Calculus 1: Chapter 3

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Chapter 3

Contents

3.1

Differential Rule:

Diffential Fomulas:

- $\frac{d}{dx}(c) = 0$
- $\frac{d}{dx}(x) = 1$
- $\frac{d}{dx}(x^n) = n \cdot x^{n-1} \to$ ***Power Rule***
- $\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}[f(x)]$
- $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

Example 0.0.1

Differentiate the following functions:

1.) $f(t) = \frac{1}{2}t^6 - 3t^4 + 1$

For the first term, we will use the ***Third and Fourth*** Rule:

$$\frac{1}{2} \cdot 6t^{t-1}.$$

For the second term, $-3t^4$, We will use the ***Third and Fifth*** Rule:

$$-3 \cdot 4t^{4-1}.$$

The last term is a constant, so according to the first rule, the Derivative of a constant is ***Zero***:

So our full equation is:

$$\begin{aligned} f\prime(x) &= \frac{1}{2} \cdot 6t^{6-1} - 3 \cdot 4t^{4-1} + 0 \\ &= 3t^5 - 12t^3. \end{aligned}$$

2.) $h(x) = (x-2)(2x+3)$

First we need to distribute out the terms:

$$\begin{aligned} h(x) &= 2x^2 + 3x - 4x - 6 \\ &= 2x^2 - x - 6. \end{aligned}$$

Now this is the function we want to diferentiate.

So $\rightarrow$

$$h\prime(x) = 2 \cdot 2x^{2-1} - 1 - 0$$
$$h\prime(x) = 4x - 1.$$

3.) $y = \frac{x^2 - 2\sqrt{x}}{x}$

So:

$$y = \frac{x^2 - 2x^{\frac{1}{2}}}{x}.$$

Since the denominator only has ***one term****, we can split the equation like:*

$$y = \frac{x^2}{x} - \frac{2x^{\frac{1}{2}}}{x}$$
$$y = x - 2x^{-\frac{1}{2}}.$$

Now:

$$\frac{dy}{dx} = 1 - 2 \cdot (-\frac{1}{2})x^{-\frac{1}{2}-1}$$
$$\frac{dy}{dx} = 1 + x^{-\frac{3}{2}}.$$

And we can even rewrite it as:

$$\frac{dy}{dx} = 1 + \frac{1}{x^{\frac{3}{2}}}.$$

4.) $V = (\sqrt{x} + \frac{1}{\sqrt[3]{x}})^2$

So:

$$V = (x^{\frac{1}{2}} + x^{\frac{-1}{3}})^2$$
$$= (x^{\frac{1}{2}})^2 + 2(x^{\frac{1}{2}})(x^{-\frac{1}{3}}) + (x^{-\frac{1}{3}})^2$$
$$= x + 2x^{\frac{1}{6}} + x^{-\frac{2}{3}}.$$

Now we find the Derivative:

$$V\prime = 1 + 2 \cdot \frac{1}{6}x^{\frac{1}{6}-1} + (\frac{-2}{3})x^{-\frac{2}{3}-1}$$
$$v\prime = 1 + \frac{1}{3}x^{-\frac{5}{6}} - \frac{2}{3}x^{-\frac{5}{3}}$$
$$v\prime = 1 + \frac{1}{3x^{\frac{5}{6}}} - \frac{2}{3x^{\frac{5}{3}}}.$$

Exponential Functions:

Recall: $(1+\frac{1}{n})^n \to e \approx 2.71828...asn \to \infty$

Definition 0.0.1: Definiton of e:

$\lim\limits_{h\to 0} \frac{e^h-1}{h} = 1$

Note:-

We'll use the above definiton to derive $\frac{d}{dx}(e^x)$

$\to$ Let $f(x) = e^x$

$$f\prime(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

So:

$$\begin{aligned} &\lim_{h\to 0} \frac{e^{x+h}-e^x}{h} \\ = &\lim_{h\to 0} \frac{e^x \cdot x^h - e^x}{h} \\ = &\lim_{h\to 0} \frac{e^x \cdot (e^h-1)}{h}. \end{aligned}$$

This function is dependent on ***h****, but* e^x *is not dependent on h, so we can pull it outside and rewrite as:*

$$e^x \cdot \lim_{h\to 0} \frac{e^h-1}{h}.$$

According to our definiton above, we can see that the right portion of this equation ***Equals 1****, Therefor we are just left with:*

$$e^x.$$

Therefore:

$$\frac{d}{dx}(e^x) = e^x.$$

Example: Find $f\prime(x)$ and $f\prime\prime(x)$ of $f(x) = e^x - x^3$

$$f\prime(x) = e^x - 3x^2.$$

$$f\prime\prime(x) = e^x - 6x.$$

Normal Line:

The normal line is perpendicular to the tangent line at the point of tangency.

$$m_{tangent} \cdot m_{normal} = -1.$$

Note:-

This definition means that the slopes are ***Opposite Recipricals***

Example: find equations of the tangent line and the normal line to the curve $y = x^4 + 8e^x$ at the point (0,8).

So we find the derivative:

$$y\prime = 4x^3 + 8e^x.$$

Then we find m_{tan}:

$$\begin{aligned} m_{tan} &= 4 \cdot 0^3 + 8e^0 \\ &= 0 + 8 \cdot 1 \\ &= 8. \end{aligned}$$

*Then we find the slope of the normal line, so we take the Reciprical of m_{tan}, so we **flip it and change the sign**:*

$$m_{normal} = -\frac{1}{8}.$$

We can check our answer using the definiton:

$$8(-\frac{1}{8}) = -1.$$

Now we find the equations of the lines:

Tangent Line:

$$\begin{aligned} y - 8 &= 8(x - 0) \\ y - 8 &= 8x \\ y &= 8x + 8. \end{aligned}$$

Normal Line:

$$\begin{aligned} y - 8 &= -\frac{1}{8}(x - 0) \\ y - 8 &= -\frac{1}{8}x \\ y &= -\frac{1}{8}x + 8. \end{aligned}$$

Example: The equation of motion of a particle is $s = t^3 - 12t$

a.) Find $v(t) = s\prime(t)$ - *Velocity*

So:

$$s\prime(t) = 3t^2 - 12.$$

B.) Find $a(t) = s\prime\prime(t)$ - *Acceleration*

So:

$$s\prime\prime(t) = 6t.$$

c.) Find the acceleration after 9 seconds

So:

$$\begin{aligned} a(9) &= 6 \cdot 9 \\ &= 54m\diagdown s^2. \end{aligned}$$

d.) Find the acceleration when the velocity is 0.

So:

Set v(t) = 0

$$\begin{aligned} 3t^2 - 12 &= 0 \\ 3t^2 &= 12 \\ t^2 &= 4 \\ t = \pm 2 \rightarrow 2 \text{ Typically we like t to be positive.} \end{aligned}$$

Now:

$$\begin{aligned} a(2) &= 6 \cdot 2 \\ &= 12m\diagdown s^2. \end{aligned}$$

3.2

The Product and Quotient Rules

Product Rule:

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)].$$

Or:

$$(f \cdot g)\prime = f \cdot g\prime + g \cdot f\prime.$$

Quotient Rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

Or:

$$\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}.$$

Example: Differentiate the following Function: **(Quotient Rule)**

1.) $y = \frac{e^x}{1+x}$

So, If:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

And:

$$f(x) = e^x \to f'(x) = e^x$$
$$g(x) = 1 + x \to g'(x) = 1.$$

Then:

$$\begin{aligned} y' &= \frac{(1+x)e^x - e^x(1)}{(1+x)^2} \\ &= \frac{e^x + xe^x - e^x}{(1+x)^2} \\ &= \frac{xe^x}{(1+x)^2}. \end{aligned}$$

Example: Differentiate The Following Function: **(Product Rule)**

2.) $R(t) = (t + e^t)(3 - \sqrt{t})$

So If:

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)].$$

And:

$$f(x) = (t + e^t) \longrightarrow f'(x) = (1 + e^t)$$
$$g(x) = (3 - t^{\frac{1}{2}}) \longrightarrow g'(x) = (0 - \frac{1}{2}t^{-\frac{1}{2}}).$$

Then:

$$R'(t) = (t + e^t)(0 - \frac{1}{2}t^{-\frac{1}{2}}) + (1 + e^t)(3 - t^{-\frac{1}{2}}).$$

Cleanup:

$$\begin{aligned} R'(t) = -\frac{1}{2}t^{\frac{1}{2}} - \frac{1}{2}e^t t^{-\frac{1}{2}} + 3 - t^{-\frac{1}{2}} + 3e^t \cdot t^{\frac{1}{2}} \\ = -\frac{3}{2}t^{\frac{1}{2}} - \frac{1}{2}e^t t^{-\frac{1}{2}} + 3 + 3e^t \cdot t^{\frac{1}{2}} \\ = -\frac{3}{2}t^{\frac{1}{2}} - \frac{e^t}{2t^{\frac{1}{2}}} + 3 + 3e^t \cdot t^{\frac{1}{2}}. \end{aligned}$$

Explanation for cleanup:

for the second equation, we just combined like terms, then for the ***third equation***, we rewrote the term with the negative power.

Example: Differentiate the following function **(Product Rule:)**

3.) $g(x) = 5e^x\sqrt{x}$

So:

$$g'(x) = (5e^x)(\frac{1}{2}x^{-\frac{1}{2}}) + (5e^x)(x^{\frac{1}{2}}).$$

From here we can simplify by pulling out common factor, $5e^x x^{-\frac{1}{2}}$

So:

$$\begin{aligned} &5e^x x^{-\frac{1}{2}}(\frac{1}{2} + x^1) \\ &= \frac{5e^x}{x^{\frac{1}{2}}} \cdot \frac{1 + 2x}{2} \\ &= \frac{5e^x(1 + 2x)}{2x^{\frac{1}{2}}}. \end{aligned}$$

Example: find $f'(x)$ and $f''(x)$

1.) $f(x) = x^8 e^x$

So:

$$f'(x) = x^8 \cdot e^x + 8x^7 \cdot e^x.$$

We can factor out an e^x

So, $f'(x)$ *is:*

$$f'(x) = e^x(x^8 + 8x^7).$$

Now:

$$\begin{aligned} f''(x) &= e^x(8x^7 + 56x^6) + (x^8 + 8x^7)(e^x) \\ &= e^x(x^8 + 8x^7 + 8x^7 + 56x^6) \\ &= e^x(x^8 + 16x^7 + 56x^6). \end{aligned}$$

Example: Differentiate (***Quotient Rule***):

$$y = \frac{x+1}{x^3 + x - 2}.$$

If:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

And:

$$\begin{aligned} f(x) = x + 1 \longrightarrow f'(x) = 1 \\ \textit{and} \\ g(x) = x^3 + x - 2 \longrightarrow g'(x) = 3x^2 + 1. \end{aligned}$$

Then:

$$\begin{aligned} y' &= \frac{(x^3 + x - 2)(1) - (x+1)(3x^2 + 1)}{(x^3 + x - 2)^2} \\ &= \frac{x^3 + x - 2 - (3x^3 + x + 3x^2 + 1)}{(x^3 + x - 2)^2} \\ &= \frac{x^3 + x - 2 - 3x^3 - x - 3x^2 - 1)}{(x^3 + x - 2)^2} \\ &= \frac{-2x^3 - 3x^2 - 3}{(x^3 + x - 2)^2}. \end{aligned}$$

Example: Find the equation of the tangent line and the normal line to the curve $y = \frac{\sqrt{x}}{x+1}$ at (4,0.4)

If:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

And:

$$f(x) = x^{\frac{1}{2}} \longrightarrow f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$
$$\textit{and}$$
$$g(x) = x + 1 \longrightarrow g'(x) = 1.$$

Then:

$$y' = \frac{(x+1)(\frac{1}{2}x^{-\frac{1}{2}}) - (x^{\frac{1}{2}})(1)}{(x+1)^2}.$$

Now m_{tan}

$$\begin{aligned} m_{tan} &= \frac{(4+1)(\frac{1}{2}\cdot 4^{-\frac{1}{2}}) - (4^{\frac{1}{2}})}{(x+1)^2} \\ &= \frac{5\cdot\frac{1}{4} - 2}{25}. \end{aligned}$$

We want to multiply by the lcd 4 to clear out the complex fraction

$$\begin{aligned} &\frac{(\frac{5}{4} - 2)\cdot 4}{25\cdot 4} \\ &= \frac{5-8}{100} \\ &= -\frac{3}{100}. \end{aligned}$$

Now to find m_{normal}, we take the Reciprical of m_{tan} and change the sign:

$$m_{norm} = \frac{100}{3}.$$

Now we want to find the equations:

Tangent Line:

$$\begin{aligned} y - 0.4 &= -0.03(x-4) \\ y - 0.4 &= -0.03x + 0.12 \\ y &= -0.03x + 0.52. \end{aligned}$$

Normal Line:

$$\begin{aligned} y - \frac{2}{5} &= \frac{100}{3}(x-4) \\ y - \frac{2}{5} &= \frac{100}{3}x - \frac{400}{3} \\ y &= \frac{100}{3}x - \frac{1994}{15}. \end{aligned}$$

Since $\frac{100}{3}$ is a repeating decimal, we stayed in fraction form.