Equivalence: State of Play

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Introduction

Synthesis:

A moderate notion of theoretical content: *Morita Equivalence*

Antithesis:

collapse of form-content division (Putnam, Realists)

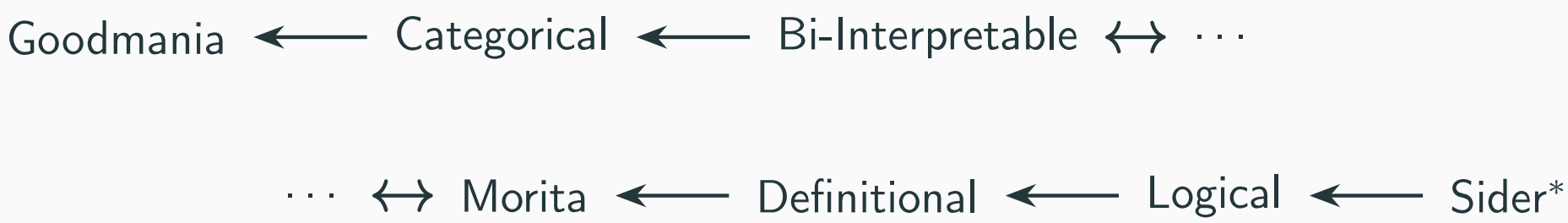
Thesis:

A theory's content is *only* its empirical consequences (Logical Positivists)

Outline

Notions of equivalence I

Spectrum of Notions of Equivalence



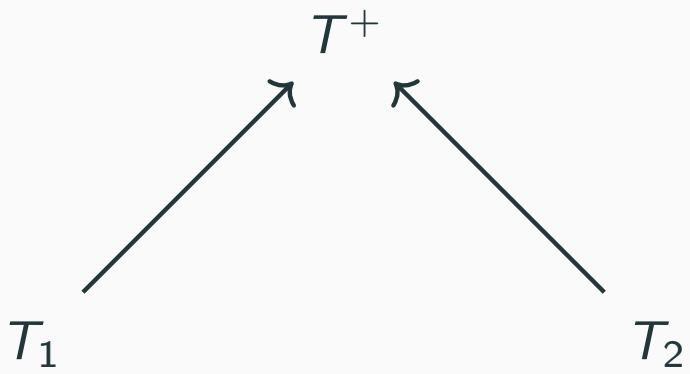
Logical equivalence

- **Defn.** (T_1, Σ_1) is **logically equivalent** to (T_2, Σ_2) just in case $\Sigma_1 = \Sigma_2$ and $T_1 \vdash \varphi$ iff $T_2 \vdash \varphi$.
- Extremely strict because T_1 and T_2 must use **same symbols**.
- Sameness of symbols is not a clear notion. E.g. is "p" the same symbol as "$\mathfrak{p}$"?

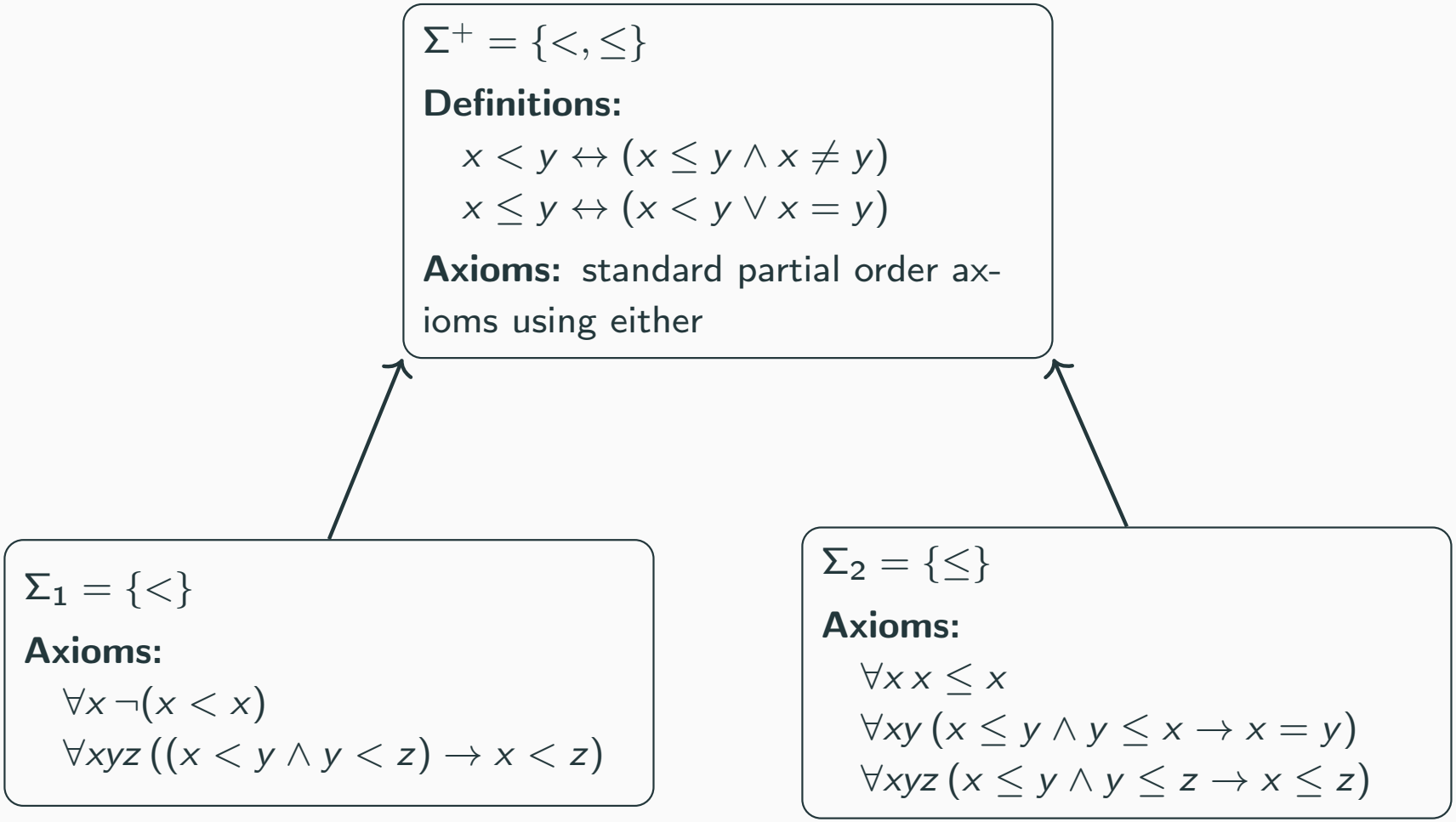
Definitional equivalence

Defn. T' is a **definitional extension** of T just in case $T' = T \cup \{\delta_1, \ldots, \delta_n\}$ where each δ_i defines a new (relation, function, constant) symbol.

Defn. T_1 and T_2 are **definitionally equivalent** just in case they have a common definitional extension T^+.



Definitional Equivalence: Partial Orders with $<$ vs. $\leq$



Morita extensions

Quotient

$$\begin{array}{c} \sigma \\ \downarrow e \\ \sigma' \end{array}$$

Subsort

$$\begin{array}{c} \sigma \\ \uparrow i \\ \sigma' \end{array}$$

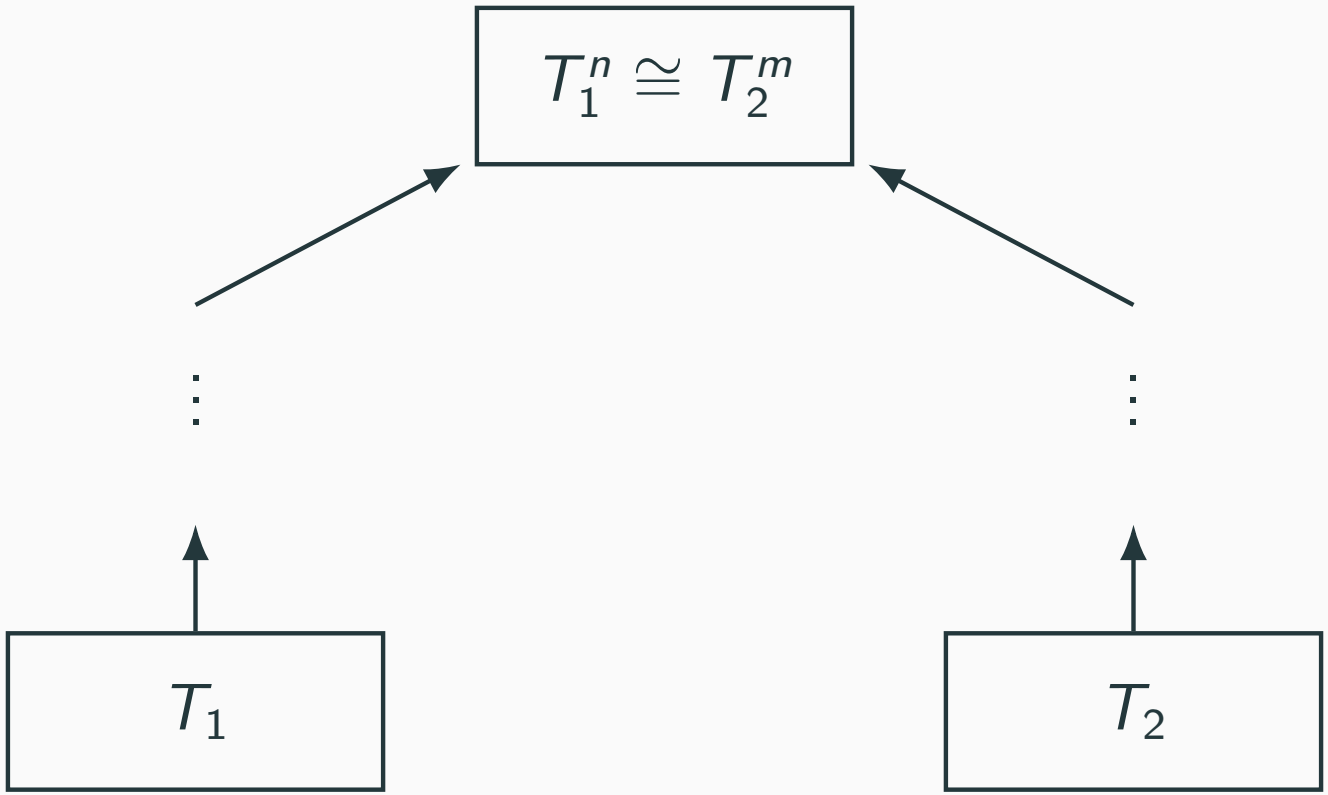
Product

$$\begin{array}{ccc} & \sigma_0 \times \sigma_1 & \\ \swarrow & & \searrow \\ \sigma_0 & & \sigma_1 \end{array}$$

Coproduct

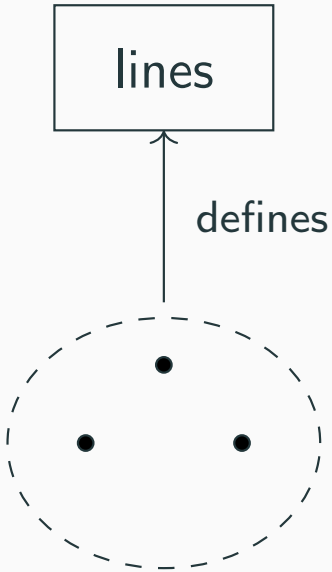
$$\begin{array}{ccc} & \sigma_0 \coprod \sigma_1 & \\ \nearrow & & \nwarrow \\ \sigma_0 & & \sigma_1 \end{array}$$

Morita equivalence



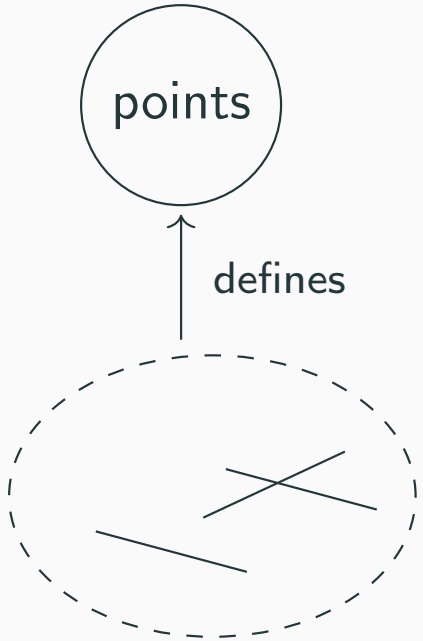
Points and lines

Equivalence classes of pairs of points



points as primitives

Equivalence classes of pairs of lines



lines as primitives

Translation Between Theories

A (generalized) **translation** from a Σ-theory T to a Σ'-theory T' is based on a **reconstrual** $F : \Sigma \to \Sigma'$, consisting of:

- A function $F_0 : S \to (S')^*$, mapping each sort to a tuple of sorts (*dimension function*).
- A variable mapping $x \mapsto \vec{x} = x_1, \ldots, x_{d(\sigma)}$, disjoint across variables.
- A domain formula D_x for each variable, natural under renaming.
- A formula $(Fp)(\vec{x}_1, \ldots, \vec{x}_n)$ interpreting each Σ-relation symbol p in Σ', also natural.

This data yields a map F from Σ-formulas to Σ'-formulas.

We say that F is a **translation** if:

$$T \vdash \varphi \quad \Rightarrow \quad T' \vdash F(\varphi)$$

Morita and Translation

- **Defn.** An **equivalence** is a pair of translations $F : T \to T'$ and $G : T' \to T$ such that $GF \cong 1_T$ and $FG \cong 1_{T'}$.
- **Prop.** T and T' are Morita equivalent iff there is an equivalence $F : T \to T'$ and $G : T' \to T$.

Examples

geometry with points	geometry with lines
nihilism	universalism
two-sorted graphs	one-sorted graphs
ZFC	ETCS

Counterexamples

- No two distinct extensions of ZF are bi-interpretable. (Enayat, 2016)

The semantic turn

Semantic

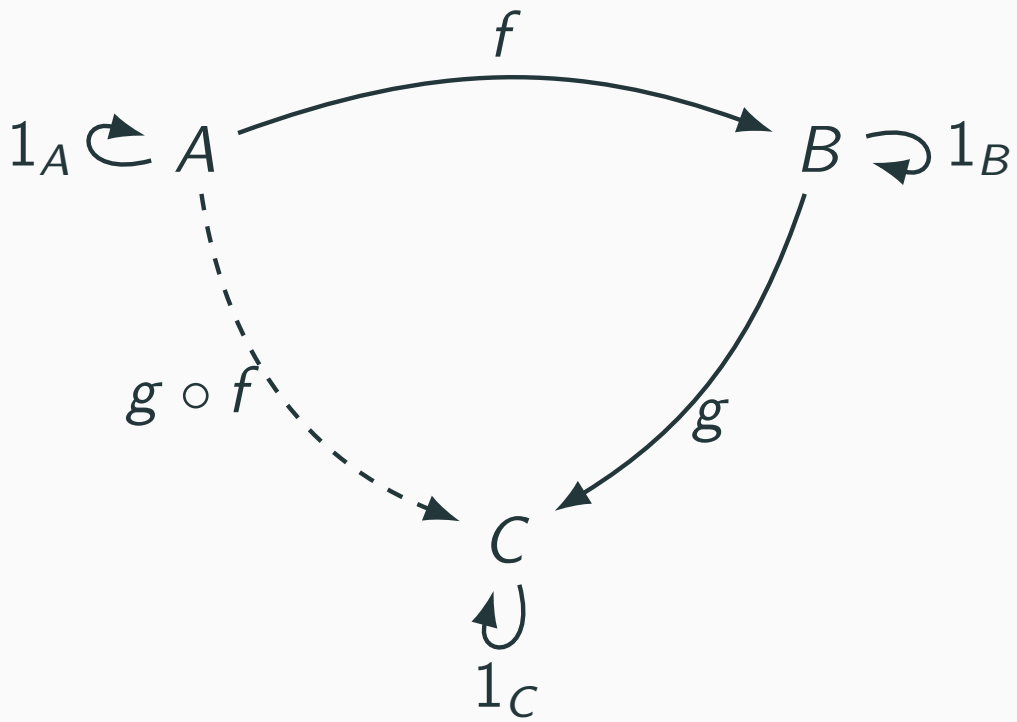
- We are considered heterodox for adopting a "syntactic" approach over a "semantic" approach. So how to make sense of equivalence if a theory is a collection of models?
- Model isomorphism criterion: T_1 is equivalent to T_2 only if each model of T_1 is isomorphic to some model of T_2.
  - North (2009): Hamiltonian mechanics is not equivalent to Lagrangian mechanics because their models are not isomorphic.
  - The MIC doesn't even follow from definitional equivalence.
- Too liberal to require only that there be a bijection between classes of models.

From semantics to syntax

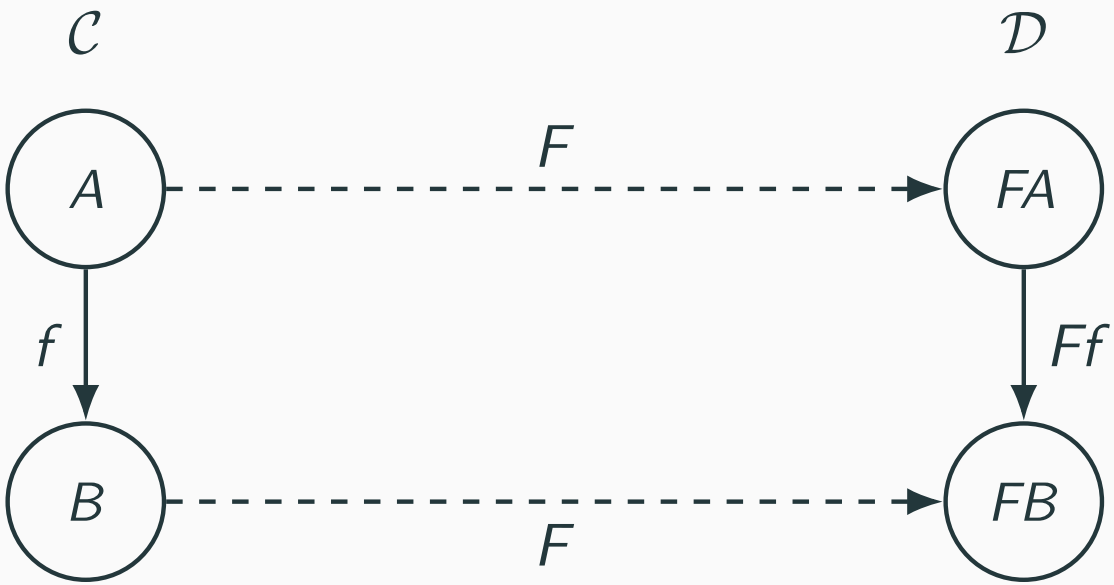
- External structure: e.g. arrows between models, nearness relations between models
- Internal structure: each model of T_2 can be "constructed from" some model of T_1.

Category theory crash course

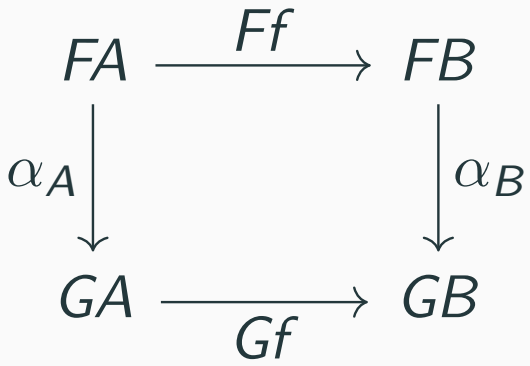
Categories



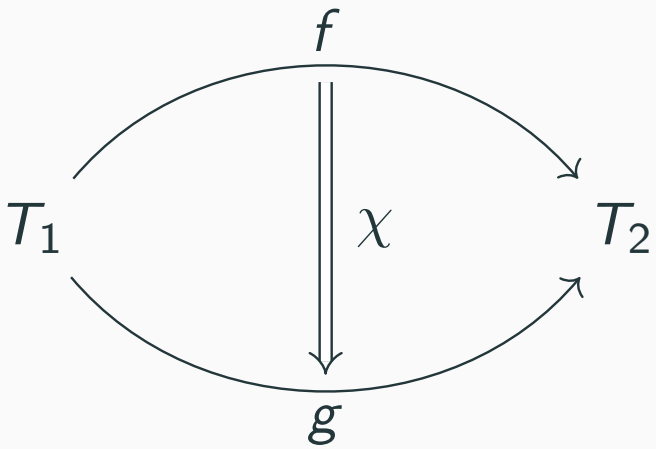
Functors



Natural transformations



2-categories



Categories of models

- Each theory T gives rise to a category $\mathrm{Mod}(T)$ of models.
- If $F : T \to T'$ is a translation then there is a functor $F^* : \mathrm{Mod}(T') \to \mathrm{Mod}(T)$.
- **Prop.** If (F, G) is an equivalence of T and T', then (F^*, G^*) is an equivalence of $\mathrm{Mod}(T)$ and $\mathrm{Mod}(T')$.
- It does not follow that if $\mathrm{Mod}(T)$ and $\mathrm{Mod}(T')$ are equivalent, then T and T' are equivalent.

Notions of equivalence II

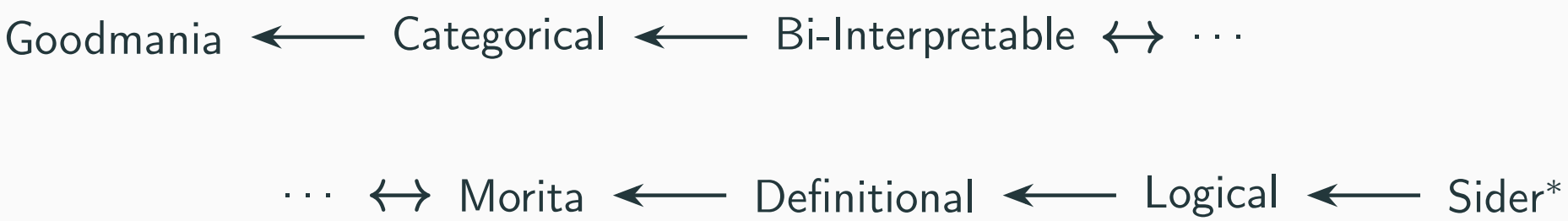
Categorical equivalence

- **Defn.** T and T' are **categorically equivalent** just in case $\mathrm{Mod}(T)$ and $\mathrm{Mod}(T')$ are equivalent categories.
- Pro: Applies even to theories that don't have a first-order formulation
- Con: The category of a theory's models might omit some of the important information about what the theory says.

Examples

Stone spaces	Boolean algebras
Coherent spaces	Rings
Relativistic spacetimes	Einstein algebras
Lagrangian mechanics	Hamiltonian mechanics

Spectrum of Notions of Equivalence



What's the point of talking about equivalence?

- Simple analogy to "baby logic"
- Rain on metaphysicians' parade?

Objections and replies

Objection: Equivalence is not a purely formal notion

"One thing we can be sure of. Whatever this structural 'isomorphism' is to be, it cannot be a purely formal notion. It cannot be, that is, an interrelationship which can be determined to hold solely on the basis of the logical form of the theories in question." (Sklar, 1982, p 93)

See: Coffey, 2014; Teitel, 2021; Butterfield, 2021

Reply

- Form versus content: What belongs to the content of a theory can (always? sometimes?) be represented formally.

Quotienting

"According to one alternative — the second 'extreme' approach to equivalence I want to discuss — we can say that theories are equivalent without saying why they are equivalent in terms of fundamentality and underlying third languages." (Sider, 2020, p 191)

Quotienting – Reply

- It's a straw person to say that there is no "reason" for saying that T and T' are equivalent. The existence of a translation scheme is a reason!

Blindness

- **Objection (Babic and Calosi):** Morita equivalence is blind to what gets constructed out of what.
- **Reply:** Granted, there still can be reason to prefer one formulation of two Morita equivalent theories.
- **Reply:** Some commitments are shown, and some are said. If we say the construction commitment, then the theories are not Morita equivalent.

Defending metaphysics

- **Objection:** Morita equivalence is part of a sinister plot against metaphysics.
- **Reply:** ME may not be consistent with, say, strong views about grounding, but ME is not intrinsically anti-realist. The permitted definition procedures call for metaphysical interpretation.

Open questions

Open questions and projects

- Is mereological nihilism equivalent to mereological universalism (when there are no restrictions on domain size)?
- Under what conditions are 4-dimensionalist theories equivalent to 3-dimensionalist theories?
- Are elementary elliptic and Euclidean geometry equivalent? (see Glymour, 1970)
- Notions of equivalence with weaker background assumptions
  - Intuitionistic logic and modal S4
  - Second order logic and many-sorted logic (see Manzano, 1996)

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