

Calculus 1: Chapter 3

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Chapter 3

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3.1

Differential Rule:

Differential Formulas:

- $\frac{d}{dx}(c) = 0$
- $\frac{d}{dx}(x) = 1$
- $\frac{d}{dx}(x^n) = n \cdot x^{n-1} \rightarrow \textbf{Power Rule}$
- $\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}[f(x)]$
- $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

Example 0.0.1

Differentiate the following functions:

1.) $f(t) = \frac{1}{2}t^6 - 3t^4 + 1$

For the first term, we will use the *Third and Fourth* Rule:

$$\frac{1}{2} \cdot 6t^{t-1}.$$

For the second term, $-3t^4$, We will use the *Third and Fifth* Rule:

$$-3 \cdot 4t^{4-1}.$$

The last term is a constant, so according to the first rule, the Derivative of a constant is *Zero*:

So our full equation is:

$$\begin{aligned} f'(x) &= \frac{1}{2} \cdot 6t^{6-1} - 3 \cdot 4t^{4-1} + 0 \\ &= 3t^5 - 12t^3. \end{aligned}$$

2.) $h(x) = (x - 2)(2x + 3)$

First we need to distribute out the terms:

$$\begin{aligned} h(x) &= 2x^2 + 3x - 4x - 6 \\ &= 2x^2 - x - 6. \end{aligned}$$

Now this is the function we want to differentiate.

So \rightarrow

$$\begin{aligned}h'(x) &= 2 \cdot 2x^{2-1} - 1 - 0 \\h'(x) &= 4x - 1.\end{aligned}$$

3.) $y = \frac{x^2 - 2\sqrt{x}}{x}$

So:

$$y = \frac{x^2 - 2x^{\frac{1}{2}}}{x}.$$

Since the denominator only has **one term**, we can split the equation like:

$$\begin{aligned}y &= \frac{x^2}{x} - \frac{2x^{\frac{1}{2}}}{x} \\y &= x - 2x^{-\frac{1}{2}}.\end{aligned}$$

Now:

$$\begin{aligned}\frac{dy}{dx} &= 1 - 2 \cdot \left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1} \\ \frac{dy}{dx} &= 1 + x^{-\frac{3}{2}}.\end{aligned}$$

And we can even rewrite it as:

$$\frac{dy}{dx} = 1 + \frac{1}{x^{\frac{3}{2}}}.$$

4.) $V = (\sqrt{x} + \frac{1}{\sqrt[3]{x}})^2$

So:

$$\begin{aligned}V &= (x^{\frac{1}{2}} + x^{-\frac{1}{3}})^2 \\ &= (x^{\frac{1}{2}})^2 + 2(x^{\frac{1}{2}})(x^{-\frac{1}{3}}) + (x^{-\frac{1}{3}})^2 \\ &= x + 2x^{\frac{1}{6}} + x^{-\frac{2}{3}}.\end{aligned}$$

Now we find the Derivative:

$$\begin{aligned}V' &= 1 + 2 \cdot \frac{1}{6}x^{\frac{1}{6}-1} + \left(\frac{-2}{3}\right)x^{-\frac{2}{3}-1} \\ v' &= 1 + \frac{1}{3}x^{-\frac{5}{6}} - \frac{2}{3}x^{-\frac{5}{3}} \\ v' &= 1 + \frac{1}{3x^{\frac{5}{6}}} - \frac{2}{3x^{\frac{5}{3}}}.\end{aligned}$$

Exponential Functions:

Recall: $(1 + \frac{1}{n})^n \rightarrow e \approx 2.71828...$ as $n \rightarrow \infty$

Definition 0.0.1: Definition of e:

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$$

Note:-

We'll use the above definition to derive $\frac{d}{dx}(e^x)$

→ Let $f(x) = e^x$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

So:

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^x \cdot e^h - e^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^x \cdot (e^h - 1)}{h}. \end{aligned}$$

This function is dependent on h , but e^x is not dependent on h , so we can pull it outside and rewrite as:

$$e^x \cdot \lim_{h \rightarrow 0} \frac{e^h - 1}{h}.$$

According to our definition above, we can see that the right portion of this equation **Equals 1**, Therefore we are just left with:

$$e^x.$$

Therefore:

$$\frac{d}{dx}(e^x) = e^x.$$

Example: Find $f'(x)$ and $f''(x)$ of $f(x) = e^x - x^3$

$$f'(x) = e^x - 3x^2.$$

$$f''(x) = e^x - 6x.$$

Normal Line:

The normal line is perpendicular to the tangent line at the point of tangency.

$$m_{\text{tangent}} \cdot m_{\text{normal}} = -1.$$

Note:-

This definition means that the slopes are ***Opposite Recipricals***

Example: find equations of the tangent line and the normal line to the curve $y = x^4 + 8e^x$ at the point (0,8).

So we find the derivative:

$$y' = 4x^3 + 8e^x.$$

Then we find m_{tan} :

$$\begin{aligned} m_{\text{tan}} &= 4 \cdot 0^3 + 8e^0 \\ &= 0 + 8 \cdot 1 \\ &= 8. \end{aligned}$$

Then we find the slope of the normal line, so we take the Reciprical of m_{tan} , so we **flip it and change the sign**:

$$m_{\text{normal}} = -\frac{1}{8}.$$

We can check our answer using the definiton:

$$8\left(-\frac{1}{8}\right) = -1.$$

Now we find the equations of the lines:

Tangent Line:

$$\begin{aligned} y - 8 &= 8(x - 0) \\ y - 8 &= 8x \\ y &= 8x + 8. \end{aligned}$$

Normal Line:

$$\begin{aligned} y - 8 &= -\frac{1}{8}(x - 0) \\ y - 8 &= -\frac{1}{8}x \\ y &= -\frac{1}{8}x + 8. \end{aligned}$$

Example: The equation of motion of a particle is $s = t^3 - 12t$

a.) Find $v(t) = s'(t)$ - *Velocity*

So:

$$s'(t) = 3t^2 - 12.$$

B.) Find $a(t) = s''(t)$ - *Acceleration*

So:

$$s''(t) = 6t.$$

c.) Find the acceleration after 9 seconds

So:

$$\begin{aligned} a(9) &= 6 \cdot 9 \\ &= 54m/s^2. \end{aligned}$$

d.) Find the acceleration when the velocity is 0.

So:

$$\text{Set } v(t) = 0$$

$$3t^2 - 12 = 0$$

$$3t^2 = 12$$

$$t^2 = 4$$

$$t = \pm 2 \rightarrow 2 \text{ Typically we like } t \text{ to be positive.}$$

Now:

$$\begin{aligned} a(2) &= 6 \cdot 2 \\ &= 12m/s^2. \end{aligned}$$

3.2

The Product and Quotient Rules

Product Rule:

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \frac{d}{dx}[g(x)] + g(x) \frac{d}{dx}[f(x)].$$

Or:

$$(f \cdot g)' = f \cdot g' + g \cdot f'.$$

Quotient Rule:

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx}[f(x)] - f(x) \frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

Or:

$$\left(\frac{f}{g} \right)' = \frac{g \cdot f' - f \cdot g'}{g^2}.$$

Example: Differentiate the following Function: (Quotient Rule)

1.) $y = \frac{e^x}{1+x}$

So, If:

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx}[f(x)] - f(x) \frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

And:

$$\begin{aligned} f(x) &= e^x \rightarrow f'(x) = e^x \\ g(x) &= 1+x \rightarrow g'(x) = 1. \end{aligned}$$

Then:

$$\begin{aligned} y' &= \frac{(1+x)e^x - e^x(1)}{(1+x)^2} \\ &= \frac{e^x + xe^x - e^x}{(1+x)^2} \\ &= \frac{xe^x}{(1+x)^2}. \end{aligned}$$

Example: Differentiate The Following Function: **(Product Rule)**

2.) $R(t) = (t + e^t)(3 - \sqrt{t})$

So If:

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \frac{d}{dx}[g(x)] + g(x) \frac{d}{dx}[f(x)].$$

And:

$$\begin{aligned} f(x) &= (t + e^t) \longrightarrow f'(x) = (1 + e^t) \\ g(x) &= (3 - t^{\frac{1}{2}}) \longrightarrow g'(x) = (0 - \frac{1}{2}t^{-\frac{1}{2}}). \end{aligned}$$

Then:

$$R'(t) = (t + e^t)(0 - \frac{1}{2}t^{-\frac{1}{2}}) + (1 + e^t)(3 - t^{-\frac{1}{2}}).$$

Cleanup:

$$\begin{aligned} R'(t) &= -\frac{1}{2}t^{\frac{1}{2}} - \frac{1}{2}e^t t^{-\frac{1}{2}} + 3 - t^{-\frac{1}{2}} + 3e^t \cdot t^{\frac{1}{2}} \\ &= -\frac{3}{2}t^{\frac{1}{2}} - \frac{1}{2}e^t t^{-\frac{1}{2}} + 3 + 3e^t \cdot t^{\frac{1}{2}} \\ &= -\frac{3}{2}t^{\frac{1}{2}} - \frac{e^t}{2t^{\frac{1}{2}}} + 3 + 3e^t \cdot t^{\frac{1}{2}}. \end{aligned}$$

Explanation for cleanup:

for the second equation, we just combined like terms, then for the **third equation**, we rewrote the term with the negative power.

Example: Differentiate the following function **(Product Rule:)**

3.) $g(x) = 5e^x \sqrt{x}$

So:

$$g'(x) = (5e^x)(\frac{1}{2}x^{-\frac{1}{2}}) + (5e^x)(x^{\frac{1}{2}}).$$

From here we can simplify by pulling out common factor, $5e^x x^{-\frac{1}{2}}$

So:

$$\begin{aligned} &5e^x x^{-\frac{1}{2}} \left(\frac{1}{2} + x^1 \right) \\ &= \frac{5e^x}{x^{\frac{1}{2}}} \cdot \frac{1 + 2x}{2} \\ &= \frac{5e^x(1 + 2x)}{2x^{\frac{1}{2}}}. \end{aligned}$$

Example: find $f'(x)$ and $f''(x)$

1.) $f(x) = x^8 e^x$

So:

$$f'(x) = x^8 \cdot e^x + 8x^7 \cdot e^x.$$

We can factor out an e^x

So, $f'(x)$ is:

$$f'(x) = e^x(x^8 + 8x^7).$$

Now:

$$\begin{aligned} f''(x) &= e^x(8x^7 + 56x^6) + (x^8 + 8x^7)(e^x) \\ &= e^x(x^8 + 8x^7 + 8x^7 + 56x^6) \\ &= e^x(x^8 + 16x^7 + 56x^6). \end{aligned}$$

Example: Differentiate (*Quotient Rule*):

$$y = \frac{x+1}{x^3+x-2}.$$

If:

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx} [f(x)] - f(x) \frac{d}{dx} [g(x)]}{[g(x)]^2}.$$

And:

$$f(x) = x+1 \longrightarrow f'(x) = 1$$

and

$$g(x) = x^3 + x - 2 \longrightarrow g'(x) = 3x^2 + 1.$$

Then:

$$\begin{aligned} y' &= \frac{(x^3 + x - 2)(1) - (x+1)(3x^2 + 1)}{(x^3 + x - 2)^2} \\ &= \frac{x^3 + x - 2 - (3x^3 + x + 3x^2 + 1)}{(x^3 + x - 2)^2} \\ &= \frac{x^3 + x - 2 - 3x^3 - x - 3x^2 - 1}{(x^3 + x - 2)^2} \\ &= \frac{-2x^3 - 3x^2 - 3}{(x^3 + x - 2)^2}. \end{aligned}$$

Example: Find the equation of the tangent line and the normal line to the curve $y = \frac{\sqrt{x}}{x+1}$ at (4,0.4)

If:

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx} [f(x)] - f(x) \frac{d}{dx} [g(x)]}{[g(x)]^2}.$$

And:

$$f(x) = x^{\frac{1}{2}} \longrightarrow f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$

and

$$g(x) = x + 1 \longrightarrow g'(x) = 1.$$

Then:

$$y' = \frac{(x+1)(\frac{1}{2}x^{-\frac{1}{2}}) - (x^{\frac{1}{2}})(1)}{(x+1)^2}.$$

Now m_{tan}

$$\begin{aligned} m_{tan} &= \frac{(4+1)(\frac{1}{2} \cdot 4^{-\frac{1}{2}}) - (4^{\frac{1}{2}})}{(x+1)^2} \\ &= \frac{5 \cdot \frac{1}{4} - 2}{25}. \end{aligned}$$

We want to multiply by the lcd 4 to clear out the complex fraction

$$\begin{aligned} &\frac{(\frac{5}{4} - 2) \cdot 4}{25 \cdot 4} \\ &= \frac{5 - 8}{100} \\ &= -\frac{3}{100}. \end{aligned}$$

Now to find m_{normal} , we take the Reciprocal of m_{tan} and change the sign:

$$m_{norm} = \frac{100}{3}.$$

Now we want to find the equations:

Tangent Line:

$$\begin{aligned} y - 0.4 &= -0.03(x - 4) \\ y - 0.4 &= -0.03x + 0.12 \\ y &= -0.03x + 0.52. \end{aligned}$$

Normal Line:

$$\begin{aligned} y - \frac{2}{5} &= \frac{100}{3}(x - 4) \\ y - \frac{2}{5} &= \frac{100}{3}x - \frac{400}{3} \\ y &= \frac{100}{3}x - \frac{1994}{15}. \end{aligned}$$

Since $\frac{100}{3}$ is a repeating decimal, we stayed in fraction form.

3.3

Derivatives of Trigonometric Functions

Pythagorn Identites:

- $\sin^2 \theta = 1 - \cos^2 \theta$
- $\cos^2 \theta = 1 - \sin^2 \theta$
- $\sin^2 \theta + \cos^2 \theta = 1$

2 Limit Formulas:

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1.$$

And:

$$\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} = 0.$$

Lets Derive $\frac{d}{dx}(\sin x)$:

$$\frac{d}{dx}(\sin x) = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h}.$$

We will refer back to the formula for $\sin(a+b) \rightarrow \sin A \cos B + \cos A \sin B$ to expand $\sin(x+h)$

So:

$$\lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}.$$

We are going to split this equation:

$$\lim_{h \rightarrow 0} \frac{\sin x \cos h - \sin x}{h} + \lim_{h \rightarrow 0} \frac{\cos x \cdot \sin h}{h}.$$

Since $\sin x$ and $\cos x$ is not changing, it is therefore a constant and we can do the following:

$$(\sin x) \left(\lim_{h \rightarrow 0} \frac{\cos h - 1}{h} \right) + (\cos x) \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right).$$

Now we can use the formulas above and we are left with:

$$\begin{aligned} 0 + \cos x \cdot 1 \\ = \cos x. \end{aligned}$$

Summary:

$$\frac{d}{dx} \sin x = \cos x.$$

Lets Derive $\frac{d}{dx}(\cos x)$:

$$\frac{d}{dx}(\cos x) = \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h}.$$

We will refer back to the formula for $\cos(A+B) \rightarrow \cos A \cos B - \sin A \sin B$ to expand $\cos(x+h)$

So:

$$\lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}.$$

Just like the one above, we are going to group the terms that have x :

$$\lim_{h \rightarrow 0} \frac{\cos x \cos h - \cos x}{h} - \lim_{h \rightarrow 0} \frac{\sin x \sin h}{h}.$$

Now we pull out the constants:

$$(\cos x) \left(\lim_{h \rightarrow 0} \frac{\cos h - 1}{h} \right) - (\sin x) \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right).$$

Now if we use the fomulas listed at the start of this section we are left with:

$$\begin{aligned} & (\cos x)(0) - (\sin x)(1) \\ & = -\sin x. \end{aligned}$$

Derivatives of Trigonometric Functions:

- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\csc x) = -\csc x \cot x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(\cot x) = -\csc^2 x$

Examples: Differentiate:

$$f(x) = \sqrt{x} \sin x.$$

If:

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \frac{d}{dx}[g(x)] + g(x) \frac{d}{dx}[f(x)].$$

And:

$$f(x) = x^{\frac{1}{2}}$$
$$g(x) = \sin x.$$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$
$$g'(x) = \cos x.$$

Then:

$$f'(x) = x^{\frac{1}{2}} \cdot \cos x + \sin x \cdot \frac{1}{2}x^{-\frac{1}{2}}$$
$$\frac{1}{2}x^{-\frac{1}{2}}(2x \cdot \cos x + \sin x)$$
$$= \frac{2x \cdot \cos x + \sin x}{2x^{\frac{1}{2}}}.$$

Example: Differentiate:

$$g(t) = 4 \sec t + \tan t.$$

So:

$$g'(t) = 4 \cdot \sec t \tan t + \sec^2 t$$
$$= 4 \cdot \frac{1}{\cos t} \cdot \frac{\sin t}{\cos t} + \frac{1}{\cos^2 t}$$
$$= 4 \cdot \frac{\sin t}{\cos^2 t} + \frac{1}{\cos^2 t}$$
$$= \frac{4 \sin t + 1}{\cos^2 t}.$$

Example:

$$y = \frac{1 - \sec x}{\tan x}.$$

If:

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx} [f(x)] - f(x) \frac{d}{dx} [g(x)]}{[g(x)]^2}.$$

And:

$$f(x) = 1 - \sec x$$
$$g(x) = \tan x.$$

$$f'(x) = \sec x \tan x$$
$$g'(x) = \sec^2 x.$$

Then:

$$\begin{aligned}
 y' &= \frac{(\tan x)(-\sec x \tan x) - (1 - \sec x)(\sec^2 x)}{\tan^2 x} \\
 &= \frac{-\sec x \tan^2 x - (\sec^2 x - \sec^3 x)}{\tan^2 x} \\
 &= \frac{-\sec x \tan^2 x - \sec^2 x + \sec^3 x}{\tan^2 x} \\
 &= \frac{-\frac{1}{\cos x} \cdot \frac{\sin^2 x}{\cos^2 x} - \frac{1}{\cos^2 x} + \frac{1}{\cos^3 x}}{\frac{\sin^2 x}{\cos^2 x}}.
 \end{aligned}$$

We need to multiply by the lcd $\cos^3 x$:

$$\frac{-\sin^2 x - \cos x + 1}{\sin^2 x \cos x}.$$

In the numerator we notice we have $1 - \sin^2 x$, which is equal to $\cos^2 x$, so:

$$\begin{aligned}
 &\frac{\cos^2 x - \cos x}{\sin^2 x \cos x} \\
 &= \frac{\cos x(\cos x - 1)}{\sin^2 x \cos x} \\
 &= \frac{\cos x - 1}{\sin^2 x}.
 \end{aligned}$$

And we can replace the denominator with $1 - \cos^2 x$:

$$\frac{\cos x - 1}{1 - \cos^2 x}.$$

And we notice that the denominator is a difference of squares, so we can factor it into:

$$\begin{aligned}
 &\frac{\cos x - 1}{(1 - \cos x)(1 + \cos x)} \\
 &= \frac{-(1 - \cos x)}{(1 - \cos x)(1 + \cos x)} \\
 &= \frac{-1}{1 + \cos x}.
 \end{aligned}$$

Limits:

Recall:

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \text{ and } \lim_{\theta \rightarrow 0} \frac{\theta}{\sin \theta} = 1.$$

Also:

$$\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} = 0.$$

Example: Find the Limit:

$$\lim_{x \rightarrow 0} \frac{\sin 4x}{\sin 6x}.$$

We want to be able to use the formulas above, so we do:

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\sin 4x}{4x} \cdot \frac{4x}{1} \cdot \frac{6x}{\sin 6x} \cdot \frac{1}{6x} \\
 = 1 \cdot 4 \cdot 1 \cdot \frac{1}{6} \\
 = \frac{2}{3}.
 \end{aligned}$$

Example: Find the Limit:

$$\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\sin \theta}.$$

To exercise the formulas above, we will rewrite as:

$$\begin{aligned}
 \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\sin \theta} \cdot \frac{\theta}{\theta} \\
 = \frac{\cos \theta - 1}{\theta} \cdot \frac{\theta}{\sin \theta} \\
 = 0 \cdot 1 \\
 = 0.
 \end{aligned}$$

Example: Find the Limit:

$$\lim_{t \rightarrow 0} \frac{\sin^2 3t}{t^2}.$$

We rewrite as:

$$\begin{aligned}
 \lim_{t \rightarrow 0} \left(\frac{\sin 3t}{t} \right)^2 \\
 = \left(\frac{\sin 3t}{3t} \cdot \frac{3}{1} \right)^2 \\
 = 1 \cdot 3^2 \\
 = 9.
 \end{aligned}$$

3.4

The Chain Rule / Differentiation Examples using the Product, Quotient, and Chain Rules

The Chain Rule:

We will use the chain rule to find Derivatives of composite functions.

Example: Find the derivative of

$$F(x) = \sqrt{4 + 3x}.$$

$F(x)$ is a composite function made up of:

$$g(x) = 4 + 3x$$

and

$$f(x) = \sqrt{x}.$$

Therefore:

$$F(x) = f(g(x)).$$

Process:

Let:

$$u = g(x) = 4 + 3x.$$

Then:

$$F(x) = f(u) \text{ and } F'(x) = f'(u) \cdot g'(x).$$

The Chain Rule (2):

If $F(x) = f(g(x))$, then:

$$F'(x) = f'(g(x)) \cdot g'(x).$$

Or:

If $y = f(u) = f(g(x))$, then:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example: Find the Derivative:

$$f(x) = (1 + x^4)^{\frac{2}{3}}.$$

So:

$$\begin{aligned} f'(x) &= \frac{2}{3}(1 + x^4)^{-\frac{1}{3}} \cdot (4x^3) \\ &= \frac{8x^3}{3(1 + x^4)^{\frac{1}{3}}}. \end{aligned}$$

Example: Differentiate the following function:

$$f(t) = \sqrt[3]{1 + \tan t}.$$

So:

$$f(t) = (1 + \tan t)^{\frac{1}{3}}.$$

Now:

$$\begin{aligned} f'(t) &= \frac{1}{3}(1 + \tan t)^{-\frac{2}{3}} \cdot (\sec^2 t) \\ &= \frac{\sec^2 t}{3(1 + \tan t)^{\frac{2}{3}}}. \end{aligned}$$

Example: Differentiate The following function:

$$y = (x^2 + 1)(\sqrt[3]{x^2 + 2}).$$

So:

$$y = (x^2 + 1)(x^2 + 2)^{\frac{1}{3}}$$

First:

$$\begin{aligned} f(x) &= (x^2 + 1) \\ f'(x) &= (2x). \end{aligned}$$

$$g(x) = (x^2 + 2)^{\frac{1}{3}}.$$

To find $g'(x)$, we will use the chain rule:

$$g'(x) = \left[\frac{1}{3}(x^2 + 2)^{-\frac{2}{3}} \cdot (2x) \right].$$

Now we use the product rule:

$$\frac{dy}{dx} = (x^2 + 1) \left[\frac{1}{3}(x^2 + 2)^{-\frac{2}{3}} \cdot (2x) \right] + (x^2 + 2)^{\frac{1}{3}} \cdot (2x)$$

From here we will factor out a GCF:

$$\begin{aligned}
 \text{if } gcf &= \frac{1}{3}(2x)(x^2 + 2)^{-\frac{2}{3}} \\
 \text{then } \frac{dy}{dx} &= \frac{1}{3}(2x)(x^2 + 2)^{-\frac{2}{3}} \left[(x^2 + 1) + 3(x^2 + 2) \right] \\
 &= \frac{2x(x^2 + 1 + 3x^2 + 6)}{3(x^2 + 2)^{\frac{2}{3}}} \\
 &= \frac{2x(4x^2 + 7)}{3(x^2 + 2)^{\frac{2}{3}}}
 \end{aligned}$$

Example: Differentiate

$$G(y) = \frac{(y-1)^4}{(y^2+2y)^5}.$$

First:

$$\begin{aligned}
 f(x) &= (y-1)^4 \\
 f'(x) &= 4(y-1)^3.
 \end{aligned}$$

$$\begin{aligned}
 g(x) &= (y^2+2y)^5 \\
 g'(x) &= 5(y^2+2y)^4 \cdot (2y+2).
 \end{aligned}$$

Now:

$$G'(y) = \frac{(y^2+2y)^5 4(y-1)^3 \cdot 1 - (y-1)^4 5(y^2+2y)^4 (2y+2)}{(y^2+2y)^{10}}.$$

From here we can factor out a GCF: $(y^2+2y)^4$:

$$\frac{dG}{dy} = \frac{(y^2+2y)^4 (y-1)^3 \cdot [4(y^2+2y) - (y-1)5 \cdot 2(y+1)]}{(y^2+2y)^{10}}.$$

We see that we can cancel out common term $(y^2+2y)^4$:

$$\begin{aligned}
 \frac{dG}{dy} &= \frac{(y-1)^3 \cdot [4y^2 + 8y - 10(y^2 - 1)]}{(y^2+2y)^6} \\
 &= \frac{dG}{dy} = \frac{(y-1)^3 \cdot (4y^2 + 8y - 10y^2 + 10)}{(y^2+2y)^6} \\
 &= \frac{dG}{dy} = \frac{(y-1)^3 \cdot (-6y^2 + 8y + 10)}{(y^2+2y)^6} \\
 &= \frac{dG}{dy} = \frac{2(y-1)^3 \cdot (-3y^2 + 4y + 5)}{(y^2+2y)^6}
 \end{aligned}$$

Example: Differentiate

$$y = \tan^2 3\theta.$$

Start by rewriting:

$$y = [\tan 3\theta]^2.$$

Now we differentiate:

$$\begin{aligned}\frac{dy}{d\theta} &= 2[\tan 3\theta] \cdot \sec^2 3\theta \cdot 3 \\ &= 6 \tan 3\theta \sec^2 3\theta.\end{aligned}$$

Example: Differentiate

$$y = x \sin\left(\frac{1}{x}\right).$$

Start by rewriting as:

$$y = x \cdot \sin x^{-1}.$$

And we can derive:

$$\begin{aligned}f(x) &= x \\ f'(x) &= 1.\end{aligned}$$

$$\begin{aligned}g(x) &= \sin x^{-1} \\ g'(x) &= \cos x^{-1} \cdot (-1x^{-2}).\end{aligned}$$

Now we can use the product rule:

$$y' = x \cdot \cos x^{-1} \cdot (-1x^{-2}) + \sin x^{-1} \cdot 1.$$

Cleanup:

$$\begin{aligned}y' &= -\cos \frac{1}{x} \cdot x^{-1} + \sin \frac{1}{x} \\ &= \frac{-\cos \frac{1}{x}}{x} + \sin \frac{1}{x} \\ &= \frac{-1}{x} \cos \frac{1}{x} + \sin \frac{1}{x}.\end{aligned}$$

Let's see what $\frac{d}{dx}(a^x)$ is using the chain rule: ($a > 0$)

We know $\frac{d}{dx}e^x = e^x$

Also Recall $a = e^{\ln a}$

Therefore:

$$a^x = (e^{\ln a})^x.$$

Which means:

$$\begin{aligned}\frac{d}{dx}(a^x) &= \frac{d}{dx}[(e^{\ln a})^x] \\ &= \frac{d}{dx}(e^{x \cdot \ln a}) \\ &= e^{x \cdot \ln a} \cdot \frac{d}{dx}(x \cdot \ln a) \\ &= e^{x \cdot \ln a} \cdot \ln a \\ &= a^x \cdot \ln a.\end{aligned}$$

Summary:

$$a^x = a^x \cdot \ln a.$$

Example: Differentiate

$$y = 10^{1-x^2}.$$

So:

$$\begin{aligned}\frac{dy}{dx} &= 10^{1-x^2} \cdot \ln 10 \cdot (-2x) \\ &= -2x \ln 10 \cdot 10^{-1-x^2}.\end{aligned}$$

Example: Differentiate

$$y = 2^{3x^2}.$$

So this is:

$$f \circ g \circ h.$$

Where:

$$\begin{aligned}f(x) &= 2^x \\ g(x) &= 3^x \\ h(x) &= x^2.\end{aligned}$$

Therefore:

$$\begin{aligned}\frac{dy}{dx} &= 2^{3^{x^2}} \cdot \ln 2 \frac{dy}{dx}(3^{x^2}) \\&= \frac{dy}{dx} = 2^{3^{x^2}} \cdot \ln 2 \cdot 3^{x^2} \cdot \ln 3 \cdot \frac{dy}{dx}(x^2) \\&= \frac{dy}{dx} = 2^{3^{x^2}} \cdot \ln 2 \cdot 3^{x^2} \cdot \ln 3 \cdot 2x \\&= 2x \cdot \ln 2 \cdot \ln 3 \cdot 2^{3^{x^2}} \cdot 3^{x^2}.\end{aligned}$$

Shortcut:

$$\begin{aligned}f(g(x)) &= \sqrt{g(x)} \\f'(x) &= \frac{1}{2}(g(x))^{-\frac{1}{2}} \cdot g'(x) \\f'(x) &= \frac{1}{2\sqrt{g(x)}} \cdot g'(x) \\&= \frac{g'(x)}{2\sqrt{g(x)}}.\end{aligned}$$

Example for shortcut:

$$f(x) = \sqrt{\sin x}.$$

$$f'(x) = \frac{\cos x}{2\sqrt{\sin x}}.$$

Example for shortcut:

$$f(x) = \sqrt{4x^3 + 7x^2}.$$

$$\begin{aligned}f'(x) &= \frac{12x^2 + 14x}{2\sqrt{4x^3 + 7x^2}} \\&= \frac{2(6x + 7)}{2\sqrt{4x^3 + 7x^2}}.\end{aligned}$$

Differentiation Examples using the Product, Quotient, and Chain Rules

Recall:

Product Rule:

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \frac{d}{dx}[g(x)] + g(x) \frac{d}{dx}[f(x)].$$

Quotient Rule:

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx}[f(x)] - f(x) \frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

The Chain Rule:

If $F(x) = f(g(x))$, then:

$$F'(x) = f'(g(x)) \cdot g'(x).$$

Or:

If $f(u) = f(g(x))$, then:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example: Differentiate the following function:

$$r = \frac{\sqrt{\theta} - 3}{\sqrt{\theta} + 3}.$$

If:

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx}[f(x)] - f(x) \frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

And:

$$f(x) = \sqrt{\theta} - 3$$

$$f'(x) = \frac{1}{2} \theta^{-\frac{1}{2}}.$$

$$g(x) = \sqrt{\theta} + 3$$

$$g'(x) = \frac{1}{2} \theta^{-\frac{1}{2}}.$$

Then:

$$\begin{aligned}
 \frac{dr}{d\theta} &= \frac{(\sqrt{\theta} + 3)(\frac{1}{2}\theta^{-\frac{1}{2}}) - (\sqrt{\theta} - 3)(\frac{1}{2}\theta^{-\frac{1}{2}})}{(\sqrt{\theta} + 3)^2} \\
 &= \frac{\frac{1}{2}\theta^{-\frac{1}{2}}(\sqrt{\theta} + 3 - \sqrt{\theta} + 3)}{(\sqrt{\theta} + 3)^2} \\
 &= \frac{\frac{1}{2}\theta^{-\frac{1}{2}}(6)}{(\sqrt{\theta} + 3)^2} \\
 &= \frac{3 \cdot \theta^{-\frac{1}{2}}}{(\sqrt{\theta} + 3)^2} \\
 &= \frac{3}{\sqrt{\theta}(\sqrt{\theta} + 3)^2}.
 \end{aligned}$$

Note:-

It's fine that we have a radical in the denominator because there was one in the original equation.

Example: Differentiate the following function:

$$p = \frac{4 + \sec q}{4 - \sec q}.$$

We will rewrite in terms of *sin* and *cos*

$$p = \frac{4 + \frac{1}{\cos q}}{4 - \frac{1}{\cos q}}.$$

Now find common denominator to clear out fractions ($\cos q$):

$$p = \frac{4 \cos q + 1}{4 \cos q - 1}.$$

Now we differentiate:

$$\begin{aligned}
 f(x) &= 4 \cos q + 1 \\
 f'(x) &= -4 \sin q.
 \end{aligned}$$

$$\begin{aligned}
 g(x) &= 4 \cos q - 1 \\
 g'(x) &= -4 \sin q.
 \end{aligned}$$

Now plug into Quotient Rule:

$$\frac{dp}{dq} = \frac{(4 \cos q - 1)(-4 \sin q) - (4 \cos q + 1)(-4 \sin q)}{(4 \cos q - 1)^2}.$$

we see we can factor out an $-4 \sin q$:

$$\begin{aligned}
 \frac{dp}{dq} &= \frac{-4 \sin q(4 \cos q - 1) - (4 \cos q + 1)(-4 \sin q)}{(4 \cos q - 1)^2} \\
 &= \frac{-4 \sin q(4 \cos q - 1 - 4 \cos q - 1)}{(4 \cos q - 1)^2} \\
 &= \frac{-4 \sin q(-2)}{(4 \cos q - 1)^2} \\
 &= \frac{8 \sin q}{(4 \cos q - 1)^2}.
 \end{aligned}$$

Example: Differentiate the following function:

$$h(x) = \left(\frac{\cos x}{1 + \sin x} \right)^4.$$

First lets figure out our Derivatives from whats within the parenthesis:

$$\begin{aligned} f(x) &= \cos x \\ f'(x) &= -\sin x. \end{aligned}$$

$$\begin{aligned} g(x) &= 1 + \sin x \\ g'(x) &= \cos x. \end{aligned}$$

We will start by using the power rule and the chain rule with the quotient rule:

$$h'(x) = 4 \left[\frac{\cos x}{1 + \sin x} \right]^3 \cdot \left[\frac{(1 + \sin x)(-\sin x) - (\cos x)(\cos x)}{(1 + \sin x)^2} \right].$$

Now we want to distribute the exponent 3, into the terms in the numerator and denominator

$$\frac{4 \cos^3 x}{(1 + \sin x)^3} \cdot \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2}.$$

We are going to factor out a -1 and bring it in front of the 4:

$$\frac{-4 \cos^3 x}{(1 + \sin x)^3} \cdot \frac{\sin x + \sin^2 x + \cos^2 x}{(1 + \sin x)^2}.$$

We know that $\sin^2 x + \cos^2 x = 1$, so:

$$\frac{-4 \cos^3 x}{(1 + \sin x)^3} \cdot \frac{\sin x + 1}{(1 + \sin x)^2}.$$

Now we can divide by common factor in the numerator:

$$\begin{aligned} \frac{-4 \cos^3 x}{(1 + \sin x)^3} \cdot \frac{1}{1 + \sin x} \\ = \frac{-4 \cos^3 x}{(1 + \sin x)^4}. \end{aligned}$$

Example: Differentiate the following function:

$$y = (e^{\cos(\frac{t}{9})})^4.$$

So by using both the product rule and the chain rule, we get:

$$y' = 4(e^{\cos \frac{t}{9}})^3 \cdot e^{\cos \frac{t}{9}} \cdot -\sin \frac{t}{9} \cdot \frac{1}{9}.$$

Cleanup:

To start, we will group all the constants, then combine the like terms.:

$$y' = -\frac{4}{9} \sin \frac{t}{9} (e^{\cos \frac{t}{9}})^4.$$

Now we will move that power of 4 to the front of cos:

$$y' = -\frac{4}{9} \sin \frac{t}{9} (e^{4 \cos \frac{t}{9}}).$$

Example: Differentiate:

$$y = \sin(4x^2 e^x).$$

So:

$$y' = \cos(4x^2 e^x)$$

Now we want to use the product rule to derive whats inside the cosine function:

$$\begin{aligned} y' &= \cos(4x^2 e^x) \cdot [8x \cdot e^x + e^x \cdot 4x^2] \\ &= e^x(4x^2 + 8x) \cos 4x^2 e^x. \end{aligned}$$

Double Prime: To make this easier to grasp we will split it into 3 parts.

First Part:

$$e^x(4x^2 e^x)(\cos 4x^2 e^x).$$

Second Part:

$$8x + 8)e^x \cos 4x^2 e^x.$$

Third Part:

$$-\sin(4x^2 e^x).$$

And then apply the chain + product rule for the stuff inside -sin, which we did in single prime above

$$(8xe^x + 4x^2 e^x).$$

and then multiply by the other 2 functions:

$$e^x(4x^2 + 8x).$$

So all together part 3 is:

$$-\sin(4x^2e^x)(8xe^x + 4x^2e^x) \cdot e^x(4x^2 + 8x).$$

So if we add it all together:

$$y'' = e^x(4x^2e^x)(\cos 4x^2e^x) + 8x + 8)e^x \cos 4x^2e^x + -\sin(4x^2e^x)(8xe^x + 4x^2e^x) \cdot e^x(4x^2 + 8x).$$

Cleanup: by factoring out terms:

$$\begin{aligned} & e^x \cdot 4x(x+2) \cos 4x^2e^x + e^x \cdot 8(x+1) \cos 4x^2e^x - [e^x(4x^2 + 8x)]^2 \sin(4x^2e^x) \\ & 4e^x \cos(4x^2e^x)[x(x+2) + 2(x+1)] - [e^x \cdot 4x(x+2)]^2 \sin(4x^2e^x) \\ & 4e^x \cos(4x^2e^x)[x(x+2) + 2(x+1)] - 16e^{2x} \cdot x^2(x+2)^2 \sin(4x^2e^x) \\ & 4e^x \cos(4x^2e^x)[x^2 + 4x + 2] - 16e^{2x} \cdot x^2(x+2)^2 \sin(4x^2e^x) \\ & 4e^x(x^2 + 4x + 2) \cos 4x^2e^x - 16e^{2x} \cdot x^2(x+2)^2 \sin(4x^2e^x) \end{aligned}$$

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