Signals & Systems

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Brief Summary of "Fouriers"

FS & FT Table

Name	*From ...*	; (	$\Rightarrow$	*To ...*	;)
CTFS	**Time**: 连续，周期性	$x(t)$	$-CTFS\rightarrow$	**Freq**: 离散，非周期	$a_k=\frac{1}{T_0}\int_{T_0}x(\tau)e^{-jk\omega_0\tau}d\tau$
Inv-CTFS	**Freq**: 离散，非周期	a_k	$-$特征表示$\rightarrow$	**Time**: 连续，周期性	$x(t)=\sum_{k=-\infty}^{+\infty}a_k\cdot e^{jk\omega_0 t}$
DTFS	**Time**: 离散，周期性	$x[n]$	$-DTFS\rightarrow$	**Freq**: 离散，周期性	$a_k=\frac{1}{N}\sum_{m=0}^{N-1}x[m]e^{-jk\omega_0 m}$
Inv-DTFS	**Freq**: 离散，周期性	a_k	$-$特征表示$\rightarrow$	**Time**: 离散，周期性	$x[n]=\sum_{k=0}^{N-1}a_k e^{jk\omega_0 n}$
CTFT	**Time**: 连续，非周期	$x(t)$	$-\mathcal{F}_{CT}\rightarrow$	**Freq**: 连续，非周期	$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$
Inv-CTFT	**Freq**: 连续，非周期	$X(j\omega)$	$-\mathcal{F}_{CT}^{-1}\rightarrow$	**Time**: 连续，非周期	$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega$
DTFT	**Time**: 离散，非周期	$x[n]$	$-\mathcal{F}_{DT}\rightarrow$	**Freq**: 连续，周期性	$X(j\omega)=\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega n}$
Inv-DTFT	**Freq**: 连续，周期性	$X(j\omega)$	$-\mathcal{F}_{DT}^{-1}\rightarrow$	**Time**: 离散，非周期	$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(j\omega)e^{j\omega n}d\omega$
DFT	**Time**: 离散，有限长	$x[n]$	$-\mathcal{F}_D\rightarrow$	**Freq**: 离散，有限长	$X[k]=\sum_{n=0}^{N-1}x[n]e^{-j\frac{2\pi k}{N}n}$
Inv-DFT	**Freq**: 离散，有限长	$X[k]$	$-\mathcal{F}_D^{-1}\rightarrow$	**Time**: 离散，有限长	$x[n]=\frac{1}{N}\sum_{n=0}^{N-1}X[k]e^{j\frac{2\pi k}{N}n}$

Tips

1. e 指数负号?
    - 时域积分/求和 $\rightarrow$ 频域，有负号
    - 频域积分/求和 $\rightarrow$ 时域，无负号

2. **积分限**范围?
    - 周期性/有限长 积分/求和 $\to$ ...，积/求一个周期上
    - 非周期无限长 积分/求和 $\to$ ...，积/求$-\infty$到∞
3. 前面**除系数**?
    - FS: 时 $\to$ 频，除系数（T_0, N）；频 $\to$ 时，不除
    - FT: 时 $\to$ 频，不除；频 $\to$ 时， 除系数（$2\pi, N$）

Overview

Basic Concepts

- *Signal*: a **function** of one or more independent variables; typically contains information about the behaviour or nature of some physical phenomena.
- *System*: responds to a particular signal input by producing output signal, **function of function**.

Continuous vs. *Discrete*

- Continuous-time: $x(t)$
- Discrete-time: $x[n]$, sequence (samples)

Transformations

- *Reflection*: $x(t) \leftrightarrow x(-t)$, $x[n] \leftrightarrow x[-n]$
- *Scaling*: $x(t) \leftrightarrow x(ct)$
- *Time-shift*: $x(t) \leftrightarrow x(t-t_0)$, $x[n] \leftrightarrow x[n-n_0]$
    - $x(t) \to x(\alpha t+\beta)$, **First Shift Then Scale**.
- Derivate: $x'(t_0) =$
    - $x'(t_0)$, if differentiable at t_0
    - $(x(t_0^+) - x(t_0^-)) \cdot \delta(t-t_0)$, o.w.
- Integration: $\int_{-\infty}^{t} x(\tau)d\tau = x * u(t) = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau$
    - convenient for computation

Even vs. *Odd*

- Even: $x(t) = x(-t)$, $x[n] = x[-n]$
- Odd: $x(t) = -x(-t)$, $x[n] = -x[-n]$
- For every signal $x(t)$, we have $x(t) = Evenx(t) + Oddx(t)$, where $Evenx(t) = \frac{1}{2}(x(t)+x(-t))$ and $Oddx(t) = \frac{1}{2}(x(t)-x(-t))$

Periodic vs. *Aperiodic*

- Periodic: $x(t) = x(t+mT)$ for $\forall$ integer m ; $x[n] = x[n+mN]$ for $\forall$ integer m
    - Fundamental period (T_0, N_0): **Smallest positive value** of T or N.
- Aperiodic: Non-periodic

Special Signals

Complex Exponential

- Euler's Formula: $e^{j\omega_0 t} = cos(\omega_0 t) + j \cdot sin(\omega_0 t)$
- Also called complex *sinusoidal* signals
  - $e^{j\omega_0 t}$ is always periodic ($T_0 = \frac{2\pi}{\omega_0}$)
  - $e^{j\Omega_0 N}$ is periodic iff $\Omega_0 N$ is multiple of 2π.

Unit Step & Impulse

- Unit Step: $u(t) = 1, t > 0$, $u[n] = 1, n \geq 0$
- Impulse: $\delta(t) = \frac{d}{dt}u(t)$, $\delta[n] = 1, n = 0$
  - Value of an impulse equals **Integral over an impulse**.

Properties of a System

Properties

- *Memoryless*: Output only depends on input **at the same time**
- *Invertible*: **Distinct input** leads to **distinct output**
- *Causal*: Output only depends on input **at the same time** or **before**
- *Stable*: **Bounded input** gives **Bounded output**
- *Time-invariant*: A **time-shift in the input** causes a **same time-shift in the output**
  - $x(t) \rightarrow y(t)$, then $x(t - t_0) \rightarrow y(t - t_0)$
- *Linearity*: Additivity and Scaling
  - $x(t) \rightarrow y(t)$, then $ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$
  - **Zero input** must give **Zero output**.
- **L.T.I**: Linear and Time-invariant

Convolution

Definition

The operation **Convolution** ($*$) is defined as $x * y(t) = \int_{-\infty}^{+\infty} x(t - \tau)y(\tau)d\tau$, which is important for L.T.I systems.

Representation of a Signal in Impulses

- $x[n] = \sum_{k=-\infty}^{+\infty} \delta[n - k]x[k] = x * \delta[n]$
- $x(t) = \int_{-\infty}^{+\infty} \delta(t - \tau)x(\tau)d\tau = x * \delta(t)$

Output of L.T.I Systems

1. We denote the *Impulse response* as $h(t)$, $\delta(t) \rightarrow h(t)$
2. According to L.T.I & Representation of signal in impulses, we have:
   - $x(t) \rightarrow y(t)$, then $y(t) = \int_{-\infty}^{+\infty} h(t - \tau)x(\tau)d\tau$
   - $x[n] \rightarrow y[n]$, then $x[n] = \sum_{k=-\infty}^{+\infty} h[n - k]x[k]$
3. Any L.T.I system's output is the **convolution of input & impulse response**

Properties of Convolution

- Commutative: $g(t) = x(t) * h(t) = h(t) * x(t)$
- Bi-linear: $(ax_1 + bx_2) * h = a(x_1 * h) + b(x_2 * h)$, $x * (ah_1 + bh_2) = a(x * h_1) + b(x * h_2)$
- Shift: $g(t - \tau) = x(t - \tau) * h(t) = x(t) * h(t - \tau)$
- Identity: Convolution on $\delta(t)$ equals itself; Identity is unique
- Associative: $x_1 * (x_2 * x_3) = (x_1 * x_2) * x_3$
- Smooth derivative: $y'(t) = x'(t) * h(t) = x(t) * h'(t)$

Calculation of Convolution

- Sliding window:
    1. Reverse the simpler one $x(t)$
    2. Record reversed $x(t)$'s jumping points
    3. Slide reversed $x(t)$, for each t, $g(t)$ is integral of multiplication
- $g(t) = \int \frac{d}{dt} x * h(t) dt = \dots$
- $g(t) = \frac{d}{dt} \int x * h(t) dt = \dots$

Eigen-function of L.T.I

Eigen-functions

- *Eigenfunction*: A signal for which the system's output is just a constant (possibly complex) times the input.
- *Eigen Basis*: If an input signal $x(t)$ can be decomposed to a weighted sum of eigenfunctions (eigen basis), then the output can be easily found.

So *goal*: Getting an **eigen basis** of an L.T.I system.

e^{st} as eigenfunction of L.T.I

1. Consider the input to be $x(t) = e^{st}$, then the output is $y(t) = \int h(\tau) e^{s(t-\tau)} d\tau = e^{st} \int h(\tau) e^{-s\tau} d\tau$
2. $H(s) = \int h(\tau) e^{-s\tau} d\tau$ is just a constant, i.e. eigenvalue for function e^{st}

Orthonormal Basis

- When s is purely imaginary ($jk\omega_0$), $e^{jk\omega_0 t}$ is *orthonormal* and *standard* among different k.
    - Definition of inner-product of perioidic functions: $< x_1(t), x_2(t) >= \frac{1}{T_0} \int_{T_0} x_1(t) x_2^*(t) dt$

Fourier Series Expansion (FS)

Expand any **periodic** function on the eigen basis mentioned above.

Expand *Continuous* & *Periodic* Functions on Eigen Basis

For any periodic function $x(t)$, ω_0 is its *fundamental frequency*.

- It can be expanded into a weighted sum of $e^{jk\omega_0 t}$, thus can enjoy the convenience of eigen basis:
    - $x(t) = \sum_{k=-\infty}^{+\infty} a_k \cdot e^{jk\omega_0 t}$, denoted as $x(t) \xleftarrow{F.S.} \rightarrow a_k$
- **Coefficients**: $a_k = \frac{1}{T_0} \int_{T_0} x(\tau) e^{-jk\omega_0 \tau} d\tau =< x(t), e^{jk\omega_0 t} >$

- Case $k = 0$ is often special!
- a_0 controls a constant 1.

Properties of F.S. Expansion

- Linearity (same T_0): $z(t) = \alpha x(t) + \beta y(t) \leftrightarrow \alpha a_k + \beta b_k$
  - Adding a constant C can be acquired by Adding C **only** on a_0
- Time-shift: $x(t - t_0) \leftrightarrow e^{-jk\omega_0 t_0} a_k$
- Reverse: $x(-t) \leftrightarrow a_{-k}$
- Scaling: $x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k \cdot e^{jk(\alpha\omega_0)t}$
- Multiplication: $x(t)y(t) \leftrightarrow h_k = \sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
- *Conjugate symmetry*: $x^*(t) \leftrightarrow a_{-k}^*$
  - If $x(t)$ is real, $a_k^* = a_{-k}$
  - IF $x(t)$ is real and even, $a_k = a_{-k} = a_k^*$
- Derivative: $\frac{d}{dt}x(t) \leftrightarrow jk\omega_0 a_k$
- Integral: $\int x(t)dt \leftrightarrow \frac{a_k}{jk\omega_0}$
- Parseval's Identity: $\frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$
  - TO BE PROVED

Expand *Discrete* & *Periodic* Functions on Eigen Basis

- $x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n}$, notice it's **not infinite**
- $a_k = \frac{1}{N}\sum_{m=0}^{N-1} x[m] e^{-jk\omega_0 m}$
  - Can be **summation over arbitrary period**!

Filters

A system that changes relative amplitude of some frequency components.

Frequency-shaping filter

- Differentiator: $H(j\omega) = j\omega$

Frequency-selective filter

- Continuous-time filters
  - *Low-pass filter*
    - $\omega = 0 \Rightarrow |H(j\omega)|$
    - $\omega \to \infty \Rightarrow |H(j\omega)| = 0$
  - *High-pass filter*
    - $\omega = 0 \Rightarrow |H(j\omega)| = 0$
    - $\omega \to \infty \Rightarrow |H(j\omega)|$
  - *Band-pass filter*
- Discrete-time filters
  - $H(e^{j\omega})$ is periodic with period 2π, therefore

- Low frequencies: ω around $0, \pm 2\pi, \pm 4\pi, \ldots$
- High frequencies: ω around $\pm\pi, \pm 3\pi, \ldots$

- *Infinite Impulse Response (IIR) filter*: Recursive
- *Finite Impulse Response (FIR) filter*: Non-recursive

Continuous-time Fourier Transform (CTFT)

Apply Fourier's idea on **Aperiodic** continuous-time functions, freq domain no longer countable but continuous.

Fourier Transform Pair

- Fourier Transform $\mathcal{F}$: $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
  - For *periodic* signals, $X(j\omega) = 2\pi \sum_{-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$, i.e. the Fourier Series
  - $X(j\omega)$ is called the "Spectrum" of $x(t)$
- Inverse F.T. $\mathcal{F}^{-1}$: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$

Dual Porperty

- $\mathcal{F}(\mathcal{F}(x(t))) = 2\pi \cdot x(-t)$
- Means that when we put the $X(j\omega)$ in *time domain*, it will produce 2π times $x(-t)$ waveform in *freq domain*

Properties of CTFT

Property	Signal	Fourier Transform
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0} \cdot X(j\omega)$
Freq Shifting	$e^{j\omega_0 t} \cdot x(t)$	$X(j(\omega - \omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time Reversal	$x(-t)$	$X(-j\omega)$
Scaling	$x(at)$	$\frac{1}{\lvert a \rvert}X(\frac{j\omega}{a})$
Convolution	$x(t) * y(t)$	$X(j\omega) \cdot Y(j\omega)$
Multiplication	$x(t) \cdot y(t)$	$\frac{1}{2\pi}(X(j\omega) * Y(j\omega))$
Time Differentiation	$\frac{dx(t)}{dt}$	$j\omega \cdot X(j\omega)$
Freq Differentiation	$-jt \cdot x(t)$	$\frac{dX(j\omega)}{d\omega}$
Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$

- **Symmetry**
  - If $x(t)$ real, then $X^*(j\omega) = X(-j\omega)$

- If $x(t)$ real and even, then $X(j\omega)$ real and even
- If $x(t)$ real and odd, then $X(j\omega)$ purely imaginary and odd

- **CAUTION**: $\delta(\frac{x-x_0}{a}) = a \cdot \delta(x - x_0)$
- **Parseval's Relation**: $\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$
  - Proof: $|x(t)|^2 \Rightarrow x(t) \cdot x^*(t)$

Normal CT Fourier Pairs

Time	Freq-domain
$\delta(t)$	1
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$
Square Pulse in time $(-T_1, T_1)$	$\frac{2\sin\omega T_1}{\omega} = 2T_1 sinc(\frac{\omega T_1}{\pi})$
$\frac{\sin(Wt)}{\pi t} = \frac{W}{\pi} sinc(\frac{Wt}{\pi})$	Square Pulse in freq $(-W, W)$
$e^{-at}u(t)$, $Re\{a\} > 0$	$\frac{1}{a+j\omega}$
$te^{-at}u(t)$, $Re\{a\} > 0$	$\frac{1}{(a+j\omega)^2}$
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $Re\{a\} > 0$	$\frac{1}{(a+j\omega)^n}$

- `sinc` function: $sinc(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$

Discrete-time Fourier Transform (DTFT)

Apply Fourier's idea on **Aperiodic** discrete-time functions.

Fourier Transform Pair

- Fourier Transform $\mathcal{F}$: $X(j\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$
  - Notice since n is an integer, $X(j\omega)$ is **periodic** on ω, with $W_0 = 2\pi$
- Inverse Fourier Transform $\mathcal{F}^{-1}$: $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(j\omega)e^{j\omega n} d\omega$

Properties of DTFT

Property	Signal	Fourier Transform
Linearity	$ax[n] + by[n]$	$aX(j\omega) + bY(j\omega)$
Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0} \cdot X(j\omega)$
Freq Shifting	$e^{j\omega_0 n} \cdot x[n]$	$X(j(\omega - \omega_0))$
Conjugation	$x^*[n]$	$X^*(-j\omega)$
Time Reversal	$x[-n]$	$X(-j\omega)$
Time Reversal	$x_{(k)}[n]$	$X(-jk\omega)$
Convolution	$x[n] * y[n]$	$X(j\omega) \cdot Y(j\omega)$
Multiplication	$x[n] \cdot y[n]$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X(j\mu) Y(j(\omega - \mu)) d\mu$
Freq Differentiation	$-jn \cdot x[n]$	$\frac{dX(e^{j\omega})}{d\omega}$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1-e^{-j\omega}} X(j\omega) + \pi X(0) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$

- **Symmetry**
  - If $x[n]$ real, then $X^*(j\omega) = X(-j\omega)$
  - If $x[n]$ real and even, then $X(j\omega)$ real and even
  - If $x[n]$ real and odd, then $X(j\omega)$ purely imaginary and odd
- **Parseval's Relation**: $\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi} \int_0^{2\pi} |X(j\omega)|^2 d\omega$

Normal DT Fourier Pairs

Time	Freq-domain
$\delta[n]$	1
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$
Square Pulse in time $(-N_1, N_1)$	$\frac{\sin[\omega(N_1+\frac{1}{2})]}{\sin(\frac{\omega}{2})}$
$\frac{\sin[Wn]}{\pi n} = \frac{W}{\pi} sinc[\frac{Wn}{\pi}]$	2π Periodic Square in freq $(-W, W)$
$a^n u(t)$, $\|a\| < 1$	$\frac{1}{1-ae^{-j\omega}}$
$(n+1)a^n u(t)$, $\|a\| < 1$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{(n+r-1)!}{n!(r-1)!} a^n u(t)$, $\|a\| < 1$	$\frac{1}{(1-ae^{-j\omega})^r}$

Discrete Fourier Transform (DFT)

Apply Fourier's idea on **Finite-length** discrete sequence.

Fourier Transform Pair

Suppose $x[n]$ is defined only on $n \in 0, 1, \ldots, N-1$

- Fourier Transform $\mathcal{F}$: $X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi k}{N}n}$
- Inverse Fourier Transform $\mathcal{F}^{-1}$: $x[n] = \frac{1}{N}\sum_{n=0}^{N-1} X[k]e^{j\frac{2\pi k}{N}n}$

Fast Fourier Transform Algorithm (FFT)

:) Basically an algorithmic thing so omitted here.

Laplace Transform (LT)

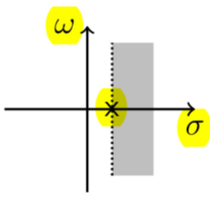
Apply CT transform to **a larger class of signals**! $s = j\omega \to \sigma + j\omega$

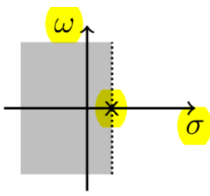
Laplace Transform Pair

- Laplace Transform: $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$
  - Unilateral (for right-sided signals): $X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$, contains impulse at zero
  - Different signals can have same LT, but different RoC!
- Inverse Laplace Transform: $x(t) = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} X(s)e^{st}ds$

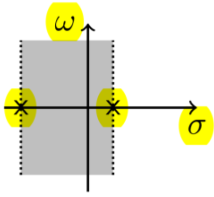
Region of Convergence (RoC)

Region of s where $\int_{-\infty}^{\infty} x(t)e^{-st}dt$ converges.

- Properties of RoC
  - Value of ω do not affect RoC. RoC **must contain strips** in s-plane.
  - If $X(s)$ is *rational* (i.e. $= \frac{P(s)}{Q(s)}$, both polynomial), RoC does not contain any pole
    - *Pole*: s s.t. $X(s) = \infty$
    - *Zero*: s s.t. $X(s) = 0$
  - $x(t)$ is finite duration and absolutely integrable $\Rightarrow$ RoC is the entire s-plane
  - $x(t)$ *right-sided*, and σ_0 in RoC $\Rightarrow$ RoC contains all s s.t. $Re(s) \geq \sigma_0$
  - $x(t)$ *left-sided*, and σ_0 in RoC $\Rightarrow$ RoC contains all s s.t. $Re(s) \leq \sigma_0$
  - $x(t)$ *two-sided* $\Rightarrow$ RoC is a strip (RoC$_R \cap$ RoC$_L$)
    - Might be empty!
- Normal Patterns
  - If $x(t)$ is *right-side* and *rational*: 

RoC $=$ **Region to the right of the rightmost pole**
  - If $x(t)$ is *left-sided* and *rational*: 

RoC $=$ **Region to the left of the leftmost pole**

- If $x(t)$ is *two-sided* and *rational*: 

RoC $=$ **Region between two consecutive poles**
- **Causal** LTI system, RoC is right-half plane
    - for $t < 0$, must have $h(t) = 0$
    - If $H(s)$ **rational, RoC is right of the right-most pole**
- **Stable** LTI system, RoC contains $j\omega$-axis
    - **Causal** LTI system, $H(s)$ rational, is **stable iff** all poles in the left to $j\omega$-axis

Properties of Laplace Transform

Property	Signal	Laplace Transform	RoC
Linearity	$ax(t) + by(t)$	$aX(s) + bY(s)$	$R_X \cap R_Y$ if no cancelation
Time Shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
s Shifting	$e^{s_0t}x(t)$	$X(s - s_0)$	R shift right by s_0
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X(\frac{s}{a})$	aR
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Convolution	$x(t) * y(t)$	$X(s) \cdot Y(s)$	At least $R_X \cap R_Y$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} X(r)Y(s-r)dr$	
Time Differenciation	$\frac{dx(t)}{dt}$	$s \cdot X(s)$	At least R
s Differenciation	$-t \cdot x(t)$	$\frac{dX(s)}{ds}$	R
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	At least $R \cap \{Re\{s\} > 0\}$

- **[Initilal & Final-Value]** For $x(t)$ is only defined on $t \geq 0$ and no impulse at $t = 0$, there is
    - $x(0^+) = \lim_{s\to\infty} s \cdot X(s)$
    - $\lim_{t\to\infty} x(t) = \lim_{s\to 0} s \cdot X(s)$
    - **Useful for checking** your LT / Inv-LT is correct

Calculation of Rational Inv-LT

Use **Partial Fraction Decomposition** and Table to easily get $x(t)$

Ex. $X(s) = \frac{4s}{(s+2)^2(s-4)}$

- Method #1: Undetermined Coefficients
    1. Decompose to partial fractions $\frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s-4}$
    2. $s = 0, -1, 1$, list equations and get A, B, C
- Method #2: Limiting Arguments
    1. Decompose to partial fractions $\frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s-4}$

2. $C = \lim_{s \to 4}(s-4) \cdot X(s) = \frac{4}{9}$
3. $B = \lim_{s \to -2}(s+2)^2 \cdot X(s) = \frac{4}{3}$, first calculate highest term's coefficient
4. $A = \lim_{s \to -2}(s+2) \cdot (X(s) - \frac{B}{(s+2)^2}) = -\frac{4}{9}$, use B to acquire A

Normal Laplace Pairs

Time	s-domain	RoC
$\delta(t)$	1	All s
$\delta(t - T_0)$	e^{-sT_0}	All s
$u(t)$	$\frac{1}{s}$	$Re\{s\} > 0$
$-u(-t)$	$\frac{1}{s}$	$Re\{s\} < 0$
$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$Re\{s\} > 0$
$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$Re\{s\} < 0$
$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$Re\{s\} > -\alpha$
$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$Re\{s\} < -\alpha$
$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$Re\{s\} > -\alpha$
$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$Re\{s\} < -\alpha$
$\cos(\omega_0 t)u(t)$	$\frac{s}{s^2+\omega_0^2}$	$Re\{s\} > 0$
$\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$Re\{s\} > 0$
$e^{-\alpha t}\cos(\omega_0 t)u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$Re\{s\} > -\alpha$
$e^{-\alpha t}\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$Re\{s\} > -\alpha$
$\frac{d^{(n)}\delta(t)}{dt^n}$	s^n	All s
$(u(t))^n$	$\frac{1}{s^n}$	$Re\{s\} > 0$

Linear Constant Coefficient (LCC) Differential Equations

- LCC differential equations can be represented by LT
  - $H(s) = \frac{Y(s)}{X(s)}$
- Need extra information like causality, stability to find the ROC and consequently the impulse response

Z-Transform (ZT)

Apply DT transform to **a larger class of signals**! $z = r \cdot e^{j\omega}$

Z-Transform Pair

- Z-Transform: $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$
    - $X(z)|_{z=e^{j\omega}} = FT\{x[n]\}$
- Inverse Z-Transform: $x[n] = \frac{1}{2\pi}\int_0^{2\pi} X(z)z^n d\omega = \frac{1}{2\pi j}\int_0^{2\pi} X(z)z^{n-1}dz$

Region of Convergence (RoC)

Region of z where $\sum_{n=-\infty}^{\infty} x[n]z^n$ converges.

- Properties of RoC
    - Value of ω do not affect RoC. RoC **must contain rings** in z-plane.
    - If $X(z)$ is *rational* (i.e. $= \frac{P(z^{-1})}{Q(z^{-1})}$, both polynomial), RoC does not contain any pole
    - $x[n]$ is finite duration $\Rightarrow$ RoC is the entire z-plane
        - If $X(z)$ contains negative powers, RoC exclude 0
        - If $X(z)$ contains positive powers, RoC exclude ∞
    - $x[n]$ *right-sided*, and r_0 in RoC $\Rightarrow$ RoC contains all z s.t. $|z| \geq r_0$
    - $x[n]$ *left-sided*, and r_0 in RoC $\Rightarrow$ RoC contains all z s.t. $|z| \leq r_0$
    - $x[n]$ *two-sided* $\Rightarrow$ RoC is a strip (RoC$_R$ $\cap$ RoC$_L$)
        - Might be empty!
- Normal Patterns
    - **Causal** LTI system, RoC is outer circle including ∞
        - for $t < 0$, must have $h[t] = 0$
        - If $H(z)$ **rational, RoC is outer of the out-most pole**
    - **Stable** LTI system, RoC contains unit circle $|z| = 1$
        - **Causal** LTI system, $H(z)$ rational, is **stable iff** all poles are within unit circle

Properties of Z-Transform

Property	Signal	Z-Transform	RoC
Linearity	$ax[n] + by[n]$	$aX(z) + bY(z)$	$R_X \cap R_Y$ if no cancelation
Time Shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R possibly add or delete zero
Scaling	$a^n x[n]$	$X(\frac{z}{a})$	aR
Time Reversal	$x[-n]$	$X(\frac{1}{z})$	$\frac{1}{R}$
Conjugation	$x^*[n]$	$X^*(z^*)$	R
Convolution	$x[n] * y[n]$	$X(z)Y(z)$	At least $R_X \cap R_Y$
z Differentiation	$-nx[n]$	$z\frac{dX(z)}{dz}$	R
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1-z^{-1}}X(z)$	At least $R \cap \lvert z\rvert > 1$

- **[Initilal-Value]** For $x[n]$ is only define on $n \geq 0$, there is

- $nx[n] = \lim_{z\to\infty} X(z)$

Calculation of Rational Inv-ZT

Same as LT! **Pay attention to z^{-1}!**

Normal Z Pairs

Time	z-domain	RoC
$\delta[n]$	1	All z
$\delta[n-m]$	z^{-m}	$z \neq 0$ if $m > 0$; $z \neq \infty$ if $m < 0$
$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
$\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
$r^n \sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$

Linear Constant Coefficient (LCC) Difference Equations

- LCC difference equations can be represented by ZT
  - $H(z) = \frac{Y(z)}{X(z)}$
- Need extra information like causality, stability to find the ROC and consequently the impulse response