How Logic Works: Solutions to Problems

Hans Halvorson

November 22, 2025

Chapter 3

Exercise 3.1

1. $\boxed{P \vdash Q \to (P \wedge Q)}$

1	(1)	P	A
2	(2)	Q	A
1,2	(3)	$P \wedge Q$	1,2 $\wedge$I
1	(4)	$Q \to (P \wedge Q)$	2,3 CP

2. $\boxed{(P \to Q) \wedge (P \to R) \vdash P \to (Q \wedge R)}$

1	(1)	$(P \to Q) \wedge (P \to R)$	A
2	(2)	P	A
1	(3)	$P \to Q$	1 $\wedge$E
1	(4)	$P \to R$	1 $\wedge$E
1,2	(5)	Q	3,2 MP
1,2	(6)	R	4,2 MP
1,2	(7)	$Q \wedge R$	5,6 $\wedge$I
1	(8)	$P \to (Q \wedge R)$	2,7 CP

3. $\boxed{P \to (Q \to R) \vdash Q \to (P \to R)}$

1	(1)	$P \to (Q \to R)$	A
2	(2)	Q	A
3	(3)	P	A
1,3	(4)	$Q \to R$	3,1 MP
1,2,3	(5)	R	4,2 MP
1,2	(6)	$P \to R$	3,5 CP
1	(7)	$Q \to (P \to R)$	2,6 CP

4. $P \rightarrow Q \vdash (Q \rightarrow R) \rightarrow (P \rightarrow R)$

1	(1)	$P \rightarrow Q$	A
2	(2)	$Q \rightarrow R$	A
3	(3)	P	A
1,3	(4)	Q	1,3 MP
1,2,3	(5)	R	2,4 MP
1,2	(6)	$P \rightarrow R$	3,5 CP
1	(7)	$(Q \rightarrow R) \rightarrow (P \rightarrow R)$	2,6 CP

5. $P \rightarrow (P \rightarrow Q) \vdash P \rightarrow Q$

1	(1)	$P \rightarrow (P \rightarrow Q)$	A
2	(2)	P	A
1,2	(3)	$P \rightarrow Q$	1,2 MP
1,2	(4)	Q	3,2 MP
1	(5)	$P \rightarrow Q$	2,4 CP

6. $P \rightarrow (Q \rightarrow R) \vdash (P \wedge Q) \rightarrow R$

1	(1)	$P \rightarrow (Q \rightarrow R)$	A
2	(2)	$P \wedge Q$	A
2	(3)	P	2 $\wedge$E
2	(4)	Q	2 $\wedge$E
1,2	(5)	$Q \rightarrow R$	1,3 MP
1,2	(6)	R	5,4 MP
1	(7)	$(P \wedge Q) \rightarrow R$	2,6 CP

7. $(P \vee Q) \rightarrow R \vdash P \rightarrow R$

1	(1)	$(P \vee Q) \rightarrow R$	A
2	(2)	P	A
2	(3)	$P \vee Q$	2 $\vee$I
1,2	(4)	R	1,3 MP
1	(5)	$P \rightarrow R$	2,4 CP

8. $\neg P \vdash \neg (P \wedge Q)$

1	(1)	$\neg P$	A
2	(2)	$P \wedge Q$	A
2	(3)	P	2 $\wedge$E
	(4)	$(P \wedge Q) \rightarrow P$	2,3 CP
1	(5)	$\neg (P \wedge Q)$	4,1 MT

9. $\neg (P \vee Q) \vdash \neg P \wedge \neg Q$

1	(1)	$\neg(P \vee Q)$	A
2	(2)	P	A
2	(3)	$P \vee Q$	2 $\vee$I
	(4)	$P \rightarrow (P \vee Q)$	2,3 CP
1	(5)	$\neg P$	4,1 MT
6	(6)	Q	A
6	(7)	$P \vee Q$	6 $\vee$I
	(8)	$Q \rightarrow (P \vee Q)$	6,7 CP
1	(9)	$\neg Q$	8,1 MT
1	(10)	$\neg P \wedge \neg Q$	5,9 $\wedge$I

10. $P \rightarrow \neg P \vdash \neg P$

1	(1)	P	A
2	(2)	$P \rightarrow \neg P$	A
1,2	(3)	$\neg P$	2,1 MP
1	(4)	$(P \rightarrow \neg P) \rightarrow \neg P$	2,3 CP
1	(5)	$\neg\neg P$	1 DN
1	(6)	$\neg(P \rightarrow \neg P)$	4,5 MT
	(7)	$P \rightarrow \neg(P \rightarrow \neg P)$	1,6 CP
2	(8)	$\neg\neg(P \rightarrow \neg P)$	2 DN
2	(9)	$\neg P$	7,8 MT

Exercise 3.4

1. $P \rightarrow Q \vdash \neg(P \wedge \neg Q)$

1	(1)	$P \rightarrow Q$	A
2	(2)	$P \wedge \neg Q$	A
2	(3)	P	2 $\wedge$E
1,2	(4)	Q	1,3 MP
2	(5)	$\neg Q$	2 $\wedge$E
1,2	(6)	$Q \wedge \neg Q$	4,5 $\wedge$I
1	(7)	$\neg(P \wedge \neg Q)$	2,6 RA

2. $\neg(P \wedge Q) \vdash \neg P \vee \neg Q$

1	(1)	$\neg(P \wedge Q)$	A
2	(2)	$\neg(\neg P \vee \neg Q)$	A
3	(3)	$\neg P$	A
3	(4)	$\neg P \vee \neg Q$	3 $\vee$I
2,3	(5)	$(\neg P \vee \neg Q) \wedge \neg(\neg P \vee \neg Q)$	4,2 $\wedge$I
2	(6)	$\neg\neg P$	3,5 RA
2	(7)	P	6 DN
8	(8)	$\neg Q$	A
8	(9)	$\neg P \vee \neg Q$	8 $\vee$I
2,8	(10)	$(\neg P \vee \neg Q) \wedge \neg(\neg P \vee \neg Q)$	9,2 $\wedge$I
2	(11)	$\neg\neg Q$	8,10 RA
2	(12)	Q	11 DN
2	(13)	$P \wedge Q$	7,12 $\wedge$I
1,2	(14)	$(P \wedge Q) \wedge \neg(P \wedge Q)$	13,1 $\wedge$I
1	(15)	$\neg\neg(\neg P \vee \neg Q)$	2,14 RA
1	(16)	$\neg P \vee \neg Q$	15 DN

3. $\neg(P \rightarrow Q) \vdash P \wedge \neg Q$

1	(1)	$\neg(P \rightarrow Q)$	A
2	(2)	$\neg P$	A
3	(3)	$\neg Q$	A
4	(4)	P	A
2,4	(5)	$P \wedge \neg P$	2,4 $\wedge$I
2,4	(6)	$\neg\neg Q$	3,5 RA
2,4	(7)	Q	6 DN
2	(8)	$P \rightarrow Q$	4,7 CP
1,2	(9)	$(P \rightarrow Q) \wedge \neg(P \rightarrow Q)$	8,1 $\wedge$I
1	(10)	$\neg\neg P$	2,9 RA
1	(11)	P	10 DN
12	(12)	Q	A
12	(13)	$P \rightarrow Q$	4,12 CP
1,12	(14)	$(P \rightarrow Q) \wedge \neg(P \rightarrow Q)$	13,1 $\wedge$I
1	(15)	$\neg Q$	12,14 RA
1	(16)	$P \wedge \neg Q$	11,15 $\wedge$I

4. $\vdash (P \rightarrow Q) \vee (Q \rightarrow P)$

1	(1)	$\neg((P \to Q) \lor (Q \to P))$	A
2	(2)	P	A
3	(3)	Q	A
2	(4)	$Q \to P$	3,2 CP
2	(5)	$(P \to Q) \lor (Q \to P)$	4 $\lor$I
1,2	(6)	$((P \to Q) \lor (Q \to P)) \land \neg((P \to Q) \lor (Q \to P))$	5,1 $\land$I
1	(7)	$\neg P$	2,6 RA
8	(8)	$\neg Q$	A
1,2	(9)	$P \land \neg P$	2,7 $\land$I
1,2	(10)	$\neg\neg Q$	8,9 RA
1,2	(11)	Q	10 DN
1	(12)	$P \to Q$	2,11 CP
1	(13)	$(P \to Q) \lor (Q \to P)$	12 $\lor$I
1	(14)	$((P \to Q) \lor (Q \to P)) \land \neg((P \to Q) \lor (Q \to P))$	13,1 $\land$I
$\varnothing$	(15)	$\neg\neg((P \to Q) \lor (Q \to P))$	1,14 RA
$\varnothing$	(16)	$(P \to Q) \lor (Q \to P)$	15 DN

5. $P \to (Q \lor R) \vdash (P \to Q) \lor (P \to R)$

1	(1)	$P \to (Q \lor R)$	A
2	(2)	$\neg((P \to Q) \lor (P \to R))$	A
3	(3)	$\neg P$	A
4	(4)	P	A
5	(5)	$\neg Q$	A
3,4	(6)	$P \land \neg P$	4,3 $\land$I
3,4	(7)	$\neg\neg Q$	5,6 RA
3,4	(8)	Q	7 DN
3	(9)	$P \to Q$	4,8 CP
3	(10)	$(P \to Q) \lor (P \to R)$	9 $\lor$I
2,3	(11)	$((P \to Q) \lor (P \to R)) \land \neg((P \to Q) \lor (P \to R))$	10,2 $\land$I
2	(12)	$\neg\neg P$	3,11 RA
2	(13)	P	12 DN
1,2	(14)	$Q \lor R$	1,13 MP
15	(15)	Q	A
15	(16)	$P \to Q$	4,15 CP
15	(17)	$(P \to Q) \lor (P \to R)$	16 $\lor$I
18	(18)	R	A
18	(19)	$P \to R$	4,18 CP
18	(20)	$(P \to Q) \lor (P \to R)$	19 $\lor$I
1,2	(21)	$(P \to Q) \lor (P \to R)$	14,15,17,18,20 $\lor$E
1,2	(22)	$((P \to Q) \lor (P \to R)) \land \neg((P \to Q) \lor (P \to R))$	21,2 $\land$I
1	(23)	$\neg\neg((P \to Q) \lor (P \to R))$	2,22 RA
1	(24)	$(P \to Q) \lor (P \to R)$	23 DN

6. $(P \land Q) \to \neg Q \vdash P \to \neg Q$

1	(1)	$(P \land Q) \to \neg Q$	A
2	(2)	P	A
3	(3)	Q	A
2,3	(4)	$P \land Q$	2,3 $\land$I
1,2,3	(5)	$\neg Q$	1,4 MP
1,2,3	(6)	$Q \land \neg Q$	3,5 $\land$I
1,2	(7)	$\neg Q$	3,6 RA
1	(8)	$P \to \neg Q$	2,7 CP

Chapter 6

Exercise 6.1

1. No logicians are celebrities. (Lx, Cx)

   $$\forall x \, (Lx \to \neg Cx)$$

   Equivalently: $\neg \exists x (Lx \land Cx)$

2. Some celebrities are not logicians. (Lx, Cx)

   $$\exists x \, (Cx \land \neg Lx)$$

3. Only students who do the homework will learn logic. (Sx, Hx, Lx)
   Either

   $$\forall x \, (Lx \to (Sx \land Hx))$$

   or (inequivalently)

   $$\forall x \, ((Sx \land Lx) \to Hx)$$

   depending on whether one intends to restrict the claim to students.

4. All rich logicians are computer scientists. (Rx, Lx, Cx)

   $$\forall x \, ((Rx \land Lx) \to Cx)$$

5. All students and professors get a discount. (Sx, Px, Dx)

   $$\forall x \, ((Sx \lor Px) \to Dx)$$

6. No logician is rich, unless she is a computer scientist. (Lx, Rx, Cx)

$$\forall x\left((Lx \wedge Rx) \rightarrow Cx\right)$$

Equivalent form: $\forall x((Lx \wedge \neg Cx) \rightarrow \neg Rx)$

7. Not all logicians are computer scientists. (Lx, Cx)

$$\neg \forall x\left(Lx \rightarrow Cx\right)$$

Often put as: $\exists x\left(Lx \wedge \neg Cx\right)$.

8. Some logicians are rich computer scientists. (Lx, Rx, Cx)

$$\exists x\left(Lx \wedge (Rx \wedge Cx)\right)$$

9. If there are rich logicians, then some logicians are computer scientists. (Rx, Lx, Cx)

$$\exists x\left(Rx \wedge Lx\right) \rightarrow \exists y\left(Ly \wedge Cy\right)$$

10. No pets except service animals are permitted in dorms. (Px, Sx, Dx)

    Can be read in a minimal way as:

$$\forall x\left((Px \wedge Dx) \rightarrow Sx\right),$$

    which says only that no non–service pets are allowed in dorms. However, ordinary policy language is typically understood more strongly: among pets, *being permitted in the dorms* and *being a service animal* coincide. That reading is captured by:

$$\forall x\left(Px \rightarrow (Dx \leftrightarrow Sx)\right).$$

    This biconditional formalization is therefore closer to the intended rule.

11. If anyone is rich, then Mary is. (Rx, m)

$$(\exists x\, Rx) \rightarrow Rm$$

Exercise 6.2

1. Mary loves everyone who loves her. (m, Lxy)

$$\forall x\left(Lxm \rightarrow Lmx\right)$$

2. Mary loves all and only those people who don't love themselves. (Lxy, m)

$$\forall x\left(Lmx \leftrightarrow \neg Lxx\right)$$

3. Everyone loves their mother. (Lxy, Mxy)

$$\forall x \forall y \, (Myx \rightarrow Lxy)$$

4. Some people love only those people who love their mother. (Lxy, Mxy)

$$\exists x \forall y (Lxy \rightarrow \forall z (Mzy \rightarrow Lyz))$$

5. Snape killed someone. (Kxy, s)

$$\exists x \, Ksx$$

6. Snape is a killer. (Kxy, s)

$$\exists x \, Ksx$$

7. Someone was killed by Snape. (Kxy, s)

$$\exists x \, Ksx$$

8. Some wizards only marry other wizards. (Wx, Mxy)

$$\exists x (Wx \wedge \forall y \, (Mxy \rightarrow Wy))$$

9. There is no greatest number. $(Nx, \ x < y)$

$$\forall x (Nx \rightarrow \exists y \, (Ny \wedge x < y))$$

10. c is the least upper bound of a and b. $(a, b, c, \ x \leq y)$

$$(a \leq c \wedge b \leq c) \ \wedge \ \forall x ((a \leq x \wedge b \leq x) \rightarrow c \leq x)$$

11. c is the greatest common divisor of a and b. $(a, b, c, Dxy, \ x \leq y)$

$$(Dca \wedge Dcb) \ \wedge \ \forall x ((Dxa \wedge Dxb) \rightarrow x \leq c)$$

Exercise 6.8

1. $\neg \exists x (Fx \wedge Gx) \vdash \forall x (Fx \rightarrow \neg Gx)$

1	(1)	$\neg \exists x (Fx \wedge Gx)$	A
2	(2)	Fa	A
3	(3)	Ga	A
2,3	(4)	$Fa \wedge Ga$	2,3 $\wedge$I
2,3	(5)	$\exists x (Fx \wedge Gx)$	4 EI
1,2,3	(6)	$\exists x (Fx \wedge Gx) \wedge \neg \exists x (Fx \wedge Gx)$	5,1 $\wedge$I
1,2	(7)	$\neg Ga$	3,6 RA
1	(8)	$Fa \rightarrow \neg Ga$	2,7 CP
1	(9)	$\forall x (Fx \rightarrow \neg Gx)$	8 UI

2. $\forall x Fx \vdash \exists x Fx$

1	(1)	$\forall x Fx$	A
1	(2)	Fa	1 UE
1	(3)	$\exists x Fx$	2 EI

3. $\forall x(Fx \rightarrow Gx), Fa \vdash \exists x Gx$

1	(1)	$\forall x(Fx \rightarrow Gx)$	A
2	(2)	Fa	A
1	(3)	$Fa \rightarrow Ga$	1 UE
1,2	(4)	Ga	3,2 MP
1,2	(5)	$\exists x Gx$	4 EI

4. $\neg Fa \vdash \exists x(Fx \rightarrow P)$

1	(1)	$\neg Fa$	A
1	(2)	$Fa \rightarrow P$	1 negative paradox
1	(3)	$\exists x(Fx \rightarrow P)$	2 EI

5. $\neg \forall x Fx \vdash \exists x(Fx \rightarrow P)$

1	(1)	$\neg \forall x Fx$	A
2	(2)	$\neg \exists x(Fx \rightarrow P)$	A
3	(3)	$Fa \rightarrow P$	A
3	(4)	$\exists x(Fx \rightarrow P)$	3 EI
2,3	(5)	$\exists x(Fx \rightarrow P) \wedge \neg \exists x(Fx \rightarrow P)$	4,2 $\wedge$I
2	(6)	$\neg(Fa \rightarrow P)$	3,5 RA
2	(7)	Fa	6 material conditional
2	(8)	$\forall x Fx$	7 UI
1,2	(9)	$\forall x Fx \wedge \neg \forall x Fx$	8,1 $\wedge$I
1	(10)	$\neg \neg \exists x(Fx \rightarrow P)$	2,9 RA
1	(11)	$\exists x(Fx \rightarrow P)$	10 DN

6. $\neg \exists x Fx \vdash \forall x(Fx \rightarrow Gx)$

1	(1)	$\neg \exists x Fx$	A
2	(2)	Fa	A
3	(3)	$\neg Ga$	A
2	(4)	$\exists x Fx$	2 EI
1,2	(5)	$\exists x Fx \wedge \neg \exists x Fx$	4,1 $\wedge$I
1,2	(6)	$\neg \neg Ga$	3,5 RA
1,2	(7)	Ga	6 DN
1	(8)	$Fa \rightarrow Ga$	2,7 CP
1	(9)	$\forall x(Fx \rightarrow Gx)$	8 UI

7. $\forall x \forall y Rxy \vdash \exists x Rxx$

1	(1)	$\forall x \forall y Rxy$	A
1	(2)	$\forall y Ray$	1 UE
1	(3)	Raa	2 UE
1	(4)	$\exists x Rxx$	3 EI

8. $P \rightarrow Fa \vdash P \rightarrow \exists x Fx$

1	(1)	$P \rightarrow Fa$	A
2	(2)	P	A
1,2	(3)	Fa	1,2 MP
1,2	(4)	$\exists x Fx$	3 EI
1	(5)	$P \rightarrow \exists x Fx$	2,4 CP

9. $\boxed{\exists x Fx \rightarrow P \vdash \forall x(Fx \rightarrow P)}$

1	(1)	$\exists x Fx \rightarrow P$	A
2	(2)	Fa	A
2	(3)	$\exists x Fx$	2 EI
1,2	(4)	P	1,3 MP
1	(5)	$Fa \rightarrow P$	2,4 CP
1	(6)	$\forall x(Fx \rightarrow P)$	5 UI

There is a typo here in the book: the direction $\forall x(Fx \rightarrow P) \vdash \exists x Fx \rightarrow P$ cannot be proven without EE, which is only introduced in the next section.

10. $\neg \exists x Fx \vdash \forall x(Fx \rightarrow P)$

1	(1)	$\neg \exists x Fx$	A
2	(2)	Fa	A
2	(3)	$\exists x Fx$	2 EI
1,2	(4)	$\exists x Fx \wedge \neg \exists x Fx$	3,1 $\wedge$I
1	(5)	$\neg Fa$	2,4 RA
1	(6)	$Fa \rightarrow P$	5 neg paradox
1	(7)	$\forall x(Fx \rightarrow P)$	6 UI

11. $\neg \exists x(Fx \rightarrow P) \vdash \forall x Fx \wedge \neg P$

1	(1)	$\neg \exists x(Fx \rightarrow P)$	A
2	(2)	$Fa \rightarrow P$	A
2	(3)	$\exists x(Fx \rightarrow P)$	2 EI
1,2	(4)	$\exists x(Fx \rightarrow P) \wedge \neg \exists x(Fx \rightarrow P)$	3,1 $\wedge$I
1	(5)	$\neg(Fa \rightarrow P)$	2,4 RA
1	(6)	$Fa \wedge \neg P$	5 material conditional
1	(7)	$\neg P$	6 $\wedge$E
1	(8)	Fa	6 $\wedge$E
1	(9)	$\forall x Fx$	8 UI
1	(10)	$\forall x Fx \wedge \neg P$	9,7 $\wedge$I

12. $\forall x Fx \rightarrow P \vdash \exists x(Fx \rightarrow P)$

1	(1)	$\forall x Fx \rightarrow P$	A
2	(2)	$\neg\exists x(Fx \rightarrow P)$	A
3	(3)	$\neg Fa$	A
3	(4)	$Fa \rightarrow P$	3 neg paradox
3	(5)	$\exists x(Fx \rightarrow P)$	4 EI
2,3	(6)	$\exists x(Fx \rightarrow P) \wedge \neg\exists x(Fx \rightarrow P)$	5,2 $\wedge$I
2	(7)	$\neg\neg Fa$	3,6 RA
2	(8)	Fa	7 DN
2	(9)	$\forall x Fx$	8 UI
1,2	(10)	P	1,9 MP
1,2	(11)	$Fb \rightarrow P$	10 pos paradox
1,2	(12)	$\exists x(Fx \rightarrow P)$	11 EI
1,2	(13)	$\exists x(Fx \rightarrow P) \wedge \neg\exists x(Fx \rightarrow P)$	12,2 $\wedge$I
1	(14)	$\neg\neg\exists x(Fx \rightarrow P)$	2,13 RA
1	(15)	$\exists x(Fx \rightarrow P)$	14 DN

Exercise 6.11

1. $\exists x Fx \vee \exists x Gx \vdash \exists x(Fx \vee Gx)$

1	(1)	$\exists x Fx \vee \exists x Gx$	A
2	(2)	$\exists x Fx$	A
3	(3)	Fa	A
3	(4)	$Fa \vee Ga$	3 $\vee$I
3	(5)	$\exists x(Fx \vee Gx)$	4 EI
2	(6)	$\exists x(Fx \vee Gx)$	2,3,5 EE
7	(7)	$\exists x Gx$	A
8	(8)	Ga	A
8	(9)	$Fa \vee Ga$	8 $\vee$I
8	(10)	$\exists x(Fx \vee Gx)$	9 EI
7	(11)	$\exists x(Fx \vee Gx)$	7,8,10 EE
1	(12)	$\exists x(Fx \vee Gx)$	1,2,6,7,11 $\vee$E

2. $\forall x(Fx \rightarrow Gx), \neg\exists x Gx \vdash \neg\exists x Fx$

1	(1)	$\forall x(Fx \rightarrow Gx)$	A
2	(2)	$\neg\exists xGx$	A
3	(3)	$\exists xFx$	A
4	(4)	Fa	A
1	(5)	$Fa \rightarrow Ga$	1 UE
1,4	(6)	Ga	5,4 MP
1,4	(7)	$\exists xGx$	6 EI
1,3	(8)	$\exists xGx$	3,4,7 EE
1,2,3	(9)	$\exists xGx \wedge \neg\exists xGx$	8,2 $\wedge$I
1,2	(10)	$\neg\exists xFx$	3,9 RA

3. $\forall x(Fx \rightarrow Gx) \vdash \exists x\neg Gx \rightarrow \exists x\neg Fx$

1	(1)	$\forall x(Fx \rightarrow Gx)$	A
2	(2)	$\exists x\neg Gx$	A
3	(3)	$\neg Ga$	A
1	(4)	$Fa \rightarrow Ga$	1 UE
1,3	(5)	$\neg Fa$	4,3 MT
1,3	(6)	$\exists x\neg Fx$	5 EI
1,2	(7)	$\exists x\neg Fx$	2,3,6 EE
1	(8)	$\exists x\neg Gx \rightarrow \exists x\neg Fx$	2,7 CP

4. $\forall x(Fx \rightarrow P) \vdash \exists xFx \rightarrow P$

1	(1)	$\forall x(Fx \rightarrow P)$	A
2	(2)	$\exists xFx$	A
3	(3)	Fa	A
1	(4)	$Fa \rightarrow P$	1 UE
1,3	(5)	P	4,3 MP
1,2	(6)	P	2,3,5 EE
1	(7)	$\exists xFx \rightarrow P$	2,6 CP

5. $P \wedge \exists xFx \vdash \exists x(P \wedge Fx)$

1	(1)	$P \wedge \exists xFx$	A
1	(2)	P	1 $\wedge$E
1	(3)	$\exists xFx$	1 $\wedge$E
4	(4)	Fa	A
1,4	(5)	$P \wedge Fa$	2,4 $\wedge$I
1,4	(6)	$\exists x(P \wedge Fx)$	5 EI
1	(7)	$\exists x(P \wedge Fx)$	3,4,6 EE

6. $\exists x(Fx \rightarrow P) \vdash \forall xFx \rightarrow P$

1	(1)	$\exists x(Fx \to P)$	A
2	(2)	$\forall xFx$	A
3	(3)	$Fa \to P$	A
2	(4)	Fa	2 UE
2,3	(5)	P	3,4 MP
3	(6)	$\forall xFx \to P$	2,5 CP
1	(7)	$\forall xFx \to P$	1,3,6 EE

Exercise 6.13

1. $\boxed{P \to \exists xFx \;\vdash\; \exists x(P \to Fx)}$

1	(1)	$P \to \exists xFx$	A
Ø	(2)	$\exists xFx \lor \neg\exists xFx$	prop taut
3	(3)	$\exists xFx$	A
4	(4)	Fa	A
4	(5)	$P \to Fa$	4 prop taut
4	(6)	$\exists x(P \to Fx)$	5 EI
3	(7)	$\exists x(P \to Fx)$	3,4,6 EE
8	(8)	$\neg\exists xFx$	A
1,8	(9)	$\neg P$	1,8 MT
1,8	(10)	$P \to Fb$	9 prop taut
1,8	(11)	$\exists x(P \to Fx)$	10 EI
1	(12)	$\exists x(P \to Fx)$	2,3,7,8,11 ∨E

2. $\boxed{\exists x(Fx \to P) \;\vdash\; \forall xFx \to P}$

1	(1)	$\exists x(Fx \to P)$	A
2	(2)	$\forall xFx$	A
3	(3)	$Fa \to P$	A
2	(4)	Fa	2 UE
2,3	(5)	P	3,4 MP
1,2	(6)	P	1,3,5 EE
1	(7)	$\forall xFx \to P$	2,6 CP

Exercise 6.14

1. $\boxed{\vdash\; \forall x(Fx \to Fx)}$

1	(1)	Fa	A
Ø	(2)	$Fa \to Fa$	1,1 CP
Ø	(3)	$\forall x(Fx \to Fx)$	2 UI

2. $\boxed{\vdash\; \forall xFx \lor \exists x\neg Fx}$

∅	(1)	$\neg\exists x\neg Fx \lor \exists x\neg Fx$	prop taut
2	(2)	$\neg\exists x\neg Fx$	A
3	(3)	$\neg Fa$	A
3	(4)	$\exists x\neg Fx$	3 EI
2,3	(5)	$\exists x\neg Fx \land \neg\exists x\neg Fx$	4,2 ∧I
2	(6)	$\neg\neg Fa$	3,5 RA
2	(7)	Fa	6 DN
2	(8)	$\forall x Fx$	7 UI
2	(9)	$\forall x Fx \lor \exists x\neg Fx$	8 ∨I
10	(10)	$\exists x\neg Fx$	A
10	(11)	$\forall x Fx \lor \exists x\neg Fx$	10 ∨I
∅	(12)	$\forall x Fx \lor \exists x\neg Fx$	1,2,9,10,11 ∨E

3. $\vdash \forall x\, \neg(Fx \land \neg Fx)$

1	(1)	$Fa \land \neg Fa$	A
∅	(2)	$\neg(Fa \land \neg Fa)$	1,1 RA
∅	(3)	$\forall x\, \neg(Fx \land \neg Fx)$	2 UI

4. $\vdash \neg\exists x(Fx \land \neg Fx)$

1	(1)	$\exists x(Fx \land \neg Fx)$	A
2	(2)	$Fa \land \neg Fa$	A
2	(3)	$\neg\exists x(Fx \land \neg Fx)$	1,2 RA
1	(4)	$\neg\exists x(Fx \land \neg Fx)$	1,2,3 EE
1	(5)	$\exists x(Fx \land \neg Fx) \land \neg\exists x(Fx \land \neg Fx)$	1,3 ∧I
∅	(6)	$\neg\exists x(Fx \land \neg Fx)$	1,5 RA

5. $\vdash \forall x\exists y(Rxy \to Rxx)$

1	(1)	Raa	A
∅	(2)	$Raa \to Raa$	1,1 CP
∅	(3)	$\exists y(Ray \to Raa)$	2 EI
∅	(4)	$\forall x\exists y(Rxy \to Rxx)$	3 UI

6. $\vdash \forall x\exists y(Rxy \to Ryx)$

1	(1)	Raa	A
∅	(2)	$Raa \to Raa$	1,1 CP
∅	(3)	$\exists y(Ray \to Rya)$	2 EI
∅	(4)	$\forall x\exists y(Rxy \to Ryx)$	3 UI

7. $\vdash \exists x(Fx \to \forall y Fy)$

1	(1)	$\neg\exists x(Fx \to \forall y Fy)$	A
2	(2)	$\neg Fa$	A
2	(3)	$Fa \to \forall y Fy$	2 prop taut
2	(4)	$\exists x(Fx \to \forall y Fy)$	3 EI
1,2	(5)	$\exists x(Fx \to \forall y Fy) \land \neg\exists x(Fx \to \forall y Fy)$	4,1 $\land$I
1	(6)	$\neg\neg Fa$	2,5 RA
1	(7)	Fa	6 DN
1	(8)	$\forall y Fy$	7 UI
1	(9)	$Fa \to \forall y Fy$	8 prop taut
1	(10)	$\exists x(Fx \to \forall y Fy)$	9 EI
1	(11)	$\exists x(Fx \to \forall y Fy) \land \neg\exists x(Fx \to \forall y Fy)$	10,1 $\land$I
$\varnothing$	(12)	$\neg\neg\exists x(Fx \to \forall y Fy)$	1,11 RA
$\varnothing$	(13)	$\exists x(Fx \to \forall y Fy)$	12 DN

8. $\vdash \exists x \forall y(Fx \to Fy)$

1	(1)	$\neg\exists x \forall y(Fx \to Fy)$	A
2	(2)	$\neg Fa$	A
2	(3)	$Fa \to Fb$	2 prop taut
2	(4)	$\forall y(Fa \to Fy)$	3 UI
2	(5)	$\exists x \forall y(Fx \to Fy)$	4 EI
1,2	(6)	$\exists x \forall y(Fx \to Fy) \land \neg\exists x \forall y(Fx \to Fy)$	5,1 $\land$I
1	(7)	$\neg\neg Fa$	2,6 RA
1	(8)	Fa	7 DN
1	(9)	$Fc \to Fa$	8 prop taut
1	(10)	$\forall y(Fc \to Fy)$	9 UI
1	(11)	$\exists x \forall y(Fx \to Fy)$	10 EI
1	(12)	$\exists x \forall y(Fx \to Fy) \land \neg\exists x \forall y(Fx \to Fy)$	11,1 $\land$I
$\varnothing$	(13)	$\neg\neg\exists x \forall y(Fx \to Fy)$	1,12 RA
$\varnothing$	(14)	$\exists x \forall y(Fx \to Fy)$	13 DN

9. $\forall x \exists y(Fx \to Gy) \vdash \exists y \forall x(Fx \to Gy)$

1	(1)	$\forall x \exists y(Fx \to Gy)$	A
$\varnothing$	(2)	$\exists y Gy \lor \neg \exists y Gy$	prop taut
3	(3)	$\exists y Gy$	A
4	(4)	Ga	A
4	(5)	$Fb \to Ga$	4 prop taut
4	(6)	$\forall x(Fx \to Ga)$	5 UI
4	(7)	$\exists y \forall x(Fx \to Gy)$	6 EI
3	(8)	$\exists y \forall x(Fx \to Gy)$	3,4,7 EE
9	(9)	$\neg \exists y Gy$	A
10	(10)	Fc	A
1	(11)	$\exists y(Fc \to Gy)$	1 UE
12	(12)	$Fc \to Gd$	A
10,12	(13)	Gd	12,10 MP
10,12	(14)	$\exists y Gy$	13 EI
9,10,12	(15)	$\exists y Gy \land \neg \exists y Gy$	14,9 $\land$I
9,12	(16)	$\neg Fc$	10,15 RA
9,12	(17)	$Fc \to Ge$	16 prop taut
1,9	(18)	$Fc \to Ge$	11,12,17 EE
1,9	(19)	$\forall x(Fx \to Ge)$	18 UI
1,9	(20)	$\exists y \forall x(Fx \to Gy)$	19 EI
1	(21)	$\exists y \forall x(Fx \to Gy)$	2,3,8,9,20 $\lor$E

10. $\vdash \forall x \exists y(Rxy \to \forall z Rxz)$

$\varnothing$	(1)	$\exists y \neg Ray \lor \neg \exists y \neg Ray$	prop taut
2	(2)	$\exists y \neg Ray$	A
3	(3)	$\neg Rab$	A
3	(4)	$Rab \to \forall z Raz$	3 prop taut
3	(5)	$\exists y(Ray \to \forall z Raz)$	4 EI
2	(6)	$\exists y(Ray \to \forall z Raz)$	2,3,5 EE
7	(7)	$\neg \exists y \neg Ray$	A
8	(8)	$\neg Rac$	A
8	(9)	$\exists y \neg Ray$	8 EI
7,8	(10)	$\exists y \neg Ray \land \neg \exists y \neg Ray$	9,7 $\land$I
7	(11)	$\neg \neg Rac$	8,10 RA
7	(12)	Rac	11 DN
7	(13)	$\forall z Raz$	12 UI
7	(14)	$Rab \to \forall z Raz$	13 prop taut
7	(15)	$\exists y(Ray \to \forall z Raz)$	14 EI
$\varnothing$	(16)	$\exists y(Ray \to \forall z Raz)$	1,2,6,7,15 $\lor$E
$\varnothing$	(17)	$\forall x \exists y(Rxy \to \forall z Rxz)$	16 UI

Exercise 6.17

$\forall x(\exists z Rxz \to \forall y Rxy), \exists x \exists y \vdash \exists x \forall y Rxy$

1	(1)	$\forall x(\exists z Rxz \to \forall y Rxy)$	A
2	(2)	$\exists x \exists y Rxy$	A
3	(3)	$\exists y Ray$	A
4	(4)	Rab	A
4	(5)	$\exists z Raz$	4 EI
1	(6)	$\exists z Raz \to \forall y Ray$	1 UE
1,4	(7)	$\forall y Ray$	6,5 MP
1,4	(8)	$\exists x \forall y Rxy$	7 EI
1,3	(9)	$\exists x \forall y Rxy$	3,4,8 EE
1,2	(10)	$\exists x \forall y Rxy$	2,3,9 EE

Question: Does it follow from these premises that $\forall x \forall y Rxy$?

Answer: No. $\circlearrowright a \longrightarrow b$

Chapter 7

Exercise 7.1

Here the proof is lengthened because of the strictness of the $=$ rules. From $a = c$ and $b = c$, we cannot immediately apply $=$E to get $a = b$.

1	(1)	$\exists x \forall y (Py \to y = x)$	A
2	(2)	$Pa \land Pb$	A
3	(3)	$\forall y (Py \to y = c)$	A
3	(4)	$Pa \to a = c$	3 UE
3	(5)	$Pb \to b = c$	3 UE
2	(6)	Pa	2 $\land$E
2	(7)	Pb	2 $\land$E
2,3	(8)	$a = c$	4,6 MP
2,3	(9)	$b = c$	5,7 MP
$\varnothing$	(10)	$b = b$	=I
2,3	(11)	$c = b$	10,9 =E
2,3	(12)	$a = b$	8,11 =E
1,2	(13)	$a = b$	1,3,12 EE
1	(14)	$(Pa \land Pb) \to a = b$	2,13 CP
1	(15)	$\forall y ((Pa \land Py) \to a = y)$	14 UI
1	(16)	$\forall x \forall y ((Px \land Py) \to x = y)$	15 UI

Exercise 7.2

1	(1)	$Fa \wedge \forall x(Fx \rightarrow x = a)$	A
2	(2)	Fb	A
1	(3)	$\forall x(Fx \rightarrow x = a)$	1 $\wedge$E
1	(4)	$Fb \rightarrow b = a$	3 UE
1,2	(5)	$b = a$	4,2 MP
6	(6)	$b = a$	A
1	(7)	Fa	1 $\wedge$E
$\varnothing$	(8)	$b = b$	=I
6	(9)	$a = b$	8,6 =E
1,6	(10)	Fb	7,9 =E
1	(11)	$Fb \leftrightarrow b = a$	2,5,6,10 CP$\times$2
1	(12)	$\forall x(Fx \leftrightarrow x = a)$	11 UI

1	(1)	$\forall x(Fx \leftrightarrow x = a)$	A
1	(2)	$Fa \leftrightarrow a = a$	1 UE
$\varnothing$	(3)	$a = a$	=I
1	(4)	Fa	2,3 MP
1	(5)	$Fb \leftrightarrow b = a$	1 UE
1	(6)	$Fb \rightarrow b = a$	5 $\wedge$E
1	(7)	$\forall x(Fx \rightarrow x = a)$	6 UI
1	(8)	$Fa \wedge \forall x(Fx \rightarrow x = a)$	4,7 $\wedge$I

Chapter 9

Exercise 9.14

1. If a formula φ contains only the connectives $\neg$ and $\wedge$, then we will say that it's a L_0-formula. To be precise, $A_1, \ldots, A_n \succ B$ means that there is a derivation of B from $A_1, \ldots, A_n$ using the $\neg$ and $\wedge$ rules, and where the connectives $\vee, \rightarrow$ do not appear in any of the formulas.

   It's true that if $A_1, \ldots, A_n, B$ are L_0-formulas, then $A_1, \ldots, A_n \vdash B$ only if $A_1, \ldots, A_n \succ B$. But the argument is complicated, because the derivation of $A_1, \ldots, A_n \vdash B$ might use other rules, and might also contain formulas with $\vee$ or $\rightarrow$. There are two ways we could try to prove the result: (1) Convert the existing derivation of $A_1, \ldots, A_n \vdash B$ into a derivation of $A_1, \ldots, A_n \succ B$. Converting steps of $\rightarrow$ and $\vee$ rules into derivations with $\neg$ and $\wedge$ rules is fairly straightforward. (E.g.: check that the L_0-equivalent of CP is derivable from the $\neg$ and $\wedge$ rules.) But there also needs to be an argument that intermediate formulas that are not L_0 can be replaced with L_0-formulas. This argument could be organized by defining a projection F of formulas onto L_0-formulas, and then by showing that $A_1 \ldots, A_n \vdash B$ implies $F(A_1), \ldots, F(A_n) \succ F(B)$. (2) Directly prove the completeness of the $\neg$ and $\wedge$ rules relative to L_0-sequents. That is, show that if $A_1, \ldots, A_n \vDash B$ then $A_1, \ldots, A_n \succ B$.