Nate Warner MATH 232 Spring 2024

Homework/Worksheet 7 - Due: Sunday, March 31

1. Find the directional derivative of the function $f(x,y) = e^x \cos(y)$ at $P\left(0, \frac{\pi}{2}\right)$ in the direction of $\mathbf{u} = \langle 0, 1 \rangle$

Remark. Let $z = f(x,y)$ be a function of two variables x and y, and assume that f_x and f_y exist and $f(x,y)$ is differentiable everywhere. Then the directional derivative of f in the direction of $\mathbf{u} = \langle u_x, u_y \rangle$ is given by

$$D_{\mathbf{u}} f(x,y) = f_x(x,y)u_x + f_y(x,y)u_y$$

We start by finding the gradient vector $\nabla f(x,y) = \langle f_x, f_y \rangle$

$$\nabla f(x,y) = \langle e^x \cos(y), -e^x \sin(y) \rangle .$$

We then compute the directional derivative $D_u f(x,y)$ as the dot product between the gradient and the unit vector

$$\begin{aligned} D_u f(x,y) &= \nabla f(x,y) \cdot \vec{\mathbf{u}} = e^x \cos(y)(0) - e^x \sin(y)(1) \\ &= -e^x \sin(y). \end{aligned}$$

With this, we can compute $D_u f\left(0, \frac{\pi}{2}\right)$

$$\begin{aligned} D_u f\left(0, \frac{\pi}{2}\right) &= -e^0 \sin\left(\frac{\pi}{2}\right) \\ &= -1(1) = -1. \end{aligned}$$

2. Find the directional derivative of the function $f(x,y) = x^2 + 2y^2$ in the direction of $\mathbf{v} = \langle \cos\theta, \sin\theta \rangle$, where $\theta = \frac{\pi}{6}$.

Again, we start by finding the gradient vector $\nabla f(x,y)$

$$\nabla f(x,y) = \langle f_x, f_y \rangle = \langle 2x, 4y \rangle .$$

We then compute the dot product against the given unit vector $\vec{\mathbf{v}} = \left\langle \cos\left(\frac{\pi}{6}\right), \sin\left(\frac{\pi}{6}\right) \right\rangle$

$$\begin{aligned} \nabla f(x,y) \cdot \vec{\mathbf{v}} &= 2x \cos\left(\frac{\pi}{6}\right) + 4y \sin\left(\frac{\pi}{6}\right) \\ &= 2x\left(\frac{\sqrt{3}}{2}\right) + 4y\left(\frac{1}{2}\right) \\ &= \sqrt{3}x + 2y. \end{aligned}$$

Thus, the directional derivative $D_u f(x,y)$, with $f(x,y) = x^2 + 2y^2$ and $\vec{\mathbf{v}} = \cos\left(\frac{\pi}{6}\right), \sin\left(\frac{\pi}{6}\right)$ is given by

$$D_v f(x,y) = \sqrt{3}x + 2y.$$

3. Find the directional derivative of the function $f(x, y) = \ln(5x+4y)$ at $P(3, 9)$ in the direction of $\mathbf{v} = \langle 6, 8 \rangle$.

Since the given vector $\vec{\mathbf{v}}$ is of $\|\vec{\mathbf{v}}\| \neq 1$, we must first divide by the norm, this gives us a unit vector in the direction of $\vec{\mathbf{v}}$ with norm one. We get $\hat{\mathbf{v}} = \frac{1}{10}\langle 6, 8 \rangle$. We now find the gradient vector $\nabla f(x, y)$

$$\nabla f(x, y) = \left\langle \frac{5}{5x+4y}, \frac{4}{5x+4y} \right\rangle.$$

From here we compute the directional derivative $D_{\hat{\mathbf{v}}} f(x, y)$

$$\begin{aligned} D_{\hat{\mathbf{v}}} f(x, y) &= \nabla f(x, y) \cdot \hat{\mathbf{v}} \\ &= \frac{1}{10}\left[\frac{5(6)}{5x+4y} + \frac{4(8)}{5x+4y}\right]. \end{aligned}$$

Thus, we have

$$\begin{aligned} D_{\hat{\mathbf{v}}}(3, 9) &= \frac{1}{10}\left[\frac{5(6)+4(8)}{5(3)+4(9)}\right] \\ &= \frac{31}{255}. \end{aligned}$$

4. Find the gradient vector of $f(x, y) = xe^y - \ln(x)$ at $P(-3, 0)$.

The gradient of $f(x, y)$ at $P(-3, 0)$ is given by

$$\begin{aligned} \nabla f(x, y) &= \left\langle e^y - \frac{1}{x}, xe^y \right\rangle \\ \nabla f(-3, 0) &= \left\langle e^0 - \frac{1}{-3}, -3e^0 \right\rangle \\ &= \left\langle \frac{4}{3}, -3 \right\rangle. \end{aligned}$$

5. Find the maximum rate of change of $f(x, y) = \cos(3x+2y)$ at $\left(\frac{\pi}{6}, -\frac{\pi}{8}\right)$ and the direction in which it occurs.

Remark. If $\nabla f(x_0, y_0) \neq 0$, then $D_{\mathbf{u}} f(x_0, y_0)$ is maximized when $\mathbf{u}$ points in the same direction as $\nabla f(x_0, y_0)$. The maximum value of $D_{\mathbf{u}} f(x_0, y_0)$ is $\|\nabla f(x_0, y_0)\|$.

Thus, we start by computing the gradient vector $\nabla f(x, y)$ and evaluating it at the point $\left(\frac{\pi}{6}, -\frac{\pi}{8}\right)$

$$\begin{aligned} \nabla f(x, y) &= \langle -3\sin(3x+2y), -2\sin(3x+2y) \rangle \\ \nabla f\left(\frac{\pi}{6}, -\frac{\pi}{8}\right) &= \left\langle -3\sin\left(\frac{\pi}{4}\right), -2\sin\left(\frac{\pi}{4}\right) \right\rangle \\ &= \left\langle \frac{-3\sqrt{2}}{2}, -\sqrt{2} \right\rangle. \end{aligned}$$

From this, we see that the maximum rate of change is at

$$\begin{aligned}\|\nabla f\left(\frac{\pi}{6}, -\frac{\pi}{8}\right)\| &= \sqrt{\left(\frac{-3\sqrt{2}}{2}\right)^2 + \left(-\sqrt{2}\right)^2} \\ &= \frac{\sqrt{26}}{2} \approx 2.5495.\end{aligned}$$

Which occurs in the direction of $\nabla f\left(\frac{\pi}{6}, -\frac{\pi}{8}\right) = \left\langle -\frac{3\sqrt{2}}{2}, -\sqrt{2}\right\rangle$

For the functions below, use the second derivative test to identify any critical points and determine whether each critical point is a maximum, minimum, saddle point, or none of these.

(a) $f(x, y) = x^2 - 6x + y^2 + 4y - 8$

(b) $f(x, y) = y^2 + xy + 3y + 2x + 3$

Remark. Let $z = f(x, y)$ be a function of two variables that is defined on an open set containing the point (x_0, y_0). The point (x_0, y_0) is called a critical point of a function of two variables f if one of the two following conditions holds:

1. $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$
2. Either $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

Let $z = f(x, y)$ be a function of two variables for which the first- and second-order partial derivatives are continuous on some disk containing the point (x_0, y_0). Suppose $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. Define the quantity

$$D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2.$$

I. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local minimum at (x_0, y_0).

II. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local maximum at (x_0, y_0).

III. If $D < 0$, then f has a saddle point at (x_0, y_0).

IV. If $D = 0$, then the test is inconclusive.

Thus, we find $\nabla f(x, y)$ and solve $\nabla f(x, y) = \vec{\mathbf{0}}$ to find the critical points

$$\begin{aligned}\nabla f(x, y) &= \langle 2x - 6, 2y + 4\rangle \\ \langle 2x - 6, 2y + 4\rangle &= \vec{\mathbf{0}} \\ \implies x &= 3 \\ \implies y &= -2.\end{aligned}$$

We see we have a critical point at $C_1(3, -2)$, calculating the discriminant we find

$$D = 2(2) - 0^2 = 4 > 0.$$

Since the discriminant is positive, we know this point must either be a local min or a local max. Since $f_{xx} > 0$, we know that the critical point $(3, -2)$ is a local min.

For part b, we again start by finding the gradient vector

$$\nabla f(x,y) = \langle y+2, 2y+x+3 \rangle .$$

This gives the following system of linear equations

$$\begin{cases} y+2 &= 0 \\ 2y+x+3 &= 0 \end{cases} \tag{1}$$

Solving the first equation for y gives $y = -2$, plugging this result into the second equation yields the x value

$$\begin{aligned} 2(-2)+x+3 &= 0 \\ \implies x &= 1. \end{aligned}$$

Thus, we have a critical point at $(1, -2)$. To further examine this point, we again calculate the discriminant

$$D = 0(2) - (1)^2 < 0.$$

Since the discriminant is negative, we conclude that the critical point (1,-2) is a saddle point by the second derivative test.

7. Use the method of Lagrange multipliers to find the maximum and minimum values of the function $f(x,y) = xy$ subject to the given constraint $4x^2 + 8y^2 = 16$.

Remark. Let f and g be functions of two variables with continuous partial derivatives at every point of some open set containing the smooth curve $g(x,y) = 0$. Suppose that f, when restricted to points on the curve $g(x,y) = 0$, has a local extremum at the point (x_0, y_0) and that $\nabla g(x_0, y_0) \neq 0$. Then there is a number λ called a Lagrange multiplier, for which

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

To find the absolute extrema, we must solve the system given by

$$\begin{cases} \nabla f(x,y) &= \lambda \nabla g(x,y) \\ g(x,y) &= 0 \end{cases} \tag{2}$$

After finding all points that satisfy this system, we plug into $f(x,y)$. The smallest value will be the absolute minimum, and the largest will be the absolute maxmimum

With this, we begin by finding $\nabla f(x,y)$ and $\nabla g(x,y)$

$$\begin{aligned} \nabla f(x,y) &= \langle y, x \rangle \\ \nabla g(x,y) &= \langle 8x, 16y \rangle . \end{aligned}$$

With this, we have the system

$$\begin{cases} \langle y, x \rangle = \lambda \langle 8x, 16y \rangle \\ 4x^2 + 8y^2 - 16 = 0 \end{cases} \tag{3}$$

Which implies the system

$$\begin{cases} y = \lambda 8x \\ x = \lambda 16y \\ 4x^2 + 8y^2 - 16 = 0 \end{cases} \tag{4}$$

Solving one and two for lambda gives

$$\begin{aligned} \lambda = \frac{y}{8x} &= \frac{x}{16y} \\ \implies 8x^2 &= 16y^2 \\ \implies x &= \pm\sqrt{2}y. \end{aligned}$$

Plugging the positive version of x into the third equation gives

$$\begin{aligned} 4(\sqrt{2}y)^2 + 8y^2 - 16 &= 0 \\ \implies 8y^2 + 8y^2 - 16 &= 0 \\ \implies 16y^2 - 16 &= 0 \\ \implies y &= \pm 1. \end{aligned}$$

Thus, we $x = +\sqrt{2}y$, $y = \pm 1$, which gives the solutions, $(-\sqrt{2}, -1)$, $(\sqrt{2}, 1)$

Similarly, we use the negative version of x to get the remaining solution.

$$\begin{aligned} 4(-\sqrt{2}y)^2 + 8y^2 - 16 &= 0 \\ \implies y &= \pm 1. \end{aligned}$$

So, when $x = -\sqrt{2}y$, we find that y is also ± 1, this gives the remaining solutions $(\sqrt{2}y, -1)$, $(-\sqrt{2}y, 1)$. To find the absolute min and max, we need to evaluate f at these points

$$\begin{aligned} f(-\sqrt{2}, -1) &= \sqrt{2} \\ f(\sqrt{2}, 1) &= \sqrt{2} \\ f(\sqrt{2}, -1) &= -\sqrt{2} \\ f(-\sqrt{2}, 1) &= -\sqrt{2}. \end{aligned}$$

Thus, we see that we have an absolute maximum at $f(-\sqrt{2}, -1) = f(\sqrt{2}, 1) = \sqrt{2}$, and an absolute minimum at $(f(\sqrt{2}, -1)) = f(-\sqrt{2}, 1) = -\sqrt{2}$