

Homework/Worksheet 7 - Due: Wednesday, November 1

1. Use the trapezoidal rule and Simpson's rule to approximate the integral

$$\int_0^2 \frac{e^x}{1+x^2} dx.$$

With $n = 10$

Trapezoidal approximation:

Remark. $T_n = \frac{h}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$ with $h = \frac{b-a}{n}$

We have,

$$\begin{aligned} x_0 = 0, \quad x_1 = \frac{1}{5}, \quad x_2 = \frac{2}{5}, \quad x_3 = \frac{3}{5}, \quad x_4 = \frac{4}{5}, \quad x_5 = 1, \quad x_6 = \frac{6}{5}, \quad x_7 = \frac{7}{5}, \quad x_8 = \frac{8}{5}, \quad x_9 = \frac{9}{5}, \quad x_{10} = 2 \\ f(x_0) = 1, \quad f(x_1) = \frac{e^{\frac{1}{5}}}{1 + \frac{1}{25}}, \quad f(x_2) = \frac{e^{\frac{2}{5}}}{1 + \frac{4}{25}}, \quad f(x_3) = \frac{e^{\frac{3}{5}}}{1 + \frac{9}{25}}, \quad f(x_4) = \frac{e^{\frac{4}{5}}}{1 + \frac{16}{25}}, \quad f(x_5) = \frac{e}{2} \\ f(x_6) = \frac{e^{\frac{6}{5}}}{1 + \frac{36}{25}}, \quad f(x_7) = \frac{e^{\frac{7}{5}}}{1 + \frac{49}{25}}, \quad f(x_8) = \frac{e^{\frac{8}{5}}}{1 + \frac{64}{25}}, \quad f(x_9) = \frac{e^{\frac{9}{5}}}{1 + \frac{81}{25}}, \quad f(x_{10}) = \frac{e^2}{5}. \end{aligned}$$

Thus:

$$\begin{aligned} T_{10} &= \frac{1}{10} \left[1 + \frac{2e^{\frac{1}{5}}}{1 + \frac{1}{25}} + \frac{2e^{\frac{2}{5}}}{1 + \frac{4}{25}} + \frac{2e^{\frac{3}{5}}}{1 + \frac{9}{25}} + \frac{2e^{\frac{4}{5}}}{1 + \frac{16}{25}} + e + \frac{2e^{\frac{6}{5}}}{1 + \frac{36}{25}} + \frac{2e^{\frac{7}{5}}}{1 + \frac{49}{25}} + \frac{2e^{\frac{8}{5}}}{1 + \frac{64}{25}} + \frac{2e^{\frac{9}{5}}}{1 + \frac{81}{25}} + \frac{e^2}{5} \right] \\ &\approx 2.6608. \end{aligned}$$

Simpson's rule approximation

Remark. $S_n = \frac{h}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$ where $h = \frac{b-a}{n}$ and $n \in 2k$ for some integer k

Thus,

$$\begin{aligned} S_{10} &= \frac{1}{10} \left[1 + \frac{4e^{\frac{1}{5}}}{1 + \frac{1}{25}} + \frac{2e^{\frac{2}{5}}}{1 + \frac{4}{25}} + \frac{4e^{\frac{3}{5}}}{1 + \frac{9}{25}} + \frac{2e^{\frac{4}{5}}}{1 + \frac{16}{25}} + 2e + \frac{2e^{\frac{6}{5}}}{1 + \frac{36}{25}} + \frac{4e^{\frac{7}{5}}}{1 + \frac{49}{25}} + \frac{2e^{\frac{8}{5}}}{1 + \frac{64}{25}} + \frac{4e^{\frac{9}{5}}}{1 + \frac{81}{25}} + \frac{e^2}{5} \right] \\ &\approx 2.6632. \end{aligned}$$

2. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

(a) $\int_0^{+\infty} \frac{1}{4+x^2} dx$

(b) $\int_e^{+\infty} \frac{1}{x \ln^2 x} dx$

(c) $\int_{-\infty}^{+\infty} \frac{e^x}{1+e^{2x}} dx$

(d) $\int_1^{+\infty} \frac{5}{x^3} dx$

2.a

$$\begin{aligned}
 & \int_0^{+\infty} \frac{1}{4+x^2} dx \\
 &= \lim_{t \rightarrow +\infty} \int_0^t \frac{1}{4+x^2} dx \\
 &= \lim_{t \rightarrow +\infty} \frac{1}{2} \tan^{-1} \frac{x}{2} \Big|_0^t \\
 &= \lim_{t \rightarrow +\infty} \frac{1}{2} \tan^{-1} \frac{t}{2} - \left(\frac{1}{2} \tan^{-1} 0 \right) \\
 &= \frac{1}{2} \lim_{t \rightarrow +\infty} \tan^{-1} \frac{t}{2} \\
 &= \frac{\pi}{4}.
 \end{aligned}$$

2.b

Let $u = \ln x$

$$du = \frac{1}{x} dx$$

When $x = t$, $u = \ln t$

When $x = e$, $u = \ln e = 1$.

$$\begin{aligned}
 & \int_e^{+\infty} \frac{1}{x \ln^2 x} dx \\
 &= \lim_{t \rightarrow +\infty} \int_e^t \frac{1}{x \ln^2 x} dx \\
 &= \lim_{t \rightarrow +\infty} \int_1^{\ln t} \frac{1}{u^2} du \\
 &= \lim_{t \rightarrow +\infty} -u^{-1} \Big|_1^{\ln t} \\
 &= \lim_{t \rightarrow +\infty} -\frac{1}{\ln(t)} + 1 \\
 &= 1.
 \end{aligned}$$

2.c

$$\begin{aligned}
& \int_{-\infty}^{+\infty} \frac{e^x}{1+(e^x)^2} dx && \text{Let } u = e^x \\
& \int_{-\infty}^0 \frac{e^x}{1+(e^x)^2} dx + \int_0^{+\infty} \frac{e^x}{1+(e^x)^2} dx && du = e^x dx. \\
& \lim_{t \rightarrow -\infty} \int_t^0 \frac{e^x}{1+(e^x)^2} dx + \lim_{t \rightarrow +\infty} \int_0^t \frac{e^x}{1+(e^x)^2} dx. && I_1 : \text{When } x = t, u = e^t \quad I_2 : \text{When } x = 0, u = 1 \\
& && \text{When } x = 0, u = 1. \quad \text{When } x = t, u = e^t.
\end{aligned}$$

 $I_1 :$

$$\begin{aligned}
& \lim_{t \rightarrow -\infty} \int_t^0 \frac{e^x}{1+(e^x)^2} dx \\
& = \lim_{t \rightarrow -\infty} \int_{e^t}^1 \frac{du}{1+u^2} \\
& = \lim_{t \rightarrow -\infty} \tan^{-1} u \Big|_{e^t}^1 \\
& = \lim_{t \rightarrow -\infty} \tan^{-1} 1 - \tan^{-1} e^t \\
& = \frac{\pi}{4} - \lim_{t \rightarrow -\infty} \tan^{-1} e^t \\
& = \frac{\pi}{4} - \lim_{t \rightarrow 0} \tan^{-1} t \quad (\text{Since } \lim_{t \rightarrow -\infty} e^t = 0) \\
& = \frac{\pi}{4} - 0 \\
& = \frac{\pi}{4}.
\end{aligned}$$

Thus, I_1 converges to $\frac{\pi}{4}$ $I_2 :$

$$\begin{aligned}
& \lim_{t \rightarrow +\infty} \int_0^t \frac{e^x}{1+(e^x)^2} dx \\
& = \lim_{t \rightarrow +\infty} \int_1^{e^t} \frac{1}{1+u^2} du \\
& = \lim_{t \rightarrow +\infty} \tan^{-1} u \Big|_1^{e^t} \\
& = \lim_{t \rightarrow +\infty} \tan^{-1} e^t - \tan^{-1} 0 \\
& = \lim_{t \rightarrow +\infty} \tan^{-1} e^t - \frac{\pi}{4} \\
& = \lim_{t \rightarrow +\infty} \tan^{-1} t - \frac{\pi}{4} \quad (\text{Since } \lim_{t \rightarrow +\infty} e^t = +\infty) \\
& = \frac{\pi}{2} - \frac{\pi}{4} \\
& = \frac{\pi}{4}.
\end{aligned}$$

Thus, I_2 also converges to $\frac{\pi}{4}$, Which means we have:

$$\begin{aligned}
I &= \frac{\pi}{4} + \frac{\pi}{4} \\
&= \frac{\pi}{2}.
\end{aligned}$$

2.d

$$\begin{aligned}
& \int_1^{+\infty} \frac{5}{x^3} dx \\
&= \lim_{t \rightarrow +\infty} \int_1^t \frac{5}{x^3} dx \\
&= \lim_{t \rightarrow +\infty} \left. -\frac{5}{2x^2} \right|_1^t \\
&= \lim_{t \rightarrow +\infty} -\frac{5}{2t^2} + \frac{5}{2} \\
&= \frac{5}{2}.
\end{aligned}$$

3. Use the Comparison Theorem to determine whether the integral

$$\int_1^{+\infty} \frac{dx}{1+\sqrt{x}} dx.$$

is convergent or divergent

Remark. Let $f(x)$ and $g(x)$ be continuous over $[a, +\infty)$, assume $0 \leq f(x) \leq g(x)$

if $\int_a^{+\infty} f(x) dx = \lim_{t \rightarrow +\infty} \int_a^t f(x) dx = +\infty$

then $\int_a^{+\infty} g(x) dx = \lim_{t \rightarrow +\infty} \int_a^t g(x) dx = +\infty$

Alternatively,

if $\int_a^{+\infty} g(x) dx = \lim_{t \rightarrow +\infty} \int_a^t g(x) dx = L$ for $L \in \mathbb{R}$

then $\int_a^{+\infty} f(x) dx = \lim_{t \rightarrow +\infty} \int_a^t f(x) dx = M$ for $M \leq L$ where $M \in \mathbb{R}$

Let $f(x)$ be $\frac{1}{1+\sqrt{x}}$, choose $g(x) = \frac{1}{\sqrt{x}}$. If $g(x)$ diverges to $+\infty$, then $f(x)$ diverges to $+\infty$

$$\begin{aligned}
& \int_1^{+\infty} \frac{1}{\sqrt{x}} dx \\
&= \lim_{t \rightarrow +\infty} \int_1^t \frac{1}{\sqrt{x}} dx \\
&= \lim_{t \rightarrow +\infty} \left. 2x^{\frac{1}{2}} \right|_1^t \\
&= \lim_{t \rightarrow +\infty} 2t^{\frac{1}{2}} - 2.
\end{aligned}$$

Thus, $f(x) = \frac{1}{1+\sqrt{x}}$ diverges to $+\infty$ since $g(x) = \frac{1}{\sqrt{x}}$ diverges to $+\infty$

4. Find a formula for the general term of the arithmetic sequence whose first term is $a_1 = 1$ such that $a_{n-1} - a_n = 17$ for $n \geq 1$.

Remark. The general form of an arithmetic sequence is of the type $a_1 + (n-1)d$ for $n \geq 1$. S.t d is the common difference defined $a_n - a_{n-1}$, and a_1 is the first term in the sequence

With this, we can deduce $a_{n-1} - a_n = 17 \rightarrow d = -17$. With a_1 of course defined as 1. Consequently, the general form of this sequence would be

$$a_n = 1 + (n-1)(-17)$$

$$a_n = 1 - 17n + 17$$

$$a_n = -17n + 18.$$

5. Find a formula for the general term of the geometric sequence whose first term is $a_1 = 1$ such that $\frac{a_{n+1}}{a_n} = 10$ for $n \geq 1$.

Remark. The general form of a geometric sequence is of the type $a_n = ar^{n-1}$ for $n \geq 1$, where r is the common ratio defined $\frac{a_n}{a_{n-1}}$

With this, we can see that the common ratio r , is defined as 10, which makes the general form:

$$a_n = 10^{n-1}.$$

6. Find a formula for the general term of the sequence $\{4, -1, \frac{1}{4}, -\frac{1}{16}, \frac{1}{64}, \dots\}$.

For $n \geq 1$ this sequence has the general form

$$a_n = 4 \left(-\frac{1}{4} \right)^{n-1}.$$

Where the common ratio $r = -\frac{1}{4}$

7. Determine whether the sequence is convergent or divergent. If it is convergent, find its limit.

(a) $a_n = \frac{4+5n^2}{1+n}$

(b) $a_n = \tan^{-1}(n^2)$

(c) $a_n = \ln\left(\frac{n+2}{n^2-3}\right)$

(d) $a_n = n \sin\left(\frac{1}{n}\right)$

(e) $a_n = \left(1 - \frac{2}{n}\right)^n$