

# Equivalence in Foundations

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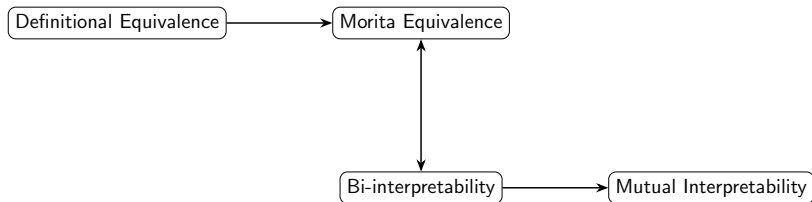
- The old consensus: Zermelo-Frankel set theory won the early 20th century battle about the foundations of mathematics
- Defeated competitors:
  - Logicism
  - Finitism
  - Intuitionism
  - Type theory

# New developments

- Category theory and the use of topos theory in various branches of pure mathematics (Grothendieck, Mac Lane, Lawvere)
- Martin-Löf type theory
- Computation
- Homotopy type theory (HoTT)
- Philosophical worries about set theory (structuralism, etc.)

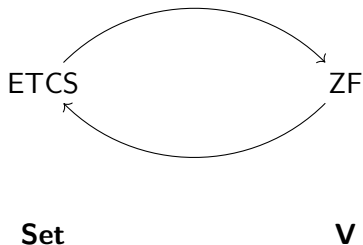
- In the 1970s and 80s, Sol Feferman argued against category-theoretic foundations for principled (philosophical) reasons
- The idea that **Set** and **Cat** are incommensurable foundations was challenged via results of Mitchell, Osiris, and Mathias
  - But what exactly did they prove?
- Steve Awodey: **Set**, **Cat**, and **Typ** are for all practical purposes interchangeable.

# What do we mean by equivalent?



Mutual interpretability does not imply bi-interpretability (Andreka and Nemeti?)

# Bi-interpretability: syntax and semantics



To clarify: Suppose that we have a functor that takes a “generic” model of  $T_2$  and returns a model of  $T_1$ , and another functor that takes a “generic” model of  $T_1$  and returns a model of  $T_2$ . Under what conditions do these two functors establish that  $T_1$  and  $T_2$  are bi-interpretable?

Adam Gajda, Michal Krynicki, and Leslaw Szerba. “A Note on Syntactical and Semantical Functions”. *Studia Logica* 46, no. 2 (1987): 177–185.

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# Shulman's theorem

- Shulman's procedure is to construct a model of ETCS from a model of ZF, and vice versa.
- What are the permitted constructions?
- In what sense does the construction need to be uniform, i.e. not dependent on specific features of a model?
- What needs to be shown about the constructions?

# Shulman's Theorem

- ① A model of ZF has a domain  $U$ . We let  $\mathcal{E}_0 = U$ , and we let  $\mathcal{E}_1$  be the subset of  $U$  consisting of “functions” (constructed as subsets of ordered pairs).
- ② Fact: the pair  $\mathcal{E}_0, \mathcal{E}_1$  forms a model of ETCS.
  - The empty set is an initial object.
  - Any singleton set is a terminal object.
  - Etc.

- We now begin with an ETCS model  $\mathcal{E}$ , and we attempt to construct a ZF model, i.e. a universe  $U$  with a relation  $\in$  that satisfies the ZF axioms.

# Construction of ZF model from ETCS model

Well-founded extensional accessible pointed graph (APG)

# Questions about Shulman's result

- The construction of APGs from a model **Set** of ETCS seems to require infinitary procedures. Is this move permitted by the standard definition of bi-interpretability?

# Type theory: Kemeny or Awodey?

# Resources

- Awodey, S. (2011) "From sets to types, to categories, to sets"
- Kemeny, J. (1949). *Type-theory vs. Set-theory*. Phd Thesis, Princeton University
- Mac Lane, S. and Moerdijk, I. *Sheaves in Geometry and Logic*
- Mathias, A. (2001) "The strength of Mac Lane set theory"  
[10.1016/s0168-0072\(00\)00031-2](https://doi.org/10.1016/s0168-0072(00)00031-2)
- M. Shulman. "Comparing material and structural set theories"

Gajda, Adam, Michal Krynicki, and Leslaw Szerba. "A Note on Syntactical and Semantical Functions". *Studia Logica* 46, no. 2 (1987): 177–185. <https://doi.org/10.1007/bf00370379>.