Challenge Exam

In what follows we assume that "$\circ$" is a binary function symbol, written in infix notation, "i" is a unary function symbol, and "e" is a constant symbol (i.e. a name). Let T be the theory with the following three axioms:

A1. $\forall x \forall y \forall z(x \circ (y \circ z) = (x \circ y) \circ z)$

A2. $\forall x(x \circ e = x)$, $\forall x(e \circ x = x)$

A3. $\forall x(x \circ i(x) = e)$, $\forall x(i(x) \circ x = e)$

Note that A2 and A3 are each a pair of axioms. When we ask you to prove $T \vdash \phi$, this means that you may cite A1,A2,A3 in your proof, such as this:

$$(n) \quad i(a) \circ a = e \qquad \text{A3, UE}$$

Here it is understood that A1,A2,A3 have no dependencies, and the final line of your proof should also have no dependencies.

Problems

Please prove the following three sequents. In your proofs, you may cut in any results proved in the textbook, lectures, psets and exams; please cite the relevant page, lecture, pset, or exam. In particular, you can use the symmetry and transitivity of equality.

1. $T \vdash \forall x \forall y((x \circ y = e) \rightarrow (y = i(x)))$
2. $T \vdash \forall x(i(i(x)) = x)$
3. $T \vdash \forall x \forall y(i(x \circ y) = i(y) \circ i(x))$