COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

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Lecture 2

Today:

- Investigate linearity of expectation and variance.
- Algorithmic application of linearity of expectation and variance.
- Introduce Markov's inequality, a fundamental concentration bound, that let us prove that a random variable lies close to its expectation with good probability.
- Learn about random hash functions, which are a key tool in randomized methods for data processing. Probabilistic analysis via linearity of expectation.

- **Expectation:** $\mathbb{E}[\mathbf{X}] = \sum_{s \in S} \Pr(\mathbf{X} = s) \cdot s.$

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- Two random variables $\mathbf{X}$, $\mathbf{Y}$ are **independent** if for all s, t, $\{\mathbf{X} = s\}$ and $\{\mathbf{Y} = t\}$ are independent events. In other words:

$$\Pr(\{\mathbf{X} = s\} \cap \{\mathbf{Y} = t\}) = \Pr(\mathbf{X} = s) \cdot \Pr(\mathbf{Y} = t).$$

When are the expectation and variance linear?

I.e., under what conditions on $\mathbf{X}$ and $\mathbf{Y}$ do we have:

$$\mathbb{E}[\mathbf{X} + \mathbf{Y}] = \mathbb{E}[\mathbf{X}] + \mathbb{E}[\mathbf{Y}]$$

and

$$\mathrm{Var}[\mathbf{X} + \mathbf{Y}] = \mathrm{Var}[\mathbf{X}] + \mathrm{Var}[\mathbf{Y}].$$

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$$\mathrm{Var}[\mathbf{X} + \mathbf{Y}] = \mathrm{Var}[\mathbf{X}] + \mathrm{Var}[\mathbf{Y}].$$

Last time we showed that linearity of expectation is true regardless of whether the random variables were independent.

$\mathbf{X}, \mathbf{Y}$: any two random variables.

LINEARITY OF VARIANCE

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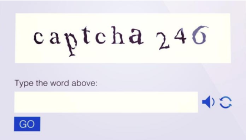
You have contracted with a new company to provide CAPTCHAS for your website.



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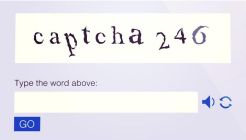

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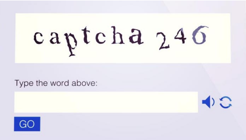

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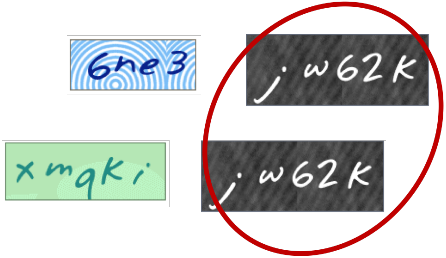
- You want to independently verify this claimed database size.
- You could make test checks until you see $1,000,000$ unique CAPTCHAS: would take $\geq 1,000,000$ checks!

AN ALGORITHMIC APPLICATION

An Idea: You run some test security checks and see if any duplicate CAPTCHAS show up. If you're seeing duplicates after not too many checks, the database size is probably not too big.



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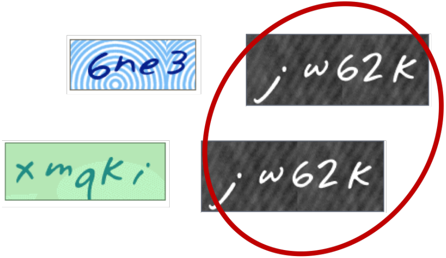

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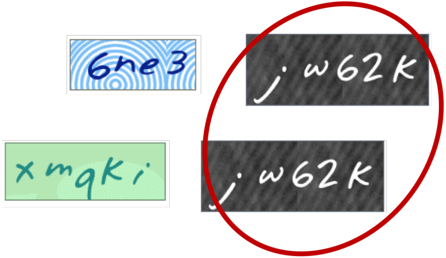

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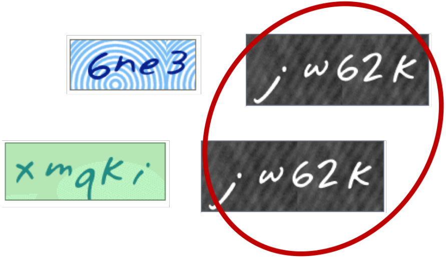

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Note that if the same CAPTCHA shows up four times this counts as $\binom{4}{2}$ duplicates.

Let $\mathbf{D}_{i,j} = 1$ if tests i and j give the same CAPTCHA, and 0 otherwise. An indicator random variable.

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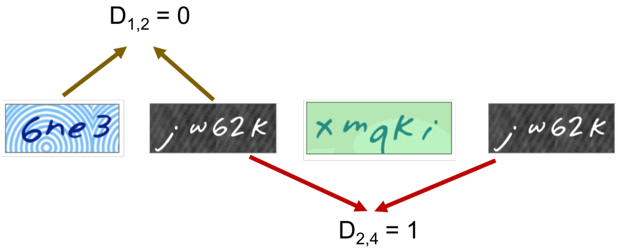

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$$\mathbb{E}[\mathbf{D}] = \sum_{i,j \in [m], i \neq j} \frac{1}{n} = \frac{\binom{m}{2}}{n} = \frac{m(m-1)}{2n}.$$

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You take $m = 1000$ samples. If the database size is as claimed ($n = 1,000,000$) then expected number of duplicates is:

$$\mathbb{E}[\mathbf{D}] = \frac{m(m-1)}{2n} = .4995$$

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Concentration Inequalities: Bounds on the probability that a random variable deviates a certain distance from its mean.

- Useful in understanding how statistical tests perform, the behavior of randomized algorithms, the behavior of data drawn from different distributions, etc.

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The larger the deviation t, the smaller the probability.

Expected number of duplicate CAPTCHAS:

$\mathbb{E}[\mathbf{D}] = \frac{m(m-1)}{2n} = .4995.$

You see $\mathbf{D} = 10$ duplicates.

n: number of CAPTCHAS in database ($n = 1000000$ claimed) , m: number of random CAPTCHAS drawn to check database size ($m = 1000$ in this example), $\mathbf{D}$: number of pairwise duplicates in m random CAPTCHAS.

BACK TO OUR APPLICATION

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Applying Markov's inequality, if the real database size is $n = 1000000$ the probability of this happening is:

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This is pretty small and you feel pretty sure the number of unique CAPTCHAS is much less than 1000000.

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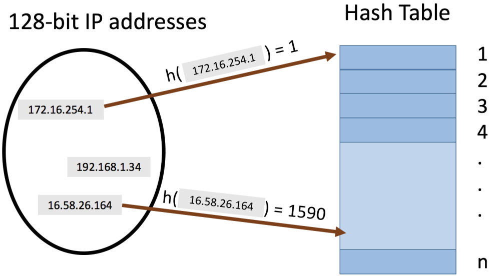
Classic Solution: Hash tables

- *Static hashing* since we won't worry about insertion and deletion today.



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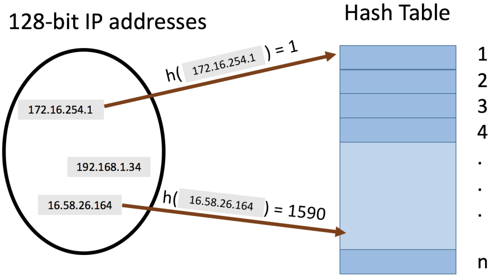

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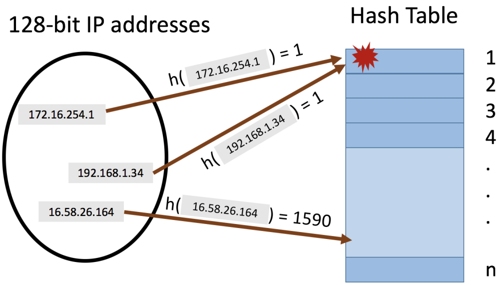

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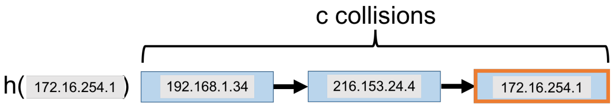
- Typically $|U| \gg n$. Many elements map to the same index.
- **Collisions:** when we insert m items into the hash table we may have to store multiple items in the same location (typically as a linked list).

Query runtime: $O(c)$ when the maximum number of collisions in a table entry is c (i.e., must traverse a linked list of size c).



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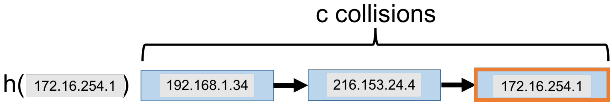

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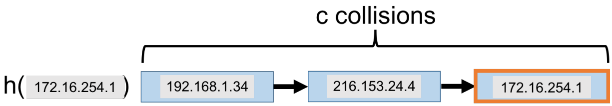

How Can We Bound c?

- In the worst case, could have $c = m$ (all items hash to the same location). In the best case, $c \approx m/n$.

Let $\mathbf{h} : U \to [n]$ be a random hash function.

- I.e., for $x \in U$, $\Pr(\mathbf{h}(x) = i) = \frac{1}{n}$ for all $i = 1, \ldots, n$ and $\mathbf{h}(x), \mathbf{h}(y)$ are independent for any two items $x \neq y$.

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- **Caveat 1:** It is *very expensive* to represent and compute such a random function. We will see how a hash function computable in $O(1)$ time function can be used instead.
- **Caveat 2:** In practice, often suffices to use hash functions like MD5, SHA-2, etc. that 'look random enough'.

LINEARITY OF EXPECTATION

Let $\mathbf{C}_{i,j} = 1$ if items i and j collide ($\mathbf{h}(x_i) = \mathbf{h}(x_j)$), and 0 otherwise. The number of pairwise duplicates is:

$$\mathbf{C} = \sum_{i,j \in [m], i \neq j} \mathbf{C}_{i,j}.$$

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Identical to the CAPTCHA analysis!

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Pretty good but we are using $O(m^2)$ space to store m items.

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TWO LEVEL HASHING

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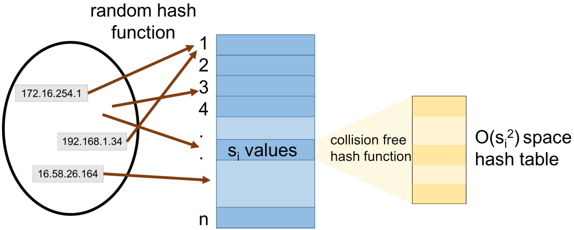

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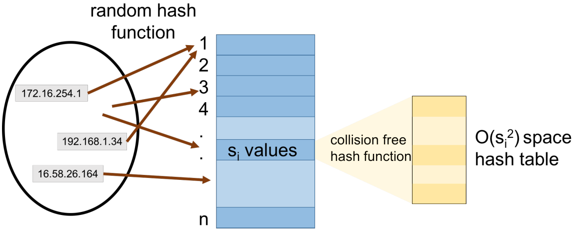

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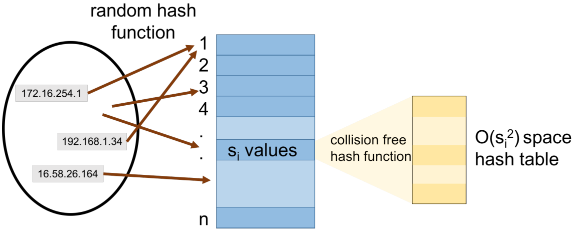

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- **Just Showed:** A random function is collision free with probability $\geq \frac{7}{8}$ so only requires checking $O(1)$ random functions in expectation to find a collision free one.

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Collisions again!

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Near optimal space with $O(1)$ query time!

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Many Applications:

- Filter spam email addresses, phone numbers, suspect IPs, duplicate Tweets.
- Quickly check if an item has been stored in a cache or is new.
- Counting distinct elements (e.g., unique search queries.)

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Efficient Alternative: Let p be a prime with $p \geq |U|$. Choose random $\mathbf{a}, \mathbf{b} \in [p]$ with $\mathbf{a} \neq 0$. Let:

$$\mathbf{h}(x) = (\mathbf{a}x + \mathbf{b} \mod p) \mod n.$$

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