Nate Warner MATH 230 September 15, 2023

Homework/Worksheet 3 - Due: Wednesday, September 20

1. Find the area between the curves $y = \cos\theta$ and $y = 0.5$, for $0 \leqslant \theta \leqslant \pi$

$$\cos\theta = 0.5$$
$$\cos\theta = \frac{1}{2}$$
$$\theta = \cos^{-1}\frac{1}{2}$$
$$\theta = \frac{\pi}{3}.$$

$$|\cos\theta - 0.5| = \begin{cases} \cos\theta - 0.5 & \text{if } 0 \leqslant \theta \leqslant \frac{\pi}{3} \\ -\cos\theta + 0.5 & \text{if } \frac{\pi}{3} < \theta \leqslant \pi \end{cases} \tag{1}$$

$$\begin{aligned}
A &= \int_a^b f(x) - g(x)\ dx \quad \text{For } f(x) \geqslant g(x) \\
&= \int_0^\pi |\cos\theta - 0.5|\ d\theta \\
&= \int_0^{\frac{\pi}{3}} \cos\theta - 0.5\ d\theta + \int_{\frac{\pi}{3}}^{\pi} -\cos\theta + 0.5\ d\theta \\
&\text{Where } I_1 = \int_0^{\frac{\pi}{3}} \cos\theta - 0.5\ d\theta \\
I_2 &= +\int_{\frac{\pi}{3}}^{\pi} -\cos\theta + 0.5\ d\theta \\
A &= I_1 + I_2 \\
I_1 &= \sin\theta - \frac{1}{2}\theta\Big|_0^{\frac{\pi}{3}} \\
&= \left(\sin\left(\frac{\pi}{3}\right) - \frac{1}{2}\left(\frac{\pi}{3}\right)\right) - \left(\sin 0\right) \\
&= \left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right) \\
&= \frac{3\sqrt{3} - \pi}{6} \\
I_2 &= \int_{\frac{\pi}{3}}^{\pi} -\cos\theta + 0.5\ d\theta \\
&= -\sin\theta + \frac{1}{2}\theta\Big|_{\frac{\pi}{3}}^{\pi} \\
&= \left(-\sin\left(\pi\right) + \frac{1}{2}\left(\pi\right)\right) - \left(-\sin\left(\frac{\pi}{3}\right) + \frac{1}{2}\left(\frac{\pi}{3}\right)\right) \\
&= \frac{\pi}{2} - \left(-\frac{\sqrt{3}}{2} + \frac{\pi}{6}\right) \\
&= \frac{\pi}{2} - \left(-\frac{3\sqrt{3} - \pi}{6}\right) \\
&= \frac{\pi}{2} + \frac{3\sqrt{3} - \pi}{6} \\
&= \frac{3\sqrt{3} + 2\pi}{6} \\
\therefore A &= \frac{3\sqrt{3} - \pi}{6} + \frac{3\sqrt{3} + 2\pi}{6} \\
&= \frac{6\sqrt{3} + \pi}{6} \\
&= \sqrt{3} + \frac{\pi}{6}.
\end{aligned}$$

2. Sketch the region enclosed by the given curves below and find its area.

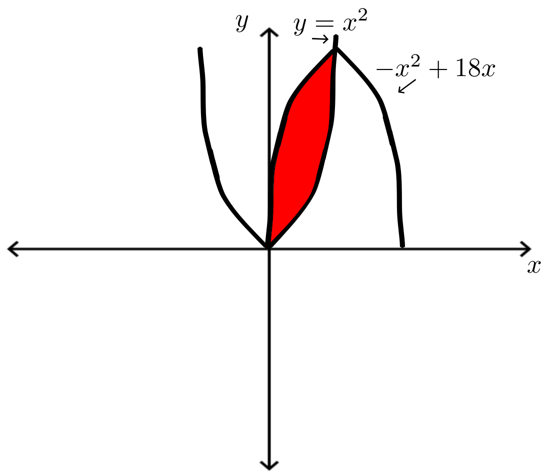
(a) $y = x^2, \quad y = -x^2 + 18x$

(b) $y = \cos x, \quad y = 2 - \cos x, \quad 0 \leqslant x \leqslant 2\pi$

(c) $y = x^3, \quad y = x^2 - 2x, \quad x = -1, \quad x = 1$

(d) $x = y^2, \quad x = y + 2$

2.a



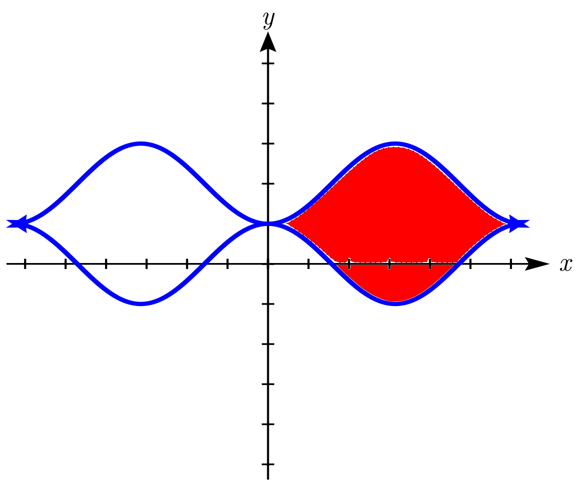
Intersection:

$$\begin{aligned} x^2 &= -x^2 + 18x \\ 2x^2 - 18x &= 0 \\ 2x(x-9) &= 0 \\ x &= 0, 9. \end{aligned}$$

Thus:

$$\begin{aligned} A &= \int_0^9 -x^2 + 18x - x^2 \; dx \\ &= \int_0^9 -2x^2 + 18x \; dx \\ &= -\frac{2}{3}x^3 + 9x^2 \Big|_0^9 \\ &= -\frac{2}{3}(9)^3 + 9(9)^2 \\ &= -\frac{1458}{3} + 729 \\ &= 243. \end{aligned}$$

2.b



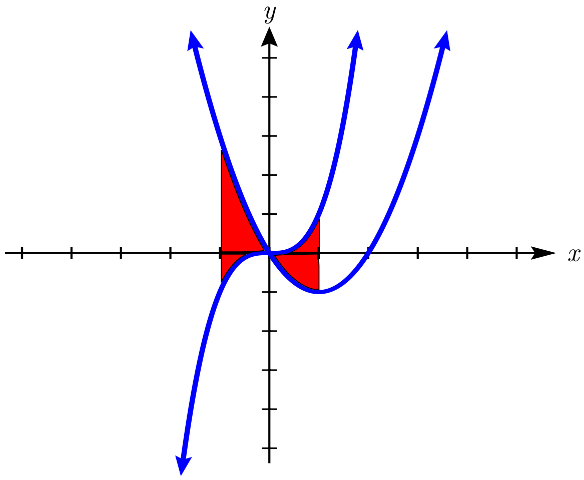
Intersection:

$$\begin{aligned} \cos x &= 2 - \cos x \\ 2\cos x &= 2 \\ \cos x &= 1 \\ x &= \cos^{-1} 1 \\ x &= 0, 2\pi. \end{aligned}$$

Thus:

$$\begin{aligned} &\int_0^{2\pi} 2 - \cos x - \cos x \; dx \\ &= \int_0^{2\pi} 2 - 2\cos x \; dx \\ &= \int_0^{2\pi} 2(1 - \cos x) \; dx \\ &= 2\left[x - \sin x\right]_0^{2\pi} \\ &= 2[2\pi - \sin(2\pi)] \\ &= 2(2\pi) \\ &= 4\pi. \end{aligned}$$

2.c



Thus:

$$\int_{-1}^{0} \; x^2 - 2x - x^3 \; dx + \int_{0}^{1} \; x^3 - (x^2 - 2x) \; dx$$

$$\text{Where: } I_1 = \int_{-1}^{0} \; x^2 - 2x - x^3 \; dx$$

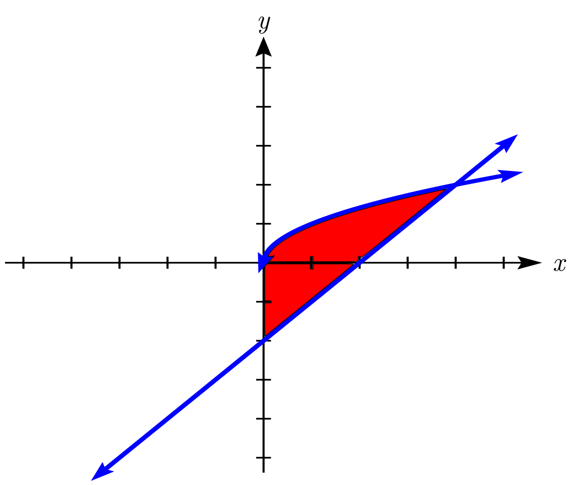
$$I_2 = \int_{0}^{1} \; x^3 - x^2 + 2x \; dx$$

$$\begin{aligned}
I_1 &= \frac{1}{3}x^3 - x^2 - \frac{1}{4}x^4 \; \Big|_{-1}^{0} \\
&= -\left(\frac{1}{3}(-1)^3 - (-1)^2 - \frac{1}{4}(-1)^4\right) \\
&= -\left(-\frac{1}{3} - 1 - \frac{1}{4}\right) \\
&= \frac{19}{12}
\end{aligned}$$

$$\begin{aligned}
I_2 &= \frac{1}{4}x^4 - \frac{1}{3}x^3 + x^2| \; \Big|_{0}^{1} \\
&= \frac{1}{4}(1)^3 - \frac{1}{3}(1)^3 + (1)^2 \\
&= \frac{1}{4} - \frac{1}{3} + 1 \\
&= \frac{11}{12}
\end{aligned}$$

$$\begin{aligned}
&\therefore A = I_1 + I_2 = \frac{19}{12} + \frac{11}{12} = \frac{30}{12} \\
&= \frac{5}{2}.
\end{aligned}$$

2.d



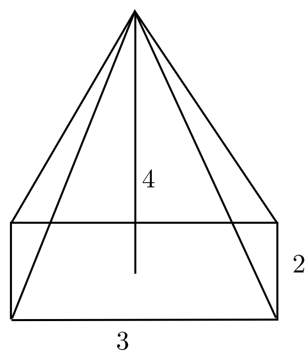
Intersection:

$$\begin{aligned}
x^{\frac{1}{2}} &= x - 2 \\
x &= (x-2)^2 \\
x &= x^2 - 4x + 4 \\
&x^2 - 5x + 4 = 0 \\
&(x-1)(x-4) \\
&\cancel{x = 1} \; nas, \; x = 4.
\end{aligned}$$

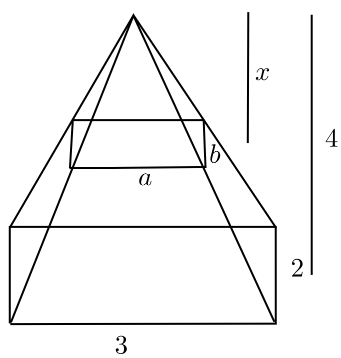
Thus:

$$\begin{aligned}
A &= \int_{0}^{4} \; x^{\frac{1}{2}} - (x - 2) \; dx \\
&= \int_{0}^{4} \; x^{\frac{1}{2}} - x + 2 \; dx \\
&= \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2 + 2x \; \Big|_{0}^{4} \\
&= \frac{16}{3}.
\end{aligned}$$

3. Find the volume of the pyramid below by using the slicing method



We can see that the base of this object is a rectangle. Thus, the cross sections will also be rectangles. With the area of the cross section being $A = ab$. If we define a cross section with some length a, some width b, and some height x, we have:



We can use proportion of similar triangles to find formulas for the lengths of a and b:

$$\frac{3}{4} = \frac{a}{x}$$
$$\frac{3}{4}x = a.$$

$$\frac{2}{4} = \frac{b}{x}$$
$$\frac{2}{4}x = b.$$

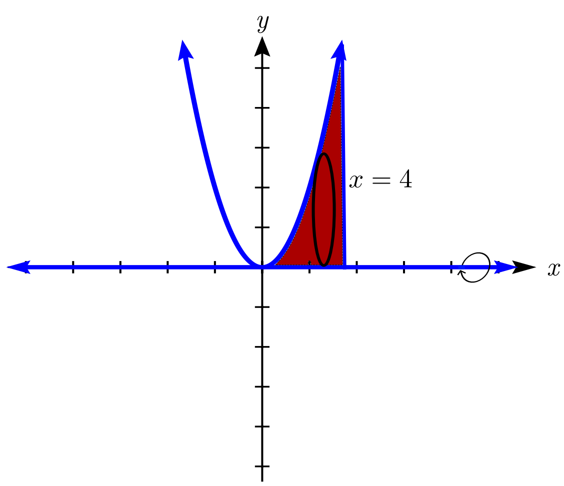
Thus we now have the formula for the area of a cross section $A(x)$, and we can use the volume equation $V = \int_a^b A(x)\ dx$ to find the volume of this shape.

$$A(x) = \left(\frac{3}{4}x\right)\left(\frac{1}{2}x\right)$$
$$= \frac{3}{8}x^2$$
$$\implies V = \int_0^4 \frac{3}{8}x^2\ dx$$
$$= \frac{3}{8}\int_0^4 x^2\ dx$$
$$= \frac{3}{8}\left[\frac{1}{3}x^3\right]_0^4$$
$$= \frac{3}{8}\left(\frac{1}{3}(4)^3\right)$$
$$= 8$$
$$\therefore V = 8 \text{ units}^3.$$

4. Find the volume of the solid obtained by rotating the region bounded by the curves below about the specified line. Sketch the region, the solid, and a typical disk or washer.

1. $y = 2x^2, \quad x = 0, \quad x = 4, \quad y = 0$; about the x-axis
2. $y = 4 - x^2, \quad y = 2 - x$; about the x-axis
3. $y = 1 + e^x, \quad x = 0, \quad x = 1, \quad y = 0$; about the x-axis
4. $y = 2x^3, \quad x = 0, \quad x = 1, \quad y = 0$; about the y-axis
5. $y = \sqrt{4 - x^2}, \quad y = 0, \quad x = 0$; about the y-axis
6. $y = \sin x, \quad y = \cos x, \quad 0 \leqslant x \leqslant \frac{\pi}{4}$; about $y = -1$

4.1

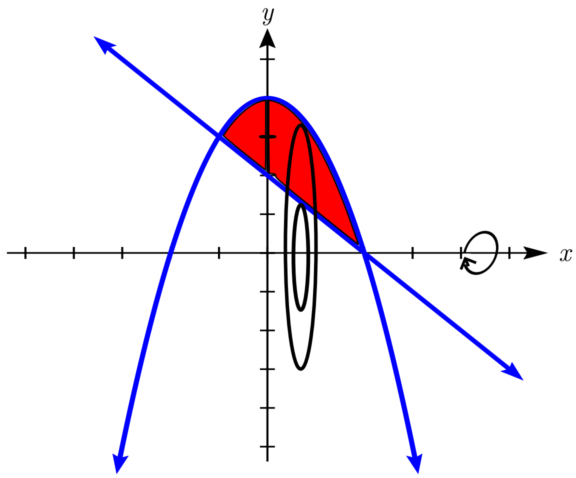


We can see from the figure that if we revolve this region around the $x - axis$, we will get a disk shaped cross section. Thus, the area of the cross section is given by $\pi(f(x))^2$, where $f(x)$ is the radius.

From this we can compute the volume:

$$\begin{aligned}
V &= \int_0^4 \pi[2x^2]^2 \ dx \\
&= \pi \int_0^4 4x^4 \ dx \\
&= 4\pi \int_0^4 x^4 \ dx \\
&= 4\pi \left[\frac{1}{5}x^5\right]_0^4 \\
&= \frac{4\pi}{5}\left((4)^5\right) \\
&= \frac{4096\pi}{5}.
\end{aligned}$$

4.2



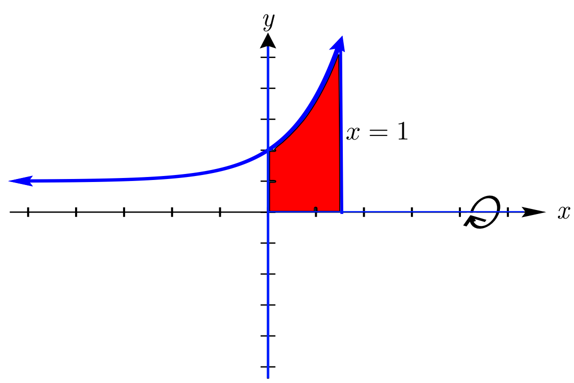
So we can from the figure that when revolved around the $x - axis$, we will end up with an annulus cross section. Where the area is given by πr^2, and the radius given by $f(x) - g(x)$, with $f(x) = 4 - x^2$ and $g(x) = 2 - x$.

Intersection:

$$\begin{aligned}
&4 - x^2 = 2 - x \\
&-x^2 + x + 2 = 0 \\
&-(x^2 - x - 2) = 0 \\
&-(x+1)(x-2) = 0 \\
&x = -1, 2.
\end{aligned}$$

$$\begin{aligned}
V &= \int_{-1}^{2} \pi\left[(4-x^2)^2 - (2-x)^2\right]\, dx \\
&= \int_{-1}^{2} \pi\left[x^4 - 8x^2 + 16 - (x^2 - 4x + 4)\right]\, dx \\
&= \pi \int_{-1}^{2} x^4 - 9x^2 + 4x + 12\; dx \\
&= \pi\left[\frac{1}{5}x^5 - 3x^3 + 2x^2 + 12x\right]_{-1}^{2} \\
&= \pi\left[\left(\frac{1}{5}(2)^5 - 3(2)^3 + 2(2)^2 + 12(2)\right)\right. \\
&\quad \left. - \left(\frac{1}{5}(-1)^5 - 3(-1)^3 + 2(-1)^2 + 12(-1)\right)\right] \\
&= \frac{72}{5} + \frac{36}{5} \\
&= \frac{108\pi}{5}.
\end{aligned}$$

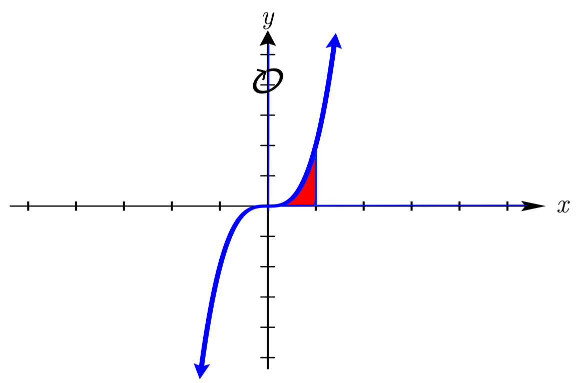
4.c



Thus:

$$\begin{aligned}
V &= \int_0^1 \pi[e^x + 1]^2\, dx \\
&= \pi \int_0^1 e^{2x} + 2e^x + 1\; dx \\
&= \pi\left[\frac{1}{2}e^{2x} + 2e^x + x\right]_0^1 \\
&= \pi\left[\left(\frac{1}{2}e^2 + 2e^1 + 1\right) - \left(\frac{1}{2}e^0 + 2e^0\right)\right] \\
&= \pi\left[\frac{e^2}{2} + 2e + 1 - \left(\frac{1}{2} + 2\right)\right] \\
&= \pi\left[\frac{e^2 + 4e + 2}{2} - \frac{5}{2}\right] \\
&= \frac{\pi e^2 + 4\pi e - 3\pi}{2}.
\end{aligned}$$

4.d

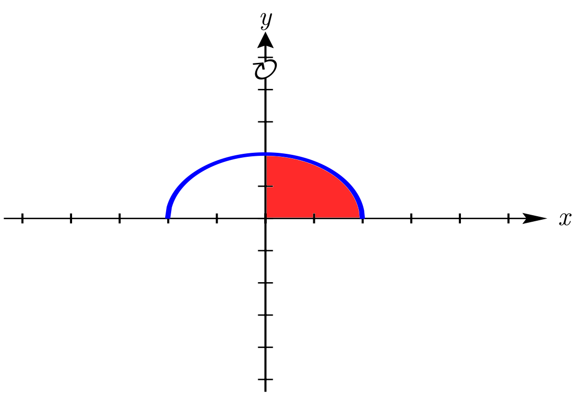


If $y = 2x^3$, then:

$$x = \left(\frac{y}{2}\right)^{\frac{1}{3}}.$$

$$\begin{aligned}
\implies V &= \int_0^2 \pi \left[\left(\frac{y}{2}\right)^{\frac{1}{3}}\right]^2 dy \\
&= \pi \int_0^2 \left(\frac{y}{2}\right)^{\frac{2}{3}} dy \\
&= \pi \int_0^2 \frac{1}{2^{\frac{2}{3}}} \cdot y^{\frac{2}{3}} dy \\
&= \frac{\pi}{4^{\frac{1}{3}}}\left[\frac{3}{5}y^{\frac{5}{3}}\right]_0^2 \\
&= \frac{3\pi}{5 \cdot 4^{\frac{1}{3}}}(2)^{\frac{5}{3}} \\
&= \frac{3\pi \cdot 32^{\frac{1}{3}}}{5 \cdot 4^{\frac{1}{3}}} \\
&= \frac{3\pi}{5} \cdot \left(\frac{32}{4}\right)^{\frac{1}{3}} \\
&= \frac{3\pi}{5} \cdot (8)^{\frac{1}{3}} \\
&= \frac{3\pi}{5} \cdot (2) \\
\therefore V &= \frac{6\pi}{5}.
\end{aligned}$$

4.e



Remark. Semi Circle with radius 2

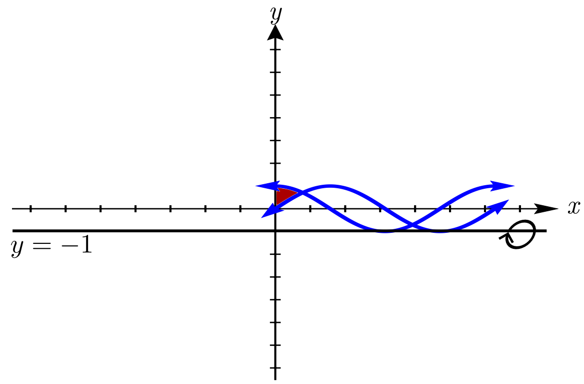
If $y = \sqrt{4 - x^2}$, then:

$$x = \sqrt{4 - y^2}.$$

Thus:

$$\begin{aligned}
\implies V &= \int_0^2 \pi \left(\sqrt{4 - y^2}\right)^2 dy \\
&= \pi \int_0^2 4 - y^2 \, dy \\
&= \pi \left[4y - \frac{1}{3}y^3\right]_0^2 \\
&= \pi \left(8 - \frac{8}{3}\right) \\
\therefore V &= \frac{16\pi}{3}.
\end{aligned}$$

4.f



Proposition. If we rotate some region R around a line that is not the x or y axis, then the radius of the disk is given by $R = f(x) + k \iff$ A.O.R is $y = -k$ else if A.O.R $y = k \to R = k - f(x)$

Thus:

$$\implies V = \int_0^{\frac{\pi}{4}} \pi\left[(\cos(x)+1)^2 - (\sin(x)+1)^2\right]\, dx.$$

$$\begin{aligned}
&= \pi \int_0^{\frac{\pi}{4}} \cos^2(x) + 2\cos(x) + 1 - (\sin^2(x) + 2\sin(x) + 1)\, dx \\
&= \pi \int_0^{\frac{\pi}{4}} \cos^2(x) + 2\cos(x) + 1 - \sin^2(x) - 2\sin(x) - 1\, dx \\
&= \pi \int_0^{\frac{\pi}{4}} \cos^2(x) - \sin^2(x) + 2\cos(x) - 2\sin(x)\, dx \\
&= \pi \int_0^{\frac{\pi}{4}} \cos(2x) + 2\cos(x) - 2\sin(x)\, dx \\
&= \pi\left[\int_0^{\frac{\pi}{4}} \cos(2x)\, dx + \int_0^{\frac{\pi}{4}} 2\cos(x) - 2\sin(x)\, dx\right].
\end{aligned}$$

Interlude. Let $I_1 = \int_0^{\frac{\pi}{4}} \cos(2x)\, dx$ and $I_2 = \int_0^{\frac{\pi}{4}} 2\cos(x) - 2\sin(x)\, dx$. Thus, V will be given be $\pi(I_1 + I_2)$

Regarding I_1 :

$$\begin{aligned}
&\text{Let } u = 2x \\
&\frac{1}{2}du = dx \\
&u(a) = 0, \quad u(b) = \frac{\pi}{2}.
\end{aligned}$$

Thus:

$$\begin{aligned}
\implies I_1 &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(u)\, du \\
&= \frac{1}{2}\left[\sin(u)\right]_0^{\frac{\pi}{2}} \\
&= \frac{1}{2}(\sin\left(\frac{\pi}{2}\right)) \\
\therefore I_1 &= \frac{1}{2}.
\end{aligned}$$

Regarding I_2:

$$\begin{aligned}
I_2 &= \int_0^{\frac{\pi}{4}} 2\cos(x) - 2\sin(x)\, dx \\
&= 2\sin(x) + 2\cos(x)\Big|_0^{\frac{\pi}{4}} \\
&= \left(2\sin\left(\frac{\pi}{4}\right) + 2\cos\left(\frac{\pi}{4}\right)\right) - \left(2\sin(0) + 2\cos(0)\right) \\
&= 2\sqrt{2} - 2.
\end{aligned}$$

Therefore:

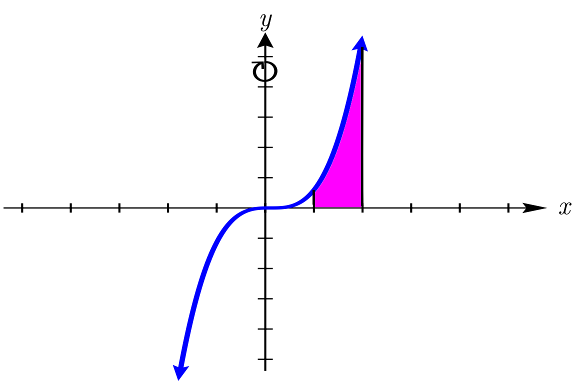
$$
\begin{aligned}
V &= \pi\left(\frac{1}{2} + 2\sqrt{2} - 2\right) \\
&= \frac{\pi}{2} + 2\pi\sqrt{2} - 2\pi \\
&= \frac{-3\pi + 4\pi\sqrt{2}}{2} \\
&= -\frac{3\pi - 4\pi\sqrt{2}}{2}.
\end{aligned}
$$

5. Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the y-axis.

(a) $y = x^3,\ y = 0,\ x = 1,\ x = 2$

(b) $y = x^2,\ y = 6x - 2x^2$

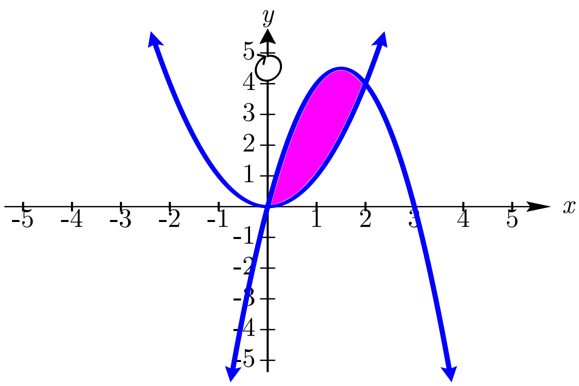
5.a:



Thus:

$$
\begin{aligned}
V &= \int_a^b 2\pi x f(x)\ dx \\
&= \int_1^2 2\pi x(x^3)\ dx \\
&= 2\pi \int_1^2 x^4\ dx \\
&= 2\pi\left[\frac{1}{5}x^5\right] \\
&= \frac{2\pi}{5}\left[2^5 - 1^5\right] \\
&= \frac{2\pi}{5}(31) \\
&= \frac{62\pi}{5}.
\end{aligned}
$$

5.b



Intersection:

$$
\begin{aligned}
x^2 &= -2x^2 + 6x \\
3x^2 - 6x &= 0 \\
3x(x-2) &= 0 \\
x &= 0, 2.
\end{aligned}
$$

Thus:

$$
\begin{aligned}
V &= \int_0^2 \; 2\pi x\Big[-2x^2 + 6x - x^2\Big]\; dx \\
&= 2\pi \int_0^2 \; x(-3x^2 + 6x)\; dx \\
&= 2\pi \int_0^2 \; -3x^3 + 6x^2 \; dx \\
&= 2\pi \int_0^2 \; -3(x^3 - 2x^2)\; dx \\
&= -6\pi \int_0^2 \; x^3 - 2x^2 \; dx \\
&= -6\pi \left[\frac{1}{4}x^4 - \frac{2}{3}x^3\right]_0^2 \\
&= -6\pi \left[4 - \frac{16}{3}\right] \\
&= -6\pi\left(-\frac{4}{3}\right) \\
&= \frac{24\pi}{3} \\
&= 8\pi.
\end{aligned}
$$