Comprehensive Compendium: Geometry, Pre-Algebra, Pre-Calculus, Trigonometry, and Calculus 1 (IN PROGRESS)

Nathan Warner

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Contents

1 Geometry

1.1 Line

- Equation

$$y = mx + b.$$

1.2 Square

- Perimeter

$$4s.$$

- Area

$$s^2.$$

1.3 Rectangle

- Perimeter

$$2w + 2l.$$

- Area

$$l \cdot w.$$

1.4 Triangle

Types:

- Right (one angles measures 90°)
- acute (no angles measure more than 90°)
- obtuse (one of the angles measures more than 90°)
- isosceles (Two sides have the same length)
- equilateral (all 3 sides have the same length)
- scalene (no sides are the same length)
- Perimeter

$$a + b + c.$$

- Area

$$\frac{1}{2}bh.$$

1.5 Circle

- Diameter

$$2r.$$

- Circumference

$$2\pi r \ or \ \pi d.$$

- Area

$$\pi r^2.$$

- Area of a sector

$$\frac{1}{2}\pi r\theta.$$

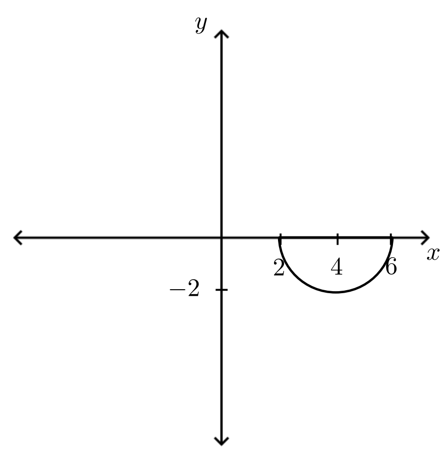
- Equation of a circle whos center is not the orgin (0,0)

$$(x-h)^2+(y-k)^2=r^2, \text{ Where (h,k) is the center of the circle and r is the radius.}$$

- equation of a circle whos center is at the orgin (0,0)

$$x^2+y^2=r^2.$$

- equation of a semicirle whos center is not at the orgin
  - say we have



We can see we have a radius of 2, and a center at (4,0), our equation for the full circle would be:

$$(x-4)^2+(y-0)^2=2^2.$$

Solving for y we get:

$$(y-0)^2=4-(x-4)^2$$
$$y=\pm\sqrt{4-(x-4)^2}.$$

And since we only have the bottom portion of this circle we **only** use the negative sign of the $\pm$

1.6 Trapezoid

- Perimeter

$$a + b + c + d.$$

- Area

$$\frac{a+b}{2}h.$$

1.7 Parallelogram

- Perimeter

$$2l + 2w.$$

- Area

$$lh.$$

1.8 Cube

- Volume

$$s^3.$$

- Surface Area

$$6s^2.$$

1.9 Square Prism

- Volume

$$a^2h.$$

- Surface Area

$$2a^2 + 4ah.$$

1.10 Rectangular Prism

- Volume

$$lwh.$$

- Surface Area

$$2lh + 2wh + 2wl.$$

1.11 Sphere

- Volume

$$\frac{4}{3}\pi r^3.$$

- Surface Area

$$4\pi r^2.$$

1.12 Cylinder

- Volume

$$\pi r^2 h.$$

- Surface Area

$$2\pi rh + 2\pi r^2.$$

1.13 Cone

- Volume

$$\frac{1}{3}\pi r^2 h.$$

- Surface Area

$$\pi r(r + \sqrt{h^2 + r^2}).$$

1.14 Square Pyramid

- Volume

$$a^2\frac{h}{3}.$$

- Surface Area

$$a^2 + 2a\sqrt{\frac{a^2}{4} + h^2}.$$

1.15 Rectangular Pyramid

- Volume

$$\frac{1}{3}lwh.$$

- Surface Area

$$lw + l\sqrt{\left(\frac{w}{2}\right)^2 + h^2} + w\sqrt{\left(\frac{l}{2}\right)^2 + h^2}.$$

2 Pre-Algebra

2.1 Types of numbers

- Integer
$$2.$$
- Rational
$$\frac{1}{2}.$$
- Irrational
$$\sqrt{2}.$$
- Prime Number
$$7.$$
- Complex
$$3 + 2i.$$

2.2 Slope

We can calculate slope with:

$$\frac{y_2 - y_1}{x_2 - x_1}.$$

And we can write the equation of the line with point-slope form:

$$y - y_1 = m(x - x_1), \text{ Where } m \text{ denotes slope.}$$

Additionally, if we know the y-intercept b, we can write the equation with slope intercept form:

$$y = mx + b.$$

2.3 Solving Inequalites

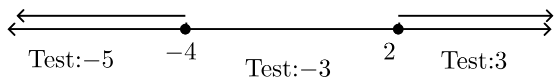
- If you divide by a negative, flip the inequality
- Quadratic Inequalites

$$\begin{aligned} x^2 + 2x - 8 &\geqslant 0 \\ (x-2)(x+4) &\geq 0 \\ x = 2 \;\; x &= -4. \end{aligned}$$

$$\begin{aligned} -5 : (-)(-) &\to + \geqslant 0 \\ -3 : (-)(+) &\to + \leqslant 0 \\ -5 : (+)(+) &\to + \geqslant 0 \\ &. \end{aligned}$$

Therefore:

$$(-\infty, -4) \cup (2, \infty).$$



2.4 Solving absolute value equations

Definition:

An absolute value equation is an equation that involves the absolute value of a variable. The absolute value of a number is its distance from zero on a number line, and it is always non-negative (or zero itself). The absolute value of a real number x is denoted as |x|.

Example 2.1

Consider the equation

$$|3x + 4| = 9.$$

- Solve for x with positive 9
- Solve for x with negative 9

So:

$$\begin{aligned} 3x + 4 &= 9 \\ x &= \frac{5}{3} \end{aligned}$$

And:

$$\begin{aligned} 3x + 4 &= -9 \\ x &= -\frac{13}{3}. \end{aligned}$$

Example 2.2

Consider the equation.

$$4|2x + 3| - 8 = 36.$$

- Get the absolute value alone
- solve for positive right side
- solve for negative right side

So:

$$\begin{aligned} |2x + 3| &= 11 \\ 2x + 3 &= 11 \\ x &= 4 \\ And: 2x + 3 &= -11 \\ x &= -7 \end{aligned}$$

2.5 Solving Systems of Equations.

Definition:

Systems of equations refer to a set of two or more equations that are considered together as a unit. These equations involve multiple variables and are interconnected, meaning the solution to the system must satisfy all the equations simultaneously. The variables in the system are typically related to each other in some way.

The two methods we will look at for solving these systems are:

- Elimination (Additon)
- Substitution

Example 2.3

Consider the system:

$$\begin{aligned} 2x + 3y &= 8 \\ 5x - 3y &= -1. \end{aligned}$$

To solve this system using **Elimination**, we will add the two equations together, so:

$$\begin{aligned} 7x &= 7 \\ x &= 1. \end{aligned}$$

Now that we have the value of x, we can pick one of the equations to plug it into to solve for y.

$$\begin{aligned} 2(1) + 3y &= 8 \\ 2 + 3y &= 8 \\ 3y &= 6 \\ y &= 2. \end{aligned}$$

Example 2.4

Consider the system:

$$\begin{aligned} 2x + 5y &= 19 \\ x - 2y &= -4. \end{aligned}$$

We can see that if we use the **elimination** method, none of our variable terms will cancel. In order to make it so our $x's$ will cancel with each other, we must multiply the equation by -2.

$$-2x + 4y = 8.$$

From here we can then use the elimination method.

Example 2.5

Now we will look at using the **Substitution method**, say we have the system:

$$\begin{aligned} y &= 5 - 2x \\ 4x + 3y &= 13. \end{aligned}$$

Using the Substitution method, we can plug our first equation into the y term in our second equation.

$$\begin{aligned} 4x + 3(5 - 2x) &= 13 \\ 4x + 15 - 6x &= 13 \\ -2x + 15 &= 13 \\ -2x &= -2 \\ x &= 1. \end{aligned}$$

Then we can plug x into our first equation.

$$\begin{aligned} y &= 5 - 2(1) \\ y &= 3. \end{aligned}$$

Example 2.6

Consider:

$$\begin{aligned} y &= 3x + 2 \\ y &= 7x - 6. \end{aligned}$$

With this system, we can replace the y in our second equation, with the first equation. So:

$$\begin{aligned} 3x + 2 &= 7x - 6 \\ 4x &= 8 \\ x &= 2. \end{aligned}$$

Then we can plug x into any of our equations to get y

$$\begin{aligned} y &= 3(2) + 2 \\ y &= 8. \end{aligned}$$

3 Algebra

3.1 Theorems of algebra

3.2 Difference Quotient

Theorem 3.1 Difference Quotient

$$\frac{f(x+h)-f(x)}{h},\ h \neq 0.$$

The difference quotient is a mathematical concept used to calculate the average rate of change of a function over a given interval. Specifically, it is defined as the change in the function's output value divided by the change in the function's input value over the interval.

3.3 Distance Formula

Theorem 3.2 Distance Formula

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

The distance formula is a mathematical formula used to calculate the distance between two points in a coordinate plane.

The distance formula can be used to calculate the distance between any two points in a two-dimensional coordinate plane, regardless of whether the points are on a straight line or a curved line. For example, if you have two points on a circle, you can use the distance formula to calculate the length of the arc between the two points. Similarly, if you have two points on a parabola, you can use the distance formula to calculate the distance between them.

3.4 Average Rate of Change

Theorem 3.3 Average Rate of Change

$$m = \frac{f(b)-f(a)}{b-a}.$$

The average rate of change is a mathematical concept used to measure the average rate at which a quantity changes over a given time interval. In calculus, the average rate of change is typically used to describe the average slope of a curve over a specific interval.

3.5 Midpoint

Theorem 3.4 Midpoint

$$(x_m, y_m) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

The midpoint formula can be used to find the coordinates of the center point of a line segment, which is the point that divides the line segment into two equal parts. This formula can be used in a variety of situations, such as when determining the center of a geometric shape, finding the average of two values, or calculating the half-way point between two locations.

The midpoint formula is also a useful tool in geometry for finding the center of a circle or the midpoint of an arc. By using the midpoint formula to calculate the center point of a circle or arc, it is possible to find its radius and diameter, as well as its position relative to other objects in the coordinate plane.

3.6 Quadratic Formula

Theorem 3.5 Quadratic Formula

$$\frac{-b \pm \sqrt{(b)^2 - 4(a)(c)}}{2(a)}.$$

3.7 Sum / Difference of squares

Theorem 3.6 Difference of Squares

Differnce of Squares:

$$(a^2 - b^2) = (a - b)(a + b).$$

Also:

$$(a^2 + b^2) = (a^2 + 2ab + b^2).$$

3.8 Sum / Differnce of Cubes:

Theorem 3.7 Sum / Difference of Cubes:

Difference of Cubes:

$$(a^3 - b^3) = (a - b)(a^2 + ab + b^2).$$

Sum of Cubes:

$$(a^3 + b^3) = (a + b)(a^2 - ab + b^2).$$

3.9 Types of Functions

- Linear (Graph is straight line) (Standard form, Point slope form, slope intercept form)

Standard Form:

$$Ax + By = C \text{ Where A and B are integers.}$$

- Quadratic

$$f(x) = x^2.$$

- Cubic

$$f(x) = x^3.$$

- Polynomial (More than 2 terms)
- Monomial (One term)
- Binomial (2 Terms)
- Trinomial (3 Terms)
- Logarithmic

$$\log_a x = y \longrightarrow a^y = x$$
$$Where\ a > 0,\ a \neq 1,\ y \neq 0.$$

$$\ln x = y \longrightarrow x = e^y.$$

- Rational
- Radical

3.10 Intercepts

- X-Int (Set function equal to zero and solve for x)
- Y-Int (plug zero in for x and solve)

3.11 Symmetry

- Even (Symmetric with respect to y axis)

$$f(-x) = f(x).$$

- Odd (Symmetric with respect to orgin)

$$f(-x) = -f(x).$$

Consider the function:

$$f(x) = x^3 - 8x.$$

If we compute $f(-x)$, we get:

$$f(-x) = -x^3 + 8x.$$

And we can show that this is the same as $-f(x)$:

$$\begin{aligned} -f(x) &= -(x^3 - 8x) \\ & -x^3 + 8x. \end{aligned}$$

Therefore the function $f(x) = x^3 - 8x$

Note:-

If a function has a constant term, the function cannot be odd.

- Symmetric with respect to x-axis

$$f(x, y) = f(x, -y).$$

3.12 Parallel or Perpendicular

- Parallel $\rightarrow$ Two lines are parallel if their slopes are the same
- Perpendicular $\rightarrow$ Two lines are Perpendicular if the product of their slopes is -1 which means the slope of one line is the negative reciprocal of the slope of the other line.

3.13 Lines (Slope and Equation)

Given a line, we can find slope by:

$$\frac{\Delta y \rightarrow Rise}{\Delta x \rightarrow Run}.$$

And we can find the equation of the line with point slope form:

$$y - y_1 = x(x - x_1).$$

A line has a standard form of

$$Ax + By = C, \text{ Where A, B and C are all integers.}$$

3.14 Circles

- Radius r
- Center (h,k)
- Equation of a circle

$$(x - h)^2 + (y - k)^2 = r^2.$$

3.15 Determine if an equation is a function of x

- each input has precisely one output
- Vertical Line test
- $\pm$ disqualifes (will not pass the vertical line test)

3.16 Function Notation

1. $(f+g)(x) = f(x) + g(x)$
2. $(f-g)(x) = f(x) - g(x)$
3. $(f \cdot g)(x) = f(x) \cdot g(x)$
4. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, assuming g(x) $\neq 0$

3.17 Domain and Range of Functions

Domain:

- Polynomial

$$D : \mathbb{R}.$$

- Rational (Set denominator $= 0$ and solve)
- Radical (Set whats inside the radical $\geqslant 0$) (If you need to factor first and get multiple x values, make number line and test points with the factored function, domain is where it is positive)
- Logarithmic (Set inside logarithm > 0.) (base domain is $(0, \infty)$)

Note:-

If the radical is in the denominator of a radical function, set whats inside radical > 0

Range:

- Look for transformations
- For a rational function, we can find the range of $f(x)$ by finding the domain of $f^{-1}(x)$.

3.18 Finding slope of secant line

We can find the slope of the secant line by utilizing the difference quotient

3.19 Piecewise Defined Functions

A function is called a **Piecewise Defined Function** if it defined by two or more functions on its domain

Ex:

$$f(x) = \begin{cases} -3x & \text{if } x < -1 \\ 0 & \text{if } x = -1 \\ 2x^2 + 1 & \text{if } x > -1 \end{cases} \tag{1}$$

3.20 Transformations

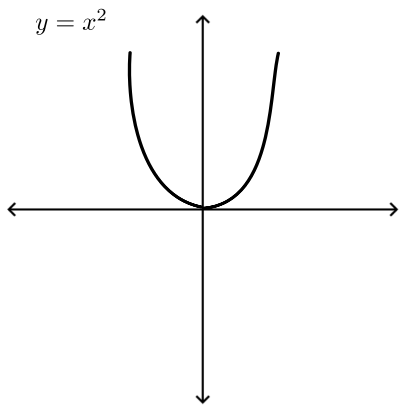
- $-f(x)$ (reflected over x-axis)
- $f(-x)$ (reflected over y axis)
- $-f(-x)$ (reflected over orgin)
- $2f(x)$ (vert stretch)
- $\frac{1}{2}f(x)$ (vert shrink)
- $f(2x)$ (Horizontal shrink)
- $f(\frac{1}{2})x$ (Hor stretch)
- $f(x-h)$ (Horizontal shift)
- $f(x)+k$ (Vertical shift)

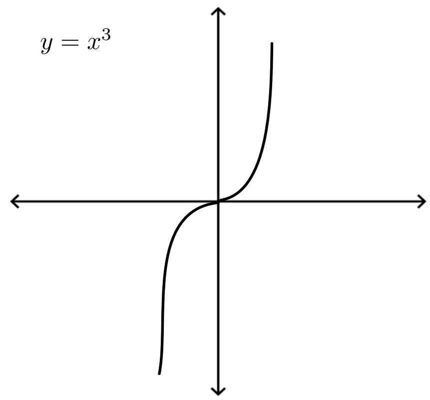
Transformations of a quadratic function

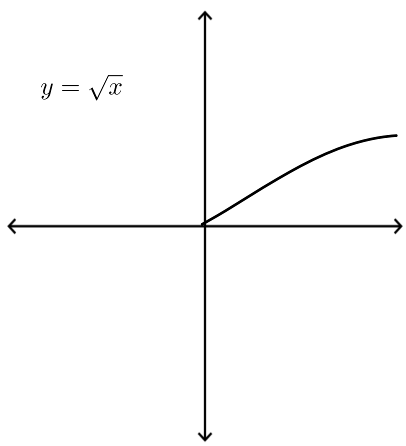
$$y = a(x-h) + k.$$

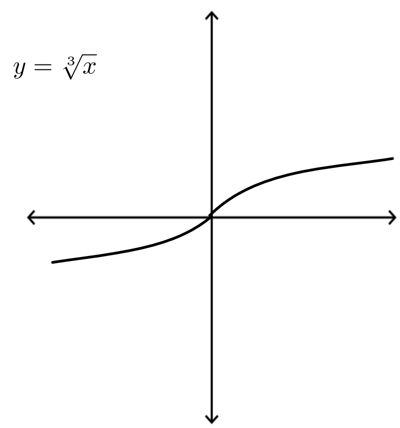
a is vertical stretch, h is horizontal shift, k is vertical shift

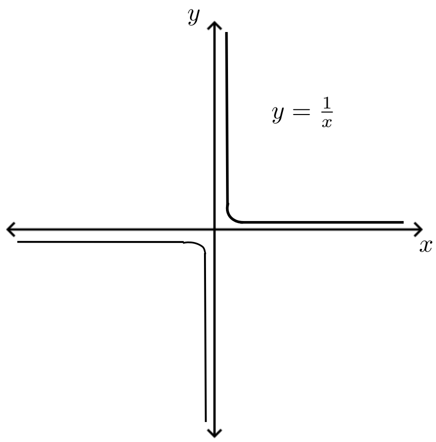
3.21 Common Functions

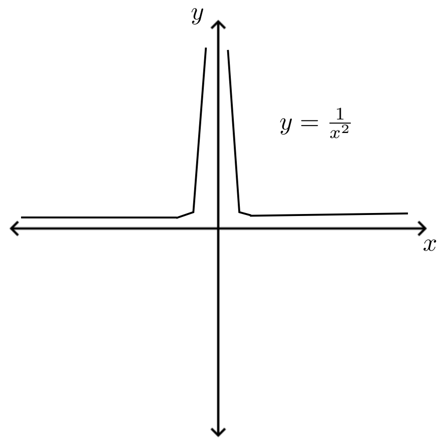


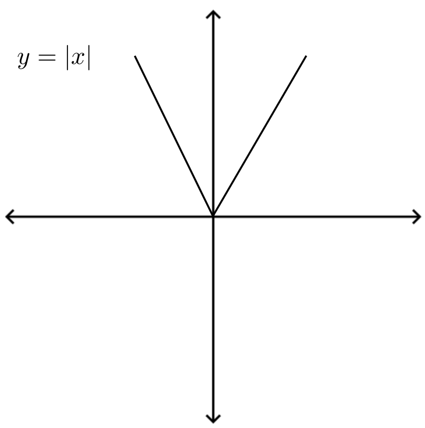







y
$y = \frac{1}{x}$
x


y
$y = \frac{1}{x^2}$
x


$y = |x|$

3.22 Linear Functions

A linear function is in the form:

$$f(x) = mx + b.$$

The graph of a linear function is a line with slope m and a y-intercept b. Its domain is the set of all real numbers.

A linear function is increasing if its slope is positive, and decreasing if its slope is negative

Note:-

We can tell if a function is linear if Δx and Δy is a constant value througout all input / outputs

3.23 Quadratic Functions and their zeros

A zero of a function $f(x)$ is a number r, such that $f(r) = 0$

- Basic factoring, simply set the function equal to 0 and solve.
- Box method (when $a > 1$ and not common factors)

  If:

  $$5x^2 - 18x + 9.$$

  Left as an exercise to the reader.

- Factor by grouping (4 terms)

  If:

  $$5v^3 - 2v^2 + 25v - 10.$$

  Then:

  $$\begin{aligned} &v^2(5v-2) + 5(5v-2) \\ &= (5v-2)(v^2+5). \end{aligned}$$

- Completing the square
  - Say we have the equation:

  $$2x^2 + 7x - 4 = 0.$$

  a.) Isolate variables
  b.) Get a's coefficient 1
  c.) $\left(\frac{b}{2}\right)^2$
  d.) Add answer from c to both sides
  e.) If you factored out a constant, you also need to multiply the right side by that constant

  $$2\left(x^2 + \frac{7}{2}\right) = 4$$
  $$\left(\frac{\frac{7}{2}}{2}\right)^2 = \frac{49}{16}.$$

So:

$$\begin{aligned}
2\left(x^2+\frac{7}{2}+\frac{49}{16}\right) &= 4+\frac{49}{16}(2)\\
2\left(x+\frac{7}{4}\right)^2 &= \frac{81}{8}\\
\left(x+\frac{7}{4}\right)^2 &= \frac{81}{16}\\
x+\frac{7}{4} &= \pm\sqrt{\frac{81}{16}}\\
x+\frac{7}{4} &= \pm\frac{9}{4}\\
\boxed{x=-4,\frac{1}{2}}.
\end{aligned}$$

3.24 Finding the zeros of a quadratic function with u Substitution

Say we have the equation:

$$(x-3)^2+5(x-3)-6=0.$$

If we let $u=x-3$, then we have:

$$\begin{aligned}
u^2+5u-6&=0\\
(u+6)(u-1)&=0\\
So\ u&=-6,1.
\end{aligned}$$

Now we substitute back in for u:

$$\begin{aligned}
x-3=-6\ and\ x-3=1\\
x=-3,\ 4.
\end{aligned}$$

3.25 Finding Where Two Functions Intersect

If given $f(x)$ and $g(x)$, What we do is set the functions equal to each other, and then solve such that the equation is in standard form.

With this new equation, if we solve for the zeros, these are the x values in which the two functions intersect. Then we plug this zero into $g(x)$ to get the corresponding y value.

Example 3.1

Consider:

$$f(x)=x+6 \quad g(x)=-x.$$

So:

$$\begin{aligned} x+6 &= -x \\ 2x+6 &= 0 \\ x &= -3. \end{aligned}$$

Then we plug -3 into $g(x)$ to get the y value.

$$g(-3) = 3.$$

Therefore our point of intersection is at $(-3, 3)$

For systems of equations, what we do is solve the system with the elimination method to get x and y.

Example 3.2
Consider:

$$\begin{aligned} x-4y &= -8 \\ 2x+3y &= -5. \end{aligned}$$

So:

$$\begin{aligned} -2(x-4y) &= -2(8) \\ -2x+8y &= 16. \end{aligned}$$

Subtracting this equation from $2x+3y=-5$ we get:

$$y = 1.$$

Now if we plug $y=1$ to one of our original equations we get that:

$$x = -4.$$

Therefore our point of intersection is at $(-4, 1)$

3.26 Quadratic Functions and their properties

A quadratic function is a function of the form $f(x) = ax^2 + bx + c$ where a, b, and c are real numbers and $a \neq 0$ The domain of a quadratic function consists of all real numbers.

A quadratic equation is an equation of the form $ax^2 + bx + c$ where a, b, and c are real numbers and $a \neq 0$

The graph of the quadratic function is called the **parabola**. More on the parabola later.

The quadratic function can be written in the form $f(x) = a(x-h)^2 + k$ (standard form) either by completing the square or by using the formulas:

$$h = -\frac{b}{2a} \text{ and } k = f(h).$$

The axis of symmetry of the parabola is $x = h$. This is the line that if we fold the parabola, it will be symmetric The vertex of the parabola is located at the point (h, k). It represents the highest point (called the maximum point) of a parabola if the parabola opens down (recall: $a < 0$). It represents the lowest point (called the minimum point) of a parabola if the parabola opens up (recall: $a > 0$).

Example 3.3

$$f(x) = 3x^2 + 6.$$

If:

- Vertex = (h, k)
- h = $\frac{-b}{2a}$
- A.O.S: x= h

Then:

$$h = \frac{0}{2(6)} = 0$$
$$k = f(0) = 6.$$

Therefore:

$$AOS: \ x = 0$$
$$Vertex: \ (0, 6) \text{ Shifted up 6 units from (0,0).}$$

3.27 Solving Quadratic Inequalites

Say we have the equation:

$$x^2 - x < 12$$
$$x^2 - x - 12 < 0$$
$$(x - 4)(x + 3) < 0.$$

So we have x-ints at (4, 0) and (-3,0), from this we can deduce that our porabola is below the x-axis from (-3,4)

3.28 Polynomial Functions and Their Models

A polynomial function is a function in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0.$$

Where $a_n, a_{n-1}..., a_1, a_0$ are real numbers and n is a nonnegative integer.

Note:-

The degree of a polynomial is the highest degree of its terms

Recall: the domain of a polynomial is $\mathbb{R}$

If a function has a negative exponent then it is **not** a polynomial.

If the degree of a function is not an integer, than it is **not** a polynomial

If the multiplicity of the zero is even, the graph **touches** the x-axis at that x-intercept, if the multiplicity of the zero is odd, the graph **crosses** the x-axis at that x-intercept

Find the max number of turning points by computing: $Degree - 1$

Example 3.4

Label the terms of the polynomial function

$$\begin{aligned} f(x) = 2x^5 - x^4 + 3x^2 - 7 \\ a_5 = 2 \\ a_4 = -1 \\ a_3 = 0 \\ a_2 = 3 \\ a_1 = 0 \\ a_0 = -7. \end{aligned}$$

Example 3.5

For the polynomial function:

- Find the degree of the polynomial
- List the real zeros and its multiplicity
- Find the x and y intercepts
- Determine whether the graph crosses or touches the x-axis at each x-intercept
- Determine the maximum number of turning points on the graph
- Determine the end behavior of the function
- Sketch a graph of the polynomial

$$f(x) = \left(x - \frac{1}{3}\right)^2 \left(x - 1\right)^3.$$

- Degree: 5 (Add the exponents)
- Zeros: $x = \frac{1}{3}, 1$
- multiplicity: 2 and 3 (Look at exponents)
- x-intercepts: $\left(\frac{1}{3}, 0\right)$ and $(1, 0)$

- y-intercepts: $f(0) = -\frac{1}{9}$, so $(0, -\frac{1}{9})$
- Touches: $(\frac{1}{3}, 0)$. Crosses: (1,0)
- Max turning points: $5 - 1 = 4$
- End behavior:
  - Since the power function is of an odd degree, the graph will resemble that of x^3, so we can determine

$$\lim_{x\to\infty} f(x) = \infty$$
$$\lim_{x\to-\infty} f(x) = -\infty.$$

3.29 Power Functions

A power function of degree n, is a monomial of the form $f(x) = ax^n$, where a is a real number, $a \neq 0$ and n is a positive integer

If n is **even**, then the following are true.

- The graph of the function is symmetric over the y-axis (The function is even)
- Domain: $\mathbb{R}$, range: [0, ∞], if a is positive and R= [-∞, 0], if a is negative
- The graph resembles the graph of x^2

If n is **odd**, then the following are true

- The graph of the function is symmetric over the orign
- Domain: $\mathbb{R}$, Range: $\mathbb{R}$
- The graph resembles the graph of x^3

3.30 Properties of Rational Functions

A Rational Function is a function of the form $\frac{g(x)}{h(x)}$, where g(x) and h(x) are polynomials and $h(x) \neq 0$

The domain of a rational function is $\mathbb{R}$, except those that make the denominator equal 0

Asymptotes:

- Vertical Asymptotes can be found by setting the denominator equal to zero and solving for x.
- Horizontal Asymptotes: If we let the highest degree of the numerator equal n, and the highest degree of the denominator equal k, then:
  1. If $n < k$, then the equation of the horizontal asymptote is $y = 0$
  2. If $n = k$, then we take the ratio of the leading coefficients
  3. If $n > k$, then the graph has no horizontal asymptote, instead the:

$$\lim_{x\to\infty \; or \; -\infty} f(x) = \infty \; or \; -\infty.$$

- Oblique (Slant) Asymptotes will occur if the degree of the numerator is precisely one higher than the degree of the denominator, to find the oblique asymptote, we must do long division

3.31 Sketching the graph of a rational function

To sketch the graph of a rational function, we must do the following:

- Factor the rational function and find the domain.
- Find the intercepts
- Find the asymptotes
- If there is a horizontal or oblique asymptote, determine if it intesects the graph. To do this, set unfactored function equal to the value of the asymptote

3.32 Polynomial and Rational Inequalties

Note: $ab < 0$ is not equivalent to $a < 0$ or $b < 0$

Procedure:

1. Get 0 on one side of the inequality and factor
2. Make a number line with the intervals from step 1 and test points between these intervals
   - If the test points lead to a true statement, that that interval is part of the solution
   - If the test points lead to a false statement, that that interval is not part of the solution
3. The solution to the inequality is the union of all the true intervals

For rational inequalities, the numbers which make the expression undefined are not part of the solution.

3.33 Synthetic Division

Consider the polynomial:

$$f(x) = \frac{3x^4 - 2x^2 + 5x^2 + 8}{x - 2}.$$

To preform **synthetic division**, we need to write out the coefficients like so. Note: because we dont have a $a_n x$ term, we will add a zero in its place. Also, our divisor is 2 because our factor we are dividing by is $x - 2$

$$\begin{array}{r|rrrrr} 2 & 3 & -2 & 5 & 0 & 8 \\ \hline & & & & & \\ \hline \end{array}.$$

From here, we want to drop down that first number, 3. Then what we do is multiply our divisor by that 3. Then take what you get from multipling and add to the term to the right of 3. Repaet these steps.

So

$$\begin{array}{r|rrrrr} 2 & 3 & -2 & 5 & 0 & 8 \\ \hline & & 6 & 8 & 26 & 52 \\ \hline & 3 & 4 & 13 & 26 & 60 \end{array}.$$

Which means our new polynomial is:

$$f(x) = 3x^3 + 4x^2 + 13x + 26 + \frac{60}{x - 2}.$$

Note that the last term is the **remainder** over the **divisor**

3.34 The real zeros of a polynomial function

If $f(x)$ and $p(x)$ are polynomials and if $p(x) \neq 0$, then there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x) = p(x) \cdot q(x) + r(x)$ where either $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $p(x)$. The polynomial $q(x)$ is the quotient, and $r(x)$ is the remainder in the division of $f(x)$ by $p(x)$.

Remainder Theorem.

If a polynomial $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

Factor Theorem.

A polynomial $f(x)$ has a factor $x - c$ if and only if $f(c) = 0$ (i.e. remainder $= 0$). That is:

- If $f(c) = 0$, then $x - c$ is a factor of $f(x)$.
- If $x - c$ is a factor of $f(x)$, then $f(c) = 0$.

Example 3.6 (Use the remainder theorem to find the remainer when $f(x)$ is divided by $x - c$. Then use the factor theorem to determine whether $x - c$ is a factor of $f(x)$)

$$f(x) = x^4 + 3x^2 - 12, \quad x + 2 \ \ (divisor).$$

So instead of long dividing with $x + 2$, the remainder theorem states that our remainder will be $f(c)$, therefore:

$$\begin{aligned} Remainder &= f(-2) = (-2)^4 + (-2)^3 - 12 \\ &= 16. \end{aligned}$$

Since this number is not zero, 16 is not a factor (by the factor theorem)

Theorem.

The maxmimum number of zeros for a polynomial equation is less than or equal to the degree of the polynomial.

Definition.

- The **Constant Term** is the term that does not contain x
- A **Variation in sign** in $f(x)$ is when two consecutive coefficients have opposite signs.

Descarte's rule of signs.

Let $f(x)$ be a polynomial with real coefficients and a nonzero constant term.

1. The number of positive real zeros of $f(x)$ either is equal to the number of variations in sign of $f(x)$ or is less than that number by an even integer.
2. The number of negative real zeros of $f(x)$ either is equal to the number of variations in sign of $f(-x)$ or is less than that number by an even integer.

Example 3.7 (Descarte's Rule of signs)

$$\begin{aligned} f(x) &= 3x^3 - 4x^2 + 3x + 7 \\ f(-x) &= -3x^3 - 4x^2 - 3x + 7. \end{aligned}$$

- Degree: 3
- Positive Solutions: 2 or 0
- Negative Solutions: 1

Rational Zeros Theorem.

If the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients and if p/q is a rational zero of $f(x)$ such that p and q have no common prime factor, then

1. the numerator p of the zero is a factor of the constant term a_0
2. the denominator q of the zero is a factor of the leading coefficient a_n.

Steps for finding the real zeros of a polynomial function

1. Use the degree of the polynomial to determine the maximum number of real zeros. For example, a polynomial of degree n can have at most n real zeros.
2. Use Descarte's Rule of Signs to determine the possible number of positive and negative zeros. Count the number of sign changes in the coefficients of $f(x)$ and $f(-x)$. The number of positive real zeros is either equal to this number or less than it by an even integer. The number of negative real zeros is either equal to this number or less than it by an even integer.
3. Use the Rational Zero Theorem to identify rational numbers that potentially could be zeros. The possible rational zeros are of the form $\pm\frac{p}{q}$, where p is a factor of the constant term and q is a factor of the leading coefficient.
4. Use Factor Theorem, synthetic division, or long division to test each potential rational zero. If a potential zero is indeed a zero, it means that the polynomial can be factored by ($x -$ potential zero).
5. Each time that a zero is found, repeat step 4 on the depressed equation. The "depressed equation" is the polynomial after dividing by a factor of ($x-$found zero). This reduces the degree of the polynomial and simplifies the search for additional zeros.
6. Factor the polynomial if possible. Once all real zeros have been found, you can use the factorization to sketch the graph of the polynomial.

Example 3.8

Use the rational zeros theroem to find all real zeros of the polynomial. Use the zeros to factor f over the real numbers

$$\begin{aligned} f(x) &= 3x^3 + 6x^2 - 15x - 30 \\ f(x) &= 3(x^3 + 2x^2 - 5x - 10) \\ f(-x) &= -3x^3 + 6x + 15x - 30. \end{aligned}$$

So:

- Degree: 3
- Max real zeros: 3
- By Decarte's rule of signs:
  - Positive real zeros: at least 1
  - Negative real zeros: either 2 or 0
- By the rational zeros theorem:

  $$\pm\left\{1, 2, 5, 10\right\}.$$

  Using synthetic divison (with factored $f(x)$), we can find that

  $$\begin{array}{r|rrrr} -2 & 1 & 2 & -5 & -10 \\ & & -2 & 0 & -10 \\ \hline & 1 & 0 & -5 & 0 \end{array}.$$

So we see that our remainder is 0, which means -2 is found to be a real zero. And we have a new polynomial that has been depressed (degree lowered by 1). So we can use the depressed polynomial to find the remaining zeros.

So our depressed polynomaial is:

$$x^2 - 5.$$

And we can set this equal to zero to find the remaining zeros.

$$\begin{aligned} x^2 - 5 &= 0 \\ x^2 &= 5 \\ x &= \pm\sqrt{5}. \end{aligned}$$

Therefore our 3 zeros are.

$$\boxed{\sqrt{5},\ -\sqrt{5},\ -2}.$$

3.35 Complex Zeros and the Fundemental theorem of algebra

Fundamental Theorem of Algebra.

If a polynomial $f(x)$ has positive degrees and complex coefficients, then $f(x)$ has at least one complex zero.

Complete Factorization Theorem for Polynomials

If $f(x)$ is a polynomial of degree $n > 0$, then there exist n complex numbers $c_1, c_2, \ldots, c_n$ such that $f(x) = a(x - c_1)(x - c_2) \ldots (x - c_n)$ where a is the leading coefficient of $f(x)$. Each number c_k is a zero of $f(x)$.

Conjugate Pairs Theorem

Let $f(x)$ be a polynomial whose coefficients are real numbers. If $r = a + bi$ is a zero of f, then the conjugate $r = a - bi$ is also a zero of f.

Note:-

$i = \sqrt{-1}$ and $i^2 = -1$

Example 3.9 (Find $f(x)$ given the zeros)

$$-3,\ 1 - 7i \quad \textit{degree}\ 3.$$

By the conjugate pairs theorem, we know that $1 + 7i$ is also a zero.

Now:

$$\begin{aligned}(x + 3)(x - (1 - 7i))(x - (1 + 7i))\\ (x + 3)(x - 1 + 7i)(x - 1 - 7i).\end{aligned}$$

And by difference of squares, which states:

$$\begin{aligned}(a + b)(a - b)\\ = (a^2 - b^2).\end{aligned}$$

We have:

$$\begin{aligned}(x + 3)((x - 1)^2 - 49i^2)\\ = (x + 3)(x^2 - 2x + 1 + 49)\\ = (x + 3)(x^2 - 2x + 50)\\ \boxed{= x^3 + x^2 + 44x + 150}.\end{aligned}$$

Example 3.10

Use the given zero to find the remaining zeros of the function

$$f(x) = x^3 + 3x^2 + 25x + 75, \quad zero: \ -5i.$$

So by the conjugate pairs theorem, we know that $5i$ is a also a zero. Furthermore, because this is a degree 3 polynomial, we know we are only missing **one** zero.

Note:-

We can use synthetic division to divide our polynomial by the know zeros, but long division will be easier.

We will write our polynomial as:

$$f(x) = (x - 5i)(x + 5i)(x - c)$$
$$f(x) = (x^2 - 25i^2)(x - c).$$

Where $(x - c)$ is the zero we are trying to find. To solve $(x - c)$, we can divide our function $f(x)$ by the two known factors.

So if we compute

$$x^2 + 0x + 25\overline{)x^3 + 3x^2 + 25x + 75}.$$

We get $(x + 3)$, with no remainder.

Now we will set the factor $(x + 3) = 0$ and solve for the missing zero

$$x + 3 = 0$$
$$x = -3.$$

So our solution set is:

$$s = \left\{ -3, \pm 5i \right\}$$
$$\boxed{f(x) = (x - 3)(x + 5i)(x - 5i)}.$$

3.36 Composite Functions

Definition.

Given two functions f and g, the composite function, denoted by $f \circ g$ (read as ”f composed with g”), is defined by $f \circ g = f(g(x))$.

The domain of $f \circ g$ is the set of all numbers x in the domain of g such that $g(x)$ is in the domain of f.

Finding domain of composite function.

Say we have the functions:

$$f(x) = \frac{4}{x+2}, \qquad g(x) = \frac{1}{x}.$$

We first have to find $(f \circ g)(x)$, and then we can determine the domain.

So:

$$\begin{aligned}(f \circ g)(x) &= \frac{4}{\frac{1}{x}+2} \\ &= \frac{4x}{1+2x}.\end{aligned}$$

Now we can set the denominator equal to zero and solve.

$$\begin{aligned}1 + 2x &= 0 \\ 2x &= -1 \\ x &= -\frac{1}{2}.\end{aligned}$$

So for our composite function:

$$x \neq 0, \quad x \neq -\frac{1}{2}.$$

Note:-

Note that the domain restrictions for $g(x)$ transfer over to the composite function, this is **not** true for $f(x)$

So in interval notation we have:

$$\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 0\right) \cup \left(0, \infty\right).$$

3.37 One-to-One Functions

Definition

A function f with domain $\mathcal{D}$ and range $\mathcal{R}$ is a one-to-one function if either of the following equivalent conditions is satisfied:

1. Whenever $a \neq b$ in $\mathcal{D}$, then $f(a) \neq f(b)$ in $\mathcal{R}$.
2. Whenever $f(a) = f(b)$ in $\mathcal{R}$, then $a = b$ in $\mathcal{D}$.

Example 3.11 (Determine whether the function is one-to-one)

$$f(x) = 2x^3 - 4.$$

So:

$$\begin{aligned} f(a) &= f(b) \\ 2a^3 - 4 &= 2b^3 - 4 \\ 2a^3 &= 2b^3 \\ a^3 &= b^3 \\ \sqrt[3]{a^3} &= \sqrt[3]{b^3} \\ a &= b \end{aligned}$$

Thus, $a = b$ so the function is **one-to-one**

Example 3.12 (Show that the function is one-to-one)

$$f(x) = \frac{4x}{x-2}.$$

So:

$$\begin{aligned} f(a) &= f(b) \\ \frac{4a}{a-2} &= \frac{4b}{b-2} \\ 4b(a-2) &= 4a(b-2) \\ 4ab - 8b &= 4ab - 8a \\ -8b &= -8a \\ b &= a \end{aligned}$$

Thus, this function is one-to-one

Horizontal Line Test.

A function f is one-to-one if and only if every horizontal line intersects the graph of f in at most one point.

3.38 Inverse Functions

Steps for finding inverse functions

1. Verify that $f(x)$ is one-to-one on its domain
2. let $y = f(x)$
3. swap x and y
4. solve for y
5. write as f^{-1}

Theorem.

Let f be a one-to-one function with domain $\mathcal{D}$ and range $\mathcal{R}$. If g is a function with domain $\mathcal{R}$ and range $\mathcal{D}$, then g is the inverse function of f if and only if both of the following conditions are true:

1. $g(f(x)) = x$ for every x in $\mathcal{D}$.
2. $f(g(y)) = y$ for every y in $\mathcal{R}$.

Domain of f^{-1} = Range of f
Range of f^{-1} = Domain of f

The graph of a function f and the graph of its inverse f^{-1} are symmetric with respect to the line $y = x$

Example 3.13 (Find the inverse function)

$$f(x) = \frac{4x}{x-2}.$$

Let's first verify that this function is one-to-one

So:

$$\begin{aligned} f(a) &= f(b) \\ \frac{4a}{a-2} &= \frac{4b}{b-2} \\ 4b(a-2) &= 4a(b-2) \\ 4ab - 8b &= 4ab - 8a \\ -8b &= -8a \\ b &= a \end{aligned}$$

Thus, this function is one-to-one

Now let's let y = f(x), interchange x and y and then solve for y.

$$
\begin{aligned}
x &= \frac{4y}{y-2} \\
x(y-2) &= 4y \\
xy - 2x &= 4y \\
xy - 4y &= 2x \\
y(x-4) &= 2x \\
y &= \frac{2x}{x-4} \\
\boxed{f^{-1} = \frac{2x}{x-4}}&.
\end{aligned}
$$

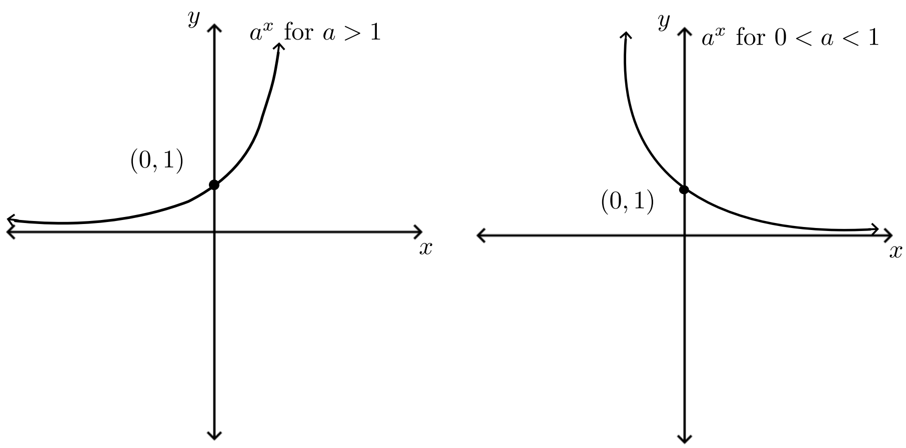
3.39 Exponential Functions

Definition.

Exponential function f with base a is written as a^x

- $\mathcal{D} : \mathbb{R}$
- $\mathcal{R} : (0, \infty)$

Figures for exponential functions.



a^x key points

$$(0, 1)$$
$$(1, a)$$
$$\left(-1, \frac{1}{a}\right).$$

And a horizontal asymptote at $y = 0$

Theorem.

The exponential function f given by $f(x) = a^x$ for $0 < a < 1$ or $a > 1$ is one-to-one. Thus, the following conditions are satisfied for real numbers x_1 and x_2.

1. If $x_1 \neq x_2$, then $a^{x_1} \neq a^{x_2}$.
2. If $a^{x_1} = a^{x_2}$, then $x_1 = x_2$.

Laws of exponents

- Product of Powers

$$x^n \cdot x^m = x^{n+m}.$$

- Quotient of Powers

$$\frac{x^n}{x^m} = x^{n-m}.$$

- Power of a Power

$$(a^n)^m = a^{n \cdot m}.$$

- Product of a Power

$$(x \cdot y)^n = x^n \cdot y^n.$$

- Product of a Quotient

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}.$$

- Negative powers

$$x^{-n} = \frac{1}{x^n}.$$

- Power of 0

$$a^0 = 1.$$

Example 3.14

$$\begin{aligned}
9^{2x} \cdot 27^{x^2} &= 3^{-1} \\
(3^2)^{2x} \cdot (3^3)^{x^2} &= 3^{-1} \\
3^{4x} \cdot 3^{3x^2} &= 3^{-1} \\
3^{4x+3x^2} &= 3^{-1} \\
4x + 3x^2 &= -1 \\
3x^2 + 4x + 1 &= 0 \\
(x+1)(x+3) &= 0 \\
x &= -3, -1.
\end{aligned}$$

3.40 Natural Exponential Function

Definition:

If n is a positive integer, then:

$$\left(1+\frac{1}{n}\right)^n \to e \approx 2.71828 \quad as\ n \to \infty.$$

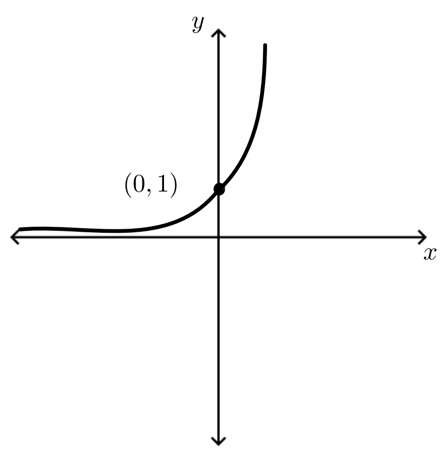
Thus:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 = e.$$

The natural exponential function f is defined by

$$f(x) = e^x \text{ for every real number x..}$$

Graph of e^x:



e^x key points

$$(0,1)$$
$$(1,e)$$
$$\left(-1,\frac{1}{e}\right).$$

Horizontal Asymptote at $y = 0$

3.41 Logarithmic Function

Recall:

1. Exponential functions are one-to-one with a horizontal asymptote at $y = 0$.
2. One-to-one functions have inverse functions.

The inverse of an exponential function is the logarithmic function.

Definition:

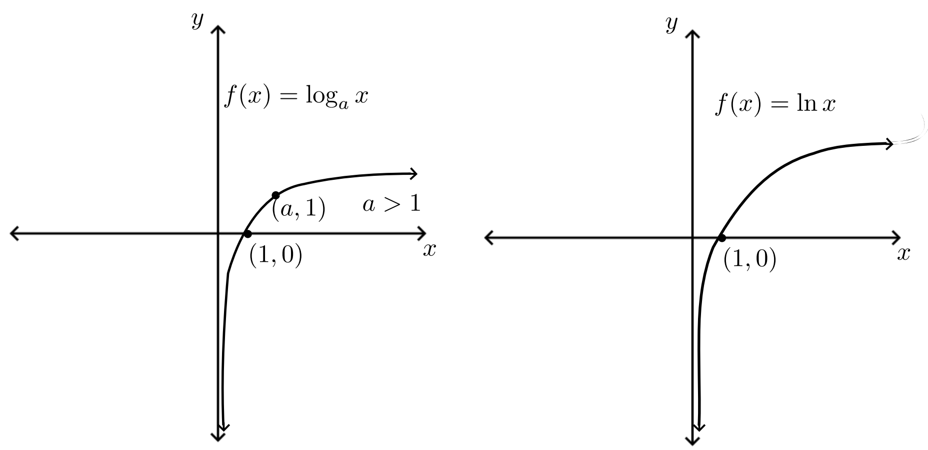
Let a be a positive real number different from 1. The logarithm of x with base a is defined by $\log_a x = y$ if and only if $x = a^y$ for every $x > 0$ and every real number y.

Domain of logarithm: $(0, \infty)$
Range of logarithm: $(-\infty, \infty)$

Common Logarithm: (Logarithm with base 10) $\log x = \log_{10} x$ for every $x > 0$

Natural Logarithm: (Logarithm base e) $\ln x = \log_e x$ for every $x > 0$

Graphs of logarithmic functions:



Key Points for $\log_a x$:

$$(1, 0)$$
$$(a, 1)$$
$$\left(\frac{1}{a}, -1\right).$$

V.A at $x = 0$

Key Points for $\ln x$:

$$(1, 0)$$
$$(e, 1)$$
$$\left(\frac{1}{e}, -1\right).$$

V.A at $x = 0$

Example 3.15 (Changing to logarithmic form)

$$3^{-4} = \frac{1}{81}$$
$$= \log_3 \frac{1}{81} = -4.$$

Example 3.16 (Changing to exponential form)

$$\log_a \frac{1}{256} = -4$$
$$a^{-4} = \frac{1}{256}.$$

Example 3.17 (Finding the exact value)

$$\log_5 \sqrt[3]{25}.$$

1. Change to exponential form
2. Solve for x

$$5^x = 25^{\frac{1}{3}}$$
$$5^x = (5^2)^{\frac{1}{3}}$$
$$5^x = 5^{\frac{2}{3}}$$
$$\boxed{x = \frac{2}{3}}.$$

Finding Domain of Logarithmic Function.

1. Set inside of logarithm strictly greater than zero.
2. Solve the inequality.
3. If you have more than one x, construct a number and test points, only positive outputs are included in the solution. Much like a radical function

Theorem.

The logarithmic function f given by $f(x) = \log_a x$ for $a \neq 1$ and $a > 0$, $x > 0$ is one-to-one.

Thus, the following conditions are satisfied for real numbers x_1 and x_2.

1. If $x_1 \neq x_2$, then $\log_a x_1 \neq \log_a x_2$.
2. If $\log_a x_1 = \log_a x_2$, then $x_1 = x_2$.

Example 3.18

$$\begin{aligned} \log_6 36 &= 5x + 3 \\ 2 &= 5x^3 \\ x &= -\frac{1}{5}. \end{aligned}$$

Example 3.19

$$\begin{aligned} 4e^{x+2} &= 5 \\ e^{x+2} &= \frac{5}{4} \\ \log_e \frac{5}{4} &= x + 2 \\ \ln \frac{5}{4} &= x + 2 \\ x &= ln\frac{5}{4} - 2. \end{aligned}$$

Properties of Logarithms.

1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\log_a (ax) = x$
4. $a^{\log_a x} = x$
5. $\log_a (u \cdot w) = \log_a u + \log_a w$
6. $\log_a \left(\frac{u}{w}\right) = \log_a u - \log_a w$
7. $\log_a (u^c) = c \cdot \log_a u$

Change of Base Formula.

If $u > 0$ and if a and b are positive real numbers different from 1, then $\log_b u = \frac{\log_a u}{\log_a b}$.

Notes

- $\log_a (u + w) \neq \log_a u + \log_a w$
- $\log_a (u - w) \neq \log_a u - \log_a w$

Example 3.20

$$2^{\log_2 x}$$
$$= x.$$

Example 3.21

$$\log_3 9 \cdot \log_8 9.$$

We can use the change of base formula to rewrite this as:

$$\left(\frac{\log_8}{\log_3}\right)\left(\frac{\log_9}{\log_8}\right)$$
$$= \frac{\log_9}{\log_3}$$
$$= \log_3 9 \to \text{Reverse change of base formula}$$
$$= 2.$$

Example 3.22

$$e^{\log_{e^2} 9}.$$

Using change of base formula.

$$\begin{aligned}
&e^{\frac{\log_e 9}{\log_e e^2}} \\
&= e^{\frac{\log_e 9}{2}} \\
&= e^{\frac{1}{2}\log_e 9} \\
&= e^{\log_e 9^{\frac{1}{2}}} \\
&= \sqrt{9} \\
&\boxed{= 3}.
\end{aligned}$$

Shortcut.

Say we have:

$$\log \frac{ab}{xyz}.$$

To expand this:

$$\begin{aligned}
&\log a + \log b - (\log x + \log y + \log z) \\
&\log a + \log b - \log x - \log y - \log z \\
&.
\end{aligned}$$

Example 3.23 (Expand)

$$\begin{aligned}
&\log \frac{\sqrt{x}}{y^4 \sqrt[3]{z}} \\
&= \log x^{\frac{1}{2}} - \log y^4 - \log z^{\frac{1}{3}} \\
&= \frac{1}{2}\log x - 4\log y - \frac{1}{3}\log z.
\end{aligned}$$

Example 3.24 (Write the expressions as a single logarithm)

$$\begin{aligned}
&5\log_a x - \frac{1}{2}\log_a (3x-4) - 3\log_a (5x+1) \\
&= \log_a x^5 - \log_a \sqrt{3x-4} - \log_a (5x+1)^3 \\
&= \log_a \left(\frac{x^5}{\sqrt{3x-4}(5x+1)^3}\right).
\end{aligned}$$

Tip.

$$\log \frac{1}{x}$$
$$= -\log x.$$

Example 3.25 (Solve the equation)

$$3\log_2 x = 2\log_2 3$$
$$= \log_2 x^3 = \log_2 3^2.$$

Since both the logarithms have the same base, we can set $x^3 = 3^2$

$$x^3 = 9$$
$$x = \sqrt[3]{9}.$$

Example 3.26 (Solve the equation)

$$\begin{aligned} \log_6(x+5) + \log_6 x &= 2 \\ = \log_6(x(x+5)) &= 2 \\ \log_6(x^2+5x) &= 2 \\ 6^2 &= x^2+5x \\ 36 &= x^2+5x \\ x^2+5x-36 &= 0 \\ (x-9)(x+4) & \\ x &= -4, 9. \end{aligned}$$

Since logarithms can't be negative numbers, only 9 is a solution.

Example 3.27 (Solve the equation)

$$\begin{aligned}
\log(57x) &= 2 + \log(x-2) \\
\log(57x) - \log(x-2) &= 2 \\
\log\left(\frac{57x}{x-2}\right) &= 2 \\
10^2 &= \frac{57x}{x-2} \\
100 &= \frac{57x}{x-2} \\
100(x-2) &= 57x \\
100x - 200 &= 57x \\
43x &= 200 \\
x &= \frac{200}{43}.
\end{aligned}$$

If we plug $\frac{200}{43}$ into the logs in our original equation, nothing comes out negative therefore this is a **valid** solution

Note:-

After solving logarithmic equations, make sure you plug solutions into original equations logarithms to make sure they are valid solutions, If you get a negative output, it is not a valid solution.

Example 3.28 (Find the exact solution, using common logarithms, and four decimal place solution, when appropriate)

$$e^{x+3} = \pi^x.$$

Since we can't get a common base on both sides, we can take the common log, or natural log of both sides.

$$\begin{aligned}
\ln e^{x+3} &= \ln \pi^x \\
x + 3 &= x \cdot \ln \pi \\
3 &= x \ln \pi - x \\
3 &= x(\ln \pi - 1) \\
x &= \frac{3}{\ln \pi - 1} \\
&\approx 20.7283.
\end{aligned}$$

Example 3.29

$$4^{2x+3} = 5^{x-2}.$$

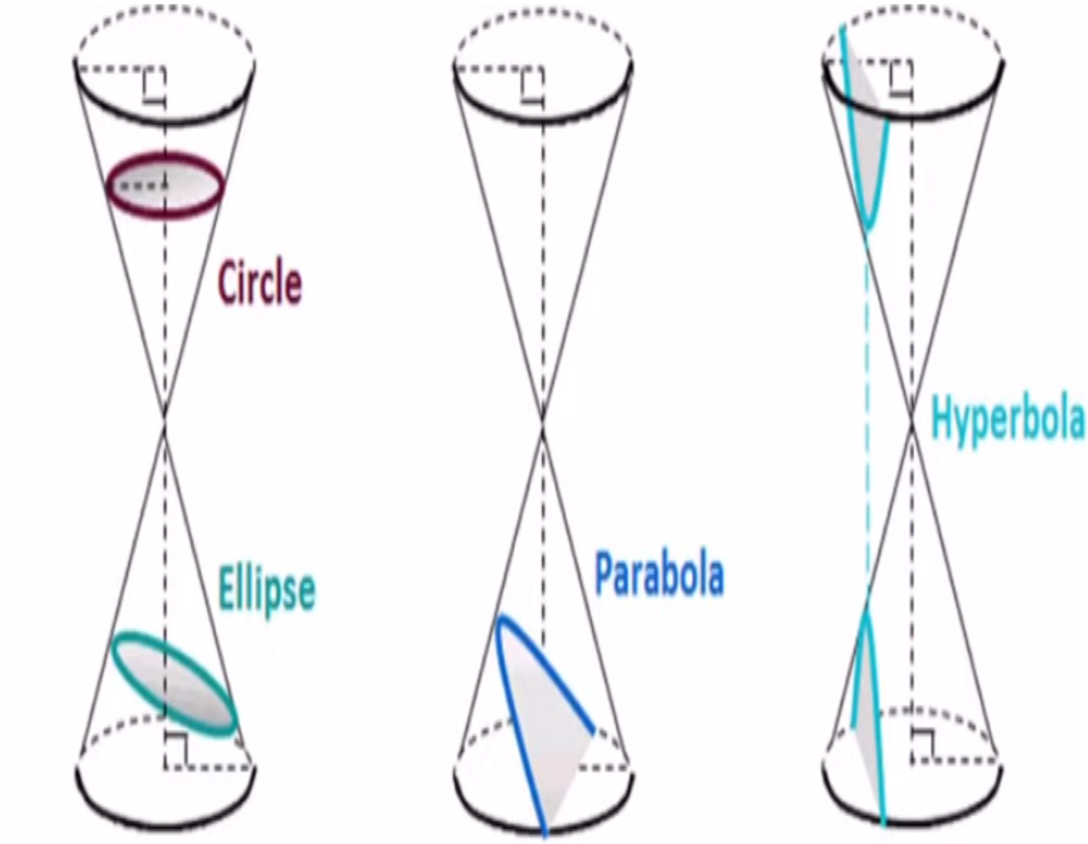
Again, take the log of both sides.

$$\begin{aligned}\log\left(4^{2x+3}\right) &= \log\left(5^{x-2}\right)\\(2x+3)\log 4 &= (x-2)\log 5\\2x\log 4 + 3\log 4 &= x\log 5 - 2\log 5.\end{aligned}$$

From here we want to get all x's on the same side.

$$\begin{aligned}3\log 4 + 2\log 5 &= 2x\log 4 - x\log 5\\3\log 4 + 2\log 5 &= x(2\log 4 - \log 5)\\x &= \frac{3\log 4 + 2\log 5}{2\log 4 - \log 5}.\end{aligned}$$

3.42 Conic Sections/The Parabola



The Parabola.

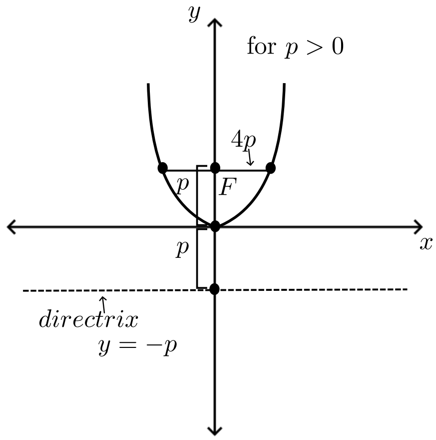
A parabola is the set of all points in a plane equidistant from a fixed point F (the focus) and a fixed line ℓ (the directrix) that lie in the plane.

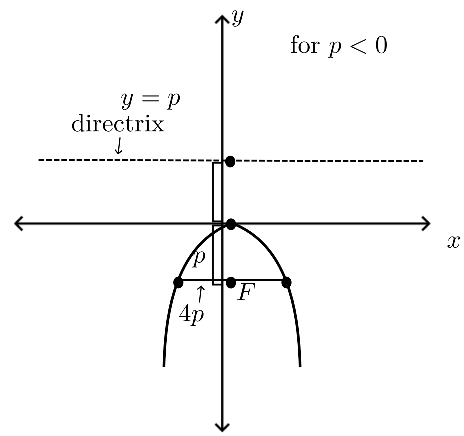
The axis of the parabola is the line through F that is perpendicular to the directrix.

The vertex of the parabola is the point V on the axis halfway from F to ℓ.

A Parabola with vertex $V(h, k)$ that opens up or down will have the standard form:

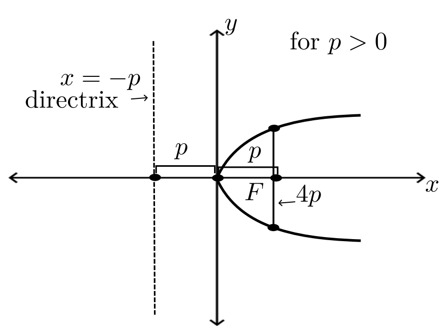
$$(x-h)^2 = 4p(y-k)$$
$$Focus: \ F(h, k+p)$$
$$Directrix: \ y = k-p$$
$$\text{Length of latus rectum (focal width)}: \ 4p.$$

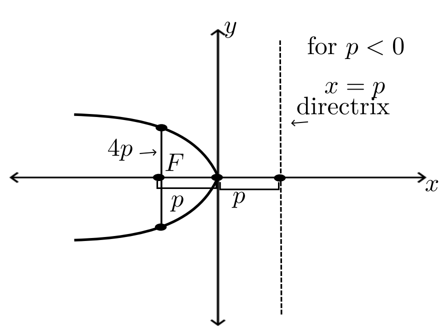




A Parabola with vertex $V(h, k)$ that left or right will have the standard form:

$$(y-k)^2 = 4p(x-h)$$
$$Focus: \ F(h+p, k)$$
$$Directrix: \ x = h-p$$
$$\text{Length of latus rectum (focal width)}: \ 4p.$$





Example 3.30

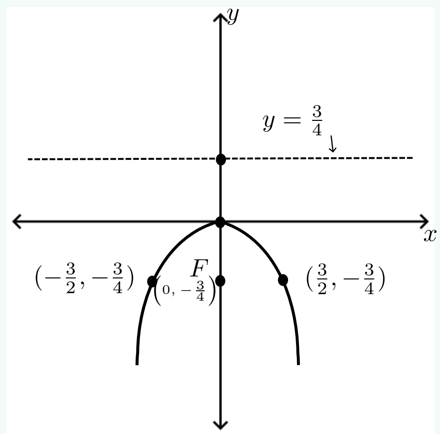
Sketch the graph and find the focus, vertex and directrix

$$x^2 = -3y.$$

So we know it is opening *down*, and:

$$\begin{aligned} 4p &= -3 \\ p &= -\frac{3}{4} \\ Vertex: &\ (0,0) \\ Focus: &\ (0, -\frac{3}{4}) \end{aligned}$$

To get the other two points, since we know that the focal width is 3, we can split that inhalf to get $\frac{3}{2}$, so our graph would look like:



Example 3.31

$$(y+1)^2 = -12(x+2).$$

Since y is the quadratic term, we know that it is either going to open left or right.

$$\begin{aligned} Vertex:&\ (-2,-1)\\ 4p &= -12\\ p &= -3. \end{aligned}$$

Since $p < 0$, we know that it is going to open left.

If:

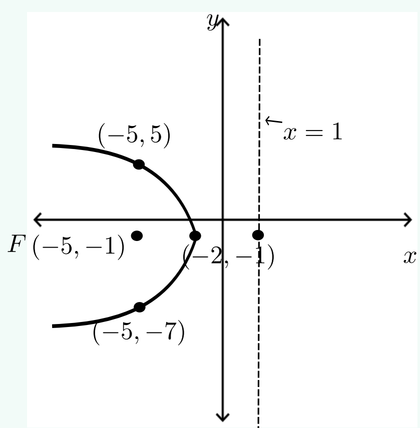
$$\begin{aligned} &F(h+4,k)\\ Then:\ &F(-5,-1). \end{aligned}$$

If:

$$\begin{aligned} Directrix:\ x &= h-p\\ Then:\ directrix\ is\ x &= -2+3\\ x &= 1. \end{aligned}$$

To find points since $4p = -12$ and $p = -3$, we know our focal width is 12, and that split in half is 6, so:

$$\begin{aligned} p_1 &= (-5,-1+6) = (-5,5)\\ p_2 &= (-5,-1-6) = (-5,-7). \end{aligned}$$



Example 3.32

$$x = 2y^2 - 6y + 7.$$

For this equation, to get in standard form $(y-h)^2 = 4p(x-k)$, we must **complete the square.**

$$\begin{aligned} x - 7 &= 2y^2 - 6y \\ x - 7 &= 2(y^2 - 3y) \\ x - 7 + \left(2\left(\frac{3}{2}\right)^2\right) &= 2\left(y^2 - 3y + \left(\frac{3}{2}\right)^2\right) \\ x - 7 + \frac{9}{2} &= 2\left(y^2 - 3y + \frac{9}{4}\right) \\ x - \frac{5}{2} &= 2\left(y - \frac{3}{2}\right)^2 \\ \left(y - \frac{3}{2}\right)^2 &= \frac{1}{2}\left(x - \frac{5}{2}\right). \end{aligned}$$

From here we can procede normally.

Example 3.33 (Find the equation of the parabola that satisfies the given conditions)

$$F(-3,-2), \quad Directrix: \ y = 1.$$

Since the directrix is a value of y, we know that this parabola is either going to open up or down. And from the fact that the directrix is **above** the focus, we can deduce that this parabola will open **Down.**

Since the Vertex is always halfway between the directrix and the focus, we can add the directrix value, 1, to the $|y|$ value of the focus to get the total distance between the two points. Then, we can divide this numebr in half to get the distance between the two points. Therefore:

$$\begin{aligned} V&\left(-3, -2 + \frac{3}{2}\right) \\ &= \left(-3, -\frac{1}{2}\right). \end{aligned}$$

Next, since the distance between the vertex and the focus is $-\frac{3}{2}$ (negative because it is opening down). We know that $p = -\frac{3}{2}$

So with all of this information, our equation would be:

$$\begin{aligned} (x+3)^2 &= 4p\left(-\frac{3}{2}\right)(y + \frac{1}{2}) \\ (x+3)^2 &= -6\left(x + \frac{1}{2}\right). \end{aligned}$$

Example 3.34

$$V(3,-2),\ \text{axis of symmetry parallel to the x-axis, and the y-intercept is 1.}$$

So since the A.O.S is a value of x, we know that this parabola will open left or right.

So our equation would be:

$$(y+2)^2 = 4p(x-3).$$

Now we can plug the y-intercept (0,1) into this equation to solve for p:

$$\begin{aligned}(1+2)^2 &= 4p(0-3)\\ 9 &= 4p(-3)\\ -3 &= 4p.\end{aligned}$$

Therefore:

$$(y+2)^2 = -3(x-3).$$

3.43 Ellipses

Definition:

An ellipse is the set of all points in a plane, the sum of whose distances from two fixed points (the foci) in the plane is a positive constant. The foci are a distance c from the center, where $c^2 = a^2 - b^2$.

The major axis of the ellipse is the longest line segment passing through the center and foci. The end points of the major axis are called the vertices of the ellipse. Vertices are a distance of a from the center.

The minor axis of the ellipse is the shortest line segment passing through the center. The length of the major axis is $2a$, and the length of the minor axis is $2b$.

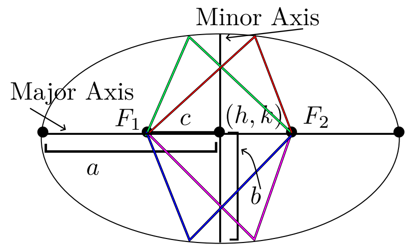
Standard Equation of an ellipse with major axis parallel to the x-axis and with center (h,k) and $c^2 = a^2 - b^2$

$$\begin{aligned}\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} &= 1,\ \text{Where } a > b > 0\\ Foci:\ & F(h \pm c, k)\\ Vertices:\ & (h \pm a, k).\end{aligned}$$

Standard Equation of an ellipse with major axis parallel to the y-axis and with center (h,k) and $c^2 = a^2 - b^2$

$$\begin{aligned}\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} &= 1,\ \text{Where } a > b > 0\\ Foci:\ & F(h, k \pm c)\\ Vertices:\ & (h, k \pm a).\end{aligned}$$

Figure: Ellipse



Consider each set of colored line segments connecting the two foci of the ellipse. If we take the sum of the line segments within a particular set, such as the green ones, we will find that their total length is equal to the combined length of all other possible line segments that can be drawn on the ellipse.

Note:-

The distance between the foci and the center is c, $c^2 = a^2 - b^2$

The endpoints of the major axis are called the **Vertices**, there are only 2 vertices on an ellipse

The distance between a vertice and the center is a, total length of the major axis is $2a$

The distance between the center and the end of the minor axis is b, total length is $2b$

Example 3.35

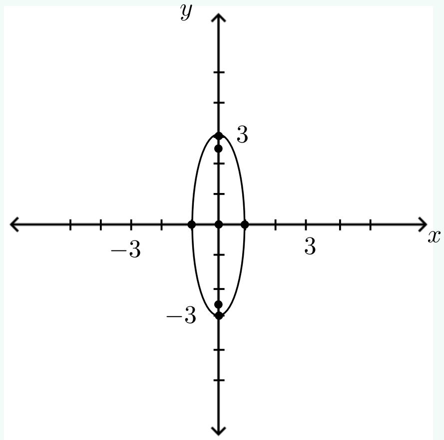
$$\begin{aligned} y^2 + 9x^2 &= 9 \\ \frac{x^2}{1} + \frac{y^2}{9} &= 1 \\ \frac{x^2}{1^2} + \frac{y^2}{3^2} &= 1 \\ x^2 + \left(\frac{y}{3}\right)^2 &= 1. \end{aligned}$$

Which means we have:

$$\begin{aligned} Center: \ & (0,0) \\ a &= 3 \\ b &= 1 \\ c^2 &= 9 - 1 \\ c &= \sqrt{8} = 2\sqrt{2}. \end{aligned}$$

Since the value under y is greater than the value under x, we know that this ellipse will have a major axis parallel to the x-axis (tall)

$$Vertices: \ (0, \pm 3)$$
$$Foci: \ (0, \pm 2\sqrt{2}).$$



Example 3.36

$$9x^2 + 4y^2 - 18x + 16y - 11 = 0.$$

So:

$$\begin{aligned}
9x^2 - 18x + 4y^2 + 16y &= 11 \\
9(x^2 - 2x) + 4(y^2 + 4y) &= 11 \\
9\left(x^2 - 2x + \left(-\frac{2}{2}\right)^2\right) + 4\left(y^2 + 4y + \left(\frac{4}{2}\right)^2\right) &= 11 + 9\big(1\big) + 4\big(4\big) \\
9(x^2 - 2x + 1) + 4(y^2 + 4y + 4) &= 36 \\
9(x-1)^2 + 4(y+2)^2 &= 36 \\
= \frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} &= 1 \\
= \frac{(x-1)^2}{2^2} + \frac{(y+2)^2}{3^2} &= 1
\end{aligned}$$

From here we can procede as normal.

Example 3.37 (find the equation of the ellipse that has its center at the orgin and satsifies the given conditions)

$$F(\pm 3, 0),\ \text{Minor axis length of 2.}$$

So since the foci lies across the x-axis, we know that this ellipse will have a major axis parallel to the x-axis.

So we would have:

$$\begin{gathered} Center:\ (0,0) \\ \frac{(x-0)^2}{a^2} + \frac{(y-0)^2}{b^2} = 1. \end{gathered}$$

If:

$$\begin{gathered} F(h \pm c, k) \\ Then:\ c = 3. \end{gathered}$$

And since:

$$\begin{aligned} Minor\ Axis := 2 \\ 2b = 2 \\ b = 1. \end{aligned}$$

Then:

$$\begin{aligned} 3^2 &= a^2 - 1^2 \\ 9 &= a^2 - 1 \\ a^2 &= 10 \\ &. \end{aligned}$$

Therefore our equation is:

$$\boxed{\frac{x^2}{10} + \frac{y^2}{1} = 1}.$$

3.44 Hyperbola

Definition:

A hyperbola is the set of all points in a plane, the difference of whose distances from two fixed points (the foci) in the plane is a positive constant. The foci are a distance c from the center, where $c^2 = a^2 + b^2$.

The vertices of the hyperbola are the points obtained at the intersection of the graph and the x-axis or y-axis. Vertices are a distance a from the center.

The transverse axis of the hyperbola is the line segment passing through the center and the vertices. The length of the transverse axis is $2a$.

The conjugate axis of the hyperbola is the line segment passing through the center and the points that are not on the hyperbola intersecting the other axis. The length of the conjugate axis is $2b$.

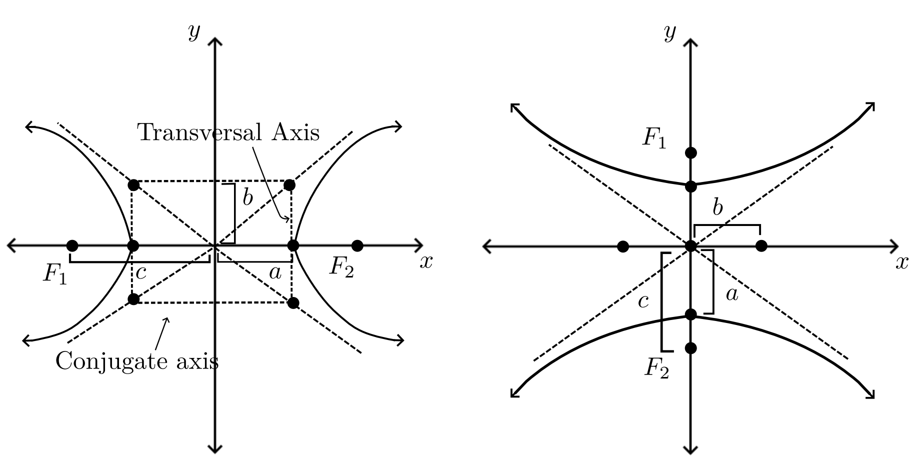
Standard Equations of a hyperbola with transversal axis parallel to the x-axis, center (h, k) and $c^2 = a^2 + b^2$.

$$\begin{aligned} \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} &= 1, \\ Foci:\ F(h \pm c, k) \\ Vertices:\ (h \pm a, k) \\ Asymptotes:\ y - k &= \pm\frac{b}{a}(x-h). \end{aligned}$$

Standard Equation of a hyperbola with transversal axis parallel to the y-axis and with center (h, k) and $c^2 = a^2 - b^2$

$$\begin{aligned} \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} &= 1, \\ Foci:\ F(h, k \pm c) \\ Vertices:\ (h, k \pm a) \\ Asymptotes:\ y - k &= \pm\frac{a}{b}(x-h). \end{aligned}$$

Figure: Hyperbola



Note:-

Whichever constant is under the positive variable is a

Example 3.38 (Consider the equation)

$$y^2 - \frac{x^2}{15} = 1.$$

Since the constant under the positive variable y^2, is 1, we know $a = 1$ and $b = \sqrt{15}$

$$\begin{aligned} a &= 1 \\ b &= \sqrt{15} \\ c^2 &= 1 + 15 \\ c &= 4. \end{aligned}$$

If:

$$\begin{aligned} F&(h, k \pm c) \\ F&(0, \pm 4). \end{aligned}$$

If:

$$\begin{aligned} V&(h, k \pm a) \\ V&(0, \pm 1). \end{aligned}$$

If:

$$\begin{aligned} Asymptote:\ y - k &= \pm\frac{a}{b}(x - h) \\ y &= \pm\frac{1}{\sqrt{15}}x. \end{aligned}$$

Example 3.39 (Find the equation of the hyperbola that has a center at the orgin and satisfies the given conditions.)

$$Vertices:\ V(\pm 4, 0),\ \text{passing through } (8, 2).$$

Since the vertices x value has the $\pm$, we know that this hyperbola is going to be opening left and right, and it has $a = 4$

So we have:

$$\frac{x^2}{4^2} - \frac{y^2}{b^2} = 1.$$

And we can plug in our given point to solve for b.

$$\begin{aligned}
\frac{(8)^2}{4^2} - \frac{(2)^2}{b^2} &= 1 \\
4 - \frac{4}{b^2} &= 1 \\
-\frac{4}{b^2} &= -3 \\
\frac{4}{b^2} &= 3 \\
3b^2 &= 4 \\
b^2 &= \frac{4}{3}.
\end{aligned}$$

Therefore:

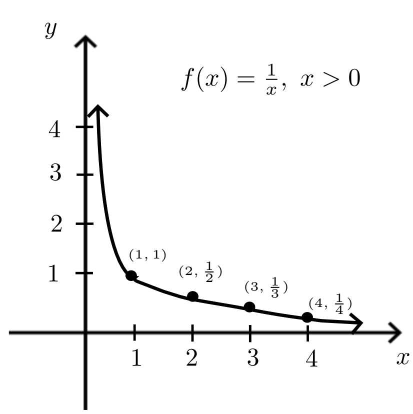
$$\boxed{\frac{x^2}{16} - \frac{y^2}{\frac{4}{3}} = 1}.$$

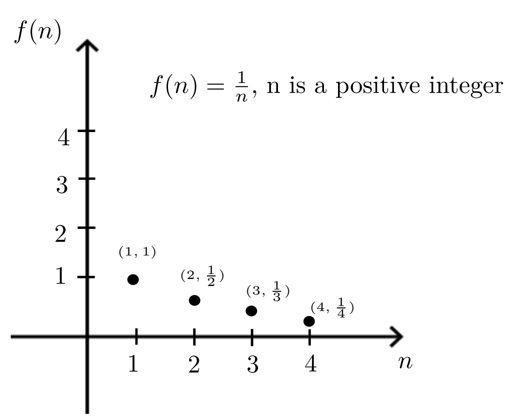
3.45 Sequences

Definition:

A sequence is a function whos domain is the set of positive integers

A sequence uses curly braces and has subscript notation with the form a_n





Note:-

The left graph is that of a **function**

The right graph is that of a **sequence**, notice it does not have a smooth curve. It only contains a series of points

Writing the first several terms of a sequence.

If:

$$\{a_n\} = \left\{\frac{n^2}{2n+1}\right\}.$$

Then:

$$\begin{aligned}
a_1 &= \frac{1^2}{2(1)+1} = \frac{1}{3} \\
a_2 &= \frac{2^2}{2(2)+1} = \frac{4}{5} \\
a_3 &= \frac{3^2}{2(3)+1} = \frac{9}{7} \\
a_4 &= \frac{4^2}{2(4)+1} = \frac{16}{9} \\
&.
\end{aligned}$$

Using Calculator (Texas-Instrument Graphing) to get the terms of a sequence.

1. 2nd $\rightarrow$ stat
2. ops $\rightarrow$ seq
3. Syntax: seq(*function*, *variable*, *start*, *stop*, *step*)
   - Ex: seq($(x^2)/(2x+1)$, x,1,6,1)
4. Optionaly: math $\rightarrow$ frac

To get table:

1. mode $\rightarrow \left[\text{func, par, pol, } \boxed{seq}\right]$
2. nMin: start value
3. $u(n)$: function
4. $u(nMin)$: value at a_1

Finding function by looking at terms in a sequence.

Consider:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots.$$

We can deduce that the function would be:

$$\{a_n\} = \left\{\frac{1}{2^n}\right\}.$$

The Factorial Symbol.

If $n \geq 0$, is an integer, the factorial symbol $n!$ is defined as follows

$$n! = n(n-1) \cdot ... \cdot 3 \cdot 2 \cdot 1, \qquad if\ n \geq 2$$
$$Or : n! = n(n-1)!.$$

3.46 Write The Terms of a Sequence Defined By Recursive Formula.

Definition:

A second way of defining a sequence is to assign a value to the first (or the first few) term(s) and specify the nth term by a formula that involves one or more of the terms preceding it.

Example 3.40 (Consider)

$$s_1 = 5, \quad s_n = 2 \cdot s_{n-1}.$$

So:

$$s_2 = 2 \cdot 5 = 10$$
$$s_3 = 2 \cdot 10 = 20$$
$$s_4 = 2 \cdot 20 = 40$$

3.47 Sequences with summation notation

Using summation notation is a short hand way of representing a sum of a sequence of terms.

For example.

$$a_1 + a_2 + a_3 + ... + a_n = \sum_{n}^{k=1} a_k \ .$$

Example 3.41 (Write out each sum)

$$\sum_{n}^{k=1} \frac{k}{k+1} \ .$$

So:

$$\frac{1}{1+1} + \frac{2}{2+1} + \frac{3}{3+1} + ... + \frac{n}{n+1}$$

Example 3.42 (Writing a sum in summation notation)

$$\left(\frac{1}{1}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{5}\right)^2.$$

So:

$$\sum_{5}^{k=1} \left(\frac{1}{k}\right)^2$$

3.48 Properties of Summations

If $\{a_n\}$ and $\{b_n\}$ are two sequences and c is a real number, then

$$\sum_{k=1}^{n} c = ca_1 + ca_2 + \cdots + ca_n = c(a_1 + a_2 + \cdots + a_n) = c\sum_{k=1}^{n} a_k$$

$$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

$$\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

$$\sum_{k=j+1}^{n} a_k = \sum_{k=1}^{n} a_k - \sum_{k=1}^{j} a_k \quad \text{where } 0 < j < n$$

Formulas for Sums of Sequences:

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{1}{2} + \frac{2^2}{2} + \frac{3^2}{2} + \cdots + \frac{n^2}{2} = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{1}{3} + \frac{2^3}{3} + \frac{3^3}{3} + \cdots + \frac{n^3}{3} = \left(\frac{n(n+1)}{2}\right)^2$$

3.49 Finding sum of sequence using calculator

1. 2nd → stat → math → sum
2. 2nd → stat → ops → seq
3. Syntax: sum(seq(*function, variable, start, end, step*))

3.50 Arithmetic Sequences.

Definition:

An arithmetic sequence is a sequence that is defined recursive as follows:

$$a_1 = 1, \quad a_n = a_{n-1} + d.$$

where d is $a_n - a_{n-1}$. In other words, d is the **common difference.**

Example 3.43 (Show that the sequence is arithmetic.)

a.)

$$4, 2, 0, -2, \ ...$$

So:

$$d = 2 - 4 = 0 - 2 = -2 - 0 = -2$$

b.)

$$\{s_n\} = \{4n - 1\}.$$

So we must find the first term:

$$s_1 = 4 \cdot 1 - 1 = 3.$$

Now we want to show that two consecutive terms in the sequence is a constant. We must show this algebraically.

Since we know that:

$$d = s_n - s_{n-1}.$$

Then:

$$\begin{aligned} d = 4n - 1 - (4(n-1) - 1) \\ 4n - 1 - (4n - 5) \\ 4n - 1 - 4n + 5 \\ d = 4. \end{aligned}$$

Note:-

Since these functions are in the form $mx+b$, we can safely say that m will be the common difference.

3.51 Find a Formula For an Arithmetic Sequence.

nth Term of an Arithetic sequence.

For an arithmetic sequence $\{a_n\}$ whose first term is a and whose common difference is d, the nth term is determined by the formula:

$$a_n = a + (n-1)d.$$

Example 3.44 (Find the twenty fourth term of the arithmetic sequence)

$$-3, 0, 3, 6, \ \ldots$$

Using the formula $a_n = a + (n-1)d$:

$$\begin{aligned} If:\ & a = -3,\ n = 24,\ d = 3 \\ Then:\ & a_{24} = -3 + (24-1)3 \\ & = 66. \end{aligned}$$

Example 3.45

The sixth term of an arithmetic sequence is 26, and the nineteenth term is 78. Find the first term and the common difference. Give a recursive formula for the sequence. What is the nth term of the sequence?

So we have:

$$26 = a + (6-1)d$$
$$78 = a + (19-1)d.$$

$$= \begin{cases} 26 = a + 5d \\ 78 = a + 18d \end{cases} \tag{2}$$

So we can see that we have a **system of equations**, with two unknows. Thus we can solve this system with **elimination.**

So Subtracting our two equations:

$$78 = a + 18d$$
$$26 = a + 5d$$

We get:

$$52 = 13d$$
$$d = 4.$$

Now that we have found d, we can plug d into one of our equations to solve for a:

$$26 = a + 5(4)$$
$$a = 6.$$

Now that we have both a and d, we can find the general formula by pluggion a and d into the general formula $a_n = a + (n-1)d$

$$a_n = 6 + (n-1)4$$
$$a_n = 6 + 4n - 4$$
$$\boxed{a_n = 4n + 2}.$$

3.52 Finding The Sum of n Terms in an Arithmetic Sequence.

Definition:

Let $\{a_n\}$ be an arithmetic sequence with first term a and common difference d. The sum S_n of the first n terms of $\{a_n\}$ is

$$S_n = \frac{n}{2}[2a + (n-1)d]$$
$$Or: \frac{n}{2}(a + a_n).$$

Example 3.46 (Find the sum of the first n terms of the sequence)

$$a_n = \{4n + 2\}.$$

If:

$$s_n = \frac{n}{2}(a + a_n).$$

Then:

$$\begin{aligned} s_n &= \frac{n}{2}(6 + 4n + 2) \\ &= \frac{n}{2}(4n + 8) \\ &= \frac{n(4n+8)}{2} \\ &= \frac{4n(n+2)}{2} \\ &= 2n(n+2). \end{aligned}$$

Suppose $n = 100$, then:

$$\begin{aligned} &2(100)(100 + 2) \\ &= 20,400. \end{aligned}$$

3.53 Geometric Sequences.

Definition:

A geometric sequence is a sequence of numbers in which each term after the first is found by multiplying the preceding term by a constant factor called the common ratio, denoted by r.

$$a_n = a, \quad a_n = ra_{n-1}.$$

We can find the common ration r with:

$$r = \frac{a_n}{a_{n-1}}.$$

Example 3.47 (Show that the sequence is geometric)

a.)

$$2, 8, 32, 128, \ ...$$

So:

$$r = \frac{8}{2} = \frac{32}{8} = \frac{128}{32} = 4$$

b.)

$$\{s_n\} = \{3^{n+1}\}.$$

Using the formula for r:

$$\begin{aligned} r &= \frac{3^{n+1}}{3^{(n-1)+1}} \\ &= \frac{3^{n+1}}{3^n} \\ &= 3^{n+1-n} \\ &= 3^1 \\ r &= 3. \end{aligned}$$

And:

$$\begin{aligned} a_1 &= 3^2 \\ &= 9. \end{aligned}$$

c.)

$$\{t_n\} = \{3(2)^n\}.$$

Using the formula for r:

$$\begin{aligned} r &= \frac{3(2^n)}{3(2^{n-1})} \\ &= \frac{2^n}{2^{n-1}} \\ &= 2^{n-(n-1)} \\ &= 2^{n-n+1} \\ &= 2^1 \\ &= 2. \end{aligned}$$

3.54 Find a Formula For a Geometric Sequence.

Definition:

For a geometric sequence $\{a_n\}$ whose first term is a and whose common ratio is r, the nth term is determined by the formula

$$a_n = a \cdot r^{n-1}, \quad r \neq 0$$

Example 3.48 (Find the ninth term of the geometric sequence)

$$3, 2, \frac{4}{3}, \frac{8}{9}, \ \ldots$$

So:

$$a = 3, \quad r = \frac{2}{3}$$

Therefore:

$$a_n = 3\left(\frac{2}{3}\right)^{n-1}$$
$$a_9 = 3\left(\frac{2}{3}\right)^{8}$$
$$.$$

To find a recursive formula for this sequence, we will use the recursive formula $a_n = ra_{n-1}$

$$a_n = \left(\frac{2}{3}\right) \cdot a_{n-1}.$$

3.55 Sum of n Terms of a Geometric Sequence

Definition:

Let $\{a_n\}$ be a geometric sequence with first term a and common ratio r, where $r \neq 0, r \neq 1$. The sum S_n of the first terms of $\{a_n\}$ is

$$S_n = \frac{a \cdot (1 - r^n)}{1 - r}, \quad r \neq 0, 1$$

Example 3.49 (Find the sum of the first n terms of the sequence)

$$3^n.$$

So:

$$\begin{aligned} a_1 &= 3^1 = 3 \\ r &= \frac{3^n}{3^{n-1}} \\ &= 3^{n(n-1)} \\ &= 3^1 \\ &= 3 \end{aligned}$$

Now using the formula $S_n = \frac{a(1-r^n)}{1-r}$:

$$S_n = \frac{3(1-3^n)}{1-3}$$
$$S_n = -\frac{3}{2}(1-3^n).$$

Now say we want the sum of the first 8 terms:

$$\begin{aligned} S_8 &= -\frac{3}{2}(1-3^8) \\ &= 9840. \end{aligned}$$

3.56 Determine Whether a Geometric Series Converges or Diverges.

Definition:

An infinite sum of the form

$$a + ar + ar^2 + \cdots + ar^{n-1}$$

with the first term a and a common ratio r, is called an infinite geometric series and is denoted by

$$\sum_{k=1}^{\infty} ar^{k-1}$$

Sum of an Infinite Geometric Series If $|r| < 1$, the sum of the infinite geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ is

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$

In the context of infinite geometric sequences, convergence and divergence refer to the behavior of the sum of the terms in the sequence as more terms are added.

If an infinite geometric sequence converges, it means that the sum of its terms approaches a finite value as more terms are added. In other words, there is a well-defined sum for the infinite series. The sum of a convergent infinite geometric series can be calculated using the formula above: $\frac{a}{1-r}$

On the other hand, if an infinite geometric sequence diverges, it means that the sum of its terms does not approach a finite value as more terms are added. In this case, the sum of the infinite series is said to be infinite or undefined. The divergence can occur if the absolute value of the common ratio is greater than or equal to 1 ($|r| >= 1$).

Determining whether an infinite geometric sequence converges or diverges depends on the value of the common ratio 'r'. If $|r|$ is less than 1, the sequence converges, and if $|r|$ is greater than or equal to 1, the sequence diverges.

Example 3.50 (Find the sum of the infinite geometric sequence.)

$$1 + \frac{1}{3} + \frac{1}{9}.$$

So:

$$a = 1$$
$$r = \frac{1}{3}.$$

And if:

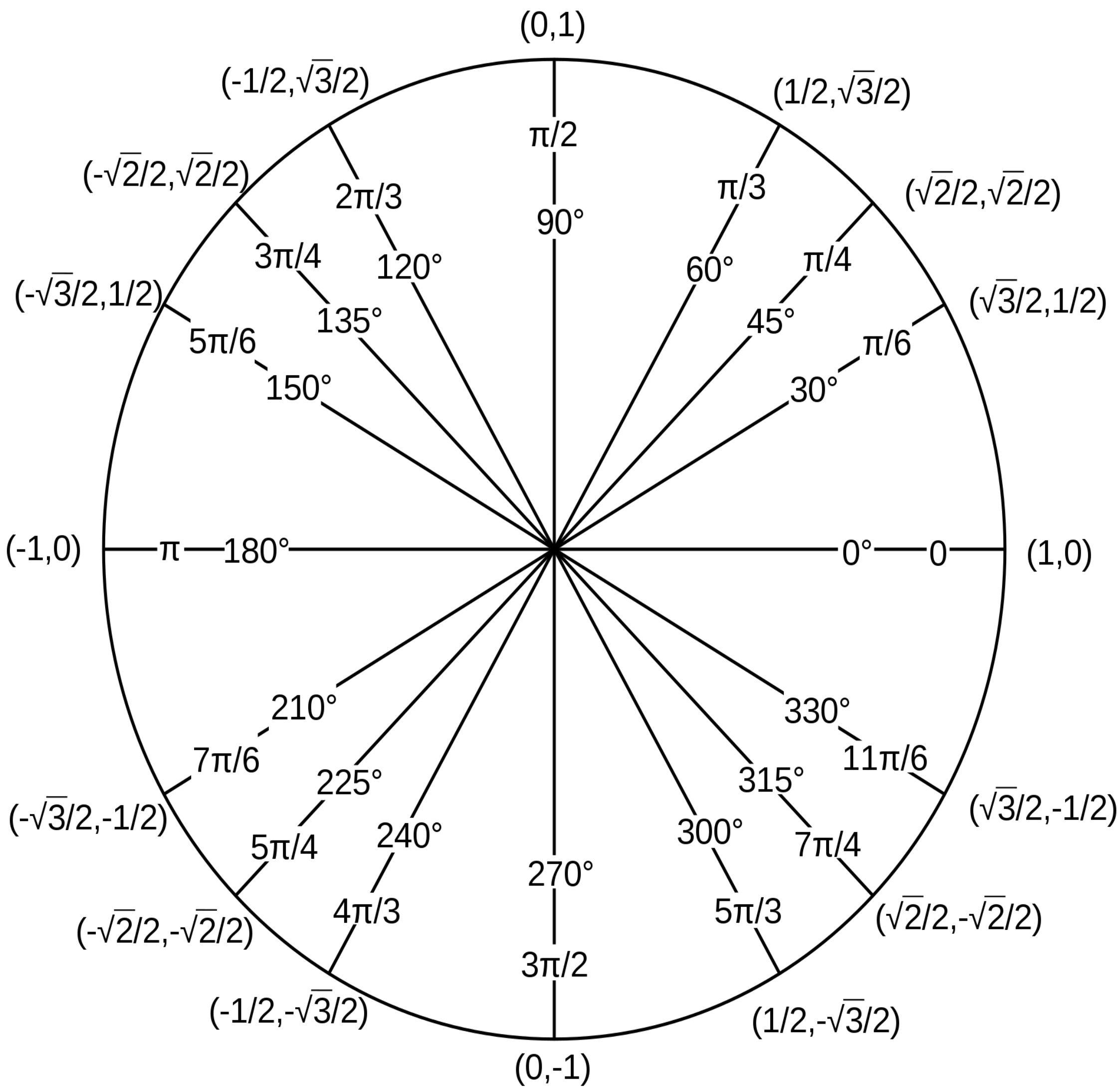
$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$

Then:

$$\frac{1}{1-\frac{1}{3}}$$
$$= \frac{2}{3}.$$

4 Trigonometry

4.1 Unit Circle Diagram



4.2 Standard Position / Central Angle

- Standard Position (Vertex at orgin and initial side coincides with positive x-axis)
- Central Angle (Positive angle whose center is at the orgin)

4.3 Linear / Angular Speed

- Linear Speed

$$v = \frac{s}{t}$$
$$and\ v = rw.$$

- Angular Speed

$$w = \frac{\theta}{t}.$$

4.4 Decimal to Degrees / Degrees to Decimal

- Convert Decimal to Degrees

$$deg + \frac{min}{60} + \frac{sec}{3600}.$$

- Convert Degrees to Decimal

If:

$$669.$$

Then:

$$29 \cdot 60 = 17$$
$$0 \cdot 60 = 24.$$

So:

$$66^\circ 17' 24''.$$

4.5 Degrees to Radians / Radians to Degrees

- Degrees to Radians

$$30^\circ \cdot \frac{\pi}{180}$$
$$= \frac{\pi}{6}.$$

- Radians to Degrees

$$\frac{\pi}{6} \cdot \frac{180}{\pi}$$
$$= 30^\circ.$$

4.6 Arc Length of a Circle

Arc length is computed with:

$$s = r \cdot \theta.$$

4.7 Trig Functions

On Unit Circle

- $\sin\theta = y$
- $\cos\theta = x$
- $\tan\theta = \frac{\sin\theta}{\cos\theta}$
- $\csc\theta = \frac{1}{\sin\theta}$
- $\sec\theta = \frac{1}{\cos\theta}$
- $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Not on Unit Circle

- $\sin\theta = \frac{y}{r}$
- $\cos\theta = \frac{x}{r}$
- $\tan\theta = \frac{y}{x}$
- $\csc\theta = \frac{r}{y}$
- $\sec\theta = \frac{r}{x}$
- $\cot\theta = \frac{x}{y}$

4.8 Periodicity

- sin, cos, csc, sec $= 2\pi$
- tan, cot $= \pi$

4.9 Even/Odd

- The cosine and Secant functions are **even**
- The sine, tangent, cosecant and cotangent functions are **odd**

4.10 Finding Values With Calculator

- $csc = \frac{1}{\sin\theta}$
- $sec = \frac{1}{\cos\theta}$
- $cot = \frac{1}{\tan\theta}$

4.11 Domain and Range of Trig Functions

- sin

$$\begin{aligned} Domain:\ & (-\infty, \infty) \\ Range:\ & [-1, 1] \\ & . \end{aligned}$$

- cos

$$\begin{aligned} Domain:\ & (-\infty, \infty) \\ Range:\ & [-1, 1]. \end{aligned}$$

- tan

$$\begin{aligned} Domain:\ & \mathbb{R} \text{ except odd integer multiples of } \frac{\pi}{2} \\ Range:\ & (-\infty, \infty). \end{aligned}$$

- csc

$$\begin{aligned} Domain:\ & \mathbb{R} \text{ except odd integer multiples of } \pi \\ Range:\ & (-\infty, -1] \cup [1, \infty]. \end{aligned}$$

- sec

$$\begin{aligned} Domain:\ & \mathbb{R} \text{ except odd integer multiples of } \frac{\pi}{2} \\ Range:\ & (-\infty, -1] \cup [1, \infty]. \end{aligned}$$

- cot

$$\begin{aligned} Domain:\ & \mathbb{R} \text{ except odd integer multiples of } \pi \\ Range:\ & (-\infty, \infty). \end{aligned}$$

4.12 Graphing

- Amplitude, Period
  - $|A|$ = Amplitude
  - T = Period

$$T = \frac{2\pi}{w}.$$

If:

$$y = -4\cos 3x.$$

Then:

$$\begin{aligned} T &= \frac{2\pi}{3} \\ A &= 4. \end{aligned}$$

- Basic Transformations

$$y = A\sin(wx - c) + d.$$

  - A = Amplitude
  - w = period
  - c = phase shift (Horizontal Shift)
  - d = vertical shift

4.13 Asymptotes of Trig Functions

Only Tan, Secant, cosecant and cotangent have Asymptotes, and they occur at:

- Tan: When $\cos\theta = 0$ at $\frac{\pi}{2} + \pi n$
- Cosecant: When $\sin\theta = 0$ at πn, where n is an integer
- Secant: When $\cos\theta = 0$ at $\frac{\pi}{2} + \pi n$, where n is an integer
- Cotangent When $\sin\theta = 0$ at πn, where n is an integer

4.14 Inverse Trig Functions

- Domain/Range (with inverse trig functions, domain and range flip, on graph, x and y flip)
- Restrictions (to make one to one)
  - $\sin^{-1}$
  $$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$
  - $\cos^{-1}$
  $$[0, \pi].$$
  - $\tan^{-1}$
  $$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$
  - $\csc^{-1}$
  $$[-\frac{\pi}{2}, \frac{\pi}{2}].$$
  - $\sec^{-1}$
  $$[0, \pi].$$
  - $\cot^{-1}$
  $$[0, \pi].$$

4.15 How to Find r

$$r^2 = x^2 + y^2$$
$$r = \sqrt{x^2 + y^2}.$$

4.16 Find Inverse of Trig Functions

1. replace f(x) with y
2. swap x and y
3. isolate siny
4. solve for y

4.17 Pythagorean Identites

- $\sin^2\theta + \cos^2\theta = 1$
- $\sin\theta = 1 - \cos\theta$
- $\sin^2\theta = 1 - \cos^2\theta$
- $\tan^2\theta + 1 = \sec^2\theta$
- $\cot^2\theta + 1 = \csc^2\theta$

4.18 Sum and Difference

- $\cos(a+b) = \cos a \cos b - \sin a \sin b$
- $\cos(a-b) = \cos a \cos b + \sin a \sin b$
- $\sin(a+b) = \sin a \cos b - \cos a \sin b$
- $\sin(a-b) = \sin a \cos b + \cos a \sin b$

4.19 Double Angle/Half Angle Formulas

- $\sin(2\theta) = 2\sin\theta\cos\theta$
- $\cos 2\theta = 1 - 2\sin^2\theta$
- $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$
- $\sin\frac{\theta}{2}$

$$\pm\sqrt{\frac{1-\cos\theta}{2}}.$$

- $\cos\frac{\theta}{2}$

$$\pm\sqrt{\frac{1+\cos\theta}{2}}.$$

- $\tan\frac{\theta}{2}$

$$\pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}.$$

4.20 Product to Sum

- $\sin a \sin b$

$$\frac{1}{2}[\cos(a-b) - \cos(a+b)].$$

- $\cos a \cos b$

$$\frac{1}{2}[\cos(a-b) + \cos(a+b)].$$

- $\sin a \cos b$

$$\frac{1}{2}[\sin(a+b) + \sin(a-b)].$$

4.21 Sum to Product

- $\sin a + \sin b$

$$2\sin\frac{a+b}{2}\cos\frac{a-b}{2}.$$

- $\sin a - \sin b$

$$2\sin\frac{a-b}{2}\cos\frac{a+b}{2}.$$

- $\cos a + \cos b$

$$2\cos\frac{a+b}{2}\cos\frac{a-b}{2}.$$

- $\cos a - \cos b$

$$-2\sin\frac{a+b}{2}\sin\frac{a-b}{2}.$$

4.22 Right Triangles

- Pythagorean Theoroem

$$a^2 + b^2 = c^2.$$

- sohcahtoa

$$sin = \frac{opp}{hyp}$$
$$cos = \frac{adj}{hyp}$$
$$tan = \frac{opp}{adj}.$$

- Complementary Angles

$$\begin{array}{ll} \sin B = \frac{b}{c} = \cos A & \csc B = \frac{c}{b} = \sec A \\ \cos B = \frac{a}{c} = \sin A & \sec B = \frac{c}{a} = \csc A \\ \tan B = \frac{b}{a} = \cot A & \cot B = \frac{a}{b} = \tan A \end{array}$$
.

4.23 Law of Sin / Cos

- cases
  - Law of sines

$$ASA$$
$$SAA$$
$$SSA$$
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

  - Law of Cosines

$$SAS$$
$$SSS$$
$$c^2 = a^2 + b^2 - 2ab\cos c.$$

4.24 Area

- S

$$s = \frac{1}{2}(a+b+c).$$

- A

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$

4.25 Hyerbolic Trig

- $\sinh x$

$$\frac{e^x - e^{-x}}{2}.$$

- $\cosh x$

$$\frac{e^x + e^{-x}}{2}.$$

- $\tanh x$

$$\frac{\sinh x}{\cosh x}.$$

- csch x

$$\frac{1}{\sinh x}.$$

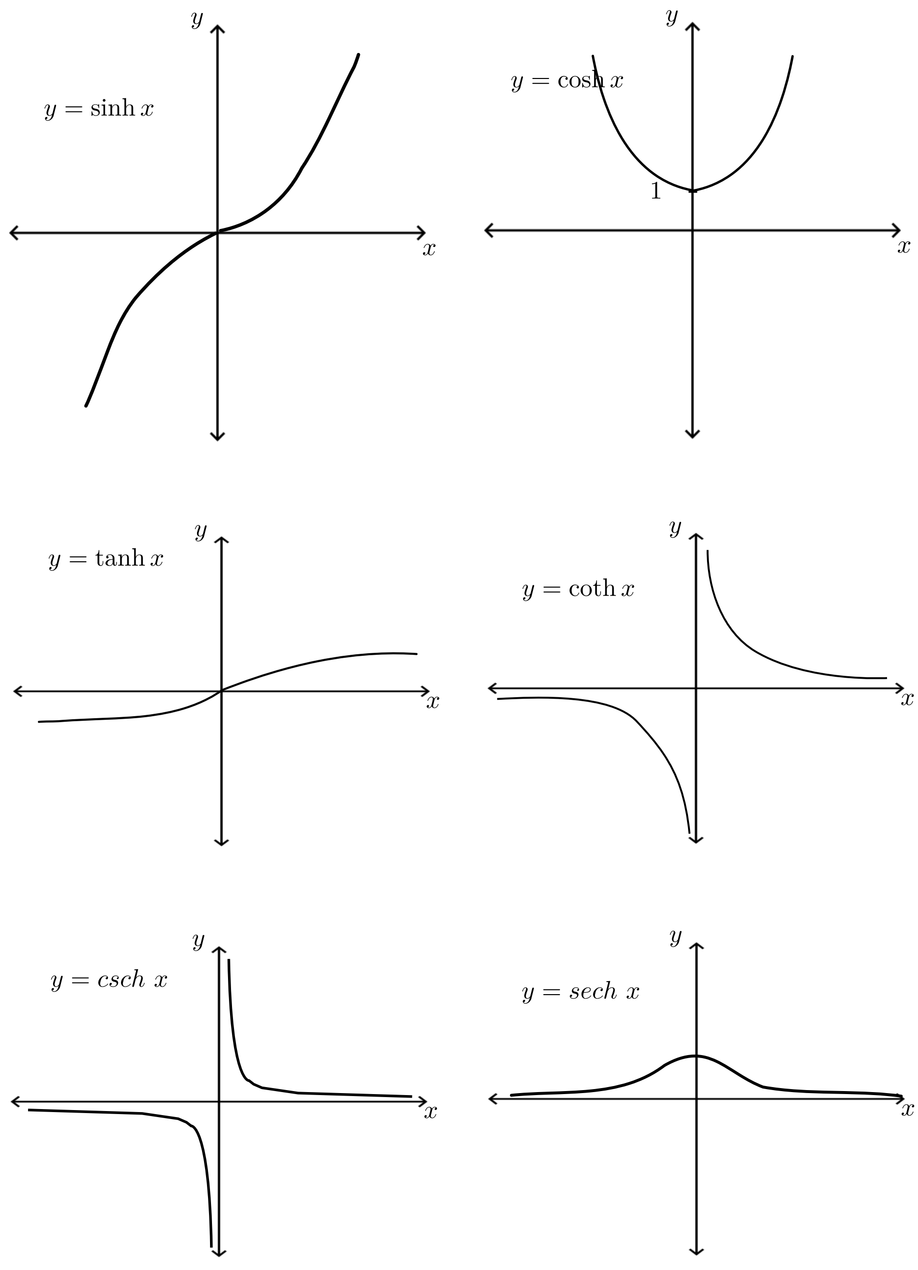
- sech x

$$\frac{1}{\cosh x}.$$

- $\coth x$

$$\frac{\cosh x}{\sinh x}.$$

4.26 Graphs of Hyperbolic Trig Functions



4.27 The Triangle

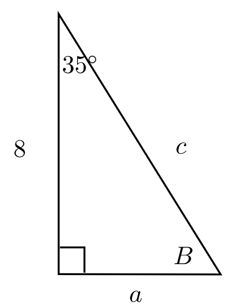
Right Triangles

- the sum of the 2 acute angles A and B must equal 90°
- SOHCAHTOA
- complementary angles theorem

Complementary Angles:

$$\begin{aligned} \sin B &= \frac{b}{c} = \cos A & \csc B &= \frac{c}{b} = \sec A \\ \cos B &= \frac{a}{c} = \sin A & \sec B &= \frac{c}{a} = \csc A \\ \tan B &= \frac{b}{a} = \cot A & \cot B &= \frac{a}{b} = \tan A \end{aligned}.$$

Say we have the triangle:



Find B:

$$90 - 35 = 55°.$$

To find a, if we choose to use the measure of A, then the function that links opposite and adjancent is the tan function, so:

$$\begin{aligned} \tan 35 &= \frac{a}{8} \\ &= 5.6. \end{aligned}$$

To find c,

$$\begin{aligned} \sin 55 &= \frac{8}{c} \\ c &= \frac{8}{\sin 55} \\ &= 9.77. \end{aligned}$$

Non Right Triangles

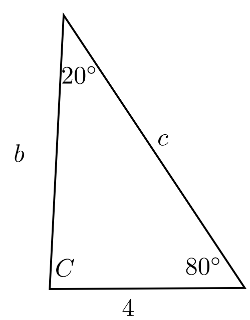
- Cases:
  - ASA
  - SAA
  - SSA

Will use the law of sines, which states:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Say we have the triangle

$$A = 20^\circ,\ b = 80^\circ,\ a = 4.$$



Use the fact that all three angles must measure to 180° to find C

$$C = 80^\circ.$$

To find b using the law of sines:

$$\begin{aligned}\frac{\sin 20}{4} &= \frac{\sin 80}{b}\\ b &= \frac{4\sin 80}{\sin 20}\\ b &= 11.52.\end{aligned}$$

Now find c:

$$\begin{aligned}\frac{\sin 20}{4} &= \frac{\sin 80}{c}\\ c &= \frac{4\sin 80}{\sin 20}\\ &= 11.52.\end{aligned}$$

Figure out if you have more than one triangle (SSA)

Say we have:

$$c = 15^\circ, c = 6, a = 10.$$

- Find the angle of the side you are given.
- compute $(180 - A_1) + 15$
- if this equates to less than 180 degrees, you have 2 triangles.
- find A_2 by computing $(180 - A_1)$

Law of cosines

The law of cosines states:

$$a^2 = b^2 + c^2 - 2bc \cdot \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cdot \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cdot \cos C.$$

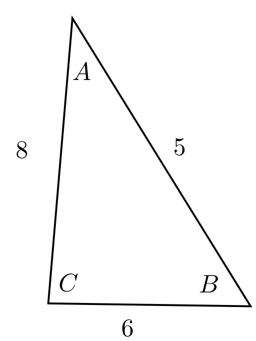
And we can rewrite to find missing angles:

$$B = \cos^{-1} \frac{b^2 - a^2 - c^2}{-2ac}.$$

Using law of cosines to solve SSS triangles

Say we have the triangle:

$$a = 6,\ b = 8,\ c = 5.$$



We can solve this triangle by using the law of cosines

To find C:

$$\begin{aligned}
5^2 &= 6^2 + 8^2 - 2(6)(8)\cos C\\
25 &= 36 + 64 - 96\cos C\\
-75 &= -96\cos C\\
\cos C &= \frac{75}{96}\\
C &= \cos^{-1}\frac{75}{96}\\
C &= 38.6.
\end{aligned}$$

And we can do the same to find B

Area of triangles

If we are given a triangle with 2 sides and a angle measurement, we can find area using the fact that:

$$A = \frac{1}{2} side \cdot side \cdot \sin{(angle)}.$$

If we are given a triangle with 3 sides known, we can find area using:

$$\begin{aligned}
s &= \frac{1}{2}(a+b+c)\\
A &= \sqrt{s(s-a)(s-b)(s-c)}.
\end{aligned}$$

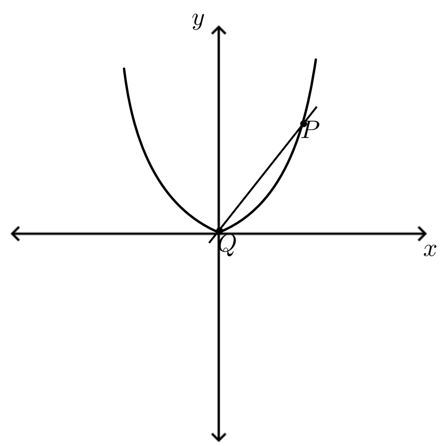
5 Calc 1

5.1 Secant Lines

- Secent lines on a graph is a line through the curve that touces at 2 places.
- To find the slope of the secent line:

$$m_{pq} = \frac{y_2 - y_1}{x_2 - x_1} \ rather, \ \frac{P_y - Q_y}{P_x - Q_x}.$$

Secant line:



5.2 Tangent Lines

- Tangent lines on a graph is a line that touches the curve exactly one time.
- Achieve an aproximation of the slope of the tangent line by moving Q closer to the tangent line and finding the slope of the secent line, we write

$$\lim_{Q \to P} m_{PQ} = m.$$

- To write the equation of the tangent line, use point slope form:

$$y - y_1 = x(x - x_1).$$

5.3 The Velocity Problem

- Average velocity is denoted by the slope of the secant line
- Instantaneous velocity is denoted by the slope of the tangent line

5.4 The Limit of a Function

- Consider the function:

$$f(x) = x^2 + 2.$$

  - If we want to investigate x=2, we can draw a table and examine what does the value of $f(x)$ approach as x approaches 2

- So we say:

$$\lim_{x \to a} f(x) = L.$$

5.5 One Sided Limits

- Approaching from the left (left hand limit):

$$\lim_{x \to a-} f(x).$$

  - Plug in a and evaluate

- Approaching from the right (right hand limit):

$$\lim_{x \to a+} f(x).$$

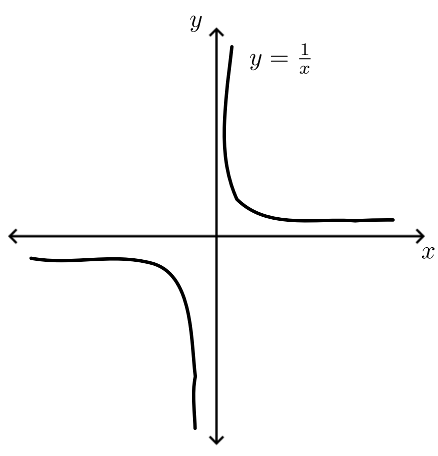
  - Plug in a and evaluate

- Know that:

$$\lim_{x \to a} f(x) = l \iff \lim_{x \to a-} f(x) = l \wedge \lim_{x \to a+} f(x) = l.$$

5.6 Infinite Limits

Consider:



We can see that:

$$\lim_{x \to 0+} f(x) = \infty.$$

And:

$$\lim_{x \to 0-} f(x) = -\infty.$$

Therefore we have determined that x=0 is a vertical asymptote, in general, we can safely say that f(x) has a vertical asymptote at a if any of the limits as $x \to a = \infty \ or - \infty$

What if we have a $\frac{Nonzero\ Constant}{Approaching\ Zero}$?

- say we have the equation:

$$\lim_{x \to 5-} \frac{x+1}{x-5}.$$

We can see that if we plug in 5, we get $\frac{6}{0}$, so since we have a $\frac{Nonzero\ Constant}{Approaching\ Zero}$, we know that our limit is either going to be $\infty\ or\ -\infty$, the way we determite the sign of infinity is as follows:

1. Plug in a number close to a in whatever direction your limit is, the sign of the output will be the sign of infinity.

Note:-

If the limit does not specify a side, test with number close to a in both directions, if the sign is not the same, the limit is DNE

5.7 Limit Laws

- $\lim_{x \to a} f(x) + g(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
- $\lim_{x \to a} f(x) - g(x) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
- $\lim_{x \to a} cf(x) = c \cdot \lim_{x \to a} f(x)$
- $\lim_{x \to a} f(x) \cdot g(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
- $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, if g(x) $\neq 0$
- $\lim_{x \to a} \left(f(x) \right)^n = \left[\lim_{x \to a} f(x) \right]^n$, where n is a positive integer
- $\lim_{x \to a} c = c$
- $\lim_{x \to a} x = a$
- $\lim_{x \to a} x^n = a^n$, where n is a positive integer
- $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$, where n is a positive integer
- $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$, where n is a positive integer

5.8 Direct Substitution Property

- You can plug in a and evaluate, if a is not in the domain of $f(x)$, you can try factoring

5.9 Continuity

For a function to have continuity ata, 3 things must be true:

1. f(x) is defined at a
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

If no. 3 is true, the function is automatically continous at a

5.10 One-Sided Continuity

- Continuity from the right:

$$\lim_{x \to a+} f(x) = f(a).$$

- Continuity from the left:

$$\lim_{x \to a-} f(x) = f(a).$$

If f and g are continuous at a, then:

- $f + g$
- $f - g$
- fg
- $\frac{f}{g}$
- cf

Are all continuous at a

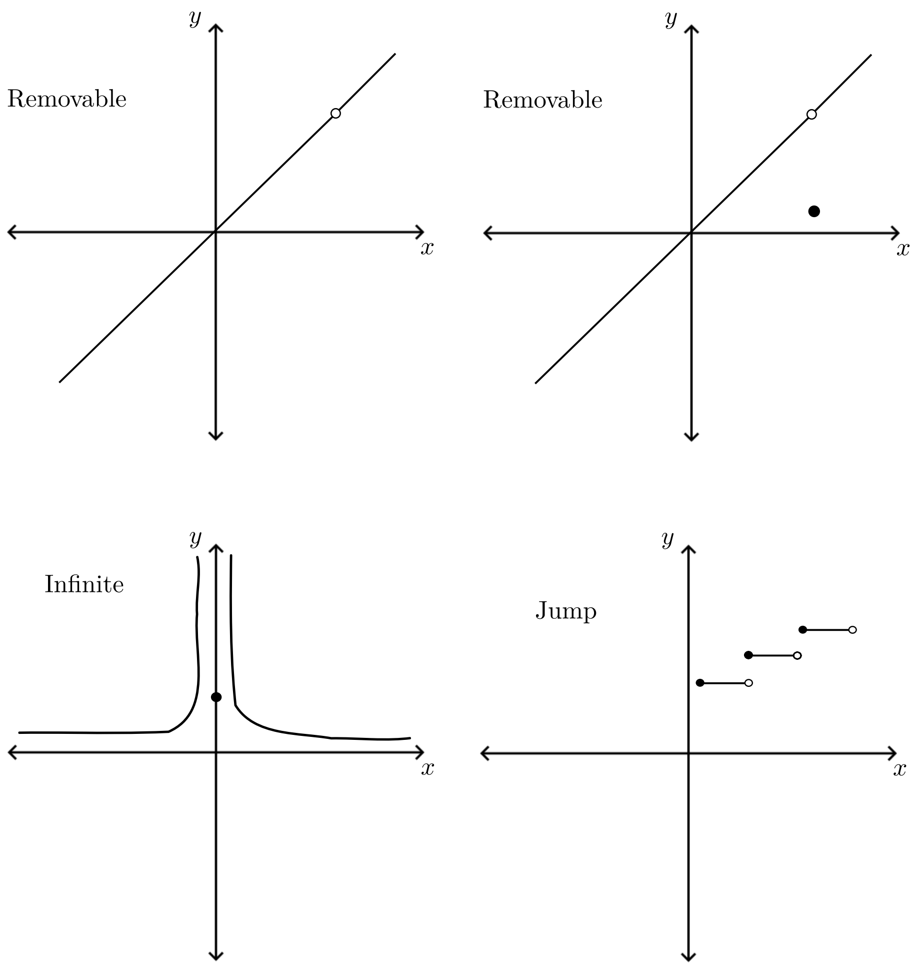
Also:

- Any polynomial is continous on its domain ($\mathbb{R}$)
- Any rational function is continous on its domain

Note:-

If $\lim_{x \to a} f(x)$ exists, Then you don't need to worry about which side the continuity is coming from

5.11 Discontinuitys



5.12 Finding where a function is Discontinuous without the graph of f

- Zeros of a rational function
- if $\lim_{x \to a} f(x) \neq f(a)$
- Piecewise functions, examine values of x where the domain flips to see if the limit is DNE

5.13 Intermediate Value Theorem

Suppose f is continuous on [a,b], Let N be any number between f(a) and f(b), where $f(a) \neq f(b)$, then:

$$\exists\ c \in (a, b) \mid f(c) = N.$$

5.14 Evaluating limits at infinity (horizontal asymptotes)

We have a horizontal Asymptote if:

$$\lim_{x\to\infty} f(x) = L$$
$$Or: \lim_{x\to-\infty} f(x) = L.$$

- Recall: If the degree of the numerator is higher than the degree of the denominator, the H.A is automatically y = 0
- For rational Functions: divide every term in the numerator and denominator by the highest degree in the denominator, then take the limit of each new term.
- Radical types like:

$$\lim_{x\to\infty} \sqrt{x^2+4x} - x.$$

  1. Conjugate (Multiply both numerator and denominator)
  2. pull out the 4
  3. divide numerate by x
  4. divide insider radical by x^2 and +x by x
  5. evaluate

- Example:

$$\lim_{x\to-\infty} \frac{\sqrt{9x^6 - x}}{x^3+1}.$$

  Since $\sqrt{x^6} = x^3$, we need to put absolute value bars and write as piecewise:

$$|x^3| = \begin{cases} x^3 & \text{if } x \geqslant 0 \\ -(x^3) & \text{if } x < 0 \end{cases} \tag{3}$$

  And since our limit is going to negative infinity, we use $-x^3$

$$\begin{aligned} \lim_{x\to-\infty} \frac{-\sqrt{\frac{9x^6-x}{x^6}}}{\frac{x^3+1}{x^3}} &= \frac{-\sqrt{9-\frac{1}{x^5}}}{1+\frac{1}{x^3}} \\ &= \frac{-\sqrt{9-0}}{1+0} \\ &= -3. \end{aligned}$$

- Limits at infinty of Polynomials
  - Say we have:

$$\lim_{x\to-\infty} 5 + 2x - x^3.$$

  - In this case, we can forget about the insignificant terms and only focus on $-x^3$, which means.

$$\begin{aligned} \lim_{x\to-\infty} -x^3 &= -(-\infty)^3 \\ &= \infty. \end{aligned}$$

- Find equation of horizontal Asymptote by looking at limtit at infinity
  - If the limit as x approaches infinity is a constant, then the equation of the H.A is y = L, where L is the constant
  - if the limit as x approches infinity of a rational function is bottom heavy then the equation of the H.A is y = 0

5.15 Derivatives and rates of change (Definitions)

- Slope of Secant Line:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$
$$or : \frac{f(a+h) - f(a)}{h}.$$

- Slope of Tangent Line:

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
$$or : \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

- Average Velocity:

$$v_{ave} = \frac{f(x) - f(x)}{x - a}.$$

- Instantaneous Velocity:

$$v_{inst} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
$$or : \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

- Speed:

$$Speed \ = |Velocity|.$$

- Derivatives Definition:

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
$$Or : f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

- Know that:

$$s(t) = postition\ function$$
$$v(t) = s'(t) = velocity\ function$$
$$a(t) = v'(t) = acceleration\ function.$$

Note:-

If f(x) is differentiable at a, then it is continous at a, the converse is not true

5.16 Differential Rule

Properties of Derivatives:

- $\frac{d}{dx}c = 0$
- $\frac{d}{dx}x = 1$
- $\frac{d}{dx}(x^n) = n \cdot x^{n-1} \rightarrow$ ***Power Rule***
- $\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}[f(x)]$
- $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

5.17 Derivatives of common functions

Exponential Functions:

- $\frac{d}{dx}e^x = e^x \cdot \frac{d}{dx}x$
- $\frac{d}{dx}a^x = a^x \cdot \ln a \cdot \frac{d}{dx}x$
- $\frac{d}{dx}\ln x = \frac{1}{x} \cdot \frac{d}{dx}x$
- $\frac{d}{dx}\log_a x = \frac{1}{x \cdot \ln a} \cdot \frac{d}{dx}x$

Trig Functions:

- $\frac{d}{dx}\sin x = \cos x$
- $\frac{d}{dx}\cos x = -\sin x$
- $\frac{d}{dx}\tan x = \sec^2 x$
- $\frac{d}{dx}\csc x = -\csc x \cot x$
- $\frac{d}{dx}\sec x = \sec x \tan x$
- $\frac{d}{dx}\cot x = -\csc^2 x$

Inverse Trig:

- $\frac{d}{dx}$arcsin(x) = $\frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}$arccos(x) = -$\frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}$arctan(x) = $\frac{1}{x^2+1}$
- $\frac{d}{dx}$arccsc(x) = $-\frac{1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}$arcsec(x) = $-\frac{1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}$arccot(x) = $-\frac{1}{x^2+1}$

Hyperbolic Trig

- $\frac{d}{dx}\sinh x = \cosh x$
- $\frac{d}{dx}\cosh x = \sinh x$
- $\frac{d}{dx}\tan x = sech^2 x$
- $\frac{d}{dx}cschx = -cschxcothx$
- $\frac{d}{dx}sechx = -sechxtanhx$
- $\frac{d}{dx}cothx = -csc^2 x$

5.18 Normal Line

- The normal line is Perpendicular to the tangent line at the point of tangency, which means:
$$m_{tan} \cdot m_{normal} = -1.$$
  - In other words, they are opposite reciprocals so flip and change sign

5.19 Find equation of tangent and normal line

Say we have the equation and point:

$$y = x^4 + 8e^x \text{ at the point (0,8)}..$$

1. Find derivative
2. Plug in x into derivative function
3. Flip m_{tan} and change the sign to find m_{normal}
4. Use point slope form to find equations

5.20 Product and Quotient Rule

- Product rule:
$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)].$$
- Quotient Rule:
$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

5.21 Chain Rule

- Know the chain rule:
  - If:
$$F(x) = f(g(x)).$$
  - Then:
$$F'(x) = f'(g(x)) \cdot g'(x).$$

5.22 Implicit Differentation

Procedure:

1. Differentiate
2. Leave $\frac{dy}{dx}$ next to terms of y that you differentiate
3. solve for $\frac{dy}{dx}$

Example:

$$\begin{aligned}
2x^3 + x^2y - xy^3 &= 2 \\
= 6x^2 + (x^2 \cdot 1 \cdot \frac{dy}{dx} + 2x \cdot y) - (x \cdot 3y^2 \cdot \frac{dy}{dx} + y^3 \cdot 1) &= 0 \\
= 6x^2 + x^2\frac{dy}{dx} + 2xy - 3xy^2\frac{dy}{dx} - y^3 &= 0 \\
= x^2\frac{dy}{dx} - 3xy^2\frac{dy}{dx} &= y^3 - 6x^2 - 2xy \\
= \frac{dy}{dx}(x^2 - 3xy^2) &= y^3 - 6x^2 - 2xy \\
= \frac{dy}{dx} &= \frac{y^3 - 6x^2 - 2xy}{x^2 - 3xy^2}.
\end{aligned}$$

5.23 Logarithmic Differentation

If you have a problem that uses chain rule, product rule, quotient rule, all together. It's best to use this Logarithmic definition

Or if you have an equation with a variable in the base, and as the exponent, like:

$$y = (\cos 5x)^x.$$

Procedure:

1. Take ln of both sides
2. Differentiate implicitly with respect to x
3. solve for y'

Example:

$$
\begin{aligned}
y &= \sqrt[4]{\frac{x^2+1}{x^2-1}} \\
\ln y &= \ln\left(\frac{x^2+1}{x^2-1}\right)^{\frac{1}{4}} \\
\frac{1}{y}\frac{dy}{dx} &= \frac{1}{4}\ln\left(\frac{x^2+1}{x^2-1}\right) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{1}{4}\ln\left(x^2+1\right) - \frac{1}{4}\ln\left(x^2-1\right) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{1}{4}\left(\frac{1}{x^2+1}\cdot 2x\right) - \frac{1}{4}\left(\frac{1}{x^2-1}\cdot 2x\right) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{x}{2(x^2+1)} - \frac{x}{2(x^2-1)} \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{x}{2(x^2+1)}\cdot\frac{x^2-1}{x^2-1} - \frac{x}{2(x^2-1)}\cdot\frac{x^2+1}{x^2+1} \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{x(x^2-1)}{2(x^2+1)(x^2-1)} - \frac{x(x^2+1)}{2(x^2-1)(x^2+1)} \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{x^3-x}{2(x^2+1)(x^2-1)} - \frac{x^3+x}{2(x^2-1)(x^2+1)} \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{x^3-x-(x^3+x)}{2(x^2+1)(x^2-1)} \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{x^3-x-x^3-x}{2(x^4-1)} \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{-2x}{2(x^4-1)} \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{-x}{x^4-1} \\
&.
\end{aligned}
$$

Solve for $\frac{dy}{dx}$ and plug in original equation for y:

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{-x}{x^4-1}\cdot y \\
\frac{dy}{dx} &= \frac{-x}{x^4-1}\cdot\sqrt[4]{\frac{x^2+1}{x^2-1}}.
\end{aligned}
$$

5.24 Rates of Change in the natural and social sciences

Velocity:

- Velocity is derivative of position function
- A particle is at rest when $v(t) = 0$
- a particle reaches its maximum height when v(t) = 0
- A particle is moving in the positive direction when $v(t) > 0$
- A particle is moving in the negative direction when $v(t) < 0$

Acceleration:

- A particle is speeding up when v(t) and a(t) have the same signs
- A particle is slowing down when v(t) and a(t) have opposite signs

5.25 Exponential Growth and Decay

Formula:

$$\begin{aligned} y &= Ce^{kt} \\ y &= Population \\ C &= \text{ } intitial \text{ } Population \\ k &= \frac{Growth \text{ } Rate}{Population} \\ t &= time. \end{aligned}$$

For example, if a bacterial culture starts with 4000 bacteria and triples every half hour.:

So:

$$y = 4000e^{kt}, \quad (\frac{1}{2}, 12000).$$

Then we can solve for k:

$$\begin{aligned} 12000 &= 4000e^{k(\frac{1}{2})} \\ 3 &= e^{\frac{k}{2}} \\ \ln 3 &= \frac{k}{2} \\ k &= 2\ln 3. \end{aligned}$$

So we have:

$$y = 4000e^{2\ln 3(t)}.$$

5.26 Newton's Law of Cooling

Newton's Law of cooling states:

$$\begin{aligned} T(t) &= t_s + Ce^{kt} \\ T(t) &= \text{ Temperature at time } t \\ t_s &= \text{ Temperature of surrounding area} \\ C &= t_0 - t_s. \end{aligned}$$

Say we have a freshly brewed cup of coffee has a temperature of $95°C$ in a $20°C$ room, five minutes later, its temerature is $88°C$

Then

$$\begin{aligned} T(t) &= 88 \\ t_s &= 20 \\ C = t_0 - t_s &= 95 - 20 = 75 \\ t &= 5. \end{aligned}$$

So our equation would be.

$$\begin{aligned} 88 &= 20 + 75e^{k(5)} \\ 68 &= 75e^{5k} \\ \frac{68}{75} &= e^{5k} \\ \ln\frac{68}{75} &= 5k \\ k &= \frac{\ln\frac{68}{75}}{5} \\ &\approx -0.0196. \end{aligned}$$

5.27 Related Rates

Many things change with item. Our goal is to find the rate at which some quantity is changing by relating it to other quantities whose rates of change are given or more easily measured

Strategy

1. Read the problem carefully
2. Draw a diagram whenever possible
3. Introduce notation
4. Express rates in terms of derivatives
5. Write an equation
6. Differentiate both sides with repect to t using Implicit Differentiation/Chain Rule
7. Solve for the unknown rate

Say the length of a rectangle is increasing at a rate of 8 cm/s and it's width is increasing at a rate of 3 cm/s. When the length is 20cm and the width is 10cm, how fast is the area increasing?

So our givens:

$$\begin{aligned}\frac{dl}{dt} &= 8\\ \frac{dw}{dt} &= 3\\ l &= 20\\ w &= 10.\end{aligned}$$

We want:

$$\frac{dA}{dt}.$$

And we know that:

$$A = l \cdot w.$$

So if we derive the area formula with Implicit differentation, and using the product rule, we get:

$$\frac{dA}{dt} = l \cdot \frac{dw}{dt} + w \cdot \frac{dl}{dt}.$$

From here we can plug in our givens:

$$\begin{aligned}\frac{dA}{dt} &= 20(3) + 10(8)\\ &= 60 + 80\\ &= 140cm^2/s.\end{aligned}$$

5.28 Linear Approximation

Formula:

$$f(x) \approx L(x) = f(a) + f'(a)(x-a).$$

Note:-

Finding the linearization of a function is the same thing as ”find the equation of the tangent line to the curve”

Procedure,

1. find derivative of function
2. plug in a to get m_{tan}
3. Get point by plugging in a to $f(x)$
4. plug everything into formula $L(x) = f(a) - f'(a)(x-a)$

If asked to use linear aproximation formula to approximate values, then:

1. set original function equal to value
2. solve for x
3. plug in for x in linear approximation formula

5.29 Differentials

Formula:

$$\begin{aligned} dy &= f'(x)dx \\ \Delta x &= dx \\ \Delta y &= (f(x+\Delta x) - f(x)). \end{aligned}$$

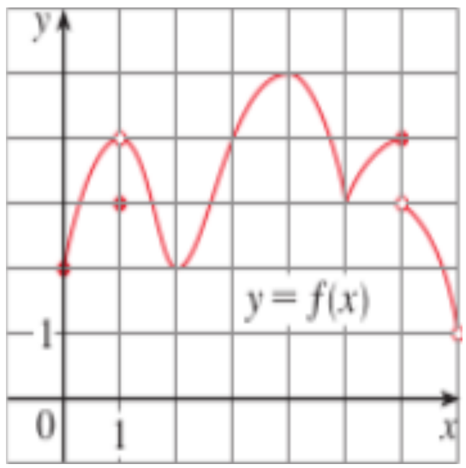
Note:-

Δy, can sometimes be difficult to find so we can use $dy \approx \Delta y$

5.30 Maximum and Minimum Values

- A function f has an absolute maximum at c if $f(c) \geqslant f(x)$ for all $x \in$ Domain of f
- A function f has an absolute minimum at k if $f(c) \leqslant f(x)$ for all $x \in$ Domain of f
- A function f has a local maximum at b if $f(b) \geqslant f(x)$ when x is near b
- A function f has a local minimum at m if $f(m) \leqslant f(x)$ when x is near m
- endpoints are not considered local min max

Cosider This Graph



Note that the textbook denotes $f(1) = 3$ as a local minimum.

5.31 Extreme Value Theorem

- If f is continuous on a closed interval [a,b], then f attains an obsulute maximum value $f(c)$ and an absolute minimum value $f(d)$ where $c, d \in [a, b]$

5.32 Fermat's Theorem

- If f has a local minimum or maximum at c, and if $f'(c)$ exists, then $f'(c) = 0$

5.33 Critical Number

- c in the domain of $f(x)$ is a critical number if $f'(c) = 0$ or if $f'(c)$ does not exist.
  Note: If f has a local max or min at c, then c is a critical number of f
- Critical number has to obey restriction

5.34 How to find the absolute maximum and minimum values of a continuous function f on a closed interval [a,b]

1. Find critical values of f in (a,b)
2. Find $f(a)$ and $f(b)$
3. Absolute Max: largest from 1.) and 2.)
4. Absolute Min: smallest from 1.) and 2.)

Rolle's Theorem:

If f(x) satisfies the following:

1. continuous on [a,b]
2. differentiable on (a,b)
3. f(a) = f(b)

Then there is a $c \in (a, b)$ such that $f'(c) = 0$

Notes:

- If rolle's theorem can be applied, just set $f'(x) = 0$ to find c, remember you are finding all c in the ***open interval***, so if c does not obey this interval, it is not a solution

5.35 The Mean Value Theorem

if f(x) satisfies the following:

1. continuous on [a,b]
2. differentiable on (a,b)

then there is a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

In other words,

$$m_{tan} = m_{sec}.$$

Then after you solve for $f'(c)$, set $f'(x)$ equal to this value. This is how you compute c.

Notes:

- If rational function, find where function is undefined, if that number is not an element of the interval, then it is continous on the closed interval
- If $f'(x)$ is defined on the open interval, then it is differentiable on the open interval
- use the theorem, then set $f'(c) = c$

What if you just have a graph?

1. Draw a secant line through a and b
2. Draw a tangent line such that the slopes are the same (they are parallel)

5.36 First Derivative Test

- If $f'(x)$ changes from + to - at c, then $f(x)$ has a local maximum at c.
- If $f'(x)$ changes from - to + at c, then $f(x)$ has a local minimumc.

5.37 Second Derivative Test

- f(x) has a local minimum at c if $f'(c) = 0$ and $f''(c) > 0$
- f(x) has a local maximum at c if $f'(c) = 0$ and $f''(c) < 0$

5.38 Domain

- Polynomial: $\mathbb{R}$
- rational: Where denominator $= 0$
- Radical: Set denominator $\geqslant 0$, if needed, make a number line and test points to see where its positive.

5.39 Intercepts

- x-ints: set function equal to zero and solve, if the function is rational, only set the numerator equal to zero
- y-ints: plug in zero for x and solve

5.40 Asymptotes

- Horizontal Asymptote, find the limit of the function as x approaches infinity and -infinty, see limits at infinity for more info on rational functions
- vertical Asymptoteo, only for rational functions, it is the zeros of the function
- Slant (oblique), if numerater degree is one higher than denominator, use long division to find equation of slant Asymptote

$$\frac{-8x^4 + 8x^3 + 4}{2x^3 - x}.$$

$$2x^3 + 0x^2 - x\overline{) - 8x^4 + 8x^3 + 4}.$$

And:

$$\frac{x^2 + 4}{x + 4}.$$

$$x + 4\overline{)x^2 + 0x + 4}.$$

5.41 Symmetric

- Even if (symmetric with respect to y axis)
  - $f(-x) = f(x)$
- Odd if (symmetric with respect to orgin)
  - $f(-x) = -f(x)$

5.42 Increasing/Decreasing (Numberline with Domain (not as critical values but still test))

- Find derivative
- set numerator equal to 0 to find critical values (make sure they are in the domain.)
- Make number line and test with $f'(x)$, ontop of the critical values, use the values are found in domain
  - ex.

$$(-\infty, -4] \cup [0, \infty].$$

  - use -4 and 0 on number line

Note: no brackets on intervals of increase/decrease

Local Min/Max

- First derivative test:
  - If $f'(x)$ switches from positive to negative, you have a local max at f(c)
  - If $f'(x)$ switches from negative to positive, you have a local min at f(c)
- Second Derivative test:
  - If $f''(c) < 0$, you have a local max at $f(c)$
  - If $f''(c) > 0$, you have a local min at $f(c)$

5.43 Abs Max and Min

1. find $f'(x)$
2. find critical values, plug into $f(x)$
3. find $f(a)$ *and* $f(b)$ from (a,b)
4. Abs max: largest form top steps
5. Abs min: smallest form top steps

5.44 Concave Up/Down and inflection points

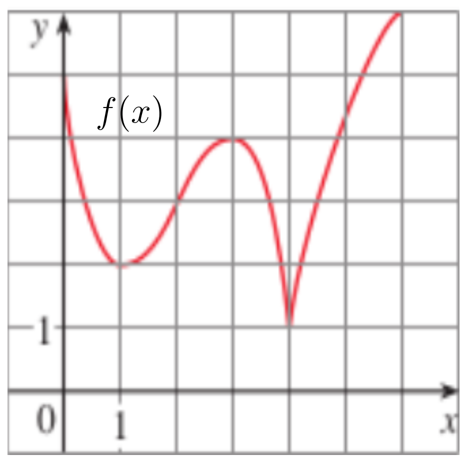
1. Find $f''(x)$
2. Find inflection points (Critical Values)
3. Test inflection points on a number line
4. If concavity changes at critical value, then plug in critical value to $f(x)$ to find inflection points

Note: If no $c \in D$, but domain is say $(-\infty, 15]$, use 15 on number line, but 15 cannot be an inflection point or numbers after used to test concavity

Find inflection points of $f(x)$, $f'(x)$, and $f''(x)$, given the graph of $f(x)$

- $f(x)$: find inflection points by locating where on the graph does it switch concavity. (x values)
- $f'(x)$: find inflection points by locating local min/maxs. (x values)
- $f''(x)$: Find inflection points by locating where it switches from positive to negative

Consider the graph of f(x):



$$
\begin{aligned}
CU &: (0,2) \\
CD &: (2,4) \cup (4,6) \\
IP &: (2,3).
\end{aligned}
$$

Note:-

Write inflection points as an ordered pair

5.45 L'Hospital's Rule

Main Concept:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

- Important concepts
  - Review limits at infinity
  - Any number other than zero or 1 to the power of negative infinity is 0
  - Any number other than zero or 1 to the power of infinity is infinity
  - The natural log of a negative number is undefined, therefore the limit as x approaches negative infinity of $\ln x$ is hence undefined
  - Same with log
  - Any contant over infinity approaches 0
- Types of indeterminate forms we do want
  - $\frac{0}{0}$
  - $\frac{\infty}{\infty}$
- Types of indeterminate forms we don't want:
  - $0\cdot\infty$
  - $\infty-\infty$
  - 0^0
  - ∞^0
  - 1^∞
- type $\infty-\infty$
  - Change to the type we want by:
    1. Common Denominators
    2. rationalizing
    3. factoring
- Types 0^0, ∞^0, 1^∞
  - Change to the type we want by:
    1. taking natural log of function or writing as exponential

Note:-

If you use natural log method, you must write final answer as $e^{\lim_{x\to\infty}\ln y}$

5.46 Curve Sketching

- Find A-G and sketch graph
  - A: Domain
  - B: Intercepts
  - C: Asymptotes
  - D: Symmetry
  - E: Local min/max
  - F: Increasing / decreasing
  - G: Concavity / Inflection Points

5.47 Optimization Problems

Strategy:

1. Read the problem carefully
2. Draw a diagram whenever possible
3. Introducte notation
4. Expess the quantity to be optimized in terms of other variables.
5. Reduce the number of variables from Step 4 to only 1 (write a function of 1 variable)
6. find the absolute minimum/maximum

Procedure:

1. Find formula
2. Get to one variable
3. Find derivative
4. Find critical values
5. Use second derivative test to test if min / max

5.48 Newton's Method

Concept:

$$x_n - \frac{f(x_n)}{f'(x_n)}.$$

What if you have a problem like:

$$\sqrt[4]{75}.$$

Rewrite as:

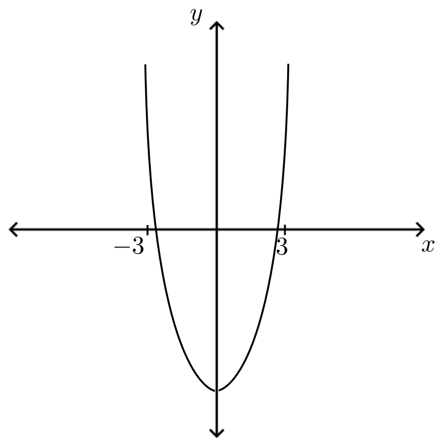
$$x^4 = 75$$
$$f(x) = x^4 - 75.$$

How do you get x_1?

$$x_1 = 3.$$

This is because we took $\sqrt[4]{81} = 3$, 81 is close to 75 so we will use that as our starting point, remember this is just an approximation.

What if it is a porabola, (ie more than one root), say we have the graph:



Then we will use 2 different x_1, at -3 and 3

5.49 Antiderivitates

- Know what it means to be an Antiderivitate and how to find them
- know the most common Antiderivitates
- understand +C
- know why

$$x^{\frac{1}{3}}$$
$$= \frac{3}{4}x^{\frac{4}{3}}.$$

- and

$$2y^2$$
$$= \frac{2}{3}y^3.$$

- Know that

$$x^{-1}$$
$$= \ln|x|.$$

Common Antiderivitates:

- Exponential
  - $x^n = \frac{x^{n+1}}{n+1} + C$
  - $\frac{1}{x} = \ln|x| + C$
  - $a^x = \frac{a^x}{\ln a} + C$
  - $\ln x = x \ln x - x + C$
  - $e^x = e^x + C$

- Trig:
  - $\sin x = -\cos x + C$
  - $\cos x = \sin x + C$
  - $\tan x = \ln|\sec x| + C$
  - $\csc x = \ln|\csc x - \cot x| + C$
  - $\sec x = \ln|\sec x + \tan x| + C$
  - $\cot x = \ln|\sin x| + C$
  - $\sin^2 x = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$
  - $\cos^2 x = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$
  - $\tan^2 x = -x + \tan x + C$
  - $\csc^2 x = -\cot x + C$
  - $\sec^2 x = \tan x + C$
  - $\cot^2 x = -x - \cot x + C$

- Hyperbolic Trig
  - $\sinh x = \cosh x + C$
  - $\cosh x = \sinh x + C$
  - $\tanh x = \ln|\cos x| + C$
  - $cschx = \ln|\tan h(\frac{1}{2}x)| + C$
  - $sechx = \tan^{-1}(\sinh(x)) + C$
  - $cothx = \ln|\sinh x| + C$
  - $csch^2 x = -\coth x + C$
  - $sech^2 x = \tanh x + C$

5.50 derivative of hyperbolic trig fuctions

- $\sinh x = \cosh x$
- $\cosh x = \sinh x$
- $\tanh x = \text{sech}^2 x$
- sech $x =$ -sech x $\tanh x$
- csch $x =$ -csch x $\coth x$
- $\coth x =$ -csch$^2 x$

lnx

- when you have f(x) like:

$$f(x) = 3x^{-2} - 7x^{-1} + 6.$$

you can see if we tried to find F(x), by adding 1 to -1, we would get:

$$\frac{-7^0}{0}.$$

Which is a problem, so instead, the Antiderivitive would be $-7\ln|x|$, this is because

$$\begin{aligned} &\frac{d}{dx} - 7\ln x \\ &= -\frac{7}{x}. \end{aligned}$$

And don't forget the absolute value bars

5.51 Sigma notation

- Know properties of summation:
  - $\sum_{i=m}^{n} c \cdot a_i = c \sum_{i=m}^{n} a_i$, where c is a constant
  - $\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$
  - $\sum_{i=m}^{n} (a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$
  - $\sum_{i=1}^{n} 1 = n$
  - $\sum_{i=1}^{n} c = c \cdot n$, where c is a constant
  - $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
  - $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
  - $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$
- Know what is means to be a telescoping sum and how to evaluate it.
- Know how to eval limits of summation, recall that if you have the limit as n -> infinty, of a rational function whos degress are equal, evaluate the limit by taking the ration of the leading coefficients.

5.52 Area and Distance Problem

- Know the definition for the area under the curve.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [\Delta x f(x_1) + \Delta x f(x_2) + ... + \Delta x f(x_n)].$$

or

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [\Delta x f(x_0) + \Delta x f(x_1) + ... + \Delta x f(x_{n-1})].$$

or

$$A = \lim_{n\to\infty} [\Delta x f(x_1^*) + ... + \Delta x f(x_n^*)].$$

Where x_i^* is any number in the ith interval.

Note:-

Δx = Base of each rectangle

Find Δx with $\frac{b-a}{n}$, on [a,b]

- And the sigma versions:
- $A = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(x_i)$ (Right endpoints)
- $A = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(x_{i-1})$ (Left endpoints)
- $A = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(x_i^*)$ (Arbitrary partiton)

Note:-

The one with the star denotes not using left or right endpoints

- Know how to find Δx

$$\Delta x = \frac{b-a}{n}.$$

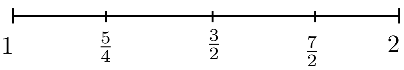
- Know how to find x_i (right endpoints)

$$x_i = a + i\Delta x.$$

- Know how to find area with a fixed number of rectangles.
  - Find Δx
  - multiply Δx by the sum of all the heights of the rectangles.

Notes for riemann sum:

- Say we are using right endpoints with $n = 4$, imagine we had the number line:
  - Since we are using right endpoints with $n = 4$, we only use $\frac{5}{4}$ onward
  - If we were using left endpoints, we would use 1 to $\frac{7}{2}$



5.53 Definite Integrals

- Know:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(x_i^*).$$

- Know riemann sum:

$$\sum_{i=1}^{n} \Delta x f(x_i).$$

- Know the properties of integrals:
  - $\int_a^b c dx = c(b-a)$
  - $\int_a^b cf(x)dx = c \cdot \int_a^b f(x)dx$
  - $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
  - $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$
  - $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$
  - if $f(x) \geqslant 0$ for all $a \leqslant x \leqslant b$, then $\int_a^b f(x)dx \geqslant 0$
  - if $f(x) \geqslant g(x)$ for all $a \leqslant x \leqslant b$, then $\int_a^b f(x)dx \geqslant \int_a^b g(x)dx$
  - if $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then $m(b-a) \leqslant \int_a^b f(x)dx \leqslant M(b-a)$

- Know how to find midpoints
  - Make number line with right endpoints, then find the middle value of each interval
  - If given an interval say, [1,3], and your asked to find midpoints, first plot number line with right endpoints. Then divide Δx by 2 and add this number to each left point. Note that Δx will remain the same when you find the riemann sum.
- Know when you have equations for geometric shapes (circle, half cirle, line)

- integral with the same bounds is **zero**
- Know how to split up integrals like

$$\int_{-1}^{2} (x - 8|x|)\ dx.$$

  - Set up piecewise

$x = 0$ *Set whats inside abs = 0 to see where sign flips.

$$= \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases} \tag{4}$$

- split integral

$$\int_{-1}^{0} x - 8(-x)\ dx + \int_{0}^{2} x - 8x\ dx.$$

- Then evaluate

5.54 The Fundemental theorem of calculus

- Know

$$Part\ 1: \ \frac{d}{dx}\int_{a}^{x} f(x)\ dt = f(x), a \leqslant x \leqslant b$$

$$Part\ 2: \ \int_{a}^{b} f(x)\ dx = F(b) - F(a)\ where\ F' = f.$$

$$and$$

$$\int_{a}^{b} f(x)\ dx = -\int_{b}^{a} f(x)\ dx.$$

- Know that you have to multiply by the derivative of the upper limit when you substitute for part 1
- Know how to split up integrals asked to find the derivative and if given something like:

$$F(x) = \int_{x}^{x^2} e^{t^7}\ dx.$$

  - so this would end up

$$F'(x) = \frac{d}{dx}\int_{x}^{0} e^{t^7}\ dt + \frac{d}{dx}\int_{0}^{x^2} e^{t^7}\ dt.$$

  - and we would have to flip our limits of integration such that our upper limit is a function of x

$$F'(x) = \frac{d}{dx} - \int_{0}^{x} e^{t^7}\ dt + \frac{d}{dx}\int_{0}^{x^2} e^{t^7}\ dt$$

$$F'(x) = e^{x^7} + e^{14} \cdot 2x$$

$$= e^{x^7} + 2xe^{14}$$

$$.$$

- Know the notation
  - If given:

$$g(x) = \int_{0}^{x} \sqrt{t^3 + t^5}\ dt.$$

  - Notation for g' (using part 1) would be:

$$g'(x) = \frac{d}{dx}\int_{0}^{x} \sqrt{t^3 + t^5}\ dt$$

$$= \sqrt{x^3 + x^5}.$$

  - definite integrals do not get +C

5.55 Indefinite Integrals

- Indefinite Integrals are essentially just asking for the Antiderivitate, do not forget +C
- review common Antiderivitates
- know how to indeterminate antiderivative for inverse trig
- Know this antiderivative:

$$F(x) = 5^x$$
$$f(x) = \frac{5^x}{\ln 5}.$$

Because:

$$\begin{aligned} &\frac{d}{dx}\frac{5^x}{\ln 5} \\ &= \frac{1}{\ln 5}\left[5^x\right] \\ &= \frac{1}{\ln 5}\left[5^x \cdot \ln 5\right] \\ &= \frac{5^x \cdot \ln 5}{\ln 5} \\ &\boxed{= 5^x}. \end{aligned}$$

5.56 Velocity Functions with integrals

If given:

$$v(t) = 3t - 8,\ 0 \leqslant t \leqslant 3.$$

- Know that to find displacement:

$$\int_0^3 3t - 8\ dt.$$

- To find distance traveled:

$$\int_0^3 |3t - 8|\ dx.$$

  - So you must write as piecewise and split up the integral (add them)

5.57 The Substitution Rule (u-sub)

If $u = g(x)$ is differentiable and its range $\in I$ and f is continuous on I, then

$$\int f(g(x))g'(x)\ dx = \int f(u)\ du.$$

Process:

1. Make a decent choice of what to let u equal
2. Change our integral from being in terms of x, to in terms of u
3. Integrate $\int f(u)\ du$
4. Change back to x

Note:-

Also include the constant in your u sub, if it is attached to the function you let u equal, Also note for rational trig functions, you can move trig functions from upstairs or downstairs based on their reciprical function, for example, a $\cos^2 x$ in the denominator can be moved upstairs as $\sec^2 x$

Notes:

- Our goal with u-sub is to let u equal some function in our composition of functions, such that if we derive that function, we get back something that is also in our integrand
- Say we have something like:

$$\begin{aligned} &\int \sec^2\theta\ d\theta \\ Let\ u &= 8\theta \\ du &= 8d\theta \\ \frac{1}{8}du &= d\theta. \end{aligned}$$

- Know that you can rewrite equations like:

$$\begin{aligned} &\int \frac{(\ln x)^{36}}{x}\ dx \\ as: &\int \frac{1}{x}(\ln x)^{36}\ dx. \end{aligned}$$

- Say we have:

$$-\frac{1}{2}\int_0^{-1} e^u\ du.$$

  - We can flip the limits of integration to remove the negative sign. So we will have:

$$\frac{1}{2}\int_{-1}^{0} e^u\ du.$$

- Look out for being able to turn antiderivative into inverse trig functions.

- Say we have:

$$\int \frac{x^7}{1+x^{16}}\ dx.$$

- Don't let $u = 1 + x^8$, we could do this by turning x^{16} *into* $(x^8)^2$, but instead, just let $u = x^8$. We do this because it ends up like:

$$\frac{1}{8}du = x^7 dx.$$

- So when we sub we get that nice arctan antiderivative, just something to look out for.

$$\frac{1}{8}\int \frac{1}{1+u^2}\ du.$$

- Note that we can write:

$$e^{2x}.$$

  - as

$$(e^x)^2.$$

Definite Integrals with u-sub

1. Find what u is going to equal
2. Find u(a) and u(b)
3. make u-sub and use u(a) and u(b) as limits of integration
4. find antiderivative and evaluate at new limits

Note:-

For definite integrals, don't sub back in for u, just evalute integral with u still subbed

5.58 Integrals of Symmetric functions:

Sometimes you will run into integrals that are either impossible, or too difficult with u Substitution. For these cases we will look at Integrals of Symmetric Functions

Even:

$$f(-x) = f(x) \ then \ \int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\ dx.$$

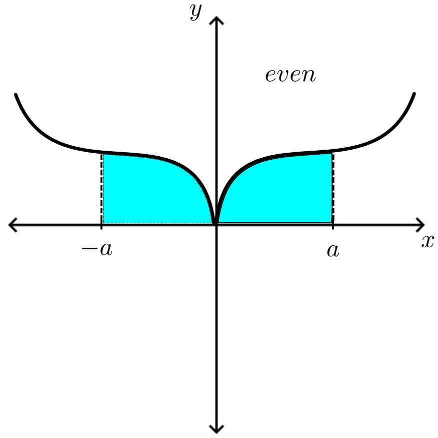
Odd:

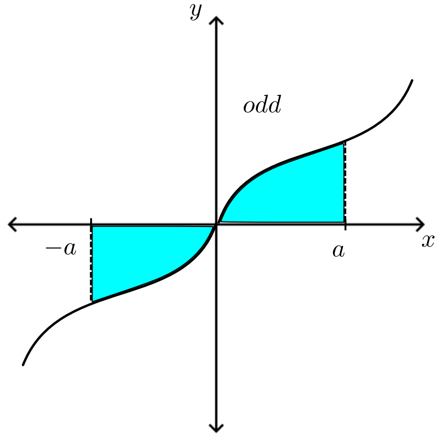
$$f(-x) = -f(x) \ then \ \int_{-a}^{a} f(x)\ dx = 0.$$

Notes:

- If a function is even, you can replace your lower limit with zero and multiply the integral by 2
- if a function is odd, then the integral equals zero.

Figures:





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