Equivalence in Foundations

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The old consensus

- Zermelo-Frankel set theory won the early 20th century battle about the foundations of mathematics.
- Defeated competitors:
  - Logicism
  - Finitism
  - Intuitionism
  - Type theory
- Philosophers take set theory as background framework for their inquiries.

  See: David K Lewis et al. (1986). *On the plurality of worlds*. Blackwell Oxford

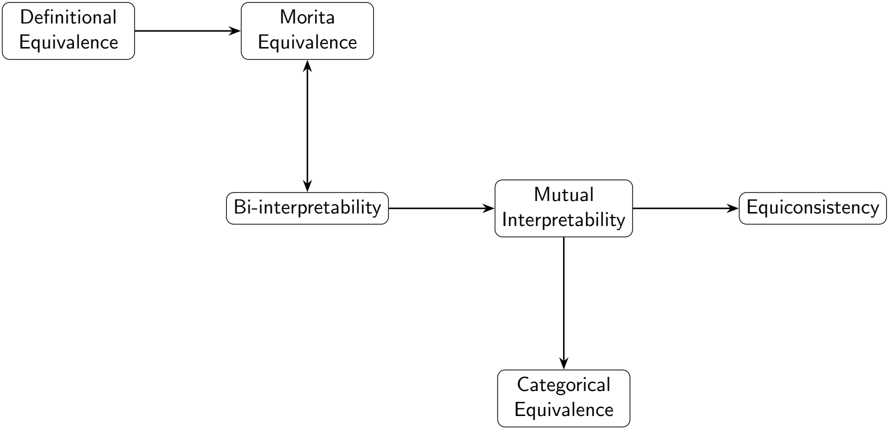
New developments

- Category theory and the use of topos theory in various branches of pure mathematics (Grothendieck, Mac Lane, Lawvere)
- Martin-Löf type theory
- Computation
- Homotopy type theory (HoTT)
- Philosophical worries about set theory (structuralism, etc.)

Is a new battle coming?

- Feferman (1969) and Feferman (1977) argue against category-theoretic foundations for principled (philosophical) reasons.
- The idea that Sets and Cats are incommensurable foundations was challenged via results of Mitchell, Osius, and Mathias
  - What exactly did they prove?
- Awodey (2009): Sets, Cats, and Types are interchangeable foundations.

What do we mean by equivalent?



Andréka and Németi (1994): Mutual interpretability does not imply bi-interpretability

TO DO: Features of theories that are not invariant under mutual interpretability

Bi-interpretability: syntax and semantics

$$\begin{tikzcd} \mathrm{ETCS} \arrow[rr, bend left] & & \mathrm{ZF} \arrow[ll, bend left] \end{tikzcd}$$

$\mathcal{E}$ $\qquad\qquad\qquad$ U

To clarify: Suppose that we have a functor that takes a "generic" model of T_2 and returns a model of T_1, and another functor that takes a "generic" model of T_1 and returns a model of T_2. Under what conditions do these two functors establish that T_1 and T_2 are bi-interpretable?

Adam Gajda, Michal Krynicki, and Leslaw Szczerba (1987). "A note on syntactical and semantical functions". In: *Studia Logica* 46.2, pp. 177–185. DOI: 10.1007/bf00370379

Topos-theoretic foundations of mathematics

Definition

An **elementary topos** $\mathcal{E}$ is a category that has the following properties:

- Finite limits.
- Exponentials: For any objects $A, B \in \mathcal{E}$, there exists an object B^A and an evaluation map $ev : B^A \times A \to B$ such that for any object C and any map $f : C \times A \to B$, there is a unique map $\lambda f : C \to B^A$ making the appropriate diagram commute.
- A subobject classifier Ω: An object Ω with a morphism $true : 1 \to \Omega$ such that for any monomorphism $m : A \to B$, there exists a unique characteristic morphism $\chi_m : B \to \Omega$ making the diagram commute.

Category Axioms

Objects and Morphisms

- Two sorts: **Objects** and **Morphisms**.
- Each morphism f has a **domain** $\mathrm{dom}(f)$ and **codomain** $\mathrm{cod}(f)$.

Composition

- For any morphisms f and g with $\mathrm{cod}(f) = \mathrm{dom}(g)$, there is a composite morphism $g \circ f$.

Associativity

- For any morphisms f, g, h: $h \circ (g \circ f) = (h \circ g) \circ f$

Identity

- For each object A, there is an identity morphism id_A.
- For any morphism f: $\mathrm{id}_{\mathrm{dom}(f)} \circ f = f \quad \text{and} \quad f \circ \mathrm{id}_{\mathrm{cod}(f)} = f$

Finite Limits

Terminal Object

- There is an object 1 (terminal object) such that for any object A, there is a unique morphism $! : A \to 1$.

Pullbacks

- For any pair of morphisms $f : A \to C$ and $g : B \to C$, there exists a pullback square:

$$\begin{tikzcd} P \arrow[r] \arrow[d] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

Topos-theoretic foundations

Element

For an object A in $\mathcal{E}$, an **element** of A is an arrow $x : 1 \to A$.

Intuitive differences between **Set** and **Cat**

In **Set**: any two sets can stand in the elementhood relationship with each other.

The question of framework

We take both ZF and ETCS as theories in many-sorted, classical, first-order logic

Shulman's Theorem

Shulman (2019) seems very close to proving bi-interpretability of ZF and ETCS.

- For each model U of ZF, there is a corresponding model of ETCS; and for each model $\mathcal{E}$ of ETCS, there is a corresponding model of ZF.
- What are the permitted constructions?
- In what sense is the construction uniform, i.e. doesn't depend on specific features of a model?
- What needs to be shown about the constructions?

From universe to topos

1. Given a model $\langle U, \in \rangle$ of ZF, let $\mathcal{E}_0 = U$, and let $\mathcal{E}_1$ be the set of functions between sets (constructed as subsets of ordered pairs).
2. Fact: the pair $\mathcal{E}_0, \mathcal{E}_1$ forms a model of ETCS.
    - The empty set is an initial object.
    - Any singleton set is a terminal object.
    - Etc.

From topos to universe

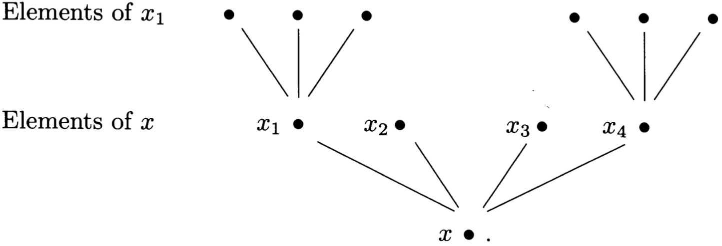
- Intuitively, the objects in $\mathcal{E}$ would become sets. But how to define the relation $A \in B$?
- So instead of taking objects in $\mathcal{E}$ as sets, we take trees:

$$t : R \rightarrowtail A \times A$$

For elements $a : 1 \to A$ and $b : 1 \to A$, we write $a \leq b$ just in case

$$\begin{tikzcd} & R \arrow[d, "t"] & \\ 1 \arrow[ur, dashed, "r"] \arrow[r, "{(a,b)}"'] & A \times A \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & A \end{tikzcd}$$

Construction of ZF model from ETCS model



Tree: A **tree** is a poset that is downward linear.

Pointed: A **pointed tree** is a tree $t : R \rightarrowtail A \times A$ and a point $e : 1 \to A$ such that $\forall x(e \leq x)$.

Accessible: A pointed tree (t, e) is **accessible** just in case: for every element $x : 1 \to A$ there is a finite R-path to the root $e : 1 \to A$.[1]

[1] This definition can be made first-order using subobjects of the natural number object in $\mathcal{E}$.

A subobject $m : S \rightarrowtail A$ is said to be **inductive** for the tree $t : R \rightarrowtail A \times A$ just in case: for any element $x : 1 \to A$, if every $y \leq x$ factors through m, then x factors through m.

Well-founded: If $m : S \rightarrowtail X$ is inductive, then m is an isomorphism.

Extensional: For any $x : 1 \to A$ and $y : 1 \to A$, if x and y have the same R-children, then $x = y$.

How do we know that there are “enough” of these trees in $\mathcal{E}$ to build an entire ZF universe?

Questions about Shulman's result

- The construction of trees from a topos $\mathcal{E}$ seems to require infinitary procedures. Is this move permitted by the standard definition of bi-interpretability?

Type theory: Kemeny or Awodey?

"It was my intention to prove the equivalence of the simple theory of types and Zermelo set-theory. Instead of this I have succeeded in proving a strong theorem from which it follows that the two systems are not equivalent under <u>any</u> reasonable definition of 'equivalent'." (Kemeny, 1949)

Further Questions

- FOL is not the most natural setting for the theory of categories. Should we use Makkai's FOLDS instead (Makkai, 1998)? And how would this affect the claim that ZF and ETCS are equivalent?

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