PHI 201: Lecture 2

Supposition & Hypothetical Reasoning

Hans Halvorson

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Deducing versus Supposing

- A new kind of rule
- A new kind of proof format

A simple example

Argument

1. If P then Q
2. If Q then R
3. Therefore, if P then R

- What licences inferring a **conditional** statement?
- Hypothetical thinking: Supposing

How to prevent mistakes when supposing

- Repaying your debts
- If P then Q
- If Q then R

Keeping track of assumptions

Rule of Assumptions (A)

Form of the rule

$n \quad (n)\ \varphi \qquad \mathsf{A}$

Explanation

- On line (n), you may write any formula φ.
- The dependency of line (n) is its own line number n.
- The justification is marked A (Assumption).

$\wedge$-Introduction ($\wedge$I)

Form of the rule

Δ	(m)	P	
Γ	(n)	Q	
Δ, Γ	(k)	$P \wedge Q$	$m, n \ \wedge \mathrm{I}$

Explanation

- If you have P on line m (with dependencies Δ), and Q on line n (with dependencies Γ), then you may infer $P \wedge Q$.
- The new line k depends on all assumptions of both lines, i.e. the union of Δ and Γ.
- The justification cites both lines: m, n $\wedge$I

$\wedge$-Elimination ($\wedge$E)

Form of the rule

Δ	(m)	$P \wedge Q$	
Δ	(k)	P	$m\ \wedge\mathrm{E}$

Explanation

- If you have $P \wedge Q$ on line m (with dependencies Δ), you may infer either conjunct.
- The new line k carries exactly the same dependency set Δ.
- The justification cites line m: m $\wedge$E.

Keeping track of dependencies

Deps	Line	Formula	Justification
1	(1)	$P \wedge Q$	A
1	(2)	P	1 $\wedge$E
1	(3)	Q	1 $\wedge$E
1	(4)	$Q \wedge P$	3,2 $\wedge$I

$\vee$-Introduction ($\vee$I)

Form of the rule

$$
\begin{array}{llll}
\Delta & (m) & P & \\
\Delta & (k) & P \vee Q & m \ \vee\mathrm{I}
\end{array}
$$

Explanation

- If you have P on line m (with dependencies Δ), you may infer a disjunction $P \vee Q$ on a new line.
- You are free to introduce any formula Q as the other disjunct.
- The new line k carries the same dependencies Δ.
- The justification cites the original line: m $\vee$E.

Modus Ponens (MP)

Form of the rule

$$
\begin{array}{lll}
\Delta & (m) & P \to Q \\
\Gamma & (n) & P \\
\Delta, \Gamma & (k) & Q \qquad\qquad\qquad\qquad m, n \text{ MP}
\end{array}
$$

Explanation

- If you have $P \to Q$ on line m (with dependencies Δ) and P on line n (with dependencies Γ), then you may infer Q.
- The new line k carries the union of dependencies: $\Delta \cup \Gamma$.
- The justification cites both lines: m, n MP.

Modus Tollens (MT)

Form of the rule

Δ	(m)	$P \to Q$	
Γ	(n)	$\neg Q$	
Δ, Γ	(k)	$\neg P$	m, n MT

Explanation

- If you have $P \to Q$ on line m (with dependencies Δ), and $\neg Q$ on line n (with dependencies Γ), then you may infer $\neg P$.
- The new line k depends on all assumptions of both lines, i.e. $\Delta \cup \Gamma$.
- The justification cites both lines: m, n MT.

Keeping track of dependencies

Deps	Line	Formula	Justification
1	(1)	$P \to (Q \to R)$	A
2	(2)	$\neg R \wedge P$	A
2	(3)	P	2 $\wedge$E
1,2	(4)	$Q \to R$	1,3 MP
2	(5)	$\neg R$	2 $\wedge$E
1,2	(6)	$\neg Q$	4,5 MT

Double Negation (DN)

Form of the rule

Δ	(m)	P	
Δ	(k)	$\neg\neg P$	m DN

Δ	(m)	$\neg\neg P$	
Δ	(k)	P	m DN

Explanation

- From P you may infer $\neg\neg P$, or from $\neg\neg P$ you may infer P.
- In either case, the dependency set Δ is preserved.
- The justification cites the relevant line: m DN.

Summary

- For all of the deducing rules (Chapter 2), the dependencies on the new line are the aggregate of the dependencies of the cited lines.
- Dependency order does not matter
  $1, 2$ is the same as $2, 1$
- Dependency duplication does not matter
  No difference between $1, 1$ and 1

Summary

Key Idea

Each proof line makes a statement:

$$\Delta \quad (n) \quad P \qquad\qquad\qquad\qquad *$$

The sentences on lines Δ logically imply P.

Watch Out!

Hint: keep an eye out for suspicious lines, for example:

$$\begin{array}{llll} 1 & (1) & P & \text{A} \\ & \vdots & & \\ 1 & (n) & P \wedge Q & \end{array}$$

Conditional proof

Conditional Proof (CP)

Form of the rule

n	(n)	P	A
Δ	(m)	Q	
$\Delta \setminus \{n\}$	(k)	$P \to Q$	n, m CP

Explanation

- Start a subproof at line n by assuming P (A).
- Derive Q on line m with dependencies Δ.
- By CP, infer $P \to Q$ on line k; its dependencies are $\Delta \setminus \{n\}$ (discharge n).

Conditional proof

Deps	Line	Formula	Justification
1	(1)	$P \rightarrow Q$	A
2	(2)	$Q \rightarrow R$	A
3	(3)	P	A
1,3	(4)	Q	1,3 MP
1,2,3	(5)	R	2,4 MP
1,2	(6)	$P \rightarrow R$	3,5 CP

Conditional proof

Deps	Line	Formula	Justification
1	(1)	$(P \wedge Q) \to R$	A
2	(2)	P	A
3	(3)	Q	A
2,3	(4)	$P \wedge Q$	2,3 ∧I
1,2,3	(5)	R	1,4 MP
1,2	(6)	$Q \to R$	3,5 CP
1	(7)	$P \to (Q \to R)$	2,6 CP

Contrapositive

Deps	Line	Formula	Justification
1	(1)	$P \to Q$	Assumption
2	(2)	$\neg Q$	A
1,2	(3)	$\neg P$	1,2 MT
1	(4)	$\neg Q \to \neg P$	2,3 CP

Proofs without premises

Deps	Line	Formula	Justification
1	(1)	$P \wedge Q$	A
1	(2)	P	1 $\wedge$E
	(3)	$(P \wedge Q) \to P$	1,2 CP

DeMorgan's rule

Deps	Line	Formula	Justification
1	(1)	$\neg(P \vee Q)$	Assumption
2	(2)	P	A
2	(3)	$P \vee Q$	2 $\vee$I
$\varnothing$	(4)	$P \to (P \vee Q)$	2,3 CP
1	(5)	$\neg P$	4,1 MT

Proofs without premises

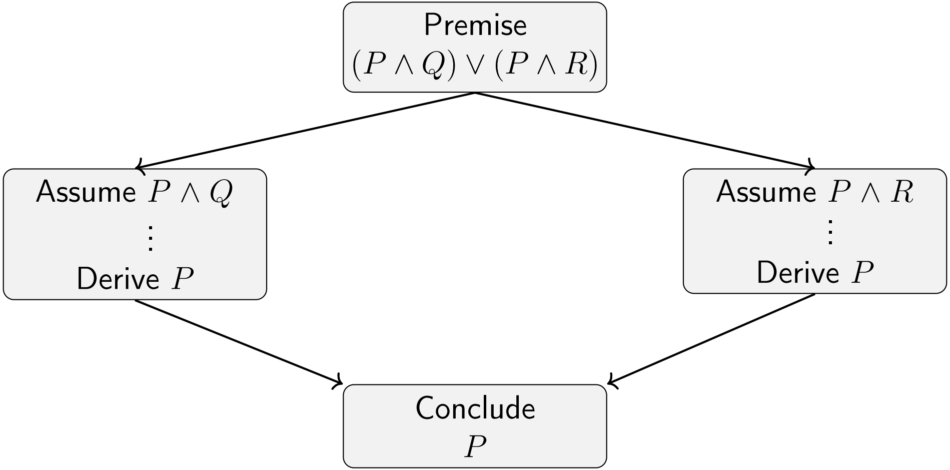
Deps	Line	Formula	Justification
1	(1)	P	A
	(2)	$P \to P$	1,1 CP

Conditional proof

Deps	Line	Formula	Justification
1	(1)	$P \wedge Q$	A
1	(2)	P	1 $\wedge$E
1	(3)	Q	1 $\wedge$E
	(4)	$P \to Q$	2,3 CP

Disjunction elimination

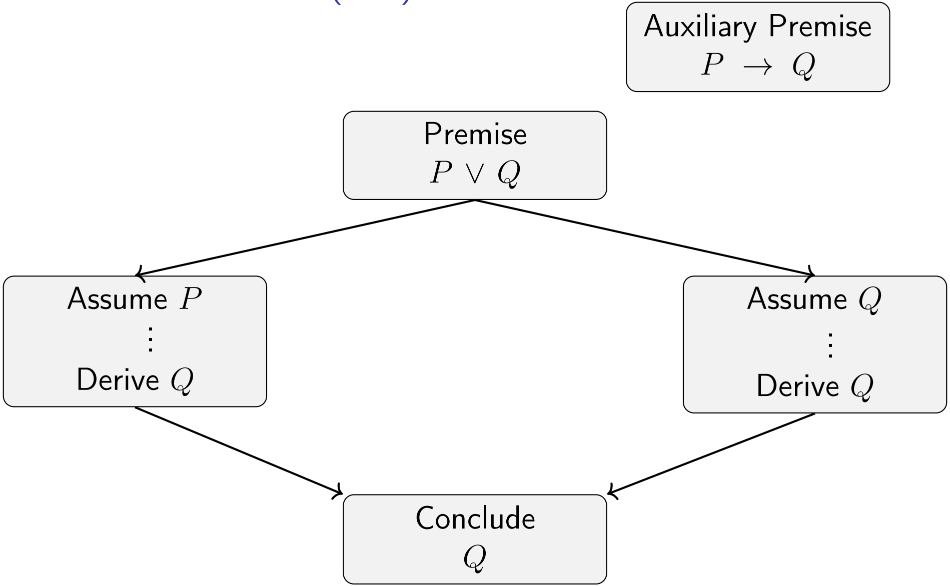
Disjunction Elimination (∨E)



Disjunction elimination

Deps	Line	Formula	Justification
1	(1)	$(P \wedge Q) \vee (P \wedge R)$	A
2	(2)	$P \wedge Q$	A
2	(3)	P	2 $\wedge$E
4	(4)	$P \wedge R$	A
4	(5)	P	4 $\wedge$E
1	(6)	P	1,2,3,4,5 $\vee$E

Disjunction Elimination (∨E)



Disjunction elimination

Deps	Line	Formula	Justification
1	(1)	$P \to Q$	A
2	(2)	$P \vee Q$	A
3	(3)	P	A
1,3	(4)	Q	1,3 MP
5	(5)	Q	A
1,2	(6)	Q	2,3,4,5,5 $\vee$E

Disjunction elimination

Deps	Line	Formula	Justification
1	(1)	$P \vee P$	A
2	(2)	P	A
1	(3)	P	1,2,2,2,2 $\vee$E

Subtleties of conditional proof

Positive paradox

Deps	Line	Formula	Justification
1	(1)	Q	A
2	(2)	P	A
1	(3)	$P \to Q$	2,1 CP

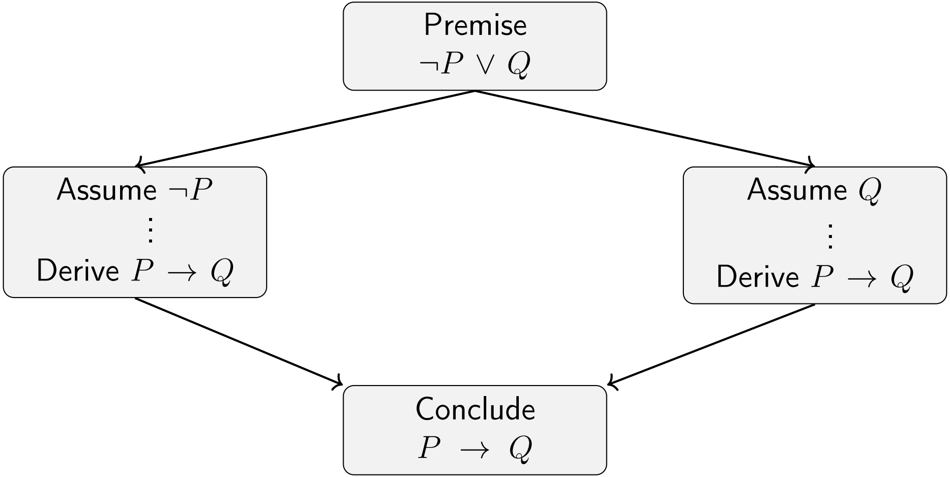
Negative paradox

Deps	Line	Formula	Justification
1	(1)	$\neg P$	A
2	(2)	P	A
3	(3)	$\neg Q$	A
2	(4)	$\neg Q \to P$	3,2 CP
1,2	(5)	$\neg\neg Q$	4,1 MT
1,2	(6)	Q	5 DN
1	(7)	$P \to Q$	2,6 CP

Ex Falso Quodlibet

Deps	Line	Formula	Justification
1	(1)	P	A
2	(2)	$\neg P$	A
3	(3)	$\neg Q$	Assumption
2	(4)	$\neg Q \to \neg P$	3,2 CP
1	(5)	$\neg\neg P$	1 DN
1,2	(6)	$\neg\neg Q$	4,5 MT
1,2	(7)	Q	6 DN

Material conditional



Summary

- From now on, proofs consist of **four** columns.
- For the deducing rules, we collect dependency numbers from the cited lines.
- **Conditional proof** allows us to derive a conditional statement from a "subproof" in which we make a new assumption.
- **Disjunction elimination** allows us to draw a conclusion from a disjunction if we can draw it from each disjunct separately.