11.1 Sequences

Sequences

A sequence can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

The number a_1 is called the first term, a_2 is the second term, and in general a_n is the nth term.

Definition of a Sequence

A **sequence** is a function f whose domain is the set of positive integers. We usually write a_n instead of the function notation $f(n)$. The values $a_1, a_2, \ldots$ are called the terms of the sequence.

Notation: The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by

$$\{a_n\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}$$

EXAMPLE 1

Some sequences can be defined by giving a formula for the nth term.

(a) $a_n = \dfrac{1}{2^n} \to \{\dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \dfrac{1}{16}, \dfrac{1}{32}, \ldots, \dfrac{1}{2^n}, \ldots\}$

(b) $\left\{\dfrac{n+1}{n}\right\}_{n=2}^{\infty} \to \{\dfrac{3}{2}, \dfrac{4}{3}, \dfrac{5}{4}, \dfrac{6}{5}, \ldots\}$

(c) $\{3, 4, 5, 6, \ldots\} = \{n+2\}_{n=1}^{\infty} = \{n\}_{n=3}^{\infty}$

(d) $\left\{\dfrac{(-1)^n \cdot 3^n}{n+1}\right\}_{n=0}^{\infty} \to \{1, -\dfrac{3}{2}, 3, -\dfrac{27}{4}, \dfrac{81}{5}, \ldots\}$

EXAMPLE 2

Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \ldots\right\}$$

SOLUTION: The signs of the terms are alternating, starting with positive, so we can use $(-1)^{n-1}$. The numerator is $n+2$ and the denominator is 5^n.

$$a_n = (-1)^{n-1}\frac{n+2}{5^n}$$

EXAMPLE 3 (Recursive Sequences)

Some sequences do not have a simple defining equation but are defined recursively. The **Fibonacci sequence** $\{f_n\}$ is defined by:

$$f_1 = 1 \quad f_2 = 1 \quad f_n = f_{n-1} + f_{n-2} \quad \text{for } n \geq 3$$

The first few terms are: $\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$

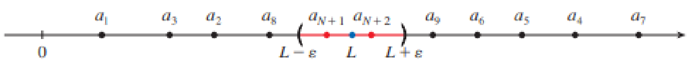
The Limit of a Sequence

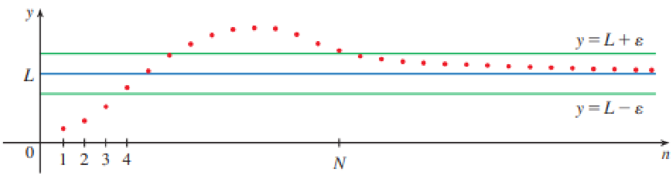
Definition of a Limit of a Sequence (Intuitive)

A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if the terms a_n get arbitrarily close to L as n becomes sufficiently large. If $\lim_{n\to\infty} a_n$ exists, the sequence **converges**. Otherwise, it **diverges**.





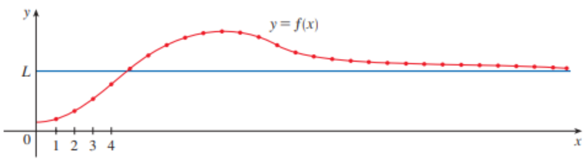
Definition of a Limit of a Sequence (Precise)

A sequence $\{a_n\}$ has the limit L if for every $\varepsilon > 0$, there is a corresponding integer N such that

$$\text{if } n > N \quad \text{then} \quad |a_n - L| < \varepsilon$$

Theorem 1

If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.



Limit Laws for Sequences (Theorem 2)

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

(a) $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

(b) $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$

(c) $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$

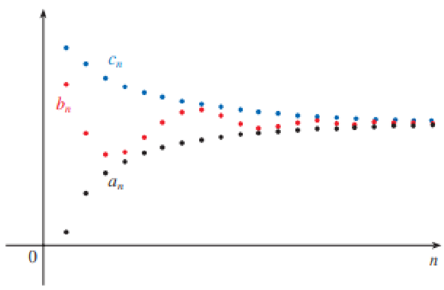
(d) $\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n)$

(e) $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$

(f) $\lim_{n\to\infty} a_n^p = [\lim_{n\to\infty} a_n]^p$ if $p > 0$ and $a_n > 0$

The Squeeze Theorem for Sequences (Theorem 3)

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.



Theorem 4

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Proof. Given $\lim_{n\to\infty} |a_n| = 0$. For any $\varepsilon > 0$, there is an integer N such that if $n > N$ then $||a_n| - 0| < \varepsilon$, which means $|a_n| < \varepsilon$. But $-|a_n| \leq a_n \leq |a_n|$. Since $|a_n| < \varepsilon$, we have $-\varepsilon < a_n < \varepsilon$, which implies $|a_n - 0| < \varepsilon$. Therefore, $\lim_{n\to\infty} a_n = 0$. $\square$

EXAMPLE 4 - 9: Calculating Limits

Ex 4: Find $\lim_{n\to\infty} \dfrac{n}{n+1}$.

SOLUTION: Divide numerator and denominator by n: $\lim_{n\to\infty} \dfrac{1}{1 + 1/n} = 1$.

Ex 5: Find $\lim_{n\to\infty} \dfrac{\ln n}{n}$.

SOLUTION: Use L'Hospital's Rule on $f(x) = \dfrac{\ln x}{x}$. $\lim_{x\to\infty} \dfrac{1/x}{1} = 0$.

Ex 6: Does $a_n = (-1)^n$ converge?

SOLUTION: No, it oscillates between 1 and -1. Divergent.

Ex 7: Find $\lim_{n\to\infty} \dfrac{(-1)^n}{n}$.

SOLUTION: $\lim_{n\to\infty} |\dfrac{(-1)^n}{n}| = \lim_{n\to\infty} \dfrac{1}{n} = 0$. By Thm 4, the limit is 0.

Ex 8: Discuss convergence of $a_n = \frac{n!}{n^n}$.
SOLUTION: Use Squeeze Theorem. $0 < a_n = \frac{1 \cdot 2 \cdots n}{n \cdot n \cdots n} \leq \frac{1}{n}$. Since $\lim_{n\to\infty} \frac{1}{n} = 0$, the limit is 0.

Theorem 4

If $\lim_{n\to\infty} |a_n| = L$, then the function f is contivuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$.

Ex 9: Evaluate $\lim_{n\to\infty} \sin(\pi/n)$.
SOLUTION: Since $\sin x$ is continuous at 0, $\lim_{n\to\infty} \sin(\pi/n) = \sin(\lim_{n\to\infty} \pi/n) = \sin(0) = 0$.

EXAMPLE 10

Discuss the convergence of the sequence $a_n = \frac{n!}{n^n}$, where $n! = 1 \cdot 2 \cdot 3 \cdots n$.
SOLUTION: Both the numerator and the denominator approach infinity as $n \to \infty$.

$$a_1 = 1 \qquad a_2 = \frac{1 \cdot 2}{2 \cdot 2} \qquad a_3 = \frac{1 \cdot 2 \cdot 3}{3 \cdot 3 \cdot 3}$$

$$a_n = \frac{1 \cdot 2 \cdot 3 \cdots n}{n \cdot n \cdot n \cdots n} = \frac{1}{n}\left(\frac{2 \cdot 3 \cdots n}{n \cdot n \cdots n}\right)$$

From this expression, it's clear that a_n is positive. We can also see that

$$0 < a_n \leq \frac{1}{n}$$

because the fraction in the parentheses is less than or equal to 1. We know that $\lim_{n\to\infty} 1/n = 0$. Therefore, by the Squeeze Theorem, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n!}{n^n} = 0$$

EXAMPLE 11

For what values of r is the sequence $\{r^n\}$ convergent?
SOLUTION: From the limits of exponential functions, we know that $\lim_{x\to\infty} r^x = \infty$ for $r > 1$ and $\lim_{x\to\infty} r^x = 0$ for $0 < r < 1$. Therefore, putting $a_n = r^n$, we have

$$\lim_{n\to\infty} r^n = \begin{cases} \infty & \text{if } r > 1 \\ 0 & \text{if } 0 < r < 1 \end{cases}$$

For $r = 1$, $\lim_{n\to\infty} 1^n = \lim_{n\to\infty} 1 = 1$. For $r = 0$, the sequence is $\{0, 0, \dots\}$ and converges to 0. If $-1 < r < 0$, then $0 < |r| < 1$, so $\lim_{n\to\infty} |r^n| = \lim_{n\to\infty} |r|^n = 0$, and $\lim_{n\to\infty} r^n = 0$. If $r \leq -1$, the sequence $\{r^n\}$ diverges.
In summary, the sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Monotonic and Bounded Sequences

Definitions

A sequence $\{a_n\}$ is called **increasing** if $a_n \leq a_{n+1}$ for all $n \geq 1$.
It is called **decreasing** if $a_n \geq a_{n+1}$ for all $n \geq 1$.
A sequence is **monotonic** if it is either increasing or decreasing.

EXAMPLE 12

Show that the sequence $a_n = \dfrac{n}{n^2+1}$ is decreasing.

SOLUTION: We must show that $a_{n+1} \leq a_n$, that is $\dfrac{n+1}{(n+1)^2+1} \leq \dfrac{n}{n^2+1}$. This inequality is equivalent to $(n+1)(n^2+1) \leq n((n+1)^2+1)$.

$$\begin{aligned} n^3+n+n^2+1 &\leq n(n^2+2n+1+1) \\ n^3+n^2+n+1 &\leq n^3+2n^2+2n \\ 1 &\leq n^2+n \end{aligned}$$

Since $n \geq 1$, this inequality is certainly true.

EXAMPLE 13

Investigate the sequence defined by the recurrence relation $a_1 = 2$, $a_{n+1} = \dfrac{1}{2}(a_n+6)$ for $n \geq 1$.
SOLUTION: We begin by computing the first few terms:

$$a_1 = 2 \quad a_2 = \frac{1}{2}(2+6) = 4 \quad a_3 = \frac{1}{2}(4+6) = 5 \quad a_4 = \frac{1}{2}(5+6) = 5.5$$

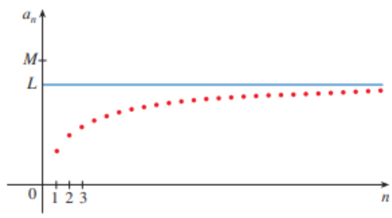
These initial terms suggest that the sequence is increasing and the terms are approaching 6.
To confirm this, use mathematical induction to show that $\{a_n\}$ is increasing and bounded above by 6.

Definitions

A sequence $\{a_n\}$ is **bounded above** if there is a number M such that $a_n \leq M$ for all $n \geq 1$.
It is **bounded below** if there is a number m such that $m \leq a_n$ for all $n \geq 1$.
If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

For instance, the sequence $a_n = n$ is bounded below ($a_n > 0$) but not above. The sequence $a_n = \dfrac{n}{n+1}$ is bounded because $0 < a_n < 1$ for all n.

We know that not every bounded sequence is convergent [for instance, the sequence $a_n = (-1)^n$ satisfies $-1 \leq a_n \leq 1$ but is divergent] and not every monotonic sequence is convergent ($a_n = n \to \infty$). But if a sequence is both bounded and monotonic, then it must be convergent.



Monotonic Sequence Theorem (Theorem 6)

Every bounded, monotonic sequence is convergent.

Proof. Let $\{a_n\}$ be an increasing sequence. Since $\{a_n\}$ is bounded, the set $S = \{a_n | n \geq 1\}$ has an upper bound. By the Completeness Axiom of the real numbers, S has a least upper bound $L = \sup S$. We will show that $\lim_{n\to\infty} a_n = L$.

Given $\varepsilon > 0$, $L - \varepsilon$ is not an upper bound for S (since L is the *least* upper bound). Therefore, there exists an integer N such that $a_N > L - \varepsilon$.

Because the sequence is increasing, we have $a_n \geq a_N$ for every $n > N$. Thus, for $n > N$, we have

$$a_n > L - \varepsilon$$

Since L is an upper bound for S, we also have $a_n \leq L$ for all n. Therefore, for $n > N$, we have

$$L - \varepsilon < a_n \leq L$$

This implies $|a_n - L| < \varepsilon$ for all $n > N$. Thus, by definition, $\lim_{n\to\infty} a_n = L$.

A similar proof can be constructed for a decreasing sequence bounded below. □

EXAMPLE 14

Investigate the sequence defined by $a_1 = 2, a_{n+1} = \frac{1}{2}(a_n + 6)$.

SOLUTION: By induction, one can show the sequence is increasing and bounded above by 6, so it converges. Let $L = \lim_{n\to\infty} a_n$. Then $L = \frac{1}{2}(L + 6)$, which gives $2L = L + 6$, so $L = 6$.

Boundedness: We show that $a_n < 6$ for all $n \geq 1$. This is true for $n = 1$ since $a_1 = 2 < 6$. Assume that $a_k < 6$ for some $k \geq 1$. Then

$$a_{k+1} = \frac{1}{2}(a_k + 6) < \frac{1}{2}(6 + 6) = 6$$

Thus $a_{n+1} < 6$ whenever $a_n < 6$. So the sequence is bounded above by 6. It is also bounded below by 2 since a_n is increasing.

Monotonicity: We show that $a_{n+1} \geq a_n$ for all $n \geq 1$.

$$a_2 - a_1 = 4 - 2 = 2 > 0$$

Assume that $a_{k+1} > a_k$ for some $k \geq 1$. Then $a_k < a_{k+1}$, so $a_k + 6 < a_{k+1} + 6$, and $\frac{1}{2}(a_k + 6) < \frac{1}{2}(a_{k+1} + 6)$. Thus $a_{k+1} < a_{k+2}$. By the principle of mathematical induction, $a_{n+1} \geq a_n$ for all n. Since the sequence $\{a_n\}$ is bounded and increasing, it is convergent by the Monotonic Sequence Theorem. The limit must be 6.