

# Equivalence in Foundations

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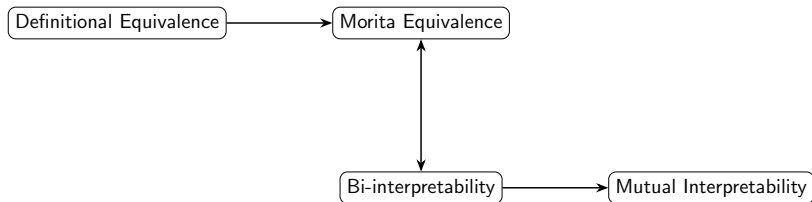
- The old consensus: Zermelo-Frankel set theory won the early 20th century battle about the foundations of mathematics
- Main competitors:
  - Finitism
  - Intuitionism
  - Type theory

# New developments

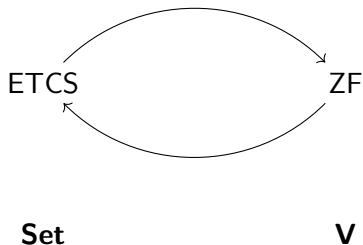
- Category theory and the use of topos theory in various branches of pure mathematics (Grothendieck, Mac Lane, Lawvere)
- Martin-Löf type theory
- Computation
- Homotopy type theory (HoTT)

- In the 1970s and 80s, Sol Feferman argued against category-theoretic foundations for principled (philosophical) reasons
- The idea that **Set** and **Cat** are incommensurable foundations was challenged via results of Mitchell, Osius, and Mathias
  - But what exactly did they prove?
- Steve Awodey: **Set**, **Cat**, and **Typ** are equivalent

# What do we mean by equivalent?



# Bi-interpretability: syntax and semantics



# Shulman's theorem

- Shulman's procedure is to construct a model of ETCS from a model of ZF, and vice versa.
- What are the permitted constructions?
- In what sense does the construction need to be uniform, i.e. not dependent on specific features of a model?
- What needs to be shown about the constructions?

# Shulman's Theorem

- ① Given  $\mathbf{V}$ , we take  $V_0$  to be its domain, and  $V_1$  the set of definable functional relations between elements of  $V_0$ .
- ② It's fairly easy to see that  $V_0$  and  $V_1$  are a model of ETCS.
  - ① The empty set is an initial object.
  - ② Any singleton set is a terminal object.



# Questions about Shulman's result

- The construction of APGs from a model **Set** of ETCS seems to require infinitary procedures. Is this move permitted by the standard definition of bi-interpretability?

# Type theory: Kemeny or Awodey?