Equivalence in Foundations

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The old consensus

- Early 20th century: Zermelo-Frankel set theory won the battle about the foundations of mathematics.
- Defeated competitors:
    - Logicism
    - Finitism
    - Intuitionism
    - Type theory
- Philosophers take set theory as background framework for their inquiries.

  See: David K Lewis et al. (1986). *On the plurality of worlds*. Blackwell Oxford

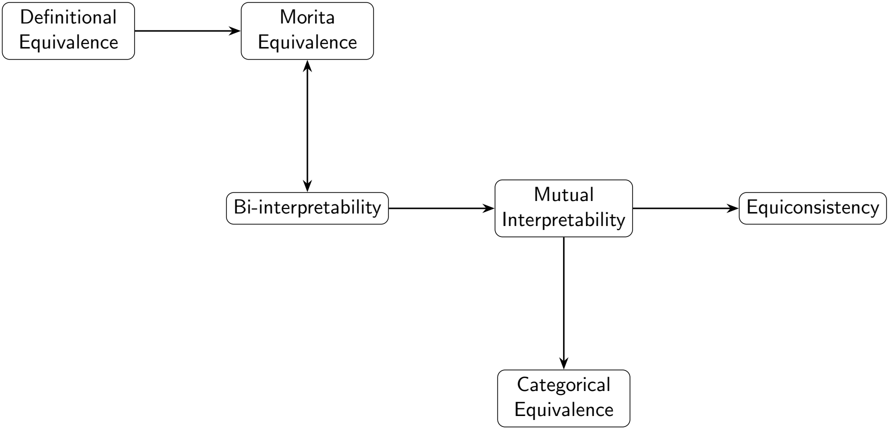
New developments

- Category theory and topos theory have proved fruitful in various branches of pure mathematics (Grothendieck, Mac Lane, Lawvere)
- Martin-Löf type theory
- Computation
- Homotopy type theory (HoTT)
- Philosophical worries about set theory (structuralism, etc.)

Is a new battle coming?

- Feferman (1969; 1977) argues against category-theoretic foundations for principled (philosophical) reasons.
- The idea that Sets and Cats are incommensurable foundations was challenged via results of Mitchell, Osius, and Mathias
  - What exactly did they prove?
- Awodey (2009): Sets, Cats, and Types are interchangeable foundations.

What do we mean by equivalent?



Andréka and Németi (1994): Mutual interpretability does not imply bi-interpretability

TO DO: Features of theories that are not invariant under mutual interpretability

Bi-interpretability: syntax and semantics

$$
\begin{tikzcd}
\mathrm{ETCS} \arrow[rr, bend left] & & \mathrm{ZF} \arrow[ll, bend left]
\end{tikzcd}
$$

$\mathcal{E}$ $\qquad\qquad$ U

To clarify: Suppose that we have a functor that takes a "generic" model of T_2 and returns a model of T_1, and another functor that takes a "generic" model of T_1 and returns a model of T_2. Under what conditions do these two functors establish that T_1 and T_2 are bi-interpretable?

Adam Gajda, Michal Krynicki, and Leslaw Szczerba (1987). "A note on syntactical and semantical functions". In: *Studia Logica* 46.2, pp. 177–185. DOI: 10.1007/bf00370379

Topos-theoretic foundations of mathematics

Definition

An **elementary topos** $\mathcal{E}$ is a category that has the following properties:

- Finite limits.
- Exponentials: For any objects $A, B \in \mathcal{E}$, there exists an object B^A and an evaluation map $ev : B^A \times A \to B$ such that for any object C and any map $f : C \times A \to B$, there is a unique map $\lambda f : C \to B^A$ making the appropriate diagram commute.
- A subobject classifier Ω: An object Ω with a morphism $true : 1 \to \Omega$ such that for any monomorphism $m : A \to B$, there exists a unique characteristic morphism $\chi_m : B \to \Omega$ making the diagram commute.

Category Axioms

Objects and Morphisms

- Two sorts: **Objects** and **Morphisms**.
- Each morphism f has a **domain** $\mathrm{dom}(f)$ and **codomain** $\mathrm{cod}(f)$.

Composition

- For any morphisms f and g with $\mathrm{cod}(f) = \mathrm{dom}(g)$, there is a composite morphism $g \circ f$.

Associativity

- For any morphisms f, g, h: $h \circ (g \circ f) = (h \circ g) \circ f$

Identity

- For each object A, there is an identity morphism id_A.
- For any morphism f: $\mathrm{id}_{\mathrm{dom}(f)} \circ f = f \quad \text{and} \quad f \circ \mathrm{id}_{\mathrm{cod}(f)} = f$

Finite Limits

Terminal Object

- There is an object 1 (terminal object) such that for any object A, there is a unique morphism $! : A \to 1$.

Pullbacks

- For any pair of morphisms $f : A \to C$ and $g : B \to C$, there exists a pullback square:

$$\begin{tikzcd} P \arrow[r] \arrow[d] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

Topos-theoretic foundations

Element

For an object A in $\mathcal{E}$, an **element** of A is an arrow $x : 1 \to A$.

Intuitive differences between **Set** and **Cat**

In **Set**: any two sets can stand in the elementhood relationship with each other.

The question of framework

We take both ZF and ETCS as theories in many-sorted, classical, first-order logic

Shulman's Theorem

Shulman (2019) seems very close to proving bi-interpretability of ZF and ETCS.

- For each model U of ZF, there is a corresponding model of ETCS; and for each model $\mathcal{E}$ of ETCS, there is a corresponding model of ZF.
- What are the permitted constructions?
- In what sense is the construction uniform, i.e. doesn't depend on specific features of a model?
- What needs to be shown about the constructions?

From universe to topos

1. Given a model $\langle U, \in \rangle$ of ZF, let $\mathcal{E}_0 = U$, and let $\mathcal{E}_1$ be the set of functions between sets (constructed as subsets of ordered pairs).
2. Fact: the pair $\mathcal{E}_0, \mathcal{E}_1$ forms a model of ETCS.
    - The empty set is an initial object.
    - Any singleton set is a terminal object.
    - Etc.

From topos to universe

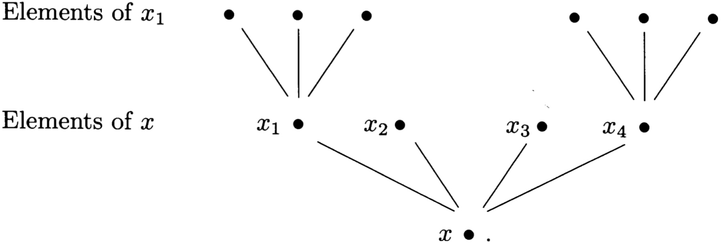
- Intuitively, the objects in $\mathcal{E}$ would become sets. But how to define the relation $A \in B$?
- So instead of taking objects in $\mathcal{E}$ as sets, we take trees:

$$t : R \rightarrowtail A \times A$$

For elements $a : 1 \to A$ and $b : 1 \to A$, we write $a \leq b$ just in case

$$\begin{tikzcd} & R \arrow[d, "t"] & \\ 1 \arrow[ur, dashed, "r"] \arrow[r, "{(a,b)}"'] & A \times A \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & A \end{tikzcd}$$

Construction of ZF model from ETCS model



Tree: A **tree** is a poset that is downward linear.

Rooted: If $t : R \rightarrowtail A \times A$ is a tree, and $e : 1 \to A$, then we say that e is the **root** of t just in case $\forall x(e \leq x)$.

Accessible: A pointed tree (t, e) is **accessible** just in case: for every element $x : 1 \to A$ there is a finite R-path to the root $e : 1 \to A$.[1]

[1] This definition can be made first-order using subobjects of the natural number object in $\mathcal{E}$.

A subobject $m : S \rightarrowtail A$ is said to be **inductive** for the tree $t : R \rightarrowtail A \times A$ just in case: for any element $x : 1 \to A$, if every $y \leq x$ factors through m, then x factors through m.

Well-founded: If $m : S \rightarrowtail X$ is inductive, then m is an isomorphism.

Extensional: For any $x : 1 \to A$ and $y : 1 \to A$, if x and y have the same R-children, then $x = y$.

How do we know that there are “enough” of these trees in $\mathcal{E}$ to build an entire ZF universe?

Questions about Shulman's result

- The construction of trees from a topos $\mathcal{E}$ seems to require infinitary procedures. Is this move permitted by the standard definition of bi-interpretability?

Type theory: Kemeny or Awodey?

"It was my intention to prove the equivalence of the simple theory of types and Zermelo set-theory. Instead of this I have succeeded in proving a strong theorem from which it follows that the two systems are not equivalent under <u>any</u> reasonable definition of 'equivalent'." (Kemeny, 1949)

Further Questions

- Dependent type theory is a more natural setting for theory of categories and the elementary theory of toposes.
  - Replace "isomorphism" with "equivalence".
- If ETCS and ZF are formalized in FOLDS (Makkai, 1995), does the equivalence result still hold?

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