COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Andrew McGregor

Lecture 17

Last Class: Low-Rank Approximation, Eigendecomposition, PCA

- For any symmetric square matrix A, we can write $A = V\Lambda V^T$ where columns of V are orthonormal eigenvectors.
- Can approximate data lying close to in a k-dimensional subspace by projecting data points into that space.
- Can find the best k-dimensional subspace via eigendecomposition applied to $\mathbf{X}^T\mathbf{X}$ (PCA).
- Measuring error in terms of the eigenvalue spectrum.

This Class: SVD and Applications

- SVD and connection to eigenvalue value decomposition.
- Applications of low-rank approximation beyond compression.

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- $\mathbf{U}$ has orthonormal columns $\vec{u}_1, \ldots, \vec{u}_r \in \mathbb{R}^n$ (left singular vectors).
- $\mathbf{V}$ has orthonormal columns $\vec{v}_1, \ldots, \vec{v}_r \in \mathbb{R}^d$ (right singular vectors).
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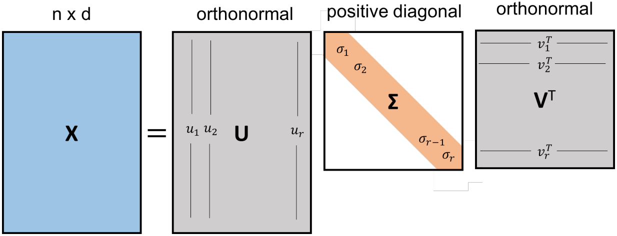

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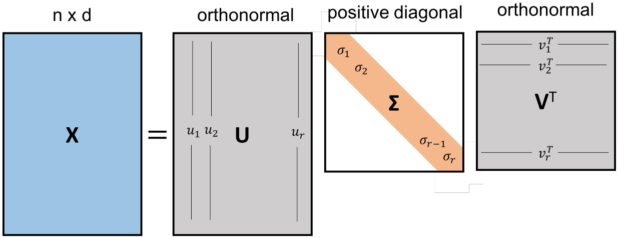

The ‘swiss army knife’ of modern linear algebra.

CONNECTION OF THE SVD TO EIGENDECOMPOSITION

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Similarly: $\mathbf{XX}^T = \mathbf{U\Sigma V}^T\mathbf{V\Sigma U}^T = \mathbf{U\Sigma}^2\mathbf{U}^T$.

The right and left singular vectors are the eigenvectors of the covariance matrix $\mathbf{X}^T\mathbf{X}$ and the gram matrix $\mathbf{XX}^T$ respectively.

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So, letting $\mathbf{V}_k \in \mathbb{R}^{d \times k}$ have columns equal to $\vec{v}_1, \ldots, \vec{v}_k$, we know that $\mathbf{X}\mathbf{V}_k\mathbf{V}_k^T$ is the best rank-k approximation to $\mathbf{X}$ (given by PCA).

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What about $\mathbf{U}_k\mathbf{U}_k^T\mathbf{X}$ where $\mathbf{U}_k \in \mathbb{R}^{n \times k}$ has columns equal to $\vec{u}_1, \ldots, \vec{u}_k$?

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Exercise: $\mathbf{U}_k\mathbf{U}_k^T\mathbf{X} = \mathbf{X}\mathbf{V}_k\mathbf{V}_k^T = \mathbf{U}_k\mathbf{\Sigma}_k\mathbf{V}_k^T$

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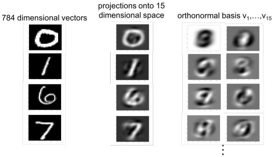
Correspond to projecting the rows (data points) onto the span of $\mathbf{V}_k$ or the columns (features) onto the span of $\mathbf{U}_k$

Row (data point) compression



Column (feature) compression

10000* bathrooms+ 10* (sq. ft.) ≈ list price

	bedrooms	bathrooms	sq.ft.	floors	list price	sale price
home 1	2	2	1800	2	200,000	195,000
home 2	4	2.5	2700	1	300,000	310,000
⋮	⋮	⋮	⋮	⋮	⋮	⋮
home n	5	3.5	3600	3	450,000	450,000

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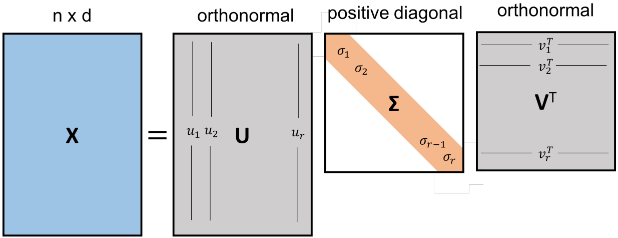

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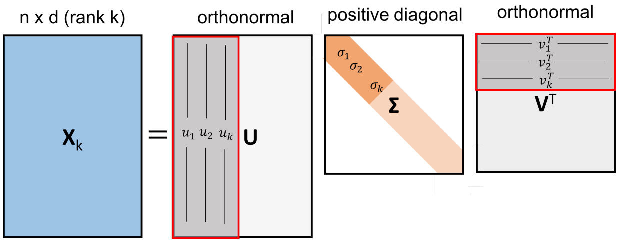

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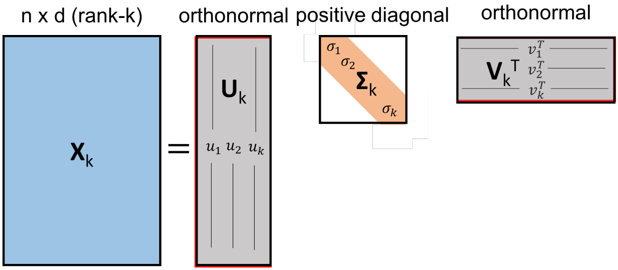

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Basic idea to prove existence of SVD

- Let $\vec{v}_1, \vec{v}_2, \ldots, \in \mathbb{R}^d$ be orthonormal eigenvectors of $\mathbf{X}^T\mathbf{X}$.
- Let $\sigma_i = \|\mathbf{X}\vec{v}_i\|_2$ and define unit vector $\vec{u}_i = \frac{\mathbf{X}\vec{v}_i}{\sigma_i}$.
- Exercise: Show $\vec{u}_1, \vec{u}_2, \ldots$ are orthonormal.
- This establishes that $\mathbf{X}\mathbf{V} = \mathbf{U}\mathbf{\Sigma}$ and that $\mathbf{V}$ and $\mathbf{U}$ have the required properties to show $\mathbf{X} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$.
- To see rest of the details, see `https://math.mit.edu/classes/18.095/2016IAP/lec2/SVD_Notes.pdf`

Rest of Class: Examples of how low-rank approximation is applied in a variety of data science applications.

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- Used for many reasons other than dimensionality reduction/data compression.

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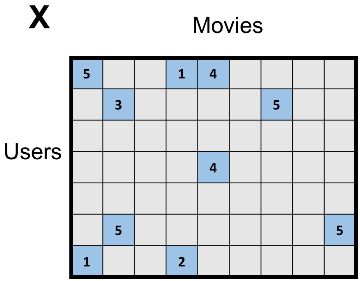

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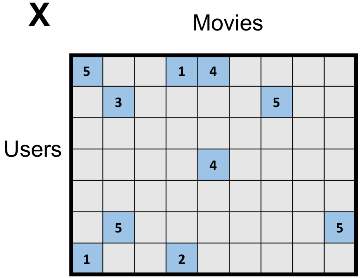

Solve: $\boldsymbol{Y} = \underset{\text{rank}-k\ \mathbf{B}}{\arg\min} \sum_{\text{observed } (j,k)} \left[\mathbf{X}_{j,k} - \mathbf{B}_{j,k}\right]^2$

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Y

Movies

Users

4.9	3.1	3	**1.1**	**3.8**	4.1	4.1	3.4	4.6
3.6	**3**	3	1.2	3.8	4.2	**5**	3.4	4.8
2.8	3	3	2.3	3	3	3	3	3.2
3.4	3	3	4	**4.1**	4.1	4.2	3	3
2.8	3	3	2.3	3	3	3	3	3.4
2.2	**5**	3	4	4.2	3.9	4.4	4	**5.3**
1	3.3	3	**2.2**	3.1	2.9	3.2	1.5	1.8

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Under certain assumptions, can show that **Y** well approximates **X** on both the observed and (most importantly) unobserved entries.

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- Documents (for topic-based search and classification)
- Words (to identify synonyms, translations, etc.)
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Usual Approach: Convert each item into a high-dimensional feature vector and then apply low-rank approximation.

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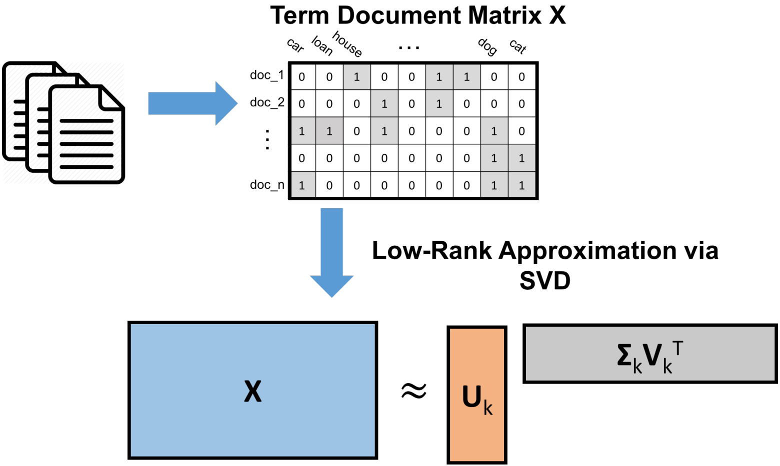

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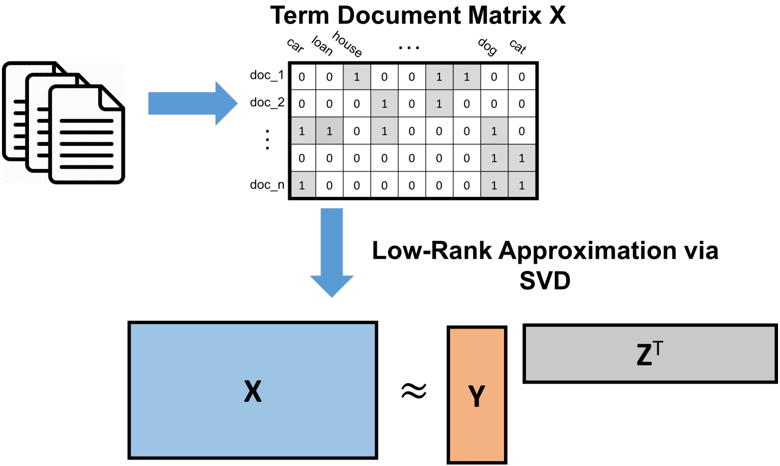

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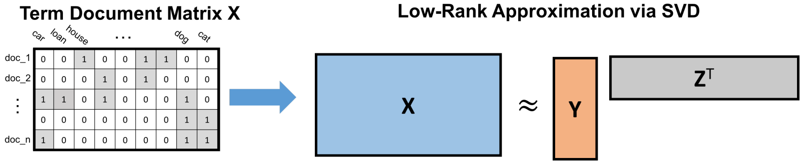

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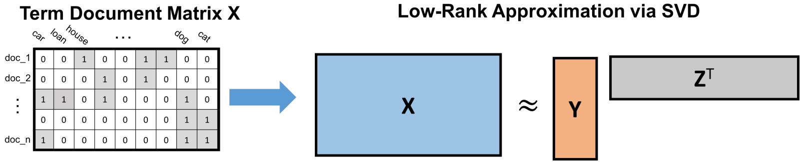

- If the error $\|\mathbf{X} - \mathbf{Y}\mathbf{Z}^T\|_F$ is small, then on average,

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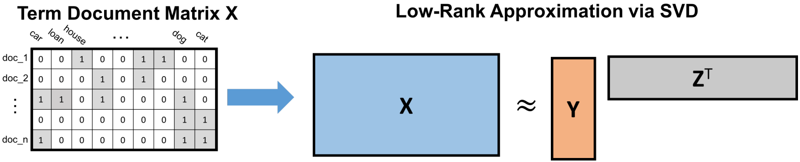

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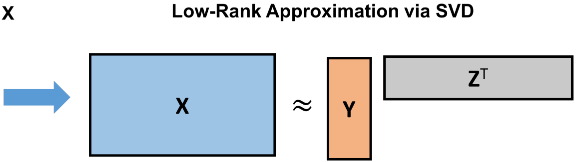
Term Document Matrix X

	car	loan	house		…			dog	cat
doc_1	0	0	1	0	0	1	1	0	0
doc_2	0	0	0	1	0	1	0	0	0
⋮	1	1	0	1	0	0	0	1	0
	0	0	0	0	0	0	0	1	1
doc_n	1	0	0	0	0	0	0	1	1



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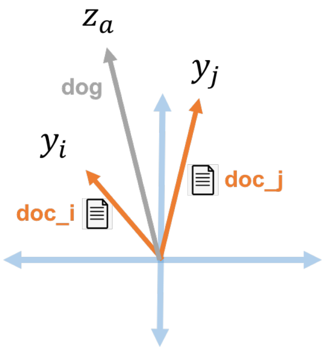

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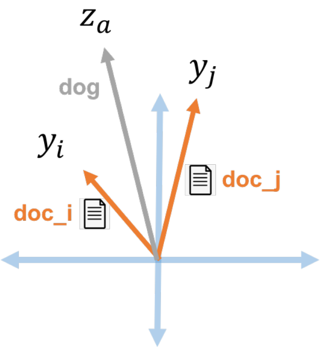

Another View: Each column of $\mathbf{Y}$ represents a 'topic'. $\vec{y}_i(j)$ indicates how much doc_i belongs to topic j. $\vec{z}_a(j)$ indicates how much $word_a$ associates with that topic.

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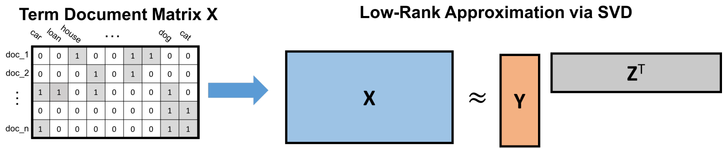

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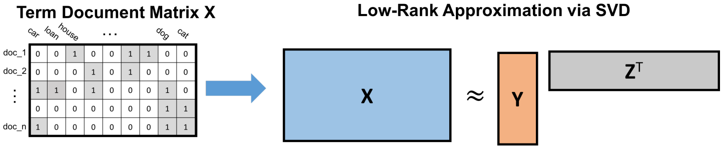

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- In an SVD decomposition we set $\mathbf{Z}^T = \mathbf{\Sigma}_k \mathbf{V}_K^T$.
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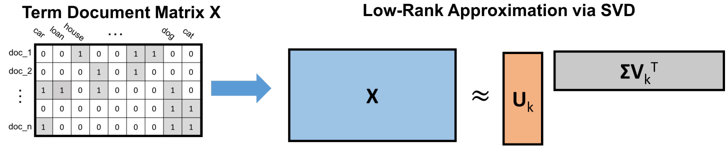

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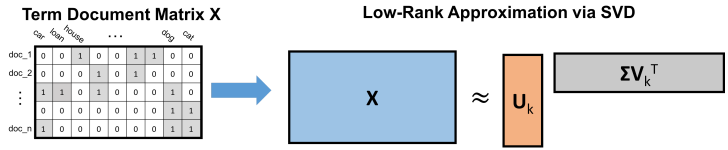

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- The best rank-k approximation of $\mathbf{X}^T\mathbf{X}$ is

$$\underset{\text{rank}-k\ \mathbf{B}}{\arg\min} \|\mathbf{X}^T\mathbf{X} - \mathbf{B}\|_F = \mathbf{V}_k\mathbf{\Sigma}_k^2\mathbf{V}_k^T = \mathbf{Z}\mathbf{Z}^T$$

LSA gives a way of embedding words into k-dimensional space.

- Embedding is via low-rank approximation of $\mathbf{X}^T\mathbf{X}$: where $(\mathbf{X}^T\mathbf{X})_{a,b}$ is the number of documents that both $word_a$ and $word_b$ appear in.

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- Replacing $\mathbf{X}^T\mathbf{X}$ with these different metrics (sometimes appropriately transformed) leads to popular word embedding algorithms: word2vec, GloVe, fastText, etc.

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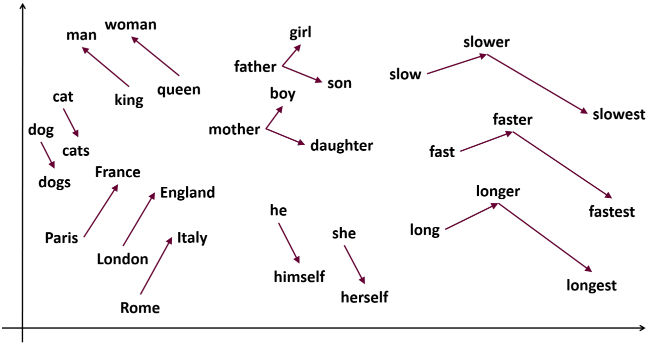

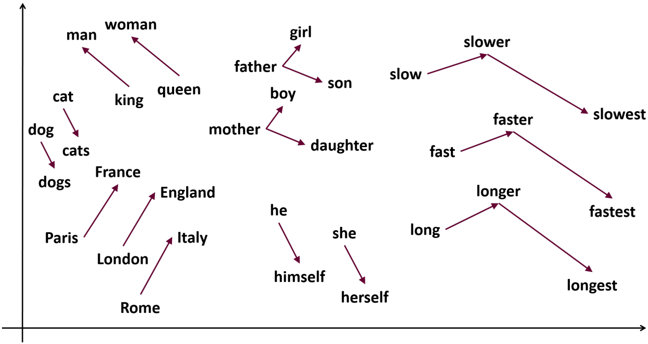

Note: word2vec is typically described as a neural-network method, but it is really just low-rank approximation of a specific similarity matrix. *Neural word embedding as implicit matrix factorization*, Levy and Goldberg.