Comprehensive Compendium:
Calculus II

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Contents

1 Calc II

1.1 Chapter 1 Key Equations

- **Mean Value Theorem For Integrals**: If $f(x)$ is continuous over an interval [a,b], then there is at least one point $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int f(x)\ dx.$$

- **Integrals resulting in inverse trig functions**

1.

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{|a|} + C.$$

2.

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C.$$

3.

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{|a|} \sec^{-1} \frac{|x|}{a} + C.$$

1.2 Chapter 2 Key Terms / Ideas

- **Finding limits of integration for region between two functions**: Usually, we want our limits of integration to be the points where the functions intersect
- A **"complex region"** between curves usually refers to an area that is not easily described by a single, continuous function over the interval of interest.
- **compound regions** are regions bounded by the graphs of functions that cross one another
- **Cross-section:** The intersection of a plane and a solid object.
- a **cylinder** is a three-dimensional shape that has two parallel, congruent bases connected by a curved surface. The bases are usually circles, but they can be other shapes as well
- The line segment connecting the centers of the two bases is called the **"axis" of the cylinder.**
- **Slicing method:** A method of calculating the volume of a solid that involves cutting the solid into pieces, estimating the volume of each piece, then adding these estimates to arrive at an estimate of the total volume; as the number of slices goes to infinity, this estimate becomes an integral that gives the exact value of the volume.
    1. Examine the solid and determine the shape of a cross-section of the solid. It is often helpful to draw a picture if one is not provided.
    2. Determine a formula for the area of the cross-section.
    3. Integrate the area formula over the appropriate interval to get the volume.
- **Solid of revolution:** A solid generated by revolving a region in a plane around a line in that plane.
- **Disk method:** A special case of the slicing method used with solids of revolution when the slices are disks.
- A **Washer (Annuli)** is a disk with holes in the center.
- **Washer method:** A special case of the slicing method used with solids of revolution when the slices are washers.
- **Method of cylindrical shells:** A method of calculating the volume of a solid of revolution by dividing the solid into nested cylindrical shells; this method is different from the methods of disks or washers in that we integrate with respect to the opposite variable.
- **Arc length:** The arc length of a curve can be thought of as the distance a person would travel along the path of the curve.
- **Surface area:** The surface area of a solid is the total area of the outer layer of the object; for objects such as cubes or bricks, the surface area of the object is the sum of the areas of all of its faces.

1.3 Chapter 2 Key Equations

- **Area between two curves, integrating on the x-axis**

$$A = \int_a^b [f(x) - g(x)]\, dx \tag{1}$$

Where $f(x) \geqslant g(x)$

$$A = \int_a^b \; [g(x) - f(x)] \; dx.$$

for $g(x) \geqslant f(x)$

- **Area between two curves, integrating on the y-axis**

$$A = \int_c^d [u(y) - v(y)]\, dy \tag{2}$$

- **Areas of compound regions**

$$\int_a^b \; |f(x) - g(x)| \; dx.$$

- **Area of complex regions**

$$\int_a^b \; f(x) \; dx + \int_b^c \; g(x) \; dx.$$

- **Slicing Method**

$$V(s) = \sum_{i=1}^{n} \; A(x_i^*) \; \Delta x = \int_a^b \; A(x) \; dx.$$

- **Disk Method along the x-axis**

$$V = \int_a^b \pi[f(x)]^2\, dx \tag{3}$$

- **Disk Method along the y-axis**

$$V = \int_c^d \pi[g(y)]^2\, dy \tag{4}$$

- **Washer Method along the x-axis**

$$V = \int_a^b \pi[(f(x))^2 - (g(x))^2]\, dx \tag{5}$$

- **Washer Method along the y-axis**

$$V = \int_c^d \pi[(u(y))^2 - (v(y))^2]\, dy \tag{6}$$

- **Radius if revolved around other line (Washer Method)**

$$If: \; x = -k$$
$$Then: \; r = Function + k.$$

$$If: \; x = k$$
$$Then: \; r = k - Function.$$

- **Method of Cylindrical Shells (x-axis)**

$$V = \int_a^b 2\pi x f(x)\, dx \tag{7}$$

- **Method of Cylindrical Shells (y-axis)**

$$V = \int_c^d 2\pi y g(y)\, dy \tag{8}$$

- **Region revolved around other line (method of cylindrical shells):**

$$If: \ x = -k$$
$$Then: \ V = \int_a^b \ 2\pi(x+k)(f(x)) \ dx.$$

$$If: \ x = k$$
$$Then: \ V = \int_a^b \ 2\pi(k-x)(f(x)) \ dx.$$

- **A Region of Revolution Bounded by the Graphs of Two Functions (method cylindrical shells)**

$$V = \int_a^b \ 2\pi x \left[f(x) - g(x)\right] \ dx.$$

- **Arc Length of a Function of x**

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \tag{9}$$

- **Arc Length of a Function of y**

$$\text{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy \tag{10}$$

- **Surface Area of a Function of x**

$$\text{Surface Area} = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx \tag{11}$$

- **Natural logarithm function**

$$\ln x = \int_1^x \frac{1}{t}\, dt \ Z \tag{12}$$

- **Exponential function**

$$y = e^x, \quad \ln y = \ln(e^x) = x \ Z \tag{13}$$

- **Logarithm Differentiation**

$$f'(x) = f(x) \cdot \frac{d}{dx} \ln\left(f'(x)\right).$$

Note: Use properties of logs before you differentiate whats inside the logarithm

1.4 Chapter 3 Key Terms

- **integration by parts**: a technique of integration that allows the exchange of one integral for another using the formula
- **integration table**: a table that lists integration formulas.
- **power reduction formula**: a rule that allows an integral of a power of a trigonometric function to be exchanged for an integral involving a lower power.
- **trigonometric integral**: an integral involving powers and products of trigonometric functions.
- **trigonometric substitution**: an integration technique that converts an algebraic integral containing expressions of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$ into a trigonometric integral.
- **partial fraction decomposition**: a technique used to break down a rational function into the sum of simple rational functions.
- **improper integral**: an integral over an infinite interval or an integral of a function containing an infinite discontinuity on the interval; an improper integral is defined in terms of a limit. The improper integral converges if this limit is a finite real number; otherwise, the improper integral diverges.

1.5 Chapter 3 Key Equations

- **Integration by parts formula**

$$\int u\,dv = uv - \int v\,du.$$

- **Integration by parts for definite integral**

$$\int_a^b u\,dv = uv\big|_a^b - \int_a^b v\,du$$

- **To integrate products involving sin(ax), sin(bx), cos(ax), and cos(bx), use the substitutions:**
  - **Sine Products**

$$\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$$

  - **Sine and Cosine Products**

$$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$$

  - **Cosine Products**

$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$$

  - **Power Reduction Formula (sine)**

$$\int \sin^n x\;dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\;dx$$
$$\int_0^{\frac{\pi}{2}} \sin^n x\;dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x\;dx.$$

  - **Power Reduction Formula (cosine)**

$$\int \cos^n x\;dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\;dx$$
$$\int_0^{\frac{\pi}{2}} \cos^n x\;dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \cos^{n-2} x\;dx.$$

  - **Power Reduction Formula (secant)**

$$\int \sec^n x\;dx = \frac{1}{n-1}\sec^{n-1} x\sin x + \frac{n-2}{n-1}\int \sec^{n-2} x\;dx$$
$$\int \sec^n x\;dx = \frac{1}{n-1}\sec^{n-2} x\tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\;dx$$

  - **Power Reduction Formula (tangent)**

$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$$

- **Trigonometric Substitution**
  - $\sqrt{a^2 - x^2}$ use $x = a\sin\theta$ with domain restriction $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
  - $\sqrt{a^2 + x^2}$ use $x = a\tan\theta$ with domain restriction $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

  - $\sqrt{x^2 - a^2}$ use $x = a\sec\theta$ with domain restriction $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$

- **Steps for fraction decomposition**
  1. Ensure $deg(Q) < deg(P)$, if not, long divide
  2. Factor denominator
  3. Split up fraction into factors
  4. Multiply through to clear denominator
  5. Group terms and equalize
  6. Solve for constants
  7. Plug constants into split up fraction
  8. Compute integral
- **Solving for constants** Either:
  - Plug in values (often the roots)
  - Equalize
- **Cases for partial fractions**
  - Non repeated linear factors
  - Repeated linear factors
  - Nonfactorable quadratic factors
- **Midpoint rule**

$$M_n = \sum_{i=1}^{n} f(m_i)\ \Delta x.$$

- **Absolute error**

$$err = \Bigg|\text{Actual} - \text{Estimated}\Bigg|.$$

- **Relative error**

$$err = \left|\frac{\text{Actual} - \text{Estimated}}{\text{Actual}}\right| \cdot 100\%.$$

- **Error upper bound for midpoint rule**

$$E_M \leqslant \frac{M(b-a)^3}{24n^2}$$

  Where M is the maximum value of the second derivative

- **Trapezoidal rule**

$$T_n \frac{1}{2}\Delta x\left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right)$$

- **Error upper bound for trapezoidal rule**

$$E_T \leqslant \frac{M(b-a)^3}{12n^2}$$

  Where M is the maximum value of the second derivative

- **Simpson's rule**

$$S_n = \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)$$

- **Error upper bound for Simpson's rule**

$$E_S \leqslant \frac{M(b-a)^5}{180n^4}$$

Where M is the maximum value of the fourth derivative

- **Finding n with error bound functions**
  1. Find $f''(x)$
  2. Find maximum values of $f''(x)$ in the interval
  3. Plug into error bound function
  4. Set value $\leqslant$ desired accuracy (ex: 0.01)
  5. Solve:
  6. If we were to truncate, we would use the ceil function $\lceil n \rceil$ DO NOT FLOOR

- **Improper integrals (Infinite interval)**
  - $\int_a^{+\infty} f(x)\ dx = \lim\limits_{t\to+\infty} \int_a^t f(x)\ dx$
  - $\int_{-\infty}^b f(x)\ dx = \lim\limits_{t\to-\infty} \int_t^b f(x)\ dx$
  - $\int_{-\infty}^{+\infty} f(x)\ dx = \int_{-\infty}^0 f(x)\ dx + \int_0^{+\infty} f(x)\ dx$

- **Improper integral (discontinuous)**
  - Let $f(x)$ be continuous on $[a,b)$, then;

$$\int_a^b f(x)\ dx = \lim_{t\to b^-} \int_a^t f(x)\ dx\ .$$

  - Let $f(x)$ be continuous on $(a,b]$, then;

$$\int_a^b f(x)\ dx = \lim_{t\to b^+} \int_t^b f(x)\ dx\ .$$

    In each case, if the limit exists, then the improper integral is said to converge. If the limit does not exist, then the improper integral is said to diverge.

  - Let $f(x)$ be continuous on $[a,b]$ except at a point $c \in (a,b)$, then;

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx.$$

    If either integral diverges, then $\int_a^b f(x)\ dx$ diverges

- **Comparison theorem** Let $f(x)$ and $g(x)$ be continuous over $[a,+\infty)$. Assume that $0 \leqslant f(x) \leqslant g(x)$ for $x \geqslant a$.
  - If $\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx = +\infty$,
    then $\int_a^{+\infty} g(x)\,dx = \lim_{t\to+\infty} \int_a^t g(x)\,dx = +\infty$.
  - If $\int_a^{+\infty} g(x)\,dx = \lim_{t\to+\infty} \int_a^t g(x)\,dx = L$, where L is a real number,
    then $\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx = M$ for some real number $M \leqslant L$.

- **P-integrals**
  - $\int_0^{+\infty} \frac{1}{x^P}\, dx = \begin{cases} \frac{1}{p-1} & \text{if p>1} \\ +\infty & \text{if p} \leqslant 1 \end{cases}$
  - $\int_0^{1} \frac{1}{x^P}\, dx = \begin{cases} \frac{1}{1-p} & \text{if p<1} \\ +\infty & \text{if p} \geqslant 1 \end{cases}$
  - $\int_a^{+\infty} \frac{1}{x^P}\, dx = \begin{cases} \frac{a^{1-p}}{p-1} & \text{if p>1} \\ +\infty & \text{if p} \leqslant 1 \end{cases}$
  - $\int_0^{a} \frac{1}{x^P}\, dx = \begin{cases} \frac{a^{1-p}}{1-p} & \text{if p<1} \\ +\infty & \text{if p} \geqslant 1 \end{cases}$

- **Bypass L'Hospital's Rule**

$$\ln(\ln(x)),\ \ln(x),\ \cdots\ x^{\frac{1}{100}},\ x^{\frac{1}{3}},\ \sqrt{x},\ 1,\ x^2,\ x^3,\ \cdots\ e^x,\ e^{2x},\ e^{3x},\ \cdots,\ e^{x^2},\ \cdots\ e^{e^x}.$$

Essentially what it means is things on the right grow faster than things on the left. Thus, if we have say:

$$\lim_{x\to\infty} \frac{x^2}{e^{2x}}.$$

We can be sure that it is zero. Because this is $x^2 \cdot e^{-2x}$. If we take $\lim_{x\to\infty} x^2 e^{-2x}$, we get $\infty \cdot 0$. As we see by the sequence e^{-2x} overrules x^2 and we can say the limit is zero.

- **Consideration for Limits**: Let $f : A \to B$ be a function defined by $x \mapsto f(x)$. If a point c lies outside the domain A, then the expression $\lim_{x\to c} f(x)$ is not meaningful, and we classify this limit as undefined. For instance, the function arcsine has a domain of $[-1, 1]$. Therefore, limits like $\lim_{x\to a} \sin^{-1}(x)$ where $a \notin [-1, 1]$ are undefined.

- **Why does**

$$\lim_{x\to 2} \tan^{-1} \frac{1}{x-2}.$$

$$\begin{aligned} &= \lim_{x\to 2^-} \tan^{-1} \frac{1}{x-2} \\ &= \lim_{x\to -\infty} \tan^{-1} x \\ &= -\pi/2. \end{aligned}$$

$$\begin{aligned} &= \lim_{x\to 2^+} \tan^{-1} \frac{1}{x-2} \\ &= \lim_{x\to +\infty} \tan^{-1} x \\ &= \frac{\pi}{2}. \end{aligned}$$