Calculus 1 Notes

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December 17, 2023

Chapter 2

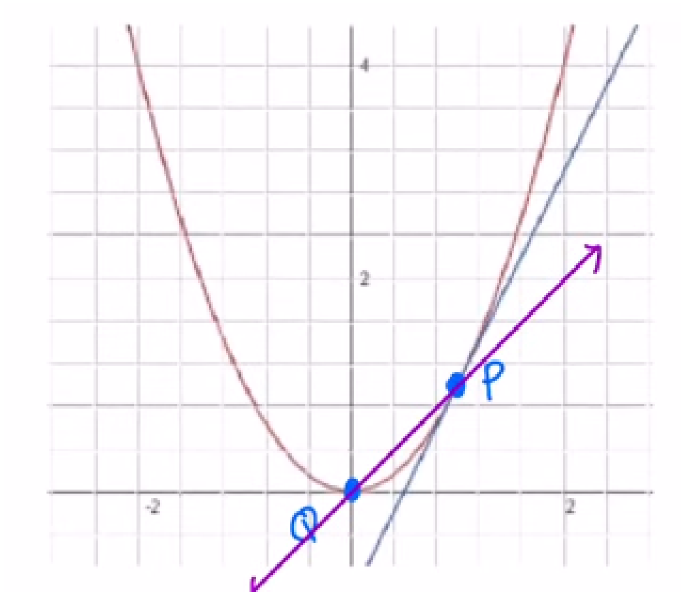
Contents

2.1: The Tangent and Velocity Problems

The Tangent Problem:

Question 1

Can we find an equation of the tangent line to $y = x^2$ at the point P(1,1)?



Explanation: .
$y = x^2$: Red parabola
Tangent line: Blue line
Secent Line: Pink line with points q and p

We are asked to get the equation of the tangent line to $y = x^2$ at the point P(1,1), However to find the equation of this line we know we need **2 things,**

- Point
- Slope

Since we only have one point, we cannot find slope. Therefore, we must use another point as an approximation and create a secent line instead. **This secent line is the pink line in the above graphic.**

So, lets use the point Q(0,0) as our second point. Now we can find slope with P(1,1), and Q(0,0).

If Slope $= \frac{y2-y1}{x2-x1}$, Then M of PQ $\longrightarrow \frac{1-0}{1-0} = 1$

Lets get a better approximation by using a point closer to the tangent line Lets use Q(0.9, 0.81)

So M of PQ $\longrightarrow \frac{1-0.81}{1-0.9} = 1.9$

Now, lets get an even closer approximation by using the point Q(0.99, 0.9801)

So, M of PQ $\longrightarrow \frac{1-0.9801}{1-0.99} = 1.99$

Notice, as the point Q gets closer to P, the slope of PQ is getting closer to 2

We write,

$$\lim_{Q \to P} \text{M of PQ} = \text{m}$$

Where **m** on the right of equation is slope of tangent line at **P**, And **M of PQ** is slope of the secent line

Now,

We will use our approximation of $m \approx 2$ to write the equation of the tangent line, using the orginial point P(1,1).

$$\begin{aligned} y - 1 &= 2\,(x - 1) \\ y - 1 &= 2x - 2 \\ y &= 2x - 1. \end{aligned}$$

The Velocity Problem:

- Average Velocity: $\frac{distance\ traveled}{time\ elapsed}$, which is represented by the slope of the secent line.
- Instantaneous Velocity = Velocity at a given instant of time, which is represented by the slope of the tangent line

Example 0.0.1

If a rock is thrown upward on the planet Mars, with a Velocity of 10 m/s, It's height in meters t seconds later is given by $y = 10t - 1.86t^2$

Question 2

Find the average Velocity over the given time intervals:

(i) [1,2] $\rightarrow$ 1 and 2 represent values of t

Substitute values into equation above

$$\begin{aligned} y(1) &= 10(1) - 1.86(1)^2 \\ &= 8.14. \end{aligned}$$

$$\begin{aligned} y(2) &= 10(2) - 1.86(2)^2 \\ &= 12.56. \end{aligned}$$

If Average Velocity = $\frac{distance\ traveled}{time\ elapsed}$ Or better yet $\frac{Change\ in\ height}{change\ in\ time}$

And we have the points (1,8.14) and (2,12.56)

Then,

$$\begin{aligned} Average\ Velocity &= \frac{12.56 - 8.14}{2 - 1} \\ &= 4.42m\diagdown s. \end{aligned}$$

(ii) [1,1.5]

Substitute values into equation above

$$\begin{aligned} y\left(1\right) &= 10\left(1\right)-1.86\left(1\right)^2 \\ &= 8.14. \end{aligned}$$

$$\begin{aligned} y\left(1.5\right) &= 10\left(1.5\right)-1.86\left(1.5\right)^2 \\ &= 10.815. \end{aligned}$$

After solving theses equations we have the points (1,8.14) and (1.5,10.815)

So,

$$\begin{aligned} Average\ Velocity &= \frac{10.815-8.14}{1.5-1} \\ &= 5.35m\diagdown s. \end{aligned}$$

2.1.1 The Limit of a Function:

Question 3

Consider the values of $f(x) = x^2 + 2$ near $x = 2$

We want to know whats going on near x=2, so we make a table

x	f(x) $= x^2+2$
0	2
1	3
1.5	4.25
1.9	5.61
2	
2.1	6.41
2.4	7.76
2.9	10.41
4	18

Now we want to look at the closet x values to 2, which is the 2 that are above and below **2**, **We observe that as x values approach 2, then f(x) values approach 6**

so we write,

$$\lim_{x \to 2} f(x) = 6.$$

Example 0.0.2

Use a table of values to estimate the limit: $\lim\limits_{x \to 0} \frac{tan3x}{tan5x}$

Rememeber the value **0** is **a** so we want to contruct our table where a is in the middle, so use values that are smaller and larger than a.

Using arbitrary values that are close to 0, we gete the table,

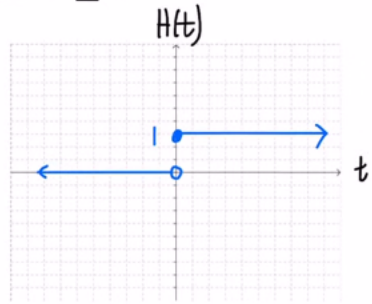
x	f(x)
−0.7	−4.56
−0.1	0.566
−0.01	0.5997
0.01	0.5997
0.1	0.566
0.7	−4.56

Now after looking at our table, we can conclude that

$$\lim_{x \to 0} \frac{tan3x}{tan5x} = 0.6.$$

One Sided Limits:

Consider $H(t) = \begin{cases} 0 & \textbf{if } t < 0 \\ 1 & \textbf{if } t \geq 0 \end{cases}$



Note:-

if there is a **minus** sign after a, that means you are approaching limit from the left if there is a **plus** sign after a, that means you are approaching limit from the right, if you see a limit with either of these, it is called a two sided limit

What is $\lim_{t \to 0-} h(t)$

So looking at the bottom line, coming from the left, as we approach 0, the y value is 0.

so $\rightarrow$

$$\lim_{t \to 0-} h(t) = 0.$$

What is $\lim_{t \to 0+} h(t)$

Given that we are approaching from the right, we are now looking at the top line, we can see that as we approach 0, y is 1

so

$$\lim_{t \to 0+} h(t) = 1.$$

Note:-

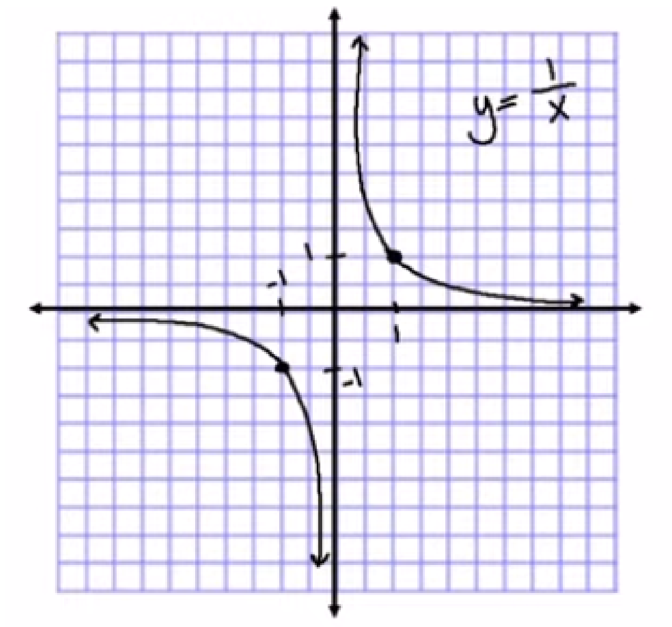
The first one is our **Left hand limit** and the bottom one is our **right hand limit** if the side we our approaching from is not specified, **we cannot find the limit, so we would say DNE**

So

$\lim_{x \to 0} f(x) = l$ **iff** (if and only if) $\lim_{x \to 0-} f(x) = L$ **and** $\lim_{x \to 0+} f(x) = L$

in other words, we can only drop the + or - after the a if the right and left hand limits are the same

Infinite Limits:



if we look at

$$\lim_{x \to 0+} f(x) = ?.$$

We notice that as we approach 0 from the right, f(x) goes to infinity

So:

$$\lim_{x \to 0+} f(x) = \infty.$$

This is also the same for x $\to$ 0$-$

So:

$$\lim_{x \to 0-} f\,(x) = \infty.$$

Note:-

x = 0 is a vertical Asymptote

In general, $x = a$ is a vertical asymptote if at least one of the following are true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = -\infty$$
$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = \infty$$
$$\lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = -\infty$$

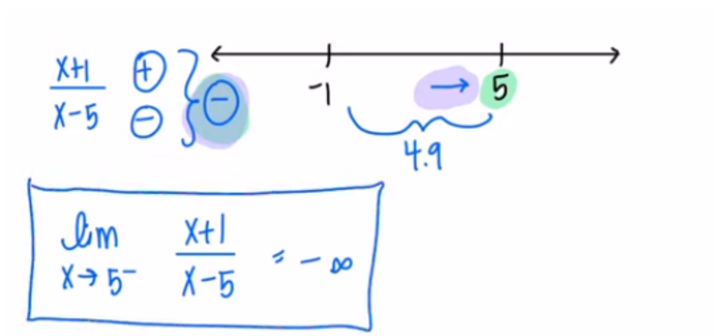
Examples: Determine the infitite limit

1.) $\lim_{x\to 5-} \frac{x+1}{x-5}$

$$\text{x + 1} \longrightarrow 6$$
$$\text{x - 5} \longrightarrow 0$$

If you have a $\frac{\textit{nonzero constant}}{\textit{approaching } 0}$ its either going to be approaching ∞ or $-\infty$ the way we find which version of infinity it will be is with either a table or a numberline

To make the numerline we want to list the zeros, so -1 and 5. Then pick a value thats close to a and approachs in the correct direction. Then plug this number into the equation and whatever sign you get will be the sign for infinity.



2.) $\lim_{x\to 5-} \frac{e^x}{(x-5)^3}$