Nate Warner MATH 353 Spring 2025

Problem set 9 - Due: Monday, March 24

1. Assume $\omega < \infty$. Show that if A^* is the antipode of A and B is any other point, then A-B-A^* and $BA^* = \omega - AB$

Proof. Assume $\omega < \infty$. Let A^* be the antipode of A in $\mathbb{P}$. Let B be any other point.

Let m be the line that contains B and A. By theorem 10.8, every line through A goes through A^* as well. Thus, $A^* \in m$. So, A, A^*, B are distinct, collinear, and by theorem 9.1, A-B-A^*

Since A-B-A^*, we have $AB + BA^* = AA^* = \omega$. Thus, $BA^* = \omega - AB$ ■

5. Suppose P is a point not on a line $m = \overleftrightarrow{AB}$, and suppose X and Y are points with A-X-P nd P-B-Y. Show that $XY < \omega$ and that $\overline{XY}$ meets m

Proof. Assume P be a point not on a line $m = \overleftrightarrow{AB}$, and let X, Y be points with A-X-P and P-B-Y.

By Ax.S, line m there exists a pair of opposite halfplanes with edge m, call them H, K. Let H be the halfplane that contains P. Let $\overleftrightarrow{AP}$ be the line through A, and P. Note that since A-X-P, $A, X \in \overleftrightarrow{AP}$. Observe that A, B, P are three noncollinear points. Thus, By proposition noncollinear, each of $AB, AP, BP < \omega$. Thus, $\overleftrightarrow{AP}$ is the unique line through A and X. Hence, $x \notin m$.

Notice that since P-B-Y, $\overrightarrow{BP}, \overrightarrow{BY}$ are opposite rays by Thm 9.6. Since $P \in H$, $B \in m$ Thm 10.3 tell us that $\text{Int}\overrightarrow{BP} \subseteq H$, $\text{Int}\overrightarrow{BY} \subseteq K$. Since $Y \in \text{Int}\overrightarrow{BY}, Y \in K$.

Further, note that $X \in \overrightarrow{AP}$ by definition of A-X-P. Since $A \in m$, $P \in H$, Thm 10.3 suggests $\text{Int}\overrightarrow{AP} \subseteq H$. Since $X \in \text{Int}\overrightarrow{AP}$, X is therefore a member of H.

Thus, we have $X \in H$, $Y \in K$. We noted previously that $X \in \overleftrightarrow{AP}$, which is the unique line through A, X and hence the only line that contains A, X. But what about Y?

First, since P-B-Y, $Y \in \overrightarrow{PB}$. Call the carrier of this ray $\overleftrightarrow{PB}$. We saw above that by proposition noncollinear, $PB < \omega$. Thus, $\overleftrightarrow{PB}$ is the unique line through P, B, and hence the unique line through P, Y. Thus, Y is contained only in this line. Since $\overleftrightarrow{AP} \neq \overleftrightarrow{PB}$, X, P, Y are three noncollinear points, and by proposition noncollinear, $XY < \omega$.

Since $X \in H$, $Y \in K$, $XY < \omega$. By the definition of opposite halfplanes with edge a line m, $\overline{XY} \cap m \neq \varnothing$. Thus, $\overline{XY}$ meets m ■

6. Let m be a line and P, Q points such that $P \notin m$, $PQ = 1$, and $PX \geqslant 2$ for all X on m. Prove that P and Q lie on the same side of m.

Proof. Assume m is a line, and P, Q are points such that $P \notin m$, $PQ = 1$, and $PX \geqslant 2$ for all X on m

By Ax.S, there exists a pair of opposite halfplanes with edge m, call them H, K. Let H be the halfplane that contains P. That is, $P \in H$.

Assume for the sake of contradiction that $Q \in m$. Since $Q \in m$, $PQ \geqslant 2$, which contradicts $PQ = 1$. Thus, $Q \notin m$.

Further assume that $Q \in K$. That is, P, Q on opposite sides of m. Then, by theorem 10.6, there exists an $X \in m$ such that P-X-Q, which implies $PX + XQ = PQ$, and thus $PX < PQ$, which again is a contradiction since $PX \geqslant 2$ and $PQ = 1$.

Thus, Q must also lie in H, and P, Q are therefore both on the same side of m ■

8. Prove Theorem 10.10

Remark. *(Theorem 10.10 (Pasch's theorem))* : Let A, B, C be three noncollinear points. Let X be a point with B-X-C, and m a line through X but not through A, B, or C. Then, exactly one of

1. m contains a point Y with A-Y-C
2. m contains a point Z with A-Z-B

Proof. Let A, B, C be three noncollinear points. Let X be a point with B-X-C, and m a line through X but not through A, B, or C.

First, we observe that since A, B, C are three noncollinear points, each of $AB, AC, BC < \omega$

By Ax.S, m there exists a pair of opposite halfplanes H, K with edge m. Since m does not go through A, A must lie in one of the halfplanes. Without loss of generality, assume $A \in H$

Consider B, C, since B-X-C, $X \in m$ we conclude by theorem 10.6 that B, C lie in opposite sides of m. Thus, either B with A in H, or C with A in H.

First, consider B with A in H. Thus, $C \in K$. Since $A \in H$, $C \in K$, $AC < \omega$, we have by the definition of opposite halfplanes, $\overline{AC} \cap m \neq \varnothing$. Thus, $\overline{AC}$ intersects m, call the point of intersection Y. By the definition of the intersection, $Y \in \overline{AC} \cap m$. Thus, $Y \in \overline{AC}$. By the definition of the segment $\overline{AC}$, A-Y-C.

Note that since $B \in H$ with A, this argument does not hold for the segment $\overline{AB}$, and we can generate no such point Z such that A-Z-B

The same argument but with $A, C \in H$, $B \notin H \implies B \in K$ generates a point Z such that A-Z-B. ■