

# Calculus 1: Chapter 3

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# Chapter 3

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**3.1**

**Differential Rule:**

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***Differential Formulas:***

- $\frac{d}{dx}(c) = 0$
- $\frac{d}{dx}(x) = 1$
- $\frac{d}{dx}(x^n) = n \cdot x^{n-1} \rightarrow \textbf{Power Rule}$
- $\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}[f(x)]$
- $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

**Example 0.0.1**

Differentiate the following functions:

1.)  $f(t) = \frac{1}{2}t^6 - 3t^4 + 1$

For the first term, we will use the ***Third and Fourth*** Rule:

$$\frac{1}{2} \cdot 6t^{6-1}.$$

For the second term,  $-3t^4$ , We will use the ***Third and Fifth*** Rule:

$$-3 \cdot 4t^{4-1}.$$

The last term is a constant, so according to the first rule, the Derivative of a constant is ***Zero***:

*So our full equation is:*

$$\begin{aligned} f'(x) &= \frac{1}{2} \cdot 6t^{6-1} - 3 \cdot 4t^{4-1} + 0 \\ &= 3t^5 - 12t^3. \end{aligned}$$

2.)  $h(x) = (x - 2)(2x + 3)$

First we need to distribute out the terms:

$$\begin{aligned} h(x) &= 2x^2 + 3x - 4x - 6 \\ &= 2x^2 - x - 6. \end{aligned}$$

Now this is the function we want to differentiate.

So  $\rightarrow$

$$h'(x) = 2 \cdot 2x^{2-1} - 1 - 0$$

$$h'(x) = 4x - 1.$$

$$\mathbf{3.)} \ y = \frac{x^2 - 2\sqrt{x}}{x}$$

So:

$$y = \frac{x^2 - 2x^{\frac{1}{2}}}{x}.$$

Since the denominator only has **one term**, we can split the equation like:

$$y = \frac{x^2}{x} - \frac{2x^{\frac{1}{2}}}{x}$$

$$y = x - 2x^{-\frac{1}{2}}.$$

Now:

$$\frac{dy}{dx} = 1 - 2 \cdot \left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1}$$

$$\frac{dy}{dx} = 1 + x^{-\frac{3}{2}}.$$

And we can even rewrite it as:

$$\frac{dy}{dx} = 1 + \frac{1}{x^{\frac{3}{2}}}.$$

$$\mathbf{4.)} \ V = \left(\sqrt{x} + \frac{1}{\sqrt[3]{x}}\right)^2$$

So:

$$V = \left(x^{\frac{1}{2}} + x^{-\frac{1}{3}}\right)^2$$

$$= (x^{\frac{1}{2}})^2 + 2(x^{\frac{1}{2}})(x^{-\frac{1}{3}}) + (x^{-\frac{1}{3}})^2$$

$$= x + 2x^{\frac{1}{6}} + x^{-\frac{2}{3}}.$$

Now we find the Derivative:

$$V' = 1 + 2 \cdot \frac{1}{6}x^{\frac{1}{6}-1} + \left(\frac{-2}{3}\right)x^{-\frac{2}{3}-1}$$

$$v' = 1 + \frac{1}{3}x^{-\frac{5}{6}} - \frac{2}{3}x^{-\frac{5}{3}}$$

$$v' = 1 + \frac{1}{3x^{\frac{5}{6}}} - \frac{2}{3x^{\frac{5}{3}}}.$$

## Exponential Functions:

**Recall:**  $(1 + \frac{1}{n})^n \rightarrow e \approx 2.71828... as n \rightarrow \infty$

### Definition 0.0.1: Definiton of e:

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$$

### Note:-

We'll use the above definiton to derive  $\frac{d}{dx}(e^x)$

→ Let  $f(x) = e^x$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

So:

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^x \cdot e^h - e^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^x \cdot (e^h - 1)}{h}. \end{aligned}$$

This function is dependent on  $h$ , but  $e^x$  is not dependent on  $h$ , so we can pull it outside and rewrite as:

$$e^x \cdot \lim_{h \rightarrow 0} \frac{e^h - 1}{h}.$$

According to our definiton above, we can see that the right portion of this equation **Equals 1**, Therefor we are just left with:

$$e^x.$$

Therefore:

$$\frac{d}{dx}(e^x) = e^x.$$

**Example:** Find  $f'(x)$  and  $f''(x)$  of  $f(x) = e^x - x^3$

$$f'(x) = e^x - 3x^2.$$

$$f''(x) = e^x - 6x.$$

## Normal Line:

The normal line is perpendicular to the tangent line at the point of tangency.

$$m_{\text{tangent}} \cdot m_{\text{normal}} = -1.$$

### Note:-

This definition means that the slopes are ***Opposite Recipricals***

**Example:** find equations of the tangent line and the normal line to the curve  $y = x^4 + 8e^x$  at the point (0,8).

So we find the derivative:

$$y' = 4x^3 + 8e^x.$$

Then we find  $m_{\text{tan}}$ :

$$\begin{aligned} m_{\text{tan}} &= 4 \cdot 0^3 + 8e^0 \\ &= 0 + 8 \cdot 1 \\ &= 8. \end{aligned}$$

Then we find the slope of the normal line, so we take the Reciprical of  $m_{\text{tan}}$ , so we **flip it and change the sign**:

$$m_{\text{normal}} = -\frac{1}{8}.$$

We can check our answer using the defintion:

$$8\left(-\frac{1}{8}\right) = -1.$$

Now we find the equations of the lines:

### Tangent Line:

$$\begin{aligned} y - 8 &= 8(x - 0) \\ y - 8 &= 8x \\ y &= 8x + 8. \end{aligned}$$

### Normal Line:

$$\begin{aligned} y - 8 &= -\frac{1}{8}(x - 0) \\ y - 8 &= -\frac{1}{8}x \\ y &= -\frac{1}{8}x + 8. \end{aligned}$$

**Example:** The equation of motion of a particle is  $s = t^3 - 12t$

a.) Find  $v(t) = s'(t)$  - *Velocity*

So:

$$s'(t) = 3t^2 - 12.$$

B.) Find  $a(t) = s''(t)$  - *Acceleration*

So:

$$s''(t) = 6t.$$

c.) Find the acceleration after 9 seconds

So:

$$\begin{aligned} a(9) &= 6 \cdot 9 \\ &= 54m/s^2. \end{aligned}$$

d.) Find the acceleration when the velocity is 0.

So:

$$\text{Set } v(t) = 0$$

$$3t^2 - 12 = 0$$

$$3t^2 = 12$$

$$t^2 = 4$$

$$t = \pm 2 \rightarrow 2 \text{ Typically we like } t \text{ to be positive.}$$

Now:

$$\begin{aligned} a(2) &= 6 \cdot 2 \\ &= 12m/s^2. \end{aligned}$$

**3.2**

**The Product and Quotient Rules**

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**Product Rule:**

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \frac{d}{dx}[g(x)] + g(x) \frac{d}{dx}[f(x)].$$

**Or:**

$$(f \cdot g)' = f \cdot g' + g \cdot f'.$$

**Quotient Rule:**

$$\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx}[f(x)] - f(x) \frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

**Or:**

$$\left( \frac{f}{g} \right)' = \frac{g \cdot f' - f \cdot g'}{g^2}.$$

**Example:** Differentiate the following Function: **(Quotient Rule)**

1.)  $y = \frac{e^x}{1+x}$

So, If:

$$\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx}[f(x)] - f(x) \frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

And:

$$\begin{aligned} f(x) &= e^x \rightarrow f'(x) = e^x \\ g(x) &= 1 + x \rightarrow g'(x) = 1. \end{aligned}$$

Then:

$$\begin{aligned} y' &= \frac{(1+x)e^x - e^x(1)}{(1+x)^2} \\ &= \frac{e^x + xe^x - e^x}{(1+x)^2} \\ &= \frac{xe^x}{(1+x)^2}. \end{aligned}$$



**Example:** Differentiate The Following Function: **(Product Rule)**

2.)  $R(t) = (t + e^t)(3 - \sqrt{t})$

So If:

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \frac{d}{dx}[g(x)] + g(x) \frac{d}{dx}[f(x)].$$

And:

$$\begin{aligned} f(x) &= (t + e^t) \longrightarrow f'(x) = (1 + e^t) \\ g(x) &= (3 - t^{\frac{1}{2}}) \longrightarrow g'(x) = (0 - \frac{1}{2}t^{-\frac{1}{2}}). \end{aligned}$$

Then:

$$R'(t) = (t + e^t)(0 - \frac{1}{2}t^{-\frac{1}{2}}) + (1 + e^t)(3 - t^{-\frac{1}{2}}).$$

Cleanup:

$$\begin{aligned} R'(t) &= -\frac{1}{2}t^{\frac{1}{2}} - \frac{1}{2}e^t t^{-\frac{1}{2}} + 3 - t^{-\frac{1}{2}} + 3e^t \cdot t^{\frac{1}{2}} \\ &= -\frac{3}{2}t^{\frac{1}{2}} - \frac{1}{2}e^t t^{-\frac{1}{2}} + 3 + 3e^t \cdot t^{\frac{1}{2}} \\ &= -\frac{3}{2}t^{\frac{1}{2}} - \frac{e^t}{2t^{\frac{1}{2}}} + 3 + 3e^t \cdot t^{\frac{1}{2}}. \end{aligned}$$

Explanation for cleanup:

for the second equation, we just combined like terms, then for the **third equation**, we rewrote the term with the negative power.

**Example:** Differentiate the following function **(Product Rule:)**

3.)  $g(x) = 5e^x \sqrt{x}$

So:

$$g'(x) = (5e^x)(\frac{1}{2}x^{-\frac{1}{2}}) + (5e^x)(x^{\frac{1}{2}}).$$

From here we can simplify by pulling out common factor,  $5e^x x^{-\frac{1}{2}}$

So:

$$\begin{aligned} &5e^x x^{-\frac{1}{2}} \left( \frac{1}{2} + x^1 \right) \\ &= \frac{5e^x}{x^{\frac{1}{2}}} \cdot \frac{1 + 2x}{2} \\ &= \frac{5e^x(1 + 2x)}{2x^{\frac{1}{2}}}. \end{aligned}$$

**Example:** find  $f'(x)$  and  $f''(x)$

1.)  $f(x) = x^8 e^x$

So:

$$f'(x) = x^8 \cdot e^x + 8x^7 \cdot e^x.$$

We can factor out an  $e^x$

So,  $f'(x)$  is:

$$f'(x) = e^x(x^8 + 8x^7).$$

Now:

$$\begin{aligned} f''(x) &= e^x(8x^7 + 56x^6) + (x^8 + 8x^7)(e^x) \\ &= e^x(x^8 + 8x^7 + 8x^7 + 56x^6) \\ &= e^x(x^8 + 16x^7 + 56x^6). \end{aligned}$$

**Example:** Differentiate (*Quotient Rule*):

$$y = \frac{x+1}{x^3+x-2}.$$

If:

$$\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx} [f(x)] - f(x) \frac{d}{dx} [g(x)]}{[g(x)]^2}.$$

And:

$$f(x) = x+1 \longrightarrow f'(x) = 1$$

and

$$g(x) = x^3 + x - 2 \longrightarrow g'(x) = 3x^2 + 1.$$

Then:

$$\begin{aligned} y' &= \frac{(x^3+x-2)(1) - (x+1)(3x^2+1)}{(x^3+x-2)^2} \\ &= \frac{x^3+x-2 - (3x^3+x+3x^2+1)}{(x^3+x-2)^2} \\ &= \frac{x^3+x-2-3x^3-x-3x^2-1}{(x^3+x-2)^2} \\ &= \frac{-2x^3-3x^2-3}{(x^3+x-2)^2}. \end{aligned}$$

**Example:** Find the equation of the tangent line and the normal line to the curve  $y = \frac{\sqrt{x}}{x+1}$  at (4,0.4)

If:

$$\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx} [f(x)] - f(x) \frac{d}{dx} [g(x)]}{[g(x)]^2}.$$

And:

$$f(x) = x^{\frac{1}{2}} \longrightarrow f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$

and

$$g(x) = x + 1 \longrightarrow g'(x) = 1.$$

Then:

$$y' = \frac{(x+1)(\frac{1}{2}x^{-\frac{1}{2}}) - (x^{\frac{1}{2}})(1)}{(x+1)^2}.$$

Now  $m_{tan}$

$$\begin{aligned} m_{tan} &= \frac{(4+1)(\frac{1}{2} \cdot 4^{-\frac{1}{2}}) - (4^{\frac{1}{2}})}{(4+1)^2} \\ &= \frac{5 \cdot \frac{1}{4} - 2}{25}. \end{aligned}$$

We want to multiply by the lcd 4 to clear out the complex fraction

$$\begin{aligned} &\frac{(\frac{5}{4} - 2) \cdot 4}{25 \cdot 4} \\ &= \frac{5 - 8}{100} \\ &= -\frac{3}{100}. \end{aligned}$$

Now to find  $m_{normal}$ , we take the Reciprocal of  $m_{tan}$  and change the sign:

$$m_{norm} = \frac{100}{3}.$$

Now we want to find the equations:

**Tangent Line:**

$$\begin{aligned} y - 0.4 &= -0.03(x - 4) \\ y - 0.4 &= -0.03x + 0.12 \\ y &= -0.03x + 0.52. \end{aligned}$$

**Normal Line:**

$$\begin{aligned} y - \frac{2}{5} &= \frac{100}{3}(x - 4) \\ y - \frac{2}{5} &= \frac{100}{3}x - \frac{400}{3} \\ y &= \frac{100}{3}x - \frac{1994}{15}. \end{aligned}$$

Since  $\frac{100}{3}$  is a repeating decimal, we stayed in fraction form.

### 3.3

## Derivatives of Trigonometric Functions

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### Pythagorn Identities:

- $\sin^2 \theta = 1 - \cos^2 \theta$
- $\cos^2 \theta = 1 - \sin^2 \theta$
- $\sin^2 \theta + \cos^2 \theta = 1$

### 2 Limit Formulas:

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1.$$

And:

$$\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} = 0.$$

### Lets Derive $\frac{d}{dx}(\sin x)$ :

$$\frac{d}{dx}(\sin x) = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h}.$$

We will refer back to the formula for  $\sin(a+b) \rightarrow \sin A \cos B + \cos A \sin B$  to expand  $\sin(x+h)$

So:

$$\lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}.$$

We are going to split this equation:

$$\lim_{h \rightarrow 0} \frac{\sin x \cos h - \sin x}{h} + \lim_{h \rightarrow 0} \frac{\cos x \cdot \sin h}{h}.$$

Since  $\sin x$  and  $\cos x$  is not changing, it is therefore a constant and we can do the following:

$$(\sin x) \left( \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} \right) + (\cos x) \left( \lim_{h \rightarrow 0} \frac{\sin h}{h} \right).$$

Now we can use the formulas above and we are left with:

$$\begin{aligned} 0 + \cos x \cdot 1 \\ = \cos x. \end{aligned}$$

**Summary:**

$$\frac{d}{dx} \sin x = \cos x.$$

**Lets Derive**  $\frac{d}{dx}(\cos x)$ :

$$\frac{d}{dx}(\cos x) = \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h}.$$

We will refer back to the formula for  $\cos(A+B) \rightarrow \cos A \cos B - \sin A \sin B$  to expand  $\cos(x+h)$

So:

$$\lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}.$$

Just like the one above, we are going to group the terms that have  $x$ :

$$\lim_{h \rightarrow 0} \frac{\cos x \cos h - \cos x}{h} - \lim_{h \rightarrow 0} \frac{\sin x \sin h}{h}.$$

Now we pull out the constants:

$$(\cos x) \left( \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} \right) - (\sin x) \left( \lim_{h \rightarrow 0} \frac{\sin h}{h} \right).$$

Now if we use the fomulas listed at the start of this section we are left with:

$$\begin{aligned} &(\cos x)(0) - (\sin x)(1) \\ &= -\sin x. \end{aligned}$$

### Derivatives of Trigonometric Functions:

- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\csc x) = -\csc x \cot x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(\cot x) = -\csc^2 x$

**Examples: Differentiate:**

$$f(x) = \sqrt{x} \sin x.$$

If:

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \frac{d}{dx}[g(x)] + g(x) \frac{d}{dx}[f(x)].$$

And:

$$f(x) = x^{\frac{1}{2}}$$

$$g(x) = \sin x.$$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$

$$g'(x) = \cos x.$$

Then:

$$f'(x) = x^{\frac{1}{2}} \cdot \cos x + \sin x \cdot \frac{1}{2}x^{-\frac{1}{2}}$$

$$\frac{1}{2}x^{-\frac{1}{2}}(2x \cdot \cos x + \sin x)$$

$$= \frac{2x \cdot \cos x + \sin x}{2x^{\frac{1}{2}}}.$$

**Example: Differentiate:**

$$g(t) = 4 \sec t + \tan t.$$

So:

$$g'(t) = 4 \cdot \sec t \tan t + \sec^2 t$$

$$= 4 \cdot \frac{1}{\cos t} \cdot \frac{\sin t}{\cos t} + \frac{1}{\cos^2 t}$$

$$= 4 \cdot \frac{\sin t}{\cos^2 t} + \frac{1}{\cos^2 t}$$

$$= \frac{4 \sin t + 1}{\cos^2 t}.$$

**Example:**

$$y = \frac{1 - \sec x}{\tan x}.$$

If:

$$\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx} [f(x)] - f(x) \frac{d}{dx} [g(x)]}{[g(x)]^2}.$$

And:

$$f(x) = 1 - \sec x$$

$$g(x) = \tan x.$$

$$f'(x) = \sec x \tan x$$

$$g'(x) = \sec^2 x.$$

Then:

$$\begin{aligned}
 y' &= \frac{(\tan x)(-\sec x \tan x) - (1 - \sec x)(\sec^2 x)}{\tan^2 x} \\
 &= \frac{-\sec x \tan^2 x - (\sec^2 x - \sec^3 x)}{\tan^2 x} \\
 &= \frac{-\sec x \tan^2 x - \sec^2 x + \sec^3 x}{\tan^2 x} \\
 &= \frac{-\frac{1}{\cos x} \cdot \frac{\sin^2 x}{\cos^2 x} - \frac{1}{\cos^2 x} + \frac{1}{\cos^3 x}}{\frac{\sin^2 x}{\cos^2 x}}.
 \end{aligned}$$

We need to multiply by the lcd  $\cos^3 x$ :

$$\frac{-\sin^2 x - \cos x + 1}{\sin^2 x \cos x}.$$

In the numerator we notice we have  $1 - \sin^2 x$ , which is equal to  $\cos^2 x$ , so:

$$\begin{aligned}
 &\frac{\cos^2 x - \cos x}{\sin^2 x \cos x} \\
 &= \frac{\cos x(\cos x - 1)}{\sin^2 x \cos x} \\
 &= \frac{\cos x - 1}{\sin^2 x}.
 \end{aligned}$$

And we can replace the denominator with  $1 - \cos^2 x$ :

$$\frac{\cos x - 1}{1 - \cos^2 x}.$$

And we notice that the denominator is a difference of squares, so we can factor it into:

$$\begin{aligned}
 &\frac{\cos x - 1}{(1 - \cos x)(1 + \cos x)} \\
 &= \frac{-(1 - \cos x)}{(1 - \cos x)(1 + \cos x)} \\
 &= \frac{-1}{1 + \cos x}.
 \end{aligned}$$

## Limits:

Recall:

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \text{ and } \lim_{\theta \rightarrow 0} \frac{\theta}{\sin \theta} = 1.$$

Also:

$$\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} = 0.$$

**Example: Find the Limit:**

$$\lim_{x \rightarrow 0} \frac{\sin 4x}{\sin 6x}.$$

We want to be able to use the formulas above, so we do:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin 4x}{4x} \cdot \frac{4x}{1} \cdot \frac{6x}{\sin 6x} \cdot \frac{1}{6x} \\ = 1 \cdot 4 \cdot 1 \cdot \frac{1}{6} \\ = \frac{2}{3}.\end{aligned}$$

**Example: Find the Limit:**

$$\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\sin \theta}.$$

*To exercise the formulas above, we will rewrite as:*

$$\begin{aligned}\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\sin \theta} \cdot \frac{\theta}{\theta} \\ = \frac{\cos \theta - 1}{\theta} \cdot \frac{\theta}{\sin \theta} \\ = 0 \cdot 1 \\ = 0.\end{aligned}$$

**Example: Find the Limit:**

$$\lim_{t \rightarrow 0} \frac{\sin^2 3t}{t^2}.$$

*We rewrite as:*

$$\begin{aligned}\lim_{t \rightarrow 0} \left( \frac{\sin 3t}{t} \right)^2 \\ = \left( \frac{\sin 3t}{3t} \cdot \frac{3}{1} \right)^2 \\ = 1 \cdot 3^2 \\ = 9.\end{aligned}$$



**3.4****The Chain Rule / Differentiation Examples using the Product, Quotient, and Chain Rules****The Chain Rule:**

We will use the chain rule to find Derivatives of composite functions.

**Example: Find the derivative of**

$$F(x) = \sqrt{4 + 3x}.$$

$F(x)$  is a composite function made up of:

$$g(x) = 4 + 3x$$

and

$$f(x) = \sqrt{x}.$$

Therefore:

$$F(x) = f(g(x)).$$

Process:

Let:

$$u = g(x) = 4 + 3x.$$

Then:

$$F(x) = f(u) \text{ and } F'(x) = f'(u) \cdot g'(x).$$

**The Chain Rule (2):**

If  $F(x) = f(g(x))$ , then:

$$F'(x) = f'(g(x)) \cdot g'(x).$$

Or:

If  $y = f(u) = f(g(x))$ , then:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

**Example:** Find the Derivative:

$$f(x) = (1 + x^4)^{\frac{2}{3}}.$$

So:

$$\begin{aligned} f'(x) &= \frac{2}{3}(1 + x^4)^{-\frac{1}{3}} \cdot (4x^3) \\ &= \frac{8x^3}{3(1 + x^4)^{\frac{1}{3}}}. \end{aligned}$$

**Example:** Differentiate the following function:

$$f(t) = \sqrt[3]{1 + \tan t}.$$

So:

$$f(t) = (1 + \tan t)^{\frac{1}{3}}.$$

Now:

$$\begin{aligned} f'(t) &= \frac{1}{3}(1 + \tan t)^{-\frac{2}{3}} \cdot (\sec^2 t) \\ &= \frac{\sec^2 t}{3(1 + \tan t)^{\frac{2}{3}}}. \end{aligned}$$

**Example:** Differentiate The following function:

$$y = (x^2 + 1)(\sqrt[3]{x^2 + 2}).$$

So:

$$y = (x^2 + 1)(x^2 + 2)^{\frac{1}{3}}$$

First:

$$\begin{aligned} f(x) &= (x^2 + 1) \\ f'(x) &= (2x). \end{aligned}$$

$$g(x) = (x^2 + 2)^{\frac{1}{3}}.$$

To find  $g'(x)$ , we will use the chain rule:

$$g'(x) = \left[ \frac{1}{3}(x^2 + 2)^{-\frac{2}{3}} \cdot (2x) \right].$$

Now we use the product rule:

$$\frac{dy}{dx} = (x^2 + 1) \left[ \frac{1}{3}(x^2 + 2)^{-\frac{2}{3}} \cdot (2x) \right] + (x^2 + 2)^{\frac{1}{3}} \cdot (2x)$$

From here we will factor out a GCF:

$$\begin{aligned}
 \text{if } gcf &= \frac{1}{3}(2x)(x^2 + 2)^{-\frac{2}{3}} \\
 \text{then } \frac{dy}{dx} &= \frac{1}{3}(2x)(x^2 + 2)^{-\frac{2}{3}} \left[ (x^2 + 1) + 3(x^2 + 2) \right] \\
 &= \frac{2x(x^2 + 1 + 3x^2 + 6)}{3(x^2 + 2)^{\frac{2}{3}}} \\
 &= \frac{2x(4x^2 + 7)}{3(x^2 + 2)^{\frac{2}{3}}}
 \end{aligned}$$

**Example: Differentiate**

$$G(y) = \frac{(y - 1)^4}{(y^2 + 2y)^5}.$$

First:

$$\begin{aligned}
 f(x) &= (y - 1)^4 \\
 f'(x) &= 4(y - 1)^3.
 \end{aligned}$$

$$\begin{aligned}
 g(x) &= (y^2 + 2y)^5 \\
 g'(x) &= 5(y^2 + 2y)^4 \cdot (2y + 2).
 \end{aligned}$$

Now:

$$G'(y) = \frac{(y^2 + 2y)^5 4(y - 1)^3 \cdot 1 - (y - 1)^4 5(y^2 + 2y)^4 (2y + 2)}{(y^2 + 2y)^{10}}.$$

From here we can factor out a GCF:  $(y^2 + 2y)^4$  :

$$\frac{dG}{dy} = \frac{(y^2 + 2y)^4 (y - 1)^3 \cdot [4(y^2 + 2y) - (y - 1)5 \cdot 2(y + 1)]}{(y^2 + 2y)^{10}}.$$

We see that we can cancel out common term  $(y^2 + 2y)^4$  :

$$\begin{aligned}
 \frac{dG}{dy} &= \frac{(y - 1)^3 \cdot [4y^2 + 8y - 10(y^2 - 1)]}{(y^2 + 2y)^6} \\
 &= \frac{dG}{dy} = \frac{(y - 1)^3 \cdot (4y^2 + 8y - 10y^2 + 10)}{(y^2 + 2y)^6} \\
 &= \frac{dG}{dy} = \frac{(y - 1)^3 \cdot (-6y^2 + 8y + 10)}{(y^2 + 2y)^6} \\
 &= \frac{dG}{dy} = \frac{2(y - 1)^3 \cdot (-3y^2 + 4y + 5)}{(y^2 + 2y)^6}
 \end{aligned}$$

**Example:** Differentiate

$$y = \tan^2 3\theta.$$

*Start by rewriting:*

$$y = [\tan 3\theta]^2.$$

*Now we differentiate:*

$$\begin{aligned}\frac{dy}{d\theta} &= 2[\tan 3\theta] \cdot \sec^2 3\theta \cdot 3 \\ &= 6 \tan 3\theta \sec^2 3\theta.\end{aligned}$$

**Example:** Differentiate

$$y = x \sin\left(\frac{1}{x}\right).$$

*Start by rewriting as:*

$$y = x \cdot \sin x^{-1}.$$

*And we can derive:*

$$\begin{aligned}f(x) &= x \\ f'(x) &= 1.\end{aligned}$$

$$\begin{aligned}g(x) &= \sin x^{-1} \\ g'(x) &= \cos x^{-1} \cdot (-1x^{-2}).\end{aligned}$$

*Now we can use the product rule:*

$$y' = x \cdot \cos x^{-1} \cdot (-1x^{-2}) + \sin x^{-1} \cdot 1.$$

**Cleanup:**

$$\begin{aligned}y' &= -\cos \frac{1}{x} \cdot x^{-1} + \sin \frac{1}{x} \\ &= \frac{-\cos \frac{1}{x}}{x} + \sin \frac{1}{x} \\ &= \frac{-1}{x} \cos \frac{1}{x} + \sin \frac{1}{x}.\end{aligned}$$

Let's see what  $\frac{d}{dx}(a^x)$  is using the chain rule: ( $a > 0$ )

We know  $\frac{d}{dx}e^x = e^x$

Also Recall  $a = e^{\ln a}$

Therefore:

$$a^x = (e^{\ln a})^x.$$

Which means:

$$\begin{aligned}\frac{d}{dx}(a^x) &= \frac{d}{dx}[(e^{\ln a})^x] \\ &= \frac{d}{dx}(e^{x \cdot \ln a}) \\ &= e^{x \cdot \ln a} \cdot \frac{d}{dx}(x \cdot \ln a) \\ &= e^{x \cdot \ln a} \cdot \ln a \\ &= a^x \cdot \ln a.\end{aligned}$$

Summary:

$$a^x = a^x \cdot \ln a.$$

**Example: Differentiate**

$$y = 10^{1-x^2}.$$

So:

$$\begin{aligned}\frac{dy}{dx} &= 10^{1-x^2} \cdot \ln 10 \cdot (-2x) \\ &= -2x \ln 10 \cdot 10^{-1-x^2}.\end{aligned}$$

**Example: Differentiate**

$$y = 2^{3x^2}.$$

So this is:

$$f \circ g \circ h.$$

Where:

$$\begin{aligned}f(x) &= 2^x \\ g(x) &= 3^x \\ h(x) &= x^2.\end{aligned}$$

Therefore:

$$\begin{aligned}\frac{dy}{dx} &= 2^{3^{x^2}} \cdot \ln 2 \frac{dy}{dx}(3^{x^2}) \\ &= \frac{dy}{dx} = 2^{3^{x^2}} \cdot \ln 2 \cdot 3^{x^2} \cdot \ln 3 \cdot \frac{dy}{dx}(x^2) \\ &= \frac{dy}{dx} = 2^{3^{x^2}} \cdot \ln 2 \cdot 3^{x^2} \cdot \ln 3 \cdot 2x \\ &= 2x \cdot \ln 2 \cdot \ln 3 \cdot 2^{3^{x^2}} \cdot 3^{x^2}.\end{aligned}$$

**Shortcut:**

$$\begin{aligned}f(g(x)) &= \sqrt{g(x)} \\ f'(x) &= \frac{1}{2}(g(x))^{-\frac{1}{2}} \cdot g'(x) \\ f'(x) &= \frac{1}{2\sqrt{g(x)}} \cdot g'(x) \\ &= \frac{g'(x)}{2\sqrt{g(x)}}.\end{aligned}$$

**Example for shortcut:**

$$f(x) = \sqrt{\sin x}.$$

$$f'(x) = \frac{\cos x}{2\sqrt{\sin x}}.$$

**Example for shortcut:**

$$f(x) = \sqrt{4x^3 + 7x^2}.$$

$$\begin{aligned}f'(x) &= \frac{12x^2 + 14x}{2\sqrt{4x^3 + 7x^2}} \\ &= \frac{2(6x + 7)}{2\sqrt{4x^3 + 7x^2}}.\end{aligned}$$

## Differentiation Examples using the Product, Quotient, and Chain Rules

**Recall:**

Product Rule:

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \frac{d}{dx}[g(x)] + g(x) \frac{d}{dx}[f(x)].$$

Quotient Rule:

$$\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx}[f(x)] - f(x) \frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

**The Chain Rule:**

If  $F(x) = f(g(x))$ , then:

$$F'(x) = f'(g(x)) \cdot g'(x).$$

Or:

If  $f(u) = f(g(x))$ , then:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

**Example:** Differentiate the following function:

$$r = \frac{\sqrt{\theta} - 3}{\sqrt{\theta} + 3}.$$

*If:*

$$\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx}[f(x)] - f(x) \frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

*And:*

$$f(x) = \sqrt{\theta} - 3$$

$$f'(x) = \frac{1}{2} \theta^{-\frac{1}{2}}.$$

$$g(x) = \sqrt{\theta} + 3$$

$$g'(x) = \frac{1}{2} \theta^{-\frac{1}{2}}.$$

Then:

$$\begin{aligned}\frac{dr}{d\theta} &= \frac{(\sqrt{\theta} + 3)(\frac{1}{2}\theta^{-\frac{1}{2}}) - (\sqrt{\theta} - 3)(\frac{1}{2}\theta^{-\frac{1}{2}})}{(\sqrt{\theta} + 3)^2} \\ &= \frac{\frac{1}{2}\theta^{-\frac{1}{2}}(\sqrt{\theta} + 3 - \sqrt{\theta} + 3)}{(\sqrt{\theta} + 3)^2} \\ &= \frac{\frac{1}{2}\theta^{-\frac{1}{2}}(6)}{(\sqrt{\theta} + 3)^2} \\ &= \frac{3 \cdot \theta^{-\frac{1}{2}}}{(\sqrt{\theta} + 3)^2} \\ &= \frac{3}{\sqrt{\theta}(\sqrt{\theta} + 3)^2}.\end{aligned}$$

**Note:-**

It's fine that we have a radical in the denominator because there was one in the original equation.

**Example: Differentiate the following function:**

$$p = \frac{4 + \sec q}{4 - \sec q}.$$

We will rewrite in terms of *sin* and *cos*

$$p = \frac{4 + \frac{1}{\cos q}}{4 - \frac{1}{\cos q}}.$$

Now find common denominator to clear out fractions ( $\cos q$ ):

$$p = \frac{4 \cos q + 1}{4 \cos q - 1}.$$

Now we differentiate:

$$\begin{aligned}f(x) &= 4 \cos q + 1 \\ f'(x) &= -4 \sin q.\end{aligned}$$

$$\begin{aligned}g(x) &= 4 \cos q - 1 \\ g'(x) &= -4 \sin q.\end{aligned}$$

Now plug into Quotient Rule:

$$\frac{dp}{dq} = \frac{(4 \cos q - 1)(-4 \sin q) - (4 \cos q + 1)(-4 \sin q)}{(4 \cos q - 1)^2}.$$

we see we can factor out an  $-4 \sin q$ :

$$\begin{aligned}\frac{dp}{dq} &= \frac{-4 \sin q(4 \cos q - 1) - (4 \cos q + 1)(-4 \sin q)}{(4 \cos q - 1)^2} \\ &= \frac{-4 \sin q(4 \cos q - 1 - 4 \cos q - 1)}{(4 \cos q - 1)^2} \\ &= \frac{-4 \sin q(-2)}{(4 \cos q - 1)^2} \\ &= \frac{8 \sin q}{(4 \cos q - 1)^2}.\end{aligned}$$



**Example: Differentiate the following function:**

$$h(x) = \left( \frac{\cos x}{1 + \sin x} \right)^4.$$

First lets figure out our Derivatives from whats within the parenthesis:

$$\begin{aligned} f(x) &= \cos x \\ f'(x) &= -\sin x. \end{aligned}$$

$$\begin{aligned} g(x) &= 1 + \sin x \\ g'(x) &= \cos x. \end{aligned}$$

We will start by using the power rule and the chain rule with the quotient rule:

$$h'(x) = 4 \left[ \frac{\cos x}{1 + \sin x} \right]^3 \cdot \left[ \frac{(1 + \sin x)(-\sin x) - (\cos x)(\cos x)}{(1 + \sin x)^2} \right].$$

Now we want to distribute the exponent 3, into the terms in the numerator and denominator

$$\frac{4 \cos^3 x}{(1 + \sin x)^3} \cdot \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2}.$$

We are going to factor out a -1 and bring it in front of the 4:

$$\frac{-4 \cos^3 x}{(1 + \sin x)^3} \cdot \frac{\sin x + \sin^2 x + \cos^2 x}{(1 + \sin x)^2}.$$

We know that  $\sin^2 x + \cos^2 x = 1$ , so:

$$\frac{-4 \cos^3 x}{(1 + \sin x)^3} \cdot \frac{\sin x + 1}{(1 + \sin x)^2}.$$

Now we can divide by common factor in the numerator:

$$\begin{aligned} \frac{-4 \cos^3 x}{(1 + \sin x)^3} \cdot \frac{1}{1 + \sin x} \\ = \frac{-4 \cos^3 x}{(1 + \sin x)^4}. \end{aligned}$$

**Example:** Differentiate the following function:

$$y = (e^{\cos(\frac{t}{9})})^4.$$

So by using both the product rule and the chain rule, we get:

$$y' = 4(e^{\cos \frac{t}{9}})^3 \cdot e^{\cos \frac{t}{9}} \cdot -\sin \frac{t}{9} \cdot \frac{1}{9}.$$

Cleanup:

To start, we will group all the constants, then combine the like terms.:

$$y' = -\frac{4}{9} \sin \frac{t}{9} (e^{\cos \frac{t}{9}})^4.$$

Now we will move that power of 4 to the front of cos:

$$y' = -\frac{4}{9} \sin \frac{t}{9} (e^{4 \cos \frac{t}{9}}).$$

**Example:** Differentiate:

$$y = \sin(4x^2 e^x).$$

So:

$$y' = \cos(4x^2 e^x)$$

Now we want to use the product rule to derive whats inside the cosine function:

$$\begin{aligned} y' &= \cos(4x^2 e^x) \cdot [8x \cdot e^x + e^x \cdot 4x^2] \\ &= e^x(4x^2 + 8x) \cos 4x^2 e^x. \end{aligned}$$

Double Prime: To make this easier to grasp we will split it into 3 parts.

First Part:

$$e^x(4x^2 e^x)(\cos 4x^2 e^x).$$

Second Part:

$$8x + 8)e^x \cos 4x^2 e^x.$$

Third Part:

$$-\sin(4x^2 e^x).$$

And then apply the chain + product rule for the stuff inside -sin, which we did in single prime above

$$(8xe^x + 4x^2 e^x).$$

and then multiply by the other 2 functions:

$$e^x(4x^2 + 8x).$$

So all together part 3 is:

$$-\sin(4x^2e^x)(8xe^x + 4x^2e^x) \cdot e^x(4x^2 + 8x).$$

So if we add it all together:

$$y'' = e^x(4x^2e^x)(\cos 4x^2e^x + 8x + 8)e^x \cos 4x^2e^x + -\sin(4x^2e^x)(8xe^x + 4x^2e^x) \cdot e^x(4x^2 + 8x).$$

Cleanup: by factoring out terms:

$$\begin{aligned} & e^x \cdot 4x(x+2) \cos 4x^2e^x + e^x \cdot 8(x+1) \cos 4x^2e^x - [e^x(4x^2 + 8x)]^2 \sin(4x^2e^x) \\ & 4e^x \cos(4x^2e^x)[x(x+2) + 2(x+1)] - [e^x \cdot 4x(x+2)]^2 \sin(4x^2e^x) \\ & 4e^x \cos(4x^2e^x)[x(x+2) + 2(x+1)] - 16e^{2x} \cdot x^2(x+2)^2 \sin(4x^2e^x) \\ & 4e^x \cos(4x^2e^x)[x^2 + 4x + 2] - 16e^{2x} \cdot x^2(x+2)^2 \sin(4x^2e^x) \\ & 4e^x(x^2 + 4x + 2) \cos 4x^2e^x - 16e^{2x} \cdot x^2(x+2)^2 \sin(4x^2e^x) \end{aligned}$$

### 3.5

## Implicit Differentiation/Derivatives of Inverse Trigonometric Functions

$y = f(x)$ , in this form,  $y$  is expressed explicitly in terms of  $x$ . Some functions are defined implicitly by a relation between  $x$  and  $y$ .

**Example:** We'll use implicit differentiation to find  $\frac{dy}{dx}$

$$2x^3 + x^2y - xy^3 = 2.$$

So:

$$6x^2 + (x^2 \cdot 1 \cdot \frac{dy}{dx} + 2x \cdot y) - (x \cdot 3y^2 \cdot \frac{dy}{dx} + y^3 \cdot 1) = 0$$

These are the Derivatives of each term, we put  $\frac{dy}{dx}$  there because in the problem, we are deriving with respect to  $x$ , so when we derive the  $y$  terms,  $y$  is not  $x$ , so we put  $\frac{dy}{dx}$ .

Cleanup:

$$6x^2 + x^2 \frac{dy}{dx} + 2xy - 3xy^2 \frac{dy}{dx} - y^3 = 0.$$

Now we are solving for  $\frac{dy}{dx}$ , so we locate all the terms with  $\frac{dy}{dx}$  and keep them on the same side of the equation:

$$x^2 \frac{dy}{dx} - 3xy^2 \frac{dy}{dx} = y^3 - 6x^2 - 2xy.$$

Now that the terms on the left all have  $\frac{dy}{dx}$ , we can factor it out:

$$\frac{dy}{dx}(x^2 - 3xy^2) = y^3 - 6x^2 - 2xy.$$

Now divide by  $x^2 - 3xy^2$ :

$$\frac{dy}{dx} = \frac{y^3 - 6x^2 - 2xy}{x^2 - 3xy^2}.$$

**Example: Differentiate:**

$$\sin x + \cos y = \sin x \cos y.$$

So:

$$\cos x + (-\sin y \cdot \frac{dy}{dx}) = \sin x(-\sin y) \cdot \frac{dy}{dx} + \cos x \cdot \cos y.$$

Cleanup:

$$\begin{aligned}
 \cos x - \sin y \frac{dy}{dx} &= -\sin x \sin y \frac{dy}{dx} + \cos x \cos y \\
 &= \sin x \sin y \frac{dy}{dx} - \sin y \frac{dy}{dx} = \cos x \cos y - \cos x \\
 &= \frac{dy}{dx} (\sin x \sin y - \sin y) = \cos x \cos y - \cos x \\
 &= \frac{dy}{dx} = \frac{\cos x \cos y - \cos x}{\sin x \sin y - \sin y} \\
 &= \frac{dy}{dx} = \frac{\cos x (\cos y - 1)}{\sin y (\sin x - 1)}.
 \end{aligned}$$

**Example:**

$$\sqrt{x+y} = 1 + x^2 y^2.$$

So:

$$\frac{1}{2\sqrt{x+y}} \left(1 + \frac{dy}{dx}\right) = 0 + (x^2 \cdot 2y \cdot \frac{dy}{dx} + y^2 \cdot 2x)$$

Now we multiply through by  $2\sqrt{x+y}$  to clear out the fraction, so:

$$\begin{aligned}
 \rightarrow \frac{dy}{dx} &= 4x^2 y \sqrt{x+y} + 4xy^2 \sqrt{x+y} \\
 \rightarrow \frac{dy}{dx} - 4x^2 y \sqrt{x+y} &= 4xy^2 \sqrt{x+y} - 1 \\
 \rightarrow \frac{dy}{dx} [1 - 4x^2 y \sqrt{x+y}] &= 4xy^2 \sqrt{x+y} - 1 \\
 \rightarrow \frac{dy}{dx} &= \frac{4xy^2 \sqrt{x+y} - 1}{1 - 4x^2 y \sqrt{x+y}}.
 \end{aligned}$$

**Example: Differentiate:**

$$\sqrt{x} + \sqrt{y} = 1.$$

y':

$$\begin{aligned}
 \frac{1}{2\sqrt{x}}(1) + \frac{1}{2\sqrt{y}}(1) \cdot \frac{dy}{dx} &= 0 \\
 \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} &= 0 \\
 \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} &= -\frac{1}{2\sqrt{x}} \\
 \left(\frac{2\sqrt{y}}{1}\right) \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} &= -\frac{1}{2\sqrt{x}} \cdot \left(\frac{2\sqrt{y}}{1}\right) \\
 \frac{dy}{dx} &= -\frac{2 \cdot \sqrt{y}}{2 \cdot \sqrt{x}} \\
 \frac{dy}{dx} &= -\frac{\sqrt{y}}{\sqrt{x}}
 \end{aligned}$$

$y''$  (need to use quotient rule):

So:

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{-(\sqrt{x} \cdot \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} - \sqrt{y} \cdot \frac{1}{2\sqrt{x}})}{(\sqrt{x})^2} \\ &= \frac{-(\frac{\sqrt{x}}{2\sqrt{y}} \cdot \frac{-\sqrt{y}}{\sqrt{x}} - \frac{\sqrt{y}}{2\sqrt{x}})}{x}\end{aligned}$$

**Note:-**

Notice: in place of  $\frac{dy}{dx}$ , we put the derivative we found to be  $y'$

Cleanup:

$$\begin{aligned}& \frac{-(\frac{1}{2} - \frac{\sqrt{y}}{2\sqrt{x}})}{x} \\ &= \frac{(-\frac{1}{2} + \frac{\sqrt{y}}{2\sqrt{x}})}{x} \\ &= \frac{(-\frac{1}{2} + \frac{\sqrt{y}}{2\sqrt{x}}) \cdot 2\sqrt{x}}{x \cdot 2\sqrt{x}} \\ &= \frac{\sqrt{x} + \sqrt{y}}{2x^{\frac{3}{2}}}.\end{aligned}$$

How  $2x^{\frac{3}{2}}$ :

$$\begin{aligned}& x^1 \cdot 2x^{\frac{1}{2}} \\ &= 2x^{\frac{1}{2}+1} \\ &= 2x^{\frac{3}{2}}.\end{aligned}$$

Look back at original equation:

$$\sqrt{x} + \sqrt{y} = \underline{1}.$$

Therefore:

$$y'' = \frac{1}{2x^{\frac{3}{2}}}.$$

## Derivatives of Inverse Trig Functions

**Recall:**

$$y = \sin^{-1} x \longrightarrow \sin y = x.$$

And Restriction is:

$$\left(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\right).$$

**Proof:** Using implicit Differentiation, derive the above  $\sin y = x$ :

$$\begin{aligned} \cos y \cdot \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\cos y}. \end{aligned}$$

From here, because:  $\sin^2 y + \cos^2 y = 1$ , we can solve for  $\cos y$  and get:

$$\cos y = \pm \sqrt{1 - \sin^2 y}.$$

And we go with the positive sign because if we notice the ***Restriction***, we are in quad 1 & 4, where sin is ***positive***

So with this we can turn:

$$\frac{dy}{dx} = \frac{1}{\cos y}.$$

Into:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - \sin^2 y}}.$$

From here if we look back at the original equation, we see that  $\sin y = x$ , so we replace  $\sin^2 y$  with  $x^2$ .

**Therefore:**

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

Using the concepts above, we know:

- $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

**Example:** Find  $\frac{dy}{dx}$

$$y = \sqrt{\tan^{-1} x}.$$

So:

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\tan^{-1} x}} \cdot \frac{1}{1+x^2}$$

Cleanup:

$$\frac{dy}{dx} = \frac{1}{2(x^2 + 1)\sqrt{\tan^{-1} x}}.$$

**Example:** find  $\frac{dy}{dx}$

$$y = \arctan \sqrt{\frac{1-x}{1+x}}.$$

So:

$$\frac{dy}{dx} = \frac{1}{1 + \left(\sqrt{\frac{1-x}{1+x}}\right)^2}$$

Then by the chain rule and the quotient rule:

$$\frac{dy}{dx} = \frac{1}{1 + \left(\sqrt{\frac{1-x}{1+x}}\right)^2} \cdot \frac{1}{2} \left(\frac{1-x}{1+x}\right)^{-\frac{1}{2}} \cdot \frac{(1+x)(-1) - (1-x)(1)}{(1+x)^2}.$$

Cleanup:

$$\frac{dy}{dx} = \frac{1}{1 + \frac{1-x}{1+x}} \cdot \frac{1}{2} \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} \left[ \frac{-1-x-1+x}{(1+x)^2} \right].$$



We are going to steal an  $(1+x)$  from the denominator of that last area and send it to the denominator in the first area to clear out the fraction, so:

$$\frac{1}{1+x+1-x} \cdot \frac{1}{2} \cdot \frac{(1+x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}} \cdot \frac{-2}{(1+x)}.$$

*now we can cancel out the  $x$ 's in the denominator of the first fraction, and cancel out the 2 from the  $\frac{1}{2}$  and the 2 from the numerator of the last fraction*

$$\begin{aligned} & -\frac{1}{2} \cdot \frac{(1+x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}} \cdot \frac{1}{(1+x)^1} \\ &= -\frac{1}{2} \cdot \frac{(1+x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}(1+x)^1} \\ &= \frac{-1}{2(1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}} \\ &= \frac{-1}{2\sqrt{(1-x)(1+x)}} \\ &= \frac{-1}{2\sqrt{1-x^2}}. \end{aligned}$$

### 3.6

## Derivatives of Logarithmic Functions:

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**Recall:**

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$

And:

$$\frac{d}{dx} e^x = e^x.$$

Also:

$$\frac{d}{dx} a^x = a^x \cdot \ln a.$$

**Proof: Derive:**

$$\frac{d}{dx} \log_a x.$$

**Process:** Let  $y = \log_a x$ , then rewrite in exponential form

$$a^y = x.$$

Now we derive,

$$\begin{aligned} a^y \cdot \ln a \cdot \frac{dy}{dx} &= 1 \\ = \frac{dy}{dx} &= \frac{1}{a^y \cdot \ln a} \\ &= \frac{1}{x \ln a}. \end{aligned}$$

**Summarize:**

$$\frac{dy}{dx} \log_a x = \frac{1}{x \cdot \ln a}.$$

**And we know:**

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$

Because it's:

$$\frac{1}{x \cdot \ln e} = \frac{1}{x \cdot 1}.$$

**Example:** Find the derivative of  $f(x) = \ln(\sin^2 x)$

So:

$$\begin{aligned} f'(x) &= \frac{1}{\sin^2 x} \cdot 2(\sin x)^1 \cdot \cos x \\ &= \frac{2 \cos x}{\sin x} \\ &= 2 \cot x \end{aligned}$$

**Example:** find the derivative of:

$$f(x) = \log_5 x e^x.$$

So:

$$\begin{aligned} f'(x) &= \frac{1}{x e^x \cdot \ln 5} (x + e^x + 1 \cdot e^x) \\ &= \frac{e^x (x + 1)}{x e^x \ln 5} \\ &= \frac{x + 1}{x \ln 5} \end{aligned}$$

**Example:** Find the derivative of:

$$F(y) = y \ln(1 + e^y).$$

So:

$$\begin{aligned} F'(y) &= y \cdot \frac{1}{1 + e^y} (0 + e^y) + \ln(1 + e^y) \cdot 1 \\ &= \frac{y e^y}{1 + e^y} + \ln(1 + e^y) \end{aligned}$$

**Note:-**

$\ln(a + b)$  cannot be simplified, because  $\neq \ln a + \ln b$

**Example:** Find the derivative of

$$y = \log_2(e^{-x} \cos \pi x).$$

So:

$$\frac{dy}{dx} = \frac{1}{e^{-x} \cos \pi x \cdot \ln 2} \cdot (e^{-x} \cdot (-\sin \pi x) \cdot \pi + (-e^{-x}) \cos \pi x)$$

Cleanup:

$$\begin{aligned} &\frac{e^{-x}(-\pi \sin \pi x - \cos \pi x)}{e^{-x} \cos \pi x \cdot \ln 2} \\ &= \frac{-\pi \sin \pi x - \cos \pi x}{\cos \pi x \ln 2}. \end{aligned}$$

## Logarithmic Differentiation:

Step 1: Take  $\ln$  of both sides

Step 2: Differentiate implicitly with respect to  $x$

Step 3: Solve for  $y'$

### Example: Differentiate

$$y = \sqrt[4]{\frac{x^2 + 1}{x^2 - 1}}.$$

So:

$$\ln y = \ln \sqrt[4]{\frac{x^2 + 1}{x^2 - 1}}$$

$$\ln y = \ln \left( \frac{x^2 + 1}{x^2 - 1} \right)^{\frac{1}{4}}$$

$$\ln y = \frac{1}{4} \ln \left( \frac{x^2 + 1}{x^2 - 1} \right)$$

$$\ln y = \frac{1}{4} \ln (x^2 + 1) - \frac{1}{4} \ln (x^2 - 1)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{4} \cdot \frac{1}{x^2 + 1} \cdot 2x - \frac{1}{4} \cdot \frac{1}{x^2 - 1} \cdot 2x$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{x}{2(x^2 + 1)} - \frac{x}{2(x^2 - 1)}$$

multiply by common denominator:

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{x}{2(x^2 + 1)} \cdot \frac{x^2 - 1}{x^2 - 1} - \frac{x}{2(x^2 - 1)} \cdot \frac{x^2 + 1}{x^2 + 1}$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{x(x^2 - 1)}{2(x^2 + 1)(x^2 - 1)} - \frac{x(x^2 + 1)}{2(x^2 - 1)(x^2 + 1)}$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{x^3 - x}{2(x^2 + 1)(x^2 - 1)} - \frac{x^3 + x}{2(x^2 - 1)(x^2 + 1)}$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{x^3 - x - (x^3 + x)}{2(x^2 + 1)(x^2 - 1)}$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{x^3 - x - x^3 - x}{2(x^4 - 1)}$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{-2x}{2(x^4 - 1)}$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{-x}{x^4 - 1}$$

Now solve for  $\frac{dy}{dx}$ ,

$$\frac{dy}{dx} = \frac{-x}{x^4 - 1} \cdot y.$$

go to the original equation and see what  $y$  is equal to:

$$\frac{dy}{dx} = \frac{-x}{x^4 - 1} \cdot \sqrt[4]{\frac{x^2 + 1}{x^2 - 1}}.$$

**Main Idea:**

If you have a problem that uses chain rule, product rule, quotient rule, all together. It's best to use this Logarithmic definition

**Example:**

$$y = (\cos 5x)^x.$$

**Note:-**

In this example, it's absolutely necessary to use the definition and steps from above because you have a variable in the base, and in the exponent

So:

$$\ln y = \ln (\cos 5x)^x$$

use the Logarithmic property and move the exponent to the front, derive by using the product rule, and the chain rule

$$\frac{1}{y} \frac{dy}{dx} = x \cdot \frac{1}{\cos 5x} \cdot (-\sin 5x) \cdot 5 + 1 \cdot \ln (\cos 5x).$$

**Cleanup:**

$$\frac{1}{y} \frac{dy}{dx} = \frac{-5 \sin 5x}{\cos 5x} + \ln \cos 5x.$$

Solve for  $\frac{dy}{dx}$  by multiplying both sides by  $y$ :

$$\frac{dy}{dx} = (-5x \tan 5x + \ln (\cos 5x)) \cdot y.$$

Now replace  $y$  with what  $y$  equals in the original equation:

$$\frac{dy}{dx} = (-5x \tan 5x + \ln (\cos 5x)) \cdot (\cos 5x)^x.$$

### 3.7

## Rates of Change in the Natural and Social Sciences

Now that we have the tools to calculate most derivatives, let's see how to apply them in the natural and social sciences

**Example:** A particle moves according to a law of motion  $s = f(t)$ ,  $t > 0$ , where  $t$  is measured in seconds and  $s$  in measured in feet

$$f(t) = t^3 - 9t^2 + 24t.$$

a.) What is the velocity at time  $t$ ?

We know that the velocity is the derivative of the function, therefore we will find  $f'(t)$

So:

$$f'(t) = 3t^2 - 18t + 24$$

b.) When is the particle at rest?

to answer this question we must set  $v(t) = 0$

So:

$$\begin{aligned} 3t^2 - 18t + 24 &= 0 \\ &= t^2 - 6t + 8 = 0 \\ (t - 2)(t - 4) &= 0 \end{aligned}$$

Therefore the particle is at rest at:

$$t = 2, 4 \text{ seconds.}$$

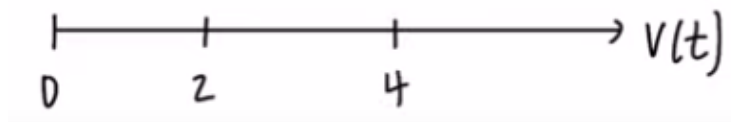
c.) When is the particle moving in positive direction?

If the particle is moving in the positive direction then  $v(t) > 0$

So:

$$3(t - 2)(t - 4) > 0$$

To solve quadratic inequalities we need to make a number line and test each interval



To start we will test the interval 0-2, so if we plug in 1 into  $(t-2)(t-4)$ , we get a negative times a negative, which will be a **positive**, therefore the velocity in this interval is positive

Now if we test the interval 2-4, with 3, we will get a negative times a positive, which is negative, therefore the velocity in the this interval is negative

Last if we test the interval for 4 onward, with 5, we will get a positive times a positive, so the velocity in this interval is positive.

**Therefore:**

The particle is moving in the positive direction in the intervals:

$$[0, 2) \cup (4, \infty].$$

**Note:-**

Since  $v(0) = 24 > 0$ , we will include zero on the interval, also notice the 2 and 4 is not inclusive

d.) What is the total distance traveled during the first 6 seconds?

In order to compute the total distance traveled:

$$\text{total distance} = |f(2) - f(0)| + |f(4) - f(2)| + |f(6) - f(4)|.$$

So:

$$\begin{aligned} &|20 - 0| + |16 - 20| + |36 - 16| \\ &= 20 + 4 + 20 \\ &= 44 \text{ ft} \end{aligned}$$

**Extra:**

$$\begin{aligned} \text{displacement} &= f(6) - f(0) \\ &= 36 - 0 \\ &= 36 \text{ ft.} \end{aligned}$$

e.) what is the acceleration of the particle at time t?

to find this, we must find the second derivative of the velocity function

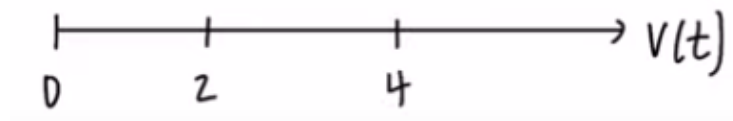
So:

$$a(t) = 6t - 18$$

f.) When is the particle speeding up, and when is it slowing down?

It is speeding up when  $v(t)$  and  $a(t)$  have the same sign, and it is slowing down when  $v(t)$  and  $a(t)$  have opposite signs

we will use the number line again, and then make a new one for  $a(t)$ :



to construct the number line of  $a(t)$ , we need to set  $a(t) = 0$

$$6t - 18 = 0$$

$$t = 3.$$



so if we plug in something smaller than 3, say **2**, then:

$$\begin{aligned} 6(2) - 18 \\ = -6. \end{aligned}$$

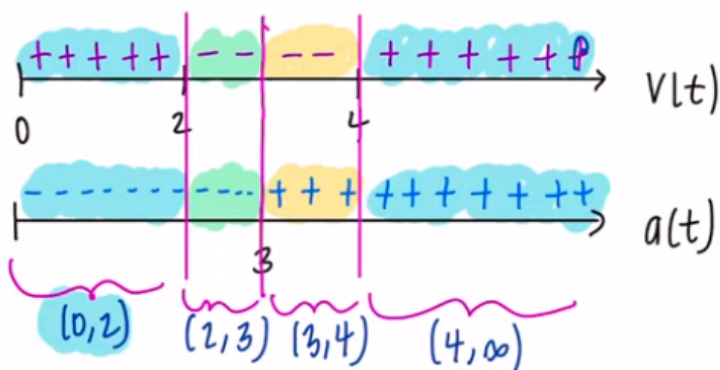
And if we plug in something bigger than 3, say 4, then:

$$\begin{aligned} 6(4) - 18 \\ = 6. \end{aligned}$$

So smaller than 3 is negative and bigger than 3 is positive.

Now if we stack our 2 number lines, and draw lines through each of the zeros, we can find when the signs are the same, and when they are opposite to determine when the particle is speeding up and slowing down.

So:



Speeding Up:  $(2, 3) \cup (4, \infty)$

Slowing Down:  $(0, 2) \cup (3, 4)$



**Example:** The height (in meters) of a projectile shot vertically upward from a point 3 feet above ground level with an initial velocity of  $40\text{ ft/s}$  is  $h = 3 + 40t - 16t^2$

- Find the velocity after 2 seconds and after 4 seconds
- When does the projectile reach its maximum height?
- What is the maximum height?
- When does it hit the ground?
- With what velocity does it hit the ground?

a.) To get  $v(t)$ , take the derivative of  $h$  and plug in values 2 and 4:

$$v(t) = h'(t) = 40 - 32t.$$

$$\begin{aligned} v(2) &= 40 - 64 \\ &= -24\text{ ft/s.} \end{aligned}$$

$$\begin{aligned} v(4) &= 40 - 128 \\ &= -88\text{ ft/s.} \end{aligned}$$

b.) Set  $v(t) = 0$

$$\begin{aligned} 40 - 32t &= 0 \\ t &= \frac{40}{32} \\ &= \frac{5}{4} \text{ seconds.} \end{aligned}$$

**Note:-**

we set the velocity function equal to zero, because when the projectile reaches its max height, it will stop moving before the velocity changes to negative

c.) Plug  $\frac{5}{4}$  into  $h$  from the original equation:

$$\begin{aligned} h\left(\frac{5}{4}\right) &= 3 + 40\left(\frac{5}{4}\right) - 16\left(\frac{5}{4}\right)^2 \\ &= 28\text{ ft.} \end{aligned}$$

d.) Set  $h$  equal to zero and solve:

$$\begin{aligned} 3 + 40t - 16t^2 &= 0 \\ &= 16t^2 - 40t - 3. \end{aligned}$$

Since this will not factor, we must use the quadratic formula:

If:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

And:

$$\begin{aligned} a &= 16 \\ b &= -40 \\ c &= -3. \end{aligned}$$

Then:

$$t = \frac{40 \pm \sqrt{40^2 - 4(16)(-3)}}{2(16)}$$

$$t = \frac{40 \pm \sqrt{1792}}{32}$$

$$t = \frac{40 \pm 16\sqrt{7}}{32}$$

$$t = \frac{5 \pm 2\sqrt{7}}{4}.$$

Since we are dealing with time, we will only report the positive value

So:

$$\frac{5 + 2\sqrt{7}}{4} \text{ seconds}$$

e.) Plug what we found for t into the velocity function:

$$v\left(\frac{5 + 2\sqrt{7}}{4}\right) = 40 - 32\left(\frac{5 + 2\sqrt{7}}{4}\right)$$

$$= 40 - 40 - 16\sqrt{7}$$

$$= -16\sqrt{7} \text{ ft/s.}$$

**Example:** if a rock is thrown vertically upward from the surface of Mars with a velocity of  $15 \text{ m/s}$ , then its height after  $t$  seconds is  $h = 15t - 1.86t^2$

a.) What is the velocity of the rock after 2 seconds?

b.) What is the velocity of the rock when its height is 25m on its way up? On its way down?

a.) Find the derivative of h and plug in 2

So:

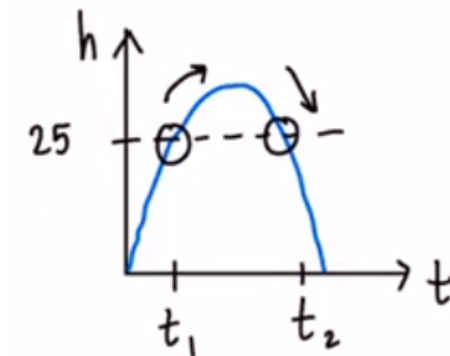
$$v(t) = h' = 15 - 3.72t$$

$$v(2) = 15 - 3.72(2)$$

$$= 7.56 \text{ m/s.}$$

b.) Set  $h(t) = 25$

Graph to visualize:



So:

$$\begin{aligned} 25 &= 15t - 1.86t^2 \\ &= 1.86t^2 - 15t - 25 \end{aligned}$$

Now we will use the quadratic formula

$$\begin{aligned} t &= \frac{15 \pm \sqrt{225 - 4(1.86)(25)}}{2(1.86)} \\ &= \frac{15 \pm \sqrt{225 - 186}}{3.72} \\ &= \frac{15 \pm \sqrt{39}}{3.72} \approx 5.71, 2.35. \end{aligned}$$

**Note:-**

The earlier time is on the way up, and the later time is on the way down.

From here we need to find the velocity at each of these times, so we plug them both into the velocity function:

$$\begin{aligned} v(2.35) &= 15 - 3.72(2.35) \\ &\approx 6.26 \text{ m/s.} \end{aligned}$$

$$\begin{aligned} v(5.71) &= 15 - 3.72(5.71) \\ &\approx -6.24 \text{ m/s.} \end{aligned}$$

Summarize:

$s(t) \rightarrow$  position fcn, also sometimes  $(h(t), x(t), y(t), f(t) \text{ etc...})$

$s'(t) = v(t)$  (velocity function)

$s''(t) = v'(t) = a(t)$  (acceleration function)

**3.8**

**Exponential Growth and Decay**

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