Practice Problems: Predicate Logic

A. Proofs

1. $Fa \vdash \forall x((a = x) \to Fx)$

1	(1)	Fa	A
2	(2)	$a = b$	A
1,2	(3)	Fb	1,2 =E
1	(4)	$a = b \to Fb$	2,3 CP
1	(5)	$\forall x(a = x \to Fx)$	4 UI

2. $\vdash \forall x \forall y\big((Fx \wedge \neg Fy) \to x \neq y\big)$

1	(1)	$Fa \wedge \neg Fb$	A
2	(2)	$a = b$	A
1	(3)	Fa	1 $\wedge$E
1	(4)	$\neg Fb$	1 $\wedge$E
1,2	(5)	Fb	3,2 =E
1,2	(6)	$Fb \wedge \neg Fb$	5,4 $\wedge$I
1	(7)	$a \neq b$	2,6 RA
$\varnothing$	(8)	$(Fa \wedge \neg Fb) \to a \neq b$	1,7 CP
$\varnothing$	(9)	$\forall y((Fa \wedge \neg Fy) \to a \neq y)$	8 UI
$\varnothing$	(10)	$\forall x \forall y((Fx \wedge \neg Fy) \to x \neq y)$	9 UI

3. $\forall x(Fx \to \exists y(Gy \wedge (y = x)) \vdash \forall x(Fx \to Gx)$

1	(1)	$\forall x(Fx \to \exists y(Gy \wedge (y = x)))$	A
2	(2)	Fa	A
1	(3)	$Fa \to \exists y(Gy \wedge (y = a))$	1 UE
1,2	(4)	$\exists y(Gy \wedge (y = a))$	3,2 MP
5	(5)	$Gb \wedge (b = a)$	A
5	(6)	Gb	5 $\wedge$E
5	(7)	$b = a$	5 $\wedge$E
5	(8)	Ga	6,7 =E
1,2	(9)	Ga	4,5,8 EE
1	(10)	$Fa \to Ga$	2,9 CP
1	(11)	$\forall x(Fx \to Gx)$	10 UI

4. $\forall y(Ray \to y = b) \vdash \exists y(Ray \wedge Gy) \to Gb$

5. $\forall x \forall y \forall z(Rxy \to \neg Ryz) \vdash \exists y \forall x \neg Rxy$

   First strategy: Assume the negation of the conclusion for RA. Use QN

1	(1)	$\forall x \forall y \forall z(Rxy \to \neg Ryz)$	A
2	(2)	$\neg\exists y \forall x \neg Rxy$	A
2	(3)	$\forall y \neg \forall x \neg Rxy$	2 QN
2	(4)	$\neg \forall x \neg Rxb$	3 UE
2	(5)	$\exists x \neg\neg Rxb$	4 QN
6	(6)	$\neg\neg Rab$	A
6	(7)	Rab	6 DN
1	(8)	$\forall y \forall z(Ray \to \neg Ryz)$	1 UE
1	(9)	$\forall z(Rab \to \neg Rbz)$	8 UE
1	(10)	$Rab \to \neg Rbc$	9 UE
1,6	(11)	$\neg Rbc$	10,7 MP
1,2	(12)	$\neg Rbc$	5,6,11 EE
1,2	(13)	$\forall x \neg Rxc$	12 UI
1,2	(14)	$\exists y \forall x \neg Rxy$	13 EI
1,2	(15)	$\exists y \forall x \neg Rxy \wedge \neg\exists y \forall x \neg Rxy$	14,2 $\wedge$I
1	(16)	$\neg\neg\exists y \forall x \neg Rxy$	2,15 RA
1	(17)	$\exists y \forall x \neg Rxy$	16 DN

Second strategy: Either there is a pair of elements with an arrow between them or not. In the first case, there cannot be an arrow into the first element. In the second case, pick anything in the domain, and there is no arrow into it. In either case, there is something that has no arrow into it.

1	(1)	$\forall x \forall y \forall z(Rxy \to \neg Ryz)$	A
$\varnothing$	(2)	$\exists y \exists x Rxy \vee \neg\exists y \exists x Rxy$	prop taut
3	(3)	$\exists y \exists x Rxy$	A
4	(4)	$\exists x Rxb$	A
5	(5)	Rab	A
6	(6)	Rca	A
1	(7)	$\forall y \forall z(Rcy \to \neg Ryz)$	1 UE
1	(8)	$\forall z(Rca \to \neg Raz)$	7 UE
1	(9)	$Rca \to \neg Rab$	8 UE
5	(10)	$\neg\neg Rab$	5 DN
1,5	(11)	$\neg Rca$	9,10 MT
1,5	(12)	$\forall x \neg Rxa$	11 UI
1,5	(13)	$\exists y \forall x \neg Rxy$	12 EI
1,4	(14)	$\exists y \forall x \neg Rxy$	4,5,13 EE
1,3	(15)	$\exists y \forall x \neg Rxy$	3,4,14 EE
16	(16)	$\neg\exists y \exists x Rxy$	A
16	(17)	$\forall y \neg\exists x Rxy$	16 QN
16	(18)	$\neg\exists x Rxa$	17 UE
16	(19)	$\forall x \neg Rxa$	18 QN
16	(20)	$\exists y \forall x \neg Rxy$	19 EI
1	(21)	$\exists y \forall x \neg Rxy$	2,3,15,16,20 $\vee$E

6. $\forall x Fx \leftrightarrow \neg\exists x\exists y Rxy \vdash \exists x\forall y\forall z(Fx \to \neg Ryz)$

   First strategy: Either $\forall x Fx$ or $\exists x\neg Fx$. In the former case, the premise gives $\neg\exists x\exists y Rxy$, and QN gives $\forall y\forall z\neg Ryz$. In the latter case, $Fa \to \neg Rbc$ by negative paradox.

7. $\vdash \forall x\exists y\forall z(\exists u Txyu \to \exists v Txzv)$

   We show how to derive the instance $\exists y\forall z(\exists u Tayu \to \exists v Tazv)$. Up to α-equivalence, the embedded conditional is of the form $\varphi(y) \to \varphi(z)$. So it's enough to prove that $\exists y\forall z(\varphi(y) \to \varphi(z))$. We have seen this before: it's a substitution instance of $\exists x\forall y(Fx \to Fy)$.

8. Show that the sentence $\forall x\exists y\forall z(Rxy \wedge \neg Ryz)$ is inconsistent.

1	(1)	$\forall x\exists y\forall z(Rxy \wedge \neg Ryz)$	A
1	(2)	$\exists y\forall z(Ray \wedge \neg Ryz)$	1 UE
3	(3)	$\forall z(Rab \wedge \neg Rbz)$	A
1	(4)	$\exists y\forall z(Rby \wedge \neg Ryz)$	1 UE
5	(5)	$\forall z(Rbc \wedge \neg Rcz)$	A
3	(6)	$Rab \wedge \neg Rbc$	3 UE
5	(7)	$Rbc \wedge \neg Rcd$	5 UE
3	(8)	$\neg Rbc$	6 $\wedge$E
5	(9)	Rbc	7 $\wedge$E
3,5	(10)	$Rbc \wedge \neg Rbc$	9,8 $\wedge$I
3,5	(11)	$P \wedge \neg P$	10 prop taut
1,3	(12)	$P \wedge \neg P$	4,5,11 EE
1	(13)	$P \wedge \neg P$	2,3,12 EE

9. $\exists x\exists y(Fx \leftrightarrow \neg Fy) \vdash \forall x\exists y(Fx \leftrightarrow \neg Fy)$

1	(1)	$\exists x\exists y(Fx \leftrightarrow \neg Fy)$	A
2	(2)	$\neg\exists y(Fa \leftrightarrow \neg Fy)$	A
2	(3)	$\forall y\neg(Fa \leftrightarrow \neg Fy)$	2 QN
2	(4)	$\neg(Fa \leftrightarrow \neg Fb)$	3 UE
2	(5)	$Fa \leftrightarrow Fb$	4 prop taut
2	(6)	$\forall z(Fa \leftrightarrow Fz)$	5 UI
7	(7)	$\exists y(Fc \leftrightarrow \neg Fy)$	A
8	(8)	$Fc \leftrightarrow \neg Fd$	A
2	(9)	$Fa \leftrightarrow Fc$	6 UE
2	(10)	$Fa \leftrightarrow Fd$	6 UE
2,8	(11)	$P \wedge \neg P$	8,9,10 prop taut
8	(12)	$\neg\neg\exists y(Fa \leftrightarrow \neg Fy)$	2,11 RA
8	(13)	$\exists y(Fa \leftrightarrow \neg Fy)$	12 DN
7	(14)	$\exists y(Fa \leftrightarrow \neg Fy)$	7,8,13 EE
1	(15)	$\exists y(Fa \leftrightarrow \neg Fy)$	1,7,14 EE
1	(16)	$\forall x\exists y(Fx \leftrightarrow \neg Fy)$	15 UI

10. $\exists x \exists y(Fx \leftrightarrow \neg Fy) \vdash \exists x \exists y(Fx \wedge \neg Fy)$

1	(1)	$\exists x \exists y(Fx \leftrightarrow \neg Fy)$	A
2	(2)	$\exists y(Fa \leftrightarrow \neg Fy)$	A
3	(3)	$Fa \leftrightarrow \neg Fb$	A
3	(4)	$(Fa \wedge \neg Fb) \vee (Fb \wedge \neg Fa)$	3 prop taut
5	(5)	$Fa \wedge \neg Fb$	A
5	(6)	$\exists y(Fa \wedge \neg Fy)$	5 EI
5	(7)	$\exists x \exists y(Fx \wedge \neg Fy)$	6 EI
8	(8)	$Fb \wedge \neg Fa$	A
8	(9)	$\exists y(Fb \wedge \neg Fy)$	8 EI
8	(10)	$\exists x \exists y(Fx \wedge \neg Fy)$	9 EI
3	(11)	$\exists x \exists y(Fx \wedge \neg Fy)$	4,5,7,8,10 $\vee$E
2	(12)	$\exists x \exists y(Fx \wedge \neg Fy)$	2,3,11 EE
1	(13)	$\exists x \exists y(Fx \wedge \neg Fy)$	1,2,12 EE