# COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Andrew McGregor

Lecture 8

**Jaccard Index:** A similarity measure between two sets.

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- **Near Neighbor Search:** Have a database of $n$ sets and given a set $A$, want to find if it has high Jaccard similarity to anything in the database. $\Omega(n)$ time with a linear scan.
- **All-pairs Similarity Search:** Have $n$ different sets and want to find all pairs with high Jaccard similarity. $\Omega(n^2)$ time if we check all pairs explicitly.

Will speed up via randomized locality sensitive hashing.

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# MINHASHING

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- Let $\mathbf{h} : U \to [0, 1]$ be a random hash function
- $\mathbf{s} := 1$
- For $x_1, \ldots, x_{|A|} \in A$
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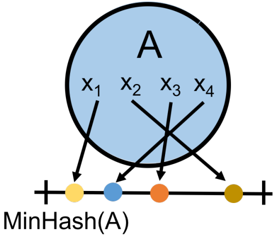

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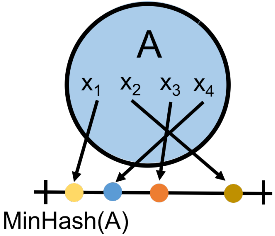


Identical to our distinct elements sketch!

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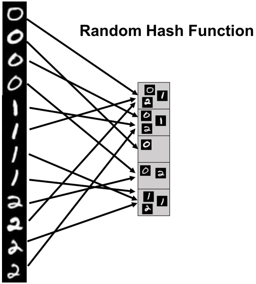

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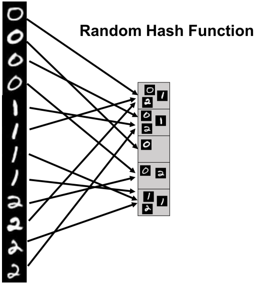


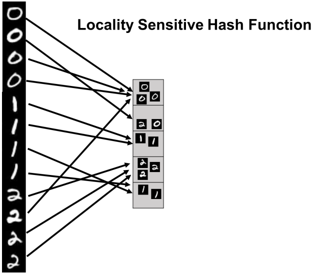

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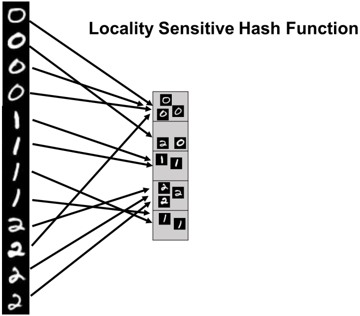

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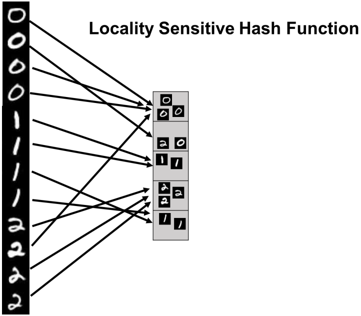


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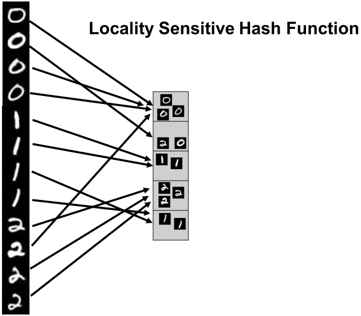


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- **All-pairs Similarity Search:** Scan through all buckets of the hash table and look for similar pairs within each bucket.

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# LSH WITH MINHASH

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Potential for a lot of false positives! Slows down search time.

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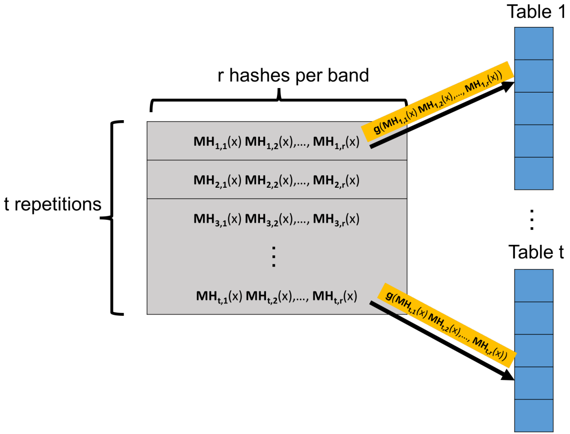


Create $t$ hash tables. Each is indexed into not with a single MinHash value, but with $r$ values, appended together. A length $r$ signature.

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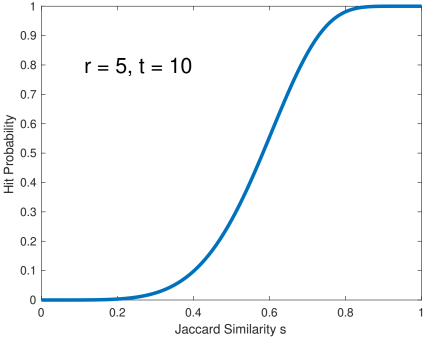

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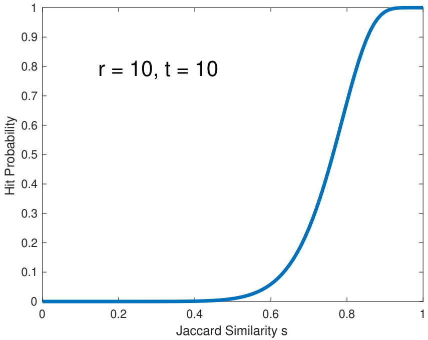

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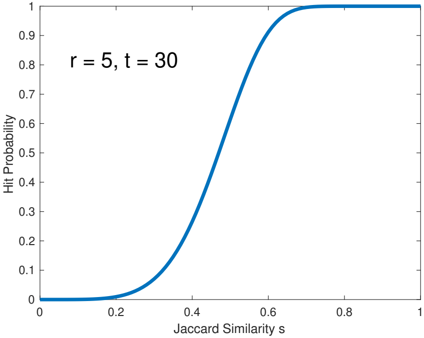

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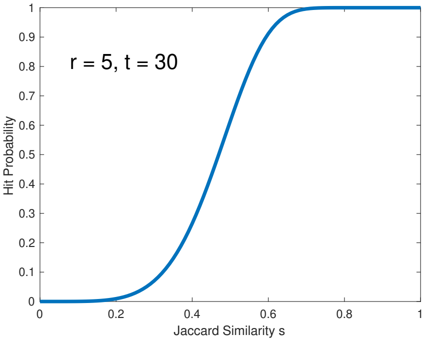


$r$ and $t$ are tuned depending on application. 'Threshold' when hit probability is $1/2$ is $\approx (1/t)^{1/r}$. E.g., $\approx (1/30)^{1/5} = .51$ in this case.

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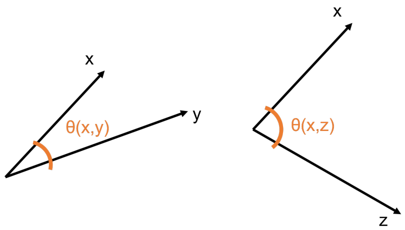

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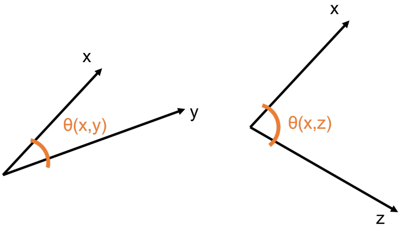


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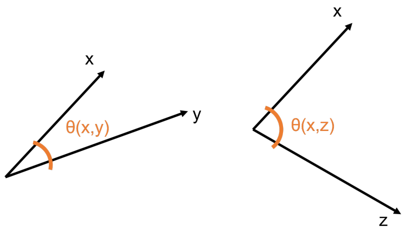


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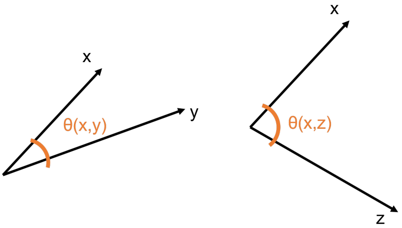


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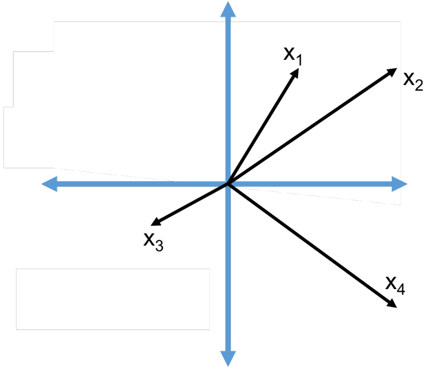

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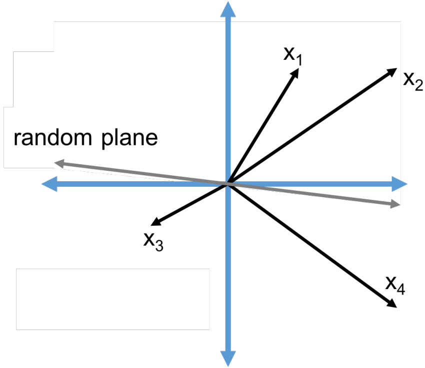

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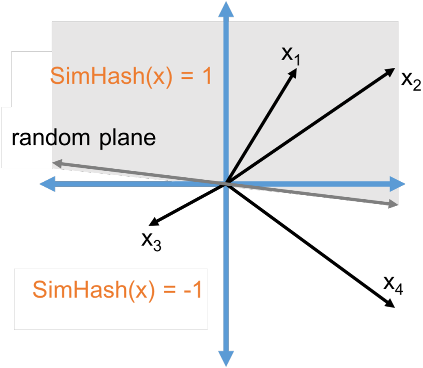

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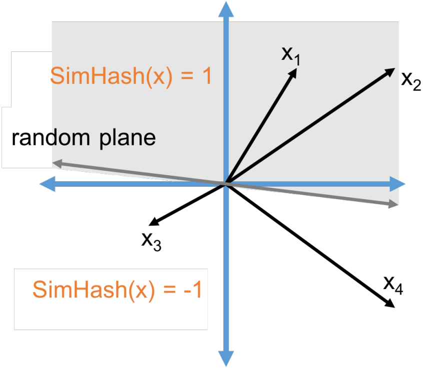


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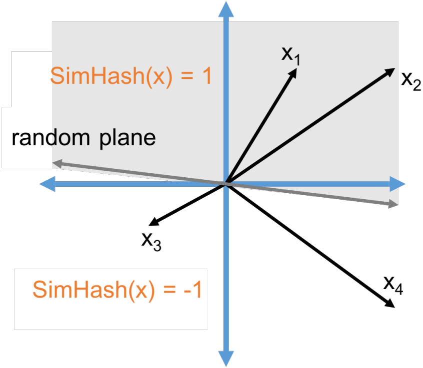


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## SIMHASH FOR COSINE SIMILARITY

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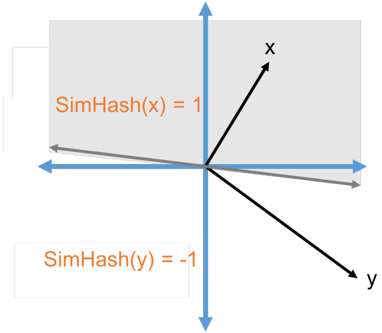

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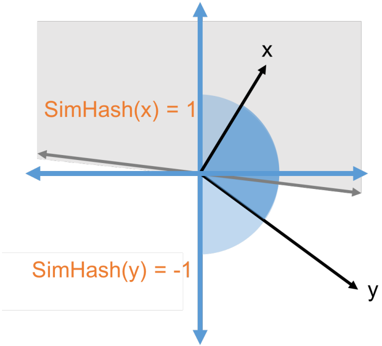

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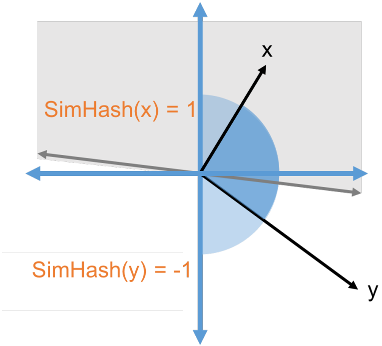


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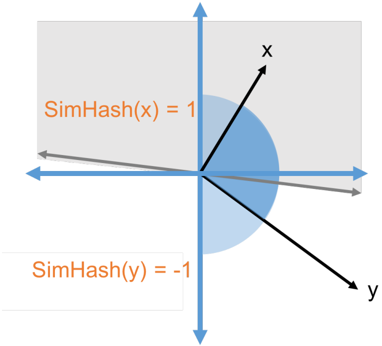


- $\Pr[SimHash(x) \neq SimHash(y)] = \frac{\theta(x,y)}{180}$
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Questions on MinHash and Locality Sensitive Hashing?