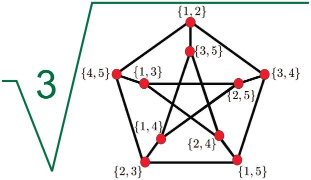

Graph Homomorphisms Through Graph Powers

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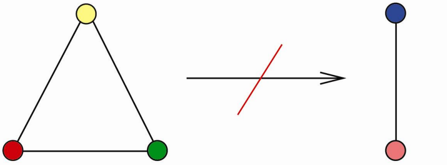
Graph Homomorphism

A homomorphism $f : G \longrightarrow H$ from a graph G to a graph H is a map $f : V(G) \longrightarrow V(H)$ such that if $uv \in E(G)$ then $f(u)f(v) \in E(H)$. Also, the existence of a homomorphism is indicated by the symbols $G \longrightarrow H$.

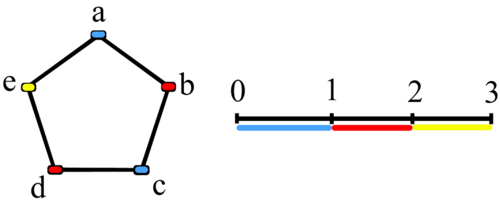
1. What is $\leq$? $G \leq H$ means that $G \longrightarrow H$.
2. What is $=$? $G \longleftrightarrow H$ ($[G] = [H]$) means that $G \longrightarrow H$ and $H \longrightarrow G$.
3. What is $<$? $G < H$ means that $G \longrightarrow H$ and there is no homomorphism from H to G.



Chromatic Number

Chromatic number

Let I be a line of length r. An r-coloring of a graph G is a mapping c which assigns to each vertex v of G an open unit length interval $c(v)$ of I, such that for every edge $uv \in E(G)$ we have $c(u) \cap c(v) = \emptyset$. The chromatic number of a graph , denoted by $\chi(G)$, is defined as,
$\chi(G) = \inf\{r : G \ admits \ an \ r - coloring\}$.

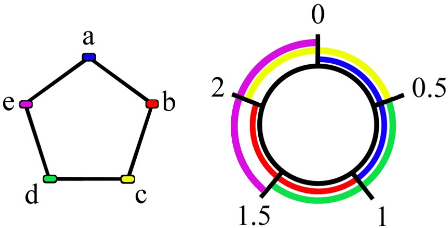


Circular Chromatic number

Let C be a circle of (Euclidean) length r. An r-circular coloring of a graph G is a mapping c which assigns to each vertex v of G an open unit length arc $c(v)$ of C, such that for every edge $uv \in E(G)$ we have $c(u) \cap c(v) = \emptyset$. The circular chromatic number of a graph , denoted by $\chi_c(G)$, is defined as, $\chi_c(G) = \inf\{r : G \textit{ admits an } r - \textit{circular coloring}\}$.

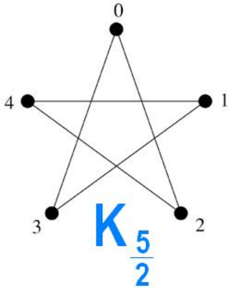
Andrew Vince

Xuding Zhu



Circular Complete Graphs

The circular complete graph $K_{\frac{p}{q}}$ has the vertex set $\{0, 1, \ldots, p-1\}$ and the edge set $\{ij : q \leq |i-j| \leq p-q\}$.



Circular chromatic number: $\chi_c(G) = \inf\{\frac{p}{q} |\ G \rightarrow K_{\frac{p}{q}}\}$

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G)$$

Power in Numbers and Graphs

Let G and H be two graphs and a, b, p, q, r and s be positive integers!

Power in Numbers:

1. Integer Power and Root Power: $a^{\frac{p}{1}}$ and $a^{\frac{1}{q}}$

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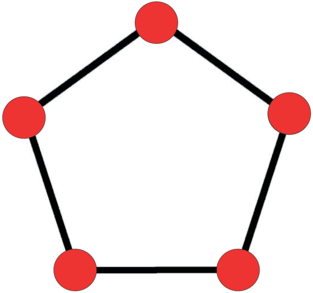
Graph Powers

For a graph G, let G^k be the kth power of G, which is obtained on the vertex set $V(G)$, by connecting any two vertices u and v for which there exists a walk of length k between u and v in G.



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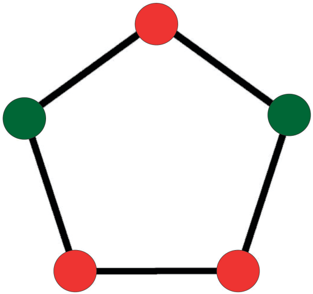

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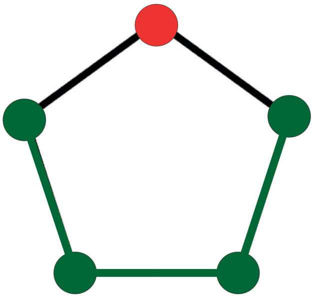

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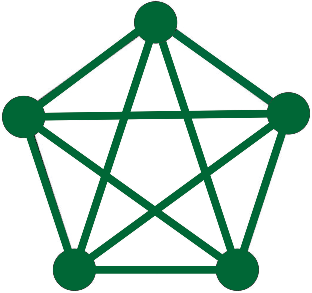

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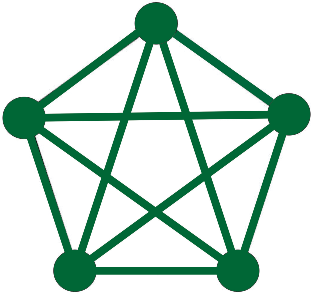

The adjacency matrix of G^k is obtained from the kth power of the adjacency matrix of G, by replacing any non-zero entries with one.

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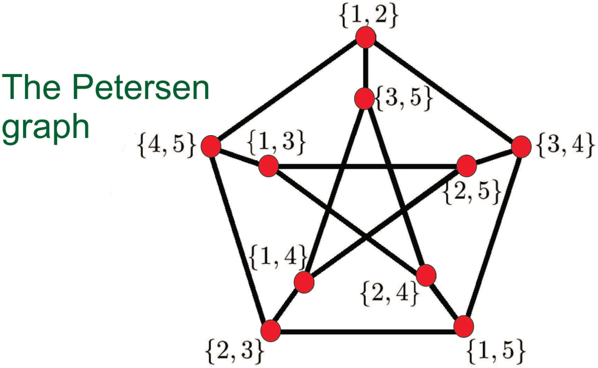

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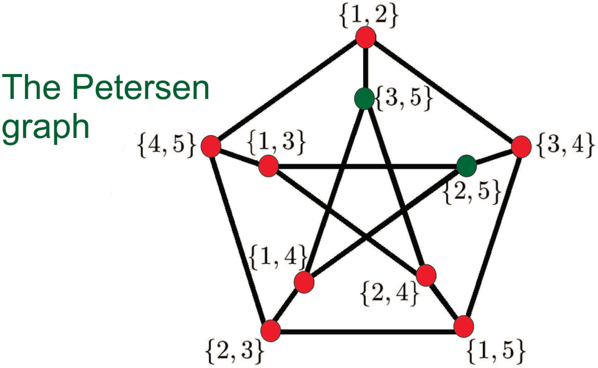

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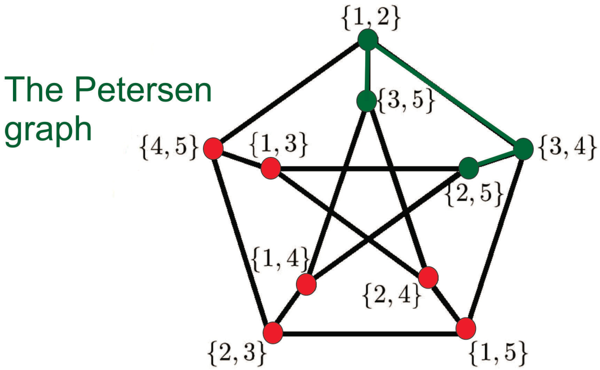

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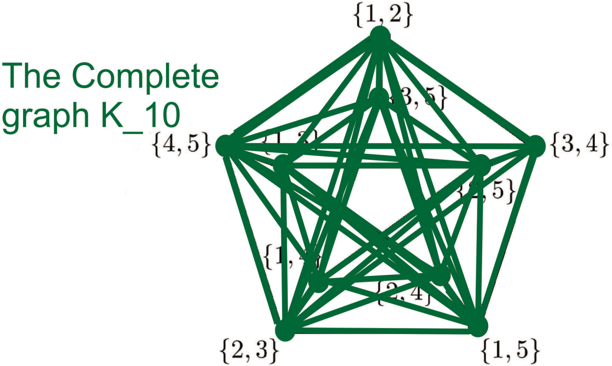

Graph Powers

Observation.

Let G and H be two simple graphs such that $\mathrm{Hom}(G, H) \neq \varnothing$. Then, for any positive integer k, $\mathrm{Hom}(G^k, H^k) \neq \varnothing$.

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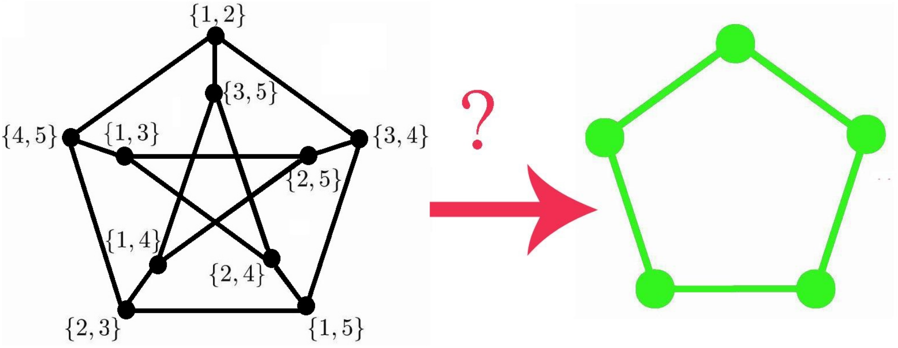

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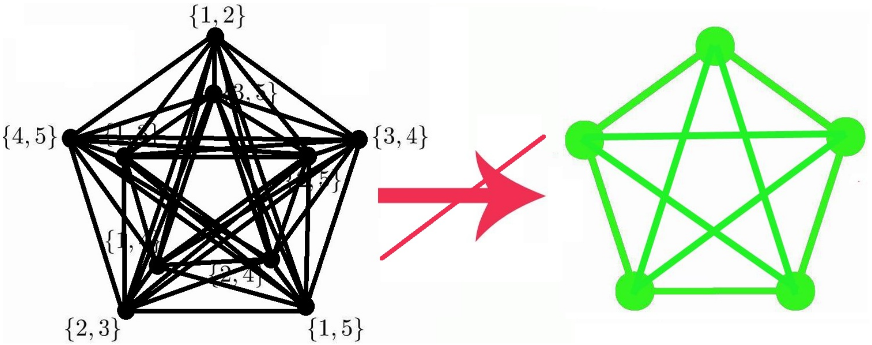
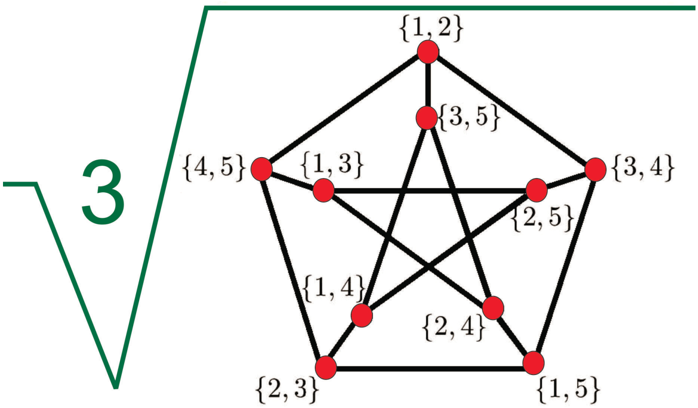


Fractional Powers



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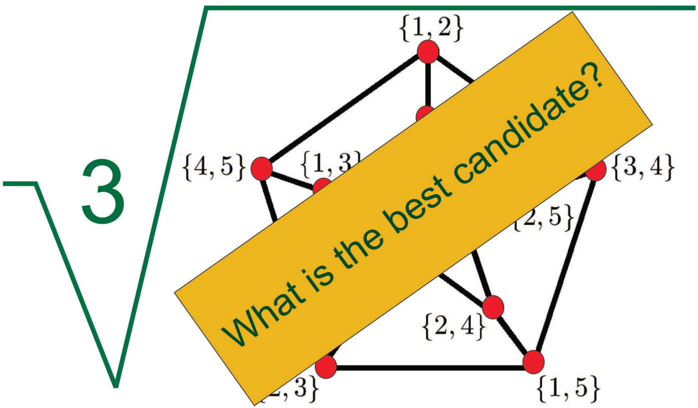

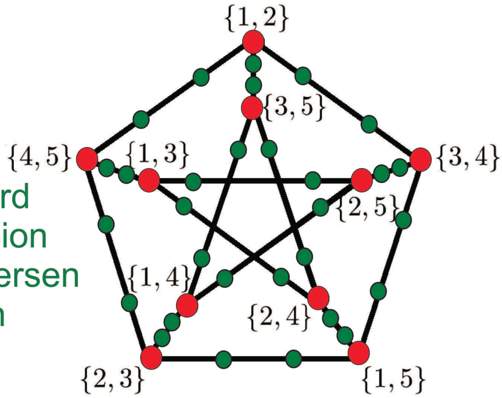

The third subdivision of the Petersen graph

Fractional Powers

1. The graph $G^{\frac{1}{r}}$ is said to be the r-subdivision of a graph G if $G^{\frac{1}{r}}$ is obtained from G by replacing each edge by a path with exactly $r-1$ inner vertices.

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Lemma.

Let G and H be two graphs. Then, $G^{\frac{1}{2s+1}} \longrightarrow H$ if and only if $G \longrightarrow H^{2s+1}$ ($G^{\frac{1}{2s+1}} \leq H$ if and only if $G \leq H^{2s+1}$).

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For any graph G, set $G^{\frac{r}{s}} := (G^{\frac{1}{s}})^r$.

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For any graph G, set $G^{\frac{r}{s}} := (G^{\frac{1}{s}})^r$.

3. A rational number r is called positive odd rational number if numerator and denominator are both positive odd integers.

Properties of Fractional Powers

H.H. and A. Taherkhani, 2010

Lemma. (Deletion Rule)

If a, b and s are positive odd integers, then $G^{\frac{sa}{sb}} \longleftrightarrow G^{\frac{a}{b}}$.

Observation.

Assume that r and s are positive odd rational number. It does not hold, in general, that $G^{rs} \longleftrightarrow (G^r)^s$.

Lemma.

If r and s are positive odd rational number, then $(G^r)^s \longrightarrow G^{rs}$.

Theorem. (The Same Base)

Let r and s be positive odd rational numbers and G be a non-bipartite graph. If $r < s < og(G)$, then $G^r < G^s$.

Observation.

$\omega(G^r)$ stands for the clique number of G^r where r is a positive odd rational number and $r < og(G)$. One can see that if r approaches to $og(G)$ from below, then $\omega(G^r)$ approaches infinity.

Blow Up a Graph

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Observation.

Let G and H be non-bipartite graphs. Then there exists a positive odd rational number $r \geq 1$ such that $H \leq G^r$.

Blow Out a Graph

Observation.

For any positive integer $a \geq 3$, there exists a graph homomorphism from $G^{\frac{1}{a}}$ to C_a.

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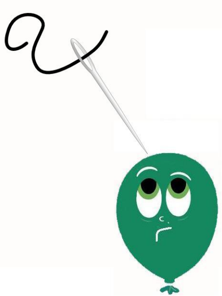

Theorem. (H.H., 2009)

For any graph G, $\chi(G^{\frac{2k+1}{3}}) \leq 3$ if and only if there exists a graph homomorphism from G to C_{2k+1}.

Odd Cycles

Theorem. (H.H., 2009)

For any graph G, $\chi(G^{\frac{2k+1}{3}}) \leq 3$ if and only if there exists a graph homomorphism from G to C_{2k+1}.

Nešetřil's Pentagon Problem, 1999

Let G be a cubic graph of sufficiently large girth, is it true that $\chi(G^{\frac{5}{3}}) \leq 3$?

Jaeger's Conjecture, 1981

Every planar graph with girth at least $4k$ has a homomorphism to C_{2k+1}.

Jaeger's Conjecture, 1981

Let P be a planar graph with girth at least $4k$. Then we have $\chi(P^{\frac{2k+1}{3}}) \leq 3$.

Observation.

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Question.

If G is a non-bipartite graph, $\chi(G^{\frac{1}{3}}) = 3$ and $\lim\limits_{r \to og(G)^-} \chi(G^r) = \infty$. Does it hold $\{\chi(G^r) |\ 0 < r < og(G),\ r \textit{ is an odd r. number }\} = \{3, 4, 5, 6 \ldots\}$?

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Definition. Power Thickness of Graph

Let G be a non-bipartite graph. What is the value of

$$\Theta(G) := \sup\{r | r = \tfrac{2t+1}{2s+1}, \chi(G^r) = \chi(G), r < og(G)\}?$$

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1. Let K_n be complete graph with $n \geq 3$ vertices. Then, $\theta(K_n) = 1$.

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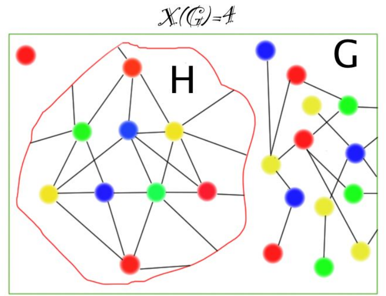
3. For any rational number $r \geq 1$, there exists a graph G with $\theta(G) = r$.

Definition.

Let G be a graph and $\chi(G) = k$. G is called a colorful graph if for any proper k-coloring of G, there exists an induced subgraph H of G such that for any $v \in V(H)$, all colors appear in the closed neighborhood of v.



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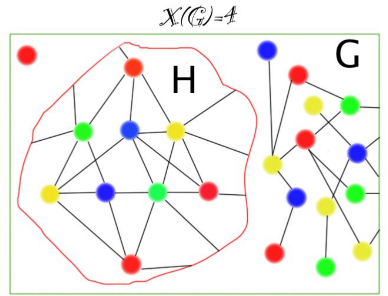

Observation.

If G is a uniquely colorable graph or $\chi(G) = \omega(G)$, G is a colourful graph. Find another example! Is the Petersen graph a colourful graph? Yes!

Theorem. (H.H. and Ali Taherkhani, 2010)

If G is a non-bipartite colorful graph, then $\theta(G) = 1$. Also, $\chi(G) = \chi_c(G)$.

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Theorem. (P.-A. Chen, 2011)

For any $m \geq 2n$, the Kneser graph $\mathrm{KG}(m, n)$ is a colorful graph.
Furthermore, $\chi_c(\mathrm{KG}(m, n)) = \chi(\mathrm{KG}(m, n))$.
Later, G.J. Chang, D.D. Liu, and X. Zhu (2012) have presented a short and interesting proof for this result.

Let G be a non-bipartite graph and Q_0 be the set of odd rational numbers.

Observation.

For any non-bipartite graph G, we have $\chi(G^{\frac{1}{3}}) = 3$.

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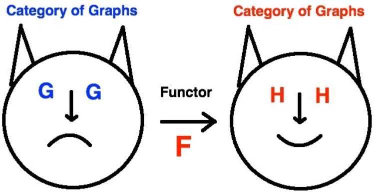
There exists a positive odd rational number $r > \frac{\chi(G)}{3(\chi(G)-2)}$ for which $\chi(G^r) = 3$ if and only if $\chi(G) \neq \chi_c(G)$.



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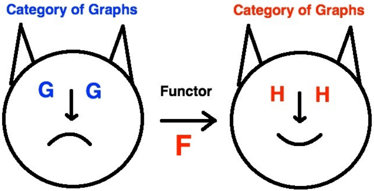

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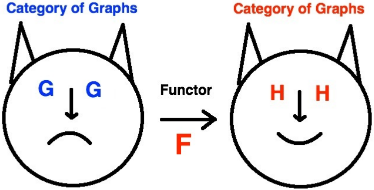

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3. For any graph G, we have

$$\chi_c(G^{\frac{1}{2s+1}}) = \frac{(2s+1)\chi_c(G)}{s\chi_c(G)+1}, \quad \chi_f(G^{\frac{1}{2s+1}}) \leq \frac{(2s+1)\chi_f(G)}{s\chi_f(G)+1}.$$

Dual Power

Theorem. (H.H. and A. Taherkhani, 2010)

Let G and H be two graphs. We have $G^{\frac{2a+1}{2b+1}} \longrightarrow H$ if and only if $G \longrightarrow H^{\widetilde{\frac{2b+1}{2a+1}}}$.

Dual Power

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Let G and H be two graphs. We have $G^{\frac{2a+1}{2b+1}} \longrightarrow H$ if and only if $G \longrightarrow H^{\overbrace{2a+1}^{2b+1}}$.

The dual power inherits (like power) some properties from power in number theory. But, for any graphs G and H, there is no reason that there exists a non-negative integer s such that $G^{\overbrace{2s+1}^{1}} \leq H$.

Theorem. (H.H., 2009)

Let m, n, and k be positive integers with $m \geq 2n$. Then $\chi(\mathrm{KG}(m,n)^{\overbrace{2k-1}^{1}}) = m - 2n + 2$. Moreover, if m is an even positive integer, then $\chi_c(\mathrm{KG}(m,n)^{\overbrace{2k-1}^{1}}) = m - 2n + 2$.

History of Dual Power

In 2004, A. Gyärfäs, T. Jensen, and M. Stiebitz defined the graph $K_n^{\frac{1}{3}}$ to show that there is a n-chromatic graph with an n-coloring where the neighbors of each color-class form an independent set.

A. Gyärfäs, T. Jensen, and M. Stiebitz

In 2005, C. Tardif defined the graph $G^{\frac{1}{3}}$ to show that the circular complete graph $K_{\frac{p}{q}}$, for $\frac{p}{q} < 4$, is a multiplicative graph, i.e., $G \times H \longrightarrow K_{\frac{p}{q}} \Longrightarrow G \longrightarrow K_{\frac{p}{q}}$ or $H \longrightarrow K_{\frac{p}{q}}$.

Claude Tardif

In 2006, G. Simonyi and G. Tardos showed that $K_n^{\frac{1}{2k+1}}$ is an n-critical graph.

Gabor Simonyi and Gabor Tardos

Problem. (Circular Chromatic Number of Planar Graphs)

In view of Four Color Theorem, one can conclude that for every planar graph P, we have $\chi(P^{\frac{3}{5}}) \leq 3$ which is equivalent to $\chi_c(P) \leq 4.5$. Prove it without Four Color Theorem!

Open Problems

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Question. (Power Thickness of Graph)

Let G be a non-bipartite graph. Set

$$\theta(G)_{\chi(G)-1} := \sup\{r | r = \frac{2t+1}{2s+1}, \chi(G^r) \leq \chi(G) - 1, r < og(G)\}.$$

Is it true that $\theta(G)_{\chi(G)-1} < 1$?

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Question. (Graph Homomorphism)

Suppose that G and H are two graphs. Also, assume that for any positive integer s, we have $G^{\frac{1}{2s+1}} \leq H^{\frac{1}{2s-1}}$. Is it true that $G \leq H$?

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