Remark. For a series $\sum\limits_{n=1}^{\infty} a_n$ with positive terms. Then

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \quad \text{(Ratio test)}$$
$$\text{Or: } \rho = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}.$$

i. If $0 \leqslant \rho < 1$, the series will converge absolutely

ii. If $\rho > 1$ or $\rho = +\infty$, the series will diverge

iii. if $\rho = 1$, the test is inconclusive

Problem 1. Test the series below for convergence using the Ratio Test.

$$\sum_{n=1}^{\infty} \frac{n^4}{0.5^n}.$$

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{\frac{(n+1)^4}{0.5^{n+1}}}{\frac{n^4}{0.5^n}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)^4}{0.5^{n+1}} \cdot \frac{0.5^n}{n^4} \right| \\
&= \lim_{n\to\infty} \frac{(n+1)^4}{0.5n^4} \\
&= 2.
\end{aligned}$$

Conclusioin. Thus, this series will diverge

Problem 2. Test the series below for convergence using the Ratio Test.

$$\sum_{n=1}^{\infty} \frac{n+2}{3^{4n+3}}.$$

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{\frac{(n+1)+2}{3^{4(n+1)+3}}}{\frac{n+2}{3^{4n+3}}} \right| \\
&= \lim_{n\to\infty} \left| \frac{\frac{n+3}{3^{4n+7}}}{\frac{n+2}{3^{4n+3}}} \right| \\
&= \lim_{n\to\infty} \left| \frac{n+3}{3^{4n+7}} \cdot \frac{3^{4n+3}}{n+2} \right| \\
&= \lim_{n\to\infty} \frac{n+3}{3^4(n+2)} \\
&= \lim_{n\to\infty} \frac{n+3}{81n+162} \\
&= \frac{1}{81}.
\end{aligned}$$

Thus, this series will converge absolutely

Problem 3. Given the series,

$$\sum_{n=1}^{\infty} \frac{n^3}{9^n} \ .$$

Use the ratio test to test for convergence

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{\frac{(n+1)^3}{9^{n+1}}}{\frac{n^3}{9^n}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)^3}{9^{n+1}} \cdot \frac{9^n}{n^3} \right| \\
&= \lim_{n\to\infty} \frac{(n+1)^3}{9n^3} \\
Test the series below for convergence using the Root Test. &= \frac{1}{9}.
\end{aligned}$$

Thus, this series will converge absolutely

Problem 4. Given the series

$$\sum_{n=1}^{\infty} \frac{8^n}{n!} \ .$$

Use ratio test to test for converge

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{\frac{8^{n+1}}{(n+1)!}}{\frac{8^n}{n!}} \right| \\
&= \lim_{n\to\infty} \left| \frac{8^{n+1}}{(n+1)!} \cdot \frac{n!}{8^n} \right| \\
&= \lim_{n\to\infty} \frac{8}{n+1} \\
&= 0.
\end{aligned}$$

Thus, this series will converge absolutely

Problem 5. Test the series below for convergence using the Root Test.

$$\sum_{n=1}^{\infty} \left(\frac{2n}{6n+4}\right)^n .$$

$$\begin{aligned}\rho &= \lim_{n\to\infty} \left| \left(\frac{(2n)^n}{(6n+4)^n} \right) \right|^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{2n}{6n+4} \\ &= \frac{1}{3}.\end{aligned}$$

Thus, this series will converge absolutely

Problem 6. Test the series below for convergence using the Root Test.

$$\sum_{n=1}^{\infty} \left(\frac{6n}{3n+2}\right)^n .$$

$$\begin{aligned}\rho &= \lim_{n\to\infty} \left| \left(\frac{(6n)^n}{(3n+2)^n} \right) \right|^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{6n}{3n+2} \\ &= 2.\end{aligned}$$

Thus, this series will diverge

Problem 7. Test the series below for convergence using the Root Test.

$$\sum_{n=1}^{\infty} \left(\frac{3n+6}{5n+3}\right)^n .$$

$$\begin{aligned}\rho &= \lim_{n\to\infty} \left| \left(\frac{(3n+6)^n}{(5n+3)^n} \right) \right|^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{3n+6}{5n+3} \\ &= \frac{3}{5}.\end{aligned}$$

Thus, this series will converge absolutely

Problem 8. Given the series

$$\sum_{n=1}^{\infty} \left(\frac{5n^2}{7n+3}\right)^n.$$

Use Root Test to test for convergence.

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \left(\frac{(7n^2)^n}{(7n+3)^n}\right) \right|^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{7n^2}{7n+3} \\
&= \lim_{n\to\infty} \frac{\frac{7n^2}{n}}{\frac{7n}{n}+\frac{3}{n}} \\
&= \lim_{n\to\infty} \frac{7n}{7+\frac{3}{n}} \\
&= \frac{\infty}{7} \\
&= \infty.
\end{aligned}$$

Thus, this series will diverge.

Problem 9. Given the series

$$\sum_{n=1}^{\infty} \frac{(\ln n)^{3n}}{n^n}.$$

Use the Root Test to test for convergence.

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{(\ln n)^{3n}}{n^n} \right|^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{\ln^3 n}{n} \\
&\overset{H}{=} \lim_{n\to\infty} \frac{3\ln^2 n}{n} \\
&\overset{H}{=} \lim_{n\to\infty} \frac{6\ln n}{n} \\
&\overset{H}{=} \lim_{n\to\infty} \frac{6}{n} \\
&= 0.
\end{aligned}$$

Thus, this series will converge absolutely.

Problem 10. Given the series

$$\sum_{n=1}^{\infty} \left(\frac{2}{e} + \frac{5}{n}\right)^n .$$

Use Root Test to test for convergence.

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \left(\frac{2}{e} + \frac{5}{n}\right)^n \right|^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{2}{e} + \frac{5}{n} \\
&= \frac{2}{e}.
\end{aligned}$$

Thus, this series will converge absolutely