COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Andrew McGregor

Lecture 24

Last Class:

- Analysis of gradient descent for optimizing convex functions.

This Class:

- Introduction to convex sets and projection functions.
- (The same) analysis of projected gradient descent for optimizing under convex functions under (convex) constraints.
- Online learning, regret, and online gradient descent.
- Application to stochastic gradient descent.

CONSTRAINED CONVEX OPTIMIZATION

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Definition – Convex Set: A set $\mathcal{S} \subseteq \mathbb{R}^d$ is convex if and only if, for any $\vec{\theta}_1, \vec{\theta}_2 \in \mathcal{S}$ and $\lambda \in [0, 1]$: $(1 - \lambda)\vec{\theta}_1 + \lambda \cdot \vec{\theta}_2 \in \mathcal{S}$

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For any convex set let $P_{\mathcal{S}}(\cdot)$ denote the projection function onto $\mathcal{S}$:

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- For $\mathcal{S} = \{\vec{\theta} \in \mathbb{R}^d : \|\vec{\theta}\|_2 \leq 1\}$ what is $P_{\mathcal{S}}(\vec{y})$?
- For $\mathcal{S}$ being a k dimensional subspace of $\mathbb{R}^d$, what is $P_{\mathcal{S}}(\vec{y})$?

Projected Gradient Descent

- Choose some initialization $\vec{\theta}_1$ and set $\eta = \frac{R}{G\sqrt{t}}$.
- For $i = 1, \ldots, t-1$
    - $\vec{\theta}_{i+1}^{(out)} = \vec{\theta}_i - \eta \cdot \vec{\nabla} f(\vec{\theta}_i)$
    - $\vec{\theta}_{i+1} = P_{\mathcal{S}}(\vec{\theta}_{i+1}^{(out)})$.
- Return $\hat{\theta} = \arg\min_{\vec{\theta}_i} f(\vec{\theta}_i)$.

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Analysis of projected gradient descent is almost identifcal to gradient descent analysis! Just need to appeal to following geometric result:

Theorem – Projection to a convex set: For any convex set $\mathcal{S} \subseteq \mathbb{R}^d$, $\vec{y} \in \mathbb{R}^d$, and $\vec{\theta} \in \mathcal{S}$,

$$\|P_{\mathcal{S}}(\vec{y}) - \vec{\theta}\|_2 \leq \|\vec{y} - \vec{\theta}\|_2.$$

Theorem – Projected GD: For convex G-Lipschitz function f, and convex set $\mathcal{S}$, Projected GD run with $t \geq \frac{R^2 G^2}{\epsilon^2}$ iterations, $\eta = \frac{R}{G\sqrt{t}}$, and starting point within radius R of $\vec{\theta}_* = \min_{\vec{\theta} \in \mathcal{S}} f(\vec{\theta})$, outputs $\hat{\theta}$ satisfying:

$$f(\hat{\theta}) \leq f(\vec{\theta}_*) + \epsilon$$

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Recall: $\vec{\theta}_{i+1}^{(out)} = \vec{\theta}_i - \eta \cdot \vec{\nabla} f(\vec{\theta}_i)$ and $\vec{\theta}_{i+1} = P_{\mathcal{S}}(\vec{\theta}_{i+1}^{(out)})$.

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Step 2: $\frac{1}{t} \sum_{i=1}^{t} f(\vec{\theta}_i) - f(\vec{\theta}_*) \leq \frac{R^2}{2\eta \cdot t} + \frac{\eta G^2}{2} \implies$ Theorem.

ONLINE GRADIENT DESCENT

In reality many learning problems are online.

- Websites optimize ads or recommendations to show users, given continuous feedback from these users.
- Spam filters are incrementally updated and adapt as they see more examples of spam over time.
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Want to minimize some global loss $L(\vec{\theta}, \mathbf{X}) = \sum_{i=1}^{n} \ell(\vec{\theta}, \vec{x}_i)$, when data points are presented in an online fashion $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ (similar to streaming algorithms)

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Stochastic gradient descent is a special case: when data points are considered a random order for computational reasons.

ONLINE OPTIMIZATION FORMAL SETUP

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- At each step, first pick (play) a parameter vector $\vec{\theta}^{(i)}$.
- Then are told f_i and incur cost $f_i(\vec{\theta}^{(i)})$.
- **Goal:** Minimize total cost $\sum_{i=1}^t f_i(\vec{\theta}^{(i)})$.

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Our analysis will make no assumptions on how $f_1, \ldots, f_t$ are related to each other!

ONLINE OPTIMIZATION EXAMPLE

Home pricing tools.

$\vec{x} = [\#baths, \#beds, \#floors \ldots]$

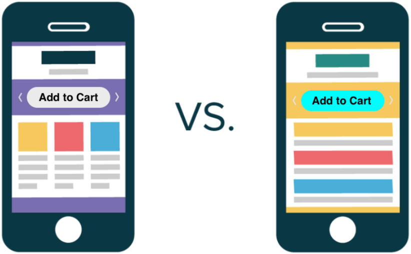
linear model

$\langle \vec{x}, \vec{\theta} \rangle$

\$275,000

- Parameter vector $\vec{\theta}^{(i)}$: coefficients of linear model at step i.
- Functions $f_1, \ldots, f_t$: $f_i(\vec{\theta}^{(i)}) = (\langle \vec{x}_i, \vec{\theta}^{(i)} \rangle - price_i)^2$ revealed when $home_i$ is listed or sold.
- Want to minimize total squared error $\sum_{i=1}^{t} f_i(\vec{\theta}^{(i)})$ (same as classic least squares regression).

UI design via online optimization.



- Parameter vector $\vec{\theta}^{(i)}$: some encoding of the layout at step i.
- Functions $f_1, \ldots, f_t$: $f_i(\vec{\theta}^{(i)}) = 1$ if user does not click 'add to cart' and $f_i(\vec{\theta}^{(i)}) = 0$ if they do click.
- Want to maximize number of purchases, i.e., minimize $\sum_{i=1}^{t} f_i(\vec{\theta}^{(i)})$.

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ϵ is called the regret and ϵ/t is the average regret.

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- This error metric is a bit unusual: Comparing online solution to best fixed solution in hindsight. ϵ can be negative!

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How small can the regret ϵ be? $\sum_{i=1}^{t} f_i(\vec{\theta}^{(i)}) \leq \sum_{i=1}^{t} f_i(\vec{\theta}^{\mathit{off}}) + \epsilon$.

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What if for $i = 1, \ldots, t$, $f_i(\theta) = |\theta - 1000|$ or $f_i(\theta) = |\theta + 1000|$ in no particular pattern? How can any online learning algorithm hope to achieve small regret?

Assume that:

- $f_1, \ldots, f_t$ are all convex.
- Each f_i is G-Lipschitz (i.e., $\|\vec{\nabla} f_i(\vec{\theta})\|_2 \leq G$ for all $\vec{\theta}$.)
- $\|\vec{\theta}^{(1)} - \vec{\theta}^{off}\|_2 \leq R$ where $\theta^{(1)}$ is the first vector chosen.

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- For $i = 1, \ldots, t$
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Theorem – OGD on Convex Lipschitz Functions: For convex G-Lipschitz $f_1, \ldots, f_t$, OGD initialized with starting point $\theta^{(1)}$ within radius R of θ^{off}, using step size $\eta = \frac{R}{G\sqrt{t}}$, has regret bounded by:

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Convexity $\implies$ **Step 1:** For all i,

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$$\begin{aligned}
\left[\sum_{i=1}^{t} f_i(\theta^{(i)}) - \sum_{i=1}^{t} f_i(\theta^{off})\right] &\leq \sum_{i=1}^{t} \frac{\|\theta^{(i)} - \theta^{off}\|_2^2 - \|\theta^{(i+1)} - \theta^{off}\|_2^2}{2\eta} + \frac{t \cdot \eta G^2}{2} \\
&= \frac{\|\theta^{(1)} - \theta^{off}\|_2^2 - \|\theta^{(t+1)} - \theta^{off}\|_2^2}{2\eta} + \frac{t \cdot \eta G^2}{2} \\
&\leq R^2/(2\eta) + t\eta G^2/2 = RG\sqrt{t}
\end{aligned}$$