11.1 Sequences

Sequences

A sequence can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

The number a_1 is called the first term, a_2 is the second term, and in general a_n is the nth term.

Definition of a Sequence

A **sequence** is a function f whose domain is the set of positive integers. We usually write a_n instead of the function notation $f(n)$. The values $a_1, a_2, \ldots$ are called the terms of the sequence.

Notation: The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by

$$\{a_n\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}$$

EXAMPLE 1

Some sequences can be defined by giving a formula for the nth term.

(a) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty} \quad a_n = \dfrac{n}{n+1} \quad \to \quad \left\{\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \ldots, \dfrac{n}{n+1}, \ldots\right\}$

(b) $\left\{\dfrac{(-1)^n(n+1)}{3^n}\right\} \quad a_n = \dfrac{(-1)^n(n+1)}{3^n} \quad \to \quad \left\{-\dfrac{2}{3}, \dfrac{3}{9}, -\dfrac{4}{27}, \dfrac{5}{81}, \ldots\right\}$

(c) $\{\sqrt{n-3}\}_{n=3}^{\infty} \quad a_n = \sqrt{n-3}, n \geq 3 \quad \to \quad \{0, 1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n-3}, \ldots\}$

(d) $\{\cos(n\pi/6)\}_{n=0}^{\infty} \quad a_n = \cos(n\pi/6), n \geq 0 \quad \to \quad \left\{1, \dfrac{\sqrt{3}}{2}, \dfrac{1}{2}, 0, -\dfrac{1}{2}, -\dfrac{\sqrt{3}}{2}, -1, \ldots\right\}$

EXAMPLE 2

Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \ldots\right\}$$

SOLUTION: The signs of the terms are alternating, starting with positive, so we can use $(-1)^{n-1}$. The numerator is $n+2$ and the denominator is 5^n.

$$a_n = (-1)^{n-1}\frac{n+2}{5^n}$$

EXAMPLE 3 (Recursive Sequences)

Some sequences do not have a simple defining equation but are defined recursively. The **Fibonacci sequence** $\{f_n\}$ is defined by:

$$f_1 = 1 \quad f_2 = 1 \quad f_n = f_{n-1} + f_{n-2} \quad \text{for } n \geq 3$$

The first few terms are: $\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$

The Limit of a Sequence

Definition of a Limit of a Sequence (Intuitive)

A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \to \infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if the terms a_n get arbitrarily close to L as n becomes sufficiently large. If $\lim_{n\to\infty} a_n$ exists, the sequence **converges**. Otherwise, it **diverges**.

Definition of a Limit of a Sequence (Precise)

A sequence $\{a_n\}$ has the limit L if for every $\varepsilon > 0$, there is a corresponding integer N such that

$$\text{if } n > N \quad \text{then} \quad |a_n - L| < \varepsilon$$

Theorem 1

If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

Limit Laws for Sequences (Theorem 2)

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

- $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$
- $\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n)$
- $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$
- $\lim_{n\to\infty} a_n^p = [\lim_{n\to\infty} a_n]^p$ if $p > 0$ and $a_n > 0$

The Squeeze Theorem for Sequences (Theorem 3)

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Theorem 4

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Proof. Given $\lim_{n\to\infty} |a_n| = 0$. For any $\varepsilon > 0$, there is an integer N such that if $n > N$ then $||a_n| - 0| < \varepsilon$, which means $|a_n| < \varepsilon$. But $-|a_n| \le a_n \le |a_n|$. Since $|a_n| < \varepsilon$, we have $-\varepsilon < a_n < \varepsilon$, which implies $|a_n - 0| < \varepsilon$. Therefore, $\lim_{n\to\infty} a_n = 0$. □

EXAMPLE 4 - 11: Calculating Limits

Ex 4: Find $\lim_{n\to\infty} \dfrac{n}{n+1}$.

SOLUTION: Divide numerator and denominator by n: $\lim_{n\to\infty} \dfrac{1}{1+1/n} = 1$.

Ex 5: Find $\lim_{n\to\infty} \dfrac{\ln n}{n}$.

SOLUTION: Use L'Hospital's Rule on $f(x) = \dfrac{\ln x}{x}$. $\lim_{x\to\infty} \dfrac{1/x}{1} = 0$.

Ex 6: Does $a_n = (-1)^n$ converge?

SOLUTION: No, it oscillates between 1 and -1. Divergent.

Ex 7: Find $\lim_{n\to\infty} \dfrac{(-1)^n}{n}$.

SOLUTION: $\lim_{n\to\infty} |\dfrac{(-1)^n}{n}| = \lim_{n\to\infty} \dfrac{1}{n} = 0$. By Thm 4, the limit is 0.

Ex 8: Discuss convergence of $a_n = \dfrac{n!}{n^n}$.

SOLUTION: Use Squeeze Theorem. $0 < a_n = \dfrac{1 \cdot 2 \cdots n}{n \cdot n \cdots n} \le \dfrac{1}{n}$. Since $\lim_{n\to\infty} \dfrac{1}{n} = 0$, the limit is 0.

Ex 9: Evaluate $\lim_{n\to\infty} \sin(\pi/n)$.

SOLUTION: Since $\sin x$ is continuous at 0, $\lim_{n\to\infty} \sin(\pi/n) = \sin(\lim_{n\to\infty} \pi/n) = \sin(0) = 0$.

Ex 10: For what values of r is $\{r^n\}$ convergent?

SOLUTION: Converges for $-1 < r \le 1$. $\lim_{n\to\infty} r^n = 0$ if $-1 < r < 1$, and $\lim_{n\to\infty} r^n = 1$ if $r = 1$.

Ex 11: The sequence defined by $a_1 = 2, a_{n+1} = \dfrac{1}{2}(a_n + 6)$ is convergent (shown in Ex 13). Find $\lim_{n\to\infty} a_n$.

SOLUTION: Let $L = \lim_{n\to\infty} a_n$. Then $\lim_{n\to\infty} a_{n+1} = L$. So $L = \dfrac{1}{2}(L+6)$, which gives $2L = L + 6$, so $L = 6$.

Monotonic and Bounded Sequences

Definitions

A sequence $\{a_n\}$ is called **increasing** if $a_n \le a_{n+1}$ for all $n \ge 1$.
It is called **decreasing** if $a_n \ge a_{n+1}$ for all $n \ge 1$.
A sequence is **monotonic** if it is either increasing or decreasing.

Definitions

A sequence $\{a_n\}$ is **bounded above** if there is a number M such that $a_n \leq M$ for all $n \geq 1$.

It is **bounded below** if there is a number m such that $m \leq a_n$ for all $n \geq 1$.

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

Monotonic Sequence Theorem (Theorem 6)

Every bounded, monotonic sequence is convergent.

Proof. Let $\{a_n\}$ be an increasing sequence. Since $\{a_n\}$ is bounded, the set $S = \{a_n | n \geq 1\}$ has an upper bound. By the Completeness Axiom of the real numbers, S has a least upper bound $L = \sup S$. We will show that $\lim_{n\to\infty} a_n = L$.

Given $\varepsilon > 0$, $L - \varepsilon$ is not an upper bound for S (since L is the *least* upper bound). Therefore, there exists an integer N such that $a_N > L - \varepsilon$.

Because the sequence is increasing, we have $a_n \geq a_N$ for every $n > N$. Thus, for $n > N$, we have

$$a_n > L - \varepsilon$$

Since L is an upper bound for S, we also have $a_n \leq L$ for all n. Therefore, for $n > N$, we have

$$L - \varepsilon < a_n \leq L$$

This implies $|a_n - L| < \varepsilon$ for all $n > N$. Thus, by definition, $\lim_{n\to\infty} a_n = L$.

A similar proof can be constructed for a decreasing sequence bounded below. $\square$

EXAMPLE 12

Show that the sequence $a_n = \dfrac{n}{n^2+1}$ is decreasing.

SOLUTION: Consider the function $f(x) = \dfrac{x}{x^2+1}$. Then $f'(x) = \dfrac{(x^2+1)\cdot 1 - x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} < 0$ for $x^2 > 1$. Thus, $f(x)$ is decreasing for $x > 1$, which shows the sequence is decreasing for $n \geq 2$.

EXAMPLE 13

Investigate the sequence defined by the recurrence relation $a_1 = 2$, $a_{n+1} = \dfrac{1}{2}(a_n + 6)$ for $n \geq 1$.

SOLUTION:

Boundedness: We show $a_n < 6$ by induction. It holds for $n = 1$ since $a_1 = 2 < 6$. Assume $a_k < 6$ for some $k \geq 1$. Then $a_{k+1} = \dfrac{1}{2}(a_k + 6) < \dfrac{1}{2}(6+6) = 6$. Thus, the sequence is bounded above by 6. It is also bounded below by 2.

Monotonicity: We show it is increasing by showing $a_{n+1} \geq a_n$ for all $n \geq 1$. We use induction. For $n = 1$, $a_2 = \dfrac{1}{2}(2+6) = 4 > a_1 = 2$. Assume $a_{k+1} > a_k$ for some $k \geq 1$. Then $a_k < a_{k+1} \Rightarrow a_k + 6 < a_{k+1} + 6 \Rightarrow \dfrac{1}{2}(a_k + 6) < \dfrac{1}{2}(a_{k+1} + 6)$, which means $a_{k+1} < a_{k+2}$. Thus, the sequence is increasing. Since the sequence is bounded and monotonic, by the Monotonic Sequence Theorem, it converges. The limit is 6, as shown in Example 11.