11.2 Series

If we try to add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$, we get an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

which is called an **infinite series** (or just a **series**) and is denoted by the symbol

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n$$

Partial Sums

Let's consider the partial sums:

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ s_n &= \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n \end{aligned}$$

These partial sums form a new sequence $\{s_n\}$, which may or may not have a limit.

Definition of a Convergent Series

Given a series $\sum_{n=1}^{\infty} a_n$, let s_n denote its nth partial sum. If the sequence of partial sums $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called **convergent** and we write:

$$\sum_{n=1}^{\infty} a_n = s$$

The number s is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

EXAMPLE 1

Suppose we know that the sum of the first n terms of the series $\sum a_n$ is $s_n = \dfrac{2n}{3n+5}$. Then the sum of the series is the limit of the sequence of partial sums:

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2n}{3n+5} = \lim_{n\to\infty} \frac{2}{3+5/n} = \frac{2}{3}$$

EXAMPLE 2 (Telescoping Sum)

Show that the series $\sum_{n=1}^{\infty} \dfrac{1}{n(n+1)}$ is convergent, and find its sum.

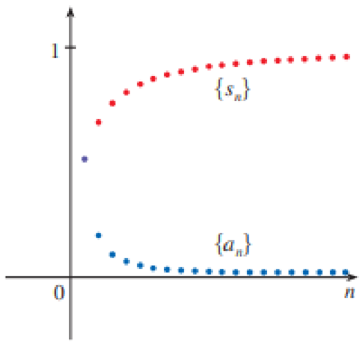
SOLUTION: Using partial fraction decomposition, we have $a_n = \dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$. The nth partial sum is:

$$s_n = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

Thus, the sum of the series is:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$$

The series converges to 1.



Geometric Series

A **geometric series** is a series of the form:

$$a + ar + ar^2 + ar^3 + \cdots = \sum_{n=1}^{\infty} ar^{n-1}, \quad a \neq 0$$

where r is the **common ratio.**

Sum of a Geometric Series

The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ is convergent if $|r| < 1$ and its sum is:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r| < 1$$

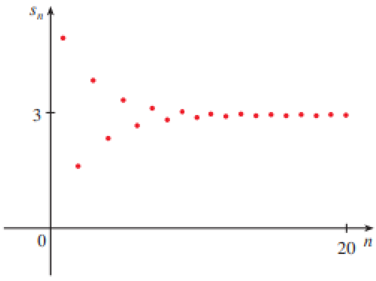
If $|r| \geq 1$, the geometric series is divergent.

EXAMPLE 3

Find the sum of the geometric series $5 - \dfrac{10}{3} + \dfrac{20}{9} - \dfrac{40}{27} + \cdots$.

SOLUTION: The first term is $a = 5$ and the common ratio is $r = -\dfrac{2}{3}$. Since $|r| = \dfrac{2}{3} < 1$, the series is convergent. The sum is:

$$s = \frac{5}{1-(-\frac{2}{3})} = \frac{5}{1+\frac{2}{3}} = \frac{5}{5/3} = 3$$



EXAMPLE 4

Is the series $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ convergent or divergent?
SOLUTION: Let's rewrite the nth term of the series in the form ar^{n-1}:

$$a_n = 2^{2n}3^{1-n} = (2^2)^n 3^{-(n-1)} = 4^n \frac{1}{3^{n-1}} = 4 \cdot \frac{4^{n-1}}{3^{n-1}} = 4\left(\frac{4}{3}\right)^{n-1}$$

This is a geometric series with $a = 4$ and common ratio $r = 4/3$. Since $r > 1$, the series **diverges**.

EXAMPLE 5

For what values of x does the series $\sum_{n=0}^{\infty} x^n$ converge?
SOLUTION: This is a geometric series with $a = 1$ and $r = x$. It converges when $|x| < 1$, that is, for $-1 < x < 1$. The sum is $\dfrac{1}{1-x}$.

EXAMPLE 6

Write the number $2.3\overline{17} = 2.3171717\ldots$ as a ratio of integers.
SOLUTION:

$$2.3\overline{17} = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \cdots$$

This contains a geometric series with first term $a = \dfrac{17}{1000}$ and common ratio $r = \dfrac{1}{100}$.

$$2.3\overline{17} = 2.3 + \frac{\frac{17}{1000}}{1-\frac{1}{100}} = \frac{23}{10} + \frac{\frac{17}{1000}}{\frac{99}{100}} = \frac{23}{10} + \frac{17}{1000} \cdot \frac{100}{99} = \frac{23}{10} + \frac{17}{990}$$

$$= \frac{23 \cdot 99 + 17}{990} = \frac{2277+17}{990} = \frac{2294}{990} = \frac{1147}{495}$$

EXAMPLE 7

Find the sum of the series $\sum_{n=0}^{\infty} x^n$, where $|x| < 1$.
SOLUTION: This is a geometric series with $a = 1$ and ratio $r = x$. Since $|x| < 1$, it converges and the

sum is:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

EXAMPLE 8 (The Harmonic Series)

The series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is called the **harmonic series** and is divergent.
SOLUTION: We show this by looking at the partial sums.

$$s_1 = 1, \quad s_2 = 1.5, \quad s_4 = s_2 + \frac{1}{3} + \frac{1}{4} > 1.5 + \frac{1}{4} + \frac{1}{4} = 2$$

$$s_8 = s_4 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 2 + 4 \cdot \frac{1}{8} = 2.5$$

In general, one can show that $s_{2^n} > 1 + n/2$. This shows that $s_n \to \infty$ as $n \to \infty$, so the harmonic series diverges.

Theorem: Test for Divergence

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Note: If $\lim_{n\to\infty} a_n = 0$, we cannot conclude that the series converges.

EXAMPLE 9

Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.
SOLUTION:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2+4} = \lim_{n\to\infty} \frac{1}{5+4/n^2} = \frac{1}{5} \neq 0$$

Since the limit is not 0, the series diverges by the Test for Divergence.

Properties of Convergent Series

If $\sum a_n$ and $\sum b_n$ are convergent series and c is a constant, then:

(i) $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

EXAMPLE 10

Find the sum of the series $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right)$.
SOLUTION: The series $\sum 1/2^n$ is a geometric series with $a = 1/2$ and $r = 1/2$, so its sum is $S = \frac{1/2}{1-1/2} = 1$. From Example 2, we know $\sum \frac{1}{n(n+1)} = 1$. Therefore, the given series converges and its sum is:

$$\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right) = 3\sum_{n=1}^{\infty}\frac{1}{n(n+1)} + \sum_{n=1}^{\infty}\frac{1}{2^n} = 3(1) + 1 = 4$$

Note: A finite number of terms does not affect the convergence or divergence of a series. For instance, suppose that we were able to show that the series $\sum_{n=4}^{\infty} \dfrac{n}{n^3+1}$ is convergent.

Since $\sum_{n=1}^{\infty} \dfrac{n}{n^3+1} = \dfrac{1}{2} + \dfrac{2}{9} + \dfrac{3}{28} + \sum_{n=4}^{\infty} \dfrac{n}{n^3+1}$ it follows that the entire series $\sum_{n=1}^{\infty} \dfrac{n}{n^3+1}$ is convergent.

Similarly, if it is known that the series $\sum_{n=N+1}^{\infty} a_n$ converges, then the full series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$ is also convergent.