Cyber-insurance & endogenous network formation

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Abstract. This paragraph shall summarize the contents of the paper in short terms.

1 Modeling Cyber-insurance

2 Model 3: Including maximum node degree and bonus

In real world networks, such as in the manufacturing industry, software development firms and many other types of business, in some scenarios a product can usually not be completed without outsourcing some of the work needed. For the manufacturer, it could be beneficial to buy certain parts from others instead of producing them on his own. A software product might need the combined knowledge from different firms. Thus the firm that outsources tasks is dependent on the other firms, and will not reach its goal before the other firms deliver their contribution. When the product is finished the company gets paid. To model this scenario we introduce a maximum node degree, m, per node, which represents the number of partners needed to complete a task. Additionally a bonus γ represents the payoff a node receives when m links are established. Except from this, the game is unchanged.

2.1 Analysis

This model is very similar to the earlier models: for nodes to connect to each other, the change in payoff still has to be positive: $U_{i+1} > U_i$. However, we also need to consider the bonus received when reaching the maximum node degree, m. To model this, we add the possible bonus divided on the number of links required to reach the bonus($\frac{\gamma}{m-i}$) in the decision process every time a node is considering to establish a link. In this way the model will change from the earlier models, because now the nodes have more incentive to connect to other nodes, and for every step closer to the goal, the nodes are more willing to accept risk than before. For example, an insured node is more likely to accept a risky link when it only needs one more link to reach the goal, compared to when it need several more links to reach the goal.

The model now introduces a risk factor, because it is not certain that the nodes will obtain enough links, and if not, they will not receive their bonus, and they are stuck with their already established connections.

To analyze this model, let us take a closer look at the four different scenarios of the game. When establishing a link between two insured nodes, the payoff the nodes will receive is as described in Eq.(1).

$$U_{i+1} = \begin{cases} \beta - I_l, & \text{if } i = 0 \\ U_i + \beta - I_l, & \text{if } i > 0 \\ U_i + \beta - I_l + \gamma, & \text{if } i = m-1 \end{cases} \tag{1}$$

As described earlier we need to include the possibility of reaching the goal in the decision, and thus for insured nodes to connect to each other, Eq.(2) has to hold.

$$\begin{aligned} U_i + \beta - I_l + \frac{\gamma}{m-i} &> U_i \\ \beta - I_l + \frac{\gamma}{m-i} &> 0 \\ \rightarrow \qquad \beta + \frac{\gamma}{m-i} &> I_l \end{aligned} \tag{2}$$

The payoff an insured node receives when connecting to a non-insured node is as follows:

$$U_{i+1} = \begin{cases} \beta - I_l - r, & \text{if } i = 0 \\ U_i + \beta - I_l - r, & \text{if } i > 0 \\ U_i + \beta - I_l - r + \gamma, & \text{if } i = m-1 \end{cases} \tag{3}$$

To establish a connection from an insured node to a non-insured one, the following has to hold:

$$\begin{aligned} U_i + \beta - I_l - r + \frac{\gamma}{m-i} &> U_i \\ \beta - I_l - r + \frac{\gamma}{m-i} &> 0 \\ \rightarrow \qquad \beta + \frac{\gamma}{m-i} - r &> I_l \end{aligned} \tag{4}$$

When a non-insured node connects to another non-insured node, this is the payoff they both will receive:

$$U_{i+1} = \begin{cases} \beta - r, & \text{if } i = 0 \\ U_i + \beta - r, & \text{if } i > 0 \\ U_i + \beta - r + \gamma, & \text{if } i = m-1 \end{cases} \tag{5}$$

To establish the connection, the following equation has to hold:

$$\begin{aligned} U_i + \beta - r + \frac{\gamma}{m-i} &> U_i \\ \beta - r + \frac{\gamma}{m-i} &> 0 \\ \rightarrow \qquad \beta + \frac{\gamma}{m-i} &> r \end{aligned} \tag{6}$$

In the case of a non-insured node wanting to establish a link with an insured node, the payoff is a strictly increasing function, see Eq.(7), and thus a non-insured node will always connect to an insured node if possible.

$$U_{i+1} = \begin{cases} \beta, & \text{if } i = 0 \\ U_i + \beta, & \text{if } i > 0 \\ U_i + \beta + \gamma, & \text{if } i = m-1 \end{cases} \tag{7}$$

2.2 Result and findings

If we want a clique of only insured nodes, we have to ensure that insured nodes connect to each other, and that they do not establish connections to non-insured nodes. We know that an insured node would want to connect to another insured node if Eq.(2) is satisfied. In the equation we see that the expected bonus per established link is increasing. Thus, if an insured node of degree zero is willing to connect to another insured node, then every insured node with a degree higher than zero would also like to connect to another insured node. To ensure that insured nodes connect to eachother this equation has to hold:

$$\beta + \frac{\gamma}{m} > I_l \tag{8}$$

We also want to ensure that insured nodes never establish links with non-insured nodes, from Eq.3 we see that this has to hold:

$$\beta + \frac{\gamma}{m-i} - r < I_l \tag{9}$$

This can be simplified, if one can ensure that the most risk willing insured node, i.e. the node with degree $m-1$, does not establish links with non-insured nodes. Then we know that no insured node with degree less than $m-1$ will establish links with non-insured nodes. From this we get equation Eq.(10).

$$\begin{aligned} \beta + \frac{\gamma}{m-(m-1)} - r &< I_l \\ \rightarrow \qquad \beta + \gamma - r &< I_l \end{aligned} \tag{10}$$

To summarize, Eq.(8) and Eq.(10) give the final limitation on the link insurance cost, Eq.(11). If this equation is satisfied, the resulting network will contain a clique of only insured nodes.

$$\beta + \gamma - r < I_l < \beta + \frac{\gamma}{m} \tag{11}$$

For this to even be possible, $\beta + \gamma - r < \beta + \frac{\gamma}{m}$, i.e. Eq.(12) has to hold. This equation reflects that as the risk to bonus ratio gets smaller, it gets more and more difficult to ensure a clique of only insured nodes. When the risk to bonus ratio is less than $1 - \frac{1}{m}$, a clique will never occur. The equation shows that a node would be more and more willing to take a risk, as the reward of doing so increases.

$$\begin{aligned} & \gamma - r < \tfrac{\gamma}{m} \\ & 1 - \frac{r}{\gamma} < \tfrac{1}{m} \\ \rightarrow \qquad & 1 - \frac{1}{m} < \tfrac{r}{\gamma} \end{aligned} \tag{12}$$

It is also useful to know when non-insured nodes connect to each other. This happens when Eq.(5) is satisfied. This equation is dependent on the node degree, and thus for the first link to be established from a non-insured node, the expected payoff has to be higher than the risk($\beta + \frac{\gamma}{m} > r$). If the risk is too high, then the non-insured node must establish links with insured nodes before it coulb be willing to establish risky links.

With these findings, an insurer can determine the outcome of the network formation game by adjusting the insurance cost parameter. If he wants a clique of only insured nodes Eq.(11) has to hold. However, it is easy to relax the condition, so that insured nodes only connect to, $j = 1, 2, 3..m$ non-insured nodes. This is done by changing Eq.(10) to $\beta + \frac{\gamma}{m-(m-j)} - r < I_l$, which gives us Eq.(13). An interesting result in this model is that due to the risk willingness among the nodes, the lower boundary, to ensure separation of insured and non-insured nodes, of the link insurance cost is higher compared to the one found in model 2.

Consequences of not reaching the required number of edges. When a node establishes a link, it does not know whether it will reach the maximum node degree, unless the current node degree is $m-1$. Hence the node might end up not reaching the desired goal. This can happen if there is not enough nodes willing to establish links. Consequently, nodes who do not reach their goal could end up with a payoff less than U_0.

$$\beta + \frac{\gamma}{j} - r < I_l \tag{13}$$

Efficiency and Stability. In this model, the incentive for establishing links is increased compared to model 2. Thus, to maintain a stable network with two cliques the cost of link establishment has to be increased. This increased incentive may result in a higher price of stability, but if every node has received its bonus, then the price of anarchy is 1. The price of anarchy is dependent on the number of nodes in both cliques, and whether its enough nodes for everyone to reach their maximum degree or not. If there are nodes that have not reached their maximum degree in both cliques, then the resulting network is not necessarily the most efficient, and we could be missing a potential payoff due to the cost constraint.

By introducing the maximum degree m we are limiting the problem of price of anarchy, because as long as m is less than the number of insured and number of non-insured nodes, there will be less links established compared to model 2, and overall fewer possible links between insured and non-insured. However, the bonus the nodes receive will contribute to inefficiency, because when nodes do not reach their maximum degree, the potential payoff that could be generated by allowing insured and non-insured nodes to connect, is greater than in model 2.

2.3 Simulation of the results

For the first simulation the parameters are set to the following: $\beta = 0.9, I_l = 0.7, r = 0.5, \gamma = 0.2$ and $m = 4$, in order to satisfy condition Eq.(11), and enable all nodes to reach their maximum degree.

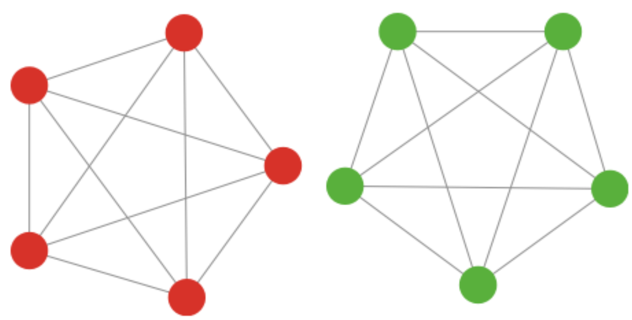

Fig. 1. Two cliques, one consisting of insured agents the other consists of non-insured. All nodes have reached their goal.

As we see in Figure 1, the results were as expected, the cost of insuring a link satisfied the conditions found earlier, and thus the result was two cliques, one consisting of only insured nodes and the other of non-insured nodes. An interesting observation is that β and r is the same as in model 2, but to ensure that only insured nodes connect to eachother, the link insurance cost had to be higher. This is to compensate for the risk the nodes now are willing to take. The price of anarchy in this scenario is 1, i.e. the socially optimal outcome.

In the second simulation we set the parameter $m = 5$, and kept the other variables unchanged. The resulting network was as expected the same as in the last simulation, but since the nodes did not reach their maximum degree, the price of anarchy is less than one. The price of anarchy can be seen in Eq.(14).

$$\begin{aligned} PoA &= \frac{\text{Sum of payoffs}}{\text{Sum of Socially optimal payoffs}} \\ PoA &= \frac{5\times4\times(0.9-0.7)+5\times4\times(0.9-0.5)}{5\times4\times(0.9-0.7)+5\times4\times(0.9-0.5)+5\times(2\times0.9-0.7-0.5+2\times0.2)} \\ PoA &= \frac{12}{17} \end{aligned} \tag{14}$$

When we changed the link insurance cost, and set it to the same value as in model 2, $I_l = 0.5$, the resulting networks change. Now we found that the insured nodes are willing to establish risky links to reach their maximum degree. Some of the resulting networks can be seen in Figure 2.3. In figure 2 the price of anarchy is 0.95, and in figure 3 the price of anarchy is 1, i.e. it has reached the socially optimal outcome.

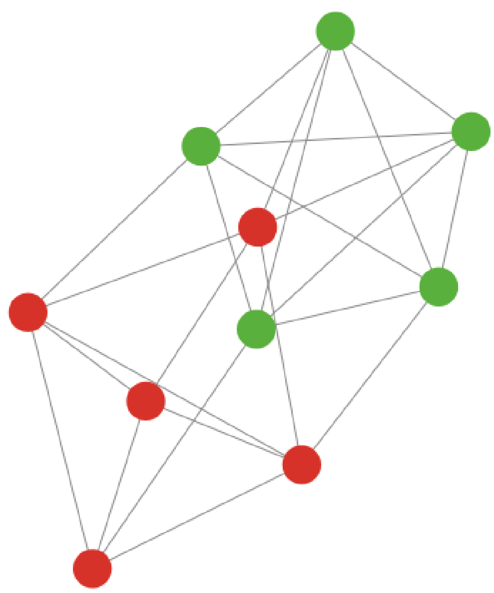

Fig. 2. One non-insured node has connected to two insured nodes.

[Two possible outcomes when insured nodes are willing to take the risk of connecting to non-insured nodes, to receive their bonus.] Two possible outcomes when insured nodes are willing to take the risk of connecting to non-insured nodes, to receive their bonus. Figure a shows a scenario where one non-insured node has connected to more than one insured node, thus not a socially optimal outcome. Figure b shows the optimal outcome.

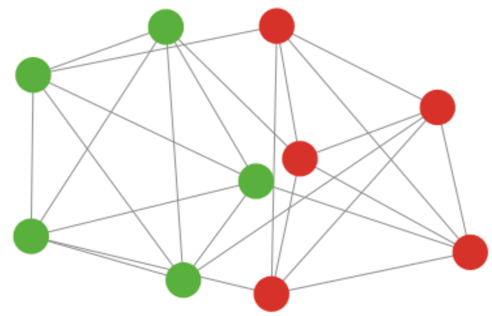

Fig. 3. Every non-insured node is connected to one insured node, this is the optimal outcome with these parameters.

3 Model 4: Including bulk insurance discount

Insurance companies often give a quantum discount when a customer purchases multiple products. From convenience stores, we are used to the slogan "buy one get one for free", and insurers tend to follow the same marketing strategy. It seems to be common for insurance companies to offer discount to their customers if the customers choose to combine some or all of their insurances with them. Several insurance companies in Norway, e.g. Sparebank 1 offers customers up to 25 % discount according to the following rules [**?**].

- 10% discount if the person has signed three different insurances
- 15% discount if the person has signed four different insurances
- 20% discount if the person has signed five or more different insurances
- Plus additional 5% discount if the person is a customer of the bank.

The insurance offered is intended for the individual market and includes among other things: travel insurance, household insurance, car insurance, house insurance, insurance of valuable items and yacht insurance.

Inspired by these kinds of discounts on insurance products, we would like to introduce a discount rate dependent on the degree of the node. In a real-world scenario where nodes have an option of acquiring insurance or not, this will make it more attractive for nodes with high degree to acquire insurance, and the discount could act as an incentive for other nodes to also acquire insurance. Therefore, this seems like a reasonable model to include. Since, if you have many links, you will pay less per link compared to a node having fewer links.

How insurance companies choose to formulate their discount rate might vary. One solution might be to follow a strict 5% discount per new connection, similar to the one from Sparebank 1, or let the discount follow a power law, or a log-function etc. We choose to follow a discount rule which directly reflects the node's degree.

3.1 Analysis

The price for adding a new link follows the equation:

$$\frac{I_l}{i+1} \tag{15}$$

Here, i is the node's current degree. This means that the more links a node establishes the cheaper the link insurance will be.

Discount model We start our analysis by applying the discount to model 2. As before we analyze the four different connection scenarios. However, because non-insured nodes do not pay any insurance, it is only the scenario where insured nodes connects to other insured nodes and insured nodes connect to non-insured nodes, that has changed compared to model 2.

When we consider links between insured nodes, we must add the discount to the conditions found in model 2. The condition for establishing links between two insured nodes is shown in Eq.(16).

$$\frac{I_l}{i+1} < \beta \tag{16}$$

For a link between insured and non-insured nodes to be established, Eq.(17) has to hold.

$$\frac{I_l}{i+1} + r < \beta \tag{17}$$

Result and findings For an insurer to be able to guarantee that the network ends up in a clique with only insured nodes, we must ensure that the most expensive link establishment, i.e. the first, to another insured node can be achieved. This gives us the same condition as in model 2, i.e. $I_l < \beta$. We also need to ensure that insured nodes do not connect to non-insured nodes, thus we get the final condition in Eq. (18), where N_I is the number of insured nodes in the network.

$$(N_I)(\beta - r) < I_l < \beta \tag{18}$$

This condition is very strong, because for it to be possible the following has to hold:$\beta - r < \frac{\beta}{N_I}$, and as the number of insured nodes gets higher this gets more and more unlikely. Thus by including bulk discount, the insurer is making it harder for himself to constrain the network formation. The reason for this is that the incentive for establishing links is higher than without discount, and therefore more links will be established.

Stability and efficiency If we compare the total payoff equation in this model, see Eq.(19), with the one in model 2 (Eq.(**??**)), we see that the cost for insured nodes has changed, and therefore the payoff achieved from links between insured nodes has increased, and so has the payoff received from potential links between insured and non-insured nodes. As we know, in a scenario where the insurer sets the cost, in a manner that makes the network end up in two cliques, the payoff received from links between insured and non-insured is zero. The potential payoff in a scenario where there are two cliques can be described like this: $(N_I N_{\overline{I}}\beta + N_I(-\sum_{i=N_I}^{N_{\overline{I}}-1} \frac{I_l}{i}))$, and as long as $(N_I N_{\overline{I}}\beta > N_I(-\sum_{i=N_I}^{N_{\overline{I}}-1} \frac{I_l}{i}))$ it would have been socially optimal to have a single clique of both insured and non-insured nodes. When the cost of establishing links decreases and the insurer forces the network formation to end up in two cliques, the price of anarchy will be higher compared to the price of anarchy in model 2. The reason for this is that the incentive for establishing links has increased, and thus for the insurer to be able to constrain the network formation, the cost of establishing links has to be higher.

$$U_{total} = (N_I(N_I-1)\beta - N_I \sum_{i=1}^{N_I-1} \frac{I_l}{i}) + (N_{\overline{I}}(N_{\overline{I}}-1)(\beta - r)) + (N_I N_{\overline{I}}\beta + N_I(-\sum_{i=N_I}^{N_{\overline{I}}-1} \frac{I_l}{i})) \tag{19}$$

Discount and Bonus model We also need to apply the discount to model 3. Still, the only scenarios that has changed in this model is the ones where insured nodes connects to other insured nodes or when insured nodes connects to non-insured nodes.

When insured nodes are considering to establish links with eachother, their payoff functions are as shown in Eq.(20).

$$U_{i+1} = \begin{cases} \beta - I_l, & \text{if } i = 0 \\ U_i + \beta - \frac{I_l}{i+1}, & \text{if } i > 0 \\ U_i + \beta - \frac{I_l}{i+1} + \gamma, & \text{if } i = m-1 \end{cases} \tag{20}$$

For insured nodes to connect to eachother Eq.(21) has to hold.

$$\begin{aligned} U_i + \beta - \frac{I_l}{i+1} + \frac{\gamma}{m-i} &> U_i \\ \beta - \frac{I_l}{i+1} + \frac{\gamma}{m-i} &> 0 \\ \rightarrow \qquad \beta + \frac{\gamma}{m-i} &> \frac{I_l}{i+1} \end{aligned} \tag{21}$$

When insured nodes are considering to connect to non-insured nodes, their payoff functions are as shown in Eq. (22).

$$U_{i+1} = \begin{cases} \beta - I_l - r, & \text{if } i = 0 \\ U_i + \beta - \frac{I_l}{i+1} - r, & \text{if } i > 0 \\ U_i + \beta - \frac{I_l}{i+1} - r + \gamma, & \text{if } i = m-1 \end{cases} \tag{22}$$

For this to happen Eq.(23) has to hold.

$$\begin{aligned} U_i + \beta - \frac{I_l}{i+1} + \frac{\gamma}{m-i} - r &> U_i \\ \rightarrow \qquad \beta + \frac{\gamma}{m-i} &> r + \tfrac{I_l}{i+1} \end{aligned} \tag{23}$$

3.2 Result and findings

We analyze the same scenario as in the other models, namely a clique of only insured nodes. The first step is to guarantee that insured nodes connect to eachother. To ensure that this happens, we need to find the condition for the lowest expected increase in payoff, i.e. at node degree zero. If nodes are willing to establish links at this point, then they will also be willing at all degrees higher than zero. At degree zero there is no discount on the insurance link cost, and thus if Eq.(8) from model 3 holds, insured nodes will connect to other insured nodes.

The condition for guaranteeing that insured nodes do not connect to non-insured nodes has changed, we know that if an insured node does not want to establish a link with a non-insured node at degree $m-1$, then neither will any insured node with degree lower than $m-1$ do so. From this we find the condition in Eq.(24)

$$\begin{aligned} U_i + \beta - \frac{I_l}{m} + \frac{\gamma}{m-(m-1)} - r &< U_i \\ \beta + \gamma - r &< \tfrac{I_l}{m} \\ \rightarrow \qquad m(\beta + \gamma - r) &< I_l \end{aligned}$$

This is a very strong condition, because the only way this can happen is if $\beta + \gamma - r < \frac{1}{m}$. This shows us that when the incentives for establishing links increase, it gets more and more difficult for the insurer to guarantee a clique of only insured nodes. The final condition for ensuring a clique of only insured nodes is shown in Eq.(24).

$$m(\beta + \gamma - r) < I_l < \beta + \frac{\gamma}{m} \tag{24}$$

The quantum discount results in an overall higher payoff for the insured nodes, since the cost of insuring a new link becomes cheaper. This means that the insured nodes will have a higher incentive to create links, making it harder for the insurer to separate insured and non-insured nodes.

We see that the problem of separating the two node types have increased compared to model 3, meaning that if we have a network where the insurer has managed to separate them, the price of anarchy is also higher compared to a similar scenario in model 3.

4 Model 5: Network externalities

In the earlier models, the experienced network effects arose only from a node's neighbours. I.e. when a node established a connection the change in utility were only dependent on fixed variables, and not dependent on the rest of the network. In many real world scenarios it is more realistic that a node will be strongly affected by the indirect connections to other nodes. Social relationships between nodes are good examples of such networks, where each person offer benefits in terms of favors, information etc.

We apply the results from the paper from Jackson and Wolinsky [**?**] and use a network formation game found in [**?**] to study indirect network effects in our model.

The benefits a player receives in this game are calculated as follows: In addition to the benefit from the direct connection, a node will also benefit from "friends of the friend", and "friends of the friends of the friend" etc. This is achieved by letting the payoff be calculated relative to the distance between the nodes. β now depends on the minimum number of hops to the node, e.g. the benefit of a direct connection is β, the benefit of a friend of a friend is β^2 etc. We want the benefit to decrease with distance, therefore we need the limitation: $0 < \beta < 1$.

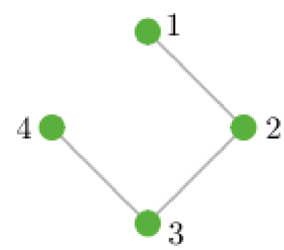

Fig. 4. Four nodes interconnected with each other.

Example: Let us consider the network shown in 4. Node 1 and node 4 in the network will receive a benefit of $\beta + \beta^2 + \beta^3$ by being connected with nodes 2 and 3. $\beta^2 + \beta^3$ represents the indirect benefits from nodes 3 and 4. Nodes 2 and 3 receive a benefit of $\beta + \beta + \beta^2$. For this network to make sense, it is important to also include some cost of having direct connections, or else the rational thing would be to establish a link with everyone. This is done as in earlier models, every node pays a cost for direct connections, but no cost for indirect connections. Thus the total payoff for a node is:

$$\sum_{j \neq i} \beta_{ij}^{d(ij)} - \sum_{j:ij \in g} c_{ij}, \tag{25}$$

Where $d(ij)$ represents the shortest path between node i and node j, and c_{ij} represents node i's cost of establishing a link between the two nodes. To simplify the model we choose a symmetric connection process where β and c is set to a fixed global value.

In the paper [**?**], the authors analyze the networks with two different approaches, one with focus on efficiency and the other on stability. The optimal network is of course both efficient and stable, but as we shall see there are some conflicts between efficiency and stability. Matthew, et.al. showed that an efficient network is:

1. *a complete graph g^N if $c < \beta - \beta^2$,*
2. *a star encompassing every node if $\beta - \beta^2 < c < \beta + \frac{(N-2)}{2}\beta^2$,*
3. *an empty network (no links) if $\beta + \frac{(N-2)}{2}\beta^2 < c$.*

The most efficient structure is a star structure which encompasses every node. A star structure has the characteristics of minimizing the average path length and uses the minimum number of links $(N-1)$ required for including every node. This structure provides the highest overall payoff for the network, but this network is not necessarily stable.

When analyzing the stability of the network, by using the definition of pairwise stability, Jackson and Wolinsky found four different stability conditions:

1. *a pairwise stable network consists of at most one (non-empty) component,*
2. *if $c < \beta - \beta^2$, the unique pairwise stable network will be a complete graph g^N,*
3. *if $\beta - \beta^2 < c < \beta$, a star encompassing every node will be pairwise stable, although not necessarily the unique pairwise stable graph,*
4. *if $\beta < c$, any pairwise stable network which is nonempty is such that each player has at least two links and is thus inefficient.*

We see that stability condition 2 is the same as efficiency condition 1, and therefore if this condition is fulfilled, the network is both stable and efficient. Condition 3 shows us why the efficient star network is not necessarily stable. If $\beta \leq c < \beta + \frac{(N-2)}{2}\beta^2$ then the efficient network will be a star, but it is not stable.

It should be noticed that it is more beneficial for a node to operate as a leaf node compared to being a center node, due to the cost of direct connections. In a star structure, a leaf node will only have to pay the cost of the link to the center node, and will benefit indirectly for each node connected to the center node. The center node will benefit from each new connection, but, the payoff will only be $\beta - c$ for each connection.

4.1 Insurance and connection game

The findings about efficiency and stability are very useful for our model, because if one has knowledge of the different variables it is possible to determine how the network will evolve. Additionally, if you are able to control the variables, you can actually determine the resulting network structure. From the referenced papers, we know that different boundaries on the link cost exists, and how the resulting stable and efficient network will be. Our earlier models show that the cost of establishing a link is the insurance cost and/or the risk cost. From this we can show that if $\beta - \beta^2 < I_l < \beta$ and $r > \beta$, a star with only insured nodes, and no connections between non-insured nodes, is both a stable and an efficient network. If $\beta - \beta^2 < I_l + r < \beta$ and $\beta - \beta^2 < I_l$ and $\beta - \beta^2 < r$, the stable and efficient network is a star consisting of both insured and non-insured nodes. If $I_l < \beta - \beta^2$ all insured nodes will connect to every other insured node, and if $r < \beta - \beta^2$ all non-insured nodes will connect to every other non-insured node. In addition if $r + I_l < \beta - \beta^2$ the resulting network will be a clique of both insured and non-insured nodes. The insurer can thus determine the formation of the network by adjusting the cost parameters.

4.2 Homogeneous symmetric connection game

From this point on, the game we will consider is a homogeneous network setting, where every node is considered to be insured. This is done because it will simplify an otherwise very complex model. We are analyzing the resulting network structure, which is easier when only considering one homogeneous cost for every node. Let us look at an example, where the parameters are set to: $\beta = 0.9, I_l = 0.5$. The resulting network from a simulation is shown in Figure 5.

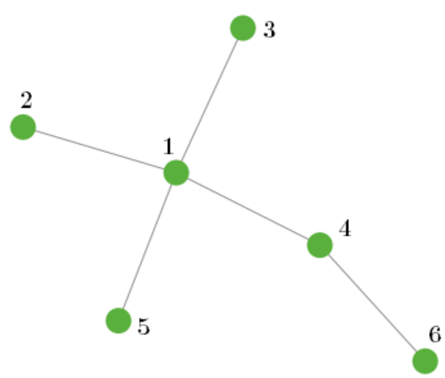

Fig. 5. The resulting network after a simulation with the parameters $\beta = 0.9, I_l = 0.5$.

As we see this is not an efficient star, but the network is stable. The efficient network would be to delete the link $4, 6$ and adding the link $1, 6$. But since we only consider one link at a time this can not be done. To show this let U_i denote the payoff of node i, the payoffs of the nodes are as described in Eq.(26).

$$\begin{aligned} U_1 &= 4\beta + \beta^2 - 4c \\ U_2 = U_3 = U_5 &= \beta + 3\beta^2 + \beta^3 - c \\ U_4 &= 2\beta + 3\beta^2 - 2c \\ U_6 &= \beta + \beta^2 + 3\beta^3 - c \end{aligned} \tag{26}$$

Node 6 would benefit from adding the link $1, 6$, but node 1 is not willing to do so, because then it must pay an extra cost, and since $\beta^2 > \beta - c$, the network is stable, but not efficient.

From this we see that, even when the most efficient and stable network is a star, we can not guarantee that the network formation game will end up in a star. This is because we only consider one link at a time, and not the whole network.

A star is not possible with high n. In the paper [**?**] the authors came up with the following proposition: Consider the symmetric connections model in the case where $\beta - \beta^2 < c < \beta$. As the number of nodes grows, the probability that a stable state (under the process where each link has an equal probability of being identified) is reached with the efficient network structure of a star goes to zero. But if a network reaches the efficient star structure, it is also pairwise stable, and will remain a star. We confirmed this when running multiple simulations. When we used few nodes the resulting network often became a star, but as the number of nodes increased the network rarely became a star.

However, the structure of the networks that evolve is very similar to a scale-free network. There are many nodes with low node degree, and few with a high node degree. One example of this is shown in Figure 6. There are only ten nodes, but the network has the properties of a scale-free network. Two nodes have a degree of 4, and the rest have a degree of one or two.

Bulk insurance. As noted before it is not preferable to be the center node, due to the cost of all the direct links. In a model with bulk insurance discount, the extra cost for the center node would decrease significantly. This could be used to increase the probability of reaching a star formation.

Using the discount formula from the previous model, we end up with Eq.(27) to achieve an efficient and stable star topology. i represents the node degree.

$$\beta - \beta^2 < \frac{i_l}{i+1} < \beta \tag{27}$$

An interesting property of the discount model is that the conditions for efficient and stable networks will change. Because when the node degree increases, the insurance cost might reach the critical degree g, and the best strategy for a node with degree g or higher, is to connect to every node, as shown in Eq.(28). The critical degree occurs when a node's optimal strategy changes from relaying on indirect connections to connecting to every node.

$$\frac{I_l}{g} < \beta - \beta^2 \tag{28}$$

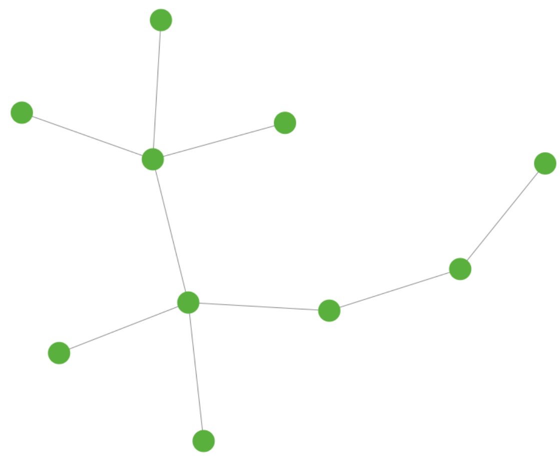

Fig. 6. The resulting network after a simulation with the parameters described earlier and 10 nodes.

This is possible when $g < n$, where n represents the number of nodes in the network. The stability condition has changed for a node with a critical degree. The stable and efficient condition for this node is, as shown earlier, to have a direct connection to every other node. Thus if we have a star topology, both the leaf nodes and the center node are stable, and the center node has been compensated for its role in the network.

Since the networks formed are similar to scale-free networks, we can calculate the probability of a node having degree g, see Eq.(29). γ is the power law parameter, as described in Chapter 4.

$$P(g) = g^{-\gamma} \tag{29}$$

When a node i reaches the critical degree g its optimal strategy is to connect to every node, since the payoff generated from direct connections is larger than any indirect connection. In general, nodes prefer to connect to nodes with high connectivity[3], and will thus prefer to connect to this node compared to nodes with a degree lower than g. In this way, nodes will connect to the node who has a degree greater than or equal to g, and remove the links to their low-degree nodes which they can instead reach through the node with high connectivity.

Let us consider a case with n nodes, and two of these nodes, i and j, have an equal degree larger than g. The rest of the nodes have a degree of one or zero. If there exists a node with degree zero, it would prefer to be connected to i or j, and so will i and j, so this will eventually happen. If a node connected to i is considering connecting to j, or vice versa, it will do so because j can offer a higher connectivity than i. Now j has a higher degree than i, and thus every

[3] A node with high degree implies a node with high connectivity.

node would prefer to connect to j over i. This will eventually result in a star formation, with j as the center node. From this we get the conjectures:

Conjecture 1 *If the critical degree ratio is low, i.e. the ratio between critical degree and number of nodes in the network, the resulting network will with high probability be a clique.*

Conjecture 2 *If the critical degree ratio is at a medium level, the resulting network will with high probability be a star.*

Conjecture 3 *If the critical degree ratio is high, the resulting network will with high probability be a star-like/scale-free structure.*

A numerical example of the boundaries between the different structures, we found from our simulation (described in the next section) with 20 nodes is the following: As seen in Figure 7 a critical degree of 1-5 applies to conjecture 1, 6-12 applies to conjecture 2 and 13-20 applies to conjecture 3.

Results and findings To prove the conjectures above, we created a simulator. The rules of the simulator are as following: Every round of the game,two random nodes, not neighbors, are selected, and asked if they would want to establish a link. The link establishment is a symmetric decision, i.e. the link is established if it result in an increased payoff for both nodes. If the link is added, we check if either of the nodes would prefer to delete some of their already existing links, this decision is asymmetric. A link will be deleted if the node will achieve a higher payoff without it. Then we ask the rest of the nodes if they would like to delete any links. This procedure is repeated as long as it is possible to add new links. The payoff function of each node is as described earlier (see Eq.(25)), except that the cost is now dependent on the degree of the node. For the simulations to be realizable, we had to set the number of nodes to 20, or else the computational time would be to high. For every critical degree, from three to nineteen, we ran 50 simulations, and noted the resulting network formation. We chose to start from critical degree equal three, since any number below would result in a clique, because it would be more beneficial to be directly connected to every node.

We know that if Eq.(27) is satisfied for all i, then the efficient and stable state is a star. But a more interesting scenario occurs when we have a graph where one or more of the nodes reaches the critical degree. -Will the final structure be scale-free, a star or simply just unstructured? The results from the simulation can be seen in Figure 7, 8 and 10. As we see from Figure 7, the probability of the resulting network being a star suddenly increases from zero to 42% at critical degree five to six, and then jumps from 42 to 70-, 86-,96-, 98% at critical degree six to nine. These results confirm our conjectures, and show that the discount can drastically increase the probability of the network ending up in a star.

From Figure 8 we can observe that the opposite is happening when the critical degree is increased; the probability of the resulting network being a clique drastically decreases. As we can see with a critical degree of seven or

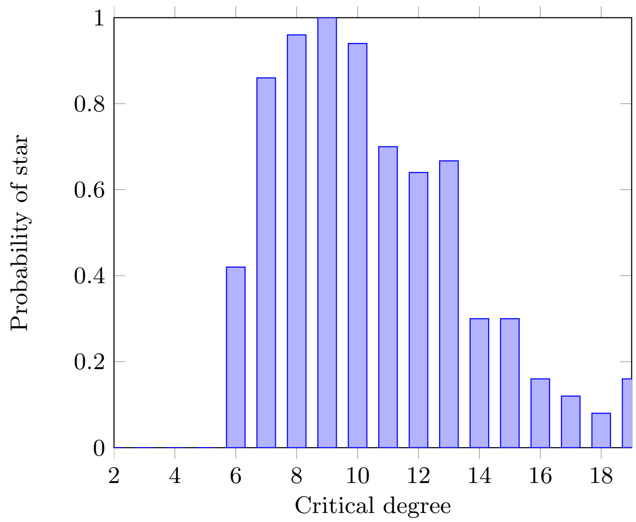

Fig. 7. Shows the probability of the network ending up in a star, given different critical degrees.

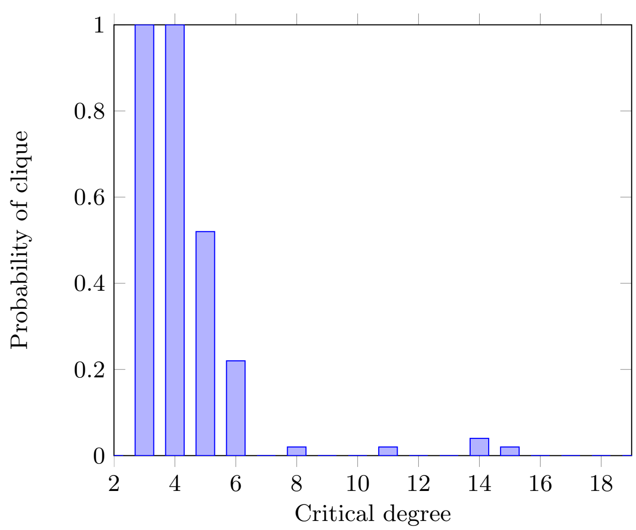

Fig. 8. Shows the probability of the network ending up in a clique, given different critical degrees.

higher, it is very unlikely that we end up with a clique. These findings support our conjectures.

An interesting comparison can be made between the emergence of a star versus a clique. Figure 9 shows a plot of the network resulting in a star and another plot of the probability for the resulting network to become a clique. As we can see, from a critical degree of five to seven, the resulting network structure, changes from almost certainly ending up in a clique, to almost certainly ending up in a star structure. The reason is as mentioned before that when the critical degree is low, the likelihood of many nodes reaching the critical degree is high. And none of these would like to delete any links. Hence we end up with a clique. The reason why we end up with star structures is because it is less likely that many nodes end up reaching the critical degree, hence most of the nodes still prefer to rely on indirect links, but the ones that reach the critical degree prefer to connect to everyone. Since the nodes with critical degree, have high connectivity, nodes will prefer to be connected with these, compared with other nodes. Nodes prefer to be connected to the ones with critical degree, the nodes with critical degree would like to connect to everyone, and thus the structure evolves into a star, with the critical degree node in the center.

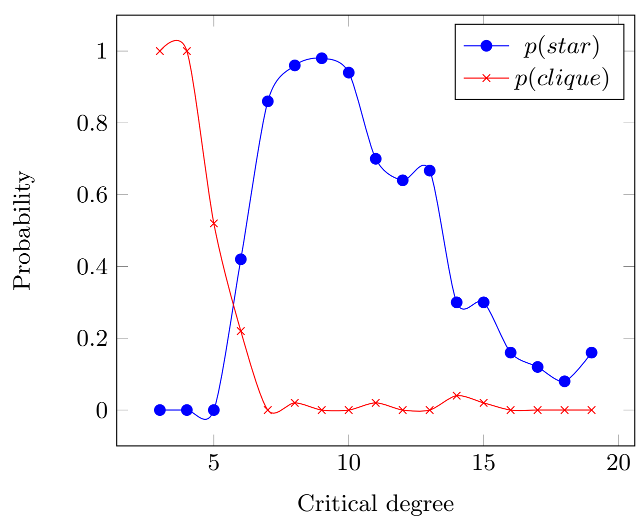

Fig. 9. Shows the comparison between the probability of the network ending up in a star (blue) or clique (red), given different critical degrees.

In Figure 7 when the critical degree gets closer to the number of nodes in the network, the probability of the network evolving into a star decreases. However, in Figure 10, we have plotted the probability of the network evolving into a network where only a few(2-4) nodes end up with a high degree, but not necessarily a critical degree. As we see, this occurs with high probability from critical degree six and up. These networks are so called scale-free networks (A-B graphs, described in the methodology chapter), because there are a few hubs, that account for most of the connectivity in the network. The reason why we end up with a scale-free network is because nodes prefer to be connected with nodes with high connectivity, and thus will delete links to nodes with low connectivity. This is very similar to the simple model that creates scale-free networks, where the probability of connecting to a node is proportional to the degree of the node.

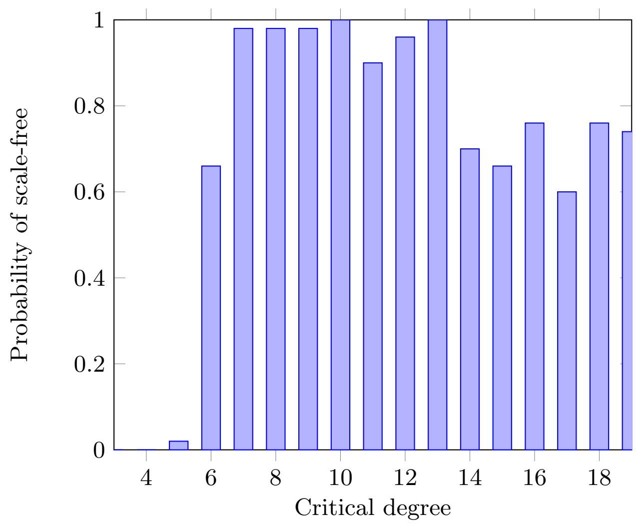

Fig. 10. Shows the probability of the network ending up in a scale-free structure, given different critical degrees.

Price of Anarchy. Another interesting thing is the average price of anarchy as function of the critical degree. The price of anarchy was calculated by taking the average total payoffs and dividing on the optimal payoff. The result can be seen in Figure 11.

We see that the price of anarchy for the first critical degrees is 1, and then decreases until degree six, and at seven it increases again. This is because at degree one to five, the socially optimal structure is a clique. At degree six, a clique and a star, are almost equally good, and at degree seven and up, a star structure is the socially optimal outcome. In other words, when the cost is low, a clique is the optimal structure, and when the cost is high a star is the optimal structure.

This further improves our findings, because we have now shown how an insurer can determine the resulting network formation by changing the cost. In addition, the formation that evolves has a price of anarchy close to 1.

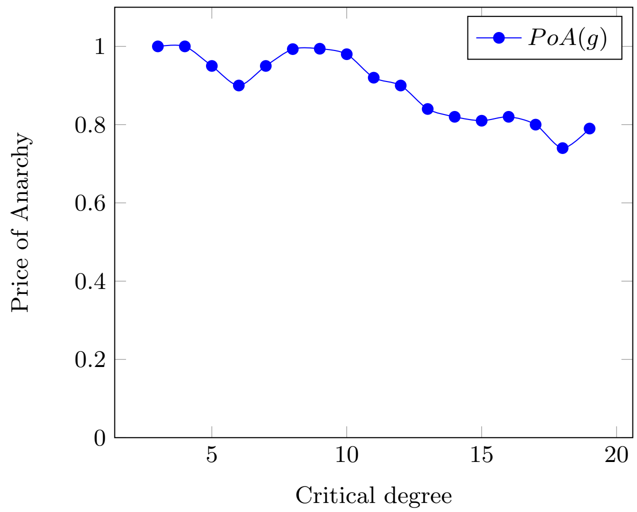

Fig. 11. Shows the price of anarchy as a function of critical degree

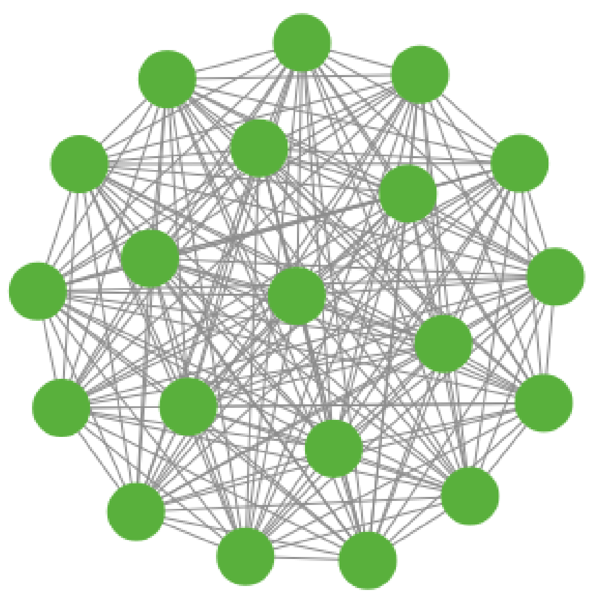

Fig. 12. A clique consisting of twenty nodes.

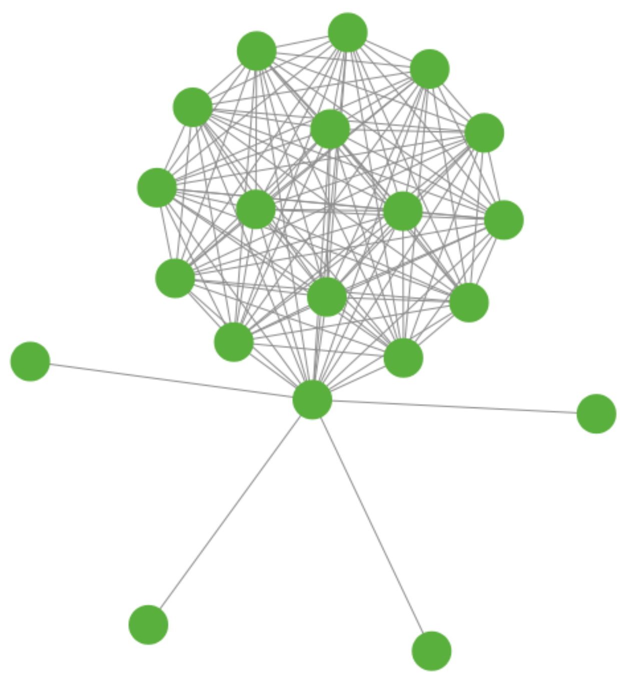

Fig. 13. A network with high average node degree, but not a clique.

Example structures from the simulation. Two different outcomes of the simulations where the critical degree is low

In Figure 4.2 we see two of the many possible outcomes when the critical degree is achieved at a low node degree. As we see, most of the nodes have reached the critical degree, and thus connected to every other node. In Figure 4.2 we see one example of a scalefree network, and the standard star network, both with twenty nodes, and results from the simulations when the critical degree was set to a value above six.

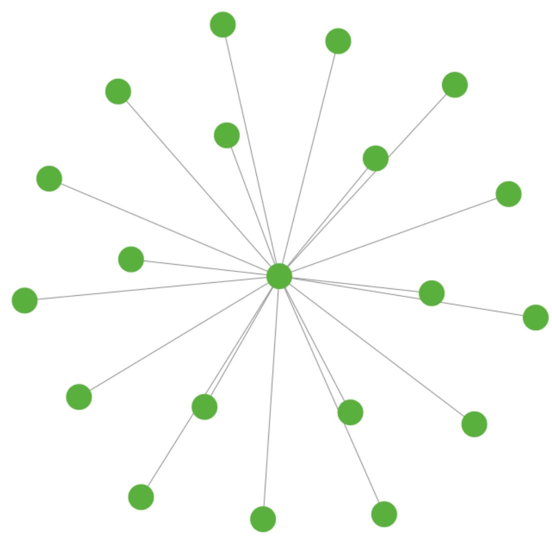

Fig. 14. A star consisting of twenty nodes

Two different outcomes from running simulations with a high critical degree.

5 Discussion

From our background study, it was revealed that the current market for cyber-insurance is far from healthy, and many have failed in attempts to establish a cyber-insurance market, also here in Norway. As described in the introduction, there are certain obstacles that are unique for cyber-insurance, and arguably these are the reasons why cyber-insurance has not emerged as expected. However, we believe that there is a need for cyber-insurance, and that our new approach of analyzing the cyber-insurance market through graphs and network formation games could help establishing and improving the market.

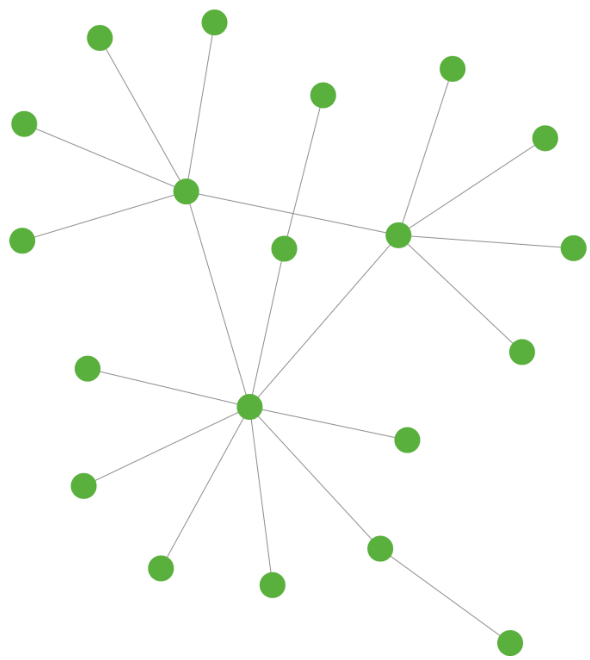

Fig. 15. A scalefree network with twenty nodes, where three nodes account for most of the connectivity.

We studied a variety of different network formation games, in order to find out if there were any superior network topologies that would fit as a cyber-insurance network, were ideally both the insurer and customers get a higher payoff from purchasing cyber-insurance. We found that star and clique networks had appropriate characteristics, not only do they have calculable fixation probability, but they could also generate better security and overall higher payoff for the nodes. With these networks in mind, we wanted to find a way of forcing networks to evolve into these structures. We found that insurers could adjust the insurance premium in order to control the formation of networks. If the price is set to the right level, networks with calculable risk will evolve, and if the insurer is able to separate the nodes into two different networks, one consisting of trusted, insured nodes, the other of non-insured nodes, the trusted nodes can even further increase their payoff, compared to a non-trusted network. The insurer now possesses a tool for setting the insurance premium properly, possible resulting in better products for both the customer and the insurer.

We created several different models, where the first model showed a very simple and naïve way for the insurer to separate insured and non-insured nodes into two cliques. To make the model more applicable to real-world scenarios, we created several models, and for each model we added some new features. To get an overview of the models we created, we refer to Figure **??**.

In model 2a we made model 1 realizable, by including the parameters: expected cost of risk, insurance cost and the benefit per link. Then we analyzed the parameters and found out when and how different network structures would evolve. By adjusting the insurance cost to the right level, the insurer can make the network formation game end up in a giant clique of both insured and non-insured nodes, or a clique of only insured nodes and another of only non-insured nodes. The condition for separating insured from non-insured nodes are: $\beta - r < I_l < \beta$, additionally if $\beta > r$, the non-insured nodes will also form a clique, and the resulting network will be two cliques. The solution is pairwise stable, since the change in payoff is linear and non-dependent on the rest of the network, when a link is added, there is no reason to remove it later. And since the resulting network consists of one or two cliques, it is not possible to add any more links. This holds for models 1, 2, 3 and 4. We also showed that when the insurer sets the cost such that the network ends up in two cliques, it is not the socially optimal, because the network will suffer from the lost benefits of connections between insured and non-insured nodes, i.e. it has a price of anarchy less than 1.

In model 2b, we showed that to be able to separate the networks into two cliques, the nodes must know the other nodes' types. Otherwise, the nodes will have an incentive to pretend to be an insured node, which will result in an untrusted network. We think it is reasonable to assume that nodes in a real world-scenario know whether their transactional partner has insurance or not, therefore we chose not to include this uncertainty in the other models.

In model 3 we applied the model to certain real-world scenarios, such as software development firms/chains, or other networks where the final product is dependent on the collaboration of multiple participants. This was done by including a bonus, which is first received when a node reaches the desired number of links (called max-degree). This made the separation process of insured and non-insured nodes more difficult for the insurer. Due to the possibility of achieving the bonus, a node will have more incentive to establish links, and is thus more accepting towards establishing links with risky nodes. The conditions for separating insured and non-insured nodes in this scenario are: $\beta+\gamma-r < I_l < \beta+\frac{\gamma}{m}$. For the separation of insured and non-insured nodes to be possible, the following has to hold: $1-\frac{1}{m} < \frac{r}{m}$. As we see, as γ and/or m increases, this gets more and more difficult to achieve.

In Model 4 we tried to implement a common feature used by insurance companies, bulk discount, in order to see how this affected the network formation. The cost of insuring a link is now dependent on the node's degree. We implemented this feature on both model 2 and 3, which resulted in even higher incentive for insured nodes to establish links with non-insured nodes. The reason is intuitive, since the cost of doing so decreases as the node degree increases. When we applied the discount on model 2, the conditions for ensuring separation of insured and non-insured nodes were: $N_I(\beta-r) < I_l < \beta$, where N_I represents the number of insured nodes in the network. This condition is very strong, because for the separation to be possible the following has to hold: $N_I(\beta-r) < \beta$. As we see, it is now more difficult for the insurer to separate insured and non-insured nodes, compared to model 2, because now the lower boundary on the insurance cost is multiplied with the number of insured nodes in the network $(N_I \times (\beta-r))$.

When applying the discount to model 3, the condition to ensure separation becomes: $m(\beta+\gamma-r) < I_l < \beta+\frac{\gamma}{m}$, and as in the other models, this further complicates the separation process for the insurer.

We also showed that the price of anarchy is even higher when applying discount to model 2. This is because the costs are decreasing, and thus when we have two separate cliques, the potential lost payoff between them will increase. When we included both bonus and discount, the calculation of price of anarchy became too complex. However, we see that the incentive for establishing links has increased, and thus the insurer has to set a higher price to compensate for this, and therefore the potential price of anarchy is even higher i.e. the more incentive for link establishment you have, the harder it gets to ensure separation of the nodes.

In our last model we applied our model 4 (discount) to an already existing model, "the symmetric connection game". In this old game it has been shown that there are three different efficient and stable networks, clique, star and an empty network, that arise under certain cost conditions. If $I_l < \beta-\beta^2$, the efficient and stable network is a clique. If $\beta-\beta^2 < I_l < \beta$ a star is both stable and efficient. If $I_l > \beta+\frac{N-2}{2}\beta^2$ an empty network is both stable and efficient. In general, a clique is the most efficient if the cost of establishing links is less than the benefit gained from indirect connections. A star is the most efficient if the cost is higher than the benefit from indirect connections, but less than the benefit of direct connections. Unfortunately, it is proved that as the number of nodes in the networks increases, the probability of the network ending up in star goes to zero. However, when we applied our insurance discount to this model, we found conjectures saying that, by setting the cost to the right level, one can with high probability ensure that either a clique, a star or a scale-free structure will evolve. This changes the connection game drastically, because now the insurer is able to force the network into three possible network formations, where the star has a fixation probability that exceeds the cliques. The insurer can use these findings to ensure that one of the beneficial structures, star or clique evolves. If the insurer is able to force a star to evolve, this can be used to drastically increase the overall security, and at the same time minimize the overall link cost.

Limitations and future work One limitation to our work, and a suggestion for future work, is to map our models and simulations to real-world networks in a more convincing way. Real-world networks are not random. Nodes may prefer to talk to nodes with high degree or low degree. In addition, the decision to use additive risk were taken due to the simplicity of the function and the fact that we do not know how a real-world risk distribution actually looks like. By introducing a complex risk function, we would only have distorted the goal of our models. i.e. suggestions for improving our models is to introduce more realistic payoff functions.

Another interesting thing to research, is the game of choosing insurance or not. In future work this should be applied to our models, but this could also possibly be too complex, and only disrupt the models.

6 Conclusion

So far, cyber-insurance has failed to reach its promising potential, and many have failed to establish a sustainable cyber-insurance market. We believe that cyber-insurance is an essential part of the internet economy, and that our new approach of analyzing the cyber-insurance market through graphs and network formation games could help improving and establishing a better market.

We surveyed literature on networks and risk, and found recent literature that showed how graphs like cliques, star, super-star, funnel and meta-funnel all have a calculable fixation probability, and that stars and funnels fixation probability exceeds the one of a clique. With these structures in mind, we created and analyzed different network formation games, and tried to find link-cost constraints, which enabled these structures to evolve.

In models one to four, we found cost constraints to separate insured and non-insured nodes into two cliques. For each model, we added some new features that made the model more applicable to real world scenarios, and for every feature added, it became more difficult for the insurer to separate the two types of nodes. This is due to the increased incentive for establishing links, and thus the nodes became more and more accepting towards risk.

In the last model, we introduced the concept of bulk insurance into an already existing network formation game, "the symmetric connection game", and showed that this enabled the insurer to determine, with high probability, when and how, cliques, stars or scale-free network would evolve. We showed that at a point, called critical degree, a node's optimal strategy would change from relying on indirect connections, to suddenly wanting to connect to everyone. If the critical degree is set to the right level, one can ensure that the different structures evolve. If the critical degree is set to a low degree, a clique will most certainly evolve, at a medium level, a star will evolve, and at a high level, a scale-free network will evolve. We proved this by performing multiple simulations, 50 simulations for every critical degree. What makes this a very interesting finding, is that in the connection game, earlier research has proven that as the number of nodes increases, the probability of the network reaching a star goes towards zero. However, by introducing a discount, that will subsidize the center node, one can drastically increase the probability of the network ending up in a star.

In summary, we have shown how insurers can determine the resulting structure of insurance networks, by adjusting the insurance cost, for several network formation games. We have also showed how insurers can be assisted in calculating the overall probability of fixation. We found these conditions for several models, with different properties that relate them to the real world and other insurance products. We believe our findings can help the cyber-insurance market evolve into a viable and better market.

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