

Calculus 2
Chapter 3

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Techniques of Integration

3.1 Integration by Parts

Definition 1:

Many students want to know whether there is a product rule for integration. There isn't, but there is a technique based on the product rule for differentiation that allows us to exchange one integral for another. We call this technique **integration by parts**.

The Integration-by-Parts Formula

If, $h(x) = f(x)g(x)$, then by using the product rule, we obtain $h'(x) = f'(x)g(x) + g'(x)f(x)$. Although at first it may seem counterproductive, let's now integrate both sides of this equation:

$$\int h'(x) dx = \int (g(x)f'(x) + f(x)g'(x)) dx.$$

This gives us

$$h(x) = f(x)g(x) = \int g(x)f'(x) dx + \int f(x)g'(x) dx.$$

Now we solve for $\int f(x)g'(x) dx$:

$$\int f(x)g'(x) dx = f(x)g(x) - \int g(x)f'(x) dx.$$

By making the substitutions $u = f(x)$ and $v = g(x)$, which in turn make $du = f'(x) dx$ and $dv = g'(x) dx$, we have the more compact form

$$\int u dv = uv - \int v du.$$

Theorem 1: Integration by Parts

Let $u = f(x)$ and $v = g(x)$ be functions with continuous derivatives. Then, the integration-by-parts formula for the integral involving these two functions is:

$$\int u \, dv = uv - \int v \, du.$$

Example 1: Using Integration by Parts

Use integration by parts with $u = x$ and $dv = \sin x \, dx$ to evaluate

$$\int x \sin x \, dx.$$

Solution: So to use the formula:

$$\int u \, dv = uv - \int v \, du.$$

We need:

$$\begin{aligned} u &= x & du &= dx \\ dv &= \sin x \, dx & v &= -\cos x. \end{aligned}$$

Thus:

$$\begin{aligned} \int x \sin x \, dx &= -x \cos x - \int -\cos x \, dx \\ &= -x \cos x + \sin x + C. \end{aligned}$$

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The natural question to ask at this point is: How do we know how to choose u and dv ? Sometimes it is a matter of trial and error; however, the acronym **LIATE** can often help to take some of the guesswork out of our choices. This acronym stands for

- **L**ogarithmic Functions
- **I**nverse Trigonometric Functions
- **A**lgebraic Functions
- **T**rigonometric Functions
- **E**xponential Functions

This mnemonic serves as an aid in determining an appropriate choice for u .

Note:-

A better version might be LIAET, where exponential and trig functions are swapped

Applying integration by parts more than once

Example 2: Evaluate

$$\int x^2 e^{3x} dx.$$

Solution: By **LIATE**, we let $u = x^2$, and $dv = e^{3x}$. Thus, we get:

$$\begin{aligned} u &= x^2 & dv &= e^{3x} \\ du &= 2x dx & v &= \frac{1}{3} e^{3x}. \end{aligned}$$

Then by theorem 1, we get:

$$\begin{aligned} \int u dv &= uv - \int v du \\ &= \int x^2 e^{3x} dx = x^2 \frac{1}{3} e^{3x} - \int \frac{1}{3} e^{3x} 2x dx \\ &= \int x^2 e^{3x} dx = x^2 \frac{1}{3} e^{3x} - \int \frac{2}{3} e^{3x} x dx \end{aligned}$$

At this point, we will notice that we still cannot evaluate the integral $\int \frac{2}{3} e^{3x} x dx$. Thus, we must apply the theorem once more.

$$\begin{aligned} \int \frac{2}{3} e^{3x} x dx \\ u = x \quad dv = \frac{2}{3} e^{3x} \\ du = dx \quad v = \frac{2}{9} e^{3x}. \end{aligned}$$

Thus:

$$\begin{aligned} \int \frac{2}{3} e^{3x} dx &= \frac{2}{9} e^{3x} x - \int \frac{2}{9} e^{3x} dx \\ &= \frac{2}{9} x e^{3x} - \frac{2}{27} e^{3x}. \end{aligned}$$

In full we have:

$$\begin{aligned} \int x^2 e^{3x} dx &= \frac{1}{3} x^2 e^{3x} - \left(\frac{2}{9} x e^{3x} - \frac{2}{27} e^{3x} \right) \\ &= \frac{1}{3} e^{3x} x^2 - \frac{2}{9} x e^{3x} + \frac{2}{27} e^{3x} + C. \end{aligned}$$



Applying Integration by Parts When LIATE Doesn't Quite Work

Example 3: Evaluate

$$\int t^3 e^{t^2} dt.$$

Solution: If we use a strict interpretation of the mnemonic **LIATE** to make our choice of u , we end up with $u = t^3$ and $dv = e^{t^2} dt$. Unfortunately, this choice won't work because we are unable to evaluate $\int e^{t^2} dt$. However, since we can evaluate $\int te^{t^2} dt$, we can try choosing $u = t^2$ and $dv = te^{t^2} dt$. With these choices we have

$$\begin{aligned} u &= t^2 & dv &= te^{t^2} \\ du &= 2t \, dt & v &= \frac{1}{2}e^{t^2}. \end{aligned}$$

Thus, we obtain:

$$\begin{aligned} \int t^3 e^{t^2} dt &= \frac{1}{2}t^2 e^{t^2} - \int \frac{1}{2}e^{t^2} 2t \, dt \\ &= \frac{1}{2}t^2 e^{t^2} - \int e^{t^2} t \, dt \\ &= \frac{1}{2}t^2 e^{t^2} - \frac{1}{2}e^{t^2} + C. \end{aligned}$$

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Example 4: Evaluate

$$\int \sin(\ln(x)) \, dx.$$

Solution: Here, we let $u = \sin(\ln(x))$ and $dv = 1dx$, so we have:

$$\begin{aligned} u &= \sin(\ln(x)) & dv &= dx \\ du &= \frac{1}{x} \cos(\ln(x)) \, dx & v &= x. \end{aligned}$$

Which gives us:

$$\int \sin(\ln(x)) \, dx = x \sin(\ln(x)) - \int \cos(\ln(x)) \, dx.$$

Which leaves us in no better shape than the original integral, so we apply the theorem once more:

$$\begin{aligned} &\int \cos(\ln(x)) \, dx \\ u &= \cos(\ln(x)) & dv &= dx \\ du &= -\frac{1}{x} \sin(\ln(x)) \, dx & v &= x. \end{aligned}$$

Thus we have:

$$\int \cos(\ln(x)) \, dx = x \cos(\ln(x)) - \int -\sin(\ln(x)) \, dx.$$

At this point, we have:

$$\begin{aligned}
 \int \sin(\ln(x)) \, dx &= x \sin(\ln(x)) - \left(x \cos(\ln(x)) - \int -\sin(\ln(x)) \, dx \right) \\
 &= x \sin(\ln(x)) - \left(x \cos(\ln(x)) + \int \sin(\ln(x)) \, dx \right) \\
 &= x \sin(\ln(x)) - x \cos(\ln(x)) - \int \sin(\ln(x)) \, dx
 \end{aligned}$$

The last integral is now the same as the original. It may seem that we have simply gone in a circle, but now we can actually evaluate the integral. To see how to do this more clearly, substitute:

$$I = \int \sin(\ln(x)) \, dx.$$

Thus, the equation becomes:

$$\begin{aligned}
 I &= x \sin(\ln(x)) - x \cos(\ln(x)) - I \\
 2I &= x \sin(\ln(x)) - x \cos(\ln(x)) \\
 I &= \frac{1}{2}x \sin(\ln(x)) - \frac{1}{2}x \cos(\ln(x)).
 \end{aligned}$$

Substituting back in for I we get:

$$\int \sin(\ln(x)) \, dx = \frac{1}{2}x \sin(\ln(x)) - \frac{1}{2}x \cos(\ln(x)) + C.$$

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Integration by parts for definite integrals

Now that we have used integration by parts successfully to evaluate indefinite integrals, we turn our attention to definite integrals. The integration technique is really the same, only we add a step to evaluate the integral at the upper and lower limits of integration.

Theorem 2: Integration by Parts for Definite Integrals

Let $u = f(x)$ and $v = g(x)$ be functions with continuous derivatives on $[a, b]$. Then:

$$\int_a^b u \, dv = uv \Big|_a^b - \int_a^b v \, du.$$

3.2 Trigonometric Integrals

In this section we look at how to integrate a variety of products of trigonometric functions. These integrals are called trigonometric integrals. They are an important part of the integration technique called trigonometric substitution, which is featured in Trigonometric Substitution. This technique allows us to convert algebraic expressions that we may not be able to integrate into expressions involving trigonometric functions, which we may be able to integrate using the techniques described in this section. In addition, these types of integrals appear frequently when we study polar, cylindrical, and spherical coordinate systems later. Let's begin our study with products of $\sin x$ and $\cos x$.

Integrating $\cos^j x \sin x$

In this case, we can perform a simple u-substitution, where we let $u = \cos x$, and from there we can evaluate.

Example 5: Evaluate

$$\begin{aligned} \int \cos^5 x \sin x \, dx \\ &= - \int u^5 \, du \\ &= -\frac{1}{6} u^6 + C \\ &= -\frac{1}{6} \cos^6 x + C. \end{aligned}$$

Integrating $\cos^j x \sin^k x$ when k is odd

In this case, we can use the trigonometric identity: $\sin^2 x = 1 - \cos^2 x$ to rewrite the expression such that using a u-substitution will work. In general:

$$\begin{aligned} \int \cos^j x \sin^k x \, dx \quad \text{s.t } k = 2l + 1, l \in \mathbb{Z} \\ &= \int \cos^j x (1 - \cos^2 x)^{\frac{k-1}{2}} \sin(x) \, dx. \end{aligned}$$

Example 6: Evaluate

$$\begin{aligned} \int \cos^2 x \sin^5 x \, dx \\ &= \int \cos^2 x (1 - \cos^2 x)^{\frac{5-1}{2}} \sin x \, dx \\ &= \int \cos^2 x (1 - \cos^2 x)^2 \sin x \, dx \\ &= \text{etc....} \end{aligned}$$

Note:-

This fact also holds for $\int \sin^j x \cos^k x$ for $k = 2l + 1, l \in \mathbb{Z}$

Integrating even powers of $\sin x$

In the next example, we see the strategy that must be applied when there are only even powers of $\sin(x)$ and $\cos(x)$. For integrals of this type, the identities

$$\sin^2(x) = \frac{1}{2} - \frac{1}{2} \cos(2x) = \frac{1 - \cos(2x)}{2}$$

and

$$\cos^2(x) = \frac{1}{2} + \frac{1}{2} \cos(2x) = \frac{1 + \cos(2x)}{2}$$

are invaluable. These identities are sometimes known as power-reducing identities and they may be derived from the double-angle identity $\cos(2x) = \cos^2(x) - \sin^2(x)$ and the Pythagorean identity $\cos^2(x) + \sin^2(x) = 1$

Example 7: Evaluate

$$\int \sin^2 x \, dx.$$

By the identity described above, we can derive:

$$\begin{aligned} \cos 2x &= 1 - 2 \sin^2 x \\ \sin^2 x &= \frac{1}{2} - \frac{1}{2} \cos 2x. \end{aligned}$$

Thus we have:

$$\begin{aligned} \int \frac{1}{2} - \frac{1}{2} \cos 2x \, dx \\ = \frac{1}{2}x - \frac{1}{4} \sin 2x + C. \end{aligned}$$

Problem-Solving Strategy: Integrating Products and Powers of $\sin x$ and $\cos x$

To integrate

$$\int \cos^j(x) \sin^k(x) dx$$

use the following strategies:

1. If k is odd, rewrite $\sin^k(x)$ as $\sin^{k-1}(x) \sin(x)$ and use the identity $\sin^2(x) = 1 - \cos^2(x)$ to rewrite $\sin^{k-1}(x)$ in terms of $\cos(x)$. Integrate using the substitution $u = \cos(x)$. This substitution makes $du = -\sin(x) dx$.
2. If j is odd, rewrite $\cos^j(x)$ as $\cos^{j-1}(x) \cos(x)$ and use the identity $\cos^2(x) = 1 - \sin^2(x)$ to rewrite $\cos^{j-1}(x)$ in terms of $\sin(x)$. Integrate using the substitution $u = \sin(x)$. This substitution makes $du = \cos(x) dx$. (Note: If both j and k are odd, either strategy 1 or strategy 2 may be used.)
3. If both j and k are even, use

$$\sin^2(x) = \frac{1}{2} - \frac{1}{2} \cos(2x)$$

and

$$\cos^2(x) = \frac{1}{2} + \frac{1}{2} \cos(2x).$$

After applying these formulas, simplify and reapply strategies 1 through 3 as appropriate.

Power reduction formulas

Theorem 3: Power reduction formulas

- **Power Reduction Formula (sine)**

$$\int \sin^n x dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x dx.$$

- **Power Reduction Formula (cosine)**

$$\int \cos^n x dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x dx.$$

- **Power Reduction Formula (secant)**

$$\begin{aligned} \int \sec^n x dx &= \frac{1}{n-1} \sec^{n-1} x \sin x + \frac{n-2}{n-1} \int \sec^{n-2} x dx \\ \int \sec^n x dx &= \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x dx \end{aligned}$$

- **Power Reduction Formula (Tangent)**

$$\int \tan^n x dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x dx.$$

Integrating products of sines and cosines of different angles

Theorem 4

To integrate products involving $\sin(ax)$, $\sin(bx)$, $\cos(ax)$, and $\cos(bx)$, use the substitutions:

- **Sine Products**

$$\sin(ax) \sin(bx) = \frac{1}{2} \cos((a-b)x) - \frac{1}{2} \cos((a+b)x)$$

- **Sine and Cosine Products**

$$\sin(ax) \cos(bx) = \frac{1}{2} \sin((a-b)x) + \frac{1}{2} \sin((a+b)x)$$

- **Cosine Products**

$$\cos(ax) \cos(bx) = \frac{1}{2} \cos((a-b)x) + \frac{1}{2} \cos((a+b)x)$$

Which are trivial if you know the trigonometric **product to sum** identities

3.3 Trigonometric Substitution

In this section, we explore integrals containing expressions of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$ and $\sqrt{x^2 - a^2}$ where the values of a are positive. We have already encountered and evaluated integrals containing some expressions of this type, but many still remain inaccessible. The technique of trigonometric substitution comes in very handy when evaluating these integrals. This technique uses substitution to rewrite these integrals as trigonometric integrals.

Integrals involving $\sqrt{a^2 - x^2}$

Consider the integral:

$$\int \sqrt{9 - x^2} \, dx.$$

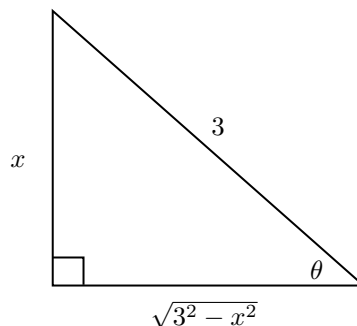
The first thing we can deduce when we see an integral of this form ($\sqrt{a^2 - x^2}$), is that the integrand looks awfully like it could be written as *Pythagoreans theorem* $a^2 + b^2 = c^2$. So, let's draw a triangle and see what we can figure out. But first, let's gather some information...

$$\text{If : } a^2 + b^2 = c^2$$

$$a = \sqrt{b^2 - c^2} \quad \text{Possibility I}$$

$$b = \sqrt{c^2 - a^2} \quad \text{Possibility II.}$$

Thus, we know our full equation $\sqrt{3^2 - x^2}$, must be either side a or b , and the terms inside the square root must be the hypotenuse, and the remaining side. When it comes to choosing which is which, we will base our reasoning on what makes things easiest...



So, we let $a = \sqrt{3^2 - x^2}$, $b = x$, and $c = 3$. The reason we choose our sides this way is because now, we can define the angle θ as $\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{x}{3}$. Thus, $x = 3 \sin \theta$

Side note: This **reference triangle** will come in handy later.

Now, since we have deduced $x = 3 \sin \theta$, we can rewrite our integral as:

$$\int \sqrt{9 - (3 \sin \theta)^2} \, dx.$$

However, we still need to account for our dx . Since we know $x = 3 \sin \theta$, dx must be $3 \cos \theta$. So our integral becomes:

$$\begin{aligned} & \int \sqrt{9 - 9 \sin^2 \theta} \, 3 \cos \theta \, d\theta \\ &= \int \sqrt{9(1 - \sin^2 \theta)} \, 3 \cos \theta \, d\theta \\ &= \int 3\sqrt{1 - \sin^2 \theta} \, 3 \cos \theta \, d\theta \\ &= 9 \int \sqrt{\cos^2 \theta} \, \cos \theta \, d\theta \\ &= 9 \int \cos^2 \theta \, d\theta. \end{aligned}$$

To understand why we can write $\sqrt{\cos^2 \theta}$ as $\cos \theta$, consider the integral involving $\sqrt{a^2 - x^2}$. For the integral to be real-valued, x must lie in the interval $[-a, a]$. When we use the substitution $x = 3 \sin \theta$, this interval corresponds to a θ domain of $[-\pi/2, \pi/2]$. However, if the problem context ensures that x is positive (for instance, due to the domain of integration or other constraints), then our θ domain narrows to $[0, \pi/2]$. In this interval, the cosine function is always non-negative, so $\sqrt{\cos^2 \theta}$ simplifies to $\cos \theta$.

Now we can evaluate the integral:

$$\begin{aligned}
 & 9 \int \cos^2 \theta d\theta \\
 &= 9 \left[\frac{1}{2} \theta + \frac{1}{4} \sin 2\theta \right] + C \\
 &= \frac{9}{2} \theta + \frac{9}{4} \sin 2\theta + C.
 \end{aligned}$$

From here, we must convert back to x's, to do this, we must revisit our reference triangle. We know:

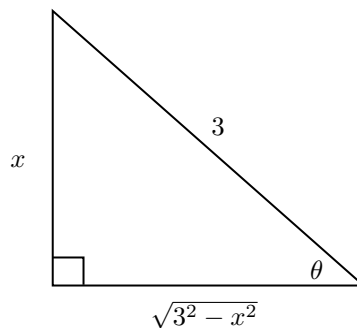
$$\sin \theta = \frac{1}{3}x.$$

Thus:

$$\begin{aligned}
 \theta &= \sin^{-1} \frac{1}{3}x \\
 \sin 2\theta &= \sin \left(2 \sin^{-1} \frac{1}{3}x \right).
 \end{aligned}$$

Therefore, our final answer is:

$$\frac{9}{2} \sin^{-1} \left(\frac{1}{3}x \right) + \frac{9}{4} \sin \left(2 \sin^{-1} \left(\frac{1}{3}x \right) \right) + C.$$



Other Forms

Now that we have walked through the process for integrals of the form $\sqrt{a^2 - x^2}$, let's take a look at the process for the other forms, specifically $\sqrt{x^2 - a^2}$ or $\sqrt{a^2 + x^2}$

When we have an integral in the form of $\sqrt{x^2 - a^2}$, we use the substitution $x = a \sec \theta$ by restricting the domain to $\left(0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$. We use the identity $\sec^2 \theta - 1 = \tan^2 \theta$ to simplify the integrand

When we have an integral in the form of $\sqrt{a^2 + x^2}$, we use the substitution $x = a \tan \theta$ by restricting the domain to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We then use the identity $1 + \tan^2 \theta = \sec^2 \theta$ to simplify the integrand

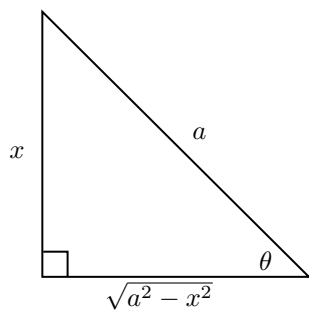
Note:-

These trigonometric substitution forms do not rely on the square root. This means we can still make the substitutions even if there is not a square root

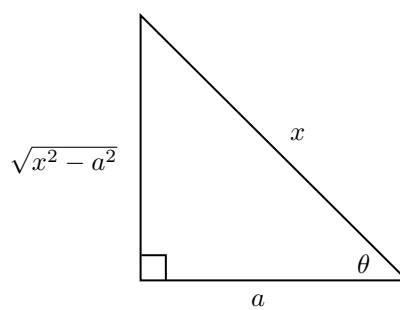
Reference Triangles

It is a good idea to be familiar with all three versions of the reference triangles, this way you don't need to expend effort deducing the sides.

Form 1: $\sqrt{a^2 - x^2}$ where $x = a \sin x$



Form 2: $\sqrt{x^2 - a^2}$ where $x = a \sec x$



Form 3: $\sqrt{a^2 + x^2}$ where $x = a \tan x$

