Theorem *(Relative error bound III).* Let A be nonsingular, $b \neq 0$, and $Ax = b$. If $(A + \delta A)(x + \delta x) = b + \delta b$, and

$$\frac{\|\delta A\|}{\|A\|} < \frac{1}{\kappa(A)},$$

then

$$\frac{\|\delta x\|}{\|x\|} \leqslant \frac{\kappa(A)\left(\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|}\right)}{1 - \kappa(A)\frac{\|\delta A\|}{\|A\|}}.$$

Remark (a)**.** Consider the quantity $q = \frac{a}{b}$, $b \neq 0$. If $c \leqslant b$, then

$$q = \frac{a}{b} \leqslant \frac{a}{c}.$$

Proof. Suppose for a moment that A nonsingular, $b \neq 0$, $Ax = b$, and $(A + \delta A)(x + \delta x) = b + \delta b$. Then, it's immediately obvious that

$$\begin{aligned}
&(A + \delta A)(x + \delta x) = b + \delta b \\
\implies& Ax + A\delta x + \delta A x + \delta A \delta x = b + \delta b \\
\implies& A\delta x + \delta A x + \delta A \delta x = \delta b \\
\implies& A\delta x = \delta b - (\delta A x + \delta A \delta x) \\
\implies& A\delta x = \delta b - \delta A\,(x + \delta x) \\
\implies& \delta x = A^{-1}\delta b - A^{-1}\delta A\,(x + \delta x) \\
\implies& \|\delta x\| = \left\|A^{-1}\delta b - A^{-1}\delta A\,(x + \delta x)\right\| \\
&\leqslant \left\|A^{-1}\delta b\right\| + \left\|-A^{-1}\delta A\,(x + \delta x)\right\| \\
&= \left\|A^{-1}\delta b\right\| + \left\|A^{-1}\delta A\,(x + \delta x)\right\| \\
&\leqslant \left\|A^{-1}\right\|\|\delta b\| + \left\|A^{-1}\right\|\|\delta A\|\,(\|x + \delta x\|) \\
&\leqslant \left\|A^{-1}\right\|\|\delta b\| + \left\|A^{-1}\right\|\|\delta A\|\,(\|x\| + \|\delta x\|) \\
&= \left\|A^{-1}\right\|\|\delta b\| + \left\|A^{-1}\right\|\|\delta A\|\|x\| + \left\|A^{-1}\right\|\|\delta A\|\|\delta x\| \\
\implies& \|\delta x\| - \left\|A^{-1}\right\|\|\delta A\|\|\delta x\| \leqslant \left\|A^{-1}\right\|\|\delta b\| + \left\|A^{-1}\right\|\|\delta A\|\|x\| \\
\implies& \|\delta x\|\left(1 - \left\|A^{-1}\right\|\|\delta A\|\right) \leqslant \left\|A^{-1}\right\|\|\delta b\| + \left\|A^{-1}\right\|\|\delta A\|\|x\| \\
\implies& \frac{\|A\|}{\|A\|}\|\delta x\|\left(1 - \left\|A^{-1}\right\|\|\delta A\|\right) \leqslant \frac{\|A\|}{\|A\|}\left(\left\|A^{-1}\right\|\|\delta b\| + \left\|A^{-1}\right\|\|\delta A\|\|x\|\right) \\
\implies& \|\delta x\|\left(1 - \|A\|\left\|A^{-1}\right\|\frac{\|\delta A\|}{\|A\|}\right) \leqslant \|A\|\left\|A^{-1}\right\|\frac{\|\delta b\|}{\|A\|} + \|A\|\left\|A^{-1}\right\|\frac{\|\delta A\|}{\|A\|}\|x\| \\
\implies& \|\delta x\|\left(1 - \kappa(A)\frac{\|\delta A\|}{\|A\|}\right) \leqslant \kappa(A)\frac{\|\delta b\|}{\|A\|} + \kappa(A)\frac{\|\delta A\|}{\|A\|}\|x\| \\
\implies& \frac{\|\delta x\|}{\|x\|}\left(1 - \kappa(A)\frac{\|\delta A\|}{\|A\|}\right) \leqslant \kappa(A)\frac{\|\delta b\|}{\|A\|\|x\|} + \kappa(A)\frac{\|\delta A\|}{\|A\|}.
\end{aligned}$$

From here, we use

$$\begin{aligned}
&Ax = b \\
\implies& \|Ax\| = \|b\| \\
\implies& \|Ax\| \leqslant \|A\|\|x\| \\
\implies& \|b\| \leqslant \|A\|\|x\|
\end{aligned}$$

and remark *(a)* to see that

$$\kappa(A)\frac{\|\delta b\|}{\|A\|\,\|x\|}+\kappa(A)\frac{\|\delta A\|}{\|A\|}\leqslant\kappa(A)\frac{\|\delta b\|}{\|b\|}+\kappa(A)\frac{\|\delta A\|}{\|A\|}.$$

Thus, it follows that

$$\begin{aligned}
\frac{\|\delta x\|}{\|x\|}\left(1-\kappa(A)\frac{\|\delta A\|}{\|A\|}\right) &\leqslant \kappa(A)\frac{\|\delta b\|}{\|A\|\,\|x\|}+\kappa(A)\frac{\|\delta A\|}{\|A\|}\\
&\leqslant \kappa(A)\frac{\|\delta b\|}{\|b\|}+\kappa(A)\frac{\|\delta A\|}{\|A\|}\\
&= \kappa(A)\left(\frac{\|\delta b\|}{\|b\|}+\frac{\|\delta A\|}{\|A\|}\right)\\
\therefore \frac{\|\delta x\|}{\|x\|} &\leqslant \frac{\kappa(A)\left(\frac{\|\delta A\|}{\|A\|}+\frac{\|\delta b\|}{\|b\|}\right)}{1-\kappa(A)\frac{\|\delta A\|}{\|A\|}}.
\end{aligned}$$

As desired. ■