On the Altermatic Number of Graphs

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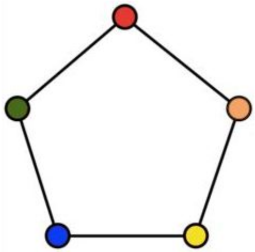
Definition (Chromatic number)

The chromatic number $\chi(G)$ of a graph G is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color.



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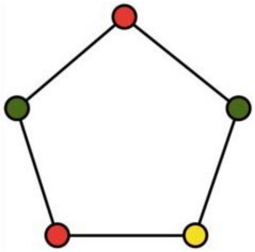

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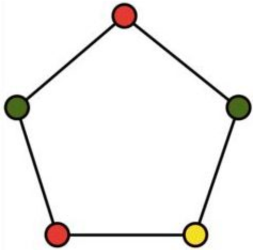
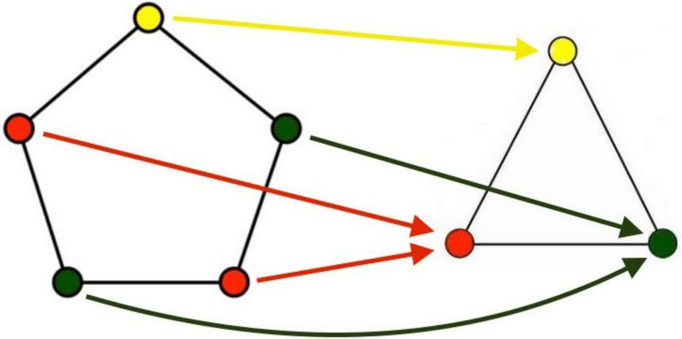


Chromatic number

It is NP-hard to compute the chromatic number of a graph!

Definition (Graph Homomorphism)

A homomorphism $f : G \longrightarrow H$ from a graph G to a graph H is a map $f : V(G) \longrightarrow V(H)$ such that if $uv \in E(G)$ then $f(u)f(v) \in E(H)$. Also, the existence of a homomorphism is indicated by the symbols $G \longrightarrow H$. Also, $G \longleftrightarrow H$ means that $G \longrightarrow H$ and $H \longrightarrow G$.



Graphs Homomorphism

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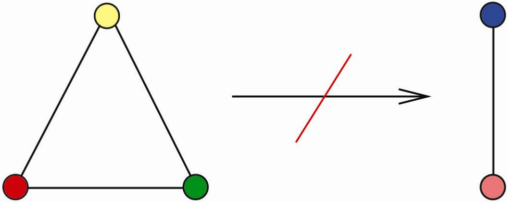

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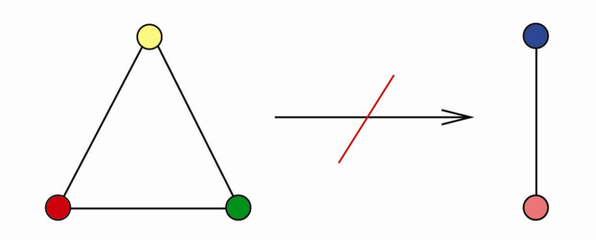
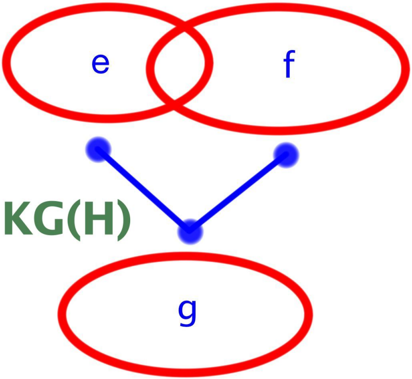


Observation

For any graph G, we have $\chi(G) = \min\{n : G \longrightarrow K_n\}$.

$V(H)=$ A ground set$=\{1,2,\ldots,n\}$

$E(H)=\{e,f,g,\ldots\}\subseteq 2^{V(H)}$

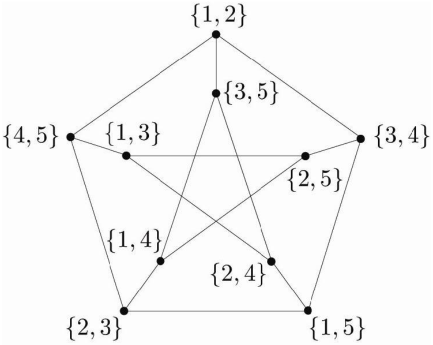


For a hypergraph H, consider the graph $\mathrm{KG}(H)$ whose vertex set is $E(H)$ and whose edge set consists of all disjoint pairs. For instance, if

$$V(H) = \{1,2,3,4,5\},$$

$$E(H) = \binom{[5]}{2},$$

then



$\mathrm{KG}(H) = \mathrm{KG}(5,2) = \text{Petersen Graph}.$

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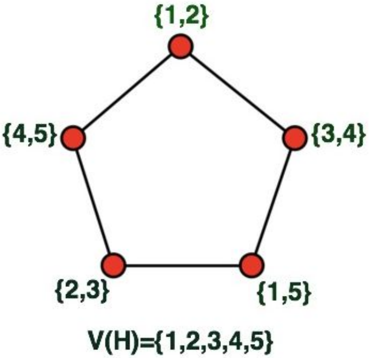
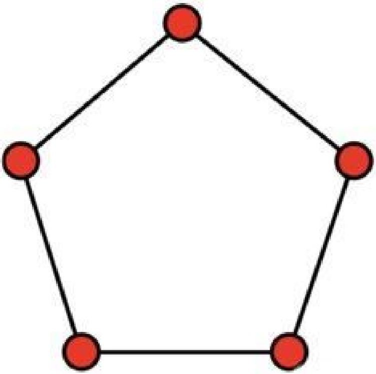
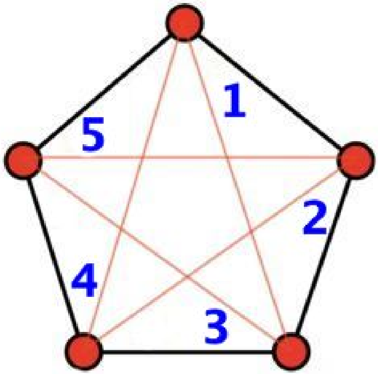
$$E(H) = \{\{1,2\},\{3,4\},\{1,5\},\{2,3\},\{4,5\}\},$$

then

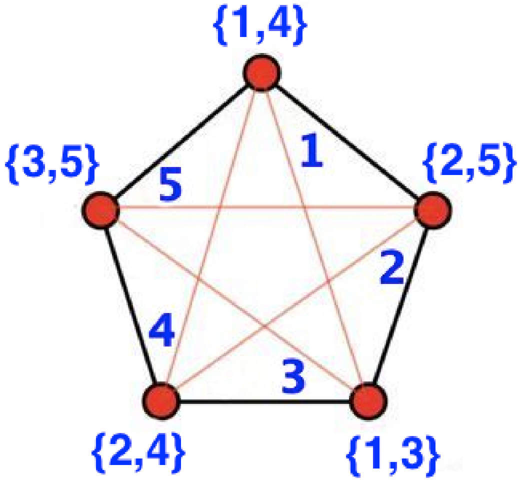
$$\mathrm{KG}(H) = C_5.$$






1
2
3
4
5

Kneser Representations of Graphs



$V(H) = \{1, 2, 3, 4, 5\}$ & $E(H) = \{\{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 5\}, \{3, 5\}\}$

1 2 3 4 5

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$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ + & + & 0 & - & + \end{array}$$

A subsequence of nonzero elements is termed an alternating subsequence if any two consecutive terms in this subsequence are different.

$V(H) = \{1, 2, 3, 4, 5\}$ & $E(H) = \{\{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 5\}, \{3, 5\}\}$

1	2	3	4	5
$+$	$+$	0	$-$	$+$
$+$	0	$+$	$-$	$+$

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1	2	3	4	5
$+$	0	$-$	$+$	0

$$\{1,4\}$$

$Alt(H) =$ The number of nonzero elements of a longest alternating subsequence which does not contain a positive OR negative hyperedge of H

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$+$	$-$	$+$	$-$	$+$
0	$+$	$-$	$+$	$-$

$\{1, 3\}$ $\{3, 5\}$

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1	2	3	4	5
$+$	$-$	0	0	$+$

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$$|V(H)| - Alt(H) = 2$$

$$V(H) = [m] = \{1, 2, \ldots, m\} \quad \& \quad E(H) = \binom{[m]}{n}$$

$$\mathrm{KG}(H) = \mathrm{KG}(m, n)$$

Theorem (L. Lovász 1978)

For any $m \geq 2n$, we have $\chi(\mathrm{KG}(m, n)) = m - 2n + 2$.

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For any $m \geq 2n$, we have $\chi(\mathrm{KG}(m, n)) = m - 2n + 2$.

1	2	3	4	$\cdots$	$2n-3$	$2n-2$	$2n-1$	$2n$	$\cdots$	$m-1$	m
$+$	$-$	$+$	$-$	$\cdots$	$+$	$-$	0	0	$\cdots$	0	0

Kneser Graph

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$$\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & \cdots & 2n-3 & 2n-2 & 2n-1 & 2n & \cdots & m-1 & m \\ + & - & + & - & \cdots & + & - & 0 & 0 & \cdots & 0 & 0 \end{array}$$

$$Alt(H) = 2n - 2$$

$$V(H) = [m] = \{1, 2, \ldots, m\} \quad \& \quad E(H) = \binom{[m]}{n}$$

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$$|V(H)| - Alt(H) \geq m - 2n + 2$$

$V(H) = \{1, 2, 3, 4, 5\}$ & $E(H) = \{\{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 5\}, \{3, 5\}\}$

1	2	3	4	5
$+$	$-$	0	$+$	$-$

$$\{1, 4\} \qquad \{2, 5\}$$

$SAlt(H) =$ The number of nonzero elements of a longest alternating subsequence which does not contain a positive AND negative hyperedge of H

$$|V(H)| - SAlt(H) + 1$$

$V(H) = \{1, 2, 3, 4, 5\}$ & $E(H) = \{\{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 5\}, \{3, 5\}\}$

1	2	3	4	5
$+$	$-$	0	$+$	0

$$\{1, 4\}$$

$SAlt(H) =$ The number of nonzero elements of a longest alternating subsequence which does not contain a positive AND negative hyperedge of H

$$|V(H)| - SAlt(H) + 1 = 3$$

Definition (Altermatic Number and Strong Altermatic Number)

The altermatic number $\zeta(G)$ and the strong altermatic number $\zeta_s(G)$ of a graph G are defined, respectively, as follows:

$$\zeta(G) = \max_H \{|V(H)| - Alt(H) : \mathrm{KG}(H) \longrightarrow G\}.$$

$$\zeta_s(G) = \max_H \{|V(H)| - SAlt(H) + 1 : \mathrm{KG}(H) \longrightarrow G\}.$$

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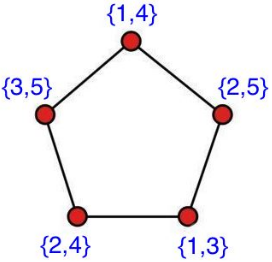
Theorem (M. Alishahi and H.H., 2013)

For any graph G, we have

$$\chi(G) \geq \zeta(G),$$

$$\chi(G) \geq \zeta_s(G).$$

The Altermatic Number of Five Cycle



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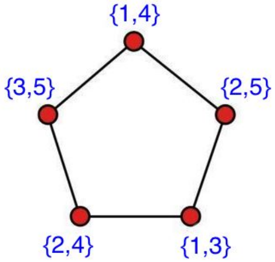
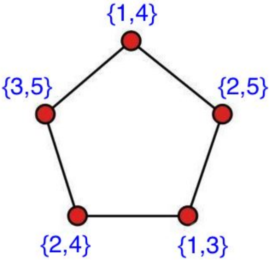


$$V(H) = \{1, 2, 3, 4, 5, a, b, c, d\}$$

$$E(H) = \{\{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 5\}, \{3, 5\}\}$$



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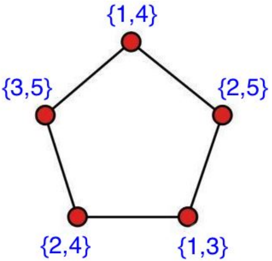
1 a 3 b 5 c 2 d 4

The Altermatic Number of Five Cycle



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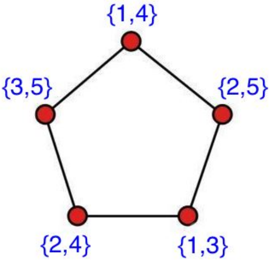
$$\begin{array}{ccccccccc} 1 & a & 3 & b & 5 & c & 2 & d & 4 \\ + & - & + & 0 & 0 & 0 & - & + & - \end{array}$$

$$\{1, 3\} \qquad \{2, 4\}$$

The Altermatic Number of Five Cycle



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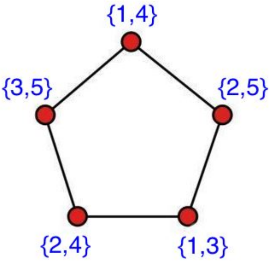
1	a	3	b	5	c	2	d	4
		0				0		



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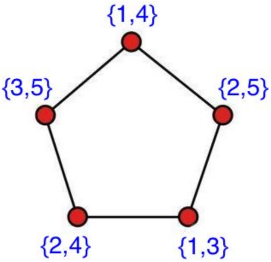
+	−	0	+	−	+	0	−	+

$$\{1, 4\}$$

The Altermatic Number of Five Cycle



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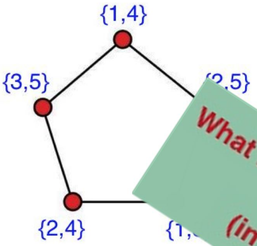
1	a	3	b	5	c	2	d	4
$+$	$-$	0	$+$	$-$	$+$	0	$-$	0

$$|V(H)| - Alt(H) = 3$$



$V(H) = \{1, 2, 3, 4, 5, a, b, c, d\}$

$= \{\{1,3\}, \{1,4\}, \{2,4\}, \{2,5\}, \{3,5\}\}$

$1 \quad a \quad 3 \quad \ldots \quad 2 \quad d \quad 4$

$+ \quad - \quad 0 \quad + \quad \ldots \quad - \quad 0$

What is the best representation for a graph?

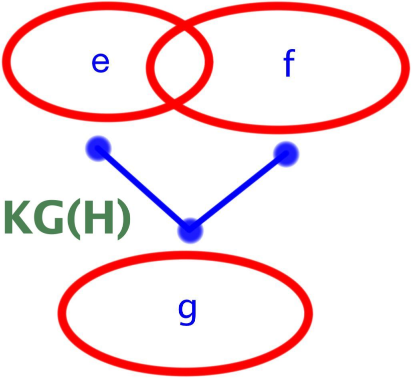
(in sense of altermatic number)

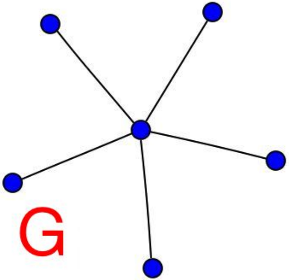
$$|V(H)| - Alt(H) \ldots 3$$

Kneser Representation

$V(H)$= A ground set=$\{1, 2, \ldots, n\}$

$E(H)$=$\{e, f, g, \ldots\}\subseteq 2^{V(H)}$

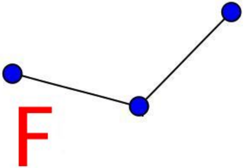




$V(H) =$ The edge set of the graph G

$E(H) =$ Every subgraph of G isomorphic to F

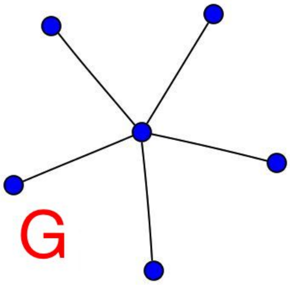
$\mathrm{KG}(H) =?$

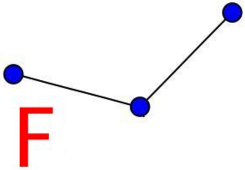


$V(H) =$ The edge set of the graph G

$E(H) =$ Every subgraph of G isomorphic to F

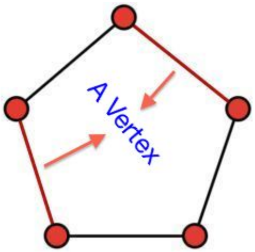
$\mathrm{KG}(H) =$ Petersen Graph $= \mathrm{KG}(G, F)$





Definition (General Kneser Graphs)

Let G be a graph and $\mathcal{F}$ be a family of graphs. By $\mathrm{KG}(G, \mathcal{F})$, we denote the general Kneser graph whose vertex set is the set of all subgraphs of G isomorphic to some member of $\mathcal{F}$ and in which two vertices are adjacent if the corresponding subgraphs are edge-disjoint.



$\mathrm{KG}(C_5, 2K_2)$

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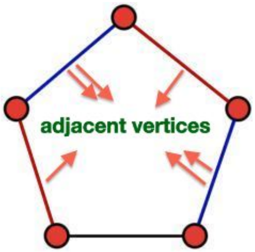

$$\mathrm{KG}(C_5, 2K_2) = C_5$$

Kneser Graphs

1. Kneser Graphs: $\mathrm{KG}(nK_2, rK_2)$, where nK_2 is a matching of size n.
2. $\chi(\mathrm{KG}(nK_2, rK_2)) = n - 2r + 2$ (L. Lovász, 1978)

A Representation For Some Graphs

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Schrijver Graphs

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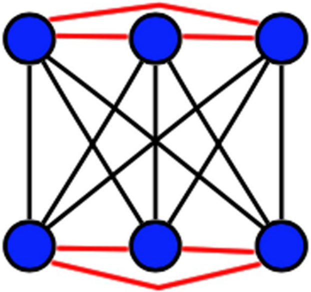
Circular Complete Graphs and Permutation Graphs

1. Circular Complete Graphs: $\mathrm{KG}(C_n, P_d)$; P_d is a path of length d.
2. Permutation Graphs: $\mathrm{KG}(K_{m,n}, rK_2)$, where $K_{m,n}$ is a complete bipartite graph.

Definition (Generalized Turán Number)

We define the generalized Turán number $\mathrm{ex}(G, \mathcal{F})$ as the largest number m such that there exists a spanning subgraph K of G with m edges which contains no subgraph isomorphic to a member of $\mathcal{F}$.



$$\mathrm{ex}(K_6, K_3) = 9$$

Turán Number

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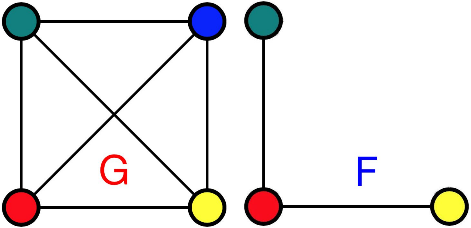

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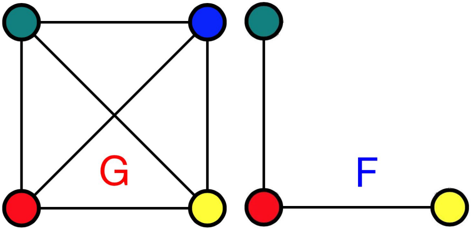
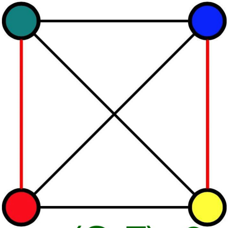
We define the generalized Turán number $\mathrm{ex}(G, \mathcal{F})$ as the largest number m such that there exists a spanning subgraph K of G with m edges which contains no subgraph isomorphic to a member of $\mathcal{F}$.



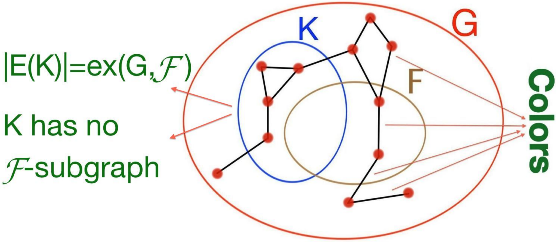


ex(G,F)=2

Upper Bound For Chromatic Number

Observation

Let G be a graph and $\mathcal{F}$ be a family of graphs. For the general Kneser graph $\mathrm{KG}(G,\mathcal{F})$, we have $\chi(\mathrm{KG}(G,\mathcal{F})) \leq |E(G)| - \mathrm{ex}(G,\mathcal{F})$.



Proof

Let K has no $\mathcal{F}$-subgraph and $|E(K)| = \mathrm{ex}(G,\mathcal{F})$. Consider an ordering for $E(G) \setminus E(K)$. Define $c : V(\mathrm{KG}(G,\mathcal{F})) \longrightarrow \{Colors\} = E(G) \setminus E(K)$ as follows. Set $c(F)$ to be the smallest edge of F in $E(G) \setminus E(K)$.

Question

Let G be a graph and $\mathcal{F}$ be a family of graphs. What is the best lower bound for the chromatic number of the general Kneser graph $\mathrm{KG}(G, \mathcal{F})$?

Lower Bound For Chromatic Number

Question

Let G be a graph and $\mathcal{F}$ be a family of graphs. What is the best lower bound for the chromatic number of the general Kneser graph $\mathrm{KG}(G,\mathcal{F})$?

Observation

$$|E(G)|-2\mathrm{ex}(G,\mathcal{F}) \leq \chi(\mathrm{KG}(G,\mathcal{F})) \leq |E(G)| - \mathrm{ex}(G,\mathcal{F}).$$

Proof

Set $\mathcal{F}'$ to be all subgraphs of G with exactly $n = \mathrm{ex}(G,\mathcal{F}) + 1$ edges. Consider a graph homomorphism $g : \mathrm{KG}(G,\mathcal{F}') \longrightarrow \mathrm{KG}(G,\mathcal{F})$. Let $m = |E(G)|$. One can check that

$\chi(\mathrm{KG}(G,\mathcal{F}')) = \chi(\mathrm{KG}(m,n)) = |E(G)| - 2\mathrm{ex}(G,\mathcal{F}) \leq \chi(\mathrm{KG}(G,\mathcal{F}))$.

Theorem (L. Lovász, 1978)

If $n \geq 2k$, for the Kneser graph $\mathrm{KG}(nK_2, kK_2)$, we have

$$\chi(\mathrm{KG}(nK_2, kK_2)) = |E(nK_2)| - 2\mathrm{ex}(nK_2, kK_2) = n - 2k + 2.$$

Chromatic Number Via Turán Number

Theorem (L. Lovász, 1978)

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Theorem (A. Schrijver, 1978)

If $n \geq 2k$, for the Schrijver graph $\mathrm{KG}(C_n, kK_2)$, we have

$$\chi(\mathrm{KG}(C_n, kK_2)) = |E(C_n)| - \mathrm{ex}(C_n, kK_2) = n - 2k + 2.$$

Chromatic Number Via Turán Number

Theorem (L. Lovász, 1978)

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$$\chi(\mathrm{KG}(C_n, kK_2)) = |E(C_n)| - \mathrm{ex}(C_n, kK_2) = n - 2k + 2.$$

Theorem (P. Frankl, 1985)

For the generalized Kneser graph $\mathrm{KG}(K_n, K_k)$, we have

$$\chi(\mathrm{KG}(K_n, K_k)) = |E(K_n)| - \mathrm{ex}(K_n, K_k) = (k-1)\binom{s}{2} + rs,$$

where $n = (k-1)s + r$, $0 \leq r < k - 1$, and n is sufficiently large.

Problem (G.O.H. Katona and Z. Tuza, 2013)

If q is a prime power and $n = q^2 + q + 1$, does the following equality hold?
$\chi(\mathrm{KG}(K_n, C_4)) = |E(K_n)| - \mathrm{ex}(K_n, C_4) = \binom{q^2+q+1}{2} - \frac{1}{2}q(q+1)^2$

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Conjecture (G.O.H. Katona and Z. Tuza, 2013)

If k is an odd integer and n is sufficiently large, then

$$\chi(\mathrm{KG}(K_n, C_k)) = |E(K_n)| - \mathrm{ex}(K_n, C_k) = \lfloor \frac{(n-1)^2}{4} \rfloor.$$

Conjectures and Problems

Problem (G.O.H. Katona and Z. Tuza, 2013)

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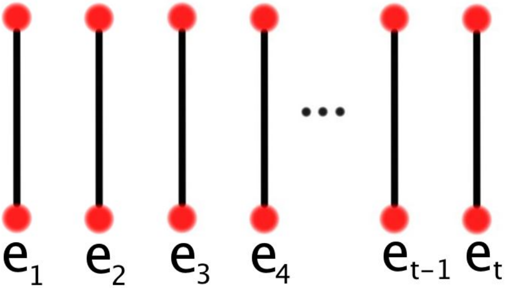
Conjecture (P. Frankl, 1985)

If $k > s \geq 2$, $n \geq 2k - s + 1$, and n is sufficiently large, then

$$\chi(\mathrm{KG}(K_{n,s}, K_{k,s})) = |E(K_{n,s})| - \mathrm{ex}(K_{n,s}, K_{k,s}),$$

where the complete hypergraph $K_{n,s}$ contains all of s-subsets of $[n]$.

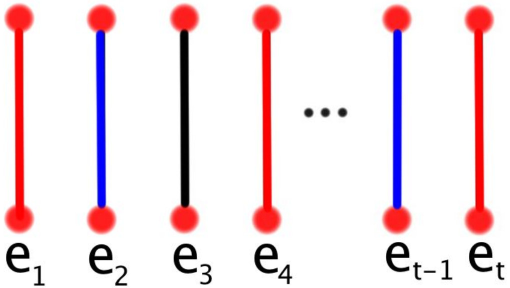
Assume that $\sigma = (e_1, e_2, \ldots, e_t)$ is an ordering of the edges of G, where $t = |E(G)|$.



Assume that $\sigma = (e_1, e_2, \ldots, e_t)$ is an ordering of the edges of G, where $t = |E(G)|$.

Definition (Alternating Coloring)

A 2-coloring of a subset $T \subseteq E(G)$ (with two colors red and blue) is called an alternating coloring (with respect to σ) for T, if we assign two colors alternatively to all members of T with respect to the ordering σ.



Assume that $\mathcal{F}$ is a family of graphs and G is a graph G.

Definition (Alternating Turán Number)

The maximum number of edges of a spanning subgraph of G such that there exists an alternating coloring for the edges of this subgraph with respect to the ordering σ and also the red subgraph **AND** the blue subgraph has no subgraph isomorphic to a member of $\mathcal{F}$ is denoted by $\mathrm{ex}_{alt}(G, \mathcal{F}, \sigma)$. Set

$$\mathrm{ex}_{alt}(G, \mathcal{F}) = \min\{\mathrm{ex}_{alt}(G, \mathcal{F}, \sigma) : \sigma \textit{ is an ordering of } E(G)\}.$$

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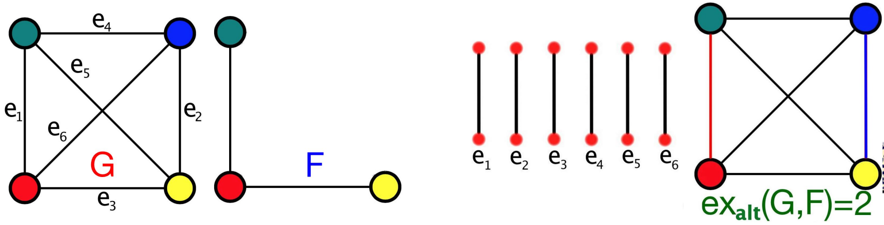

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Observation

$$\mathrm{ex}(G, \mathcal{F}) \leq \mathrm{ex}_{alt}(G, \mathcal{F}) \leq 2\mathrm{ex}(G, \mathcal{F})$$

Assume that $\mathcal{F}$ is a family of graphs and G is a graph G.

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Lemma (M. Alishahi and H.H., 2013)

$$|E(G)| - \mathrm{ex}_{alt}(G,\mathcal{F}) \leq \chi(\mathrm{KG}(G,\mathcal{F})) \leq |E(G)| - \mathrm{ex}(G,\mathcal{F}).$$

Assume that $\mathcal{F}$ is a family of graphs and G is a graph G.

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Corollary (M. Alishahi and H.H., 2013)

If $\mathrm{ex}_{alt}(G, \mathcal{F}) = \mathrm{ex}(G, \mathcal{F})$, then $\chi(\mathrm{KG}(G, \mathcal{F})) = |E(G)| - \mathrm{ex}(G, \mathcal{F})$.

Strong Alternating Turán Number

Assume that $\mathcal{F}$ is a family of graphs and G is a graph G.

Definition (Strong Alternating Turán Number)

The maximum number of edges of a spanning subgraph of G such that there exists an alternating coloring for the edges of this subgraph with respect to the ordering σ and also the red subgraph **OR** the blue subgraph has no subgraph isomorphic to a member of $\mathcal{F}$ is denoted by $\mathrm{ex}_{salt}(G, \mathcal{F}, \sigma)$. Set

$$\mathrm{ex}_{salt}(G, \mathcal{F}) = \min\{\mathrm{ex}_{salt}(G, \mathcal{F}, \sigma) : \sigma \textit{ is an ordering of } E(G)\}.$$

Strong Alternating Turán Number

Assume that $\mathcal{F}$ is a family of graphs and G is a graph G.

Definition (Strong Alternating Turán Number)

The maximum number of edges of a spanning subgraph of G such that there exists an alternating coloring for the edges of this subgraph with respect to the ordering σ and also the red subgraph **OR** the blue subgraph has no subgraph isomorphic to a member of $\mathcal{F}$ is denoted by $\mathrm{ex}_{salt}(G, \mathcal{F}, \sigma)$. Set

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Lemma (M. Alishahi and H.H., 2013)

$$|E(G)| - \mathrm{ex}_{salt}(G, \mathcal{F}) + 1 \le \chi(\mathrm{KG}(G, \mathcal{F})) \le |E(G)| - \mathrm{ex}(G, \mathcal{F}).$$

Corollary (M. Alishahi and H.H., 2013)

If $\mathrm{ex}_{salt}(G, \mathcal{F}) - 1 = \mathrm{ex}(G, \mathcal{F})$, then $\chi(\mathrm{KG}(G, \mathcal{F})) = |E(G)| - \mathrm{ex}(G, \mathcal{F})$.

Observation

$\chi(\mathrm{KG}(nK_2, rK_2)) = n - 2r + 2 = |E(nK_2)| - 2\mathrm{ex}(nK_2, rK_2)$.

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Theorem (M. Alishahi and H.H., 2013-2014)

If G is a sufficiently large dense graph or a sparse connected graph (with some conditions), then $\chi(\mathrm{KG}(G, rK_2)) = |E(G)| - ex(G, rK_2)$.

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Proof!

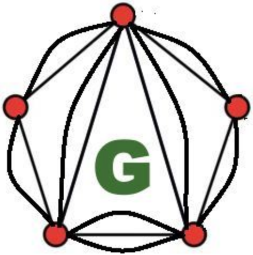
1. Present an appropriate ordering for $E(G)$.
2. In view of TutteBerge formula, we show that $\mathrm{ex}_{alt}(G, F) = \mathrm{ex}(G, F)$ or $\mathrm{ex}_{salt}(G, F) - 1 = \mathrm{ex}(G, F)$!

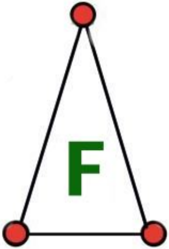
Theorem (M. Alishahi and H.H., 2013)

If G is a multigraph such that the multiplicity of each edge is at least 2 and F is a simple graph, then $\chi(\mathrm{KG}(G,F)) = |E(G)| - ex(G,F)$.

Proof!

1. Present an appropriate ordering for $E(G)$.
2. $\mathrm{ex}_{alt}(G,F) = \mathrm{ex}(G,F)$



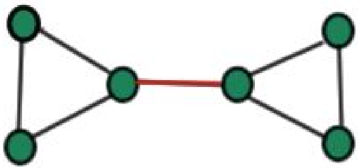


Theorem (M. Alishahi and H.H., 2014)

If G is a sufficiently large dense graph and $\mathcal{T}_n$ is the family of the spanning trees of G, then $\chi(\mathrm{KG}(G, \mathcal{T}_n)) = |\mathrm{MinimumCUT}(G)|$.

Proof!

1. Present an appropriate ordering for $E(G)$.
2. $\mathrm{ex}_{alt}(G, \mathcal{T}_n) = \mathrm{ex}(G, \mathcal{T}_n)$



$|\mathrm{MinimumCUT}(G)| = 1$

Theorem (M. Alishahi and H.H., 2013)

For any graph G, we have $\zeta(G) \leq \max\{n : K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ *is a subgraph of* $G\}$.

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Question

Is it true that for any graph G, we have $\zeta(G \vee K_n) \leq \zeta(G) + n$?

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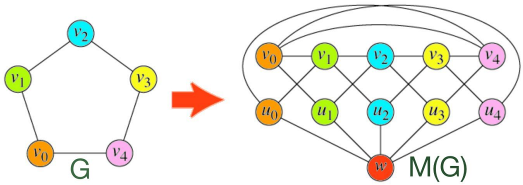
For any graph G, we have $\zeta(M(G)) \geq \zeta(G) + 1$.



Definition

Let $X = (x_1, \ldots, x_n), Y = (y_1, \ldots, y_n) \in \{-1, 0, +1\}^n$. Set $X^+ = \{i \in [n] : \ x_i = +1\}$ and $X^- = \{i \in [n] : \ x_i = -1\}$. By $X \preceq Y$, we mean $X^+ \subseteq Y^+$ and $X^- \subseteq Y^-$.

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Tucker's Lemma, 1946

Let $\lambda : \{-1, 0, +1\}^n \setminus \{(0, \ldots, 0)\} \longrightarrow \{\pm 1, \pm 2, \ldots, \pm(n-1)\}$. Also, assume that for any $X \in \{-1, 0, +1\}^n \setminus \{(0, \ldots, 0)\}$, we have $\lambda(-X) = -\lambda(X)$. Then there exist two vectors $X, Y \in \{-1, 0, +1\}^n \setminus \{(0, \ldots, 0)\}$ such that $X \preceq Y$ and also $\lambda(X) = -\lambda(Y)$.

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Z_p-Tucker Lemma (G.M. Ziegler, 2002)

Hypergraph Coloring

Definition (The Chromatic Number of Hypergraphs)

Let $H = (V(H), E(H))$ be a hypergraph. The hypergraph H is called r-colorable if there exists a map $c : V(H) \to \{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{r}\}$ such that no hyperedge is monochromatic. The chromatic number $\chi(H)$ of H is the minimum r such that H is r-colorable.

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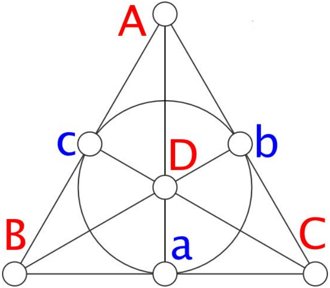
$$V(H) = \{A, B, C, D, a, b, c\}$$

$$E(H) = \{\{A, B, c\}, \{A, D, a\}, \{A, C, b\}\\ \{B, D, b\}, \{B, C, a\}, \{C, D, c\}\\ \{a, b, c\}\}$$



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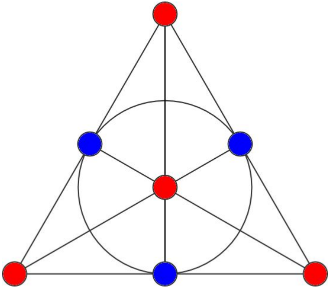

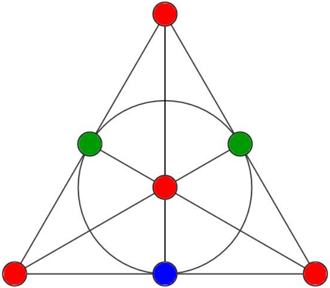

For a hypergraph H and positive integer $r \geq 2$, the Kneser hypergraph $\mathrm{KG}^r(H)$ is an r-uniform hypergraph whose vertex set is $E(H)$ and whose hyperedge set consists of all r-tuples of pairwise disjoint hyperedges of H.

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$$V(H) = \{1, 2, 3, 4, 5\},$$

$$E(H) = \binom{[5]}{2},$$

then

$$\mathrm{KG}^2(H) = \mathrm{KG}(H) = \mathrm{KG}(5, 2) = \text{Petersen Graph}.$$

For a hypergraph H and positive integer $r \geq 2$, the Kneser hypergraph $\mathrm{KG}^r(H)$ is an r-uniform hypergraph whose vertex set is $E(H)$ and whose hyperedge set consists of all r-tuples of pairwise disjoint hyperedges of H. For instance, if $r = 3$,

$$V(H) = \{1, 2, 3, 4, 5, 6\},$$

$$E(H) = \binom{[6]}{2},$$

then

$$E(\mathrm{KG}^3(H)) = \{\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}, \{\{1, 3\}, \{2, 4\}, \{5, 6\}\}, \dots\}.$$

Kneser Hypergraph

For a hypergraph H and positive integer $r \geq 2$, the Kneser hypergraph $\mathrm{KG}^r(H)$ is an r-uniform hypergraph whose vertex set is $E(H)$ and whose hyperedge set consists of all r-tuples of pairwise disjoint hyperedges of H. For

$$V(H) = [m],$$

$$E(H) = \binom{[m]}{n},$$

then denote the usual Kneser hypergraph $\mathrm{KG}^r(H)$ by

$$\mathrm{KG}^r(m, n).$$

Observation (P. Erdős, 1976)

For $m \geq rn$ and $r \geq 2$, we have $\chi(\mathrm{KG}^r(m,n)) \leq \left\lceil \frac{m-r(n-1)}{r-1} \right\rceil$.

Conjecture (P. Erdős, 1976)

If $m \geq rn$ and $r \geq 2$, then $\chi(\mathrm{KG}^r(m,n)) = \left\lceil \frac{m-r(n-1)}{r-1} \right\rceil$.

Usual Kneser Hypergraphs

Observation (P. Erdős, 1976)

For $m \geq rn$ and $r \geq 2$, we have $\chi(\mathrm{KG}^r(m,n)) \leq \left\lceil \frac{m-r(n-1)}{r-1} \right\rceil$.

Conjecture (P. Erdős, 1976)

If $m \geq rn$ and $r \geq 2$, then $\chi(\mathrm{KG}^r(m,n)) = \left\lceil \frac{m-r(n-1)}{r-1} \right\rceil$.

Theorem (N. Alon, P. Frankl, and L. Lovász, 1986)

For $m \geq rn$ and $r \geq 2$, we have $\chi(\mathrm{KG}^r(m,n)) = \left\lceil \frac{m-r(n-1)}{r-1} \right\rceil$.

$V(H) = \{1, 2, 3, 4, 5\}$ & $E(H) = \{\{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 5\}, \{3, 5\}\}$

1 2 3 4 5

$V(H) = \{1,2,3,4,5\}$ & $E(H) = \{\{1,3\},\{1,4\},\{2,4\},\{2,5\},\{3,5\}\}$

1	2	3	4	5
ω	ω	0	ω^2	ω^3

A subsequence of nonzero elements is termed an alternating subsequence if any two consecutive terms in this subsequence are different.

$V(H) = \{1, 2, 3, 4, 5\}$ & $E(H) = \{\{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 5\}, \{3, 5\}\}$

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \omega & \omega^3 & 0 & \omega^2 & \omega \end{array}$$

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1	2	3	4	5
ω	0	ω^2	ω	ω^3

$$\{1, 4\}$$

$Alt_r(H) =$ The number of nonzero elements of a longest alternating alternating subsequence with at most r signs which does not contain a positive OR negative hyperedge of H

Theorem (M. Alishahi and H.H., 2013)

For any hypergraph H and positive integer $r \geq 2$, we have

$$\chi(\mathrm{KG}^r(H)) \geq \left\lceil \frac{|V(H)| - alt_r(H)}{r-1} \right\rceil .$$

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Observation

For any positive integer $r \geq 2$ and the hypergraph $H = (V(H), E(H))$, where $V(H) = [m]$, and $E(H) = \binom{[m]}{n}$,, we have $alt_r(H) \leq r(n-1)$.

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Consequently,

Theorem (N. Alon, P. Frankl, and L. Lovász, 1986)

For $m \geq rn$ and $r \geq 2$, we have $\chi(\mathrm{KG}^r(m,n)) = \left\lceil \frac{m-r(n-1)}{r-1} \right\rceil$.

Definition (r-Colorability Defect)

Let the r-colorability defect, denoted by $\mathrm{cd}_r(H)$, be the minimum size of a subset $X \subseteq V(H)$ such that the hyperedges of H that contain no points of X is r-colorable. Precisely,
$\mathrm{cd}_r(H) = \min\{|X| : (V(H) \setminus X, \{F \in E(H) : F \cap X = \emptyset\}) \text{ is } r-\text{colorable}\}$.

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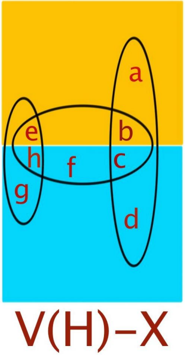

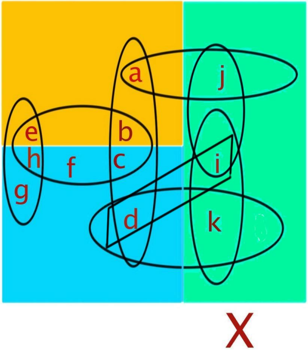

Theorem (V.L. Dol'nikov for $r = 2$ 1988, I. Kříž 1992)

For any hypergraph H and positive integer $r \geq 2$, we have $\chi(\mathrm{KG}^r(H)) \geq \left\lceil \frac{\mathrm{cd}_r(H)}{r-1} \right\rceil$.

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For any hypergraph $H = (V(H), E(H))$ and positive integer $r \geq 2$, we have $\chi(\mathrm{KG}^r(H)) \geq \left\lceil \frac{|V(H)| - alt_r(H)}{r-1} \right\rceil \geq \left\lceil \frac{\mathrm{cd}_r(H)}{r-1} \right\rceil$.

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Thank You!

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QUESTIONS?!