COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Andrew McGregor

Lecture 1

MOTIVATION FOR THIS CLASS

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- Twitter receives 6,000 tweets per second, 500 million/day. Google receives 60,000 searches per second, 5.6 billion/day.
    - How do they process them to target advertisements? To predict trends? To improve their products?

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    - How do they process them to target advertisements? To predict trends? To improve their products?
- The Large Synoptic Survey Telescope will take high definition photographs of the sky, producing 15 terabytes of data/night.
    - How do they denoise and compress the images? How do they detect anomalies such as changing brightness or position of objects to alert researchers?

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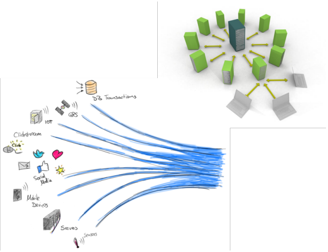
- Massive data sets require storage in a distributed manner or processing in a continuous stream.

vs.



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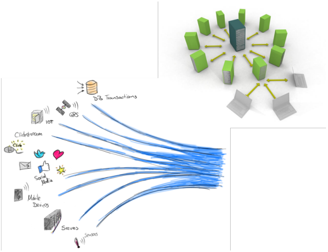

- Even 'simple' problems become very difficult in this setting.

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- How can Google estimate the number of unique search queries that are made in a given week? Given that no machine can store the full list of queries.
- When you use Shazam to identify a song from a recording, how does it provide an answer in < 10 seconds, without scanning over all ~ 8 million audio files in its database.

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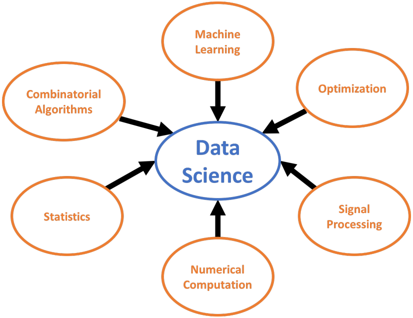

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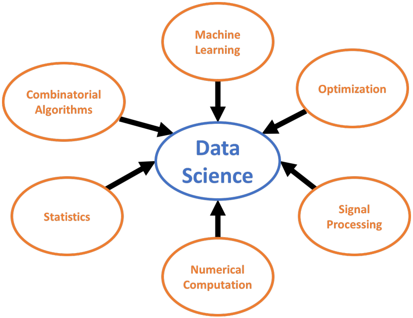

- Many techniques that aren't covered in the traditional CS algorithms curriculum.
- Emphasis on building comfort with mathematical tools that underly data science and machine learning.

WHAT WE'LL COVER

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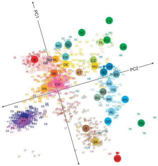
- Probability tools and concentration inequalities.
- Randomized hashing for efficient lookup, load balancing, and estimation. Bloom filters.
- Locality sensitive hashing and nearest neighbor search.
- Streaming algorithms: identifying frequent items in a data stream, counting distinct items, etc.
- Random compression of high-dimensional vectors: the Johnson-Lindenstrauss lemma, applications, and connections to the weirdness of high-dimensional geometry.

WHAT WE'LL COVER

Section 2: Spectral Methods



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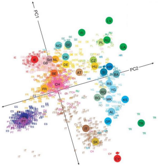

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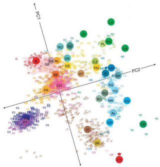
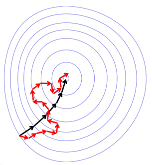


How do we identify the most important features of a dataset using linear algebraic techniques?

- Principal component analysis, low-rank approximation, dimensionality reduction.
- Singular value decomposition (SVD) and its applications to PCA, low-rank approximation, LSI, MDS, ...
- Spectral graph theory. Spectral clustering, community detection, network visualization.
- Computing the SVD on large datasets via iterative methods.

WHAT WE'LL COVER

Section 3: Optimization



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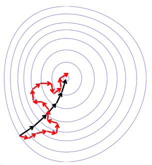

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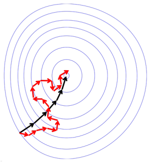

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- Stochastic and online gradient descent.
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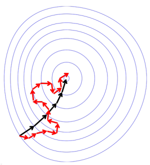

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A small taste of what you can find in COMPSCI 590OP or 690OP.

IMPORTANT TOPICS WE WON'T COVER

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  - COMPSCI 589/689: Machine Learning

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For example: Bayes' rule in conditional probability. What it means for a vector x to be an eigenvector of a matrix A, projection, greedy algorithms, divide-and-conquer algorithms.

See course webpage for lecture slides and related readings:

https://people.cs.umass.edu/~mcgregor/CS514S22/

See Moodle page for this link if you lose it.

Professor: Andrew McGregor

- Email: mcgregor@cs.umass.edu
- Office Hour: 9-10am Wednesday.

TAs:

- Shiv Shankar (sshankar@umass.edu). 3-4pm Monday.
- Weronika Nguyen (thuytrangngu@umass.edu). 11am-12pm Friday.

See Moodle page for Zoom links.

PIAZZA AND PARTICIPATION

We will use Piazza for class discussion and questions.

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You're helping yourself and others if you:

- Ask good clarifying questions and answering questions during lectures.
- Answer other students' or instructor questions on Piazza.
- Post helpful/interesting links on Piazza, e.g., resources covering class material, research articles related to class topics.

TEXTBOOKS AND MATERIALS

We will use material from two textbooks (links to free online versions on the course webpage): *Foundations of Data Science* and *Mining of Massive Datasets*, but will follow neither closely.

- I will sometimes post optional readings a few days prior to each class.
- Draft lecture notes will be posted before each class and potentially updated afterwards if necessary.

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Problem set submissions will be via Gradescope.

- See Moodle for a link to join.

WEEKLY QUIZZES

Will release a Moodle quiz each Friday. It's due Monday at 8pm.

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- Designed as a check-in that you are following the material, and to help me make adjustments as needed.
- Should take under an hour per week, open notes.
- Will sometimes include free response check-in questions to get your feedback on how the course is going, what material from the past week you find most confusing, interesting, etc.

Grade Breakdown:

- Four Problem Sets: 30%.
- Weekly Quizzes: 15%.
- Midterm: 25%. (Likely to be week before Spring Break.)
- Final: 25%. (During Final's Week.)
- Piazza Participation: 5%.

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Academic Honesty:

- A first violation cheating on a homework, quiz, or other assignment will result in a 0 on that assignment.
- A second violation, or cheating on an exam will result in failing the class.

DISABILITY SERVICES AND ACCOMMODATIONS

UMass Amherst is committed to making reasonable, effective, and appropriate accommodations to meet the needs to students with disabilities.

- If you have a documented disability **on file with Disability Services**, you may be eligible for reasonable accommodations in this course.
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I understand that people have different learning needs, home situations, etc. If something isn't working for you in the class, please reach out and let's try to work it out.

Questions?

Section 1: Randomized Methods & Sketching

SOME PROBABILITY REVIEW

Consider a random **X** variable taking values in some finite set $S \subset \mathbb{R}$. E.g., for a random dice roll, $S = \{1, 2, 3, 4, 5, 6\}$.

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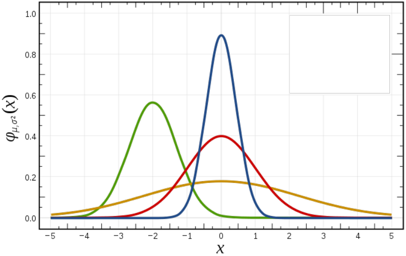
- **Expectation:** $\mathbb{E}[\mathbf{X}] = \sum_{s \in S} \Pr(\mathbf{X} = s) \cdot s.$



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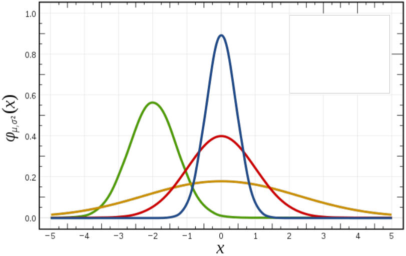

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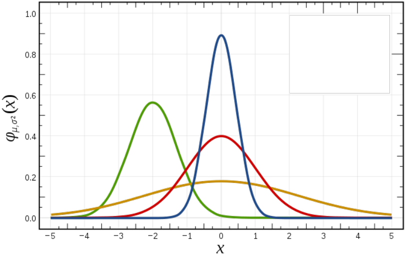

Exercise: Show that for any scalar α, $\mathbb{E}[\alpha \cdot \mathbf{X}] = \alpha \cdot \mathbb{E}[\mathbf{X}]$ and $\mathrm{Var}[\alpha \cdot \mathbf{X}] = \alpha^2 \cdot \mathrm{Var}[\mathbf{X}]$.

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Using the definition of conditional probability, independence means:

$$\frac{\Pr(A \cap B)}{\Pr(B)} = \Pr(A) \implies \Pr(A \cap B) = \Pr(A) \cdot \Pr(B).$$

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Independent Random Variables: Two random variables $\mathbf{X}$, $\mathbf{Y}$ are independent if for all s, t, $\mathbf{X} = s$ and $\mathbf{Y} = t$ are independent events. In other words:

$$\Pr(\mathbf{X} = s \cap \mathbf{Y} = t) = \Pr(\mathbf{X} = s) \cdot \Pr(\mathbf{Y} = t).$$

When are the expectation and variance linear?

I.e., under what conditions on $\mathbf{X}$ and $\mathbf{Y}$ do we have:

$$\mathbb{E}[\mathbf{X}+\mathbf{Y}] = \mathbb{E}[\mathbf{X}] + \mathbb{E}[\mathbf{Y}]$$

and

$$\mathrm{Var}[\mathbf{X}+\mathbf{Y}] = \mathrm{Var}[\mathbf{X}] + \mathrm{Var}[\mathbf{Y}].$$

$\mathbf{X}, \mathbf{Y}$: any two random variables.

LINEARITY OF EXPECTATION

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$\mathbb{E}[\mathbf{X} + \mathbf{Y}] = \mathbb{E}[\mathbf{X}] + \mathbb{E}[\mathbf{Y}]$ for any random variables **X** and **Y**.

$\mathbb{E}[\mathbf{X}+\mathbf{Y}]=\mathbb{E}[\mathbf{X}]+\mathbb{E}[\mathbf{Y}]$ for any random variables $\mathbf{X}$ and $\mathbf{Y}$.

Proof:

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(law of total probability)

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$$

LINEARITY OF VARIANCE

$$\mathrm{Var}[\mathbf{X} + \mathbf{Y}] = \mathrm{Var}[\mathbf{X}] + \mathrm{Var}[\mathbf{Y}]$$

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Together give:

LINEARITY OF VARIANCE

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&= \mathrm{Var}[\mathbf{X}] + \mathrm{Var}[\mathbf{Y}].
\end{aligned}$$