**Chapter 2:**

- 2.1: Organizing Qualitative Data
- 2.2: Organizing Quantitative Data: The Popular Displays
- 2.4: Graphical Misrepresentations of Data

**Chapter 3:**

- 3.1: Measures of Central Tendency
- 3.2: Measures of Dispersion
- 3.3: Measures of Central Tendency and Dispersion from Grouped Data
- 3.4: Measures of Position
- 3.5: The Five-Number Summary and Boxplots

**Chapter 4:**

- 4.1: Scatter Diagrams and Correlation
- 4.2: Least-Squares Regression
- 4.3: Diagnostics on the Least-Squares Regression Line
- 4.4: Contingency Tables and Association

## Learning Objectives

- **Obtain a simple random sample**
- **Explain the Sources of Bias in Sampling**
- **Describe the Characteristics of an Experiment (vocab)**
- **Explain the Steps in Designing an Experiment**
- **Explain the Completely Randomized Design**
- **Explain the Matched-Pairs Design**
- **Organize Qualitative Data in Tables**
- **Construct Bar Graphs**
- **Construct Pie Charts**
- **Organize Discrete Data in Tables**
- **Construct Histograms of Discrete Data**
- **Organize Continuous Data in Tables**
- **Construct Histograms of Continuous Data**
- **Draw Dot Plots**
- **Identify the Shape of a Distribution**
- **Describe What Can Make a Graph Misleading or Deceptive**
- **Determine the Arithmetic Mean of a Variable from Raw Data**
- **Determine the Median of a Variable from Raw Data**
- **Explain What It Means for a Statistic to be Resistant**
- **Determine the Mode of a Variable from Raw Data**
- **Determine the Range of a Variable from Raw Data**
- **Determine the Standard Deviation of a Variable from Raw Data**
- **Determine the Variance of a Variable from Raw Data**
- **Use the Empirical Rule to Describe Data That Are Bell-Shaped**
- **Approximate the Mean of a Variable from Grouped Data**
- **Compute the Weighted Mean**
- **Approximate the Standard Deviation from a Frequency Distribution**
- **Determine and Interpret z-Scores**
- **Interpret Percentiles**
- **Determine and Interpret Quartiles**
- **Determine and Interpret the Interquartile Range**
- **Check a Set of Data for Outliers**
- **Determine the Five-Number Summary**

- **Draw and Interpret Boxplots**
- **Draw and Interpret Scatter Diagrams**
- **Describe the Properties of the Linear Correlation Coefficient**
- **Compute and Interpret the Linear Correlation Coefficient**
- **4Determine Whether a Linear Relation Exists between Two Variables**
- **Explain the Difference between Correlation and Causation**
- **Find the Least-Squares Regression Line and Use the Line to Make Predictions**
- **Interpret the Slope and the y-Intercept of the Least-Squares Regression Line**
- **Compute the Sum of Squared Residuals**
- **Compute and Interpret the Coefficient of Determination**
- **Perform Residual Analysis on a Regression Model**
- **Identify Influential Observations**
- **Compute the Marginal Distribution of a Variable**
- **Use the Conditional Distribution to Identify Association Among Categorical Data**
- **Explain Simpson's Paradox**

## Vocab

- **Population:** The entire group to be studied is called the population.
- **Sample:** In statistics, it is often impractical or impossible to get access to the entire **population**, which is why we only look at a **sample.** A sample is a **subset** of the population being studied.
- **Individual:** An individual is a person or object that is a member of the population being studied.
- **Statistic:** A statistic is a numerical summary of a sample.
- **Descriptive Statistics:** Descriptive statistics consist of organizing and summarizing data. Descriptive statistics describe data through numerical summaries, tables, and graphs.
- **Inferential Statistics:** inferential Statistics uses methods that take a result from a sample, extend it to the population, and measure the reliability of the result.
- **Parameter:** A parameter is a numerical summary of a population.
- **Variables:** The characteristics of the individuals in a study. Variables vary, which means they can take on different values.
- **Constants:** Variables that do not vary. Inferential statistics is not necessary with constants.
- **Qualitative, or categorical variables** allow for the classification of individuals base on some attribute or characteristic.
- **Quantitative variables** provide numerical measures of individuals. The values of a quantitative variable can be added or subtracted and provide meaningful results.
- A **discrete variable** is a quantitative variable that has either a finite number of possible values or a countable number of possible values. A discrete variable cannot take on every possible value between any two possible values.
- A **continuous variable** is a quantitative variable that has an infinite number of possible values that are not countable. A continuous variable may take on every possible value between any two values. Continuous variables typically result from measurement. Continuous variables are often rounded. If a certain make of car gets 24 miles per gallon (mpg) of gasoline, its miles per gallon must be greater than or equal to 23.5 and less than 24.5, or $23.5 \leqslant mpg \leqslant 24.5$
- The list of observed values for a variable is **data.**
- **Qualitative data** are observations corresponding to a **qualitative variable.**
- **Quantitative data** are observations corresponding to a quantitative variable.
- **Discrete data** are observations corresponding to a discrete variable.
- **Continuous data** are observations corresponding to a continuous variable.
- **Explanatory Variable:** An explanatory variable, also known as an independent variable or predictor variable, is a variable that is manipulated or controlled by researchers in an experiment or study. It is the variable that is hypothesized to have an impact on the outcome or dependent variable.
- **Lurking variable**: An explanatory variable that was not considered in a study, but that affects the value of the response variable.
- **Response Variable**: The response variable, also known as the dependent variable or outcome variable, is the variable that is measured or observed to determine the effect or response of the explanatory variable(s). It is the variable that researchers are interested in studying or predicting.
- **Confounding:** Occurs when the effects of two or more explanatory variables are not separated. Therefore, any relation that may exist between an explanatory variable and the response variable may be due to some other variable or variables not accounted for in the study.

- **Census:** List of individuals in a population along with certain characteristics of each individual.
- **Random Sampling:** The process of using chance to select individuals from a population to be included in the sample.
- **Simple Random Sampling:** A sample of size $n$ from a population of size $N$ is obtained through simple random sampling if every possible sample of size $n$ has an equal chance of occurring. The sample is then called a simple random sample.
  - $n < N$
- **frame:** a list of all the individuals within the population.
- **Sample Without Replacement:** Once an individual is selected, the individual cannot be selected again.
- **Stratified sample**: is obtained by dividing the population into nonoverlapping groups called strata and then obtaining a simple random sample from each stratum. The individuals within each stratum should be homogenous (similar) in some way.
  - Within Stratified samples, the number of individuals sampled from each stratum should be proportional to the size of the strata in the population.
- **Systematic sample** is obtained by selecting every $k$th individual from the population. The first individual selected corresponds to a number between 1 and $k$
- **Cluster sample** is obtained by selecting all individuals within a randomly selected collection or group of individuals.
- **Convenience sample:** the individuals are easily obtained and not based on randomness.
- **Bias:** If the results of the sample are not representative of the population. Sampling bias means that the technique used to obtain the sample's individuals tends to favor one part of the population over another. Any convenience sample has sampling bias because the individuals are not chosen through a random sample.
- **Undercoverage:** Occurs when the proportion of one segment of the population is lower in a sample than it is in the population. This can result if the frame used to obtain the sample is incomplete or not representative of the population.
- **Sampling bias:** sampling bias is a bias in which a sample is collected in such a way that some members of the intended population have a lower or higher sampling probability than others. It results in a biased sample of a population in which all individuals, or instances, were not equally likely to have been selected
- **Nonresponse bias:** exists when individuals selected to be in the sample who do not respond to the survey have different opinions from those who do
  - This can be controlled with **callbacks**.
  - This can also be controlled with **rewards or incentives**
- **Response bias:** Exists when the answers on a survey do not reflect the true feelings of the respondent.
- **Open Question:** Allows the respondent to choose his or her response
- **Closed Question:** requires the respondent to choose from a list of predetermined responses
- **Nonsampling errors:** result from undercoverage, nonresponse bias, response bias, or data-entry error. Such errors could also be present in a census.
- **Sampling error:** results from using a sample to estimate information about a population. This type of error occurs because a sample gives incomplete information about a population.
- **Experiment:** is a controlled study conducted to determine the effect of varying one or more explanatory variables or **factors** has on a response variable.
- **Factor:** A variable whose effect on the response variable is to be assessed by the experimenter

- **Treatment:** Any combination of the values of the factors is called a treatment
- **Experimental Unit (or subject)** is a person, object or some other well-defined item upon which a treatment is applied
- **Control Group:** Serves as a baseline treatment that can be used to compare to other treatments.
- **Placebo:** is an innocuous medication, such as a sugar tablet, that looks, tastes, and smells like the experimental medication.
- **Blinding:** refers to nondisclosure of the treatment an experimental unit is receiving.
- **Single-blind** experiment is one in which the experimental unit (or subject) does not know which treatment he or she is receiving.
- **Double-blind** experiment is one in which neither the experimental unit nor the researcher in contact with the experimental unit knows which treatment the experimental unit is receiving.
- **Design:** To design an experiment means to describe the overall plan in conducting the experiment. Conducting an experiment requires a series of steps.
- **Blocking:** Grouping together similar experimental units and then randomly assigning the experimental units within each group to a treatment is called
- **completely randomized design:** is one in which each experimental unit is randomly assigned to a treatment.
- **matched-pairs design:** is an experimental design in which the experimental units are paired up. The pairs are selected so that they are related in some way (that is, the same person before and after a treatment, twins, husband and wife, same geographical location, and so on). There are only two levels of treatment in a matched-pairs design.
- **The relative frequency** is the proportion (or percent) of observations within a category and is found using the formula
- **Classes:** The Categories in which data is grouped
- **lower class limit:** the smallest value within the class
- **upper class limit:** the largest value within the class
- **Class Width:** is the difference between consecutive lower class limits.
- A table is **open ended** if the first class has no lower class limit or the last class has no upper class limit
- **uniform distribution:** frequency of each value of the variable is evenly spread across the values of the variable.
- **bell-shaped distribution:** highest frequency occurs in the middle and frequencies tail off to the left and right of the middle.
- **skewed right:** the tail to the right of the peak is longer than the tail to the left of the peak
- **skewed left:** tail to the left of the peak is longer than the tail to the right of the peak.
- **The arithmetic mean** of a variable is computed by adding all the values of the variable in the data set and dividing by the number of observations.
- **The population arithmetic mean**, $\mu$, (pronounced ”mew”), is a parameter that is computed using data from all the individuals in a population.
- **The sample arithmetic mean**, $\bar{x}$ (pronounced x-bar”), is a statistic that is computed using data from individuals in a sample.

- **The median** of a variable is the value that lies in the middle of the data when arranged in ascending order. We use $M$ to represent the median.
- A numerical summary of data is said to be **resistant** if observations that are extreme (very large or small) relative to the data do not affect its value substantially.
  - So the median is resistant, but the mean is not resistant.
- **The mode** of a variable is the observation of the variable that occurs most frequently in the data set.
  - If no observation occurs more than once, we say that the data have **no mode**.
- **Bimodal:** If the data has two modes
- **Multimodal:** If the data has more than two modes
- **Dispersion:** Degree to which the data are spread out.
- **Range:** The range, $r$, of a variable is the difference between the largest and smallest data value. That is, **Note:** Range is **not** resistant
- **Deviation about the mean:** a deviation refers to the difference between an individual data point and a central value, such as the mean or median. It represents how much a particular data point varies or deviates from the average or typical value in a data set.

  **Note:** The sum of the deviations about the mean always equals zero
- **Standard Deviation:** Measure of dispersion. It gives us a sense of how much the individual values deviate or differ from the average (mean) value. The standard deviation and the mean together can tell you where most of the values in your frequency distribution lie if they follow a normal distribution (Empirical Rule).
- we call $n - 1$ the **degrees of freedom** because the first $n - 1$ observations have freedom to be any value, but the $n^{th}$ observation has no freedom. It must be whatever value forces the sum of the deviations about the mean to equal zero.
- The **variance** It assesses the average squared difference between data values and the mean. Find my squaring $\sigma$ or $s$
  **Note:** The units of measure in variance are squared values. So if the variable is measured in dollars, the variance is measured in dollars squared. This makes interpreting the variance difficult.
- **Coefficient of Variation:** The coefficient of variation, CV, is defined as the ratio of the standard deviation to the mean of a data set. The CV allows for a comparison in spread by describing the amount of spread per unit mean. **Note:** When converting units of measure, the coefficient of variation is unchanged.
- **Class Midpoint:** The class midpoint is the sum of consecutive lower class limits divided by 2
- **Approximate Population Mean (if we do not have access to the original data, ie data has been grouped (classed) and frequency chart has already been made)**
- **Approximate Sample Mean (if we do not have access to the original data, ie data has been grouped (classed) and frequency chart has already been made)**
- **The weighted mean**, $\overline{x}_w$, of a variable is found by multiplying each value of the variable by its corresponding weight, adding these products, and dividing this sum by the sum of the weights. It can be expressed using the formula
- **Approximate Population Standard Deviation (if we do not have access to the original data, ie data has been grouped (classed) and frequency chart has already been made)**
- **Approximate Sample Standard Deviation (if we do not have access to the original data, ie data has been grouped (classed) and frequency chart has already been made)**

- **The z-score** represents the distance that a data value is from the mean in terms of the number of standard deviations. We find it by subtracting the mean from the data value and dividing this result by the standard deviation. **Note:** The Z-score is unitless. It has mean 0 and a standard deviation of 1
**Round z-scores to the nearest hundredth**

- The median is a special case of a general concept called the **percentile.**

- **the** $k^{th}$ **percentile,** denoted $P_k$, of a set of data is a value such that $k$ percent of the observations are less than or equal to the value.

- The most common percentiles are **quartiles**, which divide data sets into fourths, or four equal parts.

- The **interquartile range, IQR,** is the range of the middle 50% of the observations in a data set. That is, the IQR is the difference between the first and third quartiles and is found using this formula

- **Outliers:** When analyzing data, we must check for extreme observations, called outliers. Outliers can occur by chance, because of errors in the measurement of a variable, during data entry, or from errors in sampling.

- **Fences** serve as cutoff points for determining outliers.

- **bivariate data:** data in which two variables are measured on an individual. For example, we might want to know whether the amount of cola consumed per week is related to a person's bone density. The individuals would be the people in the study, and the two variables would be the amount of cola consumed weekly and bone density.

- Two variables that are linearly related are **positively associated** when above-average values of one variable are associated with above-average values of the other variable (or below-average values of one variable are associated with below-average values of the other variable). That is, two variables are positively associated if, whenever the value of one variable increases, the value of the other variable also increases.

- Two variables that are linearly related are **negatively associated** when above-average values of one variable are associated with below-average values of the other variable. That is, two variables are negatively associated if, whenever the value of one variable increases, the value of the other variable decreases.

- The **linear correlation coefficient**, or Pearson product moment correlation coefficient, is a measure of the strength and direction of the linear relation between two quantitative variables. The Greek letter $\rho$ (rho) represents the population correlation coefficient, and $r$ represents the sample correlation coefficient. We present only the formula for the sample correlation coefficient.

- **The least-squares regression line** minimizes the sum of the squared errors (or residuals). This line minimizes the sum of the squared vertical distance between the observed values of $y$ and those predicted by the line, $\hat{y}$ (read "y-hat"). We represent this as $\sum residuals^2$

- The observed distance for this club-head speed is 274 yards. The difference between the observed and predicted values of y is the error, or **residual**.

$$Residual \ = observed - predicted.$$

- **The coefficient of determination**, $R^2$, measures the proportion of total variation in the response variable that is explained by the least-squares regression line.

$$R^2 = r^2.$$

- **An influential observation** significantly affects the least-squares regression line's slope and/or y-intercept. (It also affects the value of the correlation coefficient.) Methods exist for determining whether a particular observation is influential; however, they are beyond the scope of this course. Nonetheless, we can still get a sense as to whether a particular observation is influential right now.

- the difference in our predicted value, and our actual value, is due to factors (variables) other than the club-head speed (wind speed and position of the ball on the club face, for example) and to random error. The differences just discussed are called **deviations**.

- **Total Deviation:** The deviation between the observed value, y, and mean value, $\overline{y}$, of the response variable.

$$y - \overline{y}$$
$$Or : \text{Explained Deviation} + \text{Unexplained Deviation.}$$

- **Explained Deviation:** The deviation between the predicted value, $\hat{y}$, and mean value, $\overline{y}$, of the response variable.

$$\hat{y} - \overline{y}.$$

- **Unexplained Deviation:** The deviation between the observed value, y, and predicted value, $\hat{y}$, of the response variable

$$y - \hat{y}.$$

- If a plot of the residuals against the explanatory variable shows the spread of the residuals increasing or decreasing as the explanatory variable increases, then a strict requirement of the linear model is violated. This requirement is called **constant error variance**. The statistical term for constant error variance is **homoscedasticity**
- **Simpson's Paradox**, which describes a situation in which an association between two variables inverts or goes away when a third variable is introduced to the analysis.

## Charts and Graphs:

- **A frequency distribution** lists each category of data and the number of occurrences for each category of data.
- **A relative frequency distribution** lists each category of data together with the relative frequency.
- **A bar graph** is constructed by labeling each category of data on either the horizontal or vertical axis and the frequency or relative frequency of the category on the other axis. Rectangles of equal width are drawn for each category. The height of each rectangle represents the category's frequency or relative frequency.
- **A Pareto chart** is a bar graph whose bars are drawn in decreasing order of frequency or relative frequency.
- **A pie chart** is a circle divided into sectors. Each sector represents a category of data. The area of each sector is proportional to the frequency of the category.
- A **histogram** is constructed by drawing rectangles for each class of data. The height of each rectangle is the frequency or relative frequency of the class. The width of each rectangle is the same, and the rectangles touch each other.
- We draw a **dot plot** by placing each observation horizontally in increasing order and placing a dot above the observation each time it is observed.
- The **five-number summary** of a set of data consists of the smallest data value, $Q_1$ the median, $Q_3$ and the largest data value. We use the five-number summary to learn information about the extremes of the data set. The summary is organized as follows:

  $$\text{Minimum } Q_1 \ M \ Q_3 \text{ Maximum}$$

- **A scatter diagram** is a graph that shows the relationship between two quantitative variables measured on the same individual. Each individual in the data set is represented by a point in the scatter diagram. The explanatory variable is plotted on the horizontal axis, and the response variable is plotted on the vertical axis.
  - A scatter plot is useful to determine if the presence of outliers causing an effect.
- **A marginal distribution** of a variable is a frequency or relative frequency distribution of either the row or column variable in the contingency table.
- **A conditional distribution** lists the relative frequency of each category of the response variable, given a specific value of the explanatory variable in the contingency table.

## Formulas

- **The relative frequency** is the proportion (or percent) of observations within a category.

$$Relative\ frequency = \frac{Frequency}{Sum\ of\ all\ frequency}.$$

- **The population arithmetic mean**, $\mu$, (pronounced "mew"), is a parameter that is computed using data from all the individuals in a population.

$$\mu = \frac{x_1 + x_2 + x_N}{N} = \frac{\sum x_i}{N}.$$

- **The sample arithmetic mean**, $\bar{x}$ (pronounced x-bar"), is a statistic that is computed using data from individuals in a sample.

$$\bar{x} = \frac{x_1 + x_2 + x_n}{n} = \frac{\sum x_i}{n}.$$

- **The median** of a variable is the value that lies in the middle of the data when arranged in ascending order. We use $M$ to represent the median.
  - For odd $n$:

$$M = \frac{n+1}{2}.$$

  - For even $n$:

$$M = Average\ of\ \frac{n}{2}, \frac{n}{2} + 1.$$

- **Range:** The range, $r$, of a variable is the difference between the largest and smallest data value. That is,

$$range = R = Largest\ data\ value - smallest\ data\ value.$$

- **Deviation:** a deviation refers to the difference between an individual data point and a central value, such as the mean or median. It represents how much a particular data point varies or deviates from the average or typical value in a data set. When can comopute a deviation with:

$$Individual\ data\ point - mean.$$

- **The population standard deviation** of a variable is the square root of the sum of squared deviations about the population mean divided by the number of observations in the population, $N$. The population standard deviation is symbolically represented by $\sigma$ (lowercase Greek sigma). The formula is given by:

$$\sigma = \sqrt{\frac{(x_1 - \mu)^2 + (x_2 - \mu)^2 + ... + (x_N - \mu)^2}{N}}$$
$$= \sqrt{\frac{\sum (x_i - \mu)^2}{N}}.$$

  **Note:** Standard Deviation is **not** resistant

- **The sample standard deviation**, $s$, of a variable is the square root of the sum of squared deviations about the sample mean divided by $n - 1$, where $n$ is the sample size. The formula is given as

$$s = \sqrt{\frac{(x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + ... + (x_n - \overline{x})^2}{n-1}}$$
$$= \sqrt{\frac{\sum (x_i - \overline{x})^2}{n-1}}.$$

  **Note:** Standard Deviation is **not** resistant

- **The population variance** is $\sigma^2$
- **The Sample Variance** is $s^2$
- **Approximate Population Mean (if we do not have access to the original data, ie data has been grouped (classed) and frequency chart has already been made)**

$$\begin{aligned}\mu &= \frac{\sum x_i f_i}{\sum f_i} \\ &= \frac{x_1 f_1 + x_2 f_2 + \cdots + x_n f_n}{f_1 + f_2 + \cdots + f_n}.\end{aligned}$$

where: $x_i$ is the midpoint or value of the $i$ th class
$f_i$ is the frequency of the $i$ th class
$n$ is the number of classes

- **Approximate Sample Mean (if we do not have access to the original data, ie data has been grouped (classed) and frequency chart has already been made)**

$$\begin{aligned}\overline{x} &= \frac{\sum x_i f_i}{\sum f_i} \\ &= \frac{x_1 f_1 + x_2 f_2 + ... + x_n f_n}{f_1 + f_2 + ... + f_n}.\end{aligned}$$

where: $x_i$ is the midpoint or value of the $i$ th class
$f_i$ is the frequency of the $i$ th class
$n$ is the number of classes

- **The weighted mean**, $\overline{x}_w$, of a variable is found by multiplying each value of the variable by its corresponding weight, adding these products, and dividing this sum by the sum of the weights. It can be expressed using the formula

$$\overline{x}_w = \frac{\sum w_i x_i}{w_i} = \frac{w_1 + x_1 + w_2 + x_2 + ... + w_n + x_n}{w_1 + w_2 + ... + w_n}.$$

Where: $w_i$ is the weight of the $i^{th}$ observation
$x_i$ is the value of the $i^{th}$ observation.

- **Approximate Population Standard Deviation (if we do not have access to the original data, ie data has been grouped (classed) and frequency chart has already been made)**

$$\sigma = \sqrt{\frac{\sum (x_i - \mu)^2 f_i}{\sum f_i}}.$$

Where: $x_i$ is the midpoint or value of the ith class
$f_i$ is the frequency of the $i^{th}$ class

- **Approximate Sample Standard Deviation (if we do not have access to the original data, ie data has been grouped (classed) and frequency chart has already been made)**

$$s = \sqrt{\frac{\sum (x_i - \overline{x})^2 f_i}{\sum f_i - 1}}.$$

Where: $x_i$ is the midpoint or value of the ith class
$f_i$ is the frequency of the $i^{th}$ class

- **The z-score** represents the distance that a data value is from the mean in terms of the number of standard deviations. We find it by subtracting the mean from the data value and dividing this result by the standard deviation.

– **Population Z-score**

$$z = \frac{x - \mu}{\sigma}.$$

– **Sample Z-score**

$$z = \frac{x - \overline{x}}{s}.$$

**Note:** The Z-score is unitless. It has mean 0 and a standard deviation of 1
**Round z-scores to the nearest hundredth**

- The **interquartile range, IQR,** is the range of the middle 50% of the observations in a data set. That is, the IQR is the difference between the first and third quartiles and is found using this formula

$$IQR = Q_3 - Q_1.$$

- **Fences** serve as cutoff points for determining outliers.

$$\begin{aligned} Lower\ Fence\ &= Q_1 - 1.5 \cdot IQR \\ Upper\ Fence\ &= Q_3 + 1.5 \cdot IQR. \end{aligned}$$

- The **five-number summary** of a set of data consists of the smallest data value, $Q_1$ the median, $Q_3$ and the largest data value. We use the five-number summary to learn information about the extremes of the data set. The summary is organized as follows:

$$\text{Minimum } Q_1 \ M \ Q_3 \text{ Maximum}$$

- The **linear correlation coefficient**, or Pearson product moment correlation coefficient, is a measure of the strength and direction of the linear relation between two quantitative variables. The Greek letter $\rho$ (rho) represents the population correlation coefficient, and $r$ represents the sample correlation coefficient. We present only the formula for the sample correlation coefficient.

$$r = \frac{\sum \left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right)}{n-1}.$$

Where:

$x_i$ is the $i$ th observation of the explanatory variable
$\overline{x}$ is the sample mean of the explanatory variable
$s_x$is the sample standard deviation of the explanatory variable
$y_i$ is the $i$ th observation of the response variable
$\overline{y}$ is the sample mean of the response variable
$\mathbf{S}_y$ is the sample standard deviation of the response variable
$n$ is the number of individuals in the sample

- **The least-squares regression line** minimizes the sum of the squared errors (or residuals). This line minimizes the sum of the squared vertical distance between the observed values of $y$ and those predicted by the line, $\hat{y}$ (read “y-hat”). We represent this as $\sum residuals^2$

$$\hat{y} = b_1 x + b_0.$$

Where:

$$b_1 = r \cdot \frac{s_y}{s_x} \text{ is the slope of the least-squares regression line.}$$

And:

$$b_0 = \overline{y} - b_1 \overline{x} \text{ is the y-Intercept of the least-squares regression line.}$$

- The observed distance for this club-head speed is 274 yards. The difference between the observed and predicted values of y is the error, or **residual**.

$$Residual\ = observed - predicted.$$

- **The coefficient of determination**, $R^2$, measures the proportion of total variation in the response variable that is explained by the least-squares regression line.

$$R^2 = r^2.$$

- **Total Deviation:** The deviation between the observed value, y, and mean value, $\overline{y}$, of the response variable.

$$y - \overline{y}$$
$$Or : \text{Explained Deviation} + \text{Unexplained Deviation.}$$

- **Explained Deviation:** The deviation between the predicted value, $\hat{y}$, and mean value, $\overline{y}$, of the response variable.

$$\hat{y} - \overline{y}.$$

- **Unexplained Deviation:** The deviation between the observed value, y, and predicted value, $\hat{y}$, of the response variable

$$y - \hat{y}.$$

- **Empirical Rule**
  - approximately 68% of the data within 1 standard deviation of the mean. That is, approximately 68% of the data will lie between $\mu - 1\sigma$ and $\mu + 1\sigma$

$$\mu \pm 1 \cdot \sigma.$$

  - approximately 95% of the data within 2 standard deviation of the mean. That is, approximately 95% of the data will lie between $\mu - 2\sigma$ and $\mu + 2\sigma$

$$\mu \pm 2 \cdot \sigma.$$

  - approximately 99.7% of the data within 3 standard deviation of the mean. That is, approximately 99.7% of the data will lie between $\mu - 3\sigma$ and $\mu + 3\sigma$

$$\mu \pm 3 \cdot \sigma.$$

- **Coefficient of Variation:**

$$\frac{Standard\ Deviation}{Mean}.$$

**Deciding Which Measure of Central Tendency and Dispersion to Report:**

| Shape of Distribution | Measure of Central Tendency | Measure of Dispersion |
|---|---|---|
| Symmetric | Mean | Standard Deviation |
| Skewed Left or Skewed Right | Median | Interquartile Range |

**Note:-**

For the remainder of this course, the phrase "describe the distribution" will mean to describe its shape (skewed left, skewed right, or symmetric), its center (mean or median), and its spread (standard deviation or interquartile range).

**Resistant Measures of Central Tendency:**

- **Median**

**Non-Resistant Measures of Central Tendency:**

- **Mean**
- **Mode**

**Resistant Measures of Dispersion:**

- **Quartiles**

**Non-Resistant Measures of Dispersion:**

- **Range.**
- **Standard Deviation.**
- **Variance.**

## 5-8 Subsection Titles

**Chapter 5:**

- 5.1: Probability Rules
- 5.2: The Addition Rule and Complements
- 5.3: Independence and the Multiplication Rule

**Chapter 6:**

- 6.1 Discrete Random Variables
- 6.2 The Binomial Probability Distribution

**Chapter 7:**

- 7.1: Properties of the Normal Distribution
- 7.2: Applications of the Normal Distribution
- 7.3: Assessing Normality

**Chapter 8:**

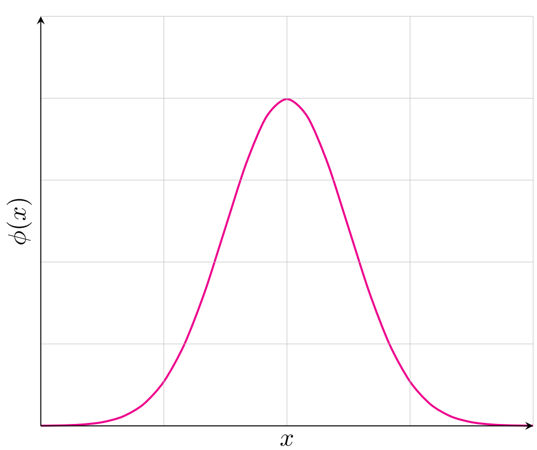
- 8.1 Distribution of the Sample Mean
- 8.2 Distribution of the Sample Proportion

## 5-8 Vocab

- **Simulation:** is a technique used to recreate a random event.
- **Random Process:** represents scenarios where the outcome of any particular trial of an experiment is unknown, but the proportion (or relative frequency) of a particular outcome is observed and approaches a specific value
- **Probability:** Is the measure of the likelihood of a random phenomenon or chance behavior occurring. It deals with experiments that yield random short-term results or outcomes yet reveal long-term predictability. The long-term proportion in which a certain outcome is observed is the probability of that outcome.
- **Outcomes:** Short term results
- **The Law of Large Numbers:** As the number of repetitions of a probability experiment increases, the proportion with which a certain outcome is observed gets closer to the probability of the outcome.
- **an experiment** is any process with uncertain results that can be repeated.
- **The sample space, $S$,** of a probability experiment is the collection of all possible outcomes for that experiment.
- **An event** is any collection of outcomes from a probability experiment. An event consists of one or more outcomes. We denote events with one outcome, sometimes called simple events, as $e_i$. In general, events are denoted using capital letters such as $E$.
- **A probability model** lists the possible outcomes of a probability experiment and each outcome's probability. A probability model must satisfy Rules 1 and 2 of the rules of probabilities.
- **An unusual event** is an event that has a low probability of occurring.
- An experiment has **equally likely outcomes** when each outcome has the same probability of occurring.
- A **subjective probability** is a probability that is determined based on personal judgment.
- **Disjoint:** Two events are disjoint if they have no outcomes in common.
- **Mutually Exclusive:** Another name for disjoint events.
- **Complement of $E$:** Let $S$ denote the sample space of a probability experiment and let $E$ denote an event. The complement of $E$, denoted $E^C$, is all outcomes in the sample space $S$ that are not outcomes in the event $E$.
- **Two events being Independent:** Two events $E$ and $F$ are independent if the occurrence of event $E$ in a probability experiment does not affect the probability of event $F$.
- **Two events being Dependent:** Two events are dependent if the occurrence of event $E$ in a probability experiment affects the probability of event $F$.
- A **random variable** is a numerical measure of the outcome of a probability experiment; so its value is determined by chance.
  - Random variables are typically denoted using capital letters such as $X$
- A **discrete random variable** has either a finite or countable number of values. The values of a discrete random variable can be plotted on a number line with space between each point. See Figure 1(a).
- A **continuous random variable** has infinitely many values. The values of a continuous random variable can be plotted on a line in an uninterrupted fashion. See Figure 1(b).
- The **probability distribution** of a discrete random variable X provides the possible values of the random variable and their corresponding probabilities. A probability distribution can be in the form of a table, graph, or mathematical formula.

- Because the mean of a random variable represents what we would expect to happen in the long run, it is also called the **expected value**, $E(X)$. The interpretation of the expected value is the same as the interpretation of the mean of a discrete random variable.
- The **binomial probability distribution** is a discrete probability distribution that describes probabilities for experiments in which there are two mutually exclusive (disjoint) outcomes. These two outcomes are generally referred to as success (such as making a free throw) and failure (such as missing a free throw). Experiments in which only two outcomes are possible are referred to as binomial experiments, provided that certain criteria are met.
- **A combination** is a collection, without regard to order and without repetition, in which $r$ objects are chosen from n distinct objects with $r \leqslant n$.
- **Trial:** Each repetition of the experiment.
- **Probability Density Function (pdf):** an equation used to compute probabilities of continuous random variables. It must satisfy the following two properties:
  1. The total area under the graph of the equation over all possible values of the random variable must equal 1
  2. The height of the graph of the equation must be greater than or equal to 0 for all possible values of the random variable. That is, the graph of the equation must lie on or above the horizontal axis for all possible values of the random variable.
- The **area under the graph of the density function** over an interval represents the probability of observing a value of the random variable in that interval.
- In mathematics, a **model** is an equation, a table, or a graph used to describe reality.
- The magenta curve in Figure 1 is a model called the **normal curve**, which is used to describe continuous random variables that are normally distributed.

*Figure 1:*




- A continuous random variable is **normally distributed**, or has a **normal probability distribution**, if its relative frequency histogram has the shape of a normal curve.
- **Inflection Points:** On a bell shaped curve (normal curve), the graph has two inflection points. At $\mu \pm \sigma$, these points are where the graph shifts from increasing at an increasing rate, to increasing at a decreasing rate
- If a normal random variable X has a mean different from 0 or a standard deviation different from 1, we can transform X into a **standard normal random variable Z** whose mean is 0 and standard deviation is 1.

- A **normal probability plot** is a graph that plots observed data versus normal scores.
- A **normal score** is the expected z-score of the data value, assuming that the distribution of the random variable is normal. The expected z-score of an observed value depends on the number of observations in the data set.
- The **sampling distribution** of a statistic is a probability distribution for all possible values of the statistic computed from a sample of size $n$
- The **sampling distribution of the sample mean** $\overline{x}$ is the probability distribution of all possible values of the random variable $\overline{x}$ computed from a sample of size $n$ from a population with mean $\mu$ and standard deviation $\sigma$.
- The standard deviation of the sampling distribution of $\overline{x}$, $\sigma_{\overline{x}}$, is called the **standard error** of the mean. **Note:** As the sample size $n$ increases, the standard error of the mean **decreases**
- **The Central Limit Theorem:** Regardless of the shape of the underlying population, the sampling distribution of $\overline{x}$ becomes approximately normal as the sample size, $n$, increases
- **The sample proportion**, $\hat{p}$, is a statistic that estimates the population proportion, $p$.

## 5-8 Formulas

- **Computing probability with the empirical method**

$$P(E) = \text{Relative frequency of E} = \frac{Frequency\ of\ E}{number\ of\ trials\ of\ experiment}.$$

- **Computing Probability With The Classical Method**
  - If an experiment has n equally likely outcomes and if the number of ways that an event E can occur is m, then the probability of E,P(E), is

$$P(E) = \frac{\text{number of ways that } E \text{ can occur}}{\text{number of possible outcomes}} = \frac{m}{n}.$$

  - So, if S is the sample space of this experiment, then

$$P(E) = \frac{N(E)}{N(E)}.$$

  where N(E) is the number of outcomes in E, and N(S) is the number of outcomes in the sample space.

- **Addition Rule for Disjoint Events:**
  - if $E$ and $F$ are disjoint (or mutually exclusive) events, then:

$$P(E\ or\ F) = P(E) + P(F).$$

- **Addition Rule for Non-Disjoint Events (General Addition Rule)**

$$P(E\ or\ F) = (P(E) + P(F)) - P(E\ and\ F).$$

- **Complement Rule:**
  - If $E$ represents any event and $E^C$ represents the complement of E, then

$$P(E^C) = 1 - P(E).$$

- **Multiplication Rule For Independent Events:**
  - If $E_1, E_2, E_3, \ldots, E_n$ are independent events, then

$$P(E_1 \text{ and } E_2 \text{ and } E_3 \text{ and } \ldots \text{ and } E_n) = P(E_1) \cdot P(E_2) \cdot \ldots \cdot P(E_n).$$

- **Notation for probability of discrete random variables**

$$P(X).$$

  - Read as:

  ”The probability that the random variable $X$ equals $x$”

- **Mean of a Discrete Random Variable**

$$\mu_X = \sum_{i=1}^{n} x \cdot P(x).$$

  Where $x$ is the value of the random variable and $P(x)$ is the probability of observing the value $x$.

- **Expected Value:**

$$E(X) = \mu_X.$$

- **Standard Deviation of a Discrete Random Variable**

$$\sigma_X = \sqrt{\sum[(x - \mu_X)^2 \cdot P(x)]}.$$

Where $x$ is the value of the random variable, $\mu_X$ is the mean of the random variable, and $P(x)$ is the probability of observing $x$.

- **Variance of discrete random variable:**

$$\mu^2.$$

- **nCk (Combinations):**
  - The $n$ objects are distinct
  - Repetition of objects is not allowed, and
  - Order is not important

$$nCk = \frac{n!}{(k!(n-k)!)}.$$

- **Binomial Probability Distribution Function**

$$P(x) = \binom{n}{x} \cdot p^x \cdot (1-p)^{n-x}, \quad x = 0, 1, 2, \ldots, n.$$

where $p$ is the probability of success.

- **Mean (or Expected Value) and Standard Deviation of a Binomial Random Variable**
  - A binomial experiment with n independent trials and probability of success p has mean, $\mu_X$, and standard deviation, $\sigma_X$, given by the formulas

$$\mu_X = np, \text{ and standard deviation, } \sigma_X = \sqrt{np(1-p)}.$$

- **Drawing normal probability plot by hand**

$$f_i = \frac{i - 0.375}{n + 0.5}.$$

Then use table to find expected zscores

- **Mean of sampling distribution**

$$\mu_{\overline{x}} = \mu.$$

- **Standard Deviation of sampling distribution**

$$\sigma_{\overline{x}} = \sqrt{\frac{N-n}{N-1}} \cdot \frac{\sigma}{\sqrt{n}}.$$

- **finite population correction factor**

$$\sqrt{\frac{N-n}{N-1}}.$$

- Suppose that a random sample of size n is obtained from a population in which each individual either does or does not have a certain characteristic. The **sample proportion**, denoted $\hat{p}$ (read "p-hat"), is given by:

$$\hat{p} = \frac{x}{n}.$$

  where $x$ is the number of individuals in the sample with the specified characteristic.

- **Mean of sampling distribution**

$$\mu_{\hat{p}} = p.$$

- **Standard Deviation of the sampling distribution:**

$$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}.$$

# 5-8 Concepts:

## Rules of Probability

**Rules of Probability:**

1. The probability of any event E, $P(E)$, must be greater than or equal to 0 and less than or equal to 1. That is, $0 \leqslant P(E) \leqslant 1$.
2. The sum of the probabilities of all outcomes must equal 1. That is, if the sample space $S = \{e_1, e_2, \ldots, e_n\}$, then

$$P(e_1) + P(e_2) + \cdots + P(e_n) = 1..$$

Consider The Table:

| Color | Probability |
|---|---|
| Brown | 0.12 |
| Yellow | 0.15 |
| Red | 0.12 |
| Blue | 0.23 |
| Orange | 0.23 |
| Green | 0.15 |

**Does this satisfy the rules?**

Rule 1 is satisfied because all probabilities are between 0 and 1, inclusive.

Rule 2 is satisfied because $\sum P(E) = 1$

## Unusual Event

Typically, an event with a probability less than 0.05 (or 5%) is considered unusual, but this cutoff point is not set in stone. The researcher and the context of the problem determine the probability that separates unusual events from not so unusual events.

## The three methods for determining the probability of an event:

- the Empirical Method
- the Classical Method
- the Subjective Method

## Rule for claiming Independence:

When we take small samples from large finite populations, we make the assumption of independence even though the events are technically dependent.

As a general rule of thumb, if the sample size $n$ is no more than 5% of the population $N$ ($n \leqslant 0.05N$), we assume independence.

## Compute "at-least" Probabilities:

Usually, when computing probabilities involving the phrase at least, use the Complement Rule.

The phrase at least means "greater than or equal to." For example, a person must be at least 17 years old to see an R-rated movie. This means that the person's age must be greater than or equal to 17 to watch the movie.

## Poll Indicates Different from our assumptions

Consider the following scenario:

*According to a poll, about 17% of adults in a country bet on professional sports. Data indicates that 47.3% of the adult population in this country is male. Assuming that betting is independent of gender, compute the probability that an adult from this country selected at random is a male and bets on professional sports.*

**Solution:**

$$\begin{aligned} P(Male\ or\ Bet\ on\ Sports) &= .17 \cdot .473 \\ &= .0804. \end{aligned}$$

Now Consider:

*The poll data indicated that % of adults in this country are males and bet on professional sports.*

**Solution:**

The assumption was incorrect and the events are **not independent**.

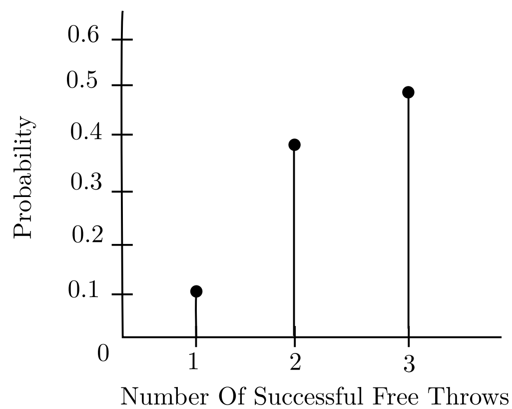
## Rules for a Discrete Probability Distribution:

Let $P(x)$ denote the probability that the random variable $X$ equals $x$; then

- $\sum P(x) = 1$
- $0 \leqslant P(x) \leqslant 1$

## Graph Discrete Probability Distributions

In the graph of a discrete probability distribution, the horizontal axis is the value of the discrete random variable and the vertical axis is the corresponding probability of the discrete random variable. When graphing a discrete probability distribution, we want to emphasize that the data are discrete. Therefore, draw the graph of discrete probability distributions using vertical lines above each value of the random variable to a height that is the probability of the random variable.




## Finding Mean of a Discrete Random Variable In Statcrunch:

1. Stat > Calculators > Custom
2. Select Values in:
3. Select Weights in (P(X)):
4. Compute

## How to Interpret the Mean of a Discrete Random Variable

The mean of a discrete random variable can be thought of as the mean outcome of the probability experiment if we repeated the experiment many times.

As the number of repetitions of the experiment increases... The mean value of the $n$ trials will approach $\mu_X$ (The difference between $\overline{x}$ and $\mu_X$ gets closer to 0 as $n$ increases)

## Life insurance policy example (expected value)

**Example: A term life insurance policy will pay a beneficiary a certain sum of money upon the death of the policy holder. These policies have premiums that must be paid annually. Suppose a life insurance company sells a $250,000 one-year term life insurance policy to a 49-year-old female for $530. According to the National Vital Statistics Report, Vol. 47, No. 28, the probability that the female will survive the year is 0.99791. Compute the expected value of this policy to the insurance company.**

**Approach:**

The experiment has two possible outcomes: survival or death. Let the random variable X represent the payout (money lost or gained), depending on survival or death of the insured. Assign probabilities to each payout and substitute these values into $\mu_X = \sum(x \cdot P(X))$

**Step 1**. Because $P(\text{survives}) = 0.99791$, $P(\text{dies}) = 0.00209$. If the client survives the year, the insurance company makes $530, or $x = +530$. If the client dies during the year, the insurance company must pay $250,000 to the client's beneficiary, but still keeps the $530 premium; so $x = \$530 - \$250,000 = -\$249,470$. The value is negative because it is money paid by the insurance company. The probability distribution is listed in Table 3.

| $x$ | $P(X)$ |
|---|---|
| $530 (survives) | 0.99791 |
| -$249,470 (dies) | 0.00209 |

**Step 2.** The expected value (from the point of view of the insurance company) of the policy is

$$\begin{aligned} E(X) &= \mu_X \\ &= \sum(x \cdot P(X)) \\ &= (\$530 \cdot 0.99791) + (-\$249470 \cdot 0.00209) \\ &= \$7.50. \end{aligned}$$

**Therefore:**

The company expects to make $7.50 for each 49-year-old female client it insures. The $7.50 profit of the insurance company is a long-term result. It does not make $7.50 on each 49-year-old female it insures; rather, the average profit per 49-year-old female insured is $7.50. Because this is a long-term result, the insurance "idea" will not work with only a few insured.

## Standard Deviation of a Random Discrete Variable In Statrunch

**Statcrunch Steps:**

1. Stat > Calculators > Custom
2. Select Values in:
3. Select Weights in (P(X)):
4. Compute

## Criteria for a Binomial Probability Experiment

An experiment is said to be a **binomial experiment** if:

1. The experiment is preformed a fixed number of times. Each repetition of the experiment is called a **trial**
2. The trials are independent. This means that the outcome of one trial will not affect the outcome of the other trials.
3. for each trial, there are two mutually exclusive outcomes, success or failure.
4. The probability of success is fixed for each trial of the experiment.

## Notation used in the binomial probability distribution:

- There are $n$ independent trials of the experiment.
- let $p$ denote the probability of success so that $1 - p$ is the probability of failure
- Let $X$ be a binomial random variable that denotes the number of successes in $n$ independent trials of the experiment. So $0 \leqslant X \leqslant n$

## three methods for obtaining binomial probabilities:

1. The binomial probability distribution formula
2. A table of binomial probabilities
3. Technology

## Using the Binomial Probability Distribution Function in Statcrunch

**Steps:**

1. Stat > Calculators > Binomial
2. Enter $n$
3. Enter $p$
4. Enter info for $P(X)$
5. Compute

## Shape of Binomial Probability Distribution for Various Values of p

The binomial probability distribution is

- skewed right if $p < 0.5$
- symmetric and approximately bell-shaped if $p = 0.5$
- skewed left of $p > 0.5$

## Shape of the Graph of a Binomial Probability Distribution for Various Values of n

- For a fixed $p$, as the number of trials $n$ in a binomial experiment increases, the probability distribution of the random variable $X$ becomes bell-shaped.
- As a rule of thumb, if $np(1 - p) \geqslant 10$, the probability distribution will be approximately bell-shaped.

**Note:-**

This result allows us to use the Empirical Rule to identify unusual observations in a binomial experiment. Recall that the Empirical Rule states that in a bell-shaped distribution, about 95% of all observations lie within 2 standard deviations of the mean. That is, about 95% of the observations lie between $-2$ and $+2$. Any observation that lies outside this interval may be considered unusual because the observation occurs less than 5% of the time.

**Example: According to CTIA, 55% of all U.S. households are wireless-only. In a simple random sample of 500 households, 301 were wireless-only. Is this result unusual?**

**Approach:**

Because $np(1-p) = 500(0.55)(1-0.55) = 123.75 \geqslant 10$, the binomial probability distribution is approximately bell-shaped. Therefore, we can use the Empirical Rule: If the observation is less than $\mu - 2\sigma$ or greater than $\mu + 2\sigma$, it is unusual.

**Solution:** We have $\mu_X = 500(0.55) = 275$ and $\sigma_X = \sqrt{np(1-p)} = \sqrt{500(0.55)(1-0.55)} = 11.1$. Now,

$$\mu_X - 2\sigma_X = 275 - 2(11.1) = 275 - 22.2 = 252.8$$
$$\textit{and}$$
$$\mu_X + 2\sigma_X = 275 + 2(11.1) = 275 + 22.2 = 297.2..$$

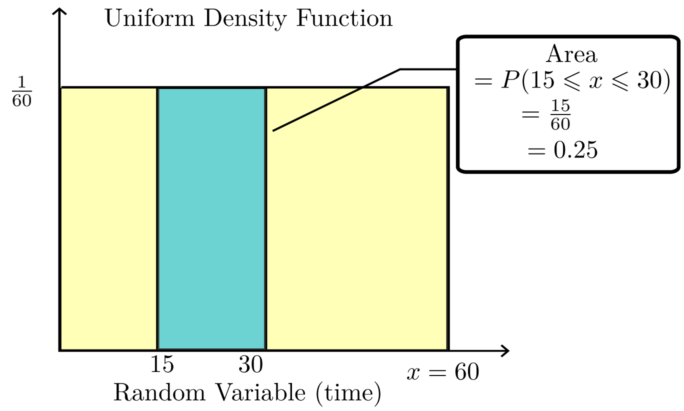
Any value less than 252.8 or greater than 297.2 is unusual; therefore, 301 is an unusual result. We should try to identify the reason for its value. Perhaps the percentage of households that are wireless-only has increased.

## Probability Density Function (pdf) Rules:

1. The total area under the graph of the equation over all possible values of the random variable must equal 1
2. The height of the graph of the equation must be greater than or equal to 0 for all possible values of the random variable. That is, the graph of the equation must lie on or above the horizontal axis for all possible values of the random variable.

## Area as a Probability:

The probability of choosing a time that is between 15 and 30 seconds after the minute is the area under the uniform density function.




## Effects of mean and standard deviation on normal curve

- as the mean increases, the graph slides to the right
- as the mean decreases, the graph slides to the left
- as the standard deviation increases, the graph flattens out and becomes wider
- as the standard deviation decreases, the graph compresses out and becomes steeper

## Properties of the Normal Curve

The normal probability density function satisfies all the requirements of probability distributions. Now we list the properties of the normal density curve. The video explains the properties.

1. Because mean = median = mode, the normal curve has a single peak and the highest point occurs at $x = \mu$.
2. The normal curve has inflection points at $\mu - \sigma$ and $\mu + \sigma$.
3. The area under the normal curve is 1.
4. The area under the normal curve to the right of $\mu$ equals the area under the normal curve to the left of $\mu$, which equals 0.5.
5. As $x$ increases without bound (gets larger and larger), the graph approaches, but never reaches, the horizontal axis. As $x$ decreases without bound (gets more and more negative), the graph approaches, but never reaches, the horizontal axis.
6. The Empirical Rule:
   - Approximately 68% of the area under the normal curve is between $x = \mu - \sigma$ and $x = \mu + \sigma$.
   - Approximately 95% of the area is between $x = \mu - 2\sigma$ and $x = \mu + 2\sigma$.
   - Approximately 99.7% of the area is between $x = \mu - 3\sigma$ and $x = \mu + 3\sigma$.

## The Role of Area in the Normal Density Function:

Suppose that a random variable $X$ is normally distributed with mean $\mu$ and standard deviation $\sigma$. The area under the normal curve for any interval of values of the random variable $X$ represents either

- The proportion of the population with the characteristic described by the interval of values,

  or

- The probability that a randomly selected individual from the population will have the characteristic described by the interval of values.

## Find and Interpret the Area under a Normal Curve

We use z-scores to find the area under a normal curve by hand. Recall that the z-score allows us to transform a random variable X with mean $\mu$ and standard deviation $\sigma$ into a random variable Z with mean 0 and standard deviation 1.

## Standardizing a Normal Random Variable:

Suppose that the random variable $X$ is normally distributed with mean $\mu$ and standard deviation $\sigma$.

Then the random variable

$$Z = \frac{X - \mu}{\sigma}.$$

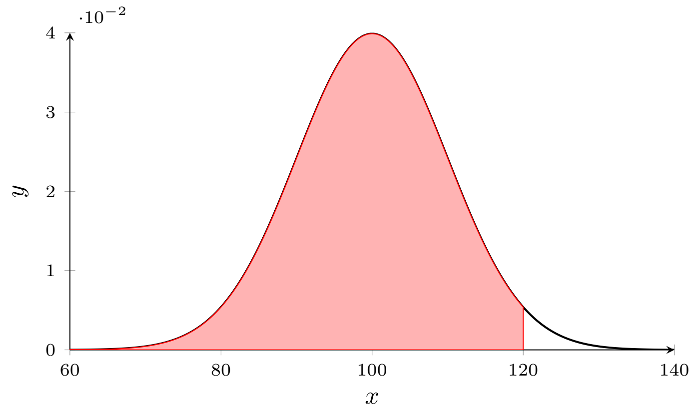
is normally distributed with mean $\mu = 0$ and standard deviation $\sigma = 1$.

The random variable $Z$ is said to have the **standard normal distribution.**

Consider an example:

IQ scores can be modeled by a normal distribution with $\mu = 100$ and $\sigma = 15$ Suppose a person has an IQ of 120

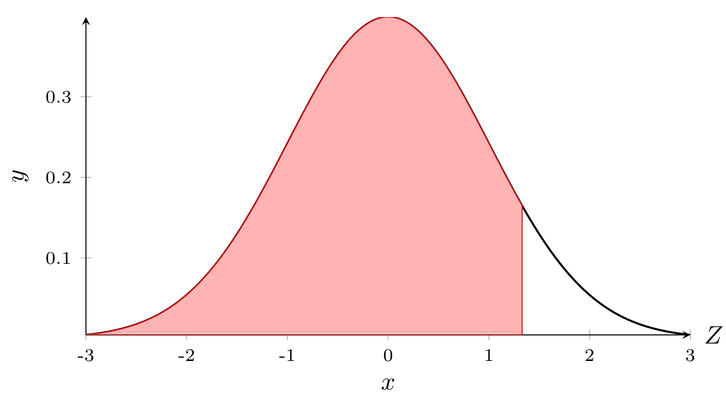
*Figure:*




Now we can transform this into a standard normal (Z distribution)

$$Z = \frac{120 - 100}{15}$$
$$= 1.33.$$

*Figure:*




Then by looking at the Table, we can determine the area in the red shaded region. Which is .9082. From this we can also determine that the area in the red region of our original distribution is also .9082

If we wanted to find the area of the unshaded region, we can use the complement rule

$$1 - .9082$$
$$= .0918.$$

**Percentiles:**

Because the area under the normal curve represents a proportion, we can also use the area to find percentile ranks of scores.

So we can say that 90.82% have an IQ $\leqslant 120$

## Finding an interpreting Area under a Normal Curve With Statcrunch:

1. Stat > Calculators > Normal
2. Input Mean
3. Input Standard Deviation
4. Input X
5. Compute

## Find the Value of a Normal Random Variable:

Often, we do not want to find the proportion, probability, or percentile given a value of a normal random variable. Rather, we want to find the value of a normal random variable that corresponds to a certain proportion, probability, or percentile. For example, we might want to know the height of a 3-year-old girl who is at the 20th percentile. Or we might want to know the scores on a standardized exam that separate the middle 90% of scores from the bottom and top 5%.

**Example: The heights of a pediatrician's 3-year-old females are approximately normally distributed, with mean 38.72 inches and standard deviation 3.17 inches. Find the height of a 3-year-old female at the $20^{th}$ percentile.**

**Approach:**

Use .2 in Statcrunch steps mentioned previously as what P(X) is equal to.

**Example: The mean incubation time of fertilized eggs is days. Suppose the incubation times are approximately normally distributed with a standard deviation of day. Determine the incubation times that make up the middle 97%**

**Solution:**

So in statcrunch, we will use values 0.05, and 0.97

## Important Notation for the Future

The notation $z_\alpha$ (pronounced "z sub alpha") is the z-score such that the area under the standard normal curve to the right of $z_\alpha$ is $\alpha$.

**Example: Find the value of** $Z_{0.21}$

**Solution:** So we want to find the area to the right of 0.21, in statcrunch:

1. Stat > Calculator > Normal
2. Mean: 0
3. Std. Dev: 1
4. $P(x \geqslant 0) = 0.21$
5. Compute!

## Drawing a Normal Probability Plot Using Statcrunch

1. Graph > QQ plot
2. Select Column
3. Select Correlation Statistic
4. Compute

**Note:-**

Normal probability plot is observed values as x and expected z-scores as y

### A Rule of Thumb for Invoking the Central Limit Theorem:

1. The shape of the distribution of the population from which the sample is drawn dictates the size of the sample required for the distribution of the sample mean to be normal.
2. The more skewed the distribution of the population is, the larger the sample size needed to invoke the Central Limit Theorem.
3. **If the distribution of the population is unknown or not normal, then the distribution of the sample mean is approximately normal provided that the sample size is greater than or equal to 30.**

### cutting the standard error of the mean in half

The sample size must be increased by a factor of four to cut the standard error in half.

### Suppose a simple random sample of size n is obtained from a population whose distribution is skewed right. As the sample size n increases, what happens to the shape of the distribution of the sample mean?

**Answer:** The distribution becomes approximately normal.

**(b) For the three probability distributions shown, rank each distribution from lowest to highest in terms of the sample size required for the distribution of the sample mean to be approximately normally distributed.**

1. **Uniform**: Lowest
2. **Normal**: Medium
3. **Skewed**: Highest

### Sampling Distribution of phat

For a simple random sample of size $n$ with a population proportion $p$,

- The shape of the sampling distribution of $\hat{p}$ is approximately normal provided $np(1-p) \geqslant 10$.
- The mean of the sampling distribution of $\hat{p}$ is:

$$\mu_{\hat{p}} = p..$$

- The standard deviation of the sampling distribution of $\hat{p}$ is:

$$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}.$$

**Note:-**

The sample size, $n$, can be no more than 5% of the population size, $N$. That is, $n \leqslant 0.05N$.

## 9-12 Subsection Titles

**Chapter 9:**

- 9.1: Estimating a Population Proportion
- 9.2: Estimating a Population Mean
- 9.3: Putting It Together: Which Procedure Do I Use?

**Chapter 10:**

- 10.1: The Language of Hypothesis Testing
- 10.2A: Hypothesis Tests on a Population Proportion with Simulation
- 10.2B: Hypothesis Tests on a Population Proportion Using the Normal Model
- 10.3: Hypothesis Tests for a Population Mean
- 10.4: Putting It Together: Which Procedure Do I Use?

**Chapter 11:**

- 11.1A: Using Randomization Techniques to Compare Two Proportions
- 11.1: Inference about Two Population Proportions: Independent Samples
- 11.3A Using Randomization Techniques to Compare Two Independent Means
- 11.3 Inference about Two Population Means: Independent Samples
- 11.4: Putting It Together: Which Procedure Do I Use?

**Chapter 12:**

- 12.1 Goodness-of-Fit Test
- 12.2 Tests for Independence and the Homogeneity of Proportions

## 9-12 Vocab

- A **point estimate** is the value of a statistic that estimates the value of a parameter.
- **A Confidence Interval** for an unknown parameter consists of an interval of numbers based on a point estimate.
- **The level of confidence** represents the expected proportion of intervals that will contain the parameter if a large number of different samples is obtained.
- **Critical Value** represents the number of standard deviations the sample statistic can be from the parameter and still result in an interval that includes the parameter.
- **The point estimate of the population mean**, $\mu$, is the sample mean, $\overline{x}$.
- **T-Interval:** confidence interval that uses the t-distribution
- A **Hypothesis** is a statement regarding a characteristic of one or more populations
- **Hypothesis Testing** is used to test statements regarding a characteristic of one or more populations **Note:** The ”heart” of hypothesis testing is making an assumption about reality, and collecting sample evidence to determine whether it contradicts the assumption.
- **Null Hypothesis** is a statement to be tested. The null hypothesis is a statement of no change, no effect or no difference and is assumed true until evidence indicates otherwise
- **alternative Hypothesis** is a statement that we are trying to find evidence to support
- Left- and right-tailed tests are referred to as **one-tailed tests**.
- Reject the null hypothesis when the null hypothesis is true. This decision would be incorrect. This type of error is called a **Type I error.**
- Do not reject the null hypothesis when the alternative hypothesis is true. This decision would be incorrect. This type of error is called a **Type II error.**
- **level of significance.** is the probability of making a type-1 error
- When observed results are unlikely under the assumption that the null hypothesis is true, the result is **statistically significant** and we reject the statement in the null hypothesis.
- A model that generates data randomly under the assumption the statement in the null hypothesis is true. Call this model the **null model**.
- A **P-value** is the probability of observing a sample statistic as extreme or more extreme than the one observed under the assumption that the statement in the null hypothesis is true. Put another way, the P-value is the likelihood or probability that a sample statistic will result in a statistic such as the one obtained if the statement in the null hypothesis is true.
- **Practical significance** refers to the idea that although small differences between the statistic and parameter stated in the null hypothesis are statistically significant, the difference may not be large enough to cause concern or be considered important.
- A sampling method is **independent** when an individual selected for one sample does not dictate which individual is to be in the second sample.
- A sampling method is **dependent** when the individuals selected to be in one sample are used to determine the individuals in the second sample. (matched pairs samples)

## 9-12 Notation/Formulas

- **Point Estimate for a Sample Proportion**

$$\hat{p} = \frac{x}{n}.$$

- The **Margin of Error** is denoted:

$$E.$$

- **Confidence Intervals for a Proportion** are of the form:

$$\text{point estimate} \pm \text{margin of error}$$

  That is:

$$\hat{p} \pm E.$$

  We have 95% confidence that the sampling proportion will lie between:

$$\hat{p} \pm 1.96\sigma_{\hat{p}}.$$

- **The level of confidence** is denoted:

$$(1-\alpha)\cdot 100\%.$$

- **Standard Error:**

$$\sigma_{\hat{p}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

- **Constructing any (1—$\alpha$)·100% Confidence Interval**

$$\hat{p} - z_{\alpha/2}\cdot\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} < p < \hat{p} + z_{\alpha/2}\cdot\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

- **Critical Value**

$$z_{\frac{\alpha}{2}}.$$

- **Constructing a (1—$\alpha$)· 100% Confidence Interval for a Population Proportion:** Suppose that a simple random sample of size $n$ is taken from a population or the data are the result of a randomized experiment. A (1—$\alpha$)·100% confidence interval for $p$ is given by the following quantities:

$$Lower\ bound:\ \hat{p} - z_{\frac{\alpha}{2}}\cdot\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$
$$Upper\ bound:\ \hat{p} + z_{\frac{\alpha}{2}}\cdot\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

  **Note:** It must be the case that $n\hat{p}(1-\hat{p}) \geqslant 10$ and $n \leqslant 0.05N$ to construct this interval. Use $\hat{p}$ in place of $p$ in the standard deviation. This is because $p$ is unknown, and $\hat{p}$ is the best point estimate of $p$.

- The **margin of error**, $E$, in a (1—$\alpha$)·100% confidence interval for a population proportion is given by

$$E = Z_{\frac{\alpha}{2}}\cdot\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$
$$or:\ \frac{Upper\ Limit\ - Lower\ Limit}{2}.$$

- **Sample Size Needed for Estimating the Population Proportion**

$$n = \hat{p}\cdot(1-\hat{p})\cdot\left(\frac{z_{\alpha/2}}{E}\right)^2.$$

  (rounded up to the next integer) where $\hat{p}$ is a prior estimate of p.

If a prior estimate of p is unavailable, the sample size required is

$$n = 0.25 \cdot \left(\frac{z_{\alpha/2}}{E}\right)^2.$$

rounded up to the next integer. The margin of error should always be expressed as a decimal when using these formulas.

- **Width**

$$2(E).$$

- **Student's t-distribution**

$$t = \frac{\overline{x} - \mu}{\frac{s}{\sqrt{n}}}.$$

- **Point estimate for a population mean**

$$\overline{x}.$$

- **Constructing a (1−$\alpha$) ·100% Confidence Interval for** $\mu$ Provided:
  - sample data come from a simple random sample or randomized experiment
  - sample size is small relative to the population size (n $\leqslant$0.05N)
  - the data come from a population that is normally distributed with no outliers, or the sample size is large

  A (1−$\alpha$)·100% confidence interval for $\mu$ is given by

$$LB := \overline{x} - t_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}$$
$$UB := \overline{x} + t_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}$$
$$.$$

  where $t_{\frac{\alpha}{2}}$ is the critical value with n−1 degrees of freedom.

- **Margin of Error:**

$$E = t_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}.$$

- **Determine the Sample Size Necessary for Estimating a Population Mean within a Given Margin of Error**

$$n = \left(\frac{z_{\frac{\alpha}{2}} \cdot s}{E}\right)^2.$$

- **Null Hypothesis**

$$H_0.$$

- **Alternative hypothesis**

$$H_1.$$

- **Level of Significance (Probability of making a type-1 error)**

$$\alpha = P(\text{Type I error}) = P(\text{rejecting } H_0 \text{ when } H_0 \text{ is true})$$
$$.$$

- **Probability of making a type-2 error**

$$\beta = P(\text{Type II error}) = P(\text{not rejecting } H_0 \text{ when } H_1 \text{ is true}).$$

- **Assumed value of the population proportion**

$$p_0.$$

- **Assumed mean of the population proportion**

$$\mu_{\hat{p}} = p_0.$$

- **Assumed Standard Error**

$$\sigma_{\hat{p}} = \sqrt{\frac{p_0(1-p_0)}{n}}.$$

- **Conditions**

$$\begin{aligned} &\text{Sample is simple random} \\ &np_0(1-p_0) \geqslant 10 \\ &and: \ n \leqslant 0.05N. \end{aligned}$$

- **Computation of test statistic**

$$z_0 = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}.$$

- **Assumed value of population mean**

$$\mu_0.$$

- **Conditions**
  - Sample is simple random
  - $n \geqslant 30$
  - $n \leqslant 0.05N$
  - If $n > 30$, we must verify that the sample comes form a normally distributed population, to do this verify with the normal probability plot (qq plot), and make a boxplot to check for outliers. If there are outliers in the data, we must not proceed.

- **Test statistic** for hypothesis test about the mean

$$t_0 = \frac{\overline{x} - \mu_0}{\frac{s}{\sqrt{n}}}.$$

- **The sampling distribution of $\hat{p}_1 - \hat{p}_2$:**

$$\hat{p}_1 = \frac{x_1}{n_1}$$
$$\hat{p}_2 = \frac{x_2}{n_2}.$$

Is approximately normal

- **Conditions**
  - Samples are obtained independently
  - $n_1\hat{p}_1(1-\hat{p}_1) \geqslant 10$
  - $n_2\hat{p}_2(1-\hat{p}_2) \geqslant 10$
  - Each sample is no more than 5% of the population

- **Mean of difference between two population proportions**

$$\mu_{\hat{p}_1-\hat{p}_2} = p_1 - p_2$$
$$Or: \ 0.$$

- **Pooled Estimate of $p$**

$$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2}.$$

- **Standard Deviation of difference between two population proportions**

$$\sigma_{\hat{p}_1-\hat{p}_2} = \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$$
$$Or: \ \sqrt{\hat{p}(1-\hat{p})} \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}.$$

- **The standardized version of $\hat{p}_1 - \hat{p}_2$ is then written as:**

$$Z_0 = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}}$$
$$Or: \ Z_0 = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}.$$

Which has an approximate standard normal distribution.

- **Constructing a $(1-\alpha)\cdot$100% Confidence Interval for the Difference between Two Population Proportions (Independent Samples)**

$$Lowerbound: \hat{p}_1 - \hat{p}_2 - z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$
$$Upperbound: \hat{p}_1 - \hat{p}_2 + z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}.$$

Notice that we do not pool the sample proportions. This is because we are not making any assumptions regarding their equality, as we did in hypothesis testing.

- **Population Mean Difference**

$$\mu_d = \mu_1 - \mu_2$$
$$Or: \ 0.$$

- **Sample Mean Difference for two dependent samples**

$$\bar{d} \text{ this is the mean of the differenced data.}$$

- **Sample Standard Deviation Difference for two dependent samples**

$$s_d \text{ this is the standard deviation of the differenced data.}$$

- **Test Statistic for difference between two dependent samples**:

$$t_0 = \frac{\bar{d} - \mu_d}{\frac{s_d}{\sqrt{n}}}.$$

- **Confidence Interval for two dependent samples**

$$Lowerbound : \bar{d} - t_{\alpha/2} \cdot \frac{s_d}{\sqrt{n}}$$
$$Upperbound : \bar{d} + t_{\alpha/2} \cdot \frac{s_d}{\sqrt{n}}.$$

The critical value $t_{\alpha/2}$ is determined using $n-1$ degrees of freedom. The values of $\bar{d}$ and $s_d$ are the mean and standard deviation of the differenced data.

**Note:-**

The interval is exact when the population is normally distributed and approximately correct for nonnormal populations, provided that $n$ is large.

- **Welch's approximate t (Sampling Distribution of the Difference of Two Means: Independent Samples with Population Standard Deviations Unknown** Suppose that a simple random sample of size $n_1$ is taken from a population with unknown mean $\mu_1$ and unknown standard deviation $\sigma_1$. In addition, a simple random sample of size $n_2$ is taken from a second population with unknown mean $\mu_2$ and unknown standard deviation $\sigma_2$. If the two populations are normally distributed or the sample sizes are sufficiently large ($n_1 \geqslant 30$ and $n_2 \geqslant 30$), then

$$t_0 = \frac{(\overline{x}_1 - \overline{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}.$$

approximately follows Student's t-distribution with the smaller of $n_1 - 1$or $n_2 - 1$ degrees of freedom, where $\overline{x}_1$ is the sample mean and $s_1$ is the sample standard deviation from population 1, and $\overline{x}_2$ is the sample mean and $s_2$ is the sample standard deviation from population 2.

- **degrees of freedom:** The degrees of freedom used to determine the P-value presented in the by-hand solution of Example 1 are conservative (this means we would require more evidence against the statement in the null hypothesis than if we use the formula for degrees freedom given below).

Results that are more accurate can be obtained using the following formula for degrees of freedom:

$$df = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)}{\frac{\left(\frac{s_1^2}{n_1}\right)^2}{n_1 - 1} + \frac{\left(\frac{s_2^2}{n_2}\right)^2}{n_2 - 1}}.$$

When using this formula to compute degrees of freedom, round down to the nearest integer to use Table VII. For hand inference, it is recommended that you use the smaller of n1−1 or n2−1 as the degrees of freedom to ease computation. However, for increased precision in determining the P-value, computer software will use the formula above when computing the degrees of freedom.

- **Constructing a (1−$\alpha$)·100% Confidence Interval for the Difference of Two Means**

$$Lower\ Bound:\ (\overline{x}_1 - \overline{x}_2) - t_{\frac{\alpha}{2}} \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$
$$Upper\ Bound:\ (\overline{x}_1 + \overline{x}_2) - t_{\frac{\alpha}{2}} \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$
$$.$$

where $t_{\frac{\alpha}{2}}$ is computed using the smaller of $n_1$−1 or $n_2$−1 degrees of freedom or using the degrees of freedom formula.

- **Compute p-values**

$$Left-tailed\ Test:\ p-value = P(Z < Z_0)$$
$$Right-tailed\ Test:\ p-value = P(Z > Z_0)$$
$$Two-tailed\ Test:\ p-value = P(Z > |Z_0|)$$
$$Or:$$
$$Left-tailed\ Test:\ p-value = P(t < t_0)$$
$$Right-tailed\ Test:\ p-value = P(t > t_0)$$
$$Two-tailed\ Test:\ p-value = P(t > |t_0|)$$
$$.$$

## Confidence interval

- for a 95% confidence interval, any sample proportion that lies withing 1.96 standard errors of the population proportion will result in a confidence interval that includes $p$. This will happen in 95% of all possible samples.
- Any sample proportion that is more than 1.96 standard errors from the population proportion will result in a confidence interval that does not contain $p$. This will happen in 5% of all possible samples (those sample proportions in the tails of the distribution).
- A confidence interval for an unknown parameter consists of an interval of numbers based on a point estimate.
- The level of confidence represents the expected proportion of intervals that will contain the parameter if a large number of different samples is obtained. The level of confidence is denoted $(1-\alpha)\cdot$ 100%.
- Whether a confidence interval contains the population parameter depends solely on the value of the sample statistic. Any sample statistic that is in the tails of the sampling distribution will result in a confidence interval that does not include the population parameter. See Figure 1. Notice that $\hat{p}_1$ results in a confidence interval that includes the population proportion, $p$. However, $\hat{p}_2$ results in a confidence interval that does not include the population proportion, $p$, because $\hat{p}_2$ is in the tails of the distribution.

## Caution

So, a 95% confidence interval does not mean that there is a 95% probability that the interval contains the parameter (such as $p$ or $\mu$). The 95% in a 95% confidence interval represents the proportion of all samples that will result in intervals that include the population proportion.

In practice, we construct only one confidence interval based on one sample. We do not know whether the sample results in a confidence interval that includes the parameter, but we do know that if we construct a 95% confidence interval, it will include the unknown parameter 95% of the time.

## Common Critical Values

| Level of confidence $(1-\alpha)\cdot 100\%$ | Area in each tail $\frac{\alpha}{2}$ | Critical Value $z_{\frac{\alpha}{2}}$ |
|---|---|---|
| 90% | 0.05 | 1.645 |
| 95% | 0.025 | 1.96 |
| 99% | 0.005 | 2.575 |

## Interpret A Confidence Interval

A $(1-\alpha)\cdot 100\%$ confidence interval indicates that $(1-\alpha)\cdot 100\%$ of all simple random samples of size $n$ from the population whose parameter is unknown will result in an interval that contains the parameter.

For example, a 90% confidence interval for a parameter suggests that 90% of all possible samples will result in an interval that includes the unknown parameter and 10% of the samples will result in an interval that does not capture the parameter.

## Computing a Confidence Level Confidence Interval With Statcrunch

1. Verify that $n\hat{p}(1-\hat{p}) \geqslant 10$
2. Verify that $n \leqslant 0.05N$
3. Stat > Proportion Stats > One Sample > with summary
4. Input no. of successes
5. Input no. of observations
6. Input confidence level
7. Method: Standard-Wald
8. Compute

## Computing a Confidence Level Confidence Interval by Hand

**Example: Constructing a Confidence Interval for a Population Proportion (By hand)**

**Problem:** In the Parent–Teen Cell Phone Survey conducted by Princeton Survey Research Associates International, 800 randomly sampled 16- to 17-year-olds living in the United States were asked whether they have ever used their cell phone to text while driving. Of the 800 teenagers surveyed, 272 indicated that they text while driving. Obtain a 95% confidence interval for the proportion of 16- to 17-year-olds who text while driving.

**Approach:** It is important to verify the requirements for constructing the confidence interval first. It must be the case that $n\hat{p}(1\hat{p}) \geqslant 10$ and the sample size is no more than 5% of the population size (n ⩽0.05N).Then, construct the confidence interval either by hand or using technology.

**Solution:**

First, we obtain $\hat{p}$

$$\hat{p} = \frac{272}{800} = .34.$$

Verify Necessary Conditions:

$$\begin{aligned} n\hat{p}(1-\hat{p}) &\geqslant 10 \\ 800(.34)(.66) = 179.52 &\geqslant 10. \end{aligned}$$

Also, we know that $n \leqslant 0.05N$:

Now, to use the formula:

$$\begin{aligned} \hat{p} &\pm E \\ Where\ E = z_{\frac{\alpha}{2}} \cdot &\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}. \end{aligned}$$

We will need to find alpha, given that the formula for confidence level is:

$$\begin{aligned}(a-\alpha)\cdot 100\%\\ 1-\alpha=0.95\\ \alpha=0.05.\end{aligned}$$

So our critical value is:

$$\begin{aligned}z_{\frac{\alpha}{2}} &= z_{\frac{0.05}{2}}\\ &= z_{0.025}.\end{aligned}$$

Using the normal distribution table, we get:

$$\alpha = 1.96.$$

Now we can compute our bounds:

$$\begin{aligned}LB := \hat{p} - z_{\frac{\alpha}{2}}\cdot\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\\ = 0.34 - 19.6\cdot\sqrt{\frac{0.34(0.66)}{800}}\\ = 0.307\\ UB = \hat{p} + z_{\frac{\alpha}{2}}\cdot\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\\ = 0.34 + 1.96\cdot\sqrt{\frac{0.34(0.66)}{800}}\\ = 0.373.\end{aligned}$$

Therefore:

$$\begin{aligned}Upper\ bound:\ 0.307 \quad Upper\ Bound:\ 0.373\\ Or: (0.307, 0.373).\end{aligned}$$

## The Effect of Level of Confidence on the Margin of Error

increasing the level of confidence increases the margin of error, resulting in a wider confidence interval.

## The Effect of Sample Size on the Margin of Error

Increasing the sample size $n$ decreases the standard error, so the margin of error decreases. Therefore, larger sample sizes will result in narrower confidence intervals.

The general relationship between the sample size and the margin of error is as follows:

Increasing the sample size by a factor M results in the margin of error decreasing by a factor of $\frac{1}{\sqrt{m}}$

### What If We Do Not Satisfy the Normality Condition?

When the normality condition is not satisfied, the proportion of intervals that capture the parameter is below the level of confidence.

### Determine the Sample Size Necessary for Estimating a Population Proportion within a Specified Margin of Error (In statcrunch)

1. Find Margin of Error (E)
2. Find Width (W)
3. Find Confidence Level
4. Find target proportion
5. Stats > Proportion Stats > One Sample > Width/Sample Size
6. Input confidence level
7. Input Target Proportion
8. Compute

### Determine the Sample Size Necessary for Estimating a Population Proportion within a Specified Margin of Error (By Hand) (WITH ESTIMATE)

**Example: Determining Sample Size**

**Problem:** An economist wants to know if the proportion of the U.S. population who commutes to work via car-pooling is on the rise. What size sample should be obtained if the economist wants an estimate within 2 percentage points of the true proportion with 90% confidence?

Assume that the economist uses the estimate of 10% obtained from the American Community Survey.

**Approach:** Since $1 - \alpha = 0.9$, we know that $\alpha = 0.10$. Use $E = 0.02$, $z_{\alpha/2} = z_{0.12} = z_{0.05} = 1.645$, and $\hat{p} = 0.10$ (the prior estimate).

**Solution:** Using the formula assuming that a prior estimate is available, $n = \hat{p}(1-\hat{p})\left(\frac{z_{\alpha/2}}{E}\right)^2$, we obtain

$$n = \hat{p}(1-\hat{p})\left(\frac{z_{\alpha/2}}{E}\right)^2 = 0.10 \times (1 - 0.10) \times \left(\frac{1.645}{0.02}\right)^2 = 608.9.$$

Round this value up to 609. So the economist must survey 609 randomly selected residents of the United States.

## Determine the Sample Size Necessary for Estimating a Population Proportion within a Specified Margin of Error (By Hand) (WITHOUT ESTIMATE)

Use 0.5 as target proportion in statrunch, or use the formula:

$$n = (0.25)\left(\frac{z_{\frac{\alpha}{2}}}{E}\right)^2.$$

## Properties of the t-distribution

1. The t-distribution is different for different degrees of freedom
2. the t-distribution is centered at 0 and is symmetric about 0
3. The area under the curve is 1. The eare uned rteh curve to the right of 0 equals the area under the curve to the left of 0, which equals 1/2
4. As t increases or decreases without bound, the graph approaches, but never equals, zero.

$$\lim_{t \to -\infty} f(x) = 0$$
$$\lim_{t \to \infty} f(x) = 0.$$

5. The area in the tails of the t-distribution is a little greater than the area in the tails of the standard normal distribution, because we are using $s$ as an estimate of $\sigma$, thereby introducing further variability into the t-statistic
6. As the sample size $n$ increases, the density curve of $t$ get closer to the standard normal density curve. This result occurs because, as the sample size increases, the values of $s$ get closer to the value of $\sigma$, by the Law of Large Numbers.

## Put the following in order for the most area in the tails of the distribution.

a.) Standard Normal Distribution

b.) Student's t-Distribution with 30 degrees of freedom.

c.) Student's t-Distribution with 50 degrees of freedom.

## A robust procedure

A confidence interval about $\mu$ can be computed for non-normal populations even though Student's t-distribution requires a normal population. This is because the procedure for constructing the confidence interval is robust—it is accurate despite minor departures from normality.

If a data set has outliers, the confidence interval is not accurate because neither the sample mean nor the sample standard deviation is resistant to outliers. Sample data should always be inspected for serious departures from normality and for outliers. This is easily done with normal probability plots and boxplots.

## Example: Constructing a confidence interval about a population mean (By Hand)

**Example: Constructing a confidence interval about a population mean (By Hand)**

**Problem:** The website fueleconomy.gov allows drivers to report the miles per gallon of their vehicle. The data in Table 3 show the reported miles per gallon of 2011 Ford Focus automobiles for 16 different owners. Treat the sample as a simple random sample of all 2011 Ford Focus automobiles. Construct a 95% confidence interval for the mean miles per gallon of a 2011 Ford Focus. Interpret the interval.

Before we begin, let's keep the following equation in mind:

$$\overline{x} \pm t_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}.$$

**Approach:** Before we begin, let's insure that the model requirements for constructing a confidence interval about a mean are satisfied by drawing a normal probability plot and boxplot.

We can confidently conclude that the data was obtained randomly, and the sample size 16 is much less than 5% of the population size

Because the sample size is small, we first need to verify that the data came from a population that is *normally distributed*, and also that the sample has not outliers. If we take a lot at the probability plot, we can see that the data is roughly linear, which insinuates that the data did indeed come from a population that is *normally distributed*, furthermore, if we examine the boxplot, it becomes obvious that there are no outliers in the data.

Using technology, we can compute the sample mean $\overline{x}$ and the sample standard deviation $s$

$$\begin{aligned} \overline{x} &= 36.8 \\ s &= 2.92. \end{aligned}$$

Now let's find the critical value for the $t - distribution$

$$\begin{aligned} (1-\alpha) &= 0.95 \\ \alpha &= 0.05 \\ \frac{\alpha}{2} &= 0.025 \end{aligned}$$

.

Thus: $z_{0.025} = 2.131$ By the t-distribution area in right tail table. From here, we can compute our bounds:

$$\begin{aligned} LB:\ \overline{x} - t_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}} \\ = 36.8 - 2.131 \cdot \frac{2.92}{\sqrt{15}} \\ = 35.24. \end{aligned}$$

$$\begin{aligned} UB:\ \overline{x} + t_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}} \\ = 36.8 + 2.131 \cdot \frac{2.92}{\sqrt{15}} \\ = 38.36. \end{aligned}$$

We are 95% confident that the mean miles per gallon of all 2011 Ford Focus cars is between 35.24 and 38.36 mpg.

## Example: Constructing a confidence interval about a population mean

**Example: Constructing a confidence interval about a population mean (Using Statcrunch)**

First, verify that the population is normally distributed by looking at the probability plot, and also that there are no outliers by looking at the boxplot

**Statrunch Steps**

1. Stat > T Stats > One Sample > with data
2. Select column containig data
3. Input confidence level
4. Compute

## Critical values for t-distribution vs critical values for z-distribution

In the above example (by hand), we found the critical value to be $t_{0.025} = 2.131$ for 15 degrees of freedom, whereas $z_{0.025} = 1.96$. The t-distribution gives a larger critical value, so the width of the interval is wider. This larger critical value using Student's t-distribution is necessary to account for the increased variability due to using $s$ as an estimate of $\sigma$.

Remember, 95% confidence refers to our confidence in the method. If we obtained 100 samples of size $n = 16$ from the population of 2011 Ford Focuses, we would expect about 95 of the samples to result in confidence intervals that include $\mu$. We do not know whether the interval in Example 3 includes $\mu$ or does not include $\mu$.

## Effects on margin of error

- As the sample size increases, the margin of error decreases.
- As the level of confidence increases, the size of the interval increases.
- Additional note: the population shape needs to be normally distributed to computer the confidence interval

## When Model Requirements Fail

- When constructing 95% confidence intervals for the mean when the parent population is right skewed and the sample size is small, the proportion of intervals that include the population mean is below 0.95.
- When constructing 95% confidence intervals for the mean when the parent population is right skewed and the sample size is small, the proportion of intervals that include the population mean approaches. 95 as the sample size, n, increases.

**What should we do if the requirements to compute a t-interval are not met?**

We could increase the sample size beyond 30 observations, or we could try to use nonparametric procedures. Nonparametric procedures typically do not require normality, and the methods are resistant to outliers. A third option is to use resampling methods, such as bootstrapping. Bootstrapping is presented later in this chapter.

## Example: Determining Sample Size (By Hand)

**Example: Determining Sample Size (By Hand)**

**Problem:** We again consider the problem of estimating the miles per gallon of a 2011 Ford Focus. How large a sample is required to estimate the mean miles per gallon within 0.5 miles per gallon with 95% confidence? Note: The sample standard deviation is s=2.92 mpg.

**Approach:** Use $n = \left(\frac{z_{\alpha/2} \cdot s}{E}\right)^2$ with $z_{\alpha/2} = z_{0.025} = 1.96$, $s = 2.92$, and $E = 0.5$ to find the required sample size.

**Solution:**

$$n = \left(\frac{z_{\alpha/2} \cdot s}{E}\right)^2 = \left(\frac{1.96 \cdot 2.92}{0.5}\right)^2 = 131.02.$$

Round 131.02 up to 132. A sample size of $n = 132$ results in an interval estimate of the population mean miles per gallon of a 2011 Ford Focus with a margin of error of 0.5 mile per gallon with 95% confidence.

## Example: Determining Sample Size (With Statcrunch)

**Example: Determining Sample Size (With Statcrunch)**

1. Find Standard Deviation
2. Stat > Z Stats > One Sample > Width/Sample Size
3. Enter Confidence Lever
4. Enter Standard Deviation
5. Enter Sample Size
6. Compute

## Determine the Appropriate Confidence Interval to Construct

Questions to ask

1. What is the variable of interest?
2. Are the conditions for constructing the interval satisfied?

For question one. *What is the variable of interest*, there are two possible situations for our purposes.

1. Qualitative variable with two outcomes: Analyze with proportions $p$
2. Quantitative variable: $\mu$

Now that we know which model we are going to construct, we need to verify that the conditions for constructing an interval are satisfied.

1. Population Proportion:
   - $n\hat{p}(1-\hat{p}) \geqslant 10$, and $n \leqslant 0.05N$
2. Mean:
   - $n \geqslant 30$
   - if $n \leqslant 30$, we need to verify that the data comes from a population that is normally distributed. Thus, we need to make a normal probability plot to verify this and make a boxplot to check for outliers.

## Steps in hypothesis testing

1. Make a statement regarding the nature of the population
2. Collect evidence (sample data) to test the statement
3. Analyze the data to assess the plausibility of the statement

## three ways to set up the null and alternative hypotheses

1. Equal versus not equal hypothesis (two-tailed test)

$$\begin{aligned} H_0 &: \text{ parameter} = \text{some value} \\ H_1 &: \text{ parameter} \neq \text{some value.} \end{aligned}$$

2. Equal versus less than (left-tailed test)

$$\begin{aligned} H_0 &: \text{ parameter} = \text{some value} \\ H_1 &: \text{ parameter} < \text{some value.} \end{aligned}$$

3. Equal versus greater than (right-tailed test)

$$\begin{aligned} H_0 &: \text{ parameter} = \text{some value} \\ H_1 &: \text{ parameter} > \text{some value.} \end{aligned}$$

## FOUR OUTCOMES FROM HYPOTHESIS TESTING

1. Reject the null hypothesis when the alternative hypothesis is true. This decision would be correct.
2. Do not reject the null hypothesis when the null hypothesis is true. This decision would be correct.
3. Reject the null hypothesis when the null hypothesis is true. This decision would be incorrect. This type of error is called a **Type I error.**
4. Do not reject the null hypothesis when the alternative hypothesis is true. This decision would be incorrect. This type of error is called a **Type II error.**

**Note:-**

The null hypothesis is never declared ”true”, either reject the null hypothesis if there is strong sample evidence against the statement in the null hypothesis, or do not reject the null hypothesis if there is not strong sample evidence against the statement in the null hypothesis

## Do we ever "accept" the null hypothesis?

No, Once the decision is made whether to reject the null hypothesis, the researcher must state his or her conclusion. It is important to recognize that we never accept the null hypothesis.

Sample evidence can never prove the null hypothesis to be true. By not rejecting the null hypothesis, we are saying that the evidence indicates that the null hypothesis could be true or that the sample evidence is consistent with the statement in the null hypothesis.

## hypothesis testing using the P-value approach.

- If the probability of getting a sample statistic as extreme or more extreme than the one obtained is small under the assumption the statement in the null hypothesis is true, reject the null hypothesis.

Of course, this leads to the question, "How low does the P-value need to be in order to reject the statement in the null hypothesis?" The answer is that it depends. The researcher must decide what represents statistically significant evidence against the statement in the null hypothesis. That said, typically we use the following rule.

**A Rule Of Thumb**

- If the P-value $< 0.05$ (the level of significance), we reject the statement in the null hypothesis.

The bottom line, however, is this. The smaller the P-value, the stronger the evidence against the statement in the null hypothesis. So, if P-value $< 0.01$, we say "strong evidence against the statement in the null hypothesis." If $0.01 <$ P-value $< 0.05$, we say "evidence against the statement in the null hypothesis." For $0.05 <$ P-value $< 0.1$, some may say "moderate evidence against the statement in the null hypothesis."

## Testing Hypotheses Regarding a Population Proportion Using Simulation

1. Verify that the variable of interest in the study is qualitative with two possible outcomes.
2. Determine the null and alternative hypotheses.
3. Build a null model that generates data randomly under the assumption the statement in the null hypothesis is true.
4. Estimate the P-value from the model in Step 3.
5. State the conclusion.

### Two-Tailed bounds

In 2000, 60% of females aged 15 or older lived alone, according to a government statistics agency. A sociologist tests whether this percentage is different today by conducting a random sample of 600 females aged 15 or older and finds that 357 are living alone. Is there sufficient evidence to conclude that the proportion has changed since 2000?

The variable of interest is whether or not a female in this country aged 15 or older lives alone. Therefore, it is qualitative with two possible outcomes.

The goal of the research is to determine whether the proportion of females in this country aged 15 or older who live alone is different from 0.6

$$H_0: \ p = .6$$
$$H_1: \ p \neq .6.$$

For the left tail:

$$p \leqslant 357.$$

For the right tail, we want to find the number of observations that are at least as extreme as 375, so:

$$600 \cdot .6 = 360$$
$$360 - 357 = 3$$
$$360 + 3 = 363.$$

$$p \geqslant 363.$$

Then find both p values with simulations and sum both p values, if the p value is less than the given value of $\alpha$, reject the null hypothesis

### Testing Hypotheses Regarding a Population Proportion Using the Normal Model

We now formalize the procedure for testing hypotheses regarding a population proportion using the normal model.

Use steps 1-5 shown below provided that:

- The sample is a simple random sample, or the result of a randomized experiment.
- $np(1-p) \geqslant 10$
- The sample values are independent of each other ($n \leqslant 0.05N$).

1. Determine the null and alternative hypotheses. The hypotheses can be structured in one of three ways:
2. Select a level of significance, $\alpha$, depending on the seriousness of making a Type I error .
3. Compute the **test statistic**

$$z_0 = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}.$$

Notice in the computation of the test statistic that we are using $p_0$ (the proportion stated in the null hypothesis) in computing the standard error rather than $\hat{p}$, as we did in constructing confidence intervals about $p$. This is because $H_0$ is assumed to be true when we are performing a hypothesis test. So the assumed mean of the distribution of $\hat{p}$ is $\mu_{\hat{p}} = p_0$, and the assumed standard error is $\sigma_{\hat{p}} = \sqrt{\frac{p_0(1-p_0)}{n}}$.

4. If $P$-value $< \alpha$, reject the null hypothesis.
5. State the conclusion

## Finding p-value with normal distribution for different tailed tests

- Two-Tailed Test
$$p\text{-value} = 2P(z > |z_0|).$$
- Left-Tailed Test
$$p\text{-value} = P(z < z_0).$$
- Right-Tailed Test
$$p\text{-value} = P(z < z_0).$$

## Two-Tailed Hypothesis Testing Using Confidence Intervals

- When testing $H_0 : p = p_0$ versus $H_1 : p \neq p_0$, if a $(1-\alpha)\cdot 100\%$ confidence interval contains $p_0$, we do not reject the null hypothesis. However, if the confidence interval does not contain $p_0$, we conclude that $p \neq p_0$ at the level of significance, $\alpha$.

## Testing Hypotheses Regarding a Mean

Use Steps 1–5 shown below provided that

- The sample is obtained by simple random sampling or the data results from a randomized experiment;
- the sample comes from a population that is normally distributed with no outliers, or the sample size, $n$, is large ($n \geqslant 30$); and
- the sampled values are independent of each other. This means that the sample size is no more than 5% of the population size ($n \leqslant 0.05N$).

**Steps:**

1. Determine the null and alternative hypotheses. The hypotheses can be structured in one of three ways:
2. Select a level of significance, $\alpha$, depending on the seriousness of making a Type I error.
3. Compute test-statistic
$$t_0 = \frac{\overline{x} - \mu_0}{\frac{s}{\sqrt{n}}}.$$
   - Two-Tailed:
$$p - value = 2P(t > |t_0|).$$
   - Left-Tailed
$$p - value = P(t < t_0).$$
   - Right-Tailed
$$p - value = P(t > t_0).$$
4. If P-value $< \alpha$, reject the null hypothesis.
5. State the conclusion.

**Note:-**

Notice that the procedure just presented requires that the population from which the sample was drawn be normal or that the sample size be large (n $\geqslant$30). The procedure is robust, so minor departures from normality will not adversely affect the results of the test. However, if the data include outliers, the procedure should not be used.

## Explain the Difference between Statistical Significance and Practical Significance

When a large sample size is used in a hypothesis test, the results may be statistically significant even though the difference between the sample statistic and mean stated in the null hypothesis may have no practical significance.

## Putting it all together: Which one do I use?

**Questions to ask yourself:**

1. What is the variable of interest?
2. Is the purpose of the study to estimate the value of a parameter or to test a statement regarding a parameter?
   - Estimate: Confidence Interval
   - Test a statment: Hypothesis Test
3. Are the conditions for performing the inference satisfied?

## Testing Hypotheses Regarding Two Proportions Using Random Assignment

1. Verify that the explanatory variable (or treatment) has two levels and the response variable in the study is qualitative with two possible outcomes.
2. Determine the null and alternative hypotheses.
3. Build a null model that randomly assigns the individuals in the study under the assumption the statement in the null hypothesis is true.
4. Estimate the P-value from the model in Step 3.
5. State the conclusion.

## Test Hypotheses Regarding Two Population Proportions from Independent Samples

To test hypotheses regarding two population proportions, $p_1$ and $p_2$, we can use the steps that follow, provided that:

- The samples are independently obtained using simple random sampling or the data result from a completely randomized experiment with two levels of treatment.
- $n_1\hat{p}_1(1-\hat{p}_1) \geqslant 10$ and $n_2\hat{p}_2(1-\hat{p}_2) \geqslant 10$.
- The sampled values are independent of each other. This means that each sample size is no more than 5% of the population size ($n_1 \leqslant 0.05N_1$ and $n_2 \leqslant 0.05N_2$). This ensures the independence necessary for a binomial experiment.

1. Determine the null and alternative hypotheses. The hypotheses can be structured in one of three ways: **Note:** $p_1$ is the population proportion for population 1, and $p_2$ is the population proportion for population 2.
2. Select a level of significance, $\alpha$, depending on the seriousness of making a Type I error.
3. Compute the test statistic.

$$z_o = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n_1} + \frac{\hat{p}(1-\hat{p})}{n_2}}},\ where\ \hat{p} = \frac{x_1 + x_2}{n_1 + n_2},\ \text{the pooled statistic of p..}$$

   Use table V to find the P-value

4. If P-value $< \alpha$, reject the null hypothesis.
5. State the conclusion.

## Test Hypotheses for a Population Mean from Matched-Pairs Data

To test hypotheses regarding the mean difference of data obtained from a dependent sample (matched-pairs data), use the following steps, provided that

- The sample is obtained by simple random sampling or the data result from a matched-pairs design experiment.
- The sample data are matched-pairs (dependent).
- The differences are normally distributed with no outliers or the sample size, $n$, is large ($n \geqslant 30$).
- The sampled values are independent of each other [that is, the sample size is no more than 5% of the population size ($n \leqslant 0.05N$)].

1. Determine the null and alternative hypotheses. The hypotheses can be structured in one of three ways:
2. Select a level of significance, $\alpha$, depending on the seriousness of making a Type I error.
3. Compute the test statistic

$$t_0 = \frac{\bar{d}}{\frac{s_d}{\sqrt{n}}}.$$

4. If P-value $< \alpha$, reject the null hypothesis.
5. State the conclusion.

### The Normality Requirement (Population mean matched-pairs)

The procedures just presented are robust, which means that minor departures from normality will not adversely affect the results of the test. If the data have outliers, however, the procedure should not be used.

Verify the assumption that the differenced data come from a population that is normally distributed by constructing a normal probability plot. Use a boxplot to determine whether there are outliers. If the normal probability plot indicates that the differenced data are not normally distributed or the boxplot reveals outliers, then either bootstrapping or nonparametric tests should be performed. Nonparametric tests are not covered in this course.

### Reject or accept with confidence interval

If the confidence interval does not include zero, reject.

### Testing Hypotheses Regarding Two Independent Means Using Random Assignment

1. Verify that the explanatory variable (or treatment) has two levels and the response variable in the study is quantitative.
2. Determine the null and alternative hypotheses. The hypotheses can be structured in one of three ways.
3. Build a null model that randomly assigns the individuals in the study to the value of the response variable under the assumption the statement in the null hypothesis is true.
4. Estimate the P-value from the model in Step 3.
5. State the conclusion

### Testing Hypotheses Regarding the Difference between Two Population Means: Independent Samples

To test hypotheses regarding two population means, $\mu_1$ and $\mu_2$, obtained from independent samples, use the following steps, provided that

- The two independent samples are obtained by simple random sampling or the data result from a completely randomized experiment with two treatments.
- The sample data come from a population that is normally distributed with no outliers or the sample sizes are large ($n_1 \geqslant 30$ and $n_2 \geqslant 30$).
- For each sample, the sample size is no more than 5% of the population size ($n \leqslant 0.05N$).

1. Determine the null and alternative hypotheses. The hypotheses are structured in one of three ways:
2. Select a level of significance, $\alpha$, depending on the seriousness of making a Type I error.
3. Compute the test statistic.

$$t_0 = \frac{(\overline{x}_1 - \overline{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}.$$

   which approximately follows Student's t-distribution. Use Table VII to approximate the P-value using the smaller of $n_1 - 1$ or $n_2 - 1$ degrees of freedom
4. If P-value $< \alpha$, reject the null hypothesis.
5. State the conclusion.

## Characteristics of the Chi-Square Distribution

- It is not symmetric.
- The shape of the chi-square distribution depends on the degrees of freedom, just like Student's t-distribution.
- As the number of degrees of freedom increases, the chi-square distribution becomes more symmetric. See Figure 1.
- The values of $\chi^2$ are nonnegative (greater than or equal to 0).

## What If the Degrees of Freedom I Need Is Not In the Table?

In studying Table VIII, notice that the degrees of freedom are numbered 1 to 30, inclusive, then 40,50,60,...,100. If the number of degrees of freedom is not in the table, choose the degrees of freedom closest to that desired. If the degrees of freedom are exactly between two values, find the mean of the values.

For example, to find the critical value corresponding to 75 degrees of freedom, compute the mean of the critical values corresponding to 70 and 80 degrees of freedom.

## The Goodness-of-Fit Test

1. Determine the null and alternative hypotheses:

$$\begin{aligned} H_0 &: \text{ The random variable follows a certain distribution.} \\ H_1 &: \text{ The random variable does not follow the distribution in the null hypothesis..} \end{aligned}$$

2. Decide on a level of significance, $\alpha$, depending on the seriousness of making a Type I error.

3. a.) Calculate the expected counts, $E_i$, for each of the $k$ categories: $E_i = n \cdot p_i$ for $i = 1, 2, \ldots, k$, where $n$ is the number of trials and $p_i$ is the probability of the $i$th category, assuming that the null hypothesis is true.

   b.) Verify that the requirements for the goodness-of-fit test are satisfied.
   - All expected counts are greater than or equal to 1 (all $E_i \geqslant 1$).
   - No more than 20% of the expected counts are less than 5.

   c.) Compute the test statistic

$$\chi_0^2 = \sum \frac{(O_i - E_i)^2}{E_i}.$$

   **Note:** $O_i$is the observed count for the $i^{th}$ category.

   d.) Use Table VIII to approximate the P-value by determining the area under the chi-square distribution with k−1 degrees of freedom to the right of the test statistic.

4. If P-value $< \alpha$, reject the null hypothesis.

5. State the conclusion.

## A Word about Conclusions of Hypothesis Tests

Whenever we are testing a hypothesis, the evidence can never prove the statement in the null hypothesis to be true. For example, in performing the goodness-of-fit test from this section, we tested

$H_0$ the distribution of household income is the same today as in 2000.

Suppose we failed to reject the null hypothesis. We would not be able to say the distribution today is the same as it was in 2000. Instead, we say that we do not have enough evidence to suggest the distribution has changed significantly.

Unfortunately, goodness-of-fit tests cannot be used to test whether sample data follow a specific distribution. We can only say the data is consistent with a distribution stated in the null hypothesis.

## The Chi-Square Test for Independence

To test hypotheses regarding the association between (or independence of) two variables in a contingency table, use the following steps.

1. Determine the null and alternative hypotheses:

$$H_0: \text{ The row variable and column variable are independent.}$$
$$H_1: \text{ The row variable and column variable are dependent..}$$

2. Decide on a level of significance, depending on the seriousness of making a Type I error.

3. (a) Calculate the expected frequencies (counts) for each cell in the contingency table.
   (b) Verify that the requirements for the test for independence are satisfied.
      i. All expected frequencies are greater than or equal to 1 (all $E_1 \geqslant 1$).
      ii. No more than 20% of the expected frequencies are less than 5.
   (c) Compute the test statistic
   (d) Use Table VIII to approximate the P-value by determining the area under the chi-square distribution with degrees of freedom to the right of the test statistic, where is the number of rows and is the number of columns. See Figure 5.

4. If P-value $< \alpha$, then reject the null hypothesis.

5. State the conclusion.

## Procedure for a test of Homogeneity

The procedures for performing a test of homogeneity are identical to those for a test of independence.

## If model requirements fail for homogeneity test fail

The requirements for performing a chi-square test are that all expected frequencies are greater than or equal to 1 and that at most 20% of the expected frequencies can be less than 5.

If these requirements are not satisfied, the researcher has one of two options:

1. combine two or more columns (or rows) to increase the expected frequencies or

2. increase the sample size.