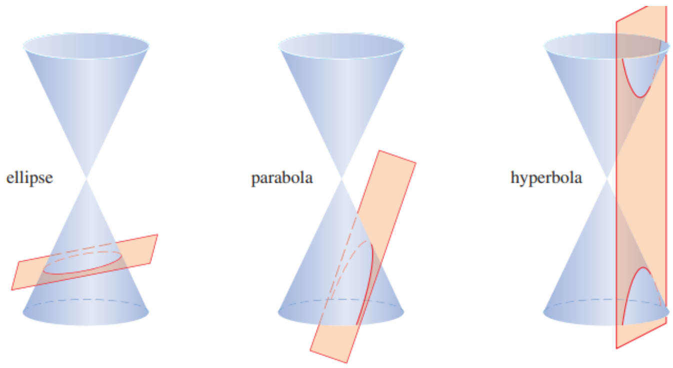
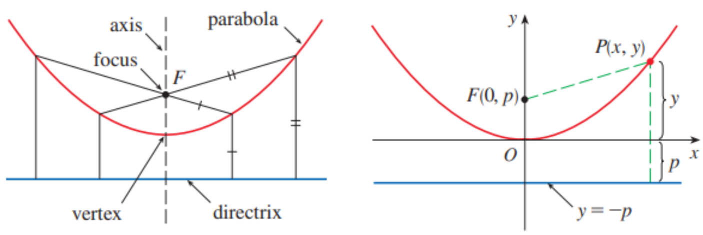
10.5 Conic Sections

In this section we give geometric definitions of parabolas, ellipses, and hyperbolas and derive their standard equations. They are called conic sections, or conics, because they result from intersecting a cone with a plane as shown in Figure 1.



Parabolas

A parabola is the set of points in a plane that are equidistant from a fixed point F called the **focus** and a fixed line called the **directrix**. The point halfway between the focus and the directrix lies on the parabola; it is called the **vertex**.
The line through the focus perpendicular to the directrix is called the axis of the parabola.



We obtain a particularly simple equation for a parabola if we place its vertex at the origin O and its directrix parallel to the x-axis. If the focus is the point $(0, p)$, then the directrix has the equation $y = -p$. If $P(x, y)$ is any point on the parabola, then the distance from P to the focus is $|PF| = \sqrt{x^2 + (y-p)^2}$ and the distance from P to the directrix is $|y + p|$.

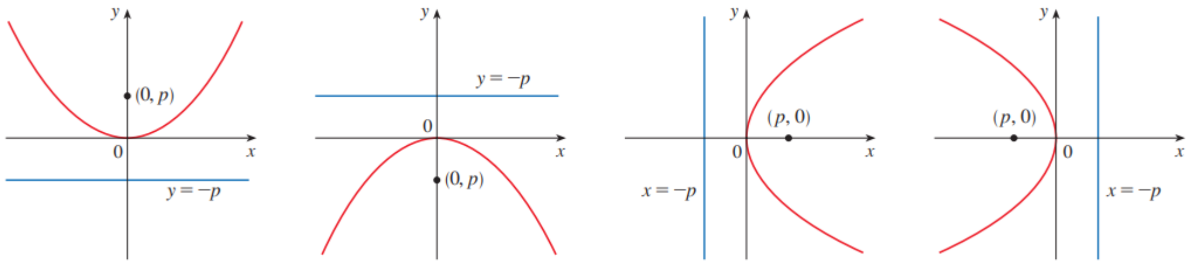
The defining property of a parabola is that these distances are equal:

$$\sqrt{x^2 + (y-p)^2} = |y + p|$$

Squaring and simplifying leads to:

$$x^2 + (y - p)^2 = (y + p)^2$$
$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2$$
$$x^2 = 4py$$

1. An equation of the parabola with focus $(0, p)$ and directrix $y = -p$ is $x^2 = 4py$. This parabola opens upward if $p > 0$ and downward if $p < 0$.
2. An equation of the parabola with focus $(p, 0)$ and directrix $x = -p$ is $y^2 = 4px$. This parabola opens to the right if $p > 0$ and to the left if $p < 0$.



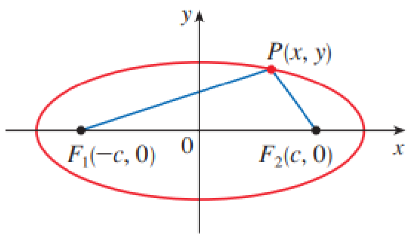
(a) $x^2 = 4py$, $p > 0$ (b) $x^2 = 4py$, $p < 0$ (c) $y^2 = 4px$, $p > 0$ (d) $y^2 = 4px$, $p < 0$

EXAMPLE 1

Find the focus and directrix of the parabola $y^2 + 10x = 0$ and sketch the graph.

Solution: If we write the equation as $y^2 = -10x$ and compare it with Equation 2, we see that $4p = -10$, so $p = -5/2$. Thus the focus is $(-5/2, 0)$ and the directrix is $x = 5/2$.

Ellipses



An ellipse is the set of points in a plane the sum of whose distances from two fixed points F_1 and F_2 is a constant. These two fixed points are called the **foci**.

To obtain the simplest equation, we place the foci on the x-axis at $(\pm c, 0)$. Let the sum of the distances from a point on the ellipse to the foci be $2a > 0$. Then a point $P(x, y)$ is on the ellipse when $|PF_1| + |PF_2| = 2a$.

This leads to the equation:

$$\begin{aligned}
\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2} &= 2a \\
\sqrt{(x-c)^2+y^2} &= 2a-\sqrt{(x+c)^2+y^2} \\
x^2-2cx+c^2+y^2 &= 4a^2-4a\sqrt{(x+c)^2+y^2}+x^2+2cx+c^2+y^2 \\
a\sqrt{(x+c)^2+y^2} &= a^2+cx \\
a^2(x^2+2cx+c^2+y^2) &= a^4+2a^2cx+c^2x^2 \\
(a^2-c^2)x^2+a^2y^2 &= a^2(a^2-c^2)
\end{aligned}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

where $b^2=a^2-c^2$. The points $(\pm a,0)$ are called the **vertices**. Since $b^2=a^2-c^2<a^2$, it follows that $b<a$. The corresponding points $(a,0)$ and $(-a,0)$ are called the **vertices** of the ellipse and the line segment joining the vertices is called the **major axis**. The line segment joining $(0,b)$ and $(0,-b)$ is the **minor axis**.

Elipse is unchanged if x is replaced by $-x$ or y is replaced by $-y$, so the ellipse is symmetric about both axes.

Notice that if the foci coincide, then $c=0$, so $a=b$ and the ellipse becomes a circle with radius $r=a=b$.

The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $a>b>0$ has foci $(\pm c,0)$, where $c^2=a^2-b^2$, and vertices $(\pm a,0)$.

The ellipse $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ where $a>b>0$ has foci $(0,\pm c)$, where $c^2=a^2-b^2$, and vertices $(0,\pm a)$.

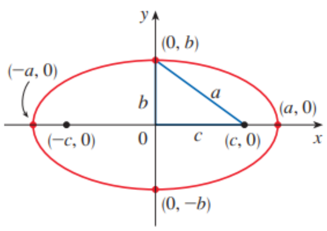


FIGURE 8
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a \geqslant b$

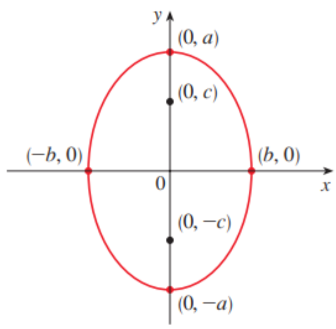


FIGURE 9
$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1,\ a \geqslant b$

EXAMPLE 2

Sketch the graph of $9x^2+16y^2=144$ and locate the foci.

Solution: Divide both sides by 144:

$$\frac{x^2}{16}+\frac{y^2}{9}=1$$

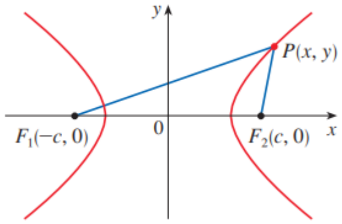
So $a^2=16$, $b^2=9$, which means $a=4, b=3$. Also, $c^2=a^2-b^2=16-9=7$, so $c=\sqrt{7}$. The foci are $(\pm\sqrt{7},0)$.

EXAMPLE 3

Find an equation of the ellipse with foci $(0,\pm 2)$ and vertices $(0,\pm 3)$.

Solution: Here $c = 2$ and $a = 3$. Then $b^2 = a^2 - c^2 = 9 - 4 = 5$. The equation is $\frac{x^2}{5} + \frac{y^2}{9} = 1$, or $9x^2 + 5y^2 = 45$.

Hyperbolas



A hyperbola is the set of all points in a plane the difference of whose distances from two fixed points F_1 and F_2 (the **foci**) is a constant.
With foci on the x-axis at $(\pm c, 0)$, the equation is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $c^2 = a^2 + b^2$. The points $(\pm a, 0)$ are the **vertices**. The hyperbola is symmetric with respect to both axes.
The **asymptotes** are $y = \pm(b/a)x$. Both branches of the hyperbola approach the asymptotes; that is, they come arbitrarily close to the asymptotes.

The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has foci $(\pm c, 0)$, vertices $(\pm a, 0)$, and asymptotes $y = \pm(b/a)x$.

The hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ has foci $(0, \pm c)$, vertices $(0, \pm a)$, and asymptotes $y = \pm(a/b)x$.

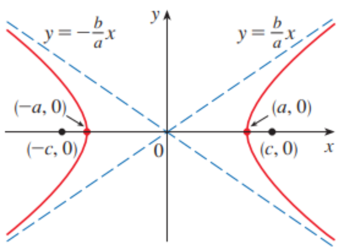


FIGURE 12
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

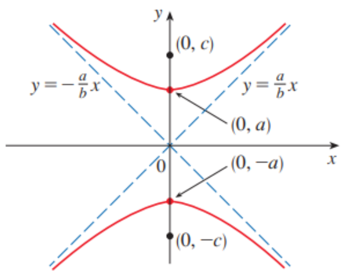


FIGURE 13
$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

EXAMPLE 4

Find the foci and asymptotes of $9x^2 - 16y^2 = 144$.

Solution: Divide by 144: $\frac{x^2}{16} - \frac{y^2}{9} = 1$. So $a = 4$, $b = 3$. $c^2 = a^2 + b^2 = 16 + 9 = 25$, so $c = 5$. Foci: $(\pm 5, 0)$. Asymptotes: $y = \pm\frac{3}{4}x$.

EXAMPLE 5

Find the foci and equation of the hyperbola with vertices $(0, \pm 1)$ and asymptote $y = 2x$.

Solution: From the vertices, $a = 1$. From the asymptote, $a/b = 2 \implies 1/b = 2 \implies b = 1/2$. Then $c^2 = a^2 + b^2 = 1 + 1/4 = 5/4$. Foci: $(0, \pm\sqrt{5}/2)$. Equation: $y^2 - 4x^2 = 1$.

Shifted Conics

We shift conics by replacing x and y with $x - h$ and $y - k$ in the standard equations.

EXAMPLE 6

Find an equation of the ellipse with foci $(2, -2)$, $(4, -2)$ and vertices $(1, -2)$, $(5, -2)$.

Solution: The center is the midpoint of the vertices: $(3, -2)$. The distance from the center to a vertex is $a = 2$. The distance from the center to a focus is $c = 1$. $b^2 = a^2 - c^2 = 4 - 1 = 3$. The equation is:

$$\frac{(x-3)^2}{4} + \frac{(y+2)^2}{3} = 1$$

EXAMPLE 7

Sketch the conic $9x^2 - 4y^2 - 72x + 8y + 176 = 0$ and find its foci.

Solution: Complete the square:

$$\begin{aligned}
(9x^2 - 72x) - (4y^2 - 8y) &= -176 \\
9(x^2 - 8x) - 4(y^2 - 2y) &= -176 \\
9(x^2 - 8x + 16) - 4(y^2 - 2y + 1) &= -176 + 9(16) - 4(1) \\
9(x-4)^2 - 4(y-1)^2 &= -176 + 144 - 4 \\
9(x-4)^2 - 4(y-1)^2 &= -36 \\
\frac{4(y-1)^2}{36} - \frac{9(x-4)^2}{36} &= 1 \\
\frac{(y-1)^2}{9} - \frac{(x-4)^2}{4} &= 1
\end{aligned}$$

This is a hyperbola with center $(4, 1)$. Here $a^2 = 9, b^2 = 4$, so $c^2 = a^2 + b^2 = 13$. Foci: $(4, 1 \pm \sqrt{13})$. Vertices: $(4, 1 \pm 3)$, which are $(4, 4)$ and $(4, -2)$. Asymptotes: $y - 1 = \pm\frac{3}{2}(x - 4)$.