

**Comprehensive Compendium:**  
Calculus II

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August 28, 2023  
United States

## Contents

<b>1</b>	<b>Calculus II</b>	<b>3</b>
1.1	Chapter 1 Definitions and Theorems . . . . .	3
1.2	Chapter 2 Definitions and Theorems . . . . .	4
1.3	Chapter 3 Definitions and Theorems . . . . .	7
1.4	Chapter 5 Definitions and Theorems . . . . .	12
1.5	Chapter 6 Definitions and Theorems . . . . .	20
1.6	Chapter 6 Problems to Remember . . . . .	27
<b>2</b>	<b>Fundemental Physics: Classical Mechanics</b>	<b>30</b>
2.1	Chapter 1: Units and Measurement . . . . .	30
2.1.1	Key Terms . . . . .	30
2.1.2	Defintions and Theorems (and important things) . . . . .	32
2.1.3	Fundemental Tables and figures . . . . .	36
2.1.4	Memorize conversions . . . . .	38
2.1.5	Problems to remember . . . . .	39
2.2	Chapter 2: Vectors . . . . .	40
2.2.1	Vocabulary . . . . .	40
2.2.2	Definitions and theorems (and important things) . . . . .	41
2.2.3	Problems to remember . . . . .	49
2.3	Chapter 3: Motion along a straight line . . . . .	50
2.3.1	Definitions and theorems . . . . .	50
2.4	Chapter 4: Motion in two and three dimensions . . . . .	54
2.4.1	Definitions and theorems . . . . .	54
2.5	Chapter 5: Newton's laws of motion . . . . .	61
2.5.1	Definitions and Theorems . . . . .	61
2.6	Chapter 6: Applications of Newton's Laws . . . . .	68

2.6.1	Definitions and theorems . . . . .	68
2.7	Chapter 7: Work and Kinetic energy . . . . .	73
2.7.1	Definitions and theorems . . . . .	73
2.8	Chapter 8: Potential Energy and Conservation of Energy . . . . .	77
2.8.1	Definitions and theorems . . . . .	77
2.9	Chapter 13: Gravitation . . . . .	81
2.9.1	Definitions and theorems . . . . .	81
2.10	Chapter 9: Linear momentum and collisions . . . . .	86
2.10.1	Definitions and Theorems . . . . .	86
2.11	Chapter 10: Fixed-axis rotation . . . . .	92
2.11.1	Definitions and theorems . . . . .	92
2.12	Chapter 11: Angular Momentum . . . . .	96
2.12.1	Definitions and Theorems . . . . .	96
<b>3</b>	<b>Calculus III</b> . . . . .	<b>98</b>
3.1	Chapter 1: Parametric equations and polar coordinates . . . . .	98
3.1.1	Definitions and Theorems . . . . .	98
3.1.2	Problems to remember . . . . .	105
3.2	Chapter 2: Vectors in Space . . . . .	107
3.2.1	Definitions and Theorems . . . . .	107
3.3	Conic sections and quadric surfaces . . . . .	118
3.3.1	Conic sections: Parabola, Ellipse, and Hyperbola . . . . .	118
3.3.2	Quadric Surfaces . . . . .	123
3.4	Chapter 3: Vector-Valued Functions . . . . .	126
3.4.1	Definitions and Theorems . . . . .	126
3.5	Chapter 4: Differentiation of Functions of Several Variables . . . . .	133
3.5.1	Definitions and Theorems . . . . .	133
3.6	Chapter 5: Multiple integration . . . . .	147
3.6.1	Definitions and theorems . . . . .	147
3.7	Chapter 6: Vector calculus . . . . .	159
3.7.1	Definitions and theorems . . . . .	159

## Calculus II

### 1.1 Chapter 1 Definitions and Theorems

- **Mean Value Theorem For Integrals:** If  $f(x)$  is continuous over an interval  $[a,b]$ , then there is at least one point  $c \in [a, b]$  such that

$$f(c) = \frac{1}{b-a} \int f(x) dx.$$

- Integrals resulting in inverse trig functions

1.

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{|a|} + C.$$

2.

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C.$$

3.

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{|a|} \sec^{-1} \frac{|x|}{a} + C.$$

## 1.2 Chapter 2 Definitions and Theorems

- Area between two curves, integrating on the x-axis

$$A = \int_a^b [f(x) - g(x)] dx \quad (1)$$

Where  $f(x) \geq g(x)$

$$A = \int_a^b [g(x) - f(x)] dx.$$

for  $g(x) \geq f(x)$

- Area between two curves, integrating on the y-axis

$$A = \int_c^d [u(y) - v(y)] dy \quad (2)$$

- Areas of compound regions

$$\int_a^b |f(x) - g(x)| dx.$$

- Area of complex regions

$$\int_a^b f(x) dx + \int_b^c g(x) dx.$$

- Slicing Method

$$V(s) = \sum_{i=1}^n A(x_i^*) \Delta x = \int_a^b A(x) dx.$$

- Disk Method along the x-axis

$$V = \int_a^b \pi[f(x)]^2 dx \quad (3)$$

- Disk Method along the y-axis

$$V = \int_c^d \pi[g(y)]^2 dy \quad (4)$$

- Washer Method along the x-axis

$$V = \int_a^b \pi[(f(x))^2 - (g(x))^2] dx \quad (5)$$

- Washer Method along the y-axis

$$V = \int_c^d \pi[(u(y))^2 - (v(y))^2] dy \quad (6)$$

- Radius if revolved around other line (Washer Method)

$$\text{If : } x = -k \\ \text{Then : } r = \text{Function} + k.$$

$$\text{If : } x = k \\ \text{Then : } r = k - \text{Function}.$$

- Method of Cylindrical Shells (x-axis)

$$V = \int_a^b 2\pi x f(x) dx \quad (7)$$

- Method of Cylindrical Shells (y-axis)

$$V = \int_c^d 2\pi y g(y) dy \quad (8)$$

- Region revolved around other line (method of cylindrical shells):

$$\text{If : } x = -k \\ \text{Then : } V = \int_a^b 2\pi(x + k)(f(x)) dx.$$

$$\text{If : } x = k \\ \text{Then : } V = \int_a^b 2\pi(k - x)(f(x)) dx.$$

- A Region of Revolution Bounded by the Graphs of Two Functions (method cylindrical shells)

$$V = \int_a^b 2\pi x [f(x) - g(x)] dx.$$

- Arc Length of a Function of x

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2} dx \quad (9)$$

- Arc Length of a Function of y

$$\text{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2} dy \quad (10)$$

- Surface Area of a Function of x (Around x)

$$\text{Surface Area} = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx \quad (11)$$

- Surface Area of a Function of x (Around y)

$$\text{Surface Area} = \int_a^b 2\pi x \sqrt{1 + [f'(x)]^2} dx \quad (12)$$

$$\text{Or: } \int_a^b 2\pi u(y) \sqrt{1 + (u'(y))^2} dy \quad (13)$$

- Natural logarithm function

$$\ln x = \int_1^x \frac{1}{t} dt \quad (14)$$

- Exponential function

$$y = e^x, \quad \ln y = \ln(e^x) = x \quad (15)$$

- Logarithm Differentiation

$$f'(x) = f(x) \cdot \frac{d}{dx} \ln(f'(x)).$$

**Note:** Use properties of logs before you differentiate whats inside the logarithm

### 1.3 Chapter 3 Definitions and Theorems

- Integration by parts formula

$$\int u \, dv = uv - \int v \, du.$$

- Integration by parts for definite integral

$$\int_a^b u \, dv = uv \Big|_a^b - \int_a^b v \, du$$

- To integrate products involving  $\sin(ax)$ ,  $\sin(bx)$ ,  $\cos(ax)$ , and  $\cos(bx)$ , use the substitutions:

– Sine Products

$$\sin(ax) \sin(bx) = \frac{1}{2} \cos((a-b)x) - \frac{1}{2} \cos((a+b)x)$$

– Sine and Cosine Products

$$\sin(ax) \cos(bx) = \frac{1}{2} \sin((a-b)x) + \frac{1}{2} \sin((a+b)x)$$

– Cosine Products

$$\cos(ax) \cos(bx) = \frac{1}{2} \cos((a-b)x) + \frac{1}{2} \cos((a+b)x)$$

– Power Reduction Formula (sine)

$$\begin{aligned} \int \sin^n x \, dx &= -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx \\ \int_0^{\frac{\pi}{2}} \sin^n x \, dx &= \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx. \end{aligned}$$

– Power Reduction Formula (cosine)

$$\begin{aligned} \int \cos^n x \, dx &= \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx \\ \int_0^{\frac{\pi}{2}} \cos^n x \, dx &= \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \cos^{n-2} x \, dx. \end{aligned}$$

– Power Reduction Formula (secant)

$$\begin{aligned} \int \sec^n x \, dx &= \frac{1}{n-1} \sec^{n-1} x \sin x + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx \\ \int \sec^n x \, dx &= \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx \end{aligned}$$

– Power Reduction Formula (tangent)

$$\int \tan^n x \, dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x \, dx$$

- Trigonometric Substitution

- $\sqrt{a^2 - x^2}$  use  $x = a \sin \theta$  with domain restriction  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- $\sqrt{a^2 + x^2}$  use  $x = a \tan \theta$  with domain restriction  $(-\frac{\pi}{2}, \frac{\pi}{2})$
- $\sqrt{x^2 - a^2}$  use  $x = a \sec \theta$  with domain restriction  $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$

- **Steps for fraction decomposition**

1. Ensure  $\deg(Q) < \deg(P)$ , if not, long divide
2. Factor denominator
3. Split up fraction into factors
4. Multiply through to clear denominator
5. Group terms and equalize
6. Solve for constants
7. Plug constants into split up fraction
8. Compute integral

- **Solving for constants** Either:

- Plug in values (often the roots)
- Equalize

- **Cases for partial fractions**

- Non repeated linear factors
- Repeated linear factors
- Nonfactorable quadratic factors

- **Midpoint rule**

$$M_n = \sum_{i=1}^n f(m_i) \Delta x.$$

- **Absolute error**

$$err = \left| \text{Actual} - \text{Estimated} \right|.$$

- **Relative error**

$$err = \left| \frac{\text{Actual} - \text{Estimated}}{\text{Actual}} \right| \cdot 100\%.$$

- **Error upper bound for midpoint rule**

$$E_M \leq \frac{M(b-a)^3}{24n^2}$$

Where  $M$  is the maximum value of the second derivative

- **Trapezoidal rule**

$$T_n \frac{1}{2} \Delta x (f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n))$$

- **Error upper bound for trapezoidal rule**

$$E_T \leq \frac{M(b-a)^3}{12n^2}$$

Where  $M$  is the maximum value of the second derivative

- **Simpson's rule**

$$S_n = \frac{\Delta x}{3} (f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n))$$

- **Error upper bound for Simpson's rule**

$$E_S \leq \frac{M(b-a)^5}{180n^4}$$

Where  $M$  is the maximum value of the fourth derivative

- **Finding  $n$  with error bound functions**

1. Find  $f''(x)$
2. Find maximum values of  $f''(x)$  in the interval
3. Plug into error bound function
4. Set value  $\leq$  desired accuracy (ex: 0.01)
5. Solve:
6. If we were to truncate, we would use the ceil function  $\lceil n \rceil$  DO NOT FLOOR

- **Improper integrals (Infinite interval)**

$$\begin{aligned} - \int_a^{+\infty} f(x) dx &= \lim_{t \rightarrow +\infty} \int_a^t f(x) dx \\ - \int_{-\infty}^b f(x) dx &= \lim_{t \rightarrow -\infty} \int_t^b f(x) dx \\ - \int_{-\infty}^{+\infty} f(x) dx &= \int_{-\infty}^0 f(x) dx + \int_0^{+\infty} f(x) dx \end{aligned}$$

- **Improper integral (discontinuous)**

- Let  $f(x)$  be continuous on  $[a, b]$ , then;

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx .$$

- Let  $f(x)$  be continuous on  $(a, b]$ , then;

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^+} \int_t^b f(x) dx .$$

In each case, if the limit exists, then the improper integral is said to converge. If the limit does not exist, then the improper integral is said to diverge.

- Let  $f(x)$  be continuous on  $[a, b]$  except at a point  $c \in (a, b)$ , then;

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx .$$

If either integral diverges, then  $\int_a^b f(x) dx$  diverges

- **Comparison theorem** Let  $f(x)$  and  $g(x)$  be continuous over  $[a, +\infty)$ . Assume that  $0 \leq f(x) \leq g(x)$  for  $x \geq a$ .
  - If  $\int_a^{+\infty} f(x) dx = \lim_{t \rightarrow +\infty} \int_a^t f(x) dx = +\infty$ ,  
then  $\int_a^{+\infty} g(x) dx = \lim_{t \rightarrow +\infty} \int_a^t g(x) dx = +\infty$ .
  - If  $\int_a^{+\infty} g(x) dx = \lim_{t \rightarrow +\infty} \int_a^t g(x) dx = L$ , where  $L$  is a real number,  
then  $\int_a^{+\infty} f(x) dx = \lim_{t \rightarrow +\infty} \int_a^t f(x) dx = M$  for some real number  $M \leq L$ .

- **P-integrals**

$$\begin{aligned}
 - \int_0^{+\infty} \frac{1}{x^p} dx &= \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ +\infty & \text{if } p \leq 1 \end{cases} \\
 - \int_0^1 \frac{1}{x^p} dx &= \begin{cases} \frac{1}{1-p} & \text{if } p < 1 \\ +\infty & \text{if } p \geq 1 \end{cases} \\
 - \int_a^{+\infty} \frac{1}{x^p} dx &= \begin{cases} \frac{a^{1-p}}{p-1} & \text{if } p > 1 \\ +\infty & \text{if } p \leq 1 \end{cases} \\
 - \int_0^a \frac{1}{x^p} dx &= \begin{cases} \frac{a^{1-p}}{1-p} & \text{if } p < 1 \\ +\infty & \text{if } p \geq 1 \end{cases}
 \end{aligned}$$

- **Bypass L'Hospital's Rule**

$$\ln(\ln(x)), \ln(x), \dots, x^{\frac{1}{100}}, x^{\frac{1}{3}}, \sqrt{x}, 1, x^2, x^3, \dots, e^x, e^{2x}, e^{3x}, \dots, e^{x^2}, \dots, e^{e^x}.$$

Essentially what it means is things on the right grow faster than things on the left.  
Thus, if we have say:

$$\lim_{x \rightarrow \infty} \frac{x^2}{e^{2x}}.$$

We can be sure that it is zero. Because this is  $x^2 \cdot e^{-2x}$ . If we take  $\lim_{x \rightarrow \infty} x^2 e^{-2x}$ , we get  $\infty \cdot 0$ . As we see by the sequence  $e^{-2x}$  overrules  $x^2$  and we can say the limit is zero.

- **Consideration for Limits:** Let  $f : A \rightarrow B$  be a function defined by  $x \mapsto f(x)$ . If a point  $c$  lies outside the domain  $A$ , then the expression  $\lim_{x \rightarrow c} f(x)$  is not meaningful, and we classify this limit as undefined. For instance, the function arcsine has a domain of  $[-1, 1]$ . Therefore, limits like  $\lim_{x \rightarrow a} \sin^{-1}(x)$  where  $a \notin [-1, 1]$  are undefined.

- **Why does**

$$\lim_{x \rightarrow 2} \tan^{-1} \frac{1}{x-2}.$$

$$\begin{aligned}
 &= \lim_{x \rightarrow 2^-} \tan^{-1} \frac{1}{x-2} && = \lim_{x \rightarrow 2^+} \tan^{-1} \frac{1}{x-2} \\
 &= \lim_{x \rightarrow -\infty} \tan^{-1} x && = \lim_{x \rightarrow +\infty} \tan^{-1} x \\
 &= -\pi/2. && = \frac{\pi}{2}.
 \end{aligned}$$

## 1.4 Chapter 5 Definitions and Theorems

- Sequence notation

$$\{a_n\}_{n=1}^{\infty}, \text{ or simply } \{a_n\}.$$

- Sequence notation (ordered list)

$$a_1, a_2, a_3, \dots, a_n, \dots$$

- Arithmetic Sequence Difference

$$d = a_n - a_{n-1}.$$

- Arithmetic sequence (common difference between subsequent terms) general form

$$\text{Index starting at 0 : } a_n = a + nd$$

$$\text{Index starting at 1 : } a_n = a + (n - 1)d$$

- Arithmetic sequence (common difference between subsequent terms) recursive form

$$a_n = a_{n-1} + d.$$

- Sum of arithmetic sequence

$$S_n = \frac{n}{2} [a + a_n]$$

$$S_n = \frac{n}{2} [2a + (n - 1)d].$$

- Geometric sequence form common ratio

$$r = \frac{a_n}{a_{n-1}}.$$

- Geometric sequence general form

$$a_n = ar^n \text{ (Index starting at 0)}$$

$$a_n = a^{n+1} \text{ (index starting at 0 and a=r)}$$

$$a_n = ar^{n-1} \text{ (Index starting at 1)}$$

$$a_n = a^n \text{ (index starting at 1 and a=r).}$$

- Geometric sequence recursive form

$$a_n = ra_{n-1}.$$

- Sum of geometric sequence (finite terms)

$$S_n = \frac{a(1 - r^n)}{1 - r} \quad r \neq 1.$$

- **Convergence / Divergence:** If

$$\lim_{n \rightarrow +\infty} a_n = L.$$

We say that the sequence converges, else it diverges

- **Formal definition of limit of sequence**

$$\lim_{n \rightarrow +\infty} a_n = L \iff \forall \varepsilon > 0, \exists N \in \mathbb{Z} \mid |a_n - L| < \varepsilon, \text{ if } n \geq N.$$

Then we can say

$$\lim_{n \rightarrow +\infty} a_n = L \text{ or } a_n \rightarrow L.$$

- **Limit of a sequence defined by a function:** Consider a sequence  $\{a_n\}$  such that  $a_n = f(n)$  for all  $n \geq 1$ . If there exists a real number  $L$  such that

$$\lim_{x \rightarrow \infty} f(x) = L,$$

then  $\{a_n\}$  converges and

$$\lim_{n \rightarrow \infty} a_n = L.$$

- **Algebraic limit laws:** Given sequences  $\{a_n\}$  and  $\{b_n\}$  and any real number  $c$ , if there exist constants  $A$  and  $B$  such that  $\lim_{n \rightarrow \infty} a_n = A$  and  $\lim_{n \rightarrow \infty} b_n = B$ , then

- $\lim_{n \rightarrow \infty} c = c$
- $\lim_{n \rightarrow \infty} ca_n = c \lim_{n \rightarrow \infty} a_n = cA$
- $\lim_{n \rightarrow \infty} (a_n \pm b_n) = \lim_{n \rightarrow \infty} a_n \pm \lim_{n \rightarrow \infty} b_n = A \pm B$
- $\lim_{n \rightarrow \infty} (a_n \cdot b_n) = (\lim_{n \rightarrow \infty} a_n) \cdot (\lim_{n \rightarrow \infty} b_n) = A \cdot B$
- $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n} = \frac{A}{B}$ , provided  $B \neq 0$  and each  $b_n \neq 0$ .

- **Continuous Functions Defined on Convergent Sequences:** Consider a sequence  $\{a_n\}$  and suppose there exists a real number  $L$  such that the sequence  $\{a_n\}$  converges to  $L$ . Suppose  $f$  is a continuous function at  $L$ . Then there exists an integer  $N$  such that  $f$  is defined at all values  $a_n$  for  $n \geq N$ , and the sequence  $\{f(a_n)\}$  converges to  $f(L)$ .

- **Squeeze Theorem for Sequences:** Consider sequences  $\{a_n\}$ ,  $\{b_n\}$ , and  $\{c_n\}$ . Suppose there exists an integer  $N$  such that

$$a_n \leq b_n \leq c_n \text{ for all } n \geq N.$$

If there exists a real number  $L$  such that

$$\lim_{n \rightarrow \infty} a_n = L = \lim_{n \rightarrow \infty} c_n,$$

then  $\{b_n\}$  converges and  $\lim_{n \rightarrow \infty} b_n = L$

- **Bounded above:** A sequence  $\{a_n\}$  is bounded above if there exists a real number  $M$  such that

$$a_n \leq M$$

for all positive integers  $n$ .

- **Bounded below:** A sequence  $\{a_n\}$  is bounded below if there exists a real number  $M$  such that

$$M \leq a_n$$

for all positive integers  $n$ .

- **Bounded:** A sequence  $\{a_n\}$  is a bounded sequence if it is bounded above and bounded below.
- **Unbounded:** If a sequence is not bounded, it is an unbounded sequence.
- **If a sequence  $\{a_n\}$  converges, then it is bounded.**
- **Increasing sequence:** A sequence  $\{a_n\}$  is increasing for all  $n \geq n_0$  if

$$a_n \leq a_{n+1} \text{ for all } n \geq n_0.$$

- **Decreasing sequence:** A sequence  $\{a_n\}$  is decreasing for all  $n \geq n_0$  if

$$a_n \geq a_{n+1} \text{ for all } n \geq n_0.$$

- **Monotone sequence:** A sequence  $\{a_n\}$  is a **monotone sequence** for all  $n \geq n_0$  if it is increasing for all  $n \geq n_0$  or decreasing for all  $n \geq n_0$
- **Monotone Convergence Theorem:** If  $\{a_n\}$  is a bounded sequence and there exists a positive integer  $n_0$  such that  $\{a_n\}$  is monotone for all  $n \geq n_0$ , then  $\{a_n\}$  converges.
- **Infinite Series form:**

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

- **Partial sum ( $k^{th}$  partial sum)**

$$S_k = \sum_{n=1}^k a_n = a_1 + a_2 + a_3 + \dots + a_k.$$

- **Convergence of infinity series notation**

For a series, say...

$$\sum_{n=1}^{\infty} a_n .$$

its convergence is determined by the limit of its sequence of partial sums. Specifically, if

$$\lim_{n \rightarrow +\infty} S_n = S \rightarrow \sum_{n=1}^{\infty} a_n = S.$$

- **Harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

Which diverges to  $+\infty$

- **Algebraic Properties of Convergent Series** Let  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  be convergent series. Then the following algebraic properties hold:

1. The series  $\sum_{n=1}^{\infty} (a_n + b_n)$  converges and

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n. \quad (\text{Sum Rule}).$$

2. The series  $\sum_{n=1}^{\infty} (a_n - b_n)$  converges and

$$\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n. \quad (\text{Difference Rule}).$$

3. For any real number  $c$ , the series  $\sum_{n=1}^{\infty} ca_n$  converges and

$$\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n. \quad (\text{Constant Multiple Rule}).$$

- **Geometric series convergence or divergence:**

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}.$$

- **Divergence test:** In the context of sequences, if  $\lim_{n \rightarrow \infty} a_n = c \neq 0$  or the limit does not exist, then the series  $\sum_{n=1}^{\infty} a_n$  is said to diverge. The converse is not true.

Because:

$$\lim_{k \rightarrow \infty} a_k = \lim_{k \rightarrow \infty} (S_k - S_{k-1}) = \lim_{k \rightarrow \infty} S_k - \lim_{k \rightarrow \infty} S_{k-1} = S - S = 0..$$

- **Integral Test Prelude:** for any integer  $k$ , the  $k$ th partial sum  $S_k$  satisfies

$$S_k = a_1 + a_2 + a_3 + \cdots + a_k < a_1 + \int_1^k f(x) dx < a_1 + \int_1^{\infty} f(x) dx..$$

and

$$S_k = a_1 + a_2 + a_3 + \cdots + a_k > \int_1^{k+1} f(x) dx..$$

- **Integral test** Suppose  $\sum_{n=1}^{\infty} a_n$  is a series with positive terms  $a_n$ . Suppose there exists a function  $f$  and a positive integer  $N$  such that the following three conditions are satisfied:

1.  $f$  positive, continuous, and decreasing on  $[N, \infty)$

2.  $f(n) = a_n$  for all integers  $n \geq N$ ,  $N \in \mathbb{Z}^+$

Then the series  $\sum_{n=1}^{\infty} a_n$  and the improper integral  $\int_N^{\infty} f(x) dx$  either both converge or both diverge..

- **P-series**  $\forall p \in \mathbb{R}$ , the series

$$\sum_{n=1}^{\infty} \frac{1}{n^P}.$$

Is called a **p-series**. Furthermore,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1. \end{cases}.$$

- **P-series extended**

$$\sum_{n=2}^{\infty} \frac{1}{n \ln(n)^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1. \end{cases}.$$

- **Remainder estimate for the integral test** Suppose  $\sum_{n=1}^{\infty} a_n$  is a convergent series with positive terms. Suppose there exists a function  $f$  and a positive integer  $M$  satisfying the following three conditions:

1.  $f$  is positive, decreasing, and continuous on  $[M, \infty)$
2.  $f(n) = a_n$  for all integers  $n \geq M$ .

Let  $S_N$  be the  $N$ th partial sum of  $\sum_{n=1}^{\infty} a_n$ . For all positive integers  $N$ ,

$$S_N + \int_{N+1}^{\infty} f(x) dx < \sum_{n=1}^{\infty} a_n < S_N + \int_N^{\infty} f(x) dx.$$

In other words, the remainder  $R_N = \sum_{n=1}^{\infty} a_n - S_N = \sum_{n=N+1}^{\infty} a_n$  satisfies the following estimate:

$$\int_{N+1}^{\infty} f(x) dx < R_N < \int_N^{\infty} f(x) dx.$$

This is known as the remainder estimate

To find a value of  $N$  such that we are within a desired margin of error, Since we know  $R_n < \int_N^{\infty} f(x) dx$ . Simply compute the improper integral and set the result  $<$  the desired error to solve for  $N$

- **Find  $a_n$  given the expression for the partial sum**

$$a_n = S_n - S_{n-1}.$$

- **telescoping series:** Telescoping series are a type of series where each term cancels out a part of another term, leaving only a few terms that do not cancel. When you sum the series, most of the terms collapse or "telescope," which simplifies the calculation of the sum. Here are some key points and generalizations you can note about telescoping series:

- Partial Fraction Decomposition
- Cancellation Pattern: In a telescoping series, look for a pattern where a term in one fraction will cancel out with a term in another fraction.
- Write out Terms
- What is left is  $S_n$ , thus the sum of the series is the  $\lim_{n \rightarrow \infty} S_n$

Try:

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}.$$

Hint, its not only the first and last terms cancel, we also have a  $\frac{1}{n}$ , when  $a_{n-1}$ : Answer is  $\frac{3}{4}$

- **Comparison test for series**

1. Suppose there exists an integer  $N$  such that  $0 \leq a_n \leq b_n$  for all  $n \geq N$ . If  $\sum_{n=1}^{\infty} b_n$  converges, then  $\sum_{n=1}^{\infty} a_n$  converges.
2. Suppose there exists an integer  $N$  such that  $a_n \geq b_n \geq 0$  for all  $n \geq N$ . If  $\sum_{n=1}^{\infty} b_n$  diverges, then  $\sum_{n=1}^{\infty} a_n$  diverges.

- **Limit Comparison Test** Let  $a_n, b_n \geq 0$  for all  $n \geq 1$ .

- If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L \neq 0$ , then  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  both converge or both diverge.
- If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$  and  $\sum_{n=1}^{\infty} b_n$  converges, then  $\sum_{n=1}^{\infty} a_n$  converges.
- If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$  and  $\sum_{n=1}^{\infty} b_n$  diverges, then  $\sum_{n=1}^{\infty} a_n$  diverges.

**Note:** Note that if  $\frac{a_n}{b_n} \rightarrow 0$  and  $\sum_{n=1}^{\infty} b_n$  diverges, the limit comparison test gives no information. Similarly, if  $\frac{a_n}{b_n} \rightarrow \infty$  and  $\sum_{n=1}^{\infty} b_n$  converges, the test also provides no information.

Consider the series

$$\sum_{n=1}^{\infty} \frac{n^4 + 6}{n^5 + 4}.$$

To find our  $b_n$  we can only focus on the leading coefficients. Thus:

$$b_n = \frac{n^4}{n^5} = \frac{1}{n}.$$

So our test...

Since  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} \neq 0 \vee +\infty$ . And  $\frac{1}{n}$  diverges, we can conclude that  $a_n$  will also diverge.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{\frac{n^4 + 6}{n^5 + 4}}{\frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} \frac{n(n^4 + 6)}{n^5 + 4} \\ &= \lim_{n \rightarrow \infty} \frac{n^5 + 6n}{n^5 + 4} \\ &= 1. \end{aligned}$$

- **Determine which series (or function) is greater**

- **Subtraction:** Given two functions  $f(x) = \frac{1}{x}$  and  $g(x) = \frac{x^4 + 6}{x^5 + 4}$ , we want to compare them by considering the function  $h(x) = f(x) - g(x)$ :

$$h(x) = f(x) - g(x) = \frac{1}{x} - \frac{x^4 + 6}{x^5 + 4}$$

To compare these directly, it would be helpful to have a common denominator:

$$h(x) = \frac{x^4 + 4 - (x^4 + 6)}{x(x^5 + 4)} = \frac{-2}{x(x^5 + 4)}$$

Now, we can see that the sign of  $h(x)$  depends on the sign of  $x$  because the denominator  $x(x^5 + 4)$  is always positive for  $x \neq 0$ . So:

- \* For  $x > 0$ ,  $h(x) < 0$ , which means  $f(x) < g(x)$ .
- \* For  $x < 0$ ,  $h(x) > 0$ , which means  $f(x) > g(x)$ .

- **Alternating Series** Any series whose terms alternate between positive and negative values is called an alternating series. An alternating series can be written in the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \dots$$

or

$$\sum_{n=1}^{\infty} (-1)^n b_n = -b_1 + b_2 - b_3 + b_4 - \dots$$

Where  $b_n > 0$  for all positive integers  $n$ .

- **alternating series test (Leibniz criterion)** An alternating series of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^n b_n$$

converges if

- $0 < b_{n+1} \leq b_n \forall n \geq 1$
- $\lim_{n \rightarrow \infty} b_n = 0$ .

**Note:** We remark that this theorem is true more generally as long as there exists some integer  $N$  such that  $0 < b_{n+1} \leq b_n$  for all  $n \geq N$ .

**Additional note:** The AST allows us to consider just the positive terms to check for these two conditions because if a series of decreasing positive terms that approach zero is alternated in sign, the alternating series will converge. This is a special property of alternating series that does not generally hold for non-alternating series.

- **Show decreasing (For the AST):** Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

So you see we have  $b_n = \frac{1}{n^2}$ . For the AST, we must show that this is decreasing. If  $b_{n+1} = \frac{1}{(n+1)^2}$ . Then we see

$$\frac{1}{(n+1)^2} < \frac{1}{n^2}.$$

Thus it is decreasing for  $n \geq 1$  ( $b_{n+1} < b_n$ ) ■

- **Remainders in alternating series** Consider an alternating series of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^n b_n,$$

that satisfies the hypotheses of the alternating series test. Let  $S$  denote the sum of the series and  $S_N$  denote the  $N$ -th partial sum. For any integer  $N \geq 1$ , the remainder  $R_N = S - S_N$  satisfies

$$|R_N| \leq b_{N+1}.$$

This tells us that if we stop at the  $N^{th}$  term, the error we are making is at most the size of the next term

- **Absolute and conditional convergence**

- A series  $\sum_{n=1}^{\infty} a_n$  exhibits absolute convergence if  $\sum_{n=1}^{\infty} |a_n|$  converges.
- A series  $\sum_{n=1}^{\infty} a_n$  exhibits conditional convergence if  $\sum_{n=1}^{\infty} a_n$  converges but  $\sum_{n=1}^{\infty} |a_n|$  diverges.
- If  $\sum_{n=1}^{\infty} |a_n|$  converges then  $\sum_{n=1}^{\infty} a_n$  converges

**Note:** if  $|a_n|$  diverges, we cannot have absolute convergence, thus we must examine to see if normal  $a_n$  converges, in which case we would have conditional convergence

**Big Note:** If a series not strictly decreasing, we can still check for absolute/conditional convergence. Take  $\sum_{n=1}^{\infty} \frac{\sin n}{3^n + 4}$  for example.

- **Ratio test** Let  $\sum_{n=1}^{\infty} a_n$  be a series with nonzero terms. Let

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| ..$$

Then:

- If  $0 \leq \rho < 1$ , then  $\sum_{n=1}^{\infty} a_n$  converges absolutely.
- If  $\rho > 1$  or  $\rho = \infty$ , then  $\sum_{n=1}^{\infty} a_n$  diverges.
- If  $\rho = 1$ , the test does not provide any information.

**Note:** The ratio test is useful for series whose terms involve factorials

- **Root test** Consider the series  $\sum_{n=1}^{\infty} a_n$ . Let

$$\rho = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} ..$$

- If  $0 \leq \rho < 1$ , then  $\sum_{n=1}^{\infty} a_n$  converges absolutely.
- If  $\rho > 1$  or  $\rho = \infty$ , then  $\sum_{n=1}^{\infty} a_n$  diverges.
- If  $\rho = 1$ , the test does not provide any information.

**Note:** The root test is useful for series whose terms involve exponentials

- **Which tests require positive terms**

- The **Integral Test:** This test applies to series where the terms come from a function that is positive, continuous, and decreasing on a certain interval. The convergence or divergence of the series is determined by the convergence or divergence of the corresponding improper integral of the function.
- The **Remainder estimate for the integral test**
- The **Comparison Test:** This test compares the terms of a series to those of another series with known convergence behavior. It requires that the terms of both series be positive or non-negative.
- The **Limit Comparison Test:** Similar to the Comparison Test, this test involves comparing the terms of two series by taking the limit of the ratio of their terms. It requires that the terms of both series be positive.
- In **alternating series**,  $b_n$  must have only positive terms

## 1.5 Chapter 6 Definitions and Theorems

- Euler definition for  $e$

$$\begin{aligned} e^a &= \lim_{n \rightarrow \infty} \left(1 + \frac{a}{n}\right)^n \\ \frac{1}{e^a} &= \lim_{n \rightarrow \infty} \left(1 + \frac{-a}{n}\right)^n \\ \frac{1}{e^a} &= \lim_{n \rightarrow \infty} \left(\frac{n}{n+a}\right)^n. \end{aligned}$$

- Other definition for  $e$

$$\begin{aligned} e &= \sum_{n=0}^{\infty} \frac{1}{n!} \\ e - 1 &= \sum_{n=1}^{\infty} \frac{1}{n!} \\ e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!}. \end{aligned}$$

- Power series: A series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots.$$

is a power series centered at  $x = 0$ .

A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots.$$

is a power series centered at  $x = a$ .

- Convergence of a Power Series

Consider the power series  $\sum_{n=0}^{\infty} c_n (x-a)^n$ . The series satisfies exactly one of the following properties:

- (i) The series converges at  $x = a$  and diverges for all  $x \neq a$ .
- (ii) The series converges for all real numbers  $x$ .
- (iii) There exists a real number  $R > 0$  such that the series converges if  $|x-a| < R$  and diverges if  $|x-a| > R$ . At the values  $x$  where  $|x-a| = R$ , the series may converge or diverge.

- A power series always converges at its center

- Radius of convergence: Consider the power series  $\sum_{n=0}^{\infty} c_n (x-a)^n$ . The set of real numbers  $x$  where the series converges is the interval of convergence. If there exists a real number  $R > 0$  such that the series converges for  $|x-a| < R$  and diverges for  $|x-a| > R$ , then  $R$  is the radius of convergence. If the series converges only at  $x = a$ , we say the radius of convergence is  $R = 0$ . If the series converges for all real numbers  $x$ , we say the radius of convergence is  $R = \infty$

- **Finding interval of convergence and radius of convergence**
  - Fact: power series is always convergent on its center
  - Use ratio test (values of  $\rho$ )
  - Use  $\rho < 1$  to find Radius of convergence
  - Test end points of interval by plugging into original series and seeing whether the series is convergent or divergent
- **If  $\rho = 0$ , the power series converges for all  $x$**
- **If  $\rho = \infty$ , the series diverges for all  $x \neq a$**
- **Combining Power Series:** Suppose that the two power series  $\sum_{n=0}^{\infty} c_n x^n$  and  $\sum_{n=0}^{\infty} d_n x^n$  converge to the functions  $f$  and  $g$ , respectively, on a common interval  $I$ .
  - (i) The power series  $\sum_{n=0}^{\infty} (c_n x^n \pm d_n x^n)$  converges to  $f \pm g$  on  $I$ .
  - (ii) For any integer  $m \geq 0$  and any real number  $b$ , the power series  $\sum_{n=0}^{\infty} b x^m c_n x^n$  converges to  $b x^m f(x)$  on  $I$ .

Eg: If we know  $\sum_{n=0}^{\infty} a_n x^n$  has  $I = (-1, 1)$ . Then

$$\begin{aligned} & \sum_{n=0}^{\infty} a_n 3^n x^n \\ &= \sum_{n=0}^{\infty} a_n (3x)^n \\ & I = (-3, 3). \end{aligned}$$
  - (iii) For any integer  $m \geq 0$  and any real number  $b$ , the series  $\sum_{n=0}^{\infty} c_n (bx^m)^n$  converges to  $f(bx^m)$  for all  $x$  such that  $bx^m$  is in  $I$ .
- **For part I, II, and III, the interval of the combined series is the smaller interval**
- **Cauchy product (Multiplying power series):** Suppose that the power series  $\sum_{n=0}^{\infty} c_n x^n$  and  $\sum_{n=0}^{\infty} d_n x^n$  converge to  $f$  and  $g$ , respectively, on a common interval  $I$ . Let

$$\begin{aligned} e_n &= c_0 d_n + c_1 d_{n-1} + c_2 d_{n-2} + \cdots + c_{n-1} d_1 + c_n d_0 \\ &= \sum_{k=0}^n c_k d_{n-k} \end{aligned}$$

Then

$$\left( \sum_{n=0}^{\infty} c_n x^n \right) \left( \sum_{n=0}^{\infty} d_n x^n \right) = \sum_{n=0}^{\infty} e_n x^n$$

and

$$\sum_{n=0}^{\infty} e_n x^n \text{ converges to } f(x) \cdot g(x) \text{ on } I.$$

The series  $\sum_{n=0}^{\infty} e_n x^n$  is known as the Cauchy product of the series  $\sum_{n=0}^{\infty} c_n x^n$  and  $\sum_{n=0}^{\infty} d_n x^n$ .

- **Sterling's Approximation**

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$

- **Gamma function (extension of the factorial function)**

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

Thus,  $n! = \Gamma(n+1)$ .

- **Cool definition for  $e^x$**

$$\begin{aligned} f'(x) &= rf(x) \\ \implies f(x) &= ce^{rx}. \end{aligned}$$

- **Term-by-Term Differentiation and Integration for Power Series.** Suppose that the power series  $\sum_{n=0}^\infty c_n(x-a)^n$  converges on the interval  $(a-R, a+R)$  for some  $R > 0$ . Let  $f$  be the function defined by the series

$$f(x) = \sum_{n=0}^\infty c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

for  $|x-a| < R$ . Then  $f$  is differentiable on the interval  $(a-R, a+R)$  and we can find  $f'$  by differentiating the series term-by-term:

$$f'(x) = \sum_{n=1}^\infty nc_n(x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots$$

for  $|x-a| < R$ . Also, to find  $\int f(x) dx$ , we can integrate the series term-by-term. The resulting series converges on  $(a-R, a+R)$ , and we have

$$\int f(x) dx = C + \sum_{n=0}^\infty \frac{c_n(x-a)^{n+1}}{n+1} = C + c_0(x-a) + \frac{c_1(x-a)^2}{2} + \frac{c_2(x-a)^3}{3} + \dots$$

for  $|x-a| < R$ .

**NOTE!** when a power series is differentiated or integrated term-by-term, it says nothing about what happens at the endpoints.

- **Uniqueness of Power Series:** Let  $\sum_{n=0}^\infty c_n(x-a)^n$  and  $\sum_{n=0}^\infty d_n(x-a)^n$  be two convergent power series such that

$$\sum_{n=0}^\infty c_n(x-a)^n = \sum_{n=0}^\infty d_n(x-a)^n$$

for all  $x$  in an open interval containing  $a$ . Then  $c_n = d_n$  for all  $n \geq 0$ .

- **When finding the Cauchy product of two power series, we include the zero term when finding the new power series general term. For integrating and differentiating, we do not**
- **Taylor and Maclaurin series:** If  $f$  has derivatives of all orders at  $x = a$ , then the Taylor series for the function  $f$  at  $a$  is

$$\sum_{n=0}^\infty \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots \quad (16)$$

The Taylor series for  $f$  at 0 is known as the Maclaurin series for  $f$ .

- **Uniqueness of Taylor series:** If a function  $f$  has a power series at  $a$  that converges to  $f$  on some open interval containing  $a$ , then that power series is the Taylor series for  $f$  at  $a$ .
- **Taylor-Macluarin Polynomials:** If  $f$  has  $n$  derivatives at  $x = a$ , then the  $n$ th Taylor polynomial for  $f$  at  $a$  is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \quad (17)$$

The  $n$ th Taylor polynomial for  $f$  at 0 is known as the  $n$ th Maclaurin polynomial for  $f$ .

- **Taylor's Theorem with Remainder:** Let  $f$  be a function that can be differentiated  $n+1$  times on an interval  $I$  containing the real number  $a$ . Let  $p_n$  be the  $n$ th Taylor polynomial of  $f$  at  $a$  and let

$$R_n(x) = f(x) - p_n(x)$$

be the  $n$ th remainder. Then for each  $x$  in the interval  $I$ , there exists a real number  $c$  between  $a$  and  $x$  such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

If there exists a real number  $M$  such that  $|f^{(n+1)}(x)| \leq M$  for all  $x \in I$ , then

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$$

$\forall x \in I$

- **Maclaurin Series/Polynomials for sine:** The Taylor series for the sine function is

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad \text{For } x \in \mathbb{R}.$$

Where  $p_n$  obeys

$$\begin{aligned} p_{2m+1} &= p_{2m+2} \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{(-1)^m x^{2m+1}}{(2m+1)!}. \end{aligned}$$

**Note:** When discussing specific polynomials, say  $P_5$  for example, we arnt talking about the first 5 terms in the series above, we are talking about the polynomial up to degree 5. Thus it would have 3 terms

- **Maclaurin Series/Polynomials for cosine:** Similar to the sine function, the Maclaurin series for the cosine function is

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad \text{For } x \in \mathbb{R}.$$

Where  $p_n$  obeys

$$\begin{aligned} p_{2m} &= p_{2m+1} \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^n \frac{x^{2m}}{(2m)!}. \end{aligned}$$

- **Maclaurin Series/Polynomials for  $e^x$ :** We find the Maclaurin series for the exponential function to be

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad \text{For } x \in \mathbb{R}.$$

**Note:** this definition is described above but now we have a way of showing its truthiness

- **Convergence of Taylor Series:** Suppose that  $f$  has derivatives of all orders on an interval  $I$  containing  $a$ . Then the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

converges to  $f(x)$  for all  $x$  in  $I$  if and only if

$$\lim_{n \rightarrow \infty} R_n(x) = 0$$

for all  $x$  in  $I$ .

**Note:** With this theorem, we can prove that a Taylor series for  $f$  at  $a$  converges to  $f$  if we can prove that the remainder  $R_n(x) \rightarrow 0$ . To prove that  $R_n(x) \rightarrow 0$ , we typically use the bound

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$$

from Taylor's theorem with remainder.

- **Using taylor series to find limits:** Consider the limit  $\lim_{x \rightarrow 0^+} \frac{\cos \sqrt{x} - 1}{2x}$ . We know we have a problem if we attempt to use the **direct substitution property**. Thus, we can substitute  $\cos(\sqrt{x})$  for its **Maclaurin series** and see what happens. We know the maclaurin series for  $\cos(x)$  is

$$\begin{aligned} \cos x &\sim \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\ \implies \cos \sqrt{x} &\sim \sum_{n=0}^{\infty} (-1)^n \frac{(\sqrt{x})^{2n}}{(2n)!} = 1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \frac{(\sqrt{x})^6}{6!} + \dots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \dots \end{aligned}$$

So we have

$$\begin{aligned} &\lim_{x \rightarrow 0^+} \frac{\left(1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \dots\right) - 1}{2x} \\ &= \lim_{x \rightarrow 0^+} \frac{\left(-\frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \dots\right)}{2x} \\ &= \lim_{x \rightarrow 0^+} \left(-\frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \dots\right) \cdot \frac{1}{2x} \\ &= \lim_{x \rightarrow 0^+} -\frac{1}{4} \\ &= -\frac{1}{4}. \end{aligned}$$

- **Multiplying a known Taylor series by some other function:** Consider  $f(x) = x \cos x$ . Since we know that the taylor series for  $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ , which converges  $\forall x \in \mathbb{R}$ . We can easily just multiply this by  $x$  to get the taylor series for  $f(x) = x \cos(x)$ . Since the Taylor series for  $\cos(x)$  converges for all real  $x$ , multiplying it by  $x$  won't affect its convergence properties. The resulting series will still converge for all  $x$ .

**Note:** The product of the Taylor series and the function will be valid only where both the series converges and the function is well-defined. Probably analyze the convergence of the product series.

- **Multiplying a known Taylor series by some other function where convergence is affected:** Consider the example above. Although this time suppose we multiply  $\cos(x)$  by  $\frac{1}{x}$  instead of  $x$ . We know the resulting taylor series must not be convergent at  $x = 0$  because  $\frac{1}{x}$  is not defined at zero. It is important to understand that we will not get this conclusion from the ratio test alone. The Ratio Test alone does not account for points where the series or its terms are not defined.

**TLDR:** Be mindful about the domain of the function you are multiplying the Taylor series by. Do not only rely on the ratio test to find points of convergence.

- **Analytic function:**

- An analytic function is infinitely differentiable within its domain.
- An analytic function can be represented by a convergent power series (like a Taylor series) around any point in its domain.
- The power series representing an analytic function not only exists but also converges to the function within a certain radius around the point of expansion

- **Maclaurin series for  $\frac{1}{1-x}$ :**

$$\frac{1}{1-x} \sim \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1.$$

- **Maclaurin series for  $\ln(1+x)$ :**

$$\ln(1+x) \sim \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \quad \text{for } |x| < 1.$$

- **Maclaurin series for  $\tan^{-1} x$ :**

$$\tan^{-1} x \sim \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{for } |x| \leq 1.$$

- **Binomial expansion for  $(1+x)^r$  for  $r \in \mathbb{Z}^+$**

$$(1+x)^r = \sum_{n=0}^r \binom{r}{n} x^n \quad r \in \mathbb{Z}^+.$$

- **Binomial expansion for  $(1+x)^r$  for  $r \in \mathbb{R}$**

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n = 1 + rx + \frac{r(r-1)}{2!} x^2 + \dots + \frac{r(r-1)\cdots(r-n+1)}{n!} x^n + \dots$$

Where

$$\binom{r}{n} = \frac{f^{(n)}(0)}{n!} = \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}.$$

**Note:** When  $n = 0$ ,  $\binom{r}{0} = 1$ , when  $n = 1$ ,  $\binom{r}{1} = r$ , when  $n = 2$ ,  $\binom{r}{2} = \frac{r(r-1)}{2!}$ , etc...

- **The binomial theorem:** For any real number  $r$ , the Maclaurin series for  $f(x) = (1+x)^r$  is the binomial series. It converges to  $f$  for  $|x| < 1$ , and we write

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n = 1 + rx + \frac{r(r-1)}{2!} x^2 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!} x^n + \cdots$$

for  $|x| < 1$ .

## 1.6 Chapter 6 Problems to Remember

- **Problem to remember (Properties of power series):** Evaluate the infinite series by identifying it as the value of an integral of a geometric series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

**Remark.** If we can find which geometric power series's integral (with some bounds) gives us the given series, we can then integrate the function representation to get the value of the original series. Consider the geometric power series

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

Suppose we then integrate the power series

$$\begin{aligned} & \int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx \\ &= \sum_{n=0}^{\infty} (-1)^n \int x^{2n} dx \\ &= \frac{1}{2n+1} x^{2n+1}. \end{aligned}$$

Now we must deduce for which bounds will the FTC give us the original series  $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ . We come to the conclusion

$$\sum_{n=0}^{\infty} (-1)^n \int_0^1 x^{2n} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

This implies we can integrate the function representation of the geometric power series we just integrated to get the value of the infinite series  $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ . Thus,

$$\int_0^1 \frac{1}{1+x^2} dx = \frac{\pi}{4}.$$

- **Problem to remember (Properties of power series):** Find the power series for  $f(x) = \ln x$  centered at  $x = 9$  by using term-by-term integration or differentiation.

*Solution.* The goal is to find a function that resembles one we know (sum of geometric series  $\frac{a}{1-x}$ ) such that if we integrate or differentiate we can get  $\ln x$ . Since we know the integral of  $\frac{1}{x}$  is  $\ln x$ , and we can easily manipulate  $\frac{1}{x}$  to be in the form  $\frac{a}{1-x}$ , we choose  $\frac{1}{x}$  to be the function to examine. Thus,

$$\frac{1}{x} = \frac{1}{9+x-9} = \frac{1}{9-(-(x-9))} = \frac{1/9}{1-\left(\frac{-(x-9)}{9}\right)}$$

$$\text{If } f(x) = \frac{a}{1-x} \sim \sum_{n=0}^{\infty} a(x^n) = a + ax + ax^2 + ax^3 + \dots$$

$$\Rightarrow f(x) = \frac{1/9}{1-\left(\frac{-(x-9)}{9}\right)} \sim \sum_{n=0}^{\infty} \frac{1}{9} \left(\frac{-(x-9)}{9}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n (x-9)^n}{9^{n+1}}.$$

Then we can throw in some integrals

$$\begin{aligned} \int f(x) dx &= \int \frac{1/9}{1 - \left(\frac{-(x-9)}{9}\right)} dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n (x-9)^n}{9^{n+1}} dx \\ \ln x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{9^{n+1}} \int (x-9)^n dx \\ \ln x &= \sum_{n=0}^{\infty} \frac{(-1)^n (x-9)^{n+1}}{(n+1)9^{n+1}} + C. \end{aligned}$$

If we let  $x = 9$

$$\begin{aligned} \ln 9 &= \sum_{n=0}^{\infty} \frac{(-1)^n (9-9)^{n+1}}{(n+1)9^{n+1}} + C \\ \ln 9 &= C. \end{aligned}$$

Thus, we have

$$f(x) = \ln 9 + \sum_{n=0}^{\infty} \frac{(-1)^n (9-9)^{n+1}}{(n+1)9^{n+1}}$$

**Note:** the "+C" is initially omitted from  $\ln(x)$  because we're considering a specific antiderivative. When you integrate the power series, you include "+C" to account for the general form of the antiderivative. The value of  $C$  is then determined using a specific condition to match the specific antiderivative you're interested in.

- **Problem to remember:** Say we want to find the power series for  $7x \ln 1+x$ . We can first find the power series for  $\ln(1+x)$

$$\frac{d}{dx} \ln(1+x) = \frac{1}{1+x} = \frac{1}{1-(-x)}.$$

We know the power series for  $\frac{1}{1-(-x)}$  is

$$\begin{aligned} &\sum_{n=0}^{\infty} (-1)^n x^n \\ \implies \int \frac{1}{1+x} &= \int \sum_{n=0}^{\infty} (-1)x^n \\ \ln(1+x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C \\ \ln(1+x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}. \end{aligned}$$

Now that we have found the power series for  $\ln(1+x)$ , to find the power series for  $7x \ln(1+x)$ ...

$$\begin{aligned} &7x \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \\ &\sum_{n=0}^{\infty} 7x \left( (-1)^n \frac{x^{n+1}}{n+1} \right) \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{7x^{n+2}}{n+1}. \end{aligned}$$

- **Problem to remember (Cumbersome taylor polynomial):** Suppose we have some function  $f$ , and we would like to find the Taylor polynomial up to degree 3. Say  $f(x) = e^{2x} \cos(x)$ . We could find each derivative up to degree 3, however, given that we know the taylor series for both  $e^{2x}$  and  $\cos(x)$ .

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \dots$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots.$$

We can find  $P_3(x)$  by multiplying these Taylor series. Thus,

$$(1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \dots)(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots).$$

And we can find  $P_3$  to be

$$P_3 = 1 + 2x + \frac{3}{2}x^2 + \frac{1}{3}x^3.$$

- **Problem to Remember (Using known Taylor series to find sum of series):**

Consider the series  $\sum_{n=0}^{\infty} (-1)^n \frac{(\frac{1}{25})^{n-3}}{2n+1}$

We notice this resembles the Taylor series for the arctangent function  $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$  for  $|x| \leq 1$ . Thus, we manipulate the series to better conform to the Taylor series for  $\tan^{-1} x$ .

$$\begin{aligned} & \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{1}{25})^{n-3}}{2n+1} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{1}{5})^{2n-6}}{2n+1} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{1}{5})^{2n} 5^6}{2n+1} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{1}{5})^{2n} 5^6 (\frac{1}{5}) 5}{2n+1} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{1}{5})^{2n+1} 5^7}{2n+1} \end{aligned}$$

.

Thus the sum will be

$$5^7 \tan^{-1} \frac{1}{5}.$$

# Fundamental Physics: Classical Mechanics

## 2.1 Chapter 1: Units and Measurement

### 2.1.1 Key Terms

- **Physics**, which comes from the Greek *phύσις*, meaning “nature,” is concerned with describing the interactions of energy, matter, space, and time to uncover the fundamental mechanisms that underlie every phenomenon.
- A **model** is a representation of something that is often too difficult (or impossible) to display directly. Although
- a **theory** is a testable explanation for patterns in nature supported by scientific evidence and verified multiple times by various groups of researchers.
- A **law** uses concise language to describe a generalized pattern in nature supported by scientific evidence and repeated experiments. Often, a law can be expressed in the form of a single mathematical equation.
- **SI units** (for the French Système International d’Unités), also known as the **metric system**.
- The Metric system may also be referred to as the **centimeter–gram–second (cgs)** system.
- **English units** (also known as the customary or imperial system). English units were historically used in nations once ruled by the British Empire and are still widely used in the United States.
- The English system may also be referred to as the **the foot–pound–second (fps)** system.
- In any system of units, the units for some physical quantities must be defined through a measurement process. These are called the **base quantities** for that system
- The base quantities units are the system’s **base units**
- All other physical quantities can then be expressed as algebraic combinations of the base quantities. Each of these physical quantities is then known as a **derived quantity** and each unit is called a **derived unit**.
- A **conversion factor** is a ratio that expresses how many of one unit are equal to another unit.
- The **dimension** of any physical quantity expresses its dependence on the base quantities as a product of symbols (or powers of symbols) representing the base quantities.
- Any quantity with a dimension that can be written so that all seven powers are zero (that is, its dimension is  $L^0 M^0 T^0 I^0 \Theta^0 N^0 J^0$ ) is called **dimensionless** (or sometimes “of dimension 1,” because anything raised to the zero power is one).
- Physicists often call dimensionless quantities **pure numbers**.
- The importance of the concept of dimension arises from the fact that any mathematical equation relating physical quantities must be **dimensionally consistent**

- Every term in an expression must have the same dimensions; it does not make sense to add or subtract quantities of differing dimension (think of the old saying: “You can’t add apples and oranges”). In particular, the expressions on each side of the equality in an equation must have the same dimensions.
  - The arguments of any of the standard mathematical functions such as trigonometric functions (such as sine and cosine), logarithms, or exponential functions that appear in the equation must be dimensionless. These functions require pure numbers as inputs and give pure numbers as outputs.
- **Estimation** means using prior experience and sound physical reasoning to arrive at a rough idea of a quantity’s value.
  - **Accuracy** is how close a measurement is to the accepted reference value for that measurement.
  - The **precision of measurements** refers to how close the agreement is between repeated independent measurements (which are repeated under the same conditions).
  - The precision of a measuring system is related to the **uncertainty** in the measurements whereas the accuracy is related to the **discrepancy** from the accepted reference value.
  - **Discrepancy** (or “measurement error”) is the difference between the measured value and a given standard or expected value.
  - Another method of expressing uncertainty is as a percent of the measured value.

### 2.1.2 Definitions and Theorems (and important things)

- Order of Magnitude (Method one)

$$m = \log_{10} l, \quad l \in \mathbb{R}.$$

For example

$$\log 450 \approx 3.$$

Thus the order of magnitude is  $10^3$

**Note:** Always round s.t  $m \in \mathbb{R}$

- Order of Magnitude (Method two): We begin by writing the number in scientific notation. For example, suppose we want to find the order of magnitude for 800, we first write

$$8 \cdot 10^2.$$

We then check to see if the first factor is greater or less than  $\sqrt{10} \approx 3$

- If it is less than  $\sqrt{10} \approx 3$ , we round it down to one
- If it is greater than  $\sqrt{10} \approx 3$ , we round it up to ten

Since 8 is greater than  $\sqrt{10} \approx 3$ , our number becomes  $10 \cdot 10^2 = 10^{2+1} = 10^3$ . Thus, we have order  $10^3$

- The SI unit for time is the **second**
- The SI unit for length is the **meter**
- The SI unit for mass is the **kilogram**
- The SI unit for weight is the **Newton**
- $10^3$  (Megagram) is also called a *metric ton*, abbreviated *t*
- **Average speed**

$$\frac{\text{distance}}{\text{time}} = \frac{d}{t}.$$

- **Dimension notation:** Square brackets

$$[r] = L.$$

- **Checking for dimensional consistency:** check that each term in a given equation has the same dimensions as the other terms in that equation and that the arguments of any standard mathematical functions are dimensionless.

- **Dimension for velocity**

$$[v] = LT^{-1}.$$

- **Dimension for acceleration**

$$[a] = LT^{-2}.$$

- **Dimensions for derivatives**

$$\left[ \frac{dv}{dt} \right] = \frac{[v]}{[t]}.$$

- **Dimensions for integrals**

$$\left[ \int v dt \right] = [v] \cdot [t]..$$

- **Volume**

$$V = AD.$$

Where  $A$  is the area and  $D$  is the depth

- **Mass**

$$M = \rho V.$$

Where  $\rho$  is the density and  $V$  is the volume

- **Density (Volume density)**

$$\rho = \frac{M}{V}.$$

Where  $M$  is the mass and  $V$  is the volume

- **Density (Area density)**

$$\rho = \frac{M}{A}.$$

Where  $M$  is the mass and  $A$  is the surface area

- **Geometric mean of bounds**

$$(order_1 \times order_2)^{0.5}.$$

- **Significant figures:**

– All non-zero numbers are Significant.

$$563 \rightarrow 3.$$

– Zeros are significant if they reside between two significant figures.

$$5002 \rightarrow 4.$$

– Leading zeros are never significant.

$$0.0056 \rightarrow 2.$$

– Trailing zeros without a decimal point are not significant

$$500 \rightarrow 1.$$

– Trailing zeros with a decimal point are significant. The decimal point indicates that the zeros are measured and are significant.

$$500.0 \rightarrow 4.$$

- **Percent uncertainty:** If a measurement  $A$  is expressed with uncertainty  $\delta A$ , the percent uncertainty is defined as

$$\text{Percent uncertainty} = \frac{\delta A}{A} \times 100\%.$$

where  $A$  is the average, and  $\delta A$  is the margin of error

**Note:** The value of the PU will take the place of the margin of error, so something like  $5.1 \text{ lbs} \pm 0.3 \text{ lbs}$  will become  $5.1 \pm 6\%$

- We can find the margin of error ( $\delta A$ ) by taking half of the range
- If the measurements going into the calculation have small uncertainties (a few percent or less), then the method of adding percents can be used for multiplication or division. This method states the percent uncertainty in a quantity calculated by multiplication or division is the sum of the percent uncertainties in the items used to make the calculation. For example, if a floor has a length of  $4.00 \text{ m}$  and a width of  $3.00 \text{ m}$ , with uncertainties of 2% and 1%, respectively, then the area of the floor is  $12.0 \text{ m}^2$  and has an uncertainty of 3%. (Expressed as an area, this is  $0.36 \text{ m}^2$  [ $12.0 \text{ m}^2 \times 0.03$ ], which we round to  $0.4 \text{ m}^2$  since the area of the floor is given to a tenth of a square meter.)
- When combining measurements with different degrees of precision with the mathematical operations of addition, subtraction, multiplication, or division, then the number of significant digits in the final answer can be no greater than the number of significant digits in the least-precise measured value. There are two different rules, one for multiplication and division and the other for addition and subtraction. There are two different rules
  - For multiplication and division, the result should have the same number of significant figures as the quantity with the least number of significant figures entering into the calculation
  - For addition and subtraction, the answer can contain no more decimal places than the least-precise measurement.
- If a quantity increases  $n\%$ , that is the same as saying that it is multiplied by a factor of

$$1 + \left( \frac{n}{100} \right).$$

- If a quantity decreases  $n\%$ , that is the same as saying that it is multiplied by a factor of

$$1 - \left( \frac{n}{100} \right).$$

- **Proportional notation:** Suppose  $A$  is proportional to  $B$ , then we say

$$A \propto B.$$

This means if  $B$  increases by some factor, then  $A$  must increase by the same factor.

In other words, the ratio of two values of  $B$  is equal to the ratio of the corresponding values of  $A$ :

$$\frac{B_2}{B_1} = \frac{A_2}{A_1}.$$

Ex: Given the circumference formula for a circle

$$c = 2\pi r.$$

We can say

$$C \propto r.$$

If the radius doubles, the circumference also doubles

- Numbers that are exact (defined) have an infinite number of significant figures because they are not measurements with any uncertainty, but rather are defined values or counts of discrete objects. For example, considering a conversion of 93.4 beats/min to beats/hour, we would compute

$$\frac{93.4 \text{ beats}}{1 \text{ m}} \cdot \frac{60 \text{ m}}{1 \text{ h}} = 5604 \text{ b/h.}$$

Since the number 60 is an exact number (it is a defined conversion factor), we should report the answer with three significant figures (since 93.4 has three). Thus, we round to 5600. To explicitly express 5600 with three sig figs, we write as  $5.60 \cdot 10^3$

- Explicitly express sig figs with scientific notation

$$\begin{aligned} 5600 &\rightarrow 2 \\ 5.60 \cdot 10^3 &\rightarrow 3. \end{aligned}$$

- Forms of scientific notation

$$10^{10} = 1.0 \cdot 10^{10} = 1.0e + 10.$$

- Decimal places in addition and subtraction: consider the example

$$501.258313 + 54.5235 + 350.257 = 906.038813.$$

We would report the value as

$$906.039.$$

Because the least precise measurement has three decimal points

- unless you are working on lab-related calculations (ie in your lab reports) or the question explicitly asks you to pay attention to sig figs, **you can err on the side of including extra ones**
- Weight is the measure of the force exerted on an object due to gravity. It is calculated as the mass of the object multiplied by the acceleration due to gravity.
- Newton

$$N = kg \cdot \frac{m}{s^2}.$$

### 2.1.3 Fundamental Tables and figures

- Known ranges of length, mass, and time

Length in Meters (m)	Masses in Kilograms (kg)	Time in Seconds (s)
$10^{-15} \text{ m} = \text{diameter of proton}$	$10^{-30} \text{ kg} = \text{mass of electron}$	$10^{-22} \text{ s} = \text{mean lifetime of very unstable nucleus}$
$10^{-14} \text{ m} = \text{diameter of large nucleus}$	$10^{-27} \text{ kg} = \text{mass of proton}$	$10^{-17} \text{ s} = \text{time for single floating-point operation in a supercomputer}$
$10^{-10} \text{ m} = \text{diameter of hydrogen atom}$	$10^{-15} \text{ kg} = \text{mass of bacterium}$	$10^{-15} \text{ s} = \text{time for one oscillation of visible light}$
$10^{-7} \text{ m} = \text{diameter of typical virus}$	$10^{-5} \text{ kg} = \text{mass of mosquito}$	$10^{-13} \text{ s} = \text{time for one vibration of an atom in a solid}$
$10^{-2} \text{ m} = \text{pinky fingernail width}$	$10^{-2} \text{ kg} = \text{mass of hummingbird}$	$10^{-3} \text{ s} = \text{duration of a nerve impulse}$
$10^0 \text{ m} = \text{height of 4 year old child}$ 	$10^0 \text{ kg} = \text{mass of liter of water}$ 	$10^0 \text{ s} = \text{time for one heartbeat}$ 
$10^2 \text{ m} = \text{length of football field}$	$10^2 \text{ kg} = \text{mass of person}$	$10^5 \text{ s} = \text{one day}$
$10^7 \text{ m} = \text{diameter of Earth}$	$10^{19} \text{ kg} = \text{mass of atmosphere}$	$10^7 \text{ s} = \text{one year}$
$10^{13} \text{ m} = \text{diameter of solar system}$	$10^{22} \text{ kg} = \text{mass of Moon}$	$10^9 \text{ s} = \text{human lifetime}$
$10^{16} \text{ m} = \text{distance light travels in a year (one light-year)}$	$10^{25} \text{ kg} = \text{mass of Earth}$	$10^{11} \text{ s} = \text{recorded human history}$
$10^{21} \text{ m} = \text{Milky Way diameter}$	$10^{30} \text{ kg} = \text{mass of Sun}$	$10^{17} \text{ s} = \text{age of Earth}$
$10^{26} \text{ m} = \text{distance to edge of observable universe}$	$10^{53} \text{ kg} = \text{upper limit on mass of known universe}$	$10^{18} \text{ s} = \text{age of the universe}$

- SI Units: Base and Derived Units

ISQ Base Quantity	SI Base Unit
Length	meter (m)
Mass	kilogram (kg)
Time	second (s)
Electrical current	ampere (A)
Thermodynamic temperature	kelvin (K)
Amount of substance	mole (mol)
Luminous intensity	candela (cd)

- Metric Prefixes

Prefix	Symbol	Meaning	Prefix	Symbol	Meaning
yotta-	Y	$10^{24}$	yocto-	y	$10^{-24}$
zetta-	Z	$10^{21}$	zepto-	z	$10^{-21}$
exa-	E	$10^{18}$	atto-	a	$10^{-18}$
peta-	P	$10^{15}$	femto-	f	$10^{-15}$
tera-	T	$10^{12}$	pico-	p	$10^{-12}$
giga-	G	$10^9$	nano-	n	$10^{-9}$
mega-	M	$10^6$	micro-	$\mu$	$10^{-6}$
kilo-	k	$10^3$	milli-	m	$10^{-3}$
hecto-	h	$10^2$	centi-	c	$10^{-2}$
deka-	da	$10^1$	deci-	d	$10^{-1}$

- Base Quantity & Symbol for Dimension

Base Quantity	Symbol for Dimension
Length	L
Mass	M
Time	T
Current	I
Thermodynamic temperature	$\Theta$
Amount of substance	N
Luminous intensity	J

**2.1.4 Memorize conversions**

- **Newton to Pounds:**

$$1N = 0.225lbs.$$

- **Pounds to newtons**

$$1lb = 4.448 N.$$

- **Length/Distance**

- 1 inch (in) = 2.54 centimeters (cm)
- 1 centimeter (cm) = 0.393701 inches (in)
- 1 foot (ft) = 0.3048 meters (m)
- 1 meter (m) = 3.28 feet (ft)
- 1 mile (mi) = 1.6 kilometers (km)
- 1 kilometer (km) = 0.621371 miles (mi)

- **Weight to mass or mass to weight**

- 1 pound = 0.453592 kilograms.
- 1 kilogram = 2.2 pounds.

### 2.1.5 Problems to remember

- **Restating mass:** Restate the mass  $1.93 \times 10^{13}$  using a metric prefix such that the resulting numerical value is bigger than one but less than 1000.

First, we must restate in terms of grams. Since  $1\text{kg} = 10^3\text{g}$ , we write

$$\begin{aligned} 1.93 \times 10^{13} \times 10^3\text{g} \\ 1.93 \times 10^{16}\text{g}. \end{aligned}$$

Since  $1\text{Pg} = 10^{15}\text{g}$ , we can write

$$1.93 \times 10^1\text{Pg}.$$

Since  $16 - 15 = 1$

- **Unit conversion:** The distance from the university to home is 10 mi and it usually takes 20 min to drive this distance. Calculate the average speed in meters per second (m/s). (Note: Average speed is distance traveled divided by time of travel.)

**Note:** There are 1609 meters in 1 mile

First, we can compute the average speed with the units given ( $\frac{\text{miles}}{\text{minute}}$ )

$$\text{Average Speed} = \frac{\text{miles}}{\text{minute}} = \frac{10}{20} = 0.5 \text{ mi/min.}$$

Now we simply convert to m/s

$$\begin{aligned} \frac{0.5 \text{ mi}}{1 \text{ min}} \times \frac{1 \text{ min}}{60 \text{ sec}} \times \frac{1609 \text{ m}}{1 \text{ mi}} \\ \approx 13 \text{ m/s.} \end{aligned}$$

- **Unit conversion:** The density of iron is  $7.86\text{g}/\text{cm}^3$  under standard conditions. Convert this to  $\text{kg}/\text{m}^3$ .

$$\begin{aligned} \frac{7.86 \text{ g}}{1 \text{ cm}^3} \times \left( \frac{100 \text{ cm}}{1 \text{ m}} \right)^3 \times \frac{1 \text{ kg}}{1000 \text{ g}} \\ = \frac{7.86(100^3)(1 \text{ kg})}{1000(1 \text{ m})} \\ = 7.86 \cdot 10^3 \text{ kg/m}^3. \end{aligned}$$

- **Proportional:** I found that if I drive my car 110 miles, I use 4 gallons of gas. If I assume that the relationship between gas guzzled and distance driven is linearly proportional, how many gallons of gas do I use if I drive 275 miles?

To answer this lets find the linear equation

$$4 \text{ gal} = k(100 \text{ mi}).$$

Where  $k$  is some arbitrary factor, Since we know the relationship is proportional, we can write

$$\begin{aligned} \frac{X}{4\text{gal}} &= \frac{k(275 \text{ mi})}{k(110 \text{ mi})} \\ X &= 10 \text{ gal.} \end{aligned}$$

## 2.2 Chapter 2: Vectors

### 2.2.1 Vocabulary

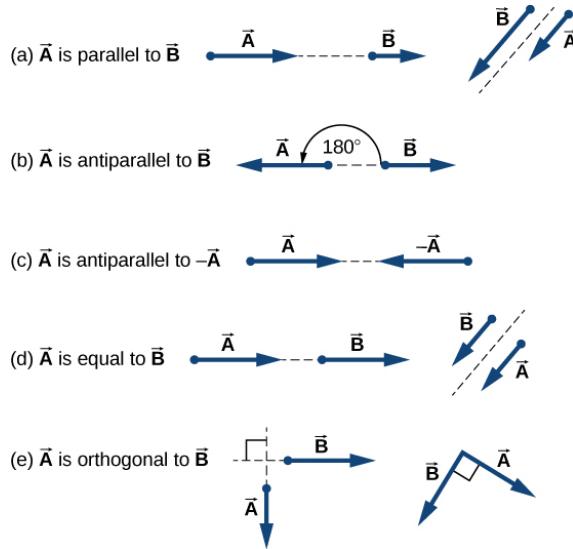
- Many familiar physical quantities can be specified completely by giving a single number and the appropriate unit. For example, “a class period lasts 50 min” or “the gas tank in my car holds 65 L” or “the distance between two posts is 100 m.” A physical quantity that can be specified completely in this manner is called a **scalar quantity**. Scalar is a synonym of “number.” Time, mass, distance, length, volume, temperature, and energy are examples of **scalar quantities**.
- possible. Physical quantities specified completely by giving a number of units (magnitude) and a direction are called **vector quantities**. Examples of vector quantities include displacement, velocity, position, force, and torque.

### 2.2.2 Definitions and theorems (and important things)

- Vectors (vector quantities) have a **number of units (magnitude)**, and a direction
- **Algebraic Operations with vectors**
  - We can **add or subtract** two vectors
  - we can **multiply** a vector by a scalar or by another vector
  - We **cannot** divide by a vector. The operation of division by a vector is not defined.
- **Vector Notation:** We denote a vector with a bold face letter with an arrow above it. For example

$$\vec{V}.$$

- **Displacement:** General term used to describe change in position.
- **The magnitude of a vector is the length of the arrow used to represent it**
- **Vector relations**



- **Scalars:** When a vector  $\vec{A}$  is multiplied by a positive scalar  $\alpha$ , the result is a new vector  $\vec{B}$  that is parallel to  $\vec{A}$ :

$$\vec{B} = \alpha \vec{A}.$$

The magnitude of this new vector  $\vec{B}$  is

$$B = |\alpha| A.$$

Where  $B$  is the magnitude of  $\vec{B}$  and  $A$  is the magnitude of  $\vec{A}$

- **Anti parallel vectors (opposing directions):** Suppose we have two vectors  $\vec{A}$  and  $\vec{B}$  of equal magnitude. If there are antiparallel, we write

$$\vec{A} = -\vec{B}.$$

- The vector sum of two (or more) vectors is called the **resultant vector** or, for short, the **resultant**
- The vector obtained by adding two vectors is called the **resultant**
- Vector Laws**

- **Commutative law:**  $\vec{A} + \vec{B} = \vec{B} + \vec{A}$
- **Associative law:**  $(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$
- **Distributive law:**  $\alpha_1 \vec{A} + \alpha_2 \vec{A} = (\alpha_1 + \alpha_2) \vec{A}$

- Vector addition**

$$\vec{A} + \vec{B}.$$

When two vectors are parallel, we can simply sum their magnitudes. However, if the vectors lie in different directions, the approach for vector addition involves finding their  $x$  and  $y$  components, summing them and then finding the magnitude.

- Vector Subtraction**

$$\vec{A} + (-\vec{B}).$$

When two vectors are aligned but point in exactly opposite directions, you can subtract their magnitudes (assuming you define one direction as positive and the other as negative) to find the net effect. This is a specific case of adding magnitudes where the direction is implicitly considered through subtraction.

- Unit Vector notation:** A unit vector in a normed vector space is a vector of length 1. We declare unit vectors with a hat instead of an arrow, consider the following example

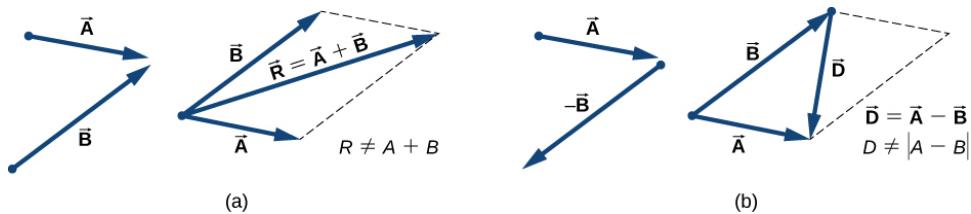
$$\hat{\mathbf{u}}.$$

We usually denote the unit vector along the positive x-axis  $\hat{i}$ , the unit vector along the positive y-axis  $\hat{j}$ , and the unit vector along the positive z-axis  $\hat{k}$

- Unit vector example:** For example, instead of saying vector  $\vec{D}_{AB}$  has a magnitude of 6.0 km and a direction of northeast, we can introduce a unit vector  $\hat{u}$  that points to the northeast and say succinctly that  $\vec{D}_{AB} = (6.0 \text{ km})\hat{u}$ . Then the southwesterly direction is simply given by the unit vector  $-\hat{u}$ . In this way, the displacement of 6.0 km in the southwesterly direction is expressed by the vector

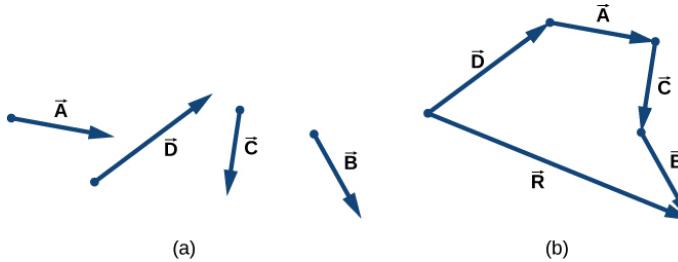
$$\vec{D}_{BA} = (-6.0 \text{ km})\hat{u}.$$

- Parallelogram rule for resultant or difference vectors in two dimensions.:**



It follows from the parallelogram rule that neither the magnitude of the resultant vector nor the magnitude of the difference vector can be expressed as a simple sum or difference of magnitudes A and B, because the length of a diagonal cannot be expressed as a simple sum of side lengths.

- tail-to-head geometric construction



- **Vector x and y components:** The  $x$  component can be denoted,  $\vec{A}_x$  the  $y$  component can be denoted  $\vec{A}_y$ . Thus, the vector can be represented as

$$\vec{A} = \vec{A}_x + \vec{A}_y.$$

- **Unit vectors of the axes:** It is customary to denote the positive direction on the  $x$ -axis by the unit vector  $\hat{i}$  and the positive direction on the  $y$ -axis by the unit vector  $\hat{j}$ . Unit vectors of the axes,  $\hat{i}$  and  $\hat{j}$ , define two orthogonal directions in the plane. The  $x$ - and  $y$ - components of a vector can now be written in terms of the unit vectors of the axes:

$$\begin{cases} \vec{A}_x &= A_x \hat{i} \\ \vec{A}_y &= A_y \hat{j} \end{cases} \quad (18)$$

- **Component form of a vector:**

$$\vec{A} = A_x \hat{i} + A_y \hat{j}.$$

- **Finding components given initial and terminal points:** If we know the coordinates  $b(x_b, y_b)$  of the origin point of a vector (where  $b$  stands for "beginning") and the coordinates  $e(x_e, y_e)$  of the end point of a vector (where  $e$  stands for "end"), we can obtain the scalar components of a vector simply by subtracting the origin point coordinates from the end point coordinates:

$$\begin{aligned} A_x &= x_e - x_b \\ A_y &= y_e - y_b. \end{aligned}$$

- **Magnitude  $A$  of a vector with components  $A_x$  and  $A_y$**

$$\begin{aligned} A^2 &= A_x^2 + A_y^2 \\ A &= \sqrt{A_x^2 + A_y^2}. \end{aligned}$$

This equation works even if the scalar components of a vector are negative.

- **Finding theta for a vector (used for direction angles)**

$$\tan \theta = \frac{A_y}{A_x}.$$

- Direction angles for vectors in the first:
- Direction angles for vectors in the second quadrant:

$$\theta_A = \theta.$$

- Direction angles for vectors in the second quadrant:

$$\theta_A = 180 - \theta.$$

- Direction angles for vectors in the second quadrant:

$$\theta_A = 180 + \theta.$$

- Direction angles for vectors in the fourth quadrant:

$$\theta_A = 360 - \theta.$$

- Finding  $A_x$  and  $A_y$  when the magnitude and direction angle are known

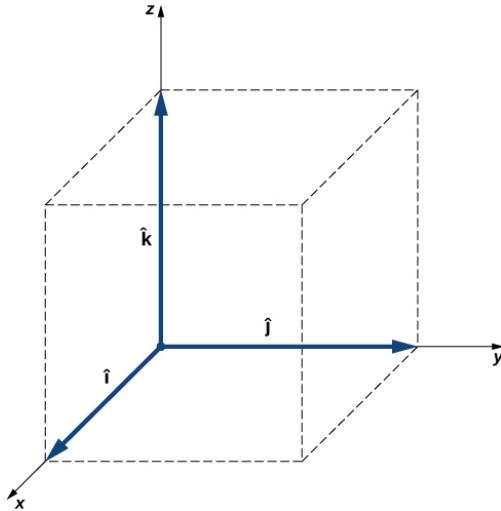
$$\begin{cases} A_x = A \cos \theta_A \\ A_y = A \sin \theta_A \end{cases} \quad (19)$$

- Polar form

$$x = r \cos \varphi$$

$$y = r \sin \varphi.$$

- Three dimensional plane



- $z$  component of a vector:

$$\vec{A}_z = A_z \hat{k}.$$

Where  $A_z$  is given by

$$z_e - z_b.$$

- **Vector in three dimensions:** A vector in three-dimensional space is the vector sum of its three vector component.

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}..$$

- **Magnitude of a vector in three dimensions**

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}..$$

- **Null vector:** Denoted by

$$\vec{0}.$$

Has all components 0. Thus,

$$\vec{0} = 0\hat{i} + 0\hat{j} + 0\hat{k}.$$

Thus, it has no direction and no length

- Two vectors  $\vec{A}$  and  $\vec{B}$  are **equal vectors** if and only if their difference is the null vector: Hence, we can write  $\vec{A} = \vec{B}$  if and only if the corresponding components of vectors  $\vec{A}$  and  $\vec{B}$  are equal:

$$\vec{A} = \vec{B} \Leftrightarrow \begin{cases} A_x = B_x \\ A_y = B_y \\ A_z = B_z \end{cases}$$

- **Components of a resultant vector:**

$$\begin{cases} R_x = A_x + B_x \\ R_y = A_y + B_y \\ R_z = A_z + B_z \end{cases} \quad (20)$$

- **components of a resultant of many vectors:** if we are to sum up  $N$  vectors  $\vec{F}_1, \vec{F}_2, \vec{F}_3, \dots, \vec{F}_N$ , where each vector is  $\vec{F}_k = F_{kx}\hat{i} + F_{ky}\hat{j} + F_{kz}\hat{k}$ , the resultant vector  $\vec{F}_R$  is

$$\begin{cases} F_{R_x} = \sum_{k=1}^N F_{kx} = F_{1x} + F_{2x} + F_{3x} + \dots + F_{Nx} \\ F_{R_y} = \sum_{k=1}^N F_{ky} = F_{1y} + F_{2y} + F_{3y} + \dots + F_{Ny} \\ F_{R_z} = \sum_{k=1}^N F_{kz} = F_{1z} + F_{2z} + F_{3z} + \dots + F_{Nz} \end{cases} \quad (21)$$

With the component form

$$\vec{F}_R = F_{R_x} \hat{i} + F_{R_y} \hat{j} + F_{R_z} \hat{k}.$$

- **Finding unit vector (Direction) of some vector:** Suppose we have some vector  $\vec{V}$ . Then

$$\hat{V} = \frac{\vec{V}}{V}.$$

- **Dot product**

$$\vec{A} \cdot \vec{B} = AB \cos \varphi.$$

Where  $\varphi$  is the angle between the vectors

- Dot product of two parallel vectors

$$\vec{A} \cdot \vec{B} = AB \cos 0^\circ = AB.$$

- Dot product of two anti-parallel vectors

$$\vec{A} \cdot \vec{B} = AB \cos 180^\circ = -AB.$$

- Dot product of two orthogonal vectors

$$\vec{A} \cdot \vec{B} = AB \cos 90^\circ = 0.$$

- Dot product of a vector with itself

$$\vec{A} \cdot \vec{A} = AA \cos 0 = A^2.$$

- **Dot products of unit vectors:** Scalar products of the unit vector of an axis with other unit vectors of axes always vanish (equals 0) because these unit vectors are orthogonal:
- **Dot product of the same unit vector:** The dot product of the same unit vector is 1

$$\hat{i} \cdot \hat{i} = i^2 = 1.$$

- Using dot product to find scalar x-component

$$\vec{A} \cdot \hat{i} = \|\vec{A}\| \|\hat{i}\| \cos \theta_A = A \cos \theta_A = A_x.$$

- Using dot product to find scalar y-component

$$\vec{A} \cdot \hat{j} = \|\vec{A}\| \|\hat{j}\| \cos (90^\circ - \theta_A) = A \sin \theta_A = A_y.$$

- **Trig complementary angles:** For a right angle triangle, the sine of the complementary angle is the cosine of the angle. And vice versa

$$\begin{aligned} \sin 90 - \theta &= \cos \theta \\ \cos 90 - \theta &= \sin \theta. \end{aligned}$$

- Dot product second method of computation

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z.$$

- Equation for  $\cos(\varphi)$

$$\cos(\varphi) = \frac{\vec{A} \cdot \vec{B}}{AB}.$$

- The **Si unit for work** is the joule (J), where

$$1 \text{ J} = 1 \text{ N} \cdot \text{m}.$$

- **The Work of a Force:** When force  $\vec{F}$  pulls on an object and when it causes its displacement  $\vec{D}$ , we say the force performs work. The amount of work the force does is the scalar product  $\vec{F} \cdot \vec{D}$ .

- **Cross Product (Vector Product):** The vector product of two vectors  $\vec{A}$  and  $\vec{B}$  is denoted by  $\vec{A} \times \vec{B}$  and is often referred to as a cross product. The vector product is a vector that has its direction perpendicular to both vectors  $\vec{A}$  and  $\vec{B}$ . In other words, vector  $\vec{A} \times \vec{B}$  is perpendicular to the plane that contains vectors  $\vec{A}$  and  $\vec{B}$ . The magnitude of the vector product is defined as

$$\|\vec{A} \times \vec{B}\| = AB \sin \varphi, \quad (22)$$

where angle  $\varphi$ , between the two vectors, is measured from vector  $\vec{A}$  (first vector in the product) to vector  $\vec{B}$  (second vector in the product), as indicated in Figure 2.29, and is between  $0^\circ$  and  $180^\circ$ .

- The **vector product vanishes** for pairs of vectors that are either **parallel** ( $\varphi = 0^\circ$ ) or **antiparallel** ( $\varphi = 180^\circ$ ) because  $\sin 0^\circ = \sin 180^\circ = 0$ .
- The cross product is **anti-commutative**

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}.$$

- **Torque:** Denoted with the greek letter "tau" is the vector product of the distance between the pivot to force with the force:

$$\vec{\tau} = \vec{R} \times \vec{F}.$$

- **the cross product has the following distributive property:**

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}.$$

- **Cross product of the same unit vectors:** The cross product of the same unit vectors is 0
- **Cross product between unit vectors**

$$\begin{cases} \hat{i} \times \hat{j} = +\hat{k} \\ \hat{j} \times \hat{i} = -\hat{k} \\ \hat{j} \times \hat{k} = +\hat{i} \\ \hat{k} \times \hat{j} = -\hat{i} \\ \hat{k} \times \hat{i} = +\hat{j} \\ \hat{i} \times \hat{k} = -\hat{j} \end{cases} \quad (23)$$

**Notice:** The cross product of two different unit vectors is always a third unit vector.

- **Computation of the cross product:**

$$\vec{C} = \vec{A} \times \vec{B} = (A_y B_z - A_z B_y) \hat{i} + (A_z B_x - A_x B_z) \hat{j} + (A_x B_y - A_y B_x) \hat{k}.$$

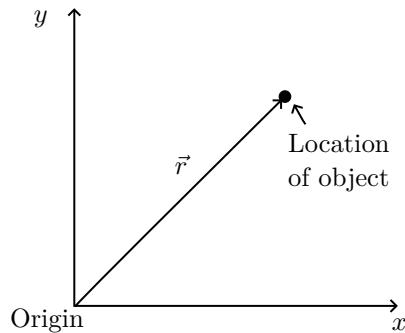
- **Average Velocity:** Velocity is the **vector quantity** of speed plus the direction. Velocity is

$$\vec{v} = \frac{\text{Displacement}}{\text{Time}} = \frac{\|\vec{d}\|}{t}.$$

Where  $\|\vec{d}\|$  is the magnitude of the displacement vector, and  $t$  is the total elapsed time.

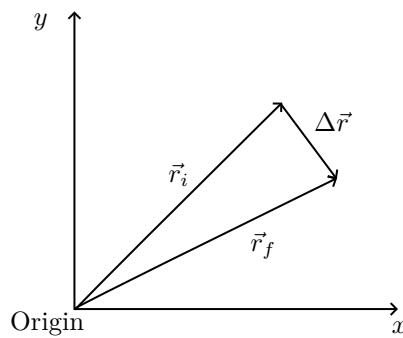
- To describe the position (location) of something, we give its distance from the origin and its direction. This vector quantity is called the **position vector**, Denoted

$$\vec{r}.$$



- **Displacement** is defined as the change in the position vector. The final position vector minus the initial position vector

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i.$$



### 2.2.3 Problems to remember

- **Unit vectors:** A long measuring stick rests against a wall in a physics laboratory with its 200-cm end at the floor. A ladybug lands on the 100-cm mark and crawls randomly along the stick. It first walks 15 cm toward the floor, then it walks 56 cm toward the wall, then it walks 3 cm toward the floor again. Then, after a brief stop, it continues for 25 cm toward the floor and then, again, it crawls up 19 cm toward the wall before coming to a complete rest (Figure 2.8). Find the vector of its total displacement and its final resting position on the stick.

If we choose the direction along the stick toward the floor as the direction of unit vector  $\hat{u}$ , then the direction toward the floor is  $+ \hat{u}$  and the direction toward the wall is  $- \hat{u}$ . The ladybug makes a total of five displacements:

$$\begin{aligned}\vec{D}_1 &= (15 \text{ cm})(+ \hat{u}), \quad \vec{D}_2 \\ &= (56 \text{ cm})(- \hat{u}), \quad \vec{D}_3 \\ &= (3 \text{ cm})(+ \hat{u}), \quad \vec{D}_4 \\ &= (25 \text{ cm})(+ \hat{u}), \quad \text{and} \quad \vec{D}_5 \\ &= (19 \text{ cm})(- \hat{u})..\end{aligned}$$

The total displacement  $\vec{D}$  is the resultant of all its displacement vectors.

The resultant of all the displacement vectors is

$$\begin{aligned}\vec{D} &= \vec{D}_1 + \vec{D}_2 + \vec{D}_3 + \vec{D}_4 + \vec{D}_5 \\ &= (15 \text{ cm})(+ \hat{u}) + (56 \text{ cm})(- \hat{u}) + (3 \text{ cm})(+ \hat{u}) + (25 \text{ cm})(+ \hat{u}) + (19 \text{ cm})(- \hat{u}) \\ &= (15 - 56 + 3 + 25 - 19) \text{ cm } \hat{u} = -32 \text{ cm } \hat{u}.\end{aligned}$$

In this calculation, we use the distributive law. The result reads that the total displacement vector points away from the 100-cm mark (initial landing site) toward the end of the meter stick that touches the wall. The end that touches the wall is marked 0 cm, so the final position of the ladybug is at the  $(100 - 32)$  cm = 68 cm mark.

## 2.3 Chapter 3: Motion along a straight line

### 2.3.1 Definitions and theorems

- **Kinematics** is a subfield of physics, developed in classical mechanics, that describes the motion of points, bodies, and systems of bodies without considering the forces that cause them to move
- **Displacement:** Displacement  $\Delta x$  is the change in position of an object:

$$\Delta x = x_f - x_0.$$

where  $\Delta x$  is displacement,  $x_f$  is the final position, and  $x_0$  is the initial position.

- We define total displacement  $\Delta x_{\text{Total}}$ , as the sum of the individual displacements, and express this mathematically with the equation

$$\Delta x_{\text{Total}} = \sum \Delta x_i.$$

- **the distance traveled** is the sum of the magnitudes of the individual displacements:

$$x_{\text{total}} = \sum_{i=1}^n |\Delta x_i|.$$

- If the details of the motion at each instant are not important, the rate is usually expressed as the **average velocity**. If  $x_1$  and  $x_2$  are the positions of an object at times  $t_1$  and  $t_2$ , respectively, then

$$\begin{aligned} \text{Average Velocity } &= \bar{v} = \frac{\text{Displacement between two points}}{\text{Time needed to make the displacement}} \\ &= \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}. \end{aligned}$$

This vector quantity is simply the total displacement between two points divided by the time taken to travel between them. The time taken to travel between two points is called the **elapsed time**  $\Delta t$

- The **instantaneous velocity** of an object is the limit of the average velocity as the elapsed time approaches zero, or the derivative of  $x$  with respect to  $t$ :

$$\begin{aligned} v(t) &= \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \\ &= \frac{d}{dt} x(t) = \frac{d\vec{r}}{dt}. \end{aligned}$$

- We can calculate the **average speed** by finding the total distance traveled divided by the elapsed time:

$$\text{Average speed } = \bar{s} = \frac{\text{Total distance}}{\text{Elapsed time}}.$$

- we can calculate the **instantaneous speed** from the magnitude of the instantaneous velocity:

$$\text{instantaneous speed } = |v(t)|.$$

- **Calculating Instantaneous Velocity** When calculating instantaneous velocity, we need to specify the explicit form of the position function  $x(t)$ . If each term in the  $x(t)$  equation has the form of  $At^n$  where  $A$  is a constant and  $n$  is an integer, this can be differentiated using the power rule to be:

$$\frac{d(At^n)}{dt} = Ant^{n-1}.$$

- **Average acceleration** is the rate at which velocity changes:

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_0}{t_f - t_0},$$

where  $\bar{a}$  is average acceleration,  $v$  is velocity, and  $t$  is time. (The bar over the  $a$  means average acceleration.)

**Note:** acceleration occurs when velocity changes in magnitude (an increase or decrease in speed) or in direction, or both.

- **Acceleration as a vector:** Acceleration is a vector in the same direction as the change in velocity,  $\Delta v$ . Since velocity is a vector, it can change in magnitude or in direction, or both. Acceleration is, therefore, a change in speed or direction, or both.

Keep in mind that although acceleration is in the direction of the change in velocity, it is not always in the direction of motion. When an object slows down, its acceleration is opposite to the direction of its motion. Although this is commonly referred to as deceleration

- **Distance over constant acceleration**

$$d = \frac{1}{2}at^2.$$

Where  $a$  is the acceleration, and  $t$  is the time

- **instantaneous acceleration**

$$a(t) = \frac{d}{dt}v(t).$$

- **Derivative of velocity function:** Suppose we have some function

$$v(t) = (20m/s)t - (10m/s^2)t^2.$$

When we find the acceleration function, we are taking the derivative of the velocity function. Thus, we are finding the change in velocity with respect to time. Consequently, our terms become

$$a(t) = 20m/s^2 - (10m/s^3)t.$$

**Notice:** How we are dividing by an additional unit of time, thus the exponents for our seconds increases by one.

- **Simplified notation:** If we take initial time to be zero, and final quantitys without subscript, then we have

$$\Delta t = t$$

$$\Delta x = x - x_0$$

$$\Delta v = v - v_0$$

- **Assumption of constant acceleration:** This assumption allows us to avoid using calculus to find instantaneous acceleration. Since acceleration is constant, the average and instantaneous accelerations are equal—that is,

$$\bar{a} = a = \text{constant}.$$

Thus, we can use the symbol  $a$  for acceleration at all times.

- **Final Position function**

$$x = x_0 + \bar{v}t.$$

- **Average velocity under constant acceleration**

$$\bar{v} = \frac{v_0 + v}{2} \quad (\text{Constant } a).$$

This reflects the fact that when acceleration is constant,  $\bar{v}$  is just the simple average of the initial and final velocities.

- **Final Velocity function under constant acceleration**

$$v = v_0 + at \quad (\text{Constant } a).$$

- **Equation for final position under constant acceleration**

$$x = x_0 + v_0 t + \frac{1}{2}at^2 \quad (\text{Constant } a).$$

When initial position and velocity are both zero, we have

$$x = \frac{1}{2}at^2 \quad (\text{Constant } a).$$

- **relationships seen in final position under constant acceleration equation**

- Displacement depends on the square of the elapsed time when acceleration is not zero.
- If acceleration is zero, then initial velocity equals average velocity  $v_0 = \bar{v}$ , and  $x = x_0 + v_0 t + \frac{1}{2}at^2$  becomes  $x = x_0 + v_0 t$ .

- **Final velocity equation (no time required)**

$$v^2 = v_0^2 + 2a(x - x_0) \quad (\text{Constant } a).$$

- **additional insights into the general relationships among physical quantities:**

- The final velocity depends on how large the acceleration is and the distance over which it acts.
- For a fixed acceleration, a car that is going twice as fast doesn't simply stop in twice the distance. It takes much farther to stop. (This is why we have reduced speed zones near schools.)

- **acceleration in terms of velocities and displacement**

$$a = \frac{v^2 - v_0^2}{2(x - x_0)}.$$

- **two-body pursuit problems**

- Find equations of motion for both bodies (with the same parameter)
- eliminate the parameter
- plug in knowns to solve for the unknown
- if **air resistance and friction are negligible**, then in a given location all objects fall toward the center of Earth with the same **constant acceleration, independent of their mass**.
- an object falling without air resistance or friction is defined to be in **free fall**
- The acceleration of free-falling objects is therefore called **acceleration due to gravity**

$$g = 9.8 \text{ m/s}^2.$$

**Note:** If we define the upward direction as positive, then  $a = g = -9.8\text{m/s}^2$ , and if we define the downward direction as positive, then  $a = g = 9.8\text{m/s}^2$

- **vertical displacement:** We denote vertical displacement with the symbol  $y$
- **Kinematic equations for objects in free fall:** We assume here that acceleration equals  $-g$  (with the positive direction upward).

$$\begin{aligned} v &= v_0 - gt \\ y &= y_0 + v_0 t - \frac{1}{2}gt^2 \\ v^2 &= v_0^2 - 2g(y - y_0). \end{aligned}$$

- **Change in velocity**

$$\Delta v = a\Delta t.$$

If you don't care about position this relates the acceleration, intervals of time and changes in velocity

- **Change in position**

$$\Delta x = \frac{1}{2}(v_f - v_i)\Delta t.$$

## 2.4 Chapter 4: Motion in two and three dimensions

### 2.4.1 Definitions and theorems

- **Location of a particle in space**

$$\begin{aligned}x &= x(t) \\y &= y(t) \\z &= z(t).\end{aligned}$$

Where  $x, y, z$  are functions of time ( $t$ ).

- **Position vector in space**

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}.$$

- **Displacement vector in space:** Suppose we have a particle. At time  $t_1$  the particle is located at  $P_1$  with position vector  $\vec{r}(t_1)$ . At some later time  $t_2$ , the particle is located at  $P_2$  with position vector  $\vec{r}(t_2)$ . The displacement vector  $\Delta\vec{r}$  is found by subtracting  $\vec{r}(t_1)$  from  $\vec{r}(t_2)$ :

$$\Delta\vec{r} = \vec{r}(t_2) - \vec{r}(t_1).$$

Thus, the displacement vector  $\Delta\vec{r} = \vec{r}(t_2) - \vec{r}(t_1)$  is the vector from  $P_1$  to  $P_2$

- **Instantaneous velocity vector in two and three dimensions:** We can do the same operation in two and three dimensions, but we use vectors. The instantaneous velocity vector is now

$$\begin{aligned}\vec{v} &= \lim_{\Delta t \rightarrow 0} \frac{\vec{r}(t - \Delta t) - \vec{r}(t)}{\Delta t} \\&= \frac{d\vec{r}}{dt}.\end{aligned}$$

- **Velocity in component form**

$$\vec{v}(t) = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}.$$

Where

$$v_x(t) = \frac{dx(t)}{dt}, \quad v_y(t) = \frac{dy(t)}{dt}, \quad v_z(t) = \frac{dz(t)}{dt}.$$

- **Average velocity in two and three dimensions:** If only the average velocity is of concern, we have the vector equivalent of the one-dimensional average velocity for two and three dimensions

$$\vec{v}_{\text{avg}} = \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1}.$$

- **The Independence of Perpendicular Motions:** When we look at the three-dimensional equations for position and velocity written in unit vector notation, we see the components of these equations are separate and unique functions of time that do not depend on one another. Motion along the x direction has no part of its motion along the y and z directions, and similarly for the other two coordinate axes. Thus, the motion of an object in two or three dimensions can be divided into separate, independent motions along the perpendicular axes of the coordinate system in which the motion takes place.

- **Independence of motion:** In the kinematic description of motion, we are able to treat the horizontal and vertical components of motion separately. In many cases, motion in the horizontal direction does not affect motion in the vertical direction, and vice versa.
- **Instantaneous Acceleration:** This acceleration vector is the instantaneous acceleration and it can be obtained from the derivative with respect to time of the velocity function. The only difference in two or three dimensions is that these are now vector quantities. Taking the derivative with respect to time  $\vec{v}(t)$ . We find

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt}.$$

- The acceleration in terms of components is

$$\vec{a}(t) = \frac{dv_x(t)}{dt}\hat{i} + \frac{dv_y(t)}{dt}\hat{j} + \frac{dv_z(t)}{dt}\hat{k}.$$

- acceleration in terms of the second derivative of the position function:

$$\vec{a}(t) = \frac{d^2x(t)}{dt^2}\hat{i} + \frac{d^2y(t)}{dt^2}\hat{j} + \frac{d^2z(t)}{dt^2}\hat{k}.$$

- **Equations for position and velocity in the two and three dimensions (8).** For simplicity I will only list the ones for the x-direction. However, keep in mind that these same equations hold for  $y$  and  $z$

- Position with initial position and average velocity:

$$x(t) = x_0 + (v_x)_{\text{avg}} t.$$

- Final velocity with initial velocity, acceleration, and time

$$v_x(t) = v_{0x} + a_x t.$$

- Position

$$x(t) = x_0 + v_{0x}t + \frac{1}{2}a_x t^2.$$

- Velocity with initial velocity, acceleration, final and initial position

$$v_x^2(t) = v_{0x}^2 + 2a_x(x - x_0).$$

- These equations can be substituted into the equations for the position and velocity vectors in component form, and velocity vector in component form without the z-component to obtain the position vector and velocity vector as a function of time in two dimensions:

$$\begin{aligned}\vec{r}(t) &= x(t)\hat{i} + y(t)\hat{j} \\ \vec{v}(t) &= v_x(t)\hat{i} + v_y(t)\hat{j}.\end{aligned}$$

- **Projectile motion** is the motion of an object thrown or projected into the air, subject only to acceleration as a result of gravity. Such objects are called **projectiles** and their path is called a **trajectory**

- **Acceleration in projectile motion:** Defining the positive direction to be upward, the components of acceleration are then very simple:

$$\begin{aligned} a_y &= -9.8 \text{ m/s}^2 \\ a_x &= 0. \end{aligned}$$

Because gravity is vertical,  $a_x = 0$ . If  $a_x = 0$ , this means the initial velocity in the  $x$  direction is equal to the final velocity in the  $x$  direction, or  $v_x = v_{0x}$

- **Kinematic equations for motion in a uniform gravitational field**

- Horizontal motion

$$v_{0x} = v_x, \quad x = x_0 + v_x t.$$

- **Average velocity (vertical)**
- Basic average velocity

$$V_{y,\text{avg}} = \frac{y - y_0}{\Delta t}.$$

- **Average velocity in special conditions**

$$V_{y,\text{avg}} = \frac{v_i + v_f}{2}.$$

**Note:** gives the same result for average velocity in scenarios of uniform acceleration, such as projectile motion or free fall, under specific conditions. This formula calculates the average of the initial and final velocities, assuming that the acceleration is constant throughout the motion. It works perfectly for the vertical component of projectile motion when considering the ascent or descent separately, or any motion where the start and end points are symmetrical in terms of velocity but in opposite directions (e.g., going up and coming back down to the same height).

- Vertical motion

$$\begin{aligned} y &= y_0 + \frac{1}{2}(v_{0y} + v_y)t \\ v_y &= v_{0y} + gt \\ y &= y_0 + v_{0y}t + \frac{1}{2}gt^2 \\ v_y^2 &= v_{0y}^2 + 2g(y - y_0). \end{aligned}$$

- **Projectile motion max height**

$$h = \frac{v_{0y}^2}{2g}.$$

This equation defines the maximum height of a projectile above its launch position and it depends only on the vertical component of the initial velocity.

- **Parabolic trajectory:** If the motion is parabolic, we should use the equation with  $t^2$  to solve for  $t$  and vertical displacement
- **Time of flight:** We can solve for the time of flight of a projectile that is both launched and impacts on a flat horizontal surface by performing some manipulations of the kinematic equations. We note the position and displacement in  $y$  must be zero

at launch and at impact on an even surface. Thus, We set the displacement in y equal to zero and find

$$T_{\text{tof}} = \frac{2(v_0 \sin \theta_0)}{g}.$$

**Note:** This is the time of flight for a projectile both launched and impacting on a flat horizontal surface. This equation does not apply when the projectile lands at a different elevation than it was launched,

- **Trajectory:** The trajectory of a projectile can be found by eliminating the time variable  $t$  from the kinematic equations for arbitrary  $t$  and solving for  $y(x)$ . We take  $x_0 = y_0 = 0$  so the projectile is launched from the origin.

$$y = (\tan \theta_0)x - \left[ \frac{g}{2(v_0 \cos \theta_0)^2} \right] x^2.$$

**Note:** This trajectory equation is of the form  $y = ax + bx^2$  which is an equation of a parabola with coefficients

$$a = \tan \theta_0, \quad b = -\frac{g}{2(v_0 \cos \theta_0)^2}.$$

- **Range:** From the trajectory equation we can also find the range, or the horizontal distance traveled by the projectile.

$$R = \frac{v_0^2 \sin(2\theta_0)}{g}.$$

- **Note:** the last three equations (**time of flight, trajectory, and range**) are derived with  $g$  as negative, thus we do not make  $g$  negative when plugging into the equation.
- **Centripetal Acceleration:** a particle moving in a circle at a constant speed has an acceleration with magnitude

$$a_c = \frac{v^2}{r}.$$

**Note:** The direction of the acceleration vector is toward the center of the circle

- **Position vector for a particle executing circular motion:** As the particle moves on the circle, its position vector sweeps out the angle  $\theta$  with the x-axis. Vector  $\vec{r}(t)$  making an angle  $\theta$  with the x-axis is shown with its components along the x- and y-axes. The magnitude of the position vector is  $A = |\vec{r}(t)|$  and is also the radius of the circle, so that in terms of its components,

$$\vec{r}(t) = A \cos(\omega t) \hat{i} + A \sin(\omega t) \hat{j}.$$

- **Average speed for circular motion:** If  $T$  is the item it takes to go around the circle once, and the distance around a circle is  $2\pi r$ , then

$$\begin{aligned} v &= \frac{2\pi r}{T} \\ \implies T &= \frac{2\pi r}{v}. \end{aligned}$$

- **Amplitude ( $A$ ):** In the context of uniform circle motion, we define the amplitude  $A$  as the radius of the circle. Thus,

$$A = r.$$

- **Angular frequency:** In the previous equation,  $\omega$  is a constant called the angular frequency of the particle.

The angular frequency has units of radians (rad) per second and is simply the number of radians of angular measure through which the particle passes per second. The angle  $\theta$  that the position vector has at any particular time is  $\omega t$

- **Angular frequency computation:** If  $T$  is the period of motion, or the time to complete one revolution ( $2\pi$  rad), then

$$\begin{aligned}\omega &= \frac{2\pi}{T} \\ \implies T &= \frac{2\pi}{\omega}.\end{aligned}$$

- **Example: Finding  $\omega$ :** A flywheel is rotating at 21 rev/s. What is the total angle, in radians, through which a point on the flywheel rotates in 37 s?

To find  $\omega$ , we must first deduce  $T$ , if we complete 21 revolutions in one second, then the time to complete one revolution is  $\frac{1}{21}s$ . Thus, we have  $\omega$  as

$$\frac{2\pi}{\frac{1}{21}} = 41\pi.$$

Which gives us

$$\begin{aligned}\theta &= \omega t \\ &= 41\pi(37s).\end{aligned}$$

- **Finding  $\theta$  with  $\omega$**

$$\theta = \omega t.$$

- **Velocity for circular motion**

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = -A\omega \sin(\omega t)\hat{i} + A\omega \cos(\omega t)\hat{j}.$$

- **Acceleration for circular motion**

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = -A\omega^2 \cos(\omega t)\hat{i} - A\omega^2 \sin(\omega t)\hat{j}.$$

From this equation we see that the acceleration vector has magnitude  $A\omega^2$  and is directed opposite the position vector, toward the origin, because  $\vec{a}(t) = -\omega^2\vec{r}(t)$ .

- **Nonuniform Circular Motion Tangential Acceleration:** Circular motion does not have to be at a constant speed. A particle can travel in a circle and speed up or slow down, showing an acceleration in the direction of the motion.

In uniform circular motion, the particle executing circular motion has a constant speed and the circle is at a fixed radius. If the speed of the particle is changing as well, then we introduce an additional acceleration in the direction tangential to the circle. Such accelerations occur at a point on a top that is changing its spin rate, or any accelerating rotor. If the speed of the particle is changing, then it has a **tangential acceleration** that is the time rate of change of the magnitude of the velocity

$$a_T = \left| \frac{d\vec{v}}{dt} \right|.$$

- **Tangential acceleration total acceleration:** The direction of tangential acceleration is tangent to the circle whereas the direction of centripetal acceleration is radially inward toward the center of the circle. Thus, a particle in circular motion with a tangential acceleration has a total acceleration that is the vector sum of the centripetal and tangential accelerations

$$\vec{a} = \vec{a}_c + \vec{a}_T.$$

Thus total acceleration can be found by finding the magnitude of this vector

$$\|\vec{a}\| = \sqrt{\vec{a}_c^2 + \vec{a}_T^2}.$$

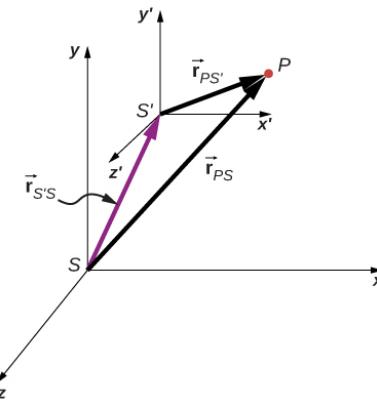
With

$$\tan \theta = \frac{y}{x}.$$

- **Reference frames:** When we say an object has a certain velocity, we must state it has a velocity with respect to a given reference frame. In most examples we have examined so far, this reference frame has been Earth.
- **Relative Motion in One Dimension Example:** Consider an example of a person sitting in a train moving east. If we choose east as the positive direction and Earth as the reference frame, then we can write the velocity of the train with respect to the Earth as  $\vec{v}_{TE} = 10 \text{ m/s } \hat{i}$  east, where the subscripts TE refer to train and Earth. Let's now say the person gets up out of her seat and walks toward the back of the train at 2 m/s. This tells us she has a velocity relative to the reference frame of the train. Since the person is walking west, in the negative direction, we write her velocity with respect to the train as  $\vec{v}_{PT} = -2 \text{ m/s } \hat{i}$ . We can add the two velocity vectors to find the velocity of the person with respect to Earth. This relative velocity is written as

$$\vec{v}_{PE} = \vec{v}_{PT} + \vec{v}_{TE}.$$

- **Relative Velocity in Two Dimensions Example:** Consider a particle  $P$  and reference frames  $S$  and  $S'$ . The position of the origin of  $S'$  as measured in  $S$  is  $\vec{r}_{S'S}$ , the position of  $P$  as measured in  $S'$  is  $\vec{r}_{PS'}$ , and the position of  $P$  as measured in  $S$  is  $\vec{r}_{PS}$ .



From this, we see

$$\vec{r}_{PS} = \vec{r}_{PS'} + \vec{r}_{S'S}.$$

- **Relative velocities (Still using previous example):** The relative velocities are the time derivatives of the position vectors. Therefore,

$$\vec{v}_{PS} = \vec{v}_{PS'} + \vec{v}_{S'S}.$$

So we see the velocity of a particle relative to  $S$  is equal to its velocity relative to  $S'$  plus the velocity of  $S'$  relative to  $S$

- **Relative accelerations (Still using same example):** We can also see how the accelerations are related as observed in two reference frames by differentiating

$$\vec{a}_{PS} = \vec{a}_{PS'} + \vec{a}_{S'S}.$$

We see that if the velocity of  $S'$  relative to  $S$  is a constant, then  $\vec{a}_{S'S} = 0$  and

$$\vec{a}_{PS} = \vec{a}_{PS'}.$$

This says the acceleration of a particle is the same as measured by two observers moving at a constant velocity relative to each other.

- **Percent difference equation:** If  $A$  is the experimental value, and  $B$  is the actual value, then the percent difference is given by

$$\text{Percent difference} = \frac{|A - B|}{A} \cdot 100\%.$$

## 2.5 Chapter 5: Newton's laws of motion

### 2.5.1 Definitions and Theorems

- **Dynamics** is the study of how forces affect the motion of objects and systems.
- **constraints of Newtonian mechanics:** Newton's laws produce a good description of motion only when the objects are moving at speeds much less than the speed of light and when those objects are larger than the size of most molecules (about  $10^{-9}m$  in diameter).
- **Intuitive definition of force:** A push or a pull—is a good place to start.

**Note:** We know that a push or a pull has both magnitude and direction (therefore, it is a vector quantity), so we can define force as the push or pull on an object with a specific magnitude and direction. Force can be represented by vectors or expressed as a multiple of a standard force.

- **Standard force:** A quantitative definition of force can be based on some standard force, just as distance is measured in units relative to a standard length. One possibility is to stretch a spring a certain fixed distance, and use the force it exerts to pull itself back to its relaxed shape—called a restoring force—as a standard. The
- **Contact forces:** Contact forces are due to direct physical contact between objects.
- **Field forces:** Field forces, however, act without the necessity of physical contact between objects. They depend on the presence of a “field” in the region of space surrounding the body under consideration.
- **Field:** You can think of a field as a property of space that is detectable by the forces it exerts.
- **The newton:** 1 N is the force needed to accelerate an object with a mass of 1 kg at a rate of  $1m/s^2$ . Hence,  $1N=1kg \cdot m/s^2$
- **Net external force:** The resultant of forces is call the net external force  $\vec{F}_{\text{net}}$  and is found by taking the vector sum of all external forces acting on an object or system

$$\vec{F}_{\text{net}} = \sum \vec{F} = \vec{F}_1 + \vec{F}_2 + \dots$$

- **Newton's first law of motion:** A body at rest remains at rest or, if in motion, remains in motion at constant velocity unless acted on by a net external force.

**Note:** Newton's first law says that there must be a cause for any change in velocity (a change in either magnitude or direction) to occur. This cause is a net external force,

- On a frictionless surface and ignoring air resistance, we can imagine an object sliding in a straight line indefinitely. Friction is thus the cause of slowing (consistent with Newton's first law). The **object would not slow down if friction were eliminated.**
- **mass** is a measure of the amount of matter in something.
- **Gravitation** is the attraction of one mass to another

**Note:** such as the attraction between yourself and Earth that holds your feet to the floor. The magnitude of this attraction is your weight, and it is a force.

- **Inertia** is the ability of an object to resist changes in its motion—in other words, to resist acceleration.

**Note:** Newton's first law is often called the **law of inertia**. The inertia of an object is measured by its mass.

- **Inertial reference frame:** In principle, we can make the net force on a body zero. If its velocity relative to a given frame is constant, then that frame is said to be inertial.

So by definition, an inertial reference frame is a reference frame in which Newton's first law is valid. Newton's first law applies to objects with constant velocity. From this fact, we can infer the following statement.

A reference frame moving at constant velocity relative to an inertial frame is also inertial. A reference frame accelerating relative to an inertial frame is not inertial.

- **Newton's first law in vector form:**

$$\vec{v} = \text{constant when } \vec{F}_{\text{net}} = \vec{0}.$$

- **static equilibrium:** Static systems do not change over time. Static equilibrium involves objects at rest
- **Dynamic equilibrium:** Dynamic systems change over time. Dynamic equilibrium involves objects in motion without acceleration,
- **a net force of zero means that an object is either at rest or moving with constant velocity,**
- **Acceleration is proportional to the net external force.** That is,

$$a \propto \sum \vec{F}.$$

- **It also seems reasonable that acceleration should be inversely proportional to the mass of the system.**

$$a \propto \frac{1}{m}.$$

In other words, the larger the mass (the inertia), the smaller the acceleration produced by a given force.

- **Experiments have shown that acceleration is exactly inversely proportional to mass, just as it is directly proportional to net external force.**
- **Newton's second law of motion** The acceleration of a system is directly proportional to and in the same direction as the net external force acting on the system and is inversely proportional to its mass. In equation form, Newton's second law is

$$\vec{a} = \frac{\vec{F}_{\text{net}}}{m},$$

where  $\vec{a}$  is the acceleration,  $\vec{F}_{\text{net}}$  is the net force, and  $m$  is the mass. This is often written in the more familiar form

$$\vec{F}_{\text{net}} = \sum \vec{F} = m\vec{a},$$

but the first equation gives more insight into what Newton's second law means. When only the magnitude of force and acceleration are considered, this equation can be written in the simpler scalar form:

$$F_{\text{net}} = ma.$$

- **Component Form of Newton's Second Law** The equations of motion in three dimensions can be represented as:

$$\sum F_x = ma_x \quad (24)$$

$$\sum F_y = ma_y \quad (25)$$

$$\sum F_z = ma_z \quad (26)$$

- **A body's mass is a measure of its inertia**
- **Newton's Second Law and Momentum:** Newton actually stated his second law in terms of momentum: "The instantaneous rate at which a body's momentum changes is equal to the net force acting on the body." ("Instantaneous rate" implies that the derivative is involved.) This can be given by the vector equation

$$\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt}.$$

Momentum was described by Newton as "quantity of motion," a way of combining both the velocity of an object and its mass.

- **Momentum:** For now, it is sufficient to define momentum  $\vec{p}$  as the product of the mass of the object  $m$  and its velocity  $\vec{v}$

$$\vec{p} = m\vec{v}.$$

- **Weight:** If air resistance is negligible, the net force on a falling object is the gravitational force, commonly called its weight  $\vec{w}$ , or its force due to gravity acting on an object of mass  $m$

Weight can be denoted as a vector because it has a direction; down is, by definition, the direction of gravity, and hence, weight is a downward force. The magnitude of weight is denoted as  $w$ .

The gravitational force on a mass is its weight. We can write this in vector form, where  $\vec{w}$  is weight and  $m$  is mass, as

$$\vec{w} = m\vec{g}.$$

In scalar form, we can write

$$w = mg.$$

- **When the net external force on an object is its weight, we say that it is in free fall**
- **Newton's third law of motion:** Whenever one body exerts a force on a second body, the first body experiences a force that is equal in magnitude and opposite in direction to the force that it exerts. Mathematically, if a body  $A$  exerts a force  $\vec{F}$  on body  $B$ , then  $B$  simultaneously exerts a force  $-\vec{F}$  on  $A$ , or in vector equation form,

$$\vec{F}_{AB} = -\vec{F}_{BA}$$

Newton's third law represents a certain symmetry in nature: Forces always occur in pairs, and one body cannot exert a force on another without experiencing a force itself.

- **Normal Force:** If the force supporting the weight of an object, or a load, is perpendicular to the surface of contact between the load and its support, this force is defined as a **normal force** and here is given by the symbol  $\vec{N}$ .

$$\vec{N} = -m\vec{g}.$$

**Note:** The word normal means perpendicular to a surface.

For objects resting on horizontal surfaces, this becomes the scalar form

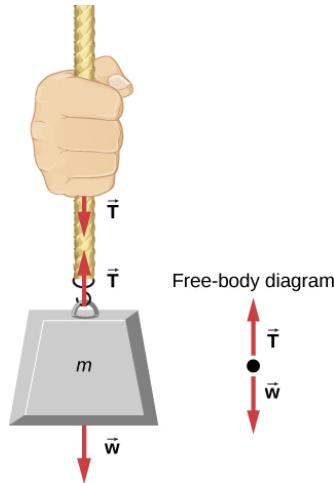
$$N = mg.$$

**Note:** The normal force can be less than the object's weight if the object is on an incline.

- **Normal force is perpendicular to the surface**
- **Tension:** A **tension** is a force along the length of a medium; in particular, it is a pulling force that acts along a stretched flexible connector, such as a rope or cable.

Any flexible connector, such as a string, rope, chain, wire, or cable, can only exert a pull parallel to its length; thus, a force carried by a flexible connector is a tension with a direction parallel to the connector.

Consider the following figure



If the 5.00-kg mass in the figure is stationary, then its acceleration is zero and the net force is zero. The only external forces acting on the mass are its weight and the tension supplied by the rope. Thus,

$$\begin{aligned} F_{\text{net}} &= T - w = 0 \\ \implies T &= w = mg. \end{aligned}$$

With this, (neglecting the mass of the rope), we see that the tension would be

$$\begin{aligned} T &= mg = (5.00\text{kg})(9.8\text{m/s}^2) \\ &= 49\text{N}. \end{aligned}$$

**Observation:** If we cut the rope and insert a spring, the spring would extend a length corresponding to a force of 49.0 N, providing a direct observation and measure of the tension force in the rope.

- **When to use normal force**

- **Contact Between Surfaces:** Use normal force in scenarios where two surfaces are in contact, such as an object resting on a table, a book lying on a shelf, or a box pushed against a wall.
- **Supporting Weight:** If an object is supported by a surface (preventing it from falling due to gravity), there's a normal force exerted by the surface on the object, equal and opposite to the component of the object's weight perpendicular to the surface.
- **Inclined Planes:** On an inclined plane, normal force acts perpendicular to the surface, supporting the component of the object's weight perpendicular to the plane. It's crucial for calculating the net force acting along the plane.

- **When to not use normal force**

- **No Direct Contact:** In scenarios where there's no direct physical contact between surfaces (e.g., objects in free fall, satellites in orbit, or a mass hanging from a string), normal force does not apply.
- **Tension-Dominated Problems:** In problems primarily involving tension (such as a tightrope walker, a hanging pendulum, or objects connected by ropes over pulleys), the focus is on tension forces rather than normal forces.

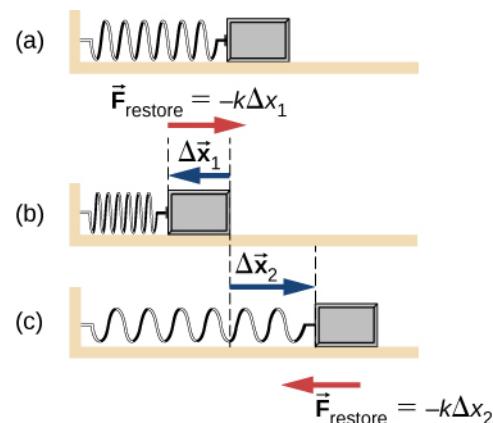
- **Tension  $T$  created when a perpendicular force ( $F_{\perp}$ ) is exerted at the middle of a flexible connector:**

$$T = \frac{F_{\perp}}{2 \sin(\theta)}.$$

The angle between the horizontal and the bent connector is represented by  $\theta$ . In this case,  $T$  becomes large as  $\theta$  approaches zero. Even

- **Hooke's law:** A spring is a special medium with a specific atomic structure that has the ability to restore its shape, if deformed. To restore its shape, a spring exerts a restoring force that is proportional to and in the opposite direction in which it is stretched or compressed. This is the statement of a law known as Hooke's law, which has the mathematical form

$$\vec{F} = -k\vec{x}.$$



A spring exerts its force proportional to a displacement, whether it is compressed or stretched.

- (a) The spring is in a relaxed position and exerts no force on the block.
- (b) The spring is compressed by displacement  $\vec{\Delta x}_1$  of the object and exerts restoring force  $-k\vec{\Delta x}_1$ .
- (c) The spring is stretched by displacement  $\vec{\Delta x}_2$  of the object and exerts restoring force  $-k\vec{\Delta x}_2$ .

- **Force on an inclined plane practice:** What force (in N) must be applied to a 250.0 kg crate on a frictionless plane inclined at  $30^\circ$  to cause an acceleration of  $7.6 \text{ m/s}^2$  up the plane? (Enter the magnitude.)

First, let's draw a diagram

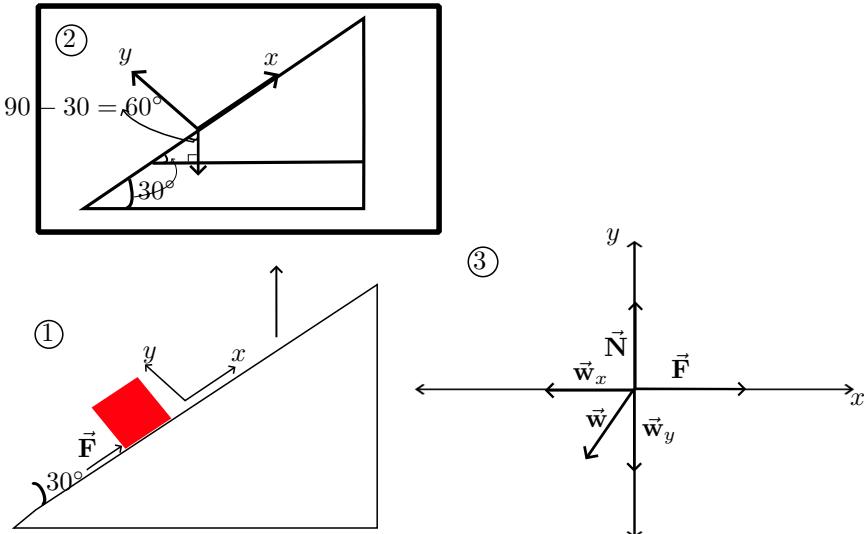


Figure 3 shows how we set up our coordinate axis, and figure 2 shows how we find the angle between the weight vector and the negative x-axis

To find the Force vector  $\vec{F}$ , we need to first find the net force in the horizontal direction. Since there is seemingly no vertical acceleration, we deduce that the net force must be zero. We find

$$W_x = \vec{w} \cos(60^\circ) = mg \cos(60^\circ)$$

Thus the net force in the horizontal direction is given by

$$\begin{aligned} F_{\text{net},x} &= \vec{F} - W_x = ma \\ \implies \vec{F} - wg \cos(60^\circ) &= ma \\ \implies \vec{F} &= ma + mg \cos(60^\circ) \\ \therefore \vec{F} &= (250)(7.6) + (250)(9.8) \cos(60^\circ) \\ &= 3125. \end{aligned}$$

- Average frictional force when stopping in car

$$|\vec{\mathbf{F}}_s| = m|(\vec{\mathbf{a}})_{\text{avg}}|.$$

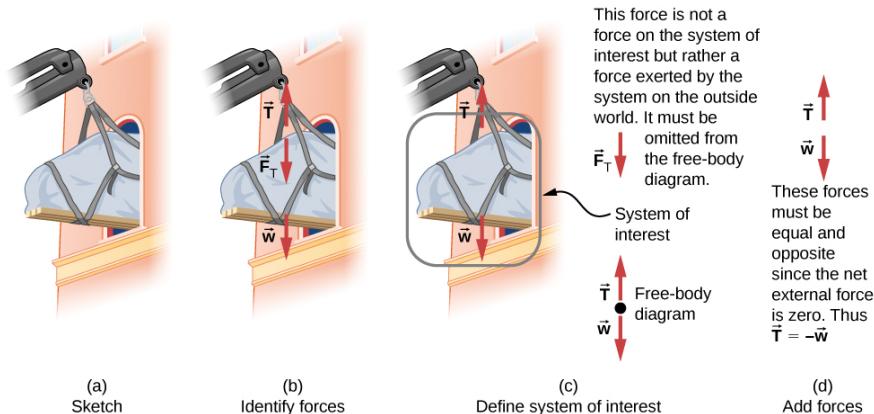
## 2.6 Chapter 6: Applications of Newton's Laws

### 2.6.1 Definitions and theorems

- **Problem solving / Omitting forces**

- Identify the physical principles involved by listing the givens and the quantities to be calculated.
- Sketch the situation, using arrows to represent all forces.
- Determine the system of interest. The result is a free-body diagram that is essential to solving the problem.
- Apply Newton's second law to solve the problem. If necessary, apply appropriate kinematic equations from the chapter on motion along a straight line.
- Check the solution to see whether it is reasonable.

Suppose we want to lift a grand piano into a second-story apartment. Consider the sketch



In this case we want to find the tension of the rope, so the system of interest is the piano. Thus, the force that the piano exerts on the rope is a force by the system and not of use. We only care about external forces for whatever system we are analyzing

- **Weight and normal force are external**

- **When tensions are equal**

- **Single Rope with No Mass and no friction:** In a system where a massless rope passes over a frictionless pulley connecting two masses, the tension throughout the rope is constant if the pulley is ideal (massless and frictionless) and the rope is massless. This is because the rope cannot exert any force by itself, and there's no mechanism (like friction or pulley mass) to change the tension in different parts of the rope.
- **Static Equilibrium:** In a scenario where the system is in static equilibrium (not moving), and if we ignore the mass of the rope or the friction in the pulley, the tension on either side of the pulley must be equal to maintain equilibrium. This is because, in equilibrium, the sum of forces in any direction must be zero, and thus the forces (tensions) pulling on either side of the pulley must balance out.

- **When Tensions are Not Equal:**

- **Rope with Mass:**

- \* If the rope has mass, the tension varies along the length of the rope. The tension is higher closer to where the heavier load is applied because it has to support more weight (including the weight of the rope itself).

- **Multiple Pulleys or Complex Systems:**

- \* In systems involving multiple pulleys or complex arrangements, tension can vary significantly throughout the system. Each segment of the rope can have different tensions due to varying angles of pull, different masses being lifted, or the mechanical advantage created by the pulleys.

- **Varying Angle Measures:**

- \* When forces are applied at different angles, especially in systems involving pulleys or objects being pulled at angles, the tension in the rope or cable can differ based on how the components of the forces contribute to the tension.

- **Acceleration:** In systems where the masses are accelerating, the tension in the rope can differ if the rope passes over a pulley that changes the direction of the tension force. If one mass is accelerating downward, it might cause the rope on its side to have more tension compared to the other side, especially if the masses or forces acting on them are not symmetrical.

- **Non-Ideal Conditions:** When the rope has mass or the pulley has friction, the tension can vary along the rope. For example, the part of the rope supporting a heavier mass will have more tension compared to the side with a lighter mass. Similarly, friction in the pulley can cause a difference in tension on either side because it requires additional force to overcome the friction, leading to higher tension on one side.

- **Friction** is a force that opposes relative motion between systems in contact.

- **Sliding friction:** is parallel to the contact surfaces between systems and is always in a direction that opposes motion or attempted motion of the systems relative to each other.

- **Static friction:** If two systems are in contact and stationary relative to one another, then the friction between them is called static friction.

- **Kinetic friction:** If two systems are in contact and moving relative to one another, then the friction between them is called kinetic friction.

- **Magnitude of static friction:** The magnitude of static friction  $f_s$  is

$$f_s \leq \mu_s N \quad (27)$$

where  $\mu_s$  is the coefficient of static friction and  $N$  is the magnitude of the normal force.

- Static friction is a responsive force that increases to be equal and opposite to whatever force is exerted, up to its maximum limit. Once the applied force exceeds  $f_s(\max)$ , the object moves. Thus,

$$f_s(\max) = \mu_s N.$$

- **Magnitude of kinetic friction:** The magnitude of kinetic friction  $f_k$  is given by

$$f_k = \mu_k N \quad (28)$$

where  $\mu_k$  is the coefficient of kinetic friction.

- the coefficients of kinetic friction are less than their static counterparts.
- Coefficient of friction is a unitless quantity with a magnitude usually between 0 and 1.0. The actual value depends on the two surfaces that are in contact.
- neither formula is accurate for lubricated surfaces or for two surfaces sliding across each other at high speeds.
- **Friction on an inclined plane:** The basic physics is the same. We usually generalize the sloping surface and call it an inclined plane but then pretend that the surface is flat.
- **Angular velocity (radial acceleration),** is the vector quantity  $\omega$ , as we know the term angular frequency refers to the scalar quantity  $\omega = \frac{2\pi}{T}$ , where  $T$  is the time it takes to complete one revolution, and  $\omega$  is given in rads/sec. This acceleration acts along the radius of the curved path and is thus also referred to as a radial acceleration.
- **Centripetal acceleration in terms of angular velocity**

$$a_c = r\omega^2.$$

- Any net force causing uniform circular motion is called a centripetal force. The direction of a centripetal force is toward the center of curvature,
- The direction of a centripetal force is toward the center of curvature, the same as the direction of centripetal acceleration.
- Equations for centripetal force

$$\begin{aligned} F_c &= ma_c \\ F_c &= m \frac{v^2}{r} \\ F_c &= mr\omega^2. \end{aligned}$$

- Centripetal force  $\vec{F}_c$  is always perpendicular to the path and points to the center of curvature,
- Friction in regards to cars and tires:
  - **Tires Rolling Without Slipping:** When a car is moving normally, and its tires are rolling without slipping, the friction between the tires and the road is static friction. Despite the car moving, the point of the tire that contacts the road is momentarily at rest relative to the road surface. This static friction is what allows the car to start moving from rest, stop, and turn without skidding. It's also the force that propels the car forward; the engine exerts a torque on the wheels, and it's the static frictional force at the contact patch of the tire that moves the car forward.
  - **Tires Slipping or Skidding:** When the tires are slipping or skidding on the road surface, the friction between the tires and the road is kinetic (sliding) friction. This occurs when the force applied (for example, through braking or accelerating too hard) exceeds the maximum static frictional force that can be developed between the tire and the road. Kinetic friction also acts when a car is drifting, where the tires are intentionally made to slip sideways.

**Note:** The key to understanding why static friction is involved with a moving car lies in the behavior of the tires in contact with the road. When a car moves and its tires are rolling without slipping, the point of contact between each tire and the road does not slide across the road surface. Instead, it momentarily "sticks" to the road before lifting off again as the tire rotates. This means that, at any given instant, the part of the tire touching the road is stationary relative to the road surface. The force preventing the tire from slipping and allowing it to roll is provided by static friction.

- **Finding theta for ideal banking (frictionless curve)**

$$\theta = \tan^{-1} \left( \frac{v^2}{rg} \right).$$

- **Drag force always opposes the motion of an object.**
- **Drag force (air resistance)** For most large objects such as cyclists, cars, and baseballs not moving too slowly, the magnitude of the drag force  $F_D$  is proportional to the square of the speed of the object.

$$F_D \propto v^2.$$

When taking into account other factors, this relationship becomes

$$F_D = \frac{1}{2} C \rho A v^2.$$

where  $C$  is the drag coefficient,  $A$  is the area of the object facing the fluid, and  $\rho$  is the density of the fluid.

This equation can also be written in a more generalized fashion as

$$F_D = bv^n.$$

Where  $b$  is a constant equivalent to  $0.5C\rho A$

**Note:** The value of the drag coefficient  $C$  is determined empirically,

- **What is terminal velocity:** Terminal velocity is the constant speed that a freely falling object eventually reaches when the resistance of the medium through which it is falling prevents further acceleration.
- **Terminal Velocity:** Consider a skydiver falling through air under the influence of gravity. Once air resistance becomes equal to the force of gravity, the net force will be zero and there will be no acceleration. At this point, the person's velocity remains constant and we say that the person has reached his terminal velocity. We have

$$mg = \frac{1}{2} C \rho A v^2$$

$$v_T = \sqrt{\frac{2mg}{\rho CA}}.$$

- **The density of air is approximately**

$$\rho = 1.21 \text{ kg/m}^3.$$

- **Stoke's Law:** For a spherical object falling in a medium, the drag force is

$$F_s = 6\pi r \eta v.$$

where  $r$  is the radius of the object,  $\eta$  is the viscosity of the fluid, and  $v$  is the object's velocity.

- A car approaches the top of a hill that is shaped like a vertical circle with a radius of 55.0m. What is the fastest speed that the car can go over the hill without losing contact with the ground?. In this case, we have  $F_y = mg - N = ma_c$ . But once the car loses contact with the road (goes past its max velocity to stay on the road), normal force goes to zero because the car would no longer be in contact with the road. Thus, the maximum velocity would be

$$\begin{aligned} F_y &= mg = a_c \\ \implies mg &= m \frac{v^2}{r} \\ \implies v &= \sqrt{g \cdot r}. \end{aligned}$$

## 2.7 Chapter 7: Work and Kinetic energy

### 2.7.1 Definitions and theorems

- **work** is done on an object when energy is transferred to the object. In other words, work is done when a force acts on something that undergoes a displacement from one position to another. Forces can vary as a function of position, and displacements can be along various paths between two points.
- **Increment of work  $dW$** : We first define the increment of work  $dW$  done by a force  $\vec{F}$  acting through an infinitesimal displacement  $d\vec{r}$  as the dot product of these two vectors:
- **Work done by a force**: We add up the contributions for infinitesimal displacements, along a path between two positions, to get the total work.

The work done by a force is the integral of the force with respect to displacement along the path of the displacement:

$$W_{AB} = \int_{\text{path AB}} \vec{F} \cdot d\vec{r}.$$

- **SI unit for work**: The units of work are units of force multiplied by units of length, which in the SI system is newtons times meters,  $N \cdot m$ . This combination is called a joule, abbreviated  $J$
- **American unit for work**: States, the unit of force is the pound (lb) and the unit of distance is the foot (ft), so the unit of work is the foot-pound (ft·lb).
- **Work done by constant forces and contact forces**: The simplest work to evaluate is that done by a force that is constant in magnitude and direction. In this case, we can factor out the force; the remaining integral is just the total displacement, which only depends on the end points A and B, but not on the path between them:

$$\begin{aligned} W_{AB} &= \vec{F} \cdot \int_A^B d\vec{r} \\ &= \vec{F} \cdot (\vec{r}_B - \vec{r}_A) \\ &= |\vec{F}| |\vec{r}_B - \vec{r}_A| \cos(\theta). \end{aligned}$$

That is, work is just  $\vec{F} \cdot \vec{d} = Fd \cos(\theta)$

- **Work of the normal force**. If the object being displaced never leaves the surface, then the displacement  $d\vec{r}$  is tangent to the surface. Since these two vectors are orthogonal, their dot product is zero. Thus, we have

$$dW_N = \vec{N} \cdot d\vec{r} = \vec{0}.$$

**Note:** If the displacement  $d\vec{r}$  did have a relative component perpendicular to the surface, the object would either leave the surface or break through it, and there would no longer be any normal contact force.

- **"Tangent to the surface"**: When the displacement is tangent to the surface, it means the movement of the object is along the surface

**Note:** This is the same as saying the displacement is parallel to the surface.

- **Work done by kinetic friction:** If the magnitude of  $\vec{f}_k$  is constant (as it would be if all the other forces on the object were constant), then the work done by friction is

$$W_{\text{fr}} = \int_A^B \vec{f}_k \cdot d\vec{r} = -f_k \int_A^B |d\vec{r}| = -f_k |l_{AB}|,$$

where  $|l_{AB}|$  is the path length on the surface.

**Note:** kinetic friction  $\vec{f}_k$  is opposite to  $d\vec{r}$  relative to the surface, so the work done by kinetic friction is negative.

- **Work done by static friction:** The force of static friction does no work in the reference frame between two surfaces because there is never displacement between the surfaces.

As an external force, static friction can do work. Static friction can keep someone from sliding off a sled when the sled is moving and perform positive work on the person.

- **the work done against a force is the negative of the work done by the force.**
- **Work done by gravity:** The work done by a constant force of gravity on an object depends only on the object's weight and the difference in height through which the object is displaced.

$$W_{\text{grav},AB} = -mg\hat{\mathbf{j}} \cdot (\vec{r}_B - \vec{r}_A) = -mg(y_B - y_A).$$

**Note:** Gravity does negative work on an object that moves upward ( $y_B > y_A$ ), or, in other words, you must do positive work against gravity to lift an object upward. Alternately, gravity does positive work on an object that moves downward ( $y_B < y_A$ ), or you do negative work against gravity to "lift" an object downward, controlling its descent so it doesn't drop to the ground. ("Lift" is used as opposed to "drop".)

- **Work Done by Forces that Vary:** In general, forces may vary in magnitude and direction at points in space, and paths between two points may be curved. The infinitesimal work done by a variable force can be expressed in terms of the components of the force and the displacement along the path,

$$dW = F_x dx + F_y dy + F_z dz.$$

- **Springs, Hooke's law, and work (perfectly elastic spring):** The force exerted by a perfectly elastic spring is governed by Hooke's law  $\vec{F} = -k\Delta\vec{x}$ , where  $k$  is the spring constant and  $\Delta\vec{x} = \vec{x} - \vec{x}_{\text{eq}}$  is the displacement from the spring's equilibrium position. This equilibrium position aligns with the spring's unstretched position in the absence of other forces or when they are neutralized. For work calculation by the spring force, the x-axis is aligned along the spring's length, with the origin at  $x_{\text{eq}} = 0$ . Thus, positive  $x$  denotes stretching and negative  $x$  compression. The work done by the spring force as  $x$  varies from  $x_A$  to  $x_B$  is calculated under this framework.

The work done by the spring from  $x_A$  to  $x_B$  is given by:

$$\begin{aligned} W_{\text{spring},AB} &= \int_A^B F_x dx = -k \int_A^B x dx \\ &= -\frac{k}{2} [x^2]_A^B = -\frac{1}{2}k(x_B^2 - x_A^2). \end{aligned}$$

**Note:** Notice that  $W_{AB}$  Depends only on the starting and ending points,  $A$  and  $B$ , and is independent of the actual path between them, as long as it starts at  $A$  and ends at  $B$ . That is, the actual path could involve going back and forth before ending.

- **Kinetic energy definition:** The kinetic energy of an object is the form of energy that it possesses due to its motion
- **Kinetic energy of a particle with mass  $m$ :**

$$K = \frac{1}{2}mv^2.$$

- **Kinetic energy for a system of particles:**

$$k = \sum \frac{1}{2}mv^2.$$

- **Kinetic energy in terms of a particles momentum (single particle):** that just as we can express Newton's second law in terms of either the rate of change of momentum or mass times the rate of change of velocity, so the kinetic energy of a particle can be expressed in terms of its mass and momentum  $\vec{p} = m\vec{v}$ , instead of its mass and velocity. Since  $v = \frac{p}{m}$ , we see that

$$k = \frac{1}{2} \frac{p^2}{m} = \frac{p^2}{2m}.$$

- **Units of kinetic energy:** The units of kinetic energy are also the units of force times distance, which are the units of work, or joules.
- **Work-energy theorem:** The net work done on a particle equals the change in the particle's kinetic energy:

$$W_{\text{net}} = K_B - K_A. \quad (30)$$

- **Average power:** we first define average power as the work done during a time interval, divided by the interval,

$$\bar{P} = \frac{\Delta W}{\Delta t}.$$

- **Instantaneous power (or just plain power):** Power is defined as the rate of doing work, or the limit of the average power for time intervals approaching zero,

$$P = \frac{dW}{dt}.$$

- **Work in constant power:** If the power is constant over a time interval, the average power for that interval equals the instantaneous power, and the work done by the agent supplying the power is

$$W = P\Delta t.$$

- **Work in varying power:** If the power during an interval varies with time, then the work done is the time integral of the power,

$$W = \int P dt.$$

- **Unit for power:** We can also define power as the rate of transfer of energy. Work and energy are measured in units of joules, so power is measured in units of joules per second, which has been given the SI name watts,

$$1J/s = 1W.$$

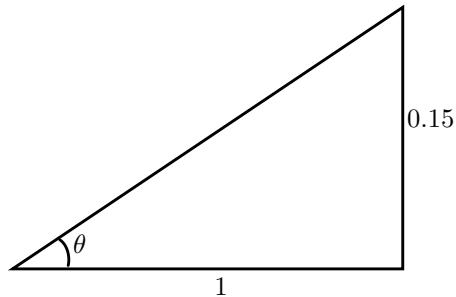
- **Horsepower:**

$$1hp = 746W.$$

- **Power in terms of forces and velocity**

$$\begin{aligned} P &= \frac{dW}{dt} = \frac{\vec{F} \cdot d\vec{r}}{dt} = \vec{F} \cdot \left( \frac{d\vec{r}}{dt} \right) \\ &= \vec{F} \cdot \vec{v}. \end{aligned}$$

- **Finding angle when given grade:** Suppose we want to find the angle associated with a 15% grade. This means for every unit increase in the x-direction, we move up 15% or 0.15.



Thus, we see

$$\begin{aligned} \tan(\theta) &= \frac{0.15}{1} \\ \theta &= \tan^{-1}(0.15) \\ &\approx 8.53^\circ. \end{aligned}$$

## 2.8 Chapter 8: Potential Energy and Conservation of Energy

### 2.8.1 Definitions and theorems

- **Potential energy definition:** Potential energy is the energy an object posses by virtue of its position in a field
- **Potential energy difference:** We define the difference of potential energy from point A to point B as the negative of the work done:

$$\Delta U_{AB} = U_B - U_A = -W_{AB}.$$

- **Potential energy:** we need to define potential energy at a given position in such a way as to state standard values of potential energy on their own, rather than potential energy differences. We do this by rewriting the potential energy function in terms of an arbitrary constant,

$$\Delta U = U(\vec{r}) - U(\vec{r}_0).$$

**Note:** the lowest height in a problem is usually defined as zero potential energy, or if an object is in space, the farthest point away from the system is often defined as zero potential energy. Then, the potential energy, with respect to zero at  $\vec{r}_0$ , is just  $U(\vec{r})$

- **Change in kinetic energy with no friction or air resistance:** As long as there is no friction or air resistance, the change in kinetic energy of the football equals negative of the change in gravitational potential energy of the football. This can be generalized to any potential energy:

$$\Delta K_{AB} = -\Delta U_{AB}.$$

- **Gravitational potential energy function**

$$U(y) = mgy + C.$$

**Note:** You can choose the value of the constant. However, for solving most problems, the most convenient constant to choose is zero for when  $y = 0$ , which is the lowest vertical position in the problem.

- **Elastic potential energy:**

$$U(x) = \frac{1}{2}kx^2 + C.$$

**Note:** If the spring force is the only force acting, it is simplest to take the zero of potential energy at  $x = 0$ , when the spring is at its unstretched length. Then, the constant is zero. (Other choices may be more convenient if other forces are acting.)

- **Conservative Force:** Conservative force, in physics, any force, such as the gravitational force between Earth and another mass, whose work is determined only by the final displacement of the object acted upon. The total work done by a conservative force is independent of the path resulting in a given displacement and is equal to zero when the path is a closed loop

- **Non-Conservative Force:** Non-conservative forces are dissipative forces such as friction or air resistance. These forces take energy away from the system as the system progresses, energy that you can't get back. These forces are path dependent; therefore it matters where the object starts and stops.
- **Work done by a conservative force:** The work done by a conservative force is independent of the path; in other words, the work done by a conservative force is the same for any path connecting two points:

$$W_{AB,\text{path-1}} = \int_A^B \text{on path-1} \vec{F}_{\text{cons}} \cdot d\vec{r} = W_{AB,\text{path-2}} = \int_A^B \text{on path-2} \vec{F}_{\text{cons}} \cdot d\vec{r}.$$

- **Work done by a non-conservative force:** The work done by a non-conservative force depends on the path taken.
- **When is a force conservative:** A force is conservative if the work it does around any closed path is zero:

$$W_{\text{closed path}} = \oint \vec{F}_{\text{cons}} \cdot d\vec{r} = 0.$$

- **Proving whether or not a force is conservative:** One answer is that the work done is independent of path if the infinitesimal work  $\vec{F} \cdot d\vec{r}$  is an exact differential, the way the infinitesimal net work was equal to the exact differential of the kinetic energy,  $dW_{\text{net}} = m\vec{v} \cdot d\vec{v} = d(\frac{1}{2}mv^2)$ ,

There are mathematical conditions that you can use to test whether the infinitesimal work done by a force is an exact differential, and the force is conservative. These conditions only involve differentiation and are thus relatively easy to apply. In two dimensions, the condition for  $\vec{F} \cdot d\vec{r} = F_x dx + F_y dy$  to be an exact differential is

$$\frac{dF_x}{dy} = \frac{dF_y}{dx}.$$

- **Non-conservative forces don't have potential energy:** we note that non-conservative forces do not have potential energy associated with them because the energy is lost to the system and can't be turned into useful work later. So there is always a conservative force associated with every potential energy.
- **The infinitesimal increment of potential energy:** force. The infinitesimal increment of potential energy is the dot product of the force and the infinitesimal displacement,

$$\begin{aligned} dU &= -\vec{F} \cdot d\vec{\ell} = -F_\ell d\ell \\ \implies F_\ell &= -\frac{dU}{d\ell}. \end{aligned}$$

Here, we chose to represent the displacement in an arbitrary direction by  $d\vec{\ell}$  so as not to be restricted to any particular coordinate direction.

In words, the component of a conservative force, in a particular direction, equals the negative of the derivative of the corresponding potential energy, with respect to a displacement in that direction.

- **Force with derivative of potential energy: One dimension**

$$\vec{F} = f_x \hat{i} = -\frac{dU}{dx} \hat{i}.$$

- **Force with derivative of potential energy: Two dimensions**

$$\vec{F} = f_x \hat{i} + f_y \hat{j} = -\frac{dU}{dx} \hat{i} + -\frac{dU}{dy} \hat{j}.$$

- **Mechanical energy:** mechanical energy is simply all the energy that an object has because of its motion and its position.

$$E = K + U.$$

Where  $E$  denotes the mechanical energy

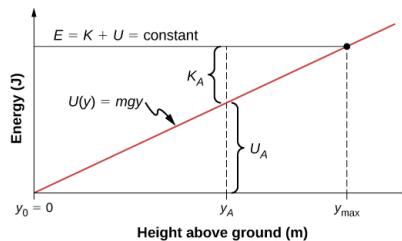
- **Conservation of Energy:** The mechanical energy  $E$  of a particle stays constant unless forces outside the system or non-conservative forces do work on it, in which case, the change in the mechanical energy is equal to the work done by the non-conservative forces:

$$W_{nc,AB} = \Delta(K + U)_{AB} = \Delta E_{AB}.$$

- **Position  $x$  as a function of time  $t$  with potential energy**

$$t = \int_{x_0}^x \frac{dx}{\sqrt{2[E - U(x)]/m}}.$$

- **Potential energy diagram example**



The line at energy  $E$  represents the constant mechanical energy of the object, whereas the kinetic and potential energies,  $K_A$  and  $U_A$ , are indicated at a particular height  $y_A$ . You can see how the total energy is divided between kinetic and potential energy as the object's height changes.

- **Max height of a particle:** Using the fact that kinetic energy must be positive, we can write  $K = E - U \geq 0 \implies U \leq E$ . If we use the gravitational potential energy reference point of zero at  $y_0$ , we can rewrite the gravitational potential energy  $U$  as  $mgy$ . Solving for  $y$  results in

$$y \leq \frac{E}{mg} = y_{\max}.$$

- **Max speed of a particle:** At ground level  $y_0$ , the potential energy is zero, and the kinetic energy and the speed are maximum.

$$U_0 = 0 = E - K_0$$

$$E = K_0 = \frac{1}{2}mv_0^2$$

$$v_0 = \pm \sqrt{\frac{2E}{m}}.$$

**Note:** The maximum speed  $\pm v_0$  gives the initial velocity necessary to reach  $y_{\max}$ , the maximum height, and  $-v_0$  represents the final velocity, after falling from  $y_{\max}$ .

- **Finding range of x.** Using the inequality  $0 \leq K \leq E$ , we can find the allowable range for x values. Considering the turning points, where all of the energy is potential, we have  $K = 0$  and  $U = E$ . For a elastic spring, we have

$$\begin{aligned} \frac{1}{2}kx^2 &= E \\ \implies x &= \pm\sqrt{\frac{2E}{k}}. \end{aligned}$$

- **Finding range of x: Example:** Suppose we have the function  $U(x) = 2(x^4 - x^2)$ , and  $E = -\frac{1}{4}$ . To find the allowable range for x values, we use the condition

$$\begin{aligned} U + K &= E \implies K = E - U \geq 0 \\ \implies -\frac{1}{4} - 2(x^4 - x^2) &\geq 0 \\ \implies 2(x^4 - x^2) &\leq -\frac{1}{4} \\ \implies 2\left(\left(x^4 - \frac{1}{2}\right)^2 - \frac{1}{4}\right) &\leq -\frac{1}{4} \\ \implies \left(x^4 - \frac{1}{2}\right)^2 &\leq \frac{-\frac{1}{4} + \frac{1}{2}}{2} \\ \implies \left(x^4 - \frac{1}{2}\right)^2 &\leq \frac{1}{8} \\ \implies -\frac{1}{\sqrt{8}} + \frac{1}{2} &\leq x^2 \leq \frac{1}{\sqrt{8}} + \frac{1}{2}. \end{aligned}$$

This represents two allowed regions,  $x_p \leq x \leq x_R$  and  $-x_R \leq x \leq -x_p$ , where  $x_p = 0.38$  and  $x_R = 0.92$  (in meters).

- **Manipulating inequalities: Squared terms:** For a non-negative, we have

$$\begin{aligned} x^2 &\leq a \\ \implies -\sqrt{a} &\leq x \leq \sqrt{a}. \end{aligned}$$

We also have

$$\begin{aligned} a &\leq x^2 \leq b \quad (\text{Suppose}) \\ \implies \sqrt{a} &\leq x \leq \sqrt{b} \quad \text{and} \quad -\sqrt{b} \leq x \leq -\sqrt{a}. \end{aligned}$$

For  $a, b$  non-negative. The implication imposed by this is that  $x$  can be within either ranges.

- **Finding equilibrium points.** To find the equilibrium points we just need to find relative extrema. Relative maxima is deemed unstable while relative minima is deemed stable

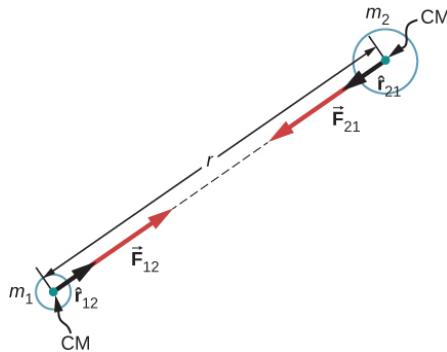
## 2.9 Chapter 13: Gravitation

### 2.9.1 Definitions and theorems

- **Newton's Law of Gravitation:** Newton's law of gravitation can be expressed as

$$\vec{F}_{12} = G \frac{m_1 m_2}{r^2} \hat{r}_{12} \quad (31)$$

where  $\vec{F}_{12}$  is the force on object 1 exerted by object 2 and  $\hat{r}_{12}$  is a unit vector that points from object 1 toward object 2.



**Note:** Notice how we take  $r$  to be the distance between the center of mass for both objects.

- **Universal gravitational constant:** The constant  $G$  is the previous equation is called the *universal gravitational constant* and Cavendish determined it to be

$$G = 6.67 \times 10^{-11} N \cdot \frac{m^2}{kg^2}$$

- **the law of gravitation applies to spherically symmetrical objects,** where the mass of each body acts as if it were at the center of the body.
- **Weight with NLOG:** We now know that this force is the gravitational force between the object and Earth. If we substitute  $mg$  for the magnitude of  $\vec{F}_{12}$  in Newton's law of universal gravitation,  $m$  for  $m_1$ , and  $M_E$  for  $m_2$ , we obtain the scalar equation

$$mg = G \frac{m M_E}{r^2} \quad (32)$$

where  $r$  is the distance between the centers of mass of the object and Earth.

**Note:** The average radius of Earth is about 6370 km. Hence, for objects within a few kilometers of Earth's surface, we can take  $r = R_E$

- **Calculating  $g$ ,** dividing the previous equation by  $m$ , we get

$$g = G \frac{M_E}{r^2} \quad (33)$$

- **Radius of the earth**  $R_E$

$$R_E = 6.37 \times 10^6 \text{ m.}$$

- **Mass of the earth:**

$$M_E = 5.95 \times 10^{24} \text{ kg.}$$

- **$g$  on the moons surface**

$$g = 1.6 \text{ m/s}^2.$$

- **The gravitational field caused by mass  $M$**

$$\vec{g} = G \frac{M}{r^2} \hat{r}.$$

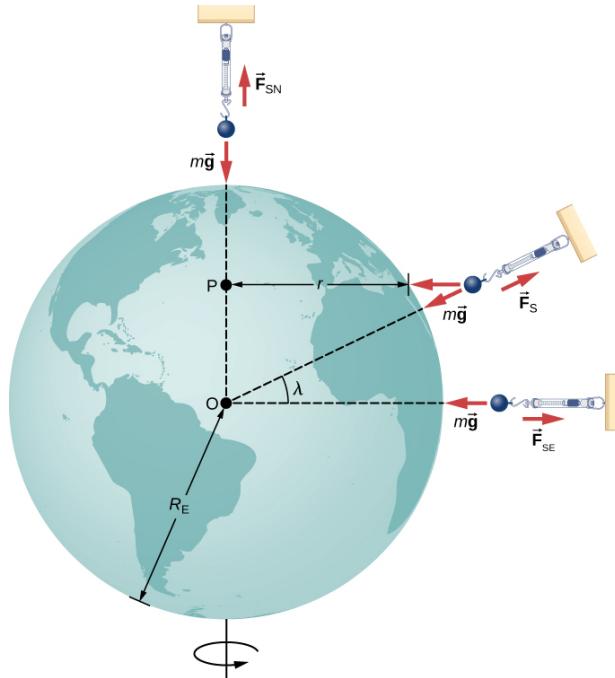
We identify this vector field  $\vec{g}$  as the *gravitational field caused by mass  $M$*

- **Weight at the equator with earths spin:**

$$\sum F = F_s - mg = ma_c.$$

Where  $a_c = -\frac{v^2}{r}$

- **Results Away from the Equator:** At any other latitude  $\lambda$ , the situation is more complicated. The centripetal acceleration is directed toward point P in the figure, and the radius becomes  $r = R_E \cos \lambda$ . The vector sum of the weight and  $\vec{F}_s$  must point toward point P, hence  $\vec{F}_s$  no longer points away from the center of Earth. (The difference is small and exaggerated in the figure.)



- **Gravity Away from the Surface:** Earlier we stated that the law of gravitation applies to spherically symmetrical objects, where the mass of each body acts as if it were at the center of the body.  $(\vec{F}_{12} = G \frac{m_1 m_2}{r^2} \hat{r}_{12})$  and  $(g = G \frac{M_E}{r^2})$ .

both equations are valid only for values of  $r \geq R_E$ . For  $r < R_E$  these equations are not valid. However, we can determine  $g$  for these cases using a principle that comes from Gauss's law,

A consequence of Gauss's law, applied to gravitation, is that only the mass *within*  $r$  contributes to the gravitational force. Also, that mass, just as before, can be considered to be located at the center. The gravitational effect of the mass *outside*  $r$  has zero net effect.

For a spherical planet with constant density, the mass within  $r$  is the density times the volume within  $r$ . This mass can be considered located at the center. Replacing  $M_E$  with only the mass within  $r$ ,  $M = \rho \times (\text{volume of a sphere})$ , and  $R_E$  with  $r$ , we get

$$g = \frac{GM_E}{R_E^2} = G\rho \left( \frac{4}{3}\pi r^3 \right) \frac{1}{r^2} = \frac{4}{3}G\rho\pi r.$$

**Note:** The value of  $g$ , and hence your weight, decreases linearly as you descend down a hole to the center of the spherical planet. At the center, you are weightless, as the mass of the planet pulls equally in all directions.

Actually, Earth's density is not constant, nor is Earth solid throughout.

- **Gravitational Potential Energy beyond Earth:** We return to the definition of work and potential energy to derive an expression that is correct over larger distances.

$$U = -\frac{GM_E m}{r}.$$

- **Conservation of Energy with Gravitation:**

$$\begin{aligned} E &= K_1 + U_1 = K_2 + U_2 \\ \implies &= \frac{1}{2}mv_1^2 - \frac{GMm}{r_1} = \frac{1}{2}mv_2^2 - \frac{GMm}{r_2}. \end{aligned}$$

**Note:** Note that we use  $M$ , rather than  $M_E$ , as a reminder that we are not restricted to problems involving Earth. However, we still assume that  $m \ll M$ . (For problems in which this is not true, we need to include the kinetic energy of both masses and use conservation of momentum to relate the velocities to each other. But the principle remains the same.)

- **Escape velocity:** *Escape velocity* is often defined to be the **minimum initial velocity** of an object that is required to escape the surface of a planet (or any large body like a moon) and never return. As usual, we assume no energy lost to an atmosphere, should there be any.

Consider the case where an object is launched from the surface of a planet with an initial velocity directed away from the planet. With the minimum velocity needed to escape, the object would just come to rest infinitely far away, that is, the object gives up the last of its kinetic energy just as it reaches infinity, where the force of gravity becomes zero. Since  $U \rightarrow 0$  as  $r \rightarrow \infty$

Thus, we find the escape velocity from the surface of an astronomical body of mass  $M$  and radius  $R$  by setting the total energy equal to zero. At the surface of the body, the object is located at  $r_1 = R$  and it has escape velocity  $v_1 = v_{\text{esc}}$ . It reaches  $r_2 = \infty$  with velocity  $v_2 = 0$ . Substituting the equation above and solving for  $v_{\text{esc}}$ , we find

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}}.$$

**Note:** Notice that  $m$  has canceled out of the equation. The escape velocity is the same for all objects, regardless of mass. Also, we are not restricted to the surface of the planet;  $R$  can be any starting point beyond the surface of the planet.

- **Not gravitationally Bound:** As noted earlier, we see that  $U \rightarrow 0$  as  $r \rightarrow \infty$ . If the total energy is zero, then as  $m$  reaches a value of  $r$  that approaches infinity,  $U$  becomes zero and so must the kinetic energy. Hence,  $m$  comes to rest infinitely far away from  $M$ . It has "just escaped"  $M$ . If the total energy is positive, then kinetic energy remains at  $r = \infty$  and certainly  $m$  does not return. When the total energy is zero or greater, then we say that  $m$  is not gravitationally bound to  $M$ .
- **Gravitationally Bonud:** On the other hand, if the total energy is negative, then the kinetic energy must reach zero at some finite value of  $r$ , where  $U$  is negative and equal to the total energy. The object can never exceed this finite distance from  $M$ , since to do so would require the kinetic energy to become negative, which is not possible. We say  $m$  is gravitationally bound to  $M$ .
- **Speed of orbit**

$$v_{\text{orbit}} = \sqrt{\frac{GM_E}{r}}.$$

- **Period of a circular orbit:**

$$T = 2\pi \sqrt{\frac{r^3}{GM_E}}.$$

- **Total energy for a circular orbit:**

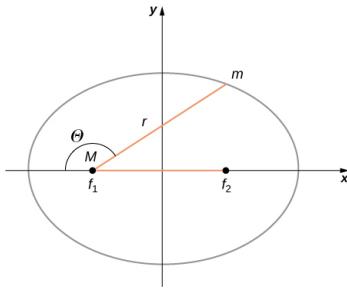
$$E = K + U = -\frac{GM_E m}{2r}.$$

For circular orbits, the magnitude of the kinetic energy is exactly one-half the magnitude of the potential energy.

- **Keplars first law:** Kepler's first law states that every planet moves along an ellipse, with the Sun located at a focus of the ellipse.
- **Form of all conics:** There are four different conic sections, all given by the equation

$$\frac{a}{r} = 1 + e \cos(\theta).$$

The variables  $r$  and  $\theta$  are shown in the figure below. In the case of an ellipse. The constants  $a$  and  $e$  are determined by the total energy and angular momentum of the satellite at a given point. The constant  $e$  is called the eccentricity. The values of  $a$  and  $e$  determine which of the four conic sections represents the path of the satellite.



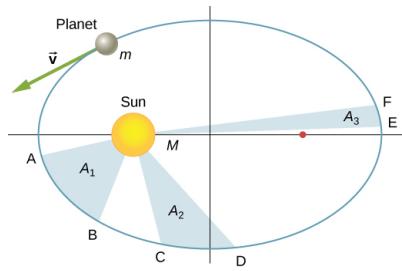
Every path taken by  $m$  is one of the four conic sections: a circle or an ellipse for bound or closed orbits, or a parabola or hyperbola for unbounded or open orbits.

- **Total energy for an elliptical orbit**

$$E = -\frac{GmM_s}{2a}.$$

Where  $M_s$  is the mass of the sun and  $a$  is the semi-major axis

- **Keplar's second law** Kepler's second law states that a planet sweeps out equal areas in equal times, that is, the area divided by time, called the areal velocity, is constant.



The time it takes a planet to move from position  $A$  to  $B$ , sweeping out area  $A_1$ , is exactly the time taken to move from position  $C$  to  $D$ , sweeping area  $A_2$ , and to move from  $E$  to  $F$ , sweeping out area  $A_3$ . These areas are the same:  $A_1 = A_2 = A_3$ .

- **semi-major axis**

$$a = \frac{1}{2}(\text{aphelion} + \text{perihelion}).$$

- **Keplar's third law:** Kepler's third law states that the square of the period is proportional to the cube of the semi-major axis of the orbit.

$$T^2 = \frac{4\pi^2}{GM}a^3.$$

- **Schwarzschild radius:** For any mass  $M$ , if that mass were compressed to the extent that its radius becomes less than the Schwarzschild radius, then the mass will collapse to a singularity, and anything that passes inside that radius cannot escape. Once inside  $R_S$ , the arrow of time takes all things to the singularity. (In a broad mathematical sense, a singularity is where the value of a function goes to infinity. In this case, it is a point in space of zero volume with a finite mass. Hence, the mass density and gravitational energy become infinite.) The Schwarzschild radius is given by

$$R_S = \frac{2GM}{c^2}.$$

**Note:** The Schwarzschild radius is also called the event horizon of a black hole.

## 2.10 Chapter 9: Linear momentum and collisions

### 2.10.1 Definitions and Theorems

- **Momentum** is a quantity of motion and is defined as the vector

$$\vec{p} = m\vec{v}.$$

So we see the momentum  $p$  of an object is the product of its mass and its velocity:

**Note:** kinetic energy. It is perhaps most useful when determining whether an object's motion is difficult or easy to change over a short time interval.

- **Impulse:** The product of a force and a time interval (over which that force acts) is called impulse, and is given the symbol  $\vec{J}$ . Formally we write

Let  $\vec{F}(t)$  be the force applied to an object over some differential time interval  $dt$ . The resulting impulse on the object is defined as

$$d\vec{J} = \vec{F}(t)dt.$$

The total impulse over the interval  $t_f - t_i$  is

$$\vec{J} = \int_{t_1}^{t_2} d\vec{J} \quad \text{or} \quad \int_{t_1}^{t_2} \vec{F}(t) dt.$$

- **Impulse with the mean value theorem:** To calculate the impulse using Equation 9.3, we need to know the force function  $F(t)$ , which we often don't. However, a result from calculus is useful here: Recall that the average value of a function over some interval is calculated by

$$f_{\text{ave}}(x) = \frac{1}{\Delta x} \int_{x_i}^{x_f} f(x) dx$$

where  $\Delta x = x_f - x_i$ . Applying this to the time-dependent force function, we obtain

$$\vec{F}_{\text{ave}} = \frac{1}{\Delta t} \int_{t_i}^{t_f} \vec{F}(t) dt.$$

Therefore, from the equation above

$$\vec{J} = \vec{F}_{\text{ave}} \Delta t.$$

**Note:** The idea here is that you can calculate the impulse on the object even if you don't know the details of the force as a function of time; you only need the average force. In fact, though, the process is usually reversed: You determine the impulse (by measurement or calculation) and then calculate the average force that caused that impulse.

- **Impulse again** To calculate the impulse, a useful result follows from writing the force in Equation 9.3 as  $\vec{F}(t) = m\vec{a}(t)$ :

$$\begin{aligned} \vec{J} &= \int_{t_i}^{t_f} \vec{F}(t) dt = m \int_{t_i}^{t_f} \vec{a}(t) dt = m[\vec{v}(t_f) - \vec{v}_i]. \\ &\therefore \vec{J} = m\Delta\vec{v}. \end{aligned}$$

- **impulse-momentum theorem:** Because  $m\vec{v}$  is the momentum of a system,  $m\Delta\vec{v}$  is the change of momentum  $\Delta\vec{p}$ . This gives us the following relation,

An impulse applied to a system changes the system's momentum, and that change of momentum is exactly equal to the impulse that was applied:

$$\vec{J} = \Delta \vec{p}.$$

- **Two crucial concepts in the impulse-momentum theorem:**

- Impulse is a vector quantity; an impulse of, say,  $-(10 \text{ N} \cdot \text{s})\hat{i}$  is very different from an impulse of  $+(10 \text{ N} \cdot \text{s})\hat{i}$ ; they cause completely opposite changes of momentum.
- An impulse does not cause momentum; rather, it causes a change in the momentum of an object. Thus, you must subtract the initial momentum from the final momentum, and—since momentum is also a vector quantity—you must take careful account of the signs of the momentum vectors.

- **Problem solving: Impulse momentum theorem**

- **Average force with momentum**

$$\vec{F}_{\text{ave}} = \frac{\Delta \vec{p}}{\Delta t}.$$

- **Force with rate of change in momentum (Newton's second law in terms of momentum)**

$$\vec{F} = \frac{d\vec{p}}{dt}.$$

This says that the rate of change of the system's momentum (implying that momentum is a function of time) is exactly equal to the net applied force (also, in general, a function of time). This is, in fact, Newton's second law, written in terms of momentum rather than acceleration.

- **Newton's third law with momentum**

$$m_1 \frac{d\vec{v}_1}{dt} = -m_2 \frac{d\vec{v}_2}{dt}.$$

- **Newton's third law with momentum if mass remains constant:** We can then pull the masses inside the derivatives (see equation above)

$$\frac{d}{dt}(m_1 \vec{v}_1) = -\frac{d}{dt}(m_2 \vec{v}_2).$$

and thus

$$\frac{d\vec{p}_1}{dt} = -\frac{d\vec{p}_2}{dt}.$$

This says that the rate at which momentum changes is the same for both objects. The masses are different, and the changes of velocity are different, but the rate of change of the product of  $m$  and  $\vec{v}$  are the same.

- **Newton's third law with momentum if mass remains constant (additive inverse version)**

$$\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = 0.$$

This says that during the interaction, although object 1's momentum changes, and object 2's momentum also changes, these two changes cancel each other out, so that the total change of momentum of the two objects together is zero.

Since the total combined momentum of the two objects together never changes, then we could write

$$\begin{aligned}\frac{d}{dt}(\vec{p}_1 + \vec{p}_2) &= 0 \\ \Rightarrow \vec{p}_1 + \vec{p}_2 &= \text{constant.}\end{aligned}$$

- **Conservation laws:** If the value of a physical quantity is constant in time, we say that the quantity is conserved.
- **Requirements for Momentum Conservation:** There is a complication, however. A system must meet two requirements for its momentum to be conserved:
  - The mass of the system must remain constant during the interaction. As the objects interact (apply forces on each other), they may transfer mass from one to another; but any mass one object gains is balanced by the loss of that mass from another. The total mass of the system of objects, therefore, remains unchanged as time passes:

$$\left[ \frac{dm}{dt} \right]_{\text{system}} = 0.$$

- The net external force on the system must be zero. As the objects collide, or explode, and move around, they exert forces on each other. However, all of these forces are internal to the system, and thus each of these internal forces is balanced by another internal force that is equal in magnitude and opposite in sign. As a result, the change in momentum caused by each internal force is cancelled by another momentum change that is equal in magnitude and opposite in direction. Therefore, internal forces cannot change the total momentum of a system because the changes sum to zero. However, if there is some external force that acts on all of the objects (gravity, for example, or friction), then this force changes the momentum of the system as a whole; that is to say, the momentum of the system is changed by the external force. Thus, for the momentum of the system to be conserved, we must have

$$\vec{F}_{\text{ext}} = \vec{0}.$$

- **Closed system:** A system of objects that meets these two requirements is said to be a closed system (also called an isolated system). Thus, the more compact way to express this is shown below.
- **Law of conservation of momentum:** The total momentum of a closed system is conserved:

$$\sum_{j=1}^N \vec{P}_j = \text{constant.}$$

This statement is called the **Law of Conservation of Momentum**

**Note:** Note that there absolutely can be external forces acting on the system; but for the system's momentum to remain constant, these external forces have to cancel, so that the net external force is zero. Billiard balls on a table all have a weight force acting on them, but the weights are balanced (canceled) by the normal forces, so there is no net force.

- **The meaning of a "system":** A **system** (mechanical) is the collection of objects in whose motion (kinematics and dynamics) you are interested. If you are analyzing the bounce of a ball on the ground, you are probably only interested in the motion of the ball, and not of Earth; thus, the ball is your system. If you are analyzing a car crash, the two cars together compose your system
- **Problem solving: Conservation of momentum:** Using conservation of momentum requires four basic steps. The first step is crucial:
  1. Identify a closed system (total mass is constant, no net external force acts on the system).
  2. Write down an expression representing the total momentum of the system before the "event" (explosion or collision).
  3. Write down an expression representing the total momentum of the system after the "event."
  4. Set these two expressions equal to each other, and solve this equation for the desired quantity.
- **Explosions:** if the object is initially motionless, then the system (which is just the object) has no momentum and no kinetic energy. After the explosion, the net momentum of all the pieces of the object must sum to zero (since the momentum of this closed system cannot change). However, the system will have a great deal of kinetic energy after the explosion, although it had none before. Thus, we see that, although the momentum of the system is conserved in an explosion, the kinetic energy of the system most definitely is not; it increases. This interaction—one object becoming many, with an increase of kinetic energy of the system—is called an **explosion**.

**Note:** Where does the energy come from? Does conservation of energy still hold? Yes; some form of potential energy is converted to kinetic energy. In the case of gunpowder burning and pushing out a bullet, chemical potential energy is converted to kinetic energy of the bullet, and of the recoiling gun. For a bow and arrow, it is elastic potential energy in the bowstring.

- **Inelastic collisions:** two or more objects collide with each other and stick together, thus (after the collision) forming one single composite object. The total mass of this composite object is the sum of the masses of the original objects, and the new single object moves with a velocity dictated by the conservation of momentum. However, it turns out again that, although the total momentum of the system of objects remains constant, the kinetic energy doesn't; but this time, the kinetic energy decreases. This type of collision is called inelastic.
- **Perfectly inelastic:** Any collision where the objects stick together will result in the maximum loss of kinetic energy (i.e.,  $K_f$  will be a minimum). Such a collision is called **perfectly inelastic**.
- **Elastic collisions:** The extreme case on the other end is if two or more objects approach each other, collide, and bounce off each other, moving away from each other at the same relative speed at which they approached each other. In this case, the total kinetic energy of the system is conserved. Such an interaction is called elastic.

- **Classifying collision types:**

- If  $0 < K_f < K_i$ , the collision is inelastic.
- If  $K_f$  is the lowest energy, or the energy lost by both objects is the most, the collision is perfectly inelastic (objects stick together).
- If  $K_f = K_i$ , the collision is elastic.

- **Problem solving: Collisions**

1. Define a closed system.
2. Write down the expression for conservation of momentum.
3. If kinetic energy is conserved, write down the expression for conservation of kinetic energy; if not, write down the expression for the change of kinetic energy.
4. You now have two equations in two unknowns, which you solve by standard methods.

- **Momentum in two dimensions:**

$$p_{f,x} = p_{1,i,x} + p_{2,i,x}$$

$$p_{f,y} = p_{1,i,y} + p_{2,i,y}.$$

- **Force of a system of particles:** Note that the change of momentum of a system is entirely due the external forces, the sum of the chaneg in momentum of internal forces is zero

$$\vec{\mathbf{F}}_{\text{ext}} = \sum_{j=1}^N \frac{d\vec{\mathbf{p}}_j}{dt} .$$

We also know

$$\vec{\mathbf{P}}_{\text{cm}} = \sum_{j=1}^N \vec{\mathbf{p}}_j .$$

Thus,

$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}_{cm}}{dt}.$$

- **Center of mass:**

$$\vec{\mathbf{r}}_{cm} = \frac{1}{M} \sum_{j=1}^N m_j \vec{\mathbf{r}}_j .$$

- **Instantaneous velocity at center of mass**

$$\begin{aligned} \vec{\mathbf{v}}_{\text{cm}} &= \frac{d}{dt} \left( \frac{1}{M} \sum_{j=1}^N m_j \vec{\mathbf{r}}_j \right) \\ &= \frac{1}{M} \sum_{j=1}^N m_j \vec{\mathbf{v}}_j . \end{aligned}$$

- **Center of Mass of Continuous Objects:** If the object in question has its mass distributed uniformly in space, rather than as a collection of discrete particles, then  $m_j \rightarrow dm$ , and the summation becomes an integral:

$$\vec{r}_{\text{cm}} = \frac{1}{M} \int \vec{r} dm.$$

- **Conservation of momentum regarding center of mass**

$$M\vec{v}_{\text{cm,f}} = M\vec{v}_{\text{cm,i}}.$$

Which implies, for a closed system with constant  $M$ , the velocities are equal

- **Rocket equation:**

$$\Delta V = u \ln \left( \frac{m_0}{m} \right).$$

Where  $\Delta v$  is the change in velocity of the rocket,  $u$  is the effective exhaust velocity (which is the average velocity at which the exhaust gases are ejected from the rocket as seen from the rocket frame),  $m_0$  is the initial mass, and  $m$  is the final mass after burning some amount of fuel.

- **Rocket in a Gravitational Field**

$$\Delta V = u \ln \left( \frac{m_0}{m} \right) - g\Delta t.$$

- **Final velocity of each object in elastic collision**

$$v_{1,f} = \frac{v_{1,i}(m_1 - m_2) + 2m_2 v_{2,i}}{m_1 + m_2}$$

$$v_{2,f} = \frac{v_{2,i}(m_2 - m_1) + 2m_1 v_{1,i}}{m_1 + m_2}$$

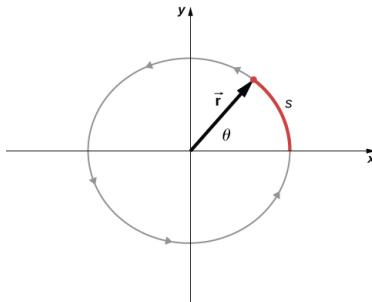
## 2.11 Chapter 10: Fixed-axis rotation

### 2.11.1 Definitions and theorems

- **Arc length of a circle**

$$s = r\theta$$

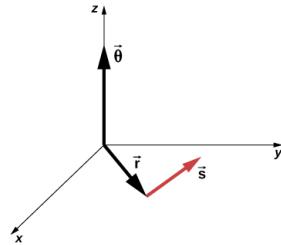
$$\implies \theta = \frac{s}{r}$$



- **Arc length with vectors**

$$\vec{s} = \vec{\theta} \times \vec{r}.$$

That is, the arc length is the cross product of the angle vector and the position vector,



- **Magnitude of angular velocity:** The magnitude of the angular velocity, denoted by  $\omega$ , is the time rate of change of the angle  $\theta$  as the particle moves in its circular path. The instantaneous angular velocity is defined as the limit in which  $\Delta t \rightarrow 0$  in the average angular velocity  $\bar{\omega} = \frac{\Delta\theta}{\Delta t}$ :

$$\omega = \lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}.$$

Units of angular velocity are  $\text{rad s}^{-1}$

- **Tangential speed  $v_t$**

$$v_t = r\omega.$$

- **Tangential speed with vectors**

$$\vec{v} = \vec{\omega} \times \vec{r}.$$

- Angular acceleration  $\alpha$

$$\alpha = \lim_{\Delta t \rightarrow 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2},$$

The units are  $\text{rad s}^{-2}$

- Tangential acceleration with cross product

$$\vec{a} = \vec{\alpha} \times \vec{r}.$$

- Tangential acceleration as scalar

$$a = \alpha r.$$

- Average angular velocity under constant acceleration

$$\bar{\omega} = \frac{\omega_0 + \omega_f}{2}.$$

- Rotation with constant angular acceleration

- Final  $\theta$

$$\theta_f = \theta_0 + \bar{\omega}t.$$

- Final  $\omega$

$$\omega_f = \omega + \alpha t.$$

- Final  $\theta$

$$\theta_f = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2.$$

- Final  $\omega$

$$\omega_f^2 = \omega_0^2 + 2\alpha(\Delta\theta).$$

- Rotational kinetic energy

$$K = \frac{1}{2}mv_t^2 = \frac{1}{2}m(\omega r)^2.$$

- Rotational kinetic energy for a rigid body

$$K = \sum_j \frac{1}{2}m_j v_j^2 = \sum_j \frac{1}{2}m_j(r_j\omega_j)^2.$$

But since  $\omega_j = \omega \forall$  masses, we have

$$K = \frac{1}{2} \left( \sum_j m_j r_j^2 \right) \omega^2.$$

- Moment of inertia

$$I = \sum_j m_j r_j^2.$$

With units  $\text{kg} \cdot \text{m}^2$

- Rotational kinetic energy using MOI notation

$$K = \frac{1}{2} I \omega^2.$$

- Moment of inertia with infinitesimally small piece of mass

$$I = \int r^2 dm.$$

- Linear mass density  $\lambda$ : Which is the mass per unit length

$$\lambda = \frac{m}{L} \implies m = \lambda L.$$

Provided the mass density of the object is uniform

- Mass differential with  $\lambda$

$$dm = d(\lambda L) = \lambda(dL).$$

- Parallel-axis theorem

Let  $m$  be the mass of an object and let  $d$  be the distance from an axis through the object's center of mass to a new axis. Then we have

$$I_{\text{parallel-axis}} = I_{\text{center of mass}} + md^2.$$

- Surface mass density (uniform): Which is the mass per unit surface area.

$$\begin{aligned} \sigma &= \frac{m}{A}, \quad \text{or} \quad \sigma A = m \\ \implies dm &= \sigma(dA). \end{aligned}$$

- Moment of inertia for compound objects

$$I_{\text{total}} = \sum_i I_i.$$

- Moment of inertia for uniform thin rod with axis through center of mass

$$I = \frac{1}{12} ML^2.$$

- Moment of inertia for uniform thin rod with axis at one of the ends

$$I = \frac{1}{3} ML^2.$$

- Moment of inertia for a uniform disk with axis at center

$$I = \frac{1}{2} m R^2.$$

- Moment of inertia for a solid cylinder

$$I = \frac{1}{2} m R^2.$$

- Moment of inertia for a solid sphere

$$I = \frac{2}{5} m R^2.$$

- Torque: A force that tends to cause rotation.

When a force  $\vec{F}$  is applied to a point  $P$  whose position is  $\vec{r}$  relative to  $O$  (Figure 10.32), the torque  $\vec{\tau}$  around  $O$  is

$$\vec{\tau} = \vec{r} \times \vec{F}.$$

With magnitude

$$\|\vec{\tau}\| = rF \sin(\theta).$$

- **Lever arm:** The quantity  $r \sin(\theta)$  from the equation above is denoted  $r_{\perp}$  and is defined to be the perpendicular distance from  $O$  to the line determined by the vector  $\vec{F}$ . Thus we can write the magnitude of the torque as

$$\|\vec{\tau}\| = r_{\perp} F.$$

- **Net torque**

$$\vec{\tau}_{\text{net}} = \sum_i \vec{\tau}_i.$$

- **Newton's second law for rotation:** If more than one torque acts on a rigid body about a fixed axis, then the sum of the torques equals the moment of inertia times the angular acceleration:

$$\tau_{\text{net}} = \sum_i \tau_i = I\alpha.$$

- **rotational work:** The total work done on a rigid body is the sum of the torques integrated over the angle through which the body rotates

$$W = \int \sum \vec{\tau} \cdot d\vec{\theta}.$$

- **Rotational work for constant  $\tau$**

$$W = \tau\theta.$$

Provided theta starts at zero

- **Incremental rotational work:**

$$dW = \left( \sum_i \tau_i \right) d\theta.$$

- **Work-energy theorem for rotation:** The work-energy theorem for a rigid body rotating around a fixed axis is

$$W_{AB} = K_B - K_A$$

where

$$K = \frac{1}{2}I\omega^2$$

and the rotational work done by a net force rotating a body from point A to point B is

$$W_{AB} = \int_{\theta_A}^{\theta_B} \left( \sum_i \tau_i \right) d\theta.$$

- **Work energy theorem for a constant  $\tau$**

$$W_{AB} = \tau(\theta_B - \theta_A) = \frac{1}{2}I(\omega_B^2 - \omega_A^2).$$

- **Power for rotational motion**

$$P = \tau\omega.$$

## 2.12 Chapter 11: Angular Momentum

### 2.12.1 Definitions and Theorems

- Velocity of pure rolling motion at center of mass (Without slipping)

$$V_{cm} = R\omega.$$

- Linear acceleration of a rolling object

$$A_{cm} = R\alpha.$$

- Distance traveled

$$d_{cm} = R\theta.$$

- Acceleration of a rolling object without slipping down an incline plane

$$a_{cm} = \frac{mg \sin(\theta)}{m + \left(\frac{I_{cm}}{r^2}\right)}.$$

- Conservation of Mechanical Energy in Rolling Motion

$$E_T = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2 + mgh.$$

In the absence of any nonconservative forces that would take energy out of the system in the form of heat, the total energy of a rolling object without slipping is conserved and is constant throughout the motion.

Energy is conserved for an object that is not slipping (has static friction)

- **Angular momentum of a particle:** The angular momentum  $\vec{l}$  of a particle is defined as the cross-product of  $\vec{r}$  and  $\vec{p}$ , and is perpendicular to the plane containing  $\vec{r}$  and  $\vec{p}$ :

$$\begin{aligned} \vec{l} &= \vec{r} \times \vec{p}. \\ \implies l &= rp \sin(\theta). \end{aligned}$$

Where  $\theta$  is the angle between  $\vec{r}$  and  $\vec{p}$ . The units of angular momentum are  $kg \cdot \frac{m^2}{s}$

- **Torque on a particle**

$$\begin{aligned} \frac{d\vec{l}}{dt} &= \sum \vec{\tau} \\ \implies \frac{d\vec{l}}{dt} &= \tau_{net}. \end{aligned}$$

- **Angular momentum of a rigid body rotating:** The angular momentum along the axis of rotation of a rigid body rotating with angular velocity  $\omega$  about the axis is

$$L = I\omega.$$

- **Law of conservation of angular momentum:** The angular momentum of a system of particles around a point in a fixed inertial reference frame is conserved if there is no net external torque around that point:

$$\frac{d\vec{L}}{dt} = 0.$$

Or

$$\vec{L} = \vec{I}_1 + \vec{I}_2 + \dots + \vec{I}_n = \text{const.}$$

This law is analogous to linear momentum being conserved when the external force on a system is zero.

- **Precession is a change in the orientation of the rotational axis of a rotating body.**
- **Precession angular velocity:**

$$\omega_P = \frac{rMg}{I\omega}.$$

# Calculus III

## 3.1 Chapter 1: Parametric equations and polar coordinates

### 3.1.1 Definitions and Theorems

- **Parametric equations, parameter:** If  $x$  and  $y$  are continuous functions of  $t$  on an interval  $I$ , then the equations

$$x = x(t) \quad \text{and} \quad y = y(t)$$

are called **parametric equations** and  $t$  is called the **parameter**.

- **Parametric Curve:** The set of points  $(x, y)$  obtained as  $t$  varies over the interval  $I$  is called the graph of the parametric equations. The graph of parametric equations is called a **parametric curve** or plane curve, and is denoted by  $C$ .
- **Eliminating the parameter:** This allows us to rewrite the two equations as a single equation relating the variables  $x$  and  $y$ . Then we can apply any previous knowledge of equations of curves in the plane to identify the curve.
  - Solve one of the equations for  $t$
  - Plug the equation for  $t$  into the equation still in terms of  $t$
  - Sketch the curve on the interval  $I$
- **Domain consideration when graphing by eliminating the parameter:** Suppose we have the equations o

$$\begin{aligned} x &= 2t^2 \\ y &= t^4 + 1. \end{aligned}$$

If we solve  $x$  for  $t$ , we get

$$t = \pm \sqrt{\frac{1}{2}x}.$$

We see that the domain of  $t$  is restricted to  $t \geq 0$ , thus, the graph of this equation will only have the positive side.

- **Parameterizing a curve** is when we start with the equation of a curve and determine a pair of parametric equations for that curve.
  - To find the first Parameterization, Define  $x(t) = t$  and  $y(t)$  as the function given, with  $t$  instead of  $x$
  - Verify there is no restriction on the domain of the original graph. Thus there is no restriction on the values of  $t$
  - To find the second parameterization, choose some function for  $x(t)$ . Ensure that the domain is the set of all real numbers
  - Plug  $x(t)$  into the original function and solve to get  $y(t)$
- **Parametric equations for a cycloid**

$$x(t) = a(t - \sin t), \quad y(t) = a(1 - \cos t).$$

- The general parametric equations for a hypocycloid are:

$$x(t) = (a - b) \cos t + b \cos \left( \frac{a - b}{b} t \right)$$

$$y(t) = (a - b) \sin t + b \sin \left( \frac{a - b}{b} t \right)$$

- **Derivatives of Parametric Equations:** Consider the plane curve defined by the parametric equations  $x = x(t)$  and  $y = y(t)$ . Suppose that  $x'(t)$  and  $y'(t)$  exist, and assume that  $x'(t) \neq 0$ . Then the derivative  $\frac{dy}{dx}$  is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}.$$

- **Second order derivatives of parametric functions**

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{\left( \frac{d}{dt} \right) \left( \frac{dy}{dx} \right)}{\frac{dx}{dt}}.$$

- **Integral involving parametric equations** Consider the non-self-intersecting plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t), \quad a \leq t \leq b$$

and assume that  $x(t)$  is differentiable. The area under this curve is given by

$$A = \int_a^b y(t)x'(t) dt.$$

- **Arc Length of a Parametric Curve:** Consider the plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t), \quad t_1 \leq t \leq t_2$$

and assume that  $x(t)$  and  $y(t)$  are differentiable functions of  $t$ . Then the arc length of this curve is given by

$$s = \int_{t_1}^{t_2} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} dt.$$

Or simply

$$s = \int_a^b \sqrt{1 + \left( \frac{dy}{dx} \right)^2} dx.$$

- **Surface area for a parametric curve:** The analogous formula for a parametrically defined curve is

$$S = 2\pi \int_a^b y(t) \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

provided that  $y(t)$  is not negative on  $[a, b]$ .

- **Parametric equations for a circle (with  $(h, k) = (0, 0)$ )**

$$x(t) = r \cos(t)$$

$$y(t) = r \sin(t).$$

For  $0 \leq t \leq 2\pi$

- **Parametric equations for the upper half of a semi-circle (with  $(h, k) = (0, 0)$ )**

$$\begin{aligned}x(t) &= r \cos(t) \\y(t) &= r \sin(t).\end{aligned}$$

For  $0 \leq t \leq \pi$

- **Parametric equations for the lower half of a semi-circle (with  $(h, k) = (0, 0)$ )**

$$\begin{aligned}x(t) &= r \cos(t) \\y(t) &= r \sin(t).\end{aligned}$$

For  $\pi \leq t \leq 2\pi$

- **Note:** A graph has a horizontal tangent line when the derivative equals zero.
- **Note:** A graph has a vertical tangent line when the derivative does not exist
- **Converting Points between Coordinate Systems:** Given a point  $P$  in the plane with Cartesian coordinates  $(x, y)$  and polar coordinates  $(r, \theta)$ , the following conversion formulas hold true:

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta, .$$

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan \theta = \frac{y}{x}..$$

These formulas can be used to convert from rectangular to polar or from polar to rectangular coordinates.

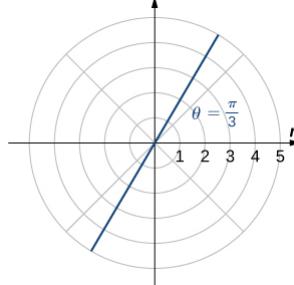
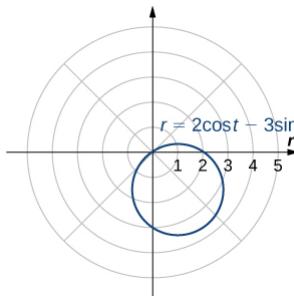
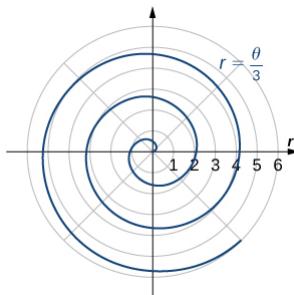
- **Sometimes finding  $\theta$  with the formula is not possible.** Consider the point  $P(0, 3)$ , using the formula  $\tan \theta = \frac{3}{0}$  is undefined. Instead, we graph the point  $(0, 3)$  and observe the angle between the  $x$  and  $y$  axis is  $\frac{\pi}{2}$
- **The polar representation of a point is not unique.** Every point in the plane has an infinite number of representations in polar coordinates. However, each point in the plane has only one representation in the rectangular coordinate system.
- The line segment starting from the center of the graph going to the right (called the positive x-axis in the Cartesian system) is the **polar axis**.
- The center point is the **pole**, or origin, of the coordinate system, and corresponds to  $r = 0$ .
- **If the value of  $r$  is positive**, move that distance along the terminal ray of the angle. **If it is negative**, move along the ray that is opposite the terminal ray of the given angle.
- **A Polar equation** has the form

$$r = f(\theta).$$

- **Plotting a Curve in Polar Coordinates**

1. Create a table with two columns. The first column is for  $\theta$ , and the second column is for  $r$ .
2. Create a list of values for  $\theta$ .
3. Calculate the corresponding  $r$  values for each  $\theta$ .
4. Plot each ordered pair  $(r, \theta)$  on the coordinate axes.
5. Connect the points and look for a pattern.

- Types of polar curves

Name	Equation	Example
Line passing through the pole with slope $\tan K$	$\theta = K$	 <p>A polar coordinate system showing concentric circles centered at the origin. A straight line passes through the origin, extending into the first quadrant. The angle between the positive r-axis and the line is labeled <math>\theta = \frac{\pi}{3}</math>. The line has a positive slope.</p>
Circle	$r = a \cos \theta + b \sin \theta$	 <p>A polar coordinate system showing concentric circles centered at the origin. A single blue circle is plotted, centered in the fourth quadrant. Its equation is given as <math>r = 2\cos\theta - 3\sin\theta</math>.</p>
Spiral	$r = a + b\theta$	 <p>A polar coordinate system showing concentric circles centered at the origin. A blue spiral curve starts at the origin and spirals outwards counter-clockwise. Its equation is given as <math>r = \frac{\theta}{3}</math>.</p>

Name	Equation	Example
Cardioid	$r = a(1 + \cos\theta)$ $r = a(1 - \cos\theta)$ $r = a(1 + \sin\theta)$ $r = a(1 - \sin\theta)$	<p>The graph shows a cardioid curve plotted in a polar coordinate system. The horizontal axis is labeled <math>r</math> and has tick marks from 1 to 7. The vertical axis is also labeled <math>r</math>. The curve is a heart shape centered at (3, 0) on the positive <math>r</math>-axis, passing through the point (3, 0).</p>
Limaçon	$r = a\cos\theta + b$ $r = a\sin\theta + b$	<p>The graph shows a limaçon curve plotted in a polar coordinate system. The horizontal axis is labeled <math>r</math> and has tick marks from 1 to 7. The vertical axis is labeled <math>r</math>. The curve is a looped shape with an inner loop, centered at approximately (1.5, 0).</p>
Rose	$r = a\cos(b\theta)$ $r = a\sin(b\theta)$	<p>The graph shows a rose curve plotted in a polar coordinate system. The horizontal axis is labeled <math>r</math> and has tick marks from 1 to 5. The vertical axis is labeled <math>r</math>. The curve has 2 petals, symmetric about the vertical axis, with its maximum radius of 3 at <math>\theta = \pi/2</math>.</p>

- For the rose, **If the coefficient of  $\theta$  is even**, the graph has twice as many petals as the coefficient. **If the coefficient of  $\theta$  is odd**, then the number of petals equals the coefficient.

- **Graphing a polar curve.**

- Make graph in rectangular system (use  $\theta$  as horizontal axis and  $r$  as vertical)
- Translate over to polar
- The circles represent values of  $r$

- **Area of a Region Bounded by a Polar Curve:** Suppose  $f$  is continuous and nonnegative on the interval  $\alpha \leq \theta \leq \beta$  with  $0 < \beta - \alpha \leq 2\pi$ . The area of the region bounded by the graph of  $r = f(\theta)$  between the radial lines  $\theta = \alpha$  and  $\theta = \beta$  is

$$A = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 d\theta = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta.$$

- **Area between two polar curves:**

$$A = \frac{1}{2} \int_{\alpha}^{\beta} f(\theta)^2 - g(\theta)^2 d\theta.$$

Where  $f(\theta) \geq g(\theta) \forall \alpha \leq \theta \leq \beta$

- **Bounds of integration for area outside some curve and inside some curve:** We find the bounds of integration the same way we found them for regularo functions, we find the points of intersection by setting the two functions equal to each other
- **Arc Length of a Curve Defined by a Polar Function:** Let  $f$  be a function whose derivative is continuous on an interval  $\alpha \leq \theta \leq \beta$ . The length of the graph of  $r = f(\theta)$  from  $\theta = \alpha$  to  $\theta = \beta$  is

$$\begin{aligned} L &= \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta \\ &= \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.. \end{aligned}$$

- **Absolute value bars:** Consider the integral

$$20 \int_0^{2\pi} \sqrt{\cos^2\left(\frac{\theta}{2}\right)} d\theta.$$

If we cancel out the sqrt and square, we must include absolute value bars. This is because  $\cos\left(\frac{\theta}{2}\right)$  can be negative on the interval  $[0, 2\pi]$ . The integral becomes

$$20 \int_0^{2\pi} \left| \cos\left(\frac{\theta}{2}\right) \right| d\theta.$$

To solve this, we use symmetry to change the bounds. Because the graph of the polar curve  $(10 + 10 \cos \theta)$  is symmetric, we can integrate instead over the range  $0, \pi$ , and multiply the result by 2.

- **Derivative of polar equaotion:** Given some function  $r = f(\theta)$ . We can find the derivative with

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta}.$$

- **Find equation of tangent line given  $r = f(\theta)$  and point  $\theta$ .**

- Find  $r$  with given  $\theta$ .
- Plug  $r$  into  $x = r \cos \theta$ ,  $y = r \sin \theta$  to get point  $P(x, y)$
- Find  $\frac{dy}{dx}$  of  $r = f(\theta)$
- Plug in  $\theta$  to get slope  $m$
- Use point slope form to find equation

### 3.1.2 Problems to remember

- **Eliminating the parameter:** Sometimes it is necessary to be a bit creative in eliminating the parameter. The parametric equations for this example are

$$x(t) = 4 \cos t, \quad y(t) = 3 \sin t.$$

Solving either equation for  $t$  directly is not advisable because sine and cosine are not one-to-one functions. However, dividing the first equation by 4 and the second equation by 3 (and suppressing the  $t$ ) gives us

$$\cos t = \frac{x}{4}, \quad \sin t = \frac{y}{3}.$$

Now use the Pythagorean identity  $\cos^2 t + \sin^2 t = 1$  and replace the expressions for  $\sin t$  and  $\cos t$  with the equivalent expressions in terms of  $x$  and  $y$ . This gives

$$\begin{aligned} \left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 &= 1 \\ \frac{x^2}{16} + \frac{y^2}{9} &= 1. \end{aligned}$$

This is the equation of a horizontal ellipse centered at the origin, with semimajor axis 4 and semiminor axis 3 as shown in the following graph.

- **Convert from cartesian to polar:** Consider the point  $P(1, 1)$ . First we find  $r$  and  $\theta$

$$\begin{aligned} r^2 &= x^2 + y^2 \\ r &= \sqrt{1^2 + 1^2} \\ r &= \sqrt{2} \\ \tan \theta &= \frac{y}{x} = \frac{1}{1} \\ \theta &= \tan^{-1} 1 \\ \theta &= \frac{\pi}{4}. \end{aligned}$$

Thus we have the point  $(\sqrt{2}, \frac{\pi}{4})$

- **Graphing a polar equation:** Graph the curve defined by the function  $r = 4 \sin \theta$ . Identify the curve and rewrite the equation in rectangular coordinates.

Because the function is a multiple of a sine function, it is periodic with period  $2\pi$ , so use values for  $\theta$  between 0 and  $2\pi$

$\theta$	$r = 4 \sin \theta$	$\theta$	$r = 4 \sin \theta$
0	0	$\pi$	0
$\frac{\pi}{6}$	2	$\frac{7\pi}{6}$	-2
$\frac{\pi}{4}$	$2\sqrt{2} \approx 2.8$	$\frac{5\pi}{4}$	$-2\sqrt{2} \approx -2.8$
$\frac{\pi}{3}$	$2\sqrt{3} \approx 3.4$	$\frac{4\pi}{3}$	$-2\sqrt{3} \approx -3.4$
$\frac{\pi}{2}$	4	$\frac{3\pi}{2}$	-4
$\frac{2\pi}{3}$	$2\sqrt{3} \approx 3.4$	$\frac{5\pi}{3}$	$-2\sqrt{3} \approx -3.4$
$\frac{3\pi}{4}$	$2\sqrt{2} \approx 2.8$	$\frac{7\pi}{6}$	$-2\sqrt{2} \approx -2.8$
$\frac{5\pi}{6}$	2	$\frac{11\pi}{6}$	-2
$2\pi$	0		

Plotting these points gives the graph of a circle. The equation  $r = 4 \sin \theta$  can be converted into rectangular coordinates by first multiplying both sides by  $r$ . This gives the equation  $r^2 = 4r \sin \theta$ . Next use the facts that  $r^2 = x^2 + y^2$  and  $y = r \sin \theta$ . This gives  $x^2 + y^2 = 4y$ . To put this equation into standard form, subtract  $4y$  from both sides of the equation and complete the square:

$$x^2 + (y^2 - 4y) = x^2 + (y^2 - 4y + 4) = x^2 + (y - 2)^2 = 0 + 4.$$

This is the equation of a circle with radius 2 and center  $(0, 2)$  in the rectangular coordinate system.

- **Finding the Arc Length of a Polar Curve:**

$$\text{Find the arc length of the polar curve } r = 2 + 2 \cos \theta.$$

When  $\theta = 0$ ,  $r = 2 + 2 \cos(0) = 4$ . Furthermore, as  $\theta$  goes from 0 to  $2\pi$ , the cardioid is traced out exactly once. Therefore, these are the limits of integration. Using  $f(\theta) = 2 + 2 \cos(\theta)$ ,  $\alpha = 0$ , and  $\beta = 2\pi$ . Thus we have

$$\begin{aligned} L &= \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta \\ &= \int_0^{2\pi} \sqrt{[2 + 2 \cos(\theta)]^2 + [-2 \sin(\theta)]^2} d\theta \\ &= \int_0^{2\pi} \sqrt{4 + 8 \cos(\theta) + 4 \cos^2(\theta) + 4 \sin^2(\theta)} d\theta \\ &= \int_0^{2\pi} \sqrt{4 + 8 \cos(\theta) + 4(\cos^2(\theta) + \sin^2(\theta))} d\theta \\ &= \int_0^{2\pi} \sqrt{8 + 8 \cos(\theta)} d\theta \\ &= 2 \int_0^{2\pi} \sqrt{2 + 2 \cos(\theta)} d\theta.. \end{aligned}$$

Next, using the identity  $\cos(2\alpha) = 2 \cos^2(\alpha) - 1$ , add 1 to both sides and multiply by 2. This gives  $2 + 2 \cos(2\alpha) = 4 \cos^2(\alpha)$ . Substituting  $\alpha = \frac{\theta}{2}$  gives  $2 + 2 \cos(\theta) = 4 \cos^2\left(\frac{\theta}{2}\right)$ , so the integral becomes

$$\begin{aligned} L &= 2 \int_0^{2\pi} \sqrt{2 + 2 \cos(\theta)} d\theta \\ &= 2 \int_0^{2\pi} \sqrt{4 \cos^2\left(\frac{\theta}{2}\right)} d\theta \\ &= 2 \int_0^{2\pi} 2 \left| \cos\left(\frac{\theta}{2}\right) \right| d\theta.. \end{aligned}$$

The absolute value is necessary because the cosine is negative for some values in its domain. To resolve this issue, change the limits from 0 to  $\pi$  and double the answer. This strategy works because cosine is positive between 0 and  $\frac{\pi}{2}$ . Thus,

$$\begin{aligned} L &= 4 \int_0^{2\pi} \left| \cos\left(\frac{\theta}{2}\right) \right| d\theta \\ &= 8 \int_0^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta \\ &= 8 \left[ 2 \sin\left(\frac{\theta}{2}\right) \right]_0^{\pi} \\ &= 16.. \end{aligned}$$

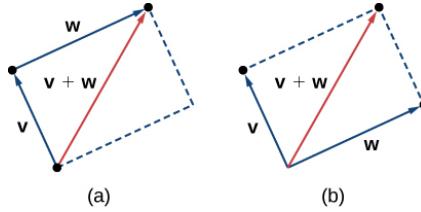
## 3.2 Chapter 2: Vectors in Space

### 3.2.1 Definitions and Theorems

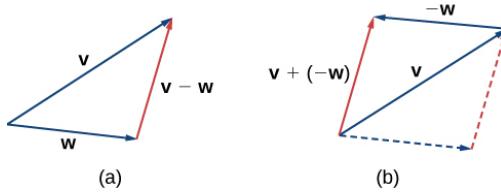
- The endpoints of the segment are called the **initial point** and the **terminal point** of the vector.
- The length of the line segment represents its magnitude. We use the notation  $\|\vec{v}\|$  to denote the magnitude of the vector  $\vec{v}$ .
- A vector with an initial point and terminal point that are the same is called the **zero vector**, denoted  $\vec{0}$ .
- Vectors with the same magnitude and direction are called **equivalent vectors**. We treat equivalent vectors as equal, even if they have different initial points. Thus, if  $\vec{v}$  and  $\vec{w}$  are equivalent, we write

$$\vec{v} = \vec{w}.$$

- Scalar Multiplication:** The product  $k\vec{v}$  of a vector  $\vec{v}$  and a scalar  $k$  is a vector with a magnitude that is  $|k|$  times the magnitude of  $\vec{v}$ , and with a direction that is the same as the direction of  $\vec{v}$  if  $k > 0$ , and opposite the direction of  $\vec{v}$  if  $k < 0$ . This is called scalar multiplication. If  $k = 0$  or  $\vec{v} = \vec{0}$ , then  $k\vec{v} = \vec{0}$ .
- Vector Addition:** The sum of two vectors  $\vec{v}$  and  $\vec{w}$  can be constructed graphically by placing the initial point of  $\vec{w}$  at the terminal point of  $\vec{v}$ . Then, the vector sum,  $\vec{v} + \vec{w}$ , is the vector with an initial point that coincides with the initial point of  $\vec{v}$  and has a terminal point that coincides with the terminal point of  $\vec{w}$ . This operation is known as vector addition.



- Vector difference:** We define  $\vec{v} - \vec{w}$  as  $\vec{v} + (-\vec{w}) = \vec{v} + (-1)\vec{w}$ . The vector  $\vec{v} - \vec{w}$  is called the vector difference. Graphically, the vector  $\vec{v} - \vec{w}$  is depicted by drawing a vector from the terminal point of  $\vec{w}$  to the terminal point of  $\vec{v}$ .



- Triangle inequality:** the length of any one side is less than the sum of the lengths of the remaining sides. So we have

$$\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|.$$

- We call a vector with its initial point at the origin a **standard-position** vector.
- **Component form of a vector:** The vector with initial point  $(0, 0)$  and terminal point  $(x, y)$  can be written in component form as

$$\vec{v} = \langle x, y \rangle.$$

The scalars  $x$  and  $y$  are called the components of  $\mathbf{v}$ .

- If a vector is in standard position, and its components are the same as the terminal point
- **Component form of a vector not in standard position:** Let  $\vec{v}$  be a vector with initial point  $(x_i, y_i)$  and terminal point  $(x_t, y_t)$ . Then we can express  $\vec{v}$  in component form as  $\vec{v} = \langle x_t - x_i, y_t - y_i \rangle$ .
- **Magnitude of vector:** If a vector is given by components  $\langle x, y \rangle$ . Then this is the vector with initial point at the origin  $(0, 0)$ , and terminal point  $(x, y)$ . We find the magnitude of the vector with

$$\|\vec{v}\| = \sqrt{x^2 + y^2}.$$

If the vector is not in standard position, and we have initial point  $(x_1, y_1)$  and terminal point  $(x_2, y_2)$ , then we find the magnitude with

$$\|\vec{v}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

- **Scalar multiplication, and vector addition (component form)** Let  $\mathbf{v} = \langle x_1, y_1 \rangle$  and  $\mathbf{w} = \langle x_2, y_2 \rangle$  be vectors, and let  $k$  be a scalar.

Scalar multiplication:  $k\mathbf{v} = \langle kx_1, ky_1 \rangle$

Vector addition:  $\mathbf{v} + \mathbf{w} = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle = \langle x_1 + x_2, y_1 + y_2 \rangle$ .

- **Properties of Vector Operations:** Let  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  be vectors in a plane. Let  $r$  and  $s$  be scalars.

1.  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$  (Commutative property)
2.  $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$  (Associative property)
3.  $\mathbf{u} + \mathbf{0} = \mathbf{u}$  (Additive identity property)
4.  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$  (Additive inverse property)
5.  $r(s\mathbf{u}) = (rs)\mathbf{u}$  (Associativity of scalar multiplication)
6.  $(r+s)\mathbf{u} = r\mathbf{u} + s\mathbf{u}$  (Distributive property)
7.  $r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$  (Distributive property)
8.  $1\mathbf{u} = \mathbf{u}, \quad 0\mathbf{u} = \mathbf{0}$  (Identity and zero properties)

**Note**  $CAAA^{-1}(ASM)D^2(I\&Z)$

- **Finding components of a vector given the magnitude and the angle  $\theta$**

$$x = \|\vec{v}\| \cos \theta$$

$$y = \|\vec{v}\| \sin \theta.$$

- **Unit vector:** A unit vector is a vector with magnitude 1. For any nonzero vector  $\vec{v}$ , we can use scalar multiplication to find a unit vector  $\vec{u}$  that has the same direction as  $\vec{v}$ . To do this, we multiply the vector by the reciprocal of its magnitude:

$$\vec{u} = \frac{1}{\|\vec{v}\|} \vec{v}.$$

- **xy-plane:**

$$\{(x, y, 0) : x, y \in \mathbb{R}\}.$$

Described by the equation  $z = 0$

- **xz-plane**

$$\{(x, 0, z) : x, z \in \mathbb{R}\}.$$

Described by the equation  $y = 0$

- **yz-plane**

$$\{(0, y, z) : y, z \in \mathbb{R}\}.$$

Described by the equation  $x = 0$

- the coordinate planes divide space between them into eight regions about the origin, called **octants**. The octants fill  $\mathbb{R}^3$  the same way that quadrants fill  $\mathbb{R}^2$ ,
- **Distance formula for three-dimensional space:** The distance  $d$  between points  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

- **Equations of planes parallel to coordinate planes**

1. The plane in space that is parallel to the xy-plane and contains point  $(a, b, c)$  can be represented by the equation  $z = c$ .
  2. The plane in space that is parallel to the xz-plane and contains point  $(a, b, c)$  can be represented by the equation  $y = b$ .
  3. The plane in space that is parallel to the yz-plane and contains point  $(a, b, c)$  can be represented by the equation  $x = a$
- A **sphere** is the set of all points in space equidistant from a fixed point, the center of the sphere, just as the set of all points in a plane that are equidistant from the center represents a circle. In a sphere, as in a circle, the distance from the center to a point on the sphere is called the *radius*.
  - **Standard equation of a sphere:** The sphere with center  $(a, b, c)$  and radius  $r$  can be represented by the equation

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

This equation is known as the **standard equation of a sphere**.

- **Properties of vectors in space:** Let  $\vec{v} = \langle x_1, y_1, z_1 \rangle$  and  $\vec{w} = \langle x_2, y_2, z_2 \rangle$  be vectors, and let  $k$  be a scalar.

- Scalar multiplication:

$$k\vec{v} = \langle kx_1, ky_1, kz_1 \rangle.$$

- Vector addition:

$$\vec{v} + \vec{w} = \langle x_1, y_1, z_1 \rangle + \langle x_2, y_2, z_2 \rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle.$$

- Vector subtraction:

$$\vec{v} - \vec{w} = \langle x_1, y_1, z_1 \rangle - \langle x_2, y_2, z_2 \rangle = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle.$$

- Vector magnitude:

$$\|\vec{v}\| = \sqrt{x_1^2 + y_1^2 + z_1^2}.$$

- Unit vector in the direction of  $\vec{v}$ :

$$\begin{aligned} \frac{1}{\|\vec{v}\|} \vec{v} &= \frac{1}{\|\vec{v}\|} \langle x_1, y_1, z_1 \rangle \\ &= \left\langle \frac{x_1}{\|\vec{v}\|}, \frac{y_1}{\|\vec{v}\|}, \frac{z_1}{\|\vec{v}\|} \right\rangle, \text{ if } \vec{v} \neq 0. \end{aligned}$$

- **The dot product:** The dot product of vectors  $\vec{u} = \langle u_1, u_2, u_3 \rangle$  and  $\vec{v} = \langle v_1, v_2, v_3 \rangle$  is given by the sum of the products of the components

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

The dot product **does not** return a new vector, the result is a **scalar**

- **Properties of the dot product** Let  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$  be vectors, and let  $c$  be a scalar.

1. Commutative property:  $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. Distributive property:  $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
3. Associative property of scalar multiplication:  $(c\vec{u}) \cdot \vec{v} = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$
4. Property of magnitude:  $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

- **Evaluating a dot product:** The dot product of two vectors is the product of the magnitude of each vector and the cosine of the angle between them:

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos \theta.$$

- **Find the measure of the angle between two nonzero vectors:**

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}.$$

**Note:** We are considering  $0 \leq \theta \leq \pi$

- **Vector Projection:** The vector projection of  $\vec{v}$  onto  $\vec{u}$  has the same initial point as  $\vec{u}$  and  $\vec{v}$  and the same direction as  $\vec{u}$ , and represents the component of  $\vec{v}$  that acts in the direction of  $\vec{u}$ .

$$\text{proj}_{\vec{u}} \vec{v} = \frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|^2} \vec{u}.$$

We say "The vector projection of  $\vec{v}$  onto  $\vec{u}$ "

- **Scalar projection notation:** This is the length of the vector projection and is denoted

$$\|\text{proj}_{\vec{u}} \vec{v}\| = \text{comp}_{\vec{u}} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|}.$$

- **Decompose some vector  $\vec{v}$  into orthogonal components such that one of the component vectors has the same direction as  $\vec{u}$** 
  - First, we compute  $\vec{p} = \text{proj}_{\vec{u}} \vec{v}$
  - Then, we define  $\vec{q} = \vec{v} - \vec{p}$
  - Check that  $\vec{q}$  and  $\vec{p}$  are orthogonal by finding  $\vec{q} \cdot \vec{p}$
- **Work:** When a constant force is applied to an object so the object moves in a straight line from point  $P$  to point  $Q$ , the work  $W$  done by the force  $\mathbf{F}$ , acting at an angle  $\theta$  from the line of motion, is given by

$$W = \vec{F} \cdot \overrightarrow{PQ} = \|\vec{F}\| \|\overrightarrow{PQ}\| \cos \theta.$$

- **Two vectors are orthogonal if**

$$\vec{u} \cdot \vec{v} = 0.$$

- **Two vectors are parallel if**

$$\exists \alpha \text{ s.t } \alpha \vec{u} = \vec{v}.$$

- **Scalar projection components of a vector**

$$\vec{v} = \langle \text{comp}_i \vec{v}, \text{comp}_j \vec{v}, \text{comp}_k \vec{v} \rangle.$$

- **Find direction angles for some vector:** Suppose we have some vector  $\vec{v}$ , to find the direction angles, we use the formula

$$\cos \theta = \frac{\vec{v} \cdot \vec{u}}{\|\vec{v}\| \|\vec{u}\|}.$$

With the unit vectors  $\hat{i}, \hat{j}, \hat{k}$ . This will give angles  $\alpha, \beta, \gamma$

- **The Cross Product:** Let  $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$  and  $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ . Then, the cross product  $\mathbf{u} \times \mathbf{v}$  is vector

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k} \\ &= \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle. \end{aligned}$$

**Note:** The cross product only works in  $\mathbb{R}^3$ , additionally, we measure the angle between  $\vec{u}$  and  $\vec{v}$  in  $\vec{u} \times \vec{v}$  from  $\vec{u}$  to  $\vec{v}$

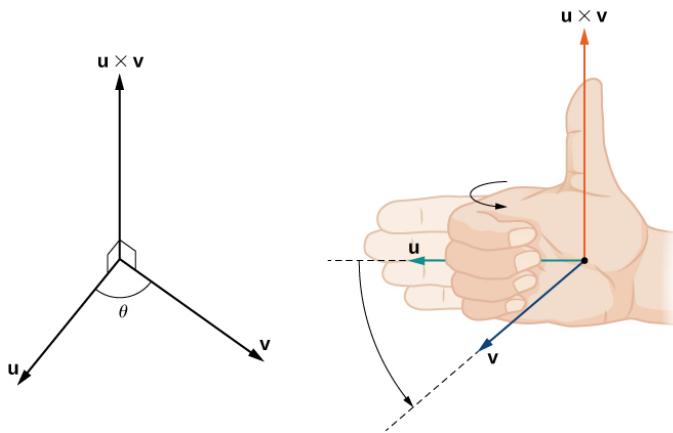
- **Cross product using matrix and determinant**, suppose we have vectors  $\vec{u}$  and  $\vec{v}$ . Then we can express them in matrix form as

$$\vec{u} \times \vec{v} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{bmatrix}.$$

Then we can find the determinant of this matrix to compute the cross product

$$\vec{u} \times \vec{v} = (u_y v_z - u_z v_y) \hat{i} - (u_x v_z - u_z v_x) \hat{j} + (u_x v_y - u_y v_x) \hat{k}.$$

- **Right hand rule for cross product:** The direction of  $\mathbf{u} \times \mathbf{v}$  is given by the right-hand rule. If we hold the right hand out with the fingers pointing in the direction of  $\mathbf{u}$ , then curl the fingers toward vector  $\mathbf{v}$ , the thumb points in the direction of the cross product, as shown.



- **Properties of the Cross Product** Let  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  be vectors in space, and let  $c$  be a scalar.

1. Anticommutative property:  $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
2. Distributive property:  $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
3. Multiplication by a constant:  $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$
4. Cross product of the zero vector:  $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
5. Cross product of a vector with itself:  $\mathbf{v} \times \mathbf{v} = \mathbf{0}$
6. Scalar triple product:  $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

**Note:** (AC)D(MC)(ZV)(IT)(TP)

- **Magnitude of the Cross Product:** Let  $\mathbf{u}$  and  $\mathbf{v}$  be vectors, and let  $\theta$  be the angle between them. Then,  $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cdot \sin \theta$ .

- **Applications of the cross product:**

- Finding a vector orthogonal to two given vectors
- Computing areas of triangles and parallelograms
- Determining the volume of the three-dimensional geometric shape made of parallelograms known as a parallelepiped:w

- **Area of a Parallelogram** If we locate vectors  $\mathbf{u}$  and  $\mathbf{v}$  such that they form adjacent sides of a parallelogram, then the area of the parallelogram is given by  $\|\mathbf{u} \times \mathbf{v}\|$ .

- **Triple Scalar Product:**

The triple scalar product of vectors  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  is  $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ .

The triple scalar product is the determinant of the  $3 \times 3$  matrix formed by the components of the vectors

- **triple scalar product identities:**

- (a)  $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = -\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v})$
- (b)  $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$

- **parallelepiped:** Let  $\mathbf{u}$  and  $\mathbf{v}$  be two vectors in standard position. If  $\mathbf{u}$  and  $\mathbf{v}$  are not scalar multiples of each other, then these vectors form adjacent sides of a parallelogram.

Now suppose we add a third vector  $\mathbf{w}$  that does not lie in the same plane as  $\mathbf{u}$  and  $\mathbf{v}$  but still shares the same initial point. Then these vectors form three edges of a parallelepiped, a three-dimensional prism with six faces that are each parallelograms,

- **Volume of a Parallelepiped:** The volume of a parallelepiped with adjacent edges given by the vectors  $\mathbf{u}$ ,  $\mathbf{u}$ , and  $\mathbf{w}$  is the absolute value of the triple scalar product:

$$V = \left| \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \right|.$$

- **Torque:** Measures the tendency of a force to produce rotation about an axis of rotation. Let  $\mathbf{r}$  be a vector with an initial point located on the axis of rotation and with a terminal point located at the point where the force is applied, and let vector  $\mathbf{F}$  represent the force. Then torque is equal to the cross product of  $\mathbf{r}$  and  $\mathbf{F}$ :

$$\tau = \mathbf{r} \times \mathbf{F}.$$

- **Choosing  $\alpha$  to make parallel vectors equal:** Suppose we have two vectors  $\mathbf{u}$  and  $\mathbf{v}$ , and  $\exists \alpha \in \mathbb{R}$  s.t  $\alpha\mathbf{v} = \mathbf{u}$ . Then

$$\alpha = \frac{\|\mathbf{u}\|}{\|\mathbf{v}\|}.$$

Or if they are **anti-parallel**

$$\alpha = -\frac{\|\mathbf{u}\|}{\|\mathbf{v}\|}.$$

- **The zero vector is considered to be parallel to all vectors**
- **vector equation of a line**

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}.$$

Where  $\mathbf{v}$  is the direction vector (vector parallel to the line),  $t$  is some scalar, and  $\mathbf{r}$ ,  $\mathbf{r}_0$  are position vectors

- **Parametric and Symmetric Equations of a Line:** A line  $L$  parallel to vector  $\mathbf{v} = \langle a, b, c \rangle$  and passing through point  $P(x_0, y_0, z_0)$  can be described by the following parametric equations:

$$x = x_0 + ta, \quad y = y_0 + tb, \quad \text{and} \quad z = z_0 + tc.$$

If the constants  $a$ ,  $b$ , and  $c$  are all nonzero, then  $L$  can be described by the symmetric equation of the line:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

**Note:** The parametric equations of a line are not unique. Using a different parallel vector or a different point on the line leads to a different, equivalent representation. Each set of parametric equations leads to a related set of symmetric equations, so it follows that a symmetric equation of a line is not unique either.

- **Vector equation of a line reworked:** Suppose we have some line, with points  $P(x_0, y_0, z_0)$ ,  $Q(x_1, y_1, z_1)$ . Where  $\mathbf{p} = \langle x_0, y_0, z_0 \rangle$  and  $\mathbf{Q} = \langle x_1, y_1, z_1 \rangle$  are the corresponding position vectors. Suppose we also have  $\mathbf{r} = \langle x, y, z \rangle$ . Then our vector equation for a line becomes

$$\mathbf{r} = \mathbf{p} + t(\vec{PQ}).$$

By properties of vectors, we get the vector equation of a line passing through points  $P$  and  $Q$  to be

$$\mathbf{r} = (1-t)\mathbf{p} + t\mathbf{q}.$$

- **Equation of a line segment between two points  $P$  and  $Q$ :** Using the result from the previous item, we find the vector equation of the line segment between  $P$  and  $Q$  is

$$\mathbf{r} = (1-t)\mathbf{p} + t\mathbf{q}, \quad 0 \leq t \leq 1.$$

Because when  $t = 0$ ,  $\mathbf{r} = \mathbf{p}$ . When  $t = 1$ ,  $\mathbf{r} = \mathbf{q}$

- **parametric equations for this line segment**

$$\begin{cases} x = x_0 + t(x_1 - x_0) \\ y = y_0 + t(y_1 - y_0) \\ z = z_0 + t(z_1 - z_0) \end{cases} \quad (34)$$

For  $0 \leq t \leq 1$

- **Distance from a Point to a Line:** Let  $L$  be a line in space passing through point  $P$  with direction vector  $\mathbf{v}$ . If  $M$  is any point not on  $L$ , then the distance from  $M$  to  $L$  is

$$d = \frac{\|\overrightarrow{PM} \times \mathbf{v}\|}{\|\mathbf{v}\|}$$

- If two lines in space are not parallel, but do not intersect, then the lines are said to be **skew** lines
- **Classifying lines in space**

		Lines Share A Common Point?	
		Yes	No
Direction Vectors Are Parallel?	Yes	Equal	Parallel but not equal
	No	Intersecting	Skew

- **Testing for parallel, intersecting, or skew:** Suppose we have two sets of parametric equations for two lines.

- First we test for parallel lines, We find their direction vectors, if their direction vectors are parallel, the lines are parallel.

- If they are not parallel, we can test for intersection. This is best explained with an example

Suppose we have the equations

$$\begin{aligned} L_1 : \quad x &= 3 + 2t & y &= 4 - t & z &= 1 + 3t \\ L_2 : \quad x &= 1 + 4s & y &= 3 - 2s & z &= 4 + 5s. \end{aligned}$$

First, we equate each set of equations

$$\begin{aligned} 3 + 2t &= 1 + 4s \\ 4 - t &= 3 - 2s \\ 1 + 3t &= 4 + 5s. \end{aligned}$$

Next, we solve the first equation for one of the variables, in this case we choose to solve for  $t$

$$t = 2s - 1.$$

Now, we want to plug this into the second equation. This gives

$$\begin{aligned} 4 - (2s - 1) &= 3 - 2s \\ 2 &= 0. \end{aligned}$$

Since we this statement is not true, we know the lines must not be intersecting. In this case, we know they must be skew

However, suppose we got some value for  $s$ , such as  $s = 1$ , we can then plug this value into the equation we got for  $t$  ( $t = 2s - 1$ ) to get a value for  $t$ , after we have a value for  $t$ , we can plug both  $s$  and  $t$  into the third equation and evaluate the truthiness of its outcome. If the outcome is true, we have a valid solution for the system of equations.

**Note:** If the lines are known to be skew, then we compute the dot product of their direction angles to check for orthogonality.

- given any three points that do not all lie on the same line, there is a unique **plane** that passes through these points. Just as a line is determined by two points, **a plane is determined by three**.
- given two distinct, intersecting lines, **there is exactly one plane containing both lines**.
- A plane is also **determined by a line and any point that does not lie on the line**.
- We say that  $\mathbf{n}$  is a normal vector, or perpendicular to the plane.
- **Vector equation of a plane:** Given a point  $P$  and vector  $\mathbf{n}$ , the set of all points  $Q$  satisfying the equation  $\mathbf{n} \cdot \vec{PQ} = 0$  forms a plane. The equation

$$\mathbf{n} \cdot \vec{PQ} = 0$$

is known as the vector equation of a plane.

- **Scalar equation of a plane:** The scalar equation of a plane containing point  $P = (x_0, y_0, z_0)$  with normal vector  $\mathbf{n} = \langle a, b, c \rangle$  is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

- **General form of the equation of a plane:** This equation (the one above) can be expressed as  $ax + by + cz + d = 0$ , where  $d = -ax_0 - by_0 - cz_0$ . This form of the equation is sometimes called the general form of the equation of a plane.
- any three points that do not all lie on the same line determine a plane. Given three such points, we can find an equation for the plane containing these points.
- **The Distance between a Plane and a Point:** Suppose a plane with normal vector  $\mathbf{n}$  passes through point  $Q$ . The distance  $d$  from the plane to a point  $P$  not in the plane is given by

$$d = \|\text{proj}_{\mathbf{n}} \vec{QP}\| = \left| \text{comp}_{\mathbf{n}} \vec{QP} \right| = \frac{|\vec{QP} \cdot \mathbf{n}|}{\|\mathbf{n}\|}.$$

- For two planes in space, we have only two possible relationships: **The two distinct planes are parallel or they intersect. When two planes are parallel, their normal vectors are parallel. When two planes intersect, the intersection is a line**
- **Finding line of intersection for two planes:** Suppose we are given the parametric equations for two planes
  - Check if their normals are parallel, if not, we know the planes must intersect
  - Find a common point that satisfies both equations (perhaps the origin)
  - Eliminate one of the variables (perhaps by adding them)
  - After elimination we should have one variable equal to one of the others, we then plug this into one of the equations to get two variables in terms of the third.
  - Define this third variable in terms of  $t$  to get  $x, y, z$  in terms of  $t$ , these equations will be the parametric equations for the line of intersection
- **find the angle formed by the intersection of two planes:** We can use normal vectors to calculate the angle between the two planes.

We can find the measure of the angle  $\theta$  between two intersecting planes by first finding the cosine of the angle, using the following equation:

$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|}.$$

We can then use the angle to determine whether two planes are parallel or orthogonal or if they intersect at some other angle.

- **Finding the distance between two parallel planes:** Let  $P(x_0, y_0, z_0)$  be a point. The distance from  $P$  to plane  $ax + by + cz + k = 0$  is given by

$$d = \frac{|ax_0 + by_0 + cz_0 + k|}{\sqrt{a^2 + b^2 + c^2}}.$$

- **Find the intersection of a plane with a line:**

- First, we get the line in parametric form, with  $t$  is the parameter.
- When then plug our parametric equations into the corresponding plane equation variables
- Solve for  $t$ ,
- Plug  $t$  into parametric equation to get point  $(x, y, z)$

### 3.3 Conic sections and quadric surfaces

#### 3.3.1 Conic sections: Parabola, Ellipse, and Hyperbola

- **Conic sections, The parabola:** A parabola is the set of all points in a plane equidistant from a fixed point  $F$  (the focus) and a fixed line  $\ell$  (the directrix) that lie in the plane.
- **Parabola that opens up:** Given a parabola opening upward with vertex located at  $(h, k)$  and focus located at  $(h, k + p)$ , where  $p$  is a constant, the equation for the parabola is given by

$$y = \frac{1}{4p}(x - h)^2 + k$$

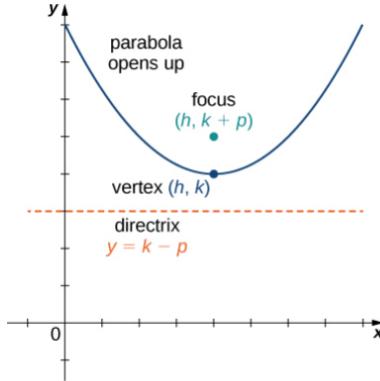
Vertex:  $(h, k)$

Focus:  $(h, k + p)$

Directrix  $y = k - p$

AOS  $x = h$ .

This is the **standard form** of a parabola.



- **Parabola that opens down:**

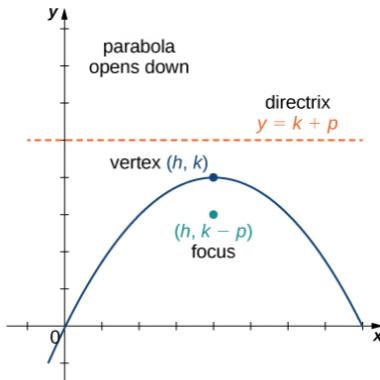
$$y = -\frac{1}{4p}(x - h)^2 + k$$

Vertex:  $(h, k)$

Focus:  $(h, k - p)$

Directrix  $y = k + p$

AOS  $x = h$ .



- Parabola opens right:

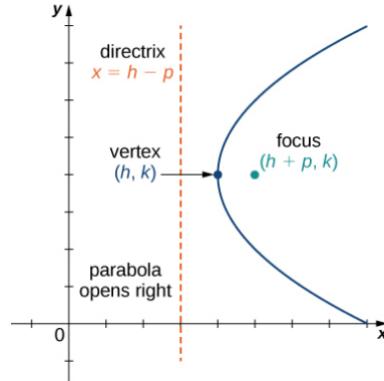
$$x = \frac{1}{4p}(y - k)^2 + h$$

Vertex:  $(h, k)$

Focus:  $(h + p, k)$

Directrix  $x = h - p$

AOS  $y = k$ .



- Parabola opens left:

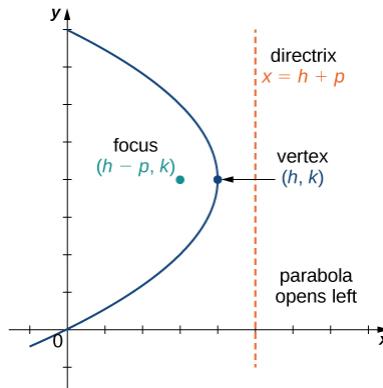
$$x = -\frac{1}{4p}(y - k)^2 + h$$

Vertex:  $(h, k)$

Focus:  $(h - p, k)$

Directrix  $x = h + p$

AOS  $y = k$ .



- **General form of a parabola:** The equation of a parabola can be written in the general form, though in this form the values of  $h$ ,  $k$ , and  $p$  are not immediately recognizable. The general form of a parabola is written as

$$ax^2 + bx + cy + d = 0 \quad \text{or} \quad ay^2 + bx + cy + d = 0.$$

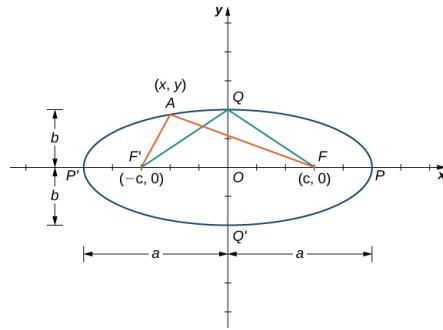
**Note:** The first equation represents a parabola that opens either up or down. The second equation represents a parabola that opens either to the left or to the right. To put the equation into standard form, use the method of completing the square.

- **Finding  $h$  and  $k$  for a parabola.**  $h$  and  $k$  for a parabola are given by

$$h = -\frac{2b}{a}$$

$$k = f(h).$$

- **Conic sections, The ellipse:** An ellipse is the set of all points for which the sum of their distances from two fixed points (the foci) is constant.



**Foci:** There are two foci

**Directrices:** There are two directrices (plural of directrix)

**Major axis:** An ellipse has two axes, one short and one long. The major axis is the longer of the two and has length  $2a$

**Minor axis:** The shorter axis has length  $2b$

**Finding  $c$  with  $a$  and  $b$ ,** to find  $c$ , we use

$$c^2 = a^2 - b^2.$$

- **Ellipse with horizontal major axis:** This type of ellipse has the form

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Center  $(h, k)$

Foci:  $(h \pm c, k)$

Directrices:  $x = h \pm \frac{a^2}{c}$ .

- **Ellipse with vertical major axis:** This type of ellipse has the form

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

Center:  $(h, k)$

Foci:  $(h, k \pm c)$

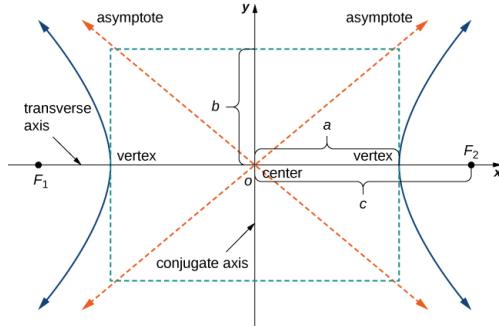
Directrices:  $y = k \pm \frac{a^2}{c}$ .

- **General form of an ellipse:** The equation of an ellipse is in general form if it is in the form

$$Ax^2 + By^2 + Cx + Dy + E = 0.$$

where  $A$  and  $B$  are either both positive or both negative. To convert the equation from general to standard form, use the method of completing the square.

- **Conic sections, the Hyperbola:** A hyperbola is the set of all points where the difference between their distances from two fixed points (the foci) is constant.



**Foci:** There are two foci

**Directrices:** There are two directrices (plural of directrix)

**Asymptotes:** There are two Asymptotes

**Transverse axis (major axis)** has length  $2a$

**Conjugate axis (minor axis)** has length  $2b$

**Finding  $c$  with  $a$  and  $b$ ,** to find  $c$ , we use

$$c^2 = a^2 + b^2.$$

- Hyperbola opening left and right (horizontal major axis)

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Center:  $(h, k)$

Foci:  $(h \pm c, k)$

$$\text{Asymptotes: } y = k \pm \frac{b}{a}(x - h)$$

$$\text{Directrices: } x = h \pm \frac{a^2}{c}.$$

- Hyperbola opening up and down (vertical major axis)

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Center:  $(h, k)$

Foci:  $(h, k \pm c)$

$$\text{Asymptotes: } y = k \pm \frac{a}{b}(x - h)$$

$$\text{Directrices: } y = k \pm \frac{a^2}{c}.$$

- **Hyperbola general form:** The equation of a hyperbola is in general form if it is in the form

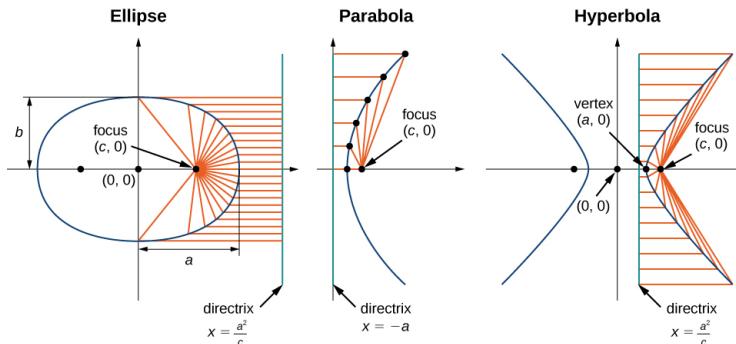
$$Ax^2 + By^2 + Cx + Dy + E = 0.$$

where  $A$  and  $B$  have opposite signs. In order to convert the equation from general to standard form, use the method of completing the square.

- **Eccentricity and directrix of conic sections:** The eccentricity  $e$  of a conic section is defined to be the distance from any point on the conic section to its focus, divided by the perpendicular distance from that point to the nearest directrix. This value is constant for any conic section, and can define the conic section as well:

- If  $e = 1$ , the conic is a parabola.
- If  $e < 1$ , it is an ellipse.
- If  $e > 1$ , it is a hyperbola.

The eccentricity of a circle is zero. The directrix of a conic section is the line that, together with the point known as the focus, serves to define a conic section. Hyperbolas and noncircular ellipses have two foci and two associated directrices. Parabolas have one focus and one directrix.



- Eccentricity for parabola

$$e = 1.$$

- Eccentricity for ellipses

$$e = \frac{c}{a}.$$

- Eccentricity for hyperbolas

$$e = \frac{c}{a}.$$

### 3.3.2 Quadric Surfaces

- A set of lines parallel to a given line passing through a given curve is known as a **cylindrical surface**, or cylinder. The parallel lines are called **rulings**.
- The **traces** of a surface are the cross-sections created when the surface intersects a plane parallel to one of the coordinate planes.
- We have learned about surfaces in three dimensions described by first-order equations; these are planes. Some other common types of surfaces can be described by second-order equations. We can view these surfaces as three-dimensional extensions of the conic sections we discussed earlier: the ellipse, the parabola, and the hyperbola. We call these graphs **quadric surfaces**.
- **Quadric surfaces** are the graphs of equations that can be expressed in the form

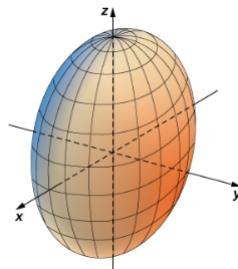
$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Jz + K = 0..$$

**Note:** When a quadric surface intersects a coordinate plane, the trace is a conic section.

- An **ellipsoid** is a surface described by an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

**Note:** Set  $x = 0$  to see the trace of the ellipsoid in the  $yz$ -plane. To see the traces in the  $xy$ - and  $xz$ -planes, set  $z = 0$  and  $y = 0$ , respectively. Notice that, if  $a = b$ , the trace in the  $xy$ -plane is a circle. Similarly, if  $a = c$ , the trace in the  $xz$ -plane is a circle and, if  $b = c$ , then the trace in the  $yz$ -plane is a circle. A sphere, then, is an ellipsoid with  $a = b = c$ .



- A **Sphere** is an ellipsoid with

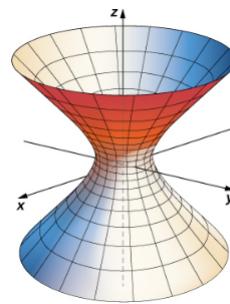
$$a = b = c.$$

**Traces:** All three traces are ellipses

- A quadric surface can have at most **two unique traces**, these traces make the name of the surface.
- **Hyperboloid of one sheet**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Two of the variables have positive coefficients and one has a negative coefficient. The axis of the surface corresponds to the variable with the negative coefficient

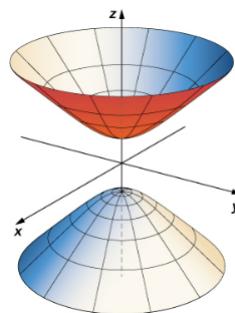


**Traces:** One ellipse and two hyperbololas

- **Hyperboloid of two sheets**

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Two of the variables have negative coefficients and one has a positive coefficient. The axis of the surface corresponds to the variable with the positive coefficient. The surface does not intersect the coordinate plane perpendicular to the axis

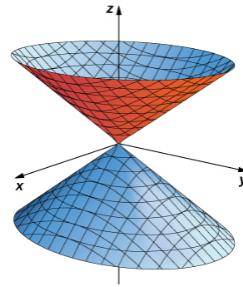


**Traces:** One ellipse (or the empty set (no trace)), and two hyperbololas

- **Elliptic cone**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$$

The axis of the surface corresponds to the variable with a negative coefficient. The traces in the coordinate planes parallel to the axis are intersecting lines



**Traces:** One ellipse and two hyperbolas.

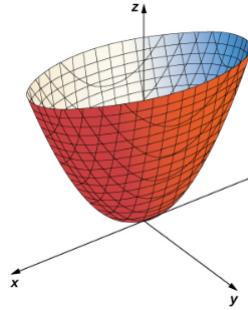
In the  $xz$ -plane, we have a pair of lines that intersect at the origin

In the  $yz$ -plane, we have a pair of lines that intersect at the origin

- **Elliptic paraboloid**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z.$$

The axis of the surface corresponds to the linear variable

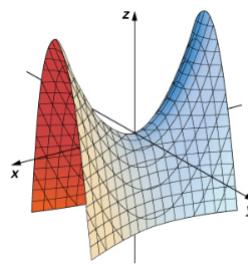


**Traces:** One ellipse and two parabolas

- **Hyperbolic paraboloid**

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z.$$

The axis of the surface corresponds to the linear variable



**Traces:** One hyperbola and two parabolas

## 3.4 Chapter 3: Vector-Valued Functions

### 3.4.1 Definitions and Theorems

- A **vector-valued function** is a function of the form

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} \quad \text{or} \quad \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k},$$

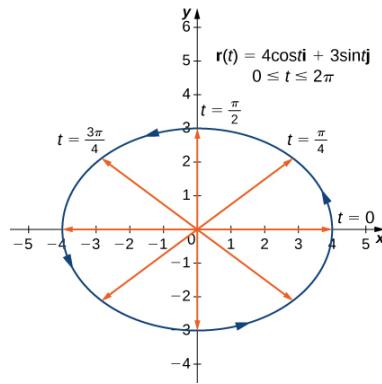
where the **component functions**  $f$ ,  $g$ , and  $h$ , are real-valued functions of the parameter  $t$ . Vector-valued functions are also written in the form

$$\mathbf{r}(t) = \langle f(t), g(t) \rangle \quad \text{or} \quad \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle.$$

In both cases, the first form of the function defines a two-dimensional vector-valued function; the second form describes a three-dimensional vector-valued function.

**Note:** The parameter  $t$  can lie between two real numbers:  $a \leq t \leq b$ . Another possibility is that the value of  $t$  might take on all real numbers. Last, the component functions themselves may have domain restrictions that enforce restrictions on the value of  $t$ . We often use  $t$  as a parameter because  $t$  can represent time.

- **Graphing vector-valued functions: Plane curve:** The graph of a vector-valued function of the form  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$  consists of the set of all  $(t, \mathbf{r}(t))$ , and the path it traces is called a **plane curve**.
- **Graphing vector-valued functions: Space curve:** The graph of a vector-valued function of the form  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$  consists of the set of all  $(t, \mathbf{r}(t))$ , and the path it traces is called a **space curve**.
- **Vector Parameterization:** Any representation of a plane curve or space curve using a vector-valued function is called a **vector parameterization** of the curve.
- **Note on graphing vector-valued functions:** When graphing a vector-valued function, we typically graph the vectors in the domain of the function in standard position, because doing so guarantees the uniqueness of the graph.
- **How to graph vector-valued functions:** As with any graph, we start with a table of values. We then graph each of the vectors in the second column of the table in standard position and connect the terminal points of each vector to form a curve



- **similarity between vector-valued functions and parameterized curves.**: Given a vector-valued function  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ , we can define  $x = f(t)$  and  $y = g(t)$ . If a restriction exists on the values of  $t$  (for example,  $t$  is restricted to the interval  $[a, b]$  for some constants  $a < b$ ), then this restriction is enforced on the parameter. The graph of the parameterized function would then agree with the graph of the vector-valued function, except that the vector-valued graph would represent vectors rather than points. Since we can parameterize a curve defined by a function  $y = f(x)$ , it is also possible to represent an arbitrary plane curve by a vector-valued function.
- **Limits of a Vector-Valued Function (Rigorous):** A vector-valued function  $\mathbf{r}$  approaches the limit  $L$  as  $t$  approaches  $a$ , written

$$\lim_{t \rightarrow a} \mathbf{r}(t) = L,$$

provided

$$\lim_{t \rightarrow a} \|\mathbf{r}(t) - L\| = 0.$$

**Note:** This is a rigorous definition of the limit of a vector-valued function. In practice, we use the following theorem:

- **Limit of a Vector-Valued Function:** Let  $f$ ,  $g$ , and  $h$  be functions of  $t$ . Then the limit of the vector-valued function  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$  as  $t$  approaches  $a$  is given by

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \left[ \lim_{t \rightarrow a} f(t) \right] \mathbf{i} + \left[ \lim_{t \rightarrow a} g(t) \right] \mathbf{j},$$

provided the limits  $\lim_{t \rightarrow a} f(t)$  and  $\lim_{t \rightarrow a} g(t)$  exist. Similarly, the limit of the vector-valued function  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$  as  $t$  approaches  $a$  is given by

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \left[ \lim_{t \rightarrow a} f(t) \right] \mathbf{i} + \left[ \lim_{t \rightarrow a} g(t) \right] \mathbf{j} + \left[ \lim_{t \rightarrow a} h(t) \right] \mathbf{k},$$

provided the limits  $\lim_{t \rightarrow a} f(t)$ ,  $\lim_{t \rightarrow a} g(t)$ , and  $\lim_{t \rightarrow a} h(t)$  exist.

- **Continuity of a vector-valued function:** Let  $f$ ,  $g$ , and  $h$  be functions of  $t$ . Then, the vector-valued function  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$  is continuous at point  $t = a$  if the following three conditions hold:

1.  $\mathbf{r}(a)$  exists.
2.  $\lim_{t \rightarrow a} \mathbf{r}(t)$  exists.
3.  $\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{r}(a)$ .

Similarly, the vector-valued function  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$  is continuous at point  $t = a$  if the following three conditions hold:

1.  $\mathbf{r}(a)$  exists.
2.  $\lim_{t \rightarrow a} \mathbf{r}(t)$  exists.
3.  $\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{r}(a)$ .

- **Example problem:** Find the vector equation that represents the curve of intersection of the cylinder  $x^2 + y^2 = 9$  and the surface  $z = x + 3y$

To start, we can easily find  $x(t)$  and  $y(t)$ .

**Proposition.**

$$\begin{aligned}x(t) &= 3 \cos t \\y(t) &= 3 \sin t.\end{aligned}$$

We can verify this simply

$$\begin{aligned}\frac{1}{3}x &= \cos t \\ \frac{1}{3}y &= \sin t.\end{aligned}$$

If  $\cos^2 t + \sin^2 t = 1$ , Then

$$\begin{aligned}\frac{1}{9}x^2 + \frac{1}{9}y^2 &= 1 \\ x^2 + y^2 &= 9.\end{aligned}$$

To find the function for  $z(t)$ , we see that our surface is already solved for  $z$ , thus we simply replace  $x$  and  $y$  with what we have for  $x(t)$  and  $y(t)$

$$\implies z(t) = 3 \cos t + 9 \sin t.$$

- **The derivative of a vector-valued function  $r(t)$**  The derivative of a vector-valued function  $\mathbf{r}(t)$  is

$$\mathbf{r}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}, \quad (35)$$

provided the limit exists. If  $\mathbf{r}'(t)$  exists, then  $\mathbf{r}$  is differentiable at  $t$ . If  $\mathbf{r}'(t)$  exists for all  $t$  in an open interval  $(a, b)$ , then  $\mathbf{r}$  is differentiable over the interval  $(a, b)$ . For the function to be differentiable over the closed interval  $[a, b]$ , the following two limits must exist as well:

$$\mathbf{r}'(a) = \lim_{\Delta t \rightarrow 0^+} \frac{\mathbf{r}(a + \Delta t) - \mathbf{r}(a)}{\Delta t} \quad (36)$$

and

$$\mathbf{r}'(b) = \lim_{\Delta t \rightarrow 0^-} \frac{\mathbf{r}(b + \Delta t) - \mathbf{r}(b)}{\Delta t}. \quad (37)$$

- **Differentiation of Vector-Valued Functions** Let  $f$ ,  $g$ , and  $h$  be differentiable functions of  $t$ .

- If  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ , then  $\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j}$ .
- If  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ , then  $\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$ .

- **Properties of the Derivative of Vector-Valued Functions** Let  $\mathbf{r}$  and  $\mathbf{u}$  be differentiable vector-valued functions of  $t$ , let  $f$  be a differentiable real-valued function of  $t$ , and let  $c$  be a scalar.

1.  $\frac{d}{dt}[\mathbf{cr}(t)] = \mathbf{cr}'(t)$ , Scalar multiple
2.  $\frac{d}{dt}[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$ , Sum and difference
3.  $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$ , Scalar product
4.  $\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}'(t) \cdot \mathbf{u}(t) + \mathbf{r}(t) \cdot \mathbf{u}'(t)$ , Dot product
5.  $\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}'(t) \times \mathbf{u}(t) + \mathbf{r}(t) \times \mathbf{u}'(t)$ , Cross product
6.  $\frac{d}{dt}[\mathbf{r}(f(t))] = \mathbf{r}'(f(t)) \cdot f'(t)$ , Chain rule
7. If  $\mathbf{r}(t) \cdot \mathbf{r}(t) = c$ , then  $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$ .

- **principal unit tangent vector** Let  $C$  be a curve defined by a vector-valued function  $\mathbf{r}$ , and assume that  $\mathbf{r}'(t)$  exists when  $t = t_0$ . A tangent vector  $\mathbf{v}$  at  $t = t_0$  is any vector such that, when the tail of the vector is placed at point  $\mathbf{r}(t_0)$  on the graph, vector  $\mathbf{v}$  is tangent to curve  $C$ . Vector  $\mathbf{r}'(t_0)$  is an example of a tangent vector at point  $t = t_0$ . Furthermore, assume that  $\mathbf{r}'(t) \neq 0$ . The principal unit tangent vector at  $t$  is defined to be

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}, \quad (38)$$

provided  $\|\mathbf{r}'(t)\| \neq 0$ .

- **the derivative provides a tangent vector to the curve represented by the function.**
- **The Fundamental Theorem of Calculus applies to vector-valued functions as well.**
- **Integrals of Vector-Valued Functions** Let  $f$ ,  $g$ , and  $h$  be integrable real-valued functions over the closed interval  $[a, b]$ . The indefinite integral of a vector-valued function  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$  is

$$\int [f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}] dt = \left[ \int f(t) dt \right] \mathbf{i} + \left[ \int g(t) dt \right] \mathbf{j} + \left[ \int h(t) dt \right] \mathbf{k}. \quad (39)$$

The definite integral of the vector-valued function is

$$\int_a^b [f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}] dt = \left[ \int_a^b f(t) dt \right] \mathbf{i} + \left[ \int_a^b g(t) dt \right] \mathbf{j} + \left[ \int_a^b h(t) dt \right] \mathbf{k}. \quad (40)$$

- **Integration constant for the integral of a vector-valued function**

$$\begin{aligned} \int [f(t)\mathbf{i} + g(t)\mathbf{j}] dt &= \left[ \int f(t) dt \right] \mathbf{i} + \left[ \int g(t) dt \right] \mathbf{j} \\ &= (F(t) + C_1)\mathbf{i} + (G(t) + C_2)\mathbf{j} \\ &= F(t)\mathbf{i} + G(t)\mathbf{j} + C_1\mathbf{i} + C_2\mathbf{j} \\ &= F(t)\mathbf{i} + G(t)\mathbf{j} + \mathbf{C}. \end{aligned}$$

where  $\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j}$ . Therefore, the integration constant becomes a constant vector.

- **Arc Length for Vector Functions**

1. **Plane curve:** Given a smooth curve  $C$  defined by the function  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ , where  $t$  lies within the interval  $[a, b]$ , the arc length of  $C$  over the interval is

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt = \int_a^b \|\mathbf{r}'(t)\| dt. \quad (41)$$

2. **Space curve:** Given a smooth curve  $C$  defined by the function  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ , where  $t$  lies within the interval  $[a, b]$ , the arc length of  $C$  over the interval is

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt = \int_a^b \|\mathbf{r}'(t)\| dt. \quad (42)$$

**Note:** Note that the formulas are defined for smooth curves: curves where the vector-valued function  $r(t)$  is differentiable with a non-zero derivative. The smoothness condition guarantees that the curve has no cusps (or corners) that could make the formula problematic.

- **Arc-length function definition:** If a vector-valued function represents the position of a particle in space as a function of time, then the arc-length function measures how far that particle travels as a function of time.
- **Arc-length function:** Let  $\mathbf{r}(t)$  describe a smooth curve for  $t \geq a$ . Then the arc-length function is given by

$$s(t) = \int_a^t \|\mathbf{r}'(u)\| du \quad (43)$$

Furthermore,  $\frac{ds}{dt} = \|\mathbf{r}'(t)\| > 0$ . If  $\|\mathbf{r}'(t)\| = 1$  for all  $t \geq a$ , then the parameter  $t$  represents the arc length from the starting point at  $t = a$ .

**Note:** If a vector-valued function represents the position of a particle in space as a function of time, then the arc-length function measures how far that particle travels as a function of time.

Since  $s(t)$  measures distance traveled as a function of time,  $s'(t)$  measures the speed of the particle at any given time.

- **Arc-length parametrization**

1. Find  $s(t)$
2. Solve  $s(t)$  for  $t$
3. Plug expression for  $t$  into  $\bar{\mathbf{r}}(t)$  to get  $\bar{\mathbf{r}}(s)$

The vector-valued function is now written in terms of the parameter  $s$ . Since the variable  $s$  represents the arc length, we call this an arc-length parameterization of the original function  $r(t)$ . One advantage of finding the arc-length parameterization is that the distance traveled along the curve starting from  $s = 0$  is now equal to the parameter  $s$

**Eg:** Suppose we have the function  $r(t) = 4 \cos t \hat{\mathbf{i}} + 4 \sin t \hat{\mathbf{j}}$  for  $t \geq 0$ . First, we find  $s(t)$

$$\begin{aligned} r'(u) &= -4 \sin(u) \hat{\mathbf{i}} + 4 \cos(u) \hat{\mathbf{k}} \\ \implies s(t) &= \int_0^t \sqrt{16 \sin^2 u + 16 \cos^2 u} du \\ &= \int_0^t 4 du = 4u \Big|_0^t = 4t. \end{aligned}$$

Now we solve for  $t$

$$s = 4t \implies t = \frac{1}{4}s.$$

Pluggin back into  $\vec{r}(t)$  we get

$$\vec{r}(s) = 4 \cos\left(\frac{1}{4}s\right) \hat{\mathbf{i}} + 4 \sin\left(\frac{1}{4}s\right) \hat{\mathbf{j}}.$$

This is the arc-length parameterization of  $\mathbf{r}(t)$ . Since the original restriction on  $t$  was given by  $t \geq 0$ , the restriction on  $s$  becomes  $\frac{s}{4} \geq 0$ , or  $s \geq 0$ .

- **Curvature:** The concept of curvature provides a way to measure how sharply a smooth curve turns. Let  $C$  be a smooth curve in the plane or in space given by  $\mathbf{r}(s)$ , where  $s$  is the arc-length parameter. The curvature  $\kappa$  at  $s$  is

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\| = \|\mathbf{T}'(s)\|. \quad (44)$$

- **Alternate formulas for curvature** If  $C$  is a smooth curve given by  $\mathbf{r}(t)$ , then the curvature  $\kappa$  of  $C$  at  $t$  is given by

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}. \quad (45)$$

If  $C$  is a three-dimensional curve, then the curvature can be given by the formula

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}. \quad (46)$$

If  $C$  is the graph of a function  $y = f(x)$  and both  $y'$  and  $y''$  exist, then the curvature  $\kappa$  at point  $(x, y)$  is given by

$$\kappa = \frac{|y''|}{[1 + (y')^2]^{3/2}}. \quad (47)$$

- **The curvature of a circle** is given by

$$\frac{1}{\text{radius}}.$$

- **Principal unit normal vector** Let  $C$  be a three-dimensional smooth curve represented by  $\mathbf{r}$  over an open interval  $I$ . If  $\mathbf{T}'(t) \neq 0$ , then the principal unit normal vector at  $t$  is defined to be

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}. \quad (48)$$

- **Principal unit binormal vector** The binormal vector at  $t$  is defined as

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t), \quad (49)$$

where  $\mathbf{T}(t)$  is the unit tangent vector.

**Note:** the binormal vector will already be of magnitude one

- **We can only find a binormal vector for a space curve, not a two dimensional curve (plane curve)**
- **Normal plane:** The unit normal vector and the binormal vector form a plane that is perpendicular to the curve at any point on the curve, called the normal plane
- These three vectors form a frame of reference in three-dimensional space called the **Frenet frame of reference (also called the TNB frame)**
- The plane determined by the vectors  $\vec{T}$  and  $\vec{N}$  forms the **osculating plane of  $C$  at any point  $P$  on the curve.**

## 3.5 Chapter 4: Differentiation of Functions of Several Variables

### 3.5.1 Definitions and Theorems

- **Function of two variables:** A function of two variables  $z = f(x, y)$  maps each ordered pair  $(x, y)$  in a subset  $D$  of the real plane  $\mathbb{R}^2$  to a unique real number  $z$ . The set  $D$  is called the domain of the function. The range of  $f$  is the set of all real numbers  $z$  that has at least one ordered pair  $(x, y) \in D$  such that  $f(x, y) = z$  as shown in the following figure.
- **Surface:** The graph of a function  $z = f(x, y)$  of two variables is called a surface.
- Given a function  $f(x, y)$  and a number  $c$  in the range of  $f$ , a **level curve of a function of two variables** for the value  $c$  is defined to be the set of points satisfying the equation  $f(x, y) = c$
- A graph of the various level curves of a function is called a **contour map**.
- Level curves are always graphed in the  $xy$ -plane, but as their name implies, **vertical traces** are graphed in the  $xz$ - or  $yz$ -planes.
- Consider a function  $z = f(x, y)$  with domain  $D \subseteq \mathbb{R}^2$ . A **vertical trace** of the function can be either the set of points that solves the equation  $f(a, y) = z$  for a given constant  $x = a$  or  $f(x, b) = z$  for a given constant  $y = b$ .
- **Limit laws for functions of two variables** Let  $f(x, y)$  and  $g(x, y)$  be defined for all  $(x, y) \neq (a, b)$  in a neighborhood around  $(a, b)$ , and assume the neighborhood is contained completely inside the domain of  $f$ . Assume that  $L$  and  $M$  are real numbers such that  $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$  and  $\lim_{(x,y) \rightarrow (a,b)} g(x, y) = M$ , and let  $c$  be a constant. Then each of the following statements holds:

**Constant Law:**

$$\lim_{(x,y) \rightarrow (a,b)} c = c \quad (4.2)$$

**Identity Laws:**

$$\lim_{(x,y) \rightarrow (a,b)} x = a \quad (4.3)$$

$$\lim_{(x,y) \rightarrow (a,b)} y = b \quad (4.4)$$

**Sum Law:**

$$\lim_{(x,y) \rightarrow (a,b)} (f(x, y) + g(x, y)) = L + M \quad (4.5)$$

**Difference Law:**

$$\lim_{(x,y) \rightarrow (a,b)} (f(x, y) - g(x, y)) = L - M \quad (4.6)$$

**Constant Multiple Law:**

$$\lim_{(x,y) \rightarrow (a,b)} (cf(x, y)) = cL \quad (4.7)$$

**Product Law:**

$$\lim_{(x,y) \rightarrow (a,b)} (f(x, y)g(x, y)) = LM \quad (4.8)$$

**Quotient Law:**

$$\lim_{(x,y) \rightarrow (a,b)} \frac{f(x, y)}{g(x, y)} = \frac{L}{M} \quad \text{for } M \neq 0 \quad (4.9)$$

**Power Law:**

$$\lim_{(x,y) \rightarrow (a,b)} (f(x,y))^n = L^n \quad (4.10)$$

for any positive integer  $n$ . **Root Law:**

$$\lim_{(x,y) \rightarrow (a,b)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} \quad (4.11)$$

for all  $L$  if  $n$  is odd and positive, and for  $L \geq 0$  if  $n$  is even and positive provided that  $f(x,y) \geq 0$  for all  $(x,y) \neq (a,b)$  in neighborhood of  $(a,b)$ .

- **Method to show that a limit does not exist.** To show that a limit exists can be quite challenging. To show that the limit exists is to show that it is the same along each and every path (infinitely many). To show that it does not exist, however, we only need to show that the limit differs for at least two unique paths

**Example:** Show that the following limit does not exist

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{2x^2 + y^2}.$$

We see that evaluating this limit directly yields an undefined result. Thus, we shall show that the limit differs among paths taken. Suppose we imagine our point at  $(0,0)$ , if we choose to go along the path  $x = 0$ , our limit becomes

$$\lim_{(0,y) \rightarrow (0,0)} \frac{-y^2}{y^2} = -1.$$

Likewise, lets go along the line  $y = 0$ , we get

$$\lim_{(x,0) \rightarrow (0,0)} \frac{x^2}{2x^2} = \frac{1}{2}.$$

Since  $-1 \neq \frac{1}{2}$ . We assert the limit does not exist at point  $(0,0)$  and hence move forward

- **L'Hospital's Rule in limits of functions of two or more variables.** It is sadly the case that we cannot use L'Hospital's Rule for limits of two variables, we can however use it when showing the limits are different among different paths. If we choose our path such that one variable vanishes, then the function is now of one variable and we are free to use L'Hospital's Rule to evaluate it.

**Example:**

$$\lim_{(x,y) \rightarrow (0,0)} \frac{3xy}{3x^2 + y^2}.$$

If we choose our path of interest to be along  $y = x$ , we get

$$\lim_{(x,x) \rightarrow (0,0)} \frac{3x^2}{4x^2} = \lim_{x \rightarrow 0} \frac{3x^2}{4x^2}.$$

We are now free to use L'Hospital's Rule, although it is clearly not necessary in this case.

- **Using polar coordinates to evaluate limits**

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2 + y^2}}.$$

We swap  $x$  and  $y$  to polar form

$$\begin{aligned} & \lim_{r \rightarrow 0} \frac{r \cos(\theta)r \sin(\theta)}{\sqrt{r^2 \cos^2(\theta) + r^2 \sin^2(\theta)}} \\ &= \lim_{r \rightarrow 0} r \cos(\theta) \sin(\theta) \\ &= 0. \end{aligned}$$

- **Squeeze theorem for multivariable limits:** Suppose we have

$$\lim_{(x,y) \rightarrow (0,0)} \frac{5x^2y}{x^2 + y^2}.$$

We want to consider the positive function, thus we examine

$$\begin{aligned} & \lim_{(x,y) \rightarrow (0,0)} \left| \frac{5x^2y}{x^2 + y^2} \right| \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{5x^2|y|}{x^2 + y^2}. \end{aligned}$$

We remark that

$$\lim_{(x,y) \rightarrow (a,b)} \left| f(x,y) \right| = \left| \lim_{(x,y) \rightarrow (a,b)} f(x,y) \right|.$$

We notice

$$0 \leq \frac{5x^2}{x^2 + y^2} \leq 5.$$

From here, we multiply all sides by  $|y|$ , note that this is why we wanted to consider the absolute value of the function, so we can peacefully manipulate the inequality

$$0 \leq \frac{5x^2|y|}{x^2 + y^2} \leq 5|y|.$$

Now, we take limits and notice the squeeze

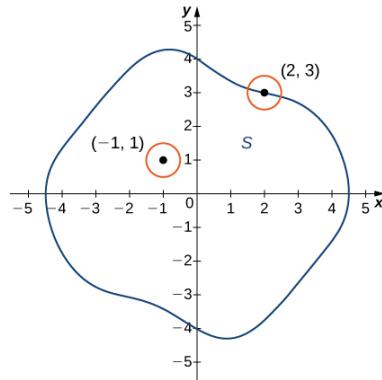
$$\lim_{(x,y) \rightarrow (0,0)} 0 \leq \frac{5x^2|y|}{x^2 + y^2} \leq \lim_{(x,y) \rightarrow (0,0)} 5|y|.$$

Thus, the limit of the middle function must be zero.

- **Limits of functions of three variables.** Similar to the functions of two variables, we can choose a path along an axis by having two of the variables be zero. Furthermore, we can go along a curve that's parametric. Thus, we set  $x, y$  and  $z$  to functions of a parameter  $t$ . This allows us to travel along a curve  $C$ . The best way to approach this is to choose the functions of  $t$  such that the degrees in both the numerator and denominator match up.

- **Interior Points and Boundary Points:** Let  $S$  be a subset of  $\mathbb{R}^2$

- (a) A point  $P_0$  is called an **interior point** of  $S$  if there is a  $\delta$ -disk centered around  $P_0$  contained completely in  $S$ .
- (b) A point  $P_0$  is called a **boundary point** of  $S$  if every  $\delta$ -disk centered around  $P_0$  contains points both inside and outside  $S$ .
- (c)  $S$  is called an **open set** if every point of  $S$  is an interior point.
- (d)  $S$  is called a **closed set** if it contains all its boundary points.
- (e) An open set  $S$  is a **connected set** if it cannot be represented as the union of two or more disjoint, nonempty open subsets
- (f) A set  $S$  is a **region** if it is open, connected, and nonempty.



- **Continuity of a function of two variables:** A function  $f(x, y)$  is continuous at a point  $(a, b)$  in its domain if the following conditions are satisfied:
  1.  $f(a, b)$  exists.
  2.  $\lim_{(x,y) \rightarrow (a,b)} f(x, y)$  exists.
  3.  $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b)$ .
- **The Sum of Continuous Functions Is Continuous**
- **The Product of Continuous Functions Is Continuous**
- **The Composition of Continuous Functions Is Continuous**
- **Sketching graphs for domain and continuity**
  - Identify domain restrictions
  - If domain restriction is an inequality, change inequality to equality and solve for  $y$
  - Graph function of  $y$ , if inequality is strict, curve should be dotted. If non-strict, solid
  - Test points outside/inside (or above/below for line) with original inequality (from domain restriction), shade regions that yield true

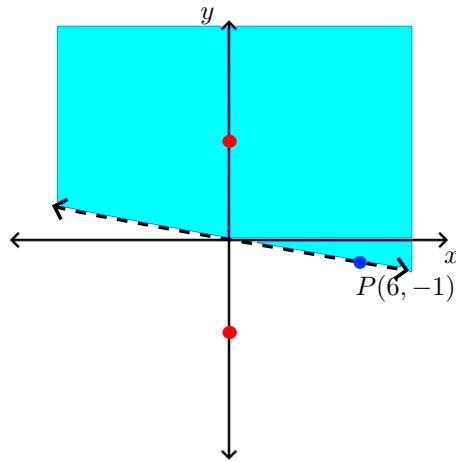
**Example:** Graph the set of points of continuity for the following function:

$$\ln(x + 6y).$$

We see

$$\begin{aligned} x + 6y &> 0 \quad (\text{strict}) \\ \implies y &= -\frac{1}{6}x. \end{aligned}$$

Graph:



The shaded region is found with test points  $T_1(0, -3)$  and  $T_2(0, 3)$  (red points). We see

$$\begin{aligned} 0 + 6(-3) &\not> 0 \\ 0 + 6(3) &> 0. \end{aligned}$$

- **Constant of proportionality**

- Direct proportionality

$$y = kx.$$

Where  $k$  is the constant of proportionality

The constant of proportionality can be determined if you know the values of the two variables at a specific point. For direct proportionality, if you know a pair of values  $(x_0, y_0)$ , you can find  $k$  by rearranging the formula:

$$k = \frac{y}{x}.$$

- Inversely proportional

$$y = \frac{k}{x}.$$

For inverse proportionality, given a pair of values  $(x_0, y_0)$ ,  $k$  can be found as

$$k = xy.$$

We use  $\propto$  notation without the constant, but when we replace it with equality, we need to introduce the constant to maintain equality

- **Limit definition of a partial derivative:** Let  $f(x, y)$  be a function of two variables. Then the partial derivative of  $f$  with respect to  $x$ , written as  $\frac{\partial f}{\partial x}$ , or  $f_x$ , is defined as

$$\frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}.$$

The partial derivative of  $f$  with respect to  $y$ , written as  $\frac{\partial f}{\partial y}$ , or  $f_y$ , is defined as

$$\frac{\partial f}{\partial y} = \lim_{k \rightarrow 0} \frac{f(x, y+k) - f(x, y)}{k}.$$

**Note:** The logic remains the same for functions of three variables

- **Fast computation of partial derivatives:** To compute partial derivatives without using the limit definition, we let the variable that we aren't interested in be a constant, this allows us to compute the derivative as if it were ordinary.
- **Interpretation of partial derivatives:** The partial derivative in the x-direction is the slope for slight movement in the x-direction, etc
- **higher-order partial derivatives:** There are four second-order partial derivatives for any function (provided they all exist):

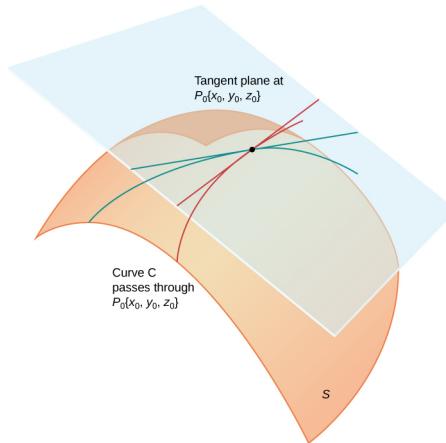
$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left[ \frac{\partial f}{\partial x} \right], \\ \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x} \left[ \frac{\partial f}{\partial y} \right], \\ \frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y} \left[ \frac{\partial f}{\partial x} \right], \\ \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y} \left[ \frac{\partial f}{\partial y} \right]..\end{aligned}$$

**Note:** in this notation, the order in which we take derivatives goes from right to left (of the denominator)

- **Higher-order partial derivatives alternate notation:** An alternative notation for each is  $f_{xx}$ ,  $f_{yx}$ ,  $f_{xy}$ , and  $f_{yy}$ , respectively. Higher-order partial derivatives calculated with respect to different variables, such as  $f_{xy}$  and  $f_{yx}$ , are commonly called mixed partial derivatives.

**Note:** with this notation, the order is from left to right

- **Tangent plane:** Let  $P_0 = (x_0, y_0, z_0)$  be a point on a surface  $S$ , and let  $C$  be any curve passing through  $P_0$  and lying entirely in  $S$ . If the tangent lines to all such curves  $C$  at  $P_0$  lie in the same plane, then this plane is called the tangent plane to  $S$  at  $P_0$ .



- **For a tangent plane to a surface to exist at a point on that surface, it is sufficient for the function that defines the surface to be differentiable at that point,**

- **Equation of a tangent plane:** Let  $S$  be a surface defined by a differentiable function  $z = f(x, y)$ , and let  $P_0 = (x_0, y_0)$  be a point in the domain of  $f$ . Then, the equation of the tangent plane to  $S$  at  $P_0$  is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

- **Linear approximation with tangent planes:** Given a function  $z = f(x, y)$  with continuous partial derivatives that exist at the point  $(x_0, y_0)$ , the linear approximation of  $f$  at the point  $(x_0, y_0)$  is given by the equation

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

- **a surface is considered to be smooth at point  $P$  if a tangent plane to the surface exists at that point.**
- **For a tangent plane to exist at the point  $(x_0, y_0)$  the partial derivatives must therefore exist at that point. However, this is not a sufficient condition for smoothness,**
- **Differentiability (book definition):** A function  $f(x, y)$  is differentiable at a point  $P(x_0, y_0)$  if, for all points  $(x, y)$  in a  $\delta$  disk around  $P$ , we can write

$$f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y) \quad (50)$$

where the error term  $E$  satisfies

$$\lim_{(x,y) \rightarrow (x_0,y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0.$$

- **Differentiability (Professor's definition):** The geometric concept of having a tangent plane at  $(x_0, y_0)$  is equivalent to the approximation  $f(x, y) \approx L(x, y)$  being sufficiently good. When this happens, we call  $f(x, y)$  differentiable at  $(x_0, y_0)$ . Formally,

$$\lim_{(x,y) \rightarrow (x_0,y_0)} E(x, y) = 0.$$

Where

$$E(x, y) = \frac{f(x, y) - L(x, y)}{|(x, y) - (x_0, y_0)|}.$$

- **Differentiability criteria:**  $f(x, y)$  is differentiable at  $(x_0, y_0)$  iff

1.  $f(x, y) = L(x) + E(x, y) \cdot |(x, y) - (x_0, y_0)|$
2.  $\lim_{(x,y) \rightarrow (x_0,y_0)} E(x, y) = 0$

Differentiability for functions of several variables (e.g.,  $f(x, y)$ ) extends the concept from single-variable calculus. A function  $f(x, y)$  is differentiable at a point  $(x_0, y_0)$  if it can be well approximated by a linear function (the tangent plane) near that point.

- **Show that a function is differentiable:** Show that

$$f(x, y) = 2x^2 - 4y.$$

Is differentiable at the point  $(2, -3)$

First, we find  $f(x, y)$ ,  $f_x(x, y)$ , and  $f_y(x, y)$ . We find

$$\begin{aligned} f(2, -3) &= 20 \\ f_x(2, -3) &= 8 \\ f_y(2, -3) &= -4. \end{aligned}$$

This implies

$$\begin{aligned} f(x, y) &= L(x) + E(x, y) \\ \implies E(x, y) &= f(x, y) - L(x, y) \\ \therefore E(x, y) &= 2x^2 - 8x + 8. \end{aligned}$$

Now we need to show that  $\lim_{(x,y) \rightarrow (x_0, y_0)} \frac{E(x, y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$ . Thus,

$$\begin{aligned} &\lim_{(x,y) \rightarrow (2, -3)} \frac{2x^2 - 8x + 8}{\sqrt{(x-2)^2 + (y+3)^2}} \\ &= \lim_{(x,y) \rightarrow (2, -3)} \frac{2(x-2)^2}{\sqrt{(x-2)^2 + (y+3)^2}} \\ &\leqslant \lim_{(x,y) \rightarrow (2, -3)} \frac{2((x-2)^2 + (y+3)^2)}{\sqrt{(x-2)^2 + (y+3)^2}} \\ &= \lim_{(x,y) \rightarrow (2, -3)} 2\sqrt{(x-2)^2 + (y+3)^2} \\ &= 0. \end{aligned}$$

Since  $E(x, y) \geq 0$  for any value of  $x$  or  $y$ , the original limit must be equal to zero. Therefore,  $f(x, y) = 2x^2 - 4y$  is differentiable at point  $(2, -3)$

- **Differentiability Implies Continuity:** Let  $z = f(x, y)$  be a function of two variables with  $(x_0, y_0)$  in the domain of  $f$ . If  $f(x, y)$  is differentiable at  $(x_0, y_0)$ , then  $f(x, y)$  is continuous at  $(x_0, y_0)$ .
- **Continuity of First Partials Implies Differentiability:** Let  $z = f(x, y)$  be a function of two variables with  $(x_0, y_0)$  in the domain of  $f$ . If  $f(x, y)$ ,  $f_x(x, y)$ , and  $f_y(x, y)$  all exist in a neighborhood of  $(x_0, y_0)$  and are continuous at  $(x_0, y_0)$ , then  $f(x, y)$  is differentiable there.
- **Total differential:** Let  $z = f(x, y)$  be a function of two variables with  $(x_0, y_0)$  in the domain of  $f$ , and let  $\Delta x$  and  $\Delta y$  be chosen so that  $(x_0 + \Delta x, y_0 + \Delta y)$  is also in the domain of  $f$ . If  $f$  is differentiable at the point  $(x_0, y_0)$ , then the differentials  $dx$  and  $dy$  are defined as

$$dx = \Delta x \quad \text{and} \quad dy = \Delta y.$$

The differential  $dz$ , also called the total differential of  $z = f(x, y)$  at  $(x_0, y_0)$ , is defined as

$$dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy.$$

- $\Delta z$

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y).$$

We use  $dz$  to approximate  $\Delta z$ , so

$$\Delta z \approx dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy.$$

With can further appromiate with

$$\begin{aligned} f(x + \Delta x, y + \Delta y) &= f(x, y) + \Delta z \\ &\approx f(x, y) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y. \end{aligned}$$

- **Chain Rule for One Independent Variable:** Suppose that  $x = g(t)$  and  $y = h(t)$  are differentiable functions of  $t$  and  $z = f(x, y)$  is a differentiable function of  $x$  and  $y$ . Then  $z = f(x(t), y(t))$  is a differentiable function of  $t$  and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt},$$

where the ordinary derivatives are evaluated at  $t$  and the partial derivatives are evaluated at  $(x, y)$ .

- **Chain Rule for Two Independent Variables:** Suppose  $x = g(u, v)$  and  $y = h(u, v)$  are differentiable functions of  $u$  and  $v$ , and  $z = f(x, y)$  is a differentiable function of  $x$  and  $y$ . Then,  $z = f(g(u, v), h(u, v))$  is a differentiable function of  $u$  and  $v$ , and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}.$$

- **Generalized Chain Rule:** Let  $w = f(x_1, x_2, \dots, x_m)$  be a differentiable function of  $m$  independent variables, and for each  $i \in \{1, \dots, m\}$ , let  $x_i = x_i(t_1, t_2, \dots, t_n)$  be a differentiable function of  $n$  independent variables. Then

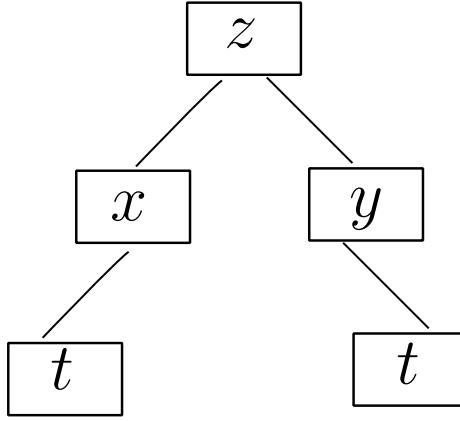
$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1} \frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2} \frac{\partial x_2}{\partial t_j} + \dots + \frac{\partial w}{\partial x_m} \frac{\partial x_m}{\partial t_j}$$

for any  $j \in \{1, 2, \dots, n\}$ .

- **Chain rule by a tree diagram.** Suppose we have some function  $z = f(x, y)$ , where  $x = g(t)$  and  $y = h(t)$ , so  $z$  has two variables  $x$  and  $y$ , both of which depend on  $t$ . To find the formula for the derivative  $\frac{dz}{dt}$ , we can create a tree diagram

The way we use the diagram is simple. We want to find  $\frac{dz}{dt}$ , which you notice is an ordinary derivative, because the leaf nodes are all the same variable. First, we start with the left side, we take derivatives all the way down, multiplying, and then the same for the right side. We sum the two sides. Thus this becomes

$$\frac{dz}{dt} = \frac{\delta z}{\delta x} \cdot \frac{dx}{dt} + \frac{\delta z}{\delta y} \cdot \frac{dy}{dt}.$$



- **Changing variables after computing derivative:** After we compute the derivative, we need to change all instances of  $x$  and  $y$  by using the equation that these variables are equal to. This way our final answer is in terms of the leaf node variables (see the tree diagram above)
- **Implicit Differentiation of a Function of Two or More Variables:** Suppose the function  $z = f(x, y)$  defines  $y$  implicitly as a function  $y = g(x)$  of  $x$  via the equation  $f(x, y) = 0$ . Then

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

provided  $\partial f/\partial y(x, y) \neq 0$ .

If the equation  $f(x, y, z) = 0$  defines  $z$  implicitly as a differentiable function of  $x$  and  $y$ , then

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z}$$

as long as  $\partial f/\partial z(x, y, z) \neq 0$ .

- **Tangent plane for a surface defined implicitly**

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

- **Direction Derivatives limit definition:** Suppose  $z = f(x, y)$  is a function of two variables with a domain of  $D$ . Let  $(a, b) \in D$  and define  $\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$ . Then the directional derivative of  $f$  in the direction of  $\mathbf{u}$  is given by

$$D_{\mathbf{u}}f(a, b) = \lim_{h \rightarrow 0} \frac{f(a + h \cos \theta, b + h \sin \theta) - f(a, b)}{h}, \quad (51)$$

provided the limit exists.

- **Directional derivatives with partial derivatives:** Let  $z = f(x, y)$  be a function of two variables  $x$  and  $y$ , and assume that  $f_x$  and  $f_y$  exist and  $f(x, y)$  is differentiable everywhere. Then the directional derivative of  $f$  in the direction of  $\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$  is given by

$$D_{\mathbf{u}}f(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta.$$

- **Gradient:** Let  $z = f(x, y)$  be a function of  $x$  and  $y$  such that  $f_x$  and  $f_y$  exist. The vector  $\nabla f(x, y)$  is called the gradient of  $f$  and is defined as

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}. \quad (52)$$

The vector  $\nabla f(x, y)$  is also written as “grad  $f$ .”

- **Divide by the norm:** If the vector that is given for the direction of the derivative is not a unit vector, then it is only necessary to divide by the norm of the vector.
- **Directional derivative with the gradient:**

$$D_u f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

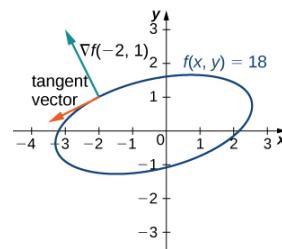
- **Properties of the Gradient:** Suppose the function  $z = f(x, y)$  is differentiable at  $(x_0, y_0)$  (Figure 4.41).

1. If  $\nabla f(x_0, y_0) = 0$ , then  $D_{\mathbf{u}} f(x_0, y_0) = 0$  for any unit vector  $\mathbf{u}$ .
2. If  $\nabla f(x_0, y_0) \neq 0$ , then  $D_{\mathbf{u}} f(x_0, y_0)$  is maximized when  $\mathbf{u}$  points in the same direction as  $\nabla f(x_0, y_0)$ . The maximum value of  $D_{\mathbf{u}} f(x_0, y_0)$  is  $\|\nabla f(x_0, y_0)\|$ .
3. If  $\nabla f(x_0, y_0) \neq 0$ , then  $D_{\mathbf{u}} f(x_0, y_0)$  is minimized when  $\mathbf{u}$  points in the opposite direction from  $\nabla f(x_0, y_0)$ . The minimum value of  $D_{\mathbf{u}} f(x_0, y_0)$  is  $-\|\nabla f(x_0, y_0)\|$ .

- **Gradient Is Normal to the Level Curve:** Suppose the function  $z = f(x, y)$  has continuous first-order partial derivatives in an open disk centered at a point  $(x_0, y_0)$ . If  $\nabla f(x_0, y_0) \neq 0$ , then  $\nabla f(x_0, y_0)$  is normal to the level curve of  $f$  at  $(x_0, y_0)$ .
- **Finding a tangent vector with normal vector:** Suppose we use the theorem above to find the vector tangent to a level curve at some point, we can then find a tangent vector by reversing the components and multiplying either one by negative one.

Example:

$$\begin{aligned} \nabla f(-2, 1) &= 9\hat{\mathbf{i}} + 18\hat{\mathbf{j}} \quad (\text{would be normal to some level curve}) \\ &- 18\hat{\mathbf{i}} - 9\hat{\mathbf{j}} \quad \text{Tangent vector at } (-2, 1). \end{aligned}$$



- **Gradient in three variables:** Let  $w = f(x, y, z)$  be a function of three variables such that  $f_x$ ,  $f_y$ , and  $f_z$  exist. The vector  $\nabla f(x, y, z)$  is called the gradient of  $f$  and is defined as

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}. \quad (53)$$

$\nabla f(x, y, z)$  can also be written as  $\text{grad } f(x, y, z)$ .

- **Directional derivative for functions of three variables (limit definition):** Suppose  $w = f(x, y, z)$  is a function of three variables with a domain of  $D$ . Let  $(x_0, y_0, z_0) \in D$  and let  $\mathbf{u} = \cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k}$  be a unit vector. Then, the directional derivative of  $f$  in the direction of  $\mathbf{u}$  is given by

$$D_{\mathbf{u}}f(x_0, y_0, z_0) = \lim_{t \rightarrow 0} \frac{f(x_0 + t \cos \alpha, y_0 + t \cos \beta, z_0 + t \cos \gamma) - f(x_0, y_0, z_0)}{t}, \quad (54)$$

provided the limit exists.

**Note:** The components of the unit vector are called the **directional cosines**

- **Directional derivative for functions of three variables (p.d definition):** Let  $f(x, y, z)$  be a differentiable function of three variables and let  $\mathbf{u} = \cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k}$  be a unit vector. Then, the directional derivative of  $f$  in the direction of  $\mathbf{u}$  is given by

$$\begin{aligned} D_{\mathbf{u}}f(x, y, z) &= \nabla f(x, y, z) \cdot \mathbf{u} \\ &= f_x(x, y, z) \cos \alpha + f_y(x, y, z) \cos \beta + f_z(x, y, z) \cos \gamma. \end{aligned}$$

- **Critical points:** Let  $z = f(x, y)$  be a function of two variables that is defined on an open set containing the point  $(x_0, y_0)$ . The point  $(x_0, y_0)$  is called a critical point of a function of two variables  $f$  if one of the two following conditions holds:

1.  $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$
2. Either  $f_x(x_0, y_0)$  or  $f_y(x_0, y_0)$  does not exist.

- **Local max / Absolute max (global max):** Let  $z = f(x, y)$  be a function of two variables that is defined and continuous on an open set containing the point  $(x_0, y_0)$ . Then  $f$  has a local maximum at  $(x_0, y_0)$  if

$$f(x_0, y_0) \geq f(x, y)$$

for all points  $(x, y)$  within some disk centered at  $(x_0, y_0)$ . The number  $f(x_0, y_0)$  is called a local maximum value. If the preceding inequality holds for every point  $(x, y)$  in the domain of  $f$ , then  $f$  has a global maximum (also called an absolute maximum) at  $(x_0, y_0)$ .

- **Local min / Absolute min (global min):** The function  $f$  has a local minimum at  $(x_0, y_0)$  if

$$f(x_0, y_0) \leq f(x, y)$$

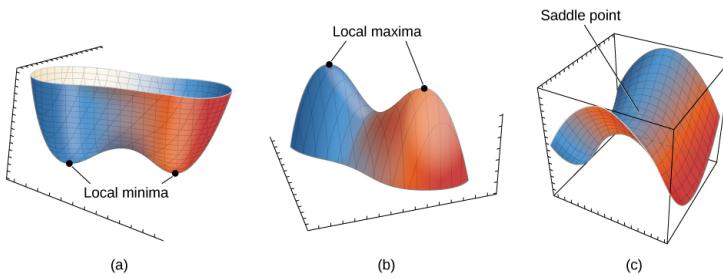
for all points  $(x, y)$  within some disk centered at  $(x_0, y_0)$ . The number  $f(x_0, y_0)$  is called a local minimum value. If the preceding inequality holds for every point  $(x, y)$  in the domain of  $f$ , then  $f$  has a global minimum (also called an absolute minimum) at  $(x_0, y_0)$ .

- **Local Extremum:** If  $f(x_0, y_0)$  is either a local maximum or local minimum value, then it is called a local extremum.
- **Fermat's Theorem for Functions of Two Variables:** Let  $z = f(x, y)$  be a function of two variables that is defined and continuous on an open set containing the point  $(x_0, y_0)$ . Suppose  $f_x$  and  $f_y$  each exists at  $(x_0, y_0)$ . If  $f$  has a local extremum at  $(x_0, y_0)$ , then  $(x_0, y_0)$  is a critical point of  $f$ .
- **Saddle point:** Given the function  $z = f(x, y)$ , the point  $(x_0, y_0, f(x_0, y_0))$  is a saddle point if both  $f_x(x_0, y_0) = 0$  and  $f_y(x_0, y_0) = 0$ , but  $f$  does not have a local extremum at  $(x_0, y_0)$ .

- **Second derivative test:** Let  $z = f(x, y)$  be a function of two variables for which the first- and second-order partial derivatives are continuous on some disk containing the point  $(x_0, y_0)$ . Suppose  $f_x(x_0, y_0) = 0$  and  $f_y(x_0, y_0) = 0$ . Define the quantity

$$D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2.$$

- I. If  $D > 0$  and  $f_{xx}(x_0, y_0) > 0$ , then  $f$  has a local minimum at  $(x_0, y_0)$ .
- II. If  $D > 0$  and  $f_{xx}(x_0, y_0) < 0$ , then  $f$  has a local maximum at  $(x_0, y_0)$ .
- III. If  $D < 0$ , then  $f$  has a saddle point at  $(x_0, y_0)$ .
- IV. If  $D = 0$ , then the test is inconclusive.



- **Extreme Value Theorem:** A continuous function  $f(x, y)$  on a closed and bounded set  $D$  in the plane attains an absolute maximum value at some point of  $D$  and an absolute minimum value at some point of  $D$ .
- **Finding extreme values:** Assume  $z = f(x, y)$  is a differentiable function of two variables defined on a closed, bounded set  $D$ . Then  $f$  will attain the absolute maximum value and the absolute minimum value, which are, respectively, the largest and smallest values found among the following:
  1. The values of  $f$  at the critical points of  $f$  in  $D$ .
  2. The values of  $f$  on the boundary of  $D$ .
- **Method of Lagrange Multipliers: One Constraint:** Let  $f$  and  $g$  be functions of two variables with continuous partial derivatives at every point of some open set containing the smooth curve  $g(x, y) = 0$ . Suppose that  $f$ , when restricted to points on the curve  $g(x, y) = 0$ , has a local extremum at the point  $(x_0, y_0)$  and that  $\nabla g(x_0, y_0) \neq 0$ . Then there is a number  $\lambda$  called a Lagrange multiplier, for which

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

- **Problem-Solving Strategy: Steps for Using Lagrange Multipliers:** Follow these steps to solve the optimization problem using Lagrange multipliers:
  1. Determine the objective function  $f(x, y)$  and the constraint function  $g(x, y)$ . Does the optimization problem involve maximizing or minimizing the objective function?
  2. Set up a system of equations using the following template:

$$\begin{aligned} \nabla f(x_0, y_0) &= \lambda \nabla g(x_0, y_0) \\ g(x_0, y_0) &= 0. \end{aligned}$$

3. Solve for  $x_0$  and  $y_0$ .

4. The largest of the values of  $f$  at the solutions found in step 3 maximizes  $f$ ; the smallest of those values minimizes  $f$ .

- **Problems with Two Constraints:** The method of Lagrange multipliers can be applied to problems with more than one constraint. In this case, the optimization function,  $w$ , is a function of three variables:

$$w = f(x, y, z)$$

and it is subject to two constraints:

$$g(x, y, z) = 0 \quad \text{and} \quad h(x, y, z) = 0.$$

There are two Lagrange multipliers,  $\lambda_1$  and  $\lambda_2$ , and the system of equations becomes

$$\begin{aligned}\nabla f(x_0, y_0, z_0) &= \lambda_1 \nabla g(x_0, y_0, z_0) + \lambda_2 \nabla h(x_0, y_0, z_0) \\ g(x_0, y_0, z_0) &= 0 \\ h(x_0, y_0, z_0) &= 0.\end{aligned}$$

- **Using gradient to find tangent plane to level surface:** Suppose we have some function  $z = f(x, y)$ , we rearrange the function such that it becomes  $F(x, y, z) = 0$ . We then use the fact that  $\nabla F(x_0, y_0, z_0)$  is normal to the surface  $F(x, y, z)$  at  $P_0$

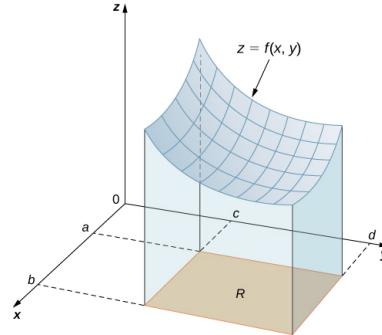
## 3.6 Chapter 5: Multiple integration

### 3.6.1 Definitions and theorems

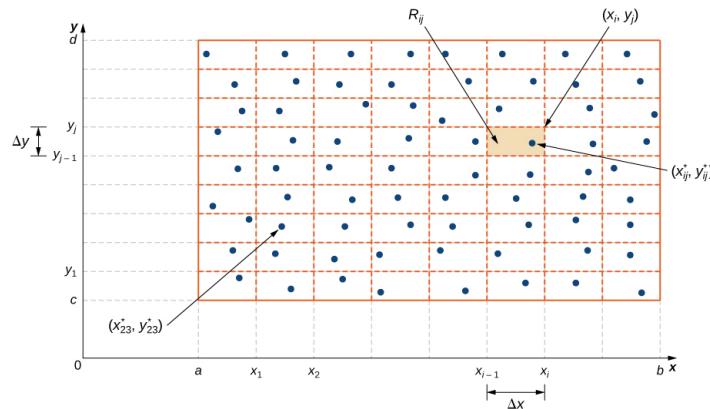
- **Double integral intro 1: Rectangular region  $R$  and solid  $S$ :** Consider a continuous function  $f(x, y) \geq 0$  of two variables defined on the closed rectangle  $R$ :

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$$

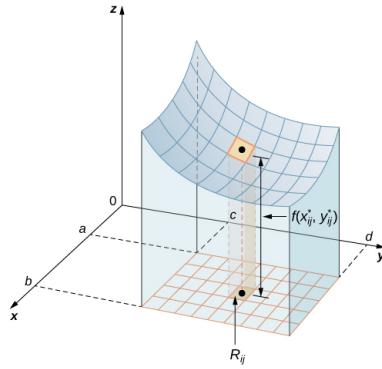
The graph of  $f$  represents a surface above the  $xy$ -plane with equation  $z = f(x, y)$  where  $z$  is the height of the surface at the point  $(x, y)$ . Let  $S$  be the solid that lies above  $R$  and under the graph of  $f$ . The base of the solid is the rectangle  $R$  in the  $xy$ -plane. We want to find the volume  $V$  of the solid  $S$ .



- **Double integral intro 2: Divisions of  $R$ :** We divide the region  $R$  into small rectangles  $R_{ij}$ , each with area  $\Delta A$  and with sides  $\Delta x$  and  $\Delta y$ . We do this by dividing the interval  $[a, b]$  into  $m$  subintervals and dividing the interval  $[c, d]$  into  $n$  subintervals. Hence,  $\Delta x = \frac{b-a}{m}$ ,  $\Delta y = \frac{d-c}{n}$ , and  $\Delta A = \Delta x \Delta y$ .



- **Double integral intro 3: volume of the subregions:** The volume of a thin rectangular box above  $R_{ij}$  is  $f(x_{ij}^*, y_{ij}^*) \Delta A$ , where  $(x_{ij}^*, y_{ij}^*)$  is an arbitrary sample point in each  $R_{ij}$  as shown in the following figure.



- **Double integral intro 4: Double Riemann sum:** Using the same idea for all the subrectangles, we obtain an approximate volume of the solid  $S$  as

$$V \approx \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A.$$

This sum is known as a double Riemann sum and can be used to approximate the value of the volume of the solid. Here, the double sum means that for each subrectangle, we evaluate the function at the chosen point, multiply by the area of each rectangle, and then add all the results.

As we have seen in the single-variable case, we obtain a better approximation to the actual volume if  $m$  and  $n$  become larger.

$$V = \lim_{m,n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A$$

or

$$V = \lim_{\Delta x, \Delta y \rightarrow 0} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A.$$

Note that the sum approaches a limit in either case, and the limit is the volume of the solid with the base  $R$ . Now we are ready to define the double integral.

- **The double integral over a rectangular region:** The double integral of the function  $f(x, y)$  over the rectangular region  $R$  in the  $xy$ -plane is defined as

$$\iint_R f(x, y) dA = \lim_{m,n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_i^*, y_j^*) \Delta A \quad (55)$$

**Note:** If  $f(x, y) \geq 0$ , then the volume  $V$  of the solid  $S$ , which lies above  $R$  in the  $xy$ -plane and under the graph of  $f$ , is the double integral of the function  $f(x, y)$  over the rectangle  $R$ . If the function is ever negative, then the double integral can be considered a “signed” volume in a manner similar to the way we defined net signed area in The Definite Integral.

**Example:** Suppose we have the surface defined by  $z = f(x, y) = 3x^2 + y$  with the region  $[0, 2] \times [0, 2]$

$$\iint_R 3x^2 + y dA = \lim_{m,n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n [(x_i^*)^2 + y_j^*] \Delta A.$$

- **Double integral exists and function is integrable:** The double integral of the function  $z = f(x, y)$  exists provided that the function  $f$  is not too discontinuous. If the function is bounded and continuous over  $R$  except on a finite number of smooth curves, then the double integral exists and we say that  $f$  is integrable over  $R$ .
- **Double integral with  $dx$  and  $dy$ :** Since  $\Delta A = \Delta x \Delta y = \Delta y \Delta x$ , we can express  $dA$  as  $dx dy$  or  $dy dx$ . This means that, when we are using rectangular coordinates, the double integral over a region  $R$  denoted by  $\iint_R f(x, y) dA$  can be written as  $\iint_R f(x, y) dx dy$  or  $\iint_R f(x, y) dy dx$ .
- **Properties of double integrals:** Assume that the functions  $f(x, y)$  and  $g(x, y)$  are integrable over the rectangular region  $R$ ;  $S$  and  $T$  are subregions of  $R$ ; and assume that  $m$  and  $M$  are real numbers.

I. The sum  $f(x, y) + g(x, y)$  is integrable and

$$\iint_R [f(x, y) + g(x, y)] dA = \iint_R f(x, y) dA + \iint_R g(x, y) dA.$$

II. If  $c$  is a constant, then  $cf(x, y)$  is integrable and

$$\iint_R cf(x, y) dA = c \iint_R f(x, y) dA.$$

III. If  $R = S \cup T$  and  $S \cap T = \emptyset$  except an overlap on the boundaries, then

$$\iint_R f(x, y) dA = \iint_S f(x, y) dA + \iint_T f(x, y) dA.$$

IV. If  $f(x, y) \geq g(x, y)$  for  $(x, y)$  in  $R$ , then

$$\iint_R f(x, y) dA \geq \iint_R g(x, y) dA.$$

V. If  $m \leq f(x, y) \leq M$ , then

$$m \times A(R) \leq \iint_R f(x, y) dA \leq M \times A(R).$$

VI. In the case where  $f(x, y)$  can be factored as a product of a function  $g(x)$  of  $x$  only and a function  $h(y)$  of  $y$  only, then over the region  $R = \{(x, y) | a \leq x \leq b, c \leq y \leq d\}$ , the double integral can be written as

$$\iint_R f(x, y) dA = \left( \int_a^b g(x) dx \right) \left( \int_c^d h(y) dy \right).$$

**Example:** Evaluate the integral  $\iint_R ye^x \cos(x) dA$  over the region  $R = \{(x, y) | 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq 1\}$ .

$$\begin{aligned} & \left( \int_0^{\frac{\pi}{2}} \cos(x) dx \right) \left( \int_0^1 e^y dy \right) \\ &= e - 1. \end{aligned}$$

- **Illustrating property V:** Over the region  $R = \{(x, y) | 1 \leq x \leq 3, 1 \leq y \leq 2\}$ , we have  $2 \leq x^2 + y^2 \leq 13$ . Find a lower and an upper bound for the integral  $\iint_R (x^2 +$

$y^2) dA$ . For a lower bound, integrate the constant function 2 over the region  $R$ . For an upper bound, integrate the constant function 13 over the region  $R$ .

$$\int_1^2 \int_1^3 2 dx dy = \int_1^2 [2x]_1^3 dy = \int_1^2 2(2) dy = 4y \Big|_1^2 = 4(2 - 1) = 4$$

$$\int_1^2 \int_1^3 13 dx dy = \int_1^2 [13x]_1^3 dy = \int_1^2 13(2) dy = 26y \Big|_1^2 = 26(2 - 1) = 26..$$

Hence, we obtain  $4 \leq \iint_R (x^2 + y^2) dA \leq 26$ .

- **Iterated integrals:** Assume  $a, b, c$ , and  $d$  are real numbers. We define an iterated integral for a function  $f(x, y)$  over the rectangular region  $R = [a, b] \times [c, d]$  as

(a)

$$\int_a^b \int_c^d f(x, y) dy dx = \int_a^b \left[ \int_c^d f(x, y) dy \right] dx.$$

(b)

$$\int_c^d \int_a^b f(x, y) dx dy = \int_c^d \left[ \int_a^b f(x, y) dx \right] dy.$$

**Note:** The notation

$$\int_a^b \left[ \int_c^d f(x, y) dy \right] dx$$

means that we integrate  $f(x, y)$  with respect to  $y$  while holding  $x$  constant. Similarly, the notation

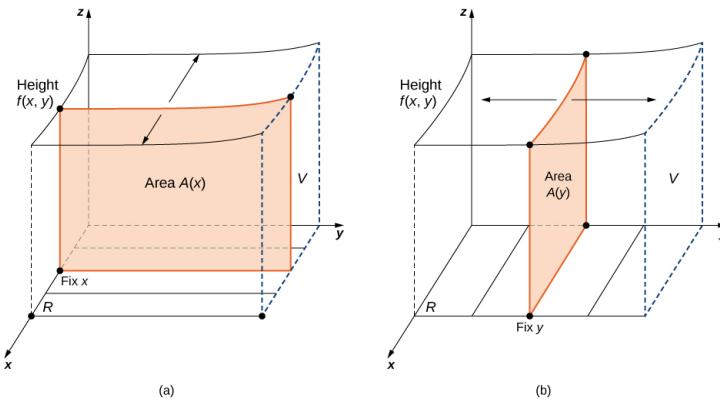
$$\int_c^d \left[ \int_a^b f(x, y) dx \right] dy$$

means that we integrate  $f(x, y)$  with respect to  $x$  while holding  $y$  constant.

- **Fubini's Theorem:** Suppose that  $f(x, y)$  is a function of two variables that is continuous over a rectangular region  $R = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$ . Then we see from the figure below that the double integral of  $f$  over the region equals an iterated integral,

$$\iint_R f(x, y) dA = \iint_R f(x, y) dx dy = \int_a^b \left( \int_c^d f(x, y) dy \right) dx = \int_c^d \left( \int_a^b f(x, y) dx \right) dy.$$

More generally, Fubini's theorem is true if  $f$  is bounded on  $R$  and  $f$  is discontinuous only on a finite number of continuous curves. In other words,  $f$  has to be integrable over  $R$ .



- **Area of a region  $R$ :** The area of the region  $R$  is given by

$$A(R) = \iint_R 1 \, dA.$$

- **Recall: Average value of a function of one variable:** the average value of a function of one variable on an interval  $[a, b]$  is given by

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x) \, dx.$$

- **Average value of a function of two variables:** The average value of a function of two variables over a region  $R$  is given by

$$\bar{f} = \frac{1}{A(R)} \iint_R f(x, y) dA.$$

- **Non rectangular region  $D$ :** Since  $D$  is bounded on the plane, there must exist a rectangular region  $R$  on the same plane that encloses the region  $D$ , that is, a rectangular region  $R$  exists such that  $D$  is a subset of  $R$  ( $D \subseteq R$ ).

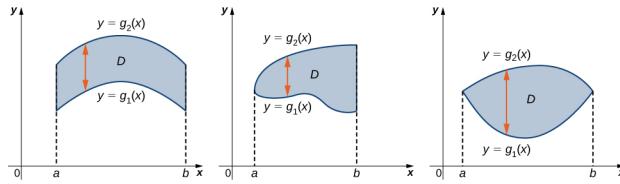
We extend the definition of the function to include all points on the rectangular region  $R$  and then use the concepts and tools from the preceding section. But how do we extend the definition of  $f$  to include all the points on  $R$ ? We do this by defining a new function  $g(x, y)$  on  $R$  as follows:

$$g(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

**Note:** we assume the boundary to be a piecewise smooth and continuous simple closed curve. We must be careful about  $g(x, y)$  and verify that  $g(x, y)$  is an integrable function over the rectangular region  $R$ . This happens as long as the region  $D$  is bounded by simple closed curves.

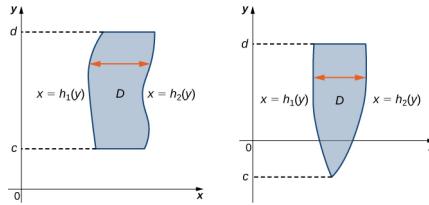
- **Types of planar bounded regions:** A region  $D$  in the  $(x, y)$ -plane is of **Type I** if it lies between two vertical lines and the graphs of two continuous functions  $g_1(x)$  and  $g_2(x)$ .

$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}.$$



A region  $D$  in the  $xy$ -plane is of **Type II** if it lies between two horizontal lines and the graphs of two continuous functions  $h_1(y)$  and  $h_2(y)$ .

$$D = \{(x, y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}.$$



- **Double Integrals over Nonrectangular Regions:** Suppose  $g(x, y)$  is the extension to the rectangle  $R$  of the integrable function  $f(x, y)$  defined on the region  $D$ , where  $D$  is inside  $R$ . Then  $g(x, y)$  is integrable and we define the double integral of  $f(x, y)$  over  $D$  by

$$\iint_D f(x, y) dA = \iint_R g(x, y) dA.$$

**Note:** The equality works because the values of  $g(x, y)$  are 0 for any point  $(x, y)$  that lies outside  $D$ , and hence these points do not add anything to the integral. However, it is important that the rectangle  $R$  contains the region  $D$ .

- **Fubini's Theorem (Strong Form):** For a function  $f(x, y)$  that is continuous on a region  $D$  of Type I, we have

$$\iint_D f(x, y) dA = \iint_D f(x, y) dy dx = \int_a^b \left[ \int_{g_1(x)}^{g_2(x)} f(x, y) dy \right] dx. \quad (56)$$

Similarly, for a function  $f(x, y)$  that is continuous on a region  $D$  of Type II, we have

$$\iint_D f(x, y) dA = \iint_D f(x, y) dx dy = \int_c^d \left[ \int_{h_1(y)}^{h_2(y)} f(x, y) dx \right] dy. \quad (57)$$

- **Decomposing Regions into Smaller Regions:** Suppose the region  $D$  can be expressed as  $D = D_1 \cup D_2$  where  $D_1$  and  $D_2$  do not overlap except at their boundaries. Then

$$\iint_D f(x, y) dA = \iint_{D_1} f(x, y) dA + \iint_{D_2} f(x, y) dA.$$

- **We can always change our region from type 1 to type 2 to make our iterated integral easier to solve**
- **Area of a plane bounded by a region  $D$ :** The area of a plane-bounded region  $D$  is defined as the double integral  $\iint_D 1 dA$ .

- **Average value of a function over a general region:** If  $f(x, y)$  is integrable over a plane-bounded region  $D$  with positive area  $A(D)$ , then the average value of the function is

$$f_{\text{ave}} = \frac{1}{A(D)} \iint_D f(x, y) dA.$$

Note that the area is  $A(D) = \iint_D 1 dA$ .

- **Fubini's Theorem for Improper Integrals:** If  $D$  is a bounded rectangle or simple region in the plane defined by  $\{(x, y) : a \leq x \leq b, g(x) \leq y \leq h(x)\}$  and also by  $\{(x, y) : c \leq y \leq d, j(y) \leq x \leq k(y)\}$ , and  $f$  is a nonnegative function on  $D$  with finitely many discontinuities in the interior of  $D$ , then

$$\iint_D f dA = \int_{x=a}^{x=b} \int_{y=g(x)}^{y=h(x)} f(x, y) dy dx = \int_{y=c}^{y=d} \int_{x=j(y)}^{x=k(y)} f(x, y) dx dy.$$

**Note:** It is very important to note that we required that the function be nonnegative on  $D$  for the theorem to work. We consider only the case where the function has finitely many discontinuities inside  $D$ .

- **Improper Integrals on an Unbounded Region:** If  $R$  is an unbounded rectangle such as  $R = \{(x, y) : a \leq x < \infty, c \leq y < \infty\}$ , then when the limit exists, we have

$$\begin{aligned} \iint_R f(x, y) dA &= \lim_{(b,d) \rightarrow (\infty, \infty)} \int_a^b \left( \int_c^d f(x, y) dy \right) dx \\ &= \lim_{(b,d) \rightarrow (\infty, \infty)} \int_c^d \left( \int_a^b f(x, y) dx \right) dy. \end{aligned}$$

$$\lim_{(b,d) \rightarrow (\infty, \infty)} \frac{1}{4} (1 - e^{-b^2})(1 - e^{-d^2}).$$

- **Joint density function:** Consider a pair of continuous random variables  $X$  and  $Y$ , such as the birthdays of two people or the number of sunny and rainy days in a month. The joint density function  $f$  of  $X$  and  $Y$  satisfies the probability that  $(X, Y)$  lies in a certain region  $D$ :

$$P((X, Y) \in D) = \iint_D f(x, y) dA.$$

Since the probabilities can never be negative and must lie between 0 and 1, the joint density function satisfies the following inequality and equation:

$$f(x, y) \geq 0 \quad \text{and} \quad \iint_{\mathbb{R}^2} f(x, y) dA = 1.$$

- **Independent random variable classification:** The variables  $X$  and  $Y$  are said to be independent random variables if their joint density function is the product of their individual density functions:

$$f(x, y) = f_1(x)f_2(y).$$

- **Probability theory expected values:** In probability theory, we denote the expected values  $E(X)$  and  $E(Y)$ , respectively, as the most likely outcomes of the events. The expected values  $E(X)$  and  $E(Y)$  are given by

$$E(X) = \iint_S xf(x, y) dA \quad \text{and} \quad E(Y) = \iint_S yf(x, y) dA,$$

where  $S$  is the sample space of the random variables  $X$  and  $Y$ .

- **Double integral in polar coordinates:** The double integral of the function  $f(r, \theta)$  over the polar rectangular region  $R$  in the  $r\theta$ -plane is defined as

$$\begin{aligned} & \iint_R f(r, \theta) dA \\ &= \lim_{m,n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(r_{ij}^*, \theta_{ij}^*) \Delta A \\ &= \lim_{m,n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(r_{ij}^*, \theta_{ij}^*) r_{ij}^* \Delta r \Delta \theta. \end{aligned}$$

- **Iterated integral for polar regions**

$$\begin{aligned} \iint_R f(r, \theta) dA &= \iint_R f(r, \theta) r dr d\theta \\ &= \int_{\theta=\alpha}^{\theta=\beta} \int_{r=a}^{r=b} f(r, \theta) r dr d\theta. \end{aligned}$$

- **Integral when changinig from  $xy$  to  $r\theta$**

$$\iint_R f(x, y) dA = \iint_R f(r \cos(\theta), r \sin(\theta)) r dr d\theta.$$

**NOTICE:** The extra  $r$  at the end of the integrand

- **Double Integrals over General Polar Regions:** If  $f(r, \theta)$  is continuous on a general polar region  $D$  as described above, then the double integral over  $D$  can be expressed as:

$$\iint_D f(r, \theta) r dr d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r, \theta) r dr d\theta$$

- **Area of a polar region**

$$A = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} r dr d\theta.$$

- **Equation of an arbitrary cone with radius  $a$  and height  $h$**

$$z = h - \frac{h}{a} \sqrt{x^2 + y^2}.$$

- **Entire  $xy$  plane region to polar region:** Suppose we have the region  $\mathbb{R}^2$ , ie the entire  $xy$ -plane. This can be seen as

$$0 \leq \theta \leq 2\pi, \quad 0 \leq r < \infty.$$

- **Triple integral:** The triple integral of a function  $f(x, y, z)$  over a rectangular box  $B$  is defined as

$$\lim_{l,m,n \rightarrow \infty} \sum_{i=1}^l \sum_{j=1}^m \sum_{k=1}^n f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta x \Delta y \Delta z = \iiint_B f(x, y, z) dV$$

if this limit exists.

**Note:** When the triple integral exists on  $B$ , the function  $f(x, y, z)$  is said to be integrable on  $B$ . Also, the triple integral exists if  $f(x, y, z)$  is continuous on  $B$ . Therefore, we will use continuous functions for our examples. However, continuity is sufficient but not necessary; in other words,  $f$  is bounded on  $B$  and continuous except possibly on the boundary of  $B$ .

The sample point  $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$  can be any point in the rectangular sub-box  $B_{ijk}$  and all the properties of a double integral apply to a triple integral. **Fubini's Theorem for Triple Integrals** If  $f(x, y, z)$  is continuous on a rectangular box  $B = [a, b] \times [c, d] \times [e, f]$ , then

$$\iiint_B f(x, y, z) dV = \int_e^f \int_c^d \int_a^b f(x, y, z) dx dy dz.$$

This integral is also equal to any of the other five possible orderings for the iterated triple integral.

- **Triple integral over general regions:** The triple integral of a continuous function  $f(x, y, z)$  over a general three-dimensional region

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

in  $\mathbb{R}^3$ , where  $D$  is the projection of  $E$  onto the  $xy$ -plane, is

$$\iiint_E f(x, y, z) dV = \iint_D \left[ \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz \right] dA.$$

- **Volume of a general region  $E$  with triple integrals:** To find the volume of a general region  $E$ , we use

$$\iiint_E dV.$$

- **Average Value of a Function of Three Variables** If  $f(x, y, z)$  is integrable over a solid bounded region  $E$  with positive volume  $V(E)$ , then the average value of the function is

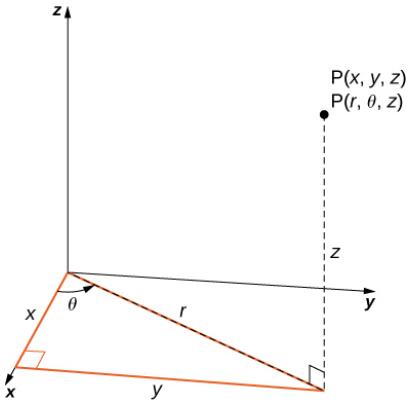
$$f_{\text{ave}} = \frac{1}{V(E)} \iiint_E f(x, y, z) dV.$$

Note that the volume is  $V(E) = \iiint_E 1 dV$ .

- **Equation of a plane with intercepts**  $(a, 0, 0), (0, b, 0), (0, 0, c)$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

- **Cylindrical coordinates:** In three-dimensional space  $\mathbb{R}^3$ , a point with rectangular coordinates  $(x, y, z)$  can be identified with cylindrical coordinates  $(r, \theta, z)$  and vice versa. We can use these same conversion relationships, adding  $z$  as the vertical distance to the point from the  $xy$ -plane as shown in the following figure.



- **Fubini's Theorem in Cylindrical Coordinates:** Suppose that  $g(x, y, z)$  is continuous on a portion of a circular cylinder  $B$ , which when described in cylindrical coordinates looks like  $B = \{(r, \theta, z) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta, c \leq z \leq d\}$ .

Then  $g(x, y, z) = g(r \cos \theta, r \sin \theta, z) = f(r, \theta, z)$  and

$$\iiint_B g(x, y, z) dV = \int_c^d \int_{\alpha}^{\beta} \int_a^b f(r, \theta, z) r dr d\theta dz.$$

**Note:** The iterated integral may be replaced equivalently by any one of the other five iterated integrals obtained by integrating with respect to the three variables in other orders.

- **Cylindrical region is a general solid:** If the cylindrical region over which we have to integrate is a general solid, we look at the projections onto the coordinate planes. Hence the triple integral of a continuous function  $f(r, \theta, z)$  over a general solid region  $E = \{(r, \theta, z) \mid (r, \theta) \in D, u_1(r, \theta) \leq z \leq u_2(r, \theta)\}$  in  $\mathbb{R}^3$ , where  $D$  is the projection of  $E$  onto the  $r\theta$ -plane, is

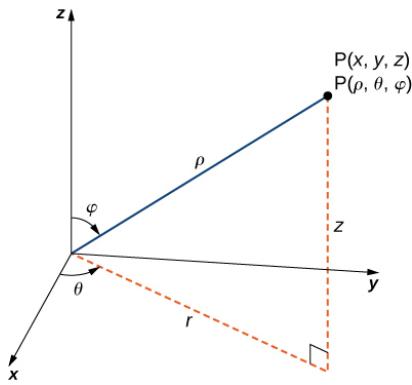
$$\iiint_E f(r, \theta, z) r dr d\theta dz = \iint_D \left[ \int_{u_1(r, \theta)}^{u_2(r, \theta)} f(r, \theta, z) dz \right] r dr d\theta.$$

In particular, if  $D = \{(r, \theta) \mid g_1(\theta) \leq r \leq g_2(\theta), \alpha \leq \theta \leq \beta\}$ , then we have

$$\iiint_E f(r, \theta, z) r dr d\theta dz = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} \int_{z=u_1(r, \theta)}^{z=u_2(r, \theta)} f(r, \theta, z) r dz dr d\theta.$$

Similar formulas exist for projections onto the other coordinate planes. We can use polar coordinates in those planes if necessary.

- **Spherical coordinates:** In three-dimensional space  $\mathbb{R}^3$  in the spherical coordinate system, we specify a point  $P$  by its distance  $\rho$  from the origin, the polar angle  $\theta$  from the positive  $x$ -axis (same as in the cylindrical coordinate system), and the angle  $\phi$  from the positive  $z$ -axis and the line  $OP$ . Note that  $\rho \geq 0$  and  $0 \leq \phi \leq \pi$ .



**Note:** Spherical coordinates are useful for triple integrals over regions that are symmetric with respect to the origin.

- **Rectangular to spherical**

$$x = \rho \sin (\varphi) \cos (\theta)$$

$$y = \rho \sin (\varphi) \sin (\theta)$$

$$z = \rho \cos (\varphi)$$

$$\rho^2 = x^2 + y^2 + z^2$$

$$\tan (\theta) = \frac{y}{x}$$

$$\varphi = \cos^{-1} \left( \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right).$$

- **Spherical to cylindrical**

$$r = \rho \sin (\varphi)$$

$$\theta = \theta$$

$$z = \rho \cos (\varphi).$$

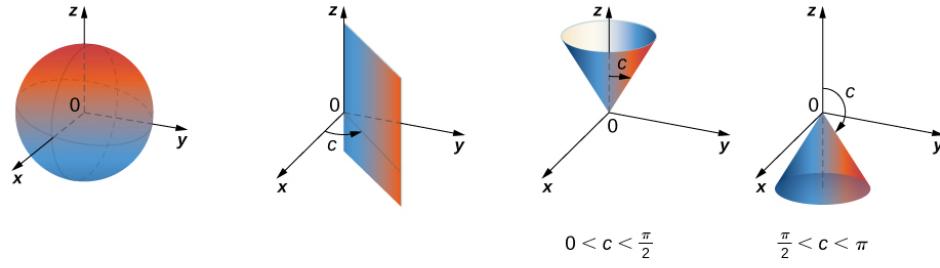
- **Cylindrical to spherical**

$$\rho = \sqrt{r^2 + z^2}$$

$$\theta = \theta$$

$$\varphi = \cos^{-1} \left( \frac{z}{\sqrt{r^2 + z^2}} \right).$$

- **solid regions that are convenient to express in spherical coordinates.**



Sphere  $\rho = c$  (constant)

Half plane  $\theta = c$  (constant)

Half cone  $\varphi = c$  (constant)

- **Fubini's Theorem for Spherical Coordinates:** If  $f(\rho, \theta, \phi)$  is continuous on a spherical solid box  $B = [a, b] \times [\alpha, \beta] \times [\gamma, \psi]$ , then

$$\iiint_B f(\rho, \theta, \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{\phi=\gamma}^{\phi=\psi} \int_{\rho=a}^{\rho=b} f(\rho, \theta, \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$

This iterated integral may be replaced by other iterated integrals by integrating with respect to the three variables in other orders.

- **Spherical region is a general solid:**
- **Jacobian:** The Jacobian of the  $C^1$  transformation  $T(u, v) = (g(u, v), h(u, v))$  is denoted by  $J(u, v)$  and is defined by the  $2 \times 2$  determinant

$$J(u, v) = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \left( \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right).$$

- A transformation  $T : G \rightarrow \mathbb{R}^2$ , defined as  $T(u, v) = (x, y)$ , is said to be a **one-to-one** transformation if no two points map to the same image point.

## 3.7 Chapter 6: Vector calculus

### 3.7.1 Definitions and theorems

- **Vector field:** A vector field is a map of vectors.

A vector field  $\mathbf{F}$  in  $\mathbb{R}^2$  is an assignment of a two-dimensional vector  $\mathbf{F}(x, y)$  to each point  $(x, y)$  of a subset  $D$  of  $\mathbb{R}^2$ . The subset  $D$  is the domain of the vector field.

A vector field  $\mathbf{F}$  in  $\mathbb{R}^3$  is an assignment of a three-dimensional vector  $\mathbf{F}(x, y, z)$  to each point  $(x, y, z)$  of a subset  $D$  of  $\mathbb{R}^3$ . The subset  $D$  is the domain of the vector field.

- **Representing a vector field:** A vector field in  $\mathbb{R}^2$  can be represented in either of two equivalent ways. The first way is to use a vector with components that are two-variable functions:

$$\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle.$$

The second way is to use the standard unit vectors:

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}.$$

A vector field is said to be continuous if its component functions are continuous.

- **Graphing a vector field:** Representing it visually by sketching it is more complex because the domain of a vector field is in  $\mathbb{R}^2$ , as is the range. Therefore, the “graph” of a vector field in  $\mathbb{R}^2$  lives in four-dimensional space. Since we cannot represent four-dimensional space visually, we instead draw vector fields in  $\mathbb{R}^2$  in a plane itself. To do this, draw the vector associated with a given point at the point in a plane. For example, suppose the vector associated with point  $(4, -1)$  is  $\langle 3, 1 \rangle$ . Then, we would draw vector  $\langle 3, 1 \rangle$  at point  $(4, -1)$ .

We should plot enough vectors to see the general shape, but not so many that the sketch becomes a jumbled mess. If we were to plot the image vector at each point in the region, it would fill the region completely and is useless. Instead, we can choose points at the intersections of grid lines and plot a sample of several vectors from each quadrant of a rectangular coordinate system in  $\mathbb{R}^2$ .

- **Radial fields:** Radial fields model certain gravitational fields and energy source fields,

In a **radial field**, all vectors either point directly toward or directly away from the origin. Furthermore, the magnitude of any vector depends only on its distance from the origin. In a radial field, the vector located at point  $(x, y)$  is perpendicular to the circle centered at the origin that contains point  $(x, y)$ , and all other vectors on this circle have the same magnitude.

- **Rotational fields:** Rotational fields model the movement of a fluid in a vortex.

In contrast to radial fields, in a rotational field, the vector at point  $(x, y)$  is tangent (not perpendicular) to a circle with radius  $r = \sqrt{x^2 + y^2}$ . In a standard rotational field, all vectors point either in a clockwise direction or in a counterclockwise direction, and the magnitude of a vector depends only on its distance from the origin.

- **Unit vector field:** A vector field  $\vec{\mathbf{F}}$  is a unit vector field if the magnitude of each vector in the field is one

- **Gradient field:** A vector field  $\mathbf{F}$  in  $\mathbb{R}^2$  or in  $\mathbb{R}^3$  is a gradient field if there exists a scalar function  $f$  such that  $\nabla f = \mathbf{F}$ .
- **Conservative vector field (Gradient field) existence:** A vector field  $\mathbf{F}$  is a conservative vector field, or a gradient field if there exists a scalar function  $f$  such that  $\nabla f = \mathbf{F}$ .
- **Potential functions:**  $f$  is called a potential function for  $\mathbf{F} \iff \nabla f = \mathbf{F}$
- **Uniqueness of Potential Functions:** Let  $\mathbf{F}$  be a conservative vector field on an open and connected domain, and let  $f$  and  $g$  be functions such that  $\nabla f = \mathbf{F}$  and  $\nabla g = \mathbf{F}$ . Then, there is a constant  $C$  such that  $f = g + C$ .
- **The Cross-Partial Property of Conservative Vector Fields:** Let  $\mathbf{F}$  be a vector field in two or three dimensions such that the component functions of  $\mathbf{F}$  have continuous first-order partial derivatives on the domain of  $\mathbf{F}$ .

If  $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$  is a conservative vector field in  $\mathbb{R}^2$ , then  $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ .

If  $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$  is a conservative vector field in  $\mathbb{R}^3$ , then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \text{and} \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}.$$

- **Evaluating a Scalar Line Integral:** Let  $f$  be a continuous function with a domain that includes the smooth curve  $C$  with parameterization  $\mathbf{r}(t)$ ,  $a \leq t \leq b$ . Then

$$\int_C f \, ds = \int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| \, dt.$$

- **Scalar Line Integral Calculation:** Let  $f$  be a continuous function with a domain that includes the smooth curve  $C$  with parameterization  $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ ,  $a \leq t \leq b$ . Then

$$\int_C f(x, y, z) \, ds = \int_a^b f(\mathbf{r}(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt.$$

Similarly,

$$\int_C f(x, y) \, ds = \int_a^b f(\mathbf{r}(t)) \sqrt{(x'(t))^2 + (y'(t))^2} \, dt.$$

if  $C$  is a planar curve and  $f$  is a function of two variables.

- **Independence of parameterization:** integral. Scalar line integrals are independent of parameterization, as long as the curve is traversed exactly once by the parameterization.
- **Arc length of a curve with line integral**

$$S = \int_C ds.$$

- **Vector line integral:** The vector line integral of vector field  $\mathbf{F}$  along oriented smooth curve  $C$  is

$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mathbf{F}(P_i^*) \cdot \mathbf{T}(P_i^*) \Delta s_i$$

if that limit exists.

Which we often write as

$$\begin{aligned} \int_C \mathbf{F} \cdot \mathbf{r}'(t) dt \\ = \int_C \mathbf{F} \cdot d\mathbf{r}. \end{aligned}$$

Where  $d\mathbf{r}$  denotes the differential

$$d\mathbf{r} = \langle x'(t), y'(t), z'(t) \rangle dt.$$

- **Another way to write vector line integral:** Let  $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$  be a two-dimensional vector field. The integral  $\int_C \mathbf{F} \cdot \mathbf{T} ds$  is sometimes written as

$$\int_C P dx + Q dy.$$

- **Orientation reversal:**

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = - \int_C \mathbf{F} \cdot d\mathbf{r}.$$

That is, reversing the orientation of a curve changes the sign of a line integral.

- **Flux:** The flux of  $\mathbf{F}$  across  $C$  is the line integral

$$\int_C \mathbf{F} \cdot \frac{\mathbf{n}(t)}{\|\mathbf{n}\|(t)} ds.$$

If  $\mathbf{F}$  is a velocity field of a fluid and  $C$  is a curve that represents a membrane, then the *flux* of  $\mathbf{F}$  across  $C$  is the quantity of fluid flowing across  $C$  per unit time, or the *rate of flow*.

- **Alternate way to write flux**

$$\int_C -Q dx + P dy.$$

- **Circulation of a vector field:** The line integral of vector field  $\mathbf{F}$  along an oriented closed curve is called the circulation of  $\mathbf{F}$  along  $C$ . Circulation line integrals have their own notation:

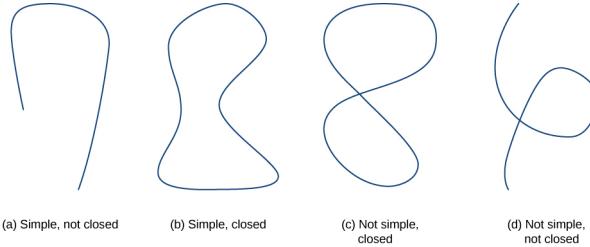
$$\oint_C \mathbf{F} \cdot \mathbf{T} ds.$$

The circle on the integral symbol denotes that  $C$  is “circular” in that it has no endpoints.

the value of the circulation  $\oint \mathbf{v} \cdot \mathbf{T} ds$  measures the tendency of the fluid to move in the direction of  $C$ .

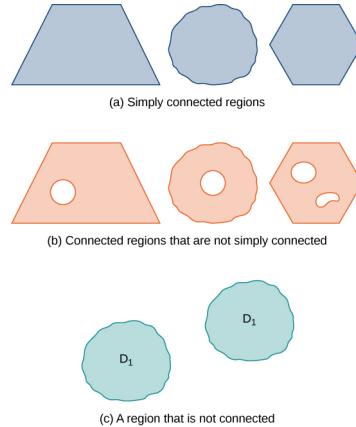
- Curve  $C$  is a **closed curve** if there is a parameterization  $\mathbf{r}(t)$ ,  $a \leq t \leq b$  of  $C$  such that the parameterization traverses the curve exactly once and  $\mathbf{r}(a) = \mathbf{r}(b)$ .
- Curve  $C$  is a **simple curve** if  $C$  does not cross itself. That is,  $C$  is simple if there exists a parameterization  $\mathbf{r}(t)$ ,  $a \leq t \leq b$  of  $C$  such that  $\mathbf{r}$  is one-to-one over  $(a, b)$ .

It is possible for  $\mathbf{r}(a) = \mathbf{r}(b)$ , meaning that the simple curve is also closed.



- A region  $D$  is a **connected region** if, for any two points  $P_1$  and  $P_2$ , there is a path from  $P_1$  to  $P_2$  with a trace contained entirely inside  $D$ .
- A region  $D$  is a **simply connected region** if  $D$  is connected and for any simple closed curve  $C$  that lies inside  $D$ , the curve  $C$  can be shrunk continuously to a point while staying entirely inside  $D$ .

All simply connected regions are connected, but not all connected regions are simply connected



- **The fundamental theorem for line integrals:** Let  $C$  be a piecewise smooth curve with parameterization  $\mathbf{r}(t), a \leq t \leq b$ .

Let  $f$  be a function of two or three variables with first-order partial derivatives that exist and are continuous on  $C$ . Then,

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

- **Path independent:** Let  $\mathbf{F}$  be a vector field with domain  $D$ . The vector field  $\mathbf{F}$  is *independent of path* (or path independent) if

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

for any paths  $C_1$  and  $C_2$  in  $D$  with the same initial and terminal points.

- **Path Independence of Conservative Fields:** If  $\mathbf{F}$  is a conservative vector field, then  $\mathbf{F}$  is independent of path.
- **The Path Independence Test for Conservative Fields:** If  $\mathbf{F}$  is a continuous vector field that is independent of path and the domain  $D$  of  $\mathbf{F}$  is open and connected, then  $\mathbf{F}$  is conservative.
- **Problem-Solving Strategy: Finding a Potential Function for a Conservative Vector Field**
  1. Integrate  $P$  with respect to  $x$ . This results in a function of the form  $g(x, y) + h(y)$ , where  $h(y)$  is unknown.
  2. Take the partial derivative of  $g(x, y) + h(y)$  with respect to  $y$ , which results in the function  $g_y(x, y) + h'(y)$ .
  3. Use the equation  $g_y(x, y) + h'(y) = Q(x, y)$  to find  $h'(y)$ .
  4. Integrate  $h'(y)$  to find  $h(y)$ .
  5. Any function of the form  $f(x, y) = g(x, y) + h(y) + C$ , where  $C$  is a constant, is a potential function for  $\mathbf{F}$ .

- **The Cross-Partial Test for Conservative Fields:** If  $\mathbf{F} = \langle P, Q, R \rangle$  is a vector field on an open, simply connected region  $D$  and the conditions

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \text{and} \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

hold throughout  $D$ , then  $\mathbf{F}$  is conservative.

- **Cross-Partial Property of Conservative Fields:** Let  $\mathbf{F} = \langle P, Q, R \rangle$  be a vector field on an open, simply connected region  $D$ . Then  $\mathbf{F}$  is conservative if and only if the following conditions are satisfied throughout  $D$ :

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \text{and} \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

- **Green's Theorem, Circulation Form:** Let  $D$  be an open, simply connected region with a boundary curve  $C$  that is a piecewise smooth, simple closed curve oriented counterclockwise. Let  $\mathbf{F} = \langle P, Q \rangle$  be a vector field with component functions that have continuous partial derivatives on  $D$ . Then,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P dx + Q dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

**Note:** Greens thoerem can only be used for a two-dimensional vector field

- **Green's Theorem, Flux Form:** Let  $D$  be an open, simply connected region with a boundary curve  $C$  that is a piecewise smooth, simple closed curve that is oriented counterclockwise. Let  $\mathbf{F} = \langle P, Q \rangle$  be a vector field with component functions that have continuous partial derivatives on an open region containing  $D$ . Then,

$$\oint_C \mathbf{F} \cdot \mathbf{N} ds = \iint_D (P_x + Q_y) dA.$$

- A **source-free field** is a field with a flux that is zero along any closed curve
- **Source-free field  $\mathbf{F} = \langle P, Q \rangle$  on a simply connected domain**

1. The flux  $\oint_C \mathbf{F} \cdot \mathbf{N} ds$  across any closed curve  $C$  is zero.

2. If  $C_1$  and  $C_2$  are curves in the domain of  $\mathbf{F}$  with the same starting points and endpoints, then  $\int_{C_1} \mathbf{F} \cdot \mathbf{N} ds = \int_{C_2} \mathbf{F} \cdot \mathbf{N} ds$ . In other words, flux is independent of path.
3. There is a stream function  $g(x, y)$  for  $\mathbf{F}$ . A stream function for  $\mathbf{F} = \langle P, Q \rangle$  is a function  $g$  such that  $P = g_y$  and  $Q = -g_x$ . Geometrically,  $\mathbf{F}(a, b)$  is tangential to the level curve of  $g$  at  $(a, b)$ . Since the gradient of  $g$  is perpendicular to the level curve of  $g$  at  $(a, b)$ , stream function  $g$  has the property  $\mathbf{F}(a, b) \cdot \nabla g(a, b) = 0$  for any point  $(a, b)$  in the domain of  $g$ . (Stream functions play the same role for source-free fields that potential functions play for conservative fields.)
4.  $P_x + Q_y = 0$