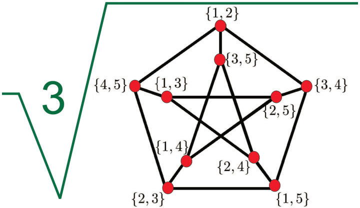

Graph Homomorphisms Through Graph Powers

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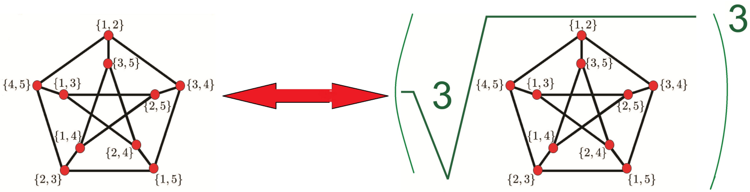
Tehran, Iran

ICRTGC-2010, Cochin, India.

August 12-15, 2010

Outline

1. What is the best choice for definition of power of graphs?



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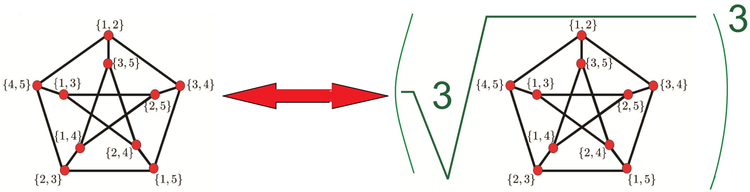

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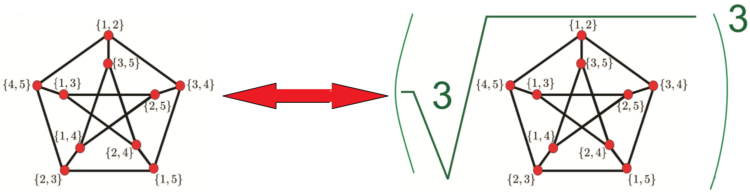

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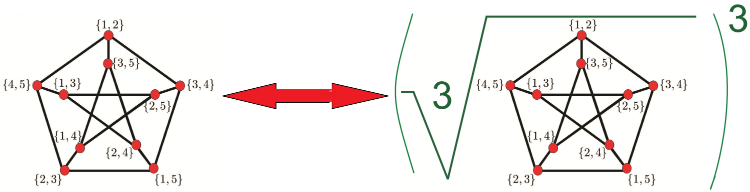

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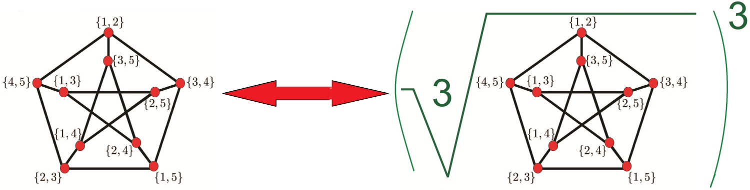

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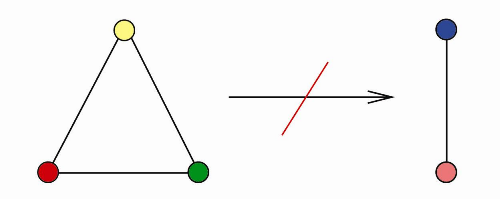
5. Applications: Graph homomorphisms, Circular coloring, and etc.

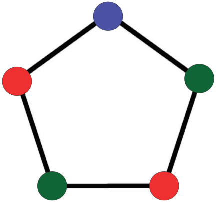
Graph Homomorphism

A homomorphism $f : G \longrightarrow H$ from a graph G to a graph H is a map $f : V(G) \longrightarrow V(H)$ such that if $uv \in E(G)$ then $f(u)f(v) \in E(H)$.



Chromatic number

How many colors does one need to color the vertices of a given graph G, so that if two vertices are connected by an edge, then they get different colors? The chromatic number of a graph G is the smallest number of colors needed to color the vertices of G.



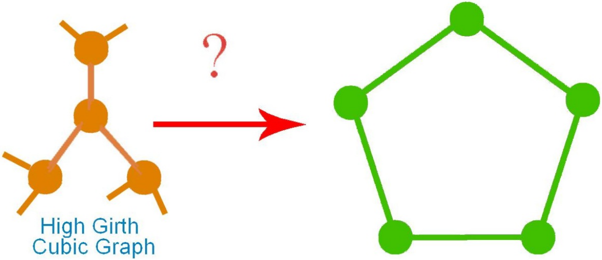
Chromatic Number of Five Cycle is three!

$$\chi(G) = \min\{n | \textit{There exists a homomorphism } \sigma : G \to K_n\}$$

Nešetřil's Pentagon problem, 1999

Let G be a cubic graph of sufficiently large girth, is it true that $\mathrm{Hom}(G, C_5) \neq \emptyset$?

Jarik Nešetřil



Nešetřil's Pentagon problem, 1999

R.L. Brooks, 1941

If in the Nešetřil's Pentagon problem C_5 is replaced by C_3, then the answer is affirmative; and in fact it is a quick consequence of Brooks' theorem.

A. Kostochka, J. Nešetřil, and P. Smolikova, 2001

If in the problem C_5 is replaced by C_{11}, then the answer is negative.

I.M. Wanless and N.C. Wormald, 2001

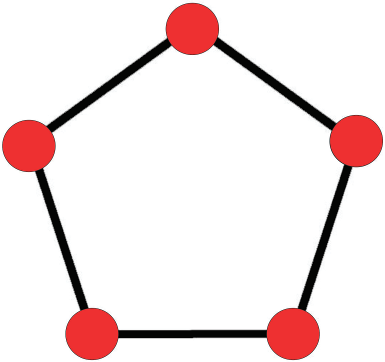
If in the problem C_5 is replaced by C_9, then the answer is negative.

H. Hatami, 2005

If in the problem C_5 is replaced by C_7, then the answer is negative.

Graph Powers

For a graph G, let G^k be the kth power of G, which is obtained on the vertex set $V(G)$, by connecting any two vertices u and v for which there exists a walk of length k between u and v in G.



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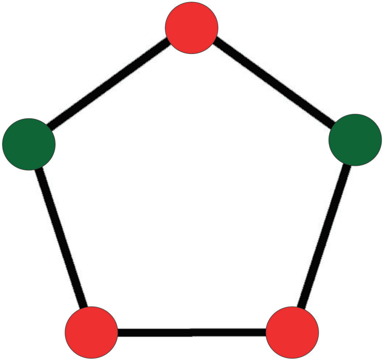

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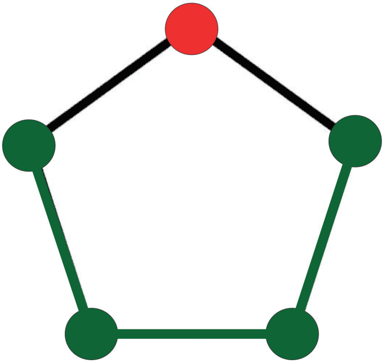

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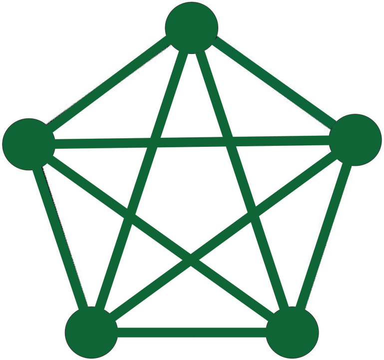

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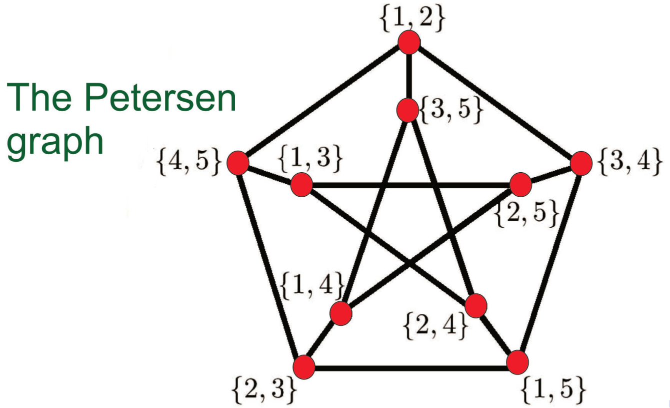

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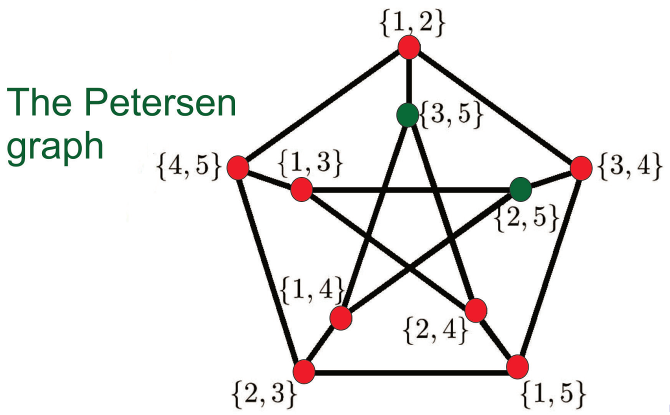

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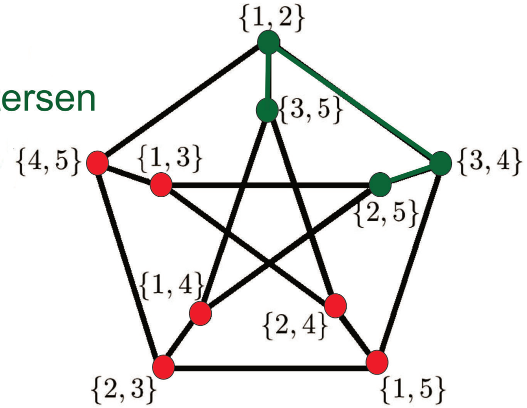
The Petersen graph



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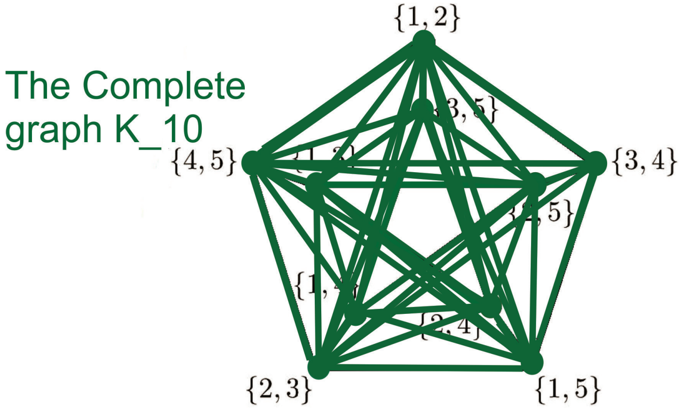

Observation.

Let G and H be two simple graphs such that $\mathrm{Hom}(G, H) \neq \varnothing$. Then, for any positive integer k, $\mathrm{Hom}(G^k, H^k) \neq \varnothing$.

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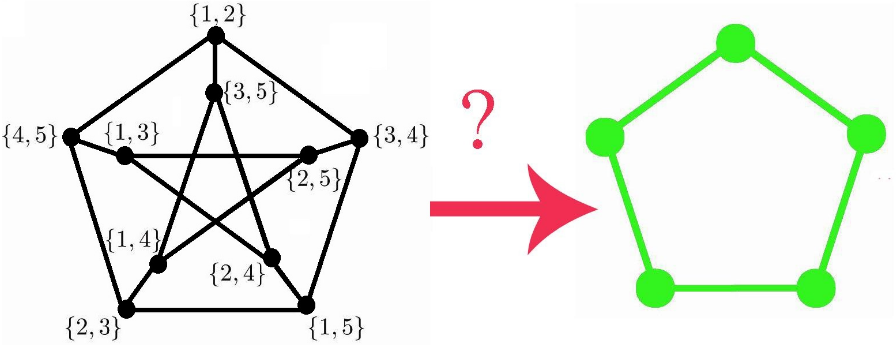

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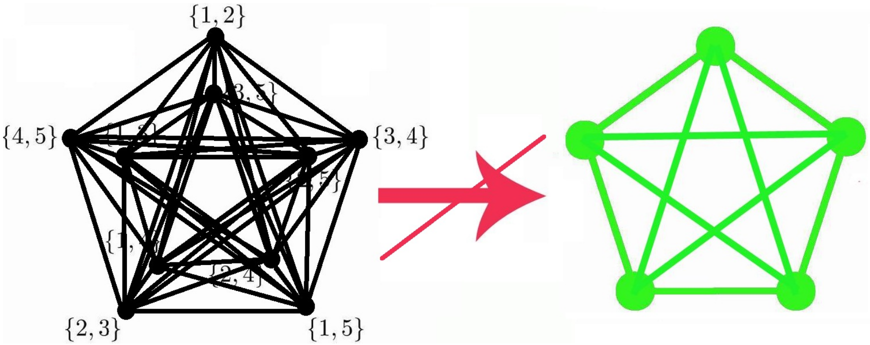

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Question. (A. Daneshgar and H. Hajiabolhassan, 2008)

Is it true that for any natural number g_0, there exists a cubic graph G whose girth is larger than g_0 and $\chi(G^3) \geq 6$?

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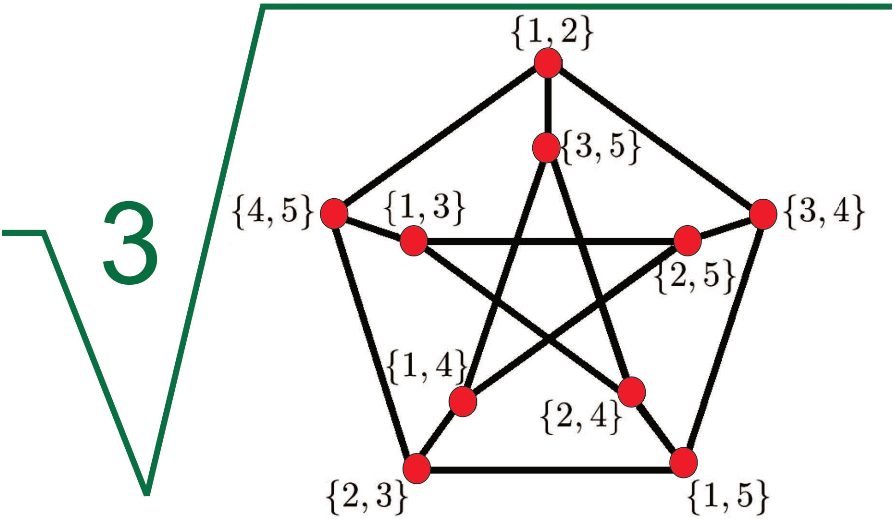
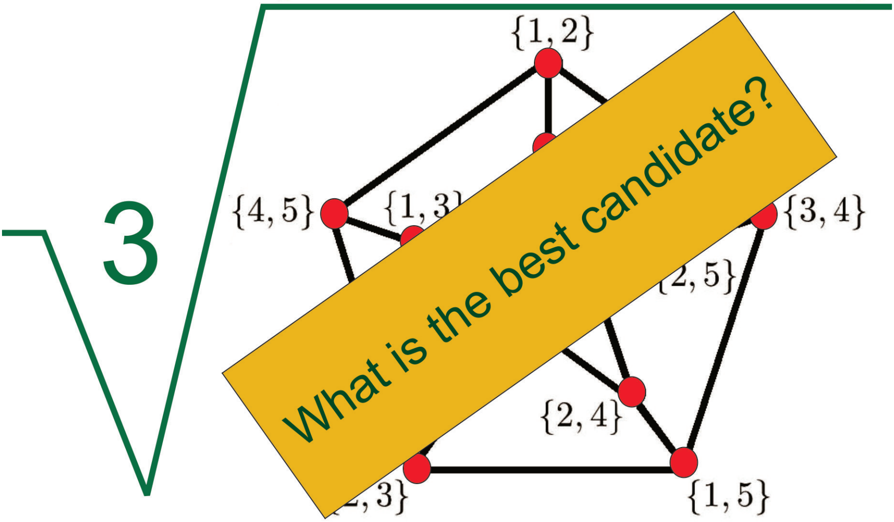
Conjecture. (Naserasr, 2007)

Let $\mathcal{P}_{2k+1}$ be the class of planar graphs of odd-girth at least $2k+1$. For every $G \in \mathcal{P}_{2k+1}$ we have $\chi(G^{2k-1}) \leq 2^{2k}$.

Fractional Powers




3
{1, 2}
{4, 5}
{1, 3}
{3, 4}
2, 5}
{2, 4}
, 3}
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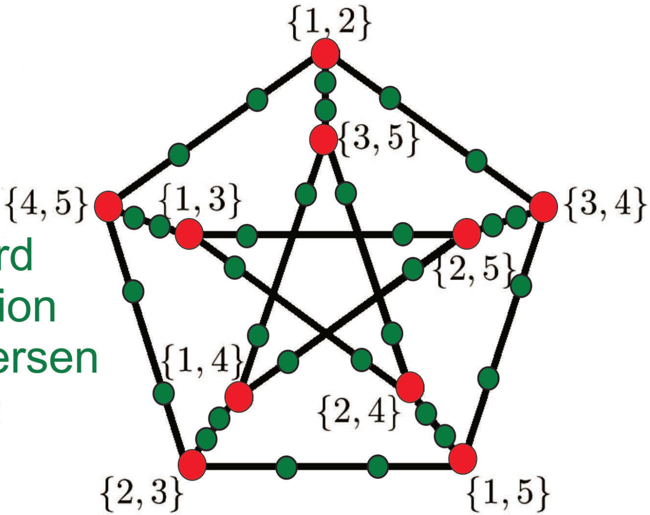
What is the best candidate?

The third subdivision of the Petersen graph



Fractional Powers

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3. For instance, if $n \geq 3$ is a positive integer, then $K_n^{\frac{7}{3}} = K_{n^2}$.
4. A rational number r is called positive odd rational number if numerator and denominator are both positive odd integers.

H. Hajiabolhassan and A. Taherkhani, 2010

Question.

Let a, b and s be positive odd integers. Is it true that $G^{\frac{sa}{sb}} \longleftrightarrow G^{\frac{a}{b}}$?

Properties of Fractional Powers

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If a, b and s are positive odd integers, then $G^{\frac{sa}{sb}} \longleftrightarrow G^{\frac{a}{b}}$.

Observation.

Assume that r and s are positive odd rational number. It does not hold, in general, that $G^{rs} \longleftrightarrow (G^r)^s$.

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Lemma.

If a, b and s are positive odd integers, then $G^{\frac{sa}{sb}} \longleftrightarrow G^{\frac{a}{b}}$.

Lemma.

Assume that r and s are positive odd rational number. If $rs < og(G)$, then $(G^r)^s \longrightarrow G^{rs}$.

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Assume that r and s are positive odd rational number. If $rs < og(G)$, then $(G^r)^s \longrightarrow G^{rs}$.

Theorem.

$G < H$ means $G \longrightarrow H$ and $H \not\longrightarrow G$. Let r and s be positive odd rational numbers and G be a non-bipartite graph. If $r < s < og(G)$, then $G^r < G^s$.

Blow Up a Graph

Observation.

$\omega(G^r)$ stands for the clique number of G^r where r is a positive odd rational number and $r < og(G)$. One can see that if r approaches to $og(G)$, then $\omega(G^r)$ approaches infinity.

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Let G and H be non-bipartite graphs. Then there exists a positive odd rational number $r \geq 1$ such that $H \longrightarrow G^r$.

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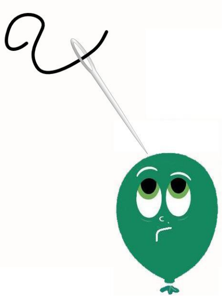

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Let G be a graph with odd girth at least $2k+1$. Then, $\chi(G^{\frac{2k+1}{3}}) \leq 3$ if and only if there exists a graph homomorphism from G to C_{2k+1}.

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Definition.

Let G be a non-bipartite graph. What is the value of

$$\Theta(G) := \sup\{r | r = \frac{2t+1}{2s+1}, \chi(G^r) = \chi(G), r < og(G)\}?$$

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2. Let n be a positive integer. Then, $\theta(C_{2n+1}) = \frac{2n+1}{3}$.
3. For any rational number $r \geq 1$, there exists a graph G with $\theta(G) = r$.

ODD CYCLES

Nešetřil's Pentagon Problem, 1999

Let G be a cubic graph of sufficiently large girth, is it true that $\chi(G^{\frac{5}{3}}) \leq 3$?

Jaeger's Conjecture, 1981

Every planar graph with girth at least $4k$ has a homomorphism to C_{2k+1}.

Jaeger's Conjecture, 1981

Let P be a planar graph with girth at least $4k$. Then we have $\chi(P^{\frac{2k+1}{3}}) \leq 3$.

Colorful Graphs

H. Hajiabolhassan and A. Taherkhani, 2010

Definition.

Let G be a graph and $\chi(G) = k$. G is called a colorful graph if for any proper k-coloring of G, there exists an induced subgraph H of G such that for any $v \in V(H)$, all colors appear in the closed neighborhood of v.

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Question.

Is it true that any graph G with $\theta(G) = 1$ is a colorful graph?

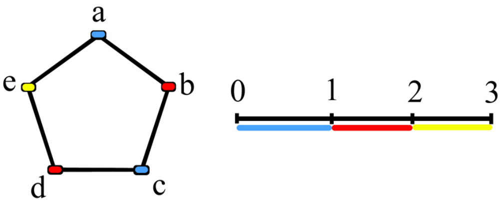
Question.

Let m and n be positive integers where $m \geq 2n$. Is the Kneser graph $\mathrm{KG}(m, n)$ a colorful graph? Is it true that $\theta(\mathrm{KG}(m, n)) = 1$?

Chromatic Number

Chromatic number

Let I be a line of length r. An r-coloring of a graph G is a mapping c which assigns to each vertex v of G an open unit length interval $c(v)$ of I, such that for every edge $uv \in E(G)$ we have $c(u) \cap c(v) = \emptyset$. The chromatic number of a graph , denoted by $\chi(G)$, is defined as,
$\chi(G) = \inf\{r : G \textit{ admits an } r - \textit{coloring}\}$.

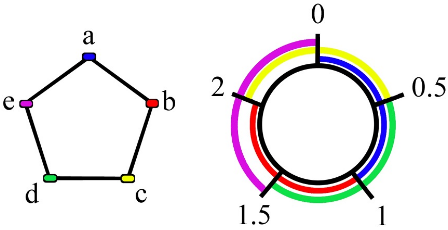


Circular Chromatic number

Let C be a circle of (Euclidean) length r. An r-circular coloring of a graph G is a mapping c which assigns to each vertex v of G an open unit length arc $c(v)$ of C, such that for every edge $uv \in E(G)$ we have $c(u) \cap c(v) = \emptyset$. The circular chromatic number of a graph , denoted by $\chi_c(G)$, is defined as, $\chi_c(G) = \inf\{r : G \textit{ admits an } r - \textit{circular coloring}\}$.

Andrew Vince

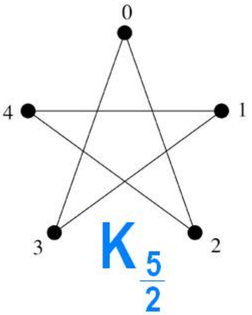
Xuding Zhu



Circular Chromatic Number

Circular Complete Graphs

The circular complete graph $K_{\frac{p}{q}}$ has the vertex set $\{0, 1, \ldots, p-1\}$ and the edge set $\{ij : q \leq |i-j| \leq p-q\}$.



Circular chromatic number: $\chi_c(G) = \inf\{\frac{p}{q} |\ G \to K_{\frac{p}{q}}\}$

Theorem.

Let G be a non-bipartite graph with chromatic number $\chi(G)$.

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1. $\chi_c(G) = \inf\{\frac{2n+1}{n-t} | \chi(G^{\frac{2n+1}{3(2t+1)}}) = 3, n > t > 0\}$.

Circular Colorings

H. Hajiabolhassan and A. Taherkhani, 2010

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3. There exists a positive odd rational number $r > \frac{\chi(G)}{3(\chi(G)-2)}$ for which $\chi(G^r) = 3$ if and only if $\chi(G) \neq \chi_c(G)$.

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Theorem.

For any rational number $\frac{p}{q} > 2$, where $q \nmid p$, we have $\theta(K_{\frac{p}{q}}) > 1$.

Definition.

Let G be a non-bipartite graph. Set
$\mu(G) := \sup\{r | r = \frac{2t+1}{2s+1}, \chi(G^r) = 3, r < og(G)\}$.

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Conjecture. (X. Zhu, 1992)

Let G and H be two graphs. Then $\chi_c(G \times H) = \min\{\chi_c(G), \chi_c(H)\}$.

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Dual Power

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1. $V(G^{\widetilde{\frac{1}{2s+1}}}) := \{(A_1, \ldots, A_{s+1})|\ A_i \subseteq V(G), |A_1| = 1, \varnothing \neq A_i \subseteq N_{i-1}(A_1)\ \ , i \leq s+1\}$, where
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2. $(A_1, \ldots, A_{s+1})(B_1, \ldots, B_{s+1}) \in E(G^{\overbrace{2s+1}^{1}})$ if for any $1 \leq i \leq s$ and $1 \leq j \leq s+1$, $A_i \subseteq B_{i+1}$, $B_i \subseteq A_{i+1}$, and every vertex of A_j is joined to every vertex of B_j.

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Definition.

For any non-negative integers r and s, define $G^{\overbrace{2s+1}^{2r+1}} := (G^{\overbrace{2s+1}^{1}})^{2r+1}$.

Dual Power

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For any $r, s, n \geq 0$, we have

$$C_{2n+1}^{\stackrel{2r+1}{\widetilde{2s+1}}} \longleftrightarrow K_{\frac{4ns+2n+2s+1}{2ns+n+s-r}}.$$

History of Dual Power

In 2004, A. Gyärfäs, T. Jensen, and M. Stiebitz defined the graph $K_n^{\frac{1}{3}}$ to show that there is a n-chromatic graph with an n-coloring where the neighbors of each color-class form an independent set.

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In 2006, G. Simonyi and G. Tardos showed that $K_n^{\frac{1}{2k+1}}$ is an n-critical graph.

Gabor Simonyi and Gabor Tardos

Properties of Dual Powers

H. Hajiabolhassan and A. Taherkhani, 2010

Lemma.

1. If a, b and s are positive odd integers, then $G^{\frac{sa}{sb}} \longleftrightarrow G^{\frac{a}{b}}$.

Lemma.

2. Assume that a, b, c and d are positive odd integers. If $ac < og(G^{\frac{1}{bd}})$, then $G^{\frac{ac}{bd}} \longrightarrow (G^{\frac{a}{b}})^{\frac{c}{d}}$.

Theorem.

3. Let a, b, c and d be positive odd integers and G be a non-bipartite graph. If $\frac{a}{b} < \frac{c}{d}$ and $c < og(G^{\frac{1}{d}})$, then $G^{\frac{a}{b}} < G^{\frac{c}{d}}$.

Dual Power

Theorem. (H. Hajiabolhassan and A. Taherkhani, 2010)

Let G and H be two graphs. Also, assume that $\frac{2a+1}{2b+1} < og(G)$ and $2b+1 < og(H^{\widetilde{\frac{1}{2a+1}}})$. We have $G^{\frac{2a+1}{2b+1}} \longrightarrow H$ if and only if $G \longrightarrow H^{\widetilde{\frac{2b+1}{2a+1}}}$.

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Theorem. (H. Hajiabolhassan, 2009)

Let $m, n,$ and k be positive integers with $m \geq 2n$. Then $\chi(KG(m,n)^{\overbrace{2k-1}^{1}}) = m - 2n + 2$. Moreover, if m is an even positive integer, then $\chi_c(KG(m,n)^{\overbrace{2k-1}^{1}}) = m - 2n + 2$.

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Theorem. (H. Hajiabolhassan and A. Taherkhani, 2010)

Let k, l, and m be positive integers where $m \geq 3$ and $\frac{2l-1}{2k-1} \leq 1$. Then $\theta(K_m^{\widetilde{\frac{2l-1}{2k-1}}}) = \frac{2k-1}{2l-1}$.

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Thank You!

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