Equivalence in Foundations

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- The old consensus: Zermelo-Frankel set theory won the early 20th century battle about the foundations of mathematics
- Defeated competitors:
    - Logicism
    - Finitism
    - Intuitionism
    - Type theory

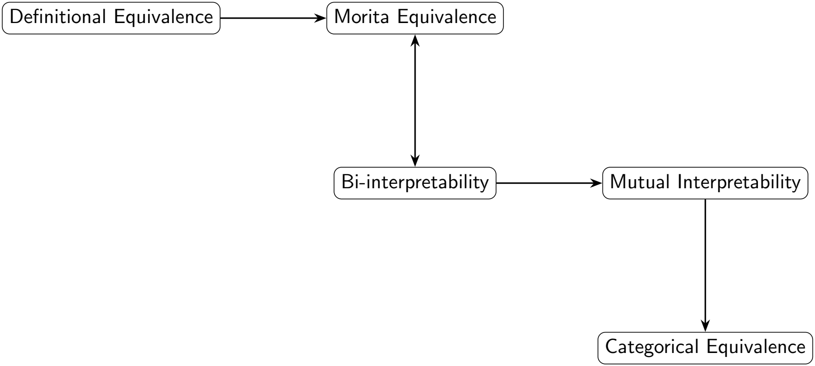
New developments

- Category theory and the use of topos theory in various branches of pure mathematics (Grothendieck, Mac Lane, Lawvere)
- Martin-Löf type theory
- Computation
- Homotopy type theory (HoTT)
- Philosophical worries about set theory (structuralism, etc.)

Is a new battle coming?

- Feferman (1969) and Feferman (1977) argue against category-theoretic foundations for principled (philosophical) reasons.
- The idea that **Set** and **Cat** are incommensurable foundations was challenged via results of Mitchell, Osius, and Mathias
  - But what exactly did they prove?
- Awodey (2009): **Set**, **Cat**, and **Typ** are for all practical purposes interchangeable.

What do we mean by equivalent?



Andréka and Németi (1994): Mutual interpretability does not imply bi-interpretability

TO DO: Features of theories that are not invariant under mutual interpretability

Bi-interpretability: syntax and semantics

$$\begin{tikzcd} \mathrm{ETCS} \arrow[r, bend left] & \mathrm{ZF} \arrow[l, bend left] \end{tikzcd}$$

Set   **V**

To clarify: Suppose that we have a functor that takes a "generic" model of T_2 and returns a model of T_1, and another functor that takes a "generic" model of T_1 and returns a model of T_2. Under what conditions do these two functors establish that T_1 and T_2 are bi-interpretable?

Adam Gajda, Michal Krynicki, and Leslaw Szczerba (1987). "A note on syntactical and semantical functions". In: *Studia Logica* 46.2, pp. 177–185. DOI: 10.1007/bf00370379

Category-theoretic foundations

Definition

An **elementary topos** $\mathcal{E}$ is a category that has the following properties:

- Finite limits.
- Exponentials: For any objects $A, B \in \mathcal{E}$, there exists an object B^A and an evaluation map $ev : B^A \times A \to B$ such that for any object C and any map $f : C \times A \to B$, there is a unique map $\lambda f : C \to B^A$ making the appropriate diagram commute.
- A subobject classifier Ω: An object Ω with a morphism $true : 1 \to \Omega$ such that for any monomorphism $m : A \to B$, there exists a unique characteristic morphism $\chi_m : B \to \Omega$ making the diagram commute.

Category Axioms

Objects and Morphisms

- Two sorts: **Objects** and **Morphisms**.
- Each morphism f has a **domain** $\mathrm{dom}(f)$ and **codomain** $\mathrm{cod}(f)$.

Composition

- For any morphisms f and g with $\mathrm{cod}(f) = \mathrm{dom}(g)$, there is a composite morphism $g \circ f$.

Associativity

- For any morphisms f, g, h: $h \circ (g \circ f) = (h \circ g) \circ f$

Identity

- For each object A, there is an identity morphism id_A.
- For any morphism f: $\mathrm{id}_{\mathrm{dom}(f)} \circ f = f \quad \text{and} \quad f \circ \mathrm{id}_{\mathrm{cod}(f)} = f$

Finite Limits

Terminal Object

- There is an object 1 (terminal object) such that for any object A, there is a unique morphism $! : A \to 1$.

Pullbacks

- For any pair of morphisms $f : A \to C$ and $g : B \to C$, there exists a pullback square:

$$\begin{tikzcd} P \arrow[r] \arrow[d] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

The question of framework

We take both ZF and ETCS as theories in many-sorted, classical, first-order logic

Shulman's theorem

Shulman (2019) is 90% of the way to proving bi-interpretability of ZF and ETCS.

- Shulman's procedure is to construct a model of ETCS from a model of ZF, and vice versa.
- What are the permitted constructions?
- In what sense does the construction need to be uniform, i.e. not dependent on specific features of a model?
- What needs to be shown about the constructions?

Shulman's Theorem

1. A model of ZF has a domain U. We let $\mathcal{E}_0 = U$, and we let $\mathcal{E}_1$ be the subset of U consisting of "functions" (constructed as subsets of ordered pairs).
2. Fact: the pair $\mathcal{E}_0, \mathcal{E}_1$ forms a model of ETCS.
    - The empty set is an initial object.
    - Any singleton set is a terminal object.
    - Etc.

- Beginning with a model $\mathcal{E}$ of ETCS, we attempt to construct a model $\langle U, \in \rangle$ of ZF.

Construction of ZF model from ETCS model

TO DO: Picture from Mac Lane and Moerdijk

Questions about Shulman's result

- The construction of APGs from a model **Set** of ETCS seems to require infinitary procedures. Is this move permitted by the standard definition of bi-interpretability?

Type theory: Kemeny or Awodey?

"It was my intention to prove the equivalence of the simple theory of types and Zermelo set-theory. Instead of this I have succeeded in proving a strong theorem from which it follows that the two systems are not equivalent under <u>any</u> reasonable definition of 'equivalent'." (Kemeny, 1949)

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