# COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Andrew McGregor

Lecture 5

- Can sometimes get tighter bounds than Markov via:

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- **Moment Generating Function:** Consider for any $t > 0$:

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# EXPONENTIAL CONCENTRATION BOUNDS

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- Weighted sum of all moments ($t$ controls the weights) and choosing $t$ appropriately lets one prove a number of very powerful exponential concentration bounds such as Chernoff, Bernstein, Hoeffding, Azuma, Berry-Esseen, etc.

**Bernstein Inequality:** Consider independent random variables $\mathbf{X}_1, \ldots, \mathbf{X}_n$ all falling in $[-M, M]$. Let $\mu = \mathbb{E}[\sum_{i=1}^n \mathbf{X}_i]$ and $\sigma^2 = \mathrm{Var}[\sum_{i=1}^n \mathbf{X}_i] = \sum_{i=1}^n \mathrm{Var}[\mathbf{X}_i]$. For any $t \geq 0$:

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- An exponentially stronger dependence on $s$!

Consider again bounding the number of heads $\mathbf{H}$ in $n = 100$ independent coin flips.

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| --- | --- | --- |
| $\Pr(\mathbf{H} \geq 60) \leq .25$ | $\Pr(\mathbf{H} \geq 60) \leq .15$ | $\Pr(\mathbf{H} \geq 60) = 0.0284$ |
| $\Pr(\mathbf{H} \geq 70) \leq .0625$ | $\Pr(\mathbf{H} \geq 70) \leq .00086$ | $\Pr(\mathbf{H} \geq 70) = .000039$ |
| $\Pr(\mathbf{H} \geq 80) \leq .04$ | $\Pr(\mathbf{H} \geq 80) \leq 3^{-7}$ | $\Pr(\mathbf{H} \geq 80) < 10^{-9}$ |

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Getting much closer to the true probability.

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**Solution:** Bloom filters (repeated random hashing). Will use much less space than a hash table.

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**Insertions:**

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**Queries:**

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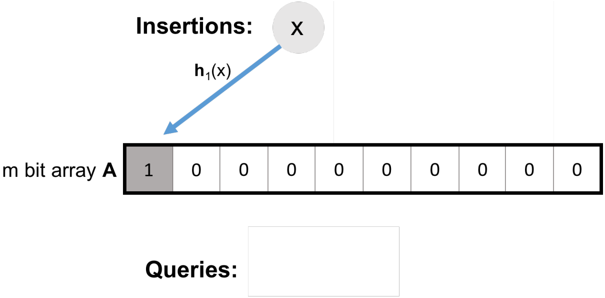

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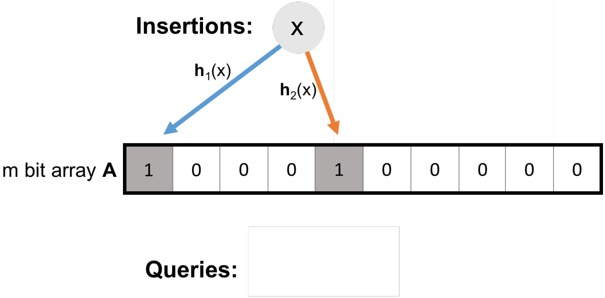

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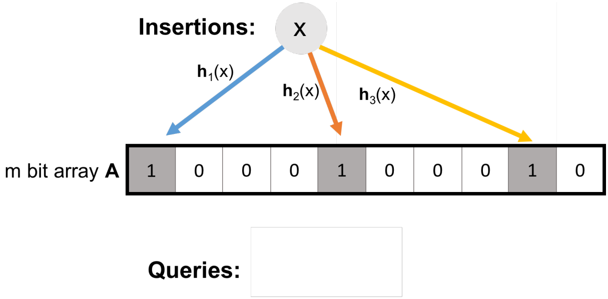

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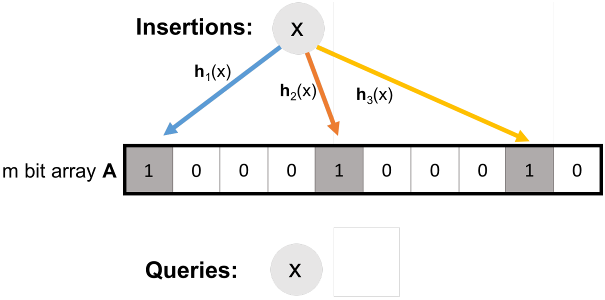

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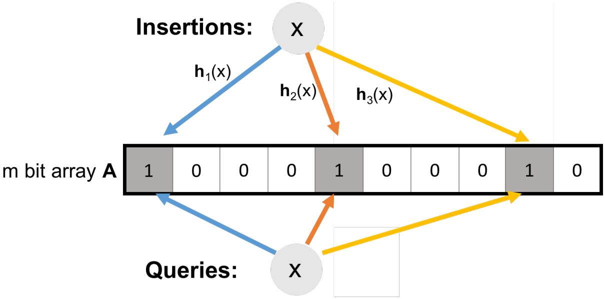

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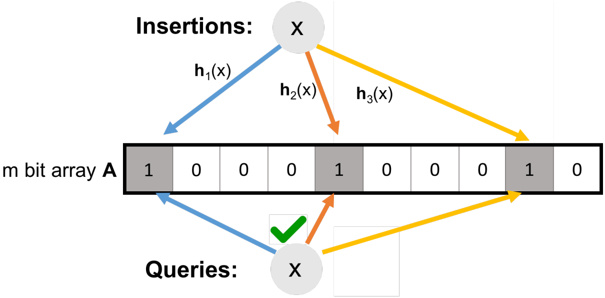

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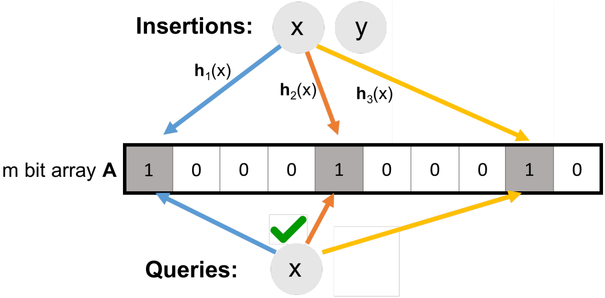

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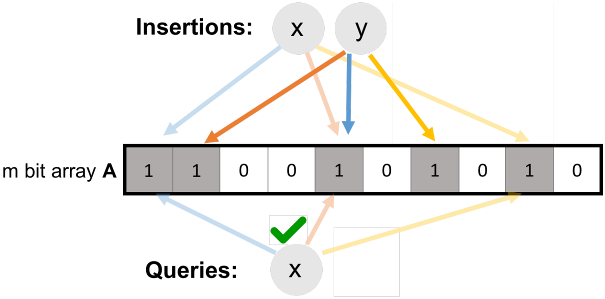

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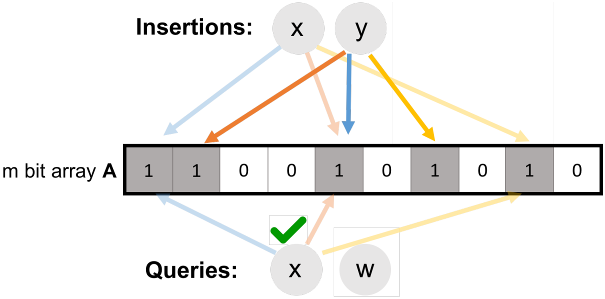

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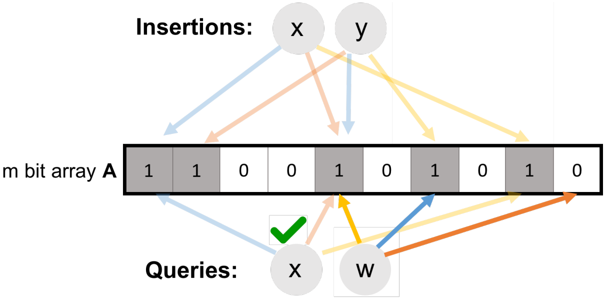

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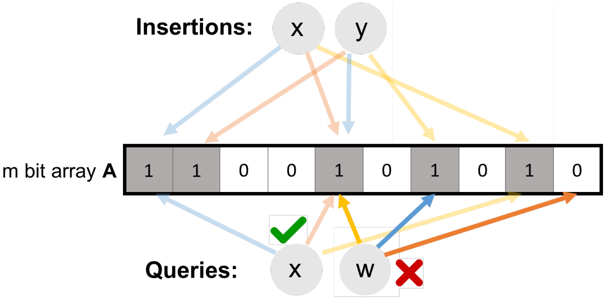
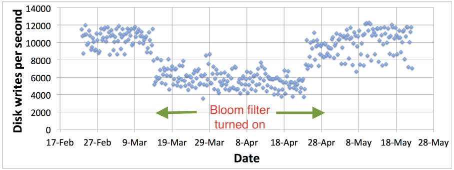



No false negatives. False positives more likely with more insertions.

Akamai (Boston-based company serving 15 – 30% of all web traffic) applies bloom filters to prevent caching of 'one-hit-wonders' – pages only visited once fill over 75% of cache.



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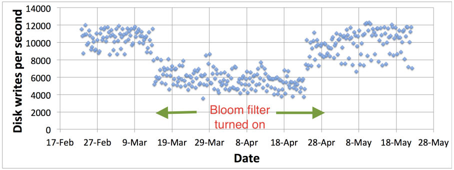


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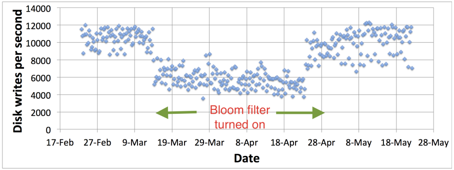


- When url $x$ comes in, if $query(x) = 1$, cache the page at $x$. If not, run $insert(x)$ so that if it comes in again, it will be cached.
- **False positive:** A new url (possible one-hit-wonder) is cached. If the bloom filter has a false positive rate of $\delta = .05$, the number of cached one-hit-wonders will be reduced by at least 95%.

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# ANALYSIS

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&= \left(1 - \frac{1}{m}\right)^{kn}
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What is the probability that after inserting $n$ elements, the $i^{th}$ bit of the array $A$ is still 0?

$$\Pr(A[i] = 0) = \left(1 - \frac{1}{m}\right)^{kn}$$

$n$: total number items in filter, $m$: number of bits in filter, $k$: number of random hash functions, $\mathbf{h}_1, \ldots \mathbf{h}_k$: hash functions, $A$: bit array, $\delta$: false positive rate.

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Let $T$ be the number of zeros in the array after $n$ inserts. Then,

$$E[T] = m\left(1 - \frac{1}{m}\right)^{kn} \approx me^{-\frac{kn}{m}}$$

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# CORRECT ANALYSIS SKETCH

If $T$ is the number of 0 entries, for a non-inserted element $w$:

$$\begin{aligned}
&\Pr(A[\mathbf{h}_1(w)] = \ldots = A[\mathbf{h}_k(w)] = 1) \\
&= \Pr(A[\mathbf{h}_1(w)] = 1) \times \ldots \times \Pr(A[\mathbf{h}_k(w)] = 1) \\
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- How small is $T/m$? Note that $\frac{T}{m} \geq \frac{m-nk}{m} \approx e^{-\frac{kn}{m}}$ when $kn \ll m$. More generally, it can be shown that $T/m = \Omega\left(e^{-\frac{kn}{m}}\right)$ via Theorem 2 of:

  `cglab.ca/~morin/publications/ds/bloom-submitted.pdf`

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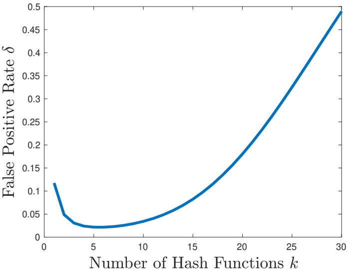

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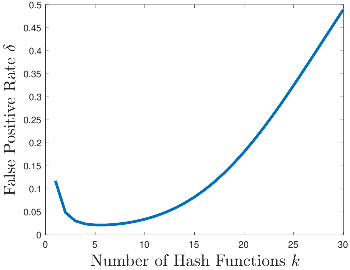


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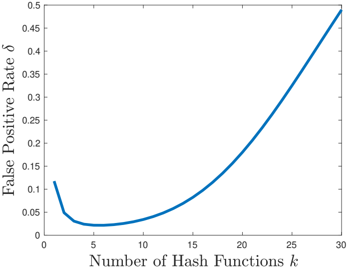


- Can differentiate to show optimal number of hashes is $k = \ln 2 \cdot \frac{m}{n}$.
- Balances between filling up the array with too many hashes and having enough hashes so that even when the array is pretty full, a new item is unlikely to have all its bits set (yield a false positive)

**Stream Processing:** Have a massive dataset $X$ with $n$ items $x_1, x_2, \ldots, x_n$ that arrive in a continuous stream. Not nearly enough space to store all the items (in a single location).

- Still want to analyze and learn from this data.

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- Compared to traditional algorithm design, which focuses on minimizing runtime, the big question here is how much space is needed to answer queries of interest.

# SOME EXAMPLES

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Google Sawzall, Facebook Presto, Apache Drill, Twitter Algebird