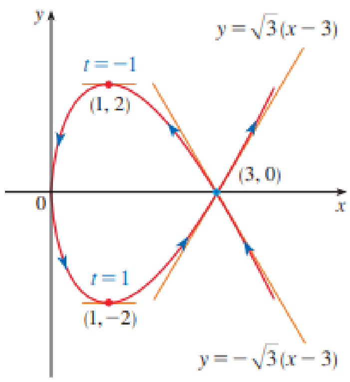

FIGURE 1

10.3 Polar Coordinates

This section introduces the polar coordinate system as an alternative to the Cartesian coordinate system for locating points in a plane.

The Polar Coordinate System

In the polar coordinate system, a point P in the plane is determined by a distance from a fixed point and an angle from a fixed ray.

Pole (or Origin): The fixed point, labeled O.

Polar Axis: The fixed ray starting at O, usually drawn horizontally to the right (corresponding to the positive x-axis).

Polar Coordinates (r, θ):

- r: The distance from O to P.
- θ: The angle between the polar axis and the line segment OP, measured in radians. Positive angles are counterclockwise, negative angles are clockwise.

If $r < 0$, the point $(-r, \theta)$ lies on the same line through O as (r, θ) but on the opposite side of O. So, $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$.

EXAMPLE 1

Plot the points whose polar coordinates are given.

(a) $(1, 5\pi/4)$

(b) $(2, 3\pi)$

(c) $(2, -2\pi/3)$

(d) $(-3, 3\pi/4)$

Solution: The points are plotted in Figure 3. In part (d) the point $(-3, 3\pi/4)$ is located three units from the pole in the fourth quadrant because the angle $3\pi/4$ is in the second quadrant and $r = -3$ is negative.

Relationship between Polar and Cartesian Coordinates

The pole corresponds to the origin and the polar axis coincides with the positive x-axis.

From Polar to Cartesian: If a point P has polar coordinates (r, θ), its Cartesian coordinates (x, y) are:

$$x = r\cos\theta \qquad y = r\sin\theta$$

From Cartesian to Polar: If a point P has Cartesian coordinates (x, y), its polar coordinates (r, θ) satisfy:

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

When converting from Cartesian to polar, care must be taken to choose θ such that (r, θ) lies in the correct quadrant.

EXAMPLE 2

Convert the point $(2, \pi/3)$ from polar to Cartesian coordinates.

Solution: Since $r = 2$ and $\theta = \pi/3$,

$$\begin{aligned} x &= 2\cos(\pi/3) = 2(1/2) = 1 \\ y &= 2\sin(\pi/3) = 2(\sqrt{3}/2) = \sqrt{3} \end{aligned}$$

The point is $(1, \sqrt{3})$ in Cartesian coordinates.

EXAMPLE 3

Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

Solution: If we choose r to be positive:

$$r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\tan\theta = -1/1 = -1$$

Since $(1, -1)$ lies in the fourth quadrant, we can choose $\theta = -\pi/4$ or $\theta = 7\pi/4$. Possible answers: $(\sqrt{2}, -\pi/4)$ or $(\sqrt{2}, 7\pi/4)$.

Note: $r^2 = x^2 + y^2, \tan\theta = y/x$ do not uniquely determine θ when x and y are given because, as θ increases through the interval $[0, 2\pi]$, each value of $\tan\theta$ occurs twice. Therefore, in converting from Cartesian to polar coordinates, it's not good enough to find r and θ that satisfy the equations.

Polar Curves

The graph of a polar equation $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

EXAMPLE 4

Sketch the curve defined by the polar equation $r = 2$.

Solution: The equation $r = 2$ means that the distance from the pole is always 2. This is a circle with center O and radius 2.

EXAMPLE 5

Sketch the curve defined by the polar equation $\theta = 1$.

Solution: The equation $\theta = 1$ means that the angle is always 1 radian. This is a straight line through the pole making an angle of 1 radian with the polar axis.

EXAMPLE 6

Sketch the curve $r = 2\cos\theta$.

Solution: We can convert to Cartesian coordinates:

$$r = 2\cos\theta \implies r^2 = 2r\cos\theta$$

$$x^2 + y^2 = 2x$$

$$x^2 - 2x + y^2 = 0$$

$$(x-1)^2 + y^2 = 1$$

This is a circle with center $(1, 0)$ and radius 1.

EXAMPLE 7

Sketch the curve $r = 1 + \sin\theta$.

Solution: We can plot points for various values of θ. This curve is a cardioid.

EXAMPLE 8

Sketch the curve $r = \cos(2\theta)$.

Solution: We can observe how r changes as θ increases. As θ goes from 0 to $\pi/4$, r decreases from 1 to 0. As θ goes from $\pi/4$ to $\pi/2$, r is negative, so this part of the curve is traced in the opposite quadrant. The pattern repeats. This curve is a four-leaved rose.

Symmetry

When sketching polar curves, it is sometimes helpful to take advantage of symmetry.

(a) **Symmetry about the polar axis:** If a polar equation is unchanged when θ is replaced by $-\theta$.

(b) **Symmetry about the pole:** If the equation is unchanged when r is replaced by $-r$, or when θ is replaced by $\theta + \pi$.

(c) **Symmetry about the vertical line $\theta = \pi/2$:** If the equation is unchanged when θ is replaced by $\pi - \theta$.

The curves sketched in Examples 6 and 8 are symmetric about the polar axis, since $\cos(-\theta) = \cos\theta$ and $\cos(-2\theta) = \cos(2\theta)$. The curves in Examples 7 and 8 are symmetric about $\theta = \pi/2$ because $\sin(\pi - \theta) = \sin\theta$ and $\cos(2(\pi - \theta)) = \cos(2\pi - 2\theta) = \cos(2\theta)$. The four-leaved rose is also symmetric about the pole.

Graphing Polar Curves with Technology

Although it's useful to be able to sketch simple polar curves by hand, we need to use a graphing calculator or computer when we are faced with a curve as complicated as the ones shown in Figures 15 and 16.

EXAMPLE 9

Graph the curve $r = \sin(8\theta/5)$.

Solution: First we need to determine the domain for θ. We ask ourselves: how many complete rotations are required until the curve starts to repeat itself? If the answer is n, then $\sin(8(\theta+2n\pi)/5) = \sin(8\theta/5+16n\pi/5)$. For the curve to repeat, $16n\pi/5$ must be an even multiple of π. This will first occur when $n = 5$. Therefore, we will graph the entire curve if we specify that $0 \leq \theta \leq 10\pi$. Figure 17 shows the resulting curve. Notice that this curve has 16 loops.

EXAMPLE 10

Investigate the family of polar curves given by $r = 1 + c\sin\theta$. How does the shape change as c changes? (These curves are called limaçons.)

Solution: Figure 18 shows computer-drawn graphs for various values of c. For $c > 1$, there is a loop that decreases in size as c decreases. When $c = 1$, the loop disappears and the curve becomes the cardioid. For c between 1 and 1/2, the cardioid's cusp is smoothed out and becomes a "dimple." When c decreases from 1/2 to 0, the limaçon is shaped like an oval. This oval becomes more circular as $c \to 0$, and when $c = 0$ the curve is just the circle $r = 1$. The remaining parts of Figure 18 show that as c becomes negative, the shapes change in reverse order. In fact, these curves are reflections about the horizontal axis of the corresponding curves with positive c.

Table 1 summarizes common polar curves:

Circles and Spirals: e.g., $r = a, r = a\sin\theta, r = a\cos\theta, r = a\theta$

Limaçons: $r = a \pm b\sin\theta, r = a \pm b\cos\theta$ (shapes depend on a and b relation)

Roses: $r = a\sin(n\theta), r = a\cos(n\theta)$ (n-leaved if n is odd, 2n-leaved if n is even)

Lemniscates: $r^2 = a^2\sin(2\theta), r^2 = a^2\cos(2\theta)$