

Review: translations and proofs

PHI 201 – Introductory Logic

November 17, 2025

Proofs

Quantifiers of the same type

$$\forall x \forall y \varphi \equiv \forall y \forall x \varphi$$

$$\exists x \exists y \varphi \equiv \exists y \exists x \varphi$$

Fact: Quantifiers of the same type commute. That is, universal quantifiers commute with each other, and existential quantifiers commute with each other.

Prenex normal form

Fact: Every sentence in predicate logic is equivalent to a sentence where all the quantifiers occur at the beginning.

Pushing negation inside

$$\neg \forall x \varphi \equiv \exists x \neg \varphi$$

$$\neg \exists x \varphi \equiv \forall x \neg \varphi$$

Equivalent formulas

For purposes of establishing equivalence, you can treat the variables inside a formula as names.

If quantifiers are stripped, and variables replaced by distinct names, then the process can be reversed to put the quantifiers back on.

Equivalent formulas

Fact: If $\varphi(a_1, \dots, a_n)$ is equivalent to $\psi(a_1, \dots, a_n)$, then $Q_1x_1 \cdots Q_nx_n \varphi(x_1, \dots, x_n)$ is equivalent to $Q_1x_1 \cdots Q_nx_n \psi(x_1, \dots, x_n)$.

Practical upshot: you can detect equivalence by looking at the formula inside quantifiers.

1	(1)	$\forall x \exists y \forall z ((Fx \wedge Gy) \rightarrow Hz)$	A
1	(2)	$\exists y \forall z ((Fa \wedge Gy) \rightarrow Hz)$	1 UE
3	(3)	$\forall z ((Fa \wedge Gb) \rightarrow Hz)$	A
3	(4)	$(Fa \wedge Gb) \rightarrow Hc$	3 UE
3	(5)	$(\neg Fa \vee \neg Gb) \vee Hc$	4 TTV
3	(6)	$\forall z ((\neg Fa \vee \neg Gb) \vee Hz)$	5 UI
3	(7)	$\exists y \forall z ((\neg Fa \vee Gy) \vee Hz)$	6 EI
1	(8)	$\exists y \forall z ((\neg Fa \vee Gy) \vee Hz)$	2,3,7 EE
1	(9)	$\forall x \exists y \forall z ((\neg Fx \vee Gy) \vee Hz)$	8 UI

$$Fy \wedge \exists x Gx \equiv \exists x (Fy \wedge Gx)$$

$$Fy \vee \exists x Gx \equiv \exists x (Fy \vee Gx)$$

$$Fy \rightarrow \exists x Gx \equiv \exists x (Fy \rightarrow Gx)$$

$$\exists x(Gx \wedge Fy) \equiv \exists xGx \wedge Fy$$

$$\exists x(Gx \vee Fy) \equiv \exists xGx \vee Fy$$

$$\exists x(Gx \rightarrow Fy) \equiv \forall xGx \rightarrow Fy$$

$$\begin{aligned}\forall y(\exists x Gx \rightarrow Fy) &\equiv \forall y \forall x (Gx \rightarrow Fy) \\ &\equiv \exists x Gx \rightarrow \forall y Gy\end{aligned}$$

$$\forall x Fx \rightarrow Gb \vdash \exists x(Fx \rightarrow Gb)$$

Test your understanding

Use what you just learned to quickly see that the following sentence is a tautology:

$$\forall y \exists x (Rxy \rightarrow \forall z Rzy)$$

Translation

Nested Quantifiers and Scope

Source note. The next few slides are based in part on Warren Goldfarb's *Deductive Logic*.

Nested quantifiers introduce a new complication:
we must determine *which quantifier governs which*.

- Quantifiers have **scope**, and one may fall inside another.
- To paraphrase, it helps to work **from the outside in**.
- First decide whether the statement as a whole is **universal** (\forall) or **existential** (\exists).
- Then paraphrase the **open formula** inside its scope.

Example Sentences

We assume—for now—that the **universe of discourse** is the class of **persons**.

Consider the following English sentences:

- 1 Every critic admires some painter.
- 2 Every critic is admired by some painter.
- 3 Every critic admires all painters.

All three sentences are **universal quantifications**.

Example (1): Every critic admires some painter

Step 1 (outer quantifier):

$$\forall x (x \text{ is a critic} \rightarrow \dots)$$

Step 2 (inner scope):

$$x \text{ admires some painter} \quad \rightsquigarrow \quad \exists y (y \text{ is a painter} \wedge x \text{ admires } y)$$

Final formalization:

$$\forall x (Cx \rightarrow \exists y (Py \wedge Axy))$$

Example (2): Every critic is admired by some painter

Paraphrase structure:

$$\forall x (Cx \rightarrow \exists y (Py \wedge Ayx))$$

Note the reversal in the predicate:

$$Ayx \equiv y \text{ admires } x.$$

Final symbolic form:

$$\forall x (Cx \rightarrow \exists y (Py \wedge Ayx))$$

Example (3): Every critic admires all painters

Step 1 (outer quantifier):

$$\forall x (Cx \rightarrow \dots)$$

Step 2 (scope):

$$x \text{ admires all painters} \quad \rightsquigarrow \quad \forall y (Py \rightarrow Axy)$$

Final formalization:

$$\forall x (Cx \rightarrow \forall y (Py \rightarrow Axy))$$

Existential sentences

There is a painter who is admired by every critic.

$\exists x(x \text{ is a painter} \wedge x \text{ is admired by every critic})$

Some critics admire all painters.

There is a critic who admires no painters.

P 's bear the relation R only to Q 's

$$\forall x(Px \rightarrow \forall y(Rxy \rightarrow Qy))$$

Conversational implicature of if and only if: “Danes only trust other Danes.”

$$\forall x(Dx \rightarrow \forall y(Txy \leftrightarrow Dy))$$

Only the F that/who are G are H .

$$\forall x(Fx \rightarrow (Hx \rightarrow Gx))$$

$$\forall x(Hx \rightarrow (Fx \wedge Gx))$$

P 's bear the relation R only to Q 's that/who are S .

a only respects Harvard professors who acknowledge that Princeton is superior.

$$\forall x((Hx \wedge Rax) \rightarrow Ax)$$

$$\forall x(Rax \rightarrow (Hx \wedge Ax))$$