Graph Colorings Via The Borsuk-Ulam Theorem

Hossein Hajiabolhassan

Joint Work With Meysam Alishahi and Frédéric Meunier

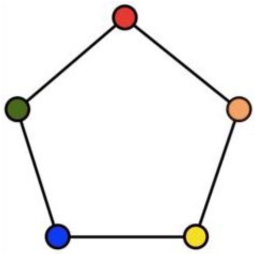
Department of Mathematical Sciences
Shahid Beheshti University
Tehran, Iran

Institute for Research in Fundamental Sciences

October 29, 2014

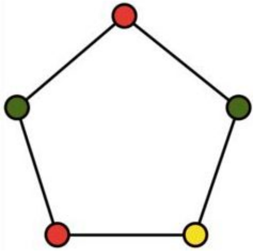
Chromatic number

The chromatic number $\chi(G)$ of a graph G is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color.



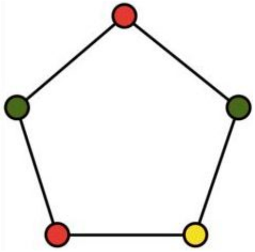
Chromatic number

The chromatic number $\chi(G)$ of a graph G is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color.



Chromatic number

The chromatic number $\chi(G)$ of a graph G is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color.

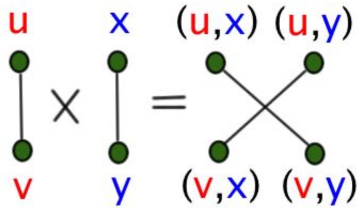


Chromatic number

It is NP-hard to compute the chromatic number of a graph!

The Categorical Product

Let G and G' be two graphs. Their categorical product $G \times G'$ is the graph whose vertex set is $V(G) \times V(G')$ and whose edge set is $E(G \times G') = \{\{(u,u'),(v,v')\} : \{u,v\} \in E(G), \{u',v'\} \in E(G')\}$.



The Categorical Product

Let G and G' be two graphs. Their categorical product $G \times G'$ is the graph whose vertex set is $V(G) \times V(G')$ and whose edge set is $E(G \times G') = \{\{(u, u'), (v, v')\} : \{u, v\} \in E(G), \{u', v'\} \in E(G')\}$.

One can see that $\chi(G \times G') \leq \min\{\chi(G), \chi(G')\}$.

Hedetniemi's Conjecture, 1966

For any two graphs G and G', $\chi(G \times G') = \min\{\chi(G), \chi(G')\}$.

Hedetniemi's Conjecture, 1966

The Categorical Product

Let G and G' be two graphs. Their categorical product $G \times G'$ is the graph whose vertex set is $V(G) \times V(G')$ and whose edge set is $E(G \times G') = \{\{(u, u'), (v, v')\} : \{u, v\} \in E(G), \{u', v'\} \in E(G')\}$.

One can see that $\chi(G \times G') \leq \min\{\chi(G), \chi(G')\}$.

Hedetniemi's Conjecture, 1966

For any two graphs G and G', $\chi(G \times G') = \min\{\chi(G), \chi(G')\}$.

Zhu's Conjecture, 1992

For any two hypergraphs H and H', $\chi(H \times H') = \min\{\chi(H), \chi(H')\}$.

Question

How is it possible to distinguish two distinct topological spaces?

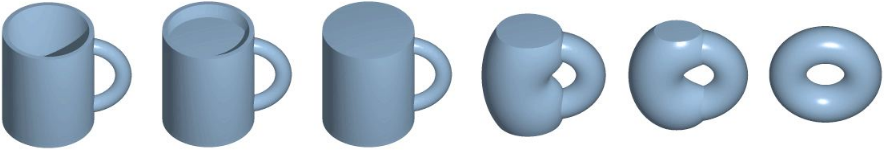
Topological Spaces

Question

How is it possible to distinguish two distinct topological spaces?

A Traditional Joke!

What is a topologist? Someone who cannot distinguish between a doughnut and a coffee cup.



Definition

Set $S^n = \{x : x \in \mathbb{R}^{n+1}, ||x|| = 1\}$.

Question

Is there a continuous mapping from S^n to $\mathbb{R}^n$? YES!

Antipodal Points

Definition

Set $S^n = \{x : x \in \mathbb{R}^{n+1}, ||x|| = 1\}$.

Question

Is there a continuous mapping from S^n to $\mathbb{R}^n$? YES!

Definition

The antipodal point of a point $x \in S^n$ is the point which is diametrically opposite to it, i.e., $-x$.

Definition

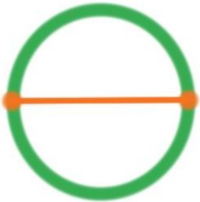
Set $S^n = \{x : x \in \mathbb{R}^{n+1}, ||x|| = 1\}$.

Question

Is there a continuous mapping from S^n to $\mathbb{R}^n$? YES!

Definition

The antipodal point of a point $x \in S^n$ is the point which is diametrically opposite to it, i.e., $-x$.

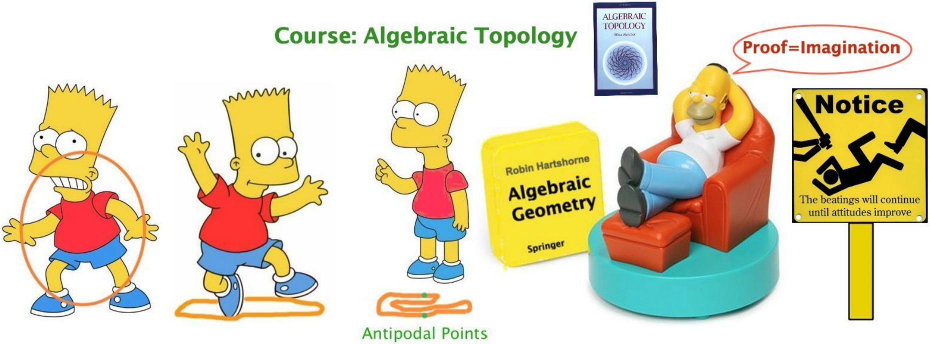


Borsuk-Ulam Theorem

- For every continuous mapping $f : S^n \longrightarrow \mathbb{R}^n$ there exists a point $x \in S^n$ with $f(x) = f(-x)$.

Borsuk-Ulam Theorem

- For every continuous mapping $f : S^n \longrightarrow \mathbb{R}^n$ there exists a point $x \in S^n$ with $f(x) = f(-x)$.



Definitions

- Let $m > n$. Is there a continuous mapping from S^m to S^n? YES!

Definitions

- Let $m > n$. Is there a continuous mapping from S^m to S^n? YES!
- Z_2-space, (Y, ω):

$$Y \xrightarrow{\omega} Y \quad \& \quad \omega^2 = id_Y.$$

- Let $m > n$. Is there a continuous mapping from S^m to S^n? YES!
- Z_2-space, (Y, ω):

$$Y \xrightarrow{\omega} Y \quad \& \quad \omega^2 = id_Y.$$

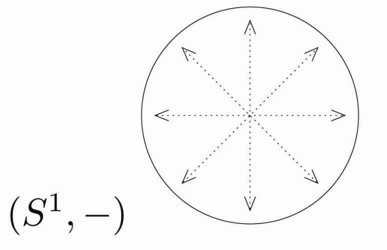
- (Y, ω) free: ω has no fixed point.

Definitions

- Let $m > n$. Is there a continuous mapping from S^m to S^n? YES!
- Z_2-space, (Y, ω):

$$Y \xrightarrow{\omega} Y \quad \& \quad \omega^2 = id_Y.$$

- (Y, ω) free: ω has no fixed point.



Definitions

- Let $m > n$. Is there a continuous mapping from S^m to S^n? YES!
- Z_2-space, (Y, ω):

$$Y \xrightarrow{\omega} Y \quad \& \quad \omega^2 = id_Y.$$

- (Y, ω) free: ω has no fixed point.
- Z_2-map $f : (X, \nu) \longrightarrow (Y, \omega)$:
- f is continuous and $f(\nu(x)) = \omega(f(x))$:
- The images of two antipodal points should be antipodal:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \nu \downarrow & & \downarrow \omega \\ X & \xrightarrow{f} & Y \end{array}$$

Theorem

(Borsuk-Ulam Theorem) For every $n \geq 0$, the following statements are equivalent, and true:

- For every continuous mapping $f : S^n \longrightarrow \mathbb{R}^n$ there exists a point $x \in S^n$ with $f(x) = f(-x)$.

Theorem

(*Borsuk-Ulam Theorem*) *For every $n \geq 0$, the following statements are equivalent, and true:*

- For every continuous mapping $f : S^n \longrightarrow \mathbb{R}^n$ there exists a point $x \in S^n$ with $f(x) = f(-x)$.
- There is no Z_2-map $f : S^{n+1} \to S^n$.

Theorem

(*Borsuk-Ulam Theorem*) *For every $n \geq 0$, the following statements are equivalent, and true:*

- For every continuous mapping $f : S^n \longrightarrow \mathbb{R}^n$ there exists a point $x \in S^n$ with $f(x) = f(-x)$.
- There is no Z_2-map $f : S^{n+1} \to S^n$.
- For any covering $U_1, \ldots, U_{n+1}$ of the sphere S^n by $n + 1$ open sets (resp. closed sets), there is at least one set containing a pair of antipodal points.

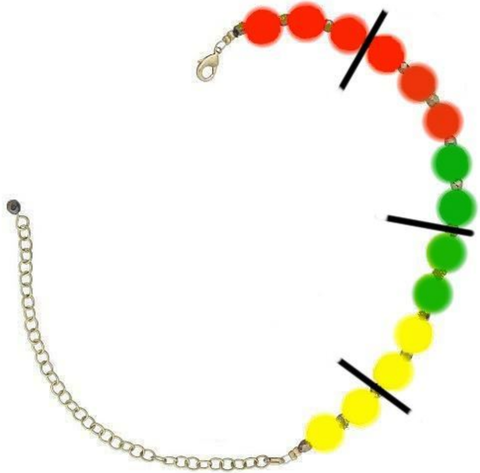
Applications

- Ham Sandwich Theorem
- Team-Splitting
- Consensus-Halving Problem (Cake Problem)
- Level of Rings $(-1 = a_1^2 + \cdots + a_n^2)$
- Graph Colorings
- Necklace Theorem

- **Necklace Theorem**. Every (open) necklace with d kinds of stones, an even number of each kind, can be divided between two thieves using no more than d cuts.

Necklace Theorem

- **Necklace Theorem**. Every (open) necklace with d kinds of stones, an even number of each kind, can be divided between two thieves using no more than d cuts.



Index and Coindex

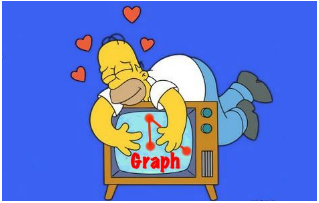
- The Z_2-index of (X,ν): $ind_{Z_2}(X) := \min\{n : X \xrightarrow{Z_2} S^n\}$.

- The Z_2-index of (X, ν): $ind_{Z_2}(X) := \min\{n : X \xrightarrow{Z_2} S^n\}$.
- Theorem. Let X and Y be Z_2 space.
  1. If $X \xrightarrow{Z_2} Y$, then $ind_{Z_2}(X) \leq ind_{Z_2}(Y)$.
  2. In other words, $ind_{Z_2}(X) > ind_{Z_2}(Y)$, then $X \not\xrightarrow{Z_2} Y$.

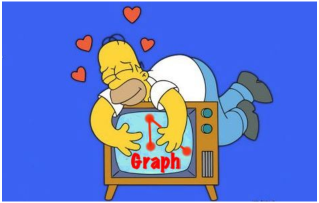
Index and Coindex

- The Z_2-index of (X, ν): $ind_{Z_2}(X) := \min\{n : X \xrightarrow{Z_2} S^n\}$.
- Theorem. Let X and Y be Z_2 space.
  1. If $X \xrightarrow{Z_2} Y$, then $ind_{Z_2}(X) \leq ind_{Z_2}(Y)$.
  2. In other words, $ind_{Z_2}(X) > ind_{Z_2}(Y)$, then $X \not\xrightarrow{Z_2} Y$.
- The Z_2-coindex of (X, ν): $Coind_{Z_2}(X) := \max\{n : S^n \xrightarrow{Z_2} X\}$.
  1. If $X \xrightarrow{Z_2} Y$, then $Coind_{Z_2}(X) \leq Coind_{Z_2}(Y)$.
  2. In other words, $Coind_{Z_2}(X) > Coind_{Z_2}(Y)$, then $X \not\xrightarrow{Z_2} Y$.

Various simplicial complexes can be assigned to a graph G such as box complexes $B(G)$ and $B_0(G)$.



Various simplicial complexes can be assigned to a graph G such as box complexes $B(G)$ and $B_0(G)$.

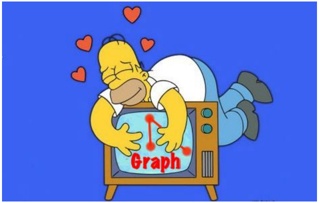


Lower Bounds for Chromatic Number

$$\chi(G) \geq \mathrm{ind}(B(G)) + 2 \geq \mathrm{ind}(B_0(G)) + 1 \geq \mathrm{coind}(B_0(G)) + 1 \geq \mathrm{coind}(B(G)) + 2$$

Topological Combinatorics

Various simplicial complexes can be assigned to a graph G such as box complexes $B(G)$ and $B_0(G)$.



Lower Bounds for Chromatic Number

$$\chi(G) \geq \mathrm{ind}(B(G)) + 2 \geq \mathrm{ind}(B_0(G)) + 1 \geq \mathrm{coind}(B_0(G)) + 1 \geq \mathrm{coind}(B(G)) + 2$$

Theorem (D. Kozlov, 2005)

For any two graphs G and G',

$$\chi(G \times G') \geq Coind(B(G \times G')) + 2 \geq \min\{Coind(B(G)), Coind(B(G'))\} + 2.$$

Definition

Let $X = (x_1, \ldots, x_n), Y = (y_1, \ldots, y_n) \in \{-1, 0, +1\}^n$. Set $X^+ = \{i \in [n] : \; x_i = +1\}$ and $X^- = \{i \in [n] : \; x_i = -1\}$. By $X \preceq Y$, we mean $X^+ \subseteq Y^+$ and $X^- \subseteq Y^-$.

Tucker's Lemma

Definition

Let $X = (x_1, \ldots, x_n), Y = (y_1, \ldots, y_n) \in \{-1, 0, +1\}^n$. Set $X^+ = \{i \in [n] : \ x_i = +1\}$ and $X^- = \{i \in [n] : \ x_i = -1\}$. By $X \preceq Y$, we mean $X^+ \subseteq Y^+$ and $X^- \subseteq Y^-$.

Tucker's Lemma, 1946

Let $\lambda : \{-1, 0, +1\}^n \setminus \{(0, \ldots, 0)\} \longrightarrow \{\pm 1, \pm 2, \ldots, \pm(n-1)\}$. Also, assume that for any $X \in \{-1, 0, +1\}^n \setminus \{(0, \ldots, 0)\}$, we have $\lambda(-X) = -\lambda(X)$. Then there exist two vectors $X, Y \in \{-1, 0, +1\}^n \setminus \{(0, \ldots, 0)\}$ such that $X \preceq Y$ and also $\lambda(X) = -\lambda(Y)$.

Tucker's Lemma

Definition

Let $X=(x_1,\dots,x_n), Y=(y_1,\dots,y_n)\in\{-1,0,+1\}^n$. Set $X^+=\{i\in[n]:\ x_i=+1\}$ and $X^-=\{i\in[n]:\ x_i=-1\}$. By $X\preceq Y$, we mean $X^+\subseteq Y^+$ and $X^-\subseteq Y^-$.

Tucker's Lemma, 1946

Let $\lambda:\{-1,0,+1\}^n\setminus\{(0,\dots,0)\}\longrightarrow\{\pm1,\pm2,\dots,\pm(n-1)\}$. Also, assume that for any $X\in\{-1,0,+1\}^n\setminus\{(0,\dots,0)\}$, we have $\lambda(-X)=-\lambda(X)$. Then there exist two vectors $X,Y\in\{-1,0,+1\}^n\setminus\{(0,\dots,0)\}$ such that $X\preceq Y$ and also $\lambda(X)=-\lambda(Y)$.

Z_p-Tucker Lemma (G.M. Ziegler, 2002)

Generalization of the Borsuk-Ulam Theorem

Tucker-Ky Fan's Lemma, 1952

Let $\lambda : \{-1, 0, +1\}^n \setminus \{(0, \dots, 0)\} \longrightarrow \{\pm 1, \pm 2, \dots, \pm m\}$. If

1. for any $X \in \{-1, 0, +1\}^n \setminus \{(0, \dots, 0)\}$, we have $\lambda(-X) = -\lambda(X)$,
2. no two vectors X and Y are such that $X \preceq Y$ and $\lambda(X) = -\lambda(Y)$,

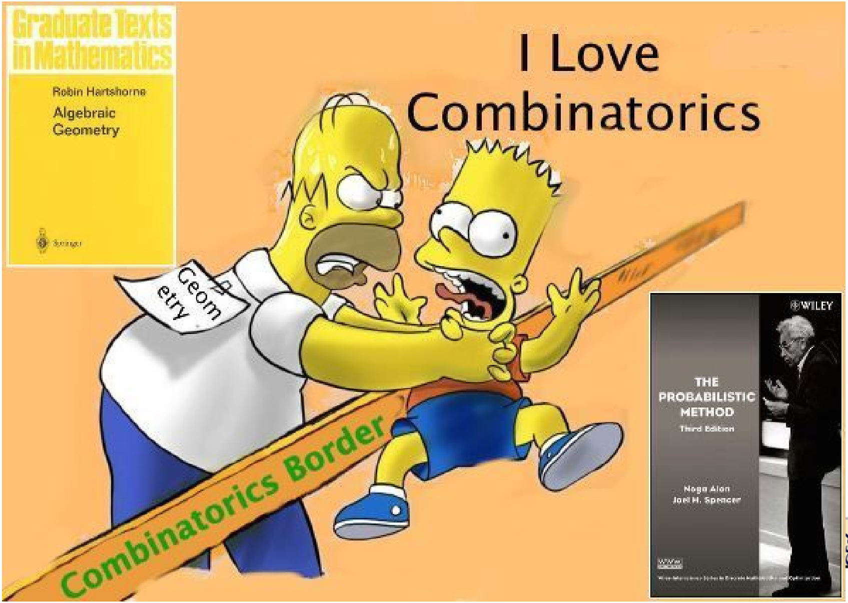
then there are n signed sets $X_1 \preceq X_2 \preceq \cdots \preceq X_n$ such that $\{\lambda(X_1), \dots, \lambda(X_n)\} = \{+c_1, -c_2, \dots, (-1)^{n-1} c_n\}$, where $1 \leq c_1 < \cdots < c_n \leq m$. In particular, $m \geq n$.

Ky Fan's Lemma, 1952

If m open (closed) subsets $F_1, F_2, \dots, F_m$ of the d-sphere S^d cover S^d and if no one of them contain a pair of antipodal points, then there exist $d + 2$ indices $l_1, l_2, \dots, l_{d+2}$, such that $1 \leq l_1 < l_2 < \cdots < l_{d+2} \leq m$ and $F_{l_1} \cap -F_{l_2} \cap \cdots \cap (-1)^{d+1} F_{l_{d+2}} \neq \varnothing$.

- Chromatic Number
- Circular Chromatic Number
- Local Chromatic Number
- Semi-Matching Chromatic Number

History


Graduate Texts in Mathematics
Robin Hartshorne
Algebraic Geometry
I Love Combinatorics
Geom etry
Combinatorics Border
WILEY
THE PROBABILISTIC METHOD
Third Edition
Noga Alon
Joel H. Spencer

History


ALG
EB
RA
Applied Math.
Geometry
INTRODUCTION TO
GRAPH THEORY
DOUGLAS B. WEST
Analysis
Math.

Kneser Representation

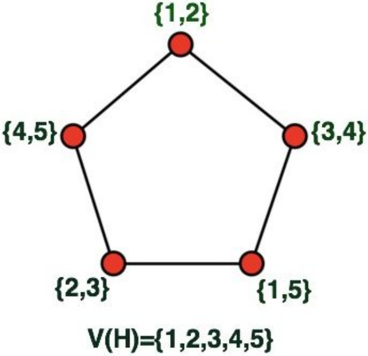
For a hypergraph H, consider the graph $\mathrm{KG}(H)$ whose vertex set is $E(H)$ and whose edge set consists of all disjoint pairs. For instance, if

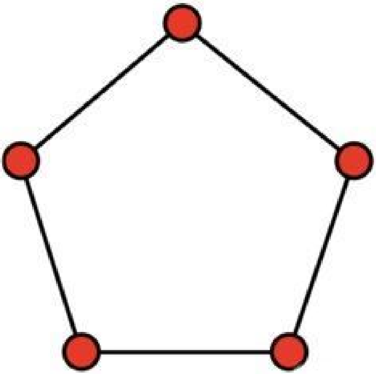
$$V(H) = \{1, 2, 3, 4, 5\},$$

$$E(H) = \{\{1, 2\}, \{3, 4\}, \{1, 5\}, \{2, 3\}, \{4, 5\}\},$$

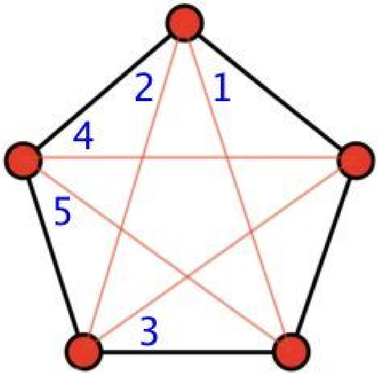
then

$$\mathrm{KG}(H) = C_5.$$

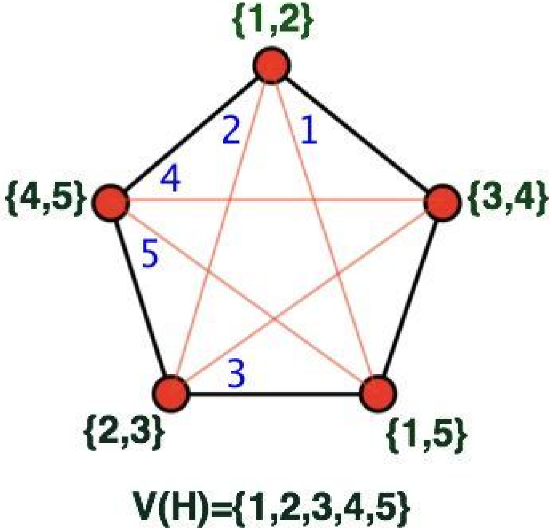




Kneser Representations of Graphs



Kneser Representations of Graphs



Altermatic Number

V(H)={1,2,3,4,5}

E(H)={{1,2},{3,4},{1,5},{2,3},{4,5}}

1 2 3 4 5

Altermatic Number

V(H)={1,2,3,4,5}

E(H)={{1,2},{3,4},{1,5},{2,3},{4,5}}

1	2	3	4	5
+	0	0	-	+

Altermatic Number

V(H)={1,2,3,4,5}

E(H)={{1,2},{3,4},{1,5},{2,3},{4,5}}

1	2	3	4	5	
+	0	0	-	+	→ 3 nonzero

Altermatic Number

V(H)={1,2,3,4,5}

E(H)={{1,2},{3,4},{1,5},{2,3},{4,5}}

1	2	3	4	5	
+	0	0	-	+	→ 3 nonzero
+	+	-	+	-	

Altermatic Number

$V(H)=\{1,2,3,4,5\}$

$E(H)=\{\{1,2\},\{3,4\},\{1,5\},\{2,3\},\{4,5\}\}$

1	2	3	4	5	
+	0	0	-	+	→ 3 nonzero
+	+	-	+	-	

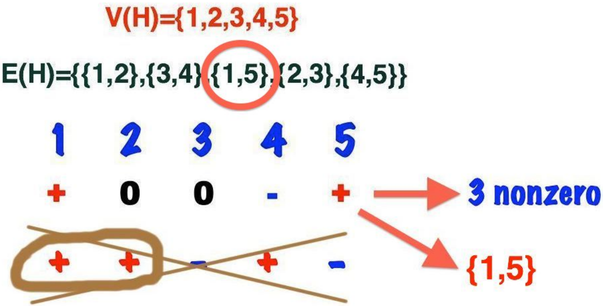
Altermatic Number

V(H)={1,2,3,4,5}

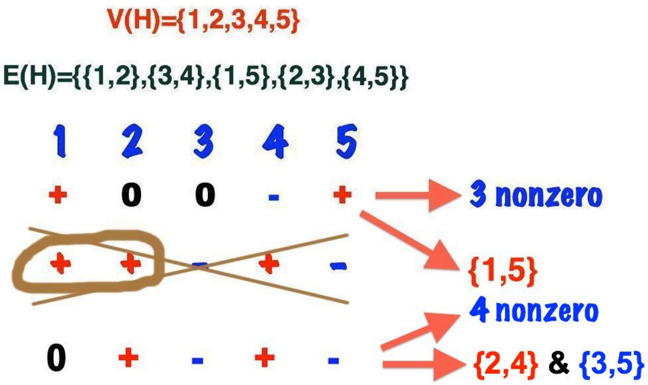
E(H)={{1,2},{3,4},{1,5},{2,3},{4,5}}

1	2	3	4	5	
+	0	0	-	+	→ 3 nonzero
+	+	-	+	-	

Altermatic Number



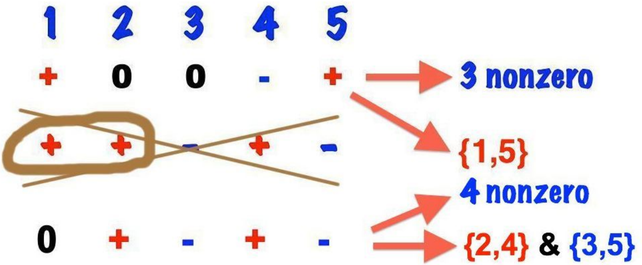
Altermatic Number



Altermatic Number

$V(H)=\{1,2,3,4,5\}$

$E(H)=\{\{1,2\},\{3,4\},\{1,5\},\{2,3\},\{4,5\}\}$

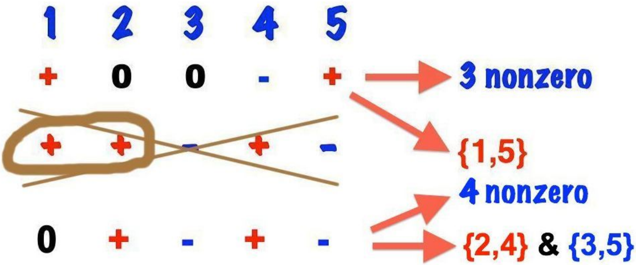


Alt(H)= The number of nonzero elements of a longest alternating subsequence which does not contain a positive OR negative hyperedge

Altermatic Number

$V(H)=\{1,2,3,4,5\}$

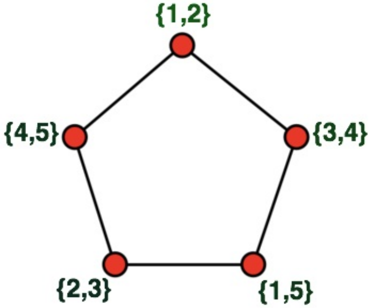
$E(H)=\{\{1,2\},\{3,4\},\{1,5\},\{2,3\},\{4,5\}\}$



Alt(H)= The number of nonzero elements of a longest alternating subsequence which does not contain a positive OR negative hyperedge

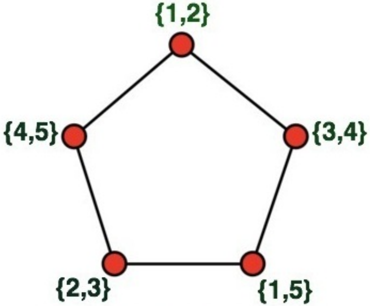
$$|V(H)|-\mathrm{Alt}(H)$$

The Altermatic Number of Five Cycle



V(H)={1,2,3,4,5}

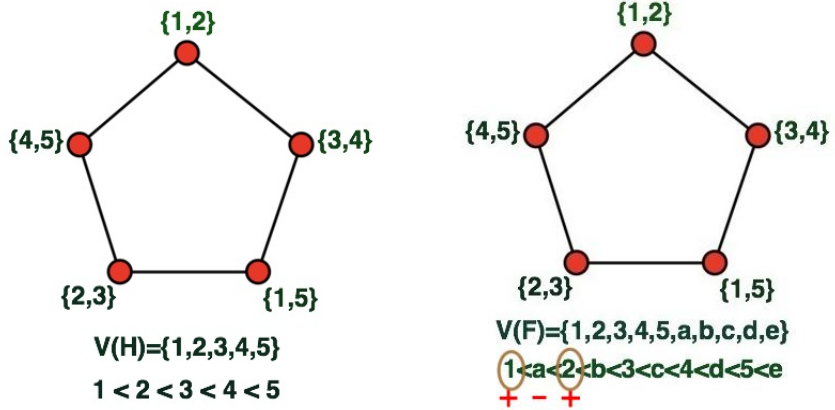
$1 < 2 < 3 < 4 < 5$



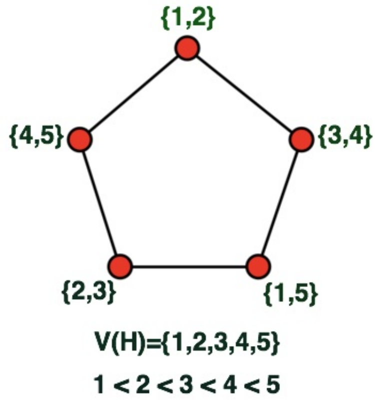
V(F)={1,2,3,4,5,a,b,c,d,e}

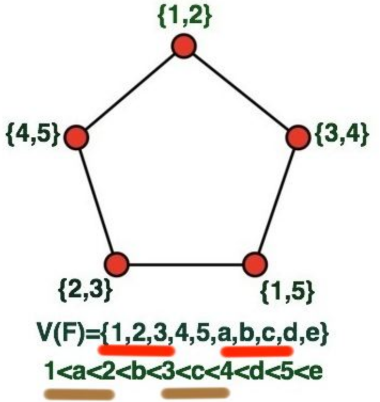
$1<a<2<b<3<c<4<d<5<e$

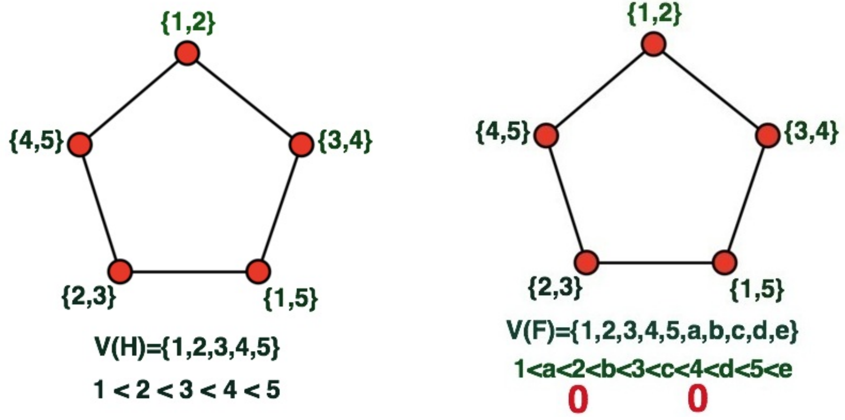
The Altermatic Number of Five Cycle



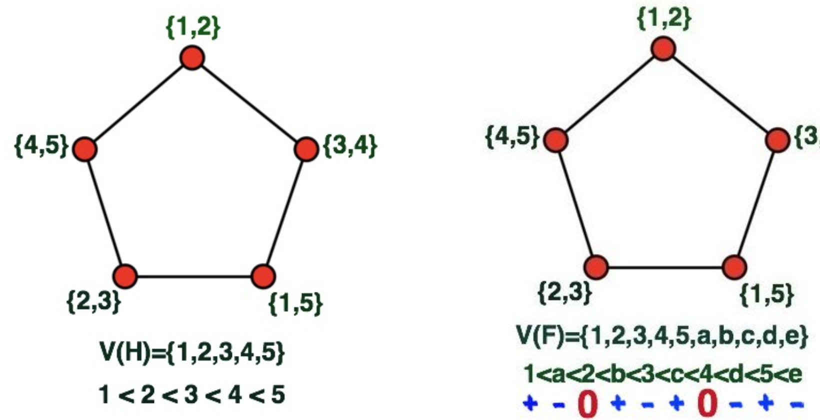
The Altermatic Number of Five Cycle



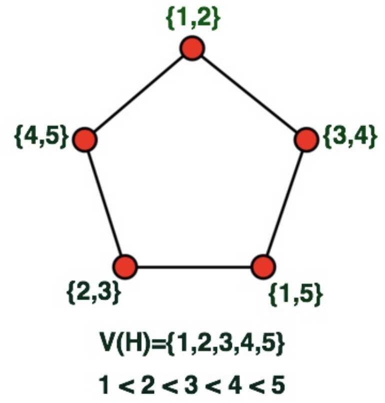


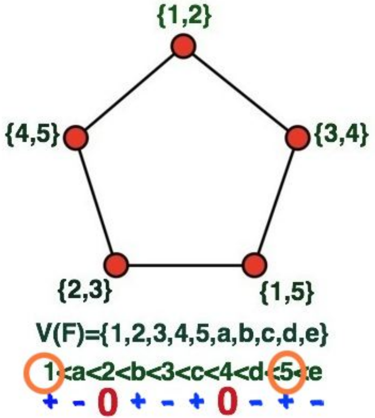

{1,2}
{4,5}
{3,4}
{2,3}
{1,5}
V(H)={1,2,3,4,5}
1 < 2 < 3 < 4 < 5
{1,2}
{4,5}
{3,4}
{2,3}
{1,5}
V(F)={1,2,3,4,5,a,b,c,d,e}
1<a<2<b<3<c<4<d<5<e
0
0

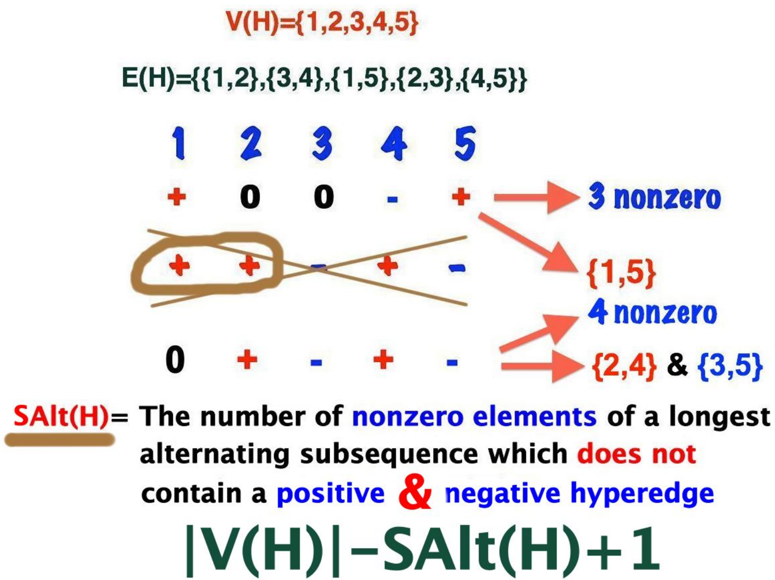
The Altermatic Number of Five Cycle



The Altermatic Number of Five Cycle






V(H)={1,2,3,4,5}
E(H)={{1,2},{3,4},{1,5},{2,3},{4,5}}
1 2 3 4 5
+ 0 0 - +
3 nonzero
{1,5}
+ + - + -
4 nonzero
0 + - + -
{2,4} & {3,5}
SAlt(H)= The number of nonzero elements of a longest alternating subsequence which does not contain a positive & negative hyperedge
|V(H)|−SAlt(H)+1

Definition

The altermatic number $\zeta(G)$ and the strong altermatic number $\zeta_s(G)$ of a graph G are defined, respectively, as follows:

$$\zeta(G) = \max_H \{|V(H)| - Alt(H) : \mathrm{KG}(H) \text{ and } G \text{ are isomorphic}\}.$$

$$\zeta_s(G) = \max_H \{|V(H)| - SAlt(H) + 1 : \mathrm{KG}(H) \text{ and } G \text{ are isomorphic}\}.$$

Altermatic Number

Definition

The altermatic number $\zeta(G)$ and the strong altermatic number $\zeta_s(G)$ of a graph G are defined, respectively, as follows:

$$\zeta(G) = \max_H \{|V(H)| - Alt(H) : \mathrm{KG}(H) \text{ and } G \text{ are isomorphic}\}.$$

$$\zeta_s(G) = \max_H \{|V(H)| - SAlt(H) + 1 : \mathrm{KG}(H) \text{ and } G \text{ are isomorphic}\}.$$

Theorem (M. Alishahi and H.H., 2013)

For any graph G, we have

$$\chi(G) \geq \zeta(G),$$

$$\chi(G) \geq \zeta_s(G).$$

Altermatic Number

Definition

The altermatic number $\zeta(G)$ and the strong altermatic number $\zeta_s(G)$ of a graph G are defined, respectively, as follows:

$$\zeta(G) = \max_H\{|V(H)| - Alt(H) : \mathrm{KG}(H) \text{ and } G \text{ are isomorphic}\}.$$

$$\zeta_s(G) = \max_H\{|V(H)| - SAlt(H) + 1 : \mathrm{KG}(H) \text{ and } G \text{ are isomorphic}\}.$$

Theorem (M. Alishahi and H.H., 2014)

For any graph G, we have

$$\chi(G) \geq \mathrm{Coind}(B_0(G)) + 1 \geq \zeta(G),$$

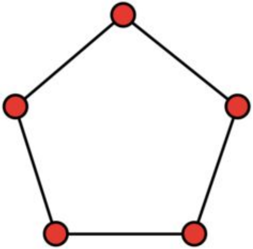
$$\chi(G) \geq \mathrm{Coind}(B(G)) + 2 \geq \zeta_s(G).$$

Definition

Let G be a graph and $\mathcal{H}$ be a family of graphs. By $\mathrm{KG}(G,\mathcal{H})$, we denote the general Kneser graph whose vertex set is the set of all subgraphs of G isomorphic to some member of $\mathcal{H}$ and in which two vertices are adjacent if the corresponding subgraphs are edge-disjoint.

Definition

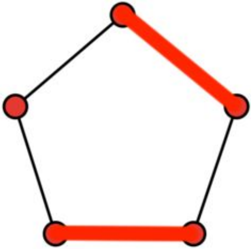
Let G be a graph and $\mathcal{H}$ be a family of graphs. By $\mathrm{KG}(G, \mathcal{H})$, we denote the general Kneser graph whose vertex set is the set of all subgraphs of G isomorphic to some member of $\mathcal{H}$ and in which two vertices are adjacent if the corresponding subgraphs are edge-disjoint.



$\mathrm{KG}(C_5, 2K_2)$

Definition

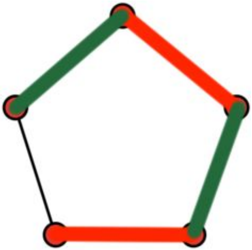
Let G be a graph and $\mathcal{H}$ be a family of graphs. By $\mathrm{KG}(G, \mathcal{H})$, we denote the general Kneser graph whose vertex set is the set of all subgraphs of G isomorphic to some member of $\mathcal{H}$ and in which two vertices are adjacent if the corresponding subgraphs are edge-disjoint.



$\mathrm{KG}(C_5, 2K_2)$

Definition

Let G be a graph and $\mathcal{H}$ be a family of graphs. By $\mathrm{KG}(G, \mathcal{H})$, we denote the general Kneser graph whose vertex set is the set of all subgraphs of G isomorphic to some member of $\mathcal{H}$ and in which two vertices are adjacent if the corresponding subgraphs are edge-disjoint.



$$\mathrm{KG}(C_5, 2K_2) = C_5$$

Definition

Let G be a graph and $\mathcal{H}$ be a family of graphs. By $\mathrm{KG}(G, \mathcal{H})$, we denote the general Kneser graph whose vertex set is the set of all subgraphs of G isomorphic to some member of $\mathcal{H}$ and in which two vertices are adjacent if the corresponding subgraphs are edge-disjoint.

A Representation For Some Graphs

1. Kneser Graphs: $\mathrm{KG}(nK_2, rK_2)$, where nK_2 is a matching of size n.
2. $\chi(\mathrm{KG}(nK_2, rK_2)) = n - 2r + 2$ (Lovász, 1978)

General Kneser Graphs

Definition

Let G be a graph and $\mathcal{H}$ be a family of graphs. By $\mathrm{KG}(G, \mathcal{H})$, we denote the general Kneser graph whose vertex set is the set of all subgraphs of G isomorphic to some member of $\mathcal{H}$ and in which two vertices are adjacent if the corresponding subgraphs are edge-disjoint.

A Representation For Some Graphs

1. Kneser Graphs: $\mathrm{KG}(nK_2, rK_2)$, where nK_2 is a matching of size n.
2. $\chi(\mathrm{KG}(nK_2, rK_2)) = n - 2r + 2$ (Lovász, 1978)
3. Schrijver Graphs: $\mathrm{KG}(C_n, rK_2)$, where C_n is a cycle of size n.
4. $\chi(\mathrm{KG}(C_n, rK_2)) = n - 2r + 2$ (Schrijver, 1978)

General Kneser Graphs

Definition

Let G be a graph and $\mathcal{H}$ be a family of graphs. By $\mathrm{KG}(G, \mathcal{H})$, we denote the general Kneser graph whose vertex set is the set of all subgraphs of G isomorphic to some member of $\mathcal{H}$ and in which two vertices are adjacent if the corresponding subgraphs are edge-disjoint.

A Representation For Some Graphs

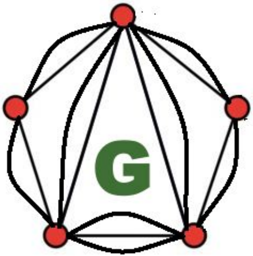
1. Kneser Graphs: $\mathrm{KG}(nK_2, rK_2)$, where nK_2 is a matching of size n.
2. $\chi(\mathrm{KG}(nK_2, rK_2)) = n - 2r + 2$ (Lovász, 1978)
3. Schrijver Graphs: $\mathrm{KG}(C_n, rK_2)$, where C_n is a cycle of size n.
4. $\chi(\mathrm{KG}(C_n, rK_2)) = n - 2r + 2$ (Schrijver, 1978)

Theorem (M. Alishahi and H.H., 2013-2014)

If rK_2 is a matching of size r and G is a sufficiently large dense graph or special sparse graph, then $\chi(\mathrm{KG}(G, rK_2)) = |E(G)| - ex(G, rK_2)$.

Theorem (M. Alishahi and H.H., 2013)

If G is a multigraph such that the multiplicity of each edge is at least 2 and H is a simple graph, then $\chi(\mathrm{KG}(G,H)) = |E(G)| - ex(G,H)$.

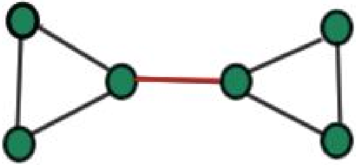


Theorem (M. Alishahi and H.H., 2013)

If G is a multigraph such that the multiplicity of each edge is at least 2 and H is a simple graph, then $\chi(\mathrm{KG}(G,H)) = |E(G)| - ex(G,H)$.

Theorem (M. Alishahi and H.H., 2014)

If G is a sufficiently large dense graph and $\mathcal{T}_n$ is the family of the spanning trees of G, then $\chi(\mathrm{KG}(G,\mathcal{T}_n)) = |\mathrm{MinimumCUT}(G)|$.



$|\mathrm{MinimumCUT}(G)| = 1$

Consider two graphs G and G'.

Theorem (M. Alishahi and H.H., 2014)

$\chi(G \times G') \geq \zeta_s(G \times G') \geq \min\{\zeta_s(G), \zeta_s(G')\}$.

Hedetniemi's Conjecture

Consider two graphs G and G'.

Theorem (M. Alishahi and H.H., 2014)

$\chi(G \times G') \geq \zeta_s(G \times G') \geq \min\{\zeta_s(G), \zeta_s(G')\}$.

Theorem (D. Kozlov, 2005)

$\chi(G\times G') \geq Coind(B(G\times G'))+2 \geq \min\{Coind(B(G)), Coind(B(G'))\}+2$

Hedetniemi's Conjecture

Consider two graphs G and G'.

Theorem (M. Alishahi and H.H., 2014)

$\chi(G \times G') \geq \zeta_s(G \times G') \geq \min\{\zeta_s(G), \zeta_s(G')\}$.

Theorem (D. Kozlov, 2005)

$\chi(G\times G') \geq Coind(B(G\times G'))+2 \geq \min\{Coind(B(G)), Coind(B(G'))\}+2$

Question

Is it true that
$Coind(B_0(G \times G')) + 1 \geq \min\{Coind(B_0(G)), Coind(B_0(G'))\} + 1$?

Hedetniemi's Conjecture

Consider two graphs G and G'.

Theorem (M. Alishahi and H.H., 2014)

$\chi(G \times G') \geq \zeta_s(G \times G') \geq \min\{\zeta_s(G), \zeta_s(G')\}$.

Theorem (D. Kozlov, 2005)

$\chi(G \times G') \geq Coind(B(G \times G')) + 2 \geq \min\{Coind(B(G)), Coind(B(G'))\} + 2$

Question

Is it true that
$Coind(B_0(G \times G')) + 1 \geq \min\{Coind(B_0(G)), Coind(B_0(G'))\} + 1$?

Theorem (H.H. and F. Meunier, 2014)

$\chi(G \times G') \geq \min\{\zeta(G), \zeta(G')\}$.

Semi-Matching Coloring

Definition:

A semi-matching coloring of a graph G is a proper coloring $c : V(G) \to \mathbb{N}$ such that for any two consecutive colors, the edges joining the colors form a matching. The minimum positive integer t for which there exists a semi-matching coloring $c : V(G) \to \{1, 2, \ldots, t\}$ is called the semi-matching chromatic number of G and denoted by $\chi_m(G)$.

Definition:

A semi-matching coloring of a graph G is a proper coloring $c : V(G) \to \mathbb{N}$ such that for any two consecutive colors, the edges joining the colors form a matching. The minimum positive integer t for which there exists a semi-matching coloring $c : V(G) \to \{1, 2, \ldots, t\}$ is called the semi-matching chromatic number of G and denoted by $\chi_m(G)$.

Conjecture (B. Omoomi and A. Pourmiri, 2008)

For any positive integers n and r, with $n \geq 2r \geq 4$, we have $\chi_l(\mathrm{KG}(nK_2, rK_2)) = 2\chi(\mathrm{KG}(nK_2, rK_2)) - 2 = 2n - 4r + 2$.

Definition:

A semi-matching coloring of a graph G is a proper coloring $c : V(G) \to \mathbb{N}$ such that for any two consecutive colors, the edges joining the colors form a matching. The minimum positive integer t for which there exists a semi-matching coloring $c : V(G) \to \{1, 2, \ldots, t\}$ is called the semi-matching chromatic number of G and denoted by $\chi_m(G)$.

Conjecture (B. Omoomi and A. Pourmiri, 2008)

For any positive integers n and r, with $n \geq 2r \geq 4$, we have $\chi_l(\mathrm{KG}(nK_2, rK_2)) = 2\chi(\mathrm{KG}(nK_2, rK_2)) - 2 = 2n - 4r + 2$.

Theorem (H.H., 2011)

Let n and r be positive integers, where $n \geq 2r \geq 4$. If $n \leq \frac{8}{3}r$, then $\chi_m(\mathrm{KG}(nK_2, rK_2)) = 2\chi(\mathrm{KG}(nK_2, rK_2)) - 2 = 2n - 4r + 2$.

References

H. Hajiabolhassan, A generalization of Kneser's conjecture. Discrete Mathematics, 2663–2668 (311), 2011.

M. Alishahi and H. Hajiabolhassan, On the chromatic number of general Kneser hypergraphs. ArXiv e-prints, February 2013.

M. Alishahi and H. Hajiabolhassan, Chromatic number via Turán number. ArXiv e-prints, December 2013.

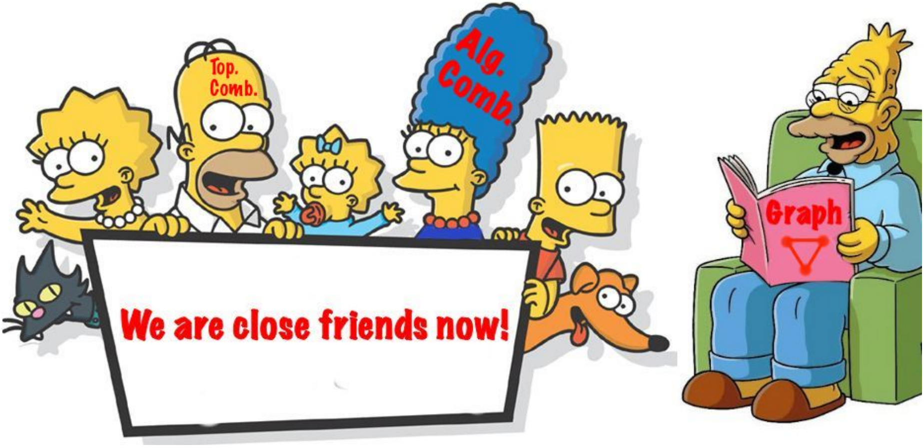
M. Alishahi and H. Hajiabolhassan, Hedetniemi's conjecture via alternating chromatic number. ArXiv e-prints, March 2014.

M. Alishahi and H. Hajiabolhassan, On chromatic number and minimum cut. ArXiv e-prints, July 2014.

H. Hajiabolhassan and F. Meunier, Hedetniemi's conjecture for Kneser hypergraphs. ArXiv e-prints, October 2014.

Thank You!

I would like to show my greatest appreciation to Professor Carsten Thomassen. Although, I can't thank him enough for his tremendous support and help.


Top.
Comb.
Alg.
Comb.
We are close friends now!
Graph

Questions?!