Chapter 2 of *World Enough and Spacetime* is the technically most demanding chapter of the book. Earman makes lots of (mostly correct) claims that take several lines of math to prove — rarely mentioning how one could reconstruct the proof. He also uses a notation that is slightly ambiguous about whether the indices are abstract (as in Malament's $a, b, c, \ldots$) or whether they are components of the tensor relative to a chart.

Nonetheless, the upshot of the chapter is clear: there is a stack of increasing spacetime structures.

spacetime	new structure	new concept
Aristotelean spacetime	worldline	absolute position
Newtonian spacetime	reference frame	absolute velocity
Galilean spacetime	affine structure	absolute acceleration
Maxwellian spacetime	affine structures	rotation
Leibnizian spacetime	temporal metric	relative acceleration
Machian spacetime	simultaneity, Euclidean metric on space	

Machian spacetime

- Assumption: M is a smooth manifold that is diffeomorphic with $\mathbb{R}^4$. i.e., there is a smooth bijection $\varphi : M \to \mathbb{R}^4$. (This assumption saves one from some nuances about global structure.)
- Motivating idea: All there is are metric relations at each moment. But we assume (contra Hume!) that we can re-identify physical objects at different times. Mathematically, this means that we permit ourselves to write $\gamma : \mathbb{R} \to M$ to represent the trajectory of a material object.
- Assumption: There is a metric g^{ab} of signature $(+ + +0)$ on M. Hence, locally, in each tangent space T_p, one can distinguish three spacelike dimensions and one timelike dimension. But since there is no **connection** on M, there is no sense to the question of whether the timelike direction in T_p is the same as the timelike direction in T_q. (Since g^{ab} is assumed to be smooth, there is a sense in which the timelike direction changes continuously. But that does *not* mean that these timelike directions can be stitched together consistently in such a way that we can define a global notion of time. The **kernel distribution** does not necessarily correspond to a smooth vector field.)
- Assumption: There is a family $\mathbb{T}$ of time functions $t : M \to \mathbb{R}$, but no particular member of $\mathbb{T}$ is privileged with regard to duration between two events. For any $t_1, t_2 \in \mathbb{T}$, there is a smooth bijection $f : \mathbb{R} \to \mathbb{R}$ such that $t_2 = f \circ t_1$ and $f' = df/dt > 0$. (The latter condition is equivalent to f being continuous and order-preserving.)

- "x and y are moving relative to each other" is definable, i.e. their distance is changing.
- "The velocity of x relative to y" is not definable.
- "x and y are accelerating relative to each other" is not definable.

- Fact: If two smooth functions $t_1, t_2 : M \to \mathbb{R}$ provide the same foliation, then they are related as $t_2 = f \circ t_1$, for a continuous, order-preserving function f.
- Assumption: $g^{ab}t_a = 0$, meaning that $g^{ab}t_a v_b = 0$ for any covector v_b. (Question: Is there a sense in which $g^{ab}t_a$ is the *projection* of t_a onto space?)

Leibnizian spacetime

- Earman does something strange here. Instead of selecting a particular time function $t \in \mathbb{T}$, he defines a temporal metric h_{ij}. He then requires that $g^{ij}h_{ij} = 0$. Of course, h_{ij} does defines temporal distances.
- Question: Couldn't we just pick a particular time function $t \in \mathbb{T}$ and then take $h_{ab} = \partial_a t \partial_b t$? Conjecture: That wouldn't reduce generality, because a metric that is non-degenerate on only one dimension is actually a product of one-forms.
- Fact: for two curves $\gamma_1, \gamma_2 : \mathbb{R} \to M$, instantaneous relative velocity is definable. Reparameterize γ_1 and γ_2 so that they are functions of the global time coordinate. Each each t, $v(t) = \gamma_1(t) - \gamma_2(t)$ is a vector in $\mathbb{R}^3$.
- Imagine three curves $\gamma_1, \gamma_2, \gamma_3$ that form a triangle with vertices of constant distance. There is no meaning to the claim that this triangle is rotating or not.

Maxwellian spacetime

- The construction here is complicated because the goal is to add a "standard of rotation" without adding a standard of absolute acceleration. Earman's idea is to fix a particular flat affine connection ∇ that is compatible with the spatial and temporal metrics. The specification of such a connection is tantamount to the specification of a notion of "straight line" in M. At this stage, absolute acceleration would be definable, and we don't want that. We just want to know if an alternative definition ∇' of "straight line" would agree with ∇ about which extended things are rotating.
- Take a spacelike vector field f^a that is constant, at each time, relative to the connection ∇. I believe this means that f^a corresponds to shifting each spatial hypersurface — i.e. we can bevel the deck in a smooth (but not necessarily linear) way.

Galilean spacetime

A covariant derivative operator ∇ takes a tensor T and produces a new tensor ∇T that has one more lower index. The intuitive interpretation is that if ξ^a is a directional vector, then $\xi^a \nabla_a T$ is the tensor representing the rate of change of T in the direction ξ.

How is it possible that (absolute) acceleration is definable while (absolute) velocity is not? Starting with the simple, binary case of "moving versus not moving" and "accelerating versus not accelerating": an affine connection (covariant derivative operator) ∇ defines a predicate on timlike curves — "γ is a geodesic". The more precise definition is that if $\vec{\gamma}|_p$ is parallel transported from p to q, then the result is $\vec{\gamma}|_q$. In short, an object (represented by a timeline curve γ) is accelerating at a point $p \in M$ just in case $\xi^a \nabla_a \xi^b \neq 0$.

Additional resources

If you want to learn more about these different spacetimes, it can be a challenging because typical physics books are focused on the current theory, viz. general relativity, which is rather different than these "classical" spacetimes.

- Weatherall, Jim (2021). Classical spacetime structure. In (Eds.), The Routledge Companion to Philosophy of Physics (pp. 33–45). Routledge.
- Lee, *Introduction to Smooth Manifolds*, chapter 19 is about foliations.
- Claude Godbillon, *Feuilletages: Études Géométriques* (1971) is the classic about foliations (although not available in English).
- "Foliations and Geometric Structures" by Barbot, Béguin, and Labourie (in Géométrie différentielle)