Calculus 1 Notes: Chapter 2

Nathan Warner

December 17, 2023

Chapter 2

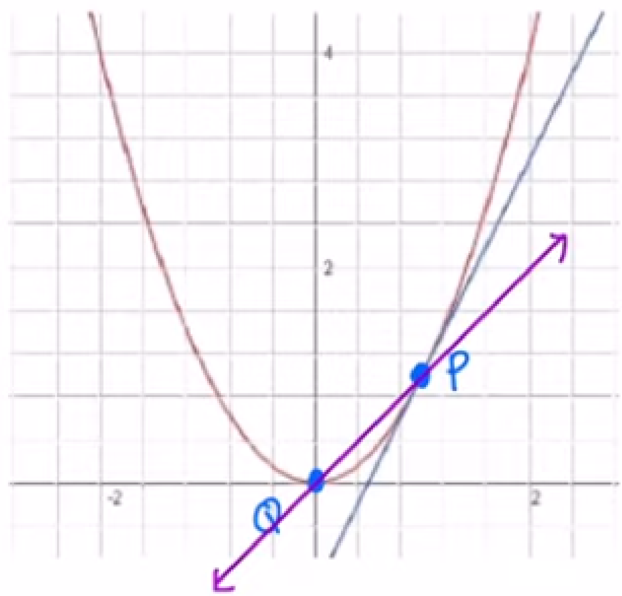
Contents

2.1: The Tangent and Velocity Problems

The Tangent Problem:

Question 1

Can we find an equation of the tangent line to $y = x^2$ at the point P(1,1)?



Explanation: .
$y = x^2$: Red parabola
Tangent line: Blue line
Secent Line: Pink line with points q and p

We are asked to get the equation of the tangent line to $y = x^2$ at the point P(1,1), However to find the equation of this line we know we need **2 things,**

- Point
- Slope

Since we only have one point, we cannot find slope. Therefore, we must use another point as an approximation and create a secent line instead. **This secent line is the pink line in the above graphic.**

So, lets use the point Q(0,0) as our second point. Now we can find slope with P(1,1), and Q(0,0).

If Slope $= \frac{y2-y1}{x2-x1}$, Then M of PQ $\longrightarrow \frac{1-0}{1-0} = 1$

Lets get a better approximation by using a point closer to the tangent line Lets use Q(0.9, 0.81)

So M of PQ $\longrightarrow \frac{1-0.81}{1-0.9} = 1.9$

Now, lets get an even closer approximation by using the point Q(0.99, 0.9801)

So, M of PQ $\longrightarrow \frac{1-0.9801}{1-0.99} = 1.99$

Notice, as the point Q gets closer to P, the slope of PQ is getting closer to 2

We write,

$$\lim_{Q \to P} \text{M of PQ} = \text{m}$$

Where **m** on the right of equation is slope of tangent line at **P**, And **M of PQ** is slope of the secent line

Now,

We will use our approximation of $m \approx 2$ to write the equation of the tangent line, using the orginial point P(1,1).

$$\begin{aligned} y - 1 &= 2\,(x - 1) \\ y - 1 &= 2x - 2 \\ y &= 2x - 1. \end{aligned}$$

The Velocity Problem:

- Average Velocity: $\frac{distance\ traveled}{time\ elapsed}$, which is represented by the slope of the secent line.
- Instantaneous Velocity = Velocity at a given instant of time, which is represented by the slope of the tangent line

Example 0.0.1

If a rock is thrown upward on the planet Mars, with a Velocity of 10 m/s, It's height in meters t seconds later is given by $y = 10t - 1.86t^2$

Question 2

Find the average Velocity over the given time intervals:

(i) [1,2] $\rightarrow$ 1 and 2 represent values of t

Substitute values into equation above

$$\begin{aligned} y(1) &= 10(1) - 1.86(1)^2 \\ &= 8.14. \end{aligned}$$

$$\begin{aligned} y(2) &= 10(2) - 1.86(2)^2 \\ &= 12.56. \end{aligned}$$

If Average Velocity = $\frac{distance\ traveled}{time\ elapsed}$ Or better yet $\frac{Change\ in\ height}{change\ in\ time}$

And we have the points (1,8.14) and (2,12.56)

Then,

$$\begin{aligned} Average\ Velocity &= \frac{12.56 - 8.14}{2 - 1} \\ &= 4.42 m \diagdown s. \end{aligned}$$

(ii) [1,1.5]

Substitute values into equation above

$$y\left(1\right) = 10\left(1\right) - 1.86\left(1\right)^{2}$$
$$= 8.14.$$

$$y\left(1.5\right) = 10\left(1.5\right) - 1.86\left(1.5\right)^{2}$$
$$= 10.815.$$

After solving theses equations we have the points (1,8.14) and (1.5,10.815)

So,

$$Average\ Velocity = \frac{10.815 - 8.14}{1.5 - 1}$$
$$= 5.35m\diagdown s.$$

2.2.1 The Limit of a Function:

Question 3

Consider the values of $f(x) = x^2 + 2$ near $x = 2$

We want to know whats going on near x=2, so we make a table

x	f(x) $= x^2+2$
0	2
1	3
1.5	4.25
1.9	5.61
2	
2.1	6.41
2.4	7.76
2.9	10.41
4	18

Now we want to look at the closet x values to 2, which is the 2 that are above and below **2**, **We observe that as x values approach 2, then f(x) values approach 6**

so we write,

$$\lim_{x\to 2} f(x) = 6.$$

Example 0.0.2

Use a table of values to estimate the limit: $\lim_{x\to 0} \frac{tan3x}{tan5x}$

Rememeber the value **0** is **a** so we want to contruct our table where a is in the middle, so use values that are smaller and larger than a.

Using arbitrary values that are close to 0, we get the table,

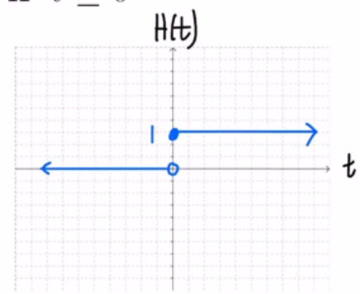
x	$f(x)$
−0.7	−4.56
−0.1	0.566
−0.01	0.5997
0.01	0.5997
0.1	0.566
0.7	−4.56

Now after looking at our table, we can conclude that

$$\lim_{x \to 0} \frac{tan3x}{tan5x} = 0.6.$$

One Sided Limits:

$$\textbf{Consider } H(t) = \begin{cases} 0 & \textbf{if } \ t < 0 \\ 1 & \textbf{if } \ t \geq 0 \end{cases}$$



Note:-

if there is a **minus** sign after a, that means you are approaching limit from the left if there is a **plus** sign after a, that means you are approaching limit from the right, if you see a limit with either of these, it is called a two sided limit

What is $\lim\limits_{t \to 0-} h(t)$

So looking at the bottom line, coming from the left, as we approach 0, the y value is 0.

so $\rightarrow$

$$\lim_{t \to 0-} h(t) = 0.$$

What is $\lim\limits_{t \to 0+} h(t)$

Given that we are approaching from the right, we are now looking at the top line, we can see that as we approach 0, y is 1

so

$$\lim_{t \to 0+} h(t) = 1.$$

Note:-

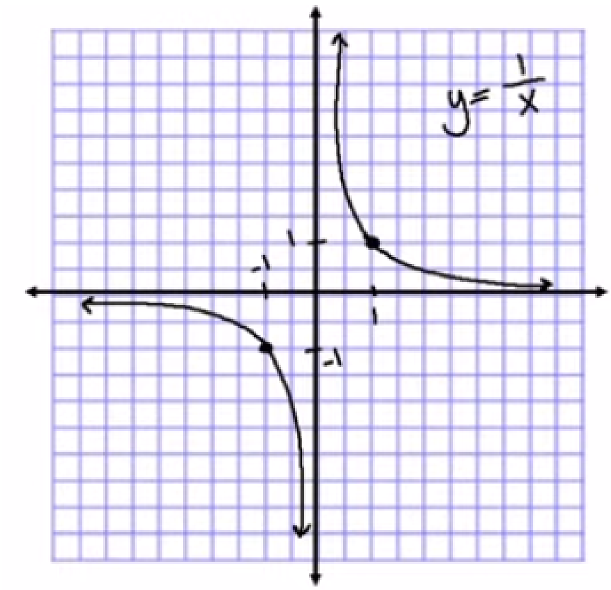
The first one is our **Left hand limit** and the bottom one is our **right hand limit** if the side we our approaching from is not specified, **we cannot find the limit, so we would say DNE**

So

$\lim\limits_{x \to 0} f(x) = l$ **iff** (if and only if) $\lim\limits_{x \to 0-} f(x) = L$ **and** $\lim\limits_{x \to 0+} f(x) = L$

in other words, we can only drop the + or - after the a if the right and left hand limits are the same

Infinite Limits:



if we look at

$$\lim_{x \to 0+} f(x) = ?.$$

We notice that as we approach 0 from the right, f(x) goes to infinity

So:

$$\lim_{x \to 0+} f(x) = \infty.$$

This is also the same for x $\to$ 0$-$

So:

$$\lim_{x \to 0-} f\left(x\right) = -\infty.$$

Note:-

x = 0 is a vertical Asymptote

In general, $x = a$ is a vertical asymptote if at least one of the following are true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = -\infty$$
$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = \infty$$
$$\lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = -\infty$$

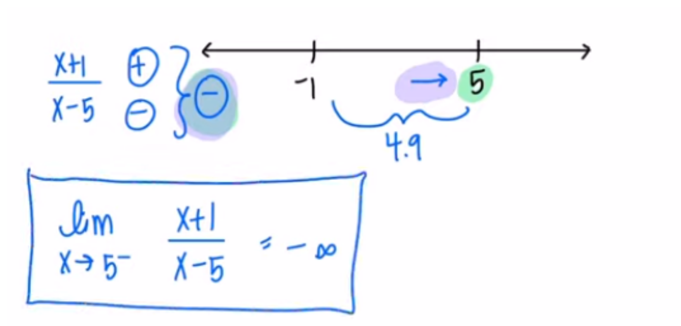
Examples: Determine the infitite limit

1.) $\lim_{x\to 5-} \frac{x+1}{x-5}$

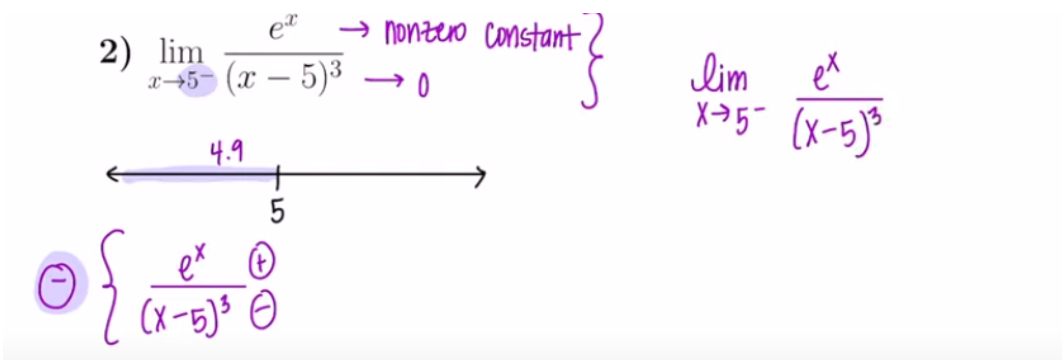
x + 1 $\longrightarrow$ 6
x - 5 $\longrightarrow$ 0

If you have a $\frac{\textit{nonzero constant}}{\textit{approaching } 0}$ its either going to be approaching ∞ or $-\infty$ the way we find which version of infinity it will be is with either a table or a numberline

To make the numerline we want to list the zeros, so -1 and 5. Then pick a value thats close to a and approachs in the correct direction. Then plug this number into the equation and whatever sign you get will be the sign for infinity.



2.) $\lim_{x\to 5-} \frac{e^x}{(x-5)^3}$



2.3: Calculating using limit laws

The limit laws:

	Limit Law in symbols	Limit Law in words
1	$\lim\limits_{x\to a}[f(x)+g(x)]=\lim\limits_{x\to a}f(x)+\lim\limits_{x\to a}g(x)$	The limit of a sum is equal to the sum of the limits.
2	$\lim\limits_{x\to a}[f(x)-g(x)]=\lim\limits_{x\to a}f(x)-\lim\limits_{x\to a}g(x)$	The limit of a difference is equal to the difference of the limits.
3	$\lim\limits_{x\to a}cf(x)=c\lim\limits_{x\to a}f(x)$	The limit of a constant times a function is equal to the constant times the limit of the function.
4	$\lim\limits_{x\to a}[f(x)g(x)]=\lim\limits_{x\to a}f(x)\cdot\lim\limits_{x\to a}g(x)]$	The limit of a product is equal to the product of the limits.
5	$\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\dfrac{\lim\limits_{x\to a}f(x)}{\lim\limits_{x\to a}g(x)}$ $\left(\text{if } \lim\limits_{x\to a}g(x)\neq 0\right)$	The limit of a quotient is equal to the quotient of the limits.
6	$\lim\limits_{x\to a}[f(x)]^n=[\lim\limits_{x\to a}f(x)]^n$	where n is a positive integer
7	$\lim\limits_{x\to a}c=c$	The limit of a constant function is equal to the constant.
8	$\lim\limits_{x\to a}x=a$	The limit of a linear function is equal to the number x is approaching.
9	$\lim\limits_{x\to a}x^n=a^n$	where n is a positive integer
10	$\lim\limits_{x\to a}\sqrt[n]{x}=\sqrt[n]{a}$	where n is a positive integer & if n is even, we assume that $a>0$
11	$\lim\limits_{x\to a}\sqrt[n]{f(x)}=\sqrt[n]{\lim\limits_{x\to a}f(x)}$	where n is a positive integer & if n is even, we assume that $\lim\limits_{x\to a}f(x)>0$

Question 4

Find the limit if $\lim\limits_{x\to 2}f(x)=4$ and $\lim\limits_{x\to -2}f(x)=-2$

$\lim\limits_{x\to 2}f(x)+5g(x)$

Solution: 😂

Using limit laws 1 and 3 we can solve this problem

$$\lim_{x\to 2}f(x)+\lim_{x\to 2}5g(x)\to \textbf{law 1}$$
$$\lim_{x\to 2}f(x)+5\lim_{x\to 2}g(x)\to \textbf{Law 3}$$
$$4+5(-2)=-6.$$

Question 5

Given $\lim_{x\to2} g(x) = -2$ $\lim_{x\to2} h(x) = 0$ find $\lim_{x\to2} \frac{g(x)}{h(x)}$

Solution: 😄

Using limit law 5 we can solve this

$$\frac{\lim_{x\to2} g(x)}{\lim_{x\to2} h(h)} = \frac{-2}{0}$$
$$\textbf{DNE}.$$

Direct Substitution Property:

Definition 0.0.1

if f is a polynomial or a rational function and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$

Example: $\lim_{x\to2} \frac{2x^2+1}{x^2+6x-4}$

a) what function is this?

Answer: 😄

This is a **rational** function

b) is 2 in the domain of the function?

Answer: 😄

if we plug in 2 in the denomonator, the function does not equal 2, so **Yes**, 2 is in the domain of this function, therefore, we can solve for f(a) and get the limit of this function

$$\begin{aligned} &\frac{2\cdot 2^2+1}{2^2+6*2-4} \\ &= \frac{9}{12} \\ &= \frac{3}{4}. \end{aligned}$$

Example 3: Evaluate the limit, if exists:

$\lim_{x\to1} \frac{x^3-1}{x^2-1}$

Solution: 😄

In this case, if we plug in 1 to the denomonator, we get 0. Therefore **a** is not in the domain of **f**. So we must attempt to find the limit of this function with **Factoring**

Review: Factoring sums or difference of cubes:

Difference of cubes: $a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$

Sum of cubes: $a^3 + b^3 = (a+b)\left(a^2 - ab - b^2\right)$

Example of difference of cubes

a) $x^3 - 8$

This is $a^3 - b^3$, Where a = x and b = 2 because $2^3 = 8$

So:

$$(x-2)\left(x^2 + 2x + 4\right).$$

Back to Example 3: So using difference of cubes we get

$$\lim_{x\to 1} \frac{(x-1)\left(x^2+x+1\right)}{(x-1)(x+1)}.$$

Now if we **cancel** out **common factors, we get:**

$$\lim_{x\to 1} \frac{\left(x^2+x+1\right)}{(x+1)}.$$

Now with this new equation, **1 is** in the domain. So we plug 1 into the new equation and get:

$$\frac{1^1+1+1}{1+1}$$
$$= \frac{3}{2}.$$

Example 4: $\lim_{h\to 0} \frac{\sqrt{9+h}-3}{h}$

Straight away, we can see that h = 0 is **not** in the domain of the function. So we want to try and get rid of this radical in the numerator by multiplying by the conjugate

So:

$$\lim_{h\to 0} \frac{\sqrt{9+h}-3}{h} \cdot \frac{\left(\sqrt{9+h}+3\right)}{\left(\sqrt{9+h}+3\right)}$$
$$= \lim_{h\to 0} \frac{9+h-9}{h\left(\sqrt{9+h}+3\right)}$$
$$= \lim_{h\to 0} \frac{h}{h\left(\sqrt{9+h}+3\right)}$$
$$= \lim_{h\to 0} \frac{1}{\sqrt{9+h}+3}$$
.

Now with this new equation, 0 is in the domain, so we can plug in 0.

$$= \frac{1}{\sqrt{9+0}+3}$$
$$= \frac{1}{6}.$$

Example 5: $\lim_{x \to 4} \frac{x^2-4x}{x^2-3x-4}$

Straight away we can see that if we plug 4 into the denomonator, we get 0. For this reason we know that 4 is not in the domain. Therefore we must factor

So:

$$\lim_{x \to 4} \frac{x\,(x-4)}{(x+1)\,(x-4)}.$$

After canceling out the common factor of $x-4$, we get the equation:

$$\lim_{x \to 4} \frac{x}{x+1}.$$

Now we can plug 4 into this new equation and get:

$$\frac{4}{5}.$$

Example 6: $\lim_{x \to -1} \frac{x^2-4x}{x^2-3x-4}$

Again we can see that -1 is not in the domain. However, with this example, if we factor out the equation and then plug -1 into our new equation, we get:

$$\frac{-1}{0}.$$

so we can see that the direct Substitution will not work. Therefore, our limit is either ∞, or DNE, Rememeber that this is the case for $\frac{nonzero\ constant}{0}$. Now we must test the equation to get the sign of ∞

First test: Left side (Testing with -1.1)

$$\lim_{x \to -1-} \frac{x}{x+1}.$$

If we plug -1.1 into the equation, we can see that both the numerator and the denomonator are negative, therefore our sign is **Positive** ∞

Second Test: Right side (testing with -0.9)

If we plug -0.9 into the equation, we can see that the numerator is negative, but the denomonator is positive. Therefore our sign is **Negative** ∞

Because the **Left and Right hand limits are not the same**, we can deduce that the limit is DNE

So:

$$\lim_{x \to -1} \frac{x^2 - 4x}{x^2 - 3x - 4}$$
$$= DNE.$$

Example 7: $\lim_{x \to -6} \frac{2x+12}{|x+6|}$

Note:-

Because we see absolute value in the denomonator, we want to rewrite as piecewise.

Review of Piecewise:

Recall:

$$f(x) = |x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases} \tag{1}$$

Example: abs as piecewise:

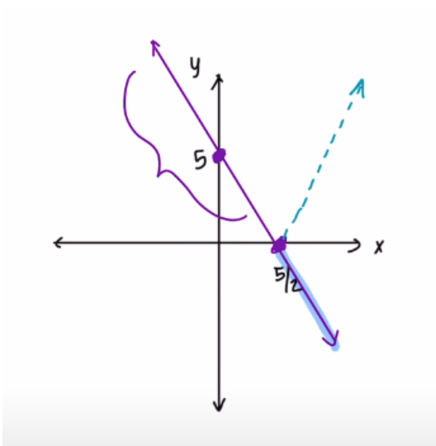
$$g(x) = |5 - 2x|.$$

First we want to figure out where the quantity inside the absolute value changes signs, to do this we set the quanity inside the absolute value **equal to 0**.

So:

$$5 - 2x = 0$$
$$x = \frac{5}{2}.$$

To visualize this, refer to this graph:



We can see that the output values beyond $\frac{5}{2}$ will be reflected about the x-axis

So to write this Algebraically, Whever the zero is for the quanity inside the absolute value, thats where we split the domain.

So:

$$g(x) = \begin{cases} 5 - 2x & \text{if } x < \dfrac{5}{2} \\ -(5 - 2x) & \text{if } x \geqslant \dfrac{5}{2} \end{cases} \tag{2}$$

Back to example 7:

We want to rewrite the denomonator as a piecewise function.

So:

$$|x + 6| = \begin{cases} x + 6 & \text{if } x \geqslant -6 \\ -(x + 6) & \text{if } x < -6 \end{cases} \tag{3}$$

Now we want to rewrite the entire equation

So:

$$\frac{2(x+6)}{|x+6|} = \begin{cases} \frac{2(x+6)}{x+6} & \text{if } x > -6 \\ \frac{2(x+6)}{-x+6} & \text{if } x < -6 \end{cases} \tag{4}$$

Now we can simplify this further by canceling out common factors x+6, and we are left with:

$$\frac{2(x+6)}{|x+6|} = \begin{cases} 2 & \text{if } x > -6 \\ -2 & \text{if } x < -6 \end{cases} \tag{5}$$

Now we can find the limit, Since the direction is not specified, we must check at both sides.

$$\lim_{x \to -6-} \frac{2x + 12}{|x + 6|} = -2.$$

The limit is -2 because if we approaching -6 from the left, we are looking at values that are smaller than -6, and if we look at our piecewise function, we can see that it would be -2 for values smaller than -6

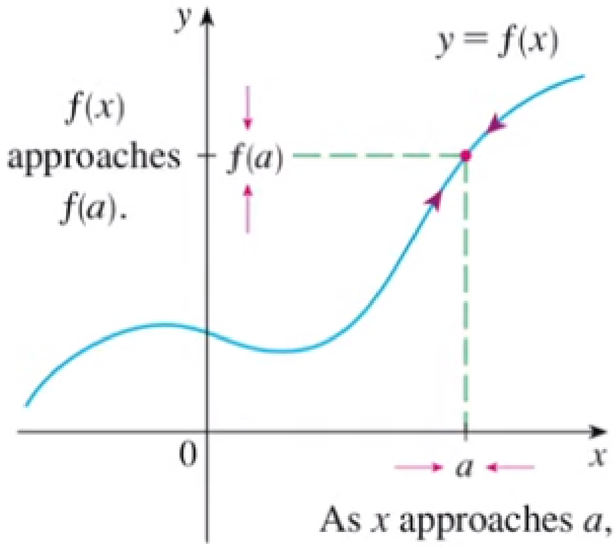
$$\lim_{x \to -6+} \frac{2x + 12}{|x + 6|} = 2.$$

Since left and right limits are not equal, this means that:

$$\lim_{x \to -6} \frac{2x + 12}{|x + 6|}$$
$$= DNE.$$

2.5: Continuity and the Intermediate Value Theorem:

Continuity:



What can we observe about f(x) at a?

- f(x) is defined at a
- $\lim_{x \to a} f(x)$ exists
- $\lim_{x \to a} f(x) = f(a)$

Definition 0.0.2

A function f is continuous at a if $\lim_{x \to a} f(x) = f(a)$

Note:-

Above 3 cases are required for f to be continuous at a, if that last bullet is true, then we know automatically that the first 2 bullets are also ***satisfied***

Example: Show f is continuous at a

$$f(x) = x^2 + \sqrt{7-x}, a = 4.$$

Remember that a is the x value we are investigating. Show we need to show that the 3 bullets above are true for this equation.

First we need to find the domain of this function and see if **4** lies within that domain.

Since this function is a polynomial function ***and a radical function***, we know that the domain of a polynomial function is $\mathbb{R}$. But because it is also a radical function, we know we must set whats inside the radical $\geqslant 0$

So if we solve the inequality:

$$x \leqslant 7.$$

Therefore the domain of this function is:

$$(-\infty, 7].$$

Note:-

Remember, when you solve for an inequality, and divide by a negative, you must ***flip the inequality.***

Since 4 $\in$ D, then f(x) is defined at a=4

Second, we need to show that $\lim_{x\to 4} f(x)$ exists. So we will first try to use direct Substitution and plug 4 into x:

$$\begin{aligned} &\lim_{x\to 4} (4)^2 + \sqrt{7-4} \\ &= 16 + \sqrt{3}. \end{aligned}$$

Third, we want to show that:

$$f(4) = \lim_{x\to 4} x^2 + \sqrt{7-4}.$$

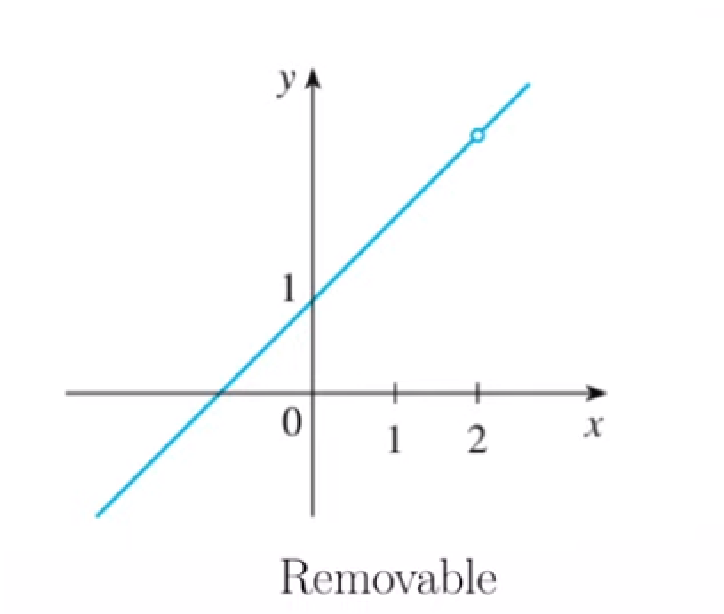
So again, like step 2, we will plug in 4 for x:

$$\begin{aligned} f(4) &= 4^2 + \sqrt{7-4} \\ &= 16 + \sqrt{3}. \end{aligned}$$

Since all 3 of these steps pass, we have shown Continuity at $x = 4$.

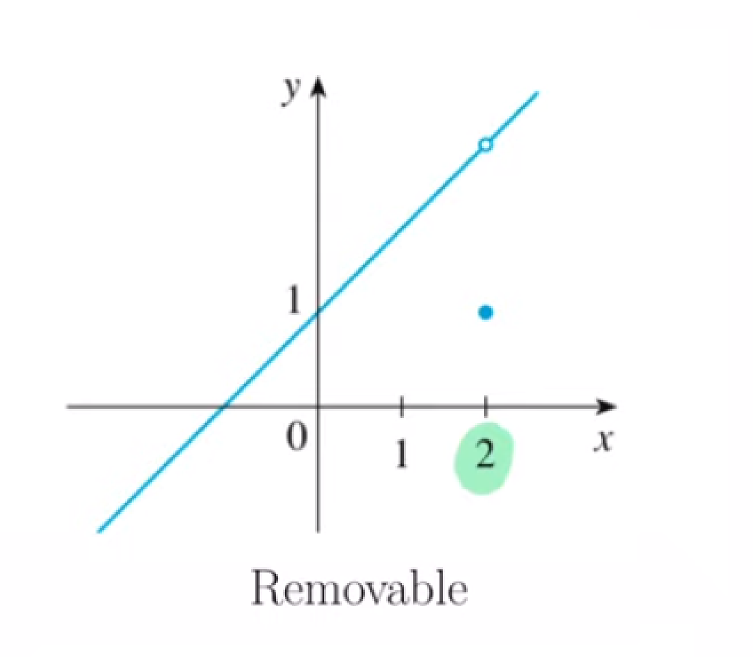
Discontinuities:

If we look at the following graph:



at we investigate a = 2, we can see that the $\lim_{x \to 2} f(x)$ does exist, **however,** f(x) is not defined at 2.

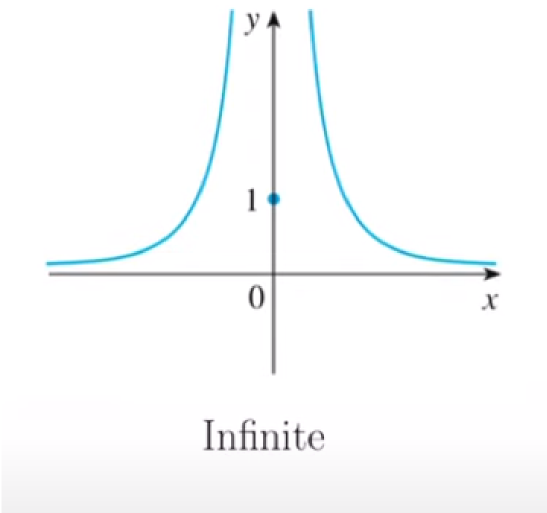
If we look at the following graph:



we can see that the $\lim_{x \to 2} f(x)$ does exist, and f(2) ***is defined***, However, $\lim_{x \to 2} f(x) \neq$ f(2)

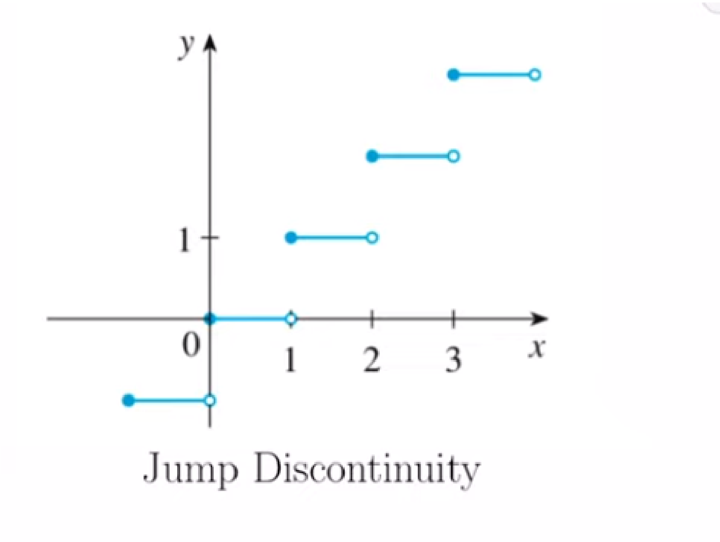
These 2 are *Removable Discontinuities* because the limit **does exist** at that point a = 2

Now if we look at: at a = 0



We can see that $\lim_{x \to 0} f(x) = \infty$, Since the limit is not a ***finite number***, that means that it is impossible for the limit to equal the function value.

Now if we look at:



We can see that the $\lim_{x \to a} f(x) = DNE$, where a is any integer

Example: Explain why f is Discontinuous at a = 1, sketch the graph of f.

$$f\left(x\right)=\begin{cases}\frac{x^2-x}{x^2-1} & \text{if } x\neq 1\\ 1 & \text{if } x=1\end{cases} \tag{6}$$

First, We will check if f(1) defined.

$$f\left(1\right)=1.$$

Second, we want to check if $\lim\limits_{x\to 1} f\left(x\right)$ exists

We can see that if we plug 1 into the denomonator, we will get an output of 0, ***therefore***, direct Substitution will not work and we must factor this equation

$$\lim_{x\to 1}\frac{x^2-x}{x^2-1}$$
$$=\lim_{x\to 1}\frac{x\left(x-1\right)}{\left(x+1\right)\left(x-1\right)}.$$

now we can cancel out the common factor of (x-1).

$$\lim_{x\to 1}\frac{x}{x+1}.$$

now if we plug in 1 to the equation we get:

$$\frac{1}{2}.$$

Third, We have to see if f(1) = $\lim\limits_{x\to 1} f\left(x\right)$, and we can see that:

$$1\neq\frac{1}{2}.$$

Therefore, f(x) is not continuous at a = 1.

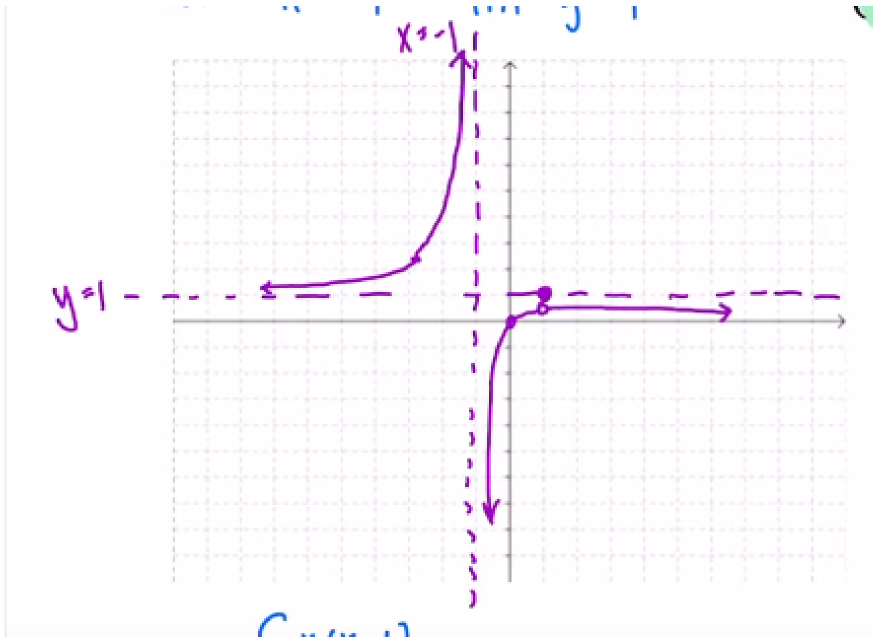
Before we make our graph, we want to make a new piecewise function with the factored version.

$$f\left(x\right)=\begin{cases}\frac{x}{x+1} & \text{if } x\neq 1\\ 1 & \text{if } x=1\end{cases} \tag{7}$$

Since the graph is Discontinuous, we will have a hole in the graph at (1, $\frac{1}{2}$), We also know that we will have a V.A at x = -1, we know this because we set the denomonator = 0 and solve. And a H.A at y = 1. This is because the degree of the numerator is the same as the degree of the denomonator, so we take the ratio of the coefficents of the leading terms of each, this is just $\frac{1}{1}$

To get additional points for the graph, we can plug in x values to the updated function. So if x = 0, we get y = 0. So we have the point (0,0), and since on the piecewise we can see that at x=1 the value is 1, we will plot an additonal point at (1,1) (this will be a point on the asymptote not connected to the line of the graph)

Graph:



Continuity from one side:

continuous from the right:

$$\lim_{x \to a+} f(x) = f(a).$$

continuous from the left:

$$\lim_{x \to a-} f(x) = f(a).$$

continuous on an interval: iff f is continuous at every number on the interval

Theorem:

If f and g are continuous at a, then

- $f + g$
- $f - g$
- fg
- $\frac{f}{g} (g \neq 0)$
- cf (c is a constant)

are also continuous at a.

Theorem

- Any polynomial is continuous on $(-\infty, \infty) = \mathbb{R}$
- Any rational function is continuous on its domain.
  (root, trig, inverse trig, log, exponential functions)

Theorem

$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$ if f is continuous at b and $\lim_{x \to a} g(x) = b$.

Theorem

If g is continuous at a and f is continuous at $g(a)$, then $(f \circ g)(x) = f(g(x))$ is continuous at a.

Example: Use continuity to evaluate.

$$\lim_{x \to \pi} \sin(x + \sin x)$$

Example: Use Continuity to Evaluate:

$$\lim_{x \to \pi} \sin\left(x + \sin x\right).$$

By the theorem, sin is continuous on its domain, which is $\mathbb{R}$

So:

$$\sin\left(\lim_{x \to \pi}\left(x + \sin x\right)\right).$$

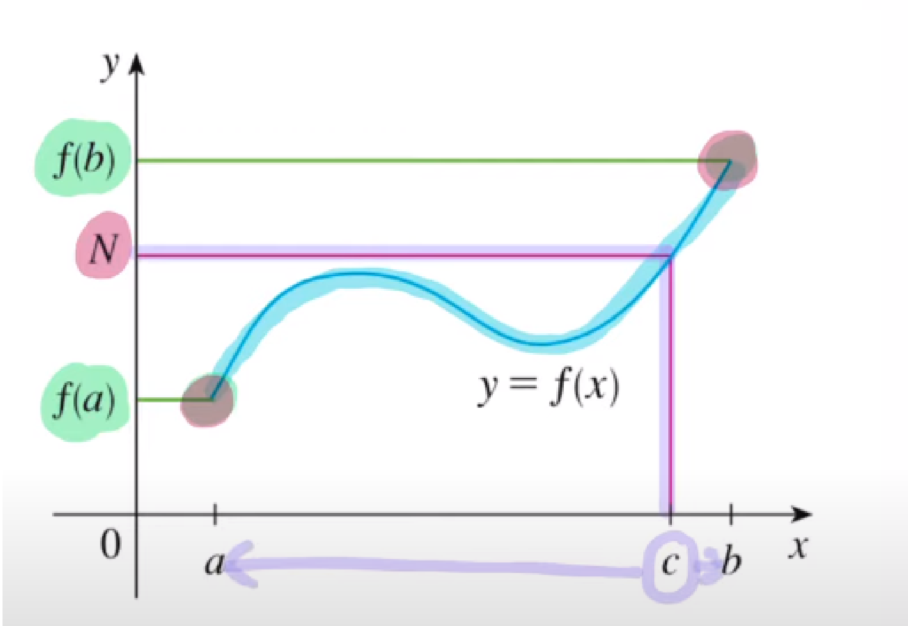
So we plug in π for x, and since $\sin\left(\pi\right) = 0$, We are just left with:

$$\begin{aligned} &\sin\left(\pi + 0\right) \\ &= 0. \end{aligned}$$

Because the sin of π is zero.

The Intermediate Value Theorem:

Suppose f is continuous on [a, b]. Let N be any number between f(a) and f(b), where f(a) $\neq$ f(b). Then There exists c $\in$ (a,b) such that f(c) = N



Example: Use the IVT to show that there is a root of f(x) = $\sqrt[3]{x} - 1 + x$ **in (0,1)**

1.) Show that f is continuous on [0,1]

By theorem, we know that f is continuous on its domain because its a sum of a polynomial and a radical function.

Because the index on the radical is odd, the domain in $\mathbb{R}$, and [0,1] lies in the domain on $\mathbb{R}$.

2.) Show that f(0) $\neq$ f(1)

so f(0):

$$\sqrt[3]{0} - 1 + 0$$
$$= -1.$$

f(1):

$$\sqrt[3]{1} - 1 + 1$$
$$= 1.$$

As you can see these 2 function values do not equal eachother

3.) Show that n $\in$ (f(0), f(1)

If they are asking for a root, then N = 0, and we can see that 0 $\in$ (-1,1)

One-Sided Continuity:

continuous from the right:

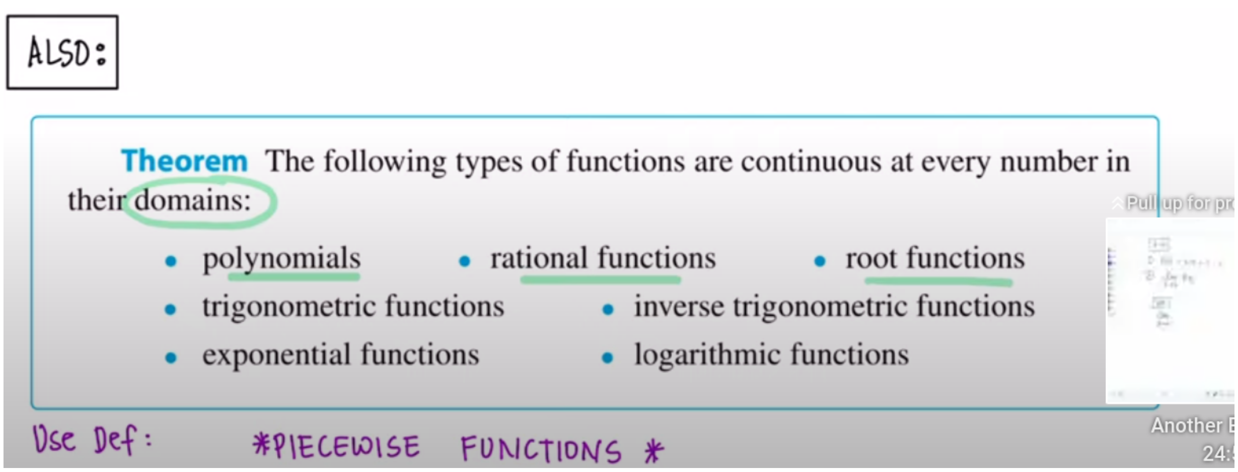
$$\lim_{x \to a+} f(x) = f(a).$$

continuous from the left:

$$\lim_{x \to a-} f(x) = f(a).$$

Note:-

If $\lim_{x \to a} f(x)$ ***Exists,*** Then you dont have to worry what side the Continuity is coming from



Note:-

You only have to be carefull with root functions if **n** is ***even***

Remember you need to check conditions 1-3 on ***piecewise functions***

Example 1: Find the numbers at which f is Discontinuous, At which of these numbers if f continuous form the right, from the left, or neither, and then sketch the graph.

$$f(x) = \begin{cases} x^2 & \text{if } x < -1 \\ x & \text{if } -1 \leqslant x < 1 \\ \frac{1}{x} & \text{if } x \geqslant 1 \end{cases} \tag{8}$$

By theorem, that on ($-\infty$, -1) and (-1,1) f(x) is a polynomial. So it is continuous.

by theorem, on (1, ∞), f(x) is a rational function, so it will be continuous on its domain. So x cannot be 0. **But (1, ∞)** does not include zero, so we are in the clear

So we only need to investigate ***2 values*, at x = -1, and x = 1**, because that is where the domain splits in the piecewise function.

So we will start by investigating x = -1

1.)

$$f(-1) = -1.$$

We plug this into the middle portion of our piecewise function because this is where the value ***-1*** falls.

2.)

$$\lim_{x \to -1} f(x).$$

We must check the limit from both sides,

Left:

$$\begin{aligned} \lim_{x \to -1-} x^2 &= (-1)^2 \\ &= 1. \end{aligned}$$

We use the first portion of the piecewise function because we are checking for values ***smaller*** than -1

Right:

$$\lim_{x \to -1+} x = -1.$$

Using the Second portion of the piecewise function

We can see that the limit from the left is not equal to the limit from the right

So we know:

$$\lim_{x \to -1} f(x) = DNE.$$

This tells us that f(x) is Discontinuities at x = -1

Notice that from step 1, we got f(-1) = -1, and this is ***the same*** as the right hand limit, so that means that f(x) is continuous from the right at x = -1

Now we will investigate at x = 1.

1.)

$$\begin{aligned} f(1) &= \frac{1}{1} \\ &= 1. \end{aligned}$$

2.)

$$\lim_{x\to 1} f(x).$$

We must check the limit from both sides,

Left:

$$\lim_{x\to 1-} x = 1$$

$$.$$

Right:

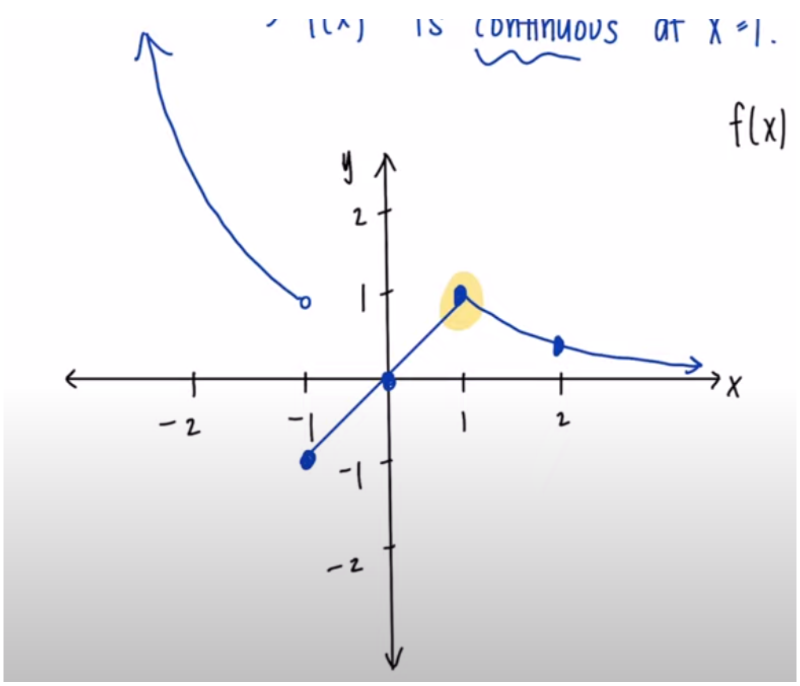
$$\begin{aligned} \lim_{x\to 1+} \frac{1}{x} &= \frac{1}{1} \\ &= 1. \end{aligned}$$

We can see that the limit from the left *is* equal to the limit from the right, So we know:

$$\lim_{x\to 1} f(x) = 1.$$

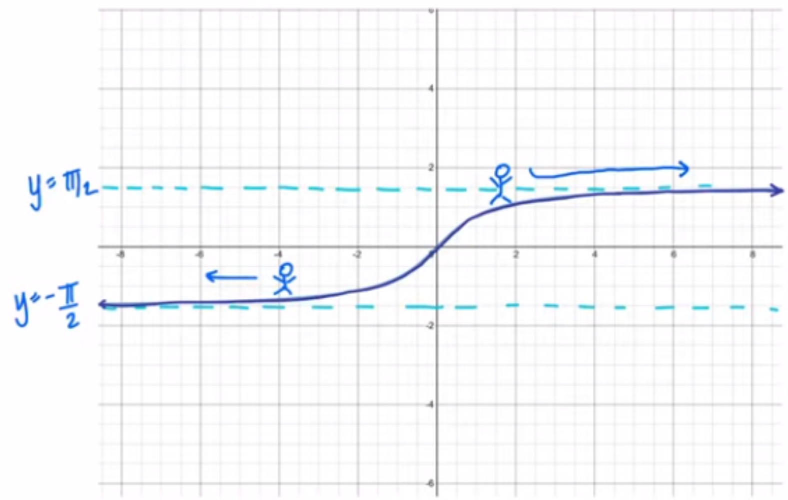
This tells us that f(x) is *continuous* at 1

Graph:



2.6: Limits at infinity, Asymptotes:

Consider the function $f(x) = \tan^{-1} x$.



$\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$ $\qquad$ $\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$

Observation: $f(x) = \tan^{-1} x$ has horizontal asymptotes at $y = \pi/2$ and $y = -\pi/2$.

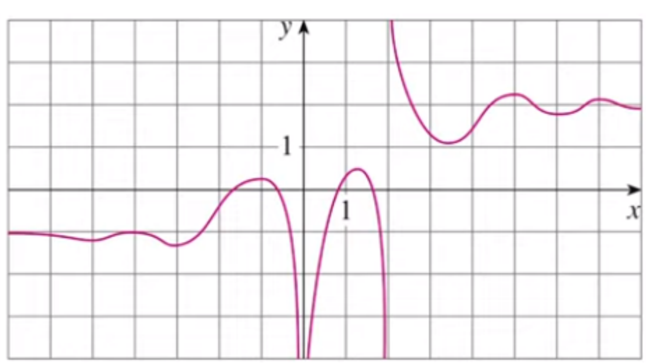
Definition: The line y = L is a Horizontal Asymptote of y = f(x) if

$$\lim_{x \to \infty} f(x) = L.$$

or

$$\lim_{x \to -\infty} f(x) = L.$$

Example: for the function g whos graph is given, find the following:



a.) $\lim_{x \to \infty} g(x)$ **= 2**

b.) $\lim_{x \to -\infty} g(x)$ **= -1**

c.) $\lim_{x \to 2} g(x)$ **= ?**

We must consider the limit from the ***left*** and ***right*** sides.

So:

$$\lim_{x \to 2+} g(x) = \infty.$$

We can see that as we approach 2 from the right, the graph is ***Increasing without bound***, so the limit is ∞

Now:

$$\lim_{x \to 2-} g(x) = -\infty.$$

We can see that as we approach 2 from the right, the graph is ***Decreasing without bound***, so the limit is - ∞

Since the limit from the left does not equal the limit from the right, the limit as x $\to$ 2 of g(x) ***DNE***

d.) $\lim_{x \to 0} g(x) =?$

Here we can see that from the left and right, the limit is approaching - ∞

Therefore,

$$\lim_{x \to 0} g(x) = -\infty.$$

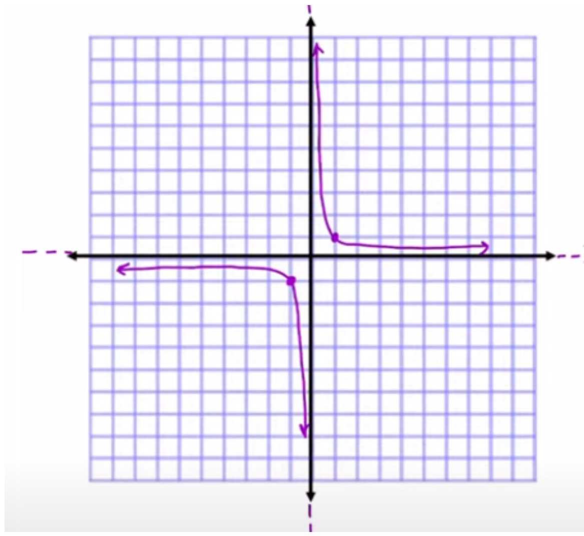
e.) $\lim_{x \to -2+} g(x)$ **= -0.4**

This limit of -0.4 is an approximation.

f.) The equations of the Asymptotes are:

H.A: $y = 2, y = -1$
V.A: $x = 0, x = 2.$

Consider: $f(x) = \frac{1}{x}$



Our points here for the ***Reciprical Function*** are (1,1) and (-1,-1)

Find:

a.) $\lim_{x \to \infty} \frac{1}{x}$

We can see that as we approach ∞ on x the graph is going towards ***zero***

Therefore:

$$\lim_{x \to \infty} \frac{1}{x} = 0.$$

b.) $\lim_{x \to -\infty} \frac{1}{x}$

We can see that as we approach - ∞ on x the graph is also going towards ***zero***

Therefore:

$$\lim_{x \to -\infty} \frac{1}{x} = 0.$$

Theorem:

$\lim_{x\to\infty} \frac{1}{x^r} = 0$ and $\lim_{x\to-\infty} \frac{1}{x^r} = 0$ if r > 0 and is a rational numebr

Examples: Find the limit:

1.) $\lim_{x\to\infty} \frac{3x+5}{x-4}$

1.) Divide by the highest degree of the denomonator.

So we are going to divide the numerator and the denomonator by $\boldsymbol{x}$

So:

$$\lim_{x\to\infty} \frac{3+\frac{5}{x}}{1-\frac{4}{x}}.$$

2.) Take the limit of each of the terms in the numerator and denominator

So:

$$\frac{\lim_{x\to\infty} 3 + \lim_{x\to\infty} \frac{5}{x}}{\lim_{x\to\infty} 1 - \lim_{x\to\infty} \frac{4}{x}}.$$

Note:-

Remember if we take the limit as x approaches ∞ of a ***constant that doesn't change***, over x, the limit ***will be zero!***

So we have:

$$\frac{3+0}{1-0}$$
$$= 3.$$

Example:

2.) $\lim_{x\to-\infty} \frac{t^2+2}{t^3+t^2-1}$

1.) Again, divide by the highest degree in the denominator, which is t^3

So we will have:

$$\lim_{t \to -\infty} \frac{\frac{1}{t} + \frac{2}{t^3}}{1 + \frac{1}{t} - \frac{1}{t^3}}.$$

Just like the previous example, if we take the limit of each of the terms in both the numerator and the denominator, we get:

$$\frac{0+0}{1+0-0}.$$

Which is just:

$$\frac{0}{1}$$
$$= 0.$$

Note:-

If the degree in the denominator is higher than the degree of the denominator, than the equation of the H.A is automatically ***y=0***

Example:

3.) $\lim_{x \to \infty} \frac{x+2}{\sqrt{9x^2+1}}$

Recall: $\sqrt{x^2} = |x|$

Which is the piecewise:

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases} \tag{9}$$

So we divide numerator and denominator by $|x|$, and we have to look at the piecewise function to retrieve the sign of x, since we are going to *positive* ∞, we will use the top portion of the piecewise. Because this is for *positive* values of x

So the numerator will be:

$$1 + \frac{2}{x}.$$

Then for the denominator, we need to divide by $\sqrt{x^2}$

So:

$$\sqrt{\frac{9x^2+1}{x^2}}.$$

Which gives us:

$$\sqrt{9+\frac{1}{x^2}}.$$

So our equation is:

$$\lim_{x\to\infty}\frac{1+\frac{2}{x}}{\sqrt{9+\frac{1}{x^2}}}.$$

Now if we take the limit of each term in the numerator and denominator, we get:

$$\frac{1+0}{\sqrt{9+0}}.$$

Which is simplifed to:

$$\frac{1}{3}.$$

Example: Negative ∞

4.) $\lim_{x\to-\infty}\frac{\sqrt{9x^6-x}}{x^3+1}$

Notice that we have $\sqrt{x^6}$, which means we have $|x^3|$

Note:-

We dont need to do absolute value with piecewise for terms like $\sqrt{x^8}$, this is because if it simplifes to x with an ***even power***, we know that any value plugged in for x will be even automatically.

So we rewrite $|3|$ as piecewise

So:

$$|3|=\begin{cases} x^3 & \text{if } x\geqslant 0 \\ -x^3 & \text{if } x<0 \end{cases} \tag{10}$$

So now we check which way the limit is going to determine which part of the piecewise function we will use, since we are going to - ∞, we will use the ***bottom*** portion of the piecewise.

So for the numerator we have:

$$\sqrt{\frac{9x^6-x}{x^6}}.$$

Which is:

$$\sqrt{\frac{9x^6}{x^6}-\frac{x}{x^6}}.$$

Which simplifies to:

$$-\sqrt{9-\frac{1}{x^5}}.$$

Notice we put a ***Negative sign*** **infront of the equation, this is because we used the bottom portion of the piecewise function, this is important**

And for the denominator, we divide by x^3, so we will have:

$$\frac{x^3}{x^3}+\frac{1}{x^3}.$$

Which simplifes to:

$$1+\frac{1}{x^3}.$$

So our full equation would be:

$$\lim_{x\to-\infty}\frac{-\sqrt{9-\frac{1}{x^5}}}{1+\frac{1}{x^3}}.$$

And now if we take the limits for each of the terms, we get:

$$\frac{-\sqrt{9-0}}{1+0}.$$

Which simplifes to:

$$-3.$$

Last Example:

5.) $\lim\limits_{x\to\infty}\frac{x^3-2x+3}{5-2x^2}$

So we divide by the highest degree in the denominator: x^2

So:

$$\lim_{x\to\infty}\frac{x-\frac{2}{x}+\frac{3}{x}}{\frac{5}{x}-2}.$$

Now we take the limit of each of the terms:

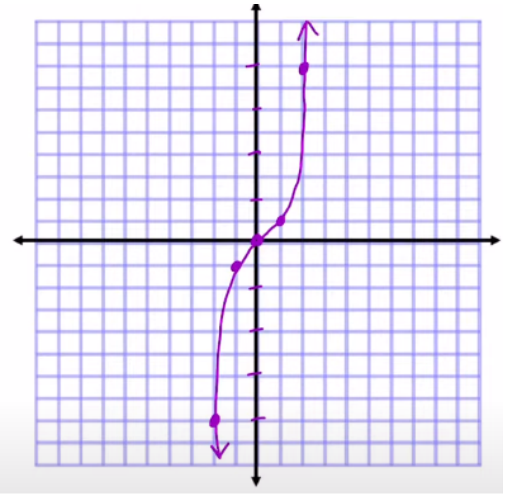
$$\frac{\infty-0+0}{0-2}.$$

Which simplifies to:

$$\frac{\infty}{-2}$$
$$=-\infty.$$

Infinite Limits at Infinity:

Consider f(x) = x^3



1.) $\lim_{x\to\infty} x^3 = \infty$

2.) $\lim_{x\to-\infty} x^3 = -\infty$

The term we use to describe this situation is ***end behavior***

Recap:

H.A:

Take $\lim_{x\to\infty} f(x)$ ***and*** $\lim_{x\to-\infty} f(x)$

If f(x) = some constant ***c***, then H.A is y = c

V.A:

x = k, where ***x - k* is a factor of the *denominator only!***

Example: Find the Horizontal Asymptotes and Vertical Asymptotes:

$$y = \frac{1+x^4}{x^2-x^4}.$$

So for ***H.A*, we want to take the limit of both *positve and negaitive infinity.***

So:

$$\lim_{x\to\infty} \frac{1+x^4}{x^2-x^4}.$$

So divide both sides by the highest degree in the denominator, x^4

$$\lim_{x\to\infty} \frac{\frac{1}{x^4}+1}{\frac{1}{x^2}-1}.$$

Now take the limits of each term and get:

$$\begin{aligned} &\frac{0+1}{0-1} \\ &= \frac{1}{-1} \\ &= -1. \end{aligned}$$

Same result for $\lim\limits_{x\to-\infty}$, **So H.A is y = -1**

Now for V.A:

Start by factoring the equation:

$$y = \frac{1+x^4}{x^2\left(1-x^2\right)}.$$

Denominator factors further by using ***Difference of squares:***

$$y = \frac{1+x^4}{x^2\left(1+x\right)\left(1-x\right)}.$$

Since we cant cancel out any common factors, this equation is fully factored and simplified, and the zeros of the Denominator is going to be the equations for our vertical Asymptotes.

So:

$$x = 0, x = -1, x = 1.$$

2.7

Derivatives and rates of change:

Back to tangent lines:

Recall:

$$m_{pq} = \frac{f(x) - f(a)}{x - a} \text{ is the slope of the } \textbf{\textit{Secent line}}.$$

And:

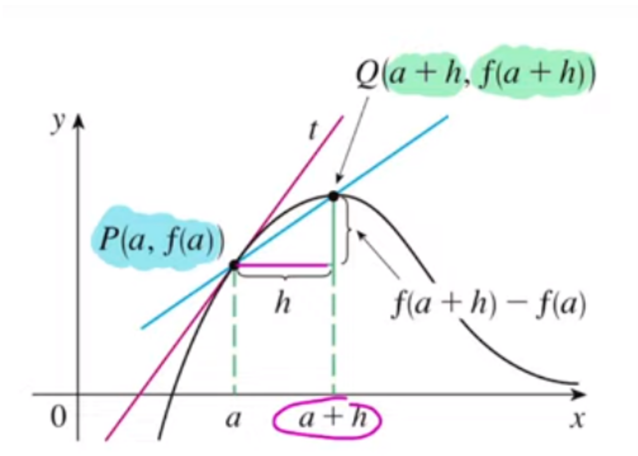
$$\lim_{q \to p} m_{pq} = m \text{ is the slope of the tangent line at point P.}$$

Another way to state that is the following:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = m.$$

Another way to expess the above slope formula is base on the following:

$$Q\left(a + h, f\left(a + h\right)\right).$$



The slope of the ***Secent Line*** would be:

$$\begin{aligned} m_{pq} &= \frac{f\left(a+h\right)-f\left(a\right)}{a+h-a} \\ &= \frac{f\left(a+h\right)-f(a)}{h}. \end{aligned}$$

Slope of the tangent line would be:

$$\lim_{h\to 0}\frac{f\left(a+h\right)-f(a)}{h}.$$

This is the ***Difference quotient***

Example: Find an equation of the tangent line to the curve $y = 2x^3 - 5x$ at the point (-1,3)

Note that P(-1,3) is (a, f(a))

So we need:

1. Point
2. Slope

We have a point, but we dont have slope since we only have ***one point***, So to find the slope of the tangent line, we will use the definiton from the previous page.

We know:

$$a = 1.$$

$$\begin{aligned} m_{tan} &= \lim_{x \to -1} \frac{f(x) - f(-1)}{x - (-1)} \\ &= \lim_{x \to -1} \frac{2x^3 - 5x - 3}{x + 1}. \end{aligned}$$

We cant use direct Substitution, so we need to factor. But we can use ***synthetic division to factor***

Synthetic Division:

$$\begin{array}{r|rrrr} -1 & 2 & 0 & -5 & -3 \\ & \downarrow & -2 & 2 & 3 \\ \hline & 2 & -2 & -3 & 0 \end{array}$$

$$\underbrace{2 \quad -2 \quad -3}_{2x^2 - 2x - 3}$$

Explanation:

2 for x^3, 0 for x^2, -5 for x, and -3, then drop down the 2, multiply by the negative one and add down. Then these are the terms.

So now:

$$\begin{aligned} &\lim_{x \to -1} \frac{\left(2x^2 - 2x - 3\right)(x + 1)}{x + 1} \\ &= \lim_{x \to -1} \left(2x^2 - 2x - 3\right). \end{aligned}$$

Note:-

Notice that we added a (x+1) in the numerator of our factored equation after synthetic division.

Now we can plug in -1 into our factored equation and output the limit.

$$\lim_{x\to -1} 2\left(-1\right)^{2} + 2\left(1\right) - 3$$
$$= 1.$$

Now we can put it all together and find the equation of the tangent line using point slope form:

$$y - 3 = 1\left(x - \left(-1\right)\right)$$
$$y - 3 = x + 1$$
$$y = x + 4.$$

Alteratively, another way to get m:

$$\lim_{h\to 0} \frac{f\left(-1+h\right) - f\left(-1\right)}{h}$$
$$= \lim_{h\to 0} \frac{2\left(-1+h\right)^{3} - 5\left(-1+h\right) - 3}{h}.$$

First, foil out $(-1+h)^3$:

$$2\left(h^{3} - 3h^{2} + 3h - 1\right)$$
$$= 2h^{3} - 6h^{2} + 6h - 2.$$

Then, distribute -5 to $(-1+h)$

$$= -5h + 5.$$

So all together we have:

$$\frac{2h^{3} - 6h^{2} + 6h - 2 - 5h + 5 - 3}{h}$$
$$= \frac{2h^{3} - 6h^{2} + h}{h}.$$

Now, factor out an h:

$$\frac{h\left(2h^{2} - 6h + 1\right)}{h}.$$

Cancel out the common factor:

$$2h^{2} - 6h + 1.$$

Now plug in zero for h

$$2\left(0\right)^{2} - 6\left(0\right) + 1$$
$$= 1.$$

So you see we got the same answer of ***1***:

Velocites:

Suppose an object moves along a straight line according to an equation of motion s = f(t)

$$v_{ave} = \frac{Displacement}{Time} = \frac{f(a+h)-f(a)}{h}.$$

For time interval $t = a$ **to** $t = a + h$

And:

$$v_{inst} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

At time $t = a$

Or:

$$\lim_{t\to a} \frac{f(t)-f(a)}{t-a}.$$

Note:-

Speed = $|Velocity|$

Example: If a ball is thrown upward, it height after t seconds is $y = 40t - 16t^2$. find the velocity when t= 3

$$v_{inst} = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h}$$
$$= \lim_{h\to 0} \frac{[40(3+h)-16(3+h)^2]-[40\cdot 3-16\cdot 3^2]}{h}.$$

Now distribute out everything:

$$\lim_{h\to 0} \frac{120+40h-144-96h-16h^2+24}{h}.$$

Combine Like terms:

$$\lim_{h\to 0} \frac{-56h-16h^2}{h}.$$

Factor out an ***h***

$$\lim_{h\to 0} \frac{h\,(-56-16h)}{h}$$
$$= \lim_{h\to 0} -56-16h.$$

Now plug in zero

$$-56-16\,(0)$$
$$= -56m\diagdown s.$$

Derivatives:

Above Formulas are given a special name:

The Derivative of a function f at a number a is:

$$f\prime(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

Or:

$$f\prime(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a}.$$

On a graph $f\prime(a) = m_{tan}$ and represents Instantaneous rate of change

Example: Find $f\prime(a)$ if f(x) = $\frac{x^2+1}{x-2}$

$$\lim_{h\to 0} \frac{\frac{(a+h)^2+1}{(a+h)-2} - \frac{a^2+1}{a-2}}{h}.$$

So we multiply by the lcd of the ***entire fraction***, which is $\frac{(a+h-2)(a-2)}{(a+h-2)(a-2)}$

$$\begin{aligned}
&\lim_{h\to 0} \frac{\left((a+h)^2+1\right)(a-2) - \left(a^2+1\right)(a+h-2)}{h\,(a+h-2)\,(a-2)} \\
&= \lim_{h\to 0} \frac{\left(a^2+2ah+h^2+1\right)(a-2) - \left(a^3+a^2h-2a^2+a+h-2\right)}{h\,(h+a-2)\,(a-2)} \\
&= \lim_{h\to 0} \frac{a^3+2a^2h+ah^2+a-2a^2-4ah-2h^2-2-a^3-a^2h+2a^2-a-h+2}{h\,(a+h-2)\,(a-2)}.
\end{aligned}$$

Now cancel out terms and combine like terms

$$\begin{aligned}
&\lim_{h\to 0} \frac{a^2h+ah^2-4ah-2h^2-h}{h(a+h-2)(a-2)} \\
&= \lim_{h\to 0} \frac{h(a^2+ah-4a-2h-1)}{h(a+h-2)(a-2)} \\
&= \lim_{h\to 0} \frac{a^2+ah-4a-2h-1}{(a+h-2)(a-2)}.
\end{aligned}$$

Now plug in zero for h

$$\begin{aligned}
&\frac{a^2+a\cdot 0-4a-2\cdot 0-1}{(a+0-2)(a-2)} \\
&= \frac{a^2-4a-1}{(a-2)^2}.
\end{aligned}$$

Example: If f(t) $= t^{-1} - t$, find the velocity and speed when $t = 5$

So we can use either formula listed above, ***Either:***

$$\lim_{h \to 0} \frac{f(5+h) - f(5)}{h}.$$

or

$$\lim_{t \to 5} \frac{f(t) - f(5)}{t-5}.$$

We are going to use the bottom one:

Note:-

t^{-1} is the same as $\frac{1}{t}$

$$\lim_{t \to 5} \frac{\frac{1}{t} - t - (\frac{1}{5} - 5)}{t-5}.$$

So we want to multiply by the lcd of the entire expression to ***clear out the fractions***, in this case the ***lcd*** is **5t**

$$\lim_{t \to 5} \frac{5 - 5t^2 - t - 25t}{(t-5)(5t)}$$
$$= \lim_{t \to 5} \frac{-5t^2 + 24t + 5}{5t(t-5)}.$$

To make things easier for the numerator, we want to factor out the negative in the ***leading term***

$$\lim_{t \to 5} \frac{-(5t^2 - 24t - 5)}{5t(t-5)}.$$

Now factor the numerator:

$$\lim_{t \to 5} \frac{-(5t+1)(t-5)}{5t(t-5)}$$
$$= \lim_{t \to 5} \frac{-(5t+1)}{5t}.$$

Now use direct Substitution

$$\frac{-(5 \cdot 5 + 1)}{5 \cdot 5}$$
$$= -\frac{26}{25}.$$

So **Velocity:** $\frac{-26}{25}$ $u \diagdown s$

And **speed:** $|\frac{-26}{25}| = \frac{26}{25}$ $u \diagdown s$

Theorems From 2.7:

Slope of Secent Line:

$$m_{pq} = \frac{f(x) - f(a)}{x - a}.$$

Or:

$$m_{pq} = \frac{f(a+h) - f(a)}{h}.$$

Slope of tangent line:

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Or:

$$m_{tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Velocites:

$$v_{ave} = \frac{Displacement}{Time} = \frac{f(x) - f(a)}{x - a}.$$

$$v_{inst} = \lim_{t \to a} \frac{f(t) - f(a)}{t - a}.$$

Or

$$v_{inst} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

$$Speed = |Velocity|.$$

Derivatives:

$$f\prime(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Or:

$$f\prime(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

2.8

The Derivatives as a Function / Finding The Derivative Using The Limit Definition:

Theorem: Earlier we constructed the derivative of f at a fixed number a

$$f\prime(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

Theorem: If we let a vary, we obtain:

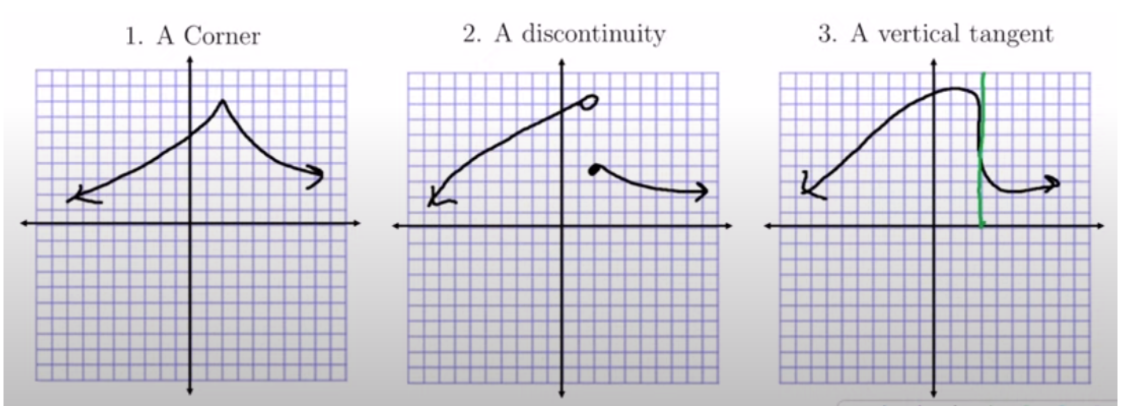
$$f\prime(a) \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

Which is a function itself, and we call it the derivative of f

Notations:

- $f\prime(x) = y\prime = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = d_x f(x)$
- We say "f is differentiable at a" if $f\prime(a)$ exists.
- We say "f is differentiable on (a, b)" if it is differentiable at every number in (a,b)

3 Cases where a function is not differentiable:



Example:

1. Find the derivative of $f(x) = x + \sqrt{x}$
2. State the domain of $f(x)$ *and* $f\prime(x)$

<u>Theorem:</u> We will use the following Theorem:

$$f\prime(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

$$\begin{aligned} &\lim_{h\to 0} \frac{(x+h+\sqrt{x+h}) - (x+\sqrt{x})}{h} \\ &= \lim_{h\to 0} \frac{x+h+\sqrt{x+h} - x - \sqrt{x}}{h} \end{aligned}.$$

Cancel out the x's in the numerator:

$$= \lim_{h\to 0} \frac{h+\sqrt{x+h} - \sqrt{x}}{h}.$$

We will split up the limit so it's not as confusing when trying to multiply by the conjugate

$$\lim_{h\to 0} \frac{h}{h} + \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}.$$

Note:-

$\lim_{h\to 0} \frac{h}{h} = 1$

So:

$$1 + \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}.$$

The 2 x's in the numerator cancel, aswell as the h's

So:

$$1 + \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}.$$

Now substitute in zero for ***h***:

$$\begin{aligned} &1 + \frac{1}{\sqrt{x+0} + \sqrt{x}} \\ &= 1 + \frac{1}{2\sqrt{x}} \end{aligned}.$$

Summerize:

$$f(x) = x + \sqrt{x} \; and \; f\prime(x) = 1 + \frac{1}{2\sqrt{x}}.$$

Domain:

Domain of f(x): [0 ∞) $\rightarrow$ Because we have the radical, we cant plug in any negative numbers

Domain of $f\prime(x)$: (0, ∞)

Note:-

The domain of $f\prime x$ may (or may not) be a subset of the domain of f(x)

Theorem 0.0.1

If f is differentiable at a, then it is continuous at a

Note:-

The converse is ***Not True***, if f is continuous at a, that *does not mean* f is differentiable at a

Higher Derivatives

1. f(x)
2. $f\prime(x)$ - 1st derivative
3. f"(x) - 2nd derivative
4. f"' - 3rd derivative

Application Problems

1. s(t) - position function
2. s'(t) - v(t) = velocity
3. s"(t) - a(t) = acceleration
4. s"'(t) - j(t) = jerk

Example: if $f(x) = 2x^3 + 5x$, find f'(x), f"(x), f"'(x) and $f^{(4)}$. State the domain of each.

$$f\prime(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

So: $f\prime(x)$

$$
\begin{aligned}
&\lim_{h\to 0}\frac{2(x+h)^3+5(x+h)-(2x^3+5x)}{h}\\
&=\lim_{h\to 0}\frac{2(x^3+h^3+3x^2h+3xh^2)+5x+5h-2x^3-5x}{h}\\
&=\lim_{h\to 0}\frac{2x^3+2h^3+6x^2h+6xh^2+5x+5h-2x^3-5x}{h}\\
&=\lim_{h\to 0}\frac{2h^3+6x^2h+6xh^2+5h}{h}\\
&=\lim_{h\to 0}\frac{h(2h^2+6x^2+6xh+5)}{h}\\
&=\lim_{h\to 0}2h^2+6x^2+6xh+5\\
&=2(0)^2+6x^2+6x(0)+5\\
&=6x^2+5.
\end{aligned}
$$

Now: $f\prime\prime(x)$**, for *__Double Prime__*, use the equation you found for** $f\prime(x)$ **as you would f(x) for *Single Prime***

$$
\begin{aligned}
f\prime\prime(x)&=\lim_{h\to 0}\frac{6(x+h)^2+5-(6x^2+5)}{h}\\
&=\lim_{h\to 0}\frac{6(x^2+h^2+2xh)+5-6x^2-5}{h}\\
&=\lim_{h\to 0}\frac{6x^2+12xh+6h^2+5-6x^2-5}{h}\\
&=\lim_{h\to 0}\frac{6h^2+12xh}{h}\\
&=\lim_{h\to 0}\frac{h(6h+12x)}{h}\\
&=\lim_{h\to 0}6h+12x\\
&=6(0)+12x\\
&=12x.
\end{aligned}
$$

$f\prime\prime\prime(x)$

$$
\begin{aligned}
f\prime\prime\prime(x)&=\lim_{h\to 0}\frac{12(x+h)-(12x)}{h}\\
&=\lim_{h\to 0}\frac{12x+12h-12x}{h}\\
&=\lim_{h\to 0}\frac{12h}{h}\\
&=\lim_{h\to 0}\frac{h(12)}{h}\\
&=\lim_{h\to 0}12=12.
\end{aligned}
$$

$f^{(4)}$

$$
\begin{aligned}
f\prime\prime\prime\prime&=\lim_{h\to 0}\frac{12-12}{h}\\
&=\lim_{h\to 0}\frac{0}{h}\\
&=\lim_{h\to 0}0=0.
\end{aligned}
$$

Summerize:

$$
\begin{aligned}
f(x) &= 2x^3 + 5x \\
f\prime(x) &= 6x^2 + 5 \\
f\prime\prime(x) &= 12x \\
f\prime\prime\prime(x) &= 12 \\
f^{(4)}(x) &= 0.
\end{aligned}
$$

<u>Domain:</u>

1. $f\prime = \mathbb{R}$
2. $f\prime\prime = \mathbb{R}$
3. $f\prime\prime\prime = \mathbb{R}$
4. $f\prime\prime\prime\prime = \mathbb{R}$

Finding the derivative using the limit definition:

Example 1: $f(x) = \frac{1}{x+3}$

$f\prime(x)$

$$f\prime(x) = \lim_{h\to 0} \frac{\frac{1}{x+h+3} - \frac{1}{x+3}}{h}$$

.

Multiply by the lcd to clear out the fractions in the numerator: LCD: $\frac{(x+h+3)(x+3)}{(x+h+3)(x+h)}$

$$
\begin{aligned}
\lim_{h\to 0} \frac{\frac{1}{x+h+3} - \frac{1}{x+3}}{h} \cdot \frac{(x+h+3)(x+3)}{(x+h+3)(x+3)} \\
= \lim_{h\to 0} \frac{x+3-(x+h+3)}{h} \\
= \lim_{h\to 0} \frac{x+3-x-h-3}{h(x+h+3)(x+3)} \\
= \lim_{h\to 0} \frac{-h}{h(x+h+3)(x+3)} \\
= \lim_{h\to 0} \frac{-1}{(x+h+3)(x+3)} \\
= \frac{-1}{(x+0+3)(x+3)} \\
= \frac{-1}{(x+3)^2}.
\end{aligned}
$$