

Existential Introduction & Elimination

PHI 201 — Introductory Logic

October 27, 2025

Overview

- Review of universal quantifier rules
- New rules for the existential quantifier:
 - Existential Introduction (**EI**)
 - Existential Elimination (**EE**)
- Practice proofs involving \exists and \forall

Existential Introduction

Rule

From a particular instance Fa , we may infer that something is F :

$$\frac{Fa}{\exists x Fx} (\text{EI})$$

1	(1)	$Fa \rightarrow Ga$	A
1	(2)	$\exists x(Fx \rightarrow Gx)$	1 EI

1	(1)	$Fa \rightarrow Ga$	A
1	(2)	$\exists x(Fx \rightarrow Ga)$	1 EI

Unlike UI, EI permits replacement of some (but not all) occurrences of a name a .

1	(1)	Raa	A
1	(2)	$\exists xRxx$	1 EI

1	(1)	Raa	A
1	(2)	$\exists yRay$	1 EI
1	(3)	$\exists x\exists yRxy$	2 EI

To show: $\neg \exists x Fx \vdash \forall x \neg Fx$

1	(1)	$\neg \exists x Fx$	A
2	(2)	Fa	A
2	(3)	$\exists x Fx$	2 EI
1,2	(4)	$\exists x Fx \wedge \neg \exists x Fx$	3,1 $\wedge I$
1	(5)	$\neg Fa$	2,4 RA
1	(6)	$\forall x \neg Fx$	5 UI

1	(1)	$\neg \forall x Fx$	A
2	(2)	$\neg \exists x \neg Fx$	A
3	(3)	$\neg Fa$	A
3	(4)	$\exists x \neg Fx$	3 EI
2,3	(5)	$\exists x \neg Fx \wedge \neg \exists x \neg Fx$	4,2 $\wedge I$
2	(6)	$\neg \neg Fa$	3,5 RA
2	(7)	Fa	6 DN
2	(8)	$\forall x Fx$	7 UI
1,2	(9)	$\forall x Fx \wedge \neg \forall x Fx$	8,1 $\wedge I$
1	(10)	$\neg \neg \exists x \neg Fx$	2,9 RA
1	(11)	$\exists x \neg Fx$	10 DN

To show: $\neg \forall x Fx \vdash \exists x(Fx \rightarrow P)$

To show $\neg \exists x Fx \vdash \forall x(Fx \rightarrow P)$

1	(1)	$\neg \exists x Fx$	A
2	(2)	Fa	A
2	(3)	$\exists x Fx$	2 EI
1,2	(4)	$\exists x Fx \wedge \neg \exists x Fx$	3,1 $\wedge I$
1	(5)	$\neg Fa$	2,4 RA
1	(6)	$Fa \rightarrow P$	5 neg par
1	(7)	$\forall x(Fx \rightarrow P)$	6 UI

Existential elimination

Idea

What can be derived from $\exists x\varphi(x)$?

Pick an arbitrary name a . If a general claim ψ can be derived from an instance $\varphi(a)$, without making any additional assumptions about a , then ψ follows from $\exists x\varphi(x)$.

Existential Elimination

$\Gamma \quad (m) \quad \exists x \varphi(x)$

$n \quad (n) \quad \varphi(a) \qquad A$

$\Delta \quad (o) \quad \psi$

$\Gamma, \Delta \setminus \{n\} \quad (p) \quad \psi \qquad m, n, o \text{ EE}$

Existential Elimination

- **Restriction:** The name a must not occur in any dependencies of the existential premise, or of the derived conclusion, except for the instance $\varphi(a)$ itself.
- Dependencies are the union of the dependencies of the existential sentence and those of the derived conclusion, minus dependency on the instance.

Example

- | | | | |
|---|-----|---------------------------|--------------|
| 1 | (1) | $\exists x(Fx \wedge Gx)$ | A |
| 2 | (2) | $Fa \wedge Ga$ | A |
| 2 | (3) | Fa | 2 $\wedge E$ |
| 2 | (4) | $\exists x Fx$ | 3 EI |
| 1 | (5) | $\exists x Fx$ | 1,2,4 EE |

Reasoning with multiple \exists

To show: $\exists x \exists y Rxy \vdash \exists y \exists x Rxy$

1	(1)	$\exists x \exists y Rxy$	A
2	(2)	$\exists y Ray$	A
3	(3)	Rab	A
3	(4)	$\exists x Rx b$	3 EI
3	(5)	$\exists y \exists x Rxy$	4 EI
2	(6)	$\exists y \exists x Rxy$	2,3,5 EE
1	(7)	$\exists y \exists x Rxy$	1,2,6 EE

Reasoning with \forall and \exists

1	(1)	$\forall x(Fx \rightarrow Gx)$	A
2	(2)	$\exists xFx$	A
3	(3)	Fa	A
1	(4)	$Fa \rightarrow Ga$	1 UE
1,3	(5)	Ga	4,3 MP
1,3	(6)	$\exists xGx$	5 EI
1,2	(7)	$\exists xGx$	2,3,6 EE

Reasoning with \forall and \exists

1	(1)	$\exists y \forall x Rxy$	A
2	(2)	$\forall x Rx b$	A
2	(3)	Rab	2 UE
2	(4)	$\exists y Ray$	3 EI
2	(5)	$\forall x \exists y Rxy$	4 UI
1	(6)	$\forall x \exists y Rxy$	1,2,5 EE

Preventing invalid inferences

1	(1)	$\exists x Fx$	A
2	(2)	$\exists x Gx$	A
3	(3)	Fa	A
4	(4)	Ga	A
3,4	(5)	$Fa \wedge Ga$	3,4 $\wedge I$
3,4	(6)	$\exists x(Fx \wedge Gx)$	5 EI

EE cannot be applied to 1,3,6 because 6 depends on 4, which contains a .

EE cannot be applied to 1,4,6 because 6 depends on 3, which contains a .

Preventing invalid inferences

1	(1)	$\forall x \exists y Rxy$	A
1	(2)	$\exists y Ray$	1 UE
3	(3)	Rab	A
3	(4)	$\forall x Rx b$	Error!

UI cannot be applied to 3 because it depends on 3, which contains a .

Quantifier order matters

$\forall x \exists y \varphi(x, y)$ follows from $\exists y \forall x \varphi(x, y)$.

But not vice versa.

Quantifier negation equivalences

$\neg \exists x Fx$ is equivalent to $\forall x \neg Fx$

$\neg \forall x Fx$ is equivalent to $\exists x \neg Fx$

“Equivalent” means mutually derivable

To show: $\forall x \neg Fx \vdash \neg \exists x Fx$

1	(1)	$\forall x \neg Fx$	A
2	(2)	$\exists x Fx$	A
3	(3)	Fa	A
1	(4)	$\neg Fa$	1 UE
1,3	(5)	$Fa \wedge \neg Fa$	3,4 $\wedge I$
1,3	(6)	$\neg \exists x Fx$	2,5 RA
1,2	(7)	$\neg \exists x Fx$	2,3,6 EE
1,2	(8)	$\exists x Fx \wedge \neg \exists x Fx$	2,7 $\wedge I$
1	(9)	$\neg \exists x Fx$	2,8 RA

Non-constructive existence proofs

How to derive $\exists x(Fx \rightarrow P)$ from $\forall x Fx \rightarrow P$?

Not possible to derive $Fa \rightarrow P$ from $\forall x Fx \rightarrow P$.