Homework/Worksheet 10 - Due: Saturday, April 27

1. Use spherical coordinates to find the volume of the ball $\rho \leqslant 3$ that is situated between the cones $\varphi = \frac{\pi}{4}$ and $\varphi = \frac{\pi}{3}$.

To find the volume of the ball, we first determine the spherical region E, which we see is given by

$$E = \{(\rho, \theta, \varphi) : \ 0 \leqslant \theta \leqslant 2\pi, \ \frac{\pi}{4} \leqslant \varphi \leqslant \frac{\pi}{3}, \ 0 \leqslant \rho \leqslant 3\}.$$

And the integral for the volume of the ball is given by

$$\begin{aligned}
&\int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \int_0^3 \rho^2 \sin(\varphi) d\rho d\varphi d\theta \\
&= \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{3} \left[\rho^3 \right|_0^3 \sin(\varphi) d\varphi d\theta \\
&= \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 9 \sin(\varphi) d\varphi d\theta \\
&= \int_0^{2\pi} -9 \left[\cos(\varphi) \right|_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\theta \\
&= \int_0^{2\pi} -9 \left(\frac{1}{2} - \frac{\sqrt{2}}{2} \right) d\theta \\
&= \int_0^{2\pi} -\frac{9}{2} + \frac{9\sqrt{2}}{2} d\theta \\
&= -\frac{9}{2}(2\pi - 0) + \frac{9\sqrt{2}}{2}(2\pi - 0) \\
&= -9\pi + 9\pi\sqrt{2}.
\end{aligned}$$

2. Convert the integral

$$\int_{-4}^{4} \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} \int_{-\sqrt{16-x^2-y^2}}^{\sqrt{16-x^2-y^2}} \left(x^2 + y^2 + z^2\right) \, dz \, dx \, dy$$

into an integral in spherical coordinates.

We clearly see that the region described by the given integral is a sphere of radius 4, with the bounds of integration for the first two integrals being a projection of this sphere onto the xy-plane. Thus, this is an elementary region regarding spherical coordinates, the conversion is trivial

$$E = \{(\rho, \theta, \varphi) : \ 0 \leqslant \theta \leqslant 2\pi, \ 0 \leqslant \varphi \leqslant \pi, \ 0 \leqslant \rho \leqslant 4\}.$$

Before we construct the new integral, we must transform the integrand using rectangular to spherical conversion formulas. We remark $\rho^2 = x^2 + y^2 + z^2$. Hence, we have the integral

$$\int_0^{2\pi} \int_0^{\pi} \int_0^3 \rho^2 \rho^2 \sin(\varphi) \, d\rho d\varphi d\theta.$$

Where $\rho^2 \sin(\varphi)$ is the Jacobian of the rectangular $\to$ spherical transformation.

$$\begin{aligned}
\implies & \int_0^{2\pi} \int_0^{\pi} \int_0^{4} \rho^4 \sin(\varphi)\, d\rho d\varphi d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} \frac{1}{5} \left[\rho^5 \Big|_0^4 \right. \sin(\varphi)\, d\varphi d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} \frac{1024}{5} \sin(\varphi)\, d\varphi d\theta \\
&= \int_0^{2\pi} -\frac{1024}{5} \left[\cos(\varphi) \Big|_0^{\pi} \right. d\theta \\
&= \int_0^{2\pi} -\frac{1024}{5} \left[\cos(\varphi) \Big|_0^{\pi} \right. d\theta \\
&= \frac{1024 \cdot 2}{5} (2\pi - 0) \\
&= \frac{4096\pi}{5}.
\end{aligned}$$

3. Convert the integral

$$\int_0^4 \int_0^{\sqrt{16-x^2}} \int_{-\sqrt{16-x^2-y^2}}^{\sqrt{16-x^2-y^2}} \left(x^2+y^2+z^2\right)^2 \, dz\, dy\, dx$$

into an integral in spherical coordinates.

This region is almost the same as the previous problem, but the region D in the xy-plane is reduced to the quarter circle (in the first quadrant). This reduces the bounds for θ to $0 \leqslant \theta \leqslant \frac{\pi}{2}$. The other bounds remain the same. Furthermore, the integrand becomes ρ^4 instead of ρ^2

$$\int_0^{\frac{\pi}{2}} \int_0^{\pi} \int_0^{4} r^6 \sin(\varphi)\, d\rho d\varphi d\theta$$

.

This is an elementary integral resulting in the quantity $\frac{16384\pi}{7}$

4. Express the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$ as triple integrals in cylindrical coordinates and spherical coordinates, respectively.

We can represent this region in cylindrical coordinates as

$$E = \{(r, \theta, z) : \; 0 \leqslant \theta \leqslant 2\pi, \; 2 \leqslant r \leqslant 4, \; -\sqrt{16-r^2} \leqslant z \leqslant \sqrt{16-r^2}\}.$$

Which gives the integral

$$\int_0^{2\pi} \int_2^{4} \int_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}} z \, dz dr d\theta.$$

Converting the given region to spherical takes some work. Unlike the cylindrical region, which was easily visualized and transformed, the transformation to spherical will be made analytically.

First, we note that the bounds for θ are quite clear, theta will range from 0 to 2π. To find the remaining bounds, we play around with the conversion forumlas and see what we can find out. The conversion formulas for rectangular $\to$ spherical are $x = \rho\sin(\varphi)\cos(\theta)$, $y = \rho\sin(\varphi)\sin(\theta)$, $z = \rho\cos(\varphi)$, and $\rho^2 = x^2 + y^2 + z^2$

$$\begin{aligned}
&x^2 + y^2 + z^2 = 16 \implies \rho = 4 \\
&x^2 + y^2 = 4 \\
\implies &4 + z^2 = 16 \\
\implies &z = \pm\sqrt{12} = \pm 2\sqrt{3}.
\end{aligned}$$

Since the region is bounded above by the sphere, which has radius 4, we know that $\rho = 4$ must be the upper bound. The lowerbound is going to take some more work. To find the lower bound, we must examine the equation of the cylinder

$$\begin{aligned}
&x^2 + y^2 = 4 \\
\implies &\rho^2\sin^2(\varphi)\cos^2(\theta) + \rho^2\sin^2(\varphi)\sin^2(\theta) = 4 \\
\implies &\rho^2\sin^2(\varphi) = 4 \\
\implies &\rho = \frac{2}{\sin(\varphi)}.
\end{aligned}$$

This must be the lower bound for the integral regarding ρ. Lastly, to find the bounds for φ, we again turn our attention to the intersection.

$$\begin{aligned}
&z = \pm 2\sqrt{3} \\
\text{with } &z = \rho\cos(\varphi) \implies \pm 2\sqrt{3} = \rho\cos(\varphi) \\
\implies &\varphi = \cos^{-1}\left(\frac{\pm 2\sqrt{3}}{4}\right) \\
&= \frac{\pi}{6}, \frac{5\pi}{6}.
\end{aligned}$$

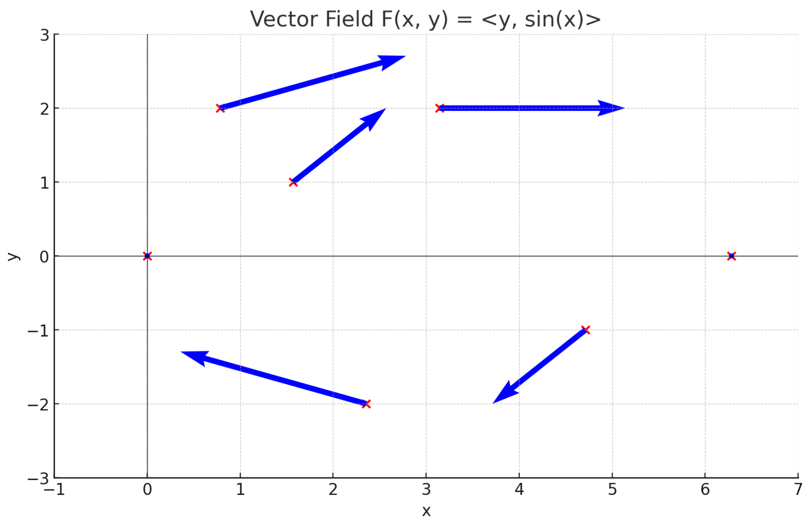
Thus, we have the integral

$$\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\int_{\frac{2}{\sin(\varphi)}}^{4} \rho^2\sin(\varphi)\,d\rho d\varphi d\theta.$$

5. Describe the vector field $F(x, y) = \langle y, \sin(x)\rangle$ by drawing some of its vectors.

To draw this vector field, we create a table with a few coordinates, then find the corresponding vector for each coordinate. We then plot the vectors

x	y	Vector $\langle y, \sin(x)\rangle$
0	0	$\langle 0, 0\rangle$
$\frac{\pi}{2}$	1	$\langle 1, 1\rangle$
π	2	$\langle 2, 0\rangle$
$\frac{3\pi}{2}$	-1	$\langle -1, -1\rangle$
2π	0	$\langle 0, 0\rangle$
$\frac{\pi}{4}$	2	$\langle 2, \frac{\sqrt{2}}{2}\rangle$
$\frac{3\pi}{4}$	-2	$\langle -2, -\frac{\sqrt{2}}{2}\rangle$



6. Find the gradient vector field of the function $f(x, y) = x \sin y + \cos y$.

To find the gradient vector field of the given function, we simply find the gradient vector $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$

$$\nabla f(x, y) = \langle \sin(y), x - \sin(y)\rangle .$$