Equivalence in Foundations

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- The old consensus: Zermelo-Frankel set theory won the early 20th century battle about the foundations of mathematics
- Defeated competitors:
  - Logicism
  - Finitism
  - Intuitionism
  - Type theory

New developments

- Category theory and the use of topos theory in various branches of pure mathematics (Grothendieck, Mac Lane, Lawvere)
- Martin-Löf type theory
- Computation
- Homotopy type theory (HoTT)
- Philosophical worries about set theory (structuralism, etc.)

- In the 1970s and 80s, Sol Feferman argued against category-theoretic foundations for principled (philosophical) reasons
- The idea that **Set** and **Cat** are incommensurable foundations was challenged via results of Mitchell, Osius, and Mathias
  - But what exactly did they prove?
- Steve Awodey: **Set**, **Cat**, and **Typ** are for all practical purposes interchangeable.

What do we mean by equivalent?

$$
\begin{tikzcd}
\text{Definitional Equivalence} \arrow[r] & \text{Morita Equivalence} \arrow[d, leftrightarrow] & \\
 & \text{Bi-interpretability} \arrow[r] & \text{Mutual Interpretability}
\end{tikzcd}
$$

Mutual interpretability does not imply bi-interpretability (Andreka and Nemeti?)

Bi-interpretability: syntax and semantics

$$
\begin{tikzcd}
\mathrm{ETCS} \arrow[r, bend left] & \mathrm{ZF} \arrow[l, bend left] \\
\mathbf{Set} & \mathbf{V}
\end{tikzcd}
$$

To clarify: Suppose that we have a functor that takes a "generic" model of T_2 and returns a model of T_1, and another functor that takes a "generic" model of T_1 and returns a model of T_2. Under what conditions do these two functors establish that T_1 and T_2 are bi-interpretable?

Adam Gajda, Michal Krynicki, and Leslaw Szczerba. "A Note on Syntactical and Semantical Functions". *Studia Logica* 46, no. 2 (1987): 177–185. https://doi.org/10.1007/bf00370379

Shulman's theorem

- Shulman's procedure is to construct a model of ETCS from a model of ZF, and vice versa.
- What are the permitted constructions?
- In what sense does the construction need to be uniform, i.e. not dependent on specific features of a model?
- What needs to be shown about the constructions?

Shulman's Theorem

1. A model of ZF has a domain U. We let $\mathcal{E}_0 = U$, and we let $\mathcal{E}_1$ be the subset of U consisting of "functions" (constructed as subsets of ordered pairs).
2. Fact: the pair $\mathcal{E}_0, \mathcal{E}_1$ forms a model of ETCS.
   - The empty set is an initial object.
   - Any singleton set is a terminal object.
   - Etc.

- We now begin with an ETCS model $\mathcal{E}$, and we attempt to construct a ZF model, i.e. a universe U with a relation $\in$ that satisfies the ZF axioms.

Construction of ZF model from ETCS model

Well-founded extensional accessible pointed graph (APG)

Questions about Shulman's result

- The construction of APGs from a model **Set** of ETCS seems to require infinitary procedures. Is this move permitted by the standard definition of bi-interpretability?

Type theory: Kemeny or Awodey?

Resources

- Awodey, S. (2011) "From sets to types, to categories, to sets"
- Kemeny, J. (1949). *Type-theory vs. Set-theory*. Phd Thesis, Princeton University
- Mac Lane, S. and Moerdijk, I. *Sheaves in Geometry and Logic*
- Mathias, A. (2001) "The strength of Mac Lane set theory" 10.1016/s0168-0072(00)00031-2
- M. Shulman. "Comparing material and structural set theories"

Gajda, Adam, Michal Krynicki, and Leslaw Szczerba. "A Note on Syntactical and Semantical Functions". *Studia Logica* 46, no. 2 (1987): 177–185. https://doi.org/10.1007/bf00370379.