10.4 Calculus in Polar Coordinates

This section applies calculus methods to polar curves, focusing on areas, arc lengths, and tangent slopes.

Area

To develop the formula for the area A of a region whose boundary is given by a polar equation, we start with the formula for the area of a sector of a circle: $A = \frac{1}{2}r^2\theta$.

Consider a region bounded by a polar curve $r = f(\theta)$ and by rays $\theta = a$ and $\theta = b$. We divide the interval $[a, b]$ into n subintervals of equal width $\Delta\theta$. These rays divide the region into n smaller regions. The area ΔA_i of the i-th region is approximated by the area of a sector of a circle with radius $f(\theta_i^*)$ and central angle $\Delta\theta$:

$$\Delta A_i \approx \frac{1}{2}[f(\theta_i^*)]^2\Delta\theta$$

Summing these approximations gives a Riemann sum. As $n \to \infty$, this sum approaches the definite integral.

Formula for Area in Polar Coordinates:

$$A = \frac{1}{2}\int_a^b [f(\theta)]^2\,d\theta \quad \text{or} \quad A = \frac{1}{2}\int_a^b r^2\,d\theta$$

EXAMPLE 1

Find the area enclosed by one loop of the four-leaved rose $r = \cos(2\theta)$.

Solution: One loop of $r = \cos(2\theta)$ is traced from $\theta = -\pi/4$ to $\theta = \pi/4$. Using symmetry and the area formula:

$$A = 2 \cdot \frac{1}{2}\int_0^{\pi/4} \cos^2(2\theta)\,d\theta$$

Applying the half-angle identity $\cos^2(u) = \frac{1+\cos(2u)}{2}$:

$$A = \int_0^{\pi/4} \frac{1+\cos(4\theta)}{2}\,d\theta = \frac{1}{2}\left[\theta + \frac{1}{4}\sin(4\theta)\right]_0^{\pi/4} = \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8}$$

EXAMPLE 2

Find the area of the region inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

Solution: Intersection points are found by $3\sin\theta = 1 + \sin\theta \implies \sin\theta = 1/2$, yielding $\theta = \pi/6$ and $\theta = 5\pi/6$. The area is the difference of the two areas between these angles:

$$\begin{aligned} A &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left[(3\sin\theta)^2 - (1+\sin\theta)^2\right]\, d\theta \\ &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta))\, d\theta \\ &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (8\sin^2\theta - 2\sin\theta - 1)\, d\theta \end{aligned}$$

Using $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$:

$$\begin{aligned} A &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left[8\left(\frac{1-\cos(2\theta)}{2}\right) - 2\sin\theta - 1\right] d\theta \\ &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3 - 4\cos(2\theta) - 2\sin\theta)\, d\theta \\ &= \frac{1}{2}\left[3\theta - 2\sin(2\theta) + 2\cos\theta\right]_{\pi/6}^{5\pi/6} = \pi \end{aligned}$$

CAUTION: Graphing polar curves is crucial for finding all intersection points. It is especially convenient to use a graphing calculator or computer to help with this task.

EXAMPLE 3

Find all points of intersection of $r = \cos(2\theta)$ and $r = 1/2$.

Solution: Equating the equations, $\cos(2\theta) = 1/2$, yields $2\theta = \pi/3, 5\pi/3, 7\pi/3, 11\pi/3, \ldots$. Thus, for $0 \le \theta < 2\pi$, intersection points are $(1/2, \pi/6)$, $(1/2, 5\pi/6)$, $(1/2, 7\pi/6)$, and $(1/2, 11\pi/6)$.

Arc Length

To find the length L of a polar curve $r = f(\theta)$ from $\theta = a$ to $\theta = b$, we use its parametric form $x = r\cos\theta$ and $y = r\sin\theta$. Differentiating with respect to θ:

$$\begin{aligned} \frac{dx}{d\theta} &= \frac{dr}{d\theta}\cos\theta - r\sin\theta \\ \frac{dy}{d\theta} &= \frac{dr}{d\theta}\sin\theta + r\cos\theta \end{aligned}$$

Squaring and summing these gives $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 + r^2$.

Formula for Arc Length in Polar Coordinates:

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

EXAMPLE 4

Find the length of the cardioid $r = 1 + \sin\theta$.

Solution: The cardioid is traced once for $0 \le \theta \le 2\pi$. With $r = 1 + \sin\theta$, we have $\frac{dr}{d\theta} = \cos\theta$.

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + (\cos\theta)^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{1 + 2\sin\theta + \sin^2\theta + \cos^2\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta \end{aligned}$$

Using the identity $1 + \sin\theta = 2\cos^2(\frac{\pi}{4} - \frac{\theta}{2})$ and careful evaluation of the absolute value, the length is 8.

Tangents

To find the slope of the tangent line dy/dx to a polar curve $r = f(\theta)$, we use its parametric form.

Formula for Slope of Tangent in Polar Coordinates:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}, \quad \text{provided } \frac{dx}{d\theta} \neq 0$$

Horizontal tangents occur when $\frac{dy}{d\theta} = 0$ (and $\frac{dx}{d\theta} \neq 0$). Vertical tangents occur when $\frac{dx}{d\theta} = 0$ (and $\frac{dy}{d\theta} \neq 0$).

Tangents at the Pole: If $r = 0$ at $\theta = \theta_0$ and $\frac{dr}{d\theta} \neq 0$, the tangent line at the pole is $\theta = \theta_0$.

EXAMPLE 5

For the cardioid $r = 1 + \sin\theta$: (a) Find the tangent slope at $\theta = \pi/3$. (b) Find horizontal and vertical tangent points.

Solution: Given $r = 1 + \sin\theta$, we have $\frac{dr}{d\theta} = \cos\theta$. (a) At $\theta = \pi/3$, substituting values into the slope formula yields $\frac{dy}{dx} = -1$.

(b) Horizontal tangents ($\frac{dy}{d\theta} = 0$): $\cos\theta(1 + 2\sin\theta) = 0$. This gives $\theta = \pi/2, 3\pi/2, 7\pi/6, 11\pi/6$. The points are $(2, \pi/2)$, $(1/2, 7\pi/6)$, and $(1/2, 11\pi/6)$.

Vertical tangents ($\frac{dx}{d\theta} = 0$): $1 - \sin\theta - 2\sin^2\theta = 0 \implies (1 + \sin\theta)(1 - 2\sin\theta) = 0$. This gives $\theta = 3\pi/2, \pi/6, 5\pi/6$. The points are $(3/2, \pi/6)$ and $(3/2, 5\pi/6)$. At $\theta = 3\pi/2$, both derivatives are zero, which corresponds to the cusp at the pole.