Nate Warner MATH 232 Spring 2024

Homework/Worksheet 8 - Due: Sunday, April 7

1. Evaluate the integrals below:

   (a) $\int_{\ln(2)}^{\ln(3)} \left(\int_0^1 \ e^{x+y} \ dy \right) \ dx$

   (b) $\int_1^e \ \int_1^e \ \frac{\sin(\ln(x))\cos(\ln(y)))}{xy} \ dx \ dy$

   (c) $\int_1^2 \ \int_0^1 \ xe^{x-y} dy \, dx$

Problem 1a.

$$\begin{aligned}
&\int_{\ln(2)}^{\ln(3)} \left(\int_0^1 \ e^{x+y} \ dy \right) \ dx \\
&= \int_{\ln 2}^{\ln 3} \ e^x \int_0^1 \ e^y \ dy \ dx \\
&= \int_{\ln 2}^{\ln 3} \ e^x [e^y]_0^1 \ dx \\
&= \int_{\ln 2}^{\ln 3} \ e^x (e-1) \ dx \\
&= e \int_{\ln 2}^{\ln 3} \ e^x \ dx - \int_{\ln 2}^{\ln 3} \ e^x \ dx \\
&= e[e^x]_{\ln 2}^{\ln 3} - e^x \Big|_{\ln 2}^{\ln 3} \\
&= e(3-2) - (3-2) \\
&= e - 1.
\end{aligned}$$

Problem 1b.

$$\begin{aligned}
&\int_1^e \ \int_1^e \ \frac{\sin(\ln(x))\cos(\ln(y)))}{xy} \ dx \ dy \\
&= \int_1^e \ \frac{\cos(\ln(y))}{y} \ \int_1^e \ \frac{\sin(\ln(x))}{x} \ dxdy \\
&= \int_1^e \ \frac{\cos(\ln(y))}{y} \ \int_0^1 \ \sin(u) \ dudy \\
&= \int_e^1 \ \frac{\cos(\ln(y))}{y} \ [\cos(u)]_0^1 dy \\
&= \int_e^1 \ \frac{\cos(\ln(y))}{y} \ [\cos(1) - 1] dy \\
&= \cos(1) \int_1^0 \ \cos(u) \ du - \int_1^0 \ \cos(u) \ du \\
&= \cos(1)(-\sin(1)) - (-\sin(1)) \\
&= \sin(1)(-\cos(1) + 1).
\end{aligned}$$

Problem 1c.

$$\begin{aligned}
&\int_1^2 \int_0^1 xe^{x-y} dy\, dx \\
&= \int_0^1 xe^x \int_1^2 e^y \ dy \ dx \\
&= \int_0^1 xe^x [e^y]_1^2 \ dx \\
&= \int_0^1 xe^x (e^2 - e) \ dx \\
&= e^2 \int_0^1 xe^x \ dx - e \int_0^1 xe^x \ dx \\
&= e^2 \left[xe^x - \int_0^1 e^x \ dx \right] - e \left[xe^x - \int_0^1 e^x \ dx \right] \\
&= e^2 \left[e - e - (-1) \right] - e \left[e - e - (-1) \right] \\
&= e^2 - e.
\end{aligned}$$

2. Find the volume of the solid under the surface $z = 2x + y^2$ and above the region bounded by $y = x^5$ and $y = x$.

We find the volume of the solid under the surface $z = 2x + y^2$ by integrating over the region $D = \{(x, y) : \ 0 \leqslant x \leqslant 1, \ x^5 \leqslant y \leqslant x\}$. The bounds of x was found by finding the points of interception between the two curves x^5 and x. That is, $x^5 - x = 0 \implies x = 0, \ x = 1$. The bounds of y were found by examining the two curves and seeing that $x^5 \leqslant x \ \forall x \in [0, 1]$.

The volume is then given by

$$\begin{aligned}
V &= \iint_D (2x + y^2) dA \\
&= \int_0^1 \int_{x^5}^x 2x + y^2 \ dy \ dx \\
&= \int_0^1 2xy + \frac{1}{3} y^3 \bigg|_{x^5}^x \ dx \\
&= \int_0^1 2x^2 + \frac{1}{3} x^3 - 2x^6 - \frac{1}{3} x^{15} \ dx \\
&= \frac{2}{3} x^3 + \frac{1}{12} x^4 - \frac{2}{7} - \frac{1}{48} x^{16} \bigg|_0^1 \\
&= \frac{2}{3} + \frac{1}{12} - \frac{2}{7} - \frac{1}{48} \\
&= \frac{149}{336}.
\end{aligned}$$

3. Find the volume of the solid under the plane $z = 3x + y$ and above the region determined by $y = x^7$ and $y = x$.

The region D for this integral is similar to the last. It is $D = \{(x, y) :\ 0 \leqslant x \leqslant 1,\ x^7 \leqslant y \leqslant x\}$. Thus the volume is given by the integral

$$\begin{aligned}
V &= \iint_D (3x + y)\, dA \\
&= \int_0^1 \int_{x^7}^{x} 3x + y \ dy \ dx \\
&= \int_0^1 3xy + \frac{1}{2}y^2 \Big|_{x^7}^{x} \ dx \\
&= \int_0^1 3x^2 + \frac{1}{2}x^2 - 3x^8 - \frac{1}{2}x^{14} \ dx \\
&= x^3 + \frac{1}{6}x^3 - \frac{1}{3}x^9 - \frac{1}{30}x^{15}\Big|_0^1 \\
&= 1 + \frac{1}{6} - \frac{1}{3} - \frac{1}{30} \\
&= \frac{4}{5}.
\end{aligned}$$

4. Find the volume of the solid bounded by the planes $x + y = 1$, $x - y = 1$, $x = 0$, $z = 0$, and $z = 10$.

To find the volume of the solid under the given parameters, we identify the three dimensional region and integrate over it.

$$\begin{aligned}
(1) :&\quad x + y = 1 \implies y = x - 1 \\
(2) :&\quad x - y = 1 \implies y = 1 - x \\
(3) :&\quad x = 0.
\end{aligned}$$

We see functions these define our region on the xy-plane. We can equate 1 and 2 to find the right bound of x values

$$\begin{aligned}
x - 1 &= 1 - x \\
\implies x &= 1.
\end{aligned}$$

Thus, our region E is defined by

$$E = \{(x, y, z) :\ 0 \leqslant x \leqslant 1,\ x - 1 \leqslant y \leqslant 1 - x,\ 0 \leqslant z \leqslant 10\}.$$

And the volume of this region is given by

$$\begin{aligned}
V &= \iiint_E dV \\
&= \int_0^1 \int_{x-1}^{1-x} \int_0^{10} dz dy dx \\
&= \int_0^1 \int_{x-1}^{1-x} z\Big|_0^{10} dy dx \\
&= \int_0^1 10 \int_{x-1}^{1-x} dy dx \\
&= \int_0^1 10 \left[y\Big|_{x-1}^{1-x} dx \right. \\
&= 10 \int_0^1 1 - x - (x-1)\, dx \\
&= 10 \int_0^1 -2x + 2\, dx \\
&= -20 \int_0^1 x - 1\, dx \\
&= -20 \left[\frac{1}{2} x^2 - x \Big|_0^1 \right. \\
&= -20 \left(\frac{1}{2} - 1 \right) \\
&= 10.
\end{aligned}$$

5. Evaluate the following integrals by changing the order of integration.

   (a) $\int_{-1}^{\frac{\pi}{2}} \int_0^{x+1} \sin(x)\, dy dx$

   (b) $\int_{-1}^{0} \int_{-\sqrt{y+1}}^{\sqrt{y+1}} y^2\, dx dy$

Problem 5a. We see that the current domain of integration is given by

$$D = \{(x, y) :\ -1 \leqslant x \leqslant \frac{\pi}{2},\ 0 \leqslant y \leqslant x + 1\}.$$

We can then convert this type 1 region to a region of type 2.

$$\therefore D = \{(x, y) :\ y - 1 \leqslant x \leqslant \frac{\pi}{2},\ 0 \leqslant y \leqslant \frac{\pi}{2} + 1\}.$$

From this it follows that our integral becomes

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}+1}\int_{y-1}^{\frac{\pi}{2}} \sin(x)\,dxdy \\
&= -\int_0^{\frac{\pi}{2}+1} \cos(x)\Big|_{\frac{\pi}{2}}^{y-1}\,dy \\
&= -\int_0^{\frac{\pi}{2}+1} \cos\left(\frac{\pi}{2}\right) - \cos(y-1)\,dy \\
&= \int_0^{\frac{\pi}{2}+1} \cos(y-1)\,dy \\
&= \sin(y-1)\Big|_{\frac{\pi}{2}+1}^{0} \\
&= \sin\left(\frac{\pi}{2}+1\right) - \sin(-1) \\
&= \sin\left(\frac{\pi}{2}\right)\cos(1) + \cos\left(\frac{\pi}{2}\right)\sin(1) + \sin(1) \\
&= 1 + \sin(1).
\end{aligned}$$

Problem 5b. We see that our current region is of type 2 and is given by

$$D = \{(x,y) :\ -1 \leqslant y \leqslant 0,\ -\sqrt{y+1} \leqslant x \leqslant \sqrt{y+1}\}.$$

We can then change this region to type 1.

$$D = \{(x,y) :\ -1 \leqslant x \leqslant 1,\ x^2 - 1 \leqslant y \leqslant 0\}.$$

Thus, we have the integral

$$\begin{aligned}
&\int_{-1}^{1}\int_{x^2-1}^{0} y^2\,dydx \\
&= \int_{-1}^{1} \frac{1}{3}\left[y^3\right]_{x^2-1}^{0}\,dx \\
&= -\frac{1}{3}\int_{-1}^{1} (x^2-1)^3\,dx \\
&= -\frac{1}{3}\int_{-1}^{1} x^6 - 3x^4 + 3x^2 - 1\,dx \\
&= -\frac{1}{3}\left[\frac{1}{7}x^7 - \frac{3}{5}x^5 + x^3 - x\right]_{-1}^{1} \\
&= \frac{32}{105}.
\end{aligned}$$