Equivalence in Foundations

Laney Gold-Rappe and Hans Halvorson

June 28, 2024

- The old consensus: Zermelo-Frankel set theory won the early 20th century battle about the foundations of mathematics
- Main competitors:
  - Finitism
  - Intuitionism
  - Type theory

New developments

- Category theory and the use of topos theory in various branches of pure mathematics (Grothendieck, Mac Lane, Lawvere)
- Martin-Löf type theory
- Computation
- Homotopy type theory (HoTT)

- In the 1970s and 80s, Sol Feferman argued against category-theoretic foundations for principled (philosophical) reasons
- The idea that **Set** and **Cat** are incommensurable foundations was challenged via results of Mitchell, Osius, and Mathias
  - But what exactly did they prove?
- Steve Awodey: **Set**, **Cat**, and **Typ** are equivalent

What do we mean by equivalent?

$$
\begin{tikzcd}
\text{Definitional Equivalence} \arrow[r] & \text{Morita Equivalence} \arrow[d, leftrightarrow] & \\
& \text{Bi-interpretability} \arrow[r] & \text{Mutual Interpretability}
\end{tikzcd}
$$

Bi-interpretability: syntax and semantics

$$\begin{tikzcd} \mathrm{ETCS} \arrow[rr, bend left] & & \mathrm{ZF} \arrow[ll, bend left] \end{tikzcd}$$

Set **V**

Shulman's theorem

- Shulman's procedure is to construct a model of ETCS from a model of ZF, and vice versa.
- What are the permitted constructions?
- In what sense does the construction need to be uniform, i.e. not dependent on specific features of a model?
- What needs to be shown about the constructions?

Shulman's Theorem

1. Given **V**, we take V_0 to be its domain, and V_1 the set of definable functional relations between elements of V_0.
2. It's fairly easy to see that V_0 and V_1 are a model of ETCS.
    1. The empty set is an initial object.
    2. Any singleton set is a terminal object.

Questions about Shulman's result

- The construction of APGs from a model **Set** of ETCS seems to require infinitary procedures. Is this move permitted by the standard definition of bi-interpretability?

Type theory: Kemeny or Awodey?