11.9 Representations of Functions as Power Series

In this section we learn how to represent certain types of functions as sums of power series by manipulating geometric series or by differentiating or integrating such a series. This is useful because representing a function as a power series gives us a way to approximate the function with polynomials.

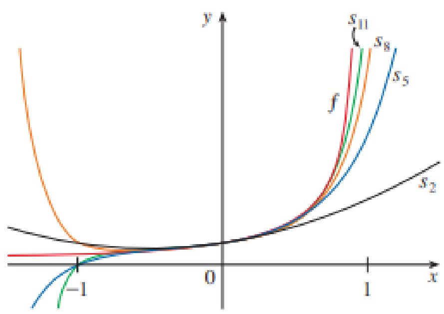
We start with the geometric series equation:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \quad |x| < 1$$

A geometric illustration is shown in picture. Because the sum of a series is the limit of the sequence of partial sums, we have

$$\frac{1}{1-x} = \lim_{n\to\infty} s_n(x)$$

$$s_n(x) = 1 + x + x^2 + ... + x^n$$



EXAMPLE 1

Express $\dfrac{1}{1+x^2}$ as the sum of a power series and find the radius of convergence.

SOLUTION: Replacing x by $-x^2$ in Equation (), we have

$$\begin{aligned}\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} &= \sum_{n=0}^{\infty}(-x^2)^n \\ &= \sum_{n=0}^{\infty}(-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots\end{aligned}$$

Because this is a geometric series, it converges when $|-x^2| < 1$, that is, $x^2 < 1$, or $|x| < 1$. Therefore the radius of convergence is $R = 1$.

EXAMPLE 2

Find a power series representation for $\dfrac{1}{x+2}$.
SOLUTION: We first rewrite the function in the form $1/(1-r)$:

$$\frac{1}{2+x} = \frac{1}{2(1+x/2)} = \frac{1}{2[1-(-x/2)]} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n$$

This series converges when $|-x/2| < 1$, that is, $|x| < 2$. So the radius of convergence is $R = 2$. The power series representation is:

$$\frac{1}{x+2} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}x^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n$$

EXAMPLE 3

Find a power series representation of $\dfrac{x^3}{x+2}$.
SOLUTION: Since this function is just x^3 times the function in Example 2, all we have to do is multiply that series by x^3:

$$\begin{aligned}\frac{x^3}{x+2} &= x^3\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^{n+3} \\ &= \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5 - \frac{1}{16}x^6 + \cdots\end{aligned}$$

This series converges for $|x| < 2$.

Differentiation and Integration of Power Series

The sum of a power series is a function $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ whose domain is the interval of convergence. We can differentiate and integrate such functions term by term.

Theorem 1: Term-by-Term Differentiation and Integration

If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(ii) $\int f(x)\,dx = C + c_0(x-a) + c_1\dfrac{(x-a)^2}{2} + c_2\dfrac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n\dfrac{(x-a)^{n+1}}{n+1}$

The radii of convergence of the power series in Equations (i) and (ii) are both R.
Note: The interval of convergence may change at the endpoints after differentiation or integration.

EXAMPLE 4

Express $\dfrac{1}{(1-x)^2}$ as a power series by differentiating $\dfrac{1}{(1-x)}$. What is the radius of convergence?
SOLUTION: We start with

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

Differenrialting both sides term by term, we get

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n$$

According to Theorem 2, the radius of convergence of the differentiated series is the same as the radius of convergence of the original series, $R = 1$.

EXAMPLE 5

Find a power series representation for $\ln(1+x)$ and its radius of convergence.
SOLUTION: We notice that, except for a factor of -1, the derivative of $\ln(1+x)$ is $1/(1-x)$.

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int (1 - x + x^2 - \cdots)\,dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + C = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$$

Letting $x = 0$, we get $\ln(1) = 0 + C$, so $C = 0$. Thus

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

The radius of convergence is the same as the original series, $R = 1$.

EXAMPLE 6

Find a power series representation for $f(x) = \tan^{-1}(x)$.
SOLUTION: We observe that $f'(x) = \dfrac{1}{1+x^2}$. From Example 1, we have

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad (|x| < 1)$$

Integrating term by term, we get

$$\begin{aligned} \tan^{-1}(x) &= \int \frac{1}{1+x^2}\,dx = \int \left(\sum_{n=0}^{\infty} (-1)^n x^{2n} \right) dx \\ &= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \end{aligned}$$

To find C, we put $x = 0$. Then $C = \tan^{-1}(0) = 0$. Therefore

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

The radius of convergence is $R = 1$.

EXAMPLE 7

(a) Evaluate $\int \frac{1}{1+x^7}\,dx$ as a power series.
(b) Use part (a) to approximate $\int_0^{0.5} \frac{1}{1+x^7}\,dx$ correct to six decimal places.
SOLUTION:

(a) Replacing x with x^7 in Equation (), we get

$$\frac{1}{1+x^7} = \frac{1}{1-(-x^7)} = \sum_{n=0}^{\infty}(-x^7)^n = \sum_{n=0}^{\infty}(-1)^n x^{7n} \quad (|x| < 1)$$

Integrating term by term:

$$\int \frac{1}{1+x^7}\,dx = \int \sum_{n=0}^{\infty}(-1)^n x^{7n}\,dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{7n+1}}{7n+1}$$

The radius of convergence is $R = 1$.

(b) Using the result from part (a):

$$\int_0^{0.5} \frac{1}{1+x^7}\,dx = \left[x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \cdots\right]_0^{0.5}$$

$$= 0.5 - \frac{(0.5)^8}{8} + \frac{(0.5)^{15}}{15} - \frac{(0.5)^{22}}{22} + \cdots$$

This is an alternating series. The terms are:

$$\begin{aligned} b_0 &= 0.5 \\ b_1 &= \frac{(0.5)^8}{8} \approx 0.00048828 \\ b_2 &= \frac{(0.5)^{15}}{15} \approx 0.00000203 \\ b_3 &= \frac{(0.5)^{22}}{22} \approx 0.00000001 \end{aligned}$$

By the Alternating Series Estimation Theorem, the error is less than b_3. To ensure six decimal place accuracy, we need the error to be less than 0.0000005. Since $b_3 < 0.0000005$, we can use the sum up to b_2:

$$\int_0^{0.5} \frac{1}{1+x^7}\,dx \approx 0.5 - \frac{(0.5)^8}{8} + \frac{(0.5)^{15}}{15} \approx 0.5 - 0.00048828 + 0.00000203 \approx 0.49951375$$

Rounding to six decimal places, we get 0.499514.