10.3 Polar Coordinates

The Polar Coordinate System

In the polar coordinate system, a point P in the plane is determined by a distance from a fixed point and an angle from a fixed ray.

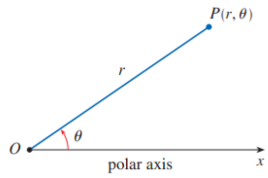


FIGURE 1

Pole (or Origin): The fixed point, labeled O.

Polar Axis: The fixed ray starting at O, usually drawn horizontally to the right (corresponding to the positive x-axis).

Polar Coordinates (r, θ)**:**

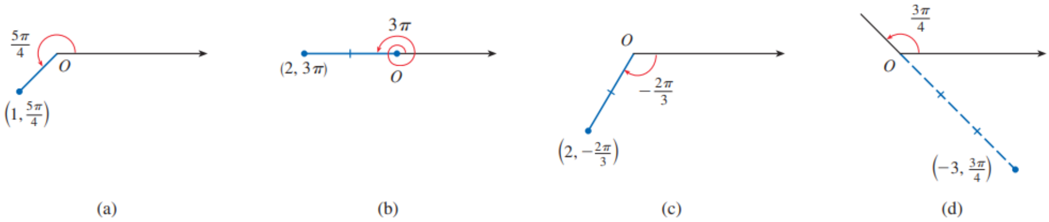
r : The distance from O to P.

θ : The angle between the polar axis and the line segment OP, measured in radians.
Positive angles are counterclockwise, negative angles are clockwise.

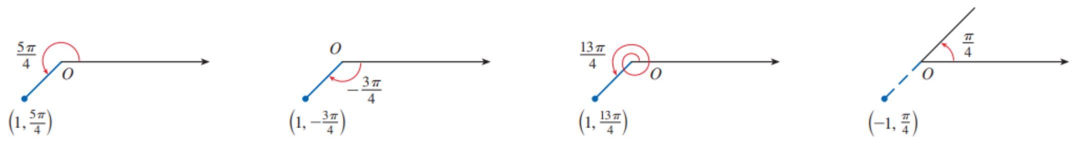
If $r < 0$, the point $(-r, \theta)$ lies on the same line through O as (r, θ) but on the opposite side of O. So, $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$.

EXAMPLE 1

Plot the points whose polar coordinates are given.
(a) $(1, 5\pi/4)$ (b) $(2, 3\pi)$ (c) $(2, -2\pi/3)$ (d) $(-3, 3\pi/4)$



Solution: The points are plotted in Figure 3. In part (d) the point $(-3, 3\pi/4)$ is located three units from the pole in the fourth quadrant because the angle $3\pi/4$ is in the second quadrant and $r = -3$ is negative.



Relationship between Polar and Cartesian Coordinates

The pole corresponds to the origin and the polar axis coincides with the positive x-axis.

From Polar to Cartesian: If a point P has polar coordinates (r, θ), its Cartesian coordinates (x, y) are:

$$x = r\cos\theta \qquad y = r\sin\theta$$

From Cartesian to Polar: If a point P has Cartesian coordinates (x, y), its polar coordinates (r, θ) satisfy:

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

When converting from Cartesian to polar, care must be taken to choose θ such that (r, θ) lies in the correct quadrant.

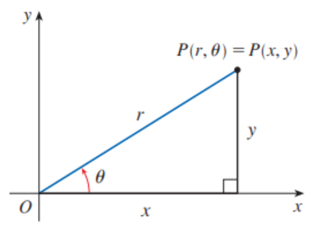


FIGURE 5

EXAMPLE 2

Convert the point $(2, \pi/3)$ from polar to Cartesian coordinates.

Solution: Since $r = 2$ and $\theta = \pi/3$,

$$x = 2\cos(\pi/3) = 2(1/2) = 1$$
$$y = 2\sin(\pi/3) = 2(\sqrt{3}/2) = \sqrt{3}$$

The point is $(1, \sqrt{3})$ in Cartesian coordinates.

EXAMPLE 3

Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

Solution: If we choose r to be positive:

$$r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\tan\theta = -1/1 = -1$$

Since $(1, -1)$ lies in the fourth quadrant, we can choose $\theta = -\pi/4$ or $\theta = 7\pi/4$. Possible answers: $(\sqrt{2}, -\pi/4)$ or $(\sqrt{2}, 7\pi/4)$.

Note: $r^2 = x^2 + y^2, \tan\theta = y/x$ do not uniquely determine θ when x and y are given because, as θ increases through the interval $[0, 2\pi]$, each value of $\tan\theta$ occurs twice.
Therefore, in converting from Cartesian to polar coordinates, it's not good enough to find r and θ that satisfy the equations.

Polar Curves

The graph of a polar equation $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

EXAMPLE 4

Sketch the curve defined by the polar equation $r = 2$.

Solution: The equation $r = 2$ means that the distance from the pole is always 2. This is a circle with center O and radius 2.

EXAMPLE 5

Sketch the curve defined by the polar equation $\theta = 1$.

Solution: The equation $\theta = 1$ means that the angle is always 1 radian. This is a straight line through the pole making an angle of 1 radian with the polar axis.

EXAMPLE 6

Sketch the curve $r = 2\cos\theta$.

Solution: We can convert to Cartesian coordinates:

$$r = 2\cos\theta \implies r^2 = 2r\cos\theta$$

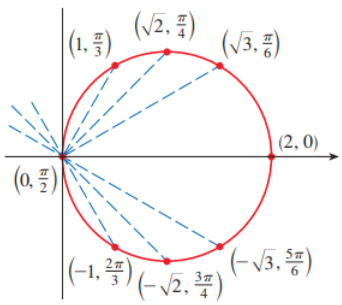
$$x^2 + y^2 = 2x$$

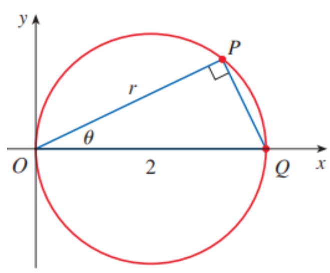
$$x^2 - 2x + y^2 = 0$$

$$(x-1)^2 + y^2 = 1$$

This is a circle with center $(1, 0)$ and radius 1.

θ	$r = 2\cos\theta$
0	2
$\pi/6$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
$\pi/2$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	-2

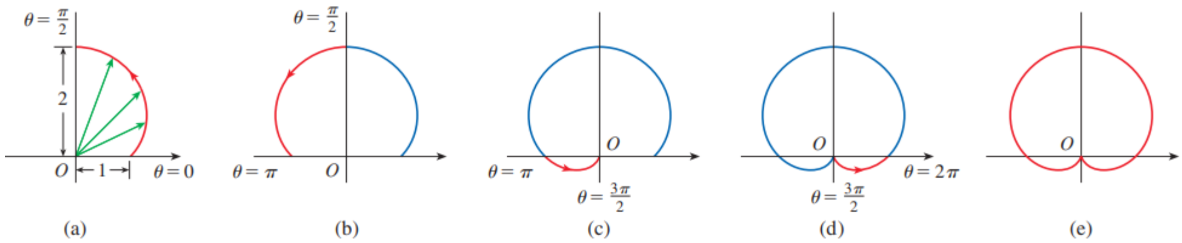




EXAMPLE 7

Sketch the curve $r = 1 + \sin\theta$.

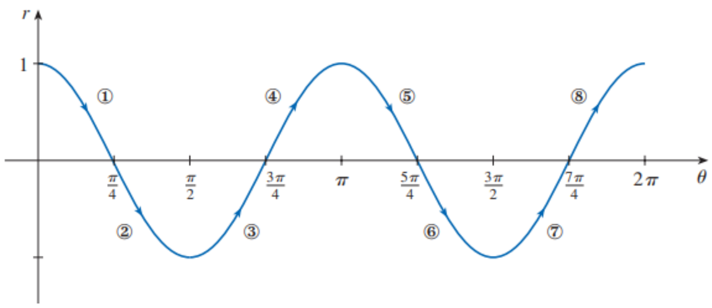
Solution: We can plot points for various values of θ. This curve is a cardioid. This enables us to read at a glance the values of r that correspond to increasing values of θ. For instance, we see that as θ increases from 0 to $\pi/2$, r (the distance from O) increases from 1 to 2, so we sketch the corresponding part of the polar curve in (a).
As θ increases from $\pi/2$ to π, r decreases from 2 to 1, so we sketch the next part of the curve as in(b).
As θ increases from π to $3\pi/2$, r decreases from 1 to 0 as shown in part (c).
Finally, as θ increases from $3\pi/2$ to 2π, r increases from 0 to 1 as shown in part (d).
If we let θ increase beyond 2π or decrease beyond 0, we would simply retrace this path. Putting together the parts of the curve from (a)–(d), we sketch the complete curve in part (e). It is called a cardioid because it's shaped like a heart.

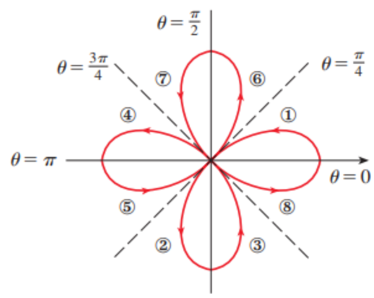


EXAMPLE 8

Sketch the curve $r = \cos(2\theta)$.

Solution: We can observe how r changes as θ increases. As θ goes from 0 to $\pi/4$, r decreases from 1 to 0. As θ goes from $\pi/4$ to $\pi/2$, r is negative, so this part of the curve is traced in the opposite quadrant. The pattern repeats. This curve is a four-leaved rose.





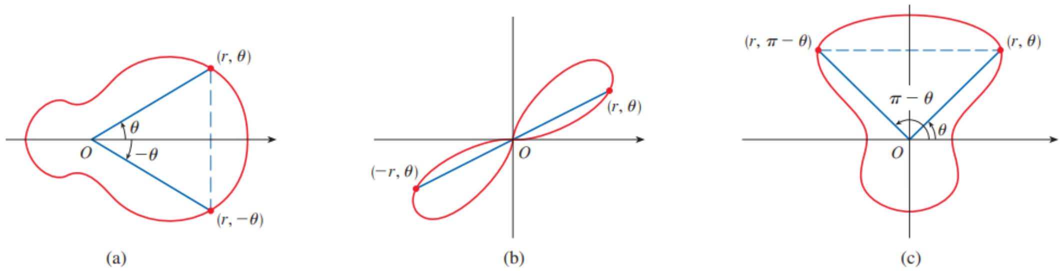
Symmetry

When sketching polar curves, it is sometimes helpful to take advantage of symmetry.

(a) **Symmetry about the polar axis:** If a polar equation is unchanged when θ is replaced by $-\theta$.

(b) **Symmetry about the pole:** If the equation is unchanged when r is replaced by $-r$, or when θ is replaced by $\theta + \pi$.

(c) **Symmetry about the vertical line $\theta = \pi/2$:** If the equation is unchanged when θ is replaced by $\pi - \theta$.



The curves sketched in Examples 6 and 8 are symmetric about the polar axis, since $\cos(-\theta) = \cos\theta$ and $\cos(-2\theta) = \cos(2\theta)$. The curves in Examples 7 and 8 are symmetric about $\theta = \pi/2$ because $\sin(\pi - \theta) = \sin\theta$ and $\cos(2(\pi - \theta)) = \cos(2\pi - 2\theta) = \cos(2\theta)$. The four-leaved rose is also symmetric about the pole.

Graphing Polar Curves with Technology

Although it's useful to be able to sketch simple polar curves by hand, we need to use a graphing calculator or computer when we are faced with a curve as complicated as the ones shown in Figures 15 and 16.

EXAMPLE 9

Graph the curve $r = \sin(8\theta/5)$.

Solution: First we need to determine the domain for θ. We ask ourselves: how many complete rotations are required until the curve starts to repeat itself?
If the answer is n, then $\sin(8(\theta + 2n\pi)/5) = \sin(8\theta/5 + 16n\pi/5)$. For the curve to repeat, $16n\pi/5$ must be an even multiple of π. This will first occur when $n = 5$.
Therefore, we will graph the entire curve if we specify that $0 \leq \theta \leq 10\pi$.
Figure 17 shows the resulting curve. Notice that this curve has 16 loops.

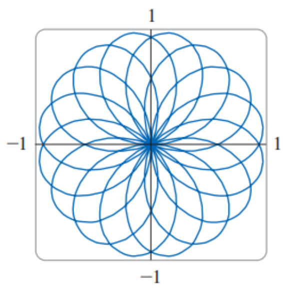


FIGURE 17
$r = \sin(8\theta/5)$

EXAMPLE 10

Investigate the family of polar curves given by $r = 1 + c\sin\theta$. How does the shape change as c changes? (These curves are called limaçons.)

Solution: Figure 18 shows computer-drawn graphs for various values of c.
For $c > 1$, there is a loop that decreases in size as c decreases.
When $c = 1$, the loop disappears and the curve becomes the cardioid.
For c between 1 and $1/2$, the cardioid's cusp is smoothed out and becomes a "dimple."
When c decreases from $1/2$ to 0, the limaçon is shaped like an oval. This oval becomes more circular as $c \to 0$, and when $c = 0$ the curve is just the circle $r = 1$.
The remaining parts of Figure 18 show that as c becomes negative, the shapes change in reverse order. In fact, these curves are reflections about the horizontal axis of the corresponding curves with positive c.

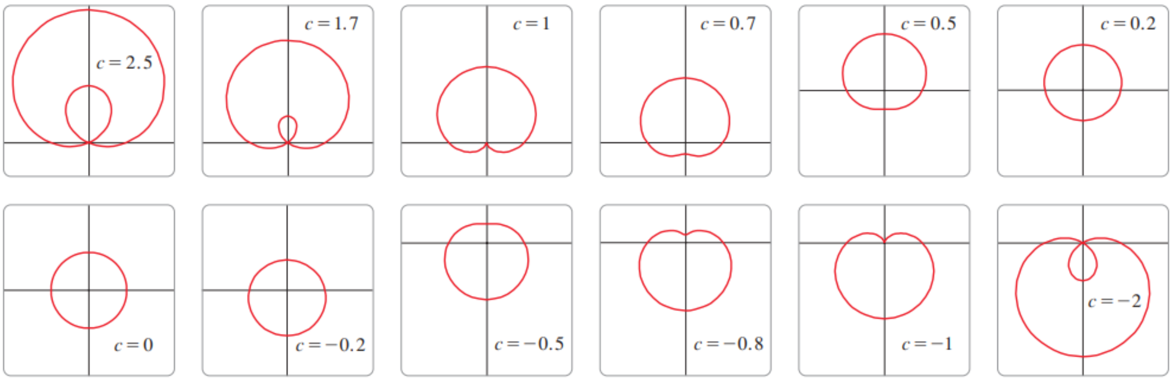


FIGURE 18 Members of the family of limaçons $r = 1 + c\sin\theta$