Chromatic Number Via Turán Number

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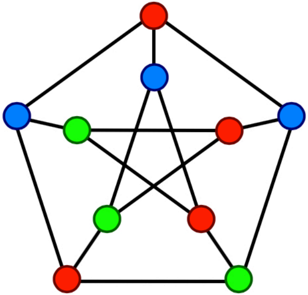
Definition (Chromatic number)

$\chi(G) = \min\{k : V(G) = V_1 \cup \cdots \cup V_k, \forall i \in [k], V_i \text{ is an independent set}\}$.

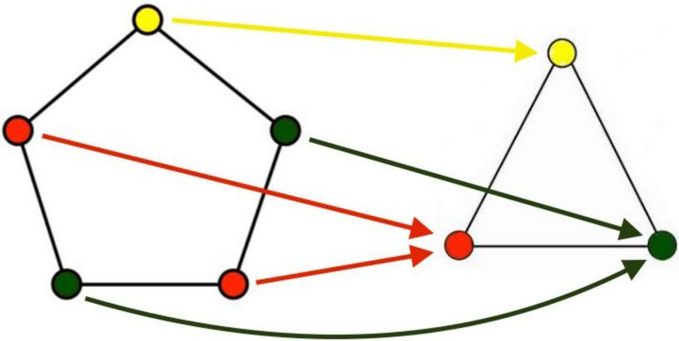


Chromatic number

It is NP-hard to compute the chromatic number of a graph!

Definition (Graph Homomorphism)

A homomorphism $f : G \longrightarrow H$ from a graph G to a graph H is a map $f : V(G) \longrightarrow V(H)$ such that if $uv \in E(G)$ then $f(u)f(v) \in E(H)$. Also, the existence of a homomorphism is indicated by the symbols $G \longrightarrow H$. Also, $G \longleftrightarrow H$ means that $G \longrightarrow H$ and $H \longrightarrow G$.



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Observation!

For any graph G, we have $\chi(G) = \min\{n : G \longrightarrow K_n\}$.

Observation!

If there exists a graph homomorphism from G into H, then $\chi(G) \leq \chi(H)$.

Problem

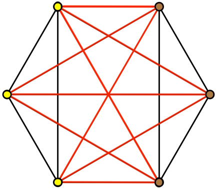
Determine the size of the largest configuration with a given property.

Definition ($\mathcal{F}$-Free Graphs)

A graph G is $\mathcal{F}$-free, if it has no subgraph isomorphic to a member of $\mathcal{F}$.

Definition (Turán Number)

The Turán number $\mathrm{ex}(K_n, F)$ is the largest number p such that there exists a spanning F-free subgraph G of K_n ($F \nsubseteq G \subseteq K_n$) with p edges.



$\mathrm{ex}(K_6, K_3) = 9$

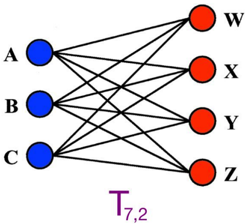
Turán Number of Complete Graphs

Definition (Turán Number)

The Turán graph $T_{n,r}$ is a complete multipartite graph formed by partitioning a set of n vertices into r parts, with sizes as equal as possible, and connecting two vertices by an edge whenever they belong to different subsets.

Theorem (Turán 1941 and Mantel 1907 for $r = 3$)

For all $r \geq 2$, the unique largest K_{r+1}-free graph on n vertices is $T_{n,r}$. In particular, $\mathrm{ex}(K_n, K_{r+1}) = |E(T_{n,r})|$.



Problem

What can one say about $\mathrm{ex}(K_n, F)$ for other graphs F?

Erdős-Stone-Simonovits Theorem

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What can one say about $\mathrm{ex}(K_n, F)$ for other graphs F?

Theorem (Erdős-Stone-Simonovits, 1946 & 1966)

If F is a fixed graph with at least one edge, then

$$\lim_{n\to\infty} \frac{\mathrm{ex}(K_n, F)}{\binom{n}{2}} = (1 - \frac{1}{\chi(F) - 1}).$$

Remark

This gives an asymptotic solution for non-bipartite graph F.

Theorem (Erdős and Rényi 1962, Erdős, Rényi, and Sós 1966)

For C_4 we have $\mathrm{ex}(K_n, C_4) = \frac{1}{2}n^{\frac{3}{2}} - o(n^{\frac{3}{2}})$. In particular, if q is a prime power and $n = q^2 + q + 1$, then $\mathrm{ex}(K_n, C_4) = \frac{1}{2}q(q+1)^2$.

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Remark

The only bipartite graph F containing a cycle for which the Turán number $\mathrm{ex}(K_n, F)$ is known (for infinitely many n) is C_4.

Turán Number of Cycles

Theorem (Erdős and Rényi 1962, Erdős, Rényi, and Sós 1966)

For C_4 we have $\mathrm{ex}(K_n, C_4) = \frac{1}{2}n^{\frac{3}{2}} - o(n^{\frac{3}{2}})$. In particular, if q is a prime power and $n = q^2 + q + 1$, then $\mathrm{ex}(K_n, C_4) = \frac{1}{2}q(q+1)^2$.

Remark

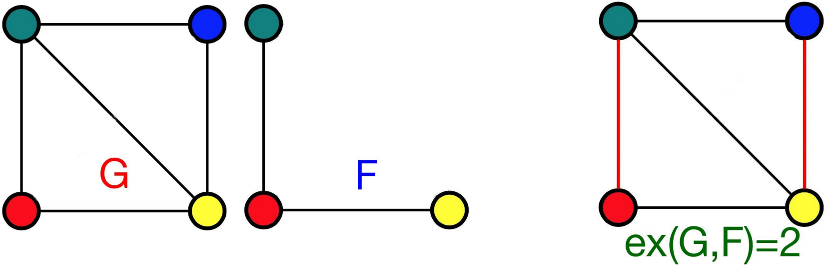
The only bipartite graph F containing a cycle for which the Turán number $\mathrm{ex}(K_n, F)$ is known (for infinitely many n) is C_4.

Theorem (Simonovits, 1966)

If r is an odd integer and n is sufficiently large, then $\mathrm{ex}(K_n, C_r) = \lfloor \frac{n^2}{4} \rfloor$.

Definition (Generalized Turán Number)

We define the generalized Turán number $\mathrm{ex}(G, \mathcal{F})$ as the largest number m such that there exists a spanning subgraph K of G with m edges which contains no subgraph isomorphic to a member of $\mathcal{F}$.



Definition (Hypergraph)

A hypergraph H is a pair $H = (V, E)$ where V is a set of elements called vertices, and E is a set of nonempty subsets of V called hyperedges.

Definition (Independence Number of Hypergraphs)

The independence number $\alpha(H)$ of a hypergraph $H = (V, E)$ is the size of a largest set of vertices containing no hyperedge of H.

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Remark (Turán number Problems $\subseteq$ Independence number Problems)

For any two graphs G and F, set V to be the edge set of G. Also, define $E \subseteq 2^V$ to be all subgraphs of G isomorphic to F. In other words, the elements of any member of E form a subgraph of G isomorphic to F. One can see that for $H_{G,F} = (V, E)$, we have $\alpha(H_{G,F}) = \mathrm{ex}(G, F)$.

$$H = (V, E):$$

$$V = \{1, 2, 3, 4, 5\} \quad \& \quad E = \{\{1,3\}, \{1,4\}, \{2,4\}, \{2,5\}, \{3,5\}\}$$

$$1 \qquad 2 \qquad 3 \qquad 4 \qquad 5$$

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$$\sigma : 1 \quad 2 \quad 3 \quad 4 \quad 5$$

Definition (Alternating Independence Number)

Let $H = (V, E)$ be a hypergraph and σ be an ordering of V. Define $\alpha_{alt}(H, \sigma)$ to be the maximum size of a subset $T \subseteq V$ such that if we assign alternatively two colors red and blue to the vertices of T (with respect to the ordering σ), then the red vertices (resp. blue vertices) form an independent set. Define

$$\alpha_{alt}(H) = \min\{\alpha_{alt}(H, \sigma) : \sigma \textit{ is an ordering of } V\}.$$

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$V=\{1,2,3,4,5\}$ & $E=\{\{1,3\},\{1,4\},\{2,4\},\{2,5\},\{3,5\}\}$

$$\sigma : 1 \quad 2 \quad 3 \quad 4 \quad 5$$

$\{1,4\}$ and $\{2,5\}$ are hyperedges of H.

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One can check that $\alpha_{alt}(H,\sigma)\geq 3$.

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$$\sigma : 1 \quad 2 \quad 3 \quad 4 \quad 5$$

This implies that $\alpha_{alt}(H,\sigma)\le 4$.

Definition (Alternating Independence Number)

Let $H = (V, E)$ be a hypergraph and σ be an ordering of V. Define $\alpha_{alt}(H, \sigma)$ to be the maximum size of a subset $T \subseteq V$ such that if we assign alternatively two colors red and blue to the vertices of T (with respect to the ordering σ), then the red vertices (resp. blue vertices) form an independent set. Define

$$\alpha_{alt}(H) = \min\{\alpha_{alt}(H, \sigma) : \sigma \textit{ is an ordering of } V\}.$$

$$V = \{1, 2, 3, 4, 5\} \quad \& \quad E = \{\{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 5\}, \{3, 5\}\}$$

$$\sigma : 1 \quad 2 \quad 3 \quad 4 \quad 5$$

This implies that $\alpha_{alt,\sigma}(H) = 3$.

Definition (Alternating Independence Number)

Let $H = (V, E)$ be a hypergraph and σ be an ordering of V. Define $\alpha_{alt}(H, \sigma)$ to be the maximum size of a subset $T \subseteq V$ such that if we assign alternatively two colors red and blue to the vertices of T (with respect to the ordering σ), then the red vertices (resp. blue vertices) form an independent set. Define

$$\alpha_{alt}(H) = \min\{\alpha_{alt}(H, \sigma) : \sigma \textit{ is an ordering of } V\}.$$

$$V = \{1, 2, 3, 4, 5\} \quad \& \quad E = \{\{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 5\}, \{3, 5\}\}$$

In fact, one can check that $\alpha_{alt}(H) = 3$

Observation!

For any hypergraph $H = (V, E)$, we have $\alpha(H) \leq \alpha_{alt}(H) \leq 2\alpha(H)$.

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For any hypergraph $H = (V, E)$, we have $\alpha(H) \leq \alpha_{alt}(H) \leq 2\alpha(H)$.

Observation!

For the four cycle $C_4 = (V, E)$, we have $\alpha_{alt}(C_4) = \alpha(C_4)$.

Proof

Let $V = \{1, 2, 3, 4\}$ and $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}\}$. Consider the ordering $\sigma :\ 1 < 3 < 2 < 4$. One can check that $\alpha_{alt}(C_4, \sigma) = 2 = \alpha(C_4)$.

$$\sigma : 1 \quad 3 \quad 2 \quad 4$$

$$\{3, 4\}$$

Definition (Strong Alternating Independence Number)

Let $H = (V, E)$ be a hypergraph and σ be an ordering of V. Define $\alpha_{alt}(H, \sigma)$ to be the maximum size of a subset $T \subseteq V$ such that if we assign alternatively two colors red and blue to the vertices of T (with respect to the ordering σ), then the red vertices or blue vertices form an independent set. Define

$$\alpha_{salt}(H) = \min\{\alpha_{salt}(H, \sigma) : \sigma \textit{ is an ordering of } V\}.$$

$V = \{1, 2, 3, 4, 5\}$ & $E = \{\{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 5\}, \{3, 5\}\}$

$$\sigma : 1 \quad 2 \quad 3 \quad 4 \quad 5$$

One can check that $\alpha_{salt}(H) = 3$.

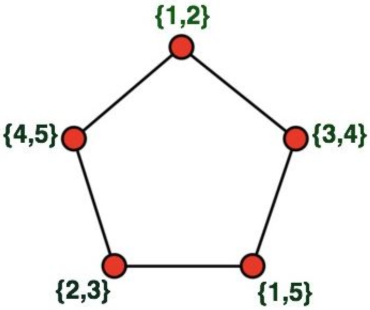
For a hypergraph $H = (V, E)$, consider the graph $\mathrm{KG}(H)$ whose vertex set is E and whose edge set consists of all disjoint pairs. For instance, if

$$V = \{1, 2, 3, 4, 5\}$$

$$E = \{\{1, 2\}, \{3, 4\}, \{1, 5\}, \{2, 3\}, \{4, 5\}\}$$

then

$$\mathrm{KG}(H) \cong C_5$$



Definition (Kneser Representations for a Graph)

A hypergraph H provides a Kneser representation for a graph G if the graph $\mathrm{KG}(H)$ is isomorphic to G.

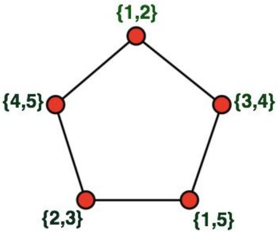
For a hypergraph $H = (V, E)$, consider the graph $\mathrm{KG}(H)$ whose vertex set is E and whose edge set consists of all disjoint pairs. Consider two hypergraphs $H = (V, E)$ and $H' = (V', E')$, where

$$V = \{1, 2, 3, 4, 5\}$$
$$V' = \{1, 2, 3, 4, 5\} \cup \{a, b, c, d, e\}$$

$$E = \{\{1, 2\}, \{3, 4\}, \{1, 5\}, \{2, 3\}, \{4, 5\}\}$$
$$E' = \{\{1, 2\}, \{3, 4\}, \{1, 5\}, \{2, 3\}, \{4, 5\}\}$$

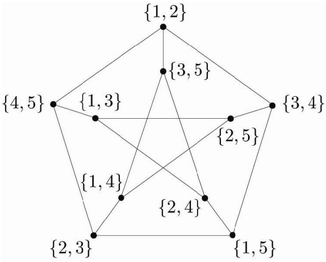
One can see that

$$\mathrm{KG}(H) \cong \mathrm{KG}(H') \cong C_5.$$



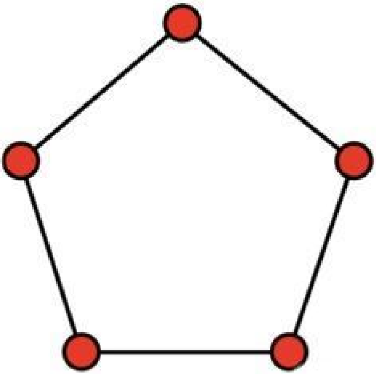
The usual Kneser graph $\mathrm{KG}(n, r)$:

1. $\mathrm{KG}(n, r) = \mathrm{KG}(H)$, where
2. $V = \{1, \ldots, n\}$
3. $E = \binom{V}{r} = \{A : A \subseteq V, |A| = r\}$
4. $\binom{V}{r}$ is the vertex set of $\mathrm{KG}(n, r)$
5. A and B are adjacent iff $A \cap B = \varnothing$



$\mathrm{KG}(5, 2) \cong$ Petersen Graph

Kneser Representations of Graphs



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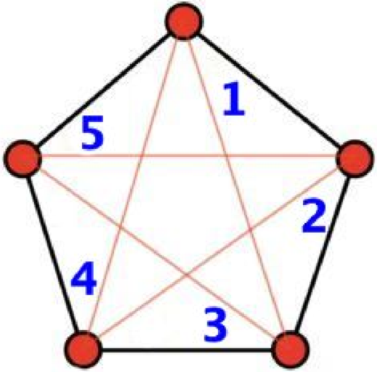

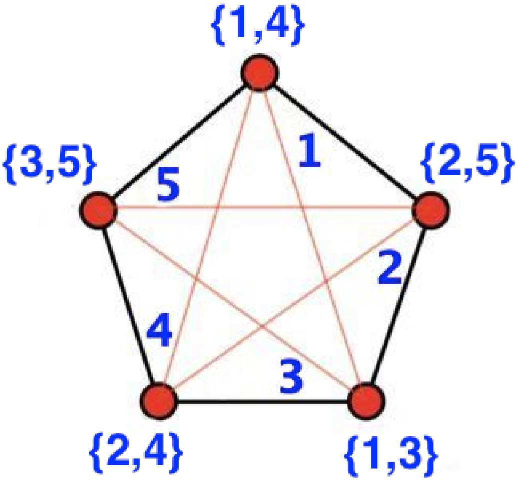
{1,4}
{3,5}
{2,5}
{2,4}
{1,3}
1
2
3
4
5

Observation!

If $H = (V, E)$ is a hypergraph, then $\chi(\mathrm{KG}(H)) \leq |V| - \alpha(H)$.

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Observation!

A vertex cover of a hypergraph is a set of vertices such that each hyperedge of the hypergraph is incident to at least one vertex of the set. The covering number $c(H)$ of a hypergraph $H = (V, E)$ is the minimum size of a vertex cover of G. It is well-known that $c(H) = |V| - \alpha(H)$.

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Proof

Let $K \subseteq V$ be a covering set of H of size $c(H) = |V| - \alpha(H)$. Consider an ordering for K. Define $c : V(\mathrm{KG}(H)) \longrightarrow K = \text{the set of Colors}$ as follows. For any $F \in E$, set $c(F)$ to be the smallest vertex of F in K.

Question

Let $H = (V, E)$ be a hypergraph. What is the best lower bound for the chromatic number of the graph $\mathrm{KG}(H)$?

Observation!

$$|V| - 2\alpha(H) \le \chi(\mathrm{KG}(H)) \le |V| - \alpha(H).$$

Proof

Consider the hypergraph $H' = (V, E')$, where E' is the set of all subsets of V with exactly $\alpha(H) + 1$ vertices. One can check that there exists a graph homomorphism $g : \mathrm{KG}(H') \longrightarrow \mathrm{KG}(H)$. Consequently,
$\chi(\mathrm{KG}(H')) = \chi(\mathrm{KG}(|V|, \alpha(H) + 1)) = |V| - 2\alpha(H) \le \chi(\mathrm{KG}(H))$.

Theorem (M. Alishahi and H.H., 2013)

If G and $\mathrm{KG}(H)$ are homomorphically equivalent, then

$$\max\{|V| - \alpha_{alt}(H), |V| - \alpha_{salt}(H) + 1\} \leq \chi(G) \leq |V| - \alpha(H)$$

Theorem (M. Alishahi and H.H., 2013)

If G and $\mathrm{KG}(H)$ are homomorphically equivalent, then

$$\max\{|V| - \alpha_{alt}(H), |V| - \alpha_{salt}(H) + 1\} \leq \chi(G) \leq |V| - \alpha(H)$$

Definition (Altermatic Number and Strong Altermatic Number)

The altermatic number $\zeta(G)$ and the strong altermatic number $\zeta_s(G)$ of a graph G are defined, respectively, as follows:

$$\zeta(G) = \max_{H=(V,E)}\{|V| - \alpha_{alt}(H) : \mathrm{KG}(H) \longleftrightarrow G\}.$$

$$\zeta_s(G) = \max_{H=(V,E)}\{|V| - \alpha_{salt}(H) + 1 : \mathrm{KG}(H) \longleftrightarrow G\}.$$

Theorem (M. Alishahi and H.H., 2013)

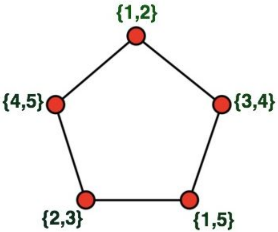
For any graph G, we have $\chi(G) \geq \zeta(G)$ and $\chi(G) \geq \zeta_s(G)$.

$$H = (V, E) \quad \& \quad H' = (V', E')$$

$$V = \{1, 2, 3, 4, 5\}$$
$$V' = \{1, 2, 3, 4, 5\} \cup \{a, b, c, d, e\}$$

$$E = \{\{1, 2\}, \{3, 4\}, \{1, 5\}, \{2, 3\}, \{4, 5\}\}$$
$$E' = \{\{1, 2\}, \{3, 4\}, \{1, 5\}, \{2, 3\}, \{4, 5\}\}$$

$$\chi(C_5) \geq \zeta(C_5) \geq |V| - \alpha_{alt}(H) = 2$$
$$\chi(C_5) \geq \zeta(C_5) \geq |V'| - \alpha_{alt}(H') = 3$$



$$H = (V, E): \quad V = [n] = \{1, 2, \ldots, n\} \quad \& \quad E = \binom{[n]}{r}$$

$$\mathrm{KG}(H) \cong \mathrm{KG}(n, r)$$

Theorem (L. Lovász 1978)

For any $n \geq 2r$, we have $\chi(\mathrm{KG}(n, r)) = n - 2r + 2$.

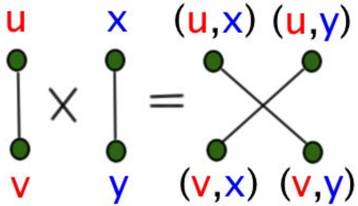
$$\sigma: \quad 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad 2r-3 \quad 2r-2 \quad 2r-1 \quad 2r \quad \cdots \quad n-1 \quad n$$

$$\alpha_{alt}(H) = 2r - 2$$

$$\chi(\mathrm{KG}(n, r)) \geq |V| - \alpha_{alt}(H) = n - 2r + 2$$

The Categorical Product

Let G and G' be two graphs. Their categorical product $G \times G'$ is the graph whose vertex set is $V(G) \times V(G')$ and whose edge set is $E(G \times G') = \{\{(u, u'), (v, v')\} : \{u, v\} \in E(G), \{u', v'\} \in E(G')\}$.



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One can see that $\chi(G \times G') \leq \min\{\chi(G), \chi(G')\}$.

Hedetniemi's Conjecture, 1966

For any two graphs G and G', $\chi(G \times G') = \min\{\chi(G), \chi(G')\}$.

Theorem (M. Alishahi and H.H., 2014)

For any two graphs G and G', we have

$$\chi(G \times G') \geq \zeta_s(G \times G') \geq \min\{\zeta_s(G), \zeta_s(G')\}.$$

Theorem (M. Alishahi and H.H., 2014)

For any two graphs G and G', we have

$$\chi(G \times G') \geq \zeta_s(G \times G') \geq \min\{\zeta_s(G), \zeta_s(G')\}.$$

Theorem (H.H. and F. Meunier, 2014)

For any two graphs G and G', we have $\chi(G \times G') \geq \min\{\zeta(G), \zeta(G')\}$.

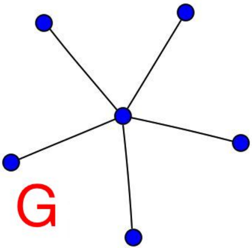
Question

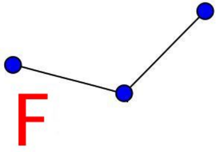
Is it true that the inequality $\zeta(G \times G') \geq \min\{\zeta(G), \zeta(G')\}$ holds for any two graphs G and G'?

G = a graph
$\mathcal{F}$ = a family of nonempty graphs

$H_{G,\mathcal{F}} = (V, E)$
V = The edge set of the graph G
E = Every subgraph of G isomorphic to a member of $\mathcal{F}$

$\mathrm{KG}(G, F) = \mathrm{KG}(H_{G,F}) \cong$ Petersen Graph





Consider the Kneser graph $\mathrm{KG}(G, F) = \mathrm{KG}(H_{G,F})$ as follows:

Kneser Graphs

Kneser Graphs: $\mathrm{KG}(nK_2, rK_2)$, where nK_2 is a matching of size n.

Schrijver Graphs

Schrijver Graphs: $\mathrm{KG}(C_n, rK_2)$, where C_n is a cycle of size n.

Circular Complete Graphs

Circular Complete Graphs: $\mathrm{KG}(C_n, P_d)$; P_d is a path of length d.

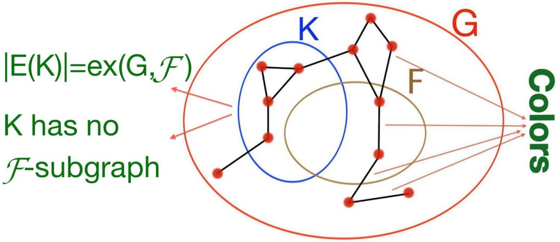
Permutation Graphs

Permutation Graphs: $\mathrm{KG}(K_{m,n}, rK_2)$, where $K_{m,n}$ is a complete bipartite

Upper Bound For Chromatic Number

Observation!

Let G be a graph and $\mathcal{F}$ be a family of nonempty graphs. For the general Kneser graph $\mathrm{KG}(G,\mathcal{F})$, we have $\chi(\mathrm{KG}(G,\mathcal{F})) \leq |E(G)| - \mathrm{ex}(G,\mathcal{F})$.



Proof

Let K be $\mathcal{F}$-free subgraph and $|E(K)| = \mathrm{ex}(G,\mathcal{F})$. Consider an ordering for $E(G) \setminus E(K)$. Define $c : V(\mathrm{KG}(G,\mathcal{F})) \longrightarrow \{Colors\} = E(G) \setminus E(K)$ as follows. Set $c(F)$ to be the smallest edge of F in $E(G) \setminus E(K)$.

Question

Let G be a graph and $\mathcal{F}$ be a family of graphs. What is the best lower bound for the chromatic number of the general Kneser graph $\mathrm{KG}(G, \mathcal{F})$?

Observation!

$$|E(G)| - 2\mathrm{ex}(G, \mathcal{F}) \leq \chi(\mathrm{KG}(G, \mathcal{F})) \leq |E(G)| - \mathrm{ex}(G, \mathcal{F}).$$

Proof

Set $\mathcal{F}'$ to be all subgraphs of G with exactly $n = \mathrm{ex}(G, \mathcal{F}) + 1$ edges. Consider a graph homomorphism $g : \mathrm{KG}(G, \mathcal{F}') \longrightarrow \mathrm{KG}(G, \mathcal{F})$. Let $m = |E(G)|$. One can check that
$\chi(\mathrm{KG}(G, \mathcal{F}')) = \chi(\mathrm{KG}(m, n)) = |E(G)| - 2\mathrm{ex}(G, \mathcal{F}) \leq \chi(\mathrm{KG}(G, \mathcal{F}))$.

Theorem (L. Lovász, 1978)

If $n \geq 2k$, for the Kneser graph $\mathrm{KG}(nK_2, kK_2)$, we have

$$\chi(\mathrm{KG}(nK_2, kK_2)) = |E(nK_2)| - 2\mathrm{ex}(nK_2, kK_2) = n - 2k + 2.$$

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Theorem (A. Schrijver, 1978)

If $n \geq 2k$, for the Schrijver graph $\mathrm{KG}(C_n, kK_2)$, we have

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Theorem (P. Frankl, 1985)

For the generalized Kneser graph $\mathrm{KG}(K_n, K_k)$, we have

$$\chi(\mathrm{KG}(K_n, K_k)) = |E(K_n)| - \mathrm{ex}(K_n, K_k) = (k-1)\binom{s}{2} + rs,$$

where $n = (k-1)s + r$, $0 \leq r < k - 1$, and n is sufficiently large.

Problem (G.O.H. Katona and Z. Tuza, 2013)

If q is a prime power and $n = q^2 + q + 1$, does the following equality hold?
$\chi(\mathrm{KG}(K_n, C_4)) = |E(K_n)| - \mathrm{ex}(K_n, C_4) = \binom{q^2+q+1}{2} - \frac{1}{2}q(q+1)^2$

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Conjecture (G.O.H. Katona and Z. Tuza, 2013)

If k is an odd integer and n is sufficiently large, then

$$\chi(\mathrm{KG}(K_n, C_k)) = |E(K_n)| - \mathrm{ex}(K_n, C_k) = \lfloor \frac{(n-1)^2}{4} \rfloor.$$

Conjectures and Problems

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Conjecture (P. Frankl, 1985)

If $k > s \geq 2$, $n \geq 2k - s + 1$, and n is sufficiently large, then

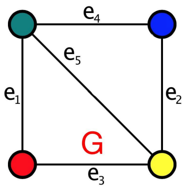
$$\chi(\mathrm{KG}(K_{n,s}, K_{k,s})) = |E(K_{n,s})| - \mathrm{ex}(K_{n,s}, K_{k,s}),$$

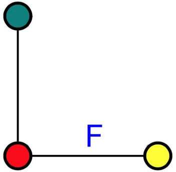
where the complete hypergraph $K_{n,s}$ contains all of s-subsets of $[n]$.

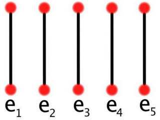
Definition (Alternating Turán Number)

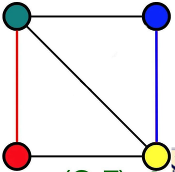
Assume that $\mathcal{F}$ is a family of graphs and G is a graph. Let σ be an ordering of $E(G)$. Define $\mathrm{ex}_{alt}(G,\mathcal{F},\sigma)$ to be the maximum number of edges of a spanning subgraph of G such that if we assign alternatively two colors red and blue to the edges of this subgraph (with respect to the ordering σ), then the red edges (resp. blue edges) form an $\mathcal{F}$-free subgraph of G. Set

$$\mathrm{ex}_{alt}(G,\mathcal{F}) = \min\{\mathrm{ex}_{alt}(G,\mathcal{F},\sigma) : \sigma \textit{ is an ordering of } E(G)\}.$$









$\mathrm{ex}_{alt}(G,F)=2$

Alternating Turán Number

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Lemma (M. Alishahi and H.H., 2013)

$$|E(G)| - \mathrm{ex}_{alt}(G,\mathcal{F}) \leq \chi(\mathrm{KG}(G,\mathcal{F})) \leq |E(G)| - \mathrm{ex}(G,\mathcal{F}).$$

Corollary (M. Alishahi and H.H., 2013)

If $\mathrm{ex}_{alt}(G,\mathcal{F}) = \mathrm{ex}(G,\mathcal{F})$, then $\chi(\mathrm{KG}(G,\mathcal{F})) = |E(G)| - \mathrm{ex}(G,\mathcal{F})$.

Definition (Strong Alternating Turán Number)

Assume that $\mathcal{F}$ is a family of graphs and G is a graph. Let σ be an ordering of $E(G)$. Define $\mathrm{ex}_{alt}(G,\mathcal{F},\sigma)$ to be the maximum number of edges of a spanning subgraph of G such that if we assign alternatively two colors red and blue to the edges of this subgraph (with respect to the ordering σ), then the red edges or blue edges form an $\mathcal{F}$-free subgraph of G. Set

$$\mathrm{ex}_{salt}(G,\mathcal{F}) = \min\{\mathrm{ex}_{salt}(G,\mathcal{F},\sigma) : \sigma \textit{ is an ordering of } E(G)\}.$$

Lemma (M. Alishahi and H.H., 2013)

$$|E(G)| - \mathrm{ex}_{salt}(G,\mathcal{F}) + 1 \le \chi(\mathrm{KG}(G,\mathcal{F})) \le |E(G)| - \mathrm{ex}(G,\mathcal{F}).$$

Corollary (M. Alishahi and H.H., 2013)

If $\mathrm{ex}_{salt}(G,\mathcal{F}) - 1 = \mathrm{ex}(G,\mathcal{F})$, then $\chi(\mathrm{KG}(G,\mathcal{F})) = |E(G)| - \mathrm{ex}(G,\mathcal{F})$.

Matching Graph $\mathrm{KG}(G, rK_2)$

Observation!

$\chi(\mathrm{KG}(nK_2, rK_2)) = n - 2r + 2 = |E(nK_2)| - 2\mathrm{ex}(nK_2, rK_2)$.

Theorem (M. Alishahi and H.H., 2013-2014)

If G is a sufficiently large dense graph or a sparse connected graph (with some conditions), then $\chi(\mathrm{KG}(G, rK_2)) = |E(G)| - ex(G, rK_2)$.

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Proof!

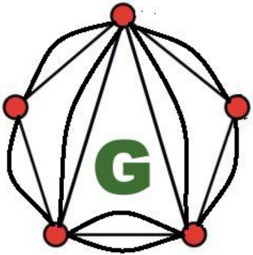
1. Present an appropriate ordering for $E(G)$.
2. In view of Tutte-Berge formula, we show that $\mathrm{ex}_{alt}(G, F) = \mathrm{ex}(G, F)$ or $\mathrm{ex}_{salt}(G, F) - 1 = \mathrm{ex}(G, F)$!

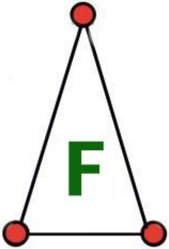
Theorem (M. Alishahi and H.H., 2013)

If G is a multigraph such that the multiplicity of each edge is at least 2 and F is a simple graph, then $\chi(\mathrm{KG}(G,F)) = |E(G)| - ex(G,F)$.

Proof!

1. Present an appropriate ordering for $E(G)$.
2. $\mathrm{ex}_{alt}(G,F) = \mathrm{ex}(G,F)$



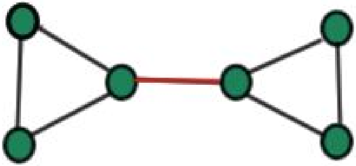


Theorem (M. Alishahi and H.H., 2014)

If G is a sufficiently large dense graph and $\mathcal{T}_n$ is the family of the spanning trees of G, then $\chi(\mathrm{KG}(G, \mathcal{T}_n)) = |\mathrm{MinimumCUT}(G)|$.

Proof!

1. Present an appropriate ordering for $E(G)$.
2. $\mathrm{ex}_{alt}(G, \mathcal{T}_n) = \mathrm{ex}(G, \mathcal{T}_n)$



$|\mathrm{MinimumCUT}(G)| = 1$

Theorem (M. Alishahi and H.H., 2013)

For any graph G, we have $\zeta(G) \leq \max\{n : K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor} \textit{ is a subgraph of } G\}$.

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Question

Is it true that for any two graphs G and H, $\zeta(G \vee H) \leq \zeta(G) + \zeta(H)$?

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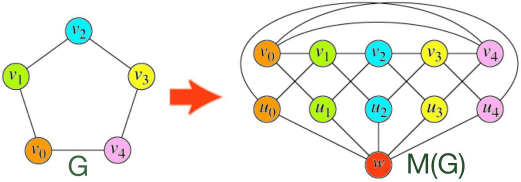
For any graph G, we have $\zeta(M(G)) \geq \zeta(G) + 1$.



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Thank You!