Nate Warner MATH 230 November 11, 2023

Homework/Worksheet 10 - Due: Wednesday, November 15

1. Find the radius of convergence and interval of conference of the series.
   (a) $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt{n}}$
   (b) $\sum_{n=1}^{\infty} \frac{nx^n}{e^n}$
   (c) $\sum_{n=1}^{\infty} \frac{10^n x^n}{n!}$
   (d) $\sum_{n=1}^{\infty} \frac{(2n)!x^n}{n^{2n}}$

Problem 1a. Using the ratio test:

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}x^{n+1}}{\sqrt{n+1}}}{\frac{(-1)^n x^n}{\sqrt{n}}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(-1)^n(-1)x^n x}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(-1)^n x^n} \right| \\
&= |x| \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} \\
&= |x|.
\end{aligned}$$

Consequently, we have convergence when $|x| < 1$, and divergence for $|x| > 1$

$$\begin{aligned}
&\implies -1 < x < 1 \\
&\therefore R = 1.
\end{aligned}$$

Testing the endpoints we see

When $x = -1$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{\sqrt{n}} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{\sqrt{n}} \\
&= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} .
\end{aligned}$$

Since this is a p-series with $p = \frac{1}{2}$, we have divergence for $x = -1$

When $x = 1$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} .$$

Using the AST we get

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{\sqrt{n}} \\
&= 0.
\end{aligned}$$

Thus, we have convergence when $x = 1$

Conclusion. For the power series $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt{n}}$, $R = 1$, $I = (-1, 1]$

Problem 1b. By the ratio test

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{(n+1)x^n x}{e^n e}\cdot\frac{e^n}{nx^n}\right| \\ &= |x|\lim_{n\to\infty}\frac{n+1}{en} \\ &= \frac{|x|}{e}.\end{aligned}$$

Consequently, we have convergence when $\frac{1}{e}|x| < 1$, and divergence for $\frac{1}{e}|x| > 1$

$$\begin{aligned}&\implies -e < x < e \\ &\therefore R = e.\end{aligned}$$

Testing the endpoints we see

When $x = -e$

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{n(-e)^n}{e^n} \\ &= \sum_{n=1}^{\infty} n(-1)^n\ .\end{aligned}$$

Which is divergent

When $x = e$

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{ne^n}{e^n} \\ &= \sum_{n=1}^{\infty} n\ .\end{aligned}$$

Which is also divergent

Conclusion. The power series $\sum\limits_{n=1}^{\infty}\frac{nx^n}{e^n}$ has $R = e$, $I = (-e, e)$

Problem 1c. Using the ratio test

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{10^n 10x^n x}{n!(n+1)}\cdot\frac{n!}{x^n 10^n}\right| \\ &= 10|x|\lim_{n\to\infty}\frac{1}{n+1} \\ &= 0 \\ &\therefore R = \infty.\end{aligned}$$

Conclusion. The power series $\sum\limits_{n=1}^{\infty}\frac{10^n x^n}{n!}$ has $R = \infty$ and $I = \mathbb{R} \implies$ convergence $\forall\ x \in \mathbb{R}$

Problem 1d. Using the ratio test

$$\rho = \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)(2n)!x^n x}{n^{2n}n^2}\cdot\frac{n^{2n}}{(2n)!x^n}\right|$$
$$= |x|\lim_{n\to\infty}\frac{(2n+2)(2n+1)}{n^2}$$
$$= 4|x|.$$

Consequently, we have convergence when $4|x| < 1$, and divergence for $4|x| > 1$

$$\implies -\frac{1}{4} < x < \frac{1}{4}$$
$$\therefore R = \frac{1}{4}.$$

Testing the endpoints we see

When $x = -\frac{1}{4}$

$$\sum_{n=1}^{\infty}\frac{(2n)!\left(-\frac{1}{4}\right)^n}{n^{2n}}$$
$$= \sum_{n=1}^{\infty}\frac{(2n)!(-1)^n\left(\frac{1}{4}\right)^n}{n^{2n}}\ .$$

Using Sterling's approximation $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \implies (2n)! \approx \sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}$ we can rewrite the series as

$$\sum_{n=1}^{\infty}\frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}(-1)^n\left(\frac{1}{4}\right)^n}{n^{2n}}$$
$$= \sum_{n=1}^{\infty}\frac{\sqrt{4\pi n}\left(\frac{n^2}{e^2}\right)^n(-1)^n}{n^{2n}}$$
$$= \sum_{n=1}^{\infty}\sqrt{4\pi n}\left(\frac{1}{e^{2n}}\right)(-1)^n$$

.

Which we see is an alternating series. Thus,

$$\lim_{n\to\infty}\sqrt{4\pi n}\left(\frac{1}{e^{2n}}\right)$$
$$= \lim_{n\to\infty}\frac{2\sqrt{\pi}\sqrt{n}}{e^{2n}}$$
$$\overset{H}{=} \lim_{n\to\infty}\frac{2\sqrt{\pi}}{2e^{2n}2\sqrt{n}}$$
$$= \lim_{n\to\infty}\frac{\sqrt{\pi}}{2e^{2n}\sqrt{n}}$$
$$= 0.$$

Thus, we have convergence at $x = -\frac{1}{4}$

When $x = \frac{1}{4}$, again we use Sterling's approximation to get the new series

$$\sum_{n=1}^{\infty}\frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}\left(\frac{1}{4}\right)^n}{n^{2n}}$$
$$= \sum_{n=1}^{\infty}\frac{1}{e^{2n}}\sqrt{4\pi n}\ .$$

Using the root test

$$\rho = \lim_{n\to\infty}\left|\frac{\sqrt{4\pi(n+1)}}{e^{2n}e^2}\cdot\frac{e^{2n}}{\sqrt{4\pi n}}\right|$$
$$= \lim_{n\to\infty}\frac{\sqrt{4\pi n+4\pi}}{e^2}\cdot\frac{1}{2\sqrt{\pi n}}$$
$$= \lim_{n\to\infty}\frac{\sqrt{4(\pi n+\pi)}}{e^2}\cdot\frac{1}{2\sqrt{\pi n}}$$
$$= \lim_{n\to\infty}\frac{2\sqrt{\pi n+\pi}}{e^2}\cdot\frac{1}{2\sqrt{\pi n}}$$
$$= \frac{1}{e^2}\lim_{n\to\infty}\frac{\sqrt{\pi n+\pi}}{\sqrt{\pi n}}$$
$$= \frac{1}{e^2}.$$

Thus, we also have convergence at $x = \frac{1}{4}$

Conclusion. The power series $\sum\limits_{n=1}^{\infty}\frac{(2n)!x^n}{n^{2n}}$ has $R = \frac{1}{4}$ and $I = [-\frac{1}{4}, \frac{1}{4}]$

2. Find the power series for each function with the given center a, and identify its interval of convergence.
   (a) $f(x) = \frac{1}{x};\ a = 1$
   (b) $f(x) = \frac{1}{1-x^2};\ a = 0$
   (c) $f(x) = \frac{1}{2-x};\ a = 1$

Remark. A function of the form $\frac{1}{1-x}$ has the power series $\sum\limits_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ...$. A function of the form $\frac{1}{1-(x-a)}$ has a power series $\sum\limits_{n=0}^{\infty} (x-a)^n$ where a is the center. Additionally, these series converge when $|x| < 1$ and $|x-a| < 1$ respectively

Problem 2a. We can rewrite the function as

$$f(x) = \frac{1}{1-(1-x)}.$$

Which means we have the power series

$$\sum_{n=0}^{\infty} (x-1)^n = 1 + (x-1) + (x-1)^2 + (x-1)^3 + ... \text{ for } |x-a| < 1.$$

This implies the series will converge for

$$\begin{aligned} &|x-1| < 1 \\ &\implies -1 < x-1 < 1 \\ &\implies 0 < x < 2. \end{aligned}$$

Conclusion. When $x = 0$, the series $\sum\limits_{n=0}^{\infty} (-1)^n$ diverges. When $x = 2$ we have $\sum\limits_{n=0}^{\infty} 1$, which also diverges. Therefore $I = (0, 2)$

Problem 2b. For the function

$$f(x) = \frac{1}{1-x^2}.$$

We have the power series

$$\sum_{n=0}^{\infty} x^{2n} \text{ for } |x^2| < 1.$$

This implies the series will converge for

$$\begin{aligned} &|x^2| < 1 \\ &\implies x^2 < 1 \\ &\implies -1 < x < 1. \end{aligned}$$

Conclusion. When $x = -1$ the series $\sum\limits_{n=0}^{\infty} 1$ diverges. For $x = 1$, the series will also diverge. Thus $I = (-1, 1)$

Problem 2d. We can rewrite the function as

$$f(x) = \frac{1}{2-x}$$
$$= \frac{1}{2(1-\frac{x}{2})}.$$

This implies we have the power series

$$\sum_{n=0}^{\infty} \frac{1}{2}(\frac{x}{2})^n \quad \text{for } \left|\frac{x}{2}\right| < 1$$
$$= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} x^n \quad \text{for } \left|\frac{x}{2}\right| < 1.$$

This implies the series will converge for

$$-2 < x < 2.$$

Conclusion. When $x = -2$ the series $\sum_{n=0}^{\infty} \frac{1}{2}(-1)^n$ will diverge. For $x = 2$, the series $\sum_{n=0}^{\infty} \frac{1}{2}$ will diverge. Thus we have $I = (-2, 2)$