Discrete Structures

Number Theory

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Contents

Number Theory

1 Introduction

Definition 1. Number theory is the branch of mathematics that deals with the properties and relationships of numbers, especially the positive integers.

2 Parity

Definition 2. The **Parity** of an integer is its attribute of being even or odd

Even integers are in the form

$$2k, \quad \text{where } k \text{ is an integer.}$$

Odd Integers are in the form

$$2k + 1, \quad \text{where k is an integer.}$$

Proposition 1. Zero is an even integer. Let's consider the integer *zero*. By the common and standard definition, zero is considered an even integer. This is based on the property mentioned above: an even integer can be written in the form 2k, where k is an integer. When you plug in k = 0, you get 2 * 0 = 0, which is an even integer. In this sense, zero fits the definition of an even number. If you were to apply the definition of an odd integer, which is in the form 2k + 1, and plug in k = 0, you would indeed get 2 * 0 + 1 = 1. This would suggest that zero is odd. However, this is not the definition that is traditionally used.

Suppose we have the integer -101, how might we show that this is an odd integer? Well, let's see if we can derive an equation from the rule $2k + 1,$ where k is an integer

$$2(-51) + 1 = -101.$$

Thus, -101 is an odd integer

Now let's suppose we have the equation:

$$6a + 8b = 1, \quad \text{where a and b are integers.}$$

What result might this yield? Even or Odd?

To further examine this, we can use algebra. So:

$$2(3a + 4b) = 1$$

.

Since we know that a and b are integers, then we also know that $3a + 4b$ must also an integer. Thus, our equation is in the form $2k + 1$, where k is an integer. This means that this equation must produce an odd integer

Now suppose we have the equation

$$4a^2b.$$

Might this result in an even, or odd integer. To find out we can again utilize some algebra. So:

$$2(2a^2b), \quad \text{where } a \text{ and } b \text{ are integers.}$$

And since we know a and b are both integers, the product of $2a^2b$ must also be an integer. Thus, our equation is in the form $2k$, where k is an integer and this equation must yield an even integer.

3 Divisibility

Definition 1. Suppose we have $n, d \in \mathbb{Z}$ where $d \neq 0$. Then $n|d \iff \exists k \in \mathbb{Z} \mid n = dk$. Where $n|d$ is read "n divides d". If this theorem holds for any arbitrary integers n, d, then we say "n divides d"

Let's suppose we have

$$\frac{6}{18}.$$

Divisibility Rules.

1. Divisible by 1: All integers are divisible by 1
2. Divisible by 2: If the last digit of the integer is even.
3. Divisible by 3: If the sum of the digit's numbers are divisible by 3.
4. Divisible by 4: If the last 2 digits are divisible by 4
5. Divisible by 5: If the last digit is either 0 or 5.
6. Divisible by 6: If it is divisible by both 2 and 3. (For divisibility by 2 and 3, check rule 2 and 3)
7. Divisible by 7: If you double the last digit and subtract it from the rest of the number and the answer is either:
   - 0
   - divisible by 7
8. Divisible by 8: If the last three digits are divisible by 8.
9. Divisible by 9: If the sum of the digits are divisible by 9

10. Divisible by 10: If the number ends in 0.
11. Divisible by 11: Add and subtract digits in an alternating pattern (add first, subtract second, add third, etc). Then the answer must be either:
    - 0
    - Divisible by 11
12. Divisible by 12: If the number is both divisible by 3 and 4. (check divisibility rules for 3 and 4)

4 Prime Numbers

Definition 1. Prime Numbers are integers greater than 1 whose only factors are 1 and the number itself.

Definition 2. Composite Numbers are integers numbers greater than 1 and not prime

Definition 3. Fundamental theorem of arithmetic (Prime factorization theorem) states that Any integer greater than 1 is either a prime number, or can be written as a unique product of prime numbers (ignoring the order).

5 Prime factorization

Definition 1. Prime factorization is finding which prime numbers multiply together to make the original number.

Suppose we have the composite number 20. Which prime numbers multiply to make 20?

$$2 \cdot 2 \cdot 5.$$

So we say that the prime factors of 20 are $2 \cdot 2 \cdot 5$

To easily find the prime factorization of any composite number, we must follow these steps:

1. Find the smallest prime number that divides into our composite number.
2. Repeat steps until you are left with 1

For example, say we want to find the prime factorization of 50

$$2|50 = 25$$
$$5|25 = 5$$
$$5|5 = 1.$$

So in this case our prime factors are $2 \cdot 5 \cdot 5$

Definition 2. The **Unique factorization theorem** States that there's only one set of possible prime factors that can create a composite number

6 GCD and LCM

Definition 1. The greatest common divisor (GCD) of two nonzero integers a and b is the greatest positive integer d such that d is a divisor of both a and b

Suppose we have $a = 6$ $b = 10$. If we list out the divisors of both numbers:

$$6: \ 1, 2, 3, 6$$
$$10: \ 1, 2, 5, 10.$$

Then we can clearly see that the GCD is 2. Thus, $gcd(10, 6) = 2$

Definition 2. A **Multiple** of a number is a number that is the product of a given number and some other natural number

Definition 3. The **Least Common Multiple (LCD)** is the smallest multiple that two or more numbers have in common

Suppose we want to find

$$LCD(10, 4).$$

So we can list the multiples of both numbers and then find the smallest multiple that is common between both numbers. So:

$$4: \ 4, 8, 12, 16, 20, 24, ...$$
$$10: \ 10, 20, 30, 40, ...$$

So we can see that:

$$LCD(10, 4) = 20.$$