Nate Warner MATH 230 September 28, 2023

Homework/Worksheet 4 - Due: Wednesday, October 4

1.) Find the length of the functions below over the given interval. If you cannot evaluate the integral exactly, use technology to approximate it.

(a) $y = x^{\frac{3}{2}}$ from $(1, 1)$ to $(8, 4)$

(b) $y = \frac{1}{3}(x^2 - 2)^{\frac{3}{2}}$ from $x = 2$ to $x = 4$

(c) $y = \frac{x}{3} + \frac{1}{4x}$ from $x = 1$ to $x = 4$

1.a

$$\frac{d}{dx}x^{\frac{3}{2}} = \frac{3}{2}x^{\frac{1}{2}}.$$

Thus:

$$\begin{aligned}
s &= \int_1^8 \sqrt{1 + \left(\frac{3}{2}x^{\frac{1}{2}}\right)^2}\, dx \\
&= \int_1^8 \sqrt{1 + 3x}\, dx \\
&\text{Let } u = 1 + 3x \\
&du = 3dx \\
&\frac{1}{3}du = dx \\
&u(a) = 4 \\
&u(b) = 25 \\
&\frac{1}{3}\int_4^{25} \sqrt{u}\, du \\
&= \frac{1}{3}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_4^{25} \\
&= \frac{2}{9}\left[u^{\frac{3}{2}}\right]_4^{25} \\
&= \frac{2}{9}\left[25^{\frac{3}{2}} - 4^{\frac{3}{2}}\right] \\
&= \frac{2}{9}\left[125 - 8\right] \\
&= 26.
\end{aligned}$$

1.b

$$
\begin{aligned}
&\frac{1}{3}\left[\frac{d}{dx}(x^2-2)^{\frac{3}{2}}\right]\\
&\frac{1}{3}\left[\frac{3}{2}(x^2-2)^{\frac{1}{2}}\right]\cdot 2x\\
&= x(x^2-2)^{\frac{1}{2}}.
\end{aligned}
$$

Thus:

$$
\begin{aligned}
s &= \int_2^4 \sqrt{1+(x(x^2-2)^{\frac{1}{2}})^2}\ dx\\
&= \int_2^4 \sqrt{1+x^2(x^2-2)}\ dx\\
&= \int_2^4 \sqrt{1+x^4-2x^2}\ dx\\
&= \int_2^4 \sqrt{(x^2-1)^2}\ dx.
\end{aligned}
$$

Since we know the domain is nonnegative, we can rewrite as:

$$
\begin{aligned}
&\int_2^4 x^2-1\ dx\\
&= \frac{1}{3}x^3 - x\ \Big|_2^4\\
&= \left(\frac{1}{3}(4)^3-4\right)-\left(\frac{1}{3}(2)^3-2\right)\\
&= \frac{64}{3}-4-\left(\frac{8}{3}-2\right)\\
&= \frac{50}{3}.
\end{aligned}
$$

1.c

$$\frac{d}{dx}\frac{1}{3}x^3 + \frac{1}{4}x^{-1}$$
$$= x^2 - \frac{1}{4}x^{-2}.$$

Thus:

$$\begin{aligned}
s &= \int_1^4 \sqrt{1 + \left(x^2 - \frac{1}{4x^2}\right)^2}\ dx \\
&= \int_1^4 \sqrt{1 + x^4 - 2\left(\frac{1}{4}\right) + \frac{1}{16x^4}}\ dx \\
&= \int_1^4 \sqrt{1 + x^4 - \frac{1}{2} + \frac{1}{16x^4}}\ dx \\
&= \int_1^4 \sqrt{x^4 + \frac{1}{16x^4} + \frac{1}{2}}\ dx \\
&= \int_1^4 \sqrt{(x^2)^2 + \frac{1}{(4x^2)^2} + \frac{1}{2}}\ dx \\
&= \int_1^4 \sqrt{\left(x^2 + \frac{1}{4x^2}\right)^2}\ dx \\
&= \int_1^4 \sqrt{\left(\frac{4x^4+1}{4x^2}\right)^2}\ dx.
\end{aligned}$$

Since the domain is *nonnegative*, we can rewrite as:

$$\begin{aligned}
&\int_1^4 \frac{4x^4+1}{4x^2}\ dx \\
&= \int_1^4 x^2 + \frac{1}{4x^2}\ dx \\
&= \int_1^4 x^2 + \frac{1}{4}x^{-2}\ dx \\
&= \frac{1}{3}x^3 - \frac{1}{4}x^{-1}\ \Big|_1^4 \\
&= \left(\frac{1}{3}(4)^3 - \frac{1}{4}(4)^{-1}\right) - \left(\frac{1}{3}(1)^3 - \frac{1}{4}(1)^{-1}\right) \\
&= \frac{339}{16}.
\end{aligned}$$

2.) Find the surface area of the volume generated by revolving the curve $y = x^3$, $0 \leqslant x \leqslant 1$, around the x-axis.

$$\frac{d}{dx}x^3 = 3x^2.$$

Thus:

$$\begin{aligned} sa &= \int_0^1 \ 2\pi x^3 \sqrt{1+(3x^2)^2} \ dx \\ &= 2\pi \int_0^1 \ x^3\sqrt{1+9x^4} \ dx. \end{aligned}$$

Thus:

Let $u = 1 + 9x^4$

$du = 36x^3 dx$

$\frac{1}{36}du = x^3 dx$

$u(a) = 1$

$u(b) = 10.$

$$\begin{aligned} sa &= \frac{2\pi}{36}\int_1^{10} \ u^{\frac{1}{2}} \ du \\ &= \frac{\pi}{18}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^{10} \\ &= \frac{\pi}{27}\left[\left((10)^{\frac{3}{2}} - 1\right)\right] \\ &= \frac{10^{\frac{3}{2}}\pi}{27} - \frac{\pi}{27} \\ &= \frac{10^{\frac{3}{2}}\pi - \pi}{27}. \end{aligned}$$

3.) Find the surface area of the volume generated by revolving the curve $y = 3x^4$, $0 \leqslant x \leqslant 1$, around the y-axis.

Derivative:

$$\frac{d}{dx}3x^4 = 12x^3.$$

Thus:

$$\begin{aligned} sa &= \int_0^1 \ 2\pi x\sqrt{1+(12x^3)^2} \ dx \\ &= 2\pi \int_0^1 \ x\sqrt{1+144x^6} \ dx \\ &\approx 15.8264. \end{aligned}$$

4.) Evaluate the following integrals

(a) $\int \frac{(\ln(x))^2}{x}\ dx$

(b) $\int_0^{\frac{\pi}{4}}\ \tan x\ dx$

4.a)

$$\int \frac{(\ln x)^2}{x}\ dx$$
$$\text{Let } u = \ln x$$
$$du = \frac{1}{x}dx$$
$$\implies \int u^2\ du$$
$$= \frac{1}{3}u^3 + C$$
$$= \frac{1}{3}(\ln x)^3 + C$$
$$= \frac{1}{3}\ln^3 x + C.$$

4.b

Integrate:

$$\int \tan x\ dx$$
$$= \int \frac{\sin x}{\cos x}\ dx$$
$$\text{Let } u = \cos x$$
$$du = -\sin x dx$$
$$-du = \sin x dx$$
$$\implies \int u^{-1}\ du$$
$$= -\ln|u| + C$$
$$= -\ln|\cos x| + C$$
$$= \ln|\frac{1}{\cos x}| + C$$
$$= \ln|\sec x| + C.$$

Thus:

$$\ln|\sec x|\ \Big|_0^{\frac{\pi}{3}}$$
$$= \ln|\sec\frac{\pi}{3}| - \ln|\sec 0|$$
$$= \ln 2 - \ln 1$$
$$= \ln 2.$$

5.) Compute the derivative, $\frac{dy}{dx}$ of the following functions.

(a) $y = e^{\sin x}$

(b) $y = xe^{x}$

(c) $y = \frac{x^{-1}}{\ln x}$

5.a)

$$\begin{aligned}
&\frac{d}{dx} e^{\sin x} \\
&= e^{\sin x} \cdot \frac{d}{dx} \sin x \\
&= \cos x e^{\sin x}.
\end{aligned}$$

5.b)

By the product rule:

$$\begin{aligned}
&\frac{d}{dx} xe^{x} \\
&= xe^{x} + e^{x} \\
&= e^{x}(x+1).
\end{aligned}$$

5.c

$$\begin{aligned}
&\frac{d}{dx} \frac{1}{x \ln x} \\
&\frac{1}{x \ln x} \cdot \frac{d}{dx} \ln\left(\frac{1}{x \ln x}\right) \\
&= \frac{1}{x \ln x} \cdot \frac{d}{dx} \left[-\ln(x \ln x)\right] \\
&= \frac{1}{x \ln x} \cdot \frac{d}{dx} \left[-\ln x + \ln(\ln(x))\right] \\
&= \frac{1}{x \ln x} \left[-\frac{1}{x} + \frac{1}{x \ln x}\right] \\
&= \frac{1}{x \ln x} \left[-\frac{\ln x + 1}{x \ln x}\right] \\
&= -\frac{\ln x + 1}{x^2 \ln^2 x}.
\end{aligned}$$