Comprehensive Compendium:
Calculus II

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1 Calc II

1.1 Chapter 1 Key Equations

- **Mean Value Theorem For Integrals**: If $f(x)$ is continuous over an interval [a,b], then there is at least one point $c \in [a, b]$ such that
$$f(c) = \frac{1}{b-a} \int f(x)\ dx.$$

- **Integrals resulting in inverse trig functions**

1. $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{|a|} + C.$$

2. $$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C.$$

3. $$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{|a|} \sec^{-1} \frac{|x|}{a} + C.$$

1.2 Chapter 2 Key Terms / Ideas

- **Finding limits of integration for region between two functions**: Usually, we want our limits of integration to be the points where the functions intersect
- A **"complex region"** between curves usually refers to an area that is not easily described by a single, continuous function over the interval of interest.
- **compound regions** are regions bounded by the graphs of functions that cross one another
- **Cross-section:** The intersection of a plane and a solid object.
- a **cylinder** is a three-dimensional shape that has two parallel, congruent bases connected by a curved surface. The bases are usually circles, but they can be other shapes as well
- The line segment connecting the centers of the two bases is called the **"axis" of the cylinder.**
- **Slicing method:** A method of calculating the volume of a solid that involves cutting the solid into pieces, estimating the volume of each piece, then adding these estimates to arrive at an estimate of the total volume; as the number of slices goes to infinity, this estimate becomes an integral that gives the exact value of the volume.
  1. Examine the solid and determine the shape of a cross-section of the solid. It is often helpful to draw a picture if one is not provided.
  2. Determine a formula for the area of the cross-section.
  3. Integrate the area formula over the appropriate interval to get the volume.
- **Solid of revolution:** A solid generated by revolving a region in a plane around a line in that plane.
- **Disk method:** A special case of the slicing method used with solids of revolution when the slices are disks.
- A **Washer (Annuli)** is a disk with holes in the center.
- **Washer method:** A special case of the slicing method used with solids of revolution when the slices are washers.
- **Method of cylindrical shells:** A method of calculating the volume of a solid of revolution by dividing the solid into nested cylindrical shells; this method is different from the methods of disks or washers in that we integrate with respect to the opposite variable.
- **Arc length:** The arc length of a curve can be thought of as the distance a person would travel along the path of the curve.
- **Surface area:** The surface area of a solid is the total area of the outer layer of the object; for objects such as cubes or bricks, the surface area of the object is the sum of the areas of all of its faces.

1.3 Chapter 2 Key Equations

- **Area between two curves, integrating on the x-axis**

$$A = \int_a^b [f(x) - g(x)]\, dx \tag{1}$$

Where $f(x) \geqslant g(x)$

$$A = \int_a^b \ [g(x) - f(x)] \ dx.$$

for $g(x) \geqslant f(x)$

- **Area between two curves, integrating on the y-axis**

$$A = \int_c^d [u(y) - v(y)]\, dy \tag{2}$$

- **Areas of compound regions**

$$\int_a^b \ |f(x) - g(x)| \ dx.$$

- **Area of complex regions**

$$\int_a^b \ f(x) \ dx + \int_b^c \ g(x) \ dx.$$

- **Slicing Method**

$$V(s) = \sum_{i=1}^{n} \ A(x_i^*) \ \Delta x = \int_a^b \ A(x) \ dx.$$

- **Disk Method along the x-axis**

$$V = \int_a^b \pi[f(x)]^2\, dx \tag{3}$$

- **Disk Method along the y-axis**

$$V = \int_c^d \pi[g(y)]^2\, dy \tag{4}$$

- **Washer Method along the x-axis**

$$V = \int_a^b \pi[(f(x))^2 - (g(x))^2]\, dx \tag{5}$$

- **Washer Method along the y-axis**

$$V = \int_c^d \pi[(u(y))^2 - (v(y))^2]\, dy \tag{6}$$

- **Radius if revolved around other line (Washer Method)**

$$\begin{aligned} If: \ x &= -k \\ Then: \ r &= Function + k. \end{aligned}$$

$$\begin{aligned} If: \ x &= k \\ Then: \ r &= k - Function. \end{aligned}$$

- **Method of Cylindrical Shells (x-axis)**

$$V = \int_a^b 2\pi x f(x)\, dx \tag{7}$$

- **Method of Cylindrical Shells (y-axis)**

$$V = \int_c^d 2\pi y g(y)\, dy \tag{8}$$

- **Region revolved around other line (method of cylindrical shells):**

$$If:\ x = -k$$
$$Then:\ V = \int_a^b\ 2\pi(x+k)(f(x))\ dx.$$

$$If:\ x = k$$
$$Then:\ V = \int_a^b\ 2\pi(k-x)(f(x))\ dx.$$

- **A Region of Revolution Bounded by the Graphs of Two Functions (method cylindrical shells)**

$$V = \int_a^b\ 2\pi x\left[f(x) - g(x)\right]\ dx.$$

- **Arc Length of a Function of x**

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \tag{9}$$

- **Arc Length of a Function of y**

$$\text{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy \tag{10}$$

- **Surface Area of a Function of x**

$$\text{Surface Area} = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx \tag{11}$$

- **Natural logarithm function**

$$\ln x = \int_1^x \frac{1}{t}\, dt\ Z \tag{12}$$

- **Exponential function**

$$y = e^x, \quad \ln y = \ln(e^x) = x\ Z \tag{13}$$

- **Logarithm Differentiation**

$$f'(x) = f(x) \cdot \frac{d}{dx} \ln\left(f'(x)\right).$$

Note: Use properties of logs before you differentiate whats inside the logarithm

1.4 Chapter 3 Key Terms

- **absolute error**: if B is an estimate of some quantity having an actual value of A, then the absolute error is given by $|A - B|$.
- **computer algebra system (CAS)**: technology used to perform many mathematical tasks, including integration.
- **improper integral**: an integral over an infinite interval or an integral of a function containing an infinite discontinuity on the interval; an improper integral is defined in terms of a limit. The improper integral converges if this limit is a finite real number; otherwise, the improper integral diverges.
- **integration by parts**: a technique of integration that allows the exchange of one integral for another using the formula
- **integration table**: a table that lists integration formulas.
- **midpoint rule**: a rule that uses a Riemann sum of the form
- **numerical integration**: the variety of numerical methods used to estimate the value of a definite integral, including the midpoint rule, trapezoidal rule, and Simpson's rule.
- **partial fraction decomposition**: a technique used to break down a rational function into the sum of simple rational functions.
- **power reduction formula**: a rule that allows an integral of a power of a trigonometric function to be exchanged for an integral involving a lower power.
- **relative error**: error as a percentage of the absolute value, given by
- **Simpson's rule**: a rule that approximates $\int_a^b f(x)\,dx$ using the integrals of a piecewise quadratic function. The approximation S_n to $\int_a^b f(x)\,dx$ is given by
- **trapezoidal rule**: a rule that approximates $\int_a^b f(x)\,dx$ using trapezoids.
- **trigonometric integral**: an integral involving powers and products of trigonometric functions.
- **trigonometric substitution**: an integration technique that converts an algebraic integral containing expressions of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$ into a trigonometric integral.

1.5 Chapter 3 Key Equations

- **Integration by parts formula**

$$\int u\,dv = uv - \int v\,du.$$

- **Integration by parts for definite integral**

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

- **To integrate products involving sin(ax), sin(bx), cos(ax), and cos(bx), use the substitutions:**
  - **Sine Products**

$$\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$$

  - **Sine and Cosine Products**

$$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$$

  - **Cosine Products**

$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$$

  - **Power Reduction Formula (sine)**

$$\int \sin^n x\ dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\ dx.$$

  - **Power Reduction Formula (cosine)**

$$\int \cos^n x\ dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\ dx.$$

  - **Power Reduction Formula (secant)**

$$\int \sec^n x\ dx = \frac{1}{n-1}\sec^{n-1} x\sin x + \frac{n-2}{n-1}\int \sec^{n-2} x\ dx$$
$$\int \sec^n x\ dx = \frac{1}{n-1}\sec^{n-2} x\tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\ dx$$

  - **Power Reduction Formula (tangent)**

$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$$

- **Trigonometric Substitution**
  - $\sqrt{a^2-x^2}$ use $x = a\sin\theta$ with domain restriction $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
  - $\sqrt{a^2+x^2}$ use $x = a\tan\theta$ with domain restriction $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
  - $\sqrt{x^2-a^2}$ use $x = a\sec\theta$ with domain restriction $\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right)$

- **Midpoint rule**

$$M_n = \sum_{i=1}^{n} f(m_i)\Delta x$$

- **Trapezoidal rule**

$$T_n = \frac{1}{2}\Delta x\left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right)$$

- **Simpson's rule**

$$S_n = \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)$$

- **Error bound for midpoint rule**

$$\text{Error in } M_n \leqslant \frac{M(b-a)^3}{24n^2}$$

- **Error bound for trapezoidal rule**

$$\text{Error in } T_n \leqslant \frac{M(b-a)^3}{12n^2}$$

- **Error bound for Simpson's rule**

$$\text{Error in } S_n \leqslant \frac{M(b-a)^5}{180n^4}$$