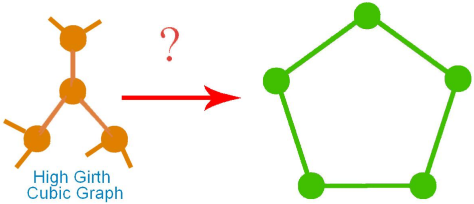
On Nešetřil's Pentagon Problem



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Workshop on Graph Homomorphisms and Related Topics
Nova Louka, Czech Republic

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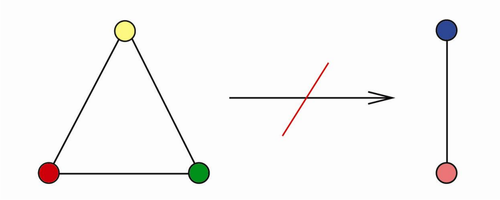
Graphs Homomorphism

- A homomorphism $f : G \longrightarrow H$ from a graph G to a graph H is a map $f : V(G) \longrightarrow V(H)$ such that if $uv \in E(G)$ then $f(u)f(v) \in E(H)$.



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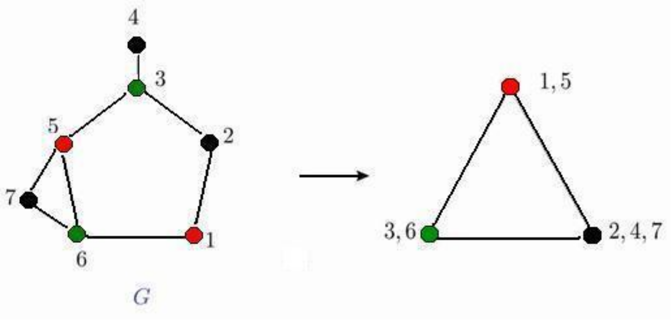
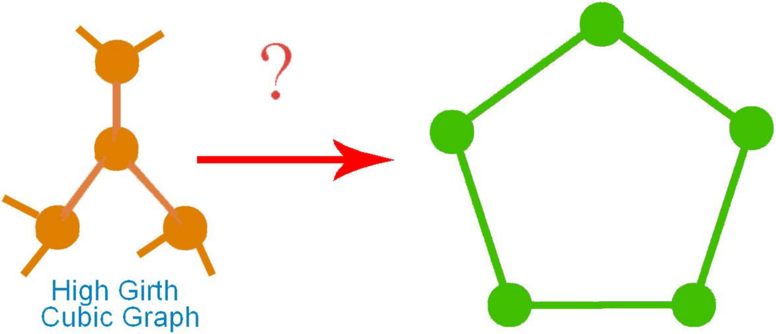


Nešetřil's Pentagon problem

- Let G be a cubic graph of sufficiently large girth, is it true that $\mathrm{Hom}(G, C_5) \neq \emptyset$?



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- If in the Nešetřil's Pentagon problem C_5 is replaced by C_3, then the answer is affirmative; and in fact it is a quick consequence of Brooks' theorem.
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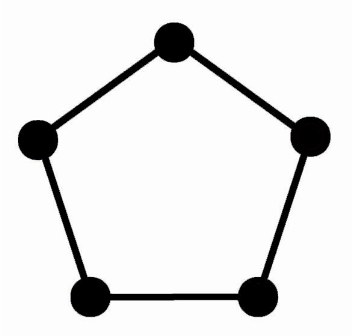
Graph Powers

For a graph G, let G^k be the kth power of G, which is obtained on the vertex set $V(G)$, by connecting any two vertices u and v for which there exists a walk of length k between u and v in G.



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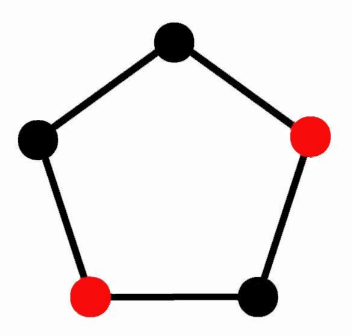

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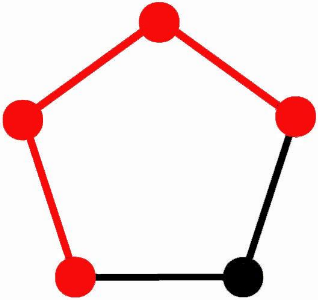

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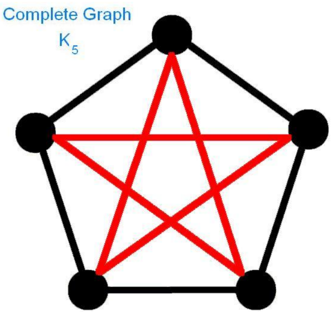

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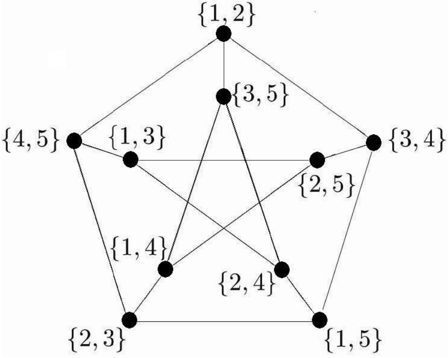

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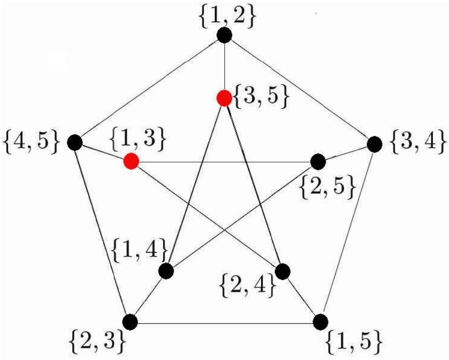

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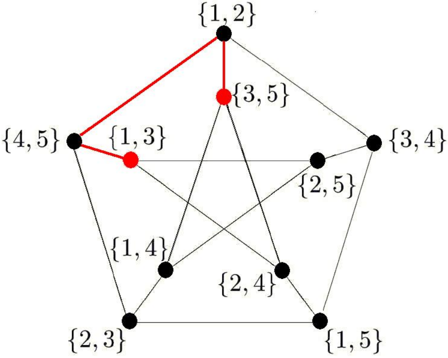

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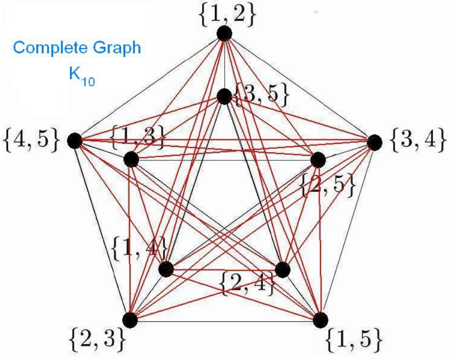

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- Lemma. Let G and H be two simple graphs such that $\mathrm{Hom}(G, H) \neq \varnothing$. Then, for any positive integer k, $\mathrm{Hom}(G^k, H^k) \neq \varnothing$.

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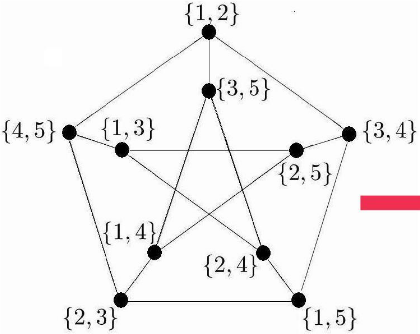

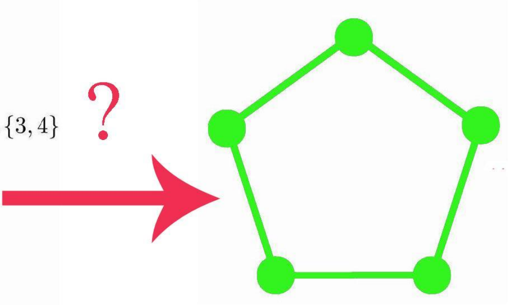

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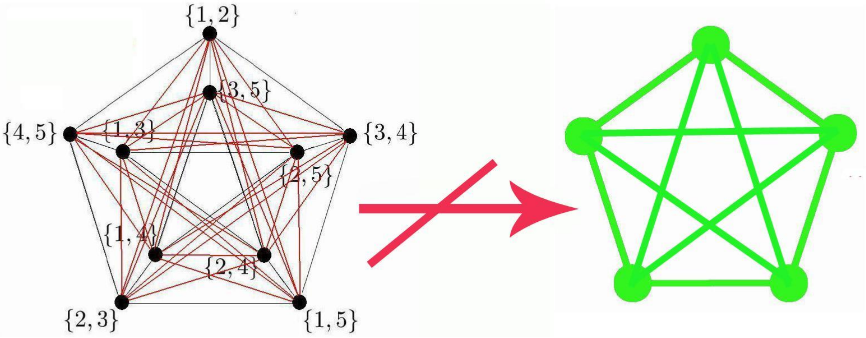

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- If G is a cubic graph and $\mathrm{Hom}(G, C_5) \neq \varnothing$. Then, $\mathrm{Hom}(G^3, K_5) \neq \varnothing$.
- Question. (A. Daneshgar and H. Hajiabolhassan) Is it true that for any natural number g_0, there exists a cubic graph G whose girth is larger than g_0 and $\chi(G^3) \geq 6$?
- What is the exact value of $\max\limits_{g(G)\geq g} \chi(G^3)$? (maximum is taken over all cubic graphs with girth at least g)

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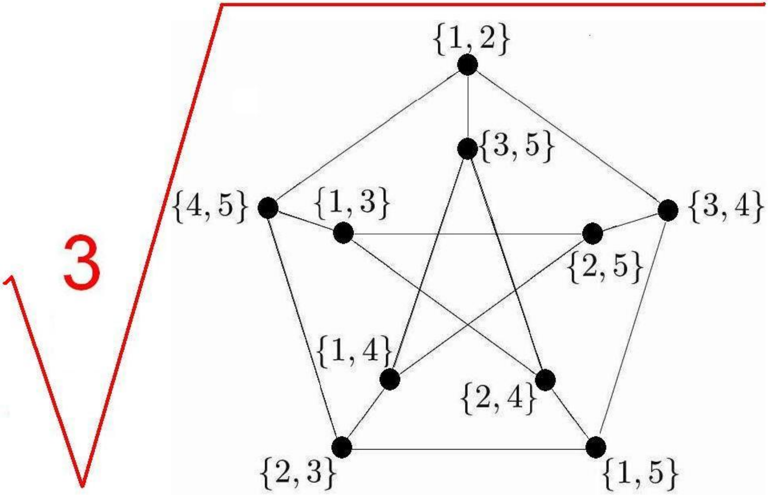

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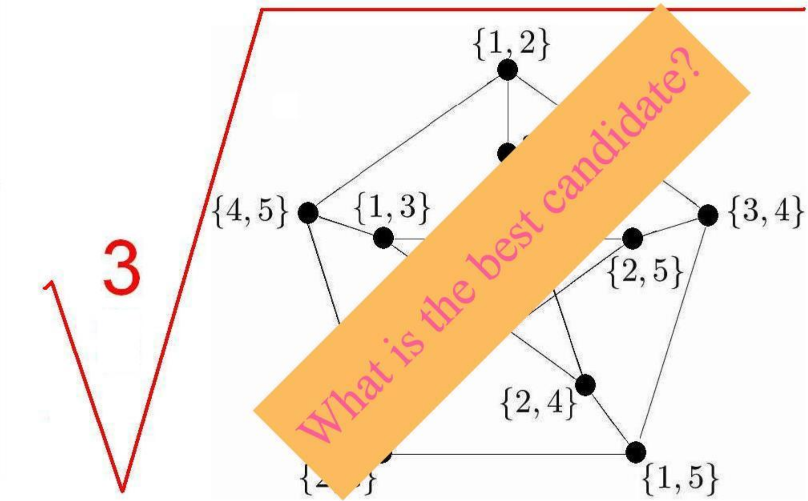

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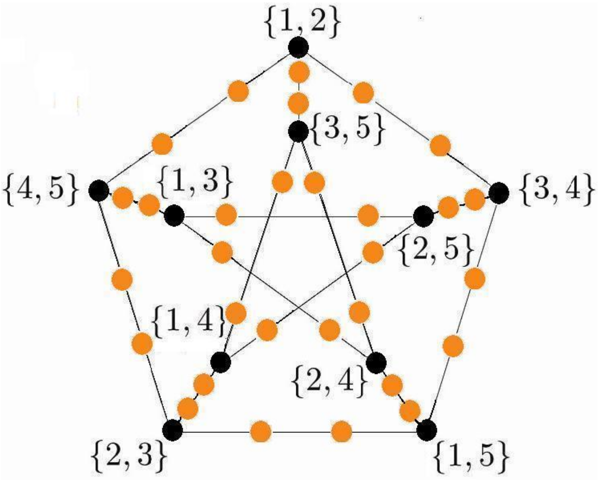

Fractional Powers

- The graph $G^{\frac{1}{t}}$ is said to be the t-subdivision of a graph G if G is obtained from G by replacing each edge by a path with exactly $t-1$ inner vertices.
- Note that $G^{\frac{1}{1}}$ is isomorphic to G. Also, $(G^{\frac{1}{t}})^{\frac{1}{s}}$ is isomorphic to $G^{\frac{1}{st}}$.
- For any graph G, set $G^{\frac{t}{s}} := (G^{\frac{1}{s}})^{t}$.
- For instance, if $n \geq 3$ is a positive integer, then $K_n^{\frac{7}{3}} = K_{n^2}$. Moreover, $K_n^{\frac{6r+1}{2r+1}} = K_{rn^2-rn+n}$.
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Blow Up a Graph

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- Let G and H be non-bipartite graphs. Then there exists a positive odd rational number $r \geq 1$ such that $H \longrightarrow G^r$.

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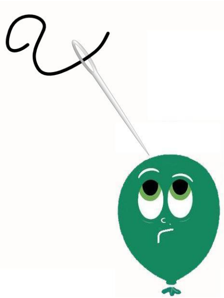

Properties of Fractional Powers

H. Hajiabolhassan and A. Taherkhani

- If a, b and s are non-negative integer, then $G^{\frac{(2s+1)a}{(2s+1)b}} \longleftrightarrow G^{\frac{a}{b}}$.
- Let r be a positive odd rational number and a is a positive odd integer. If $ra < og(G)$, then $(G^r)^a \longleftrightarrow G^{ra}$.
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- Let G and H be two graphs where $2s+1 < og(H)$. Then, $G^{\frac{1}{2s+1}} \longrightarrow H$ if and only if $G \longrightarrow H^{2s+1}$.
- $G < H$ means $G \longrightarrow H$ and $H \not\longrightarrow G$.
- Let r and s be positive odd rational numbers and G be a non-bipartite graph. If $r < s < og(G)$, then $G^r < G^s$.

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Power Thickness

- (H. Hajiabolhassan) Let G be a graph with odd girth at least $2k+1$. Then, $\chi(G^{\frac{2k+1}{3}}) \leq 3$ if and only if there exists a graph homomorphism from G to C_{2k+1}.
- Let G be a non-bipartite graph. What is the value of

$$\Theta(G) := \sup\{r | r = \frac{2t+1}{2s+1}, \chi(G^r) = \chi(G), r < og(G)\}?$$

- Let K_n be complete graph with $n \geq 3$ vertices. Then, $\theta(K_n) = 1$.
- Let n be a positive integer. Then, $\theta(C_{2n+1}) = \frac{2n+1}{3}$.
- Let G and H be two non-bipartite graphs with $\chi(G) = \chi(H)$. If $G \longrightarrow H$, then $\theta(G) \geq \theta(H)$.

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Odd Cycles

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- (Nešetřil's Pentagon Problem) Let G be a cubic graph of sufficiently large girth, is it true that $\Theta(G) \leq \frac{5}{3}$?
- (Jaeger's Conjecture) Every planar graph with girth at least $4k$ has a homomorphism to C_{2k+1}.
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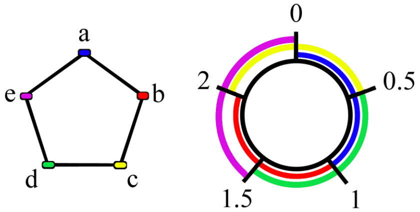
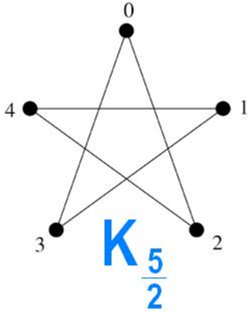
Circular chromatic number

- Let C be a circle of (Euclidean) length r. An r-circular coloring of a graph G is a mapping c which assigns to each vertex v of G an open unit length arc $c(v)$ of C, such that for every edge $uv \in E(G)$ we have $c(u) \cap c(v) = \emptyset$. The circular chromatic number of a graph , denoted by $\chi_c(G)$, is defined as, $\chi_c(G) = \inf\{r : G \ admits \ an \ r - circular \ coloring\}$.



Circular Complete Graphs

- The circular complete graph $K_{\frac{p}{q}}$ has the vertex set $\{0, 1, \ldots, p-1\}$ and the edge set $\{ij : q \leq |i-j| \leq p-q\}$.



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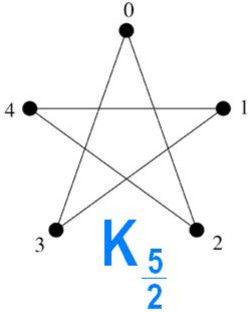

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Circular Colorings

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- Let G be a non-bipartite graph with chromatic number $\chi(G)$.
  - If r is a positive odd rational number and $r \leq \frac{\chi(G)}{3(\chi(G)-2)}$, then $\chi(G^r) = 3$.
  - There exists a positive odd rational number $r > \frac{\chi(G)}{3(\chi(G)-2)}$ for which $\chi(G^r) = 3$ if and only if $\chi(G) \neq \chi_c(G)$.
- $\chi_c(G) = \inf\{\frac{2n+1}{n-t} | \chi(G^{\frac{2n+1}{3(2t+1)}}) = 3, n > t > 0\}$.
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SUBDIVISION AND CIRCULAR COLORINGS

- For any positive integer $a \geq 3$. There exists a graph homomorphism from $G^{\frac{1}{a}}$ to C_a.
- Let G be a non-bipartite graph.
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- (A. Daneshgar and H. H.) If $\sigma \in \mathrm{Hom}(G, C_{2n+1})$, then,

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References

A. Daneshgar and H. Hajiabolhassan, Graph homomorphims through random walks.
Journal of Graph Theory, 44 (2003), 15–38.

A. Daneshgar and H. Hajiabolhassan, Density and power graphs in graph homomorphism problem.
Discrete Mathematics, 308 (2008), 4027-4030.

H. Hajiabolhassan, On colorings of graph powers.
Discrete Mathematics, 309 (2009), 4299-4305.

H. Hajiabolhassan and A. Taherkhani, Graph powers and graph homomorphisms.
Manuscript 2008, arXiv: 0808.0362v3.

Thank You!