COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Andrew McGregor

Lecture 20

Spectral Graph Partitioning

- Focus on separating graphs with small but relatively balanced cuts.
- Connection to second smallest eigenvector of graph Laplacian.
- Today: Provable guarantees for stochastic block model.

- To partition a graph, find the eigenvector of the Laplacian with the second smallest eigenvalue. Partition nodes based on whether corresponding value in eigenvector is positive/negative.

$$\vec{v}_{n-1} = \operatorname*{arg\,min}_{\vec{v} \in \mathbb{R}^n, \|\vec{v}\|=1, \vec{v}^T \vec{1}=0} \vec{v}^T L \vec{v}$$

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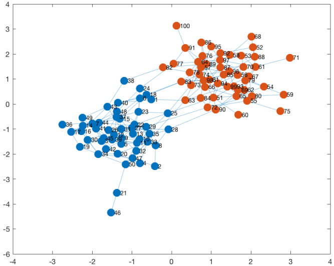
- We argued this "should" partition graph along a small cut that separates the graph into large pieces.
- Haven't given formal guarantees; it's difficult for general input graphs. But can consider randoms "natural" graphs...

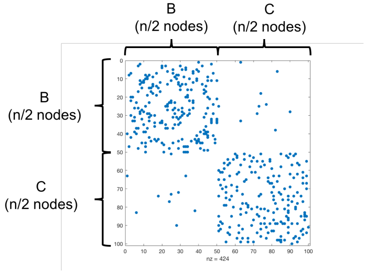
Stochastic Block Model (Planted Partition Model): Let $G_n(p, q)$ be a distribution over graphs on n nodes, split randomly into two groups B and C, each with $n/2$ nodes.

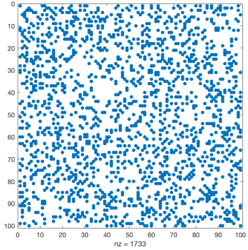
- Any two nodes in the same group are connected with probability p (including self-loops).
- Any two nodes in different groups are connected with prob. $q < p$.
- Connections are independent.



Let G be a stochastic block model graph drawn from $G_n(p,q)$.

- Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be the adjacency matrix of G, ordered in terms of group ID.

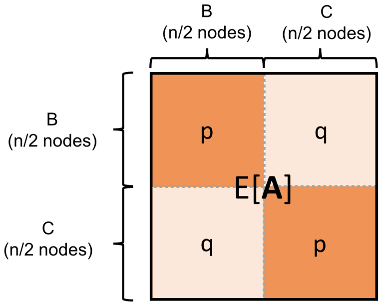




$G_n(p,q)$: stochastic block model distribution. B, C: groups with $n/2$ nodes each. Connections are independent with probability p between nodes in the same group, and probability q between nodes not in the same group.

EXPECTED ADJACENCY SPECTRUM

Letting G be a stochastic block model graph drawn from $G_n(p,q)$ and $\mathbf{A} \in \mathbb{R}^{n\times n}$ be its adjacency matrix. $(\mathbb{E}[\mathbf{A}])_{i,j} = p$ for i, j in same group, $(\mathbb{E}[\mathbf{A}])_{i,j} = q$ otherwise.



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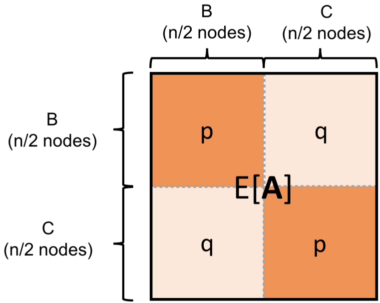

What is rank($\mathbb{E}[\mathbf{A}]$)?
What are the eigenvectors and eigenvalues of $\mathbb{E}[\mathbf{A}]$?

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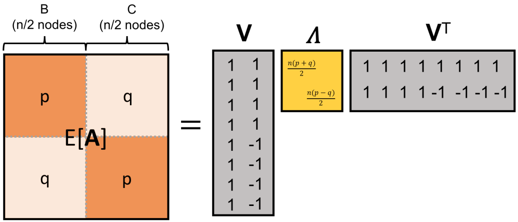

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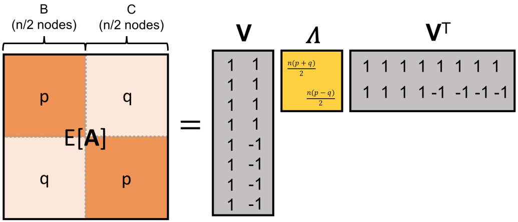

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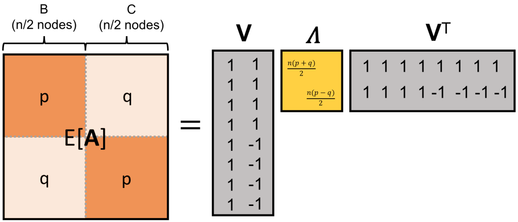
- Can show that for $G \sim G_n(p, q)$, $\mathbf{A}$ is "close" to $\mathbb{E}[\mathbf{A}]$ in some appropriate sense (matrix concentration inequality).



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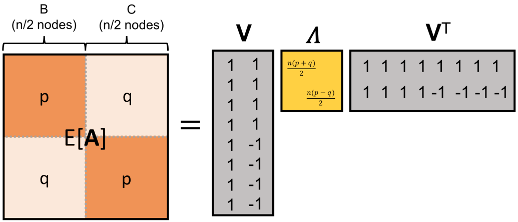

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When rows/columns aren't sorted by ID, second eigenvector is e.g., $[1, -1, 1, -1, \ldots, 1, 1, -1]$ and entries give community ids.

Letting G be a stochastic block model graph drawn from $G_n(p, q)$, $\mathbf{A} \in \mathbb{R}^{n \times n}$ be its adjacency matrix and $\mathbf{L}$ be its Laplacian, what are the eigenvectors and eigenvalues of $\mathbb{E}[\mathbf{L}]$?

Upshot: The second smallest eigenvector of $\mathbb{E}[\mathbf{L}]$ is $\chi_{B,C}$ – the indicator vector for the cut between the communities.

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- If the matrices $\mathbf{A}$ and $\mathbf{L}$ were exactly equal to their expectation, partitioning using this eigenvector (i.e., spectral clustering) would exactly recover the two communities B and C.

Expected Laplacian Spectrum

Upshot: The second smallest eigenvector of $\mathbb{E}[\mathbf{L}]$ is $\chi_{B,C}$ – the indicator vector for the cut between the communities.

- If the matrices $\mathbf{A}$ and $\mathbf{L}$ were exactly equal to their expectation, partitioning using this eigenvector (i.e., spectral clustering) would exactly recover the two communities B and C.

How do we show that a matrix is close to its expectation? Matrix concentration inequalities.

- Analogous to scalar concentration inequalities like Markovs, Chebyshevs, Bernsteins.
- Random matrix theory is a very recent and cutting edge subfield of mathematics that is being actively applied in computer science, statistics, and ML.

Matrix Concentration Inequality: If $p \geq O\left(\frac{\log^4 n}{n}\right)$, then with high probability

$$\|\mathbf{A} - \mathbb{E}[\mathbf{A}]\|_2 \leq O(\sqrt{pn}).$$

where $\|\cdot\|_2$ is the matrix spectral norm (operator norm).

For any $\mathbf{X} \in \mathbb{R}^{n \times d}$, $\|\mathbf{X}\|_2 = \max_{z \in \mathbb{R}^d : \|z\|_2 = 1} \|\mathbf{X}z\|_2$.

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For the stochastic block model application, we want to show that the second eigenvectors of $\mathbf{A}$ and $\mathbb{E}[\mathbf{A}]$ are close. How does this relate to their difference in spectral norm?

Davis-Kahan Eigenvector Perturbation Theorem: Suppose $\mathbf{A}, \overline{\mathbf{A}} \in \mathbb{R}^{d \times d}$ are symmetric with $\|\mathbf{A} - \overline{\mathbf{A}}\|_2 \leq \epsilon$ and eigenvectors $v_1, v_2, \ldots, v_d$ and $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_d$. Letting $\theta(v_i, \bar{v}_i)$ denote the angle between v_i and $\bar{v}_i$, for all i:

$$\sin[\theta(v_i, \bar{v}_i)] \leq \frac{\epsilon}{\min_{j \neq i} |\lambda_i - \lambda_j|}$$

where $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of $\overline{\mathbf{A}}$.

The errors get large if there's eigenvalues with similar magnitudes.

APPLICATION TO STOCHASTIC BLOCK MODEL

Claim 1 (Matrix Concentration): For $p \geq O\left(\frac{\log^4 n}{n}\right)$,

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Claim 2 (Davis-Kahan): For $p \geq O\left(\frac{\log^4 n}{n}\right)$,

$$\sin\theta(v_2, \bar{v}_2) \leq \frac{O(\sqrt{pn})}{\min_{j \neq i} |\lambda_i - \lambda_j|}$$

$\mathbf{A}$ adjacency matrix of random stochastic block model graph. p: connection probability within clusters. $q < p$: connection probability between clusters. n: number of nodes. $v_2, \bar{v}_2$: second eigenvectors of $\mathbf{A}$ and $\mathbb{E}[\mathbf{A}]$ respectively.

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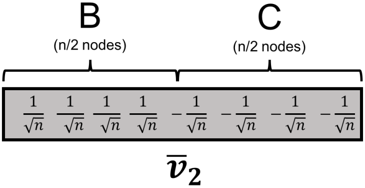

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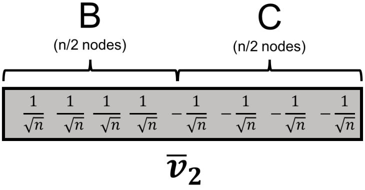

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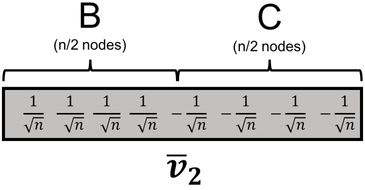

- Every i where $v_2(i)$, $\bar{v}_2(i)$ differ in sign contributes $\geq \frac{1}{n}$ to $\|v_2 - \bar{v}_2\|_2^2$.
- So they differ in sign in at most $O\left(\frac{p}{(p-q)^2}\right)$ positions.

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Upshot: If G is a stochastic block model graph with adjacency matrix $\mathbf{A}$, if we compute its second large eigenvector v_2 and assign nodes to communities according to the sign pattern of this vector, we will correctly assign all but $O\left(\frac{p}{(p-q)^2}\right)$ nodes.