Graphs Homomorphisms and Spectral Conditions

Hossein Hajiabolhassan

Department of Mathematical Sciences

Shahid Beheshti University
Tehran, Iran
hhaji@sbu.ac.ir

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Definition

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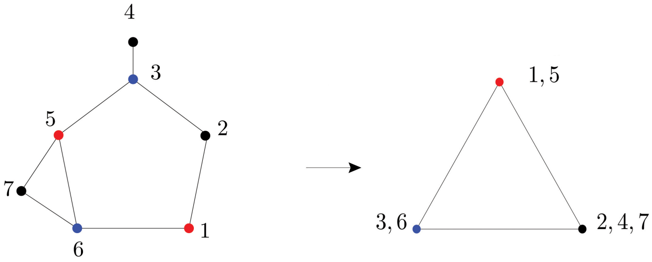

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- We order the eigenvalues of A_G, L_G, and T_G, respectively, as follows
  - $\alpha_1^G \geq \alpha_2^G \geq \cdots \geq \alpha_n^G,$
  - $\lambda_1^G \leq \lambda_2^G \leq \cdots \leq \lambda_n^G,$
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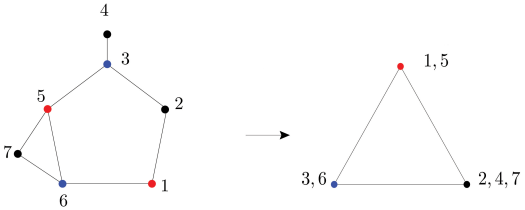

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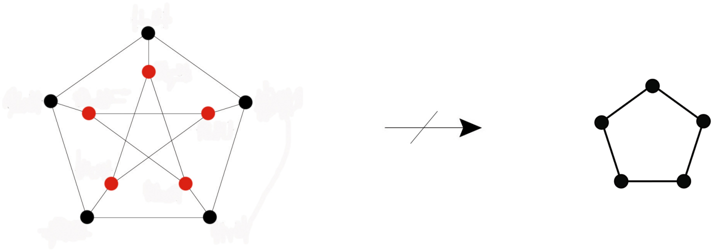

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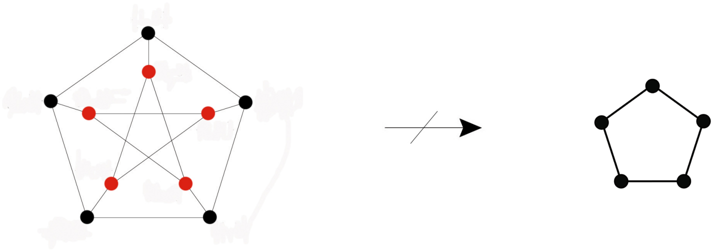

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- Let G be a graph with n vertices. If $\overline{d} = \frac{2|E(G)|}{n}$ then,

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$$b < v \quad \Rightarrow \quad \max\left(\frac{v}{\lambda_n^G}, \frac{v-2}{\mu_{n-1}^G}\right) \leq \frac{r}{\lambda} \leq \min\left(\frac{v}{\lambda_2^G}, \frac{v-2}{\mu_1^G}\right).$$

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Professor J. Nesetril & O. Serra
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Thank You!