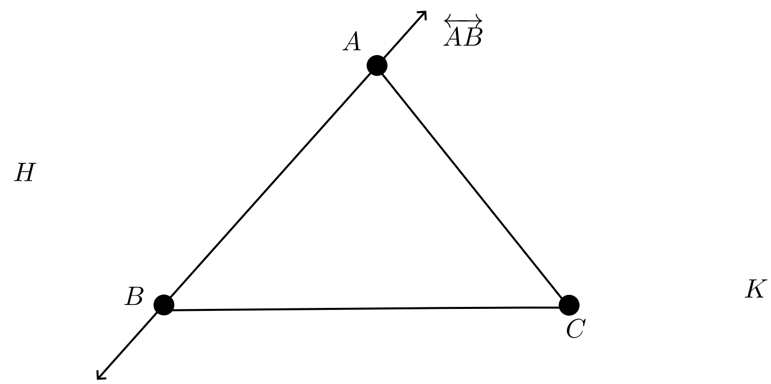
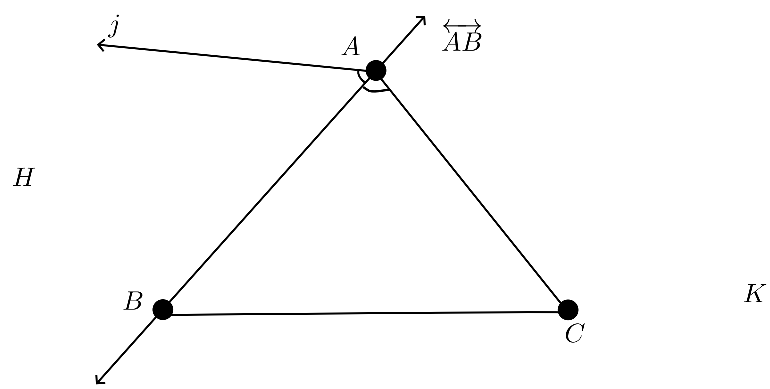
8. Suppose that A, B, C are three noncollinear points. Prove that there exsits a fourth point D not on $\overleftrightarrow{AB}$ so that $\triangle ABC \cong \triangle ABD$

Proof. Suppose that A, B, C are three noncollinear points. If $D = C$, then $\triangle ABC \cong \triangle ABD$ under the correspondence $ABC \leftrightarrow ABD$ trivially, so we may assume that $C \neq D$.

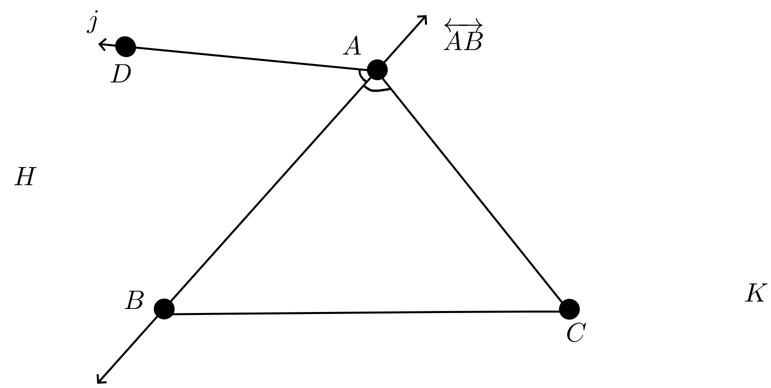
By Ax.S, $\overleftrightarrow{AB}$ generates a pair of opposite halfplanes H, K with edge $\overleftrightarrow{AB}$. Since A, B, C noncollinear, $C \notin \overleftrightarrow{AB}$. Let K be the halfplane that contains C



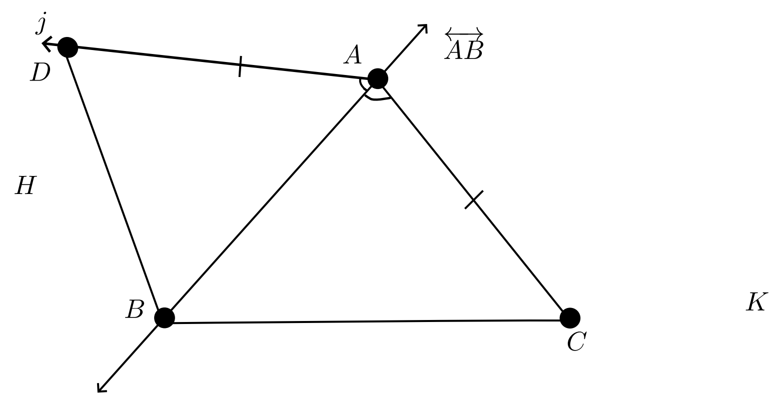
Consider the angle measure $\overrightarrow{AB}\overrightarrow{AC} = \angle BAC$. By theorem 12.3, there are two rays j, k with endpoint A and angle measure $\overrightarrow{AB}j = \overrightarrow{AB}k = \overrightarrow{AB}\overrightarrow{AC}$. Let k be one the side that contains C (K), by Theorem 11.6, $k = \overrightarrow{AC}$. Since $k^0 \subseteq K, j^0 \subseteq H$.



Consider the distance AC, by Theorem 8.6, each ray with endpoint A has a unique point X such that $AX = AC$, on ray $\overrightarrow{AC}$, $X = C$. Call the point in ray j D



Since $D \in H$ implies $D \notin \overleftrightarrow{AB}$, A, B, D are three noncollinear points, and a triangle $\triangle ABD$ is formed.



Consider the correspondence $ABC \leftrightarrow ABD$ between the vertices of triangles $\triangle ABC$ and $\triangle ABD$

Since $AD = AC$, $AB = AB$, and $\angle BAD = \angle BAC$, we have $\overline{AD} \cong \overline{AC}, \overline{AB} \cong \overline{AB}, \underline{\angle BAC} \cong \underline{\angle BAD}$, and Ax.SAS implies that $\triangle ABC \cong \triangle ABD$. ■

9. Suppose that $\omega < \infty$, that P, Q, R are noncollinear points with $P^* =$ antipode of P, and that $\angle PQR = 30$ and $\angle PRQ = 150$. Prove that $\triangle P^*QR \cong \triangle PRQ$

Proof. Suppose that $\omega < \infty$, that P, Q, R are noncollinear points with $P^* =$ antipode of P, and that $\angle PQR = 30$, and $\angle PRQ = 150$

First, we note that $QR = RQ$, which implies $\overline{QR} = \overline{RQ} \cong \overline{RQ}$

Next, by Theorem 11.8, we have that $\overrightarrow{RP}$-$\overrightarrow{RQ}$-$\overrightarrow{RP'}$. By Coroll. 9.8, $\overrightarrow{RP'} = \overrightarrow{RP^*}$, so $\overrightarrow{RP}$-$\overrightarrow{RQ}$-$\overrightarrow{RP^*}$. Thus,

$$\overrightarrow{RP}\overrightarrow{RQ} + \overrightarrow{RQ}\overrightarrow{RP^*} = \overrightarrow{RP}\overrightarrow{RP^*} = 180.$$

Since $\overrightarrow{RP}\overrightarrow{RQ} = \angle PRQ$, and $\overrightarrow{RQ}\overrightarrow{RP^*} = \angle QRP^*$, we have

$$\begin{aligned} &\angle PRQ + \angle QRP^* = 180 \\ &\implies \angle QRP^* = 180 - \angle PRQ = 180 - 150 = 30. \end{aligned}$$

So, $\angle PQR = \angle P^*RQ$, and thus $\underline{\angle PQR} \cong \underline{\angle P^*RQ}$

Similarly $\overrightarrow{QP}$-$\overrightarrow{QR}$-$\overrightarrow{QP^*}$ by Theorem 11.8 and Coroll.9.8, so

$$\overrightarrow{QP}\overrightarrow{QR} + \overrightarrow{QR}\overrightarrow{QP^*} = \overrightarrow{QP}\overrightarrow{QP^*} = 180.$$

Since $\overrightarrow{QP}\overrightarrow{QR} = \angle PQR$, and $\overrightarrow{QR}\overrightarrow{QP^*} = \angle RQP^*$, we have that

$$\begin{aligned} &\angle PQR + \angle RQP^* = 180 \\ &\quad \implies \angle RQP^* = 180 - \angle PQR = 180 - 30 = 150. \end{aligned}$$

Thus, $\angle P^*QR = 150$, which means $\angle P^*QR = \angle PRQ = 150$, and therefore $\underline{\angle P^*QR} \cong \underline{\angle PRQ}$

So, under the correspondence $PRQ \leftrightarrow P^*QR$ between the vertices of triangles $\triangle PRQ$ and $\triangle P^*QR$, by Theorem 13.1 (ASA), we have that

$$\triangle PRQ \cong \triangle P^*QR.$$

■