logic precept 2

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Review pset1

- Minor problems with translation: never write a non-atomic sentence as if it were atomic
- Minor problems with proofs: each rule applies to a sentence type, and can only be applied to entire sentences of that type

  E.g. DN applied to $P \to \neg\neg Q$ does not give $P \to Q$

  E.g. MP cannot be applied to $P \vee (Q \to R)$
- The harder problems in part C

```
(1) P∨(Q∧R)   A
(2) P∨Q       1 ∧E
```

```
Line (2) is not a valid application of ∧E. To apply ∧E to a line,
the formula on that line needs to be a conjunction. But the
formula on line (1) is a disjunction.
```

```
This new rule would be bad. Consider, for example, the following
"proof" that would be permitted by the new rule.

(1) (P∧Q)→R    A
(2) P→R        new rule

But we do not consider such inferences to be valid. For example,
let P = "I want a new bicycle" and Q = "I am able to buy a new
bicycle", and R = "I buy a new bicycle". Then the premise could be
true while the conclusion is false -- so the argument is not
valid.
```

Warmup: Old proof rules in new form

Exercise 2.5 ($\wedge$E, $\wedge$I, $\vee$I, MP, MT, DN)

1. $P \wedge (Q \wedge R) \dashv\vdash (P \wedge Q) \wedge R$

```
1  (1) p&(q&r)   A
1  (2) p         1 &E
1  (3) q&r       1 &E
1  (4) q         3 &E
1  (5) r         3 &E
1  (6) p&q       2,4 &I
1  (7) (p&q)&r   6,5 &I
```

Here we see, among other things, that and introduction tells us to combine dependencies, but that we don't need to include redundancies. In particular, we write just `1` as dependency in lines 6 and 7 instead of `1,1` or `1,1,1`.

2. $P \dashv\vdash P \wedge P$
3. $P \to \neg Q, Q \vdash \neg P$
4. $\neg\neg P \vdash \neg\neg P \wedge (P \vee Q)$
5. $\neg(P \wedge Q) \to R, \neg R \vdash P$
6. $P \to (Q \wedge R), A \to \neg R, P \vdash \neg A$
7. $\neg P \to \neg Q, Q \vdash P$
8. $P \vdash \neg\neg(P \vee Q)$

New proof rules

We will work in blocks. First block ends in problems A1 and A2 on the pset. The focus is on "pure" applications of CP.

$P \to R \vdash (P \wedge Q) \to R$

$P \to Q \vdash (R \to P) \to (R \to Q)$

$P \to Q \vdash (Q \to R) \to (P \to R)$

(A1) $P \to Q \vdash P \to (Q \vee R)$

(A2) $P \to (Q \to R) \vdash Q \to (P \to R)$

Second block ends in problems A3 and A4. The focus is on the "contrapositive maneuver".

$P \to Q \vdash \neg Q \to \neg P$

$\vdash (P \wedge Q) \to P$

(A3) $\neg P \vdash \neg(P \wedge Q)$

$\vdash P \to (P \vee Q)$

(A4) $\neg(P \vee Q) \vdash \neg P$

Third block ends in problems A5 and A6. We come back to this if we have time.

$P \vdash (P \to Q) \to Q$

(A5) $P \vdash (P \to \neg P) \to \neg P$

The fourth block ends in problems B1 and B2.

$(P \wedge Q) \vee (P \wedge R) \vdash P$

$P \vee Q \vdash Q \vee P$

(B1) $P \vee (Q \wedge R) \vdash P \vee Q$

The fifth block ends in problems B3 and B4.

$\neg P \vdash P \to Q$

$P, \neg P \vdash Q$

(B3) $P \vee Q, \neg P \vdash Q$

(B4) $(P \to R) \wedge (Q \to R) \vdash (P \vee Q) \to R$

Evaluating proofs

Exercise Which of the following proofs with CP is correct? If a proof is not correct, explain what is wrong with it, and say whether there is a simple fix, or whether it is fatally flawed. (The following "proofs" use a slightly different notation – one that is easier to input via keyboard. Hopefully the relation between the two notations will become clear from the context.)

```
1   (1) p&q     A
1   (2) p       1 &E
1   (3) q       2 &E
    (4) p>q     2,3 CP
```

```
1  (1) q     A
2  (2) p     A
1  (3) p>q   2,1 CP
```

Exercise Which of the following proofs with or-elimination is correct? If a proof is not correct, explain what is wrong with it, and say whether there is a simple fix, or whether it is fatally flawed. (The following "proofs" use a slightly different notation – one that is easier to input via keyboard. Hopefully the relation between the two notations will become clear from the context.)

```
1  (1) p|p    A
```

```
2  (2) p      A
1  (3) p      1,2,2,2,2 |E
```

```
1   (1) p|q    A
2   (2) p      A
3   (3) q      A
2,3 (4) p&q    2,3 &I
2,3 (5) p      4 &E
1   (6) p      1,2,2,3,5 |E
```

Additional practice problems

$P \vdash Q \to (P \wedge Q)$

$(P \to Q) \wedge (P \to R) \vdash P \to (Q \wedge R)$

$P \to (P \to Q) \vdash P \to Q$

$(P \vee Q) \to R \vdash P \to R$

$P \to (Q \to R), P \to Q \vdash P \to R$

$P \to (Q \to R) \vdash (P \to Q) \to (P \to R)$

$P \to Q \vdash \neg Q \to \neg P$

$(P \wedge Q) \to R \vdash P \to (Q \to R)$

$P \to (Q \to R) \vdash P \to (\neg R \to \neg Q)$

$(P \to Q) \to P \vdash (P \to Q) \to Q$ (Hint: Not as difficult as it looks. Assume $(P \to Q) \to P$ and $P \to Q$. The latter can be used both as the antecedent of a conditional, and as a conditional itself.)

$\vdash (P \wedge Q) \to (Q \wedge P)$

$\vdash Q \to (P \to Q)$

$\vdash Q \to (P \to P)$

$(P \to Q) \to P, Q \vdash P$

$(P \to Q) \to P \vdash \neg P \to P$

$(P \to R) \wedge (Q \to R) \vdash (P \vee Q) \to R$

$P \vee (Q \vee R) \dashv\vdash (P \vee Q) \vee R$

$P \wedge (Q \vee R) \dashv\vdash (P \wedge Q) \vee (P \wedge R)$

$P \vee (Q \wedge R) \dashv\vdash (P \vee Q) \wedge (P \vee R)$

$\neg P \vee Q \;\dashv\vdash\; P \to Q$

$\neg P \vee \neg Q \;\vdash\; \neg(P \wedge Q)$

$P \to Q \;\vdash\; \neg(P \wedge \neg Q)$

$\neg(P \wedge Q) \;\vdash\; \neg P \vee \neg Q$

$\neg(P \to Q) \;\vdash\; P \wedge \neg Q$

$\vdash (P \to Q) \vee (Q \to P)$

$P \to (Q \vee R) \;\vdash\; (P \to Q) \vee (P \to R)$

$(P \wedge Q) \to \neg Q \;\vdash\; P \to \neg Q$

$P \wedge \neg Q \;\vdash\; \neg(P \to Q)$

$\vdash\; ((P \to Q) \to P) \to P$ (Hint: One possibility is to first prove $\vdash P \vee \neg P$, and then argue by cases. The first case is easy if you remember "positive paradox". For the second case, remember "negative paradox", i.e. that $\neg P$ implies $P \to Q$.)

$P \to (Q \vee R) \;\vdash\; \neg R \to (\neg Q \to \neg P)$

$P \to \neg Q \;\vdash\; (P \wedge Q) \to R$

$P \to \neg Q \;\vdash\; P \to (Q \to R)$

$\neg(P \to Q) \;\vdash\; P \to \neg Q$

$P \;\dashv\vdash\; (P \wedge Q) \vee (P \wedge \neg Q)$

$P \to (Q \to R) \;\dashv\vdash\; (P \to Q) \to (P \to R)$

$P \to \neg P \;\dashv\vdash\; \neg P$

$P \to (Q \to \neg Q) \;\dashv\vdash\; P \to \neg Q$

$(P \to Q) \to (P \to \neg Q) \;\dashv\vdash\; P \to \neg Q$

$P \;\dashv\vdash\; P \wedge (Q \vee \neg Q)$

$P \;\dashv\vdash\; P \vee (Q \wedge \neg Q)$

$(P \to Q) \to Q \;\vdash\; (Q \to P) \to P$

$(P \to Q) \to R \;\vdash\; (P \to R) \to R$

$(P \to R) \to R \;\dashv\vdash\; P \vee R$ (Hint: it's easy to derive $\neg P \to R$ from the sentence on the left.)

$(P \to Q) \to P \;\dashv\vdash\; P$ (Hint: assume $\neg P$ and derive $P \to Q$.)