Exam 1

Nathan Warner

Northern Illinois University

Computer Science
Northern Illinois University
United States

Contents

Axioms

Axiom of distance: For all points P, Q

1. $PQ \geqslant 0$
2. $PQ = 0 \iff P = Q$
3. $PQ = QP$

Axioms of incidence

1. There are at least two different lines
2. Each line contains at least two different points
3. Each pair of points are together in at least one line
4. Each pair of points P, Q, with $PQ < \omega$ are together in at most one line

Betweenness of points axiom (Ax. BP): If A, B, C are distinct, collinear points, and if $AB + BC \leqslant \omega$, then there exists a betweenness relation among A, B, C

What this is really saying is that if **any** of $AB + BC$, $BA + AC$, $AC + CB$ is $\leqslant \omega$, then there is a betweenness relation.

Note: If Ax.BP is true for a plane $\mathbb{P}$, and if $AB + BC \leqslant \omega$ for distinct collinear A, B, C, then there is a betweenness relation, but not necessarily A-B-C

When $\omega = \infty$, then for any distinct collinear A, B, C, $AB + BC < \infty = \omega$, so there will be a betweenness relation

Quadrichotomy Axiom for Points (Ax.QP): If A, B, C, X are distinct, collinear points, and if A-B-C. Then, at least one of the following must hold

$$X\text{-}A\text{-}B, \quad A\text{-}X\text{-}B, \quad B\text{-}X\text{-}C, \quad \text{or} \quad B\text{-}C\text{-}X$$

Thus, Ax.QP says that whenever A-B-C (say on line ℓ), then any other point X on line ℓ is in either $\overrightarrow{BA}$ or $\overrightarrow{BC}$. That is,

$$\ell = \overrightarrow{BA} \cup \overrightarrow{BC}$$

Nontriviality Axiom (Ax.N): For any point A on a line ℓ there exists a point B on ℓ with $0 < AB < \omega$

This axiom is true for the planes in which $\omega = \infty$ ($\mathbb{E}$, $\mathbb{M}$, $\mathbb{H}$, $\mathbb{G}$, $\mathbb{R}^3$, $\hat{\mathbb{E}}$, ws)

This axiom is also true for $\mathbb{S}$ and Fano, where $\omega < \infty$

Real ray Axiom (Ax.RR): For any ray $\overrightarrow{AB}$, and for any real number s with $0 \leqslant s \leqslant \omega$, there is a point X in $\overrightarrow{AB}$ with $AX = s$

Separation Axiom Ax.S: for each line m, there exists a pair of opposite halfplanes with edge m.

Definitions

- **Definition (Endpoints)**. Point A is called an endpoint of ray $\overrightarrow{AB}$
- **Definition (Interior points and length for a segment):** Given a segment $\overline{AB}$, A and B are called its endpoints. All other points of $\overline{AB}$ are called **Interior points** of $\overline{AB}$

  Distance AB is called the **length** of $\overline{AB}$

  The interior of $\overline{AB}$, denoted $\text{Int}\overline{AB}$ or $\overline{AB}^0$, means the set of all interior points of $\overline{AB}$. That is, $\text{Int}\overline{AB} = \overline{AB}^0 = \{X : A\text{-}X\text{-}B\}$
- **Definition**. Assume $\omega < \infty$. Let A be a point on a line m. The unique point A_m^* on m such that $AA_m^* = \omega$ is called the **antipode** of A on m. Thus,

$$\begin{cases} A, A_m^* \text{ are on m, } AA_m^* = \omega \\ \text{and } A\text{-}X\text{-}A_m^* \text{ for all other points } X \text{ on } m \end{cases}$$

- **Definition (interior points of a ray)**: Let $h = \overrightarrow{AB}$ be a ray. All points of h that are not endpoints of h are called *interior points* of h.

  The *interior* of h is the set of all interior points of h, and is denoted by h°, $\overline{AB}^\circ$, or Int $\overrightarrow{AB}$.
- **Definition (Opposite rays)**: Two rays with the same endpoint whose union is a line are called **opposite rays**
- **Notation:** Denote the ray opposite to ray h by h'. So, $\overrightarrow{AB}'$ means the ray opposite $\overrightarrow{AB}$
- **Definition**: Let H, K be opposite halfplanes with edge m. Two points in the same halfplane are said to be on the **same side** of m.
- **Definition**: A^* is called the **antipode** of A

Theorems

- **Theorem 6.1 (Symmetry of betweenness)**. For a general plane $\mathbb{P}$ with points, lines, distance, and satisfy the seven axioms, $A - B - C \iff C - B - A$
- **Theorem 6.2 (UMT)**: If $A - B - C$ then $B - A - C$ and $A - C - B$ are false.
- **Theorem 7.6**: For any point A on a line ℓ there exists a point C not on ℓ with $0 < AC < \omega$
- **Triangle inequality for the line**: If A, B, C are any three distinct, collinear points, then

$$AB + BC \geqslant AC$$

- **Rule of insertion**:
  - If A-B-C and A-X-B, then A-X-B-C
  - If A-B-C and B-X-C, then A-B-X-C
- **Theorem 8.1**: If $\omega = \infty$, then $\mathbb{D} = [0, \infty)$; if $\omega < \infty$, then $\mathbb{D} = [0, \omega]$
- **Theorem 8.2** Each segment, ray, and line has infinitely many points.
- **Theorem 8.3**. If $X \neq Y$ are points different from A on ray $\overrightarrow{AB}$, then one of A-X-Y or A-Y-X is true.
- **Theorem 8.4**. If C is any point on ray $\overrightarrow{AB}$ with $0 < AC < \omega$, then $\overrightarrow{AC} = \overrightarrow{AB}$
- **Theorem 8.6 (UDR)** For any ray $\overrightarrow{AB}$ and any real number s with $0 \leqslant s \leqslant \omega$, there is a **unique** point X on $\overrightarrow{AB}$ with $AX = s$. X is in $\overline{AB}$ if and only if $s \leqslant AB$
- **Theorem 9.1 (Antipode on line theorem)**: Let A be a point on a line m (in a plane with the 11 axioms). Assume that $\omega < \infty$. Then, there exists a unique point A^*_m on m such that $AA^*_m = \omega$. Further, if X is any other point on m, then A-X-A^*_m
- **Theorem 9.2 (Almost-uniqueness for Quadrichotomy)**: Suppose that A, B, C, X are distinct points on a line m, and that $A - B - C$. Then ***exactly one*** of the following holds:

  $$X - A - B, \quad A - X - B, \quad B - X - C, \quad B - C - X$$

  with the ***only exception*** that both $X - A - B$ and $B - C - X$ are true when $\omega < \infty$ and $X = B^*_m$.

  (Note that $B^*_m - A - B$ and $B - C - B^*_m$ ***are both true*** by Thm. 9.1)
- **Theorem 9.4**. If h is a ray with two endpoints A and P, then $\omega < \infty$ and $P = A^*_m$, where m is the carrier of h ($h \subseteq m$).
- **Theorem 9.6 (Opposite ray theorem)**: If B-A-C, then $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays

Also, for $m = \overleftrightarrow{AB}$

$$\overrightarrow{AB} \cap \overrightarrow{AC} = \begin{cases} \{A\} & \text{if } \omega = \infty \\ \{A, A_m^*\} & \text{if } \omega < \infty \end{cases}$$

- **Corollary 9.7**: Each ray has a unique opposite ray.
- **Corollary 9.8**: Let A, B be points on line m with $0 < AB < \omega < \infty$. Then $\overrightarrow{AB'} = \overrightarrow{AB_m^*}$
- **Corollary 9.9**: Let A, B be points on line m with $0 < AB < \omega < \infty$. Then, $m = \overline{AB} \cup \overline{BA_m^*} \cup \overline{A_m^* B_m^*} \cup \overline{B_m^* A}$, with the interiors of these segments being disjoint.
- **Theorem 9.10**: Let A, B be points on line m with $0 < AB < \omega < \infty$. Let $C \neq A, B, A_m^*, B_m^*$ be another point on m. Then there is no betweenness relation for A, B, C if and only if $C \in \overline{A_m^* B_m^*}^0$
- **Definition**. A subset S of $\mathbb{P}$ is **convex** if for each pair of points $X \neq Y$ in S with $XY < \omega$, $\overline{XY} \subseteq S$ holds.
- **Theorem 10.1**: If S_1 and S_2 are convex sets in $\mathbb{P}$, then so is $S_1 \cap S_2$
- **Theorem 10.2**: Segments, rays, and lines are convex.
- **Definition**: A pair of sets H, K in $\mathbb{P}$ is called **opposed around a line** m if
  - $H, K \neq \varnothing$
  - H, K are convex
  - $H \cap K = \varnothing$
  - $H \cup K = \mathbb{P} - m$
- **Theorem 10.3** Let H, K be sets opposed around a line m in $\mathbb{P}$. Suppose that A, C are points so that $C \in m$, $A \in H$, $AC < \omega$. Then, $\mathrm{Int}\overrightarrow{CA} \subseteq H$, and $\mathrm{Int}\overrightarrow{CA'} \subseteq K$
- **Corollary 10.4**: let H, K be sets opposed around a line m, let A, B be points not on m, with A-X-B for some point $X \in m$. Then, A, B lie one in each of H and K, in some order.
- **Definition**: Let m be a line. Sets H, K are called **opposite halfplanes with edge** m if:

$$\begin{gathered} H, K \text{ are opposed around } m \text{, and whenever } X \in H, Y \in K \text{ and } XY < \omega, \\ \text{then, } \overline{XY} \cap m \neq \varnothing \end{gathered}$$

- **Theorem 10.5**: Suppose that m is a line so that there exists a pair H, K of opposite half planes with edge m. Suppose also that $\omega < \infty$ and A is a point on m. If B is any point in $\mathbb{P}$ with $AB = \omega$, then $B \in m$ (so $B = A_m^*$, and there is only one point B in all of $\mathbb{P}$ with $AB = \omega$)

In other words, let H, K be opposite halfplanes with edge a line m, let $A \in m$, $\omega < \infty$. If $B \in \mathbb{P}$, $AB = \omega$, then $B \in m$, and B unique in $\mathbb{P}$

- **Theorem 10.6**: Suppose that there is a pair H, K of opposite halfplanes with edge m. Let $A \neq B$ be points not on m. Then,

$$A, B \text{ lie one in each of } H, K \iff \text{there is a point } X \text{ on } m \text{ such that } A\text{-}X\text{-}B$$

- **Corollary 10.7 (Needs proof)**: Suppose that there is a pair H, K of opposite halfplanes with edge a line m. Then, H, K is the only pair of sets opposed around m.

- **Theorem 10.8**: Suppose that $\omega < \infty$. For each point A, there is exactly one point A^* in $\mathbb{P}$ with $AA^* = \omega$. Also, every line through A goes through A^* as well.

- **Corollary 10.9**: Suppose that $\omega < \infty$. For any line m and point P, there are just two possibilities:

$$\begin{cases} P, P^* & \text{both on } m \\ P, P^* & \text{on opposite sides of } m \end{cases}$$

- **Theorem 10.10 (Pasch's Axioms) (needs proof)**: Let A, B, C be three non-collinear points. Let X be a point with B-X-C, and m a line through X but not through A, B, or C. Then, exactly one of

  1. m contains a point Y with A-Y-C
  2. m contains a point Z with A-Z-B

- **Theorem 10.11**: Assume that $\omega < \infty$. Then, any two distinct lines must have a point (in fact, a pair of antipodes) in common.

Propositions

- **Proposition 6.3**
  (a) $\overline{AB}$ lies in one line, the line $\overleftrightarrow{AB}$
  (b) $\overline{AB} = \overline{BA}$
  (c) If $x \in \overline{AB}$, with $X \neq B$, then $AX < AB$

- **Proposition 6.4**: Let A, B, C, D be collinear points with $0 < AB < \omega$, $0 < CD < \omega$, and $\overline{AB} = \overline{CD}$, then
  (a) Either $\{A, B\} = \{C, D\}$ or $\{A, B\} \cap \{C, D\} = \varnothing$
  (b) $AB = CD$

- **Proposition 7.1**: If A-B-C and A-C-D, then A, B, C, D are distinct and collinear

- **Proposition 7.2** If A-B-C-D, then A, B, C, D are distinct and collinear, and D-C-B-A

- **Proposition 7.5**: If $X \neq Y$ are points distinct from A or ray $\overrightarrow{AB}$, then at least one of A-X-Y or A-Y-X or X, Y in $\overline{AB}$ is true.

- **Important fact**: Suppose X is a point on a ray $\overrightarrow{AB}$ in a general plane.
  1. If A-X-B then $AX < AB$
  2. If A-B-X then $AX > AB$
  3. IF $X = B$ then $AX = AB$

- **Proposition 8.11** Let A, B be any two points on line m, with $0 < AB < \omega$. Then, there exists a point C on m with C-A-B and $CB < \omega$.

- **Proposition 8.5**: A ray has at most two endpoints

- **Proposition 8.7**: Let $\overline{AB}$ be a segment and $X, Y \in \overline{AB}$. Then, $XY \leqslant AB$, and if $XY = AB$, then $\{X, Y\} = \{A, B\}$

- **Proposition 8.8** If $\overline{AB} = \overline{CD}$, then $\{A, B\} = \{C, D\}$

- **Proposition 8.9**: In each segment $\overline{AB}$ there is a unique point M, called the **midpoint** of $\overline{AB}$, with the property that $AM = \frac{1}{2}AB$. Further, $AM = MB$

- **Proposition 9.3**: Assume $\omega < \infty$. Let A, B be points on line m with $0 < AB < \omega$. Then
  (a) $\overrightarrow{AB} = \overline{AB} \cup \overline{BA_m^*}$ and $\overline{AB}^\circ \cap \overline{BA_m^*}^\circ = \varnothing$.
  (b) $\overrightarrow{AB} = \overrightarrow{A_m^* B}$, so that if A is an endpoint of a ray with carrier m, then so is A_m^*.

- **Proposition between** Let $\overrightarrow{AB}$ and $\overrightarrow{AC}$ be opposite rays, and points $X \in \mathrm{Int}\overrightarrow{AB}$, $Y \in \mathrm{Int}\overrightarrow{AC}$ with $AX + AY \leqslant \omega$, then X-A-Y

- **Proposition Noncollinear**: If A, B, C are three noncollinear points (not all on the same line), then AB, AC, BC all less than ω.