STOKASTISKA PROCESSER, FMS 041, VT-01
LÖSNINGAR TILL ÖVNINGSUPPGIFTER I STATIONÄRA STOKASTISKA PROCESSER

Kapitel 2: Stationära processer

201. $\rho(X_1, X_2) = \frac{C(X_1,X_2)}{\sqrt{V(X_1)V(X_2)}} = -\frac{4.5}{\sqrt{5 \cdot 10}} = -0.636.$

202. a. $E(X_1 - 2X_2) = 1 - 2 = -1$
     b. $C(X_1 - 2X_2, X_1 - 2X_2 + X_3) = V(X_1) + 4V(X_2) = 5$

203. $m_X(t) = E(1.2e_t + 0.9e_{(t-1)}) = 2.1m$

$r_X(s,t) = C(1.2e_s + 0.9e_{s-1}, 1.2e_t + 0.9e_{t-1}) =$

$$= \begin{cases} (1.2^2 + 0.9^2)\sigma^2 = 2.25\sigma^2, & s = t \\ 1.2 \cdot 0.9\sigma^2 = 1.08\sigma^2, & s = t \pm 1 \\ 0, & |s-t| \geq 2 \end{cases}$$

204. (a) ty $r_X(1)$ tycks vara positiv och $r_X(-3)$ tycks vara negativ (??).

205.

$$\begin{aligned} m_Y(k) &= E(X_1 + \ldots + X_k) = 1500k \\ r_Y(k,j) &= C(X_1 + \ldots + X_k, X_1 + \ldots + X_j) \overset{k \leq j}{=} \\ &= V(X_1) + \ldots + V(X_k) = k \cdot 10^4 \end{aligned}$$

Symmetri $\Rightarrow$ $r_Y(k,j) = 10^4 \cdot \min(k,j)$. $V(Y_k) = 10^4 k$, Y_k är approximativt fördelad som $N(1500k, 100\sqrt{k})$ om k är stort.

206.

$$\begin{aligned} r(s,t) &= C(X(s), X(t)) \overset{s \leq t}{=} C(X(s), X(s) + \underbrace{X(t) - X(s)}_{\text{oberoende av} X(s)}) \\ &= C(X(s), X(s)) = V(X(s)) = \lambda s \end{aligned}$$

Symmetri $\Rightarrow r(s,t) = \lambda \cdot \min(s,t)$.

$\rho(s,t) = \frac{r(s,t)}{\sqrt{r(s,s) \cdot r(t,t)}} = \frac{\lambda \min(s,t)}{\sqrt{\lambda s \lambda t}} = \min(\sqrt{s/t}, \sqrt{t/s})$.

207.

$$\begin{aligned} r_Y(1,2) &= C(Y_1, Y_2) = C(X_1, 0.5X_1 + X_2) \\ &= 0.5C(X_1, X_1) + C(X_1, X_2) = 0.5\sigma^2 + 0 = 0.5\sigma^2 \\ r_Y(2,3) &= C(Y_2, Y_3) = C(0.5X_1 + X_2, 0.5(0.5X_1 + X_2) + X_3) \\ &= C(0.5X_1 + X_2, 0.25X_1 + 0.5X_2 + X_3) \\ &= 0.125V(X_1) + 0 + 0 + 0 + 0.5V(X_2) + 0 \\ &= \sigma^2(0.5^3 + 0.5) = 0.625\sigma^2 \\ r_Y(1,2) &\neq r_Y(2,3) \Rightarrow Y_k \quad \text{är icke-stationär} \\ r_Y(3,4) &= C(Y_3, Y_4) \\ &= C(0.25\,X_1 + 0.5\,X_2 + X_3, 0.125\,X_1 + 0.25\,X_2 + 0.5\,X_3 + X_4) \\ &= E(0.25 \cdot 0.125\,X_1^2) + E(0.5 \cdot 0.25\,X_2^2) + E(0.5\,X_3^2) \\ &= \frac{21}{32}\sigma^2 = 0.65625\,\sigma^2 \end{aligned}$$

208.
$$
\begin{aligned}
C(Y(n), Z(n+\tau)) &= C(X(n) + X(n-1), \\
&\quad 3X(n+\tau) - 2X(n-1+\tau) + X(n-2+\tau)) \\
&= \begin{cases} 3\sigma^2 - 2\sigma^2 = \sigma^2, & \tau = 0 \\ -2\sigma^2 + \sigma^2 = -\sigma^2, & \tau = 1 \\ \sigma^2 = \sigma^2, & \tau = 2 \\ 3\sigma^2 = 3\sigma^2, & \tau = -1 \\ 0 = 0, & \text{för övrigt.} \end{cases}
\end{aligned}
$$

209. a. $E(Y(t)) = \lambda(t+1) - \lambda t = \lambda.$

$$
\begin{aligned}
r_Y(s,t) &= C(X(s+1) - X(s), X(t+1) - X(t)) = \\
&= \begin{cases} 0 \text{ om } s+1 \leq t \text{ (oberoende ökningar)} \\ C((X(s+1) - X(t)) + (X(t) - X(s)), \\ \quad (X(t+1) - X(s+1) + (X(s+1) - X(t))) \\ \quad = C(X(s+1) - X(t), X(s+1) - X(t)) \\ \quad = s+1-t \text{ om } s \leq t \leq s+1 \\ t+1-s \text{ om } t \leq s \leq t+1 \\ 0 \text{ om } t+1 \leq s \end{cases}
\end{aligned}
$$

c. Ja, processen är svagt stationär.

210. Nej, ty $V(Y(t)) = V(X(t)) = \lambda t$

211. a. $m_{X+Y}(t) = E(X(t) + Y(t)) = E(X(t)) + E(Y(t)) = m_X + m_Y$

b.

$$
\begin{aligned}
r_{X+Y}(s,t) &= C(X(s) + Y(s), X(t) + Y(t)) \\
&= r_X(s,t) + r_Y(s,t) + C(X_s, Y_t) + C(X_t, Y_s) \\
&= r_X(s,t) + r_Y(s,t)
\end{aligned}
$$

Slutsats: sann.

c. $a + b \Rightarrow X_t + Y_t$ är svagt stationär.

$$
\begin{aligned}
\rho_{X+Y}(\tau) &= \frac{r_{X+Y}(\tau)}{r_{X+Y}(0)} = \frac{r_X(\tau) + r_Y(\tau)}{r_X(0) + r_Y(0)} \\
&\overset{\text{i allmänhet}}{\neq} \frac{r_X(\tau)}{r_X(0)} + \frac{r_Y(\tau)}{r_Y(0)} = \rho_X(\tau) + \rho_Y(\tau)
\end{aligned}
$$

$\Rightarrow$ ej alltid sann.

d. $r_{X-Y}(s,t) = C(X_s - Y_s, X_t - Y_t) = r_X(s,t) + r_Y(s,t) \Rightarrow$ ej alltid sann.

212. $m_X(t) = E(Y \sin t) = \sin t \cdot E(Y) = 0$

$r_X(s,t) = C(Y \sin s, Y \sin t) = E(Y^2) \sin s \cdot \sin t = \sin s \cdot \sin t$ ty $E(Y^2) = 1$

Slutsats: $V(X(t)) = (\sin t)^2$, är tidsvariabel $\Rightarrow X(t)$ är ej svagt stationär.

213. a.

$$
\begin{aligned}
Y(t) &= Y_1 \sin(2\pi f t) + Y_2 \cos(2\pi f t), \quad 0 \leq t \leq \tau \\
E(Y(t)) &= E(Y_1) \sin(2\pi f t) + E(Y_2) \cos(2\pi f t) = 0 \\
C(Y(s), Y(t)) &= C(Y_1 \sin(2\pi f s) + Y_2 \cos(2\pi f s),
\end{aligned}
$$

$$
\begin{aligned}
& Y_1 \sin(2\pi f t) + Y_2 \cos(2\pi f t)) \\
= \;& \sin(2\pi f s) \cdot \sin(2\pi f t) \cdot V(Y_1) + 0 + 0 \\
& + \cos(2\pi f s) \cdot \cos(2\pi f t) \cdot V(Y_2) \\
= \;& \left\{ V(Y_1) = V(Y_2) = \frac{1}{3} \right\} = \frac{1}{3} \cdot \cos(2\pi f(s-t))
\end{aligned}
$$

b. Ja, svagt stationär.

Kapitel 3: Spektralframställning

301. a. $R(f) = \sqrt{\frac{\pi}{\alpha}} \cdot \exp(-\frac{(2\pi f)^2}{4\alpha})$

b. $R(f) = \frac{\alpha}{\alpha^2+(\omega_0-2\pi f)^2} + \frac{\alpha}{\alpha^2+(\omega_0+2\pi f)^2}$

c.

$$
\begin{aligned}
R(f) &= \alpha_0^2 \cdot \delta(f) + \frac{\alpha_1^2}{2} \cdot \mathcal{F}\{\frac{1}{2} \cdot e^{i2\pi f_0 \tau} + \frac{1}{2} \cdot e^{-i2\pi f_0 \tau}\} \\
&= \alpha_0^2 \cdot \delta(f) + \frac{\alpha_1^2}{4}(\delta(f-f_0) + \delta(f+f_0))
\end{aligned}
$$

d. $R(f) = \{\text{tabell}\} = \pi \cdot e^{-2\pi|f|}$

302. a - g
b - f
c - e
d - h

303. $\frac{A}{(1+(2\pi f)^2)\cdot(4+(2\pi f)^2)} = \frac{A}{3} \cdot \left(\frac{1}{1+(2\pi f)^2} - \frac{1}{4+(2\pi f)^2}\right) \overset{\mathcal{F}^{-1}}{\longmapsto} \frac{A}{3}\left(\frac{1}{2}e^{-|\tau|} - \frac{1}{4}e^{-2|\tau|}\right) = r(\tau)$

304. 1: b och c   2: a och d

305.

$$
\begin{aligned}
R(f) &= 2.3 + 0.6\cos 2\pi f - 1.6\cos 4\pi f, \quad -\frac{1}{2} < f \le \frac{1}{2} \\
R'(f) &= 1.6 \cdot 4\pi \cdot \sin 4\pi f - 0.6 \cdot 2\pi \cdot \sin 2\pi f, \quad -\frac{1}{2} < f \le \frac{1}{2} \\
R'(f) &= 0 \Rightarrow \sin 2\pi f = \frac{6.4}{0.6} \cdot \sin 2\pi f \cdot \cos 2\pi f, \quad -\frac{1}{2} < f \le \frac{1}{2} \\
&\Rightarrow f = 0 \text{ eller } f \approx 0.24; \quad \max \quad \text{för } f \approx 0.24.
\end{aligned}
$$

306.

$$
\begin{aligned}
Y(n) &= 2 \cdot X(n) + X(n-1) - 2 \cdot X(n-2) \\
m_Y(n) &= 2 \cdot m + m - 2 \cdot m = m \\
r_Y(k,j) &= C(2X_k + X_{k-1} - 2X_{k-2}, 2X_j + X_{j-1} - 2X_{j-2}) \\
&\overset{k=j}{=} r_X(0) \cdot (2 \cdot 2 + 1 \cdot 1 + 2 \cdot 2) = 9 \cdot \sigma^2 \\
&\overset{k=j-1}{=} r_X(0) \cdot (2 \cdot 1 - 1 \cdot 2) = 0 \\
&\overset{k=j-2}{=} r_X(0) \cdot (-2 \cdot 2) = -4 \cdot \sigma^2 \\
&\overset{k<j-2}{=} 0
\end{aligned}
$$

Slutsats:

$$r_Y(\tau) = \begin{cases} 9 \cdot \sigma^2 & \tau = 0 \\ 0 & \tau = \pm 1 \\ -4 \cdot \sigma^2 & \tau = \pm 2 \\ 0 & \text{för övrigt} \end{cases}$$

$$\sum |r(\tau)| < \infty \Rightarrow R(f) = \sum_{-\infty}^{\infty} r(\tau) \cdot e^{-i2\pi f \cdot \tau}$$

Slutsats:

$R_Y(f) = 9 \cdot \sigma^2 - 4\sigma^2(e^{i2\pi f \cdot 2} + e^{-i2\pi f \cdot 2}) = \sigma^2 \cdot (9 - 8 \cdot \cos(4\pi f));\ \frac{1}{2} < f \leq \frac{1}{2}$

307. $R_X(f) = \sum_{k=-\infty}^{\infty} e^{-i2\pi fk} r_X(k) = A + 2B \cdot \cos(4\pi f), -\frac{1}{2} < f \leq \frac{1}{2}$. $R_X(f)$ ska vara:

1) symmetrisk (sant!)
2) integrerbar (sant!)
3) icke-negativ, dvs

$$A + 2B \geq 0 \text{ och } A - 2B \geq 0$$

eftersom $\cos(4\pi f)$ varierar mellan -1 och $+1$ då f går från $-1/2$ till $+1/2$. Kravet kan alternativt skrivas $A \geq 2|B|$.

308.

$$\begin{aligned} R(f) &= \int_{-\infty}^{\infty} g(\tau) e^{-i2\pi f\tau} d\tau = \ldots = \frac{1}{(\pi f)^2}\left(\frac{\sin(2\pi f)}{2\pi f} - \cos(2\pi f)\right) \\ R(1) &= -1/\pi^2 < 0 \Rightarrow R \text{ ej spektraltäthet} \Rightarrow g(\tau) \text{ ej kovariansfunktion} \end{aligned}$$

309. a. För att undvika aliaseffekt måste $R_X(f)$ vara koncentrerad till intervallet $(-\frac{1}{2d}, \frac{1}{2d})$, dvs $\frac{1}{2d} \geq 1 \Rightarrow d \leq \frac{1}{2}$.

b. X_s = samplad process med $T_s = 1 = d$:

$$\begin{aligned} R_{X_s}(f) &= \frac{1}{d} \sum_{k=-\infty}^{\infty} R_X\left(\frac{f+k}{d}\right) = \left(\text{för } -\frac{1}{2} < f \leq \frac{1}{2}\right) \\ &= \begin{cases} \cos^2(f \cdot \frac{\pi}{2}) + \cos^2((f-1)\frac{\pi}{2}) & f \geq 0 \\ \cos^2(f \cdot \frac{\pi}{2}) + \cos^2((f+1)\frac{\pi}{2}) & f \leq 0 \end{cases} \\ &= \begin{cases} \cos^2(f \cdot \frac{\pi}{2}) + \sin^2(f \cdot \frac{\pi}{2}) = 1 & f \geq 0 \\ \sin^2((f+1)\frac{\pi}{2}) + \cos^2((f+1)\frac{\pi}{2}) = 1 & f \leq 0 \end{cases} \\ &= 1 \text{ för alla } |f| \leq \frac{1}{2}. \end{aligned}$$

Med $T_s = \frac{1}{5} = d$ blir $R_{X_s}(f) = R_X(f)$.

310. Man har $r_Y(\tau) = \frac{2}{1+(2\pi\tau)^2}$ och $X_k = Y(kd)$. $g(\tau) = \frac{2}{1+\tau^2} \Rightarrow G(f) = 2\pi \cdot e^{-2\pi|f|}$. $r_X(\tau) = g(2\pi\tau)$ ger

$$\begin{aligned} R_Y(f) &= \frac{1}{2\pi} \cdot 2\pi \cdot e^{-2\pi \cdot |f|/2\pi} = e^{-|f|} \\ R_X(f) &\overset{|f| \leq \frac{1}{2}}{=} \frac{1}{d} \sum_{k=-\infty}^{\infty} R_Y\left(\frac{f+k}{d}\right) = \frac{1}{d} \sum_{k=-\infty}^{\infty} e^{-|f+k|/d} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{d}\left(e^{-|f|/d}+\sum_{k=1}^{\infty}e^{-\frac{k}{d}-\frac{f}{d}}+\sum_{k=-\infty}^{-1}e^{\frac{k}{d}+\frac{f}{d}}\right)\\
&= \frac{1}{d}\cdot\left(e^{-|f|/d}+e^{-\frac{f}{d}}\cdot\frac{e^{-1/d}}{1-e^{-1/d}}+e^{\frac{f}{d}}\cdot\frac{e^{-1/d}}{1-e^{-1/d}}\right)\\
&= \frac{1}{d}\left(e^{-|f|/d}+\frac{1}{e^{1/d}-1}\left(e^{\frac{f}{d}}+e^{-\frac{f}{d}}\right)\right),\quad -\frac{1}{2}<f\le\frac{1}{2}
\end{aligned}
$$

311. Distorderad effekt $\approx 4\times$ (en svansyta). (Faktorn 4 motiveras av att vikningseffekten påverkar både de höga och de låga frekvenserna genom att ta bort effekt från de höga och ge till de låga.)

Svansyta $= \int_{1/2d}^{\infty} R_Y(f)df = \int_{1/2d}^{\infty} e^{-2\pi f}df = \left(-\frac{1}{2\pi}e^{-2\pi f}\right)_{1/2d}^{\infty} = \frac{1}{2\pi}e^{-\pi/d}$.

Hela effekten $= 2\cdot\int_0^{\infty} e^{-2\pi f}df = \frac{1}{\pi}$

Slutsats: $\frac{4}{2\pi}e^{-\pi/d} \le 0.001\cdot\frac{1}{\pi} \Leftrightarrow e^{\pi/d}\ge 2000 \Leftrightarrow \frac{\pi}{d}\ge \ln\ 2000 \Leftrightarrow d\le\frac{\pi}{\ln 2000} = 0.413$

Kapitel 4: Normalprocesser

401. $X(t)\in N(0,1)\Rightarrow P(X(t)>2) = 1-\Phi(2) = 0.02275$.

$$
\begin{aligned}
V(X(t)+X(t+\frac{1}{2})) &= V(X(t))+2C(X(t),X(t+\frac{1}{2}))+V(X(t+\frac{1}{2}))\\
&= 1+2\cdot\frac{1}{2}+1 = 3
\end{aligned}
$$

$\Rightarrow X(t)+X(t+\frac{1}{2})\in N(0,\sqrt{3})$. $P(X(t)+X(t+\frac{1}{2})>2) = 1-\Phi(\frac{2}{\sqrt{3}}) = 0.124$.

$V(X(t)+X(t+1)) = V(X(t))+V(X(t+1)) = 2$ ty variablerna är oberoende $\Rightarrow X(t)+X(t+1)\in N(0,\sqrt{2})$. $P(X(t)+X(t+1)>2) = 1-\Phi\left(\frac{2}{\sqrt{2}}\right) = 0.079$.

402. $\{Y(t)\}$ är en normalprocess ty linjärkombinationer är normalfördelade.

Kovariansfunktionen för $Y(t)$ är

$$
\begin{aligned}
r_Y(s,t) &= C(X(s)-0.4X(s-2),X(t)-0.4X(t-2))\\
&= r(t-s)+0.16r(t-s)-0.4r(t-s-2)-0.4r(t-s+2),
\end{aligned}
$$

och den beror bara av $t-s$. Dessutom är $m_Y(t)$ konstant, så att Y-processen är svagt stationär. Eftersom den är en normalprocess är den också strikt stationär.

403. Sätt $Y = X_1/2 - X_2 + X_3/2$. Man får $E(Y) = 0$ och

$$
\begin{aligned}
V(Y) &= C\left(\frac{X_1}{2}-X_2+\frac{X_3}{2},\frac{X_1}{2}-X_2+\frac{X_3}{2}\right)\\
&= r(0)\left(\frac{1}{4}+1+\frac{1}{4}\right)+r(1)\left(-\frac{1}{2}\cdot 2-\frac{1}{2}\cdot 2\right)+r(2)\left(2\cdot\frac{1}{4}\right)\\
&= 3-3+\frac{3}{5} = 0.6.
\end{aligned}
$$

Slutsats: $Y\in N(0,\sqrt{0.6})$ och $P(Y<-1) = \Phi(-\frac{1}{\sqrt{0.6}}) = \Phi(-1.291)\approx 0.0985$.

404. $m_Z(t) = E\left(\frac{Y_t - Y_{t/2}}{\sqrt{t}}\right) = 0$

$$\begin{aligned}
r_Z(s,t) &= C\left(\frac{Y_s - Y_{s/2}}{\sqrt{s}}, \frac{Y_t - Y_{t/2}}{\sqrt{t}}\right) \\
&= \frac{1}{\sqrt{s \cdot t}}\left(r(s,t) - r(s,\frac{t}{2}) - r(\frac{s}{2},t) + r(\frac{s}{2},\frac{t}{2})\right) \\
&= \begin{cases} \frac{1}{\sqrt{s\cdot t}}\left(s - s - \frac{s}{2} + \frac{s}{2}\right) = 0 & \text{för } s \leq \frac{t}{2} \\ \frac{1}{\sqrt{s\cdot t}}\left(s - \frac{t}{2} - \frac{s}{2} + \frac{s}{2}\right) = \frac{1}{\sqrt{st}}\left(s - \frac{t}{2}\right) & \text{för } \frac{t}{2} \leq s \leq t \\ \frac{1}{\sqrt{st}}\left(t - \frac{t}{2} - \frac{s}{2} + \frac{t}{2}\right) = \frac{1}{\sqrt{st}}\left(t - \frac{s}{2}\right) & \text{för } t \leq s \leq 2t \\ 0 & \text{för } s > 2t \end{cases}
\end{aligned}$$

Slutsats: processen är ej svagt stationär.

Speciellt: $r(t,t) = V(Z(t)) = \frac{t/2}{\sqrt{st}} \overset{s=t}{=} 1/2.$

Slutsats: $Z(t) \in N\left(0, \sqrt{\frac{1}{2}}\right)$

405. $m(t) = \lambda \cdot \int_{-\infty}^{\infty} g(x)\,dx = \lambda \int_0^a kx\,dx = \lambda k \cdot \left[\frac{x^2}{2}\right]_0^a = \frac{\lambda k \cdot a^2}{2}$

$$\begin{aligned}
r(\tau) &= \lambda \cdot \int_{-\infty}^{\infty} g(x) \cdot g(x-\tau)\,dx \overset{0\leq\tau\leq a}{=} \lambda \cdot \int_\tau^a kx \cdot k(x-\tau)\,dx \\
&= \lambda k^2 \cdot \int_\tau^a \left(x^2 - \tau x\right)\,dx = \lambda k^2 \cdot \left[\frac{x^3}{3} - \tau \cdot \frac{x^2}{2}\right]_\tau^a \\
&= \lambda k^2 \cdot \left(\frac{a^3}{3} - \frac{\tau a^2}{2} - \frac{\tau^3}{3} + \frac{\tau^3}{2}\right) \\
&= \begin{cases} \lambda k^2 \left(\frac{a^3}{3} - \frac{a^2}{2}|\tau| + \frac{|\tau|^3}{6}\right) & |\tau| \leq a \\ 0 & |\tau| \geq a \end{cases}
\end{aligned}$$

406. $m = 10^4 \cdot \int_0^\infty 10^{-2} e^{-t}\,dt = 100$. För $\tau \geq 0$ är

$$\begin{aligned}
r(\tau) &= 10^4 \cdot \int_\tau^\infty 10^{-2} e^{-t} \cdot 10^{-2} \cdot e^{-(t-\tau)} dt = e^{-\tau} \cdot \int_\tau^\infty e^{-2(t-\tau)} dt \\
&= e^{-\tau} \cdot \int_0^\infty e^{-2t} dt = e^{-\tau}/2.
\end{aligned}$$

Vid varje tidpunkt t består $X(t)$ av tiotusentals bidrag, ty varje bidrag är $\leq 10^{-2}$ och de ska summeras till ca 100. Slutsats: $X(t)$ ungefär normal.

$$\begin{aligned}
E(X(t+1) - X(t)) &= 0 \\
V(X(t+1) - X(t)) &= C(X(t+1) - X(t), X(t+1) - X(t)) \\
&= V(X(t+1)) + V(X(t)) - 2C(X(t), X(t+1)) \\
&= \frac{1}{2} + \frac{1}{2} - 2 \cdot \frac{1}{2} e^{-1} = 1 - e^{-1}
\end{aligned}$$

Slutsats: $X(t+1) - X(t) \in N\left(0, \sqrt{1 - e^{-1}}\right)$ och därför

$P(X(t+1) - X(t) > 2) = 1 - \Phi\left(\frac{2}{\sqrt{1-e^{-1}}}\right) = 0.006$

407. $C(X_s, X_t) = E(X_s X_t) - E(X_s)\cdot E(X_t) = E(Z_s Z_{s-1} Z_t Z_{t-1})$ ty $E(X_t) = E(Z_t Z_{t-1}) = E(Z_t)\cdot E(Z_{t-1}) = 0$. Man finner

$$C(X_s, X_t) = \begin{cases} E(Z_{t+1}\cdot Z_t^2 \cdot Z_{t-1}) = 0 & \text{för } s-1=t \\ 0 & \text{för } s+1=t \\ 0 & \text{för } \mid s-t \mid \neq 1 \end{cases}$$

Slutsats: X_s och X_t är okorrelerade. De är däremot ej oberoende. Det ser man enklast genom att beräkna $C(X_s^2, X_t^2)$ för $s = t+1$:

$$\begin{aligned} C(X_{t+1}^2, X_t^2) &= E(X_{t+1}^2 \cdot X_t^2) - E(X_{t+1}^2)\cdot E(X_t^2) \\ &= E(Z_{t+1}^2 \cdot Z_t^4 \cdot Z_{t-1}^2) - E(Z_{t+1}^2 \cdot Z_t^2)\cdot E(Z_t^2 \cdot Z_{t-1}^2) \\ &= E(Z_{t+1}^2)E(Z_t^4)E(Z_{t-1}^2) - E(Z_{t+1}^2)(E(Z_t^2))^2 E(Z_{t-1}^2) \\ &= 1\cdot 3\cdot 1 - 1\cdot 1^2 \cdot 1 = 2 \neq 0 \end{aligned}$$

X_t är ej normalfördelad.

Kapitel 5: Filtrering

501. $E\left(\int_0^1 X(t)\,dt\right) = \int_0^1 m(t)\,dt = 0$

$$\begin{aligned} V\left(\int_0^1 X(t)\,dt\right) &= C\left(\int_0^1 X(s)\,ds, \int_0^1 X(t)\,dt\right) = \int_{s=0}^1 \int_{t=0}^1 r_X(s,t)\,ds\,dt \\ &= 2\cdot \int_{t=0}^1 \int_{s=0}^t \frac{1}{1+(s-t)^2}\,ds\,dt \\ &= 2\cdot \int_{t=0}^1 [\arctan(s-t)]_{s=0}^t \;dt \\ &= 2\cdot \int_{t=0}^1 \arctan t\,dt = 2[t\cdot \arctan t]_0^1 - 2\cdot \int_0^1 t\cdot \frac{1}{1+t^2}\,dt \\ &= \frac{\pi}{2} - (\ln(1+t^2))_0^1 = \frac{\pi}{2} - \ln 2 + \ln 1 \approx 0.878 \end{aligned}$$

Slutsats: $\int_0^1 X(t)dt \in N\left(0, \sqrt{0.878}\right) = N(0, 0.937\Big)$ och alltså $P\left(\int_0^1 X(t)\,dt < -1\right) = \Phi\left(\frac{-1}{0.937}\right) = 1 - \Phi(1.067) = 0.14$

502. a. Filtret är kausalt och

$$\begin{aligned} Y(t) &= \int_{t-2}^t X(u)\,du \overset{u=t-v}{=} -\int_2^0 X(t-v)\,dv \\ &= \int_0^2 X(t-v)\,dv = \int_{-\infty}^{\infty} h(v)\cdot X(t-v)\,dv \end{aligned}$$

om

$$h(v) = \begin{cases} 1 & 0 \le v \le 2 \\ 0 & \text{för övrigt} \end{cases}$$

b.

$$\begin{aligned} H(f) &= \int_{-\infty}^{\infty} h(t)\cdot e^{-i2\pi f t}\,dt = \int_0^2 e^{-i2\pi f t}\,dt \\ &= \left[\frac{e^{-i2\pi f t}}{-i2\pi f}\right]_0^2 = \frac{1-e^{-i4\pi f}}{i2\pi f} \quad \text{då } f \neq 0 \\ H(0) &= 2 \end{aligned}$$

c. $R_Y(f) = |H(f)|^2 \cdot R_X(f)$
Ur Tabell över Fouriertransformer får man om $r_X(\tau) = \max(0, 1 - |\tau|)$ att

$$R_X(f) = \begin{cases} 1 & f = 0 \\ \frac{2}{(2\pi f)^2} \cdot (1 - \cos(2\pi f)) & f \neq 0 \end{cases}$$

Med

$$|H(f)|^2 = \frac{(1 - e^{-i4\pi f})(1 - e^{i4\pi f})}{4\pi^2 f^2} = \frac{2(1 - \cos(4\pi f))}{4\pi^2 f^2}$$

blir

$$R_Y(f) = \begin{cases} 4 & f = 0 \\ \frac{4(1-\cos(2\pi f))(1-\cos(4\pi f))}{(2\pi f)^4} & f \neq 0 \end{cases}$$

503. a. $m_Y = m_X H(0) = 0$ och $R_Y(f) = |H(f)|^2 R_X(f)$ Medeleffekten hos $Y(t)$ blir $E(Y^2(t)) = \int |H(f)|^2 \cdot R_X(f)\, df = 2 \cdot \int_{f_0-\Delta f/2}^{f_0+\Delta f/2} R_X(f)\, df$

b. $R_X(f_0) \approx \frac{1}{\Delta f} \int_{f_0-\frac{\Delta f}{2}}^{f_0+\frac{\Delta f}{2}} R_X(f)\, df = \frac{1}{2\Delta f} E(Y^2(t))$

504. $r_{X,Y}(\tau) = \int_{-\infty}^{\infty} h(u) r_X(\tau - u)\, du = \int_0^1 \frac{1}{1+(\tau-u)^2}\, du = [-\arctan(\tau - u)]_0^1 = \arctan\tau - \arctan(\tau - 1)$

505. $h(u) = \begin{cases} \delta(u) + e^{-u}, & u \geq 0 \\ 0 & \text{för övrigt} \end{cases}$

$$\begin{aligned} H(f) &= \int_{-\infty}^{+\infty} h(u) e^{-i2\pi f u}\, du = \int_0^{\infty} (\delta(u) + e^{-u}) e^{-i2\pi f u}\, du \\ &= 1 + \int_0^{\infty} e^{-(1+i2\pi f)u} du = \frac{2 + i2\pi f}{1 + i2\pi f} \end{aligned}$$

$R_Y(f) = |H(f)|^2 R_X(f) = \frac{4+(2\pi f)^2}{1+(2\pi f)^2} \cdot \frac{4}{4+(2\pi f)^2} = \frac{4}{1+(2\pi f)^2} \xmapsto{\mathcal{F}^{-1}} r_Y(\tau) = 2e^{-|\tau|}$

506. Partialbråksuppdelning ger

$$\begin{aligned} R_Y(f) &= |H(f)|^2 R_X(f) = \frac{(2\pi f)^2}{(1 + (2\pi f)^2)(1 + 0.25(2\pi f)^2)} \\ &= \frac{A}{1 + (2\pi f)^2} + \frac{B}{1 + 0.25(2\pi f)^2} \end{aligned}$$

Konstanterna A och B bestäms av

$$\begin{cases} A + B = 0 \\ A/4 + B = 1 \end{cases} \iff \begin{cases} A = -\frac{4}{3} \\ B = \frac{4}{3} \end{cases}$$

dvs $r_Y(\tau) = -\frac{2}{3} e^{-|\tau|} + \frac{4}{3} e^{-2|\tau|}$

507. $Y(t) = \int h(u) X(t - u)\, du = \int (\delta_0(u) - \delta_1(u)) X(t - u)\, du$
$= X(t) - X(t - 1)$

$$\begin{aligned} P(Y(1) > 3 + Y(0)) &= P(X(1) - X(0) > 3 + X(0) - X(-1)) \\ &= P\,(\underbrace{X(1) + X(-1) - 2X(0)}_{Z} > 3) \\ &= P(Z > 3) = 1 - P(Z \leq 3) \\ &= 1 - \Phi(\frac{3}{\sqrt{12/5}}) \approx 0.026 \end{aligned}$$

ty $\{X(t)\}$ är en normalprocess $\Longrightarrow Z$ är normalfördelad.

$$\begin{aligned}
E(Z) &= E(X(1)) + E(X(-1)) - 2E(X(0)) = 3 + 3 - 2 \cdot 3 = 0 \\
V(Z) &= V(X(1)) + V(X(-1)) + (-2)^2 V(X(0)) + 2C(X(1), X(-1)) \\
&\quad +2 \cdot (-2) C(X(1),\ X(0)) + 2 \cdot 1(-2) C(X(-1), X(0)) \\
&= r_X(0) + r_X(0) + 4r_X(0) + 2r_X(2) - 4r_X(1) - 4r_X(-1) \\
&= 1 + 1 + 4 + 2 \cdot \frac{1}{5} - 4 \cdot \frac{1}{2} - 4 \cdot \frac{1}{2} = \frac{12}{5}.
\end{aligned}$$

508. $X_k + aX_{k-1} = e_k,\ V(e_k) = \sigma^2.$

Yule-Walker ekvationerna ger:

$$\begin{cases} r(0) + ar(1) = \sigma^2 \\ r(1) + ar(0) = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} 10 + a \cdot 5 = \sigma^2 \\ 5 + a \cdot 10 = 0 \end{cases}$$

ty $r(0) = 10, r(1) = 5$

Slutsats: $a = -\frac{1}{2},\ \sigma^2 = 7.5$

509. $X_t + 0.25X_{t-2} = e_t$

Yule-Walker ekvationerna ger:

$r(0) + 0.25r(2) = 15$

$r(k) + 0.25r(k-2) = 0 \qquad k = 1, 2, \ldots$

Speciellt blir

$r(1) \cdot 1.25 = 0 \ \Rightarrow\ r(1) = 0$

$r(k) = -0.25 \cdot r(k-2) \qquad k \geq 2$

och

$r(0) - \frac{1}{16} r(0) = 15 \ \Rightarrow\ r(0) = 16$

Den allmänna lösningen blir då

$r(2k) = (-0.25)^{|k|} \cdot 16,\ r(2k+1) = 0$

Spektraltätheten blir

$R_X(f) = \frac{15}{|1 + 0.25 \cdot e^{-i4\pi f}|^2} = \frac{15}{1.0625 + 0.5 \cdot \cos 4\pi f},\ -\frac{1}{2} < f \leq \frac{1}{2}$

510. $m_X = m(1 + 2 - 1) = 2m$

$$r_X(\tau) = \begin{cases} 6\sigma^2 & \tau = 0 \\ 2\sigma^2 & \tau = \pm 1 \\ -2\sigma^2 & \tau = \pm 2 \\ -\sigma^2 & \tau = \pm 3 \\ 0 & \text{för övrigt} \end{cases}$$

$R_X(f) = \sum r(k) \cdot e^{-i2\pi fk} = \sigma^2 \cdot (6 + 4\cos 2\pi f - 4\cos 4\pi f - 2\cos 6\pi f)$

511. $(1 - 1.559 + 0.81) \cdot E(X(t)) = 3.5 \Rightarrow E(X(t)) = \frac{3.5}{0.251} = 13.94$

$X(t) - 13.94$ är en AR(2)-process.

$V(X(t)) = r(0)$ fås från Yule-Walker ekvationerna:

$$\begin{cases} r(0) - 1.559\ r(1) + 0.81\ r(2) = 4 \\ r(1) - 1.559\ r(0) + 0.81\ r(1) = 0 \\ r(2) - 1.559\ r(1) + 0.81\ r(0) = 0 \end{cases}$$

Lösning: $r(0) = 45.062, r(1) = 38.813, r(2) = 24.009$

512. $Y(m) = X(2m) = 0.8X(2m-1) + e(2m)$ där $X(2m-1) = 0.8X(2m-2) + e(2m-1) = 0.8Y(m-1) + e(2m-1)$

Slutsats: $Y(m) = 0.64Y(m-1) + \underbrace{0.8e(2m-1) + e(2m)}_{u(m)}$

Det är lätt att se att $u(m)$ är okorrelerade, och att $u(m)$ är okorrelerat med $Y(m-1), Y(m-2), \ldots$

Slutsats: $Y(m)$ är en AR(1)-process

$V(u(m)) = 0.64V(e(2m-1)) + V(e(2m)) = 1.64 \cdot 2 = 3.28$

Slutsats: $a = -0.64, \sigma^2 = 3.28$

513. $r(k) = 0$ för $k \geq 3 \Rightarrow q = 2$.

$$
\begin{aligned}
r(0) &= C(e(t) + b_1 e(t-1) + b_2 e(t-2), e(t) + b_1 e(t-1) + b_2 e(t-2)) \\
&= 1 + b_1^2 + b_2^2 = \frac{5}{4} \\
r(1) &= C(e(t+1) + b_1 e(t) + b_2 e(t-1), e(t) + b_1 e(t-1) + b_2 e(t-2)) \\
&= b_1 + b_1 b_2 = 0 \\
r(2) &= C(e(t+2) + b_1 e(t+1) + b_2 e(t), e(t) + b_1 e(t-1) + b_2 e(t-2)) \\
&= b_2 = \frac{1}{2} \\
&\Rightarrow \begin{cases} b_1 = 0 \\ b_2 = 1/2 \end{cases} \quad H(f) = \sum_{k=0}^{2} b_k e^{-i2\pi fk} = 1 + \frac{1}{2}e^{-i4\pi f} \\
R_X(f) &= R_e(f)|H(f)|^2 = (1 + \frac{1}{2}e^{-i4\pi f})(1 + \frac{1}{2}e^{i4\pi f}) = \frac{5}{4} + \cos 4\pi f
\end{aligned}
$$

514. $r_k = C(X_{t+k}, e_t) = 0$ för $k < 0$ enligt definition. För övrigt blir

$$
r_k = C(X_{t+k}, e_t) = C(0.4X_{t+k-1} + e_{t+k}, e_t) = \begin{cases} 0.4r_{k-1} + 2, & k = 0 \\ 0.4r_{k-1}, & k > 0 \end{cases}
$$

Slutsats: $r_k = \begin{cases} 0, & k < 0 \\ 2 \cdot 0.4^k, & k \geq 0 \end{cases}$

515. $R_Y(f) = |H(f)|^2 \cdot 3 = \frac{3}{1+(2\pi f)^2}, r_Y(0) = \int_{-\infty}^{+\infty} R_Y(f)\, df$

Samplingintervallet d bestäms ur villkoret

$2\int_{1/2d}^{\infty} R_Y(f) df \leq 0.05 \cdot 2\int_0^{+\infty} R_Y(f)\, df$

$\int_0^{+\infty} \frac{3}{1+(2\pi f)^2}\, df = \left(\frac{3}{2\pi} \arctan 2\pi f\right)_0^{+\infty} = \frac{3}{4}$

$\int_{1/2d}^{\infty} \frac{3}{1+(2\pi f)^2} df = \frac{3}{4} - \frac{3}{2\pi} \arctan \frac{\pi}{d}$, dvs $\arctan \frac{\pi}{d} > 0.475\pi \Rightarrow d < 0.247$

516.

$$
\begin{aligned}
R_X(f) &= \pi \cdot e^{-2\pi|f|} \\
R_{X'}(f) &= (2\pi f)^2 \cdot R_X(f) = \pi \cdot (2\pi f)^2 \cdot e^{-2\pi|f|} \\
r_X(\tau) &\overset{tabell}{=} \pi \cdot \frac{4\pi}{(2\pi)^2 + (2\pi\tau)^2} = \frac{1}{1+\tau^2}
\end{aligned}
$$

$$
\begin{aligned}
r_{X'}(\tau) &= -r"_X(\tau) = -\frac{d^2}{d\tau^2}\left((1+\tau^2)^{-1}\right) = \frac{d}{d\tau}(2\tau(1+\tau^2)^{-2}) \\
&= \left(\frac{2}{(1+\tau^2)^2} - \frac{8\tau^2}{(1+\tau^2)^3}\right) = \frac{2-6\tau^2}{(1+\tau^2)^3}
\end{aligned}
$$

517. a.

$$
\begin{aligned}
L(\varepsilon) &= E\{(\frac{X(t)-X(t-\varepsilon)}{\varepsilon} - X'(t))^2\} \\
&= V(\frac{1}{\varepsilon}(X(t)-X(t-\varepsilon)) - X'(t)) \\
&= \frac{1}{\varepsilon^2}(2r(0)-2r(\varepsilon)) + r_{X'}(0) - \frac{2}{\varepsilon}\underbrace{r_{XX'}(0)}_{=0} + \frac{2}{\varepsilon}r_{XX'}(\varepsilon) \\
&= \frac{1}{\varepsilon^2}(2-2e^{-\varepsilon^2/2}) + 1 + \frac{2}{\varepsilon}\underbrace{r'_X(\varepsilon)}_{-\varepsilon e^{-\varepsilon^2/2}} \\
&= \frac{2}{\varepsilon^2}(1-e^{-\varepsilon^2/2}) + 1 - 2e^{-\varepsilon^2/2}, \text{ ty}
\end{aligned}
$$

$$
\begin{cases}
r_{X'}(\tau) = -r''_X(\tau) = e^{-\tau^2/2} - \tau^2 e^{-\tau^2/2} \\
r_{XX'}(\tau) = r'_X(\tau) = -\tau e^{-\tau^2/2}
\end{cases}
$$

$$
\begin{aligned}
L(\varepsilon) &\approx \frac{2}{\varepsilon^2}(1-(1-\varepsilon^2/2+\varepsilon^4/8)) + 1 - 2(1-\varepsilon^2/2) \\
&= \varepsilon^2 - \varepsilon^2/4 = \frac{3}{4}\varepsilon^2
\end{aligned}
$$

b. $L(0.1) = 0.00748$ (jämför approximativt värde 0.0075)
$Y(t) = \frac{1}{0.1}(X(t)-X(t-0.1)) - X'(t) \in N(0, \sqrt{L(0.1)})$

$$
\begin{aligned}
P(|Y(t)| < 0.1) &= P(-0.1 < Y(t) < 0.1) \\
&= \Phi\left(\frac{0.1}{\sqrt{L(0.1)}}\right) - \Phi\left(-\frac{0.1}{\sqrt{L(0.1)}}\right) \\
&= 2\Phi\left(\frac{0.1}{\sqrt{L(0.1)}}\right) - 1 \approx 0.759
\end{aligned}
$$

518. a. $H(f) = \frac{1}{1+12(i2\pi f)+8(i2\pi f)^2} = \frac{1}{1-32\pi^2 f^2 + i24\pi f}$

b.

$$
\begin{aligned}
R_Y(f) &= |H(f)|^2 \cdot R_X(f) = \\
&= \frac{1+f^2}{(1-32\pi^2 f^2)^2 + (24\pi f)^2} \qquad (|f| \le 5) \\
&= \frac{1+f^2}{1+512\pi^2 f^2 + 1024\pi^4 f^4} \qquad (|f| \le 5)
\end{aligned}
$$

519. $P(Y(t) - 2Y'(t) > 1) = P(\underbrace{X(t) + 3X'(t) - 10X''(t)}_{Z(t)\in N(m,\sigma)} > 1)$

$$
\begin{cases}
m &= E(Z(t)) = 5 \\
\sigma^2 &= V(X(t) + 3X'(t) - 10X''(t)) \\
&= V(X(t)) + 9V(X'(t)) + 100V(X''(t)) - 20C(X(t), X''(t))
\end{cases}
$$

ty $X(t)$ och $X'(t)$ oberoende, $X'(t)$ och $X''(t)$ oberoende.

Skriv $r_X(\tau) = e^{-\tau^2/2} = 1 - \tau^{2/2} + \tau^4/8 + \dots$

$$\begin{aligned} V(X'(t)) &= -r''(0) = +1 \\ V(X''(t)) &= r^{IV}(0) = 3 \\ C(X(t), X''(t)) &= r''(0) = -1 \end{aligned}$$

Slutsats: $\sigma^2 = 1 + 9 \cdot 1 + 100 \cdot 3 - 20 \cdot (-1) = 330$ så att $Z(t) \in N(5, \sqrt{330})$ vilket ger $P(Z(t) > 1) = 1 - \Phi\left(\frac{1-5}{\sqrt{330}}\right) = \Phi(0.222) = 0.59$

520. a. $r_{X'}(\tau) = -r''_X(\tau) = (1 - \tau^2)e^{-\tau^2/2}$

b. $r_{X,X'}(\tau) = C(X(t), X'(t+\tau)) = r'_X(\tau) = -\tau e^{-\tau^2/2}$

c. $Y(t) = X(t+0.5) - X(t) - 0.5X'(t), \quad E(Y(t)) = 0$

$$\begin{aligned} E(Y^2(t)) &= V(Y(t)) = C(Y(t), Y(t)) \\ &= C(X(t+0.5) - X(t) - 0.5X'(t), \\ &\qquad X(t+0.5) - X(t) - 0.5X'(t)) \\ &= r_X(0) - r_X(0.5) - 0.5r_{X,X'}(-0.5) - r_X(0.5) \\ &\quad + r_X(0) + 0.5r_{X,X'}(0) - 0.5r_{X,X'}(-0.5) \\ &\quad + 0.5r_{X,X'}(0) + 0.25r_{X'}(0) \\ &= 2 - 2e^{-1/8} - 0.5e^{-1/8} + 0 + 0.25 = 2.25 - 2.5e^{-1/8} \end{aligned}$$

$$\begin{aligned} P(|Y(t)| > \frac{1}{4}) &= 1 - P(|Y(t)| \le \frac{1}{4}) \\ &= 1 - (2\Phi(\frac{\frac{1}{4}}{\sqrt{2.25 - 2.5e^{-1/8}}}) - 1) \\ &\approx 2 - 2\Phi(1.195) \approx 0.232 \end{aligned}$$

521. Det första filtrets frekvensfunktion är

$$H_1(f) = 10\left(1 - e^{-i2\pi f/10}\right)$$

Slutsats:

$$\begin{aligned} R_{ut}(f) &= |H_1(f)|^2 \cdot |H(f)|^2 \cdot R(f) \\ &= 100\left(\left(1 - \cos\frac{2\pi f}{10}\right)^2 + \sin^2\frac{2\pi f}{10}\right) \cdot \frac{1}{1 + (\pi f)^2} \cdot \frac{1}{1 + 0.01(2\pi f)^2} \\ &= \frac{200\left(1 - \cos\frac{2\pi f}{10}\right)}{(1 + (\pi f)^2) \cdot (1 + 0.01(2\pi f)^2)} \end{aligned}$$

522.

$$\begin{aligned} r_{X,Y}(\tau) &= C(X(n), Y(n+\tau)) \\ &= C(X(n), X(n+\tau) + X(n+\tau-1) + X(n+\tau-2)) \\ &= r_X(\tau) + r_X(\tau-1) + r_X(\tau-2) \end{aligned}$$

så att

$$r_{X,Y} = \begin{cases} 0 & \text{för} \quad \tau \leq -3 \\ 1 & \text{för} \quad \tau = -2 \\ 3 & \text{för} \quad \tau = -1 \\ 6 & \text{för} \quad \tau = 0 \\ 7 & \text{för} \quad \tau = 1 \\ 6 & \text{för} \quad \tau = 2 \\ 3 & \text{för} \quad \tau = 3 \\ 1 & \text{för} \quad \tau = 4 \\ 0 & \text{för} \quad \geq 5 \end{cases}$$

523. a. $r_{X,Y}(k) = C(X_n, Y_{n+k}) = \begin{cases} 1 & k = 0 \\ 0 & \text{för övrigt} \end{cases}$

$R_{X,Y}(f) = \sum_k r_{X,Y}(k) e^{-i2\pi fk} = 1 \qquad -\frac{1}{2} < f \leq \frac{1}{2}$

$A_{X,Y}(f) = 1, \qquad \Phi_{X,Y}(f) = 0$

$R_X(f) = R_Y(f) = 1, \qquad -\frac{1}{2} < f \leq 1/2$

Slutsats: $\kappa^2(f) = 1 \qquad -\frac{1}{2} < f \leq 1/2$

I fortsättningen skriver vi ej ut "$-\frac{1}{2} < f \leq 1/2$".

b. $r_{X,Y}(k) = C(X_n, X_{n+k-1}) = \begin{cases} 1 & k = 1 \\ 0 & \text{för övrigt} \end{cases}$

$R_{X,Y}(f) = e^{-i2\pi f}$

$A_{X,Y}(f) = 1$

$\Phi_{X,Y}(f) = -2\pi f$

$\kappa^2(f) = 1$ på samma sätt som (a)

c. $r_{X,Y}(k) = \begin{cases} 1 & k = 2 \\ 0 & \text{för övrigt} \end{cases}$

$R_{X,Y}(f) = e^{-i2\cdot 2\pi f}$

$A_{X,Y}(f) = 1$

$\Phi_{X,Y}(f) = -4\pi f$

$\kappa^2(f) = 1$

d. $r_{X,Y}(k) = C(X_n, X_{n+k} + X_{n+k-1}) \begin{cases} 1 & k = 0 \text{ eller } 1 \\ 0 & \text{för övrigt} \end{cases}$

$R_{X,Y}(f) = 1 + e^{-i2\pi f} = 1 + \cos(2\pi f) - i\sin(2\pi f)$

$A_{X,Y}(f) = \sqrt{(1 + \cos 2\pi f)^2 + \sin^2(2\pi f)} = \sqrt{2(1 + \cos(2\pi f))}$

$\Phi_{X,Y}(f) = \arctan\left(\frac{-\sin(2\pi f)}{1 + \cos(2\pi f)}\right) = -\pi f$

$r_Y(k) = \begin{cases} 2 & k = 0 \\ 1 & k = \pm 1 \\ 0 & \text{för övrigt} \end{cases}$

$R_Y(f) = 2(1 + \cos(2\pi f))$

$\kappa^2(f) = \frac{A^2_{X,Y}(f)}{R_X(f) R_Y(f)} = 1$

524. $Y_k = \sum_n h_{k-n} X_n + Z_k \Rightarrow$

$$\begin{aligned} r_{X,Y}(k) &= C(X_m, Y_{m+k}) = C\left(X_m, \sum_n h_{m+k-n} X_n + Z_{m+k}\right) \\ &= C\left(X_m, \sum_n h_{m+k-n} X_n\right) \end{aligned}$$

dvs samma med eller utan Z_k

Slutsats: också samma $R_{X,Y}(f), A_{X,Y}(f), \Phi_{X,Y}(f), \qquad R_X(f) = 1$

a.

$$\begin{aligned}
Y_m &= X_m + Z_m \\
r_Y(k) &= C(X_m + Z_m, X_{m+k} + Z_{m+k}) = \begin{cases} 1+\sigma^2 & k=0 \\ 0 & \text{för övrigt} \end{cases} \\
R_Y(f) &= 1+\sigma^2 \\
\kappa^2(f) &= \frac{1^2}{1\cdot(1+\sigma^2)} = \frac{1}{1+\sigma^2} < 1
\end{aligned}$$

b.

$$\begin{aligned}
Y_m &= X_{m-1} + Z_m \\
r_Y(k) &= C(X_{m-1} + Z_m, X_{m+k-1} + Z_{m+k}) = \begin{cases} 1+\sigma^2 & k=0 \\ 0 & \text{för övrigt} \end{cases} \\
R_Y(f) &= 1+\sigma^2, \qquad \kappa^2(f) = \frac{1}{1+\sigma^2}
\end{aligned}$$

c. samma som (b)

d.

$$\begin{aligned}
Y_m &= X_m + X_{m-1} + Z_m \\
r_Y(k) &= C(Y_m, Y_{m+k}) \\
&= C(X_m + X_{m-1} + Z_m, X_{m+k} + X_{m+k-1} + Z_{m+k}) \\
&= \begin{cases} 2+\sigma^2 & k=0 \\ 1 & k=\pm 1 \\ 0 & \text{för övrigt} \end{cases} \\
R_Y(f) &= 2+\sigma^2 + 2\cos(2\pi f) \\
\kappa^2(f) &= \frac{2+2\cos(2\pi f)}{2+\sigma^2+2\cos(2\pi f)} = 1 - \frac{1}{1+\frac{2}{\sigma^2}(1+\cos(2\pi f))}
\end{aligned}$$

525. $R_X(f) = \sum_{k=-\infty}^{\infty} r_X(k) e^{-i2\pi f k} = 1 + \cos 2\pi f$

a.

$$\begin{aligned}
r_{X,Y}(k) &= C(X_m, X_{m+k} + Z_{m+k}) \\
&= C(X_m, X_{m+k}) = r_X(k) \\
R_{X,Y}(f) &= R_X(f) = 1 + \cos(2\pi f) \\
r_Y(k) &= C(X_m + Z_m, X_{m+k} + Z_{m+k}) = r_X(k) + r_Z(k) \\
R_Y(f) &= 1 + \cos 2\pi f + \sigma^2
\end{aligned}$$

$$\Phi_{X,Y}(f) = 0$$

$$\begin{aligned}
\kappa^2(f) &= \frac{(1+\cos(2\pi f))^2}{(1+\cos(2\pi f))\cdot(1+\sigma^2+\cos(2\pi f))} \\
&= 1 - \frac{1}{1+\frac{1}{\sigma^2}(1+\cos 2\pi f)}
\end{aligned}$$

b.

$$
\begin{aligned}
r_{X,Y}(k) &= C(X_m, X_{m+k-1}) = r_X(k-1) = \begin{cases} 1 & k=1 \\ 0.5 & k = 0 \text{ eller } 2 \\ 0 & \text{för övrigt} \end{cases} \\
R_{X,Y}(f) &= 0.5 + e^{-i2\pi f} + 0.5 \cdot e^{-i \cdot 2 \cdot 2\pi f} = e^{-i2\pi f} \cdot (1 + \cos 2\pi f) \\
A_{X,Y}(f) &= 1 + \cos 2\pi f \\
\Phi_{X,Y}(f) &= -2\pi f \\
r_X(k) &= C(X_{m-1} + Z_m, X_{m+k-1} + Z_{m+k}) = r_X(k) + r_Z(k) \\
R_Y(f) &= R_X(f) + \sigma^2 = 1 + \cos 2\pi f + \sigma^2 \\
\kappa^2 &= \frac{(1+\cos 2\pi f)^2}{(1+\cos 2\pi f)(1 + \cos 2\pi f + \sigma^2)} \\
&= 1 - \frac{1}{1 + \frac{1}{\sigma^2}(1 + \cos 2\pi f)}
\end{aligned}
$$

526. a.

$$
\begin{aligned}
r_{X,Y}(\tau) &= C\left(X(t), \int_0^\infty h(s)X(t+\tau-s)\,ds + Z(t+\tau)\right) \\
&= \int_0^\infty h(s) r_X(\tau - s)\,ds \qquad \text{(faltning)}
\end{aligned}
$$

b. $R_{X,Y}(f) = H(f) \cdot R_X(f)$

c. $r_X(\tau) = \sigma^2 \cdot \delta_0(\tau), \qquad R_X(f) = \sigma^2 \qquad -\infty < f < \infty$

$r_{X,Y}(\tau) = \sigma^2 \cdot h(\tau), \qquad R_{X,Y}(f) = \sigma^2 \cdot H(f)$

527.

$$
\begin{aligned}
Y_t &= \int_{-\infty}^{+\infty} h(t-u)X(u)\,du + Z_t; \quad h(t) = \begin{cases} e^{-\beta t} & t \geq 0 \\ 0 & t < 0 \end{cases} \\
H(f) &= \int_{-\infty}^{+\infty} e^{-i2\pi f u} h(u)\,du = \frac{1}{\beta + i2\pi f} \\
R_{X,Y}(f) &= H(f) R_X(f) \qquad \text{(ty } X \text{ och } Z \text{ oberoende)} \\
&= \frac{1}{\beta + i2\pi f} \frac{2\alpha}{\alpha^2 + (2\pi f)^2} = \frac{2\alpha}{\alpha^2 + (2\pi f)^2} \frac{\beta - i2\pi f}{\beta^2 + (2\pi f)^2} \\
A_{X,Y}(f) &= |R_{X,Y}(f)| = \frac{2\alpha}{\alpha^2 + (2\pi f)^2} \frac{1}{\sqrt{\beta^2 + (2\pi f)^2}} \\
\Phi_{X,Y}(f) &= -\arctan \frac{2\pi f}{\beta}
\end{aligned}
$$

Kapitel 6: Inferens för stationära processer

601. $m^* = \frac{1}{n} \sum_{k=1}^{n} x_k = \frac{20.6}{1.2} = 1.717$

$x_k - m^*$:

0.783	−0.317	0.083	−0.417	−0.617	0.283
−0.417	−0.217	−0.017	0.183	0.083	0.583

$$
r^*(0) = \frac{1}{n} \sum_{k=1}^{n} (x_k - m^*)^2 = \frac{1.9568}{12} = 0.1631
$$

$$r^*(2) = \frac{1}{n}\sum_{k=1}^{n-2}(x_k - m^*)(x_{k+2} - m^*) = \frac{0.2966}{12} = 0.0247$$

$$\rho^*(2) = \frac{r^*(2)}{r^*(0)} = 0.1514$$

602. a.

$$\begin{aligned} V(Y(n)) &= \frac{1}{N^2}(Nr_X(0) + 2(N-1)r_X(1)) \\ &= \frac{1}{N^2}(1.2N + N - 1) = \frac{1}{N^2}(2.2N - 1) \le 0.1 \end{aligned}$$

ger $N^2 - 22N + 10 \ge 0$ dvs $N \ge 22$.

$$V(Z(n)) = \frac{1}{M^2} \cdot Mr_X(0) = \frac{1.2}{M} \le 0.1$$

dvs $M \ge 12$.

b. $r_X(1) > 0$ men $r_X(2) = 0$. $Y(n)$ innehåller alltså positivt korrelerade s.v., medan $Z(n)$ innehåller okorrelerade s.v. Det behövs därför fler termer i $Y(n)$ än i $Z(n)$ för att variansen ska bli liten.

603. $m_X = E(X_n) = m + 0.5(m_X - m)$, Slutsats: $m_X = m$

a. $\{X_k - m\}$ är en AR(1)-process med $a_1 = -0.5$, dvs

$$r_X(\tau) = \frac{\sigma^2}{1-(-0.5)^2}0.5^{|\tau|} = \frac{4\sigma^2}{3}0.5^{|\tau|}$$

b. Felaktigt antagande: okorrelerade mätvärden

$$\begin{aligned} V(m_2^*) &= \frac{1}{4}V(X_1 + X_2) = \frac{1}{4}(2r_X(0) + 2r_X(1)) = \sigma^2 \\ V(m_4^*) &= \frac{1}{16}V(X_1 + X_2 + X_3 + X_4) \\ &= \frac{1}{16}(4r_X(0) + 6r_X(1) + 4r_X(2) + 2r_X(3)) = \frac{11}{16}\sigma^2 \end{aligned}$$

Variansen reduceras till $\frac{11}{16}$ av den första.

c. Ja, eftersom $\sum_{-\infty}^{+\infty} |r_X(\tau)| = \frac{4}{3}\sigma^2(1 + 2\frac{\frac{1}{2}}{1-\frac{1}{2}}) = 4\sigma^2$

604. $V(m^*) = \frac{1}{9}(V(Y_1) + V(Y_2) + V(Y_3) + 2C(Y_1,Y_2) + 2C(Y_1,Y_3) + 2C(Y_2,Y_3))$
Alt a : $V(m_a^*) = \frac{1}{9}(3e^{-0} + 3 \cdot 2 \cdot e^{-d}) = \frac{1}{9}(3 + 6e^{-d})$
Alt b : $V(m_b^*) = \frac{1}{9}(3e^{-0} + 2 \cdot 2e^{-d} + 2e^{-2d}) = \frac{1}{9}(3 + 4e^{-d} + 2e^{-2d})$
$V(m_a^*) - V(m_b^*) = \frac{1}{9}(2e^{-d} - 2e^{-2d}) > 0$ $(d > 0)$ dvs placering b bäst!

605. a. $m = 0 \Rightarrow r_X^*(\tau) = \frac{1}{n}\sum_{i=1}^{n-\tau} x_i x_{i+\tau}$

$$r_X^*(0) = \frac{1}{15}\sum_{i=1}^{15} x_i^2 \approx 0.64, \quad r_X^*(1) = \frac{1}{15}\sum_{i=1}^{14} x_i x_{i+1} \approx -0.12.$$

b. Yule-Walker ekvationerna ger:

$$\begin{cases} r_X(0) + a_1 r_X(1) = V(e_t) \\ r_X(1) + a_1 r_X(0) = 0 \end{cases} \iff \begin{cases} a_1 = -\frac{r_X(1)}{r_X(0)} \\ V(e_t) = r_X(0) - \frac{(r_X(1))^2}{r_X(0)} \end{cases}$$

Insättning av skattningar i (a) ger $a_1^* \approx 0.19$ och $V(e_t)^* \approx 0.62$.

606. a. $E(m_1^*) = \frac{1}{4}\int_0^4 E(X(t))\,dt = m \Rightarrow$ vvr
$E(m_2^*) = \frac{1}{4}\sum_1^4 E(X(k)) = m \Rightarrow$ vvr

b.

$$
\begin{aligned}
V(m_1^*) &= C\left(\frac{1}{4}\int_0^4 X(s)\,ds, \frac{1}{4}\int_0^4 X(t)\,dt\right) \\
&= \frac{1}{16}\int_0^4\int_0^4 r(t-s)\,ds\,dt = \frac{1}{8}\int_0^4\int_t^4 e^{-\alpha(s-t)}\,ds\,dt \\
&= \frac{1}{8}\int_{t=0}^4 (e^{\alpha t}\cdot \underbrace{\int_t^4 e^{-\alpha s} ds}_{\frac{1}{\alpha}(e^{-\alpha t}-e^{-4\alpha})})\,dt = \frac{1}{8\alpha}\int_0^4 \left(1-e^{-4\alpha}\cdot e^{\alpha t}\right)\,dt = \\
&= \frac{1}{8\alpha}\left(4-e^{-4\alpha}\cdot\frac{1}{\alpha}\left(e^{4\alpha}-1\right)\right) = \frac{1}{8\alpha}\left(4-\frac{1}{\alpha}+\frac{1}{\alpha}e^{-4\alpha}\right) \\
&= \begin{cases} 0.3773 & \alpha = 1 \\ 0.2188 & \alpha = 2 \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
V(m_2^*) &= C\left(\frac{1}{4}\sum_1^4 X_k, \frac{1}{4}\sum_1^4 X_m\right) \\
&= \frac{1}{16}\sum_{k=1}^4\sum_{m=1}^4 r(m-k) = \frac{1}{16}\left(4r(0)+6r(1)+4r(2)+2r(3)\right) \\
&= \frac{1}{8}\left(2+3e^{-\alpha}+2e^{-2\alpha}+e^{-3\alpha}\right) \\
&= \begin{cases} 0.4280 & \alpha = 1 \\ 0.3056 & \alpha = 2 \end{cases}
\end{aligned}
$$

Skillnaden är störst för $\alpha = 2$. Då fluktuerar processen mest, och man får mest kunskap om processernas väntevärde om man använder $X(t)$ också mellan tidpunkterna $1, 2, 3$ och 4.

607. a. $m^* = \overline{x} = \frac{1}{10}\sum_1^{10} x_t = 0.7347$

b. Sökt $I_m = (m^* \pm \lambda_{\alpha/2} D(m^*))$

$$
\begin{aligned}
V(m^*) &= V\left(\frac{1}{10}\sum_{t=1}^{10} x_t\right) = \frac{1}{100}C\left(\sum_1^{10} x_t, \sum_1^{10} x_s\right) = \\
&= \frac{1}{100}\left(10r_X(0)+2\cdot 9r_X(1)+2\cdot 8r_X(2)+\ldots+2\cdot r_X(9)\right)
\end{aligned}
$$

Beräkna $r_X(k)$: Konstatera först $r_X(k) = r_Y(k)$ (jämför $V(X+c) = V(X)$). Yule-Walker ekvationerna ger (med $Y(t) = X(t) - m$):

$$
\begin{cases} r_Y(k) + 0.25 r_Y(k-1) = 0, & k > 0 \\ r_Y(0) + 0.25 r_Y(1) = \sigma^2 = V(e_t) = 1 \end{cases}
$$

dvs

$$
\begin{aligned}
r_Y(0) &= \frac{1}{1-0.25^2} = \frac{16}{15} \quad (\text{sätt } k = 1) \\
r_Y(k) &= -0.25 r_Y(k-1) = (-0.25)^k r_Y(0) = (-0.25)^k\frac{16}{15}, \quad k > 0, \\
V(m^*) &= \frac{1}{100}\frac{1}{15}(10\cdot 16 - 18\cdot 4 + 16\cdot 1 - 14\cdot\frac{1}{4} + \ldots - 2\cdot\frac{1}{16384}) \\
&= 0.06741333... \\
D(m^*) &= 0.25964
\end{aligned}
$$

$$\begin{aligned}
\lambda_{\alpha/2} &= \lambda_{0.025} = 1.9600 \\
I_m &= (m^* \pm \lambda_{\alpha/2} D(m^*)) = 0.7347 \pm 1.9600 \cdot 0.2596 \\
&= 0.7347 \pm 0.5089 = (0.2258, 1.2436)
\end{aligned}$$

608. a. $E(R^*_{\text{per}}(f)) \approx R(f)$,
$V(R^*_{\text{per}}(f)) \approx R^2(f), \quad f \neq 0, \pm 1/2$ då $n \to \infty$

b. Genom utjämning med frekvensfönster,

$$R^*(f) = \int K_n(f-u) R^*_{per}(u)\, du$$

eller genom medelvärdesbildning över många delintervall.

c. $V(R^*_{mv}(f)) \approx \frac{L}{n} R^2(f), \quad f \neq 0, \pm 1/2$

Kapitel 7: Tillämpningar

701. a. Om T är observationstidpunkten, så ges det anpassade filtret av

$$h(k) = s(T-k) = \begin{cases} 2 & \text{för} \quad k = T-1 \\ 3 & \text{för} \quad k = T \\ 1 & \text{för} \quad k = T+1 \\ 0 & \text{för övrigt} \end{cases}$$

vilket är kausalt för T=1, t ex.

b. Vi separerar filtrerad signal, $s_u(k)$ (om den sänts) och filtrerat brus, $N_u(k)$. Vi får för $k = T$:

$$s_u(T) = \sum_k h(k)s(T-k) = \sum h^2(k) = 14$$

Vidare:

$$\begin{aligned}
N_u(T) &= \sum_k h(k)N(T-k) = h(T-1)\cdot N(1) + h(T)N(0) + \\
&\quad + h(T+1)N(-1) = 2N(1) + 3N(0) + N(-1) \\
E(N_u(T)) &= 0 \\
V(N_u(T)) &= V(N(k))\cdot(2^2+3^2+1^2) = 14 \cdot 0.83
\end{aligned}$$

Slutsats: $N_u(T) \in N(0, 3.4088)$

Välj beslutströskel $\frac{s_u(T)}{2} = 7$. Då gäller (för båda feltyperna):

$$P(\text{fel}) = 1 - \Phi(\frac{7}{3.4088}) = 0.02.$$

702.

$$\begin{aligned}
h(u) &= cs(T-u) = \begin{cases} A & 0 \leq u \leq T \\ 0 & \text{för övrigt} \end{cases} \\
s_u &= cA^2T \\
E(N_u^2(T)) &= N_0 c^2 A^2 T
\end{aligned}$$

Miss:

$$P(Y(T) \leq k | Y(T) \in N(s_u(T), cA\sqrt{N_0 T})) = \Phi(\frac{k - s_u(T)}{cA\sqrt{N_0 T}}) = 0.005$$

$$\Rightarrow \quad k = \lambda_{0.995} cA\sqrt{N_0 T} + s_u(T)$$

Falskt alarm:

$$P(Y(T) > k|Y(T) = N_u(T)) = 1 - \Phi(\frac{k}{cA\sqrt{N_0T}}) \leq 0.01$$

så att kravet blir

$$\begin{aligned}\Phi(\frac{\lambda_{0.995} \cdot cA\sqrt{N_0T} + cA^2T}{cA\sqrt{N_0T}}) &\geq 0.99 \\ \lambda_{0.995} + \frac{A\sqrt{T}}{\sqrt{N_0}} &\geq \lambda_{0.01}\end{aligned}$$

dvs

$$\frac{A\sqrt{T}}{\sqrt{N_0}} \geq \lambda_{0.01} - \lambda_{0.995} \approx 4.9$$

703.

$$\begin{aligned}S_U(t) &= 0 \quad \text{eller} \quad 1, \qquad N_U(t) \in N(0, \sqrt{0.1}) \\ P(\text{fel}) &= P(\text{fel} \mid 1 \text{ sänd}) \cdot P(1 \text{ sänd}) + P(\text{fel} \mid 0 \text{ sänd}) \cdot P(0 \text{ sänd}) \\ &= P(1 + N_U(t) \leq 0.5) \cdot P(1) + P(0 + N_U(t) > 0.5) \cdot P(0) \\ &= \Phi\left(-\frac{0.5}{\sqrt{0.1}}\right) \cdot P(1) + \Phi\left(-\frac{0.5}{\sqrt{0.1}}\right) \cdot P(0) \\ &= \Phi\left(-\frac{0.5}{\sqrt{0.1}}\right) = 1 - \Phi(1.58) = 1 - 0.9429 = 0.0571\end{aligned}$$

Medelantal fel per sekund är $0.0571 \cdot 9600$

Slutsats: Medeltid mellan fel är $\frac{1}{0.0571 \cdot 9600} = 0.001824$ sek $= 1.82$ ms.

704. Signalanpassat filter med $h(t) = s(T - t)$

$$s(t) = \begin{cases} \frac{t}{\varepsilon} & 0 \leq t \leq \varepsilon \\ 1 & \varepsilon \leq t \leq T - \varepsilon \\ \frac{T-t}{\varepsilon} & T - \varepsilon \leq t \leq T \\ 0 & \text{för övrigt} \end{cases}$$

$\Longrightarrow$

$$h(t) = \begin{cases} \frac{T-t}{\varepsilon} & T - \varepsilon \leq t \leq T \\ 1 & \varepsilon \leq t \leq T - \varepsilon \\ \frac{t}{\varepsilon} & 0 \leq t \leq \varepsilon \\ 0 & \text{för övrigt} \end{cases}$$

$$\begin{aligned}\text{SNR} &= \frac{1}{R_0} \int_0^T s^2(T-t)\,dt = \frac{1}{R_0} \int_0^T h^2(t)\,dt \\ &= \frac{1}{R_0} \left(\left[\frac{t^3}{3\varepsilon^2} \right]_0^{\varepsilon} + [t]_{\varepsilon}^{T-\varepsilon} + \left[\frac{(T-t)^3}{-3\varepsilon^2} \right]_{T-\varepsilon}^{T} \right) = \frac{1}{R_0}(T - \frac{4}{3}\varepsilon)\end{aligned}$$

705. a. Efter filtret gäller

Om signalen sänts:

$$\begin{aligned} Y(3) &= \sum_k h(3-k)s(k) + \sum_k h(3-k)N(k) \\ &= 2 + N(0) + N(1) \in N(2, \sqrt{2}) \end{aligned}$$

Om signalen ej sänts:

$$Y(k) = N(0) + N(1) \in N(0, \sqrt{2})$$

Alltså:

$$\begin{aligned} P(\text{tro att signalen sänd} \mid \text{ej sänd}) &= P(N(0) + N(1) > 1) \\ &= 1 - \Phi(\frac{1}{\sqrt{2}}) = 0.240 \end{aligned}$$

b. Optimalt filter med beslutstidpunkt T

$$h(k) = s(T-k) = \begin{cases} 1 & \text{för } k = T-3, T-2, T-1, T \\ 0 & \text{för övrigt} \end{cases}$$

Kausalt, t ex om $T = 3$

Om signalen sänds: $Y(3) = 4 + \sum_0^3 N(k) \in N(4, 2)$

Om signalen ej sänds: $Y(3) = \sum_0^3 N(k) \in N(0, 2)$

Beslut: $Y(3) \begin{cases} > 2 : \text{sänd} \\ \leq 2 : \text{ej sänd} \end{cases}$

$$\begin{aligned} P(\text{tro att signalen sänd} \mid \text{ej sänd}) &= P(Y(3) > 2 \mid Y(3) \in N(0,2)) \\ &= 1 - \Phi(\frac{2}{2}) = 0.1587. \end{aligned}$$

706.

$$\begin{aligned} H(f) &= \frac{R_X(f)}{R_X(f) + R_N(f)}, \\ R_Y(f) &= |H(f)|^2 (R_X(f) + R_N(f)) = \frac{R_X^2(f)}{R_X(f) + R_N(f)} \\ R_X(f) &= \frac{1}{1+f^2}, \quad R_N(f) = \frac{10}{10^2 + f^2} \\ R_Y(f) &= \left(\frac{1}{1+f^2}\right)^2 / \left(\frac{1}{1+f^2} + \frac{10}{10^2+f^2}\right) = \frac{1}{1+f^2} - \frac{0.1}{1.1 + 0.11 f^2} \end{aligned}$$

Slutsats: $r_Y(\tau) = \pi e^{-2\pi|\tau|} - \frac{10}{11}\frac{\pi}{\sqrt{10}} e^{-2\pi\sqrt{10}|\tau|}$

707. a. $H(f) = \frac{R_S(f)}{R_S(f) + R_N(f)} = \frac{1}{1 + 100/|f|}, \qquad 100 \leq |f| \leq 1000$

$$\begin{aligned} \text{SNR}_{\text{MAX}} &= \frac{\int R_S(f)\,df}{\int \frac{R_S(f) R_N(f)}{R_S(f) + R_N(f)}}\, df \\ \int R_S(f)\,df &= 2\int_{100}^{1000} 1\,df = 1800 \\ \int \frac{R_s(f) R_N(f)}{R_s(f) + R_N(f)}\, df &= 2\int_{100}^{1000} \frac{1 \cdot 100/f}{1 + 100/f}\, df = 2\int \frac{100}{100 + f} df \end{aligned}$$

$$\begin{aligned} &= 200[\ln(100+f)]_{100}^{1000} = 200(\ln 1100 - \ln 200) \\ &= 340.9 \end{aligned}$$

$$\mathrm{SNR_{MAX}} = 5.28$$

b.

$$G(f) = 1; \quad 100 \le f \le 1000; \quad \int R_S(f)\,df = 1800$$

$$\begin{aligned} E\{(Y(t)-S(t))^2\} &= E\{(S_u(t)-S(t))^2\} + E\{N_u(t)^2\} \\ &= E\{N_u(t)^2\} \end{aligned}$$

$$\begin{aligned} \int R_N(f)df &= 2\int_{100}^{1000} \frac{100}{f}\,df \\ &= 200(\ln(f))_{100}^{1000} = 200(\ln 1000 - \ln 100) = 460.52, \end{aligned}$$

SNR ≈ 3.91

708. a. För $t \in [0,T]$ gäller

$$\begin{aligned} S_u(t) &= \int_0^t 2(T-s)ds \\ &= 2tT - 2\frac{t^2}{2} = 2tT - t^2 = t(2T-t) \end{aligned}$$

Slutsats:

$$s_u(t) = \begin{cases} 0 & t < 0 \\ t(2T-t) & 0 \le t \le T \\ T^2 & t > T \end{cases}$$

Man får då

$$\begin{aligned} V\left(\int_0^t N(u)du\right) &= V_0 \cdot \int_0^t \int_0^t \delta(u-v)dudv = \\ &= V_0 \cdot \int_0^t 1\,dv = V_0 \cdot t \end{aligned}$$

$$\mathrm{SNR} = \frac{t^2(2T-t)^2}{V_0 \cdot t} = \frac{1}{V_0} \cdot t(2T-t)^2 \qquad \text{för } 0 \le t \le T.$$

$$\begin{aligned} \frac{d\mathrm{SNR}}{dt} &= \frac{1}{V_0}((2T-t)^2 + 2t(t-2T)) \\ &= \frac{1}{V_0}(4T^2 + t^2 - 4Tt + 2t^2 - 4tT) \\ &= \frac{1}{V_0}(3t^2 - 8Tt + 4T^2) = \frac{3}{V_0}\left(t^2 - \frac{8T}{3}t + \frac{4T^2}{3}\right) \end{aligned}$$

$$\begin{aligned} \text{Nollställen} &= \frac{4T}{3} \pm \sqrt{\frac{16T^2}{9} - \frac{12T^2}{9}} = \frac{4T \pm 2T}{3} \\ &= \frac{2T}{3} \;\left(\text{eller } \frac{6T}{3} = 2T\right) \quad \text{ty} \quad 0 \le t \le T \end{aligned}$$

$$\frac{d^2\mathrm{SNR}}{dt^2} = \frac{3}{V_0}\left(2t - \frac{8T}{3}\right) < 0 \quad \text{för} \quad t = \frac{2T}{3}$$

Slutsats: max för $\frac{2T}{3}$

b. $h(t) = \begin{cases} cs(T_s - t) = c2(T - T_s + t), & T_s - T \le t \le T_s \\ 0 & \text{för övrigt} \end{cases}$

Kausalt för $T_s \ge T$

$S_u(T_s) = \int_{T_s-T}^{T_s} h(u)s(T_s - u)\,du = c\int_{T_s-T}^{T_s} 4(T - T_s + u)^2\,du = \frac{c4T^3}{3}$

$V(\int_{T_s-T}^{T_s} h(u)N(T_s - u)\,du) = V_0 \int_{T_s-T}^{T_s}\int_{T_s-T}^{T_s} h(u)h(v)\delta(u - v)\,du\,dv$

$= V_0 \int_{T_s-T}^{T_s} h^2(u)\,du = V_0 c^2 \int_{T_s-T}^{T_s} 4(T - T_s + u)^2\,du = V_0 c^2 \frac{4T^3}{3}$

$\text{SNR} = \frac{c^2 16 T^6 3}{9 V_0 c^2 4 T^3} = \frac{4}{3}\frac{T^3}{V_0}$

709. a. $h(t) = c \cdot s(T - t) = \begin{cases} c \cdot A \cdot e^{-b(T-t)} & -\infty < t \le T \\ 0 & t > T \end{cases}$

Observera att detta filter är icke-kausalt.

$$\begin{aligned} \text{SNR}_{\text{MAX}} &= \frac{1}{N_0}\int_{-\infty}^{\infty} s^2(T-u)du = \\ &= \frac{1}{N_0}A^2 \cdot \int_{-\infty}^{T} e^{-b(T-t)\cdot 2}dt \overset{u=t-T}{=} \\ &= \frac{A^2}{N_0}\int_{-\infty}^{0} e^{2bu}du = \frac{A^2}{2bN_0} \end{aligned}$$

b. Sätt $h_1(t) = \begin{cases} c \cdot A \cdot e^{-b(T-t)} & 0 \le t \le T \\ 0 & t < 0, \quad t > T \end{cases}$

Nu är filtret kausalt.

$$\begin{aligned} \text{SNR} &= \frac{s_u^2(T)}{E(N_u^2(T))} \\ s_u(T) &= \int_{-\infty}^{\infty} h_1(T-t)s(t)dt = \int_0^T c \cdot A^2 \cdot e^{-2bt}dt \\ &= \frac{cA^2}{2b}(1 - e^{-2bT}) \\ N_u(T) &= \int_{-\infty}^{\infty} h_1(T-t) \cdot N(t)dt \\ E(N_u^2(T)) &= V(N_u(T)) \\ &= C\left(\int_{-\infty}^{\infty} h_1(T-s)\cdot N(s)ds, \int_{-\infty}^{\infty} h_1(T-t)\cdot N(t)dt\right) \\ &= \int\int h_1(T-s)\cdot h,\ (T-t)\cdot N_0\delta(s-t)dsdt \\ &= N_0 \cdot \int h_1(T-t)(\int h_1\ (T-s)\delta(s-t)ds)dt \\ &= N_0 \int h_1(T-t)\cdot h_1\ (T-t)dt = \int_0^T c^2A^2e^{-2bt}\cdot N_0 dt \\ &= \frac{N_0c^2A^2}{2b}(1 - e^{-2bT}) \\ \text{SNR} &= \frac{c^2A^4(1-e^{-2bT})^2/(2b)^2}{N_0c^2A^2(1-e^{-2bT})/2b} = \frac{A^2(1-e^{-2bT})}{N_0 \cdot 2b} \end{aligned}$$

c. Sök ett T så att $\frac{SNR}{SNR_{MAX}} > 0.99$. Man får

$1 - e^{-2bT} > 0.99 \Leftrightarrow e^{-2bT} < 0.01 \Leftrightarrow e^{2bT} > 100 \Leftrightarrow 2bT > \ln 100$

Slutsats: $T > \frac{\ln 100}{2b} = \frac{2.30}{b}$

710. $R_X = 1/(1 + 0.81 + 2 \cdot 0.9 \cdot \cos 2\pi f), \quad -\frac{1}{2} < f \leq 1/2$

$$
\begin{aligned}
R_Y(f) &= 1.81 + 1.8 \cdot \cos 2\pi f, \quad -\frac{1}{2} < f \leq 1/2 \\
H(f) &= \frac{R_X(f)}{R_X(f) + R_Y(f)} = \frac{1}{4.276 + 6.516 \cos 2\pi f + 3.24(\cos 2\pi f)^2}
\end{aligned}
$$

711. (a) Sann; (b) Falsk; (c) Falsk; (d) Falsk.