11.10 Taylor and Maclaurin Series

This section investigates which functions have power series representations and how to find such representations. It focuses on Taylor series and Maclaurin series.

Definitions of Taylor Series and Maclaurin Series

Suppose a function f can be represented by a power series centered at a:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n \quad |x-a| < R$$

By evaluating $f(a)$ and its successive derivatives $f'(a), f''(a), f'''(a), \ldots$, we can determine the coefficients c_n.

$$\begin{aligned}
f(a) &= c_0 \\
f'(a) &= c_1 \\
f''(a) &= 2c_2 \\
f'''(a) &= 2 \cdot 3c_3 = 3!c_3 \\
&\vdots \\
f^{(n)}(a) &= n!c_n
\end{aligned}$$

Solving for c_n gives the following theorem:

Theorem 5

If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \quad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Substituting this formula back into the series gives the **Taylor series** of the function f at a:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

For the special case $a = 0$, the Taylor series becomes the **Maclaurin series**:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots$$

Note 1: Having a Taylor series doesn't guarantee the series sums to $f(x)$.
Note 2: A power series representation at a is unique; it must be the Taylor series. Representations found in Section 11.9 are Taylor series.

EXAMPLE 1

Confirm the Maclaurin series for $f(x) = \dfrac{1}{1-x}$.
SOLUTION: We found $f^{(n)}(0) = n!$. Therefore,

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{n!}{n!} = 1$$

The Maclaurin series is $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$, which matches the geometric series representation, valid for $|x| < 1$.

EXAMPLE 2

Find the Maclaurin series for $f(x) = e^x$ and its radius of convergence.
SOLUTION: Since $f^{(n)}(x) = e^x$ for all n, we have $f^{(n)}(0) = e^0 = 1$ for all n. The Maclaurin series is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Using the Ratio Test:

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \frac{|x|}{n+1} \to 0 \text{ as } n \to \infty$$

The series converges for all x, so the radius of convergence is $R = \infty$.

When Is a Function Represented by Its Taylor Series?

Let $T_n(x)$ be the n-th degree Taylor polynomial of f at a:

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} (x-a)^i$$

The remainder is defined as $R_n(x) = f(x) - T_n(x)$. A function f is equal to the sum of its Taylor series if $f(x) = \lim_{n\to\infty} T_n(x)$.

Theorem 5

If $f(x) = T_n(x) + R_n(x)$, where T_n is the n-th degree Taylor polynomial of f at a, and if

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x - a| < R$, then f is equal to the sum of its Taylor series on the interval $|x - a| < R$.

To show $\lim_{n\to\infty} R_n(x) = 0$, we often use Taylor's Inequality.

Theorem 5

If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| \le d$$

A useful limit for applying Taylor's Inequality:

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad \text{for every real number } x$$

EXAMPLE 3

Prove that e^x is equal to the sum of its Maclaurin series.
SOLUTION: Let $f(x) = e^x$. Then $f^{(n+1)}(x) = e^x$. For $|x| \le d$, $|f^{(n+1)}(x)| = e^x \le e^d$. Take $M = e^d$. Taylor's Inequality gives:

$$|R_n(x)| \le \frac{e^d}{(n+1)!}|x|^{n+1} \quad \text{for } |x| \le d$$

Using Equation (),

$$\lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} = e^d \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

By the Squeeze Theorem, $\lim_{n\to\infty} |R_n(x)| = 0$, so $\lim_{n\to\infty} R_n(x) = 0$ for all x. By Theorem 8, e^x equals the sum of its Maclaurin series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x$$

Setting $x = 1$, we get

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

EXAMPLE 4

Find the Taylor series for $f(x) = e^x$ at $a = 2$.
SOLUTION: We have $f^{(n)}(2) = e^2$ for all n. The Taylor series is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$$

It can be shown that $\lim_{n\to\infty} R_n(x) = 0$, so

$$e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n \quad \text{for all } x$$

Taylor Series of Important Functions

EXAMPLE 5

Find the Maclaurin series for $\sin x$ and prove that it represents $\sin x$ for all x.
SOLUTION: Let $f(x) = \sin x$. The derivatives at 0 follow a pattern: $0, 1, 0, -1, 0, 1, 0, -1, \ldots$. The

Maclaurin series is:

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Since $|f^{(n+1)}(x)| = |\pm \sin x|$ or $|\pm \cos x|$, we have $|f^{(n+1)}(x)| \leq 1$ for all x. Take $M = 1$ in Taylor's Inequality:

$$|R_n(x)| \leq \frac{1}{(n+1)!}|x|^{n+1}$$

Using Equation (), $\lim_{n\to\infty} \dfrac{|x|^{n+1}}{(n+1)!} = 0$. Thus $\lim_{n\to\infty} R_n(x) = 0$ for all x.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{for all } x$$

EXAMPLE 6

Find the Maclaurin series for $\cos x$.

SOLUTION: Differentiate the Maclaurin series for $\sin x$ (Equation) term by term:

$$\begin{aligned}
\cos x = \frac{d}{dx}(\sin x) &= \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \\
&= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}
\end{aligned}$$

The radius of convergence is $R = \infty$.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \text{for all } x$$

EXAMPLE 7

Represent $f(x) = \sin x$ as the sum of its Taylor series centered at $\pi/3$.

SOLUTION: Calculate derivatives at $a = \pi/3$: $f(\pi/3) = \sqrt{3}/2$, $f'(\pi/3) = 1/2$, $f''(\pi/3) = -\sqrt{3}/2$, $f'''(\pi/3) = -1/2$, etc. The Taylor series is:

$$\frac{\sqrt{3}}{2} + \frac{1}{2 \cdot 1!}\left(x - \frac{\pi}{3}\right) - \frac{\sqrt{3}}{2 \cdot 2!}\left(x - \frac{\pi}{3}\right)^2 - \frac{1}{2 \cdot 3!}\left(x - \frac{\pi}{3}\right)^3 + \cdots$$

This can be written as:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{3}}{2(2n)!}\left(x - \frac{\pi}{3}\right)^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2(2n+1)!}\left(x - \frac{\pi}{3}\right)^{2n+1}$$

This representation is valid for all x.

EXAMPLE 8

Find the Maclaurin series for $f(x) = (1+x)^k$, where k is any real number.

SOLUTION: Calculate derivatives: $f(0) = 1$, $f'(0) = k$, $f''(0) = k(k-1)$, $f'''(0) = k(k-1)(k-2)$, ...,

$f^{(n)}(0) = k(k-1)\cdots(k-n+1)$. The Maclaurin series is:

$$\sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

Using binomial coefficient notation $\binom{k}{n} = \dfrac{k(k-1)\cdots(k-n+1)}{n!}$ for $n \geq 1$ and $\binom{k}{0} = 1$, the series is $\sum_{n=0}^{\infty} \binom{k}{n} x^n$. The Ratio Test gives:

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\binom{k}{n+1} x^{n+1}}{\binom{k}{n} x^n} \right| = \left| \frac{k-n}{n+1} x \right| \to |x| \text{ as } n \to \infty$$

The series converges if $|x| < 1$ and diverges if $|x| > 1$. Radius of convergence is $R = 1$.

Theorem 5

If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \cdots$$

Convergence at endpoints $x = \pm 1$ depends on k.

EXAMPLE 9

Find the Maclaurin series for $f(x) = \dfrac{1}{\sqrt{4-x}}$ and its radius of convergence.
SOLUTION: Rewrite $f(x)$:

$$\frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{4(1-x/4)}} = \frac{1}{2\sqrt{1-x/4}} = \frac{1}{2}\left(1 + \left(-\frac{x}{4}\right)\right)^{-1/2}$$

Use the binomial series with $k = -1/2$ and x replaced by $-x/4$:

$$\begin{aligned}
\frac{1}{\sqrt{4-x}} &= \frac{1}{2} \sum_{n=0}^{\infty} \binom{-1/2}{n} \left(-\frac{x}{4}\right)^n \\
&= \frac{1}{2} \left[1 + \left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!} \left(-\frac{x}{4}\right)^2 + \cdots \right] \\
&= \frac{1}{2} \left[1 + \frac{1}{8}x + \frac{1 \cdot 3}{2! 8^2} x^2 + \frac{1 \cdot 3 \cdot 5}{3! 8^3} x^3 + \cdots + \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n! 8^n} x^n + \cdots \right]
\end{aligned}$$

The series converges for $|-x/4| < 1$, which means $|x| < 4$. The radius of convergence is $R = 4$.

Theorem 5

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

New Taylor Series from Old

We can find Taylor series for new functions by manipulating known series (from Table 1) via substitution, multiplication, division, addition, subtraction, differentiation, and integration.

EXAMPLE 10

Find the Maclaurin series for (a) $f(x) = x\cos x$ and (b) $f(x) = \ln(1+3x^2)$.
SOLUTION: (a) Multiply the series for $\cos x$ by x:

$$x\cos x = x\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!} \quad (R = \infty)$$

(b) Replace x by $3x^2$ in the series for $\ln(1+x)$:

$$\ln(1+3x^2) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(3x^2)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n x^{2n}}{n}$$

Converges for $|3x^2| < 1$, i.e., $|x| < 1/\sqrt{3}$. $R = 1/\sqrt{3}$.

EXAMPLE 11

Find the function represented by the power series $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{n!}$.
SOLUTION: Rewrite the series as $\sum_{n=0}^{\infty} \frac{(-2x)^n}{n!}$. This is the Maclaurin series for e^u with $u = -2x$. So the function is e^{-2x}.

EXAMPLE 12

Find the sum of the series $\dfrac{1}{1\cdot 2} - \dfrac{1}{2\cdot 2^2} + \dfrac{1}{3\cdot 2^3} - \dfrac{1}{4\cdot 2^4} + \cdots$.

SOLUTION: The series is $\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{1}{n2^n} = \sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{(1/2)^n}{n}$. This matches the series for $\ln(1+x)$ with $x = 1/2$. The sum is $\ln(1+1/2) = \ln(3/2)$.

EXAMPLE 13

(a) Evaluate $\int e^{-x^2}\,dx$ as an infinite series.

(b) Evaluate $\int_0^1 e^{-x^2}\,dx$ correct to within an error of 0.001.

SOLUTION: (a) Replace x with $-x^2$ in the series for e^x:

$$e^{-x^2} = \sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

Integrate term by term:

$$\int e^{-x^2}\,dx = C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)n!} = C + x - \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots$$

(b) Evaluate the definite integral using the series from part (a) with C=0:

$$\int_0^1 e^{-x^2}\,dx = \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216} - \cdots\right]_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \cdots \approx 1 - 0.33333 + 0.10000 - 0.02381 + 0.00463 - \cdots$$

This is an alternating series. The first few terms are $1, -0.33333, 0.10000, -0.02381, 0.00463, -0.00059, \ldots$. The term $b_5 = 1/(11\cdot 5!) = 1/1320 \approx 0.00076$. The term $b_6 = 1/(13\cdot 6!) = 1/9360 \approx 0.00011$. The error using s_5 is less than $b_6 \approx 0.00011 < 0.001$.

$$\int_0^1 e^{-x^2}\,dx \approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \frac{1}{1320} \approx 0.7467$$

(Using s_4 gives error less than $b_5 \approx 0.00076 < 0.001$.)

$$\int_0^1 e^{-x^2}\,dx \approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} \approx 0.7475$$

EXAMPLE 14

Evaluate $\lim_{x\to 0} \dfrac{e^x - 1 - x}{x^2}$.
SOLUTION: Use the Maclaurin series for e^x:

$$\begin{aligned}
\frac{e^x - 1 - x}{x^2} &= \frac{(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots) - 1 - x}{x^2} \\
&= \frac{\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots}{x^2} \\
&= \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots
\end{aligned}$$

Therefore,

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \left(\frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots \right) = \frac{1}{2!} = \frac{1}{2}$$

Multiplication and Division of Power Series

Power series can be multiplied and divided like polynomials.

EXAMPLE 15

Find the first three nonzero terms in the Maclaurin series for (a) $e^x \sin x$ and (b) $\tan x$.
SOLUTION: (a) Multiply the series for e^x and $\sin x$:

$$e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$= (1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots)(x - \frac{1}{6}x^3 + \cdots)$$

$$= 1(x - \frac{1}{6}x^3 + \cdots) + x(x - \frac{1}{6}x^3 + \cdots) + \frac{1}{2}x^2(x - \frac{1}{6}x^3 + \cdots) + \frac{1}{6}x^3(x - \cdots) + \cdots$$

$$= x - \frac{1}{6}x^3 + x^2 - \frac{1}{6}x^4 + \frac{1}{2}x^3 - \frac{1}{12}x^5 + \frac{1}{6}x^4 + \cdots$$

$$= x + x^2 + \left(-\frac{1}{6} + \frac{1}{2}\right)x^3 + \left(-\frac{1}{6} + \frac{1}{6}\right)x^4 + \cdots$$

$$= x + x^2 + \frac{1}{3}x^3 + \cdots$$

(b) Divide the series for $\sin x$ by the series for $\cos x$ using long division:

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots}{1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots}$$

Performing the long division:

$$
\begin{array}{r}
x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots \\
1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots \overline{) x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots} \\
-(x - \frac{1}{2}x^3 + \frac{1}{24}x^5 - \cdots) \\
\hline
\frac{1}{3}x^3 - \frac{1}{30}x^5 + \cdots \\
-(\frac{1}{3}x^3 - \frac{1}{6}x^5 + \cdots) \\
\hline
\frac{2}{15}x^5 + \cdots
\end{array}
$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$$