COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Andrew McGregor

Lecture 12

Distributional JL Lemma: Let $\boldsymbol{M} \in \mathbb{R}^{m \times d}$ have independent $\mathcal{N}(0, 1/m)$ entries. If we set $m = O\left(\frac{\log(1/\delta)}{\epsilon^2}\right)$, then for any $y \in \mathbb{R}^d$, with probability at least $1 - \delta$

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- Remains to show that $\|\tilde{y}\|_2^2$ is tightly concentrated around $\|y\|_2^2$.

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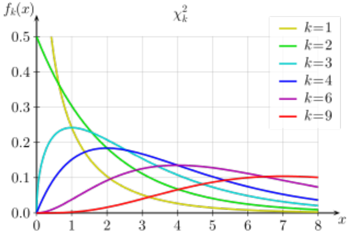

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Gives the distributional JL Lemma and thus the classic JL Lemma!

JL LEMMA IS ESSENTIALLY OPTIMAL

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- What is the largest set of mutually orthogonal unit vectors in d-dimensional space? Answer: d.
- How large can a set of unit vectors in d-dimensional space be that have all pairwise dot products $|\langle x, y \rangle| \leq \epsilon$? Answer: $2^{\Omega(\epsilon^2 d)}$.

An exponentially large set of random vectors will be nearly pairwise orthogonal with high probability!

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- If $t = \frac{1}{2}e^{\epsilon^2 d/12}$, using a union bound over $\binom{t}{2} \leq \frac{1}{8}e^{\epsilon^2 d/6}$ possible pairs, with probability $\geq 3/4$ all will be nearly orthogonal.

We won't prove it but this is essentially optimal: In d dimensions, there can be at most $2^{O(\epsilon^2 d)}$ nearly orthogonal unit vectors.

Recall: The Johnson Lindenstrauss lemma states that if $\boldsymbol{M} \in \mathbb{R}^{m \times d}$ is a random matrix (linear map) with $m = O\left(\frac{\log n}{\epsilon^2}\right)$, for $x_1, \ldots, x_n \in \mathbb{R}^d$ with high probability, for all i, j:

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Implies: If $x_1, \ldots, x_n$ are nearly orthogonal unit vectors in d-dimensions (with pairwise dot products bounded by $\epsilon/8$), then

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$$\|x_i - x_j\|_2^2 = \|x_i\|_2^2 + \|x_j\|_2^2 - 2\langle x_i, x_j\rangle \ .$$

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CONNECTION TO DIMENSIONALITY REDUCTION

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- Tells us that the JL lemma is optimal up to constants.

Central Limit Theorem

INTERPRETATION AS A CENTRAL LIMIT THEOREM

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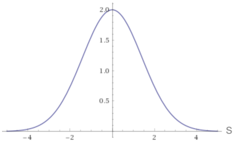

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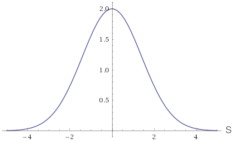

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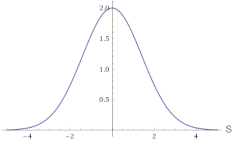

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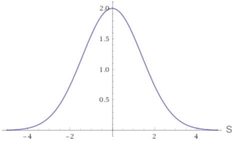

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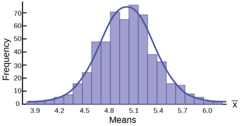
Central Limit Theorem Interpretation: Bernstein's inequality gives a quantitative version of the CLT. The distribution of the sum of *bounded* independent random variables can be upper bounded with a Gaussian (normal) distribution.



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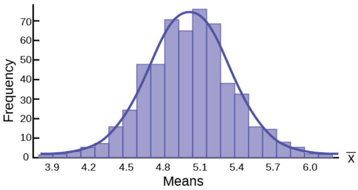

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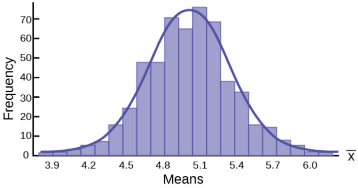

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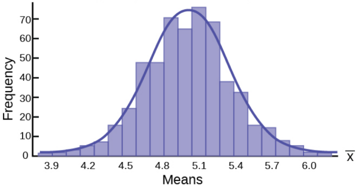

- Why is the Gaussian distribution is so important in statistics, science, ML, etc.?
- Many random variables can be approximated as the sum of a large number of small and roughly independent random effects. Thus, their distribution looks Gaussian by CLT.