9.5 Linear Equations

This section introduces a method for solving a class of first-order differential equations that are not necessarily separable: linear differential equations. These equations are frequently encountered in various scientific applications.

Linear Differential Equations

A first-order linear differential equation is one that can be written in the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x)$ and $Q(x)$ are continuous functions on a given interval.
Linear equations can be solved by multiplying both sides by a suitable function called an integrating factor. The integrating factor $I(x)$ is given by:

$$I(x) = e^{\int P(x)\,dx}$$

Derivation of the Integrating Factor

To solve $\frac{dy}{dx} + P(x)y = Q(x)$, we want to find a function $I(x)$ such that when we multiply the entire equation by $I(x)$, the left side becomes the derivative of a product, specifically $\frac{d}{dx}[I(x)y]$. Multiplying by $I(x)$ gives:

$$I(x)\frac{dy}{dx} + I(x)P(x)y = I(x)Q(x)$$

From the product rule, we know that:

$$\frac{d}{dx}[I(x)y] = I(x)\frac{dy}{dx} + I'(x)y$$

Comparing the two expressions, we must have $I(x)P(x)y = I'(x)y$, which simplifies to:

$$I'(x) = I(x)P(x)$$

This is a separable differential equation for $I(x)$. Separating variables and integrating gives:

$$\int \frac{1}{I(x)}\,dI = \int P(x)\,dx \quad \Longrightarrow \quad \ln|I(x)| = \int P(x)\,dx$$

Exponentiating both sides gives the integrating factor. After multiplying the original equation by $I(x)$, we get:

$$\frac{d}{dx}[I(x)y] = I(x)Q(x)$$

Integrating both sides and solving for y yields the general solution:

$$y(x) = \frac{1}{I(x)}\left[\int I(x)Q(x)\,dx + C\right]$$

To solve a linear differential equation $y' + P(x)y = Q(x)$

1. Identify $P(x)$ and $Q(x)$.
2. Calculate the integrating factor $I(x) = e^{\int P(x)\,dx}$.
3. Multiply both sides of the differential equation by $I(x)$.
4. Recognize the left side as $\frac{d}{dx}[I(x)y]$.
5. Integrate both sides with respect to x.

Example 1: Solving a Linear Differential Equation

Solve the differential equation $\frac{dy}{dx} + 3x^2y = 6x^2$.

Solution: This is a linear equation with $P(x) = 3x^2$ and $Q(x) = 6x^2$. The integrating factor is:

$$I(x) = e^{\int 3x^2\,dx} = e^{x^3}$$

Multiplying the equation by $I(x)$ and recognizing the left side as a product rule derivative gives:

$$\frac{d}{dx}(e^{x^3}y) = 6x^2e^{x^3}$$

Integrating both sides and solving for y, we get:

$$y = 2 + Ce^{-x^3}$$

Example 2: Initial-Value Problem

Find the solution of the initial-value problem $x^2y' + xy = 1$, for $x > 0$, with $y(1) = 2$.

Solution: First, write the equation in standard form: $y' + \frac{1}{x}y = \frac{1}{x^2}$. Here, $P(x) = 1/x$ and $Q(x) = 1/x^2$. The integrating factor is:

$$I(x) = e^{\int \frac{1}{x}\,dx} = e^{\ln|x|} = x \quad \text{(since } x > 0\text{)}$$

Multiplying by x gives $\frac{d}{dx}(xy) = \frac{1}{x}$. Integrating both sides gives $xy = \ln x + C$. Using the initial condition $y(1) = 2$, we find $C = 2$. The solution is:

$$y = \frac{\ln x + 2}{x}$$

Example 3: Solutions Involving Non-Elementary Integrals

Solve $y' + 2xy = 1$.

Solution: With $P(x) = 2x$ and $Q(x) = 1$, the integrating factor is $I(x) = e^{\int 2x\,dx} = e^{x^2}$. Multiplying by $I(x)$ gives:

$$\frac{d}{dx}(e^{x^2}y) = e^{x^2}$$

Integrating both sides, we get:

$$e^{x^2}y = \int e^{x^2}\,dx + C$$

The integral $\int e^{x^2}\,dx$ cannot be expressed in terms of elementary functions, but it is a valid function. The solution is:

$$y = e^{-x^2}\int e^{x^2}\,dx + Ce^{-x^2}$$

Figure 1 shows the graphs of several members of the family of solutions in Example 1. Notice that they all approach 2 as $x \to \infty$.

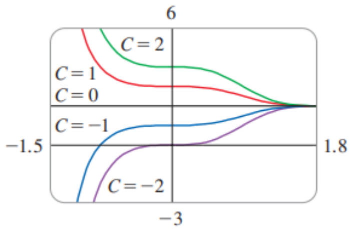

FIGURE 1

The solution of the initial-value problem in Example 2 is shown in Figure 2.

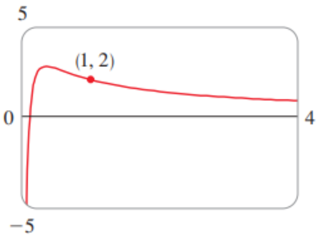

FIGURE 2

Even though the solutions of the differential equation in Example 3 are expressed in terms of an integral, they can still be graphed by a computer (Figure 3).

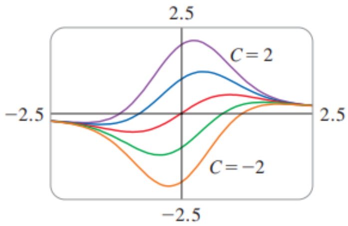

FIGURE 3

Application to Electric Circuits

First-order linear differential equations model the current $I(t)$ in a simple series circuit. Kirchhoff's Law gives the equation:

$$L\frac{dI}{dt} + RI = E(t)$$

Example 4: Current with a Constant Voltage

A circuit has $R = 12\Omega$, $L = 4H$, a constant voltage $E(t) = 60V$, and $I(0) = 0$. Find $I(t)$.

Solution: The differential equation is $4\frac{dI}{dt} + 12I = 60$, which simplifies to $\frac{dI}{dt} + 3I = 15$. The integrating factor is e^{3t}. The solution process yields:

$$I(t) = 5 + Ce^{-3t}$$

Using the initial condition $I(0) = 0$, we find $C = -5$. The current is:

$$I(t) = 5(1 - e^{-3t})$$

Example 5: Current with a Variable Voltage

The circuit is the same as in Example 4 ($R = 12\Omega, L = 4H$), but with a variable voltage $E(t) = 60\sin(30t)$. Find $I(t)$ assuming $I(0) = 0$.

Solution: The differential equation is $\frac{dI}{dt} + 3I = 15\sin(30t)$. The integrating factor is e^{3t}.

$$\frac{d}{dt}(e^{3t}I) = 15e^{3t}\sin(30t)$$

Using the integral formula $\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin(bx) - b\cos(bx))$, we integrate both sides:

$$e^{3t}I = 15\left(\frac{e^{3t}}{3^2 + 30^2}(3\sin(30t) - 30\cos(30t))\right) + C$$

$$e^{3t}I = \frac{5}{101}e^{3t}(\sin(30t) - 10\cos(30t)) + C$$

The general solution for the current is:

$$I(t) = \frac{5}{101}(\sin(30t) - 10\cos(30t)) + Ce^{-3t}$$

Using the initial condition $I(0) = 0$:

$$0 = \frac{5}{101}(0 - 10) + C \implies C = \frac{50}{101}$$

Thus, the final solution is:

$$I(t) = \frac{5}{101}(\sin(30t) - 10\cos(30t)) + \frac{50}{101}e^{-3t}$$

Figure 5 shows how the current in Example 4 approaches its limiting value.

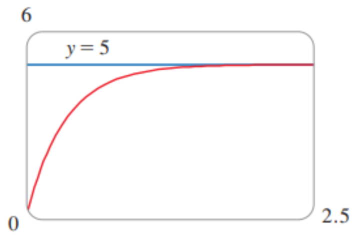

FIGURE 5

Figure 6 shows the graph of the current when the battery is replaced by a generator.

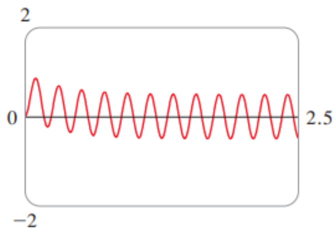

FIGURE 6