10.6 Conic Sections in Polar Coordinates

This section provides a unified treatment of parabolas, ellipses, and hyperbolas by defining them in terms of a single focus, a directrix, and a property called eccentricity. This approach leads to a simple and convenient polar equation for these conic sections.

A Unified Description of Conics

A conic section can be defined geometrically using a fixed point (the focus) and a fixed line (the directrix).

Let F be a fixed point (the focus) and l be a fixed line (the directrix) in a plane. Let e be a fixed positive number (the eccentricity). The set of all points P in the plane such that the ratio of the distance from F to the distance from l is the constant e is a conic section.

$$\frac{|PF|}{|Pl|} = e$$

The conic is:

(a) an ellipse if $e < 1$

(b) a parabola if $e = 1$

(c) a hyperbola if $e > 1$

Proof Derivation

To derive a polar equation, we place the focus F at the origin (the pole) and the directrix parallel to the y-axis and d units to the right, so its equation is $x = d$. For any point P with polar coordinates (r, θ):

The distance from the focus F (origin) to P is $|PF| = r$.

The distance from P to the directrix l is $|Pl| = d - r\cos\theta$.

The defining condition $|PF| = e|Pl|$ becomes:

$$r = e(d - r\cos\theta) \tag{1}$$

Squaring both sides of this polar equation and converting to rectangular coordinates, we get:

$$\begin{aligned} x^2 + y^2 &= e^2(d-x)^2 = e^2(d^2 - 2dx + x^2) \\ \Rightarrow (1-e^2)x^2 + 2de^2x + y^2 &= e^2d^2 \end{aligned}$$

After completing the square, we have:

$$\left(x + \frac{e^2d}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{e^2d^2}{(1-e^2)^2}$$

If $e < 1$, we recognize this as the equation of an ellipse:

$$\frac{(x-h)^2}{a^2} + \frac{y^2}{b^2} = 1$$

where:

$$h = -\frac{e^2 d}{1-e^2}, \quad a^2 = \frac{e^2 d^2}{(1-e^2)^2}, \quad b^2 = \frac{e^2 d^2}{1-e^2}$$

From Section 10.5, the foci of an ellipse are at a distance c from the center, where:

$$c^2 = a^2 - b^2 = \frac{e^4 d^2}{(1-e^2)^2}$$

This shows that:

$$c = \frac{e^2 d}{1-e^2} = -h$$

Solving for r from Equation (1), we get:

$$r(1 + e\cos\theta) = ed \Rightarrow r = \frac{ed}{1 + e\cos\theta}$$

This equation shows how the shape of the conic depends on the eccentricity e.

Polar Equations of Conics

Depending on the location of the directrix relative to the focus at the origin, there are four standard forms for the polar equation of a conic.

A polar equation of the form

$$r = \frac{ed}{1 \pm e\cos\theta} \quad \text{or} \quad r = \frac{ed}{1 \pm e\sin\theta}$$

represents a conic section with eccentricity e. The conic is an ellipse if $e < 1$, a parabola if $e = 1$, or a hyperbola if $e > 1$. The orientation of the conic is determined by the trigonometric function and the sign in the denominator:

- $1 + e\cos\theta$: Directrix $x = d$ (to the right of the focus)
- $1 - e\cos\theta$: Directrix $x = -d$ (to the left of the focus)
- $1 + e\sin\theta$: Directrix $y = d$ (above the focus)
- $1 - e\sin\theta$: Directrix $y = -d$ (below the focus)

EXAMPLE 1

Find a polar equation for a parabola that has its focus at the origin and whose directrix is the line $y = -6$.

SOLUTION: Using Theorem 6, we have a parabola, so the eccentricity is $e = 1$. The directrix is $y = -6$, so we use the form $r = \frac{ed}{1-e\sin\theta}$ with $d = 6$. The equation is:

$$r = \frac{6}{1 - \sin\theta}$$

EXAMPLE 2

A conic is given by the polar equation $r = \frac{10}{3-2\cos\theta}$. Find the eccentricity, identify the conic, locate the directrix, and sketch the conic.

SOLUTION: First, we write the equation in the standard form by dividing the numerator and denominator by 3:

$$r = \frac{10/3}{1 - \frac{2}{3}\cos\theta}$$

Comparing this to the standard forms in Theorem 6, we can identify:

- **Eccentricity:** $e = 2/3$
- **Conic Identification:** Since $e < 1$, the conic is an ellipse.
- **Directrix:** The equation is in the form with $1 - e\cos\theta$, which corresponds to a directrix $x = -d$. We have $ed = 10/3$, so $d = (10/3)/(2/3) = 5$. The directrix is the line $x = -5$.
- **Sketching:**
  - $\theta = 0, r = \frac{10}{3-2} = 10$
  - $\theta = \pi/2, r = \frac{10}{3} \approx 3.33$
  - $\theta = \pi, r = \frac{10}{3+2} = 2$
  - $\theta = 3\pi/2, r = \frac{10}{3} \approx 3.33$

EXAMPLE 3

Sketch the conic $r = \frac{12}{2+4\sin\theta}$.

SOLUTION: First, write the equation in standard form by dividing by 2:

$$r = \frac{6}{1 + 2\sin\theta}$$

- **Eccentricity:** $e = 2$
- **Conic Identification:** Since $e > 1$, the conic is a hyperbola.
- **Directrix:** The form is $1 + e\sin\theta$, so the directrix is $y = d$. We have $ed = 6$, so $d = 3$. The directrix is $y = 3$.
- **Sketching:**
  - $\theta = \pi/2, r = \frac{12}{2+4} = 2$
  - $\theta = 3\pi/2, r = \frac{12}{2-4} = -6$

  Asymptotes occur when the denominator is zero: $1 + 2\sin\theta = 0 \Rightarrow \sin\theta = -1/2$, which occurs at $\theta = 7\pi/6$ and $11\pi/6$.

EXAMPLE 4

If the ellipse of Example 2 is rotated through an angle $\pi/4$ about the origin, find a polar equation and graph the resulting ellipse.

SOLUTION: To rotate a curve through an angle α, we replace θ with $(\theta - \alpha)$ in its polar equation. The original equation is $r = \frac{10}{3-2\cos\theta}$. Replacing θ with $(\theta - \pi/4)$, the new equation is:

$$r = \frac{10}{3 - 2\cos(\theta - \pi/4)}$$

The graph is the ellipse from Example 2, rotated counterclockwise by $\pi/4$ about its left focus (the origin).

Kepler's Laws and Planetary Motion

In 1609, Johannes Kepler formulated three laws of planetary motion based on astronomical data.

1. A planet revolves around the sun in an elliptical orbit with the sun at one focus.
2. The line joining the sun to a planet sweeps out equal areas in equal times.
3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

For astronomical calculations, the polar equation of an ellipse can be expressed in terms of its semimajor axis a and eccentricity e.

Polar Equation of an Elliptical Orbit:

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

This equation describes an ellipse with the sun at the focus (origin), semimajor axis a, and eccentricity e. The points in the orbit closest to and farthest from the sun are the perihelion and aphelion, respectively.

- Perihelion distance ($\theta = 0$): $r = a(1-e)$
- Aphelion distance ($\theta = \pi$): $r = a(1+e)$

EXAMPLE 5

(a) Find an approximate polar equation for the elliptical orbit of the Earth around the Sun (at one focus) given that the eccentricity is about 0.017 and the length of the major axis is about 2.99×10^8 km.

(b) Find the distance from the Earth to the Sun at perihelion and at aphelion.

SOLUTION: (a) The length of the major axis is $2a = 2.99 \times 10^8$ km, so the semimajor axis is $a = 1.495 \times 10^8$ km. The eccentricity is $e = 0.017$.

Using the polar equation:

$$r = \frac{a(1-e^2)}{1+e\cos\theta} = \frac{1.495 \times 10^8(1-0.017^2)}{1+0.017\cos\theta} \approx \frac{1.49 \times 10^8}{1+0.017\cos\theta}$$

(b)

- Perihelion distance: $a(1-e) \approx (1.495 \times 10^8)(1-0.017) \approx 1.47 \times 10^8$ km
- Aphelion distance: $a(1+e) \approx (1.495 \times 10^8)(1+0.017) \approx 1.52 \times 10^8$ km