11.8 Definition of Power Series

A power series is a type of series that depends on a variable, x. It resembles a polynomial but with infinitely many terms. Power series are central to many applications of calculus, including solving differential equations.

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A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n's are constants called the coefficients of the series.
More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called a **power series centered at** a or a **power series about** a.

For a given x, a power series is a series of constants that we can test for convergence or divergence. A power series may converge for some values of x and diverge for other values.
The sum of the series is a function $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ whose domain is the set of all x for which the series converges. Notice that f resembles a polynomial.
For what values of x is the series $\sum_{n=0}^{\infty} x^n$ convergent?
This is a geometric series with $a = 1$ and $r = x$. It converges when $|x| < 1$, that is, for $-1 < x < 1$. The sum is $\dfrac{1}{1-x}$.

EXAMPLE 1

For what values of x does the series $\sum_{n=1}^{\infty} \dfrac{(x-3)^n}{n}$ converge?

SOLUTION: Let $a_n = \dfrac{(x-3)^n}{n}$. We use the Ratio Test.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{n}{n+1}(x-3) \right| = |x-3| \lim_{n\to\infty} \frac{n}{n+1} \\
&= |x-3| \cdot 1 = |x-3|
\end{aligned}$$

By the Ratio Test, the series converges if $|x-3| < 1$ and diverges if $|x-3| > 1$. This means the series converges if $-1 < x-3 < 1$, which is $2 < x < 4$. The series diverges if $x < 2$ or $x > 4$.

The Ratio Test is inconclusive when $|x-3| = 1$, so we must test the endpoints $x = 2$ and $x = 4$. If $x = 4$, the series becomes $\sum_{n=1}^{\infty} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series, which is divergent. If $x = 2$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which converges by the Alternating Series Test.
Thus the given power series converges for $2 \leq x < 4$.

EXAMPLE 2

For what values of x is the series $\sum_{n=0}^{\infty} n! x^n$ convergent?
SOLUTION: We use the Ratio Test. If $x \neq 0$,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right| = \lim_{n\to\infty} (n+1)|x| = \infty$$

The series diverges for all $x \neq 0$. When $x = 0$, the series is $\sum 0 = 0$, which converges.
Thus the series converges only when $x = 0$.

EXAMPLE 3

For what values of x does the series $\sum_{n=1}^{\infty} \frac{x^n}{2n!}$ converge?
SOLUTION: Let $a_n = x^n/2n!$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(2n+2)!} \cdot \frac{2n!}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = |x| \lim_{n\to\infty} \frac{1}{n+1} = |x| \cdot 0 = 0$$

Since this limit is 0 for all x, and $0 < 1$, the series converges for all values of x by the Ratio Test.

Theorem: Convergence of a Power Series

For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, there are only three possibilities:

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

The number R in case (iii) is called the **radius of convergence**.
By convention, $R = 0$ in case (i) and $R = \infty$ in case (ii).
The **interval of convergence** is the set of values of x for which the series converges.

EXAMPLE 4

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \dfrac{(-3)^n x^n}{\sqrt{n+1}}$.

SOLUTION: Let $a_n = \dfrac{(-3)^n x^n}{\sqrt{n+1}}$.

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n}\right| \\
&= \lim_{n\to\infty} \left|-3x\sqrt{\frac{n+1}{n+2}}\right| = 3|x| \lim_{n\to\infty} \sqrt{\frac{1+1/n}{1+2/n}} \\
&= 3|x| \cdot 1 = 3|x|
\end{aligned}$$

By the Ratio Test, the series converges if $3|x| < 1$ and diverges if $3|x| > 1$. Thus it converges if $|x| < 1/3$ and diverges if $|x| > 1/3$. The radius of convergence is $R = 1/3$.

Now we test the endpoints $x = 1/3$ and $x = -1/3$. If $x = 1/3$, the series is $\sum_{n=0}^{\infty} \dfrac{(-3)^n (1/3)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{\sqrt{n+1}}$. This converges by the Alternating Series Test ($b_n = 1/\sqrt{n+1}$ is decreasing and approaches 0).
If $x = -1/3$, the series is $\sum_{n=0}^{\infty} \dfrac{(-3)^n(-1/3)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \dfrac{1^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \dfrac{1}{\sqrt{n+1}}$. This is the series $\sum_{m=1}^{\infty} 1/\sqrt{m}$ (let $m = n+1$), which is a p-series with $p = 1/2 < 1$, so it diverges.

Therefore the interval of convergence is $[-1/3, 1/3)$.

EXAMPLE 5

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \dfrac{n(x+2)^n}{3^{n+1}}$.

SOLUTION: Let $a_n = \dfrac{n(x+2)^n}{3^{n+1}}$.

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n}\right| \\
&= \lim_{n\to\infty} \left|\frac{n+1}{n}\frac{x+2}{3}\right| = \frac{|x+2|}{3} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) \\
&= \frac{|x+2|}{3}
\end{aligned}$$

Using the Ratio Test, the series converges if $|x+2|/3 < 1$ and diverges if $|x+2|/3 > 1$. So it converges if $|x+2| < 3$. The radius of convergence is $R = 3$. The inequality $|x+2| < 3$ can be written $-3 < x+2 < 3$, which means $-5 < x < 1$.

We test the endpoints $x = -5$ and $x = 1$. If $x = -5$, the series is $\sum_{n=0}^{\infty} \dfrac{n(-3)^n}{3^{n+1}} = \sum_{n=0}^{\infty} \dfrac{n(-1)^n 3^n}{3 \cdot 3^n} = \dfrac{1}{3}\sum_{n=0}^{\infty}(-1)^n n$. This series diverges by the Test for Divergence since $\lim_{n\to\infty}(-1)^n n$ does not exist.
If $x = 1$, the series is $\sum_{n=0}^{\infty} \dfrac{n(3)^n}{3^{n+1}} = \sum_{n=0}^{\infty} \dfrac{n3^n}{3 \cdot 3^n} = \dfrac{1}{3}\sum_{n=0}^{\infty} n$. This series also diverges by the Test for Divergence since $\lim_{n\to\infty} n = \infty$.

Therefore the interval of convergence is $(-5, 1)$.