

11.11 Applications of Taylor Polynomials

Approximating Functions by Polynomials

Suppose that $f(x)$ is equal to the sum of its Taylor series at a :

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

In Section 11.10 we introduced the notation $T_n(x)$ for the n -th partial sum of this series, called the **n -th degree Taylor polynomial** of f at a .

$$T_n(x) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x-a)^i = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Since $T_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$, T_n can be used as an approximation to f : $f(x) \approx T_n(x)$.

Notice that the first-degree Taylor polynomial

$$T_1(x) = f(a) + f'(a)(x-a)$$

is the same as the linearization of f at a (from Section 2.9). The derivatives of T_n at a agree with those of f up to order n .

To determine the accuracy of the approximation $f(x) \approx T_n(x)$, we must estimate the absolute value of the remainder:

$$|R_n(x)| = |f(x) - T_n(x)|$$

There are three possible methods for estimating the error:

1. Use a calculator or computer to graph $|R_n(x)|$.
2. If the series is an alternating series, use the Alternating Series Estimation Theorem.
3. In all cases, use Taylor's Inequality.

Theorem 11.10.9: Taylor's Inequality

If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1} \quad \text{for } |x-a| \leq d$$

EXAMPLE 1

- (a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.
- (b) How accurate is this approximation when $7 \leq x \leq 9$?

SOLUTION: (a) First, we compute the derivatives and evaluate them at $a = 8$:

$$\begin{aligned} f(x) &= x^{1/3} & f(8) &= 2 \\ f'(x) &= \frac{1}{3}x^{-2/3} & f'(8) &= \frac{1}{3 \cdot 8^{2/3}} = \frac{1}{12} \\ f''(x) &= -\frac{2}{9}x^{-5/3} & f''(8) &= -\frac{2}{9 \cdot 8^{5/3}} = -\frac{2}{9 \cdot 32} = -\frac{1}{144} \end{aligned}$$

The second-degree Taylor polynomial is:

$$\begin{aligned} T_2(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ &= 2 + \frac{1}{12}(x-8) + \frac{-1/144}{2}(x-8)^2 \\ &= 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 \end{aligned}$$

The approximation is $\sqrt[3]{x} \approx T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$.

(b) We use Taylor's Inequality with $n = 2$ and $a = 8$. The remainder $R_2(x)$ satisfies:

$$|R_2(x)| \leq \frac{M}{3!}|x-8|^3$$

where $|f'''(x)| \leq M$. For x in the interval $[7, 9]$, we must find the maximum value of $|f'''(x)|$.

$$|f'''(x)| = \left| \frac{10}{27}x^{-8/3} \right| = \frac{10}{27x^{8/3}}$$

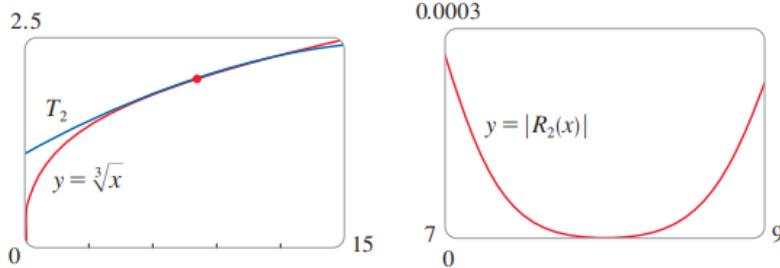
This function is decreasing for $x > 0$, so its maximum value on $[7, 9]$ occurs at $x = 7$.

$$|f'''(x)| \leq \frac{10}{27 \cdot 7^{8/3}} < 0.0021$$

We can take $M = 0.0021$. Also, for $7 \leq x \leq 9$, we have $-1 \leq x-8 \leq 1$, so $|x-8| \leq 1$. Using Taylor's Inequality:

$$|R_2(x)| \leq \frac{0.0021}{3!}(1)^3 = \frac{0.0021}{6} < 0.0004$$

Thus, if $7 \leq x \leq 9$, the approximation is accurate to within 0.0004.



EXAMPLE 2

(a) What is the maximum error possible in using the approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

when $-0.3 \leq x \leq 0.3$? Use this approximation to find $\sin 12^\circ$ correct to six decimal places.

(b) For what values of x is this approximation accurate to within 0.00005?

SOLUTION: (a) The error in approximating $\sin x$ by the first three terms ($T_5(x)$) is at most the absolute value of the first neglected term:

$$|R_5(x)| \leq \left| -\frac{x^7}{7!} \right| = \frac{|x|^7}{5040}$$

If $-0.3 \leq x \leq 0.3$, then $|x| \leq 0.3$, so the error is smaller than:

$$\frac{(0.3)^7}{5040} < 4.3 \times 10^{-8}$$

To find $\sin 12^\circ$, we first convert to radians:

$$12^\circ = 12 \left(\frac{\pi}{180} \right) = \frac{\pi}{15} \text{ rad}$$

Using the approximation:

$$\sin \left(\frac{\pi}{15} \right) \approx \left(\frac{\pi}{15} \right) - \frac{(\pi/15)^3}{3!} + \frac{(\pi/15)^5}{5!} \approx 0.20791169$$

The error is less than 4.3×10^{-8} , so, correct to six decimal places, $\sin 12^\circ \approx 0.207912$.

(b) We want the error to be smaller than 0.00005:

$$\frac{|x|^7}{5040} < 0.00005$$

Solving this inequality for $|x|$:

$$|x|^7 < 5040(0.00005) = 0.252$$

$$|x| < (0.252)^{1/7} \approx 0.821$$

So the approximation is accurate to within 0.00005 when $|x| < 0.82$.

(Using Taylor's Inequality: The approximation is $T_6(x) = T_5(x)$. The next derivative is $f^{(7)}(x) = -\cos x$. Thus $|f^{(7)}(x)| \leq 1$ for all x . We take $M = 1$.

$$|R_6(x)| \leq \frac{M}{(6+1)!} |x|^{6+1} = \frac{|x|^7}{7!} = \frac{|x|^7}{5040}$$

This gives the same estimate. Graphical methods (Figures 4 and 5 in the text) also confirm this result.)

