Some rules of Probability and Statistics

Def P is a **probability measure** if
1. $0 \leq P(A) \leq 1$
2. $P(\Omega) = 1$
3. $A \cap B = \emptyset \;\Rightarrow\; P(A \cup B) = P(A) + P(B)$

for all events $A \subset \Omega$ where Ω is the sample space.

Basic principle of counting <u>If</u> an experiment can be divided into j subexperiments where the
first has n_1 outcomes
second has n_2 outcomes
$\vdots$
jth has n_j outcomes
<u>then</u> the experiment totally has $n_1 \cdot n_2 \cdots n_j$ outcomes.

Addition theorem $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Def A and B are **independent events** if $P(A \cap B) = P(A)P(B)$

Def The **conditional probability** of A given B is $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

Bayes theorem <u>If</u> $A_1, \ldots, A_n$ is a partition of Ω
(i.e. $i \neq j \Rightarrow A_i \cap A_j = \emptyset$ and $\bigcup_{i=1}^{n} A_i = \Omega$).
<u>then</u> $P(A_k|B) = \dfrac{P(B|A_k)P(A_k)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}$ for each $k = 1, 2, \ldots, n$.

Combinatorics The number of ways to choose k elements among n possible, without replacement and without respect to the order, is

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 2\cdot 1}$$

Random variables

X discrete: **Probability fcn**: $p(x) = P(X = x)$
Distribution fcn: $P(X \leq a) = F(a) = \sum_{x \leq a} p(x)$,

X cont.: **Density fcn**: $f(x) = \frac{d}{dx} P(X \leq x)$
Distribution fcn: $P(X \leq a) = F(a) = \int_{-\infty}^{a} f(x)\, dx$.

Expected value and variance

X discrete: The **expected value** of X: $\mu = E(X) = \sum_{x\in\Omega} x\,p(x)$.
Variance of X: $\sigma^2 = V(X) = \sum_{x\in\Omega}(x-\mu)^2\,p(x)$.

X cont: **Expected value** of X: $\mu = E(X) = \int_{x\in\Omega} x\,f(x)\,dx$.
Variance of X: $\sigma^2 = V(X) = \int_{x\in\Omega}(x-\mu)^2\,f(x)\,dx$.

Covariance of X and Y: $C(X,Y) = E((X-\mu_x)(Y-\mu_y))$
Correlation of X and Y: $\rho = \frac{C(X,Y)}{\sqrt{V(X)V(Y)}}$

Standard dev. $\sigma = \sqrt{V(X)}$.

Linearity: $E(aX+bY) = a\,E(X) + b\,E(Y)$ for all random variables X and Y and real numbers a and b.
If X, Y indep. then $V(aX+bY) = a^2V(X) + b^2V(Y)$.

Rules: $E(g(X)) = \int_{\mathbb{R}} g(x)\,f(x)\,dx$
$V(X) = E(X^2) - (E(X))^2$
$C(X,Y) = \int\int_{\mathbb{R}^2} xy\,f(x,y) - E(X)E(Y)$
$C(\sum_i a_iX_i, \sum_k b_kY_k) = \sum_i\sum_k a_ib_kC(X_i,Y_k)$

Normal distribution denoted by $\boldsymbol{N(\mu,\sigma^2)}$ where μ is the expected value and σ^2 is the variance
$N(0,1)$ is called **standard normal distribution** with density function $\Phi(x)$
<u>If</u> $X \in N(\mu,\sigma^2)$ <u>then</u> $P(X \le x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
<u>Symmetry</u>: $\Phi(-x) = 1 - \Phi(x)$
<u>Probabilities</u>: $P(a \le X \le b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$

Def The random variables $X_1, X_2, \ldots, X_n$ are a **sample** of X if all variables, X_i, is distributed as X, $i = 1,\ldots,n$, and all variables are independent of each other at all levels.

CLT (Central Limit Theorem)

<u>If</u> $X_1,\ldots,X_n$ is a sample where
$E(X_i) = \mu$ and $V(X_i) = \sigma^2$, $i = 1,\ldots,n$

<u>then</u> $P\left(\frac{\sqrt{n}}{\sigma}(\bar{X}-\mu) \le x\right) \to \Phi(x)$ as $n \to \infty$.
This implies that $\sum_{i=1}^n X_i$ is approximately $N(n\mu, n\sigma^2)$ and $\bar{X}$ is approximately $N(\mu, \sigma^2/n)$ for large n.

Approximations

Distribution	Condition	Approximative distribution
$Bin(n,\pi)$	$n \ge 10$ and $\pi < 0.1$	$Po(n\pi)$
$Bin(n,p)$	$np(1-p) \ge 10$	$N(np, np(1-p))$
$Poi(\lambda)$	$\lambda \ge 15$	$N(\lambda,\lambda)$

Inference

Def A point estimator, θ^*, of a parameter θ is **unbiased** if $E(\theta^*) = \theta$.
If θ_1^* and θ_2^* are unbiased estimators of θ, then θ_1^* is **better** (or **more efficient**) than θ_2^* om $V(\theta_1^*) < V(\theta_2^*)$.

Point estimation <u>If</u> $E(X) = \mu$, $V(X) = \sigma^2$ and $X_1, \ldots, X_n$ is a sample of X
<u>then</u> unbiased point estimators of μ and σ^2 are:

$$\hat{\mu} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$\hat{\sigma}^2 = S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \mu^2$$

if μ is known

$$\hat{\sigma}^2 = S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{1}{n-1}\left(\sum_{i=1}^{n} X_i^2 - n\,\bar{X}^2\right)$$

if μ is unknown.

Distributions, expected values and variances

	X	$p(x), f(x)$	$E(X)$	$V(X)$
Discrete distributions	Bern(p)	$px + (1-p)(1-x)$ $x = 0, 1$	p	$p(1-p)$
	Bin(n, p)	$\binom{n}{x} p^x (1-p)^{n-x}$ $x = 0, 1, 2, \ldots, n$	np	$np(1-p)$
	Poi(λ)	$e^{-\lambda}\lambda^x / x!$ $x = 0, 1, 2, \ldots$	λ	λ
	Geo(π)	$(1-\pi)^{x-1}\pi$ $x = 1, 2, 3, \ldots$	$1/\pi$	$(1-\pi)/\pi^2$
Cont. distributions	R(a, b)	$1/(b-a)$ $a \le x \le b$	$(a+b)/2$	$(a-b)^2/12$
	Exp(λ)	$\lambda e^{-\lambda x}$ $x > 0$	$1/\lambda$	$1/\lambda^2$
	N(μ, σ)	$(\sigma\sqrt{2\pi})^{-1} e^{-(x-\mu)^2/(2\sigma^2)}$ $x \in \mathbb{R}$	μ	σ^2

Confidence intervals Assume $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ are independent and normally distributed $N(\mu_X, \sigma)$ and $N(\mu_Y, \sigma)$ respectively. Then a $100(1-\alpha)\%$ confidenceinterval for the parameter θ is:

θ	*Conf. int.*	*Remark*
μ_X	$\bar{x} \pm \lambda_{\alpha/2}\frac{\sigma}{\sqrt{m}}$	σ known
μ_X	$\bar{x} \pm t_{\alpha/2}(m-1)\frac{s}{\sqrt{m}}$	σ unknown
σ^2	$\left(0\,,\ (m-1)s^2/(\chi^2_{1-\alpha}(m-1))\right)$	
$\mu_X - \mu_Y$	$\bar{\Delta} \pm t_{\alpha/2}(m+n-2)s_\Delta\sqrt{\frac{1}{m}+\frac{1}{n}}$	σ okänd $\bar{\Delta} = \bar{x} - \bar{y}$ $s^2_\Delta = \frac{(m-1)s^2_X+(n-1)s^2_Y}{m+n-2}$

Hypothesis testing Assume $X_1, \ldots, X_n$ is a sample of a random variable with distribution F with parameter θ. To test the hypothesis

$$\begin{cases} H_0 & : \ \theta = \theta_0 \quad \text{(nullhypothesis)} \\ H_1 & : \ \theta \in \Theta \quad \text{(alternative hypothesis)} \end{cases}$$

the test statistic $U = U(X_1, \ldots, X_n)$ is used, and the critical region C_α which corresponds to Θ according to the distribution F_U of U under H_0 at level α of significance. The test rule is

$$\begin{cases} \text{Reject } H_0 \text{ if } u \in C_\alpha \\ \text{Do not reject } H_0 \text{ if } u \notin C_\alpha \end{cases}$$

θ	H_0	H_1	u	F_U	C_α
π if $n\pi_0(1-\pi_0) > 5$	$\pi = \pi_0$	$\pi < \pi_0$	$\frac{\sqrt{n}(p-\pi_0)}{\sqrt{\pi_0(1-\pi_0)}}$	$N(0,1)$	$u < -\lambda_\alpha$
		$\pi > \pi_0$			$u > \lambda_\alpha$
		$\pi \neq \pi_0$			$\lvert u\rvert > \lambda_{\alpha/2}$
π_1, π_2 if $n_1p_1(1-p_1)>5$ and $n_2p_2(1-p_2)>5$	$\pi_1 = \pi_2$	$\pi_1 < \pi_2$	$\frac{p_1 - p_2}{\sqrt{p(1-p)(\frac{1}{n_1}+\frac{1}{n_2})}}$	$N(0,1)$	$u < -\lambda_\alpha$
		$\pi_1 > \pi_2$			$u > \lambda_\alpha$
		$\pi_1 \neq \pi_2$			$\lvert u\rvert > \lambda_{\alpha/2}$
μ if σ is known	$\mu = \mu_0$	$\mu < \mu_0$	$\frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma}$	$N(0,1)$	$\{u < -\lambda_\alpha\}$
		$\mu > \mu_0$			$\{u > \lambda_\alpha\}$
		$\mu \neq \mu_0$			$\{\lvert u\rvert > \lambda_{\alpha/2}\}$
μ if σ is unknown	$\mu = \mu_0$	$\mu < \mu_0$	$\frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma}$	$t(n-1)$	$\{u < -t_\alpha(n-1)\}$
		$\mu > \mu_0$			$\{u > t_\alpha(n-1)\}$
		$\mu \neq \mu_0$			$\{\lvert u\rvert > t_{\alpha/2}(n-1)\}$
σ	$\sigma = \sigma_0$	$\sigma < \sigma_0$	$\frac{(n-1)s^2}{\sigma_0^2}$	$\chi^2(n-1)$	$\{u < \chi^2_{1-\alpha}\}$
		$\sigma > \sigma_0$			$\{u > \chi^2_\alpha\}$
		$\sigma \neq \sigma_0$			$\{u < \chi^2_{1-\alpha/2}\} \cup \{u > \chi^2_{\alpha/2}\}$
$\mu_X - \mu_Y$ if σ is known	$\mu_X = \mu_Y$	$\mu_X < \mu_Y$	$\frac{\bar{x}-\bar{y}}{\sigma\sqrt{\frac{1}{m}+\frac{1}{n}}}$	$N(0,1)$	$\{u < -\lambda_\alpha\}$
		$\mu_X > \mu_Y$			$\{u > \lambda_\alpha\}$
		$\mu_X \neq \mu_Y$			$\{\lvert u\rvert > \lambda_{\alpha/2}\}$
$\mu_X - \mu_Y$ if σ is unknown	$\mu_X = \mu_Y$	$\mu_X < \mu_Y$	$\frac{\bar{x}-\bar{y}}{s_p\sqrt{\frac{1}{m}+\frac{1}{n}}}$	$t(m+n-2)$	$\{u < -t_{\alpha/2}(m+n-2)\}$
		$\mu_X > \mu_Y$			$\{u > t_\alpha(m+n-2)\}$
		$\mu_X \neq \mu_Y$			$\{\lvert u\rvert > t_{\alpha/2}(m+n-2)\}$
F where $E_i = NP(X \in I_i \lvert H_0)$	$F = F_0$	$F \neq F_0$	$\sum_i \frac{(O_i - E_i)^2}{E_i}$	$\chi^2(k-1)$	$\{u > \chi^2_\alpha(k-1)\}$

Assume $X_1, \ldots, X_n$ is a sample of $X \in Bern(p)$. For testing H_0 against H_1, reject H_0 at level α of significance if $\alpha_0 < \alpha$ where

H_0	H_1	u	F_U	α_0
$p = p_0$	$p < p_0$	$\sum_i x_i$	$\mathrm{Bin}(n, p_0)$	$P(U < u \mid H_0)$
	$p > p_0$			$P(U > u \mid H_0)$
	$p \neq p_0$			$\begin{cases} 2P(U < u \mid H_0) & \text{om } u < np_0 \\ 2P(U > u \mid H_0) & \text{om } u > np_0 \end{cases}$

Type I error is to reject H_0 when H_0 is true. $P(\text{Type I error}) = \alpha$.
Type II error is to not reject H_0 when H_1 är true. $P(\text{Type II errorl}) = \beta$.
Test **power** is the probability to reject H_0 when H_1 is true, i.e. $1 - \beta$.

Normal distribution values

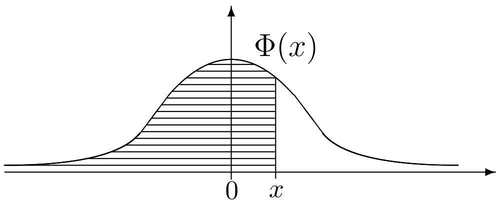

Table over values of $\Phi(x) = P(X \leq x)$ where $X \in N(0,1)$. For $x < 0$, use the relation $\Phi(x) = 1 - \Phi(-x)$.

x	+0.00	+0.01	+0.02	+0.03	+0.04	+0.05	+0.06	+0.07	+0.08	+0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

x	+0.0	+0.1	+0.2	+0.3	+0.4	+0.5	+0.6	+0.7	+0.8	+0.9
3	0.9987	0.9990	0.9993	0.9995	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000

Percentiles:

Some values of λ_α such that $P(X > \lambda_\alpha) = \alpha$ where $X \in N(0,1)$

α	λ_α	α	λ_α
0.1	1.281552	0.005	2.575829
0.05	1.644854	0.001	3.090232
0.025	1.959964	0.0005	3.290527
0.01	2.326348	0.0001	3.719016

t percentiles

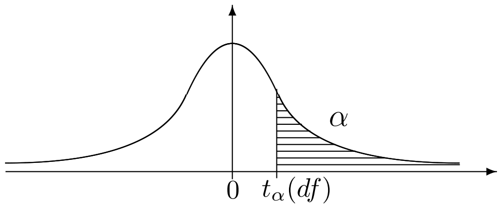

Table over values of $t_\alpha(df)$.

df \ α	0.25	0.10	0.05	0.025	0.02	0.01	0.005	0.001
1	1.0000	3.0777	6.3138	12.7062	15.8945	31.8205	63.6567	318.3088
2	0.8165	1.8856	2.9200	4.3027	4.8487	6.9646	9.9248	22.3271
3	0.7649	1.6377	2.3534	3.1824	3.4819	4.5407	5.8409	10.2145
4	0.7407	1.5332	2.1318	2.7764	2.9986	3.7470	4.6041	7.1732
5	0.7267	1.4759	2.0150	2.5706	2.7565	3.3649	4.0322	5.8934
6	0.7176	1.4398	1.9432	2.4469	2.6122	3.1427	3.7074	5.2076
7	0.7111	1.4149	1.8946	2.3646	2.5168	2.9980	3.4995	4.7853
8	0.7064	1.3968	1.8595	2.3060	2.4490	2.8965	3.3554	4.5008
9	0.7027	1.3830	1.8331	2.2622	2.3984	2.8214	3.2498	4.2968
10	0.6998	1.3722	1.8125	2.2281	2.3593	2.7638	3.1693	4.1437
12	0.6955	1.3562	1.7823	2.1788	2.3027	2.6810	3.0545	3.9296
14	0.6924	1.3450	1.7613	2.1448	2.2638	2.6245	2.9768	3.7874
17	0.6892	1.3334	1.7396	2.1098	2.2238	2.5669	2.8982	3.6458
20	0.6870	1.3253	1.7247	2.0860	2.1967	2.5280	2.8453	3.5518
25	0.6844	1.3163	1.7081	2.0595	2.1666	2.4851	2.7874	3.4502
30	0.6828	1.3104	1.6973	2.0423	2.1470	2.4573	2.7500	3.3852
50	0.6794	1.2987	1.6759	2.0086	2.1087	2.4033	2.6778	3.2614
100	0.6770	1.2901	1.6602	1.9840	2.0809	2.3642	2.6259	3.1737

χ^2 percentiles

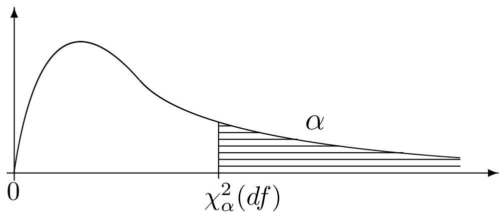

Table over values of $\chi^2_\alpha(df)$.

df \ α	0.999	0.995	0.99	0.95	0.05	0.01	0.005	0.001
1	0.0000	0.0000	0.0002	0.0039	3.8415	6.6349	7.8794	10.8276
2	0.0020	0.0100	0.0201	0.1026	5.9915	9.2103	10.5966	13.8155
3	0.0243	0.0717	0.1148	0.3518	7.8147	11.3449	12.8382	16.2662
4	0.0908	0.2070	0.2971	0.7107	9.4877	13.2767	14.8603	18.4668
5	0.2102	0.4117	0.5543	1.1455	11.0705	15.0863	16.7496	20.5150
6	0.3811	0.6757	0.8721	1.6354	12.5916	16.8119	18.5476	22.4577
7	0.5985	0.9893	1.2390	2.1673	14.0671	18.4753	20.2777	24.3219
8	0.8571	1.3444	1.6465	2.7326	15.5073	20.0902	21.9550	26.1245
9	1.1519	1.7349	2.0879	3.3251	16.9190	21.6660	23.5894	27.8772
10	1.4787	2.1559	2.5582	3.9403	18.3070	23.2093	25.1882	29.5883
12	2.2142	3.0738	3.5706	5.2260	21.0261	26.2170	28.2995	32.9095
14	3.0407	4.0747	4.6604	6.5706	23.6848	29.1412	31.3193	36.1233
17	4.4161	5.6972	6.4078	8.6718	27.5871	33.4087	35.7185	40.7902
20	5.9210	7.4338	8.2604	10.8508	31.4104	37.5662	39.9968	45.3147
25	8.6493	10.5197	11.5240	14.6114	37.6525	44.3141	46.9279	52.6197
30	11.5880	13.7867	14.9535	18.4927	43.7730	50.8922	53.6720	59.7031
50	24.6739	27.9907	29.7067	34.7643	67.5048	76.1539	79.4900	86.6608
100	61.9179	67.3276	70.0649	77.9295	124.342	135.807	140.169	149.449

Values of the Poisson distribuion

Table over values of $F(x) = P(X \leq x)$ where $X \in Po(\lambda)$.

λ	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0.5	0.607	0.910	0.986	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1	0.368	0.736	0.920	0.981	0.996	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	0.135	0.406	0.677	0.857	0.947	0.983	0.995	0.999	1.000	1.000	1.000	1.000	1.000	1.000
3	0.050	0.199	0.423	0.647	0.815	0.916	0.966	0.988	0.996	0.999	1.000	1.000	1.000	1.000
4	0.018	0.092	0.238	0.433	0.629	0.785	0.889	0.949	0.979	0.992	0.997	0.999	1.000	1.000
5	0.007	0.040	0.125	0.265	0.440	0.616	0.762	0.867	0.932	0.968	0.986	0.995	0.998	0.999
6	0.002	0.017	0.062	0.151	0.285	0.446	0.606	0.744	0.847	0.916	0.957	0.980	0.991	0.996

Values of the Binomial distribution

Table over values of $P(x) = P(X \leq x)$ where $X \in Bin(n, p)$.
For $p > 0.5$, use the relation $P(X \leq x) = P(Y \geq n-x)$ where $Y \in Bin(n, 1-p)$.

n	p	0	1	2	3	4	5	6	7	8	9	10
3	0.1	0.729	0.972	0.999	1.000	–	–	–	–	–	–	–
	0.2	0.512	0.896	0.992	1.000	–	–	–	–	–	–	–
	0.3	0.343	0.784	0.973	1.000	–	–	–	–	–	–	–
	0.4	0.216	0.648	0.936	1.000	–	–	–	–	–	–	–
	0.5	0.125	0.500	0.875	1.000	–	–	–	–	–	–	–
4	0.1	0.656	0.948	0.996	1.000	1.000	–	–	–	–	–	–
	0.2	0.410	0.819	0.973	0.998	1.000	–	–	–	–	–	–
	0.3	0.240	0.652	0.916	0.992	1.000	–	–	–	–	–	–
	0.4	0.130	0.475	0.821	0.974	1.000	–	–	–	–	–	–
	0.5	0.062	0.312	0.688	0.938	1.000	–	–	–	–	–	–
5	0.1	0.590	0.919	0.991	1.000	1.000	1.000	–	–	–	–	–
	0.2	0.328	0.737	0.942	0.993	1.000	1.000	–	–	–	–	–
	0.3	0.168	0.528	0.837	0.969	0.998	1.000	–	–	–	–	–
	0.4	0.078	0.337	0.683	0.913	0.990	1.000	–	–	–	–	–
	0.5	0.031	0.188	0.500	0.812	0.969	1.000	–	–	–	–	–
6	0.1	0.531	0.886	0.984	0.999	1.000	1.000	1.000	–	–	–	–
	0.2	0.262	0.655	0.901	0.983	0.998	1.000	1.000	–	–	–	–
	0.3	0.118	0.420	0.744	0.930	0.989	0.999	1.000	–	–	–	–
	0.4	0.047	0.233	0.544	0.821	0.959	0.996	1.000	–	–	–	–
	0.5	0.016	0.109	0.344	0.656	0.891	0.984	1.000	–	–	–	–
7	0.1	0.478	0.850	0.974	0.997	1.000	1.000	1.000	1.000	–	–	–
	0.2	0.210	0.577	0.852	0.967	0.995	1.000	1.000	1.000	–	–	–
	0.3	0.082	0.329	0.647	0.874	0.971	0.996	1.000	1.000	–	–	–
	0.4	0.028	0.159	0.420	0.710	0.904	0.981	0.998	1.000	–	–	–
	0.5	0.008	0.062	0.227	0.500	0.773	0.938	0.992	1.000	–	–	–
8	0.1	0.430	0.813	0.962	0.995	1.000	1.000	1.000	1.000	1.000	–	–
	0.2	0.168	0.503	0.797	0.944	0.990	0.999	1.000	1.000	1.000	–	–
	0.3	0.058	0.255	0.552	0.806	0.942	0.989	0.999	1.000	1.000	–	–
	0.4	0.017	0.106	0.315	0.594	0.826	0.950	0.991	0.999	1.000	–	–
	0.5	0.004	0.035	0.145	0.363	0.637	0.855	0.965	0.996	1.000	–	–
9	0.1	0.387	0.775	0.947	0.992	0.999	1.000	1.000	1.000	1.000	1.000	–
	0.2	0.134	0.436	0.738	0.914	0.980	0.997	1.000	1.000	1.000	1.000	–
	0.3	0.040	0.196	0.463	0.730	0.901	0.975	0.996	1.000	1.000	1.000	–
	0.4	0.010	0.071	0.232	0.483	0.733	0.901	0.975	0.996	1.000	1.000	–
	0.5	0.002	0.020	0.090	0.254	0.500	0.746	0.910	0.980	0.998	1.000	–
10	0.1	0.349	0.736	0.930	0.987	0.998	1.000	1.000	1.000	1.000	1.000	1.000
	0.2	0.107	0.376	0.678	0.879	0.967	0.994	0.999	1.000	1.000	1.000	1.000
	0.3	0.028	0.149	0.383	0.650	0.850	0.953	0.989	0.998	1.000	1.000	1.000
	0.4	0.006	0.046	0.167	0.382	0.633	0.834	0.945	0.988	0.998	1.000	1.000
	0.5	0.001	0.011	0.055	0.172	0.377	0.623	0.828	0.945	0.989	0.999	1.000

Trigonometrics

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\cos\alpha = \frac{1}{2}(e^{i\alpha} + e^{-i\alpha}) \qquad \sin\alpha = \frac{1}{2i}(e^{i\alpha} - e^{-i\alpha}) \qquad \tan\alpha = \frac{e^{i\alpha} - e^{i\alpha}}{i(e^{i\alpha} + e^{-i\alpha})}$

Some special sums and series

Binomial theorem: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

If $|a| < 1$ then $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$, $\sum_{k=1}^{\infty} k a^k = \frac{a}{(1-a)^2}$ and $\sum_{k=1}^{\infty} \frac{a^k}{k} = -\ln(1-a)$

Taylor series: $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \quad \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$