Logic Precepts Workbook

PHI 201 – Introductory Logic

Fall 2025

Contents

1 Conditional proof and $\lor$ elimination

Warmup: Deducing

Exercise 2.5 ($\land$E, $\land$I, $\lor$I, MP, MT, DN)

$P \to \neg Q, Q \vdash \neg P$

$\neg\neg P \vdash \neg\neg P \land (P \lor Q)$

$\neg(P \land Q) \to R, \neg R \vdash P$

$\neg P \to \neg Q, Q \vdash P$

$P \vdash \neg\neg(P \lor Q)$

New proof rules

We will work in blocks. The first block leads to problem A1.

$P \to Q \vdash P \to (Q \lor R)$

$(P \lor Q) \to R \vdash P \to R$

$P \to Q \vdash (R \to P) \to (R \to Q)$

$P \to Q \vdash (Q \to R) \to (P \to R)$

(A1) $P \to (Q \to R) \vdash Q \to (P \to R)$

The second block leads to A2. The focus is on the "contrapositive maneuver".

$P \to Q \vdash \neg Q \to \neg P$

$\vdash P \to (P \lor Q)$

$\neg(P \lor Q) \vdash \neg P$

$\vdash (P \land Q) \to P$

(A2) $\neg P \vdash \neg(P \land Q)$

The third block leads to A3 and A4.

$P \vdash (P \to Q) \to Q$

(A3) $P \vdash (P \to \neg P) \to \neg P$

(A4) $Q \vdash \neg(Q \to \neg Q)$

The fourth block leads to problem B1.

$P \vee Q \vdash Q \vee P$

$P \vee (Q \wedge R) \vdash (P \wedge Q) \vee (P \wedge R)$

(B1) $P \wedge (Q \vee R) \vdash (P \wedge Q) \vee (P \wedge R)$

The fifth block leads to problem B2.

$P, \neg P \vdash Q$

(B2) $P \vee Q, \neg P \vdash Q$

Evaluating proofs

Exercise Which of the following proofs with CP is correct? If a proof is not correct, explain what is wrong with it, and say whether there is a simple fix, or whether it is fatally flawed.

Deps	Line	Formula	Justification
1	(1)	$P \wedge Q$	A
1	(2)	P	1 $\wedge$E
1	(3)	Q	2 $\wedge$E
	(4)	$P \to Q$	2,3 CP

Deps	Line	Formula	Justification
1	(1)	Q	A
2	(2)	P	A
1	(3)	$P \to Q$	2,1 CP

Exercise Explain what is wrong with the following "proof".

Deps	Line	Formula	Justification
1	(1)	$P \vee Q$	A
2	(2)	P	A
3	(3)	Q	A
2,3	(4)	$P \wedge Q$	2,3 $\wedge$I
2,3	(5)	P	4 $\wedge$E
1	(6)	P	1,2,2,3,5 $\vee$E

Additional practice problems

$\neg P \vee \neg Q \dashv\vdash \neg(P \wedge Q)$

$P \to (P \to Q) \vdash P \to Q$

$(P \vee Q) \to R \vdash P \to R$

$P \to (Q \to R), P \to Q \vdash P \to R$

$P \to (Q \to R) \vdash (P \to Q) \to (P \to R)$

$(P \to Q) \to P \vdash (P \to Q) \to Q$

$(P \to Q) \to P \vdash \neg P \to P$

$(P \to R) \wedge (Q \to R) \vdash (P \vee Q) \to R$

$P \vee (Q \vee R) \dashv\vdash (P \vee Q) \vee R$

$P \wedge (Q \vee R) \dashv\vdash (P \wedge Q) \vee (P \wedge R)$

$P \vee (Q \wedge R) \dashv\vdash (P \vee Q) \wedge (P \vee R)$

$\neg P \vee Q \dashv\vdash P \to Q$

$\neg(P \to Q) \dashv\vdash P \wedge \neg Q$

$\vdash (P \to Q) \vee (Q \to P)$

$P \to (Q \vee R) \vdash (P \to Q) \vee (P \to R)$

$\vdash ((P \to Q) \to P) \to P$ (Hint: One possibility is to first prove $\vdash P \vee \neg P$, and then argue by cases. The first case is easy if you remember "positive paradox". For the second case, remember "negative paradox", i.e. that $\neg P$ implies $P \to Q$.)

$P \to (Q \vee R) \vdash \neg R \to (\neg Q \to \neg P)$

$P \to \neg P \dashv\vdash \neg P$

$(P \to Q) \to Q \vdash (Q \to P) \to P$

$(P \to Q) \to R \vdash (P \to R) \to R$

$(P \to R) \to R \dashv\vdash P \vee R$ (Hint: derive $\neg P \to R$ from the sentence on the left.)

$(P \to Q) \to P \dashv\vdash P$ (Hint: assume $\neg P$ and derive $P \to Q$.)

2 Reductio ad absurdum

Proofs

Review of $\vee$-elimination

1. $(P \to Q) \vee (P \to R) \vdash P \to (Q \vee R)$
2. $\neg P \vee \neg Q \vdash \neg(P \wedge Q)$
3. $\neg P \vee Q \vdash P \to Q$

Reductio ad Absurdum

1. $P \to Q \vdash \neg(P \wedge \neg Q)$
2. $\neg(P \to Q) \vdash \neg Q$
3. pset1 $\neg(P \to Q) \vdash Q \to R$
4. $\neg(P \vee Q) \vdash \neg P$
5. pset2 $P \to Q \vdash \neg P \vee Q$
6. pset3 $P \to (Q \vee R) \vdash (P \to Q) \vee R$

Challenge problem: Pierce's law

$\vdash ((P \to Q) \to P) \to P$

Truth tables

Key Concepts

- arguments: valid, invalid
- counterexample
- truth-value
- main connective
- sentences (syntactic): atomic, conjunction, negation, disjunction, conditional, biconditional
- sentences (semantic): tautology, inconsistency, contingency
- two sentences: equivalent, inconsistent, independent

For arguments

Determine whether the following arguments are valid or not. Explain your answer by showing the existence of a row of a truth table, or by pointing to a full truth table, or something of the sort. Your answer should be articulated in English prose so that it can convince anyone else who is familiar with truth tables.

1. $P \to (Q \vee R) \vdash (P \to Q) \vee R$
2. $\vdash (P \leftrightarrow Q) \vee (P \leftrightarrow R) \vee (Q \leftrightarrow R)$
3. $P \to (Q \to R) \vdash (P \wedge Q) \to R$
4. $P \to R \vdash (P \vee Q) \to R$
5. $(P \leftrightarrow Q) \leftrightarrow R \vdash P \vee R$
6. $\vdash (P \to Q) \vee (Q \to R)$

Sentence classification (syntactic)

Exercise. What is the **main connective** of each of the following formulas?

1. $\neg(P \to Q)$
2. $\neg P \to Q$
3. $\neg(P \to \neg Q)$
4. $(P \wedge Q) \vee \neg(P \to Q)$
5. $((P \to Q) \to P) \to P$

Sentence classification (semantic)

Exercise. Classify each of the following sentences as tautology, inconsistency, or contingency.

1. $(P \to \neg P) \to \neg P$
2. $(P \wedge Q) \vee (\neg P \wedge \neg Q)$
3. $(P \wedge (Q \wedge \neg R)) \vee (\neg P \wedge (\neg Q \wedge R))$
4. $(P \leftrightarrow Q) \leftrightarrow R$

5. $(P \wedge Q) \vee \neg(P \rightarrow Q)$
6. $((P \rightarrow Q) \rightarrow R) \rightarrow Q$

Exercise. Show that if B is a tautology, then $A \wedge B$ is logically equivalent to A.

Exercise. Show that if B is an inconsistency, then $A \vee B$ is logically equivalent to A.

For multiple sentences

Exercise: What is the semantic relationship between $(P \wedge Q)$ and $\neg(P \rightarrow Q)$?

Exercise: If $\phi \wedge \psi$ is a contingency, then what are the possibilities for ϕ and ψ?

Exercise: If ϕ is a tautology, then what are the possibilities for $\phi \wedge \psi$? What are the possibilities for $\phi \vee \psi$?

3 Universal quantifier rules

Review

1. What kinds of sentences are there in predicate logic?
2. What is the difference between a **formula** and a **sentence**?
3. What is an **instance** of a universal sentence?
4. What is the restriction on UI?

Proofs

Warmup Problems

1. $\forall x(Fx \to Gx) \ \vdash \ \forall xFx \to \forall xGx$
2. $\forall x(Px \to Qx),\ \forall x(Qx \to Rx) \ \vdash \ \forall x(Px \to Rx)$
3. $P \to \forall xFx \ \vdash \ \forall x(P \to Fx)$
4. $\forall x\forall yRxy \ \vdash \ \forall y\forall xRxy$

Pset Problems

1. $\forall x(Fx \to \forall yGy) \ \vdash \ \forall x\forall y(Fx \to Gy)$
2. $\forall x\forall y(Fx \to Gy) \ \vdash \ \forall x(Fx \to \forall yGy)$
3. $\vdash \ \forall x(\forall yRxy \to Rxx)$

Translation

Exercise

How do you symbolize the following?

1. All F are G.
2. No F are G.
3. Some F are G.
4. Some F are not G.

Exercise

Use F for "is French", G for "is German", C for "is Canadian", Lxy for "x likes y", a for Alice, and b for Bob. How would you symbolize:

1. Alice likes Canadians.
2. Alice likes Bob only if Bob likes Canadians.
3. Alice likes Bob only if he likes her.
4. Alice is a German who likes Canadians.
5. Alice is French only if she doesn't like Canadians.
6. Alice likes only those people who don't like Canadians.
7. Someone likes only those people who like Canadians.
8. French people only like Canadians who don't like Germans.
9. Some French people like only those Germans who don't like themselves.

4 Theories

Translation

1. Mary is the only student who didn't miss any questions on the exam.
2. All professors except a are boring.
3. There is no greatest prime number.
4. The smallest prime number is even.
5. For each natural number, there is a unique next-greater natural number.
6. There are at least two Ivy League universities in New York state.

Proofs with equality

1. $Fa \vdash \forall x((x = a) \to Fx)$
2. $\forall x((x = a) \to Fx) \vdash Fa$
3. $\exists x \forall y(x = y) \vdash \forall x \forall y(x = y)$

Partial order

In real life, rigorous proofs are rarely written with line numbers, dependencies, or named justifications. But the idea is to give the reader enough information so that s/he could reconstruct such a proof.

1. Write down a predicate logic sentence that expresses the claim that every two elements have a least upper bound.
2. Give an example of a partially ordered set in which that sentence is false.
3. Prove (informally) that if any two elements have a least upper bound, then so do any three elements.
4. We say that $\leq$ is a serial relation just in case $\forall x \exists y(x \leq y \wedge x \neq y)$. Is there a *finite* partially ordered set that satisfies the serial axiom?

Set theory

For sets a and b, we write $a \subseteq b$ for the claim that $\forall x(x \in a \to x \in b)$.

We let $a \cap b$ be the set defined by $\forall x((x \in a \cup b) \leftrightarrow (x \in a \land x \in b))$.

1. Show that if $a \subseteq b$ and $b \subseteq c$ then $a \subseteq c$.
2. Show that $a \subseteq b$ if and only if $a \cap b = a$.

5 Models

1. What does it mean for a predicate logic sentence to be *consistent*?

2. Suppose that we had an algorithm that determined whether sentences are consistent. Explain how we could use this algorithm to determine if arguments are valid.

3. Consider the following English sentences, along with the two possible translations into predicate logic. Are the two PL sentences logically equivalent? Does one imply the other? How does this information bear on your judgment about which is the best translation?

   (a) Only students who do the homework will learn logic. (Sx, Hx, Lx)

   $$\forall x(Lx \to (Sx \land Hx))$$

   $$\forall x((Sx \land Lx) \to Hx)$$

   (b) There is some student who respects only those professors who respect all students. (Sx, Px, Rxy)

   $$\exists x(Sx \land \forall y(Rxy \to (Py \land \forall z(Sz \to Ryz))))$$

   $$\exists x(Sx \land \forall y((Py \land Rxy) \to \forall z(Sz \to Ryz)))$$

4. Explain why the sentence $\exists x(Mx \to Dx)$ is *not* a good translation of "There is a melancholy Dane."

5. Provide models to show that the following sequents are invalid:

   (a) $\forall x(Fx \lor Gx) \vdash \forall xFx \lor \forall xGx$

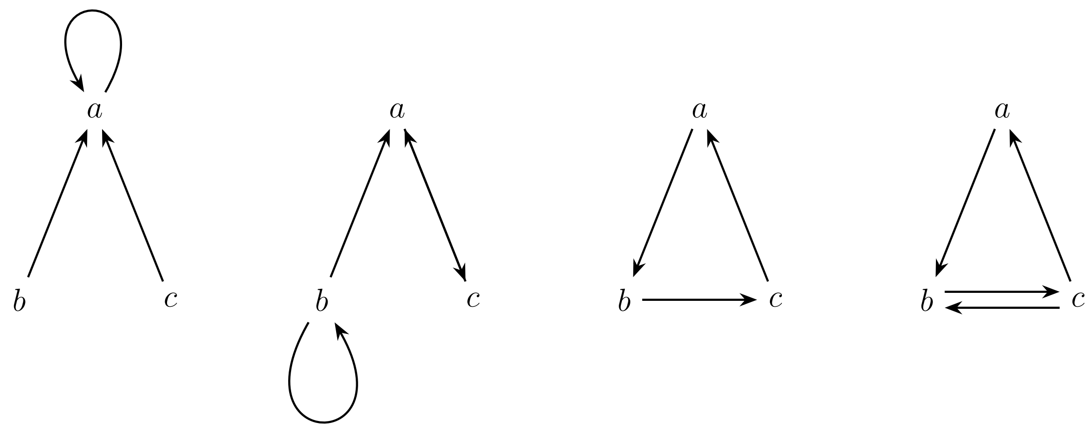
   (b) $\forall xFx \to \forall xGx \vdash \forall x(Fx \to Gx)$

   (c) $\exists x(Fx \to P) \vdash \exists xFx \to P$

6. The EE rule requires that the arbitrary name that is used in the instance of the existential formula does *not* appear in (a) the existential formula, (b) the auxiliary assumptions used to derive the conclusion, and (c) the conclusion itself. Explain why dropping any one of these three restrictions would lead to an unsound rule.

7. Which of the following sentences are true in which of the diagrams below.

   (a) $\forall x \forall y (Rxy \rightarrow Ryy)$
   (b) $\forall x \exists y (Rxy \wedge Ryx)$
   (c) $\exists x \forall y (Rxy \rightarrow \exists z Ryz)$
   (d) $\forall x \exists y (Rxy \wedge \forall z (Ryz \rightarrow Rxz))$
   (e) $\exists x \exists y (Rxy \wedge \neg Ryx)$



12. A theory of propositional logic

Prove something interesting about sentences

Let Γ be the collection of all sentences built with P, Q and the connectives $\neg$ and $\leftrightarrow$. Show that every sentence in Γ is *even* in the sense that its truth table has an even number of zeroes (and ones).

Fill in steps of soundness argument

Fill in steps of completeness argument