logic pset3

Resources: Lecture 3 and Chapters 3 and 5 of *How Logic Works*. (Note that we are skipping over Chapter 4 for now.)

A. Proofs

Use any of the rules of inference, including reductio ad absurdum, to prove the following sequents.

1. $\neg(P \to Q) \vdash Q \to R$

1	(1)	$\neg(P \to Q)$	A
2	(2)	Q	A
3	(3)	$\neg R$	A
4	(4)	P	A
2	(5)	$P \to Q$	4,2 CP
1,2	(6)	$(P \to Q) \wedge \neg(P \to Q)$	5,1 $\wedge$I
1,2	(7)	$\neg\neg R$	3,6 RA
1,2	(8)	R	7 DN
1	(9)	$Q \to R$	2,8 CP

2. $P \to Q \vdash \neg P \vee Q$

1	(1)	$P \to Q$	A
2	(2)	$\neg(\neg P \vee Q)$	A
3	(3)	Q	A
3	(4)	$\neg P \vee Q$	3 $\vee$I
2,3	(5)	$(\neg P \vee Q) \wedge \neg(\neg P \vee Q)$	4,2 $\wedge$I
2	(6)	$\neg Q$	3,5 RA
1,2	(7)	$\neg P$	1,6 MT
1,2	(8)	$\neg P \vee Q$	7 $\vee$I
1,2	(9)	$(\neg P \vee Q) \wedge \neg(\neg P \vee Q)$	8,2 $\wedge$I
1	(10)	$\neg\neg(\neg P \vee Q)$	2,9 RA
1	(11)	$\neg P \vee Q$	10 DN

3. $P \to (Q \vee R) \vdash (P \to Q) \vee R$

   My strategy here is to assume the negation of the conclusion for reductio ad absurdum. Following the same pattern as DeMorgan's, we get $\neg(P \to Q)$ and $\neg R$. The former implies $P \wedge \neg Q$. So we have $P, P \to (Q \vee R), \neg Q$ and $\neg R$. These form an inconsistent set.

   There are other strategies that might be more intelligible, but yield a longer proof. For example, $P \to (Q \vee R)$ implies $\neg P \vee (Q \vee R)$, which implies $(\neg P \vee Q) \vee R$, which implies $(P \to Q) \vee R$.

1	(1)	$P \to (Q \vee R)$	A
2	(2)	$\neg((P \to Q) \vee R)$	A
3	(3)	$\neg P$	A
4	(4)	P	A
3,4	(5)	$P \wedge \neg P$	3,4 $\wedge$I
6	(6)	$\neg Q$	A
3,4	(7)	$\neg\neg Q$	6,5 RA
3,4	(8)	Q	7 DN
3	(9)	$P \to Q$	4,8 CP
3	(10)	$(P \to Q) \vee R$	9 $\vee$I
2,3	(11)	$((P \to Q) \vee R) \wedge \neg((P \to Q) \vee R)$	10,2 $\wedge$I
2	(12)	$\neg\neg P$	3,11 RA
2	(13)	P	12 DN
1,2	(14)	$Q \vee R$	1,13 MP
15	(15)	Q	A
15	(16)	$P \to Q$	4,15 CP
15	(17)	$(P \to Q) \vee R$	16 $\vee$I
18	(18)	R	A
18	(19)	$(P \to Q) \vee R$	18 $\vee$I
1,2	(20)	$(P \to Q) \vee R$	14,15,17,18,19 $\vee$E
1,2	(21)	$((P \to Q) \vee R) \wedge \neg((P \to Q) \vee R)$	20,2 $\wedge$I
1	(22)	$\neg\neg((P \to Q) \vee R)$	2,21 RA
1	(23)	$(P \to Q) \vee R$	22 DN

B. Truth tables

1. Use truth table reasoning to show that $P \vee (Q \wedge R) \vDash P \vee Q$. You don't have to display a full truth table, but if you do, explain how the table demonstrates the result.

   Consider a line L of the truth table on which $P \vee (Q \wedge R)$ is true. In this case either P is true on L, or $Q \wedge R$ is true on L. In the former case, $P \vee Q$ is true on L. In the latter case, Q is also true on L and hence $P \vee Q$ is true on L. In either case, $P \vee Q$

is true on L. Since L was an arbitrary line of a truth table, whenever $P \vee (Q \wedge R)$ is true, $P \vee Q$ is also true.

2. Use truth table reasoning to show that $P \to (Q \vee R) \not\models P \to Q$.

   Consider the line L where P and R are true, but Q is false. In that case $Q \vee R$ is true, and hence $P \to (Q \vee R)$ is true. But since P is true while Q is false, $P \to Q$ is false. Thus, there is a scenario in which $P \to (Q \vee R)$ is true while $P \to Q$ is false.

3. Use truth table reasoning to show that the following "proof" must have a mistake.

1	(1)	$P \vee Q$	A
2	(2)	P	A
3	(3)	Q	A
2,3	(4)	$P \wedge Q$	2,3 $\wedge$I
2,3	(5)	P	4 $\wedge$E
1	(6)	P	1,2,2,3,5 $\vee$E

   Consider line (6), which asserts that $P \vee Q \vdash P$. It is clear that $P \vee Q \not\models P$ since Q could be true while P is false. By the *soundness* of our proof system, $P \vee Q \nvdash P$. Therefore line (6) cannot be part of any correctly written proof. (In fact, line (6) does not calculate dependency numbers correctly. An application of $\vee$E to lines $1, 2, 2, 3, 5$ should result in dependencies $1, 2$.)