Calculus II
Chapter 1: Integration

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Contents

Preface

Much of this chapter has been omitted, as it is mostly review and discussed in other documents. Topics of discussion in chapter I include:

- Approximating Areas

- The Definite Integral

- The Fundamental Theorem of Calculus

- Integration Formulas and the Net Change Theorem

- Substitution

- Integrals Involving Exponential and Logarithmic Functions

This document will **only** contain notes for section *1.7: Integrals Resulting in Inverse Trigonometric Functions*

Chapter I: Integration

1.7: Integrals Resulting in Inverse Trigonometric Functions

Recall that trigonometric functions are not one-to-one unless the domains are restricted. When working with inverses of trigonometric functions, we always need to be careful to take these restrictions into account. Also in Derivatives, we developed formulas for derivatives of inverse trigonometric functions. The formulas developed there give rise directly to integration formulas involving inverse trigonometric functions.

The following integration formulas yield inverse trigonometric functions.

1.

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{|a|} + C.$$

2.

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C.$$

3.

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{|a|} \sec^{-1} \frac{|u|}{a} + C.$$

If we Assume $a > 0$, then it is acceptable to drop the absolute value bars.

The following example can be found in exercise 429 of chapter 1.7.

Example 1.

$$\int_0^{\frac{1}{2}} \frac{\tan\left(\sin^{-1} t\right)}{\sqrt{1-t^2}}\, dt$$

.

Proposition. $\tan\left(\sin^{-1} x\right) = \frac{x}{\sqrt{1-x^2}}$

Proof. Show that $\tan\left(\sin^{-1} x\right) = \frac{x}{\sqrt{1-x^2}}$
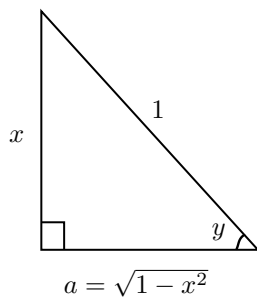
$$y = \sin^{-1} x$$
$$\sin y = x$$
$$\sin y = \frac{x}{1}$$
$$Where \ \sin x = \frac{opp}{hyp}.$$

Now if we draw a right triangle

Then we can solve for the unknown side c:

$$x^2 + a^2 = 1^2$$
$$a^2 = 1 - x^2$$
$$a = \sqrt{1-x^2}.$$

Thus:

$$\tan x = \frac{opp}{adj}$$
$$\tan y = \frac{x}{\sqrt{1-x^2}}$$
$$\implies \tan\left(\sin^{-1} x\right) = \frac{x}{\sqrt{1-x^2}}.$$

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Using this, we can compute the integral.

$$\int_0^{\frac{1}{2}} \frac{\tan\left(\sin^{-1} t\right)}{\sqrt{1-t^2}}\, dt$$
$$= \int_0^{\frac{1}{2}} \frac{\frac{t}{\sqrt{1-t^2}}}{\sqrt{1-t^2}}\, dt$$
$$= \int_0^{\frac{1}{2}} \frac{t}{\sqrt{1-t^2}\sqrt{1-t^2}}\, dt$$
$$= \int_0^{\frac{1}{2}} \frac{t}{1-t^2}\, dt.$$

$$\text{Let } u = 1 - t^2$$
$$du = -2t \ dt$$
$$-\frac{1}{2}du = t \ dt$$
$$u(a) = 1 - (0)^2 = 1$$
$$u(b) = 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}.$$

$$= -\frac{1}{2}\int_1^{\frac{3}{4}} u^{-1} \ du$$
$$= -\frac{1}{2}\left[\ln|u|\right]_1^{\frac{3}{4}}$$
$$= -\frac{1}{2}\left(\ln\left|\frac{3}{4}\right| - \ln|1|\right)$$
$$= -\frac{1}{2}\ln\left|\frac{3}{4}\right|.$$

0.1 Integrals Resulting in Other Inverse Trigonometric Functions

There are six inverse trigonometric functions. However, only three integration formulas are noted in the rule on integration formulas resulting in inverse trigonometric functions because the remaining three are negative versions of the ones we use. The only difference is whether the integrand is positive or negative. Rather than memorizing three more formulas, if the integrand is negative, simply factor out -1 and evaluate the integral using one of the formulas already provided.

1 Additional Information from chapter I

1.1 The Mean Value Theorem for Integrals

The Mean Value Theorem for Integrals states that a continuous function on a closed interval takes on its average value at some point in that interval.

Definition 1. If $f(x)$ is continuous over an interval $[a, b]$, then there is at least one point $c \in [a, b]$ such that

$$f(c) = \frac{1}{b - a} \int_a^b f(x) \; dx.$$

This formula can also be stated as:

$$\int_a^b f(x) \; dx = f(c)(b - a).$$

1.2 Integration Formulas Involving Logarithmic Functions

1.

$$\int \log_a x = \frac{x}{\ln a}(\ln x - 1) + C.$$

2.

$$\int \ln x \; dx = x \ln x - x + C.$$