

Lectures 12.x

PHI 201 – Introductory Logic

November 24, 2025

Remaining tasks

- 1 Figure out how to prove harder things (more reliably)
 - Example: $\vdash \exists x \forall y (Fx \rightarrow Fy)$
 - Idea: Convert semantic intuition into proof
- 2 Learn how to reason *about* propositional logic
 - New axiom schema: mathematical induction
 - Main results: soundness, completeness

A theory *about* propositional logic

- You'll keep using logic, but most of you won't study logic again in a formal setting like this one.
- But learning about how logic works will help you become better at doing logic.
 - Analogy to an athlete and understanding physiology and nutrition.
 - But that analogy fails to capture the fascinating fact that studying logic is another use of logic.

- To formalize a theory in predicate logic, one chooses some basic vocabulary (names, predicates, relations, functions).
- We are going to be talking about **sentences**, **sequents**, **valuations**, etc. So, for example, we would have a predicate symbol $\text{Sent}(x)$ to mean that x is a sentence, and a relation symbol $\text{Seq}(x_1, \dots, x_n, y)$ to mean that there is a valid proof whose last line has formula y with dependencies x_1, \dots, x_n .

Mathematical induction

- The interesting theorems about propositional logic involve claims about infinite sets. For example:
For every sentence φ , φ is provably equivalent to a sentence in which \wedge does not occur.
- But our infinite sets are generated from a finite number of cases by a finite number of rules. There is a special method of proof for such sets: **mathematical induction**.

Aside: Function symbols

An n -ary function symbol f combines with n terms to give another term.

Terms

- Base case: Variables and names are terms.
- Inductive case: If t_1, \dots, t_n are terms, and f is an n -ary function symbol, then $f(t_1, \dots, t_n)$ are terms.

Induction inference rule for arithmetic

$$\varphi(0)$$

base case

$$\forall x(\varphi(x) \rightarrow \varphi(x + 1))$$

inductive step

$$\forall x \varphi(x)$$

conclusion

Fact: Every number is either even or odd.

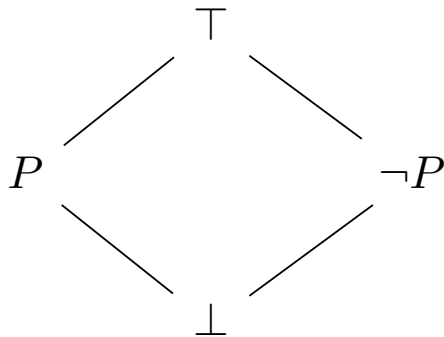
$$\varphi(x) \equiv (\exists y(x = y + y) \vee \exists z(x = z + z + 1))$$

Induction on the construction of sentences

Derivation rule for $\{\vee, \neg\}$ sentences

- | | | |
|-------|---|------------------------------------|
| (1) | Atomic sentences have property X . | <i>base case</i> |
| (2) | If φ and ψ have property X , then $\varphi \vee \psi$ has property X . | <i>induction \vee</i> |
| (3) | If φ has property X , then $\neg\varphi$ has property X . | <i>induction \neg</i> |
| <hr/> | | |
| (C) | Every sentence built from atomics using \vee and \neg has property X . | <i>conclusion</i> |

Fact: Every sentence built from the atomic sentence P , using \vee and \neg , is provably equivalent to one of the four sentences in the diamond:



Fact: Every sentence built from P , using all propositional connectives, is provably equivalent to a sentence that only contains \vee and \neg .

Truth functions

Unary truth-functions

A unary truth-function is a map from $\{0, 1\}$ to $\{0, 1\}$. There are exactly **4** possibilities:

- identity: $0 \mapsto 0, 1 \mapsto 1$
- flip: $0 \mapsto 1, 1 \mapsto 0$
- constant 0: $0, 1 \mapsto 0$
- constant 1: $0, 1 \mapsto 1$

Each can be expressed with our connectives.

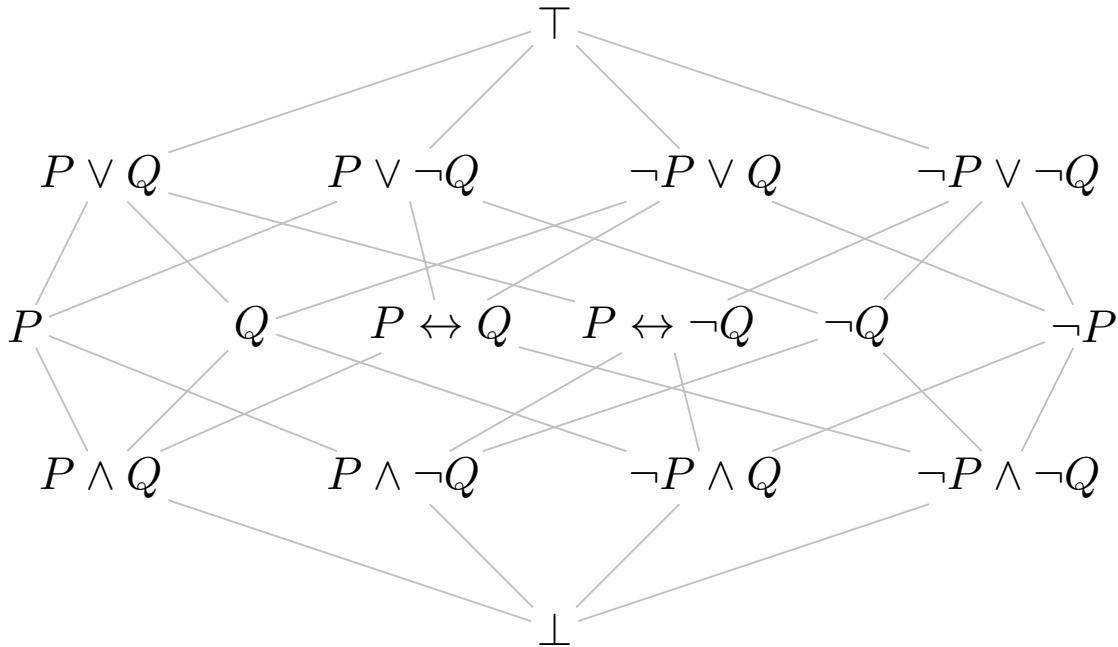
Binary truth-functions

A binary truth-function is a map from $\{0, 1\} \times \{0, 1\}$ to $\{0, 1\}$.

There are 4 elements of $\{0, 1\} \times \{0, 1\}$.

Binary truth-functions correspond one-to-one to subsets of $\{0, 1\} \times \{0, 1\}$.

There are $2^4 = 16$ binary truth functions.



Expressive completeness

We say that a set Γ of connectives is **expressively complete** just in case every truth function can be expressed in terms of Γ .

Fact: The set $\{\neg, \wedge\}$ is expressively complete.

$$P \rightarrow Q \equiv \neg(P \wedge \neg Q)$$

$$P \vee Q \equiv \neg(\neg P \wedge \neg Q)$$

Fact: The set $\{\wedge\}$ is not expressively complete.

How do I know? Any sentence built from \wedge alone has a 0 in its truth-table.

Base case: Atomic sentences have zeroes in their truth tables.

Inductive step: If φ and ψ have zeroes in their truth tables, then $\varphi \wedge \psi$ has a zero in its truth table.