# COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Andrew McGregor

Lecture 14

**Next Few Classes: Low-rank approximation, the SVD, and principal component analysis (PCA).**

- Reduce $d$-dimensional data points to a smaller dimension $m$.
- Like JL, compression is linear, i.e., by applying a matrix.
- Chose matrix taking into account structure of dataset.
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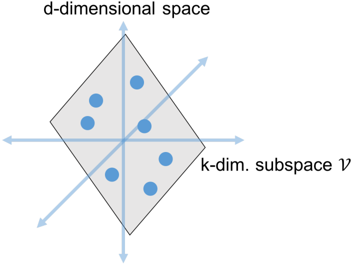
Will be using a fair amount of linear algebra: orthogonal basis, column/row span, eigenvectors, etc, Assume that data points $\vec{x}_1, \ldots, \vec{x}_n$ lie in any $k$-dimensional subspace $\mathcal{V}$ of $\mathbb{R}^d$.



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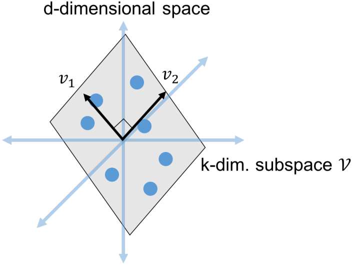
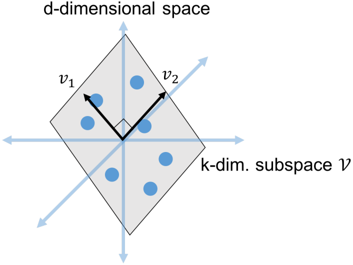



**Claim:** Let $\vec{v}_1, \ldots, \vec{v}_k$ be an orthonormal basis for $\mathcal{V}$ and $\mathbf{V} \in \mathbb{R}^{d \times k}$ be the matrix with these vectors as its columns. For all $\vec{x}_i, \vec{x}_j$:

$$\|\mathbf{V}^T \vec{x}_i - \mathbf{V}^T \vec{x}_j\|_2 = \|\vec{x}_i - \vec{x}_j\|_2.$$

# EMBEDDING WITH ASSUMPTIONS

Assume that data points $\vec{x}_1, \ldots, \vec{x}_n$ lie in any $k$-dimensional subspace $\mathcal{V}$ of $\mathbb{R}^d$.

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- $\mathbf{V}^T \in \mathbb{R}^{k\times d}$ is a linear embedding of $\vec{x}_1, \ldots, \vec{x}_n$ into $k$ dimensions with no distortion.

**Claim:** Let $\vec{v}_1, \ldots, \vec{v}_k$ be an orthonormal basis for $\mathcal{V}$ and $\mathbf{V} \in \mathbb{R}^{d\times k}$ be the matrix with these vectors as its columns. For all $\vec{y} \in \mathcal{V}$:

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Proof:

# DOT PRODUCT TRANSFORMATION

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- So $\|\vec{y}\|_2^2 = \vec{c}^T\vec{c} = \|\mathbf{V}^T\vec{y}\|_2^2$.

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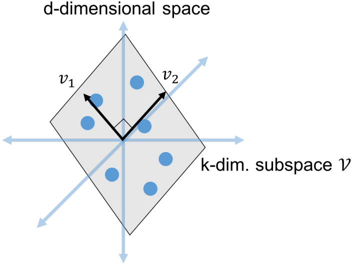

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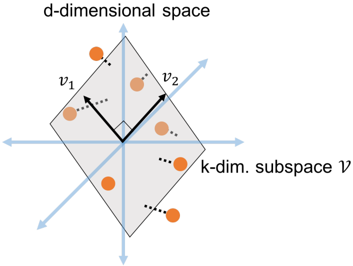

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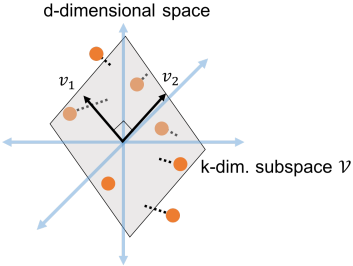


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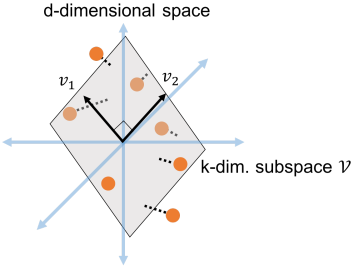


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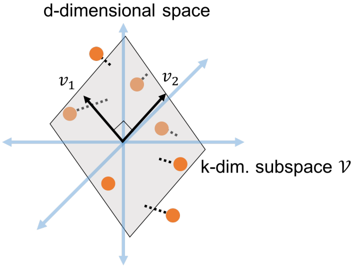


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- How do we find $\mathcal{V}$ and $\mathbf{V}$?
- How good is the embedding?

**Claim:** $\vec{x}_1, \ldots, \vec{x}_n$ lie in a $k$-dimensional subspace $\mathcal{V} \Leftrightarrow$ the data matrix $\mathbf{X} \in \mathbb{R}^{n \times d}$ has rank $\leq k$.

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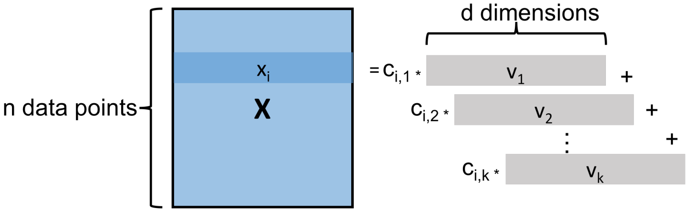
- Letting $\vec{v}_1, \ldots, \vec{v}_k$ be an orthonormal basis for $\mathcal{V}$, can write any $\vec{x}_i$ as:

$$\vec{x}_i = \mathbf{V}\vec{c}_i = c_{i,1} \cdot \vec{v}_1 + c_{i,2} \cdot \vec{v}_2 + \ldots + c_{i,k} \cdot \vec{v}_k.$$

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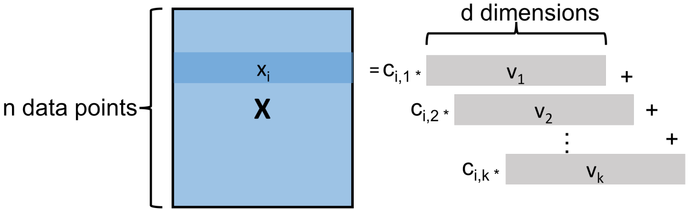
- So $\vec{v}_1, \ldots, \vec{v}_k$ span the rows of $\mathbf{X}$ and thus $\text{rank}(\mathbf{X}) \leq k$.

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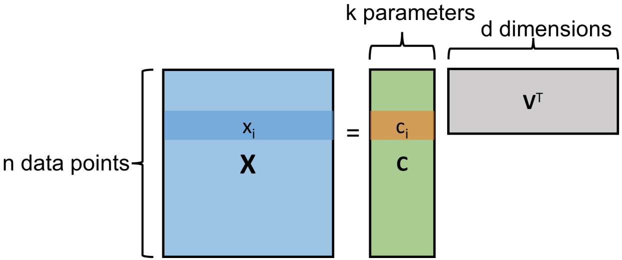


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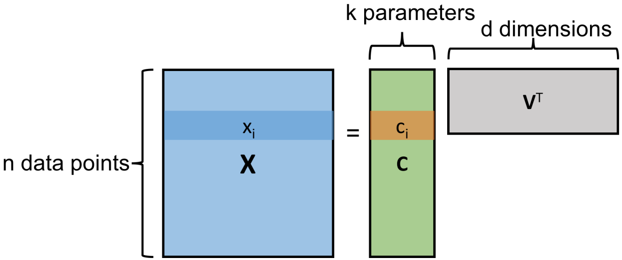


- $\mathbf{X}$ can be represented by $(n+d) \cdot k$ parameters vs. $n \cdot d$.

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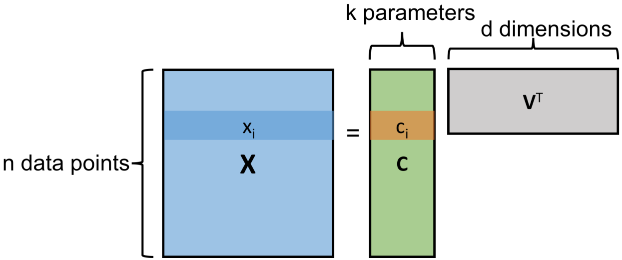


- $\mathbf{X}$ can be represented by $(n+d) \cdot k$ parameters vs. $n \cdot d$.
- The rows of $\mathbf{X}$ are spanned by $k$ vectors: the columns of $\mathbf{V}$ $\implies$ the columns of $\mathbf{X}$ are spanned by $k$ vectors: the columns of $\mathbf{C}$.

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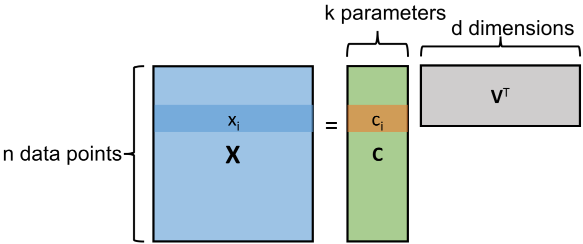


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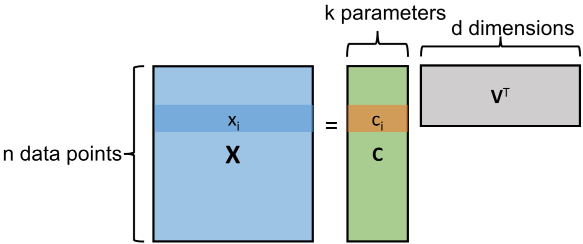


**Exercise:** What is this coefficient matrix **C**? **Hint:** Use that $\mathbf{V}^T\mathbf{V} = \mathbf{I}$.

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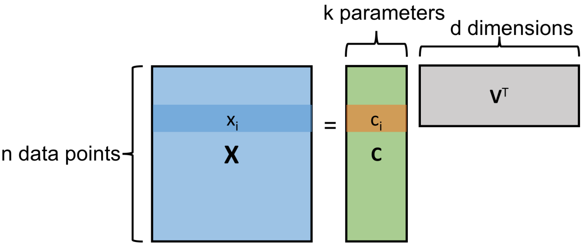


**Exercise:** What is this coefficient matrix $\mathbf{C}$? **Hint:** Use that $\mathbf{V}^T\mathbf{V} = \mathbf{I}$.

- $\mathbf{X} = \mathbf{C}\mathbf{V}^T \implies \mathbf{X}\mathbf{V} = \mathbf{C}\mathbf{V}^T\mathbf{V}$

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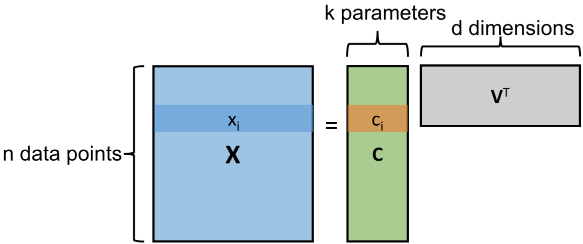


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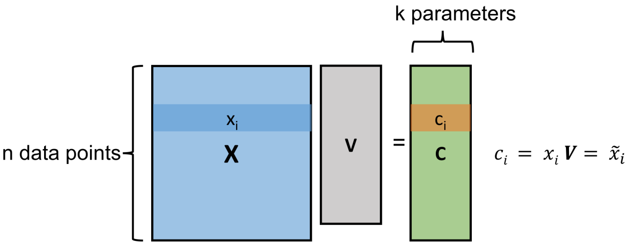


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# PROJECTION VIEW

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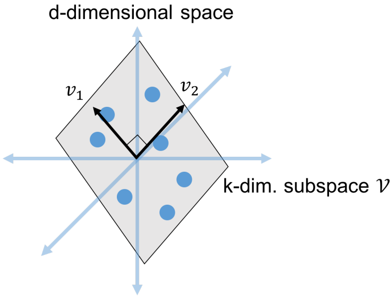


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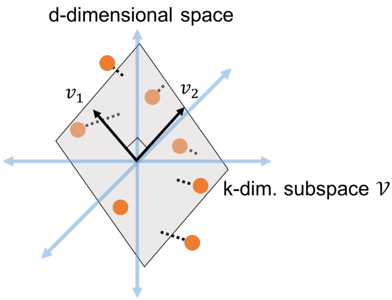


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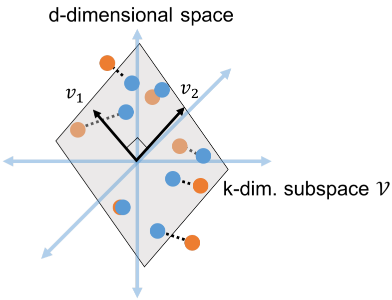


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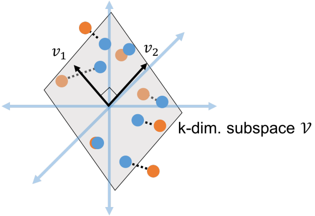


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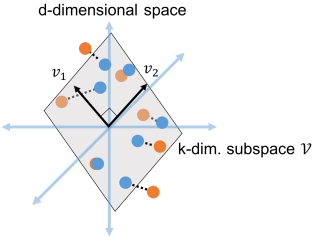


**Note:** $\mathbf{X}\mathbf{V}\mathbf{V}^T$ has rank $k$. It is a low-rank approximation of $\mathbf{X}$.

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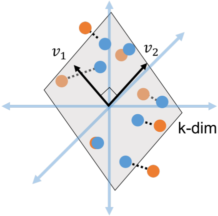


**Note:** $\mathbf{X}\mathbf{V}\mathbf{V}^T$ has rank $k$. It is a low-rank approximation of $\mathbf{X}$.

$$\mathbf{X}\mathbf{V}\mathbf{V}^{\mathbf{T}} = \underset{\mathbf{B} \text{ with rows in } \mathcal{V}}{\arg\min} \|\mathbf{X} - \mathbf{B}\|_F^2 = \underset{\mathbf{B} \text{ with rows in } \mathcal{V}}{\arg\min} \sum_{i,j} (\mathbf{X}_{i,j} - \mathbf{B}_{i,j})^2.$$

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**So Far:** If $\vec{x}_1, \ldots, \vec{x}_n$ lie close to a $k$-dimensional subspace $\mathcal{V}$ with orthonormal basis $\mathbf{V} \in \mathbb{R}^{d \times k}$, the data matrix can be approximated as:

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This is the closest approximation to $\mathbf{X}$ with rows in $\mathcal{V}$ (i.e., in the column span of $\mathbf{V}$).

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- Letting $(\mathbf{X}\mathbf{V}\mathbf{V}^T)_i, (\mathbf{X}\mathbf{V}\mathbf{V}^T)_j$ be the $i^{th}$ and $j^{th}$ projected data points, $\|(\mathbf{X}\mathbf{V}\mathbf{V}^T)_i - (\mathbf{X}\mathbf{V}\mathbf{V}^T)_j\|_2 = \|[(\mathbf{X}\mathbf{V})_i - (\mathbf{X}\mathbf{V})_j]\mathbf{V}^T\|_2 = \|[(\mathbf{X}\mathbf{V})_i - (\mathbf{X}\mathbf{V})_j]\|_2$.

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- Can use $\mathbf{X}\mathbf{V} \in \mathbb{R}^{n \times k}$ as a compressed approximate data set.

$\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$: data points, $\mathbf{X} \in \mathbb{R}^{n \times d}$: data matrix, $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^d$: orthogonal basis for subspace $\mathcal{V}$. $\mathbf{V} \in \mathbb{R}^{d \times k}$: matrix with columns $\vec{v}_1, \ldots, \vec{v}_k$.

# LOW-RANK APPROXIMATION

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$$\mathbf{X} \approx \mathbf{X}\mathbf{V}\mathbf{V}^T.$$

This is the closest approximation to $\mathbf{X}$ with rows in $\mathcal{V}$ (i.e., in the column span of $\mathbf{V}$).

- Letting $(\mathbf{X}\mathbf{V}\mathbf{V}^T)_i, (\mathbf{X}\mathbf{V}\mathbf{V}^T)_j$ be the $i^{th}$ and $j^{th}$ projected data points, $\|(\mathbf{X}\mathbf{V}\mathbf{V}^T)_i - (\mathbf{X}\mathbf{V}\mathbf{V}^T)_j\|_2 = \|[(\mathbf{X}\mathbf{V})_i - (\mathbf{X}\mathbf{V})_j]\mathbf{V}^T\|_2 = \|[(\mathbf{X}\mathbf{V})_i - (\mathbf{X}\mathbf{V})_j]\|_2$.
- Can use $\mathbf{X}\mathbf{V} \in \mathbb{R}^{n \times k}$ as a compressed approximate data set.

Key question is how to find the subspace $\mathcal{V}$ and correspondingly $\mathbf{V}$.

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