Nate Warner MATH 353 Spring 2025

Problem set 10 - Due: Friday, March 28

1. Assume $\omega < \infty$. Suppose that A, B, C are points with $AC < \omega$ and A-B-C. Let X be any point not on $\overleftrightarrow{AC}$ and let A^* be the antipode of A. Prove that $\overrightarrow{XB}$-$\overrightarrow{XC}$-$\overrightarrow{XA^*}$

Proof. Assume $\omega < \infty$, A, B, C are points with $AC < \omega$, and A-B-C. Let X be any point not on $\overleftrightarrow{AC}$, and let A^* be the antipode of A.

First, let $\overrightarrow{XA} = n, \overrightarrow{XB} = j, \overrightarrow{XC} = \ell, \overrightarrow{XA^*} = k$. Thus, we aim to show that j-ℓ-k. We first note that by Ax.C, we have

$$n\text{-}j\text{-}\ell.$$

Next, we observe that by theorem 9.1, A-C-A^*, and by Ax.C, $\overrightarrow{XA}$-$\overrightarrow{XC}$-$\overrightarrow{XA^*}$, or n-ℓ-k. Thus, we have

$$n\text{-}j\text{-}\ell \quad \text{and} \quad n\text{-}\ell\text{-}k.$$

Which by the rule of insertion, gives us

$$n\text{-}j\text{-}\ell\text{-}k.$$

Which yields j-ℓ-$k = \overrightarrow{XB}$-$\overrightarrow{XC}$-$\overrightarrow{XA^*}$ as desired ■

2. Prove Theorem 11.9

Remark. (*Theorem 11.9 Almost uniqueness of quadrichotomy for rays*): Suppose that a, b, c, r are distinct rays in a pencil P, and that a-b-c. Then, **exactly** one of

$$r\text{-}a\text{-}b \quad a\text{-}r\text{-}b \quad b\text{-}r\text{-}c \quad b\text{-}c\text{-}r$$

With the exception that both r-a-b and b-c-r are true when $r = b'$

(*Dual of Theorem 8.3*): Let $x \neq y$ by rays distinct from ray a on the fan $\overrightarrow{ab}$. Then, exactly one of the following relations must hold.

$$a\text{-}x\text{-}y \quad \text{or} \quad a\text{-}y\text{-}x.$$

Proof: We proceed by dualizing the proof of theorem 9.2.

By Axiom.QR, at least one of

$$r\text{-}a\text{-}b \quad a\text{-}r\text{-}b \quad b\text{-}r\text{-}c \quad b\text{-}c\text{-}r.$$

Suppose we have a-r-b. Then, a-b-c and the rule of insertion yields a-r-b-c

So, a-r-b and r-b-c are true. Which, by the UMT guarantees that both b-r-c and b-c-r are false.

Next, suppose that b-r-c is true. Then, a-b-c and the rule of insertion yields a-b-r-c. So, a-b-r and b-r-c are true, and by the UMT, all three of r-a-b, a-r-b, b-c-r are false. Thus, none of the other three relations hold.

So, if more than one of r-a-b, a-r-b, b-r-c, b-c-r holds, they must be exactly r-a-b and b-c-r

Assume that r-a-b and b-c-r are true. Suppose toward a contradiction that $br < 180$. Then, fan $\overrightarrow{br}$ is defined, and r-a-b, b-c-r implies a, c are in $\overrightarrow{br}$. By the dual of theorem 8.3 (stated above), one of

$$b\text{-}a\text{-}c \quad \text{or} \quad b\text{-}c\text{-}a$$

is true. But, this contradicts a-b-c by the UMT.

Therefore, $br = 180$, hence $r = b'$. ■

3. Prove Theorem 11.10

Remark. *(Theorem 11.10: Opposite Fan Theorem).* Let p, q, r be rays in pencil P such that q-p-r. Then, $\overrightarrow{pq} \cup \overrightarrow{pr} = P$, and $\overrightarrow{pq} \cap \overrightarrow{pr} = \{p, p'\}$

Proof. p, q, r are together in the unique pencil P. Further, q-p-r implies $pq, pr < qr \leqslant 180$, so fans $\overrightarrow{pq}, \overrightarrow{pr}$ are defined.

If $x \neq p, q, r$ is in pencil P, then ax.QR says one of

$$x\text{-}q\text{-}p \quad q\text{-}x\text{-}p \quad p\text{-}x\text{-}r \quad p\text{-}r\text{-}x$$

must be satisfied. In other words, x is in $\overrightarrow{pq}$ or $\overrightarrow{pr}$. So, $P \subseteq \overrightarrow{pq} \cup \overrightarrow{pr}$. Hence, $P = \overrightarrow{pq} \cup \overrightarrow{pr}$

Since $\overrightarrow{pq}$ and $\overrightarrow{pr}$ have the same endpoint, and $\overrightarrow{pq} \cup \overrightarrow{pr} = P$, $\overrightarrow{pq}$ and $\overrightarrow{pr}$ are opposite rays

What about $\overrightarrow{pq} \cap \overrightarrow{pr}$? q-p-r implies not p-q-r or p-r-q, so $q \notin \overrightarrow{pr}$, and $r \notin \overrightarrow{pq}$. So, neither q nor r is in $\overrightarrow{pq} \cap \overrightarrow{pr}$

Let x be any ray $\neq p, q, r$ in P. Suppose $X \in \overrightarrow{pq} \cap \overrightarrow{pr}$

$$\begin{aligned} x \in \overrightarrow{pq} &\implies x\text{-}q\text{-}p \text{ or } q\text{-}x\text{-}p \\ x \in \overrightarrow{pr} &\implies p\text{-}x\text{-}r \text{ or } p\text{-}r\text{-}x. \end{aligned}$$

So two are true. Theorem 11.10 applied to q-p-r and ray x implies it must be q-x-p and p-r-x, with $x = p'$. Thus, $\overrightarrow{pq} \cap \overrightarrow{pr} = \{p, p'\}$ ■