11.4 The Comparison Tests

In the comparison tests, the idea is to compare a given series with a series that is known to be convergent or divergent.

The Direct Comparison Test

The Direct Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is **convergent** and $a_n \le b_n$ for all n, then $\sum a_n$ is also **convergent**.

(ii) If $\sum b_n$ is **divergent** and $a_n \ge b_n$ for all n, then $\sum a_n$ is also **divergent**.

Standard series for use with the comparison tests:

- A **p-series** $\sum 1/n^p$ converges if $p > 1$ and diverges if $p \le 1$.
- A **geometric series** $\sum ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \ge 1$.

Proof. Let $s_n = \sum_{i=1}^{n} a_i$, $t_n = \sum_{i=1}^{n} b_i$, and $t = \sum_{n=1}^{\infty} b_n$. Since both series have positive terms, the sequences $\{s_n\}$ and $\{t_n\}$ are increasing.

(i) Since $\sum b_n$ is convergent, the sequence $\{t_n\}$ is convergent and thus bounded. So there is a number M such that $t_n \le M$ for all n. Since $a_i \le b_i$ for all i, we have $s_n \le t_n$. Therefore $s_n \le M$ for all n. This means that $\{s_n\}$ is increasing and bounded above and so converges by the Monotonic Sequence Theorem. Thus $\sum a_n$ converges.

(ii) If $\sum b_n$ is divergent, then $t_n \to \infty$ (since $\{t_n\}$ is increasing). But $a_i \ge b_i$, so $s_n \ge t_n$. Thus $s_n \to \infty$. Therefore $\sum a_n$ diverges. $\square$

EXAMPLE 1

Determine whether the series $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ converges or diverges.

SOLUTION: For large n the dominant term in the denominator is $2n^2$, so we compare the given series with the series $\sum 5/(2n^2)$. Observe that

$$\frac{5}{2n^2+4n+3} < \frac{5}{2n^2}$$

because the left side has a bigger denominator. We know that

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent because it's a constant times a p-series with $p = 2 > 1$. Therefore the given series is **convergent** by the Direct Comparison Test.

EXAMPLE 2

Test the series $\sum_{k=1}^{\infty} \frac{\ln k}{k}$ for convergence or divergence.
SOLUTION: We compare it with the harmonic series. Observe that $\ln k > 1$ for $k \geq 3$ and so

$$\frac{\ln k}{k} > \frac{1}{k} \quad \text{for } k \geq 3$$

We know that $\sum 1/k$ is divergent (p-series with $p = 1$). Thus the given series is **divergent** by the Direct Comparison Test.

The Limit Comparison Test

The Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Proof. Let m and M be positive numbers such that $m < c < M$. Because a_n/b_n is close to c for large n, there is an integer N such that

$$m < \frac{a_n}{b_n} < M \quad \text{for } n > N$$

This can be rewritten as

$$mb_n < a_n < Mb_n \quad \text{for } n > N$$

If $\sum b_n$ converges, then $\sum Mb_n$ also converges. From the right side of the inequality, we see that $\sum a_n$ converges by the Comparison Test.

If $\sum b_n$ diverges, then $\sum mb_n$ also diverges. From the left side of the inequality, we see that $\sum a_n$ diverges by the Comparison Test. $\square$

EXAMPLE 3

Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ for convergence or divergence.
SOLUTION: We use the Limit Comparison Test with $a_n = \frac{1}{2^n-1}$ and $b_n = \frac{1}{2^n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(2^n-1)}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n-1} = \lim_{n\to\infty} \frac{1}{1-1/2^n} = 1 > 0$$

Since this limit exists and $\sum 1/2^n$ is a convergent geometric series, the given series **converges** by the Limit Comparison Test.

EXAMPLE 4

Determine whether the series $\sum_{n=1}^{\infty} \dfrac{2n^2+3n}{\sqrt{5+n^5}}$ converges or diverges.

SOLUTION: The dominant part of the numerator is $2n^2$ and the dominant part of the denominator is $\sqrt{n^5} = n^{5/2}$. This suggests taking

$$a_n = \frac{2n^2+3n}{\sqrt{5+n^5}} \qquad b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{5+n^5}} \cdot \frac{n^{1/2}}{2} = \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{2\sqrt{5+n^5}} = \lim_{n\to\infty} \frac{2+3/n}{2\sqrt{5/n^5+1}} = \frac{2+0}{2\sqrt{0+1}} = 1$$

Since $\sum b_n = 2\sum 1/n^{1/2}$ is divergent (p-series with $p = 1/2 < 1$), the given series **diverges** by the Limit Comparison Test.

Estimating Sums

If we have used the Direct Comparison Test to show that a series $\sum a_n$ converges by comparison with a series $\sum b_n$, then we may be able to estimate the sum $\sum a_n$ by comparing remainders. Let $R_n = s - s_n$ and $T_n = t - t_n$. Since $a_n \le b_n$ for all n, we have $R_n \le T_n$.

EXAMPLE 5

Use the sum of the first 100 terms to approximate the sum of the series $\sum \frac{1}{n^3+1}$. Estimate the error involved in this approximation.

SOLUTION: Since $\dfrac{1}{n^3+1} < \dfrac{1}{n^3}$, the given series is convergent by the Direct Comparison Test. The remainder T_n for the comparison series $\sum 1/n^3$ was estimated using the Remainder Estimate for the Integral Test. We found that

$$T_n \le \int_n^{\infty} \frac{1}{x^3} dx = \frac{1}{2n^2}$$

Therefore the remainder R_n for the given series satisfies

$$R_n \le T_n \le \frac{1}{2n^2}$$

With $n = 100$ we have

$$R_{100} \le \frac{1}{2(100)^2} = 0.00005$$

Using a calculator, we find that

$$\sum_{n=1}^{\infty} \frac{1}{n^3+1} \approx \sum_{n=1}^{100} \frac{1}{n^3+1} \approx 0.6864538$$

with error less than 0.00005.