Section 11.1 Infinite Sequences

Infinite Sequences

A sequence is a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \dots, a_n, \dots$$

The number a_1 is the first term, a_2 is the second term, and in general, a_n is the nth term. A sequence can be defined as a function f whose domain is the set of positive integers, where we write a_n instead of $f(n)$.
Notation: The sequence $\{a_1, a_2, a_3, \dots\}$ is denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

EXAMPLE 1: Defining Sequences with a Formula

(a) $a_n = \frac{1}{2^n} \to \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots, \frac{1}{2^n}, \dots\}$

(b) $\left\{\frac{n+1}{n}\right\}_{n=2}^{\infty} \to \{\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \dots\}$

(c) $\{3, 4, 5, 6, \dots\} = \{n+2\}_{n=1}^{\infty} = \{n\}_{n=3}^{\infty}$

(d) $\left\{\frac{(-1)^n \cdot 3^n}{n+1}\right\}_{n=0}^{\infty} \to \{1, -\frac{3}{2}, 3, -\frac{27}{4}, \frac{81}{5}, \dots\}$
Note: The $(-1)^n$ factor creates terms that alternate in sign.

EXAMPLE 2: Finding a Formula for a Sequence

Given sequence: $\{\frac{5}{3}, -\frac{25}{4}, \frac{125}{5}, -\frac{625}{6}, \frac{3125}{7}, \dots\}$
General term: $a_n = (-1)^{n-1} \cdot \frac{5^n}{n+2}$

EXAMPLE 3: Sequences without a Simple Defining Equation

(a) $\{p_n\}$, where p_n is the world population on January 1 of year n.

(b) $\{a_n\}$, where a_n is the nth decimal digit of e: $\{7, 1, 8, 2, 8, 1, 8, 2, 8, 4, 5, \dots\}$

(c) Fibonacci sequence $\{f_n\}$ defined by: $f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$.
First terms: $\{1, 1, 2, 3, 5, 8, 13, 21, \dots\}$

The Limit of a Sequence

Intuitive Definition: $\lim_{n\to\infty} a_n = L$ if a_n gets arbitrarily close to L as n increases. If this limit exists, the sequence converges; otherwise, it diverges.

Precise Definition of a Limit

For every $\varepsilon > 0$, there exists N such that $n > N \Rightarrow |a_n - L| < \varepsilon$.

Properties of Convergent Sequences

Theorem

If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ for integers n, then $\lim_{n\to\infty} a_n = L$.

- **Corollary:** If $r > 0$, then $\lim_{n\to\infty} \frac{1}{n^r} = 0$.

Limit Laws for Sequences

If $\{a_n\}$ and $\{b_n\}$ converge:

- $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty}(c \cdot a_n) = c \cdot \lim_{n\to\infty} a_n$
- $\lim_{n\to\infty}(a_n \cdot b_n) = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n)$
- $\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{\lim_{n\to\infty} b_n}{\lim_{n\to\infty} a_n}$, provided denominator $\neq 0$
- **Power Law:** $\lim_{n\to\infty}(a_n)^p = [\lim_{n\to\infty} a_n]^p$, if $p > 0$ and $a_n > 0$.

Squeeze Theorem for Sequences

Theorems

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

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If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

EXAMPLES 4-11: Finding Limits

- **Ex 4:** $\lim_{n\to\infty} \frac{n+1}{n} = 1$
- **Ex 6:** $\lim_{n\to\infty} \frac{n}{\ln(n)} = \infty$ using $f(x) = \frac{x}{\ln(x)}$
- **Ex 7:** $a_n = (-1)^n$ diverges (oscillates between 1 and -1)
- **Ex 8:** For $a_n = \frac{(-1)^n}{n}$, since $|a_n| = \frac{1}{n} \to 0$, we have $a_n \to 0$.
- **Ex 11:** $\{r^n\}$ converges if $-1 < r \leq 1$. The limit is 0 if $-1 < r < 1$, and 1 if $r = 1$.

Monotonic and Bounded Sequences

Definition:

- **Increasing:** $a_n < a_{n+1}$
- **Decreasing:** $a_n > a_{n+1}$
- **Monotonic:** either increasing or decreasing
- **Bounded Above:** $\exists M$ such that $a_n \leq M$
- **Bounded Below:** $\exists m$ such that $m \leq a_n$

- **Bounded:** both above and below

Monotonic Sequence Theorem

Every bounded, monotonic sequence converges.

EXAMPLE 14: Using the Monotonic Sequence Theorem

Given: $a_1 = 2, a_{n+1} = \frac{1}{2}(a_n + 6)$

- **Show increasing:** by induction, show $a_{n+1} > a_n$.
- **Show bounded:** by induction, show $a_n < 6$.

Conclusion: The sequence is increasing and bounded, therefore it converges.
Limit: Let $\lim_{n\to\infty} a_n = L$. Then $L = \frac{1}{2}(L + 6) \Rightarrow 2L = L + 6 \Rightarrow L = 6$.