Homework/Worksheet 4 - Due: Friday, February 23

1. Find the domain of the vector function $\mathbf{r}(t) = t^2\ \mathbf{i} + \sqrt{t-3}\ \mathbf{j} + \frac{3}{2t+1}\ \mathbf{k}$.

First, we define

$$x(t) = t^2$$
$$y(t) = \sqrt{t-3}$$
$$z(t) = \frac{3}{2t+1}.$$

Next, next define the domains of each function. The intersection of the domains will be the domain of $\vec{\mathbf{r}}(t)$

$$d(x(t)) = \mathbb{R}$$
$$d(y(t)) = t - 3 \geqslant 0 \implies t \geqslant 3$$
$$d(z(t)) = 2t + 1 \neq 0 \implies t \neq -\frac{1}{2}.$$

Conclusion 1. From this, we have the overall domain

$$d(\vec{\mathbf{r}}(t)) : \ \{t \mid t \geqslant 3\}.$$

2. Evaluate the limit $\lim_{t\to 1}\left(\frac{t^2-1}{t-1}\mathbf{i} + \sqrt{t+3}\mathbf{j} + \frac{\sin(\pi t)}{\ln t}\mathbf{k}\right)$.

Remark. Let f, g, and h be functions of t. The limit of the vector-valued function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ as t approaches a is given by

$$\lim_{t\to a}\mathbf{r}(t) = \left[\lim_{t\to a} f(t)\right]\mathbf{i} + \left[\lim_{t\to a} g(t)\right]\mathbf{j} + \left[\lim_{t\to a} h(t)\right]\mathbf{k},$$

provided the limits $\lim_{t\to a} f(t)$, $\lim_{t\to a} g(t)$, and $\lim_{t\to a} h(t)$ exist.

Thus, if we define

$$f(t) = \frac{t^2-1}{t-1}$$
$$g(t) = \sqrt{t+3}$$
$$h(t) = \frac{\sin(\pi t)}{\ln(t)}.$$

We can find the limit of each function as $t \to 1$

$$\lim_{t\to 1} f(t) = \lim_{t\to 1}\frac{t^2-1}{t-1} = \lim_{t\to 1}\frac{(t-1)(t+1)}{t-1} = \lim_{t\to 1} t+1 = 2$$
$$\lim_{t\to 1} g(t) = \lim_{t\to 1}\sqrt{t+3} = \sqrt{4} = 2$$
$$\lim_{t\to 1} h(t) = \lim_{t\to 1}\frac{\sin(\pi t)}{\ln(t)} \stackrel{H}{=} \lim_{t\to 1}\frac{\pi\cos(\pi t)}{\frac{1}{t}} = \lim_{t\to 1}\pi t\cos(\pi t) = -\pi.$$

Conclusion 2. Thus, we have the limit for the vector function

$$\lim_{t\to 1} \vec{\mathbf{r}}(t) = 2\ \hat{\mathbf{i}} + 2\ \hat{\mathbf{j}} - \pi\ \hat{\mathbf{k}}.$$

3. Find the derivative of the vector function $\mathbf{r}(t) = te^t\mathbf{i} + t\ln t\mathbf{j} + \sin(3t)\mathbf{k}$.

Remark. Let f, g, and h be differentiable functions of t. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then $\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$.

Thus, we define

$$\begin{aligned} f(t) &= te^t \\ g(t) &= t\ln(t) \\ h(t) &= \sin(3t). \end{aligned}$$

Next, we find the derivative with respect to t of each function

$$\begin{aligned} f'(t) &= te^t + e^t \\ g'(t) &= t \cdot \frac{1}{t} + \ln(t) = 1 + \ln(t) \\ h'(t) &= 3\cos(3t). \end{aligned}$$

Conclusion 3. Therefore we have

$$\vec{\mathbf{r}}'(t) = \left(te^t + e^t\right)\ \hat{\mathbf{i}} + \left(1 + \ln(t)\right)\ \hat{\mathbf{j}} + \left(3\cos(3t)\right)\ \hat{\mathbf{k}}.$$

4. For the vector-valued functions below, find a tangent parametric equations for the tangent line to the curve at the given point.

(a) $\mathbf{r}(t) = \cos 2t\mathbf{i} + 2\sin t\mathbf{j} + t^2\mathbf{k}$; $t = \frac{\pi}{2}$

(b) $\mathbf{r}(t) = \ln(t+1)\mathbf{i} + t\cos 2t\mathbf{j} + 2^t\mathbf{k}$; $t = 0$

Remark. To find the parametric equation of a line, we need a direction vector and a point. To find the parametric equations for a line tangent to a curve at some point t, we use $\vec{\mathbf{r}}'(t)$ as the direction vector. This is due to the fact that for any position on our curve given by $\vec{\mathbf{r}}(t)$, the derivative $\vec{\mathbf{r}}'(t)$ will be a vector tangent to that point. To find a point suitable for the parametric equations, we use $\vec{\mathbf{r}}(t)$.

4a. First, we find our point at $\vec{\mathbf{r}}(t)$. In this case, $\vec{\mathbf{r}}\left(\frac{\pi}{2}\right)$

$$\begin{aligned} \vec{\mathbf{r}}\left(\frac{\pi}{2}\right) &= \cos(\pi)\ \hat{\mathbf{i}} + 2\sin\left(\frac{\pi}{2}\right)\ \hat{\mathbf{j}} + \left(\frac{\pi}{2}\right)^2\ \hat{\mathbf{k}} \\ &= -1\ \hat{\mathbf{i}} + 2\ \hat{\mathbf{j}} + \left(\frac{\pi^2}{4}\right)\ \hat{\mathbf{k}}. \end{aligned}$$

Thus, we have the point $P\left(-1, 2, \frac{\pi^2}{4}\right)$. Next, we find the direction vector by finding $\vec{\mathbf{r}}'\left(\frac{\pi}{2}\right)$

$$\begin{aligned}
\vec{\mathbf{r}}'(t) &= -2\sin(2t)\ \hat{\mathbf{i}} + 2\cos(t)\ \hat{\mathbf{j}} + 2t\ \hat{\mathbf{k}} \\
\vec{\mathbf{r}}\left(\frac{\pi}{2}\right) &= -2\sin(\pi)\ \hat{\mathbf{i}} + 2\cos\left(\frac{\pi}{2}\right)\ \hat{\mathbf{j}} + \pi\ \hat{\mathbf{k}} \\
&= \pi\hat{\mathbf{k}}.
\end{aligned}$$

Thus, we have the direction vector $\langle 0, 0, \pi\rangle$

Conclusion 4. The tangent line to the curve at the point $t = \frac{\pi}{2}$ is given by the parametric equations

$$\begin{aligned}
x(\tau) &= -1 \\
y(\tau) &= 2 \\
z(\tau) &= \frac{\pi^2}{4} + \pi\tau.
\end{aligned}$$

Problem 4b. Again, we find $\vec{\mathbf{r}}(t)$ and $\vec{\mathbf{r}}'(t)$

$$\begin{aligned}
\vec{\mathbf{r}}(0) &= 0\ \hat{\mathbf{i}} + 0\ \hat{\mathbf{j}} + \hat{\mathbf{k}} \\
\vec{\mathbf{r}}'(t) &= \frac{1}{t+1}\ \hat{\mathbf{i}} - 2t\sin(2t) + \cos(2t)\ \hat{\mathbf{j}} + 2^t \ln 2\ \hat{\mathbf{k}} \\
\vec{\mathbf{r}}'(0) &= 1\ \hat{\mathbf{i}} + \ln 2\ \hat{\mathbf{k}}.
\end{aligned}$$

Conclusion 5. Thus, we have the point (0,0,1) and the direction vector $\langle 1, 0, \ln 2\rangle$, which gives the parametric equations

$$\begin{aligned}
x(\tau) &= t \\
y(\tau) &= 0 \\
z(\tau) &= 1 + \ln(2)\tau.
\end{aligned}$$

5. Suppose that the acceleration function, initial velocity, and initial position of a particle are $\mathbf{a}(t) = -5\cos t\mathbf{i} - 5\sin t\mathbf{j}$, $\mathbf{v}(0) = 9\mathbf{i} + 2\mathbf{j}$, and $\mathbf{r}(0) = 5\mathbf{i}$, respectively. Find $\mathbf{v}(t)$ and $\mathbf{r}(t)$.

To find the velocity vector, we integrate the acceleration vector

$$\begin{aligned}
\vec{\mathbf{v}}(t) = \int \vec{\mathbf{a}}(t) &= -5\int \cos(t)\ dt\ \hat{\mathbf{i}} - 5\int \sin(t)\ dt\ \hat{\mathbf{j}} \\
&= -5\sin(t) + C_1\ \hat{\mathbf{i}} + 5\cos(t)\ + C_2\ \hat{\mathbf{j}}.
\end{aligned}$$

We use the fact that $\vec{\mathbf{v}}(0) = 9\ \hat{\mathbf{i}} + 2\ \hat{\mathbf{j}}$ to find the constants of integration

$$\begin{aligned}
-5\sin(0) + C_1\ \hat{\mathbf{i}} = 9\ \hat{\mathbf{i}} &\implies C_1 = 9 \\
5\cos(0) + C_2\ \hat{\mathbf{j}} = 2\ \hat{\mathbf{j}} &\implies C_2 = -3.
\end{aligned}$$

Thus, the velocity vector is given by

$$\vec{\mathbf{v}}(t) = -5\sin(t) + 9\ \hat{\mathbf{i}} + 5\cos(t) - 3\ \hat{\mathbf{j}}.$$

We then integrate the velocity vector to find the position vector $\vec{\mathbf{r}}(t)$

$$\begin{aligned}\vec{\mathbf{r}}(t) &= \int \mathbf{v}(\vec{\mathbf{t}})\ dt = -5\int \sin(t)\ dt\ \hat{\mathbf{i}} + 5\int \cos(t)\ dt\ \hat{\mathbf{j}} \\ &= 5\cos(t) + C_1\ \hat{\mathbf{i}} + 5\sin(t)\ + C_2\ \hat{\mathbf{j}}.\end{aligned}$$

We then use the fact that $r(0) = 5\ \hat{\mathbf{i}}$ to find the constants of integration

$$\begin{aligned}5\cos(0) + C_1\ \hat{\mathbf{i}} = 5\ \hat{\mathbf{i}} &\implies C_1 = 0 \\ 5\sin(0) + C_2\ \hat{\mathbf{j}} = 0\ \hat{\mathbf{j}} &\implies C_2 = 0.\end{aligned}$$

Conclusion 6. Thus, we have

$$\begin{aligned}\vec{\mathbf{v}}(t) &= -5\sin(t) + 9\ \hat{\mathbf{i}} + 5\cos(t) - 3\ \hat{\mathbf{j}} \\ \vec{\mathbf{r}}(t) &= 5\cos(t)\ \hat{\mathbf{i}} + 5\sin(t)\ \hat{\mathbf{j}}.\end{aligned}$$