COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Andrew McGregor

Lecture 3

Last Class We Covered:

- Markov's inequality: the most fundamental concentration bound.
- Algorithmic applications of Markov's inequality, linearity of expectation, and indicator random variables:
    - Counting collisions to estimate CAPTCHA database size.
    - Counting collisions to understand the runtime of hash tables with random hash functions.

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  - Counting collisions to estimate CAPTCHA database size.
  - Counting collisions to understand the runtime of hash tables with random hash functions.
- Collision counting is closely related to the birthday paradox.

Today:

- Finish up random hash functions and hash tables.
- See an application of random hashing to load balancing in distributed systems.
- Through this application learn about:
  - Chebyshev's inequality, which strengthens Markov's inequality.
  - The union bound, for understanding the probabilities of correlated random events.

TWO LEVEL HASHING

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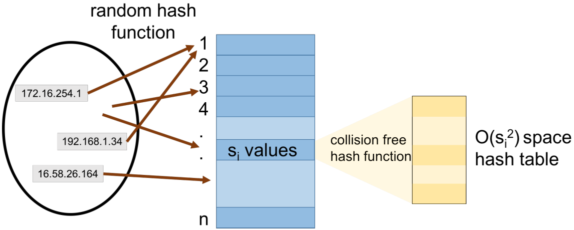

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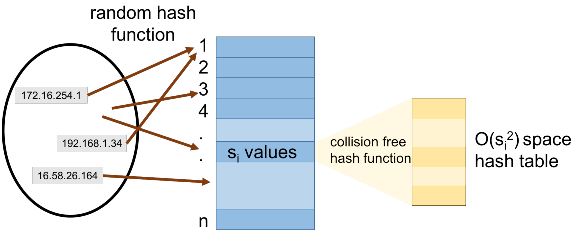

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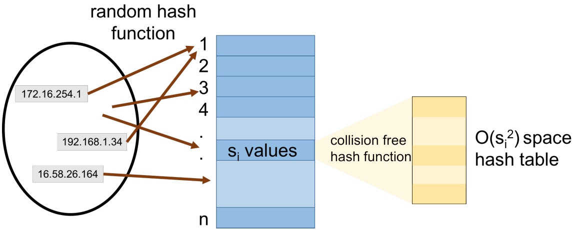

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- **Just Showed:** A random function is collision free with probability $\geq \frac{7}{8}$ so can just generate a random hash function and check if it is collision free.

SPACE USAGE

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x_j, x_k: stored items, n: hash table size, $\mathbf{h}$: random hash function, $\mathbf{S}$: space usage of two level hashing, $\mathbf{s}_i$: # items stored in hash table at position i.

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Collisions again!

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- For $j \neq k$, $\mathbb{E}\left[\mathbb{I}_{\mathbf{h}(x_j)=i} \cdot \mathbb{I}_{\mathbf{h}(x_k)=i}\right] = \frac{1}{n^2}$.

Total Expected Space Usage: (if we set $n = m$)

$$\mathbb{E}[\mathbf{S}] = n + \sum_{i=1}^{n} \mathbb{E}[\mathbf{s}_i^2] \leq n + n \cdot 2 = 3n = 3m.$$

Near optimal space with $O(1)$ query time!

x_j, x_k: stored items, m: # stored items, n: hash table size, $\mathbf{h}$: random hash function, $\mathbf{S}$: space usage of two level hashing, $\mathbf{s}_i$: # items stored at pos i.

EFFICIENTLY COMPUTABLE HASH FUNCTION

So Far: we have assumed a **fully random hash function** $\mathbf{h}(x)$ with $\Pr[\mathbf{h}(x) = i] = \frac{1}{n}$ for $i \in 1, \ldots, n$ and $\mathbf{h}(x), \mathbf{h}(y)$ independent for $x \neq y$.

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- To compute a random hash function we have to store a table of x values and their hash values. Would take at least $O(m)$ space and $O(m)$ query time if we hash m values. Making our whole quest for $O(1)$ query time pointless!

x	**h(x)**
x_1	45
x_2	1004
x_3	10
⋮	⋮
x_m	12

EFFICIENTLY COMPUTABLE HASH FUNCTIONS

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Efficient Alternative: Let p be a prime with $p \geq |U|$. Choose random $\mathbf{a}, \mathbf{b} \in [p]$ with $\mathbf{a} \neq 0$. Let:

$$\mathbf{h}(x) = (\mathbf{a}x + \mathbf{b} \mod p) \mod n.$$

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Remember: A fully random hash function is both 2-universal and pairwise independent. But it is not efficiently implementable.

NEXT STEP

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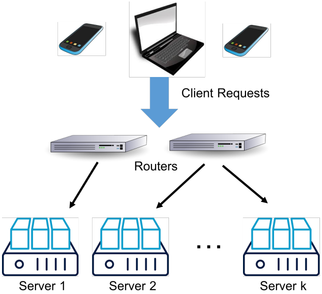
2. Then we'll show how a simple twist on Markov's can give a much stronger result.

Randomized Load Balancing:



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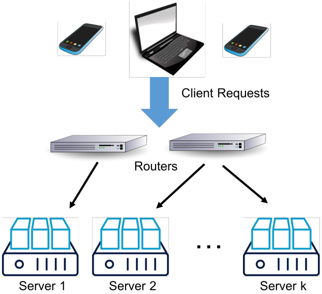

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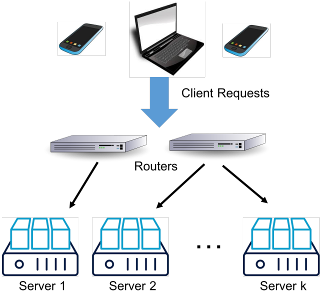

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Not great...half the servers may be overloaded.

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$$\Pr(|\mathbf{X} - \mathbb{E}[\mathbf{X}]| \geq t) \leq \frac{\mathrm{Var}[\mathbf{X}]}{t^2}.$$

(by plugging in the random variable $\mathbf{X} - \mathbb{E}[\mathbf{X}]$)

LOAD BALANCING VARIANCE

We can write the number of requests assigned to server i, $\mathbf{R}_i$ as:

$$\mathbf{R}_i = \sum_{j=1}^{n} \mathbf{R}_{i,j}$$

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- Overload probability is extremely small when $k \ll n$!
- Might seem counterintuitive – bound gets worse as k grows.
- When k is large, the number of requests each server sees in expectation is very small so the law of large numbers doesn't 'kick in'.

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