> 6a. Show that if $\omega = \infty$, $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$

Proof. Assume A, B are distinct collinear, and $\omega = \infty$. Since $\omega = \infty$, $AB < \omega$. Thus, $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are well defined. Further, A, B are together in a unique line. Namely, the line $\overleftrightarrow{AB}$.

Let X exist on the line $\overleftrightarrow{AB}$. If $X = A$, then $X \in \overrightarrow{AB}$ and $X \in \overrightarrow{BA}$ by definition of $\overrightarrow{AB}$ and $\overrightarrow{BA}$. Similarly, if $X = B$, then $X \in \overrightarrow{AB}$ and $X \in \overrightarrow{BA}$. For the following argument, we can therefore assume that $X \neq A$ or B.

Since $\omega = \infty$, it is guaranteed that $AB + BX \leqslant \omega = \infty$. Thus, by Ax.BP, there exists a betweenness relation among A, B, X, and exactly one of the following must be satisfied

$$A\text{-}X\text{-}B \tag{1}$$
$$A\text{-}B\text{-}X \tag{2}$$
$$B\text{-}A\text{-}X \tag{3}$$

We examine these cases separately. If A-X-B, then $X \in \overrightarrow{AB}$ and $X \in \overrightarrow{BA}$. If A-B-X, then $X \in \overrightarrow{AB}$. Lastly, if B-A-X, then $X \in \overrightarrow{BA}$

In any case, $X \in \overleftrightarrow{AB}$ implies X is either in $\overrightarrow{AB}$ or $\overrightarrow{BA}$ or both.

Therefore, $\overleftrightarrow{AB} = \overrightarrow{AB} \cup \overrightarrow{BA}$ ∎

> 12. Suppose that A, B, C are three distinct, collinear points such that $AC \leqslant \frac{1}{2}AB$ and $BC \leqslant \frac{1}{2}AB$. Prove that A-C-B and $AC = BC = \frac{1}{2}AB$

Proof. Assume A, B, C are three distinct, collinear points such that $AC \leqslant \frac{1}{2}AB$, and $BC \leqslant \frac{1}{2}AB$

By the definition of ω, $AC, BC, AB \leqslant \omega$. Since $AC \leqslant \frac{1}{2}AB$, and $BC \leqslant \frac{1}{2}AB$, we have

$$AC + BC \leqslant \frac{1}{2}AB + \frac{1}{2}AB \leqslant AB \leqslant \omega$$

Observe that since $AC + BC \leqslant AB$, both AC and $BC = CB$ must be less than AB. That is, $AC, BC < AB$ since by Ax.D2, $AC, BC, AB \neq 0$.

Since $AC + BC \leqslant \omega$, by Ax.BP, there is a betweenness relation among A, B, C. One of the following must hold

$$A\text{-}B\text{-}C$$
$$B\text{-}A\text{-}C$$
$$A\text{-}C\text{-}B$$

Assume the relation is A-B-C. Then, we have $AB + BC = AC$ which implies $AB, BC < AC$. But this contradicts the fact that $AC < AB$. Thus, the relation is not A-B-C.

Assume the relation is B-A-C. Then, $BA + AC = BC$, or equivalently $AB + AC = BC$. This contradicts the fact that $BC < AB$. Thus, this must also not be the relation.

Therefore, the relation we have is A-C-B.

Next, we aim to show that $AC = BC = \frac{1}{2}AB$. Since A-C-B was established, we have $AC + BC = AB$, solving for BC, we get

$$BC = AB - AC$$

Since AC is bounded above by $\frac{1}{2}AB$. That is, $AC \leqslant \frac{1}{2}AB$, it must be that $AB - AC \geqslant AB - \frac{1}{2}AB$. Thus,

$$BC = AB - AC \geqslant AB - \frac{1}{2}AB$$
$$\therefore BC \geqslant \frac{1}{2}AB$$

But, since we know that $BC \leqslant \frac{1}{2}AB$, the only way both $BC \leqslant \frac{1}{2}AB$ and $BC \geqslant \frac{1}{2}AB$ can be satisfied is if $BC = \frac{1}{2}AB$. Now, since $BC = \frac{1}{2}AB$, we have

$$AC + BC = AB \implies AC + \frac{1}{2}AB = AB$$
$$\therefore AC = AB - \frac{1}{2}AB = \frac{1}{2}AB$$

Thus, we conclude that $AC = BC = \frac{1}{2}AB$ ■