COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Andrew McGregor

Lecture 6

Distinct Elements (Count-Distinct) Problem: Given a stream $x_1, \ldots, x_n$, output the number of distinct elements in the stream.

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Google Sawzall, Facebook Presto, Apache Drill, Twitter Algebird

DISTINCT ELEMENTS IDEAS

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- Let $\mathbf{h} : U \to [0, 1]$ be a random hash function (with a real valued output)
- $s := 1$
- For $i = 1, \ldots, n$
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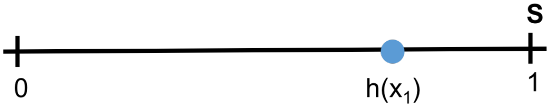

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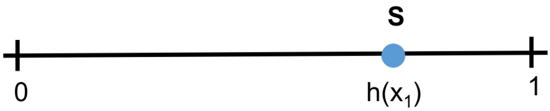

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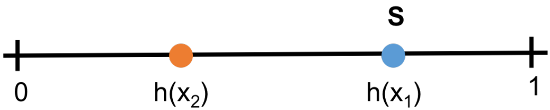

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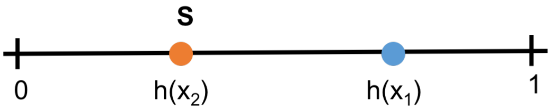

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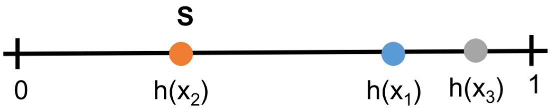

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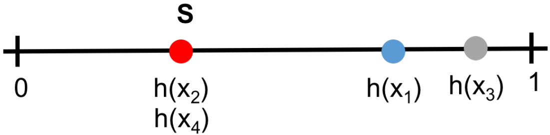

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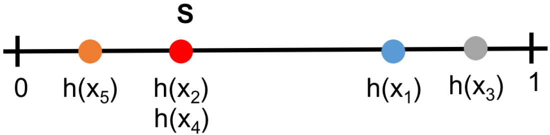

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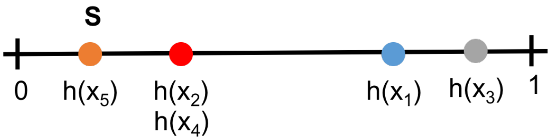

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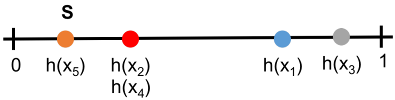

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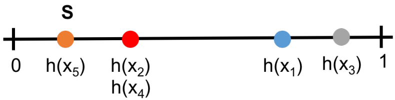

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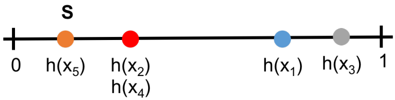

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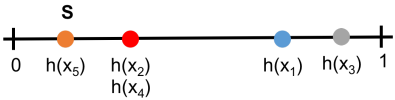

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- Same idea as Flajolet-Martin algorithm and HyperLogLog, except they use discrete hash functions.

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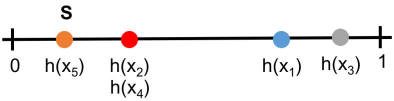

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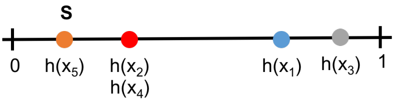

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PERFORMANCE IN EXPECTATION

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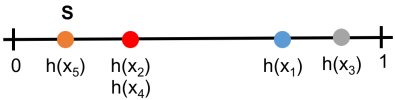

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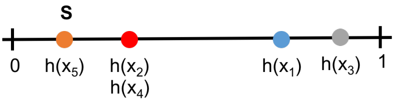

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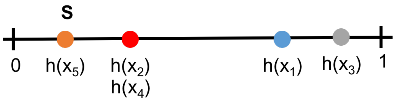

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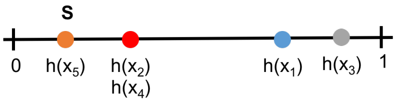

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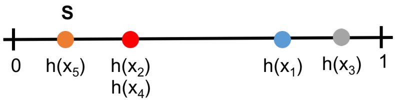

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- **Approximation is robust:** if $|\mathbf{s} - \mathbb{E}[\mathbf{s}]| \le \epsilon \cdot \mathbb{E}[\mathbf{s}]$ for any $\epsilon \in (0, 1/2)$,

$$(1 - 4\epsilon)d \le \widehat{\mathbf{d}} \le (1 + 4\epsilon)d$$

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IMPROVING PERFORMANCE

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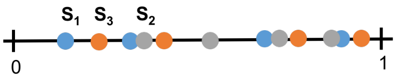

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$$\mathrm{Var}[\mathbf{s}_j] \leq \frac{1}{(d+1)^2}$$

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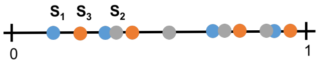
Hashing for Distinct Elements:

- Let $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_k : U \to [0, 1]$ be random hash functions
- $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k := 1$
- For $i = 1, \ldots, n$
  - For j=1,..., k, $\mathbf{s}_j := \min(\mathbf{s}_j, \mathbf{h}_j(x_i))$
- $\mathbf{s} := \frac{1}{k} \sum_{j=1}^{k} \mathbf{s}_j$
- Return $\widehat{\mathbf{d}} = \frac{1}{\mathbf{s}} - 1$



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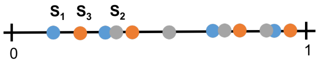

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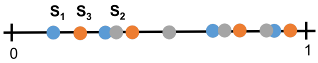

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- $\delta = 5\%$ failure rate gives a factor 20 overhead in space complexity.

IMPROVED FAILURE RATE

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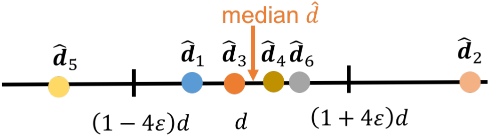

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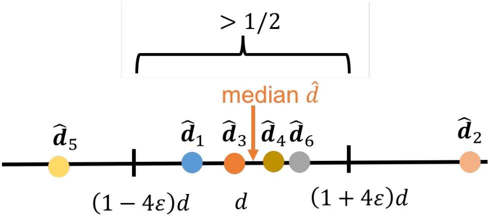

- If $> 1/2$ of trials fall in $[(1-4\epsilon)d, (1+4\epsilon)d]$, then the median will.

- $\widehat{\mathbf{d_1}}, \ldots, \widehat{\mathbf{d_t}}$ are the outcomes of the t trials, each falling in

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with probability at least $3/4$. Let $\widehat{\mathbf{d}} = \textit{median}(\widehat{\mathbf{d}}_1, \ldots, \widehat{\mathbf{d}}_t)$.

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Upshot: The median of $t = O(\log(1/\delta))$ independent runs of the hashing algorithm for distinct elements returns

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No dependence on the number of distinct elements d or the number of items in the stream n! Both can be very large.

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No dependence on the number of distinct elements d or the number of items in the stream n! Both can be very large.

A note on the median: The median is often used as a robust alternative to the mean, when there are outliers (e.g., heavy tailed distributions, corrupted data).

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DISTINCT ELEMENTS IN PRACTICE

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Note: Careful averaging of estimates from multiple hash functions.

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- Set the maximum # of trailing zeros to the maximum in the two sketches.

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