# On Cover-Free Family

Hossein Hajiabolhassan

Joint work with Farokhlagha Moazami

Department of Mathematical Sciences
Shahid Beheshti University, G.C.
Tehran, Iran

IUT Combinatorics Day
Isfahan University of Technology
Isfahan, Iran

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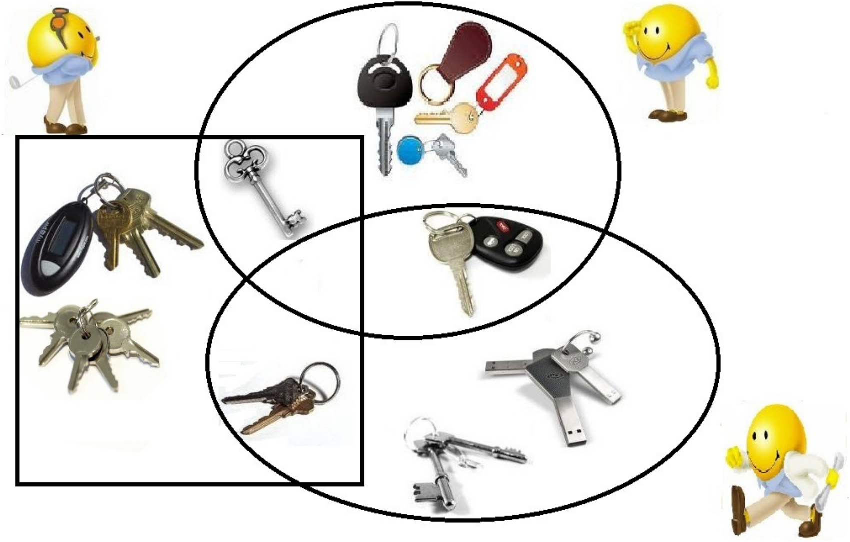
# Outline

1. Key Distribution Scheme
2. Key Predistribution Pattern
3. Cover-Free Family
4. Group Testing
5. Frameproof Code
6. Biclique Covering

**Definition**

A Key Predistribution Scheme is a mechanism of distributing information among a set of users in such away that every user in a group in some specified family is able to compute individually a common key associated with that group.

# Key Predistribution Scheme



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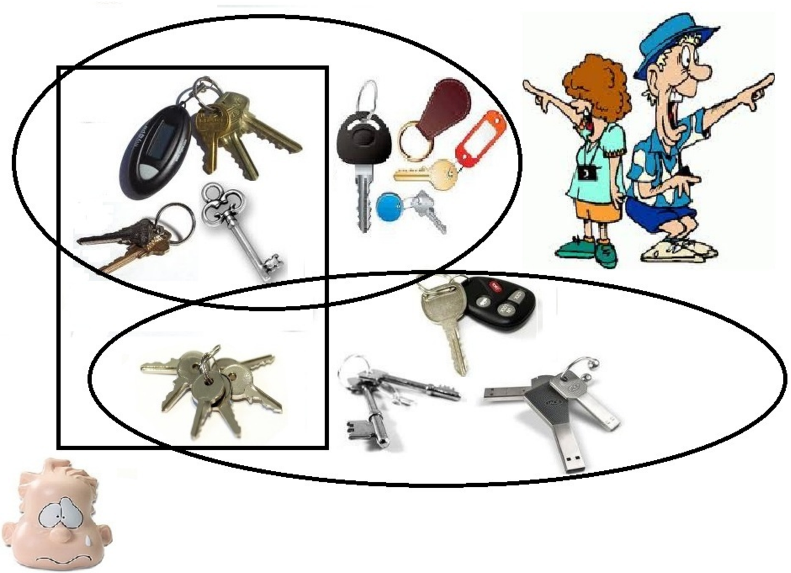
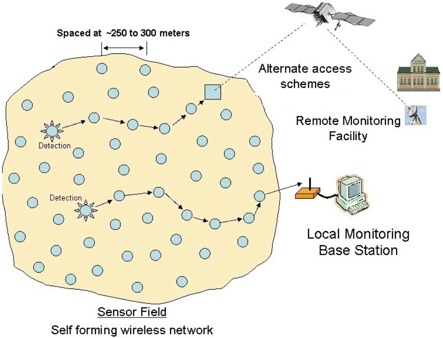
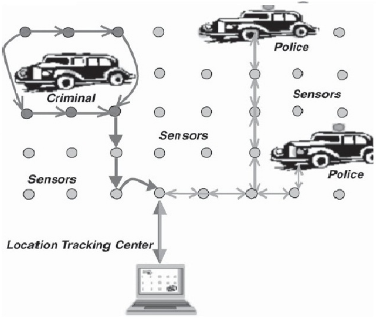


# Wireless Sensor Network




Law enforcement–national security applicatio

- Sensor network can measure various physical characteristic, such as sound, temperature, pressure, etc. They monitor and collect various information.
- The sensor nodes in DSNs should be able to communicate with each other in order to relay or accumulate secret information.

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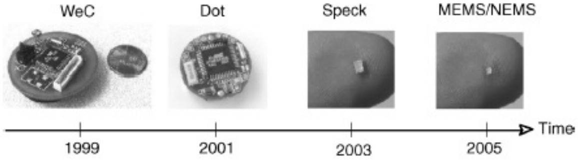


## Properties

- Unpredictable topology.
- Limited battery power.
- Limited memory.
- Limited computational and communication capability.
- Large number of sensor.
- Wireless sensors are not tamper resistant.

# Blom scheme

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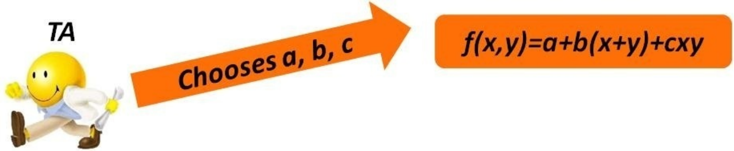

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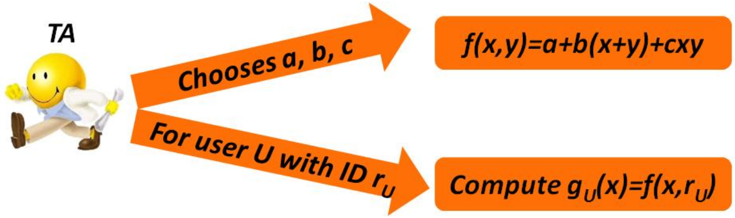
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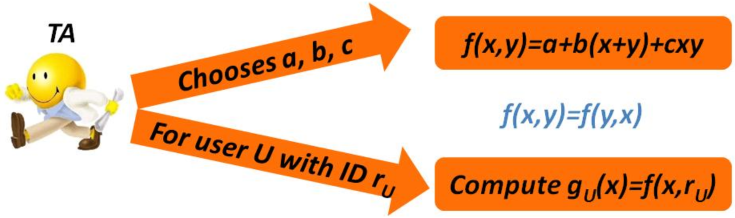
Chooses a, b, c
f(x,y)=a+b(x+y)+cxy
For user U with ID $r_U$
Compute $g_U(x)=f(x,r_U)$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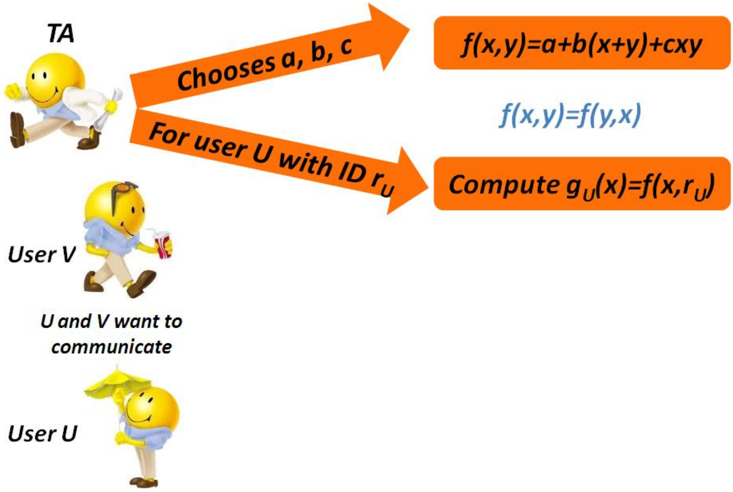
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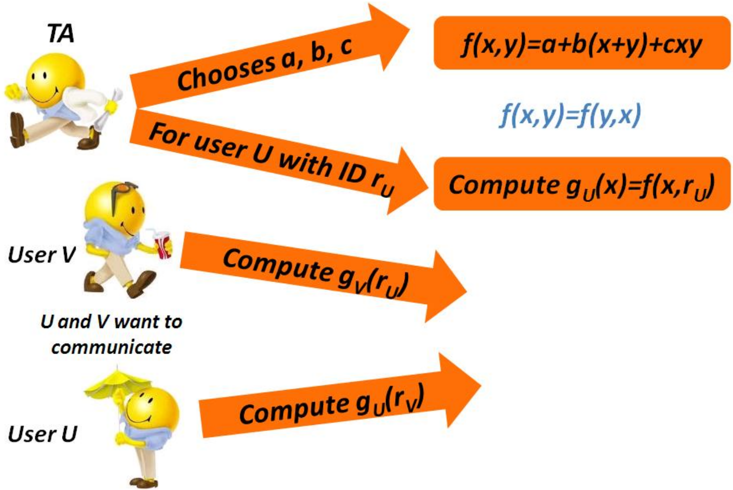

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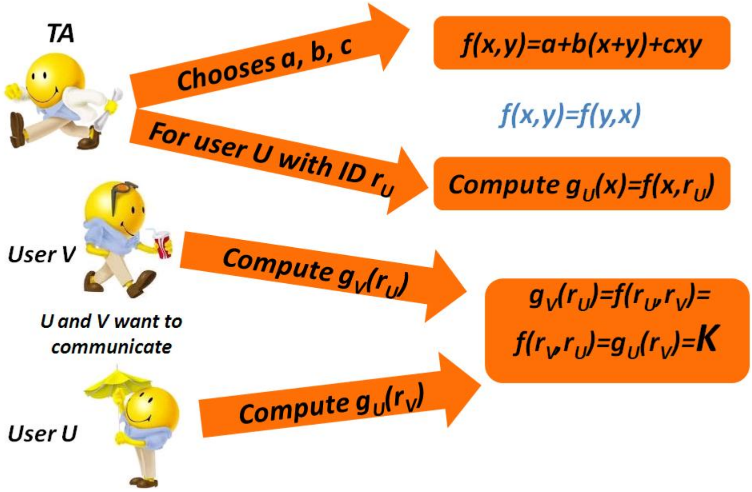

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4. The common key of Alice and Bob is: $W^t DV = (V^t DW)^t = V^t DW$.

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- $$M = \begin{array}{c|cccc} & u_1 & u_2 & u_j & u_v \\ \hline k_1 & 0 & 1 & 1 & 0 \\ k_2 & 1 & 1 & 1 & 1 \\ & & & & \\ k_i & 1 & 0 & 0 & 1 \\ & & & & \\ k_n & 0 & 1 & 1 & 0 \end{array}$$

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1. For a key distribution pattern $M$ and a subset of users $P \subseteq \mathcal{U}$ , define $keys(P) = \{k_i : M(i,j) = 1, \ \forall U_j \in P\}$, i.e., $keys(P) = \bigcap_{U_i \in P} keys(U_i)$.

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4. Unconditional Security: There is an element $i$ where

$$k_i \in Keys(P) \setminus \bigcup_{U_j \in F} Keys(U_j).$$

# Mitchell-Piper Key Distribution Patterns

A Mitchell-Piper $(r, w)$-KDP (or more briefly, an $(r, w)$-KDP) is a KDP in which there is a key for every group of $r$ users, and each such key is secure against any disjoint coalition of at most $w$ users.

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$k_1$

$k_n$

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\begin{array}{c|cccccc}
 & u_1 & u_{i_1} & u_{i_2} & & u_{i_r} & u_t \\
\hline
k_1 & & & & & & \\
 & & & & & & \\
k_i & & 1 & 1 & & 1 & \\
 & & & & & & \\
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k_1 & & & & & & & & \\
k_i & & 1 & 0 & 1 & 0 & 0 & 1 & \\
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## Definition

A set system is a pair $(X, \mathcal{B})$, where $X$ is a finite set of elements called points and $\mathcal{B}$ is a set of subsets of $X$ called blocks.

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$$\bigcap_{l \in L} B_l \nsubseteq \bigcup_{m \in M} B_m.$$

Incidence matrix of an $(r, w)$-cover free family

$$\begin{array}{cccccc} & x_1 & x_2 & & & x_n \\ B_1 & & & & & \\ \vdots & & & & & \\ B_t & & & & & \end{array}$$

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# Generalized Cover-Free Family

## Definition

Let $w$ and $r$ be positive integers, a set system $(X, \mathcal{B})$ where $|X| = n$ and $\mathcal{B} = \{B_1, \ldots, B_t\}$ is called an $(r, w; d) - CFF(n, t)$ if for any two sets of indices $L, M \subseteq [t]$ such that $L \cap M = \varnothing$, $|L| = r$, and $|M| = w$, we have

$$| \bigcap_{l \in L} B_l \setminus \bigcup_{m \in M} B_m | \geq d.$$

# History

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2004 Stinson and Wei generalized the definition of cover-free family.

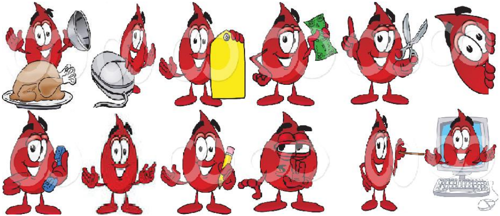
# Group Testing

In combinatorial mathematics, group testing is a set of problems with the objective of reducing the cost of identifying certain elements of a set. In fact, we like to find a small number of defective/interesting items from a large set.

# Group Testing

1. Given $v$ items with at most $d$ positive ones.

# Group Testing



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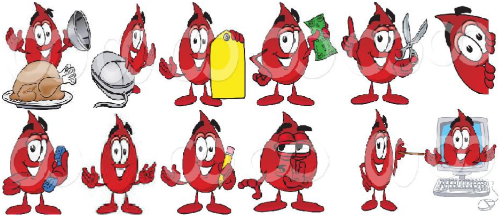

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6. Non-adaptive group testing needs more tests, but shorter time.
7. In molecular biology, non-adaptive group testing is usually taken.

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 & & & & \vdots & & & & \\
B_i & - & 0 & + & \cdots & 0 & - & \longrightarrow & + \\
 & & & & \vdots & & & & \\
B_j & 0 & - & - & \cdots & 0 & - & \longrightarrow & -
\end{array}
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 & & & & \vdots & & & & \\
B_j & 0 & - & - & \cdots & 0 & - & \longrightarrow & - \\
 & & & & \vdots & & & & \\
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Assume that $x_i$ is a negative sample and $x_{i_1}, \ldots, x_{i_d}$ are positive samples.

$$
\begin{array}{c|cccc}
 & k_1 & k_2 & k_j & k_b \\
\hline
u_1 & & & & \\
u_{i_1} & & & 0 & \\
u_i & & & 1 & \\
u_{i_d} & & & 0 & \\
u_v & & & &
\end{array}
$$

The incidence matrix of a $(1, d) - CFF(b, v)$

$$
\begin{array}{cc|ccccc}
 & & x_1 & x_{i_1} & x_i & x_{i_d} & x_v \\
 & & u_1 & u_{i_1} & u_i & u_{i_d} & u_v \\
\hline
B_1 & k_1 & & & & & \\
B_2 & k_2 & & & & & \\
B_j & k_j & & 0 & 1 & 0 & \\
B_b & k_b & & & & &
\end{array}
\quad \rightarrow \quad -
$$

This is a $d$-NAGTA$(v; b)$

# Frameproof Codes

# Frameproof Codes


■ = "detectable positions"

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *pirate #1* | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| #2 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| #3 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 |
| #4 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 |
| Attacked Content | 1 | 0/1 | 1 | 0 | 0/1 | 0 | 1 | 0/1 | 0/1 | 0 | 0/1 | 1 |

■ = "detectable positions"

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *pirate #1* | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| #2 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| #3 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 |
| #4 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 |
| Attacked Content | 1 | 0/1 | 1 | 0 | 0/1 | 0 | 1 | 0/1 | 0/1 | 0 | 0/1 | 1 |

**Marking Assumption (D. Boneh and J. Shaw, 1998)**

Pirates detect fingerprint positions by finding differences in their copies. They make changes only in the detectable positions.

Suppose $C = \{w^{(u_1)}, w^{(u_2)}, \ldots, w^{(u_d)}\} \subseteq \Gamma$. Let $U(C)$ be the set of undetectable bit positions for $C$. Set

$$F(C) = \{x \in \{0,1\}^v : x|_{U(C)} = w^{(u_i)}|_{U(C)} \text{ for all } w^{(u_i)} \in C\}.$$

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## Definition

Suppose that $\Gamma$ is a $(v, t)$-code. $\Gamma$ is said to be an *r-secure frameproof code* if for any $C_1, C_2 \subseteq \Gamma$ such that $|C_1| \leq r$, $|C_2| \leq r$ and $C_1 \cap C_2 = \varnothing$, we have that $F(C_1) \cap F(C_2) = \varnothing$. We will say that $\Gamma$ is an $r - SFPC(v, t)$ for short.

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**Definition**

Suppose that $\Gamma$ is a $(v,t)$-code. $\Gamma$ is said to be an $r$-secure frameproof code if for any $C_1, C_2 \subseteq \Gamma$ such that $|C_1| \leq r$, $|C_2| \leq r$ and $C_1 \cap C_2 = \varnothing$, we have that $F(C_1) \cap F(C_2) = \varnothing$. We will say that $\Gamma$ is an $r - SFPC(v,t)$ for short.

**Theorem. (D.R. Stinson, Tran van Trung, R. Wei, 2000)**

If there exists an $r - SFPC(v,t)$, then there exists an $(r,r) - CFF(2v,t)$.

**Definition**

A $t-(v,k,\lambda)$ packing design is a set system $(X,\mathcal{B})$, where $|X|=v, |\mathcal{B}|=k$ for every $B\in\mathcal{B}$, and every $t$-subset of $X$ occurs in at most $\lambda$ blocks in $\mathcal{B}$.

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**Theorem. (Wei, 2006)**

If there exists a $t-(v,k,1)$ packing design having $b$ blocks, then there exists a $(r,1;d)-CFF(v,b)$, where $r=\lfloor (k-d-2)/(t-1)\rfloor$.

# Bounds

- $N((r, w; d), t)$ denote the minimum number of points in any $(r, w; d) - CFF$ having $t$ blocks.

# Bounds

- $N((r,w;d),t)$ denote the minimum number of points in any $(r,w;d)-CFF$ having $t$ blocks.

**Theorem. (D.R. Stinson and R. Wei, 2004)**

Let $r$, $w$, and $t$ be positive integers where $t \geq r+w$. Then

$$N((r,w;d),t) \geq 2c\frac{\binom{w+r}{w}}{\log(w+r)}\log t + \frac{1}{2}c\binom{w+r}{w}(d-1)$$

**Theorem. (D.R. Stinson and R. Wei, 2004)**

TheoremLet $r$, $w$, and $t$ be positive integers where $t \geq r+w$. Then

$$N((r,w;d),t) \geq 0.7c\frac{\binom{w+r}{w}(w+r)}{\log(w+r)}\log t + \frac{1}{2}c\binom{w+r}{w}(d-1)$$

**Definition**

A biclique cover of a graph $G$ is a collection of bicliques (complete bipartite graphs) of $G$ such that each edge of $G$ is in at least one of the bicliques. The number of bicliques in a minimum biclique covering of $G$ is called the biclique covering number of $G$ and denoted by $bc(G)$.

### Definition

A biclique cover of a graph $G$ is a collection of bicliques (complete bipartite graphs) of $G$ such that each edge of $G$ is in at least one of the bicliques. The number of bicliques in a minimum biclique covering of $G$ is called the biclique covering number of $G$ and denoted by $bc(G)$.

### Definition

A $d$- biclique cover of a graph $G$ is a collection of bicliques of $G$ such that each edge of $G$ is in at least $d$ of the bicliques. The number of bicliques in a minimum $d$-biclique covering of $G$ is called the $d$-biclique covering number of $G$ and denoted by $bc_d(G)$.

# Biclique covering

**Definition**

For $0 < w \leq r \leq t$, the bi-intersection graph $I_t(r,w)$ is a bipartite graph whose vertices are the $w$- and $r$-subsets of a $t$-element set where two vertices are adjacent if and only if their intersection is empty.

**Theorem. (H. H. and F. Moazami, 2010)**

For every positive integer $r$, $w$, $d$ and $t$, where $t \geq r + w$ we have

$$N((r,w;d),t) = bc_d(I_t(r,w)).$$

**Theorem. (H. H. and F. Moazami, 2010)**

For every positive integer $r$, $w$, $d$ and $t$, where $t \geq r + w$ we have

$$N((r,w),t) \geq \binom{r+w-2}{r-1} \mathcal{R}(t-r-w+2).$$

**Theorem. (H. H. and F. Moazami, 2010)**

For every integer $0 \leq s < w \leq r$ and $t \geq r + w$,

$$N((r,w;d),t) \geq \sum_{i=0}^{s} \binom{s}{i} N((r-i, w-s+i; d), t-s).$$

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# Thank You!

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