Nate Warner MATH 353 Spring 2025

**Problem set 5 - Due: Friday, Feb 14**

1. Show that for any three points $A, B, C$ on any line in $\mathbb{H}$,

$$A\text{-}B\text{-}C \text{ in E} \iff A\text{-}B\text{-}C \text{ in } \mathbb{H}$$

We prove in two parts

(a) $A$-$B$-$C \in \mathbb{E} \implies A$-$B$-$C \in \mathbb{H}$

(b) $A$-$B$-$C \in \mathbb{H} \implies A$-$B$-$C \in \mathbb{E}$

***Proof*** We begin by proving part (a). Assume $A$-$B$-$C$ is true in $\mathbb{E}$ for three distinct collinear points $A, B, C$. Thus,

$$AB + BC + AC$$

For $A$-$B$-$C$ (B between A and C) in the hypebolic plane (Poincare model), We require $d_{\mathbb{H}}(AB) + d_{\mathbb{H}}(BC) = d_{\mathbb{H}}(AC)$. That is,

$$\ln\left(\frac{e(AN)e(BM)}{e(AM)e(BN)}\right) + \ln\left(\frac{e(BN)e(CM)}{e(BM)e(CN)}\right) = \ln\left(\frac{e(AN)e(CM)}{e(AM)e(CN)}\right)$$

We have

$$\begin{aligned}
&\ln\left(\frac{e(AN)e(BM)}{e(AM)e(BN)}\right) + \ln\left(\frac{e(BN)e(CM)}{e(BM)e(CN)}\right) \\
&= \ln(e(AN)) + \ln(e(BM)) - \ln(e(AM)) - \ln(e(BN)) \\
&+ \ln(e(BN)) + \ln(e(CM)) - \ln(BM) - \ln(CN) \\
&= \ln(e(AN)) - \ln(e(AM)) + \ln(e(CM)) - \ln(e(CN)) \\
&= \ln(e(AN)) + \ln(e(CM)) - \ln(e(AM)) - \ln(e(CN)) \\
&= \ln\left(\frac{e(AN)e(CM)}{e(AM)e(CN)}\right) = d_{\mathbb{H}}(AC)
\end{aligned}$$

Thus, $A$-$B$-$C$ in $\mathbb{E}$ implies $A$-$B$-$C$ in $\mathbb{H}$. Similarly, $B$-$A$-$C$ in $\mathbb{E}$ implies $B$-$A$-$C$ in $\mathbb{H}$, and $A$-$C$-$B$ in $\mathbb{E}$ implies $A$-$C$-$B$ in $\mathbb{H}$

By the UMT, since $A$-$B$-$C$ occurs in $\mathbb{E}$, both $B$-$A$-$C$ and $A$-$C$-$B$ will not occur. Exactly one of them will occur, and each relation in $\mathbb{E}$ implies the same relation happens in $\mathbb{H}$

(b) If $A$-$B$-$C$ happens in $\mathbb{H}$, then by the UMT the other two do not. But since $A, B, C$ are distinct and collinear, one of them must occur in $\mathbb{E}$, so only $A$-$B$-$C$ will be true in $\mathbb{E}$ by the UMT ■

2. Show that in example 6.1, the relations $A$-$C$-$B$, $A$-$D$-$B$, $C$-$A$-$D$, and $C$-$B$-$D$ hold

We have distances

| | $A$ | $B$ | $C$ | $D$ | $E$ |
|---|---|---|---|---|---|
| $A$ | 0 | 3 | 1 | 2 | 4 |
| $B$ | 3 | 0 | 2 | 1 | 4 |
| $C$ | 1 | 2 | 0 | 3 | 4 |
| $D$ | 2 | 1 | 3 | 0 | 4 |
| $E$ | 4 | 4 | 4 | 4 | 0 |

We have

$$\begin{aligned} AC + CB = 1 + 2 = 3 = AB &\implies A\text{-}C\text{-}B \\ AD + DB = 2 + 1 = 3 = AB &\implies A\text{-}D\text{-}B \\ CA + AD = 1 + 2 = 3 = CD &\implies C\text{-}A\text{-}D \\ CB + BD = 2 + 1 = 3 = CD &\implies C\text{-}B\text{-}D \end{aligned}$$

■

3. Assume the first seven axioms. Suppose that $A, B, X, Y$ are distinct, collinear points such that the distance between any two of them is less than $\omega$ and such that $Y \in \overline{AB}$, $X \in \overrightarrow{AB}, X \notin \overline{AB}$, and $B \in \overline{XY}$. Prove that $Y \in \overline{AX}$

***Proof.*** Assume $A, B, X, Y$ are distinct, collinear points such that the distance between two of them is less than $\omega$. Further, assume that $Y \in \overline{AB}$, $X \in \overrightarrow{AB}$, $X \notin \overline{AB}$, and $B \in \overline{XY}$. We aim to show that $Y \in \overline{AX}$. More specifically, that $A$-$Y$-$X$, or $AY + YX + AX$

Since the distance between any two of the given points is less than $\omega$, all rays and segments involving any pair of points are well defined. Using the given information, we have

$$\begin{aligned} Y \in \overline{AB} &\implies A\text{-}Y\text{-}B \implies AY + YB = AB && (1) \\ X \in \overrightarrow{AB} &\implies A\text{-}X\text{-}B \text{ or } A\text{-}B\text{-}X \\ X \notin \overline{AB} &\implies \neg(A\text{-}X\text{-}B) \implies A\text{-}B\text{-}X \implies AB + BX = AX && (2) \\ B \in \overline{XY} &\implies X\text{-}B\text{-}Y \implies XB + BY = XY && (3) \end{aligned}$$

Observe that since $AY + YB = AB$, and $AB + BX = AX$, we have $AY + YB + BX = AX$. Next, notice that $XB + BY = XY \implies BX + YB = YX$ by distance axiom 3. Since these distances are just real numbers, we can rearrange the expression as $YB = YX - BX$. We can then plug this expression into $AY + YB + BX = AX$ to get

$$\begin{aligned} AY + YB + BX &= AX \\ \implies AY + YX - BX + BX &= AX \\ \implies AY + YX &= AX \end{aligned}$$

Which, by the definition of betweenness, $A$-$Y$-$X$. Which, by the definition of the segment $\overline{AX} = \{P : A\text{-}P\text{-}X\}$, means that $y \in \overline{AX}$ ■

4. Construct an example of a plane $\mathbb{P}$ that satisfies the first seven axioms, with a ray $\overrightarrow{AB}$ and points $X \neq Y$ in $\overrightarrow{AB}$ such that $AX = AY$

Let $\mathbb{P} = \{A, B, X, Y\}$, $\mathbb{L} = \{A, B, X, Y\}, \{X, Y\}$, with distances

| | $A$ | $B$ | $X$ | $Y$ |
|---|---|---|---|---|
| $A$ | 0 | 2 | 1 | 1 |
| $B$ | 2 | 0 | 1 | 1 |
| $X$ | 1 | 1 | 0 | 3 |
| $Y$ | 1 | 1 | 3 | 0 |

Which satisfies distance axioms

1. $PQ \geqslant 0$
2. $PQ = 0 \iff P = Q$
3. $PQ = QP$

And incidence axioms

(a) At least two lines

(b) Each line contains at least two different points

(c) Each pair of points are together in at least one line

(d) Each pair of points $P, Q$ with $PQ < \omega$ are together in at most one line