Calculus II

Chapter 2: Applications of Integration

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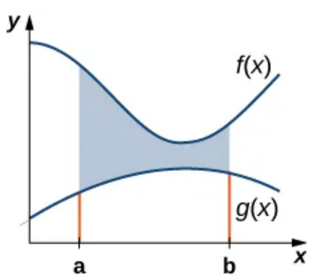
Applications of Integration

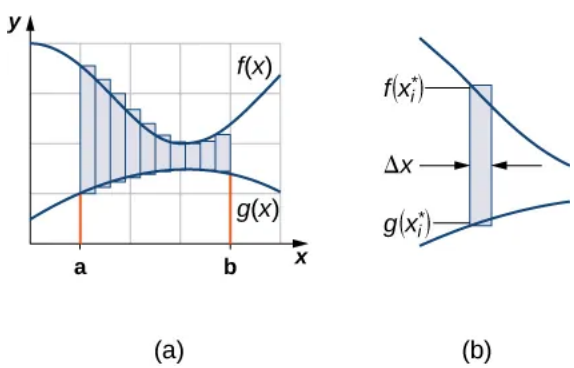
2.1: Areas between Curves

Area of a Region between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$ such that $f(x) \geqslant g(x)$ on $[a, b]$. We want to find the area between the graphs of the functions, as shown in the following figure.



As we did before, we are going to partition the interval on the x-axis and approximate the area between the graphs of the functions with rectangles. So, for $i = 0, 1, 2, \ldots, n$, let $P = \{x_i\}$ be a regular partition of $[a, b]$. Then, for $i = 1, 2, \ldots, n$, choose a point $x_i^* \in [x_{i-1}, x_i]$, and on each interval $[x_{i-1}, x_i]$ construct a rectangle that extends vertically from $g(x_i^*)$ to $f(x_i^*)$. Figure 2.3(a) shows the rectangles when x_i^* is selected to be the left endpoint of the interval and $n = 10$.



The height of each individual rectangle is $f(x_i^*) - g(x_i^*)$ and the width of each rectangle is Δx. Adding the areas of all the rectangles, we see that the area between the curves is approximated by

$$A \approx \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\, \Delta x.$$

This is a Riemann sum, so we take the limit as $n \to \infty$ and we get

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\, \Delta x = \int_a^b [f(x) - g(x)]\; dx.$$

These findings are summarized in the following theorem.

Theorem 1

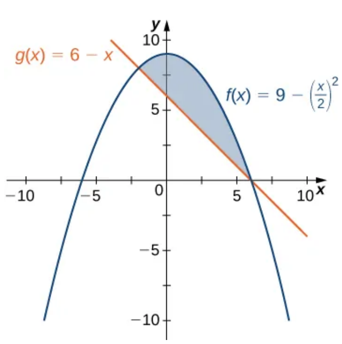
Let $f(x)$ and $g(x)$ be continuous functions such that $f(x) \geqslant g(x)$ over an interval $[a, b]$. Let R denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of R is given by

$$A = \int_a^b [f(x) - g(x)]\; dx.$$

Quite often, though, we want to define our interval of interest based on where the graphs of the two functions intersect. This is illustrated in the following example.

Example 1

If R is the region bounded above by the graph of the function $f(x) = 9 - \left(\frac{x}{2}\right)^2$ and below by the graph of the function $g(x) = 6 - x$, find the area of region R.



To find our limits of integration, we first need to find where the two functions intersect.

$$\begin{aligned}
&= 9 - \left(\frac{x}{2}\right)^2 = 6 - x \\
&= 9 - \frac{x^2}{4} = 6 - x \\
&= \frac{36 - x^2}{4} = 6 - x \\
&= 36 - x^2 = 24 - 4x \\
&= -x^2 + 4x + 12 \\
&= -(x^2 - 4x - 12) = 0 \\
&= -(x - 6)(x + 2) \\
&= x = 6, -2.
\end{aligned}$$

Thus, we need to integrate from x=-2 to 6

By Theorem 1: $A = \int_a^b \left[f(x) - g(x)\right]\, dx$, we find that:

$$A = \frac{64}{3}\; Units^2.$$

Areas of Compound Regions

So far, we have required $f(x) \geqslant g(x)$ over the entire interval of interest, but what if we want to look at regions bounded by the graphs of functions that cross one another? In that case, we modify the process we just developed by using the absolute value function.

Theorem 2

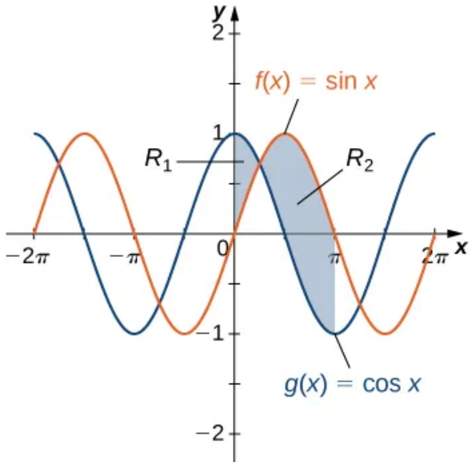
Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$. Let R denote the region between the graphs of $f(x)$ and $g(x)$, and be bounded on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of R is given by

$$\int_a^b |f(x) - g(x)| \, dx.$$

In practice, applying this theorem requires us to break up the interval [a,b] and evaluate several integrals, depending on which of the function values is greater over a given part of the interval. We study this process in the following example.

Example 2

If R is the region between the graphs of the functions $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[0, \pi]$, find the area of region R.



By theorem 2, we can find the area A, in the region R with the following integral:

$$\begin{aligned} &\int_a^b |f(x) - g(x)| \, dx \\ &= \int_0^\pi |\sin x - \cos x| \, dx \end{aligned}$$

.

First, we need to find where the sign of the expression changes:

$$\begin{aligned} &= \sin x - \cos x = 0 \\ &= \frac{\sin x}{\cos x} = \frac{\cos x}{\cos x} \\ &= \tan x = 1 \\ x &= \tan^{-1} 1, \qquad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ x &= \frac{\pi}{4}. \end{aligned}$$

Thus:

$$\sin x - \cos x = \begin{cases} \sin x - \cos x & \text{if } x \geqslant \frac{\pi}{4} \\ -(\sin x - \cos x) & \text{if } x < \frac{\pi}{4} \end{cases} \tag{1}$$

So our integral is now:

$$\begin{aligned}
&\int_0^{\frac{\pi}{4}} -(\sin x - \cos x)dx + \int_{\frac{\pi}{4}}^{\pi} \sin x - \cos x \ dx \\
&= \int_0^{\frac{\pi}{4}} \cos x - \sin x \ dx + \int_{\frac{\pi}{4}}^{\pi} \sin x - \cos x \ dx \\
&Where: \ I_1 = \int_0^{\frac{\pi}{4}} \cos x - \sin x \ dx \\
&I_2 = \int_{\frac{\pi}{4}}^{\pi} \sin x - \cos x \ dx \\
&A = I_1 + I_2.
\end{aligned}$$

Computing both integrals we get:

$$\begin{aligned}
I_1 &= \int_0^{\frac{\pi}{4}} \cos x - \sin x \ dx \\
&= \sin x + \cos x \Big]_{\frac{\pi}{4}}^{0} \\
&= \left(\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)\right) - \left(\sin(0) + \cos 0 \right) \\
&= \sqrt{2} - 1.
\end{aligned}$$

$$\begin{aligned}
&= I_2 = \int_{\frac{\pi}{4}}^{\pi} \sin x - \cos x \ dx \\
&= -\cos x - \sin x \Big]_{\pi}^{\frac{\pi}{4}} \\
&= -\Big[\cos x + \sin x \Big]_{\frac{\pi}{4}}^{\pi} \\
&= -\left[\left(\cos \pi + \sin \pi \right) - \left(\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)\right)\right] \\
&= -\left[-1 - \sqrt{2} \right] \\
&= 1 + \sqrt{2}.
\end{aligned}$$

Thus:

$$\begin{aligned}
A &= I_1 + I_2 \\
&= \sqrt{2} - 1 + 1 + \sqrt{2} \\
&= 2\sqrt{2} \quad Units^2.
\end{aligned}$$

Finding the Area of a Complex Region

Definition 1:

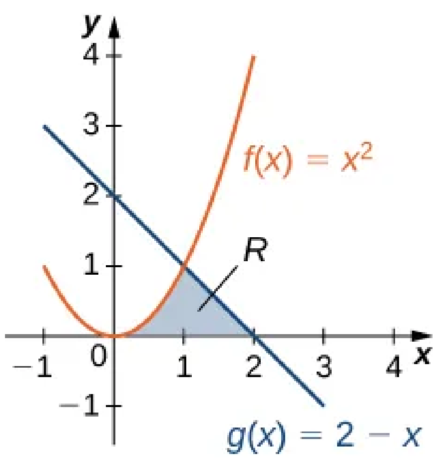
a "complex region" between curves usually refers to an area that is not easily described by a single, continuous function over the interval of interest.

Consider the following example:

Example 3

If R is the region between the graphs of the functions $f(x) = x^2$ and $g(x) = 2 - x$ over the interval $[0, 2]$, find the area of region R.

Figure 2.7



As with Example 2.3, we need to divide the interval into two pieces. The graphs of the functions intersect at $x = 1$ (set $f(x) = g(x)$ and solve for x), so we evaluate two separate integrals: one over the interval $[0, 1]$ and one over the interval $[1, 2]$.

Over the interval $[0, 1]$, the region is bounded above by $f(x) = x^2$ and below by the x-axis, so we have

$$A_1 = \int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}.$$

Over the interval $[1, 2]$, the region is bounded above by $g(x) = 2 - x$ and below by the x-axis, so we have

$$A_2 = \int_2^1 (2 - x)\,dx = \left[2x - \frac{x^2}{2}\right]_2^1 = \frac{1}{2}.$$

Adding these areas together, we obtain

$$A = A_1 + A_2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

The area of the region is $\frac{5}{6}$ units^2.

Regions Defined with Respect to y

In Example 3, we had to evaluate two separate integrals to calculate the area of the region. However, there is another approach that requires only one integral. What if we treat the curves as functions of y, instead of as functions of x?

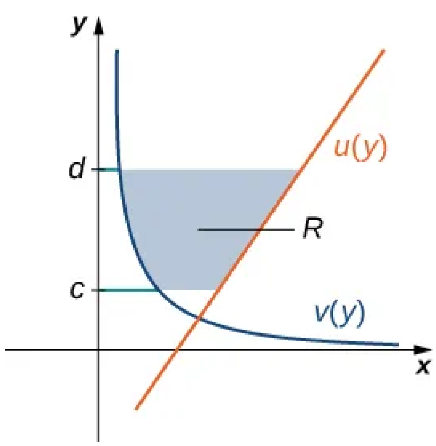
Review Figure 2.7. Note that the left graph, shown in red, is represented by the function $y = f(x) = x^2$. We could just as easily solve this for x and represent the curve by the function $x = v(y) = \sqrt{y}$.

Note:-

(Note that $x = -\sqrt{y}$ is also a valid representation of the function $y = f(x) = x^2$ as a function of y. However, based on the graph, it is clear we are interested in the positive square root.)

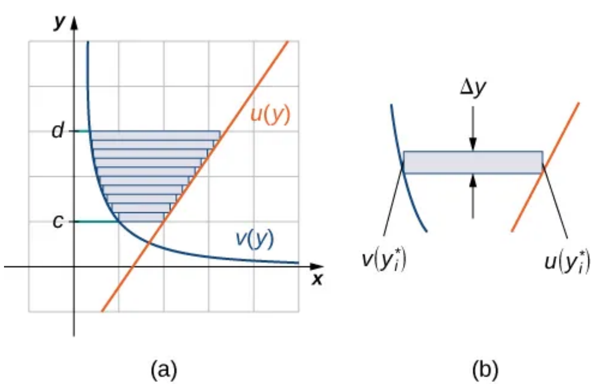
Similarly, the right graph is represented by the function $y = g(x) = 2 - x$, but could just as easily be represented by the function $x = u(y) = 2 - y$.

When the graphs are represented as functions of y, we see the region is bounded on the left by the graph of one function and on the right by the graph of the other function. Therefore, if we integrate with respect to y, we need to evaluate one integral only. Let's develop a formula for this type of integration.



Let $u(y)$ and $v(y)$ be continuous functions over an interval $[c, d]$ such that $u(y) \geqslant v(y)$ for all $y \in [c, d]$. We want to find the area between the graphs of the functions, as shown in the following figure.

Figure 2.9



This time, we are going to partition the interval on the y-axis and use horizontal rectangles to approximate the area between the functions. So, for $i = 0, 1, 2, \ldots, n$, let $Q = \{y_i\}$ be a regular partition of $[c, d]$. Then, for $i = 1, 2, \ldots, n$, choose a point $y_i^* \in [y_{i-1}, y_i]$, then over each interval $[y_{i-1}, y_i]$ construct a rectangle that extends horizontally from $v(y_i^*)$ to $u(y_i^*)$.

Note:-

Figure 2.9(a) shows the rectangles when y_i^* is selected to be the lower endpoint of the interval and $n = 10$. Figure 2.9(b) shows a representative rectangle in detail.

The height of each individual rectangle is Δy and the width of each rectangle is $u(y_i^*) - v(y_i^*)$. Therefore, the area between the curves is approximately

$$A \approx \sum_{i=1}^{n} [u(y_i^*) - v(y_i^*)]\Delta y$$

This is a Riemann sum, so we take the limit as $n \to \infty$, obtaining

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [u(y_i^*) - v(y_i^*)]\Delta y = \int_c^d [u(y) - v(y)]\, dy$$

These findings are summarized in the following theorem.

Theorem 3

Let $u(y)$ and $v(y)$ be continuous functions such that $u(y) \geqslant v(y)$ for all $y \in [c, d]$. Let R denote the region bounded on the right by the graph of $u(y)$, on the left by the graph of $v(y)$, and above and below by the lines $y = d$ and $y = c$, respectively. Then, the area of R is given by

$$A = \int_c^d [u(y) - v(y)]\, dy$$

2.2 Determining Volumes by Slicing

Volume and the Slicing Method

Just as area is the numerical measure of a two-dimensional region, volume is the numerical measure of a three-dimensional solid. Most of us have computed volumes of solids by using basic geometric formulas. The volume of a rectangular solid, for example, can be computed by multiplying length, width, and height: $V = l \times w \times h$.

The formulas for the volume of a sphere $\left(V = \frac{4}{3}\pi r^3\right)$, a cone $\left(V = \frac{1}{3}\pi r^2 h\right)$, and a pyramid $\left(V = \frac{1}{3}Ah\right)$ have also been introduced. Although some of these formulas were derived using geometry alone, all these formulas can be obtained by using integration.

We can also calculate the volume of a cylinder. Although most of us think of a cylinder as having a circular base, such as a soup can or a metal rod, in mathematics the word cylinder has a more general meaning. To discuss cylinders in this more general context, we first need to define some vocabulary.

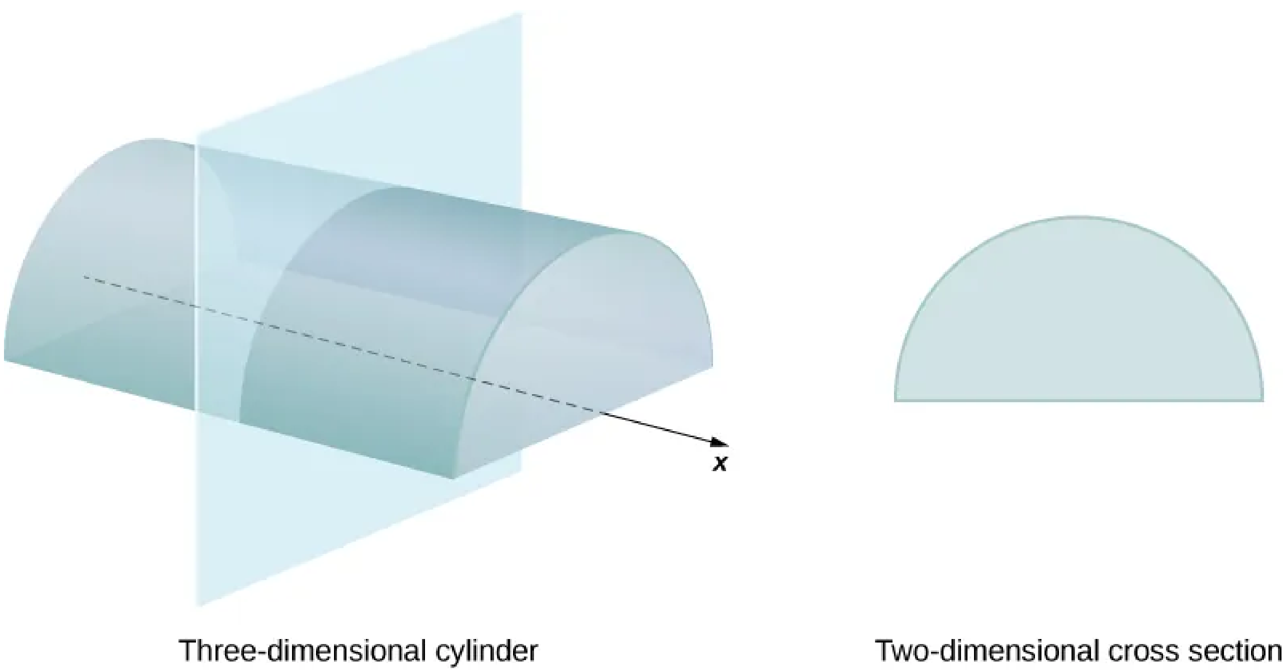
Definition 2:

We define the **cross-section** of a solid to be the intersection of a plane with the solid. A **cylinder** is defined as any solid that can be generated by translating a plane region along a line perpendicular to the region, called the **axis** of the cylinder. Thus, all cross-sections perpendicular to the axis of a cylinder are identical.

In the case of a right circular cylinder (soup can), this becomes $V = \pi r^2 h$.

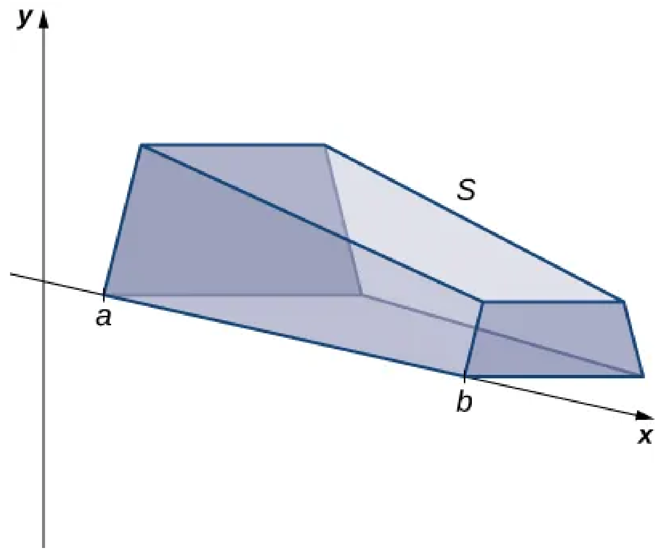
The solid shown in Figure 2.11 is an example of a cylinder with a noncircular base. To calculate the volume of a cylinder, then, we simply multiply the area of the cross-section by the height of the cylinder: $V = A \cdot h$.

Figure 2.11



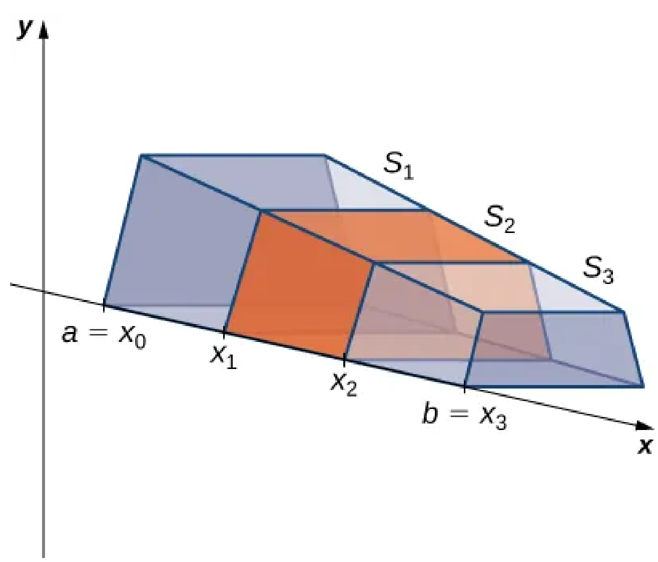
If a solid does not have a constant cross-section (and it is not one of the other basic solids), we may not have a formula for its volume. In this case, we can use a definite integral to calculate the volume of the solid. We do this by slicing the solid into pieces, estimating the volume of each slice, and then adding those estimated volumes together. The slices should all be parallel to one another, and when we put all the slices together, we should get the whole solid. Consider, for example, the solid S shown in Figure 2.12, extending along the x-axis.

Figure 2.12 *A solid with a varying cross-section.*



We want to divide S into slices perpendicular to the x-axis. As we see later in the chapter, there may be times when we want to slice the solid in some other direction—say, with slices perpendicular to the y-axis. The decision of which way to slice the solid is very important. If we make the wrong choice, the computations can get quite messy. Later in the chapter, we examine some of these situations in detail and look at how to decide which way to slice the solid. For the purposes of this section, however, we use slices perpendicular to the x-axis.

Because the cross-sectional area is not constant, we let $A(x)$ represent the area of the cross-section at point x. Now let $P = \{x_0, x_1, \ldots, x_n\}$ be a regular partition of $[a, b]$, and for $i = 1, 2, \ldots, n$, let S_i represent the slice of S stretching from x_{i-1} to x_i. The following figure shows the sliced solid with $n = 3$.



Finally, for $i = 1, 2, \ldots, n$, let x_i^* be an arbitrary point in $[x_{i-1}, x_i]$. Then the volume of slice S_i can be estimated by $V(S_i) \approx A(x_i^*)\Delta x$. Adding these approximations together, we see the volume of the entire solid S can be approximated by

$$V(s) = \sum_{i=1}^{n} A(x_i^*)\ \Delta x.$$

By now, we can recognize this as a Riemann sum, and our next step is to take the limit as $n \to \infty$. Thus, we have:

$$V(s) = \sum_{i=1}^{n} A(x_i^*)\ \Delta x = \int_a^b A(x)\ dx.$$

Thus, we can summarize our findings as:

Theorem 4: Slicing Method

$$V(s) = \sum_{i=1}^{n} A(x_i^*)\ \Delta x = \int_a^b A(x)\ dx.$$

Definition 3:

The technique we have just described is called the **slicing method**. To apply it, we use the following strategy.

1. Examine the solid and determine the shape of a cross-section of the solid. It is often helpful to draw a picture if one is not provided.
2. Determine a formula for the area of the cross-section.
3. Integrate the area formula over the appropriate interval to get the volume.

Note:-

For now, we assume the slices are perpendicular to the x-axis. Therefore, the area formula is in terms of x and the limits of integration lie on the x-axis. However, the problem-solving strategy shown here is valid regardless of how we choose to slice the solid.

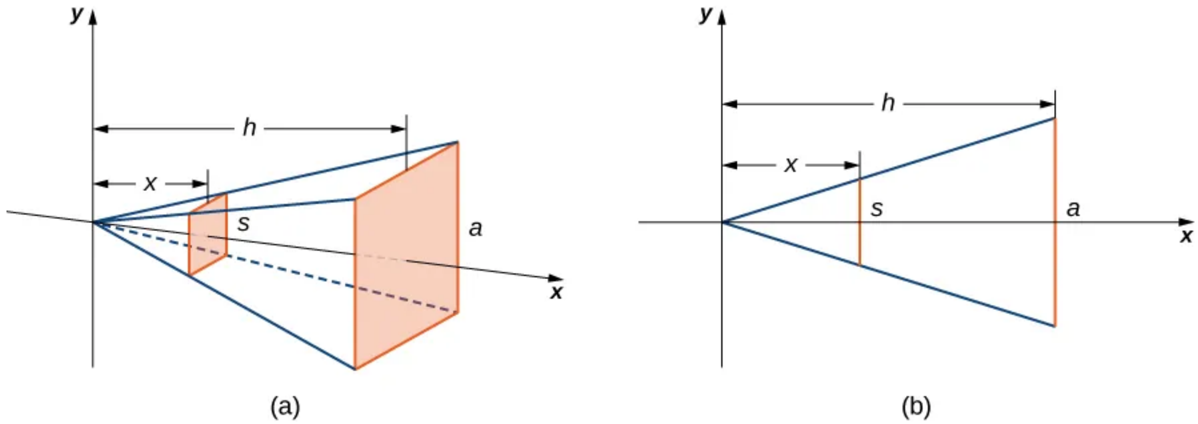
Example 4: Deriving the Formula for the Volume of a Pyramid

We know from geometry that the formula for the volume of a pyramid is $V = \frac{1}{3}Ah$ If the pyramid has a square base, this becomes $V = \frac{1}{3}a^2h$ where a denotes the length of one side of the base. We are going to use the slicing method to derive this formula.

Solution:

We want to apply the slicing method to a pyramid with a square base. To set up the integral, consider the pyramid shown in Figure 2.14, oriented along the x-axis.

Figure 2.14: *A pyramid with a square base is oriented along the x-axis. (b) A two-dimensional view of the pyramid is seen from the side.*



We first want to determine the shape of a cross-section of the pyramid. We know the base is a square, so the cross-sections are squares as well (step 1). Now we want to determine a formula for the area of one of these cross-sectional squares. Looking at Figure 2.14(b), and using a proportion, since these are similar triangles, we have

$$\frac{s}{a} = \frac{x}{h} \quad s = \frac{ax}{h}.$$

Therefore, the area of one of the cross-sectional squares is

$$A(x) = s^2 = \left(\frac{ax}{h}\right)^2 \quad \text{(Step 2.)}.$$

Then we find the volume of the pyramid by integrating from 0 to h (step 3):

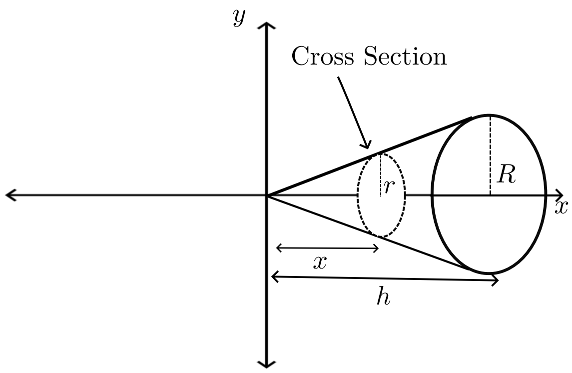
$$\begin{aligned} V &= \int_0^h A(x)\,dx \\ &= \int_0^h \left(\frac{ax}{h}\right)^2 dx \\ &= \int_0^h \frac{a^2x^2}{h^2}\,dx \\ &= \frac{a^2}{h^2}\int_0^h x^2\,dx \\ &= \left[\frac{a^2}{h^2}\left(\frac{1}{3}x^3\right)\right]_h^0 \\ &= \frac{1}{3}a^2h. \end{aligned}$$

This is the formula we were looking for.

Example 5

Use the slicing method to derive the formula for the volume of a cone ($V = \frac{1}{3}\pi r^2 h$)

Let's begin by construing a diagram:



Now our goal is find a function $A(x)$, which represents the area of the cross section (in this case, a circle), as our cross section gets smaller and smaller. Then, like we did with the square pyramid, we can use the fact that these our similar triangles to find the equation for r. Thus:

$$\frac{r}{x} = \frac{R}{h}$$
$$r = \frac{Rx}{h}.$$

Since we know that the area of a circle is πr^2, we can deduce that our $A(x)$ will be:

$$A(x) = \pi r^2$$
$$A(x) = \pi\left(\frac{Rx}{h}\right)^2.$$

Now by the slice method, which states: $V = \int_a^b\ A(x)\ dx$, we have:

$$\begin{aligned}
V &= \int_0^h\ \pi\left(\frac{Rx}{h}\right)^2\ dx \\
V &= \int_0^h\ \pi\frac{R^2x^2}{h^2}\ dx \\
V &= \frac{\pi R^2}{h^2}\int_0^h\ x^2\ dx \\
&= \frac{\pi R^2}{h^2}\left[\frac{1}{3}x^3\right]_h^0 \\
&= \frac{\pi R^2}{h^2}\left(\frac{1}{3}h^3\right) \\
&= \frac{1}{3}\pi R^2 h.
\end{aligned}$$

Solids of Revolution