Discrete Structures
Relations

Nathan Warner

Computer Science
Northern Illinois University

August 23, 2023
United States

Contents

Relations

1 The language of relations

Definition 1. A relation is the relationship between two or more set of values

Suppose we have two sets A and B, and $A \subseteq B$, then A and B are said to be **related**, because there is some attribute that binds them together. We can make a similar argument for $x > y$ for some x and y. This is the general idea behind relations.

Suppose we have the sets:

$$A = \{1, 2, 3\} \quad B = \{2, 3, 4\}.$$

And we can create a relationship by saying: *x is related to y* $\iff$ $x < y$, which is longhand for $x \ R \ y$.

$$\begin{array}{l} 1 \ R \ 2 \checkmark \\ 1 \ R \ 3 \checkmark \\ 1 \ R \ 4 \checkmark \\ 2 \not R \ 2 \\ 2 \ R \ 3 \checkmark \\ 2 \ R \ 4 \checkmark \\ 3 \not R \ 2 \\ 3 \not R \ 3 \\ 3 \ R \ 4 \checkmark \end{array}.$$

And we can write it in terms of **ordered pairs**

$$\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}.$$

2 Relations on sets

Let a and b be sets. A relation R from A to B is a subset of $A \times B$. Given an ordered pair (x, y) in $A \times B$, x is related to y by R, written $x \ \ R \ \ y$, if, and only if, (x, y) is in R. The set A is called the domain of R and the set B is called its co-domain

Suppose we have the sets:

$$A = \{1, 2, 3\} \quad B = \{1, 3, 5, 6\}.$$

Suppose the relation S means $x < y$, then we have:

$$x \ S \ y = \{(1, 3), (1, 5), (1, 6), (2, 3), (2, 5), (2, 6), (3, 5), (3, 6)\}.$$

Note:-

Notice here we are using S to denote our relation, deduce that use of R is not strictly enforced, we can use any letter.

3 Inverse of Relations

Definition 2. Let R be a relation from A to B. Define the inverse relation R^{-1} from B to A as follows:

$$R^{-1} = \{(y, x) \in B \times A \mid (x, y) \in R\}.$$

Suppose we have the sets:

$$A = \{2, 3, 4\} \quad B = \{5, 6, 8\}$$
$$Where: \; x \;\; R \;\; y \iff x | y.$$

Then we have:

$$R = \{(2, 6), (2, 8), (3, 6), (4, 8)\}$$
$$R^{-1} = \{(6, 2), (8, 2), (6, 3), (8, 4)\}.$$

4 Reflexivity, Symmetry, and Transitivity

Definition 3. A binary relation R on a set A is said to be reflexive if every element is related to itself. Formally, a relation R is reflexive if for every $a \in A$, the pair (a, a) is in R.

$$\forall a \in A,\ (a, a) \in R\ .$$

Definition 4. A binary relation R on a set A is said to be symmetric if the relation holds in both directions between any two elements that are related. Formally, R is symmetric if for every $(a, b) \in R$, (b, a) is also in R.

$$\forall (a, b) \in R, (b, a) \in R.$$

Definition 5. A binary relation R on a set A is said to be transitive if the existence of a relation from one element to a second, and from the second element to a third, implies the existence of a relation from the first element to the third. Formally, R is transitive if for every $(a, b) \in R$ and $(b, c) \in R$, (a, c) is also in R.

$$\forall (a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R.$$

5 Properties of equality and less than

5.1 Equality Relation

1. **Reflexive**: For all a, $a = a$.
2. **Symmetric**: If $a = b$, then $b = a$.
3. **Transitive**: If $a = b$ and $b = c$, then $a = c$.

5.2 Less Than Relation

1. **Irreflexive**: For all a, it is not the case that $a < a$.
2. **Asymmetric**: If $a < b$, then it is not the case that $b < a$.
3. **Transitive**: If $a < b$ and $b < c$, then $a < c$.

6 Equivalence Relation

Definition 6. ***R* is an equivalence relation iff *R* is:**

1. Reflexive
2. Symmetric
3. Transitive

7 Equivalence Classes

Definition 7. Let A be a set and R be an equivalence relation on A. For each element in A, the **equivalence class** of a, denoted [a] and called the **class of a** for short, is the set of all elements x in A such that x is related to a by R

$$[a] = \{x \in A \mid x \;\; R \;\; a\}.$$

Suppose we have:

$$A = \{0, 1, 2, 3, 4\}$$
$$R = \{(0,0), (0,4), (1,1), (1,3), (2,2), (3,1), (3,3), (4,0), (4,4)\}.$$

Then an example of a few **equivalence classes** would be:

$$[0] = \{0, 4\}$$
$$[1] = \{1, 3\}$$
$$etc....$$