Metatheory

PHI 201 – Introductory Logic

Week 11

Remaining tasks

1. Figure out how to prove harder things (more reliably)
    - Example: $\vdash \exists x \forall y (Fx \to Fy)$
    - Idea: Convert semantic intuition into proof
2. Learn how to reason *about* propositional logic
    - New axiom schema: mathematical induction
    - Main results: soundness, completeness

A theory *about* propositional logic

- You'll keep using logic, but most of you won't study logic again in a formal setting like this one.
- But learning about how logic works will help you become better at doing logic.
  - Analogy to an athlete and understanding physiology and nutrition.
  - But that analogy fails to capture the fascinating fact that studying logic is another use of logic.

- To formalize a theory in predicate logic, one chooses some basic vocabulary (names, predicates, relations, functions).
- We are going to be talking about sentences, sequents, valuations, etc. So, for example, we would have a predicate symbol Sent(x) to mean that x is a sentence, and a relation symbol Seq$(x_1, \ldots, x_n, y)$ to mean that there is a valid proof whose last line has formula y with dependencies $x_1, \ldots, x_n$.

Mathematical induction

- The interesting theorems about propositional logic involve claims about infinite sets. For example:

  For every sentence φ, φ is provably equivalent to a sentence in which $\wedge$ does not occur.

- But our infinite sets are generated from a finite number of cases by a finite number of rules. There is a special method of proof for such sets: mathematical induction.

Aside: Function symbols

An n-ary function symbol f combines with n terms to give another term.

Terms

- Base case: Variables and names are terms.
- Inductive case: If $t_1, \ldots, t_n$ are terms, and f is an n-ary function symbol, then $f(t_1, \ldots, t_n)$ are terms.

Induction inference rule for arithmetic

$\varphi(0)$	base case
$\forall x(\varphi(x) \to \varphi(x+1))$	inductive step
$\forall x\, \varphi(x)$	conclusion

Fact: Every number is either even or odd.

$$\varphi(x) \;\equiv\; (\exists y(x = y + y) \vee \exists z(x = z + z + 1))$$

Induction on the construction of sentences

Derivation rule for $\{\vee, \neg\}$ sentences

(1)	Atomic sentences have property X.	*base case*
(2)	If φ and ψ have property X, then $\varphi \vee \psi$ has property X.	*induction* $\vee$
(3)	If φ has property X, then $\neg\varphi$ has property X.	*induction* $\neg$
(C)	Every sentence built from atomics using $\vee$ and $\neg$ has property X.	*conclusion*

Fact: Every sentence built from the atomic sentence P, using $\vee$ and $\neg$, is provably equivalent to one of the four sentences in the diamond:

$$
\begin{tikzcd}
 & \top \arrow[dl, dash] \arrow[dr, dash] & \\
P \arrow[dr, dash] & & \neg P \arrow[dl, dash] \\
 & \bot &
\end{tikzcd}
$$

Fact: Every sentence built from P, using all propositional connectives, is provably equivalent to a sentence that only contains $\vee$ and $\neg$.

Truth functions

Unary truth-functions

A unary truth-function is a map from $\{0,1\}$ to $\{0,1\}$. There are exactly **4** possibilities:

- identity: $0 \mapsto 0, \; 1 \mapsto 1$
- flip: $0 \mapsto 1, \; 1 \mapsto 0$
- constant 0: $0,1 \mapsto 0$
- constant 1: $0,1 \mapsto 1$

Each can be expressed with our connectives.

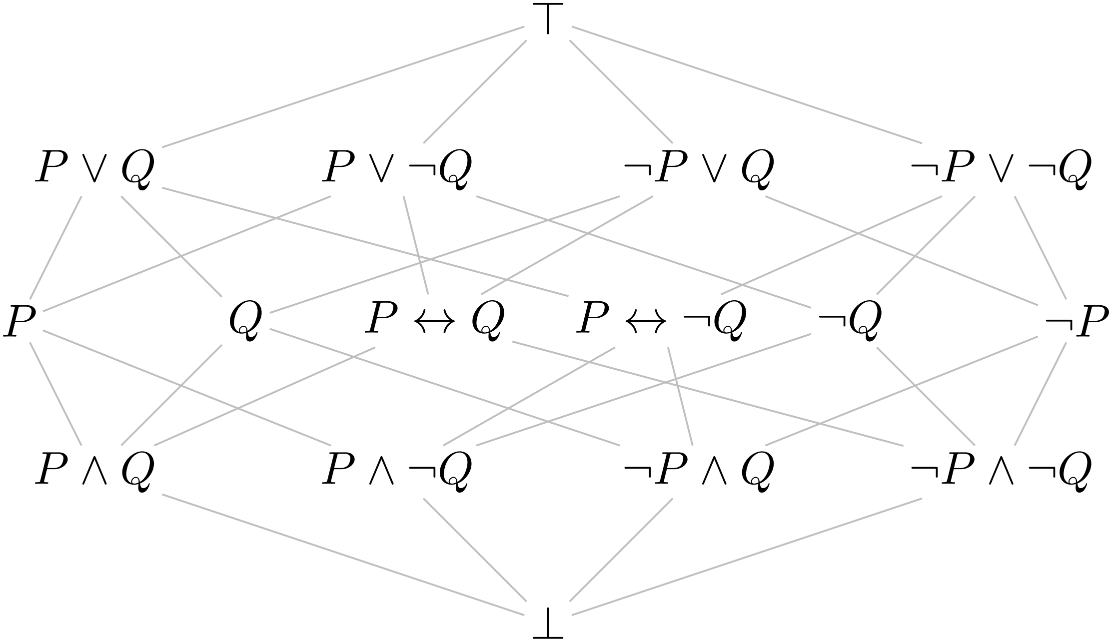
Binary truth-functions

A binary truth-function is a map from $\{0,1\} \times \{0,1\}$ to $\{0,1\}$.

There are 4 elements of $\{0,1\} \times \{0,1\}$.

Binary truth-functions correspond one-to-one to subsets of $\{0,1\} \times \{0,1\}$.

There are $2^4 = 16$ binary truth functions.


$\top$
$P \lor Q$
$P \lor \neg Q$
$\neg P \lor Q$
$\neg P \lor \neg Q$
P
Q
$P \leftrightarrow Q$
$P \leftrightarrow \neg Q$
$\neg Q$
$\neg P$
$P \land Q$
$P \land \neg Q$
$\neg P \land Q$
$\neg P \land \neg Q$
$\bot$

Expressive completeness

We say that a set Γ of connectives is **expressively complete** just in case every truth function can be expressed in terms of Γ.

Fact: The set $\{\neg, \wedge\}$ is expressively complete.

$$\begin{aligned} P \to Q \;&\equiv\; \neg(P \wedge \neg Q) \\ P \vee Q \;&\equiv\; \neg(\neg P \wedge \neg Q) \end{aligned}$$

Fact: The set $\{\wedge\}$ is not expressively complete.

How do I know? Any sentence built from $\wedge$ alone has a 0 in its truth-table.

Base case: Atomic sentences have zeroes in their truth tables.

Inductive step: If φ and ψ have zeroes in their truth tables, then $\varphi \wedge \psi$ has a zero in its truth table.