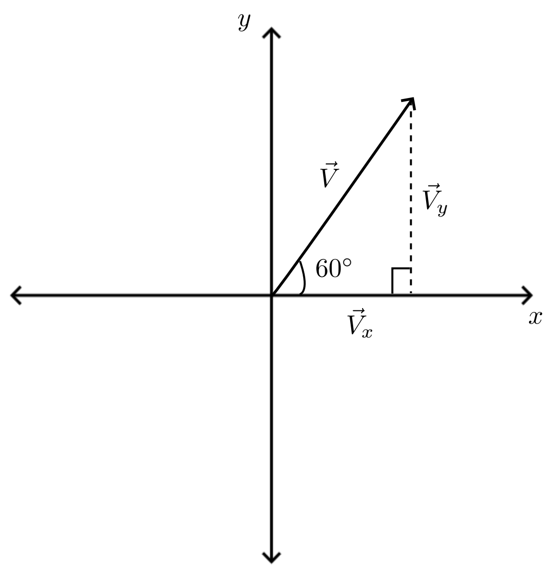
Homework/Worksheet 3 - Due: Wednesday, February 7

1. A football thrown by a quarterback has an initial speed of 70 mph and an angle of elevation of 60°. Determine the velocity vector in mph and express it in component form. (Round to two decimal places.)

First, lets construct a figure



Since we know $\|\vec{V}\| = 70mph$, we can use properties of right triangles to find $\vec{V}_x$ and $\vec{V}_y$

$$\cos\theta = \frac{\text{opp}}{\text{hyp}} \implies \text{hyp}\cos\theta = \text{opp}$$
$$\sin\theta = \frac{\text{adj}}{\text{hyp}} \implies \text{hyp}\sin\theta = \text{adj}.$$

With these findings, we can find the components of our velocity vector.

$$\vec{V}_x = \|\vec{V}\|\cos\theta = 70mph\cos 60° = 35mph$$
$$\vec{V}_y = \|\vec{V}\|\sin\theta = 70mph\sin 60° = 60.62mph.$$

Conclusion. Thus, the components for the velocity vector are

$$\vec{V} = (35\hat{\imath} + 60.62\hat{\jmath}) \text{ mph}.$$

Where $\hat{\imath}$ is the unit vector along the positive x-axis, and $\hat{\jmath}$ is the unit vector along the positive y-axis.

2. Let $\mathbf{u} = \langle 1, 1, 0\rangle$, $\mathbf{v} = \langle 0, 1, -1\rangle$. Find the magnitude of the vectors $\mathbf{u} - \mathbf{v}$ and $-2\mathbf{v}$.

To find $\vec{u}$ - $\vec{v}$, we simply subtract their components.

$$\begin{aligned}\vec{u} - \vec{v} &= \langle 1-0, 1-1, 0-(-1)\rangle \\ &= \langle 1, 0, 1\rangle.\end{aligned}$$

From this, we can find $\|\vec{u} - \vec{v}\|$

$$\begin{aligned}\|\vec{u} - \vec{v}\| &= \sqrt{1^2 + 0^2 + 1^2} \\ &= \sqrt{2}.\end{aligned}$$

Next, we find the vector corresponding to $-2\vec{v}$

By this, we have

$$\begin{aligned}-3\vec{v} &= \langle -3(0), -3(1), -3(-1)\rangle \\ &= \langle 0, -3, 3\rangle \\ &\implies \|\langle 0, -3, 3\rangle\| = \sqrt{0^2 + (-3)^2 + 3^2} \\ &= \sqrt{18} = 3\sqrt{2}.\end{aligned}$$

3. Find a vector $\vec{u}$ in the same direction of the vector $\vec{v} = \langle 2, 4, 1\rangle$, whose magnitude is 15, that is, $\|\vec{u}\| = 15$

First, we find $\|\vec{v}\|$

$$\begin{aligned}\|\vec{v}\| &= \sqrt{2^2 + 4^2 + 1^2} \\ &= \sqrt{21}.\end{aligned}$$

Next, we find $\hat{u}$ in the direction of $\vec{v}$

$$\begin{aligned}\hat{u} &= \frac{1}{\|\vec{v}\|}\vec{v} \\ &= \frac{1}{\sqrt{21}}\langle 2, 4, 1\rangle \\ &= \left\langle \frac{2}{\sqrt{21}}, \frac{4}{\sqrt{21}}, \frac{1}{\sqrt{21}} \right\rangle.\end{aligned}$$

This gives us a vector $\hat{u}$ in the direction of $\vec{v}$ with magnitude 1. When then manipulate $\hat{u}$ s.t the magnitude becomes 15. To do this, we multiply by a scalar of 15.

$$\begin{aligned}15\hat{u} &= 15\left\langle \frac{2}{\sqrt{21}}, \frac{4}{\sqrt{21}}, \frac{1}{\sqrt{21}} \right\rangle \\ &= \left\langle \frac{30}{\sqrt{21}}, \frac{60}{\sqrt{21}}, \frac{15}{\sqrt{21}} \right\rangle.\end{aligned}$$

Conclusion. The vector $\vec{u}$ in the direction of $\vec{v}$ with magnitude 15 is

$$\left\langle \frac{30}{\sqrt{21}}, \frac{60}{\sqrt{21}}, \frac{15}{\sqrt{21}} \right\rangle.$$

4. Find the angle between the vectors $\vec{a} = \langle 0, -1, -3\rangle$ and $\vec{b} = \langle 2, 3, -1\rangle$

For this, we use the formula

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}.$$

Which gives

$$\begin{aligned}\cos\theta &= \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}\\ \cos\theta &= \frac{-3+3}{\|\vec{a}\|\|\vec{b}\|}\\ \theta &= \cos^{-1} 0\\ &= \frac{\pi}{2} = 90^\circ.\end{aligned}$$

Conclusion. Thus, the angle between these two vectors is $\frac{\pi}{2} = 90^\circ$

5. Let $\vec{u} = \langle 2, 4, 0\rangle$ and $\vec{v} = \langle 0, 4, 2\rangle$

   (a) Find the component form of vector $w = \text{proj}_{\vec{u}}\vec{v}$ that represents the projection of $\vec{v}$ onto $\vec{u}$.

   (b) Write the decomposition $\vec{v} = \vec{w} + \vec{q}$, where $\vec{w}$ is the projection of $\vec{v}$ onto $\vec{u}$ and $\vec{q}$ is a vector orthogonal to the direction of $\vec{u}$

First, we find $\vec{w} = \text{proj}_{\vec{u}}\vec{v}$

$$\begin{aligned}\vec{w} = \text{proj}_{\vec{u}}\vec{v} &= \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\vec{u}\\ &= \frac{16}{20}\langle 2, 4, 0\rangle\\ &= \left\langle \frac{32}{20}, \frac{64}{20}, 0\right\rangle\\ &= \left\langle \frac{8}{5}, \frac{16}{5}, 0\right\rangle.\end{aligned}$$

Next, we define $\vec{q} = \vec{v} - \vec{w}$. Where $\vec{q}$ is the vector orthogonal to the direction of $\vec{u}$

$$\begin{aligned}\vec{q} &= \left\langle 0 - \frac{8}{5}, 4 - \frac{16}{5}, 2 - 0\right\rangle\\ &= \left\langle -\frac{8}{5}, \frac{4}{5}, 2\right\rangle.\end{aligned}$$

Conclusion. The projection of $\vec{v}$ onto $\vec{u}$ is given by $\vec{w} = \left\langle \frac{8}{5}, \frac{16}{5}, 0\right\rangle$, and the vector orthogonal to the direction of $\vec{u}$ is given by $\vec{q} = \left\langle -\frac{8}{4}, \frac{4}{5}, 2\right\rangle$

6. Let $A(2,-3,4)$, $B(0,1,2)$, $C(-1,2,0)$

   (a) Find a vector orthogonal to both $\vec{u} = \vec{AB}$ and $\vec{v} = \vec{AC}$

   (b) Find the area of parallelogram $ABCD$ with adjacent sides $\vec{AB}$ and $\vec{AC}$

   (c) Find the area of the triangle ABC.

First, we find the vectors $\vec{u} = \vec{AB}$ and $\vec{v} = \vec{AC}$

$$\begin{aligned}\vec{u} &= \langle 0-2, 1-(-3), 2-4\rangle \\ &= \langle -2, 4, -2\rangle \\ \vec{v} &= \langle -1-2, 2-(-3), 0-4\rangle \\ &= \langle -3, 5, -4\rangle .\end{aligned}$$

To find a vector orthogonal to both $\vec{u}$ and $\vec{v}$, we compute the cross product $\vec{u} \times \vec{v}$

$$\begin{aligned}\vec{u} \times \vec{v} &= \langle u_y v_z - u_z v_y, u_x v_z - u_z v_x, u_x v_y - u_y v_x\rangle \\ &= \langle 4(-4) - (-2)(5), (-2)(-4) - (-2)(-3), (-2)(5) - 4(-3)\rangle \\ &= \langle -16+10, 8-6, -10+12\rangle \\ &= \langle -6, 2, 2\rangle .\end{aligned}$$

Next, we find the area of parallelogram $ABCD$ with adjacent sides $\vec{AB}$ and $\vec{AC}$ by finding the magnitude of the cross product.

$$\begin{aligned}\|\vec{u} \times \vec{v}\| &= \sqrt{(-6)^2 + 2^2 + 2^2} \\ &= \sqrt{44}.\end{aligned}$$

Finally, the area of a triangle formed by the two vectors is half the area of the parallelogram formed by the same vectors. Thus we have

$$\frac{1}{2}\sqrt{44} = \frac{\sqrt{44}}{2}.$$