Theories

PHI 201 — Introductory Logic

Lecture 9

The utopian vision of symbolic logic

- Two original hopes for symbolic logic.
  1. It provides a universal language for science.
  2. It dissolves philosophical pseudo-problems.
- While this doesn't work out so easily in practice, there is a sense in which all "theories" in mathematics can be formalized in predicate/relational logic.

Equality

Equality is a special relation

- Equality is a binary relation which we write as an infix rather than as a prefix

$$c = d,\ \exists x(x = d),\ \forall y \exists x(x = y)$$

- Using "$=$" allows us to express many new things.

At least n

$$\exists x \exists y (x \neq y)$$

$$\exists x \exists y \exists z ((x \neq y \wedge x \neq z) \wedge y \neq z)$$

At most n

$$\forall x \forall y (x = y)$$

$$\forall x \forall y \forall z ((x = y \vee x = z) \vee y = z)$$

Exactly n

$$\exists x \exists y (x \neq y \wedge \forall z (z = x \vee z = y))$$

$$\begin{aligned} \exists x\, \exists y\, \exists z (((x \neq y \wedge x \neq z) \wedge y \neq z) \\ \wedge \forall w\, ((w = x \vee w = y) \vee w = z)) \end{aligned}$$

There is a unique P

$$\exists x(Px \land \forall y(Py \to x = y))$$

Definite descriptions

Superlatives

"There is a tallest student."

$$\exists x \forall y (x \neq y \to Txy)$$

This sentence entails uniqueness only because we implicitly assume that "taller than" is asymmetric.

$$\forall x \forall y (Txy \to \neg Tyx)$$

1	(1)	$\exists x \forall y(x \neq y \to Txy)$	A
2	(2)	$\forall y(a \neq y \to Tay)$	A
3	(3)	$\forall y(b \neq y \to Tby)$	A
4	(4)	$a \neq b$	A
2	(5)	$a \neq b \to Tab$	2 UE
3	(6)	$b \neq a \to Tba$	3 UE

Inference rules for equality

$$
\begin{array}{rcll}
\Gamma & \text{(m)} & \varphi(a) & \\
\Delta & \text{(n)} & a = b & \\
\Gamma, \Delta & \text{(o)} & \varphi(b) & \qquad\qquad \text{m,n} =\!\text{E}
\end{array}
$$

To show: $a = b, b = c \vdash a = c$

1	(1)	$a = b$	A
2	(2)	$b = c$	A
1,2	(3)	$a = c$	2,1 $=$E

Inference rules for equality

(m) $a = a$ $\qquad\qquad\qquad\qquad =\mathsf{I}$

To show: $a = b \vdash b = a$

1	(1)	$a = b$	A
	(2)	$a = a$	$=$I
1	(3)	$b = a$	2,1 $=$E

The only one

Alice respects nobody but Bob.

$$Rab \wedge \forall x(Rax \rightarrow x = b)$$

$$\forall x(Rax \leftrightarrow x = b)$$

Order relations

transitive:

$$\forall x \forall y \forall z((x \leq y \wedge y \leq z) \rightarrow x \leq z)$$

reflexive:

$$\forall x(x \leq x)$$

antisymmetric:

$$\forall x \forall y((x \leq y \wedge y \leq x) \rightarrow x = y)$$

linear:

$$\forall x \forall y((x \leq y) \vee (y \leq x))$$

- What's a sentence that is true of the natural numbers $1, 2, 3, \ldots$ but false of the integers $\ldots, -2, -1, 0, 1, 2, \ldots$?
- What's a sentence that is true of the integers but false of the rational numbers?

Set theory

extensionality

$$\forall x \forall y (x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y))$$

empty set

$$\exists z \forall x (x \notin z)$$

Uniqueness of the emptyset

1	(1)	$\forall x(x \notin a)$	A
2	(2)	$\forall x(x \notin b)$	A
1	(3)	$c \notin a$	1 UE
1	(4)	$c \in a \to c \in b$	3 neg par

pairing

$$\forall x \forall y \exists z \forall w (w \in z \leftrightarrow (w = x \vee w = y))$$

separation: For every formula $\varphi(x, b_1, \dots, b_n)$,

$$\forall y \exists z \forall x \Big(x \in z \ \leftrightarrow \ (x \in y \wedge \ \varphi(x, b_1, \dots, b_n)) \Big).$$

Existence and uniqueness of intersections

	(1)	$\exists z \forall x(x \in z \leftrightarrow (x \in a \land x \in b))$	sep
2	(2)	$\forall x(x \in c \leftrightarrow (x \in a \land x \in b))$	A
	(3)	$\forall y \forall y'(\forall x(x \in y \leftrightarrow x \in y') \to y = y')$	ext
4	(4)	$\forall x(x \in d \leftrightarrow (x \in a \land x \in b))$	A
2,4	(5)	$\forall x(x \in c \leftrightarrow x \in d)$	2,4
	(6)	$\forall x(\in c \leftrightarrow x \in d) \to c = d$	3 UE
2,4	(8)	$c = d$	6,5 MP