Calculus 1: Chapter 3

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Chapter 3

Contents

3.1

Differential Rule:

Diffential Fomulas:

- $\frac{d}{dx}(c) = 0$
- $\frac{d}{dx}(x) = 1$
- $\frac{d}{dx}(x^n) = n \cdot x^{n-1} \rightarrow$ ***Power Rule***
- $\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}[f(x)]$
- $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

Example 0.0.1
Differentiate the following functions:

1.) $f(t) = \frac{1}{2}t^6 - 3t^4 + 1$

For the first term, we will use the ***Third and Fourth*** Rule:

$$\frac{1}{2} \cdot 6t^{t-1}.$$

For the second term, $-3t^4$, We will use the ***Third and Fifth*** Rule:

$$-3 \cdot 4t^{4-1}.$$

The last term is a constant, so according to the first rule, the Derivative of a constant is ***Zero***:

So our full equation is:

$$\begin{aligned} f\prime(x) &= \frac{1}{2} \cdot 6t^{6-1} - 3 \cdot 4t^{4-1} + 0 \\ &= 3t^5 - 12t^3. \end{aligned}$$

2.) $h(x) = (x-2)(2x+3)$

First we need to distribute out the terms:

$$\begin{aligned} h(x) &= 2x^2 + 3x - 4x - 6 \\ &= 2x^2 - x - 6. \end{aligned}$$

Now this is the function we want to diferentiate.

So $\rightarrow$

$$h\prime(x) = 2 \cdot 2x^{2-1} - 1 - 0$$
$$h\prime(x) = 4x - 1.$$

3.) $y = \frac{x^2 - 2\sqrt{x}}{x}$

So:

$$y = \frac{x^2 - 2x^{\frac{1}{2}}}{x}.$$

Since the denominator only has ***one term****, we can split the equation like:*

$$y = \frac{x^2}{x} - \frac{2x^{\frac{1}{2}}}{x}$$
$$y = x - 2x^{-\frac{1}{2}}.$$

Now:

$$\frac{dy}{dx} = 1 - 2 \cdot (-\frac{1}{2})x^{-\frac{1}{2}-1}$$
$$\frac{dy}{dx} = 1 + x^{-\frac{3}{2}}.$$

And we can even rewrite it as:

$$\frac{dy}{dx} = 1 + \frac{1}{x^{\frac{3}{2}}}.$$

4.) $V = (\sqrt{x} + \frac{1}{\sqrt[3]{x}})^2$

So:

$$V = (x^{\frac{1}{2}} + x^{\frac{-1}{3}})^2$$
$$= (x^{\frac{1}{2}})^2 + 2(x^{\frac{1}{2}})(x^{-\frac{1}{3}}) + (x^{-\frac{1}{3}})^2$$
$$= x + 2x^{\frac{1}{6}} + x^{-\frac{2}{3}}.$$

Now we find the Derivative:

$$V\prime = 1 + 2 \cdot \frac{1}{6}x^{\frac{1}{6}-1} + (\frac{-2}{3})x^{-\frac{2}{3}-1}$$
$$v\prime = 1 + \frac{1}{3}x^{-\frac{5}{6}} - \frac{2}{3}x^{-\frac{5}{3}}$$
$$v\prime = 1 + \frac{1}{3x^{\frac{5}{6}}} - \frac{2}{3x^{\frac{5}{3}}}.$$

Exponential Functions:

Recall: $(1 + \frac{1}{n})^n \to e \approx 2.71828...asn \to \infty$

Definition 0.0.1: Definiton of e:

$\lim\limits_{h \to 0} \frac{e^h - 1}{h} = 1$

Note:-

We'll use the above definiton to derive $\frac{d}{dx}(e^x)$

$\to$ Let $f(x) = e^x$

$$f\prime(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

So:

$$\begin{aligned} &\lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \\ &= \lim_{h \to 0} \frac{e^x \cdot x^h - e^x}{h} \\ &= \lim_{h \to 0} \frac{e^x \cdot (e^h - 1)}{h}. \end{aligned}$$

*This function is dependent on **h**, but* e^x *is not dependent on h, so we can pull it outside and rewrite as:*

$$e^x \cdot \lim_{h \to 0} \frac{e^h - 1}{h}.$$

*According to our definiton above, we can see that the right portion of this equation **Equals 1**, Therefor we are just left with:*

$$e^x.$$

Therefore:

$$\frac{d}{dx}(e^x) = e^x.$$

Example: Find $f\prime(x)$ and $f\prime\prime(x)$ of $f(x) = e^x - x^3$

$$f\prime(x) = e^x - 3x^2.$$

$$f\prime\prime(x) = e^x - 6x.$$

Normal Line:

The normal line is perpendicular to the tangent line at the point of tangency.

$$m_{tangent} \cdot m_{normal} = -1.$$

Note:-

This definition means that the slopes are ***Opposite Reciprицals***

Example: find equations of the tangent line and the normal line to the curve $y = x^4 + 8e^x$ at the point (0,8).

So we find the derivative:

$$y\prime = 4x^3 + 8e^x.$$

Then we find m_{tan}:

$$\begin{aligned} m_{tan} &= 4 \cdot 0^3 + 8e^0 \\ &= 0 + 8 \cdot 1 \\ &= 8. \end{aligned}$$

*Then we find the slope of the normal line, so we take the Reciprical of m_{tan}, so we **flip it and change the sign**:*

$$m_{normal} = -\frac{1}{8}.$$

We can check our answer using the definiton:

$$8(-\frac{1}{8}) = -1.$$

Now we find the equations of the lines:

Tangent Line:

$$\begin{aligned} y - 8 &= 8(x - 0) \\ y - 8 &= 8x \\ y &= 8x + 8. \end{aligned}$$

Normal Line:

$$\begin{aligned} y - 8 &= -\frac{1}{8}(x - 0) \\ y - 8 &= -\frac{1}{8}x \\ y &= -\frac{1}{8}x + 8. \end{aligned}$$

Example: The equation of motion of a particle is $s = t^3 - 12t$

a.) Find $v(t) = s\prime(t)$ - *Velocity*

So:

$$s\prime(t) = 3t^2 - 12.$$

B.) Find $a(t) = s\prime\prime(t)$ - *Acceleration*

So:

$$s\prime\prime(t) = 6t.$$

c.) Find the acceleration after 9 seconds

So:

$$\begin{aligned} a(9) &= 6 \cdot 9 \\ &= 54m\diagdown s^2. \end{aligned}$$

d.) Find the acceleration when the velocity is 0.

So:

$$\text{Set v(t) = 0}$$

$$\begin{aligned} 3t^2 - 12 &= 0 \\ 3t^2 &= 12 \\ t^2 &= 4 \\ t = \pm 2 \rightarrow 2 \text{ Typically we like t to be positive.} \end{aligned}$$

Now:

$$\begin{aligned} a(2) &= 6 \cdot 2 \\ &= 12m\diagdown s^2. \end{aligned}$$

3.2

The Product and Quotient Rules

Product Rule:

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)].$$

Or:

$$(f \cdot g)\prime = f \cdot g\prime + g \cdot f\prime.$$

Quotient Rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

Or:

$$\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}.$$

Example: Differentiate the following Function: **(Quotient Rule)**

1.) $y = \frac{e^x}{1+x}$

So, If:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

And:

$$f(x) = e^x \to f'(x) = e^x$$
$$g(x) = 1 + x \to g'(x) = 1.$$

Then:

$$\begin{aligned} y' &= \frac{(1+x)e^x - e^x(1)}{(1+x)^2} \\ &= \frac{e^x + xe^x - e^x}{(1+x)^2} \\ &= \frac{xe^x}{(1+x)^2}. \end{aligned}$$

Example: Differentiate The Following Function: **(Product Rule)**

2.) $R(t) = (t + e^t)(3 - \sqrt{t})$

So If:

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)].$$

And:

$$f(x) = (t + e^t) \longrightarrow f'(x) = (1 + e^t)$$
$$g(x) = (3 - t^{\frac{1}{2}}) \longrightarrow g'(x) = (0 - \frac{1}{2}t^{-\frac{1}{2}}).$$

Then:

$$R'(t) = (t + e^t)(0 - \frac{1}{2}t^{-\frac{1}{2}}) + (1 + e^t)(3 - t^{-\frac{1}{2}}).$$

Cleanup:

$$\begin{aligned} R'(t) = -\frac{1}{2}t^{\frac{1}{2}} - \frac{1}{2}e^t t^{-\frac{1}{2}} + 3 - t^{-\frac{1}{2}} + 3e^t \cdot t^{\frac{1}{2}} \\ = -\frac{3}{2}t^{\frac{1}{2}} - \frac{1}{2}e^t t^{-\frac{1}{2}} + 3 + 3e^t \cdot t^{\frac{1}{2}} \\ = -\frac{3}{2}t^{\frac{1}{2}} - \frac{e^t}{2t^{\frac{1}{2}}} + 3 + 3e^t \cdot t^{\frac{1}{2}}. \end{aligned}$$

Explanation for cleanup:

for the second equation, we just combined like terms, then for the ***third equation***, we rewrote the term with the negative power.

Example: Differentiate the following function **(Product Rule:)**

3.) $g(x) = 5e^x\sqrt{x}$

So:

$$g'(x) = (5e^x)(\frac{1}{2}x^{-\frac{1}{2}}) + (5e^x)(x^{\frac{1}{2}}).$$

From here we can simplify by pulling out common factor, $5e^x x^{-\frac{1}{2}}$

So:

$$\begin{aligned} 5e^x x^{-\frac{1}{2}}(\frac{1}{2} + x^1) \\ = \frac{5e^x}{x^{\frac{1}{2}}} \cdot \frac{1 + 2x}{2} \\ = \frac{5e^x(1 + 2x)}{2x^{\frac{1}{2}}}. \end{aligned}$$

Example: find $f'(x)$ and $f''(x)$

1.) $f(x) = x^8 e^x$

So:

$$f'(x) = x^8 \cdot e^x + 8x^7 \cdot e^x.$$

We can factor out an e^x

So, $f'(x)$ *is:*

$$f'(x) = e^x(x^8 + 8x^7).$$

Now:

$$\begin{aligned} f''(x) &= e^x(8x^7 + 56x^6) + (x^8 + 8x^7)(e^x) \\ &= e^x(x^8 + 8x^7 + 8x^7 + 56x^6) \\ &= e^x(x^8 + 16x^7 + 56x^6). \end{aligned}$$

Example: Differentiate (***Quotient Rule***):

$$y = \frac{x+1}{x^3 + x - 2}.$$

If:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

And:

$$\begin{aligned} f(x) = x + 1 \longrightarrow f'(x) = 1 \\ \textit{and} \\ g(x) = x^3 + x - 2 \longrightarrow g'(x) = 3x^2 + 1. \end{aligned}$$

Then:

$$\begin{aligned} y' &= \frac{(x^3 + x - 2)(1) - (x+1)(3x^2 + 1)}{(x^3 + x - 2)^2} \\ &= \frac{x^3 + x - 2 - (3x^3 + x + 3x^2 + 1)}{(x^3 + x - 2)^2} \\ &= \frac{x^3 + x - 2 - 3x^3 - x - 3x^2 - 1)}{(x^3 + x - 2)^2} \\ &= \frac{-2x^3 - 3x^2 - 3}{(x^3 + x - 2)^2}. \end{aligned}$$

Example: Find the equation of the tangent line and the normal line to the curve $y = \frac{\sqrt{x}}{x+1}$ at (4,0.4)

If:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

And:

$$\begin{aligned} f(x) = x^{\frac{1}{2}} \longrightarrow f'(x) &= \frac{1}{2}x^{-\frac{1}{2}} \\ &\textit{and} \\ g(x) = x+1 \longrightarrow g'(x) &= 1. \end{aligned}$$

Then:

$$y' = \frac{(x+1)(\frac{1}{2}x^{-\frac{1}{2}}) - (x^{\frac{1}{2}})(1)}{(x+1)^2}.$$

Now m_{tan}

$$\begin{aligned} m_{tan} = \frac{(4+1)(\frac{1}{2}\cdot 4^{-\frac{1}{2}}) - (4^{\frac{1}{2}})}{(x+1)^2} \\ = \frac{5\cdot\frac{1}{4} - 2}{25}. \end{aligned}$$

We want to multiply by the lcd 4 to clear out the complex fraction

$$\begin{aligned} &\frac{(\frac{5}{4}-2)\cdot 4}{25\cdot 4} \\ &= \frac{5-8}{100} \\ &= -\frac{3}{100}. \end{aligned}$$

Now to find m_{normal}, we take the Reciprical of m_{tan} and change the sign:

$$m_{norm} = \frac{100}{3}.$$

Now we want to find the equations:

Tangent Line:

$$\begin{aligned} y - 0.4 &= -0.03(x-4) \\ y - 0.4 &= -0.03x + 0.12 \\ y &= -0.03x + 0.52. \end{aligned}$$

Normal Line:

$$\begin{aligned} y - \frac{2}{5} &= \frac{100}{3}(x-4) \\ y - \frac{2}{5} &= \frac{100}{3}x - \frac{400}{3} \\ y &= \frac{100}{3}x - \frac{1994}{15}. \end{aligned}$$

Since $\frac{100}{3}$ is a repeating decimal, we stayed in fraction form.

3.3

Derivatives of Trigonometric Functions

Pythagorn Identites:

- $\sin^2\theta = 1 - \cos^2\theta$
- $\cos^2\theta = 1 - \sin^2\theta$
- $\sin^2\theta + \cos^2\theta = 1$

2 Limit Formulas:

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$

And:

$$\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = 0.$$

Lets Derive $\frac{d}{dx}(\sin x)$:

$$\frac{d}{dx}(\sin x) = \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h}.$$

We will refer back to the formula for $\sin(a+b) \to \sin A\cos B + \cos A\sin B$ to expand $\sin(x+h)$

So:

$$\lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}.$$

We are going to split this equation:

$$\lim_{h\to 0}\frac{\sin x\cos h - \sin x}{h} + \lim_{h\to 0}\frac{\cos x\cdot\sin h}{h}.$$

Since $\sin x$ *and* $\cos x$ *is not changing, it is therefore a constant and we can do the following:*

$$(\sin x)\left(\lim_{h\to 0}\frac{\cos h - 1}{h}\right) + (\cos x)\left(\lim_{h\to 0}\frac{\sin h}{h}\right).$$

Now we can use the formulas above and we are left with:

$$\begin{aligned} &0 + \cos x\cdot 1 \\ &= \cos x. \end{aligned}$$

Summary:

$$\frac{d}{dx}\sin x = \cos x.$$

Lets Derive $\frac{d}{dx}(\cos x)$:

$$\frac{d}{dx}(\cos x) = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}.$$

We will refer back to the formula for $\cos(A+B) \to \cos A\cos B - \sin A\sin B$ to expand $\cos(x+h)$

So:

$$\lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h}.$$

Just like the one above, we are going to group the terms that have x:

$$\lim_{h\to 0}\frac{\cos x\cos h - \cos x}{h} - \lim_{h\to 0}\frac{\sin x\sin h}{h}.$$

Now we pull out the constants:

$$(\cos x)\left(\lim_{h\to 0}\frac{\cos h - 1}{h}\right) - (\sin x)\left(\lim_{h\to 0}\frac{\sin h}{h}\right).$$

Now if we use the fomulas listed at the start of this section we are left with:

$$\begin{aligned}(\cos x)(0) - (\sin x)(1)\\ = -\sin x.\end{aligned}$$

Deriviatives of Trigonometric Functions:

- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\csc x) = -\csc x\cot x$
- $\frac{d}{dx}(\sec x) = \sec x\tan x$
- $\frac{d}{dx}(\cot x) = -\csc^2 x$

Examples: Differentiate:

$$f(x) = \sqrt{x}\sin x.$$

If:

$$\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)].$$

And:

$$\begin{aligned} f(x) &= x^{\frac{1}{2}} \\ g(x) &= \sin x. \end{aligned}$$

$$\begin{aligned} f'(x) &= \frac{1}{2}x^{-\frac{1}{2}} \\ g'(x) &= \cos x. \end{aligned}$$

Then:

$$\begin{aligned} f'(x) &= x^{\frac{1}{2}} \cdot \cos x + \sin x \cdot \frac{1}{2}x^{-\frac{1}{2}} \\ &\frac{1}{2}x^{-\frac{1}{2}}(2x \cdot \cos x + \sin x) \\ &= \frac{2x \cdot \cos x + \sin x}{2x^{\frac{1}{2}}}. \end{aligned}$$

Example: Differentiate:

$$g(t) = 4 \sec t + \tan t.$$

So:

$$\begin{aligned} g'(t) &= 4 \cdot \sec t \tan t + \sec^2 t \\ &= 4 \cdot \frac{1}{\cos t} \cdot \frac{\sin t}{\cos t} + \frac{1}{\cos^2 t} \\ &= 4 \cdot \frac{\sin t}{\cos^2 t} + \frac{1}{\cos^2 t} \\ &= \frac{4 \sin t + 1}{\cos^2 t}. \end{aligned}$$

Example:

$$y = \frac{1 - \sec x}{\tan x}.$$

If:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

And:

$$\begin{aligned} f(x) &= 1 - \sec x \\ g(x) &= \tan x. \end{aligned}$$

$$\begin{aligned} f'(x) &= \sec x \tan x \\ g'(x) &= \sec^2 x. \end{aligned}$$

Then:

$$
\begin{aligned}
y' &= \frac{(\tan x)(-\sec x \tan x) - (1-\sec x)(\sec^2 x)}{\tan^2 x} \\
&= \frac{-\sec x \tan^2 x - (\sec^2 x - \sec^3 x)}{\tan^2 x} \\
&= \frac{-\sec x \tan^2 x - \sec^2 x + \sec^3 x}{\tan^2 x} \\
&= \frac{-\frac{1}{\cos x} \cdot \frac{\sin^2 x}{\cos^2 x} - \frac{1}{\cos^2 x} + \frac{1}{\cos^3 x}}{\frac{\sin^2 x}{\cos^2 x}}.
\end{aligned}
$$

We need to multiply by the lcd $\cos^3 x$*:*

$$\frac{-\sin^2 x - \cos x + 1}{\sin^2 x \cos x}.$$

In the numerator we notice we have $1 - \sin^2 x$*, which is equal to* $\cos^2 x$*, so:*

$$
\begin{aligned}
&\frac{\cos^2 x - \cos x}{\sin^2 x \cos x} \\
&= \frac{\cos x(\cos x - 1)}{\sin^2 x \cos x} \\
&= \frac{\cos x - 1}{\sin^2 x}.
\end{aligned}
$$

And we can replace the denominator with $1 - \cos^2 x$*:*

$$\frac{\cos x - 1}{1 - \cos^2 x}.$$

And we notice that the denominator is a difference of squares, so we can factor it into:

$$
\begin{aligned}
&\frac{\cos x - 1}{(1-\cos x)(1+\cos x)} \\
&= \frac{-(1-\cos x)}{(1-\cos x)(1+\cos x)} \\
&= \frac{-1}{1+\cos x}.
\end{aligned}
$$

Limits:

Recall:

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \text{ and } \lim_{\theta \to 0} \frac{\theta}{\sin\theta} = 1.$$

Also:

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0.$$

Example: Find the Limit:

$$\lim_{x \to 0} \frac{\sin 4x}{\sin 6x}.$$

We want to be able to use the fomulas above, so we do:

$$\begin{aligned}\lim_{x\to 0} \frac{\sin 4x}{4x} \cdot \frac{4x}{1} \cdot \frac{6x}{\sin 6x} \cdot \frac{1}{6x} \\ = 1 \cdot 4 \cdot 1 \cdot \frac{1}{6} \\ = \frac{2}{3}.\end{aligned}$$

Example: Find the Limit:

$$\lim_{\theta\to 0} \frac{\cos\theta - 1}{\sin\theta}.$$

To exercise the formulas above, we will rewrite as:

$$\begin{aligned}\lim_{\theta\to 0} \frac{\cos\theta - 1}{\sin\theta} \cdot \frac{\theta}{\theta} \\ = \frac{\cos\theta - 1}{\theta} \cdot \frac{\theta}{\sin\theta} \\ = 0 \cdot 1 \\ = 0.\end{aligned}$$

Example: Find the Limit:

$$\lim_{t\to 0} \frac{\sin^2 3t}{t^2}.$$

We rewrite as:

$$\begin{aligned}\lim_{t\to 0} \left(\frac{\sin 3t}{t}\right)^2 \\ = \left(\frac{\sin 3t}{3t} \cdot \frac{3}{1}\right)^2 \\ = 1 \cdot 3^2 \\ = 9.\end{aligned}$$

3.4

The Chain Rule / Differentiation Examples using the Product, Quotient, and Chain Rules

The Chain Rule:

We will use the chain rule to find Deriviatives of composite functions.

Example: Find the derivative of

$$F(x) = \sqrt{4+3x}.$$

F(x) is a composite function made up of:

$$g(x) = 4 + 3x$$
$$and$$
$$f(x) = \sqrt{x}.$$

Therefore:

$$F(x) = f(g(x)).$$

Process:

Let:

$$u = g(x) = 4 + 3x.$$

Then:

$$F(x) = f(u) \; and \; F'(x) = f'(u) \cdot g'(x).$$

The Chain Rule (2):

If F(x) = f(g(x)), then:

$$F'(x) = f'(g(x)) \cdot g'(x).$$

Or:

If y = f(u) = f(g(x)), then:

$$\frac{dy}{dx} = \frac{dy}{dx} \cdot \frac{du}{dx}.$$

Example: Find the Derivative:

$$f(x) = (1 + x^4)^{\frac{2}{3}}.$$

So:

$$\begin{aligned} f'(x) &= \frac{2}{3}(1 + x^4)^{-\frac{1}{3}} \cdot (4x^3) \\ &= \frac{8x^3}{3(1 + x^4)^{\frac{1}{3}}}. \end{aligned}$$

Example: Differentiate the following function:

$$f(t) = \sqrt[3]{1 + \tan t}.$$

So:

$$f(t) = (1 + \tan t)^{\frac{1}{3}}.$$

Now:

$$\begin{aligned} f'(t) &= \frac{1}{3}(1 + \tan t)^{-\frac{2}{3}} \cdot (\sec^2 t) \\ &= \frac{\sec^2 t}{3(1 + \tan t)^{\frac{2}{3}}}. \end{aligned}$$

Example: Differentiate The following function:

$$y = (x^2 + 1)(\sqrt[3]{x^2 + 2}).$$

So:

$$y = (x^2 + 1)(x^2 + 2)^{\frac{1}{3}}$$

First:

$$\begin{aligned} f(x) &= (x^2 + 1) \\ f'(x) &= (2x). \end{aligned}$$

$$g(x) = (x^2 + 2)^{\frac{1}{3}}.$$

To find $g'(x)$, we will use the chain rule:

$$g'(x) = [\frac{1}{3}(x^2 + 2)^{-\frac{2}{3}} \cdot (2x)].$$

Now we use the ***product rule:***

$$\frac{dy}{dx} = (x^2 + 1)[\frac{1}{3}(x^2 + 2)^{-\frac{2}{3}} \cdot (2x)] + (x^2 + 2)^{\frac{1}{3}} \cdot (2x)$$

From here we will factor out a GCF:

$$\begin{aligned} if\ gcf &= \frac{1}{3}(2x)(x^2+2)^{-\frac{2}{3}} \\ then\ \frac{dy}{dx} &= \frac{1}{3}(2x)(x^2+2)^{-\frac{2}{3}}\Big[(x^2+1)+3(x^2+2)\Big] \\ &= \frac{2x(x^2+1+3x^2+6)}{3(x^2+2)^{\frac{2}{3}}} \\ &= \frac{2x(4x^2+7)}{3(x^2+2)^{\frac{2}{3}}} \end{aligned}$$

.

Example: Differentiate

$$G(y) = \frac{(y-1)^4}{(y^2+2y)^5}.$$

First:

$$\begin{aligned} f(x) &= (y-1)^4 \\ f'(x) &= 4(y-1)^3. \end{aligned}$$

$$\begin{aligned} g(x) &= (y^2+2y)^5 \\ g'(x) &= 5(y^2+2y)^4 \cdot (2y+2). \end{aligned}$$

Now:

$$G'(y) = \frac{(y^2+2y)^5 4(y-1)^3 \cdot 1 - (y-1)^4 5(y^2+2y)^4(2y+2)}{(y^2+2y)^{10}}$$

.

From here we can factor out a GCF: $(y^2+2y)^4$ *:*

$$\frac{dG}{dy} = \frac{(y^2+2y)^4(y-1)^3 \cdot [4(y^2+2y) - (y-1)5 \cdot 2(y+1)]}{(y^2+2y)^{10}}.$$

We see that we can cancel out common term $(y^2+2y)^4$ *:*

$$\begin{aligned} \frac{dG}{dy} &= \frac{(y-1)^3 \cdot [4y^2+8y-10(y^2-1)]}{(y^2+2y)^6} \\ = \frac{dG}{dy} &= \frac{(y-1)^3 \cdot (4y^2+8y-10y^2+10)}{(y^2+2y)^6} \\ = \frac{dG}{dy} &= \frac{(y-1)^3 \cdot (-6y^2+8y+10)}{(y^2+2y)^6} \\ = \frac{dG}{dy} &= \frac{2(y-1)^3 \cdot (-3y^2+4y+5)}{(y^2+2y)^6} \end{aligned}$$

.

Example: Differentiate

$$y = \tan^2 3\theta.$$

Start by rewriting:

$$y = [\tan 3\theta]^2.$$

Now we differentiate:

$$\begin{aligned} \frac{dy}{d\theta} &= 2[\tan 3\theta] \cdot \sec^2 3\theta \cdot 3 \\ &= 6 \tan 3\theta \sec^2 3\theta. \end{aligned}$$

Example: Differentiate

$$y = x \sin(\frac{1}{x}).$$

Start by rewriting as:

$$y = x \cdot \sin x^{-1}.$$

And we can derive:

$$\begin{aligned} f(x) &= x \\ f'(x) &= 1. \end{aligned}$$

$$\begin{aligned} g(x) &= \sin x^{-1} \\ g'(x) &= \cos x^{-1} \cdot (-1x^{-2}). \end{aligned}$$

Now we can use the product rule:

$$y' = x \cdot \cos x^{-1} \cdot (-1x^{-2}) + \sin x^{-1} \cdot 1.$$

Cleanup:

$$\begin{aligned} y' &= -\cos\frac{1}{x} \cdot x^{-1} + \sin\frac{1}{x} \\ &= \frac{-\cos\frac{1}{x}}{x} + \sin\frac{1}{x} \\ &= \frac{-1}{x}\cos\frac{1}{x} + \sin\frac{1}{x}. \end{aligned}$$

Let's see what $\frac{d}{dx}(a^x)$ is using the chain rule: (a>0)

We know $\frac{d}{dx}e^x = e^x$

Also Recall $a = e^{ln\ a}$

Therefore:

$$a^x = (e^{ln\ a})^x.$$

Which means:

$$\begin{aligned}\frac{d}{dx}(a^x) &= \frac{d}{dx}[(e^{ln\ a})^x]\\ &= \frac{d}{dx}(e^{x \cdot ln\ a})\\ &= e^{x \cdot ln\ a} \cdot \frac{d}{dx}(x \cdot \ln a)\\ &= e^{x \cdot \ln a} \cdot \ln a\\ &= a^x \cdot \ln a.\end{aligned}$$

Summary:

$$a^x = a^x \cdot \ln a.$$

Example: Differentiate

$$y = 10^{1-x^2}.$$

So:

$$\begin{aligned}\frac{dy}{dx} &= 10^{1-x^2} \cdot \ln 10 \cdot (-2x)\\ &= -2x \ln 10 \cdot 10^{-1-x^2}.\end{aligned}$$

Example: Differentiate

$$y = 2^{3^{x^2}}.$$

So this is:

$$f \circ g \circ h.$$

Where:

$$\begin{aligned}f(x) &= 2^x\\ g(x) &= 3^x\\ h(x) &= x^2.\end{aligned}$$

Therefore:

$$
\begin{aligned}
\frac{dy}{dx} &= 2^{3^{x^2}} \cdot \ln 2 \frac{dy}{dx}(3^{x^2}) \\
= \frac{dy}{dx} &= 2^{3^{x^2}} \cdot \ln 2 \cdot 3^{x^2} \cdot \ln 3 \cdot \frac{dy}{dx}(x^2) \\
= \frac{dy}{dx} &= 2^{3^{x^2}} \cdot \ln 2 \cdot 3^{x^2} \cdot \ln 3 \cdot 2x \\
&= 2x \cdot \ln 2 \cdot \ln 3 \cdot 2^{3^{x^2}} \cdot 3^{x^2}.
\end{aligned}
$$

Shortcut:

$$
\begin{aligned}
f(g(x)) &= \sqrt{g(x)} \\
f'(x) &= \frac{1}{2}(g(x))^{-\frac{1}{2}} \cdot g'(x) \\
f'(x) &= \frac{1}{2\sqrt{g(x)}} \cdot g'(x) \\
&= \frac{g'(x)}{2\sqrt{g(x)}}.
\end{aligned}
$$

Example for shortcut:

$$f(x) = \sqrt{\sin x}.$$

$$f'(x) = \frac{\cos x}{2\sqrt{\sin x}}.$$

Example for shortcut:

$$f(x) = \sqrt{4x^3 + 7x^2}$$

.

$$
\begin{aligned}
f'(x) &= \frac{12x^2 + 14x}{2\sqrt{4x^3 + 7x^2}} \\
&= \frac{2(6x + 7)}{2\sqrt{4x^3 + 7x^2}}.
\end{aligned}
$$

Differentiation Examples using the Product,Quotient, and Chain Rules

Recall:

Product Rule:

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)].$$

Quotient Rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

The Chain Rule:

If F(x) = f(g(x)), then:

$$F'(x) = f'(g(x)) \cdot g'(x).$$

Or:

If f(u) = f(g(x)), then:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example: Differentiate the following function:

$$r = \frac{\sqrt{\theta} - 3}{\sqrt{\theta} + 3}.$$

If:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

And:

$$f(x) = \sqrt{\theta} - 3$$
$$f'(x) = \frac{1}{2}\theta^{-\frac{1}{2}}.$$

$$g(x) = \sqrt{\theta} + 3$$
$$g'(x) = \frac{1}{2}\theta^{-\frac{1}{2}}.$$

Then:

$$\begin{aligned}\frac{dr}{d\theta} &= \frac{(\sqrt{\theta}+3)(\frac{1}{2}\theta^{-\frac{1}{2}}) - (\sqrt{\theta}-3)(\frac{1}{2}\theta^{-\frac{1}{2}})}{(\sqrt{\theta}+3)^2} \\ &= \frac{\frac{1}{2}\theta^{-\frac{1}{2}}(\sqrt{\theta}+3-\sqrt{\theta}+3)}{(\sqrt{\theta}+3)^2} \\ &= \frac{\frac{1}{2}\theta^{-\frac{1}{2}}(6)}{(\sqrt{\theta}+3)^2} \\ &= \frac{3\cdot\theta^{-\frac{1}{2}}}{(\sqrt{\theta}+3)^2} \\ &= \frac{3}{\sqrt{\theta}(\sqrt{\theta}+3)^2}.\end{aligned}$$

Note:-

It's fine that we have a radical in the denominator because there was one in the original equation.

Example: Differentiate the following function:

$$p = \frac{4+\sec q}{4-\sec q}.$$

We will rewrite in terms of ***sin and cos***

$$p = \frac{4+\frac{1}{\cos q}}{4-\frac{1}{\cos q}}.$$

Now find common denominator to clear out fractions ($\cos q$):

$$p = \frac{4\cos q+1}{4\cos q-1}.$$

Now we differentiate:

$$\begin{aligned}f(x) &= 4\cos q+1 \\ f'(x) &= -4\sin q.\end{aligned}$$

$$\begin{aligned}g(x) &= 4\cos q = 1 \\ g'(x) &= -4\sin q.\end{aligned}$$

Now plug into Quotient Rule:

$$\frac{dp}{dq} = \frac{(4\cos q-1)(-4\sin q) - (4\cos q+1)(-4\sin q)}{(4\cos q-1)^2}.$$

we see we can factor out an $-4\sin q$:

$$\begin{aligned}\frac{dp}{dq} &= \frac{-4\sin q(4\cos q-1)-(4\cos q+1)}{(4\cos q-1)^2} \\ &= \frac{-4\sin q(4\cos q-1-4\cos q-1)}{(4\cos q-1)^2} \\ &= \frac{-4\sin q(-2)}{(4\cos q-1)^2} \\ &= \frac{8\sin q}{(4\cos q-1)^2}.\end{aligned}$$

Example: Differentiate the following function:

$$h(x) = \left(\frac{\cos x}{1 + \sin x}\right)^4.$$

First lets figure out our Deriviatives from whats withing the parenthesis:

$$f(x) = \cos x$$
$$f'(x) = -\sin x.$$

$$g(x) = 1 + \sin x$$
$$g'(x) = \cos x.$$

We will start by using the power rule and the chain rule with the quotient rule:

$$h'(x) = 4\left[\frac{\cos x}{1 + \sin x}\right]^3 \cdot \left[\frac{(1 + \sin x)(-\sin x) - (\cos x)(\cos x)}{(1 + \sin x)^2}\right].$$

Now we want to distribute the exponent ***3****, into the terms in the numerator and denominator*

$$\frac{4\cos^3 x}{(1 + \sin x)^3} \cdot \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2}.$$

We are going to factor out a -1 and bring it infront of the 4:

$$\frac{-4\cos^3 x}{(1 + \sin x)^3} \cdot \frac{\sin x + \sin^2 x + \cos^2 x}{(1 + \sin x)^2}.$$

We know that $\sin^2 x + \cos^2 x = 1$*, so:*

$$\frac{-4\cos^3 x}{(1 + \sin x)^3} \cdot \frac{\sin x + 1}{(1 + \sin x)^2}.$$

Now we can divide by common factor in the numerator:

$$\frac{-4\cos^3 x}{(1 + \sin x)^3} \cdot \frac{1}{1 + \sin x}$$
$$= \frac{-4\cos^3 x}{(1 + \sin x)^4}.$$

Example: Differentiate the following function:

$$y = (e^{\cos\left(\frac{t}{9}\right)})^4.$$

So by using both the product rule and the chain rule, we get:

$$y' = 4(e^{\cos\frac{t}{9}})^3 \cdot e^{\cos\frac{t}{9}} \cdot -\sin\frac{t}{9} \cdot \frac{1}{9}.$$

Cleanup:

To start, we will group all the constants, then combine the like terms.:

$$y' = -\frac{4}{9}\sin\frac{t}{9}(e^{\cos\frac{t}{9}})^4.$$

Now we will move that power of 4 to the front of cos:

$$y' = -\frac{4}{9}\sin\frac{t}{9}(e^{4\cos\frac{t}{9}}).$$

Example: Differentiate:

$$y = \sin(4x^2e^x).$$

So:

$$y' = \cos(4x^2e^x)$$

Now we want to use the product rule to derive whats inside the cosine function:

$$\begin{aligned} y' &= \cos(4x^2e^x) \cdot [8x \cdot e^x + e^x \cdot 4x^2] \\ &= e^x(4x^2 + 8x)\cos 4x^2e^x. \end{aligned}$$

Double Prime: *To make this easier to grasp we will split it into 3 parts.*

First Part:

$$e^x(4x^2e^x)(\cos 4x^2e^x).$$

Second Part:

$$8x + 8)e^x \cos 4x^2e^x.$$

Third Part:

$$-\sin(4x^2e^x).$$

And then apply the chain + product rule for the stuff inside -sin, which we did in single prime above

$$(8xe^x + 4x^2e^x).$$

and then multiply by the other 2 functions:

$$e^x(4x^2 + 8x).$$

So all together part 3 is:

$$-\sin(4x^2e^x)(8xe^x + 4x^2e^x) \cdot e^x(4x^2 + 8x).$$

So if we add it all together:

$$y'' = e^x(4x^2e^x)(\cos 4x^2e^x) + 8x + 8)e^x \cos 4x^2e^x + -\sin(4x^2e^x)(8xe^x + 4x^2e^x) \cdot e^x(4x^2 + 8x).$$

Cleanup: by factoring out terms:

$$\begin{aligned}
e^x \cdot 4x(x+2)\cos 4x^2e^x + e^x \cdot 8(x+1)\cos 4x^2e^x - [e^x(4x^2+8x)]^2 \sin(4x^2e^x) \\
4e^x \cos(4x^2e^x)[x(x+2) + 2(x+1)] - [e^x \cdot 4x(x+2)]^2 \sin(4x^2e^x) \\
4e^x \cos(4x^2e^x)[x(x+2) + 2(x+1)] - 16e^{2x} \cdot x^2(x+2)^2 \sin(4x^2e^x) \\
4e^x \cos(4x^2e^x)[x^2 + 4x + 2] - 16e^{2x} \cdot x^2(x+2)^2 \sin(4x^2e^x) \\
4e^x(x^2 + 4x + 2)\cos 4x^2e^x - 16e^{2x} \cdot x^2(x+2)^2 \sin(4x^2e^x) \\
.
\end{aligned}$$

3.5

Implicit Differentiation/Derivatives of Inverse Trigonometric Functions

y = f(x), in this form, y is expressed explicity in terms of x. Some functions are defined implicitly by a relation between x and y.

Example: We'll use implicit differentation to find $\frac{dy}{dx}$

$$2x^3 + x^2y - xy^3 = 2.$$

So:

$$6x^2$$