Nate Warner MATH 434 Fall 2025

Problem set 2 - Due: Wednesday, October 15

1.7.10. Let

$$A = \begin{bmatrix} 2 & 1 & -1 & 3 \\ -2 & 0 & 0 & 0 \\ 4 & 1 & -2 & 6 \\ -6 & -1 & 2 & -3 \end{bmatrix}, \qquad b = \begin{bmatrix} 13 \\ -2 \\ 24 \\ -14 \end{bmatrix}.$$

(a) Calculate the appropriate (four) determinants to show that A can be transformed to (nonsingular) upper-triangular form by operations of type 1 only. (By the way, this is strictly an academic exercise. In practice one never calculates these determinants in advance.)

(b) Carry out the row operations of type 1 to transform the system $Ax = b$ to an equivalent system $Ux = y$, where U is upper triangular. Save the multipliers for use in Exercise 1.7.18.

(c) Carry out the back substitution on the system $Ux = y$ to obtain the solution of $Ax = b$. Don't forget to check your work.

Remark. Let $A \in \mathbb{R}^{n\times n}$. A admits an LU factorization $A = LU$ where L is unit lower triangular and U is upper triangular if and only if all leading principal submatrices are nonsingular. ■

a.) So, we check that

$$\det\left([2]\right),\ \det\left(\begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix}\right),\ \det\left(\begin{bmatrix} 2 & 1 & -1 \\ -2 & 0 & 0 \\ 4 & 1 & -2 \end{bmatrix}\right),\ \det\left(\begin{bmatrix} 2 & 1 & -1 & 3 \\ -2 & 0 & 0 & 0 \\ 4 & 1 & -2 & 6 \\ -6 & -1 & 2 & -3 \end{bmatrix}\right)$$

are all nonzero. We see that

$$\begin{aligned}
\det\left([2]\right) &= 2 \neq 0, \\
\det\left(\begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix}\right) &= 2(0) - (1)(-2) = 2 \neq 0, \\
\det\left(\begin{bmatrix} 2 & 1 & -1 \\ -2 & 0 & 0 \\ 4 & 1 & -2 \end{bmatrix}\right) &= -1 \cdot -2(1(-2) - (-1)(1)) = -1 \cdot 2(-2+1) = -2 \neq 0, \\
\det\left(\begin{bmatrix} 2 & 1 & -1 & 3 \\ -2 & 0 & 0 & 0 \\ 4 & 1 & -2 & 6 \\ -6 & -1 & 2 & -3 \end{bmatrix}\right) &= -1 \cdot -2 \det\left(\begin{bmatrix} 1 & -1 & 3 \\ 1 & -2 & 6 \\ -1 & 2 & -3 \end{bmatrix}\right) \\
&= 2\left(1(-2(-3) - 6(2)) - (-1)(1(-3) - 6(-1)) + 3(1(2) - (-2)(-1))\right) \\
&= 2(-3) = -6 \neq 0.
\end{aligned}$$

Thus, all leading principal submatrices are nonsingular and A can be transformed to nonsingular upper-triangular form by operations of type 1 only.

b.) We use Gaussian Elimination on the augmented system $[A|b] \to [U|y]$. We have

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 3 & 13 \\ -2 & 0 & 0 & 0 & -2 \\ 4 & 1 & -2 & 6 & 24 \\ -6 & -1 & 2 & -3 & 14 \end{array}\right].$$

The opertions to get $a_{21} = a_{31} = a_{41} = 0$ are

$$\begin{aligned} &- (-1)r_1 + r_2 \to r_2' \\ &- 2r_1 + r_3 \to r_3' \\ &- (-3)r_1 + r_4 \to r_4'. \end{aligned}$$

Thus, $m_{21} = -1$, $m_{31} = 2$, $m_{41} = -3$ and the system becomes

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 3 & 13 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 1 & -2 & 6 & 24 \\ 0 & -1 & 2 & -3 & 14 \end{array}\right].$$

1.7.18. Solve the linear system $Ax = \hat{b}$, where A is as in Exercise 1.7.10 and

$$\hat{b} = \begin{bmatrix} 12 \\ -8 \\ 21 \\ -26 \end{bmatrix}^T.$$

Use the L and U that you calculated in Exercise 1.7.10.

1.7.26. Use the inner-product formulation to calculate the LU decomposition of the matrix A in Exercise 1.7.10

1.7.34. In this exercise you will show that performing an elementary row operation of type 1 is equivalent to left multiplication by a matrix of a special type. Suppose $\tilde{A}$ is obtained from A by adding m times the jth row to the ith row.

1. Show that $\tilde{A} = MA$, where M is the triangular matrix obtained from the identity matrix by replacing the zero by an m in the (i, j) position. For example, when $i > j$, M has the form

$$M = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & m & 1 & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix}.$$

   Notice that this is the matrix obtained by applying the type 1 row operation directly to the identity matrix. We call M an *elementary matrix of type 1*.

2. Show that $\det(M) = 1$ and $\det(\tilde{A}) = \det(A)$. Thus we see (again) that $\tilde{A}$ is nonsingular if and only if A is.

3. Show that M^{-1} differs from M only in that it has $-m$ instead of m in the (i, j) position. M^{-1} is also an elementary matrix of type 1. To which elementary operation does it correspond?

1.7.36. Suppose $\tilde{A}$ is obtained from A by multiplying the ith row by the nonzero constant c.

1. Find the form of the matrix M (an *elementary matrix of type 3*) such that $\tilde{A} = MA$.
2. Find M^{-1} and state its function as an elementary matrix.
3. Find $\det(M)$ and determine the relationship between $\det(\tilde{A})$ and $\det(A)$. Deduce that $\tilde{A}$ is nonsingular if and only if A is.

1.8.4. Let

$$A = \begin{bmatrix} 2 & 2 & -4 \\ 1 & 1 & 5 \\ 1 & 3 & 6 \end{bmatrix}, \qquad b = \begin{bmatrix} 10 \\ -2 \\ -5 \end{bmatrix}.$$

Use Gaussian elimination with partial pivoting (by hand) to find matrices L and U such that U is upper triangular, L is unit lower triangular with $|l_{ij}| \leqslant 1$ for all $i > j$, and $LU = \tilde{A}$, where $\tilde{A}$ can be obtained from A by making row interchanges. Use your LU decomposition to solve the system $Ax = b$.

1.8.9. Let A be the matrix in Exercise 1.8.4. Determine matrices P, L, and U with the properties stated in Theorem 1.8.8, such that $A = P^T LU$

1.8.12. Write an algorithm that implements Gaussian elimination with partial pivoting. Store L and U over A, and save a record of the row interchanges.

2.1.10. Prove that the 1-norm is a norm.

2.1.13. Prove that the ∞-norm is a norm.

2.1.17.

1. Let A be a positive definite matrix, and let R be its Cholesky factor, so that $A = R^T R$. Verify that for all $x \in \mathbb{R}^n$,
$$\|x\|_A = \|Rx\|_2.$$
2. Using the fact that the 2-norm is indeed a norm on $\mathbb{R}^n$, prove that the A-norm is a norm on $\mathbb{R}^n$.

2.2.6.

(a) Show that $\kappa(A) = \kappa(A^{-1})$

(b) Show that for any nonzero scalar c, $\kappa(cA) = \kappa(A)$

2.2.15. Let us take another look at the ill-conditioned matrices

$$A = \begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} -998 & 999 \\ 999 & -1000 \end{bmatrix}$$

from Example 2.2.8. Notice that

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1999 \\ 1997 \end{bmatrix}. \tag{2.2.16}$$

If we use the ∞-norm to measure lengths, the magnification factor

$$\frac{\|Ax\|_\infty}{\|x\|_\infty}$$

is 1999, which equals $\|A\|_\infty$. Thus $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a vector that is magnified maximally by A.

Since the amount by which a vector is magnified depends only on its direction and not on its length, we say that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in a *direction of maximum magnification* by A.

Equivalently we can say that $\begin{bmatrix} 1999 \\ 1997 \end{bmatrix}$ lies in a *direction of minimum magnification.*

Looking now at A^{-1}, we note that

$$A^{-1} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1997 \\ -1999 \end{bmatrix}.$$

The magnification factor $\frac{\|A^{-1}x\|_\infty}{\|x\|_\infty}$ is 1999, which equals $\|A^{-1}\|_\infty$, so $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is in a direction of maximum magnification by A^{-1}. Equivalently

$$A \begin{bmatrix} 1997 \\ -1999 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \tag{2.2.17}$$

and $\begin{bmatrix} 1997 \\ -1999 \end{bmatrix}$ is in a direction of minimum magnification by A.

We will use the vectors in (2.2.16) and (2.2.17) to construct a spectacular example. Suppose we wish to solve the system

$$\begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1999 \\ 1997 \end{bmatrix}. \tag{2.2.18}$$

That is, $Ax = b$, where $b = \begin{bmatrix} 1999 \\ 1997 \end{bmatrix}$. Then by (2.2.16) the unique solution is

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Now suppose that we solve instead the slightly perturbed system

$$\begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 1998.99 \\ 1997.01 \end{bmatrix}. \tag{2.2.19}$$

This is $\hat{A}x = b + \delta b$, where $\delta b = \begin{bmatrix} -0.01 \\ 0.01 \end{bmatrix} = 0.01 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, which is in a direction of maximum magnification by A^{-1}. By (2.2.17), $A\delta x = \delta b$, where $\delta x = \begin{bmatrix} 19.97 \\ -19.99 \end{bmatrix}$. Therefore $\hat{x} = x + \delta x = \begin{bmatrix} 20.97 \\ -18.99 \end{bmatrix}$. Thus the nearly identical problems (2.2.18) and (2.2.19) have very different solutions.

Repeat the proof of Theorem 2.3.3.

Remark. Theorem 2.3.3. Let A be nonsingular, let $b \neq 0$, and let x and $\hat{x} = x + \delta x$ be solutions of $Ax = b$ and $(A + \delta A)\hat{x} = b$, respectively. Then,

$$\frac{\|\delta x\|}{\|\hat{x}\|} \leqslant \kappa(A) \frac{\|\delta A\|}{\|A\|}. \tag{2.3.4}$$

Proof. Rewriting the equation $(A + \delta A)\hat{x} = b$ as $Ax + A\delta x + \delta A\hat{x} = b$, using the equation $Ax = b$, and reorganizing the resulting equation, we obtain

$$\delta x = -A^{-1}\delta A\hat{x}.$$

Thus

$$\|\delta x\| \leqslant \|A^{-1}\| \, \|\delta A\| \, \|\hat{x}\|. \tag{2.3.5}$$

Dividing through by $\|\hat{x}\|$ and using the definition $\kappa(A) = \|A\| \, \|A^{-1}\|$, we obtain the desired result.