On Some Applications of Algebraic Topology in Combinatorics:

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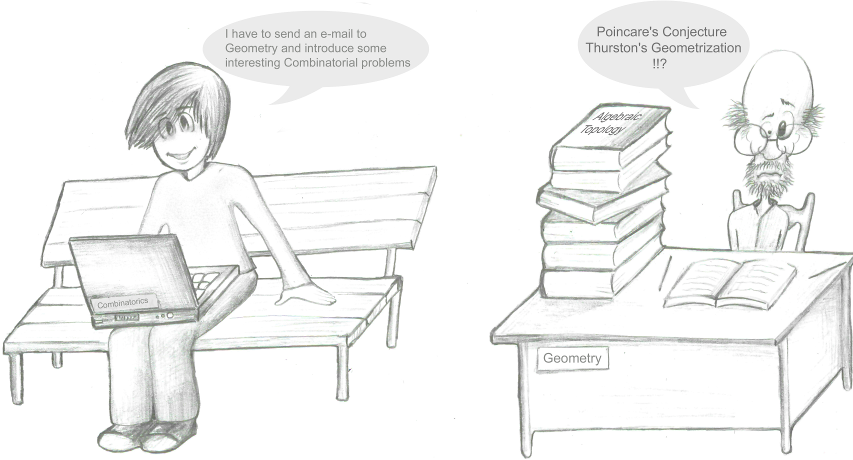
History


I have to send an e-mail to Geometry and introduce some interesting Combinatorial problems
Combinatorics
Poincare's Conjecture
Thurston's Geometrization
!!?
Algebraic Topology
Geometry

Topological Methods

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- Topology of graphs and its applications to group theory.

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Basic Idea

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$$X \xrightarrow{\nu} X \quad \& \quad \nu^2 = id_X .$$

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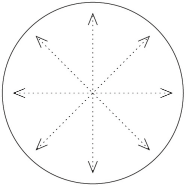
$(S^1, -)$ 

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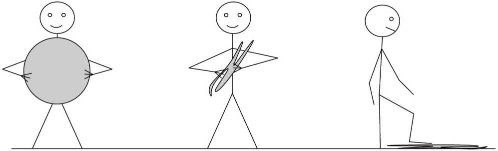
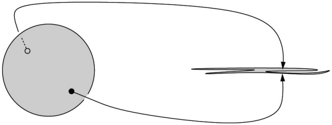


Then there are two points on the surface of the ball that were diametrically opposite (antipodal) and now are lying on top of one another!



▸ CH

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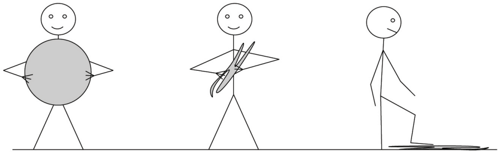

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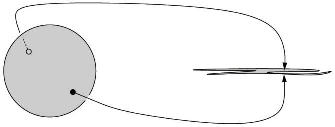

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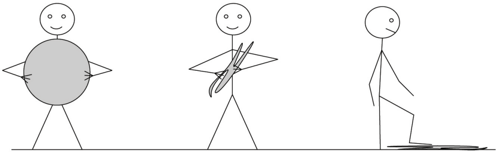

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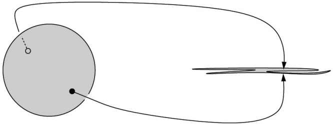

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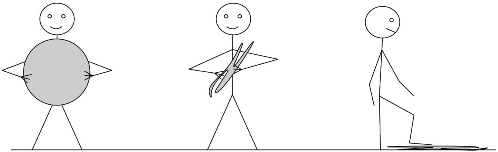

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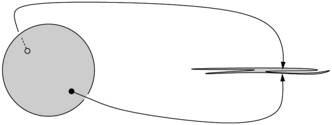

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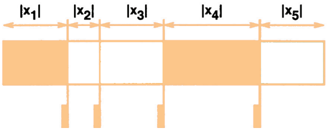

Figure: A cut-set represented by $(-0.2, +0.1, +0.2, -0.3, +0.2)$. The portion A_+ is the union of the white pieces; A_- is the union of the shaded pieces.

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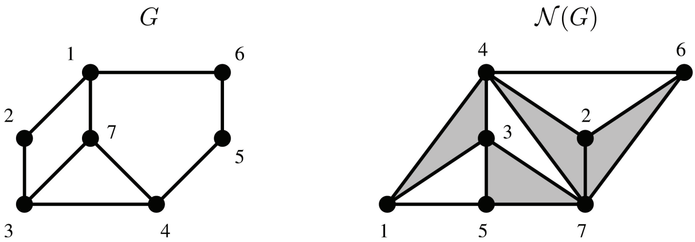

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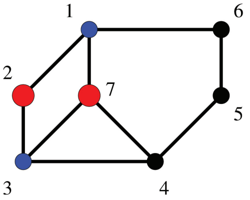

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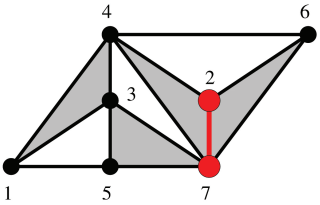

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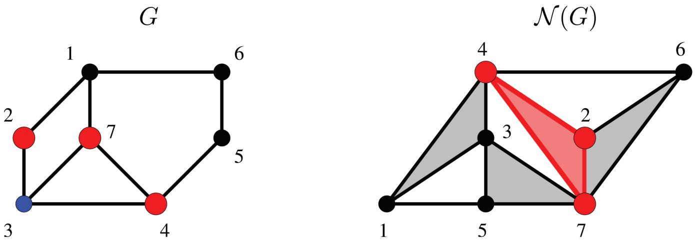

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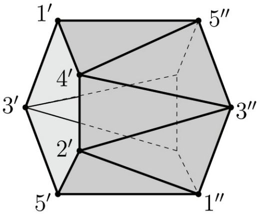

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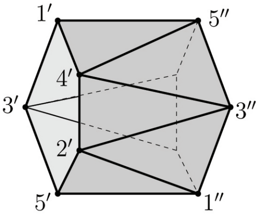

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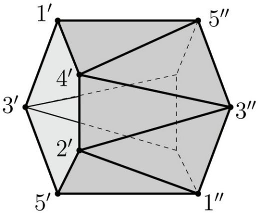

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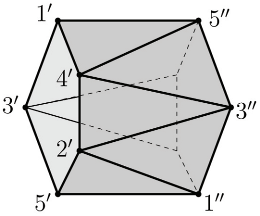

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Definition

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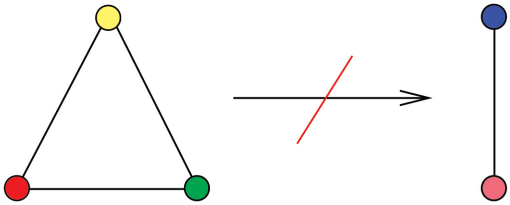

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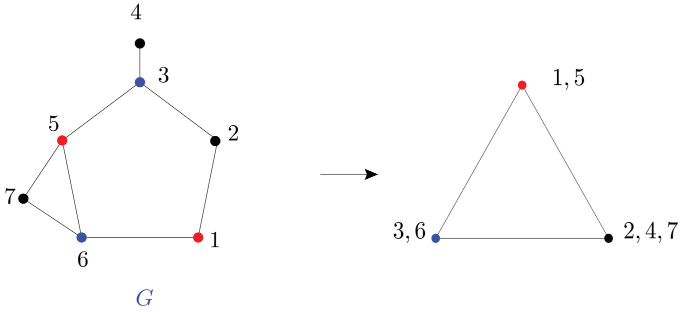

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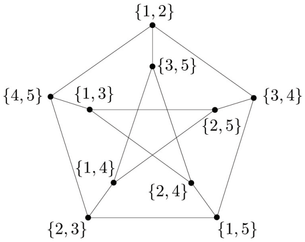

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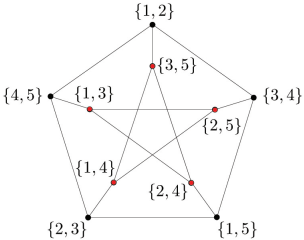

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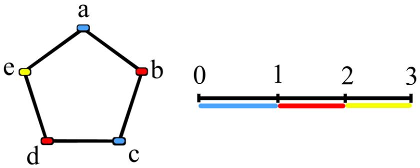

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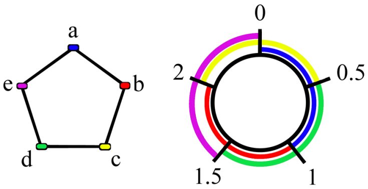

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I would like to acknowledge Mr Ehsan & Ali Kamalinejad for their helps.

Thank You!