Assignment 3A

- **Problem 1 (50 points):** Given below is a differential equation:

$$\frac{dx}{dt} = -x^3 + \sin(t) \tag{1}$$

with the initial condition $x = 0$ at $t = 0$.

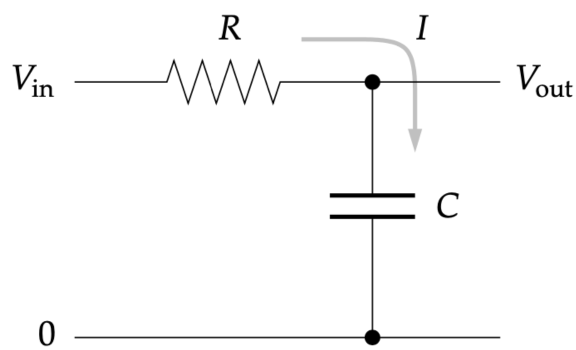
(a) Write a program that uses the midpoint method (2nd order Runge–Kutta method) to solve the differential equation.

(b) Write a program that uses the 4th order Runge–Kutta method to solve the differential equation.

(c) For both programs, calculate from $t = 0$ to $t = 10$ using a reasonable number of steps. Plot the results for each of the methods, including the one from the previous assignment for Euler's method. Do you see improved accuracy with higher-order Runge–Kutta methods? Why?

- **Problem 2 (50 points):** A low-pass filter.

Here is a simple electronic circuit with one resistor and one capacitor:



This circuit acts as a low-pass filter: you send a signal in on the left, and it comes out filtered on the right.

Using Ohm's law and the capacitor law, and assuming that the output load has very high impedance so that a negligible amount of current flows through it, we can write down the equations governing this circuit as follows. Let I be the current that flows through R and into the capacitor, and let Q be the charge on the capacitor. Then:

$$IR = V_{\text{in}} - V_{\text{out}}, \quad Q = CV_{\text{out}}, \quad I = \frac{dQ}{dt}.$$

Substituting the second equation into the third, then substituting the result into the first equation, we find that:

$$\frac{dV_{\text{out}}(t)}{dt} + \frac{1}{RC}V_{\text{out}}(t) = \frac{1}{RC}V_{\text{in}}(t)\,,$$

or, equivalently:

$$\boxed{\frac{dV_{\text{out}}(t)}{dt} = \frac{1}{RC}\left(V_{\text{in}}(t) - V_{\text{out}}(t)\right)}\,. \tag{2}$$

In this assignment you will solve this equation, calculating numerically $V_{\text{out}}(t)$ for a given $V_{\text{in}}(t)$.

(a) Write a program (or modify a previous program) to solve this equation for $V_{\text{out}}(t)$ using the 4th-order Runge–Kutta method when the input signal is a square-wave with frequency 1 and amplitude 1.

Use the program to make plots of the output of the filter circuit from $t = 0$ to $t = 10$ when $RC = 0.01$, 0.1, and 1, with the initial condition $V_{\text{out}}(0) = 0$. where $V_{\text{in}}(t)$ is a square wave defined as:

$$V_{\text{in}}(t) = \begin{cases} 1, & \text{if } \lfloor 2t \rfloor \text{ is even,} \\ -1, & \text{if } \lfloor 2t \rfloor \text{ is odd.} \end{cases}$$

For clarity, see the graph of $V_{\text{in}}(t)$ in Figure 1.

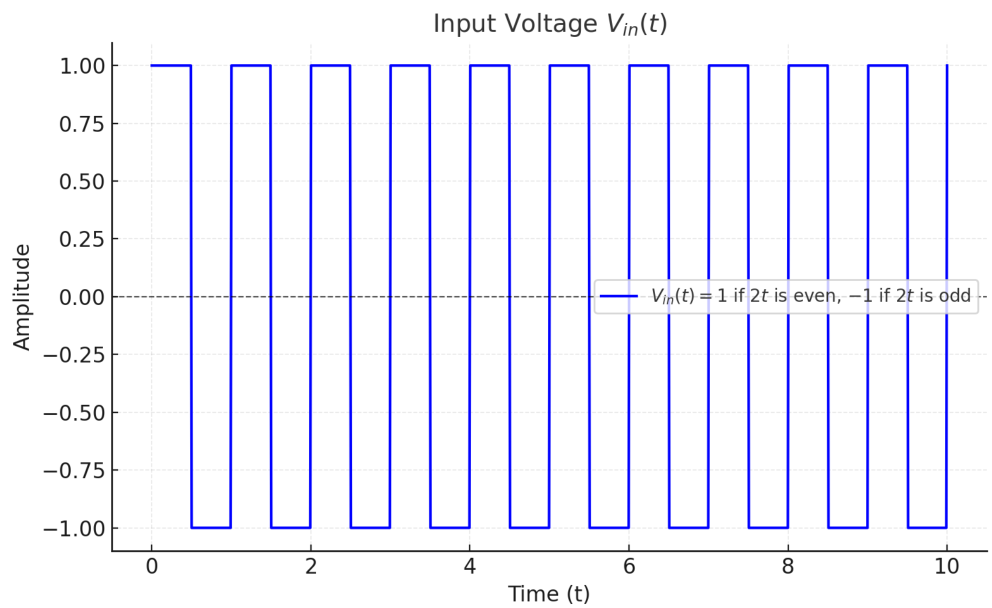


Figure 1: Input Voltage $V_{\text{in}}(t)$ as a square wave.

Note: You will have to decide on a suitable value of h for your calculation. Small values give more accurate results, but the program will take longer to run. Try a variety of different values and choose one for your final calculations that seems sensible to you.

(b) Compare the large-time behavior of $V_{\text{out}}(t)$ with the expected periodic solution calculated using the Fourier series for $V_{\text{in}}(t)$. The Fourier series expansion of the square wave (approximated by truncating the series up to a chosen value of N) is given by:

$$V_{\text{in}}(t) \simeq \sum_{n=1}^{N} w_n \sin(2\pi n t) \,,$$

where $w_n = \frac{4}{\pi n}$ for odd positive n and 0 for all other n.

Using this expansion, the periodic part of $V_{\text{out}}(t)$ can be approximated as:

$$\bar{V}_\infty \simeq \sum_{n=1,3,5,\ldots}^{N} \frac{\alpha w_n}{\alpha^2 + \omega_0^2 n^2} \left(-\omega_0 n \cos(\omega_0 n t) + \alpha \sin(\omega_0 n t)\right),$$

where $\alpha = (RC)^{-1}$, and $\omega_0 = 2\pi$.

Write a program to compute $\bar{V}_\infty(t)$ using this formula for a chosen number of terms N in the series. Use the same values of RC as in part (a). Compare these results with the numerical solution you obtained earlier.

Note: This is *not* the general solution, but only the periodic component capturing the long-time behavior. The most general solution is the sum of the periodic part captured by the Fourier series and a generic solution of the homogeneous equation. Therefore, it is of the form:

$$V_{out}(t) = \bar{V}_\infty(t) + A\,e^{-\alpha t},$$

where A depends on the initial condition.

(c) Based on the graphs produced by your program, describe what you see and explain what the circuit is doing for different values of RC.

Note: A program similar to the one you are working on in this Assignment is running inside most stereos and music players, to create the effect of the "bass" control. In the old days, the bass control on a stereo would have been connected to a real electronic low-pass filter in the amplifier circuitry, but these days there is just a computer processor that simulates the behavior of the filter in a manner similar to your program.