Nate Warner MATH 336 Spring 2026

Problem set 5 - Due: Monday, February 16

1.

$$x^2y' = xy + y^2.$$

We have

$$x^2y' = xy + y^2 \implies x^2\ dy - (xy + y^2)\ dx = 0.$$

Notice that this is homogeneous, so let $y = xv$, then $y' = xv' + v$, and

$$\begin{aligned} x^2y' = xy + y^2 &\implies x^2(xv' + v) = x^2v + x^2v^2 \\ \implies x^3v' + x^2v = x^2v + x^2v^2 &\implies x^3v' = x^2v^2 \\ \implies xv' = v^2 &\implies \frac{1}{v^2}\ dv = \frac{1}{x}\ dx. \end{aligned}$$

So,

$$\begin{aligned} \int \frac{1}{v^2}\ dv = \int \frac{1}{x}\ dx &\implies -\frac{1}{v} = \ln(|x|) + C \\ \implies v = \frac{1}{C - \ln|x|}&. \end{aligned}$$

Now, we switch back to y, so

$$\frac{y}{x} = \frac{1}{C - \ln(|x|)} \implies y = \frac{x}{C - \ln(|x|)}.$$

2.

$$xyy' = y^2 + x\sqrt{4x^2 + y^2}.$$

We have

$$xy\ dy - (y^2 + x\sqrt{4x^2 + y^2})\ dx = 0.$$

Let $M(x, y) = -(y^2 + x\sqrt{4x^2} + y^2)$, and $N(x, y) = xy$. We check for homogeneity.

$$N(\lambda x, \lambda y) = (\lambda x)(\lambda y) = \lambda^2 xy,$$

and

$$\begin{aligned} M(\lambda x, \lambda y) &= -((\lambda y)^2 + (\lambda x)\sqrt{4(\lambda x)^2 + (\lambda y)^2}) \\ &= -(\lambda^2y^2 + \lambda x\sqrt{4\lambda^2x^2 + \lambda^2y^2}) \\ &= -(\lambda^2y^2 + \lambda^2x\sqrt{4x^2 + y^2}). \end{aligned}$$

Thus, they are homogeneous of the same degree, and the DE is therefore homogeneous. So, let $y = xv$, then $y' = xv' + v$, and

$$\begin{aligned} xyy' = y^2 + x\sqrt{4x^2 + y^2} &\implies x(xv)(xv' + v) = x^2v^2 + x\sqrt{4x^2 + x^2v^2} \\ \implies x^3vv' + x^2v^2 = x^2v^2 + x\sqrt{4x^2 + x^2v^2} &\implies x^3vv' = x\sqrt{4x^2 + x^2v^2} \\ \implies x^3vv' = x(x)\sqrt{4 + v^2} = x^2\sqrt{4 + v^2} &\implies xvv' = \sqrt{4 + v^2}. \end{aligned}$$

So, we can separate to get

$$\frac{v}{\sqrt{4+v^2}}\ dv = \frac{1}{x}\ dx.$$

Thus,

$$\int \frac{v}{\sqrt{4+v^2}}\ dv = \int \frac{1}{x}\ dx = \ln|x| + C.$$

For the LHS integral, Let $u = 4+v^2$, then $\frac{1}{2}\ du = v\ dv$, so

$$\int \frac{v}{\sqrt{4+v^2}}\ dv = \frac{1}{2}\int \frac{1}{\sqrt{u}}\ du = \frac{1}{2}(2)\sqrt{u} = \sqrt{4+v^2}.$$

Thus,

$$\sqrt{4+v^2} = \ln|x| + C.$$

Then,

$$\sqrt{4+\left(\frac{y}{x}\right)^2} = \ln\left(|x|\right) + C$$

is an implicit solution.

3.

$$x^2y' + 2xy = 5y^3.$$

We have

$$y' + \frac{2}{x}y = \frac{5}{x^2}y^3.$$

Which is a Bernoulli DE, so let $v = y^{1-3} = y^{-2}$, then $\frac{dv}{dx} = -\frac{2}{y^3}y'$, so $y' = -\frac{1}{2}y^3v'$. Then,

$$\begin{aligned} -\frac{1}{2}y^3v' + \frac{2}{x}y = \frac{5}{x^2}y^3 &\implies -\frac{1}{2}v' + \frac{2}{x}y^{-2} = \frac{5}{x^2} \\ &\implies v' - \frac{4}{x}v = -\frac{10}{x^2}, \end{aligned}$$

which is linear in v. Now, define

$$\mu(x) = e^{-4\int \frac{1}{x}\ dx} = e^{-4\ln(x)} = \frac{1}{x^4}.$$

So,

$$\left(\frac{1}{x^4}v\right)' = -\frac{10}{x^2}\left(\frac{1}{x^4}\right) = -\frac{10}{x^6}.$$

Thus,

$$\begin{aligned} \frac{1}{x^4}v &= -10\int x^{-6}\ dx = -\frac{10}{-5}x^{-5} + C = \frac{2}{x^5} + C \\ \implies v &= x^4\left(\frac{2}{x^5} + C\right) = \frac{2}{x} + Cx^4 \\ \implies \frac{1}{y^2} &= \frac{2}{x} + Cx^4. \end{aligned}$$

This is an implicit solution.

4.

$$xy' + 6y = 3xy^{\frac{4}{3}}.$$

We have

$$y' + \frac{6}{x}y = 3y^{\frac{4}{3}}.$$

So, let $v = y^{1-\frac{4}{3}} = y^{-\frac{1}{3}}$, then $v' = -\frac{1}{3}y^{-\frac{4}{3}}y'$, so $y' = -3y^{\frac{4}{3}}v'$. Then,

$$\begin{aligned} -3y^{\frac{4}{3}}v' + \frac{6}{x}y = 3y^{\frac{4}{3}} &\implies -3v' + \frac{6}{x}y^{-\frac{1}{3}} = 3 \\ &\implies v' - \frac{2}{x}v = -1. \end{aligned}$$

Notice that this is linear in v, so define

$$\mu(x) = e^{-2\int \frac{1}{x}\,dx} = e^{-2\ln(x)} = \frac{1}{x^2}.$$

Then,

$$\left(\frac{1}{x^2}v\right)' = -\frac{1}{x^2}.$$

So,

$$\frac{1}{x^2}v = -\int \frac{1}{x^2}\,dx \implies \frac{1}{x^2}v = \frac{1}{x} + C.$$

Thus,

$$v = x^2\left(\frac{1}{x} + C\right) = x + Cx^2.$$

Therefore,

$$\frac{1}{y^{\frac{1}{3}}} = x + Cx^2$$

is an implicit solution.

5.

$$(3x^2 + 2y^2)\,dx + (4xy + 6y^2)\,dy = 0.$$

We can check if this DE is exact by letting $M(x, y) = 3x^2 + 2y^2$, and $N(x, y) = 4xy + 6y^2$ and checking if $M_y = N_x$. We have

$$M_y = 4y, \quad N_x = 4y.$$

Thus, the DE is exact. Now, we can find $F(x, y)$ by integrating M with respect to x, since $M = F_x$. So,

$$F(x, y) = \int M\,dx = \int (3x^2 + 2y^2)\,dx = x^3 + 2xy^2 + h(y).$$

Now, we differentiate with respect to y,

$$\frac{\partial F}{\partial y} = 4xy + h'(y).$$

If we set this equal to $F_y = N$, we see that

$$4xy + h'(y) = 4xy + 6y^2.$$

So, $h'(y) = 6y^2$. Therefore,

$$h(y) = \int h'(y)\ dy = \int 6y^2\ dy = 2y^3 + K.$$

Thus,

$$F(x, y) = x^3 + 2xy^2 + 2y^3 + K.$$

But, $F(x, y) = C_0$, so

$$x^3 + 2xy^2 + 2y^3 + K = C_0 \implies x^3 + 2xy^2 + 2y^3 = C$$

Is an implicit solution.

6.

$$\left(x^3 + \frac{y}{x}\right)\ dx + (y^2 + \ln(x))\ dy = 0.$$

We check $M_y(x, y) = \left(x^3 + \frac{y}{x}\right)_y$ against $N_x(x, y) = (y^2 + \ln(x))_x$

$$M_y = \frac{1}{x}, \quad N_x = \frac{1}{x}.$$

Thus, the DE is exact. So,

$$F(x, y) = \int M\ dx = \int \left(x^3 + \frac{y}{x}\right)\ dx = \frac{1}{4}x^4 + y\ln(x) + h(y).$$

So,

$$F_y = \ln(x) + h'(y).$$

Thus,

$$\ln(x) + h'(y) = y^2 + \ln(x) \implies h'(y) = y^2,$$

so $h(y) = \frac{1}{3}y^3$. Now, we have that

$$F(x, y) = \frac{1}{4}x^4 + y\ln(x) + \frac{1}{3}y^3 + K.$$

So, since $F(x, y) = C_0$,

$$\frac{1}{4}x^4 + y\ln(x) + \frac{1}{3}y^3 = C$$

is an implicit solution.

7.

$$(1 + ye^{xy})\ dx + (2y + xe^{xy})\ dy = 0.$$

We have

$$M_y = e^{xy} + xye^{xy}, \quad N_x = e^{xy} + xye^{xy}.$$

Thus, the DE is exact, and

$$F(x, y) = \int (1 + ye^{xy})\ dx = x + \frac{y}{y}e^{xy} + h(y) = x + e^{xy} + h(y).$$

Thus,

$$F_y = xe^{xy} + h'(y).$$

So,

$$xe^{xy} + h'(y) = 2y + xe^{xy} \implies h'(y) = 2y \implies h(y) = y^2 + K.$$

Thus,

$$F(x, y) = x + e^{xy} + y^2 + K,$$

and

$$x + e^{xy} + y^2 = C$$

is an implicit solution.