Graph Colorings via The Borsuk-Ulam Theorem

Hossein Hajiabolhassan

Department of Mathematical Sciences

Shahid Beheshti University
Tehran, Iran

Workshop On Topological Combinatorics
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Tehran, Iran

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Basic Idea

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Combinatorial object G

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Simplicial Map

- Let K and L be simplicial complexes, and let f be a map from vertices of K to L such that whenever the vertices $v_0, \ldots, v_t$ span a simplex of K, the points , $f(v_0), \ldots, f(v_t)$, are vertices of a simplex of L. We call f a simplicial map and it is easy to see that f can be extended to a continuous map $||f|| : ||K|| \longrightarrow ||L||$.

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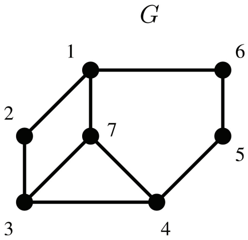

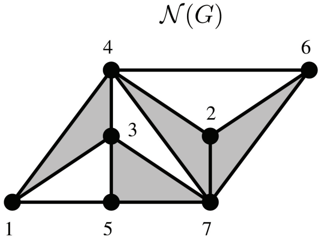

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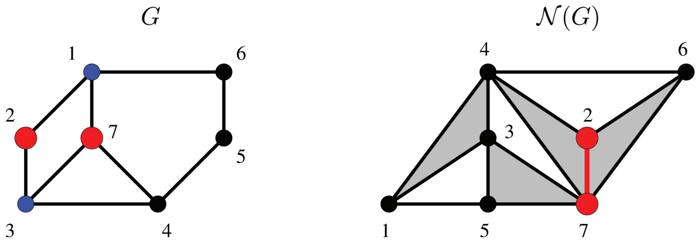

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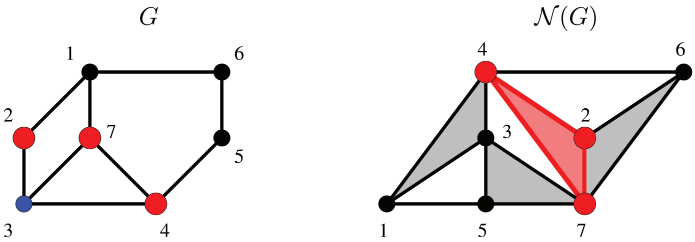

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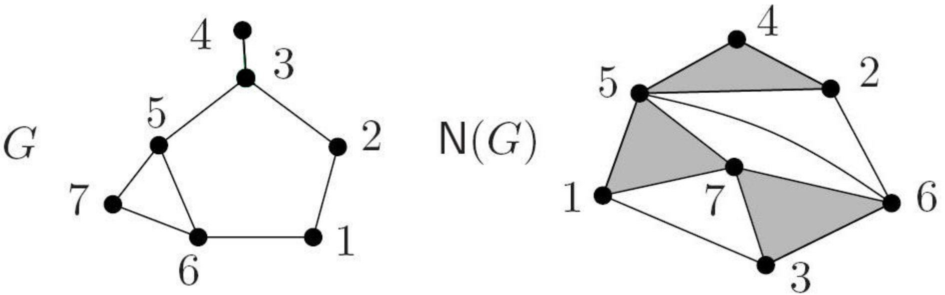

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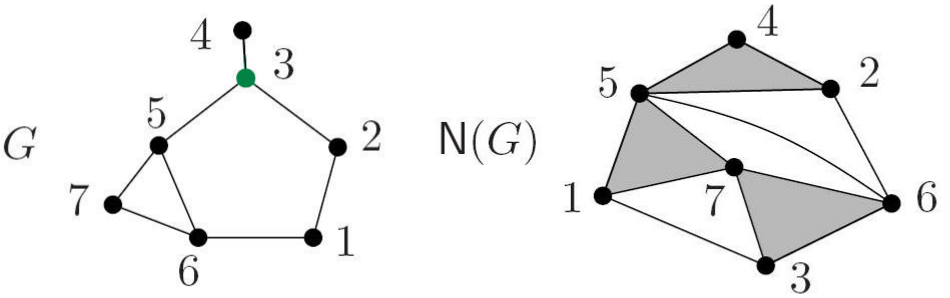

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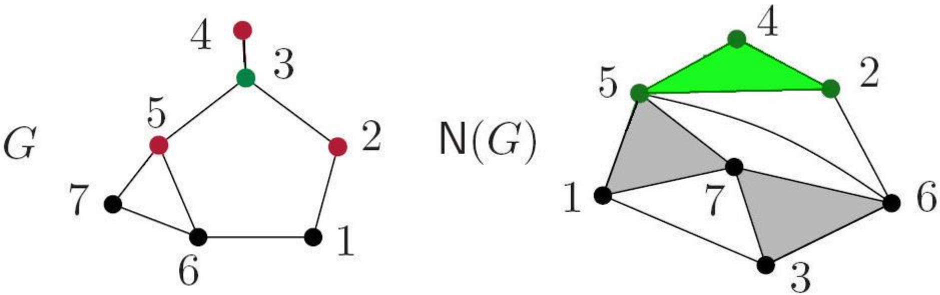

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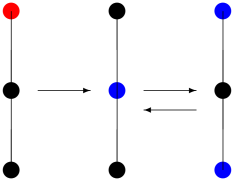

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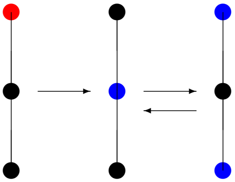

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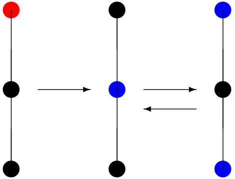

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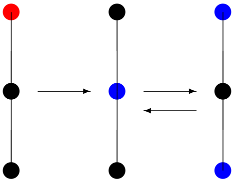

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- Let $L(G) \subseteq \mathcal{N}_1(G)$ where $V(L(G)) := \{CN(A) \neq \emptyset : A \subseteq V(G)\}$.
- $(L(G), CN)$ is a free Z_2-space.

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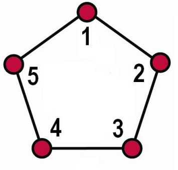

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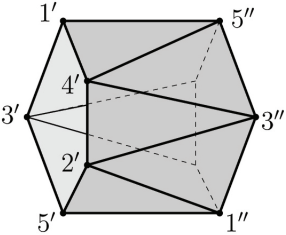

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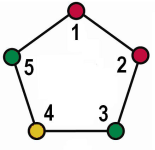

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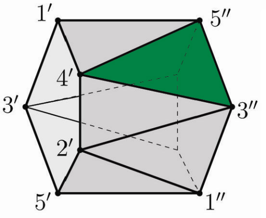

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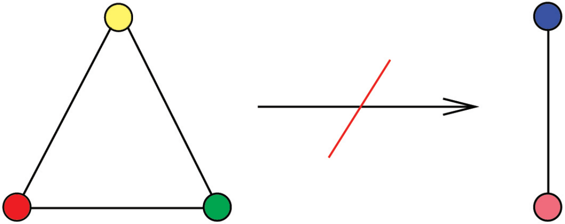

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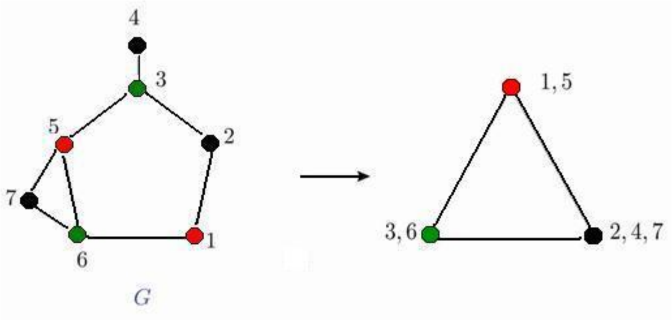

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- Local chromatic number:
  $\psi(G) = \min\{r | \exists n,\ \textit{there exists a homomorphism } \sigma : G \to U(n,r)\}.$

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- Let K and L be simplicial complexes, and let f be a map from vertices of K to L such that whenever the vertices $v_0, \ldots, v_t$ span a simplex of K, the points , $f(v_0), \ldots, f(v_t)$, are vertices of a simplex of L. We call f a simplicial map and it is easy to see that f can be extended to a continuous map $||f|| : ||K|| \longrightarrow ||L||$.

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- If there exists a homomorphism $f : G \longrightarrow H$ then there are simplicial maps $N(f) : \mathcal{N}(G) \longrightarrow \mathcal{N}(H)$ and $B(f) : B(G) \longrightarrow B(H)$.

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- $B(K_n) \overset{Z_2}{\leftrightarrow} S^{n-2}$ & $ind(B(K_n)) = n - 2$.
- (Lovasz, 1978) For any graph G, $\chi(G) \geq ind(B(G)) + 2$.
- (Sarkaria) For any graph G, $\chi(G) \geq ind(susp(B(G)) + 1$.

Colorful $K_{l,m}$ Theorem

- Let G be a finite graph with $t = coind(susp(B(G))) + 1$ and let c be an arbitrary proper coloring of G by colors $1, 2$, etc and let $A, B \subseteq [t]$ form a bipartition of the color set, i.e., $A \cup B = [t]$ and $A \cap B = \emptyset$. Then there exists a complete bipartite subgraph $K_{l,m}$ of G with sides L, M such that $|L| = l = |A|, |M| = m = |B|$, and $\{c(v) : v \in L\} = A$, and $\{c(v) : v \in M\} = B$. In particular, this $K_{l,m}$ is completely multicolored at c.

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Thank You!