COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Andrew McGregor

Lecture 4

Last Class:

- 2-Level Hashing Analysis (linearity of expectation and Markov's inequality)
- 2-universal and pairwise independent hash functions
- Chebyshev: $\Pr(|\mathbf{X} - \mathbb{E}[\mathbf{X}]| \geq t) \leq \mathrm{Var}[\mathbf{X}]/t^2$

LAST TIME

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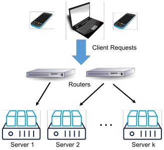
This Time:

- Random hashing for load balancing. Motivating:
  - Stronger concentration inequalities: Chebyshev's inequality, exponential tail bounds, and their connections to the law of large numbers and central limit theorem.
  - The union bound.

Randomized Load Balancing:



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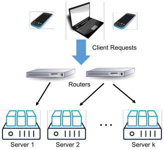

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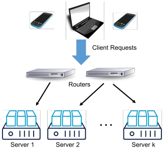

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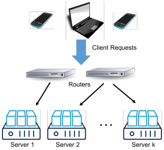

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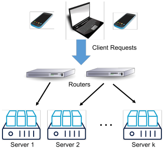

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- Suppose each server can handle at most $\mathbb{E}[\mathbf{R}_i] = n/k$ requests
- By Markov's inequality, $\Pr[\mathbf{R}_i \geq 2\mathbb{E}[\mathbf{R}_i]] \leq 1/2$.
- By Chebyshev's inequality, $\Pr[\mathbf{R}_i \geq 2\mathbb{E}[\mathbf{R}_i]] \leq \frac{\mathrm{Var}[\mathbf{R}_i]}{\mathbb{E}[\mathbf{R}_i]^2} < \frac{k}{n}$.

What is the probability that the maximum server load exceeds $2 \cdot \mathbb{E}[\mathbf{R}_i] = \frac{2n}{k}$. I.e., that some server is overloaded if we give each $\frac{2n}{k}$ capacity?

n: total number of requests, k: number of servers randomly assigned requests, $\mathbf{R}_i$: number of requests assigned to server i. $\mathbb{E}[\mathbf{R}_i] = \frac{n}{k}$. $\mathrm{Var}[\mathbf{R}_i] = n/k$.

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How do we do this? Note that $\mathbf{R}_1, \ldots, \mathbf{R}_k$ are correlated in a somewhat complex way.

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THE UNION BOUND

Union Bound: For any random events $A_1, A_2, ..., A_k$,

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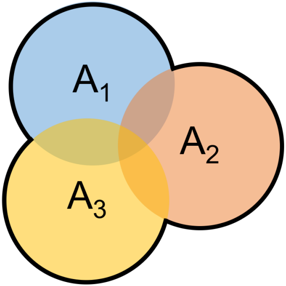

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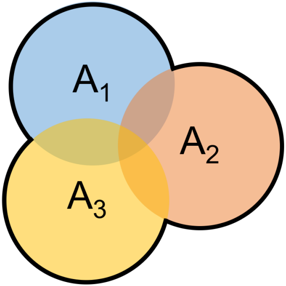

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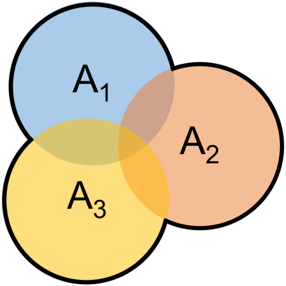

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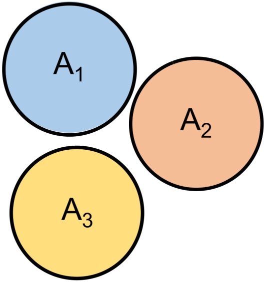

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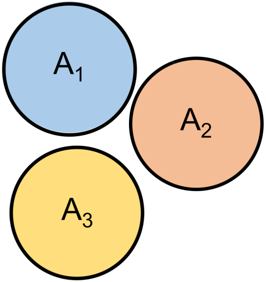

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On the first problem set, you will prove the union bound, as a consequence of Markov's inquality.

APPLYING THE UNION BOUND

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&\leq \sum_{i=1}^{k} \Pr\left(\left[\mathbf{R}_i \geq \frac{2n}{k}\right]\right) && \text{(Union Bound)} \\
&\leq \sum_{i=1}^{k} \frac{k}{n} && \text{(Bound from Chebyshev's)}
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As long as $k \ll \sqrt{n}$, the maximum server load will be small (compared to the expected load) with good probability.

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BACK TO CHEBYSHEV'S INEQUALITY

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X: any random variable, t, s: any fixed numbers.

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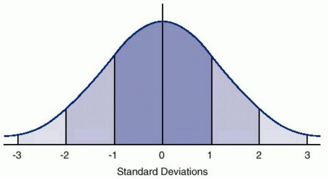
What is the probability that $\mathbf{X}$ falls s standard deviations from it's mean?



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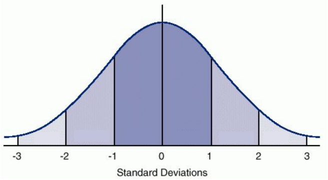

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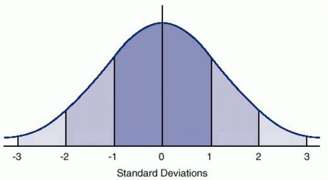

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Why is this so powerful?

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- Cannot show from vanilla Markov's inequality.

SERVER LOAD AND LAW OF LARGE NUMBERS

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- There are many requests routed to a relatively small number of servers so the load seen on each server is close to what is expected via law of large numbers.

n: total number of requests, k: number of servers randomly assigned requests.

Questions on union bound, Chebyshev's inequality, random hashing?

We flip $n = 100$ independent coins, each are heads with probability $1/2$ and tails with probability $1/2$. Let $\mathbf{H}$ be the number of heads.

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Markov's:

$\Pr(\mathbf{H} \geq 60) \leq .833$

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$\mathbf{H}$ has a simple Binomial distribution, so can compute these probabilities exactly.

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TIGHTER CONCENTRATION BOUNDS

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- What if we just apply Markov's inequality to even higher moments?

Consider any random variable $\mathbf{X}$:

$$\Pr(|\mathbf{X} - \mathbb{E}[\mathbf{X}]| \geq t) = \Pr\left((\mathbf{X} - \mathbb{E}[\mathbf{X}])^4 \geq t^4\right)$$

A FOURTH MOMENT BOUND

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- Apply Fourth Moment Bound: $\Pr(|\mathbf{H} - \mathbb{E}[\mathbf{H}]| \geq t) \leq \frac{1862.5}{t^4}$.

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- We will explore the basic proof approach in homework.

Bernstein Inequality: Consider independent random variables $\mathbf{X}_1, \ldots, \mathbf{X}_n$ all falling in $[-M, M]$. Let $\mu = \mathbb{E}[\sum_{i=1}^n \mathbf{X}_i]$ and $\sigma^2 = \mathrm{Var}[\sum_{i=1}^n \mathbf{X}_i] = \sum_{i=1}^n \mathrm{Var}[\mathbf{X}_i]$. For any $t \geq 0$:

$$\Pr\left(\left|\sum_{i=1}^n \mathbf{X}_i - \mu\right| \geq t\right) \leq 2\exp\left(-\frac{t^2}{2\sigma^2 + \frac{4}{3}Mt}\right).$$

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Assume that $M = 1$ and plug in $t = s \cdot \sigma$ for $s \leq \sigma$.

Bernstein Inequality: Consider independent random variables $\mathbf{X}_1, \ldots, \mathbf{X}_n$ all falling in [-1,1]. Let $\mu = \mathbb{E}[\sum_{i=1}^n \mathbf{X}_i]$ and $\sigma^2 = \text{Var}[\sum_{i=1}^n \mathbf{X}_i] = \sum_{i=1}^n \text{Var}[\mathbf{X}_i]$. For any $s \geq 0$:

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- An exponentially stronger dependence on s!

COMPARISION TO CHEBYSHEV'S

Consider again bounding the number of heads **H** in $n = 100$ independent coin flips.

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H: total number heads in 100 random coin flips. $\mathbb{E}[\mathbf{H}] = 50$.

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Getting much closer to the true probability.

$\mathbf{H}$: total number heads in 100 random coin flips. $\mathbb{E}[\mathbf{H}] = 50$.

A useful variation of the Bernstein inequality for binary (indicator) random variables is:

Chernoff Bound (simplified version): Consider independent random variables $\mathbf{X}_1, \ldots, \mathbf{X}_n$ taking values in $\{0, 1\}$. Let $\mu = \mathbb{E}[\sum_{i=1}^n \mathbf{X}_i]$. For any $\delta \geq 0$

$$\Pr\left(\left|\sum_{i=1}^{n} \mathbf{X}_i - \mu\right| \geq \delta\mu\right) \leq 2\exp\left(-\frac{\delta^2\mu}{2+\delta}\right).$$

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As δ gets larger and larger, the bound falls of exponentially fast.