Boat lab

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Contents

1 System description

The model[1] of the system that will be used in this report has a state vector given by $\mathbf{x} = \begin{bmatrix}\xi_w \; \psi_w \; \psi \; r \; b\end{bmatrix}^T$, where

- ψ - Is the average heading without wave disturbance.
- ψ_w - Is a high-frequency component due to wave disturbance.
- $\dot{\xi}_w = \psi_w$
- r - Is the rate of change for the average heading without wave disturbance.
- b - Bias to the rudder.

The model which will be used is stated as

$$\dot{\xi}_w = \psi_w, \tag{1a}$$
$$\dot{\psi}_w = -{\omega_0}^2\xi_w - 2\lambda\omega_0\psi_w + K_w\omega_w, \tag{1b}$$
$$\dot{\psi} = r, \tag{1c}$$
$$\dot{r} = -\frac{1}{T}r + \frac{K}{T}(\delta - b), \tag{1d}$$
$$\dot{b} = \omega_b, \tag{1e}$$
$$y = \psi + \psi_w + v, \tag{1f}$$

where y is the measured heading (compass measurement). w_b, w_w and v are white noise processes. Further, the system can be written as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u + \mathbf{E}\mathbf{w}, \quad y = \mathbf{C}\mathbf{x} + v \tag{2}$$

with $\mathbf{x} = \begin{bmatrix}\xi_w \; \psi_w \; \psi \; r \; b\end{bmatrix}^T$, $u = \delta$ and $\mathbf{w} = \begin{bmatrix}w_w \; w_b\end{bmatrix}$. δ is the rudder angle relative to the BODY frame. The purpose of the model is to estimate the course angle (also referred to as the compass angle in this report) without the wave disturbance. Therefore, the ship is modelled as a system not affected by waves and include the disturbance only in the measurement. Lastly, the current b only affects the rudder angle, as shown in (1d).

[1] The complete derivation of the system is given in [3].

2 Part I

The main objective for this part is to find a mathematical model for the ship. This is done by finding the transfer function from the input δ to the output ψ. Further, the parameters in the model will be decided under both smooth and rough weather conditions.

2.1 Problem 1a

Want to calculate the transfer function from δ to ψ, $H(s) = \frac{\psi}{\delta}(s)$, parameterized by T and K. Taking the derivative of (1c) gives

$$\ddot{\psi} = \dot{r}, \tag{3}$$

$$\ddot{\psi} = -\frac{1}{T} + \frac{K}{T}(\delta - b). \tag{4}$$

Since no disturbances are assumed, it is given from (1e) that $\dot{b} = 0$. A further assumption is that all initial conditions are zero. These two assumptions implies that $b = 0$. By using this result, combined with (1c), (4) gives

$$\ddot{\psi} = -\frac{1}{T}\dot{\psi} + \frac{K}{T}\delta. \tag{5}$$

Taking the Laplace transform of (5) yields

$$s^2\psi(s) = -\frac{1}{T}s\psi(s) + \frac{K}{T}\delta(s). \tag{6}$$

By the use of algebraic manipulation, (6) can be rewritten as

$$H(s) = \frac{\psi}{\delta}(s) = \frac{K}{s(1+Ts)}. \tag{7}$$

2.2 Problem 1b

The goal is to determine the constants K and T in smooth weather conditions (i.e. no disturbances). This is done by applying two different sine waves on the input respectively. The simulink model is shown in fig. 1. The amplitude of the output signal will then be equal to the amplitude of the transfer function found in section 2.1. This amplitude can be expressed mathematically as

$$\mid H(j\omega) \mid = \mid \frac{K}{j\omega(1+T\omega)} \mid = \frac{K}{\omega\sqrt{1+T^2\omega^2}}. \tag{8}$$

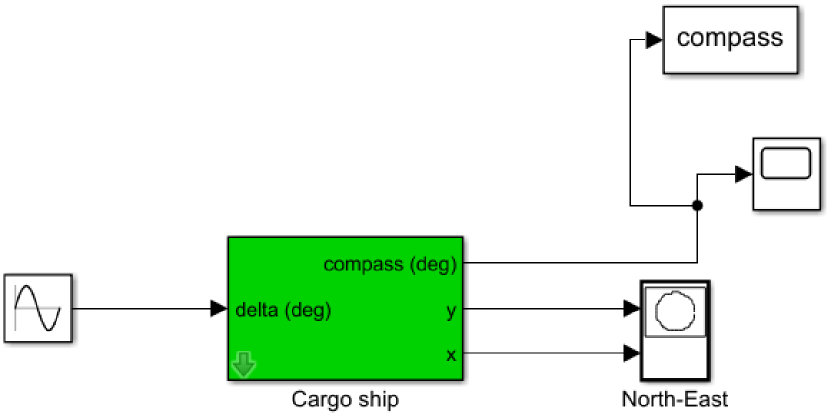


Figure 1: Applying a sine wave to the ship model.

By using MATLAB 1, the amplitude of the output signals shown in fig. 2a and fig. 2b, $A_1 =\mid H(j\omega_1) \mid$ and $A_2 =\mid H(j\omega_2), \mid$ were found to be 29.3582 and 0.8491 respectively. This results in two equations

$$\frac{K}{\omega_1\sqrt{1+T^2\omega_1^2}} = A_1, \tag{9a}$$

$$\frac{K}{\omega_2\sqrt{1+T^2\omega_2^2}} = A_2. \tag{9b}$$

The following equations for T and K can be derived from (9)

$$T = \sqrt{\frac{A_2^2\omega_2^2 - A_1^2\omega_1^2}{A_1^2\omega_1^4 - A_2^2\omega_2^4}}, \tag{10a}$$

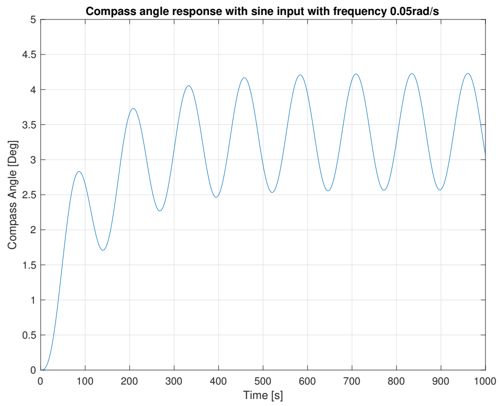
$$K = A_1\omega_1\sqrt{1+T^2\omega_1^2}. \tag{10b}$$

Inserted values gives

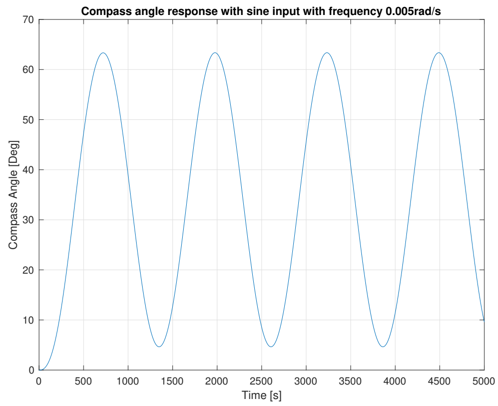
$$T = 70.55\ [s], \tag{11a}$$

$$K = 0.16\ [s^{-1}]. \tag{11b}$$

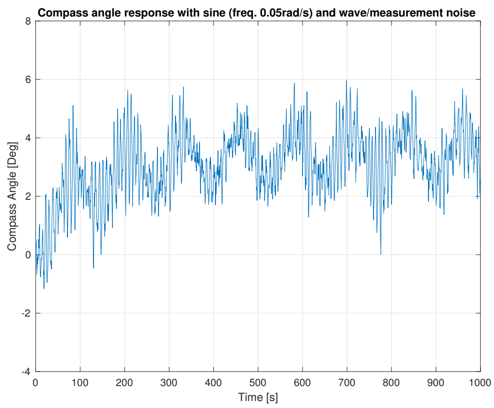
The units for the parameters are found by looking at (4). It is known that $\ddot{\psi}$ has unit $[\frac{rad}{s^2}]$, which means that K and T must have units equal to $[s^{-1}]$ and $[s]$ respectively.



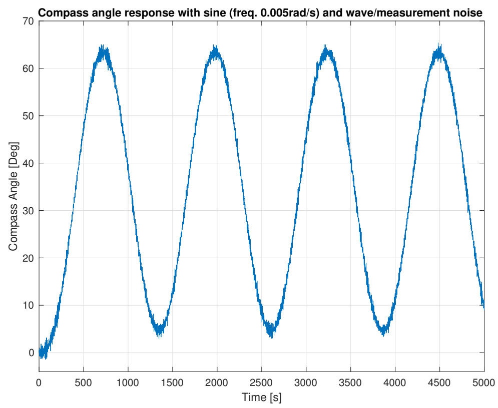
(a) Compass angle without noise, $\omega = 0.05 \frac{rad}{s}$.



(b) Compass angle without noise, $\omega = 0.005 \frac{rad}{s}$.



(c) Compass angle with noise, $\omega = 0.05 \frac{rad}{s}$.



(d) Compass angle with noise, $\omega = 0.005 \frac{rad}{s}$.

Figure 2: Compass angle with and without wave and measurement noise and with a sine input at different frequencies.

```matlab
% The "to workspace"-block is named "compass"
maxValue = max(compass.signals.value);
minValue = min(compass.signals.value(1000:5000);
% The sampling time was implemented due to the ... sine wave showing abnormalities in the start ... phase
amplitude = (maxValue - minValue) / 2;
```

MATLAB 1: Code implementation on how the amplitude was found.

2.3 Problem 1c

Similar to section 2.2, the goal is to determine the constants K and T. However, this time the weather conditions are rough (i.e. waves and measurement noise). Because of the disturbances affecting the peak values, shown in fig. 2c and fig. 2d, the values for A_1 and A_2 can not be found using MATLAB 1. The MATLAB script gave amplitude values which resulted in a complex time constant. Therefore, the amplitude values were manually read from the graph. In hindsight, instead of manually reading out the values, an estimate of the correct peak values could be calculated by taking the average of all the peak values. The amplitude values could then be calculated from the estimated peak values. However, the amplitude values for A_1 and A_2 were read to be 29.5 and 1 respectively. This resulted in the following parameters

$$T = 55.6 \ [s] \tag{12a}$$

$$K = 0.15 \ [s^{-1}] \tag{12b}$$

Generally, it is difficult to get good estimates of the parameters under rough weather conditions. The output is characterized by noise, something which makes it almost impossible to read the correct amplitude values. In addition, the equation for T is sensitive for changes in both A_1 and A_2. Small variations in the amplitudes will result in large variations for T. Rough weather conditions will therefore complicate the process of finding good estimates for the parameters.

2.4 Problem 1d

The step response of the model was plotted with the parameters found in section 2.2. No disturbances is assumed for both the model and the ship. The two step responses are shown in fig. 3. As seen in the figure, the model's response is somewhat slower than the ship's response. However, the model is overall a good estimation. A small deviation between the model and the ship occurs first after 1000 seconds have passed. After 5000 seconds the deviation between the ship and the model is approximately 50 degrees. Both observations are acceptable.

To improve the model, other values for T where tested. By looking at the plot, it could be intuitive to decrease T. This is because the model has a somewhat slower response than the ship. A smaller T resulted in a smaller gap between the ship and model after 5000 s. However, the changed T resulted in a deviation between the ship and the model for small time values. It was therefore concluded that the model could not be improved appreciable by tuning the parameters.

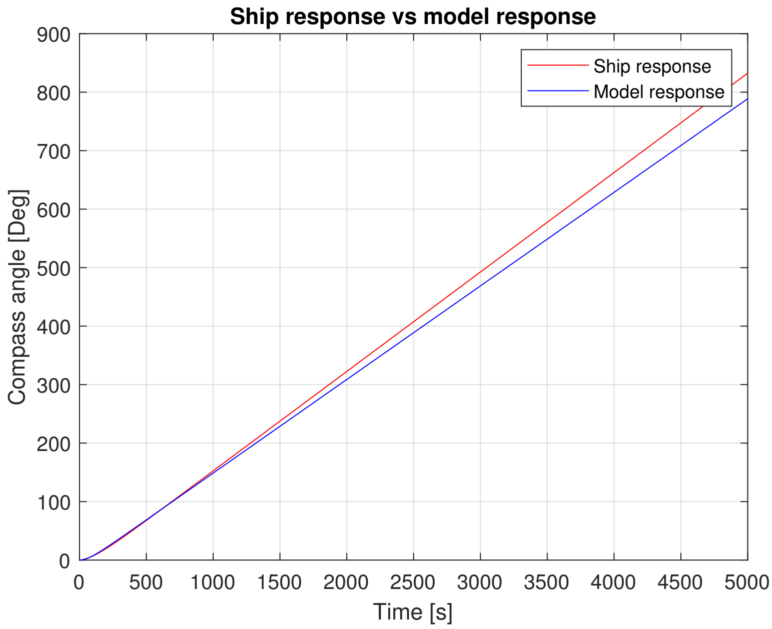


Figure 3: Step response of the model and the ship.

3 Part II

In this section, both an estimation and the analytical expression for the Power Spectral Density function of ψ_w is found.

3.1 Problem 2a

An influential wave force on the compass measurement, ψ_w, was applied to the system and the power spectral density function, $S_{\psi_\omega}(\omega)$, was estimated using a sampling frequency of $10Hz$. The power spectral density intensity for each frequency was found using the MATLAB function for the Welch's power spectral density estimate, shown in line 3 in MATLAB 2. Where x is the size of the waves in radians and fs is the frequency which was set to $10Hz$. Since fs was given in Hz, the resulting pxx and f was scaled with $\frac{1}{2\pi}$ and 2π respectively to obtain correct units. The PSD estimation in fig. 4 shows the correlation between pxx and fs.

The power spectral density shows how the power is distributed over different frequencies.

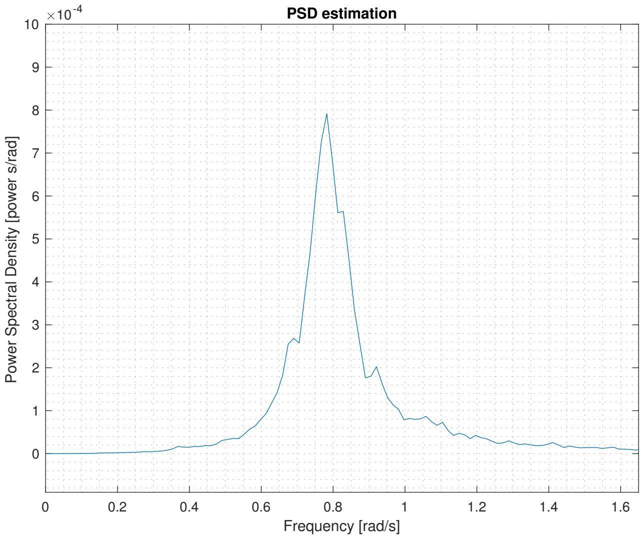


Figure 4: Power spectral density plotted with frequency.

3.2 Problem 2b

First, the task is to find an analytical expression for the transfer function of the wave response model. The transfer function is from w_w to ψ_w, $\frac{\psi_w}{w_w}(s)$. Taking the derivative of (1b) gives

$$\ddot{\psi}_w = -{\omega_0}^2\dot{\xi}_w - 2\lambda\omega_0\dot{\psi}_w + K_w\dot{w}_w. \tag{13}$$

Inserting (1a) into (13) results in

$$\ddot{\psi}_w = -{\omega_0}^2\psi_w - 2\lambda\omega_0\dot{\psi}_w + K_w\dot{w}_w. \tag{14}$$

A Laplace transform of (14), assuming initial conditions are zero, yields

$$s^2\psi_w(s) = -{\omega_0}^2\psi_w(s) - 2\lambda\omega_0 s\psi_w(s) + K_w s w_w(s). \tag{15}$$

By the use of algebraic manipulation, (15) can be rewritten as

$$G(s) = \frac{\psi_w}{w_w}(s) = \frac{K_w s}{s^2 + 2\lambda\omega_0 s + {\omega_0}^2}. \tag{16}$$

Secondly, it is desirable to find an analytical expression for the Power Spectral Density function. The given situation can be described as white noise w_w being coloured by a filter. The output ψ_w will therefore only contain parts of the input's frequencies. An illustration of the situation is shown in fig. 5. Knowing that the input is Fourier transformable, the input spectrum is simply modified by $G(jw)$ when going through the filter. The equation relating the input and the ouput spectral functions is given by [1, p.106]

$$P_{\psi_w}(\omega) = P_{w_w}(\omega)G(j\omega)G(-j\omega). \tag{17}$$

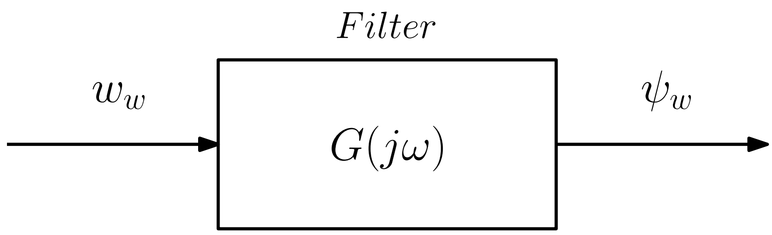


Figure 5: Colouring of a white noise process.

It is stated that w_w is a zero mean white noise process with unity variance. Denoting the white noise spectral amplitude as A gives

$$P_{w_w}(j\omega) = A. \tag{18}$$

The unity variance of w_w is equivalent to setting $A = 1$, thus

$$P_{w_w}(j\omega) = 1. \tag{19}$$

This simplifies (17) to

$$P_{\psi_w}(\omega) = G(j\omega)G(-j\omega) = \frac{(K_\omega \omega)^2}{\omega^4 + \omega_0^4 + 2\omega^2 {\omega_0}^2(2\lambda^2 - 1)}. \tag{20}$$

This result makes the precondition for creating a Kalman filter (i.e. all noises are white) still valid. The coloured noise can simply be seen on as white noise passing through a filter. This filter can then be put in series with the system.

3.3 Problem 2c

The modal peak frequency ω_0 is found by investigating at which frequency the Power Spectral Density in fig. 2b has its maximum value. The corresponding intensity is denoted as σ^2. By using MATLAB 2 on fig. 4, the parameters were found to be

$$\omega_0 = 0.7823\ [rad/s] \quad \text{and} \quad \sigma^2 = 7.9191 \cdot 10^{-4}\ [\frac{power \cdot s}{rad}] \tag{21}$$

```
fs = 10;
x = (psi_w);
[pxx,f] = pwelch(x(2,:)*pi/180,4096,[],[],fs);
pxx1 = pxx/(2*pi);
f1 = f * 2*pi;
%%Finding max value with corr. frequency%%
LineH = get(gca, 'children');
Value = get(LineH, 'YData');
Time = get(LineH, 'XData');
[maxValue, maxIndex] = max(Value);
maxTime = Time(maxIndex);
display(maxTime);
```

MATLAB 2: Code implementation showing how ω_0 and σ^2 were found.

3.4 Problem 2d

Different responses of the mathematical model of the power spectral density function presented in (20) was found by adjusting λ. The best fitting value of λ was found by plotting the mathematical model with the estimation. This was done repeatedly with different values of λ. With this procedure of trial and error, $\lambda = 0.08$ was found relatively quickly to be the best fit. This is shown in fig. 6, where the mathematical model is plotted against the estimation with different λ values. K_ω was defined as $2\lambda\omega_0\sigma$, with values of ω_0 and σ given in (21).

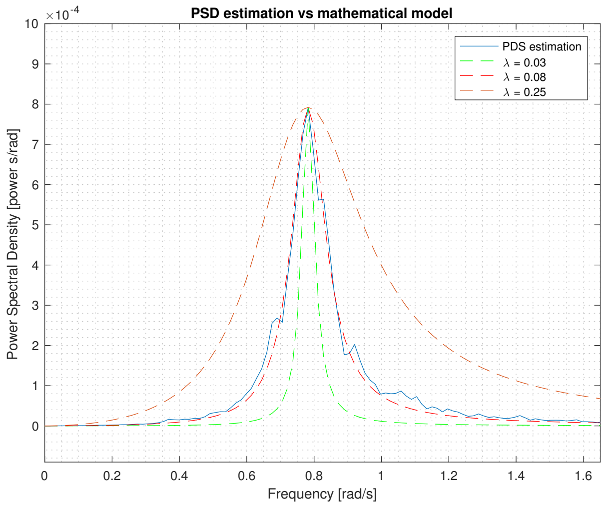


Figure 6: PSD estimation vs mathematical model with different values of λ.

4 Part III

In this section, an autopilot for the ship is designed. The ship model in the simulations only holds for small deviations in compass values, so the reference value is set equal 30 degrees. Note in this section that the plots all have the rudder angle saturated, due to the rudder angle being constrained at ±45.

4.1 Problem 3a

It is desired to design a PD controller based on (7) with crossover frequency $w_c = 0.1$ such that the phase margin is 50°. Firstly, the controllers derivation time T_d was set equal to the time constant T from (12a). Thus the new open loop transfer function became

$$H_0 = H_{pd} H_{\psi\delta} = \frac{K K_{pd}}{s(1 + T_f s)}. \tag{22}$$

The gain K_{pd} was found by solving the amplitude response of the system, which is given by

$$\log |H_0(jw)| = \log K + \log K_{pd} - \log |jw| - \log |1 + jwT_f|. \tag{23}$$

Evaluated at the crossover frequency, $\log |H_0|$ is equal to 0 dB. This yields

$$\log K_{pd} = -\log K + \log w_c + \log |1 + jw_c T_f|. \tag{24}$$

In a limited PD controller the goal is to raise the phase of the system for a certain band of frequencies. Therefore it is convention to choose $T_f = \alpha T_d$, where α is a number between 0 and 1. Thus, by looking at the bode plot in MATLAB, the constant α was tweaked until the open loop system had the desired value for the phase margin. MATLAB found the values to be

$$K_{pd} = 0.8161, \tag{25a}$$

$$T_f = \alpha T_d = 0.1190 \cdot 70.5500 = 8.3954. \tag{25b}$$

which gave the bode plot as shown in fig. 7. An implementation of the PD-controller is shown in fig. 8.

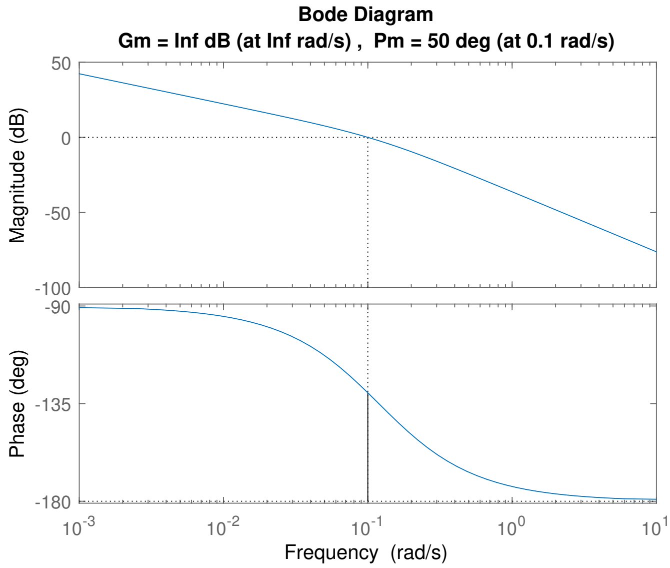


Figure 7: Bode Plot with desired phase margin.

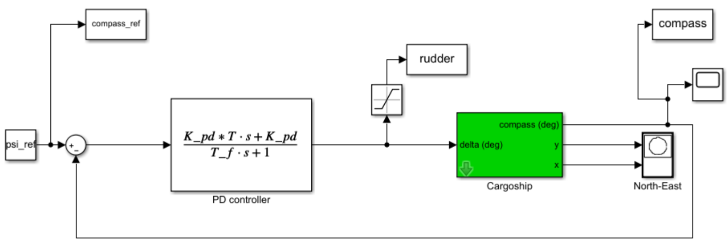


Figure 8: Implementation of PD-controller.

4.2 Problem 3b

With only measurement noise, the auto pilot behaves nicely and reaches the desired reference compass angle. As seen in fig. 9, the system is fast and does not oscillate.

However, a dip in the response can be seen after approximately 50 s. This might be due to the derivative effect of the controller. Since a limited derivation effect is implemented, the derivative term has its own time constant. Therefore, the effect of the derivation term is not seen before approximately 50 s. Ultimately, the dip prevents an overshoot, thus making the system more robust.

Two other possible reasons for the dip are the saturation in the rudder input and non-linearities in the ship model. Simulating without saturation on the input, results in the dip being removed and the response becoming oscillatory. On the other hand, the ship model could contain non-linearities.

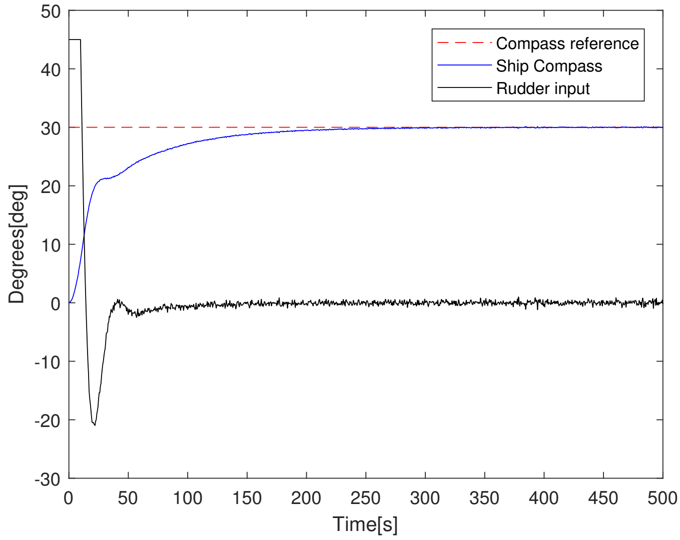


Figure 9: Response of auto pilot with only measurement noise.

Non-linearities are often the origin of "weird" responses.

The main advantage of the PD controller is the increase in robustness it provides. By lifting the phase in certain bands of frequencies (in this case around $0.1\frac{rad}{s}$) it makes it possible to increase the gain of the system such that the response is faster, without the sacrifice to the robustness. With a normal P controller on the other hand, an increase in the gain would to a much larger extent decrease the robustness. In this particular case, the PD controller also cancels out the time constant, T, of the system. Thus the dynamics of the system is fully decided by T_f. Again, this would not be possible with just a P controller, and the dynamics of the system would be bound by T. Note that the time constants found in (12a) and the model (7) are estimates, and the ship may therefore behave slightly different to what is expected by choosing $T_d = T$.

4.3 Problem 3c

With a current disturbance, the auto pilot never reaches the desired reference. As seen in fig. 10 there is a stationary deviation between the compass angle of the ship, ψ and the desired angle. Thus it can be concluded the auto pilot does not work. This is due to the fact that current is a constant disturbance and since there is not a pure integrator in the controller, it will not be able to counteract its effect.

Although the auto pilot fails in this case, note that the input to the rudder angle in fig. 10 is almost equal to the one in fig. 9. The difference is that the rudder angle for this scenario reaches a non-zero stationary value

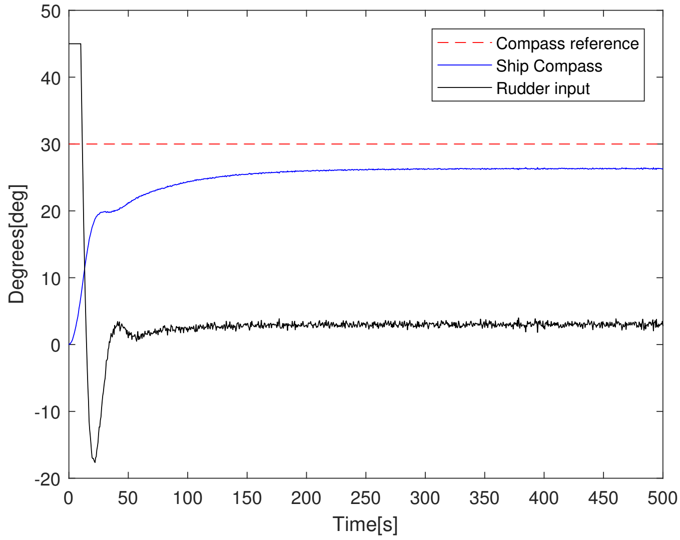


Figure 10: Response of auto pilot with current disturbance and measurement noise

value. This is due to the deviation between the set-point angle and the compass angle of the ship. The ship will try to reach its set-point, but the current disturbance will counteract this attempt. Ultimately, the compass angle of the ship will be unchanged and thus have a stationary deviation.

Since the current is a constant disturbance, it does not exploit the weakness of a PD controller, and does not lead to an increase in noise to the rudder. Thus, by adding an integrator to this controller as well, the stationary deviation could be removed and the auto pilot would work again.

4.4 Problem 3d

With a wave disturbance, the response becomes as seen in fig. 11. Notice that the autopilot, though not optimal, roughly follows the reference for the compass angle. The rudder input, on the other hand, does not act as desired. This is due to the fact that the wave disturbance exploits the weakness of the PD controller. Its high frequent noise is amplified due to the PD controller taking its derivative. This results in large and high-frequent variations in the rudder input. Overall, it could be said that the ship tries to counteract the effect each wave has on the compass angle. A solution to this problem could be to add a low-pass filter in the feedback loop. In this way, all the high-frequent oscillations would be filtered out.

From a physical perspective, counteracting the effect of each wave is hard to achieve and also a huge waste of power. From a mechanical perspective, the rudder may take damage from such large and sudden variations, mak-

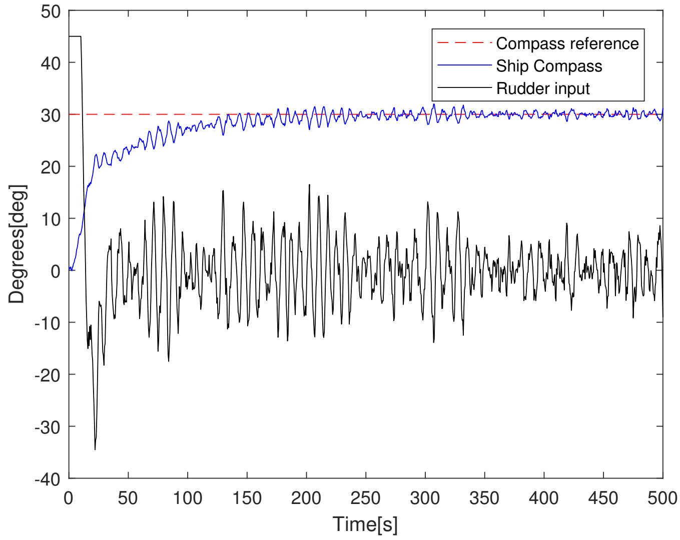


Figure 11: Response of auto pilot with wave disturbance and measurement noise

ing it a non-sustainable solution. Thus the system cannot be said to work satisfactory although the auto pilot roughly follows the reference.

5 Part IV

The goal for this part is to check if the system given in section 1 is observable in different scenarios. The different scenarios include no disturbances, only current disturbance, only wave disturbance and both wave disturbance and current disturbance.

5.1 Problem 4a

As stated in section 1, the model of the ship can be written on the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u + \mathbf{E}\mathbf{w}, \quad y = \mathbf{C}\mathbf{x} + v, \tag{26}$$

with

$$\mathbf{x} = \begin{bmatrix} \xi_w \\ \psi_w \\ \psi \\ r \\ b \end{bmatrix}, \quad u = \delta \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} w_w \\ w_b \end{bmatrix}. \tag{27}$$

Equation (1) gives

$$\dot{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -{\omega_0}^2 & -2\lambda\omega_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{T} & -\frac{K}{T} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{=\mathbf{A}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{K}{T} \\ 0 \end{bmatrix}}_{=\mathbf{B}} u + \underbrace{\begin{bmatrix} 0 & 0 \\ K_w & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{=\mathbf{E}} \mathbf{w}, \tag{28a}$$

$$y = \underbrace{\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \end{bmatrix}}_{=\mathbf{C}} \mathbf{x} + v. \tag{28b}$$

5.2 Problem 4b

The observability matrix for a system is generally given by[2]

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ . \\ . \\ \mathbf{CA}^{n-1} \end{bmatrix}. \tag{29}$$

As shown in (29), the observability matrix is not directly affected by the disturbance matrix **E** nor the input matrix **B**. However, several of the states

[2] [2, p. 197]

in the state space model represents disturbances (e.g ψ_w and ξ_w). Therefore, by excluding or including different disturbances in the model, the **A** matrix and the **C** matrix will change. This can again affect the observability of the system.

By assuming no disturbances, it is given that $b = 0$. Further, since ψ_w is a high-frequency component due to wave disturbance, the assumption also implies that $\psi_w = 0$. The state ξ_w is only dependent on ψ_w, so $\xi_w = 0$. Ultimately, this gives a new state vector

$$\mathbf{x} = \begin{bmatrix} \psi \\ r \end{bmatrix}. \tag{30}$$

Based on the new state vector, the state space model from (27) is reduced to

$$\dot{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{T} \end{bmatrix}}_{=\mathbf{A}} \mathbf{x} + \begin{bmatrix} 0 \\ \frac{K}{T} \end{bmatrix} u, \tag{31a}$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{=\mathbf{C}} \mathbf{x} + v. \tag{31b}$$

The observability matrix given by (29) can be calculated using MATLAB. The MATLAB function $obsv(A, C)$ gives

$$\mathcal{O} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{32}$$

The observability matrix has full column rank, $rank(\mathcal{O}) = n = 2$. Therefore, the system is observable [3] without disturbances.

5.3 Problem 4c

Assuming only current disturbance means that $\psi_w = \xi_w = 0$, similar as in section 5.2. However, the current affects the bias on the rudder angle. This means that $b \neq 0$. The state vector in this scenario is therefore

$$\dot{\mathbf{x}} = \begin{bmatrix} \psi \\ r \\ b \end{bmatrix}, \tag{33}$$

which gives the state space model

$$\dot{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{1}{T} & -\frac{K}{T} \\ 0 & 0 & 0 \end{bmatrix}}_{=\mathbf{A}} \mathbf{x} + \begin{bmatrix} 0 \\ \frac{K}{T} \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w_b, \tag{34a}$$

[3] [2, p. 197]

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{=\mathbf{C}} \mathbf{x} + v. \tag{34b}$$

The MATLAB function $obsv(A, C)$ gives

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -0.0142 & -0.0023 \end{bmatrix} \tag{35}$$

Also here, the observability matrix has full column rank. The system is therefore observable with current disturbance.

5.4 Problem 4d

Assuming only wave disturbance means that $b = 0$. Since ψ_w is a high-frequency component due to wave disturbance, and ξ_w is dependent on ψ_w alone, it is given that both $\psi_w \neq 0$ and $\xi_w \neq 0$. In this scenario, the state vector is

$$\dot{\mathbf{x}} = \begin{bmatrix} \xi_w \\ \psi_w \\ \psi \\ r \end{bmatrix}. \tag{36}$$

This gives the following state space model

$$\dot{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -{\omega_0}^2 & -2\lambda\omega_0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{1}{T} \end{bmatrix}}_{=\mathbf{A}} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{K}{T} \end{bmatrix} u + \begin{bmatrix} 0 \\ K_w \\ 0 \\ 0 \end{bmatrix} w_w, \tag{37a}$$

$$y = \underbrace{\begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}}_{=\mathbf{C}} \mathbf{x} + v. \tag{37b}$$

By using $\omega_0 = 0.78$ and $\lambda = 0.08$, as found in section 3.3 and section 3.4 respectively, the resulting observability matrix is

$$\mathcal{O} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -0.6084 & -0.1248 & 0 & 1 \\ 0.0759 & -0.5928 & 0 & -0.0142 \\ 0.3607 & 0.1499 & 0 & 0.0002 \end{bmatrix}. \tag{38}$$

The observability matrix has full column rank. Thus, the system is observable with wave disturbances.

5.5 Problem 4e

Assuming both wave disturbance and current disturbance, the state space model is given by the one found in section 5.1. The corresponding observability matrix is

$$\mathcal{O} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ -0.6084 & -0.1248 & 0 & 1 & 0 \\ 0.0759 & -0.5928 & 0 & -0.0142 & -0.0023 \\ 0.3607 & 0.1499 & 0 & 0.002 & 3 \cdot 10^{-5} \\ -0.0926 & 0.3460 & 0 & -3 \cdot 10^{-6} & -5 \cdot 10^{-7} \end{bmatrix}. \tag{39}$$

The observablity matrix has full column rank. This means that the system is observable with both current disturbance and wave disturbance.

Ultimately, the system is observable in all scenarios regarding different disturbances. This means that, in all scenarios, the knowledge of the input u and output y over an interval $[0, t_1]$ where t_1 is finite, is sufficient to determine uniquely the initial state $\mathbf{x}(0)$. This trait lays the foundation for the development of an estimator, since it is possible to determine $\mathbf{x}$ by having knowledge of the input and output.

6 Part V

In this section a discrete Kalman filter is designed. The filter is used in both feed back and feed forward control of the system. The different responses with the Kalman filter are presented and discussed.

6.1 Problem 5a

The exact discretization of the continuous system given in (28) is obtained by using numerical discretization in MATLAB. The sample frequency used was $10Hz$ and the system was inserted with the following values

$$\begin{aligned} K &= 0.16, \\ \omega_0 &= 0.7823, \\ \sigma &= \sqrt{0.00079191}, \\ \lambda &= 0.08, \\ T &= 70.55, \\ K_w &= 2\lambda\omega_0\sigma = 0.0035. \end{aligned}$$

The discrete state space model is given by

$$\dot{\mathbf{x}}[k+1] = \mathbf{A_d}\mathbf{x}[k] + \mathbf{B_d}u[k] + \mathbf{E_d}\mathbf{w}[k], \tag{41a}$$

$$y[k] = \mathbf{C_d}\mathbf{x}[k] + v[k], \tag{41b}$$

where the discretized system matrices are

$$\mathbf{A_d} = \begin{bmatrix} 1.0 & 0.099 & 0 & 0 & 0 \\ -0.061 & 0.98 & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 0.1 & -1.1\cdot 10^{-5} \\ 0 & 0 & 0 & 1.0 & -2.3\cdot 10^{-4} \\ 0 & 0 & 0 & 0 & 1.0 \end{bmatrix}, \tag{42a}$$

$$\mathbf{B_d} = \begin{bmatrix} 0 \\ 0 \\ 1.1\cdot 10^{-5} \\ 2.3\cdot 10^{-4} \\ 0 \end{bmatrix}, \tag{42b}$$

$$\mathbf{E_d} = \begin{bmatrix} 1.8\cdot 10^{-5} & 0 \\ 3.5\cdot 10^{-4} & 0 \\ 0 & -3.8\cdot 10^{-7} \\ 0 & -1.1\cdot 10^{-5} \\ 0 & 0.1 \end{bmatrix}, \tag{42c}$$

$$\mathbf{C_d} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \end{bmatrix}. \tag{42d}$$

The discretized matrices were obtained using code shown in MATLAB 3. The values used in the continouos matrices are given in section 5.1.

```
%Defining values
K = 0.16;
omega_0 = 0.7823;
sigma = sqrt(0.00079191);
lambda = 0.08;
T = 70.55;
Kw = 2*lambda*omega_0*sigma;

%Defining the continouos system
A = [0 1 0 0 0; -(omega_0)^2 -2*lambda*omega_0 ...
    0 0 0; 0 0 0 1 0; 0 0 0 -1/T -K/T; 0 0 0 0 0];
B = [0; 0; 0; K/T; 0];
C = [0 1 1 0 0];
E = [0 0; Kw 0; 0 0; 0 0; 0 1];
T1 = 10;

%Discretization
[ad, bd] = c2d(A, B, 1/T1);
[ad, ed] = c2d(A, E, 1/T1);
cd = C;
```

MATLAB 3: Discretization of the continuous system

6.2 Problem 5b

The model of the ship has different configurations for different disturbances. Thus, the measurement noise can be estimated by simply inserting zero input into the model with only measurement noise turned on. This gave the plot as shown in fig. 12. Then, by saving the values to the workspace and using the MATLAB script shown in MATLAB 4, the variance of the measurement was estimated to be 0.0020 deg^2 ($6.092 \cdot 10^{-7}$ rad^2).

```
measurement_noise = var(compass.signals.values);
```

MATLAB 4: Code implementation showing the variance of the measurement noise was found.

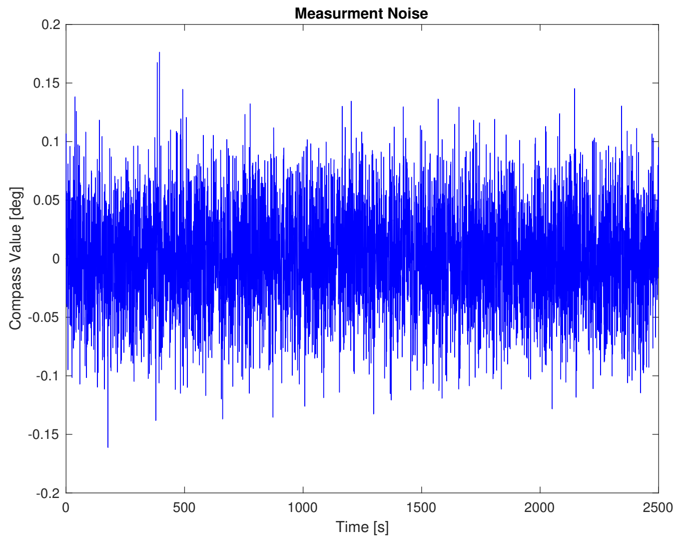


Figure 12: Measurement noise.

6.3 Problem 5c

Equations for the discrete Kalman filter are given in [4]. The filter is initialized with the prior estimate $\hat{\mathbf{x}}_0^-$ and its error covariance $\mathbf{P}_0^-$ given by [3]. Further, the filter was implemented in MATLAB with the recursive formula presented in fig. 13. The implementation is shown in MATLAB 5.

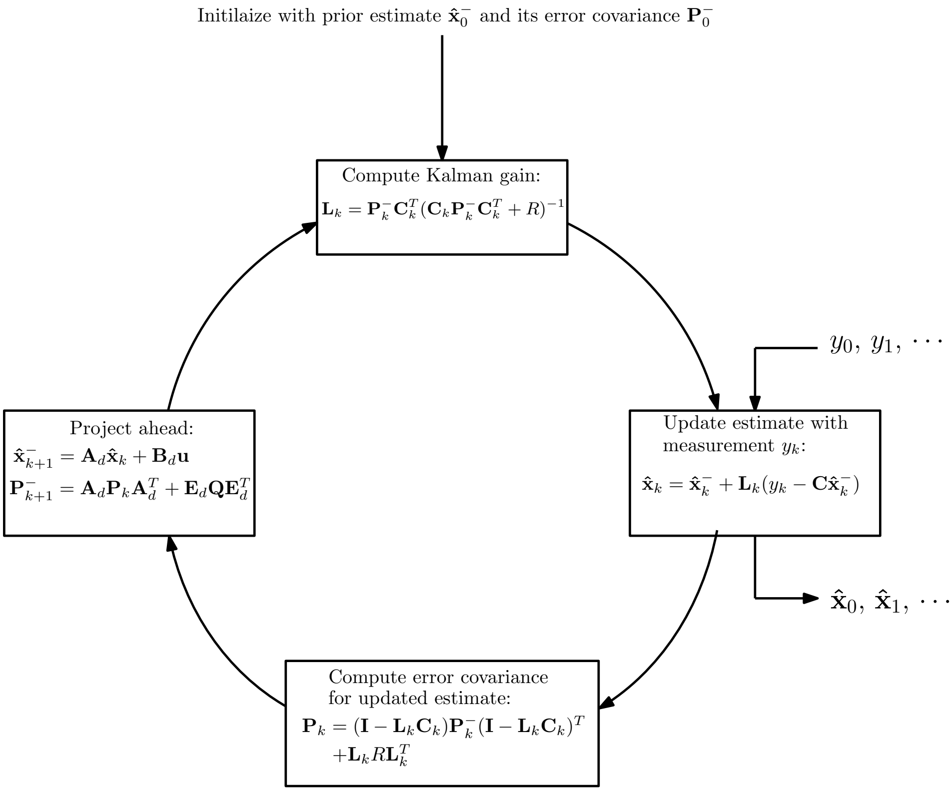


Figure 13: Visual representation of the discrete Kalman filter procedure loop.

```matlab
function [psi,b] = ...
   Kalman_filter(compass_measurement, ...
   rudder_input, data)
%ALGORITHM FOR COMPUTING KALMAN FILTER%

%%Defining init flag
persistent init_flag ad bd Cd ed Q_k R ...
   prior_P_covar prior_x_est I
%%Defining the needed starting conditions
if isempty(init_flag)
    [ad, bd, Cd, ed, Q_k, R, prior_P_covar, ...
       prior_x_est, I] = deal(data.ad, data.bd, ...
       data.Cd, data.ed, data.Q_k, data.R, ...
       data.prior_P_covar, data.prior_x_est, ...
       data.I);
    init_flag = 1;
end

%%Computing Kalman gain
L_k = prior_P_covar * transpose(Cd) * (Cd * ...
   prior_P_covar * transpose(Cd) + R)^(-1);

%%Updating estimate with measurement y_k
x_est = prior_x_est + L_k * ...
   (compass_measurement - Cd * prior_x_est);

%%Computing error covariance for updated estimate
P_k = (I - L_k * Cd) * prior_P_covar * ...
   transpose(I - L_k * Cd) + L_k * R * ...
   transpose(L_k);

%%Project ahead
prior_x_est = ad * x_est + bd * rudder_input;
prior_P_covar = ad * P_k * transpose(ad) + ed * ...
   Q_k * transpose(ed);

%%Outputs
psi = x_est(3);
b = x_est(5);
end
```

MATLAB 5: Code implementation for discrete Kalman filter with estimated compass angle and bias output.

Among other parameters, the function takes in a parameter *data*. This parameter is defined as a *struct* and is used to contain the values for the different matrices in the system, as can be seen in MATLAB 6.

```
% Defining the correct matrices
cd = [0 1 1 0 0];
ad = [0.997 0.0993 0 0 0;-0.0608 0.985 0 0 0;0 ...
    0 1 0.0999 -1.13e-05;0 0 0 0.999 -0.000227;0 ...
    0 0 0 1];
bd = [0;0;1.13e-05;2.279e-04;0];
Q_k = [30 0; 0 10^(-6)];
I = [1 0 0 0 0; 0 1 0 0 0; 0 0 1 0 0; 0 0 0 1 ...
    0; 0 0 0 0 1];
R = 0.0020*pi^2/180^2;
ed = [1.753e-05 0;3.50e-04 0;0 -3.78e-07;0 ...
    -1.13e-05;0 0.100];
prior_x_est = [0; 0; 0; 0; 0];
prior_P_covar = [1 0 0 0 0; 0 0.013 0 0 0; 0 0 ...
    pi^2 0 0; 0 0 0 1 0; 0 0 0 0 2.5*10^(-3)];

% Put the different matrices in a struct
% This struct is used in the Kalman filter
data = struct('ad', ad, 'bd', bd, 'Cd', Cd, ...
    'ed', ed, 'Q_k', Q_k, 'I', I, 'R',R, ...
    'prior_P_covar',prior_P_covar, ...
    'prior_x_est', prior_x_est);
```

MATLAB 6: Definition of data struct.

The different values in the struct are dealt to the persistent variables in the MATLAB function block during initialization, that is when the $init\_flag$ is an empty variable. Furthermore, the outputs of the function block are set to be scalars, ψ and b respectively. Since the compass measurement and input are continuous processes, a zero-order hold was put on both the output of the process and the output of the controller. The sampling frequency of the hold blocks were also set to 10 Hz. Memory blocks were put on the output of the Kalman filter to prevent algebraic loops. How the different blocks are implemented is shown in fig. 14.

6.4 Problem 5d

As opposed to the controller in section 4.3, the new controller has a feed forward of the estimated bias. The estimation of the bias is done by the

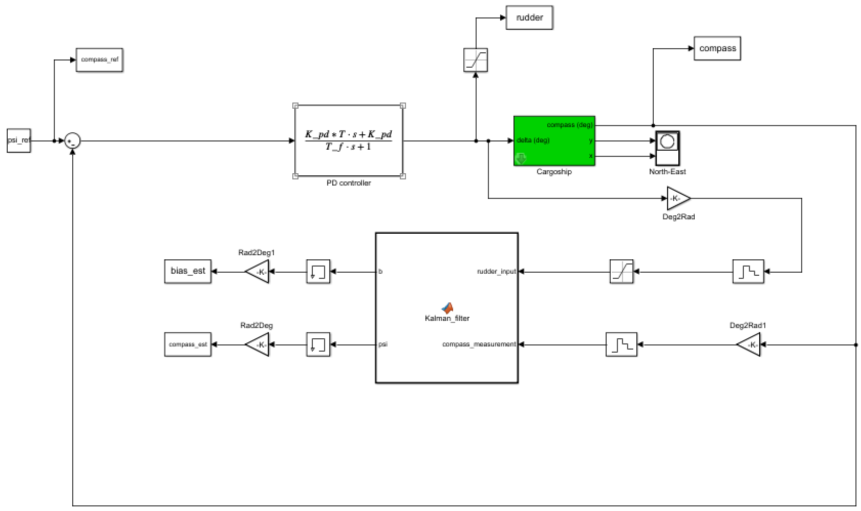


Figure 14: Implementation of Kalman function block.

Kalman filter. Since the estimated bias is equal to the steady-state error and added to the input signal, the feed forward cancels out the stationary deviation from fig. 10. The response is shown in fig. 15, and is satisfactory compared to the response in fig. 10.

As seen in fig. 16, the two rudder inputs differ from each other when the bias has a non-zero value. This is due to the feed forward, where the bias is added to the rudder angle. When the bias has a positive non-zero value, the rudder input is increased. This can be interpreted as a "counterweight" to the current disturbance. By increasing the rudder input, the effect of the disturbance will be cancelled.

After the reference for the compass angle has been met, the error in the closed loop will be zero. The input will then single-handedly be decided by the feed forward. This is shown in fig. 16 where the rudder input and the bias will be equal to each other after approximately 200 s.

In the scenario where the feed forward is not present, the rudder input is decided by the output of the PD controller alone. When the bias is a non-zero value, the rudder input will not be large enough to overcome the bias. Thus, resulting in a response with a stationary deviation between the compass angle reference and measurement. The rudder input will, similar to the rudder input with feed forward, station itself on a non-zero value equal to the bias. However, the value should in this case be interpreted as the error between the compass angle reference and measurement, which equals the bias.

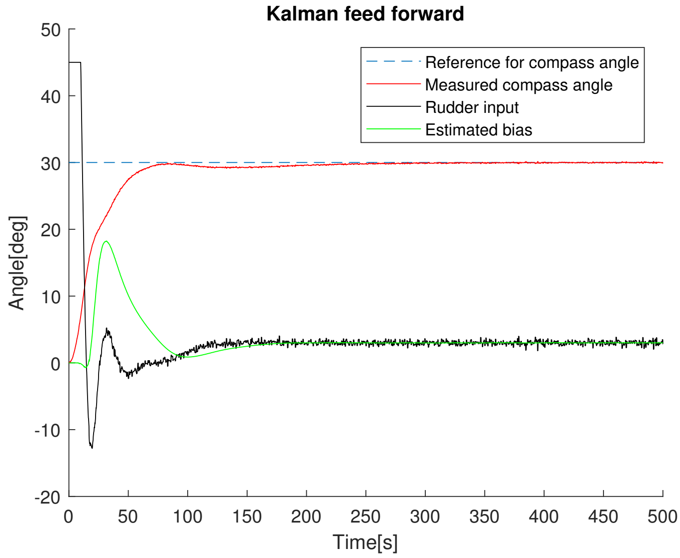


Figure 15: Response with feed forward from Kalman filter.

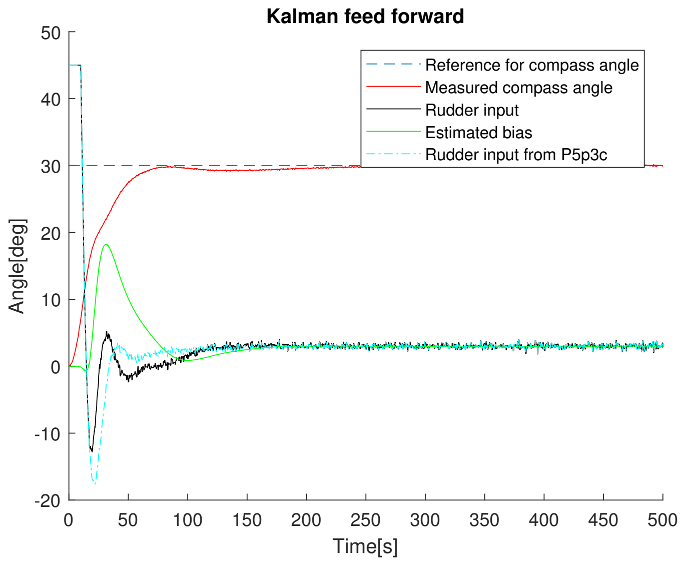


Figure 16: Comparison between inputs with and without feed forward.

6.5 Problem 5e

The autopilot in this section uses the filtered compass angle obtained from the discrete Kalman filter in its feedback instead of the measured compass angle described in section 6.3. A simulink implementation is shown if fig. 19.

The ship is affected by measurement, wave and current noise. Since the wave disturbance is a high-frequent component, the compass angle measurement consisting of the compass angle, wave disturbance and measurement noise will fluctuate excessively. The measurement noise is also contributing to this fluctuation. To reduce this problem, wave filtration is implemented. The measurement $y = \psi + \psi_w + v$ is replaced by the estimated compass angle $\hat{\psi}$ in the feedback loop. Thus, only the dynamics of the estimated compass angle together with the feed forward of the bias will influence the rudder input. The estimation will filter out the high-frequent component of the wave disturbance. However, the low-frequent wave disturbance will still have an impact on the controller, where the ship will be "rocked" by the waves. This "wavy" response can be seen in the estimation of the compass angle in fig. 17 and fig. 18a.

The ship's performance with the discrete Kalman filter is better than the performance described in section 4.4. With the same disturbances in both scenarios (wave and measurement noise), the autopilot with the filtered wave feedback had a better response. The different performances can be seen in fig. 11 and fig. 17. The autopilot with the discrete Kalman filter feedback (fig. 17) is faster than the one with measured feedback and the rudder input is considerably les and smoother. A small overshoot of the compass angle can be seen in fig. 17, but this is preferred over the slow response in fig. 11.

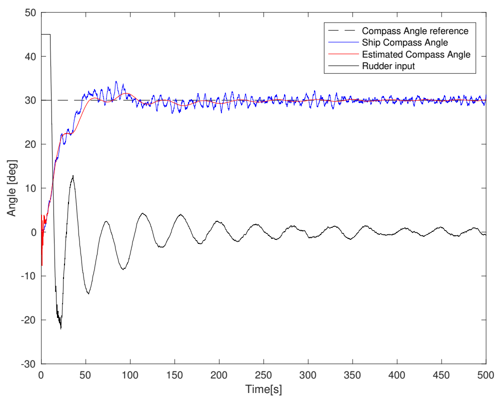
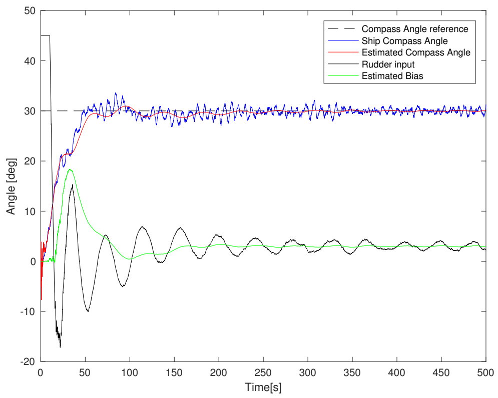
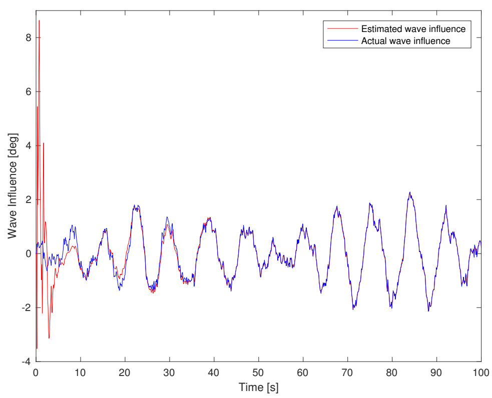


Figure 17: The ship's performance with Kalman filtering and without current noise.

The current noise is from here turned back on. After wave filtration, the ship's response is still fast and the estimated compass angle is not noisy, as can be seen in fig. 18a. It is still oscillating a little, especially from 50 to 250 seconds of the simulation. The rudder input behaves in the same manner as in fig. 17, but does not reach the same negative extremum. The estimated bias is as expected related to the compass angle and the rudder input, and is descending to a stationary non-zero value as the system steadies. This will be beneficial for the ship in a physical sense as well. The rudder will have fewer sudden changes in input, which will not tear it down as quickly.



(a) Performance of the autopilot.



(b) Actual and estimated wave influence

Figure 18: Autopilot with discrete Kalman filtering and with measurement, wave and current noise.

The actual and the estimated wave influence is also plotted, as seen in fig. 18b. The estimated wave influence lags the actual wave influence, which is expected as the measurement is used to calculate the estimate. However, this lagging "disappears" eventually as it catches up with the measured signal. When finding the estimated wave influence, the measured noise and current noise was turned off and the rudder input was set to zero. This was done to investigate the wave influence on the system only. It is worth noting the spiky parts early in the plots in fig. 18. The spikes are easier to see in fig. 18b, but they are also present in fig. 18a. Their origin is lack of measurement data when calculating the estimations. As more data points for the wave influence are gathered, the estimation will follow the actual data. The estimated high frequency wave influence is similar to the measured wave influence which indicates a good estimator.

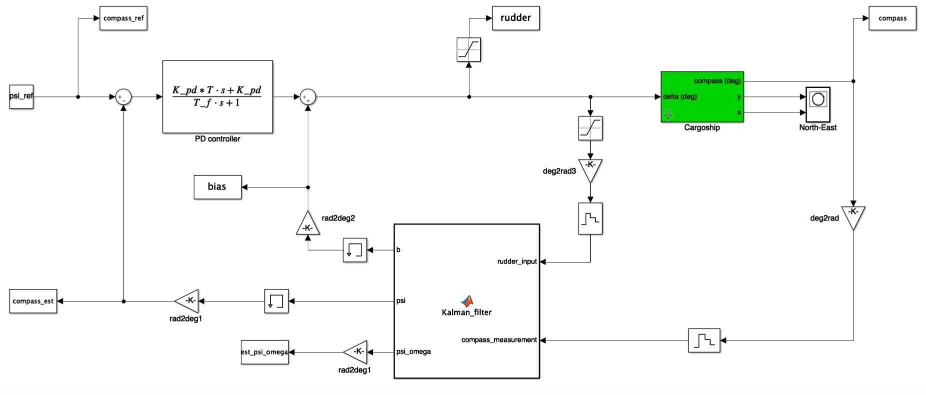


Figure 19: The simulink diagram of the system with Kalman filtered feedback.

6.6 Problem 5f

The reason for including a $\mathbf{Q}$ matrix is to compensate for process noise, a phenomenon which can be interpreted as uncertainties in the model. Therefore, by setting a too small $\mathbf{Q}$, the filter will be overconfident in its prediction model. On the other hand, if $\mathbf{Q}$ is set too large, the filter will be influenced by the noise process in an exaggerated manner and thus perform sub-optimally.

As seen in fig. 13, the $\mathbf{Q}$ matrix directly affects the a priori estimate of the error covariance $\mathbf{P}_k^-$. That is, the estimate of the error covariance without taking the measurement into consideration. The equation for $\mathbf{P}_k^-$ is given by

$$\mathbf{P}_k^- = \mathbf{A}_d \mathbf{P}_{k-1} \mathbf{A}_d^T + \mathbf{E}_d \mathbf{Q} \mathbf{E}_d{}^T. \tag{43}$$

Shown in (43), a larger $\mathbf{Q}$ matrix will result in a larger a priori estimate of the error covariance. This is equivalent to having a process with a lot of process noise and/or a model with considerable uncertainties. The size of the error before taking any measurements could therefore fluctuate between greater values, something which the larger $\mathbf{P}_k^-$ matrix represents. Further, the Kalman gain is given by

$$\mathbf{L}_k = \mathbf{P}_k^- \mathbf{C}_k^T (\mathbf{C}_k \mathbf{P}_k^- \mathbf{C}_k^T + R)^{-1} \tag{44}$$

where the gain is directly influenced by $\mathbf{P}_k^-$. From (44) it is shown that a larger $\mathbf{P}_k^-$ will result in a larger Kalman gain. It is known that a larger Kalman gain can be interpreted as taking the measurements more into account in the calculation of the estimation. A correlation can then be seen between the $\mathbf{Q}$ matrix and the Kalman gain. A large $\mathbf{Q}$ matrix represents noise processes and uncertainties in the model, and the corresponding large Kalman gain represents that the estimator then will base the estimation more on the measurements instead of the insufficient model.

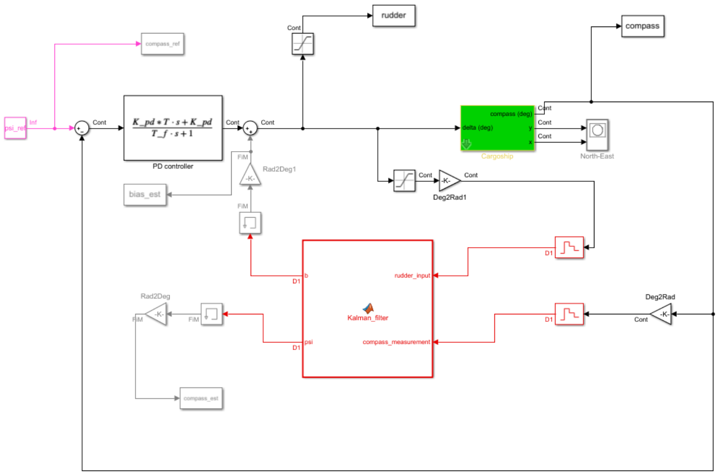


Figure 20: Simulink with feedforward from Kalman filter

As mentioned, tuning the $\mathbf{Q}$ matrix determines how much the Kalman filter "trusts" the model. The upper left element of the matrix represents the variance of the w_w and the lower right element of the matrix represents the variance of w_b. In fig. 21 and fig. 22, different tunings have been tested with the same parameters as the rest of the model in the previous exercises. In fig. 21 the controller consists of a feedback loop of the measurement and a feed forward of the estimated bias, $\hat{b}$, from the Kalman filter, as shown in fig. 20. The plotted responses and their corresponding rudder inputs, δ, vary dependently on the elements in $\mathbf{Q}$. When the variance of w_b is low, the estimated state corresponding to the bias disturbance is mainly based of the model described in section 1. Notice that with high $E[w_b^2]$, the rudder input oscillates over a short period. This is due to the Kalman filter taking into account uncertainties in the model, and reacting as if the bias was varying over a larger spectrum. In addition, notice how the filter with $E[w_w^2] = 0$ acts as if there are waves present due to the modeled ψ_w. Since the bias isn't based on the model alone ($E[w_b] \neq 0$), the bias is calculated as larger while the waves are modeled, thus making the input negative over a longer time period. Ultimately this makes the controller slower. When the variance of w_w is high, the estimated state corresponding to this disturbance is mainly based on y. Since there is no wave disturbance turned on, they do not affect the filter and therefore not the bias.

When using the estimated state $\hat{\psi}$ instead of ψ in the feedback loop the goal is to make the controller react less to the high frequent waves. A high variance in w_w makes the Kalman filter base its calculations around the measurements. Thus, in periods where the other disturbances do not have such an effect, the controller will predict the wave pattern well. On the other hand it also takes the other disturbances more into consideration, such as

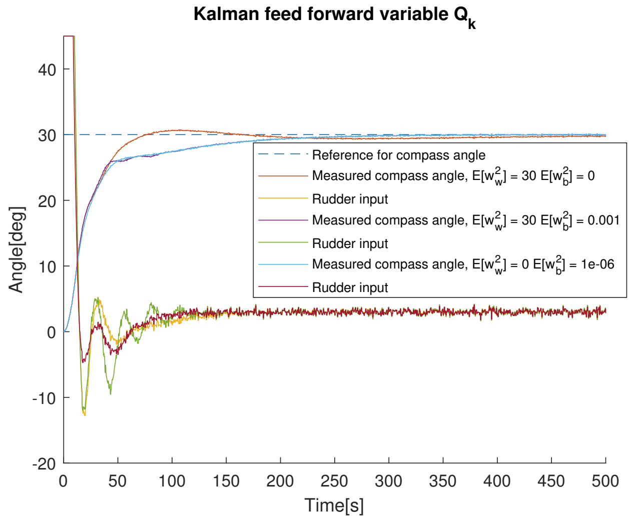


Figure 21: Different responses to different $\mathbf{Q}$ tuning and their inputs.

the current and measurement noise and predict these as waves. Notice how the response with low $E[w_w^2]$ has a faster response than the one with high $E[w_w^2]$ in fig. 22. This is due to the other, more well modeled disturbances. However, since the wave model has large uncertainties, the response with low variance will think the ship is actually altering its course when being hit with a lot of waves. This is not the case with the high variance response, since it takes into account the measurement to a much larger extent.

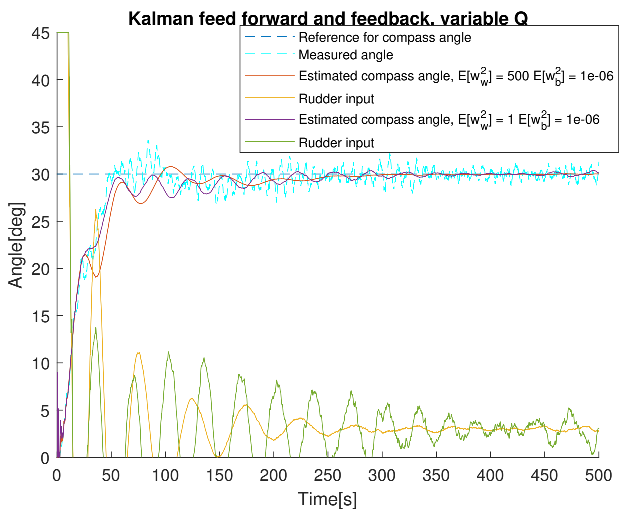


Figure 22: Different responses to different $\mathbf{Q}$ tuning and their inputs.

7 Conclusion

This report has shown how current and waves affect the behaviour of a ship. A PD controller was implemented and the controller was able to control the ship when no disturbances were present.

However, a current disturbance resulted in a steady state error between the ship's compass angle and the desired reference. The wave disturbance, on the other hand, resulted in rapid changes in the rudder input angle. From a physical perspective, this would lead to wear and tear on the rudder which is sub-optimal.

It was then confirmed that the system was observable in all scenarios regarding different disturbances. Thus the mentioned problems were solved by implementing a Kalman filter. The steady state error caused by the current disturbance was cancelled out by the use of a feed forward loop of the estimated bias. A filtered feedback, where both the measurement noise and the high frequency component of the wave disturbance were filtered out, smoothed out the rudder operation.

Ultimately, the Kalman filter contributed to the ship having a desirable behaviour. No offset between the reference and the compass was achieved combined with the rudder operating smoothly.

References

[1] R. G. Brown and P. Y. C. Hwang. *Introduction to Random Signals and Applied Kalman Filtering, 4th edt.* John Wiley Sons, Inc, 2012.

[2] Chi-Tsong Chen. *Linear System Theory and Design.* Oxford University Press, Incorporated, 2014.

[3] *Discrete Kalman Filter Applied to a Ship Autopilot.* `https://ntnu.blackboard.com/bbcswebdav/pid-420179-dt-content-rid-18213414_1/courses/194_TTK4115_1_2018_H_1/boat_assignment%282%29.pdf`. Accessed:

[4] *Fossen2011_kalman_filter.* `https://ntnu.blackboard.com/bbcswebdav/pid-498852-dt-content-rid-18213466_1/courses/194_TTK4115_1_2018_H_1/Fossen2011_kalman_filter.pdf`. Accessed: