Nate Warner MATH 230 November 11, 2023

Homework/Worksheet 9 - Due: Wednesday, November 15

1. Use the comparison test to determine whether the series is convergent or divergent

   (a) $\sum\limits_{n=1}^{\infty} \frac{1}{2n-1}$

   (b) $\sum\limits_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$

   (c) $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$

Remark. Suppose we have two series a_n, b_n and $\exists\ N \in \mathbb{Z}^+$ s.t $0 \leqslant a_n \leqslant b_n \ \ \forall\ n \geqslant N$. If b_n converges then a_n will also converge. Conversely, if $a_n \geqslant b_n \geqslant 0 \ \ \forall\ n \geqslant N$, and b_n diverges, then a_n will also diverge

Problem 1.a. If we let b_n be the series $\sum\limits_{n=1}^{\infty} \frac{1}{2n}$. We may conject that this series will diverge since it is know that the harmonic series $\sum \frac{1}{n}$ diverges, and multiplying a series by a constant factor will not affect the convergence or divergence. Furthermore,

$$\frac{1}{2n-1} > \frac{1}{2n}.$$

Conclusion. Thus, since $\sum\limits_{n=1}^{\infty} \frac{1}{2n}$ diverges, we can conclude that $\sum\limits_{n=1}^{\infty} \frac{1}{2n-1}$ will also diverge

Problem 1.b Let b_n be the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ Since we know the sine function produces outputs in the range [-1,1], the sine function squared will produce outputs withing the range [0,1]. However, since we are only considering integer values $[1, \infty)$, $\sin^2 n$ will only produce outputs $(0, 1)$. This is because the sine function has outputs of 1 at $\frac{\pi}{2} + 2k\pi,\ k \in \mathbb{Z}$, and outputs of 0 at $k\pi,\ k \in \mathbb{Z}$,

Problem 1.b: Let $b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$. We know The function $\sin^2(x)$ yields values in the range [0,1], as $\sin(x)$ varies between -1 and 1. For integer values n in the range [1, ∞), $\sin^2(n)$ will produce values in the interval (0,1). This is because $\sin(x)$ equals 1 at $\frac{\pi}{2} + 2k\pi$ and 0 at $k\pi$, where k is an integer, and these points are not integers. Thus we can conclude

$$\frac{\sin^2 n}{n^2} < \frac{1}{n^2}.$$

Conclusion. Since we know by the p-series $\frac{1}{n^2}$ will converge, $\frac{\sin^2 n}{n^2}$ will also converge

Problem 1.c Let b_n be the series $\sum_{n=1}^{\infty} \frac{1}{n+1}$. We know this series will diverge because it is just the harmonic series $\frac{1}{n}$ shifted over by 1. We can deduce that $\frac{1}{\sqrt{n^2+1}} > \frac{1}{n+1}$ by looking at their reciprocals

$$\begin{aligned}
&\sqrt{n^2+1} < n+1 \\
&n^2+1 < (n+1)^2 \\
&n^2+1 < n^2+2n+1 \\
&n^2 < n^2+2n.
\end{aligned}$$

Conclusion. Since this is clearly a true statement, then by the reciprocal identify for inequalities, which states if $0 \leqslant a \leqslant b$, then $\frac{1}{a} \geqslant \frac{1}{b}$ it holds that $\frac{1}{\sqrt{n^2+1}} > \frac{1}{n+1}$. and since we know $\sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges, then $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ will also diverge.

2. Use the Limit Comparison Test to determine whether the series is convergent or divergent.

   (a) $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$

   (b) $\sum_{n=1}^{\infty} \frac{1}{4^n-3^n}$

   (c) $\sum_{n=1}^{\infty} (1-\cos\frac{1}{n})$

Remark. Suppose we have two series $a_n,\ b_n$ where $a_n,\ b_n \geqslant 0\ \forall\ n \geqslant 1$. Then if

- $\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$ or $+\infty$. Then a_n and b_n either both converge or both diverge
- $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then if b_n converges, so does a_n
- $\lim_{n\to\infty} \frac{a_n}{b_n} = +\infty$, then if b_n diverges, so does a_n

Problem 2.a Let $b_n = \frac{1}{n^2}$, which by the p-series, converges

$$\begin{aligned}
&\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\ln\left(1+\frac{1}{n^2}\right)}{\frac{1}{n^2}} \\
&= \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n^2}\right)}{n^{-2}} \\
&\overset{H}{=} \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n^2}} \cdot -\frac{2}{n^3}}{-\frac{2}{n^3}} \\
&= \lim_{n\to\infty} \frac{-2n^3}{-2n^3\left(1+\frac{1}{n^2}\right)} \\
&= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n^2}\right)} \\
&= 1.
\end{aligned}$$

Conclusion. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so does $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$

Problem 2.b Choose $b_n = \frac{1}{4^n} = \left(\frac{1}{4}\right)^n$, which is a geometric series with $|r| < 1$ and thus converges

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\frac{1}{4^n-3^n}}{\frac{1}{4^n}} \\
&= \lim_{n\to\infty} \frac{4^n}{4^n - 3^n} \\
&= \lim_{n\to\infty} \frac{\frac{4^n}{4^n}}{\frac{4^n}{4^n} - \frac{3^n}{4^n}} \\
&= \lim_{n\to\infty} \frac{1}{1 - \left(\frac{3}{4}\right)^n} \\
&= 1.
\end{aligned}$$

Thus, since $\sum\limits_{n=1}^{\infty} \frac{1}{4^n}$ converges, so does $\sum\limits_{n=1}^{\infty} \frac{1}{4^n-3^n}$

Problem 2.c Choose $b_n = \frac{1}{n^2}$, which we know converges

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1-\cos\left(\frac{1}{n}\right)}{\frac{1}{n^2}} \quad (\text{Indeterminate } \frac{0}{0}) \\
&\overset{H}{=} \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)\cdot -\frac{1}{n^2}}{-\frac{2}{n^3}} \\
&= \lim_{n\to\infty} \frac{n^3\sin\left(\frac{1}{n}\right)}{2n^2} \\
&= \frac{1}{2}\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) \\
&= \frac{1}{2}\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{n^{-1}} \quad (\text{Indeterminate } \frac{0}{0}) \\
&\overset{H}{=} \frac{1}{2}\lim_{n\to\infty} \frac{\cos\left(\frac{1}{n}\right)\cdot -\frac{1}{n^2}}{-\frac{1}{n^2}} \\
&= \frac{1}{2}\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) \\
&= \frac{1}{2}\cos 0 \\
&= \frac{1}{2}.
\end{aligned}$$

Conclusion. Since $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$ or $+\infty$, since b_n converges, so does a_n