Calculus 1 Notes

Nathan Warner

December 17, 2023

Chapter 2

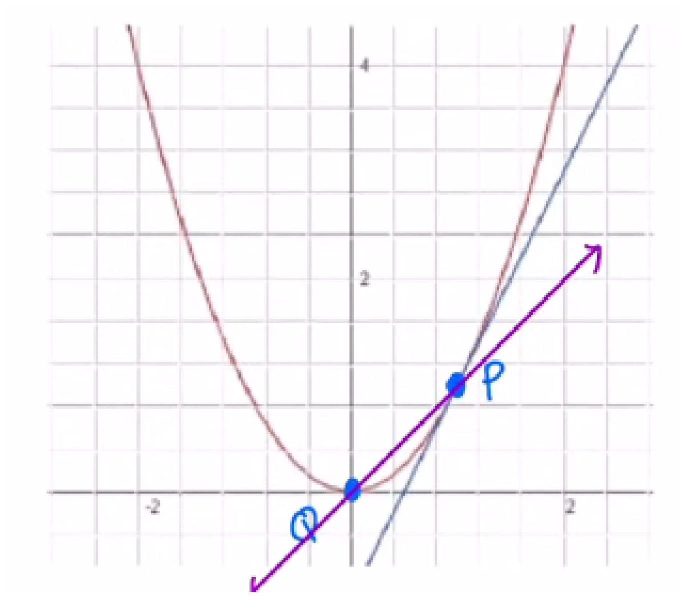
Contents

2.1: The Tangent and Velocity Problems

The Tangent Problem:

Question 1

Can we find an equation of the tangent line to $y = x^2$ at the point P(1,1)?



Explanation: .
$y = x^2$: Red parabola
Tangent line: Blue line
Secent Line: Pink line with points q and p

We are asked to get the equation of the tangent line to $y = x^2$ at the point P(1,1), However to find the equation of this line we know we need **2 things,**

- Point
- Slope

Since we only have one point, we cannot find slope. Therefore, we must use another point as an approximation and create a secent line instead. **This secent line is the pink line in the above graphic.**

So, lets use the point Q(0,0) as our second point. Now we can find slope with P(1,1), and Q(0,0).

If Slope $= \frac{y2-y1}{x2-x1}$, Then M of PQ $\longrightarrow \frac{1-0}{1-0} = 1$

Lets get a better approximation by using a point closer to the tangent line Lets use Q(0.9, 0.81)

So M of PQ $\longrightarrow \frac{1-0.81}{1-0.9} = 1.9$

Now, lets get an even closer approximation by using the point Q(0.99, 0.9801)

So, M of PQ $\longrightarrow \frac{1-0.9801}{1-0.99} = 1.99$

Notice, as the point Q gets closer to P, the slope of PQ is getting closer to 2

We write,

$$\lim_{Q \to P} \text{M of PQ} = \text{m}$$

Where **m** on the right of equation is slope of tangent line at **P**, And **M of PQ** is slope of the secent line

Now,

We will use our approximation of $m \approx 2$ to write the equation of the tangent line, using the orginial point P(1,1).

$$\begin{aligned} y - 1 &= 2\,(x - 1) \\ y - 1 &= 2x - 2 \\ y &= 2x - 1. \end{aligned}$$

The Velocity Problem:

- Average Velocity: $\frac{distance\ traveled}{time\ elapsed}$, which is represented by the slope of the secent line.
- Instantaneous Velocity = Velocity at a given instant of time, which is represented by the slope of the tangent line

Example 0.0.1

If a rock is thrown upward on the planet Mars, with a Velocity of 10 m/s, It's height in meters t seconds later is given by $y = 10t - 1.86t^2$

Question 2

Find the average Velocity over the given time intervals:

(i) [1,2] $\rightarrow$ 1 and 2 represent values of t

Substitute values into equation above

$$\begin{aligned} y\left(1\right) &= 10\left(1\right) - 1.86\left(1\right)^2 \\ &= 8.14. \end{aligned}$$

$$\begin{aligned} y\left(2\right) &= 10\left(2\right) - 1.86\left(2\right)^2 \\ &= 12.56. \end{aligned}$$

If Average Velocity $= \frac{distance\ traveled}{time\ elapsed}$ Or better yet $\frac{Change\ in\ height}{change\ in\ time}$

And we have the points (1,8.14) and (2,12.56)

Then,

$$\begin{aligned} Average\ Velocity &= \frac{12.56 - 8.14}{2 - 1} \\ &= 4.42m\diagdown s. \end{aligned}$$

(ii) [1,1.5]

Substitute values into equation above

$$y\left(1\right) = 10\left(1\right) - 1.86\left(1\right)^2$$
$$= 8.14.$$

$$y\left(1.5\right) = 10\left(1.5\right) - 1.86\left(1.5\right)^2$$
$$= 10.815.$$

After solving theses equations we have the points (1,8.14) and (1.5,10.815)

So,

$$Average\ Velocity = \frac{10.815 - 8.14}{1.5 - 1}$$
$$= 5.35m\diagdown s.$$

2.1.1 The Limit of a Function:

Question 3

Consider the values of $f(x) = x^2 + 2$ near $x = 2$

We want to know whats going on near x=2, so we make a table

x	f(x) $= x^2+2$
0	2
1	3
1.5	4.25
1.9	5.61
2	
2.1	6.41
2.4	7.76
2.9	10.41
4	18

Now we want to look at the closet x values to 2, which is the 2 that are above and below **2**, **We observe that as x values approach 2, then f(x) values approach 6**

so we write,

$$\lim_{x \to 2} f(x) = 6.$$

Example 0.0.2

Use a table of values to estimate the limit: $\lim\limits_{x \to 0} \frac{tan3x}{tan5x}$

Rememeber the value **0** is **a** so we want to contruct our table where a is in the middle, so use values that are smaller and larger than a.

Using arbitrary values that are close to 0, we get the table,

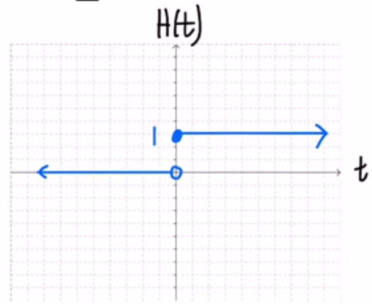
x	f(x)
−0.7	−4.56
−0.1	0.566
−0.01	0.5997
0.01	0.5997
0.1	0.566
0.7	−4.56

Now after looking at our table, we can conclude that

$$\lim_{x \to 0} \frac{tan3x}{tan5x} = 0.6.$$

One Sided Limits:

$$\textbf{Consider } H(t) = \begin{cases} 0 & \textbf{if } t < 0 \\ 1 & \textbf{if } t \geq 0 \end{cases}$$



Note:-

if there is a **minus** sign after a, that means you are approaching limit from the left if there is a **plus** sign after a, that means you are approaching limit from the right, if you see a limit with either of these, it is called a two sided limit

What is $\lim_{t \to 0-} h(t)$

So looking at the bottom line, coming from the left, as we approach 0, the y value is 0.

so $\rightarrow$

$$\lim_{t \to 0-} h(t) = 0.$$

What is $\lim_{t \to 0+} h(t)$

Given that we are approaching from the right, we are now looking at the top line, we can see that as we approach 0, y is 1

so

$$\lim_{t \to 0+} h(t) = 1.$$

Note:-

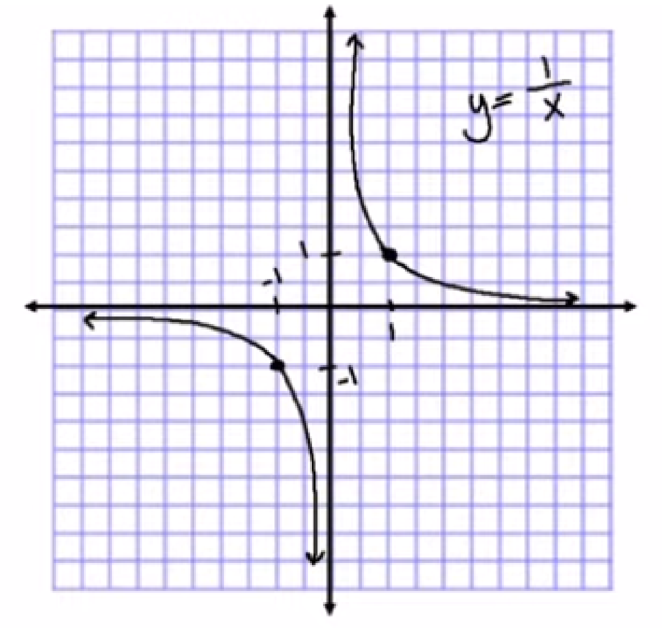
The first one is our **Left hand limit** and the bottom one is our **right hand limit** if the side we our approaching from is not specified, **we cannot find the limit, so we would say DNE**

So

$\lim_{x \to 0} f(x) = l$ **iff** (if and only if) $\lim_{x \to 0-} f(x) = L$ **and** $\lim_{x \to 0+} f(x) = L$

in other words, we can only drop the + or - after the a if the right and left hand limits are the same

Infinite Limits:



if we look at

$$\lim_{x \to 0+} f(x) = ?.$$

We notice that as we approach 0 from the right, f(x) goes to infinity

So:

$$\lim_{x \to 0+} f(x) = \infty.$$

This is also the same for x $\to$ 0$-$

So:

$$\lim_{x \to 0-} f\left(x\right) = \infty.$$

Note:-

x = 0 is a vertical Asymptote

In general, $x = a$ is a vertical asymptote if at least one of the following are true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = -\infty$$
$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = \infty$$
$$\lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = -\infty$$

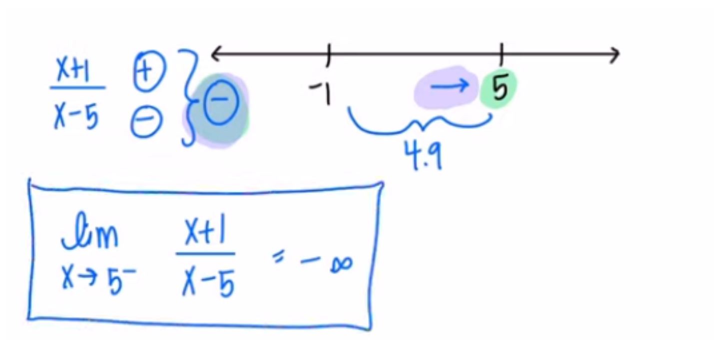
Examples: Determine the infitite limit

1.) $\lim\limits_{x\to 5-} \frac{x+1}{x-5}$

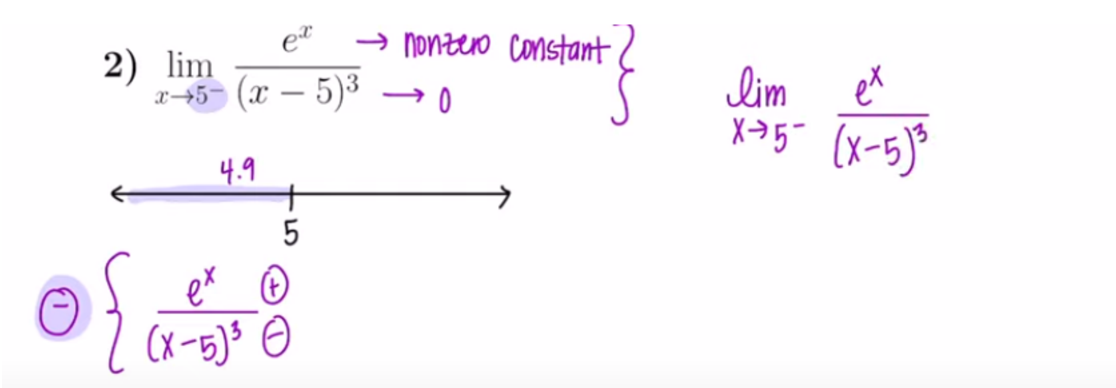
$$\text{x + 1} \longrightarrow 6$$
$$\text{x - 5} \longrightarrow 0$$

If you have a $\frac{\textit{nonzero constant}}{\textit{approaching } 0}$ its either going to be approaching ∞ or $-\infty$ the way we find which version of infinity it will be is with either a table or a numberline

To make the numerline we want to list the zeros, so -1 and 5. Then pick a value thats close to a and approachs in the correct direction. Then plug this number into the equation and whatever sign you get will be the sign for infinity.



2.) $\lim\limits_{x\to 5-} \frac{e^x}{(x-5)^3}$



2.3: Calculating using limit laws

The limit laws:

	Limit Law in symbols	Limit Law in words
1	$\lim\limits_{x\to a}[f(x)+g(x)] = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$	The limit of a sum is equal to the sum of the limits.
2	$\lim\limits_{x\to a}[f(x)-g(x)] = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x)$	The limit of a difference is equal to the difference of the limits.
3	$\lim\limits_{x\to a} cf(x) = c\lim\limits_{x\to a} f(x)$	The limit of a constant times a function is equal to the constant times the limit of the function.
4	$\lim\limits_{x\to a}[f(x)g(x)] = \lim\limits_{x\to a} f(x)\cdot \lim\limits_{x\to a} g(x)]$	The limit of a product is equal to the product of the limits.
5	$\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} \quad \left(\text{if } \lim\limits_{x\to a} g(x) \neq 0\right)$	The limit of a quotient is equal to the quotient of the limits.
6	$\lim\limits_{x\to a}[f(x)]^n = [\lim\limits_{x\to a} f(x)]^n$	where n is a positive integer
7	$\lim\limits_{x\to a} c = c$	The limit of a constant function is equal to the constant.
8	$\lim\limits_{x\to a} x = a$	The limit of a linear function is equal to the number x is approaching.
9	$\lim\limits_{x\to a} x^n = a^n$	where n is a positive integer
10	$\lim\limits_{x\to a}\sqrt[n]{x} = \sqrt[n]{a}$	where n is a positive integer & if n is even, we assume that $a > 0$
11	$\lim\limits_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)}$	where n is a positive integer & if n is even, we assume that $\lim\limits_{x\to a} f(x) > 0$

Question 4

Find the limit if $\lim\limits_{x\to 2} f(x) = 4$ and $\lim\limits_{x\to -2} f(x) = -2$

$\lim\limits_{x\to 2} f(x) + 5g(x)$

Solution: ☺

Using limit laws 1 and 3 we can solve this problem

$$\lim_{x\to 2} f(x) + \lim_{x\to 2} 5g(x) \to \textbf{law 1}$$
$$\lim_{x\to 2} f(x) + 5\lim_{x\to 2} g(x) \to \textbf{Law 3}$$
$$4 + 5(-2) = -6.$$

Question 5

Given $\lim_{x \to 2} g(x) = -2$ $\lim_{x \to 2} h(x) = 0$ find $\lim_{x \to 2} \frac{g(x)}{h(x)}$

Solution: 😊

Using limit law 5 we can solve this

$$\frac{\lim_{x \to 2} g(x)}{\lim_{x \to 2} h(h)} = \frac{-2}{0}$$
$$\textbf{DNE}.$$

Direct Substitution Property:

Definition 0.0.1

if f is a polynomial or a rational function and a is in the domain of f, then $\lim_{x \to a} f(x) = f(a)$

Example: $\lim_{x \to 2} \frac{2x^2+1}{x^2+6x-4}$

a) what function is this?

Answer: 😊

This is a **rational** function

b) is 2 in the domain of the function?

Answer: 😊

if we plug in 2 in the denomonator, the function does not equal 2, so **Yes**, 2 is in the domain of this function, therefore, we can solve for f(a) and get the limit of this function

$$\begin{aligned} &\frac{2 \cdot 2^2 + 1}{2^2 + 6 * 2 - 4} \\ &= \frac{9}{12} \\ &= \frac{3}{4}. \end{aligned}$$

Example 3: Evaluate the limit, if exists:

$\lim_{x \to 1} \frac{x^3-1}{x^2-1}$

Solution: 😊

In this case, if we plug in 1 to the denomonator, we get 0. Therefore **a** is not in the domain of **f**. So we must attempt to find the limit of this function with **Factoring**

Review: Factoring sums or difference of cubes:

Difference of cubes: $a^3 - b^3 = (a-b)\left(a^2+ab+b^2\right)$

Sum of cubes: $a^3 + b^3 = (a+b)\left(a^2-ab-b^2\right)$

Example of difference of cubes

a) $x^3 - 8$

This is $a^3 - b^3$, Where a = x and b = 2 because $2^3 = 8$

So:

$$(x-2)\left(x^2+2x+4\right).$$

Back to Example 3: So using difference of cubes we get

$$\lim_{x\to 1} \frac{(x-1)\left(x^2+x+1\right)}{(x-1)(x+1)}.$$

Now if we **cancel** out **common factors, we get:**

$$\lim_{x\to 1} \frac{\left(x^2+x+1\right)}{(x+1)}.$$

Now with this new equation, **1 is** in the domain. So we plug 1 into the new equation and get:

$$\begin{aligned} &\frac{1^1+1+1}{1+1} \\ &= \frac{3}{2}. \end{aligned}$$

Example 4: $\lim\limits_{h\to 0} \frac{\sqrt{9+h}-3}{h}$

Straight away, we can see that h = 0 is **not** in the domain of the function. So we want to try and get rid of this radical in the numerator by multiplying by the conjugate

So:

$$\begin{aligned} &\lim_{h\to 0} \frac{\sqrt{9+h}-3}{h} \cdot \frac{\left(\sqrt{9+h}+3\right)}{\left(\sqrt{9+h}+3\right)} \\ &= \lim_{h\to 0} \frac{9+h-9}{h\left(\sqrt{9+h}+3\right)} \\ &= \lim_{h\to 0} \frac{h}{h\left(\sqrt{9+h}+3\right)} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{9+h}+3} \\ &. \end{aligned}$$

Now with this new equation, 0 is in the domain, so we can plug in 0.

$$= \frac{1}{\sqrt{9+0}+3}$$
$$= \frac{1}{6}.$$

Example 5: $\lim_{x \to 4} \frac{x^2-4x}{x^2-3x-4}$

Straight away we can see that if we plug 4 into the denomonator, we get 0. For this reason we know that 4 is not in the domain. Therefore we must factor

So:

$$\lim_{x \to 4} \frac{x\,(x-4)}{(x+1)\,(x-4)}.$$

After canceling out the common factor of $x-4$, we get the equation:

$$\lim_{x \to 4} \frac{x}{x+1}.$$

Now we can plug 4 into this new equation and get:

$$\frac{4}{5}.$$

Example 6: $\lim_{x \to -1} \frac{x^2-4x}{x^2-3x-4}$

Again we can see that -1 is not in the domain. However, with this example, if we factor out the equation and then plug -1 into our new equation, we get:

$$\frac{-1}{0}.$$

so we can see that the direct Substitution will not work. Therefore, our limit is either ∞, or DNE, Rememeber that this is the case for $\frac{nonzero\ constant}{0}$. Now we must test the equation to get the sign of ∞

First test: Left side (Testing with -1.1)

$$\lim_{x \to -1-} \frac{x}{x+1}.$$

If we plug -1.1 into the equation, we can see that both the numerator and the denomonator are negative, therefore our sign is **Positive** ∞

Second Test: Right side (testing with -0.9)

If we plug -0.9 into the equation, we can see that the numerator is negative, but the denomonator is positive. Therefore our sign is **Negative** ∞

Because the **Left and Right hand limits are not the same**, we can deduce that the limit is DNE

So:

$$\lim_{x \to -1} \frac{x^2 - 4x}{x^2 - 3x - 4}$$
$$= DNE.$$

Example 7: $\lim\limits_{x \to -6} \frac{2x+12}{|x+6|}$

Note:-

Because we see absolute value in the denomonator, we want to rewrite as piecewise.

Review of Piecewise:

Recall:

$$f(x) = |x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases} \tag{1}$$

Example: abs as piecewise:

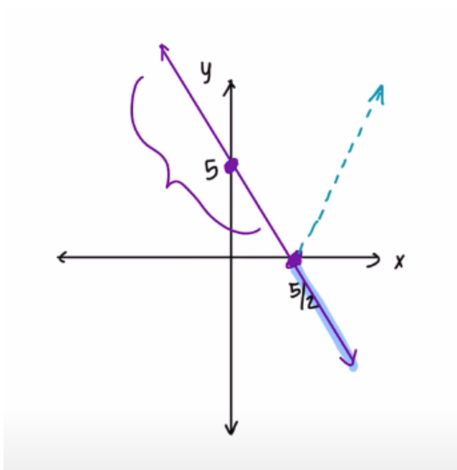
$$g(x) = |5 - 2x|.$$

First we want to figure out where the quantity inside the absolute value changes signs, to do this we set the quanity inside the absolute value **equal to 0**.

So:

$$5 - 2x = 0$$
$$x = \frac{5}{2}.$$

To visualize this, refer to this graph:



We can see that the output values beyond $\frac{5}{2}$ will be reflected about the x-axis

So to write this Algebraically, Whever the zero is for the quanity inside the absolute value, thats where we split the domain.

So:

$$g(x) = \begin{cases} 5 - 2x & \text{if } x < \dfrac{5}{2} \\ -(5 - 2x) & \text{if } x \geqslant \dfrac{5}{2} \end{cases} \tag{2}$$

Back to example 7:

We want to rewrite the denomonator as a piecewise function.

So:

$$|x + 6| = \begin{cases} x + 6 & \text{if } x \geqslant -6 \\ -(x + 6) & \text{if } x < -6 \end{cases} \tag{3}$$

Now we want to rewrite the entire equation

So:

$$\frac{2(x+6)}{|x+6|} = \begin{cases} \frac{2(x+6)}{x+6} & \text{if } x > -6 \\ \frac{2(x+6)}{-x+6} & \text{if } x < -6 \end{cases} \tag{4}$$

Now we can simplify this further by canceling out common factors x+6, and we are left with:

$$\frac{2(x+6)}{|x+6|} = \begin{cases} 2 & \text{if } x > -6 \\ -2 & \text{if } x < -6 \end{cases} \tag{5}$$

Now we can find the limit, Since the direction is not specified, we must check at both sides.

$$\lim_{x \to -6-} \frac{2x + 12}{|x + 6|} = -2.$$

The limit is -2 because if we approaching -6 from the left, we are looking at values that are smaller than -6, and if we look at our piecewise function, we can see that it would be -2 for values smaller than -6

$$\lim_{x \to -6+} \frac{2x + 12}{|x + 6|} = 2.$$

Since left and right limits are not equal, this means that:

$$\lim_{x \to -6} \frac{2x + 12}{|x + 6|}$$
$$= DNE.$$

Squeeze Theorem