Undergraduate Topics in Mathematics (4)
Proof writing, The theory of sets, Axiomatic geometry, Numerical analysis

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Contents

Proofs

1.1 Intro to proof writing, intuitive proofs

- **Intro to definitions, propositions and proofs: the chessboard problem**: Suppose you have a chessboard (8×8 grid of squares) and a bunch of dominoes (2×1 block of squares), so each domino can perfectly cover two squares of the chessboard.

  Note that with 32 dominoes you can cover all 64 squares of the chessboard. There are many different ways you can place the dominoes to do this, but one way is to cover the first column by 4 dominoes end-to-end, cover the second column by 4 dominoes, and so on

  Math runs on definitions, so let's give a name to this idea of covering all the squares. Moreover, let's not define it just for 8×8 boards — let's allow the definition to apply to boards of other dimensions

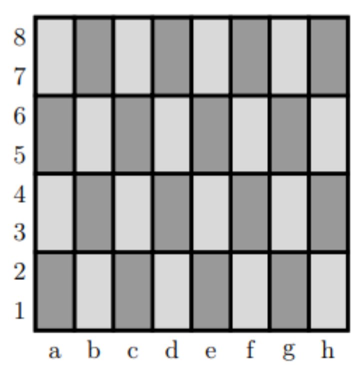
  Definition. A perfect cover of an $m \times n$ board with 2×1 dominoes is an arrangement of those dominoes on the chessboard with no squares left uncovered, and no dominoes stacked or left hanging off the end.

  As we demonstrated above, there exist perfect covers of the 8×8 chessboard. This is a book about proofs, so let's write this out as a proposition (something which is true and requires proof) and then let's write out a formal proof of this fact.

  Proposition. There exists a perfect cover of an 8×8 chessboard.

  This proposition is asserting that "there exists" a perfect cover. To say "there exists" something means that there is at least one example of it. Therefore, any proposition like this can be proven by simply presenting an example which satisfies the statement.

  Proof. Observe that the following is a perfect cover.



  We have shown by example that a perfect cover exists, completing the proof. ■

We typically put a small box at the end of a proof, indicating that we have completed our argument. This practice was brought into mathematics by Paul Halmos, and it is sometimes called the Halmos tombstone

One apocryphal story is that Halmos regarded proofs as living until proven. Once proven, they have been defeated — killed. And so he wrote a little tombstone to conclude his proof

What if I cross out the bottom-left and top-left squares, can we still perfectly cover the 62 remaining squares?

As you can probably already see, the answer is yes. For example, the first column can now be covered by 3 dominoes and the other columns can be covered by 4 dominoes each.

What if I cross out just one square, like the top-left square? Can this be perfectly covered?

The answer is no

Proposition. If one crosses out the top-left square of an 8×8 chessboard, the remaining squares can not be perfectly covered by dominoes.

Proof Idea. The idea behind this proof is that one domino, wherever it is placed, covers two squares. And two dominoes must cover four squares. And three cover six. In general, the number of squares covered — 2, 4, 6, 8, 10, etc. — is always an even number. This insight is the key, because the number of squares left on this chessboard is 63— an odd number

Proof. Since each domino covers 2 squares and the dominoes are non-overlapping, if one places our k dominoes on the board, then they will cover $2k$ squares, which is always an even number. Therefore, a perfect cover can only cover an even number of squares. Notice, though, that the board has 63 remaining squares, which is an odd number. Thus, it can not be perfectly covered.

What if I take an 8×8 chessboard and cross out the top-left and the bottom-right squares? Then can it be covered by dominoes?

Proposition. If one crosses out the top-left and bottom-right squares of an 8×8 chessboard, the remaining squares can not be perfectly covered by dominoes.

Proof. Observe that the chessboard has 62 remaining squares, and since every domino covers two squares, if a perfect cover did exist it would require

$$\frac{62}{2} = 31 \text{ dominoes.}$$

Also observe that every domino on the chessboard covers exactly one white square and exactly one black square

Thus, whenever you place 31 non-overlapping dominoes on a chessboard, they will collectively cover 31 white squares and 31 black squares.

Next observe that since both of the crossed-out squares are white squares, the remaining squares consist of 30 white squares and 32 black squares. Therefore, it is impossible to have 31 dominoes cover these 62 squares. ■

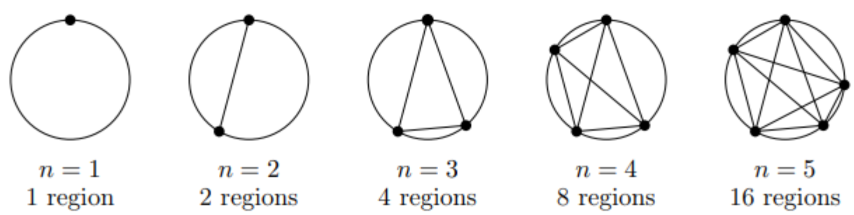
- **Naming Results**: So far, all of our results have been called "propositions." Here's the run-down on the naming of results:

  - A theorem is an important result that has been proved.
  - A proposition is a result that is less important than a theorem. It has also been proved.
  - A lemma is typically a small result that is proved before a proposition or a theorem, and is used to prove the following proposition or theorem.
  - A corollary is a result that is proved after a proposition or a theorem, and which follows quickly from the proposition or theorem. It is often a special case of the proposition or theorem.

  All of the above are results that have been proved — a conjecture, though, has not.

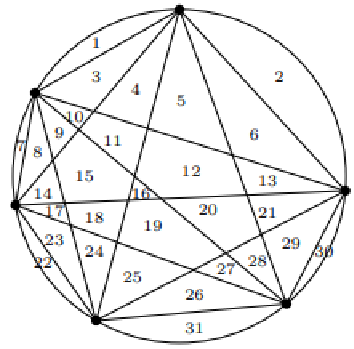
  - A conjecture is a statement that someone guesses to be true, although they are not yet able to prove or disprove it.

- **Conjectures and counterexamples**: As an example of a conjecture, suppose you were investigating how many regions are formed if one places n dots randomly on a circle and then connects them with lines.



At this point, if you were to conjecture how many regions there will be for the $n = 6$ case, your guess would probably be 32 regions — the number of regions certainly seems to be doubling at every step. In fact, if it kept doubling, then with a little more thought you might even conjecture a general answer: that n randomly placed dots form 2^{n-1} regions;

Surprisingly, this conjecture would be incorrect. One way to disprove a conjecture is to find a counterexample to it. And as it turns out, the $n = 6$ case is such a counterexample



$n = 6$

31 regions

This counterexample also underscores the reason why we prove things in math. Sometimes math is surprising. We need proofs to ensure that we aren't just guessing at what seems reasonable. Proofs ensure we are always on solid ground. Further, proofs help us understand why something is true — and that understanding is what makes math so fun

Lastly, we study proofs because they are what mathematicians do

- **The pingeonhole principle**

  Principle. The principle has a simple form and a general form. Assume k and n are positive integers

  Simple form: If $n+1$ objects are placed into n boxes, then at least one box has at least two objects in it.

  General form: If $kn+1$ objects are placed into n boxes, then at least one box has at least $k+1$ objects in it.

  Birthday example: If there are 330 million people in the united states, how many U.S. residents are guaranteed to have the same birthday according to the pigeonhole principle?

  To determine this, let's see what would happen if each date of the year had exactly the same number of people born on it

$$\frac{330 \times 10^6}{366} = 901,639.344.$$

  Since 901,639.344 people are born on an average day of the year, we should be able round up and say that at least one day of the year has had at least 901,640 people born on it. That is, with the pigeonhole principle we should be able to prove that there are at least 901,640 people in the USA with the same birthday

Solution. Imagine you have one box for each of the 366 dates of the (leap) year, and each person in the U.S. is considered an object. Put each person in the box corresponding to their birthday. By the general form of the pigeonhole principle (with $n = 366$ and $k = 901,639$ and thus $k+1 = 901,640$), any group of

$$(901,639)(366) + 1.$$

people is guaranteed to contain 901,640 people which have the same birthday.

- **Another pingeonhole example**:

  Proposition. Given any five numbers from the set $\{1,2,3,4,5,6,7,8\}$, two of the chosen numbers will add up to 9.

  We may think to start by listing the pairs that sum to 9. We have

  $$\begin{gathered} 1+8 \\ 2+7 \\ 3+6 \\ 4+5. \end{gathered}$$

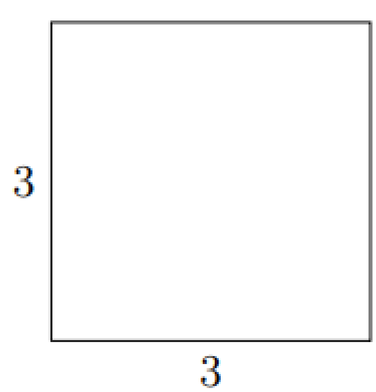
  And of course $8+1, 7+2, ..$ etc. We see we have four sums, we choose these sums as our boxes. If each of the four sums is a box, and each number is an object, then we are placing five objects into four boxes

  Proof. Let one box correspond to the numbers 1 and 8, a second box correspond to 2 and 7, another to 3 and 6, and a final box to 4 and 5. Notice that each of these pairs adds up to 9.

  Given any five numbers from $\{1,2,3,4,5,6,7,8\}$, place each of these five numbers in the box to which it corresponds; for example, if your first number is a 6, then place it in the box labeled "3 and 6." Notice that we just placed five numbers into four boxes. Thus, by the simple form of the pigeonhole principle, there must be some box which contains two numbers in it. These two numbers add up to 9, as desired

- **Another pingeonhole example**:

  Proposition. Given any collection of 10 points from inside the following square (of side-length 3), there must be at least two of these points which are of distance at most $\sqrt{2}$



Proof. Divide the 3×3 square into nine 1×1 boxes. Placing 10 arbitrary points amongst the boxes gaurantees that at least one box will have at least two points. We observe that the farthest these two points can be from each other is when they sit in two corners such that a diagonal line through the box hits both points. The length of this line is given by

$$\sqrt{1^2 + 1^2} = \sqrt{2}.$$

Thus, we observe that the maximum distance of these two points is $\sqrt{2}$ ■

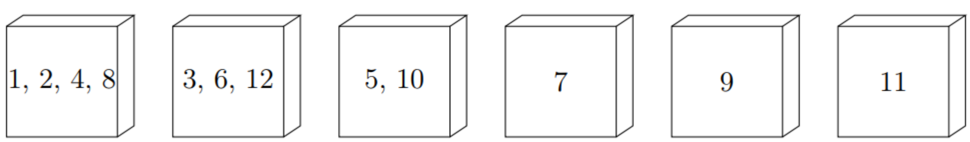
- **Another pingeonhole example**:

Proposition. Given any 101 integers from $\{1, 2, 3, ..., 200\}$, at least one of these numbers will divide another

Solution. As we ponder about how to construct 100 boxes from the properties of the set, we may wonder how the even and odd members partition this set. Call $S = \{1, 2, 3, ..., 200\}$, $E = \{2, 4, 6, ..., 200\}$, and $O = \{1, 3, 5, ..., 199\}$. Note that $E \cup O = S$. We notice that these two sets are arithmetic sequences, each with difference two. If $a_n = a_1 + (n-1)d$, then

$$\begin{aligned} n &= \frac{a_n - a_1}{2} + 1 \\ \implies n &= 100. \end{aligned}$$

Let's make the odd numbers are boxes. We note that any even number ℓ can be written as $\ell = 2^k m$, where m is odd, and k is the highest power of two that divides ℓ. Thus, in box m, we place any number of the form $2^k m$



For any pair of numbers in the same box, the smaller divides the larger. Picking 101 numbers from the set S, and only 100 boxes... by the pigeonhole principle we must have atleast two numbers in the same box, and thus the smaller divides the larger. ■.

Formal proof. Proof. For each number n from the set $\{1, 2, 3, \ldots, 200\}$, factor out as many 2's as possible, and then write it as $n = 2^k \cdot m$, where m is an odd number. So, for example, $56 = 2^3 \cdot 7$, and $25 = 2^0 \cdot 25$. Now, create a box for each odd number from 1 to 199; there are 100 such boxes.

Remember that we are given 101 integers and we want to find a pair for which one divides the other. Place each of these 101 integers into boxes based on this rule:

If the integer is n, then place it in Box m if $n = 2^k \cdot m$ for some k.

For example, $72 = 2^3 \cdot 9$ would go into Box 9, because that's the largest odd number inside it.

Since 101 integers are placed in 100 boxes, by the pigeonhole principle (Principle 1.5) some box must have at least 2 integers placed into it; suppose it is Box m. And suppose these two numbers are $n_1 = 2^k \cdot m$ and $n_2 = 2^\ell \cdot m$, and let's assume the second one is the larger one, meaning $\ell > k$. Then we have now found two integers where one divides the other; in particular n_1 divides n_2, because:

$$\frac{n_2}{n_1} = \frac{2^\ell \cdot m}{2^k \cdot m} = 2^{\ell - k}.$$

This completes the proof.■

- **Another pigeonhole example**

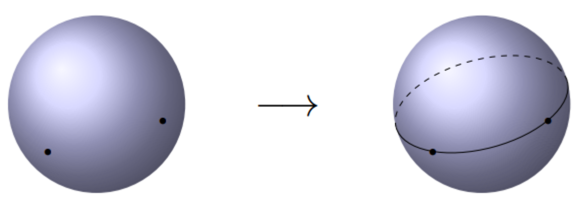
  Proposition. Suppose G is a graph with $n \geqslant 2$ vertices. Then G contains two vertices which have the same degree.

  We start by observing that the minimum degree is zero, and the maxmium is $n-1$. It could happen that a vertex is connected to no other vertices, and a vertex could be connected to all other vertices. If a vertex is connected to all other vertices, than it has degree $n-1$, because it has an edge going to all vertices but itself. Thus, we have our boxes. But you may notice that we have n boxes for n vertices. This may seem like a problem, but after some thought you may see that it is not possible for the zero box and the $n-1$ box to both be used for a specific graph G. Thus, we have only $n-1$ boxes for n vertices.

  The rest of the proof is left as an exercise for the reader.

- **Classic Geometry Theorem**. Given any two points on the sphere, there is a great circle that passes through those two points.

  Given a sphere, there are infinitely many ways to cut it in half, and each of these paths of the knife is called a great circle



- **Final pigeonhole example**

  Proposition. If you draw five points on the surface of an orange in marker, then there is always a way to cut the orange in half so that four points (or some part of the point) all lie on one of the halves.

***Proof*.** Consider an orange with five points drawn on it. Pick any two of these points, and call them p and q. By the Classic Geometry Theorem, there exists a great circle passing through these points; angle your knife to cut along this great circle. Because the points are drawn in marker, they are wide enough so that part of these two points appear on both halves.

Now consider the remaining three points and the two halves that you just cut the orange into. Consider these three points to be objects and the halves to be boxes; by the simple form of the pigeonhole principle, at least two of these three points are on the same orange half. These two, as well a portion of p and of q, give four points or partial points, as desired ■

1.2 Direct proofs

- **Fact about integers**: The sum of integers is an integer, the difference of integers is an integer, and the product of integers is an integer. Also, every integer is either even or odd.

  We are calling these facts because, while they are true and one could prove them, we will not be proving them here

- **Even and odd integers**: An integer n is *even* if $n = 2k$ for some integer k

  An integer n is *odd* if $n = 2k + 1$ for some integer k

- **Sum of two even integers**

  Proposition. The sum of two even integers is even

  Proof. Assume n and m are even integers, then $n = 2a$, and $m = 2b$ for some integers a and b. Furthermore,

  $$n + m = 2a + 2b = 2(a + b).$$

  Since the sum of two integers is itself an integer, then we have two times an integer, which satisfies the definition of an even number. Hence, the sum $n+m$ is even, where n and m are even. $\int$

- **More on propositions**: We can rewrite our propositions to take the form

  > if *statement* is true, then *other statement* is also true

  For example,

  > if m and n are even, then $m + n$ is also even

  Another way to summarize such statements is this:

  > *some statement* is true implies *some other statement* is true.

  Which allows us to use the implies symbol $\implies$. For example,

  > m and n being even $\implies$ $m + n$ is even

  We have the general form $P \implies Q$, where P and Q are statements

  However, when writing formally, like when writing up the final draft of your homework, these symbols are rarely used. You should write out solutions with words, complete sentences, and proper grammar. Pick up any of your math textbooks, or look online at math research articles, and you will find that such practices are standard.

- **The structure of direct proofs**: A direct proof is a way to prove a "$P \Rightarrow Q$" proposition by starting with P and working your way to Q. The "working your way to Q" stage often involves applying definitions, previous results, algebra, logic, and techniques. Here is the general structure of a direct proof:

Proposition. $P \implies Q$

Proof. Assume P

Explain what P means by applying definitions and/or other results

$\vdots$ Apply algebra,

$\vdots$ logic techniques.

Hey look, that's what Q means

Therefore Q ■

- **Proof by cases**: A related proof strategy is proof by cases. This is a "divide and conquer" strategy where one breaks up their work into two or more cases

The below example of proof by cases will also give us more practice with direct proofs involving definitions. Indeed, when you break up a problem in two parts, those two parts still need to be proven, and a direct proof is often the way to tackle each of those parts

Proposition. If n is an integer, then $n^2 + n + 6$ is even.

Proof. Assume n is an integer, then either n is even or it is odd.

Case I. Assume n is even, then $n = 2m$ for some integer m. Thus, we have

$$\begin{aligned} n^2 + n + 6 &= (2m)^2 + 2m + 6 \\ &= 4m^2 + 2m + 6 \\ &= 2(2m^2 + m + 3). \end{aligned}$$

Observe that $2m^2 + m + 3 \in \mathbb{Z}$. Thus, we have two times an integer, which satisfies the definition of an even number.

Case 2. Assume n is odd, then $n = 2m + 1$ for some integer m. Thus,

$$\begin{aligned} n^2 + n + 6 &= (2m + 1)^2 + 2m + 1 + 6 \\ &= 4m^2 + 4m + 1 + 2m + 7 \\ &= 4m^2 + 6m + 8 \\ &= 2(2m^2 + 3m + 4). \end{aligned}$$

Since m is an integer, $2m^2 + 3m + 4$ is an integer, and we again have two times an integer, which is an even integer.

We have shown that $n^2 + n + 6$ is even whether n is even or odd. Combined, this shows that $n^2 + n + 6$ is even for all integers n ■

- **Proof by exhaustion (brute force proof)**: A proof by cases cuts up the possibilities into more manageable chunks. If the theorem refers to a collection of elements and your proof is simply checking each element individually, then it is called a *proof by exhaustion* or a *brute force proof*

- **Divisibility**: An integer a is said to divide an integer b if $b = ak$ for some integer k. When a does divide b, we write $a \mid b$, and when a does not divide b, we write $a \nmid b$.

  Note: A common mistake is to see something like "$2 \mid 8$" and think that this equals 4. The expression "$a \mid b$" is either true or false

  Remark. $a \mid 0$ for any integer a, because $0 = a \cdot 0$ for every such a

  $0 \nmid b$ for any nonzero integer b, because for any such b, we have $b \neq 0 \cdot k$ for any integer k

- **The transitive property of divisibility**:

  Proposition. Let a, b, and c be integers, if $a \mid b$ and $b \mid c$, then $a \mid c$

  Proof. Assume a, b, and c are integers. Further assume that $a \mid b$, and $b \mid c$

  By the definition of divisibility, $a \mid b$ and $b \mid c$ implies $b = ak$ for some integer k, and $c = bs$ for some integer s

  If $a \mid c$, we require that $c = ar$ for some integer r

$$
\begin{aligned}
& b = ak \\
\implies & c = (ak)s \\
\implies & c = a(ks).
\end{aligned}
$$

  Since k and s are integers, then their product ks is itself an integer. Let $r = ks$. Then $c = ar$, which is precisely the definition of divisiblity, and we conclude that $a \mid c$. ■

- **The division algorithm**:

  Theorem. For all integers a and m with $m > 0$, there exist unique integers q and r such that

$$a = mq + r.$$

  Where $0 \leqslant r < m$. We call q the *quotient* and r the *remainder*

- **Common divisor, greatest common divisor**: Let a and b be integers. If $c \mid a$ and $c \mid b$, then c is said to be a common divisor of a and b.

  The greatest common divisor of a and b is the largest integer d such that $d \mid a$ and $d \mid b$. This number is denoted $\gcd(a, b)$.

Note that there is one pair of integers that does not have a greatest common divisor; if $a = 0$ and $b = 0$, then every positive integer d is a common divisor of a and b. This means that no divisor is the greatest divisor, since you can always find a bigger one. Thus, in this one case, $gcd(a, b)$ does not exist

- **Bezout's identity**: If a and b are positive integers, then there exist integers k and ℓ such that

$$\text{gcd}\,(a, b) = ak + b\ell.$$

As an example, suppose $a = 12$ and $b = 20$, then $\text{gcd}\,(12, 20) = 4$, and we have

$$\begin{aligned} 4 &= 12k + 20\ell \\ \implies \ell &= \frac{1}{5} - \frac{3}{5}k. \end{aligned}$$

Let $k = 2$, then we see $\ell = -1$. We see that there are infinitely many solutions, $k = 2, \ell = -1$ is just one of them. Nevertheless, this theorem simply says that at least one solution must exist.

Proof. Assume a and b are fixed positive integers, notice that the expression $ax + by$ can take many values for integers x and y. Let d be the *smallest positive integer* that $ax + by$ can be equal. Let k and ℓ be the x and y that obtain this d. That is,

$$d = ak + b\ell.$$

We now must show that d is a common divisor of a and b, and then that it is the *greatest common divisor*

Part 1 (common divisor). d is a common divisor of a and b if $d \mid a$ and $d \mid b$. To see that $d \mid a$, we examine the division algorithm. We know that there exsits unique integers q and r such that

$$a = dq + r.$$

With $0 \leqslant r < d$. We have

$$\begin{aligned} r &= a - dq \\ &= a - (ak + b\ell)q \\ &= a - akq - b\ell q \\ &= a(1 - kq) + b(-\ell q). \end{aligned}$$

Observe that $1 - kq$, and $-\ell q$ are both integers, Since r is written in the form $ax + by$, $0 \leqslant r < d$, and d is the smallest positive integer that this form can produce (with the given a, b), it must be that $r = 0$. Thus,

$$a = dq + 0 = dq.$$

And we see that $d \mid a$. A similar argument will show that $d \mid b$ as well. This proves that d is a common divisor of a and b.

Part 2 (gcd). Assume that d' is some other common divisor of a and b. We must show that $d' \leqslant d$. If d' is a common divisor of a and b, then $d' \mid a$ and $d' \mid b$, which implies $a = d'n$, and $b = d'm$, for some integers n and m. If $d = ak + b\ell$, then

$$\begin{aligned} d &= d'nk + d'm\ell \\ &= d'(nk + m\ell) \\ \implies d' &= \frac{d}{nk + m\ell}. \end{aligned}$$

Since $n, k, m, \ell \in \mathbb{Z}$, it follows that $nk + m\ell \in \mathbb{Z}$. Thus, $d' \leqslant d$.

Therefore, we have shown that d is not only a common divisor of a and b, but that it is also the largest, and hence the *gcd*. Thus,

$$\gcd(a, b) = d = ak + b\ell.$$

■

A corollary from this result is that $\gcd(ma, mb) = m \gcd(a, b)$. If $\gcd(a, b) = ak + b\ell$, we have

$$\begin{aligned} \gcd(ma, mb) &= mak + mb\ell \\ &= m(ak + b\ell) \\ &= m \gcd(a, b). \end{aligned}$$

- **Modulo and congruence**: For integers a, r, and m, we say that a is congruent to r modulo m and we write $a \equiv r \pmod{m}$ if $m \mid (a - r)$.

  For example, $18 \equiv 4 \pmod 7$ because $18 = 7(2) + 4$, we see that $7 \mid (18 - 4)$

  If a divided by m leaves a remainder of r, then $a \equiv r \pmod{m}$. However, this is not the only way to have $a \equiv r \pmod{m}$ — it is not required that r be the remainder when a is divided by m; all that is required is that a and r have the same remainder when divided by m. For example:

  $$18 = 11 \pmod 7.$$

- **Properties of modular congruence**: Assume that a, b, c, d and m are integers, $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$. Then

  (i) $a + c \equiv b + d \pmod{m}$

  (ii) $a - c \equiv b - d \pmod{m}$

  (iii) $a \cdot c \equiv b \cdot d \pmod{m}$

  Proof of property *i*. Assume that $a \equiv b \pmod{m}$, and $c \equiv d \pmod{m}$, we must show that $a + c \equiv b + d \pmod{m}$

  If $a \equiv b \pmod{m}$, then $m \mid a - b$, which implies $a - b = mk$ for some $k \in \mathbb{Z}$. Similarly, $c \equiv d \pmod{m} \implies m \mid c - d \implies c - d = m\ell$, for some $\ell \in \mathbb{Z}$. Adding these two equations yields

  $$\begin{aligned} (a - b) + (c - d) &= mk + m\ell \\ \implies (a + c) - (b + d) &= m(k + \ell). \end{aligned}$$

Since $k + \ell \in \mathbb{Z}$, then by the definition of divisibility

$$m \mid (a + c) - (b + d).$$

Which then by the definition of congruence

$$a + c \equiv b + d \pmod{m}.$$

■

Proof of property *iii*. Assume $a \equiv b \pmod{m}$, and $c \equiv d \pmod{m}$

> From above we know it follows that $a - b = mk$, and $c - d = m\ell$, for $k, \ell \in \mathbb{Z}$. If $ac \equiv bd \pmod{m}$, it must be that $ac - bd = ms$, for some $s \in \mathbb{Z}$. Let's see if we can derive $ac - bd$ in terms of what we know, namely $a - b$ and $c - d$. Amazingly,
>
> $$\begin{aligned} ac - bd &= (a - b)c + (c - d)b \\ &= mkc + m\ell b \\ &= m(kc + \ell b). \end{aligned}$$

It then follows that

$$m \mid ac - bd.$$

Thus,

$$ac \equiv bd \pmod{m}.$$

■

- **Prime and composite integers**: An integer $p \geqslant 2$ is prime if its only positive divisors are 1 and p. An integer $n \geqslant 2$ is composite if it is not prime. Equivalently, n is composite if it can be written as $n = st$, where s and t are integers and $1 < s, t < n$.

  Note: To be clear, "$1 < s, t < n$" means that both s and t are between 1 and n.

- **Properties of primes and divisibility**:

  Lemma. Let a, b and c be integers, and let p be a prime:

  (i) If $p \nmid a$, then $\gcd(p, a) = 1$.
  (ii) If $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$.
  (iii) If $p \mid bc$, then $p \mid b$ or $p \mid c$ (or both).

  Proof of property *i*. Assume that p does not divide a, then p cannot possibly be a common divisor of a and p, because it is not a divisor of a.

  Since $p \in \mathbb{P}$[1], then the only divisors of p are one and itself. Thus, the only option left is one. Hence, the greatest common divisor is one. ■

[1] Where $\mathbb{P}$ is the family of primes

Proof of property *ii*. Assume $a \mid bc$, and $\gcd(a, b) = 1$. Then, $bc = ar$ for some integer r, and by Bezout's identity, there exist some integers k, ℓ such that

$$\begin{aligned} \gcd(a,b) &= ak + b\ell \\ \implies 1 &= ak + b\ell. \end{aligned}$$

If $a \mid c$, we require $c = as$, for some integer s. If we multiply the above expression by c, we get

$$c = cak + cb\ell.$$

Since we assumed $a \mid bc$, then it must be that $bc = ar$, for $r \in \mathbb{Z}$. Thus, we have

$$\begin{aligned} c &= cak + ar\ell \\ &= a(ck + r\ell). \end{aligned}$$

Since $c, k, r, \ell \in \mathbb{Z}$, the expression $ck + r\ell$ is also an integer, and by the definition of divisibility, it must be that $a \mid c$ ■

Proof of property *iii*. Assume that $p \mid bc$. Then there are two cases, either $p \mid b$, or $p \nmid b$.

Case I. If $p \mid b$, then the statement is true and we are done

Case II. If $p \nmid b$, then by property i, it must be that $\gcd(p, b) = 1$. By property ii, if $p \mid bc$, and $\gcd(p, b) = 1$, then it must be that $p \mid c$. ■.

- **More on properties of congruence**: We return to congruence to examine the statement

$$ak \equiv bk \pmod{m} \overset{?}{\implies} a \equiv b \pmod{m}.$$

Proposition (*modular cancellation law*). Let a, b, k, m be integers. If $ak \equiv bk \pmod{m}$, and $\gcd(m, k) = 1$, then $a \equiv b \pmod{m}$

Proof. Assume $ak \equiv bk \pmod{m}$, and $\gcd(m, k) = 1$, then $m \mid ak - bk$, and $ak - bk = m\ell$, for some integer ℓ.

If $a \equiv b \pmod{m}$, then $m \mid a - b$, and $a - b = mr$, for some integer r. Since $ak \equiv bk \pmod{m}$, then it must be that

$$\begin{aligned} ak - bk &= m\ell \\ \implies k(a-b) &= m\ell \\ \implies a - b &= \frac{m\ell}{k}. \end{aligned}$$

Thus, we require $\frac{\ell}{k}$ to be an integer, it then follows that the proposition holds true.

We know that if $a \mid bc$, and $\gcd(a, b) = 1$, then $a \mid c$. Thus, since $k \mid m\ell$, and $\gcd(m, k) = 1$, it must be that $k \mid \ell$. Hence, $\frac{\ell}{k} \in \mathbb{Z}$, and

$$a - b = m\left(\frac{\ell}{k}\right).$$

And by the definition of divisibility, $m \mid a - b$, which implies $a \equiv b \pmod{m}$ ■.

- **Fermat's little theorem**: If a is an integer and p is a prime which does not divide a, then

$$a^{p-1} \equiv 1 \pmod{p}.$$

Proof. Assume that a is an integer and p is a prime which does not divide a. We begin by proving that when taken modulo p,

$$\{a, 2a, 3a, \dots, (p-1)a\} \equiv \{1, 2, 3, \dots, p-1\}.$$

To do this, observe that the set on the right has every residue modulo p except 0, and each such residue appears exactly once. Therefore, since both sets have $p-1$ elements listed, in order to prove that the left set is the same as the right set, it suffices to prove this:

1. No element in the left set is congruent to 0, and
2. Each element in the left set appears exactly once.

In doing so, we will twice use the modular cancellation law (Proposition 2.18) to cancel out an a, and so we note at the start that by Lemma 2.17 part (i) we have $\gcd(p, a) = 1$.

Step 1. First we show that none of the terms in $\{a, 2a, 3a, \dots, (p-1)a\}$, when considered modulo p, are congruent to 0. To do this, we will consider an arbitrary term ia, where i is anything in $\{1, 2, 3, \dots, p-1\}$. Indeed, if we did have some

$$ia \equiv 0 \pmod{p},$$

which is equivalent to

$$ia \equiv 0a \pmod{p},$$

then by the modular cancellation law (Proposition 2.18) we would have

$$i \equiv 0 \pmod{p}.$$

That is, in order to have $ia \equiv 0 \pmod{p}$, that would have to have $i \equiv 0 \pmod{p}$. Therefore we are done with Step 1, since no i from $\{1, 2, 3, \dots, p-1\}$ is congruent to 0 modulo p.

Step 2. Next we show that every term in $\{a, 2a, 3a, \dots, (p-1)a\}$, when considered modulo p, does not appear more than once in that set. Indeed, if we did have

$$ia \equiv ja \pmod{p},$$

for i and j from $\{1, 2, 3, \dots, p-1\}$, then by the modular cancellation law (Proposition 2.18) we have

$$i \equiv j \pmod{p}.$$

And since i and j are both from the set $\{1, 2, 3, \dots, p-1\}$, this means that $i = j$. In other words, each term in $\{a, 2a, 3a, \dots, (p-1)a\}$ is not congruent to any other term from that set — it is only congruent to itself. This completes Step 2.

We have succeeded in proving that when taken modulo p,

$$\{a, 2a, 3a, \ldots, (p-1)a\} \equiv \{1, 2, 3, \ldots, p-1\},$$

even though the numbers in these sets may be in a different order. But since the order does not matter when multiplying numbers, we see that

$$a \cdot 2a \cdot 3a \cdot 4a \cdot \cdots \cdot (p-1)a \equiv 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot (p-1) \pmod{p}.$$

Then, since $\gcd(2, p) = 1$ by Lemma 2.17 part (i), by the modular cancellation law (Proposition 2.18) we may cancel a 2 from both sides:

$$a \cdot 3a \cdot 4a \cdot \cdots \cdot (p-1)a \equiv 1 \cdot 3 \cdot 4 \cdot \cdots \cdot (p-1) \pmod{p}.$$

Then, since $\gcd(3, p) = 1$ by Lemma 2.17 part (i), by the modular cancellation law (Proposition 2.18) we may cancel a 3 from both sides:

$$a \cdot a \cdot 4a \cdot \cdots \cdot (p-1)a \equiv 1 \cdot 4 \cdot \cdots \cdot (p-1) \pmod{p}.$$

Continuing to do this for the $4, 5, \ldots, (p-1)$ on each side (each of which has a greatest common divisor of 1 with p, by Lemma 2.17 part (i)), by the modular cancellation law (Proposition 2.18) we obtain

$$\underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{p-1 \text{ copies}} \equiv 1 \pmod{p},$$

which is equivalent to what we sought to prove:

$$a^{p-1} \equiv 1 \pmod{p}.$$

- **Bonus proof**:

  Proposition. If x and y are positive integers, and $x \geqslant y$, then $\sqrt{x} \geqslant \sqrt{y}$

  Proof. Assume x and y are positive integers, and $x \geqslant y$. Then

$$\begin{aligned} & x \geqslant y \\ \implies & x - y \geqslant 0 \\ & . \end{aligned}$$

  Since $x, y \geqslant 0$, $\sqrt{x^2} = |x| = x$, and $\sqrt{y^2} = |y| = y$. Thus,

$$\begin{aligned} & x - y \geqslant 0 \\ \implies & \sqrt{x^2} - \sqrt{y^2} \geqslant 0 \\ \implies & (\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y}) \geqslant 0 \\ \implies & \sqrt{x} - \sqrt{y} \geqslant 0 \qquad \blacksquare. \end{aligned}$$

- **The AM-GM inequality**:

  Theorem (*AM-GM inequality*). If $x, y \geqslant 0 \in \mathbb{Z}$, then $\sqrt{xy} \leqslant \frac{x+y}{2}$

Proof. Assume $x, y \geqslant 0 \in \mathbb{Z}$. Consider

$$0 \leqslant (x-y)^2.$$

Which we know to be true, squaring an integer is always positive, and we know $x - y$ to be an integer. It then follows that

$$0 \leqslant x^2 - 2xy + y^2.$$

If we add $4xy$ to both sides, we get

$$\begin{aligned} & 4xy \leqslant x^2 + 2xy + y^2 \\ \implies & 4xy \leqslant (x+y)^2 \end{aligned}$$
$$.$$

Now let's take the square root of both sides

$$2\sqrt{xy} \leqslant |x+y|.$$

Since $x, y \geqslant 0$, $|x+y| = x+y$. Thus,

$$\begin{aligned} 2\sqrt{xy} &\leqslant x+y \\ \therefore \sqrt{xy} &\leqslant \frac{x+y}{2}. \end{aligned}$$

Note: Some of the steps taken in this proof may seem a bit random, but if we start at the proposition $\sqrt{xy} \leqslant \frac{x+y}{2}$ and work backwards algebraically, we see

$$\begin{aligned} \sqrt{xy} &\leqslant \frac{x+y}{2} \\ 2\sqrt{xy} &\leqslant x+y \\ 4xy &\leqslant (x+y)^2 \\ 4xy &\leqslant x^2 + 2xy + y^2 \\ 0 &\leqslant x^2 + 2xy + y^2 - 4xy \\ 0 &\leqslant x^2 - 2xy + y^2 \\ 0 &\leqslant (x-y)^2. \end{aligned}$$

We see that we have derived a starting point, and were just working backwards in the proof.

1.3 Sets

- **Vacuous truth**: a vacuous truth is a conditional or universal statement (a universal statement that can be converted to a conditional statement) that is true because the antecedent cannot be satisfied.[1] It is sometimes said that a statement is vacuously true because it does not really say anything. For example, the statement "all cell phones in the room are turned off" will be true when no cell phones are present in the room. In this case, the statement "all cell phones in the room are turned on" would also be vacuously true, as would the conjunction of the two: "all cell phones in the room are turned on and turned off", which would otherwise be incoherent and false.

- **Review: Proper subset**: If $A = B$, then $A \subseteq B$. In the case that $A \subseteq B$ and $A \neq B$, we say that A is a proper subset of B. the correct notation for this is "$A \subset B$."

- **Proving** $A \subseteq B$

  Definition. Suppose A and B are sets. If every element in A is also an element of B, then A is a subset of B, which is denoted $A \subseteq B$

  Note: For every set B, it is true that $\varnothing \subseteq B$. To see it, first note that, because there are no elements in $\varnothing$, it would be true to say "for any $x \in \varnothing$, x is a purple elephant that speaks German." It's vacuously[2] true! You certainly can't disprove it, right? You can't present to me any element in $\varnothing$ that is not a purple elephant that speaks German.

  By this reasoning, I could switch out "is a purple elephant that speaks German" for any other statement, and it would still be true! And this includes the subset criteria: if $x \in \varnothing$, then $x \in B$, which by definition means that $\varnothing \subseteq B$. Again, you certainly can not present to me any $x \in \varnothing$ which is not also an element of B, can you?

  in order to prove that $A \subseteq B$, what we would have to show is this:

  $$\text{If } x \in A, \text{ then } x \in B.$$

  In other words, for any arbitrary element in A, that same element is also in B

  Proposition. It is the case that

  $$\{n \in \mathbb{Z} : \ 12 \mid n\} \subseteq \{n \in \mathbb{Z} : \ 3 \mid n\}.$$

  Proof. Let $A = \{n \in \mathbb{Z} : \ 12 \mid n\}$, and $B = \{n \in \mathbb{Z} : \ 3 \mid n\}$. Assume $a \in A$

  > Since $a \in A$, then $12 \mid a$, which implies $a = 12k$, for some $k \in \mathbb{Z}$. If $a \in B$, then $3 \mid a \implies a = 3\ell$
  >
  > Since $a = 12k$, and $a = 3\ell$, then $12k = 3\ell \implies \ell = 4k$. Thus, we have
  >
  > $$a = 3(4k).$$
  >
  > Which by the definition of divisiblity, and since $4k \in \mathbb{Z}$, we have $3 \mid a$.
  >
  > Therefore, $a \in B$ ■

[2] A statement is vacuously true if it asserts something about all elements of the empty set.

- **Proving** $A = B$: Recall that, for sets A and B, to say that "$A = B$" is to say that these two sets contain *exactly* the same elements. Said differently, it means these two things:

  1. Every element in A is also in B (which means $A \subseteq B$), and
  2. Every element in B is also in A (which means $B \subseteq A$).

  Indeed, a slick way to prove that $A = B$ is to prove both $A \subseteq B$ and $B \subseteq A$, both of which can be done using the approach discussed above.

- **Review of set operations**:

  – The *union* of sets A and B is the set $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

  – The *intersection* of sets A and B is the set $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

  – Likewise, if $A_1, A_2, A_3, \ldots, A_n$ are all sets, then the union of all of them is the set
  $$A_1 \cup A_2 \cup \cdots \cup A_n = \{x : x \in A_i \text{ for some } i\}.$$
  This set is also denoted
  $$\bigcup_{i=1}^{n} A_i.$$

  – Likewise, if $A_1, A_2, A_3, \ldots, A_n$ are all sets, then the intersection of all of them is the set
  $$A_1 \cap A_2 \cap \cdots \cap A_n = \{x : x \in A_i \text{ for all } i\}.$$
  This set is also denoted
  $$\bigcap_{i=1}^{n} A_i.$$

  Assume A and B are sets and "$x \notin B$" means that x is not an element of B.

  – The *subtraction* of B from A is $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$.

  – If $A \subseteq U$, then U is called a *universal set* of A. The *complement* of A in U is $A^c = U \setminus A$.

  Furthermore,

  – The *power set* of a set A is $\mathcal{P}(A) = \{X : X \subseteq A\}$.

  – The *cardinality* of a set A is the number of elements in the set, and it is denoted $|A|$.

  Assume A and B are sets, The Cartesian product of A and B is
  $$A \times B = \{(a, b) : \ a \in Aandb \in B\}..$$

- **More on power sets**:

  Proposition. Suppose A and B are sets. If $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then $A \subseteq B$.

  Proof. Assume A and B are sets, and $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

  > Choose $x \in \mathcal{P}(A)$, which means $x \subseteq A$. Since $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, it follows that $x \in \mathcal{P}(B)$, which means $x \subseteq B$. Let $x = A$, since $A \in \mathcal{P}(A)$. Since $x \subseteq B$, then $A \subseteq B$

  Therefore, $A \subseteq B$ $\blacksquare$

- **De Morgan's law**:

  Theorem. Suppose A and B are subsets of a universal set U. Then,

  $$(A \cup B)^C = A^C \cap B^C. \tag{1}$$

  And

  $$(A \cap B)^C = A^C \cup B^C. \tag{2}$$

  Proof (1). Assume A and B are subsets of a universal set U, since $(A \cup B)^C$, and $A^C \cap B^C$ are sets, we show equality by showing $(A \cup B)^C \subseteq A^C \cap B^C$, and $A^C \cap B^C \subseteq (A \cup B)^C$. It then follows that $(A \cup B)^C = A^C \cap B^C$

  > Choose $x \in (A \cup B)^C$, by the definition of the complement, we have $x \notin (A \cup B)$, which by the definition of the union means x cannot be in A, and it cannot be in B. In other words, $x \notin A$ and $x \notin B \implies x \in A^C$ and $x \in B^C$. Therefore,
  >
  > $$x \in A^C \cap B^C.$$
  >
  > Which by the definition of the subset, means $(A \cup B)^C \subseteq A^C \cap B^C$
  >
  > Next, let $x \in A^C \cap B^C$, then $x \in A^C$ and $x \in B^C$, which means $x \notin A$ and $x \notin B$, which implies $x \notin (A \cup B) \implies x \in (A \cup B)^C$.
  >
  > Therefore, since $x \in A^C \cap B^C \implies x \in (A \cup B)^C$, by the definition of a subset, we have $A^C \cap B^C \subseteq (A \cup B)^C$

  Since both $(A \cup B)^C \subseteq A^C \cap B^C$, and $A^C \cap B^C \subseteq (A \cup B)^C$, it must be the case that $(A \cup B)^C = A^C \cap B^C$ $\blacksquare$

  It should be addressed that this proof can be done by simply manipulating the set builder notation. We have

  $$\begin{aligned} A^C \cap B^C &= \{x \in \mathbb{R} : \ x \in A^C \text{ and } x \in B^C\} \\ &= \{x \in \mathbb{R} : \ x \notin A \text{ and } x \notin B\} \\ &= \{x \in \mathbb{R} : \ x \notin (A \cup B)\} \\ &= \{x \in \mathbb{R} : \ x \in (A \cup B)^C\}. \end{aligned}$$

  $\blacksquare$

- **Proving** $a \in A$: Consider the set $\{x \in S : \ P(x)\}$, where $P(x)$ is some condition on x

Given a set of this form, if you are presented with a specific a and you wish to prove that $a \in A$, then you must show that

1. $a \in S$
2. $P(a)$ is true

For example, Let $A = \{(x, y) \in \mathbb{Z} \times \mathbb{N} : \ x \equiv y \pmod 5\}$, then $(17, 2) \in A$

Proof. First, note that $(17, 2) \in \mathbb{Z} \times \mathbb{N}$ because $17 \in \mathbb{Z}$, and $2 \in \mathbb{N}$, Next, observe that

$$17 \equiv 2 \pmod 5.$$

Because $5 \mid (17 - 2)$

- **Indexed Families of Sets**: Consider a set $\mathcal{F}$, If every element of $\mathcal{F}$ is itself a set, then $\mathcal{F}$ is called a *family of sets*. Then, one can ask questions about such a family, — like, what is the union of all of the sets in $\mathcal{F}$. That is,

$$\bigcup_{S \in \mathcal{F}} S = \{x : \ x \in S \text{ for some } S \in \mathcal{F}\}.$$

Likewise,

$$\bigcap_{S \in \mathcal{F}} S = \{x : \ x \in S \text{ for every } S \in \mathcal{F}\}.$$

- **Bonus example I**.

Proposition. It is the case that

$$\{n \in \mathbb{Z} : \ 12 \mid n\} = \{n \in \mathbb{Z} : \ 3 \mid n\} \cap \{n \in \mathbb{Z} : \ 4 \mid n\}.$$

Proof. Let $A = \{n \in \mathbb{Z} : \ 12 \mid n\}$, $B = \{n \in \mathbb{Z} : \ 3 \mid n\}$, and $C = \{n \in \mathbb{Z} : \ 4 \mid n\}$

Part i.) Choose $x \in A$, we then have $12 \mid x$, and $x = 12k$, for some $k \in \mathbb{Z}$. Thus,

$$x = 12k = 3(4k) = 4(3k).$$

Which by the definition of divisibility implies both $3 \mid x$ and $4 \mid x$, since both $4k$ and $3k \in \mathbb{Z}$. Hence, $x \in B \cap C$

Part ii.) Choose $x \in B \cap C$, then both $x = 3r$ and $x = 4s$, for $r, s \in \mathbb{Z}$. We have

$$3r = 4s.$$

Which implies $3 \mid 4s$, since $r \in \mathbb{Z}$. Because $3 \in \mathbb{P}$, we know that either $3 \mid 4$ or $3 \mid s$. Since it is clear that $3 \nmid 4$, it must be the case that $3 \mid s$, and thus $s = 3\ell$ for an integer ℓ. It then follows that

$$x = 4s = 4(3\ell) = 12\ell.$$

Which by the definition of divisibility implies $12 \mid x$, and thus $x \in A$

Since choosing an $x \in A \implies x \in B \cap C$, it must be that $A \subseteq B \cap C$, and choosing an $x \in B \cap C \implies x \in A$, it must also be that $B \cap C \subseteq A$. With these two facts, we can assert that $A = B \cap C$ ■

- **The Cardinality of the Power Set**: Suppose A is a set with n elements. How many subsets of A are there? Said differently, what is $|P(A)|$?

We could check the first few cases by hand

A	$\lvert A\rvert = n$	$\lvert\mathcal{P}(A)\rvert$
$\{1\}$	1	2
$\{1,2\}$	2	4
$\{1,2,3\}$	3	8
$\{1,2,3,4\}$	4	16

It sure looks like if $|A| = n$, then $|P(A)| = 2^n$. Why would this be true? There is actually a pretty slick way to see it. Every subset of $\{1, 2, 3\}$ can be thought of by asking whether or not each element is included in the subset. For example, $\{1, 3\}$ can be thought of as $\langle$yes, no, yes$\rangle$, since 1 was included, 2 was not, and 3 was.

Suppose you're trying to generate a subset of $\{1, 2, 3\}$. You could think about doing so by asking three yes/no questions, the answers to which uniquely determine your set. With 2 options for the first element, 2 for the second, and 2 for the third, in total there are $2 \times 2 \times 2 = 8$ ways to answer the three questions, and hence 8 subsets!

With n straight yes/no questions, there are $2 \times 2 \times \cdots \times 2 = 2^n$ ways to answer the questions, each corresponding uniquely to a subset of A. Thus, if $|A| = n$, then $|P(A)| = 2^n$.

- **A consequence of the above fact**:

  Proposition. Given any $A \subseteq \{1, 2, 3, \ldots, 100\}$ for which $|A| = 10$, there exist two different subsets $X \subseteq A$ and $Y \subseteq A$ for which the sum of the elements in X is equal to the sum of the elements in Y.

  For example, consider the set $\{6, 23, 30, 39, 44, 46, 62, 73, 90, 91\}$, If we let

  $$X = \{6, 23, 46, 73, 90\} \text{ and } Y = \{30, 44, 73, 91\}.$$

  then the elements in both sets sum to 238:

  Proof. We prove this fact using the pigeonhole principle. Consider the smallest and largest possible subset sums. If $A = \varnothing \subseteq \{1, 2, 3, ..., 100\}$, then the sum is 0. If $A = \{91, 92, 93, 94, 95, 96, 97, 98, 99, 100\}$, then the subset sum is 955. Thus, there are no more than 956 possible subset sums for the set $A \subseteq \{1, 2, 3, ..., 100\}$, for which $|A| = 10$.

  Consider 956 boxes, each representing a unique subset sum. Since we have $2^{|A|} = 2^{10} = 1024$ subsets and only 956 boxes to place each subset in, there must be a box containing two subsets A, which means they must have the same sum $\blacksquare$.

- **The symmetric difference of sets**. The *symmetric difference* of two sets A and B, denoted $A \Delta B$, or $A \ominus B$, is the set which contains the elements which are either in set A or in set B but not in both

1.4 Induction

- **Dominoes**: Consider a line of dominoes, perfectly arranged, just waiting to be knocked over. Dominoes stacked up like this have the following properties:

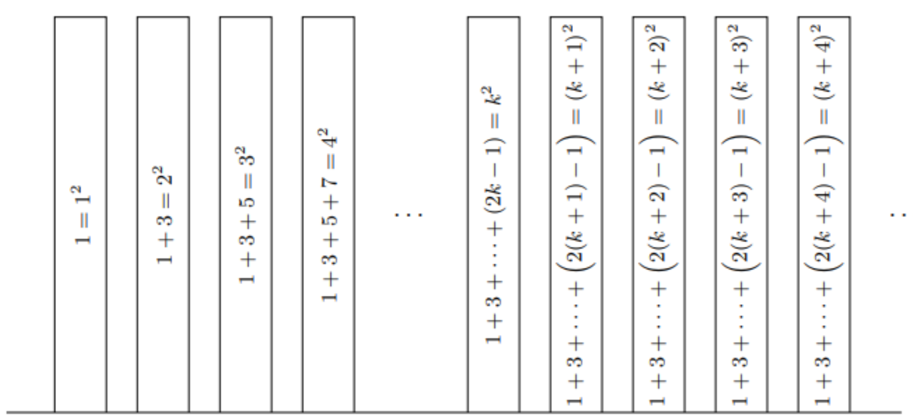
  1. If you give the first domino a push, it will fall (in particular, it will fall into the second domino, knocking it over).
  2. Moreover, every domino, when it's knocked over, falls into the next one and knocks it over.

  Given these two properties, it must be the case that if you knock over the first domino, then every domino will eventually fall. The first premise gets the process going, as it implies that the first domino will fall. And then the second premise keeps it going: Applying the second premise means that the falling first domino will cause the second domino to fall. Applying the second premise again means that the second falling domino will cause the third domino to fall. Applying the second premise again means that the third falling domino will cause the fourth domino to fall. And so on.

- **Sum of the first n odd numbers**: Take a look at the following

$$
\begin{aligned}
1 &= 1 = 1^2 \\
1+3 &= 4 = 2^2 \\
1+3+5 &= 9 = 3^2 \\
1+3+5+7 &= 16 = 4^2 \\
1+3+5+7+9 &= 25 = 5^2 \\
1+3+5+7+9+11 &= 36 = 6^2 \\
1+3+5+7+9+11+13 &= 49 = 7^2.
\end{aligned}
$$

  It sure looks like the sum of the first n odd numbers is n^2. But how can we prove that it's true for every one of the infinitely many n? The trick is to use the domino idea. Imagine one domino for each of the above statements.



  Suppose we do the following:

  – Show that the first domino is true (this is trivial, since obviously $1 = 1^2$).
  – Show that any domino, if true, implies that the following domino is true too

Given these two, we may conclude that all the dominoes are true. It's exactly the same as noting that all the dominoes from earlier will fall. This is a slick way to prove infinitely many statements all at once, and it is called the *principle of mathematical induction*, or, when among friends, it is simply called *induction*.

- **Induction**: Consider a sequence of mathematical statements, $S_1, S_2, S_3, ...$
  - Suppose S_1 is true, and
  - Suppose, for each $k \in \mathbb{N}$, if S_k is true then S_{k+1} is true.

  Then, S_n is true for every $n \in \mathbb{N}$.

- **Induction framework**:

  Proposition. $S_1, S_2, S_3, ...$ are all true

  Proof. *General setup or assumptions if needed*

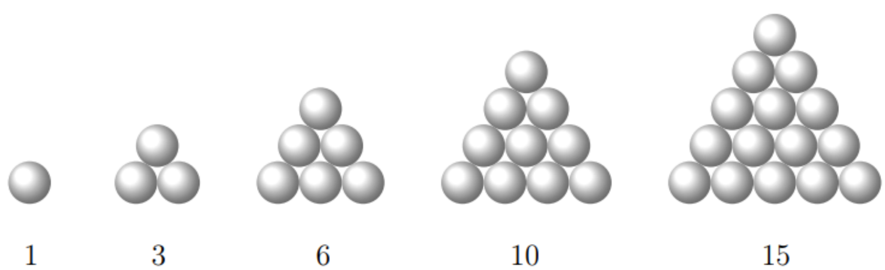
  Base case. $\langle\langle$Demonstration that S_1 is true$\rangle\rangle$

  Inductive hypothesis. Assume that S_k is true

  Induction step. $\langle\langle$Proof that S_k implies $S_{k+1}\rangle\rangle$

  Conclusion. Therefore, by induction, all the S_n are true. ∎

- **Induction example 1**: Let's simply sum the first n natural numbers: $1+2+3+4+$ ůůů $+ n$. These sums are called the triangular numbers since they can be pictured as the number of balls in the following triangles.



Proposition. For any $n \in \mathbb{N}$, $\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

Proof. We proceed by induction

Base case: The base case is when $n = 1$, and

$$1 = \frac{1(1+1)}{2} = 1.$$

Inductive hypothesis: Let $k \in \mathbb{N}$, assume

$$1 + 2 + 3 + ... + k = \frac{k(k+1)}{2}.$$

Inductive step: We aim to show that the result holds for $k+1$. Thus,

$$1 + 2 + 3 + ... + k + k + 1 = \frac{(k+1)((k+1)+1)}{2}.$$

We have

$$\begin{aligned} 1 + 2 + 3 + ... + k + k + 1 &= \frac{(k+1)(k+2)}{2} \\ \implies \frac{k(k+1)}{2} + k + 1 &= \frac{(k+1)(k+2)}{2} \\ \implies \frac{k^2 + k + 2k + 1}{2} &= \frac{k^2 + 2k + k + 2}{2}. \end{aligned}$$

Therefore, by induction, $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$ $\blacksquare$

- **Induction example 2**:

Proposition. Let S_n be the sum of the first n natural numbers. Then, for any $n \in \mathbb{N}$,

$$S_n + S_{n+1} = (n+1)^2.$$

We will prove this proposition twice. The first proof is a direct proof, the second will be by induction.

Direct proof. We have

$$\begin{aligned} S_n + S_{n+1} &= \frac{n(n+1)}{2} + \frac{(n+1)((n+1)+1)}{2} \\ &= \frac{n^2+n}{2} + \frac{n^2+2n+n+2}{2} \\ &= \frac{n^2+n+n^2+3n+2}{2} \\ &= \frac{2n^2+4n+2}{2} \\ &= \frac{2(n^2+2n+1)}{2} \\ &= n^2+2n+1 \\ &= (n+1)^2 \quad \blacksquare. \end{aligned}$$

Proof by induction. We proceed by induction

Base case: The base case is when $n = 1$, and

$$S_1 + S_2 = 1 + 3 = 4 = (1 + 1)^2.$$

as desired

Inductive hypothesis. Let $k \in \mathbb{N}$, and assume that

$$S_k + S_{k+1} = (k + 1)^2.$$

Inductive step. We aim to prove that the result holds for $k + 1$. That is,

$$S_{k+1} + S_{k+2} = (k + 2)^2.$$

For this, we use the fact that S_{k+1} is the sum of the first $k + 1$ natural numbers, thus we can write it as $S_k + (k+1)$. Likewise, $S_{k+2} = S_{k+1} + (k + 2)$. Thus,

$$\begin{aligned} S_{k+1} + S_{k+2} &= S_k + (k + 1) + S_{k+1} + (k + 2) \\ &= S_k + S_{k+1} + 2k + 3 \\ &= (k + 1)^2 + 2k + 3 \\ &= k^2 + 2k + 1 + 2k + 3 \\ &= k^2 + 4k + 4 \\ &= (k + 2)^2. \end{aligned}$$

Conclusion. Therefore, by induction, the proposition holds for all $n \in \mathbb{N}$ ■

- **A quick note about induction**: For some proof techniques, adding a sentence at the end of your proof is nice but not required. For induction, though, it really is required. You can prove that the first domino will fall, and you can prove that each domino — if fallen— will knock over the next domino, but why does this mean they all fall? Because induction says so! Until you say "by induction. . . " your work will not officially prove the result

- **Induction example 3.**

Proposition. For every $n \in N$, the product of the first n odd natural numbers equals $\frac{(2n)!}{2^n n!}$. That is,

$$1 \cdot 3 \cdot 5 \cdot ... \cdot (2n - 1) = \frac{(2n)!}{2^n n!}.$$

Proof. We proceed by induction.

Base case: The base case occurs when $n = 1$,

$$1 = \frac{(2(1))!}{2^1 1!} = 1.$$

As desired

Inductive hypothesis. Let $k \in \mathbb{N}$, assume

$$1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-1) = \frac{(2k)!}{2^k k!}.$$

Inductive step. We aim to prove that the result holds for $k+1$. Thus, we wish to show

$$\begin{aligned} 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-1) \cdot (2(k+1)-1) &= \frac{(2(k+1))!}{2^{k+1}(k+1)!} \\ &= \frac{(2k+2)!}{2^{k+1}(k+1)!}. \end{aligned}$$

By the inductive hypothesis, we have

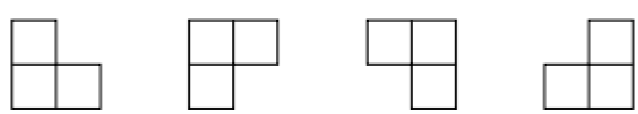
$$\begin{aligned} 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-1) \cdot (2k+1) &= \frac{(2k)!}{2^k k!}(2k+1) \\ &= \frac{(2k)!(2k+1)}{2^k k!} \\ &= \frac{(2k+1)!}{2^k k!} \\ &= \frac{(2k+1)!}{2^k k!} \cdot \frac{(2k+2)}{(2k+2)} \\ &= \frac{(2k+2)!}{2^k k!(2k+2)} \\ &= \frac{(2k+2)!}{2^k k! \cdot 2(k+1)} \\ &= \frac{(2k+2)!}{2^{k+1}(k+1)!}. \end{aligned}$$

Therefore, by induction, the proposition holds for all $n \in \mathbb{N}$ ■

- **Induction example 4**.

Proposition. For every $n \in \mathbb{N}$, if any one square is removed from a $2^n \times 2^n$ chessboard, the result can be perfectly covered with L-shaped tiles.

The tiles cover three squares and look like this:

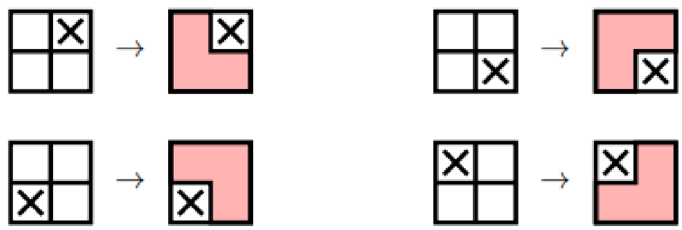


Since the proposition refers to something being true "for every $n \in \mathbb{N}$," that's a pretty good indication that induction is the way to proceed. The base case (when $n = 1$) will be fine. For the inductive hypothesis, we will be assuming that any $2^k \times 2^k$ board, with one square removed, can be perfectly covered by L-shaped tiles.

In the induction step we are going to consider a $2^{k+1} \times 2^{k+1}$ board — a board that is twice as big in each dimension— with one square missing.

Proof. We proceed by induction

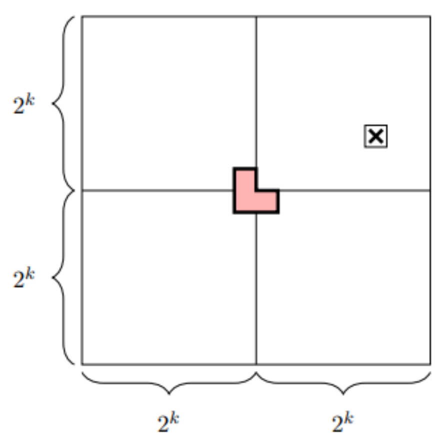
Base Case. The base case is when $n = 1$, and among the four possible squares that one can remove from a 2×2 chessboard, each leaves a chessboard which can be perfectly covered by a single L-shaped tile:



Inductive Hypothesis. Let $k \in \mathbf{N}$, and assume that if any one square is removed from a $2^k \times 2^k$ chessboard, the result can be perfectly covered with L–shaped tiles.

Induction Step. Consider a $2^{k+1} \times 2^{k+1}$ chessboard with any one square removed. Cut this chessboard in half vertically and horizontally to form four $2^k \times 2^k$ chessboards. One of these four will have a square removed, and hence, by the induction hypothesis, can be perfectly covered.

Next, place a single L-shaped tile so that it covers one square from each of the other three 2^k»2^k chessboards, as shown in the picture below.



Each of these other three $2^k \times 2^k$ chessboards can be perfectly covered by the inductive hypothesis, and hence the entire $2^{k+1} \times 2^{k+1}$ chessboard can be perfectly covered.

Conclusion. By induction, for every $n \in \mathbb{N}$, if any one square is removed from a $2^n \times 2^n$ chessboard, the result can be perfectly covered with L-shaped tiles.

- **Another note about induction**: So far, in all of our examples we proved that a statement holds from all $n \in \mathbb{N}$. The base case was $n = 1$ and in the inductive hypothesis we assumed that the result holds for some $k \in \mathbb{N}$.

  There are times where one instead wants to prove that a statement holds for only the natural numbers past some point. For example, it is possible to prove the p-test by induction, a result that you might remember from your calculus class:

  $$\sum_{i=1}^{\infty} \frac{1}{i^n} \text{ converges for all integers } n \geqslant 2.$$

  To prove this result, the base case would be $n = 2$ and in the inductive hypothesis we would assume that the result holds for some $k \in \{2, 3, 4, 5, \ldots\}$.

  At other times, you may want to prove that a result holds for more than just the natural numbers. For example, a result from combinatorics is that

  $$\sum_{i=1}^{n} \binom{n}{i} = 2^n \text{ holds for all integers } n \geqslant 0.$$

  Here, the base case is $n = 0$, and the inductive hypothesis is the assumption that this holds for some $k \in \{0, 1, 2, 3, \ldots\}$.

- **Strong induction idea**: The idea behind strong induction is that at the point when the 100th domino is the next to get knocked down, you know for sure that all of the first 99 dominoes have fallen, not just the 99th. Likewise, when you are proving some sequence of statements $S_1, S_2, S_3, S_4, \ldots$, instead of just assuming that S_k is true in order to prove S_{k+1}, why not just assume that $S_1, S_2, \ldots, S_k$ are all true in order to prove S_{k+1} — because by the time you are proving S_{k+1}, you have shown them all to be true!

- **Strong induction**: Consider a sequence of mathematical statements, $S_1, S_2, S_3, ...$

  - Suppose S_1 is true, and
  - Suppose, for any $k \in \mathbb{N}$, if $S_1, S_2, \ldots, S_k$ are all true, then S_{k+1} is true.

  Then S_n is true for every $n \in \mathbb{N}$.

  Note: In regular induction, you essentially use S_1 to prove S_2, and then S_2 to prove S_3, and then S_3 to prove S_4, and so on. With strong induction, you use S_1 to prove S_2, and then S_1 and S_2 to prove S_3, and then S_1, S_2, and S_3 to prove S_4, and so on.

- **Fundemental theorem of arithmetic**: If n is an integer and $n \geqslant 2$, then n is either prime or composite. An integer p is prime if $p \geqslant 2$ and its only positive divisors are 1 and p. A positive integer $n \geqslant 2$ that is not prime is called composite, and is therefore one that can be written as $n = st$, where s and t are integers smaller than n but larger than 1. And with that, it is time for a really big and important result.

  Theorem 4.8 (Fundamental Theorem of Arithmetic). Every integer $n \geqslant 2$ is either prime or a product of primes.

  Proof. We proceed by strong induction

Base case. The base case occurs when $n = 2$. Observe that $2 \in \mathbb{P}$

Inductive hypothesis. Let $k \in \mathbb{N}$ such that $k \geqslant 2$. Assume that the integers $2, 3, 4, ..., k$ are either prime or a product of primes.

Induction step. Next, we consider $k + 1$. We aim to show that $k + 1$ is either prime or a product of primes. Since $k + 1$ is larger than one, it is either prime or composite. Consider these two cases separately. Case 1 is that $k + 1$ is prime. In this case, our goal is achieved.

Case 2 is that $k+1$ is composite; that is, $k+1$ has positive factors other than one and itself. Say, $k + 1 = st$, where s, t are positive integers greater than zero, and

$$1 < s < k + 1 \quad 1 < t < k + 1.$$

By the inductive hypothesis, both s and t can be written as a product of primes, say

$$s = p_1 \cdot p_2 \cdot ... \cdot p_m$$
$$t = q_1 \cdot q_2 \cdot ... \cdot q_\ell.$$

Where each $p_i, q_j \in \mathbb{P}$, then

$$k + 1 = st = (p_1 \cdot p_2 \cdot ... \cdot p_m)(q_1 \cdot q_2 \cdot ... \cdot q_\ell).$$

Is written as a product of primes

Note that if s or t where prime, then m or ℓ would be one. Say s was prime, then $s = p_1$

Conclusion. By strong induction, every positive integer larger than 2 can be written as a product of primes.

- **Chocolate bar example**:

Proposition. Suppose you have a chocolate bar that is an $m \times n$ grid of squares. The entire bar, or any smaller rectangular piece of that bar, can be broken along the vertical or horizontal lines separating the squares.

The number of breaks to break up that chocolate bar into individual squares is precisely $mn - 1$.

Proof. We proceed by strong induction

Base case: The base case occurs when $n = 1$, which is an 1×1 chocolate bar. Since the number of breaks needed to break the bar into individual squares is clearly zero, we have

$$0 = 1(1) - 1 = 0.$$

As desired

Inductive hypothesis: Let $k \in \mathbb{N}$, assume that all bars with at most k squares satisfy the proposition.

Induction step: Consider now any bar with $k + 1$ squares, suppose this bar has dimensions $m \times n$. Consider an arbitrary first break, and suppose the two smaller bars have a squares and b squares, respectively. Note that we must have $a + b = mn$, because the number of squares in the smaller bars must add up to the number of squares in the original $m \times n$ bar.

By the inductive hypothesis, the bar with a squares will require $a1$ breaks to completely break it up, and the bar with b breaks will require $b1$ breaks. Therefore, to break up the $m \times n$ bar, we must make a first break, followed by $(a1) + (b1)$ additional breaks. The total number of breaks is then

$$\begin{aligned} 1 + (a - 1) + (b - 1) &= a + b - 1 \\ &= mn - 1. \end{aligned}$$

And $mn1$ is indeed one less than the number of squares in the $m \times n$ bar.

Conclusion. By strong induction, a chocolate bar of any size requires one break less than its number of squares to break it up into individual squares ■

Note: What if the pieces were in the shape of a triangle? If it had T squares would it still require $T - 1$ breaks?

What about other shapes? What if there are pieces missing in the middle? Interestingly, the answer is $T - 1$ no matter the bar's shape, and even if pieces are missing! As long as each of your "breaks" divides one chunk into two, that's the answer.

Here is some intuition for that: No matter the shape, the bar starts out as a single "chunk" of chocolate, and after your sequence of breaks the bar is broken into T chunks of chocolate — the T individual squares. How many breaks does it take to move from 1 chunk to T chunks? Notice that every break increases the number of chunks by 1. So after 1 break, there will be 2 chunks. After 2 breaks, there will be 3 chunks. And so on. Thus, after $T - 1$ breaks there will be T chunks, which is why $T - 1$ breaks is guaranteed to be the answer, no matter which shape you started with.

- **Multiple base cases**: When proving the $(k + 1)$st case within the induction step, strong induction allows you to apply not just the kth step, but any of the steps $1, 2, 3, \ldots, k$. In the previous two examples, you had no idea which earlier steps you will need, so it was vital that you assumed them all. At times, though, you really only need, say, the previous two steps. The kth step is perhaps not enough, but the $(k - 1)$st step and the kth step is guaranteed to be enough.

If you rely on the two previous steps, then that is analogous to saying that it takes the previous two dominoes to knock over the next one. Thus, if you knock over dominoes 1 and 2, then they will collectively knock over the third. Then, since the second and third have fallen, those two will collectively knock over the fourth. Then the third and fourth will knock over the fifth. And so on. Thus, the induction relies on two base cases, because without knocking over the first two the third won't fall and the process won't begin

Example:

Proposition. Every $n \in N$ with $n \geqslant 11$ can be written as $2a + 5b$ for some natural numbers a and b.

Base Cases. In the induction step, we will need two cases prior, so we show two base cases here: $n = 11$ and $n = 12$. Both of these can be written as asserted:

$$11 = 2 \cdot 3 + 5 \cdot 112 = 2 \cdot 1 + 5 \cdot 2.$$

Inductive Hypothesis. Assume that for some integer $k \geqslant 12$, the results hold for

$$n = 11, 12, 13, \ldots, k.$$

Induction Step. We aim to prove the result for $k + 1$. By the inductive hypothesis,

$$k - 1 = 2a + 5b$$

for some $a, b \in \mathbb{N}$. Adding 2 to both sides,

$$k + 1 = 2(a + 1) + 5b.$$

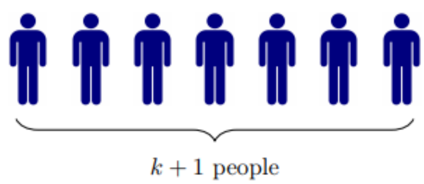
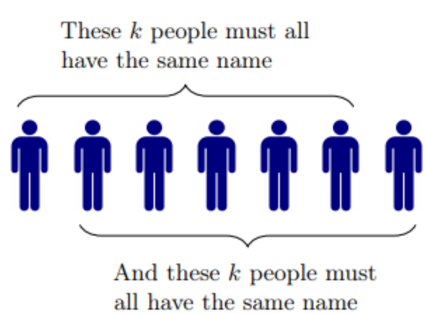
Observe that $(a + 1) \in \mathbb{N}$ and $b \in \mathbb{N}$, proving that this is indeed a representation of $(k + 1)$ in the desired form.

Conclusion. Therefore, by strong induction, every integer $n \geqslant 11$ can be written as the proposition asserts. ∎

- **False proofs with induction**:

Proposition. Everyone on Earth has the same name

Fake Proof. We will consider groups of n people at a time, and by induction we will "prove" that for every $n \in \mathbb{N}$, every group of n people must have everyone with the same name.

> **Base Case.** If $n = 1$, then of course everyone in the group has the same name, since there's only one person in the group!
>
> **Inductive Hypothesis.** Let $k \in \mathbb{N}$, and assume that any group of k people all have the same name.
>
> **Induction Step.** Consider a group of $k + 1$ people.
>
> 
>
>
> But notice that we can look at the first k of these people and then the last k of these people, and to each of these groups we can apply the inductive hypothesis:
>
> 
>
>
> And the only way that this can all happen, is if all $k + 1$ people have the same name.

Conclusion. This "proves" by induction that for every $n \in N$, every group of n people must have the same name. So if you let n be equal to the number of people on Earth, this "proves" that everyone has the same name.

For $k + 1$ people, the proof assumes that you can take the first k people and the last k people, and both of these subsets must have the same name because the induction hypothesis applies to them individually.

However, this reasoning fails when $k+1=2$. For $k+1=2$, the first subset has one person, and the second subset also has one person. These subsets do not overlap, so there is no logical connection ensuring that these two people share the same name.

The induction relies on overlapping subsets of k people to conclude that all $k+1$ people must have the same name. However, this overlap only works if $k+1>2$, meaning the proof doesn't actually establish the result for $k+1=2$, which breaks the induction chain. Without the foundation for $n=2$, the argument fails for all larger n.

- **Induction bonus example 1**.

  Lemma 4.13. For every $n \in \mathbb{N}_0$,

  $$1+2+4+8+...+2^n = 2^{n+1}-1.$$

  For example,

  $$\begin{aligned} 1 &= 2^1-1 \\ 1+2 &= 2^2-1 \\ 1+2+4 &= 2^3-1 \\ 1+2+4+8 &= 2^4-1. \end{aligned}$$

  Base case. The base case occurs when $n=1$, we have

  $$1 = 2^1-1 = 1.$$

  As desired

  Inductive hypothesis. Let $k \in \mathbb{N}_0$, assume that

  $$1+2+4+...+2^k = 2^{k+1}-1.$$

  Induction step. We wish to show that the result holds for $k+1$. That is,

  $$1+2+4+...+2^k+2^{k+1} = 2^{(k+1)+1}-1 = 2^{k+2}-1.$$

  By the inductive hypothesis, we have

  $$\begin{aligned} 1+2+4+...+2^k+2^{k+1} &= 2^{k+1}-1+2^{k+1} \\ &= 2(2^{k+1})-1 \\ &= 2^{k+2}-1. \end{aligned}$$

  As desired

  Therefore, by induction, the proposition holds for all $n \in \mathbb{N}_0$

- **Induction bonus example 2**. **Proof.** We proceed by strong induction. **Base Case.** Our base case is when $n=1$. Note that 1 can be written as 2^0, and this is the only way to write 1 as a sum of distinct powers of 2, because all other powers of 2 are larger than 1.

Inductive Hypothesis. Let $k \in \mathbb{N}$, and assume that each of the integers $1, 2, 3, \ldots, k$ can be expressed as a sum of distinct powers of 2 in precisely one way.

Induction Step. We now aim to show that $k + 1$ can be expressed as a sum of distinct powers of 2 in precisely one way.

Let 2^m be the largest power of 2 such that $2^m \leqslant k + 1$. We now consider two cases: the first is if $2^m = k + 1$, and the second is if $2^m < k + 1$.

Case 1: $2^m = k + 1$. If this occurs, then 2^m itself is a way to express $k + 1$ as a (one-term) sum of distinct powers of 2. Moreover, there is no other way to express $k + 1$ as a sum of distinct powers of 2, because by Lemma 4.13 all smaller powers of 2 sum to $2^m - 1 = k$. Thus, even by including all smaller powers of 2, we are unable to reach $k + 1$. So, in Case 1, there is precisely one such expression for $k + 1$.

Case 2: $2^m < k + 1$. In order to apply the inductive hypothesis, we will consider $(k + 1) - 2^m$. First, note that $(k + 1) - 2^m$ is less than 2^m, because otherwise $k + 1$ would have two copies of 2^m within it, implying that $2^m + 2^m \leqslant k + 1$. However, since $2^m + 2^m = 2 \cdot 2^m = 2^{m+1}$, this would mean $2^{m+1} \leqslant k + 1$. This can't be, since 2^m was chosen to be the largest power of 2 that is at most $k + 1$. Thus, it must be the case that $(k + 1) - 2^m < 2^m$.

Next, by the inductive hypothesis, $(k + 1) - 2^m$ can be expressed as a sum of distinct powers of 2 in precisely one way, and since $(k + 1) - 2^m < 2^m$, this unique expression for $(k + 1) - 2^m$ will not contain a 2^m. Thus, by adding a 2^m to it, we obtain an expression for $k + 1$ as a sum of powers of 2. And this expression is unique because $(k + 1) - 2^m$ is unique according to the inductive hypothesis, and the 2^m portion is unique because, again by Lemma 4.13, even if you summed all of the smaller powers of 2, you will not reach 2^m.

Conclusion. By strong induction, every $n \in \mathbb{N}$ can be expressed as a sum of distinct powers of 2 in precisely one way. □

- **Induction bonus example 3.**

Theorem 4.15 (*The binomial theorem*). For $x, y \in \mathbb{R}$, and $n \in \mathbb{N}_0$

$$(x + y)^n = \sum_{m=0}^{n} \binom{n}{m} x^{n-m} y^m.$$

Here, when $n \geqslant m$, the binomial coefficient $\binom{n}{m}$ is defined to be

$$\binom{n}{m} = \frac{n!}{m!(n - m)!},$$

which one can show is always an integer. The binomial coefficients can also be defined combinatorially: $\binom{n}{m}$ is equal to the number of ways to choose m elements from an n-element set; in fact, $\binom{n}{m}$ is read "n choose m." For example,

$$\binom{4}{2} = 6$$

because there are six subsets of the set $\{1, 2, 3, 4\}$ containing two elements:

$$\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}.$$

Binomial coefficients can be computed iteratively using *Pascal's rule*, which says that

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r},$$

as well as the fact that

$$\binom{n}{0} = 1 \quad \text{and} \quad \binom{n}{n} = 1 \quad \text{for all } n \in \mathbb{N}_0.$$

A beautiful way to combine these facts is called *Pascal's triangle*:

$$\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
\quad = \quad
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Indeed, we can even prove the binomial theorem by induction, by making use of Pascal's rule. Here is a sketch of that proof:

Proof sketch. The base case is when $n = 0$, and indeed $(x + y)^0 = 1$. The next couple cases are more interesting, and you can check that $(x + y)^1 = x + y$ and $(x+y)^2 = x^2 + 2xy + y^2$ do indeed match the theorem. The inductive hypothesis will be

$$(x + y)^k = x^k + \binom{k}{1} x^{k-1} y + \binom{k}{2} x^{k-2} y^2 + \cdots + \binom{k}{k-1} x y^{k-1} + y^k.$$

For the induction step, we perform easy algebra, then apply the inductive hypothesis, then perform hard algebra, then apply Pascal's rule:

$$(x + y)^{k+1} = (x + y)(x + y)^k$$

$$= (x + y) \left[x^k + \binom{k}{1} x^{k-1} y + \binom{k}{2} x^{k-2} y^2 + \cdots + \binom{k}{k-1} x y^{k-1} + y^k \right]$$

$$= x^{k+1} + \left[\binom{k}{0} \right] x^k y + \left[\binom{k}{1} \right] x^{k-1} y^2 + \cdots + \left[\binom{k}{k} \right] x y^k + y^{k+1}$$

$$= x^{k+1} + \binom{k+1}{1} x^k y + \binom{k+1}{2} x^{k-1} y^2 + \cdots + \binom{k+1}{k} x y^k + y^{k+1}.$$

And that—a few boring algebraic details omitted—is the proof.

The binomial theorem tells us that in order to expand $(x+y)^5$ you can just look at the 5th row of Pascal's triangle (where the top element counts as the 0th row, so the 5th row is 1 5 10 10 5 1):

$$(x+y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5.$$

Moreover, by plugging in special values for x and y, all sorts of neat identities pop out. There are loads of examples of this, but here are just three:

- By plugging in $x = 1$, $y = 1$, we prove $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.
- By plugging in $x = 2$, $y = 1$, we prove $3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k$.
- By plugging in $x = -1$, $y = 1$, we prove $0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$.

1.5 Logic

- **Statements**: A statement is a sentence or mathematical expression that is either true or false. If the logic is valid and the statements are true, then it is called sound

  Every theorem/proposition/lemma/corollary is a (true) statement; Every conjecture is a statement (of unknown truth value); and Every incorrect calculation is a (false) statement.

- **Open sentence**: A related notion is that of an *open sentence*, which refers to sentences or mathematical expressions that:

  1. do not have a truth value,
  2. depend on some unknown, like a variable x or an arbitrary function f, and
  3. when the unknown is specified, the open sentence becomes a statement (and thus has a truth value).

  Their truth value depends on the specific value of x or f that is chosen.

  Typically, we use capital letters for statements, like P, Q and R. Open sentences are often written the same, or perhaps like $P(x)$, $Q(x)$ or $R(x)$ when one wishes to emphasize the variabl

- **And, or, not**: Let P and Q be statements or open sentences.

  1. $P \wedge Q$ means "P and Q".
  2. $P \vee Q$ means "P or Q (or both)".
  3. $\sim P$ means "not P".

- **Implies, iff**: Let P and Q be statements or open sentences.

  1. $P \implies Q$ means "P implies Q".
  2. $P \iff Q$ means "P if and only if Q".

  Let's now discuss a subtle aspect of implications: Translating them to and from English. Language can be complicated,[3] and we in fact have many different ways in English to say "P implies Q." Here are some examples:

  - If P, then Q
  - Q if P
  - P only if Q
  - Q whenever P
  - Q, provided that P
  - Whenever P, then also Q
  - P is a sufficient condition for Q
  - For Q, it is sufficient that P
  - For P, it is necessary that Q

  For example, "If it is raining, then the grass is wet" has the same meaning as "The grass is wet if it is raining." These also mean the same as "The grass is wet whenever it is raining" or "For the grass to be wet, it is sufficient that it is raining."

[3] Language nuances can make logical translation challenging.

Next, here are some ways to say "P if and only if Q":

- P is a necessary and sufficient condition for Q.
- For P, it is necessary and sufficient that Q.
- P is equivalent to Q.
- If P, then Q, and conversely.
- P implies Q and Q implies P.
- Shorthand: P iff Q.
- Symbolically: $(P \implies Q) \wedge (Q \implies P)$.

The fact that "P implies Q" is the same as "If P, then Q" or "Q if P" is sometimes intuitive to students. But the fact that these are all the same as "P only if Q" is often confusing. Most people's guts tell them that "P implies Q" should be the same as "Q only if P."

The answer is "P only if Q", and the way to think about it is that "P implies Q" means that whenever P is true, Q must also be true. And "P only if Q" means that P can only be true if Q is true...that is, whenever P is true, it must be the case that Q is also true...that is, $P \implies Q$.

- **Conditional, biconditional statements**: Now, if P and Q are statements, then "$P \implies Q$" and "$P \iff Q$" are also statements, meaning they must also be either true or false. The statement $P \implies Q$ is called a conditional statement, whereas $P \iff Q$ is called a biconditional statement. These are minor definitions, but the following is an important definition.

- **Converse**: The *converse* of $P \implies Q$ is $Q \implies P$

  Note: If $P \implies Q$, it is not necessarily the case that $Q \implies P$

- **Truth tables for and, or, and not**: A truth table models the relationship between the truth values of one or more statements, and that of another

P	Q	$P \wedge Q$
True	True	True
True	False	False
False	True	False
False	False	False

For for "P and Q" to be a true statement, both P and Q must be independently true

Here's how the truth values for P and for Q affect the truth value for $P \vee Q$.

P	Q	$P \vee Q$
True	True	True
True	False	True
False	True	True
False	False	False

It is sufficient that either P is true or that Q is true (or both).

Finally, here is how the truth values for P affects that of $\neg P$.

P	$\neg P$
True	False
False	True

In order for "not P" to be true, it is required that P be false. By applying this reasoning twice, this also implies that $\sim\sim P$ and P always have the same truth value.

One last example shows how we proceed with more complicated statements

P	Q	$P \vee Q$	$P \wedge Q$	$\neg(P \wedge Q)$	$(P \vee Q) \wedge \neg(P \wedge Q)$
True	True	True	True	False	False
True	False	True	False	True	True
False	True	True	False	True	True
False	False	False	False	True	False

- **De Morgan's Logic Laws**: Take a loot at the truth tables for $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$, side by side:

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$
True	True	True	False
True	False	False	True
False	True	False	True
False	False	False	True

P	Q	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$
True	True	False	False	False
True	False	False	True	True
False	True	True	False	True
False	False	True	True	True

Since the final columns are the same, if one is true, the other is true; if one is false, the other is false; that is, there is no way to select P and Q without these two agreeing. When two statements have the same final column in their truth tables, like in the example above, they are said to be logically equivalent (one is true if and only if the other is true), which we denote with an " $\iff$ " symbol. De Morgan's logic law, for example, can be written like this:

$$\neg(P \wedge Q) \iff (\neg P \vee \neg Q)$$

"P and Q are not both true" is the same as "P is false or Q is false."

Theorem: If P and Q are statements, then

$$\neg(P \wedge Q) \iff \neg P \vee \neg Q \quad \text{and} \quad \neg(P \vee Q) \iff \neg P \wedge \neg Q.$$

- ***P*, *Q*, and their names**: In logical statements involving P and Q, the terms P and Q are referred to as propositions or statements. Depending on the logical operator used, they may also have more specific names:
  1. **In a conjunction ($P \wedge Q$):**
     - P and Q are called **conjuncts**.
  2. **In a disjunction ($P \vee Q$):**
     - P and Q are called **disjuncts**.
  3. **In an implication ($P \implies Q$):**
     - P is called the **antecedent** (or **hypothesis**, **premise**).

- Q is called the **consequent** (or **conclusion**).

4. **In a biconditional ($P \iff Q$):**
   - P and Q are called **equivalents** (since $P \iff Q$ means P and Q are logically equivalent).
5. **In negation ($\neg P$):**
   - P is simply the proposition being negated.

- **Implications**: We call the conditional statemests, $P \implies Q$ *implications*. They are called implications because they express a logical relationship where one statement (the premise, P) "implies" or leads to another statement (the conclusion, Q). The word "implication" comes from the Latin root *implicare*, meaning "to entwine" or "to involve," reflecting the idea that P is connected to Q.

  A biconditional statement combines two implications, $P \implies Q$ AND $Q \implies P$

- **Truth Tables with Implications**: Consider the truth table for the implication $P \implies Q$

P	Q	$P \implies Q$
True	True	True
True	False	False
False	True	True
False	False	True

The results of the first two rows are trivial, but the last two may be hard to grasp.

Why is the implication true if the assumption, P, is false? It's kind of like how we said that this is true: "If $x \in \varnothing$, then x is a purple elephant that speaks German." Since there is nothing in the empty set, if you suppose $x \in \varnothing$, you can then claim anything you want about x and it is inherently true — you certainly cannot present to me any element in the empty set that is not a purple elephant that speaks German. In the set theory chapter, we called such a claim *vacuously true*.

Likewise, in a universe where P is true, the statement $P \implies Q$ has some real meaning that needs to be proven or disproven: Does P being true imply Q is true, or not? But in a universe where P is not true, it claims nothing, and hence $P \implies Q$ is *vacuously true*.

"If unicorns exist, then they can fly" can certainly not be considered false, because unicorns do not exist, so any claim about them is considered vacuously true. Indeed, the way to falsify that proposition would be to locate a unicorn that cannot fly, which is impossible to do. Every unicorn in existence can indeed fly! Also, every unicorn in existence cannot fly! Neither can be disproven!

Let's now consider the truth table for the statement $P \iff Q$

P	Q	$P \iff Q$
True	True	True
True	False	False
False	True	False
False	False	True

We can see this by writing $P \iff Q$ as $(P \implies Q) \wedge (Q \implies P)$

- **Quantifiers**: Consider the sentence

$$n \text{ is even}$$

Which is not a statement because it is neither true nor false. One way to turn a sentence like this into a statement is to give n a value. For example,

$$\text{If } n = 5, \text{ then } n \text{ is even}$$

What I'd like to discuss now are two other basic ways to turn "n is even" into a statement: add quantifiers. A quantifier is an expression which indicates the number (or quantity) of our objects

$$\forall\ n \in \mathbb{N},\ n \text{ is even}$$
$$\exists\ n \in \mathbb{N} \text{ such that } n \text{ is even}$$

Where $\forall$ means "for all", and $\exists$ means "there exists". The symbol $\forall$ is known as the *universal quantifer*. Whereas $\exists$ is known as the *existential quantifier.*

Note: We also have $\nexists$ "there does not exist", and $\exists!$ "there exists a unique"

- **Rules of negating**: We have the following rules for negating statements
  - $\neg\wedge = \vee$
  - $\neg\vee = \wedge$
  - $\neg\forall = \exists$
  - $\neg\exists = \forall$

Consider the statement, R : for every real number x, there is some real number y such that $y^3 = x$. Symbolically, we have

$$\forall\ x \in \mathbb{R},\ \exists\ y \in \mathbb{R} \text{ such that } y^3 = x.$$

Then,

$$\neg(\forall\ x \in \mathbb{R},\ \exists\ y \in \mathbb{R} \text{ such that } y^3 = x).$$

Is equivalent to the statement

$$\exists\ x \in \mathbb{R}, \text{ such that } \forall\ y \in \mathbb{R},\ y^3 \neq x.$$

- **Negations with implications**: First, recall the truth table for $P \implies Q$

P	Q	$P \implies Q$
True	True	True
True	False	False
False	True	True
False	False	True

The only way for $P \implies Q$ to be false is for both P to be true and for Q to be false. This shows that

$$\neg(P \implies Q) \Leftrightarrow P \wedge \neg Q.$$

Consider the statement

$$S: \ \forall\, n \in \mathbb{N}, (3 \mid n) \implies (6 \mid n).$$

Then,

$$\begin{aligned}\neg S : &\neg(\forall\, n \in \mathbb{N}, (3 \mid n) \implies (6 \mid n)) \\ &\Leftrightarrow \exists\, n \in \mathbb{N} \text{ such that } (3 \mid n) \wedge (6 \nmid n).\end{aligned}$$

- **The contrapositive (and the inverse)**: The *contrapositive* of $P \implies Q$ is $\neg Q \implies \neg P$

  Note: The *inverse* of $P \implies Q$ is $\neg P \implies \neg Q$

  Theorem: An implication is logically equivalent to its contrapositive. That is,

  $$P \implies Q \Leftrightarrow \neg Q \implies \neg P.$$

  The truth table easily verifys this

- **Proving quantified statements: Existential proofs**: To prove an existence statement, it suffices to exhibit an example satisfying the criteria. The above strategy is called a constructive proof — you literally construct an example. There are also non-constructive ways to prove something exists. Often (but not always!) non-constructive proofs make use of some other theorem.

- **Proving quantified statements: Universal proofs**: To prove a universal statement, it suffices to choose an arbitrary case and prove it works there. We have seen several examples of this. For example, if you were asked to prove that "For every odd number n, it follows that $n + 1$ is even," your proof wouldn't explicitly check 1 and 3 and 5 and so on. Rather, you would say "Since n is odd, $n = 2a + 1$ for some $a \in \mathbb{Z}$." Then you would note that

  $$n + 1 = (2a + 1) + 1 = 2(a + 1)$$

  is even. The point here is that by letting $n = 2a + 1$, you were essentially selecting an arbitrary odd number, and operating on that. Every odd number can be written in that form, and every odd number can have 1 added to it and then factored like we did. Since our n was completely arbitrary, everything we did could be applied to any particular odd number. Proving something holds for an arbitrary element of a set, proves that it in turn holds for every element in that set.

- **Proving biconditional statements**: In order to prove a statement in the form $P \implies Q$, we must prove both directions. That is, $P \implies Q$ and $Q \implies P$

1.6 Proof using the contrapositive

- **Proof outline**:

  Proposition. $P \implies Q$

  Proof. We will use the contrapositive. Assume not-Q

  $\langle\langle$ An explanation of what not-Q means $\rangle\rangle$, use definitions, and/or other results

  $$\vdots \quad \text{Apply algebra,}$$

  $$\vdots \quad \text{logic, techniques.}$$

  $\langle\langle$ Hey look, that's what not-P means $\rangle\rangle$

  Therefore not-P

  Since not-$Q \implies$ not-P, by the contrapositive $P \implies Q$ ■

- **Contrapositive proof 1.**

  Proposition. Suppose $n \in \mathbb{N}$, if n^2 is odd, then n is odd.

  Proof. We will use the contrapositive. The statement, $\forall\ n \in \mathbb{N},\ n^2 = 2k+1 \implies n = 2\ell+1,\ k,\ell \in \mathbb{Z}$ has the logically equivalent contrapositive $\forall n \in \mathbb{N},\ n \neq 2\ell+1 \implies n^2 \neq 2k+1$. Since $n \in \mathbb{N}$, if n, n^2 is not odd, then it must be even. Thus, the statement becomes $\forall n \in \mathbb{N},\ n = 2\ell \implies n^2 = 2k,\ k,\ell \in \mathbb{N}$ which becomes much easier to proof. For some extra practice negating statements, here is the negation

  $$\begin{aligned}&\neg(\forall n \in \mathbb{N},\ n^2 = 2k+1 \implies n = 2\ell+1,\ k,\ell \in \mathbb{N})\\ &= \exists n \in \mathbb{N} \text{ such that } n^2 = 2k+1 \wedge n \neq 2\ell+1.\end{aligned}$$

  Recall $\neg(P \implies Q) = P \wedge \neg Q$

  Assume $n \in \mathbb{N}$, and that n is even. Since n is even, it must be that $n = 2\ell$, for some integer ℓ. Squaring both sides, we get

  $$\begin{aligned}n^2 &= (2\ell)^2\\ &= 4\ell^2 = 2(2\ell^2).\end{aligned}$$

  Since $\ell \in \mathbb{Z}$, we know $2\ell^2 \in \mathbb{Z}$, and thus n^2 is even.

  Therefore, since n not being odd implies n^2 is also not odd, we have shown by the contrapositive that if n^2 is odd, n is also odd ■

- **Contrapositive proof 2.**

  Proposition. Suppose $n \in N$. Then, n is odd if and only if $3n+5$ is even

Proof. We will prove this in two parts

Part 1: If n is odd then $3n+5$ is even. Assume $n \in \mathbb{N}$ is odd, then $n = 2k+1$, for $k \in \mathbb{N}_0$. Thus,

$$\begin{aligned} 3n+5 &= 3(2k+1)+5 \\ &= 6k+3+5 = 6k+8 \\ &= 2(3k+4). \end{aligned}$$

Thus even.

Part 2: $3n+5$ being even implies n is odd. We prove this by use of the contrapositive. The given statement has the following contrapositive...

$$n = 2k \implies 3n+5 = 2\ell+1, \ k, \ell \in \mathbb{N}_0.$$

Thus,

$$\begin{aligned} 3n+5 &= 3(2k)+5 \\ &= 6k+5 = 6k+4+1 \\ &= 2(3k+2)+1. \end{aligned}$$

Thus odd.

Since $P \implies Q$, and $Q \implies P$, it must be that $P \iff Q$ is true. Thus, we assert for $n \in \mathbb{N}$, n is odd if and only if $3n+5$ is even.

- **Contrapositive proof 3.**:

  Proposition. Let $a, b \in \mathbb{Z}$, and $p \in \mathbb{P}$. If $p \nmid ab$, then $p \nmid a$ and $p \nmid b$ **Proof.** Suppose $a, b \in \mathbb{Z}$ and p is a prime. We will use the contrapositive. Suppose that it is not true that $p \nmid a$ and $p \nmid b$. By the logic form of De Morgan's law (Theorem 5.9), this is equivalent to saying it is not true that $p \nmid a$ *or* it is not true that $p \nmid b$. That is, $p \mid a$ *or* $p \mid b$. Let's consider these two cases separately.

Case 1. Suppose $p \mid a$, which by the definition of divisibility (Definition 2.8) means that $a = pk$ for some $k \in \mathbb{Z}$. Thus,

$$ab = (pk)b = p(kb).$$

Since $k, b \in \mathbb{Z}$, also $(kb) \in \mathbb{Z}$. And so, by the definition of divisibility (Definition 2.8), $p \mid ab$.

Case 2. Suppose $p \mid b$, which by the definition of divisibility (Definition 2.8) means that $b = p\ell$ for some $\ell \in \mathbb{Z}$. Thus,

$$ab = a(p\ell) = b(a\ell).$$

Since $a, \ell \in \mathbb{Z}$, also $(a\ell) \in \mathbb{Z}$. And so, by the definition of divisibility (Definition 2.8), $p \mid ab$.

In either case, we concluded that $p \mid ab$, which is equivalent to saying that it is not true that $p \nmid ab$.

We proved that if it is not true that $p \nmid a$ and $p \nmid b$, then it is not true that $p \nmid ab$. Hence, by the contrapositive, this implies that if $p \mid ab$, then $p \mid a$ and $p \mid b$. $\square$

Note: Mathematicians have agreed that we should be allowed to skip essentially-identical cases

If you have two cases, like $p \mid a$ and $p \mid b$, and there is literally no mathematical distinction between them, then you are allowed to say "without loss of generality, assume $p \mid a$." This allows you to skip the "$p \mid b$" case entirely.

Condensed, Elder-Approved Proof. Suppose $a, b \in \mathbb{Z}$ and p is a prime. We will use the contrapositive. Suppose that it is not true that $p \nmid a$ and $p \nmid b$. By the logic form of De Morgan's law (Theorem 5.9), this is equivalent to saying it is not true that $p \nmid a$ *or* it is not true that $p \nmid b$. That is, $p \mid a$ *or* $p \mid b$. Without loss of generality, assume $p \mid a$.

By the definition of divisibility (Definition 2.8), this means that $a = pk$ for some $k \in \mathbb{Z}$. Thus,

$$ab = (pk)b = p(kb).$$

Since $k, b \in \mathbb{Z}$, also $(kb) \in \mathbb{Z}$. And so, by the definition of divisibility (Definition 2.8), $p \mid ab$.

We proved that if it is not true that $p \nmid a$ and $p \nmid b$, then it is not true that $p \nmid ab$. Hence, by the contrapositive, this implies that if $p \mid ab$, then $p \mid a$ and $p \mid b$. $\square$

- **Contrapositive proof 4.**

  Proposition. Let $a, b, n \in \mathbb{N}$. If $36a \not\equiv 36b \pmod{n}$, then $n \nmid 36$

  Proof idea. The fact that this proposition says a lot of things are not happening is one indication that the contrapositive could be worthwhile. The contrapositive states For $a, b, n \in \mathbb{N}$, If $n \mid 36$, then $36a \equiv 36b \pmod{n}$

Proof. Assume $a, b, n \in \mathbb{N}$, and $n \mid 36$. In this case, we have $36 = nk$, for $k \in \mathbb{Z}$. We require $36a - 36b = n\ell$, for $\ell \in \mathbb{Z}$. We then examine the quantity $36a - 36b$. Since $36 = nk$, we have

$$\begin{aligned} 36a - 36b &= nka - nkb \\ &= n(ka - kb). \end{aligned}$$

Which is precisely the definition of divisibility, since it is clear that $ka - kb \in \mathbb{Z}$. Thus, we have $n \mid 36a - 36b$, and by the definition of modular congruence $36a \equiv 36b \pmod{n}$.

Therefore, by the contrapositive, $36a \not\equiv 36b \pmod{n}$ implies that $n \nmid 36$ ■

- **Lemma 6.6** This lemma has two parts

  (i) If $m \in \mathbb{Z}$, then $m^2 + m$ is even

  (ii) If $a \in \mathbb{Z}$, and a^2 is even, then a is even

  This proof is trivial and will not be shown. Proving i is simply a proof by cases. To prove ii, we can use the contrapositive, instead proving that if a is odd, then a^2 is odd. Which, by the contrapositive shows that if a^2 is even, then a must also be even.

- **Contrapositive proof 5.**

  Proposition. If a is an odd integer, then $x^2 + x - a^2 = 0$ has no integer solution.

  Proof idea. We will use the contrapositive, which states if $x^2 + x - a^2 = 0$ has an integer solution, then a is even.

  Note: Negating Q in this case ($x^2 + x - a^2 = 0$ has no integer solution) does not given $x^2 + x - a^2 \neq 0$... It is important to question what it means for the given statement to be false in order to properly negate. The negation of the statement is "it is false that $x^2 + x - a^2 = 0$ has no integer solutions", which must mean that some integer m exists such that $m^2 + m - a^2 = 0$.

  Proof. Suppose that a is an odd integer. We will use the contrapositive. Assume that it is false that $x^2 + x - a^2 = 0$ has no integer solutions; that is, assume that there is some integer m such that

$$m^2 + m - a^2 = 0.$$

  By the quadratic formula[9] and then some algebra,

$$\begin{aligned} m &= \frac{-1 \pm \sqrt{1^2 - 4(1)(-a^2)}}{2(1)} \\ m &= \frac{-1 \pm \sqrt{1 + 4a^2}}{2} \\ 2m &= -1 \pm \sqrt{1 + 4a^2} \\ 2m + 1 &= \pm\sqrt{1 + 4a^2} \\ 4m^2 + 4m + 1 &= 1 + 4a^2 \\ m^2 + m &= a^2.. \end{aligned}$$

Next, observe that $m^2 + m$ is guaranteed to be even, by Lemma 6.6 part (i). Thus, since we just deduced that $m^2 + m = a^2$, this means that a^2 must be even. And since a is an integer, a^2 being even implies that a is even, by Lemma 6.6 part (ii). In particular, this means that a is not odd.

We have shown that if it is false that $x^2 + x - a^2 = 0$ has no integer solutions, then it is also false that a is an odd integer. By the contrapositive, if a is an odd integer, then $x^2 + x - a^2 = 0$ has no integer solution. $\square$

1.7 Contradiction

- **The idea**: The big idea is this: If you start with something true and apply correct logic to it, you will never arrive at something false. So it can't be true that Carmen stole the bag, if that would imply the falsity that she can be in two places at once. Indeed, if your assumptions imply something false, then something you assumed had to be false as well.

  Suppose we had a theorem $P \implies Q$. Throughout the problem, we assume P to be true. The goal is to show that Q is also true. By the truth tables, either Q is true or $\neg Q$ is true, not both. This gives two options.

  1. P is true and Q is true $(P \wedge Q)$
  2. P is true and $\neg Q$ is true $(P \wedge \neg Q)$

  If $P \wedge \neg Q$ implies anything false, that can't be the correct option. That is, it must be $P \wedge Q$. Thus, we have shown $P \implies Q$

  Notice that the only way that $P \implies Q$ can be false is if P is true and Q is false.

P	Q	$P \implies Q$
True	True	True
True	False	False
False	True	True
False	False	True

  Thus, this is the only case we have to rule out in order to prove our theorem: that $P \implies Q$ is true. So, if you assume that P is true and Q is false, and manage to use that to deduce a contradiction, then you will have ruled out the one and only bad case, which in turn means that the theorem must be true!

  In other words, if $P \wedge \neg Q$ cannot be, then it must be that $P \implies Q$

- **Contradiction example 1.**

  Proposition. There does not exist a largest natural number

  Proof Idea. One quick note: This proposition is not phrased explicitly as "$P \implies Q$," but you are probably starting to see how to rephrase propositions in this form. For example, this proposition could instead be stated as: "If N is the set of natural numbers, then N does not have a largest element." Or, equivalently: "If N is larger than every natural number, then $N \notin \mathbb{N}.''Or, equivalently: ``IfN$ is a natural number, then there exists a natural number larger than $N.$"

  For our proof by contradiction, we will assume that there *is* a largest natural number, and then deduce a contradiction. There are several ways to do this, but one way is to assume that N is the largest and then show that $N + 1$ must be larger—if it weren't, we could deduce that $0 \geqslant 1$, which is clearly a contradiction. Here's that:

  Proof. Assume for a contradiction that there is a largest element of $\mathbb{N}$, and call this number N. Being larger than every other natural number, N has the property that $N \geqslant m$ for all $m \in \mathbb{N}$.

Observe that since $N \in \mathbb{N}$, also $(N+1) \in \mathbb{N}$. And so, by assumption,

$$N \geqslant N+1.$$

Subtracting N from both sides,

$$0 \geqslant 1.$$

This is a contradiction[1] since we know that $0 < 1$, and therefore there must not be a largest element of $\mathbb{N}$. $\square$

- **Contradiction example 2.**

  Proposition. There does not exist a smallest positive rational number.

  Proof. Assume for the sake of contradiction that there does exist a smallest positive rational number. Call this number q. Since $q \in \mathbb{Q}$, we have

  $$q = \frac{a}{b}.$$

  Where $a, b \in \mathbb{Z}$, and $a, b > 0$. Since q is the smallest, than for all $r \in \mathbb{Q}$, we have $q \leqslant r$. Let $r = \frac{a}{2b}$. Then,

  $$\begin{aligned} \frac{a}{b} &\leqslant \frac{a}{2b} \\ \implies 2ab &\leqslant ab \\ \implies 2 &\leqslant 1. \end{aligned}$$

  This is a contradiction, since we know $2 > 1$. It must be that there is no smallest positive rational number.

- **Proof by contradiction general form**:

  Proposition. $P \implies Q$

  Proof. Assume for the sake of contradiction P and $\neg Q$

  $\langle\langle$ An explanation of what these mean $\rangle\rangle$

  $\vdots$ Apply algebra,

  $\vdots$ logic, techniques.

  $\langle\langle$ Hey look, that contradicts something we know to be true $\rangle\rangle$

  We obtained a contradiction, therefore $P \implies Q$ $\blacksquare$

- **Proof by contradiction example 3.**

  Proposition. If A, B are sets, then $A \cap (B \setminus A) = \varnothing$

Proof. Assume for the sake of contradiction, that $A \cap (B \setminus A) \neq \varnothing$

> Since $A \cap (B \setminus A) \neq \varnothing$, then $\exists x \in A \cap (B \setminus A)$. Thus, $x \in A \wedge x \in (B \setminus A)$. Rewrite $B \setminus A$ as $B \cap A^C$. Thus, $x \in B \ \wedge \ x \in A^C$. Since $x \in A^C$, it must be that $x \notin A$. Thus, we have $x \in A$, $x \in B$, and $x \notin A$

Therefore, since $x \in A$ and $x \notin A$ is a contradiction, it must be that if A, and B are sets, then $A \cap (B \setminus A) = \varnothing$ ■.

- **Proof by contradiction example 4.**

Proposition. There does not exists integers m, n such that $15m + 35n = 1$

Proof. Assume for the sake of contradiction there does exist integers m, n such that $15m + 35n = 1$, since $m, n \in \mathbb{Z}$, $3m + 7n \in \mathbb{Z}$, but

$$\begin{aligned} 15m + 35n &= 1 \\ \implies 3m + 7n &= \frac{1}{5}. \end{aligned}$$

Since $3m + 7n \notin \mathbb{Z}$, we have a contradiction. Thus, it must be that there does not exist integers m, n such that $15m + 35n = 1$.

Alternatively, we could have done

$$\begin{aligned} 15m + 35n &= 1 \\ \implies 5(3m + 7n) &= 1. \end{aligned}$$

Which implies $5 \mid 1$. But it is clearly the case that $5 \nmid 1$, since there exists no $k \in \mathbb{Z}$ such that $1 = 5k$. Thus, another way to arrive at a contradiction. ■

- **Proof by contradiction example 5.**

Proposition. There are infinitely many primes.

Proof. Suppose for the sake of contradiction that there are finitely many primes, say k in total. Let $p_1, p_2, p_3, ..., p_k$ be the complete list. Consider the number $N = p_1 \cdot p_2 \cdot p_3 \cdot ... \cdot p_k$. Next, consider $N + 1$. That is, $p_1 p_2 p_3 ... p_k + 1$. Either $N + 1$ is prime or it is composite, we consider both cases separately

<u>Case 1: $N + 1$ is prime.</u> In this case, $N + 1$ is prime and greater than all the p_is we have previously considered. Thus, we have found a new prime.

<u>Case 2: $N + 1$ is composite.</u> We begin by showing that no such p_i divides $N + 1$. Because we know that $p_i \mid N$, we have

$$N \equiv 0 \pmod{p_i}.$$

Adding one to both sides, we get

$$N + 1 \equiv 1 \pmod{p_i}.$$

Hence, it must be that $p_i \nmid N + 1$. Since p_i was arbitrary, this shows that none of our k primes divide $N + 1$

We assumed that $p_1, p_2, \ldots, p_k$ was the complete list of prime numbers. And recall that $N+1$ is assumed to be composite, which means it is a product of primes. But since none of the p_i divide $N+1$, there must be some other prime number, q, which divides $N+1$. And hence, we have again found a new prime.

In either case, we have contradicted the claim that $p_1, p_2, \ldots, p_k$ was an exhaustive list of the prime numbers. Therefore, there must be infinitely many primes. ■

- **Proof by contradiction example 6.**

  Proposition The number $\sqrt{2}$ is irrational

  Proof. Assume for a contradiction that $\sqrt{2}$ is rational. Then there must be some non-zero integers p and q where
  $$\sqrt{2} = \frac{p}{q}.$$
  Moreover, we may assume that this fraction is written in *lowest terms*, meaning that p and q have no common divisors. Then,
  $$\sqrt{2}q = p.$$
  By squaring both sides,
  $$2q^2 = p^2.$$
  Since $q^2 \in \mathbb{Z}$, by the definition of divisibility, this implies that $2 \mid p^2$, and hence $2 \mid p$ by Lemma 2.17 part (iii). By a second application of the definition of divisibility, this means that $p = 2k$ for some non-zero integer k. Plugging this in:
  $$\begin{aligned} 2q^2 &= p^2, \\ 2q^2 &= (2k)^2, \\ 2q^2 &= 4k^2, \\ q^2 &= 2k^2 \end{aligned}$$
  Therefore, $2 \mid q^2$, and hence $2 \mid q$, again by Lemma 2.17 part (iii). But this is a contradiction: We had assumed that p and q had no common factors, and yet we proved that 2 divides each. Therefore, $\sqrt{2}$ cannot be rational, meaning it is irrational.

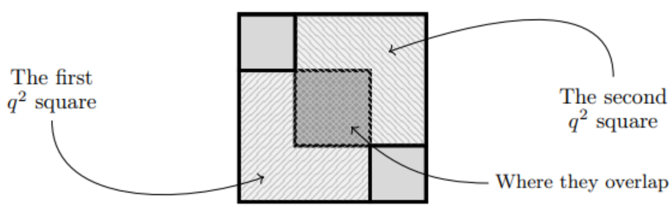
  The following is a geometric proof that $\sqrt{2} \in \bar{\mathbb{Q}}$. Recall that $\bar{\mathbf{Q}}$ is the set of irrational numbers.

  Assume for a contradiction that $\sqrt{2} = \frac{p}{q}$ where $p, q \in \mathbb{N}$ and the fraction is written in lowest terms. This implies that
  $$2q^2 = p^2,$$
  but this time let's think about this as
  $$p^2 = 2q^2.$$
  Or, better yet,
  $$p^2 = q^2 + q^2.$$
  Since p and q are integers, p^2 represents the area of a square with side length p, and each q^2 represents the area of a square with side length q.

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
\quad = \quad
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Recall that $\sqrt{2} = \frac{p}{q}$ was written in lowest terms. In particular, this means that there do not exist any smaller integers a and b for which $\sqrt{2} = \frac{a}{b}$. Our contradiction will be to find such a and b.

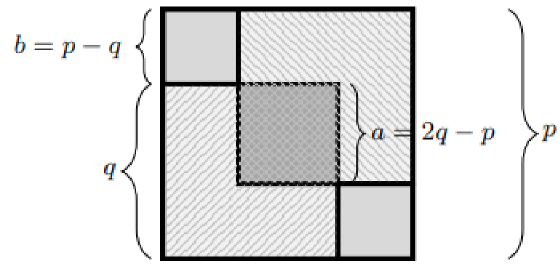
Getting back to the squares above, we are now going to imagine each square is a piece of paper and we are going to place the two q^2 squares on top of the p^2 square. If one q^2 square is placed in the lower-left, and the other is placed in the upper-right, this happens



Notice that there is one square region in the middle that was covered twice, and two small squares in the upper-left and lower-right that were not covered at all. And remember: The amount of area in the p^2 square is equal to the amount of area in the two q^2 squares. Therefore, the area that was covered twice must equal the area that was not covered at all! Let's suppose the middle square has dimensions $a \times a$, and the two corner squares have dimensions $b \times b$. Then, this reasoning shows that

$$\boxed{a^2} = \boxed{b^2} + \boxed{b^2}$$

And those a and b must also be integers, since they are the difference of integers from the overlap picture:



We had assumed that p and q were the smallest integers for which $\sqrt{2} = \frac{p}{q}$, and yet the above image shows that a and b are also integers, and since $a^2 = b^2 + b^2$, which implies $2b^2 = a^2$, we have $2 = \frac{a^2}{b^2}$. And so, finally, by taking the square root of each side, we see that

$$\sqrt{2} = \frac{a}{b}.$$

We have shown that a and b are integers with the above property. The picture above also shows that a is smaller than p, and b is smaller than q. Combined, this contradicts our assumption that p and q are the smallest integers where $\sqrt{2} = \frac{p}{q}$.

- **The irrational numbers**: The fact that irrational numbers exist explains why we need the real numbers $\mathbb{R}$—the rational numbers $\mathbb{Q}$ are clearly not enough! Next, note that while $\sqrt{2}$ is not a ratio of integers, it is a root of $x^2 - 2 = 0$, which is a polynomial with integer coefficients.

  Big Question: Is every irrational number a root of a polynomial with integer coefficients?

  Big Answer: Nope! In 1844, Joseph Liouville proved that

  $$\sum_{k=1}^{\infty} \frac{1}{10^{k!}} = 0.110001000000000000000001000\ldots$$

  is not the root of any polynomial with integer coefficients.

  The irrational numbers were thus partitioned into *algebraic numbers*, which are the roots of such polynomials, and *transcendental numbers*, which are not. Today, π and e are the most famous numbers which have been proved to be transcendental.

- **Proof of the halting problem**:

  Theorem. Assume that P is an arbitrary program and i is a possible input of P; we write $P(i)$ to be the result of plugging input i into the program P. There does not exist a program $H(P(i))$ which determines whether $P(i)$ will eventually halt.

  Proof. Assume for a contradiction that such a program H did exist. Create a new program $T(x)$; its input, x, is itself a program with some input. Now, we define the program $T(x)$ as follows:

```
Input: A program x, with its own input
Run H(x)
if H(x) answers Program x will halt then
    begin an infinite loop
else halt
```

The program T is designed to run counter to x: If the input program x was going to halt, then T begins an infinite loop. And if the input program was going to run forever, then T says to halt

The program T accepts as input any program. And since T is itself a program, we are allowed to *plug T into itself!* What is the result? Well, since $T(T)$ is a program, like any program either $T(T)$ contains an infinite loop or it does not. Let's consider each of these two cases.

<u>Case 1:</u> Observe that if $T(T)$ has an infinite loop, then like all programs with infinite loops, it will not halt — but by looking at the above pseudocode for T, it is clear that if $T(T)$ has an infinite loop, then it will halt! This is a contradiction.

<u>Case 2:</u> Conversely, if $T(T)$ does not have an infinite loop, then like all programs without an infinite loop it must eventually halt — but by looking at the above pseudocode for T, it is clear that if $T(T)$ will eventually halt, then it will begin an infinite loop which will prevent it from halting! This is again a contradiction.

Whether T does or does not have an infinite loop, we have reached a contradiction. And since T was built from H, our assumption that there exists a halting program H must have been incorrect. This concludes the proof. ■

- **Proof by contradiction example 7**:

  Proposition. Every natural number is interesting

  Proof. Assume for a contradiction that not every natural number is interesting. Then, there must be a smallest uninteresting number, which we call n. But being the smallest uninteresting number is a very interesting property for a number to have! So n is both uninteresting and interesting, which gives the contradiction. Therefore, every natural number must be interesting. ■

- **Proof by minimal counterexample**: We proved that every natural number is interesting. The way we did this was by assuming for a contradiction that not every number is interesting. Under this assumption, there exist uninteresting natural numbers, and so there must exist a smallest uninteresting natural number.

  Despite it being a silly example, there is an important idea behind it which is sometimes called *proof by minimal counterexample*. Consider a theorem which asserts something is true for every natural number, and you are attempting to prove it by contradiction. Then you would assume for a contradiction not every natural number satisfies the result — that is, you're assuming there is at least one counterexample. Well, among all of the counterexamples, one of them must be the smallest. And thinking about that smallest counterexample — such as the smallest uninteresting number — can at times be a powerful variant of proof by contradiction.

  We used strong induction to prove the fundamental theorem of arithmetic. But there's another slick proof of this theorem that uses a proof by minimal counterexample

  Theorem (*Fundemental theorem of arithmetic*). Every integer $n \geqslant 2$ is either prime or a product of primes.

Recall that every integer $n \geqslant 2$ is either prime or composite, and being composite means it is a product of smaller integers

Proof. Assume for a contradiction that this is not true. Then there must be a minimal counterexample; let's say N is the smallest natural number at least 2 which is neither prime nor the product of primes. The fact that it is not prime means that it is composite: $N = ab$ for some $a, b \in \{2, 3, \ldots, N-1\}$.

We now make use of the fact that N is assumed to be the minimal counterexample to this result — which means that everything smaller than N must satisfy the result. In particular, since a and b are smaller than this smallest counterexample, a and b must each be prime or a product of primes.

And this gives us a contradiction: Since $N = ab$, if a and b are each prime or a product of primes, then their product — which equals N — must be as well. This contradicts our assumption that N was a counterexample, completing the proof.

Another way to think about this proof is that it argues that if N were a counterexample, then since $N = ab$, it can't possibly be that both a and b are primes or a product of primes, since as we just saw, that would produce a contradiction. And therefore, it must be the case that either a or b is also a counterexample. This implies that every counterexample produces a smaller counterexample — every N produces an a or a b. But this is a contradiction, since you can not repeatedly find smaller and smaller natural numbers — at some point you reach the bottom.

- **Proof of the division algorithm**

  Theorem (*The division algorithm*): For all integers a and m with $m > 0$, there exist unique integers q and r such that

  $$a = mq + r,$$

  where $0 \leqslant r < m$.

  Proof. **Existence.** First, note that if $a = 0$, then by simply choosing $q = 0$ and $r = 0$, the theorem follows. Thus, we may assume that $a \neq 0$.

  Next, we will argue that if the theorem holds for all positive a, then it also holds for all negative a. Indeed, assume that $a > 0$, and suppose a and m can be expressed as

  $$a = mq + r,$$

  where $0 \leqslant r < m$. Then, $-a$ has an expression as well. In particular, if we let $q' = -q - 1$ and $r' = m - r$, then

  $$mq' + r' = m(-q-1) + (m-r) = -mq - m + m - r = -(mq + r) = -a.$$

  Therefore, for these integers q' and r',

  $$-a = mq' + r',$$

  where $0 \leqslant r' < m$. Because of this, any expression for $a > 0$ immediately produces one for $-a$. Thus, we need only prove the case where a is a positive integer.

We will implement a proof by minimal counterexample in order to prove the case where a is positive. Fix any $m > 0$, and assume for a contradiction that not every $a \in \mathbb{N}$ satisfies the theorem, which in turn means that there is a smallest a for which the theorem fails. Consider three cases.

Case 1: $a < m$. In this case, we can simply let $q = 0$ and $r = a$, and we have obtained

$$a = m \cdot q + r,$$

with $0 \leqslant r < m$, and the theorem is satisfied.

Case 2: $a = m$. In this case, we can simply let $q = 1$ and $r = 0$, and we have obtained

$$a = m \cdot q + r,$$

with $0 \leqslant r < m$, and the theorem is satisfied.

Case 3: $a > m$. Recall that the theorem assumes that $m > 0$, and so in this case we have $a > m > 0$. In particular, note that $a > a - m$ and also $a - m > 0$.

Since a is the smallest positive counterexample to this theorem, and $a - m$ is both positive and less than a, the integer $a' = a - m$ must satisfy this theorem! That is, there must exist integers d and s for which

$$(a - m) = m \cdot d + s,$$

with $0 \leqslant s < m$. By moving the m on the left side over,

$$a = m \cdot d + s + m.$$

By factoring,

$$a = m \cdot (d + 1) + s.$$

Thus, by letting $q = d+1$ and $r = s$, we have shown that our smallest counterexample is not a counterexample at all:

$$a = m \cdot q + r,$$

with $0 \leqslant r < m$. Since there cannot exist a smallest counterexample, there cannot exist any counterexample. Thus, for each a and m, there must exist a q and r as the theorem asserts.

Uniqueness. Assume for a contradiction that for our fixed a and m, the q and r are not unique. That is, assume there exist two different representations of a:

$$a = mq + r \quad \text{and} \quad a = mq' + r',$$

where $q, r, q', r' \in \mathbb{Z}$ and $0 \leqslant r, r' < m$. Then,

$$mq + r = mq' + r'.$$

By some algebra, we find:

$$r - r' = mq' - mq,$$

which means

$$r - r' = m(q' - q).$$

Since q and q' are integers, so is $q - q'$ (by Fact 2.1), which means the above expression matches the definition of divisibility (Definition 2.8)! That is, $m \mid (r - r')$.

Notice that since $0 \leqslant r, r' < m$, the difference $r - r'$ would have these restrictions:

$$-m < r - r' < m.$$

And the only number in this range which is divisible by m is zero. That is, $r - r' = 0$, or $r = r'$.

Next, since $r = r'$, the fact that $r - r' = m(q - q')$ implies that

$$0 = m(q - q').$$

Since $m > 0$, we may divide both sides by m, which means $0 = q - q'$, or $q = q'$.

We assumed that

$$a = mq + r \quad \text{and} \quad a = mq' + r'$$

were two different representations of a and m, but we have proven that $q = q'$ and $r = r'$, proving that they are in fact the same representation, giving the contradiction and concluding the proof.

1.8 Functions

- **The definition of a function**: Given a pair of sets A and B, suppose that each element $x \in A$ is associated, in some way, to a unique element of B, which we denote $f(x)$. Then f is said to be a function from A to B. This is often denoted $f : A \to B$.

  Furthermore, A is called the **domain** of f, and B is called the **codomain** of f.

  The set $\{f(x) : x \in A\}$ is called the **range** of f.

- **The *Existence*, and *uniqueness* property of functions**: When discussing functions, the ideas of existence and uniqueness will come up repeatedly. We defined a function $f : A \to B$ to be a rule which sends each $x \in A$ to some $f(x) \in B$. What this means is that $f(x)$ must exist (it must be equal to some $b \in B$), and it must be unique (it must be equal to only one $b \in B$).

  For example, defining $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \ln(x)$ fails the *existence* requirement of functions, because the natural logarithm function $\ln(x)$ is not defined for negative values of x or $x = 0$. his means that the function $\ln(x)$ would fail the requirement of existence for all elements in the domain $\mathbb{R}$.

  To make $f(x) = \ln(x)$ a valid function, we must adjust the domain to only include values for which $\ln(x)$ is defined. The correct domain is $(0, \infty)$, the set of positive real numbers. Thus, we would write

  $$f : (0, \infty) \to \mathbb{R}.$$

  A "function" that fails the uniqueness requirement of functions would assign a single element in the domain to more than one element in the codomain.

  Consider a rule $f : A \to B$ defined as

  $$f(x) = \begin{cases} b_1 & \text{if } x = a \\ b_2 & \text{if } x = a \end{cases}.$$

  Where $b_1 \neq b_2$, and $a \in A$. This rule clearly violates the *uniqueness* criterion, and is therefore not a function.

  In high school you were probably taught the *vertical line test* to check whether a graph corresponds to a function. The vertical line test says that if every vertical line hits the graph in one (existence) and only one (uniqueness) spot, then the graph corresponds to a function

- **Injections, Surjections and Bijections**: A function $f : A \to B$ is injective (or one-to-one) if $f(a_1) = f(a_2)$ implies that $a_1 = a_2$.

  The contrapositive of the second half states, A function $f : A \to B$ is *injective* if $a_1 \neq a_2$ implies that $f(a_1) \neq f(a_2)$

  A function $f : A \to B$ is surjective (or onto) if, for every $b \in B$, there exists some $a \in A$ such that $f(a) = b$

  Let's take a look at another way to define this same idea, by again applying the contrapositive (and doing a little rearranging).

A function $f: A \to B$ is surjective (or onto) if there does not exist any $b \in B$ for which $f(a) \neq b$ for all $a \in A$.

When defining a function $f : A \to B$, the ideas of existence and uniqueness were focused on A — for every $x \in A$, we demanded that $f(x)$ exist and be unique. To be injective and surjective, the attention shifts to B. To be surjective means that B has an existence criterion (for every $b \in B$, there exists some $a \in A$ that maps to it). And to be injective means that B has a uniqueness-type criterion (for every $b \in B$, there is at most one $a \in A$ that maps to it).

A function $f: A \to B$ is *bijective* if it is both injective and surjective.

Defining a function $f : A \to B$ placed existence and uniqueness criteria on A. If f is both injective and surjective, then this adds existence and uniqueness criteria to B. Thus, if f is a bijection, then it has these criteria on both sides: Every $a \in A$ is mapped to precisely one $b \in B$, and every $b \in B$ is mapped to by precisely one $a \in A$. In effect, this pairs up each element of A with an element of B; namely, a is paired with $f(a)$ in this way.

- **Proving $x$jectiveness for $x \in$ {in,sur,bi}**: Based on its definition, this is the outline to prove a function is injective.

Proposition. $f: A \to B$ is an injection

Proof. Assume $x, y \in A$, and $f(x) = f(y)$

$\vdots$ Apply algebra,

$\vdots$ logic, techniques.

Therefore, $x = y$

Since $f(x) = f(y)$ implies $x = y$, f is injective ■

Alternatively, one could use the contrapositive, which would mean one starts by assuming $x \neq y$, and then concludes that $f(x) \neq f(y)$.

Next, here's the outline for a surjective proof.

Proposition. $f: A \to B$ is a surjection

Proof. Assume $b \in B$

$\vdots$ Magic to find an $a \in A$

$\vdots$ where $f(a) = b$.

Since every $b \in B$ has an $a \in A$ where $f(a) = b$, f is surjective ■

- **Proving jectiveness examples**

- $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$ is not injective, surjective, or bijective.
- $g : \mathbb{R}^+ \to \mathbb{R}$ where $g(x) = x^2$ is injective, but not surjective or bijective.
- $h : \mathbb{R} \to \mathbb{R}^+$ where $h(x) = x^2$ is surjective, but not injective or bijective.
- $k : \mathbb{R}^+ \to \mathbb{R}^+$ where $k(x) = x^2$ is injective, surjective, and bijective.

Proof (part a). Observe that $f(-2) = f(2) = 4$, while $-2 \neq 2$. Thus, f is not injective. Next, notice that $f(x) = x^2 > 0$. Thus, there is no such $a \in \mathbb{R}$ such that $f(a) = -4$. Since -4 is in the codomain and is not hit, f is not surjective. Since f is not both injective and surjective, it is therefore not bijective.

Part b. Let $a_1, a_2 \in \mathbb{R}^+$, assume $g(a_1) = g(a_2)$. Thus,

$$\begin{aligned} a_1^2 &= a_2^2 \\ \implies a_1 &= \pm a_2. \end{aligned}$$

But, for all $a \in \mathbb{R}^+$, $a > 0$. Thus, $a_1 = a_2$ and g is injective. Observe that again there is no such value in the domain of g such that $g(x) = -4$. Since -4 is in the codomain of g, it is not surjective, and is therefore not bijective.

Part c. Observe that $h(-2) = h(2) = 4$, while $-2 \neq 2$. Thus, h is not injective. Further, let $b \in \mathbb{R}^+$, then

$$\begin{aligned} h(a) &= b \\ \implies a^2 &= b \\ \implies a &= \pm b. \end{aligned}$$

But, the codomain is restricted to positive values, thus $a = b$ and h is surjective. Since h is not injective, it is not bijective.

Part d. Let $a_1, a_2 \in \mathbb{R}^+$, assume $f(a_1) = f(a_2)$, which implies

$$\begin{aligned} a_1^2 &= a_2^2 \\ \implies a_1 &= \pm a_2. \end{aligned}$$

Again, since the domain is restricted to positive values, we have $a_1 = a_2$ and f is injective. Next, let $b \in \mathbb{R}^+$, then

$$\begin{aligned} f(a) &= b \\ \implies a^2 &= b \\ \implies a &= \pm b. \end{aligned}$$

But since the codomain is restricted to positive values, $a = b$ and the function is surjective. Since the function is both onto and one-to-one, the function is bijective (invertible). ■

- **Proving jectiveness example 2.** Show $f : (\mathbb{Z} \times \mathbb{Z}) \to (\mathbb{Z} \times \mathbb{Z})$, with $f(x, y) = (x + 2y, 2x + 3y)$ is a bijection.

Proof. First, we show injectiveness. Let $(a,b),(c,d) \in \mathbb{Z}^2$. Assume $f(a,b) = f(c,d)$. Thus,

$$\begin{aligned}
&(a+2b, 2a+3b) = (c+2d, 2c+3d) \\
\implies &\begin{cases} a+2b &= c+2d \\ 2a+3b &= 2c+3d \end{cases} \\
\implies &\begin{cases} a+2b-2a-3b &= 0 \\ 2a+3b-2c-3d &= 0 \end{cases}.
\end{aligned}$$

We then solve this system,

$$\left.\begin{array}{cccc|c} 1 & 2 & -1 & -2 & 0 \\ 2 & 3 & -2 & -3 & 0 \end{array}\right. \implies \left.\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right..$$

Which implies

$$\begin{cases} a &= c \\ b &= d \end{cases}$$

As desired. Thus, f is injective. Next, let $(c,d) \in \mathbb{Z}^2$. Require $f(a,b) = (c,d)$ for some $(a,b) \in \mathbb{Z}^2$. Thus,

$$\begin{aligned}
&(a+2b, 2a+3b) = (c,d) \\
\implies &\begin{cases} a+2b &= c \\ 2a+3b &= d \end{cases}.
\end{aligned}$$

Solving this system yields

$$\left.\begin{array}{cc|c} 1 & 2 & c \\ 2 & 3 & d \end{array}\right. \implies \left.\begin{array}{cc|c} 1 & 0 & -3c+2d \\ 0 & 1 & 2c-d \end{array}\right..$$

Thus, $(a,b) = (-3c+2d, 2c-d)$ and the function is surjective. Because the function is both injective and surjective, it is therefore bijective.

Alternatively, observe that $f: \mathbb{Z}^2 \to \mathbb{Z}^2$, $f(x,y) = (x+2y, 2x+3y)$ is given by the matrix representation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Thus, since A is square, we can simply check its determinant. [4]

$$\det \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = 1(2) - 2(3) = -1.$$

Since $\det(A) \neq 0$, the function is invertible

- **The func-y pigeonhole principle**:

  Theorem 8.10 (The func-y pigeonhole principle): Suppose A and B are finite sets and $f: A \to B$ is any function.

  (a) If $|A| > |B|$, then f is not injective.

  (b) If $|A| < |B|$, then f is not surjective.

[4] Common linear algebra W

Proof. **Part (a).** Consider each element in A to be an object and each element of B to be a box. Given an $a \in A$, place object a into box b if $f(a) = b$. Since there are more objects than boxes, by the pigeonhole principle at least one box has at least two objects in it. That is, $f(a_1) = f(a_2)$ for some distinct a_1 and a_2, implying that f is not injective.

Part (b). Since f is a function, each $a \in A$ is mapped to only one $b \in B$. Thus, k elements in A can map to at most k elements of B. And so the $|A|$ elements in A can map to at most $|A|$ elements in B. However, since $|A| < |B|$, there must be some elements not hit, meaning that f is not surjective.

It is again useful to think about what the contrapositive tells us:

(a) If f is injective, then $|A| \leqslant |B|$.

(b) If f is surjective, then $|A| \geqslant |B|$.

Viewing the statements this way is beneficial for another reason: It demonstrates clearly that in order for f to be a bijection—meaning an injection and a surjection—we would need $|A| = |B|$.

It is also worth mentioning that this theorem still holds true in the case that $|A|$ and/or $|B|$ are infinite.[5]

- **The Composition**: Let A, B, and C be sets, $g : A \to B$, and $f : B \to C$. Then the composition function is denoted $f \circ g$ and is defined as follows:

$$(f \circ g) : A \to C \quad \text{where} \quad (f \circ g)(a) = f(g(a)).$$

Suppose

$$g: \ \mathbb{R} \to \mathbb{R}, \ g(x) = x + 1$$
$$f: \ \mathbb{R} \to \mathbb{R}^+, \ f(x) = x^2.$$

Then,

$$(f \circ g): \ \mathbb{R} \to \mathbb{R}^+, \ (f \circ g)(x) = (x+1)^2.$$

- **Property of injective functions under composition**:

Theorem 8.13. Suppose A, B and C are sets, $g: \ A \to B$ is injective, and $f: \ B \to C$ is injective. Then $f \circ g$ is injective

Proof. Since $(f \circ g): \ A \to C$, to show that is an injection we must show that for all $a_1, a_2 \in A$, $(f \circ g)(a_1) = (f \circ g)(a_2)$ implies $a_1 = a_2$. Assume $a_1, a_2 \in A$, and $(f \circ g)(a_1) = (f \circ g)(a_2)$. Using the definition of the composition, we have

$$f(g(a_1)) = f(g(a_2)).$$

Since f is injective, we know that for any $b_1, b_2 \in B$, $f(b_1) = f(b_2)$ implies $b_1 = b_2$. Since $g(a_1), g(a_2) \in B$, we have

$$g(a_1) = g(a_2).$$

Likewise, since g is injective, it must be that $a_1 = a_2$

[5] But proving this to be the case would take us too far afield.

Thus, we have shown that for any $a_1, a_2 \in A$, if $(f \circ g)(a_1) = (f \circ g)(a_2)$, then $a_1 = a_2$. Therefore, $(f \circ g)$ is an injection. ■

- **Property of surjective functions under composition**:

  Theorem 8.14: Suppose A, B and C are sets, $g : A \to B$ is surjective, and $f : B \to C$ is surjective. Then $f \circ g$ is surjective.

  Proof. Since $(f \circ g) : A \to C$, to show that $f \circ g$ is surjective, we must show that for all $c \in C$, there exists some $a \in A$ such that $(f \circ g)(a) = c$. To start, since f is surjective, then for all $c \in C$, there exists some $b \in B$ such that $f(b) = c$. Further, we know that g is surjective. Thus, for all $b \in B$, there exists some $a \in A$ such that $g(a) = b$.

  Thus, for an arbitrary $c \in C$, we have found an $a \in A$ such that

  $$(f \circ g)(a) = f(g(a)) = f(b) = c.$$

  Completing the proof ■

- **A corollary from the above two results**: Suppose A, B and C are sets, $g : A \to B$ is bijective, and $f : B \to C$ is bijective. Then $f \circ g$ is bijective.

  Proof. By Theorem 8.13, $f \circ g$ is an injection. By Theorem 8.14, $f \circ g$ is a surjection. Thus, by the definition of a bijection (Definition 8.7), $f \circ g$ is a bijection.

- **Note about compositions**: Notice that in our definition of function composition (Definition 8.11) we had functions g and f where $g : A \to B$, and $f : B \to C$. Notice that we don't really need the codomain of g to equal the domain of f. If we had $g : A \to B$ and $f : D \to C$ where $B \subseteq D$, that would be enough (for the definition, and for these last two theorems). As long as $g(a)$ is a part of f's domain, then $f(g(a))$ will make sense, which is all we need.

- **Identity function and invertibility**: For a set A, the identity function on A is the function

  $$i_A : A \to A \text{ where } i_A(x) = x \text{ for every } x \in A$$

  The inverse of a function $f : A \to B$, if it exists, is the function $f^{-1} : B \to A$ such that $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.

  For example, if $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x + 1$, then $f^{-1} : \mathbb{R} \to \mathbb{R}$ is the function $f^{-1}(x) = x - 1$. To see this, simply note that

  $$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f(x - 1) = (x - 1) + 1 = x$$

  and

  $$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(x + 1) = (x + 1) - 1 = x.$$

- **Arctan and the natural logarithm**: this is a great opportunity to mention a couple important functions — $\arctan(x)$ and $\ln(x)$ — which are defined as the inverses to other important function.

  - If $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is the tangent function, then its inverse is defined to be $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$, and is called the arctangent function.[6]
  - If $\exp : \mathbb{R} \to \mathbb{R}^+$ is the exponential function (that is, $\exp(x) = e^x$), then its inverse is defined to be $\ln : \mathbb{R}^+ \to \mathbb{R}$, and is called the natural logarithm function.

- **When does an inverse exist**:

6

Theorem: A function $f:\ A \to B$ is invertible if and only if f is a bijection.

Proof. First, suppose that $f : A \to B$ is invertible. We will prove that f is both an injection and a surjection, which will prove that f is a bijection. To see that f is a surjection, choose any $b \in B$. We aim to find an $a \in A$ such that $f(a) = b$. To this end, let $a = f^{-1}(b)$, which exists and is in A because $f^{-1} : B \to A$. Now simply observe that the definition of an invertible function (Definition 8.16) implies

$$f(a) = f(f^{-1}(b)) = b.$$

This proves that f is a surjection.

To see that f is an injection, let $a_1, a_2 \in A$ and assume $f(a_1) = f(a_2)$. Note that $f(a_1)$ (and hence $f(a_2)$, since they're equal) is an element of B due to the fact that $f : A \to B$. And so, since $f^{-1} : B \to A$, we may apply f^{-1} to both sides:

$$f(a_1) = f(a_2)$$

$$f^{-1}(f(a_1)) = f^{-1}(f(a_2))$$

$$a_1 = a_2,$$

by the definition of the inverse. Thus, f is an injection. And since we already showed that f is a surjection, it must be a bijection. This concludes the forward direction of the theorem.

As for the backwards direction, assume that f is a bijection. For $b \in B$, we will now define $f^{-1}(b)$ like this:

$$f^{-1}(b) = a \quad \text{if} \quad f(a) = b.$$

That is, we are defining f^{-1} to act as an inverse from B to A should act, without yet claiming that f^{-1} is a function. Our goal now is to demonstrate that this definition of f^{-1} satisfies the conditions to be a function, which would prove that f is invertible. To do so, recall that to be a function there is an existence condition ($f^{-1}(b)$ must be equal to some $a \in A$) and a uniqueness condition ($f^{-1}(b)$ must be equal to only one $a \in A$). We will check these separately.

Existence: Let $b \in B$. Since f is surjective, there must be some $a \in A$ such that $f(a) = b$. Hence, by our definition of f^{-1}, we have $f^{-1}(b) = a$. We have shown that for every $b \in B$ there exists at least one $a \in A$ for which $f^{-1}(b) = a$, which concludes the existence portion of this argument.

Uniqueness: Suppose $f^{-1}(b) = a_1$ and $f^{-1}(b) = a_2$, for some $b \in B$ and $a_1, a_2 \in A$. By the definition of f^{-1}, this means that $f(a_1) = b$ and $f(a_2) = b$. But since f is injective, this means that $a_1 = a_2$. We have shown that $f^{-1}(b)$ can not be equal to two different elements of A, which concludes the uniqueness portion of this argument.

Combined, these two parts show that $f^{-1} : B \to A$ is a function, hence proving that f is invertible.

We have proved the forwards and backwards directions of Theorem 8.17, which completes its proof. □

- **The image and inverse image**: Let $f : A \to B$ be a function, and assume $X \subseteq A$ and $Y \subseteq B$. The *image* of A is

$$f(X) = \{y \in B : y = f(x) \text{ for some } x \in X\},$$

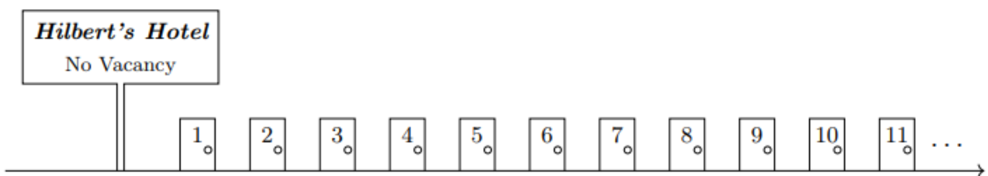
and the *inverse image* of Y is

$$f^{-1}(Y) = \{x \in A : f(x) \in Y\}.$$

- **The bijection principle**:

Principle (*The bijection principle.*) Two sets have the same size if and only if there is a bijection between them.

- **Hilbert's hotel**: We begin by talking about the set of problems related to the so-called Hilbert's Hotel. Assume that there is a hotel, called Hilbert's Hotel, which has infinitely many rooms in a row.



  - Assume every room has someone in it, and so the "No Vacancy" sign has been turned on. With most hotels, this would mean that if someone else arrives at the hotel, they will not be given a room. But this isn't the case with Hilbert's Hotel. If, for $n \in \mathbb{N}$, the patron in room n moves to room $n + 1$, then nobody is left without a room and suddenly room 1 is completely open! So the new customer can go to room 1. We created a room out of nothing![7]
  - Now imagine 2 people arrived at the hotel. Can we accommodate them? Certainly! Now, just have everyone move from room n to room $n + 2$. This leaves rooms 1 and 2 open to the newcomers, and we are again good-to-go.
  - What if, however, we have infinitely many people lined up wanting a room? Can we accommodate all of them? Yes! We still can! Just have the person in room n move to room $2n$. Then all of the odd-numbered rooms are vacant and the infinite line of people can take these rooms.

The first point of this exercise is to simply realize that weird stuff can happen when dealing with the infinite. The second point, though, is to realize that each time the people switched rooms, those same exact people got new rooms. So in the first example when they each just moved one room down, that should mean that there are just as many rooms from 1 to ∞ as there are from 2 to ∞. . . And likewise for the others.

- **Cardinality and infinite sets**:

[7]

Example There are the same number of natural numbers as there are natural numbers larger than 1 (that is, $|\mathbb{N}| = |\{2, 3, 4, \dots\}|$). What's the bijection that shows this? Let

$$f : \mathbb{N} \to \{2, 3, 4, \dots\} \quad \text{where} \quad f(n) = n + 1.$$

In other (non-)words, this is the pairing

$$1 \leftrightarrow 2 \quad 2 \leftrightarrow 3 \quad 3 \leftrightarrow 4 \quad 4 \leftrightarrow 5 \quad \dots$$

The Moral. Two sets can have the same size even though one is a proper subset of the other.

Example. There are the same number of natural numbers as even natural numbers (that is, $|\mathbb{N}| = |2\mathbb{N}|$). What's the bijection that shows this? Let

$$f : \mathbb{N} \to \{2, 4, 6, 8, \dots\} \quad \text{where} \quad f(n) = 2n.$$

In other (non-)words, this is the pairing

$$1 \leftrightarrow 2 \quad 2 \leftrightarrow 4 \quad 3 \leftrightarrow 6 \quad 4 \leftrightarrow 8 \quad \dots$$

The Moral. Two sets can have the same size even though one is a proper subset of the other and the larger one even has *infinitely many more elements* than the smaller one.[8]

And in a similar way, one can prove that $|\mathbb{N}| = |\mathbb{Z}|$. Indeed, a bijection $f : \mathbb{N} \to \mathbb{Z}$ can be given by following this pattern:

$$f(1) = 0, \quad f(2) = 1, \quad f(3) = -1, \quad f(4) = 2, \quad f(5) = -2, \quad f(6) = 3, \quad \dots$$

One way to write such a function is this:

$$f : \mathbb{N} \to \mathbb{Z} \quad \text{where} \quad f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even;} \\ -\frac{(n-1)}{2} & \text{if } n \text{ is odd.} \end{cases}$$

[8]

Elementary fields, groups, and rings

- **Modular congruence and congruence classes**: Recall that two integers a and b are said to be congruent modulo n if they leave the same remainder when divided by n. Mathematically, this is written as

  $$a \equiv b \pmod{n}.$$

  Which means

  $$n \mid a - b.$$

  When an integer a is divided by n

  $$a = q_1 n + r_1 \quad \text{with} 0 \leqslant r_1 < n.$$

  Similarly, for an integer b divided by n

  $$b = q_2 n + r \quad \text{with} 0 \leqslant r_2 < n.$$

  Subtracting b from a

  $$a - b = (q_1 - q_2)n + (r_1 - r_2). \tag{1}$$

  If $n \mid (a - b)$,

  $$a - b = nk, \quad k \in \mathbb{Z}.$$

  By (1) above, we have

  $$(q_1 - q_2)n + (r_1 - r_2) = kn.$$

  For this to hold, we require $r_1 - r_2$ to be a multiple of n, since $q_1 - q_2$ is already a multiple of n. Since r_1, r_2 satisfy $0 \leqslant r_1, r_2 < n$. It must be that $-n < r_1 - r_2 < n$. In this case, for n to divide $r_1 - r_2$. It must be that

  $$r_1 - r_2 = 0.$$

  Which implies $r_1 = r_2$. Hence, a and b have the same remainder when divided by n when $n \mid a - b$.

  A congruence class modulo n is the set of all integers that are congruent to a particular integer a modulo n. This set is denoted as

  $$[a]_n = \{x \in \mathbb{Z} \mid x \equiv a \pmod{n}\}.$$

  For example, $[0]_3$ is

  $$\{x \in \mathbb{Z} : \ x \equiv 0 \pmod{3}\}$$

  .

  Which is the integers x such that $3 \mid x - 0$. In other words, it describes the set of integers that are divisible by 3.

  The set $[1]_3$ is the set

  $$[1]_3 = \{x \in \mathbb{Z} : \ x \equiv 1 \pmod{3}\}.$$

  Which implies $3 \mid x - 1$, and thus $x = 3k + 1$, for $k \in \mathbb{Z}$. In words, it is the set of integers that leave a remainder of one when divided by three.

The modulus n partitions the integers into n distinct congruence classes:

$$[0]_n, [1]_n, ..., [n-1]_n.$$

Every integer belongs to exactly one of these classes.

Arithmetic operations can be performed within the framework of congruence classes

- **Addition**: If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then

$$a + c \equiv b + d \pmod{n}.$$

- **Multiplication**: If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then

$$ac \equiv bd \pmod{n}.$$

- **Groups**: A group is a collection of objects G, together with one operation $\oplus$, which has the following properties:
  - **Associativity**: $a \oplus (b \oplus c) = (a \oplus b) \oplus c$
  - **Identity**: There is an element $e \in G$ such that $e \oplus g = g \oplus e = g$ for all $g \in G$
  - **Inverse**: For every $g \in G$, there exists $g^{-1} \in G$ such that $g \oplus g^{-1} = g^{-1} \oplus g = e$

  For example, $\mathbb{Z}$ is a group under addition.
  - **Associativity**: Two integers a, b are associative, $a + (b + c) = (a + b) + c$
  - **Identity**: Zero is the identity element, since $0 \in \mathbb{Z}$ and $0 + a = a + 0 = a$
  - **Inverse**: $a + (-a) = (-a) + a = 0$

  Note: A group is said to be *abelian* if it is commutative under its operation. In other words, $x \oplus y = y \oplus x$ for all $x, y \in G$

- **Rings**: A ring is a set R, together with two operations $\oplus$ and $*$, which has the following properties
  - R is an abelian group under $\oplus$
  - R is associative under $*$
  - The operation $*$ distributes over $\oplus$

$$\begin{aligned} a * (b \oplus c) &= (a * b) \oplus a * c \\ (a \oplus b) * c &= (a * c) \oplus (b * c). \end{aligned}$$

  For example, $\mathbb{Z}$ is a ring under addition and multiplication. First note that $\mathbb{Z}$ is an abelian group under addition. Further, for $a, b \in \mathbb{Z}$, $a \cdot b = b \cdot a$.

  $1 \in \mathbb{Z}$ is the identity, $1 \cdot a = a \cdot 1 = a$ for all $a \in \mathbb{Z}$, and we know that multiplication distributes over addition

$$\begin{aligned} a \cdot (b + c) &= a \cdot b + a \cdot c \\ (a + b) \cdot c &= a \cdot c + b \cdot c. \end{aligned}$$

- **Fields**: A field is a set F, together with two operations $\oplus$ and $*$, which has the following properties
  - F is a commutative ring under $\oplus$ and $*$

- Every nonzero $f \in F$ has a multplicative inverse, that is, some element $g \in F$ for which

$$f * g = g * f = 1.$$

The sets $\mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ under addition and multiplication are examples of fields. The set of integers $\mathbb{Z}$ is not. Although it is a commutative ring under addition and multiplication, not every element has a multiplicative inverse. For example, there is no such $a \in \mathbb{Z}$ such that $2 \cdot g = 1$

- **Vector spaces**: A vector space is a set of vectors V, together with a set of scalars F, with the following properties
  - V is a abelian group under vector addition
  - F is a field under multiplication
  - For each $s \in F$, and $\mathbf{v} \in V$, scalar multiplication gives a unique element $s \cdot \mathbf{v} \in V$
  - Additional properties

$$\begin{aligned} 1\mathbf{v} &= \mathbf{v} \\ a(b\mathbf{v}) &= (ab)\mathbf{v} \\ a(\mathbf{u}+\mathbf{v}) &= a\mathbf{u} + a\mathbf{v} \\ (a+b)\mathbf{v} &= a\mathbf{v} + b\mathbf{v}. \end{aligned}$$

Combinatorics

3.1 Introduction

- **What is combinatorics?**: Combinatorics is a collection of techniques and a language for the study of finite or countably infinite discrete structures. Given a set of elements and possibly some structure on that set, typical questions are
  - Does a specific arrangement of the elements exists?
  - How many such arrangemets are there?
  - What properties do these arrangements have?
  - Which one of the arrangemetns is maximal, minimal, or optimal according to some criterion?
- **Counting the number of subsets for a set**: Let $[n] = \{1, 2, ..., n\}$, and let $f(n)$ be the number of subsets of $[n]$. Then $f(n) = 2^n$. For any particular subset of $[n]$, each element is either in that subset or not. Thus, to construct a subset, we have to make one of two choices for each element of $[n]$. Furthermore, these choices are independent of each other. Hence, the total number of choices, and consequently the total number of subsets is

$$\underbrace{2 \times 2 \times ... \times 2}_{n} = 2^n.$$

- **Number of subsets without consecutive integers**: For a sequence $[n] = \{1, ..., n\}$ we can count the number of subsets given by $f(n)$, that do not contain consecutive integers with the recurrence relation

$$f(n) = f(n-1) + f(n-2).$$

  We consider two cases

  1. n in not included in the subsets
  2. n is included in the subsets. In this case, we build the subsets considering the subsequence $[n-2] = \{1, ..., n-2\}$. Note that if we include n, we must exclude $n-1$, because $n-1$ and n are consecutive, this will become cleare in the upcoming example.

  Consider the sequence $[n] = \{1, 2, 3, 4\}$. By the relation above,

$$f(4) = f(3) + f(2).$$

  Before we are able to compute this, we must define our base cases.

$$f(n) = \begin{cases} 3 & \text{if } n = 2 \\ 2 & \text{if } n = 1 \end{cases}.$$

  If $n = 2$, we have $\{1, 2\}$, and the allowed subsets are $\varnothing, \{1\}, \{2\}$. If we have $n = 1$, the subsets are $\{\varnothing, \{1\}\}$. Thus

$$\begin{aligned} f(4) &= f(3) + f(2) = f(2) + f(1) + f(2) \\ &= 3 + 2 + 3 = 8. \end{aligned}$$

Let's explicitly break up the given sequence so we can see whats going on. In the first case, n is excluded, thus the sequence becomes $\{1, 2, 3\}$. If n is included, the sequence becomes $\{1, 2\}$, where we build the subsets of $\{1, 2\}$, and then add 4 to each one. Thus,

$$\begin{aligned}\{1,2,3\} + \{1,2\} &= \{1,2,3\} + \varnothing + \{1\} + \{2\} \\ &= \{1,2,3\} + \{4\} + \{1,4\} + \{2,4\}.\end{aligned}$$

Since the sequence $\{1, 2, 3\}$ in not a base case, we must split this one up aswell, we have

$$\begin{aligned}\{1,2,3\} &= \{1,2\} + \{1\} + \{4\} + \{1,4\} + \{2,4\} \\ &= \varnothing + \{1\} + \{2\} + \varnothing + \{1\} + \{4\} + \{1,4\} + \{2,4\} \\ &= \varnothing + \{1\} + \{2\} + \{3\} + \{1,3\} + \{4\} + \{1,4\} + \{2,4\}.\end{aligned}$$

Thus, we conclude all "good" subsets of $[n]$ either have n or don't have n. The ones that don't have n are exactly the "good" subsets of $[n-1]$. The "good" subsets of $[n]$ that include n are exactly the "good" subsets of $[n-2]$ together with n. Thus $f(n) = f(n-1) + f(n-2)$ ■

3.2 Induction and recurrence relations

- **Principal of Mathematical Induction**: Given an infinite sequence of propositions

$$P_1, P_2, P_3, ...P_n, ...,.$$

In order to prove that all of them are true, it is enough to show two things

1. **The base case:** P_1 is true
2. **The inductive step**: For all positive integers k, if P_k is true, then so is P_{k+1}

Example: Show that

$$1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}.$$

Base case:

$$1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1.$$

Inductive step: P_k is given by

$$1 + 2 + 3 + ... + k = \frac{k(k+1)}{2}.$$

P_{k+1} is given by

$$1 + 2 + 3 + ... + k + k + 1 = \frac{k + 1(k+2)}{2}.$$

If $1 + 2 + 3 + ... + k = \frac{k(k+1)}{2}$, then

$$\begin{aligned} 1 + 2 + 3 + ... + k + k + 1 &= \frac{k + 1(k+2)}{2} \\ \frac{k(k+1)}{2} + k + 1 &= \frac{k + 1(k+2)}{2} \\ \frac{k(k+1) + 2k + 2}{2} &= \frac{k^2 + 3k + 2}{2} \\ \frac{k^2 + 3k + 2}{2} &= \frac{k^2 + 3k + 2}{2}. \end{aligned}$$

Thus, we have showed that $P_k \implies P_{k+1}$ ■.

Note: Our aim is not to directly prove P_{k+1}, but to prove that P_k implies P_{k+1}. In the inductive step we assume P_k to be true, then show under this assumption, P_{k+1} is also true.

- **Understanding gauss's formula for the sum of the first n natural numbers**: Suppose we want to find the sum $1+2+3+...+(n-1)+n$. We could have discovered the formula that we proved above by first writing the sum twice

$$\begin{aligned} &1 + 2 + 3 + ... + (n-1) + n \\ &n + (n-1) + (n-2) + ... + 2 + 1. \end{aligned}$$

The sum of the two numbres in each column is $n + 1$, and there are n columns, so the total sum is $n(n+1)$, it then follows that the actual sum is $\frac{1}{2}n(n+1)$

- **Trianglular numbers**: The sequence of integers

$$\begin{aligned}
&1 && 3 = 1+2 \\
&6 = 1+2+3 \\
&10 = 1+2+3+4 \\
&15 = 1+2+3+4+5 \\
&....
\end{aligned}$$

  Are called *triangular numbers*. If you were to make a triangle of dots out of the sum, where the highest number is the base, the second highest is the layer ontop of the base, etc, you would form a triangle.

- **Strong induction**: Given an infinite sequence of propositions

$$P_1, P_2, P_3, ..., P_n.$$

  In order to demonstrate that all of them are true, it is enough to know two things.

  1. **The base case**: P_1 is true
  2. **The inductive step**: For all integers $k \geqslant 1$, if $P_1, P_2, P_3, ..., P_k$ are true, then so is P_{k+1}

- **Pingala-fibonacci numbers**: Define a sequence of positive integers as follows: $F_0 = 0, F_1 = 1$, and for $n = 2, 3, ...$ we have

$$F_n = F_{n-2} + F_{n-1}.$$

  This sequence is also known as *the fibonacci sequence*.

- **Lucas numbers**: Change the initial values on the fibonacci sequence. Let $L_0 = 2, L_1 = 1$, and $L_n = L_{n-2} + L_{n-1}$. Then, we get the *Lucas numbers*

$$2, 1, 3, 4, 7, 11, 18, 29, 47, ...$$

$$\mathcal{L}.$$