How Logic Works: Solutions to Problems

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Chapter 3

Exercise 3.1

1. $P \vdash Q \to (P \wedge Q)$

1	(1)	P	A
2	(2)	Q	A
1,2	(3)	$P \wedge Q$	1,2 $\wedge$I
1	(4)	$Q \to (P \wedge Q)$	2,3 CP

2. $(P \to Q) \wedge (P \to R) \vdash P \to (Q \wedge R)$

1	(1)	$(P \to Q) \wedge (P \to R)$	A
2	(2)	P	A
1	(3)	$P \to Q$	1 $\wedge$E
1	(4)	$P \to R$	1 $\wedge$E
1,2	(5)	Q	3,2 MP
1,2	(6)	R	4,2 MP
1,2	(7)	$Q \wedge R$	5,6 $\wedge$I
1	(8)	$P \to (Q \wedge R)$	2,7 CP

3. $P \to (Q \to R) \vdash Q \to (P \to R)$

1	(1)	$P \to (Q \to R)$	A
2	(2)	Q	A
3	(3)	P	A
1,3	(4)	$Q \to R$	3,1 MP
1,2,3	(5)	R	4,2 MP
1,2	(6)	$P \to R$	3,5 CP
1	(7)	$Q \to (P \to R)$	2,6 CP

4. $P \to Q \vdash (Q \to R) \to (P \to R)$

1	(1)	$P \to Q$	A
2	(2)	$Q \to R$	A
3	(3)	P	A
1,3	(4)	Q	1,3 MP
1,2,3	(5)	R	2,4 MP
1,2	(6)	$P \to R$	3,5 CP
1	(7)	$(Q \to R) \to (P \to R)$	2,6 CP

5. $P \to (P \to Q) \vdash P \to Q$

1	(1)	$P \to (P \to Q)$	A
2	(2)	P	A
1,2	(3)	$P \to Q$	1,2 MP
1,2	(4)	Q	3,2 MP
1	(5)	$P \to Q$	2,4 CP

6. $P \to (Q \to R) \vdash (P \wedge Q) \to R$

1	(1)	$P \to (Q \to R)$	A
2	(2)	$P \wedge Q$	A
2	(3)	P	2 $\wedge$E
2	(4)	Q	2 $\wedge$E
1,2	(5)	$Q \to R$	1,3 MP
1,2	(6)	R	5,4 MP
1	(7)	$(P \wedge Q) \to R$	2,6 CP

7. $(P \vee Q) \to R \vdash P \to R$

1	(1)	$(P \vee Q) \to R$	A
2	(2)	P	A
2	(3)	$P \vee Q$	2 $\vee$I
1,2	(4)	R	1,3 MP
1	(5)	$P \to R$	2,4 CP

8. $\neg P \vdash \neg(P \wedge Q)$

1	(1)	$\neg P$	A
2	(2)	$P \wedge Q$	A
2	(3)	P	2 $\wedge$E
	(4)	$(P \wedge Q) \to P$	2,3 CP
1	(5)	$\neg(P \wedge Q)$	4,1 MT

9. $\neg(P \vee Q) \vdash \neg P \wedge \neg Q$

1	(1)	$\neg(P \vee Q)$	A
2	(2)	P	A
2	(3)	$P \vee Q$	2 $\vee$I
	(4)	$P \rightarrow (P \vee Q)$	2,3 CP
1	(5)	$\neg P$	4,1 MT
6	(6)	Q	A
6	(7)	$P \vee Q$	6 $\vee$I
	(8)	$Q \rightarrow (P \vee Q)$	6,7 CP
1	(9)	$\neg Q$	8,1 MT
1	(10)	$\neg P \wedge \neg Q$	5,9 $\wedge$I

10. $P \rightarrow \neg P \vdash \neg P$

1	(1)	P	A
2	(2)	$P \rightarrow \neg P$	A
1,2	(3)	$\neg P$	2,1 MP
1	(4)	$(P \rightarrow \neg P) \rightarrow \neg P$	2,3 CP
1	(5)	$\neg\neg P$	1 DN
1	(6)	$\neg(P \rightarrow \neg P)$	4,5 MT
	(7)	$P \rightarrow \neg(P \rightarrow \neg P)$	1,6 CP
2	(8)	$\neg\neg(P \rightarrow \neg P)$	2 DN
2	(9)	$\neg P$	7,8 MT

Exercise 3.4

1. $P \rightarrow Q \ \vdash \ \neg(P \wedge \neg Q)$

1	(1)	$P \rightarrow Q$	A
2	(2)	$P \wedge \neg Q$	A
2	(3)	P	2 $\wedge$E
1,2	(4)	Q	1,3 MP
2	(5)	$\neg Q$	2 $\wedge$E
1,2	(6)	$Q \wedge \neg Q$	4,5 $\wedge$I
1	(7)	$\neg(P \wedge \neg Q)$	2,6 RA

2. $\neg(P \wedge Q) \ \vdash \ \neg P \vee \neg Q$

1	(1)	$\neg(P \wedge Q)$	A
2	(2)	$\neg(\neg P \vee \neg Q)$	A
3	(3)	$\neg P$	A
3	(4)	$\neg P \vee \neg Q$	3 $\vee$I
2,3	(5)	$(\neg P \vee \neg Q) \wedge \neg(\neg P \vee \neg Q)$	4,2 $\wedge$I
2	(6)	$\neg\neg P$	3,5 RA
2	(7)	P	6 DN
8	(8)	$\neg Q$	A
8	(9)	$\neg P \vee \neg Q$	8 $\vee$I
2,8	(10)	$(\neg P \vee \neg Q) \wedge \neg(\neg P \vee \neg Q)$	9,2 $\wedge$I
2	(11)	$\neg\neg Q$	8,10 RA
2	(12)	Q	11 DN
2	(13)	$P \wedge Q$	7,12 $\wedge$I
1,2	(14)	$(P \wedge Q) \wedge \neg(P \wedge Q)$	13,1 $\wedge$I
1	(15)	$\neg\neg(\neg P \vee \neg Q)$	2,14 RA
1	(16)	$\neg P \vee \neg Q$	15 DN

3. $\neg(P \to Q) \vdash P \wedge \neg Q$

1	(1)	$\neg(P \to Q)$	A
2	(2)	$\neg P$	A
3	(3)	$\neg Q$	A
4	(4)	P	A
2,4	(5)	$P \wedge \neg P$	2,4 $\wedge$I
2,4	(6)	$\neg\neg Q$	3,5 RA
2,4	(7)	Q	6 DN
2	(8)	$P \to Q$	4,7 CP
1,2	(9)	$(P \to Q) \wedge \neg(P \to Q)$	8,1 $\wedge$I
1	(10)	$\neg\neg P$	2,9 RA
1	(11)	P	10 DN
12	(12)	Q	A
12	(13)	$P \to Q$	4,12 CP
1,12	(14)	$(P \to Q) \wedge \neg(P \to Q)$	13,1 $\wedge$I
1	(15)	$\neg Q$	12,14 RA
1	(16)	$P \wedge \neg Q$	11,15 $\wedge$I

4. $\vdash (P \to Q) \vee (Q \to P)$

1	(1)	$\neg((P \to Q) \vee (Q \to P))$	A
2	(2)	P	A
3	(3)	Q	A
2	(4)	$Q \to P$	3,2 CP
2	(5)	$(P \to Q) \vee (Q \to P)$	4 $\vee$I
1,2	(6)	$((P \to Q) \vee (Q \to P)) \wedge \neg((P \to Q) \vee (Q \to P))$	5,1 $\wedge$I
1	(7)	$\neg P$	2,6 RA
8	(8)	$\neg Q$	A
1,2	(9)	$P \wedge \neg P$	2,7 $\wedge$I
1,2	(10)	$\neg\neg Q$	8,9 RA
1,2	(11)	Q	10 DN
1	(12)	$P \to Q$	2,11 CP
1	(13)	$(P \to Q) \vee (Q \to P)$	12 $\vee$I
1	(14)	$((P \to Q) \vee (Q \to P)) \wedge \neg((P \to Q) \vee (Q \to P))$	13,1 $\wedge$I
$\varnothing$	(15)	$\neg\neg((P \to Q) \vee (Q \to P))$	1,14 RA
$\varnothing$	(16)	$(P \to Q) \vee (Q \to P)$	15 DN

5. $P \to (Q \vee R) \vdash (P \to Q) \vee (P \to R)$

1	(1)	$P \to (Q \vee R)$	A
2	(2)	$\neg((P \to Q) \vee (P \to R))$	A
3	(3)	$\neg P$	A
4	(4)	P	A
5	(5)	$\neg Q$	A
3,4	(6)	$P \wedge \neg P$	4,3 $\wedge$I
3,4	(7)	$\neg\neg Q$	5,6 RA
3,4	(8)	Q	7 DN
3	(9)	$P \to Q$	4,8 CP
3	(10)	$(P \to Q) \vee (P \to R)$	9 $\vee$I
2,3	(11)	$((P \to Q) \vee (P \to R)) \wedge \neg((P \to Q) \vee (P \to R))$	10,2 $\wedge$I
2	(12)	$\neg\neg P$	3,11 RA
2	(13)	P	12 DN
1,2	(14)	$Q \vee R$	1,13 MP
15	(15)	Q	A
15	(16)	$P \to Q$	4,15 CP
15	(17)	$(P \to Q) \vee (P \to R)$	16 $\vee$I
18	(18)	R	A
18	(19)	$P \to R$	4,18 CP
18	(20)	$(P \to Q) \vee (P \to R)$	19 $\vee$I
1,2	(21)	$(P \to Q) \vee (P \to R)$	14,15,17,18,20 $\vee$E
1,2	(22)	$((P \to Q) \vee (P \to R)) \wedge \neg((P \to Q) \vee (P \to R))$	21,2 $\wedge$I
1	(23)	$\neg\neg((P \to Q) \vee (P \to R))$	2,22 RA
1	(24)	$(P \to Q) \vee (P \to R)$	23 DN

6. $(P \wedge Q) \to \neg Q \vdash P \to \neg Q$

1	(1)	$(P \wedge Q) \to \neg Q$	A
2	(2)	P	A
3	(3)	Q	A
2,3	(4)	$P \wedge Q$	2,3 $\wedge$I
1,2,3	(5)	$\neg Q$	1,4 MP
1,2,3	(6)	$Q \wedge \neg Q$	3,5 $\wedge$I
1,2	(7)	$\neg Q$	3,6 RA
1	(8)	$P \to \neg Q$	2,7 CP

Chapter 6

Exercise 6.1

1. No logicians are celebrities. (Lx, Cx)

$$\forall x\,(Lx \to \neg Cx)$$

Equivalently: $\neg\exists x(Lx \wedge Cx)$

2. Some celebrities are not logicians. (Lx, Cx)

$$\exists x\,(Cx \wedge \neg Lx)$$

3. Only students who do the homework will learn logic. (Sx, Hx, Lx)
Either

$$\forall x\,(Lx \to (Sx \wedge Hx))$$

or (inequivalently)

$$\forall x\,((Sx \wedge Lx) \to Hx)$$

depending on whether one intends to restrict the claim to students.

4. All rich logicians are computer scientists. (Rx, Lx, Cx)

$$\forall x\,((Rx \wedge Lx) \to Cx)$$

5. All students and professors get a discount. (Sx, Px, Dx)

$$\forall x\,((Sx \vee Px) \to Dx)$$

6. No logician is rich, unless she is a computer scientist. (Lx, Rx, Cx)

$$\forall x\,((Lx \land Rx) \to Cx)$$

   Equivalent form: $\forall x((Lx \land \neg Cx) \to \neg Rx)$

7. Not all logicians are computer scientists. (Lx, Cx)

$$\neg\forall x\,(Lx \to Cx)$$

   Often put as: $\exists x\,(Lx \land \neg Cx)$.

8. Some logicians are rich computer scientists. (Lx, Rx, Cx)

$$\exists x\,(Lx \land (Rx \land Cx))$$

9. If there are rich logicians, then some logicians are computer scientists. (Rx, Lx, Cx)

$$\exists x\,(Rx \land Lx) \to \exists y\,(Ly \land Cy)$$

10. No pets except service animals are permitted in dorms. (Px, Sx, Dx)
    Can be read in a minimal way as:

$$\forall x\,((Px \land Dx) \to Sx),$$

    which says only that no non–service pets are allowed in dorms. However, ordinary policy language is typically understood more strongly: among pets, *being permitted in the dorms* and *being a service animal* coincide. That reading is captured by:

$$\forall x\,(Px \to (Dx \leftrightarrow Sx)).$$

    This biconditional formalization is therefore closer to the intended rule.

11. If anyone is rich, then Mary is. (Rx, m)

$$(\exists x\, Rx) \to Rm$$

Exercise 6.2

1. Mary loves everyone who loves her. (m, Lxy)

$$\forall x\,(Lxm \to Lmx)$$

2. Mary loves all and only those people who don't love themselves. (Lxy, m)

$$\forall x\,(Lmx \leftrightarrow \neg Lxx)$$

3. Everyone loves their mother. (Lxy, Mxy)

$$\forall x \forall y\,(Myx \to Lxy)$$

4. Some people love only those people who love their mother. (Lxy, Mxy)

$$\exists x \forall y (Lxy \to \forall z (Mzy \to Lyz))$$

5. Snape killed someone. (Kxy, s)

$$\exists x\, Ksx$$

6. Snape is a killer. (Kxy, s)

$$\exists x\, Ksx$$

7. Someone was killed by Snape. (Kxy, s)

$$\exists x\, Ksx$$

8. Some wizards only marry other wizards. (Wx, Mxy)

$$\exists x (Wx \wedge \forall y\,(Mxy \to Wy))$$

9. There is no greatest number. $(Nx,\ x < y)$

$$\forall x (Nx \to \exists y\,(Ny \wedge x < y))$$

10. c is the least upper bound of a and b. $(a, b, c,\ x \leq y)$

$$(a \leq c \wedge b \leq c) \ \wedge\ \forall x((a \leq x \wedge b \leq x) \to c \leq x)$$

11. c is the greatest common divisor of a and b. $(a, b, c, Dxy,\ x \leq y)$

$$(Dca \wedge Dcb) \ \wedge\ \forall x((Dxa \wedge Dxb) \to x \leq c)$$

Exercise 6.8

1. $\neg \exists x (Fx \wedge Gx) \vdash \forall x (Fx \to \neg Gx)$

1	(1)	$\neg \exists x (Fx \wedge Gx)$	A
2	(2)	Fa	A
3	(3)	Ga	A
2,3	(4)	$Fa \wedge Ga$	2,3 $\wedge$I
2,3	(5)	$\exists x (Fx \wedge Gx)$	4 EI
1,2,3	(6)	$\exists x (Fx \wedge Gx) \wedge \neg \exists x (Fx \wedge Gx)$	5,1 $\wedge$I
1,2	(7)	$\neg Ga$	3,6 RA
1	(8)	$Fa \to \neg Ga$	2,7 CP
1	(9)	$\forall x (Fx \to \neg Gx)$	8 UI

2. $\forall xFx \vdash \exists xFx$

1	(1)	$\forall xFx$	A
1	(2)	Fa	1 UE
1	(3)	$\exists xFx$	2 EI

3. $\forall x(Fx \to Gx), Fa \vdash \exists xGx$

1	(1)	$\forall x(Fx \to Gx)$	A
2	(2)	Fa	A
1	(3)	$Fa \to Ga$	1 UE
1,2	(4)	Ga	3,2 MP
1,2	(5)	$\exists xGx$	4 EI

4. $\neg Fa \vdash \exists x(Fx \to P)$

1	(1)	$\neg Fa$	A
1	(2)	$Fa \to P$	1 negative paradox
1	(3)	$\exists x(Fx \to P)$	2 EI

5. $\neg\forall xFx \vdash \exists x(Fx \to P)$

1	(1)	$\neg\forall xFx$	A
2	(2)	$\neg\exists x(Fx \to P)$	A
3	(3)	$Fa \to P$	A
3	(4)	$\exists x(Fx \to P)$	3 EI
2,3	(5)	$\exists x(Fx \to P) \land \neg\exists x(Fx \to P)$	4,2 $\land$I
2	(6)	$\neg(Fa \to P)$	3,5 RA
2	(7)	Fa	6 material conditional
2	(8)	$\forall xFx$	7 UI
1,2	(9)	$\forall xFx \land \neg\forall xFx$	8,1 $\land$I
1	(10)	$\neg\neg\exists x(Fx \to P)$	2,9 RA
1	(11)	$\exists x(Fx \to P)$	10 DN

6. $\neg\exists xFx \vdash \forall x(Fx \to Gx)$

1	(1)	$\neg\exists xFx$	A
2	(2)	Fa	A
3	(3)	$\neg Ga$	A
2	(4)	$\exists xFx$	2 EI
1,2	(5)	$\exists xFx \land \neg\exists xFx$	4,1 $\land$I
1,2	(6)	$\neg\neg Ga$	3,5 RA
1,2	(7)	Ga	6 DN
1	(8)	$Fa \to Ga$	2,7 CP
1	(9)	$\forall x(Fx \to Gx)$	8 UI

7. $\forall x\forall yRxy \vdash \exists xRxx$

1	(1)	$\forall x \forall y Rxy$	A
1	(2)	$\forall y Ray$	1 UE
1	(3)	Raa	2 UE
1	(4)	$\exists x Rxx$	3 EI

8. $P \to Fa \vdash P \to \exists x Fx$

1	(1)	$P \to Fa$	A
2	(2)	P	A
1,2	(3)	Fa	1,2 MP
1,2	(4)	$\exists x Fx$	3 EI
1	(5)	$P \to \exists x Fx$	2,4 CP

9. $\exists x Fx \to P \vdash \forall x(Fx \to P)$

1	(1)	$\exists x Fx \to P$	A
2	(2)	Fa	A
2	(3)	$\exists x Fx$	2 EI
1,2	(4)	P	1,3 MP
1	(5)	$Fa \to P$	2,4 CP
1	(6)	$\forall x(Fx \to P)$	5 UI

There is a typo here in the book: the direction $\forall x(Fx \to P) \vdash \exists x Fx \to P$ cannot be proven without EE, which is only introduced in the next section.

10. $\neg\exists x Fx \vdash \forall x(Fx \to P)$

1	(1)	$\neg\exists x Fx$	A
2	(2)	Fa	A
2	(3)	$\exists x Fx$	2 EI
1,2	(4)	$\exists x Fx \wedge \neg\exists x Fx$	3,1 $\wedge$I
1	(5)	$\neg Fa$	2,4 RA
1	(6)	$Fa \to P$	5 neg paradox
1	(7)	$\forall x(Fx \to P)$	6 UI

11. $\neg\exists x(Fx \to P) \vdash \forall x Fx \wedge \neg P$

1	(1)	$\neg\exists x(Fx \to P)$	A
2	(2)	$Fa \to P$	A
2	(3)	$\exists x(Fx \to P)$	2 EI
1,2	(4)	$\exists x(Fx \to P) \wedge \neg\exists x(Fx \to P)$	3,1 $\wedge$I
1	(5)	$\neg(Fa \to P)$	2,4 RA
1	(6)	$Fa \wedge \neg P$	5 material conditional
1	(7)	$\neg P$	6 $\wedge$E
1	(8)	Fa	6 $\wedge$E
1	(9)	$\forall x Fx$	8 UI
1	(10)	$\forall x Fx \wedge \neg P$	9,7 $\wedge$I

12. $\forall xFx \to P \vdash \exists x(Fx \to P)$

1	(1)	$\forall xFx \to P$	A
2	(2)	$\neg\exists x(Fx \to P)$	A
3	(3)	$\neg Fa$	A
3	(4)	$Fa \to P$	3 neg paradox
3	(5)	$\exists x(Fx \to P)$	4 EI
2,3	(6)	$\exists x(Fx \to P) \wedge \neg\exists x(Fx \to P)$	5,2 $\wedge$I
2	(7)	$\neg\neg Fa$	3,6 RA
2	(8)	Fa	7 DN
2	(9)	$\forall xFx$	8 UI
1,2	(10)	P	1,9 MP
1,2	(11)	$Fb \to P$	10 pos paradox
1,2	(12)	$\exists x(Fx \to P)$	11 EI
1,2	(13)	$\exists x(Fx \to P) \wedge \neg\exists x(Fx \to P)$	12,2 $\wedge$I
1	(14)	$\neg\neg\exists x(Fx \to P)$	2,13 RA
1	(15)	$\exists x(Fx \to P)$	14 DN

Exercise 6.11

1. $\exists xFx \vee \exists xGx \vdash \exists x(Fx \vee Gx)$

1	(1)	$\exists xFx \vee \exists xGx$	A
2	(2)	$\exists xFx$	A
3	(3)	Fa	A
3	(4)	$Fa \vee Ga$	3 $\vee$I
3	(5)	$\exists x(Fx \vee Gx)$	4 EI
2	(6)	$\exists x(Fx \vee Gx)$	2,3,5 EE
7	(7)	$\exists xGx$	A
8	(8)	Ga	A
8	(9)	$Fa \vee Ga$	8 $\vee$I
8	(10)	$\exists x(Fx \vee Gx)$	9 EI
7	(11)	$\exists x(Fx \vee Gx)$	7,8,10 EE
1	(12)	$\exists x(Fx \vee Gx)$	1,2,6,7,11 $\vee$E

2. $\forall x(Fx \to Gx), \neg\exists xGx \vdash \neg\exists xFx$

1	(1)	$\forall x(Fx \to Gx)$	A
2	(2)	$\neg\exists xGx$	A
3	(3)	$\exists xFx$	A
4	(4)	Fa	A
1	(5)	$Fa \to Ga$	1 UE
1,4	(6)	Ga	5,4 MP
1,4	(7)	$\exists xGx$	6 EI
1,3	(8)	$\exists xGx$	3,4,7 EE
1,2,3	(9)	$\exists xGx \wedge \neg\exists xGx$	8,2 $\wedge$I
1,2	(10)	$\neg\exists xFx$	3,9 RA

3. $\forall x(Fx \to Gx) \vdash \exists x\neg Gx \to \exists x\neg Fx$

1	(1)	$\forall x(Fx \to Gx)$	A
2	(2)	$\exists x\neg Gx$	A
3	(3)	$\neg Ga$	A
1	(4)	$Fa \to Ga$	1 UE
1,3	(5)	$\neg Fa$	4,3 MT
1,3	(6)	$\exists x\neg Fx$	5 EI
1,2	(7)	$\exists x\neg Fx$	2,3,6 EE
1	(8)	$\exists x\neg Gx \to \exists x\neg Fx$	2,7 CP

4. $\forall x(Fx \to P) \vdash \exists xFx \to P$

1	(1)	$\forall x(Fx \to P)$	A
2	(2)	$\exists xFx$	A
3	(3)	Fa	A
1	(4)	$Fa \to P$	1 UE
1,3	(5)	P	4,3 MP
1,2	(6)	P	2,3,5 EE
1	(7)	$\exists xFx \to P$	2,6 CP

5. $P \wedge \exists xFx \vdash \exists x(P \wedge Fx)$

1	(1)	$P \wedge \exists xFx$	A
1	(2)	P	1 $\wedge$E
1	(3)	$\exists xFx$	1 $\wedge$E
4	(4)	Fa	A
1,4	(5)	$P \wedge Fa$	2,4 $\wedge$I
1,4	(6)	$\exists x(P \wedge Fx)$	5 EI
1	(7)	$\exists x(P \wedge Fx)$	3,4,6 EE

6. $\exists x(Fx \to P) \vdash \forall xFx \to P$

1	(1)	$\exists x(Fx \to P)$	A
2	(2)	$\forall xFx$	A
3	(3)	$Fa \to P$	A
2	(4)	Fa	2 UE
2,3	(5)	P	3,4 MP
3	(6)	$\forall xFx \to P$	2,5 CP
1	(7)	$\forall xFx \to P$	1,3,6 EE

Exercise 6.13

1. $P \to \exists xFx \vdash \exists x(P \to Fx)$

1	(1)	$P \to \exists xFx$	A
$\varnothing$	(2)	$\exists xFx \lor \neg\exists xFx$	prop taut
3	(3)	$\exists xFx$	A
4	(4)	Fa	A
4	(5)	$P \to Fa$	4 prop taut
4	(6)	$\exists x(P \to Fx)$	5 EI
3	(7)	$\exists x(P \to Fx)$	3,4,6 EE
8	(8)	$\neg\exists xFx$	A
1,8	(9)	$\neg P$	1,8 MT
1,8	(10)	$P \to Fb$	9 prop taut
1,8	(11)	$\exists x(P \to Fx)$	10 EI
1	(12)	$\exists x(P \to Fx)$	2,3,7,8,11 $\lor$E

2. $\exists x(Fx \to P) \vdash \forall xFx \to P$

1	(1)	$\exists x(Fx \to P)$	A
2	(2)	$\forall xFx$	A
3	(3)	$Fa \to P$	A
2	(4)	Fa	2 UE
2,3	(5)	P	3,4 MP
1,2	(6)	P	1,3,5 EE
1	(7)	$\forall xFx \to P$	2,6 CP

Exercise 6.14

1. $\vdash \forall x(Fx \to Fx)$

1	(1)	Fa	A
$\varnothing$	(2)	$Fa \to Fa$	1,1 CP
$\varnothing$	(3)	$\forall x(Fx \to Fx)$	2 UI

2. $\vdash \forall xFx \lor \exists x\neg Fx$

$\varnothing$	(1)	$\neg\exists x \neg Fx \vee \exists x \neg Fx$	prop taut
2	(2)	$\neg\exists x \neg Fx$	A
3	(3)	$\neg Fa$	A
3	(4)	$\exists x \neg Fx$	3 EI
2,3	(5)	$\exists x \neg Fx \wedge \neg\exists x \neg Fx$	4,2 $\wedge$I
2	(6)	$\neg\neg Fa$	3,5 RA
2	(7)	Fa	6 DN
2	(8)	$\forall x Fx$	7 UI
2	(9)	$\forall x Fx \vee \exists x \neg Fx$	8 $\vee$I
10	(10)	$\exists x \neg Fx$	A
10	(11)	$\forall x Fx \vee \exists x \neg Fx$	10 $\vee$I
$\varnothing$	(12)	$\forall x Fx \vee \exists x \neg Fx$	1,2,9,10,11 $\vee$E

3. $\vdash \forall x \neg(Fx \wedge \neg Fx)$

1	(1)	$Fa \wedge \neg Fa$	A
$\varnothing$	(2)	$\neg(Fa \wedge \neg Fa)$	1,1 RA
$\varnothing$	(3)	$\forall x \neg(Fx \wedge \neg Fx)$	2 UI

4. $\vdash \neg\exists x(Fx \wedge \neg Fx)$

1	(1)	$\exists x(Fx \wedge \neg Fx)$	A
2	(2)	$Fa \wedge \neg Fa$	A
2	(3)	$\neg\exists x(Fx \wedge \neg Fx)$	1,2 RA
1	(4)	$\neg\exists x(Fx \wedge \neg Fx)$	1,2,3 EE
1	(5)	$\exists x(Fx \wedge \neg Fx) \wedge \neg\exists x(Fx \wedge \neg Fx)$	1,3 $\wedge$I
$\varnothing$	(6)	$\neg\exists x(Fx \wedge \neg Fx)$	1,5 RA

5. $\vdash \forall x \exists y(Rxy \to Rxx)$

1	(1)	Raa	A
$\varnothing$	(2)	$Raa \to Raa$	1,1 CP
$\varnothing$	(3)	$\exists y(Ray \to Raa)$	2 EI
$\varnothing$	(4)	$\forall x \exists y(Rxy \to Rxx)$	3 UI

6. $\vdash \forall x \exists y(Rxy \to Ryx)$

1	(1)	Raa	A
$\varnothing$	(2)	$Raa \to Raa$	1,1 CP
$\varnothing$	(3)	$\exists y(Ray \to Rya)$	2 EI
$\varnothing$	(4)	$\forall x \exists y(Rxy \to Ryx)$	3 UI

7. $\vdash \exists x(Fx \to \forall y Fy)$

1	(1)	$\neg\exists x(Fx \to \forall yFy)$	A
2	(2)	$\neg Fa$	A
2	(3)	$Fa \to \forall yFy$	2 prop taut
2	(4)	$\exists x(Fx \to \forall yFy)$	3 EI
1,2	(5)	$\exists x(Fx \to \forall yFy) \land \neg\exists x(Fx \to \forall yFy)$	4,1 $\land$I
1	(6)	$\neg\neg Fa$	2,5 RA
1	(7)	Fa	6 DN
1	(8)	$\forall yFy$	7 UI
1	(9)	$Fa \to \forall yFy$	8 prop taut
1	(10)	$\exists x(Fx \to \forall yFy)$	9 EI
1	(11)	$\exists x(Fx \to \forall yFy) \land \neg\exists x(Fx \to \forall yFy)$	10,1 $\land$I
$\varnothing$	(12)	$\neg\neg\exists x(Fx \to \forall yFy)$	1,11 RA
$\varnothing$	(13)	$\exists x(Fx \to \forall yFy)$	12 DN

8. $\vdash \exists x\forall y(Fx \to Fy)$

1	(1)	$\neg\exists x\forall y(Fx \to Fy)$	A
2	(2)	$\neg Fa$	A
2	(3)	$Fa \to Fb$	2 prop taut
2	(4)	$\forall y(Fa \to Fy)$	3 UI
2	(5)	$\exists x\forall y(Fx \to Fy)$	4 EI
1,2	(6)	$\exists x\forall y(Fx \to Fy) \land \neg\exists x\forall y(Fx \to Fy)$	5,1 $\land$I
1	(7)	$\neg\neg Fa$	2,6 RA
1	(8)	Fa	7 DN
1	(9)	$Fc \to Fa$	8 prop taut
1	(10)	$\forall y(Fc \to Fy)$	9 UI
1	(11)	$\exists x\forall y(Fx \to Fy)$	10 EI
1	(12)	$\exists x\forall y(Fx \to Fy) \land \neg\exists x\forall y(Fx \to Fy)$	11,1 $\land$I
$\varnothing$	(13)	$\neg\neg\exists x\forall y(Fx \to Fy)$	1,12 RA
$\varnothing$	(14)	$\exists x\forall y(Fx \to Fy)$	13 DN

9. $\forall x\exists y(Fx \to Gy) \vdash \exists y\forall x(Fx \to Gy)$

1	(1)	$\forall x \exists y(Fx \to Gy)$	A
$\varnothing$	(2)	$\exists y Gy \vee \neg\exists y Gy$	prop taut
3	(3)	$\exists y Gy$	A
4	(4)	Ga	A
4	(5)	$Fb \to Ga$	4 prop taut
4	(6)	$\forall x(Fx \to Ga)$	5 UI
4	(7)	$\exists y \forall x(Fx \to Gy)$	6 EI
3	(8)	$\exists y \forall x(Fx \to Gy)$	3,4,7 EE
9	(9)	$\neg\exists y Gy$	A
10	(10)	Fc	A
1	(11)	$\exists y(Fc \to Gy)$	1 UE
12	(12)	$Fc \to Gd$	A
10,12	(13)	Gd	12,10 MP
10,12	(14)	$\exists y Gy$	13 EI
9,10,12	(15)	$\exists y Gy \wedge \neg\exists y Gy$	14,9 $\wedge$I
9,12	(16)	$\neg Fc$	10,15 RA
9,12	(17)	$Fc \to Ge$	16 prop taut
1,9	(18)	$Fc \to Ge$	11,12,17 EE
1,9	(19)	$\forall x(Fx \to Ge)$	18 UI
1,9	(20)	$\exists y \forall x(Fx \to Gy)$	19 EI
1	(21)	$\exists y \forall x(Fx \to Gy)$	2,3,8,9,20 $\vee$E

10. $\vdash \forall x \exists y(Rxy \to \forall z Rxz)$

$\varnothing$	(1)	$\exists y \neg Ray \vee \neg\exists y \neg Ray$	prop taut
2	(2)	$\exists y \neg Ray$	A
3	(3)	$\neg Rab$	A
3	(4)	$Rab \to \forall z Raz$	3 prop taut
3	(5)	$\exists y(Ray \to \forall z Raz)$	4 EI
2	(6)	$\exists y(Ray \to \forall z Raz)$	2,3,5 EE
7	(7)	$\neg\exists y \neg Ray$	A
8	(8)	$\neg Rac$	A
8	(9)	$\exists y \neg Ray$	8 EI
7,8	(10)	$\exists y \neg Ray \wedge \neg\exists y \neg Ray$	9,7 $\wedge$I
7	(11)	$\neg\neg Rac$	8,10 RA
7	(12)	Rac	11 DN
7	(13)	$\forall z Raz$	12 UI
7	(14)	$Rab \to \forall z Raz$	13 prop taut
7	(15)	$\exists y(Ray \to \forall z Raz)$	14 EI
$\varnothing$	(16)	$\exists y(Ray \to \forall z Raz)$	1,2,6,7,15 $\vee$E
$\varnothing$	(17)	$\forall x \exists y(Rxy \to \forall z Rxz)$	16 UI

Exercise 6.17

$\forall x(\exists zRxz \to \forall yRxy), \exists x\exists y \vdash \exists x\forall yRxy$

1	(1)	$\forall x(\exists zRxz \to \forall yRxy)$	A
2	(2)	$\exists x\exists yRxy$	A
3	(3)	$\exists yRay$	A
4	(4)	Rab	A
4	(5)	$\exists zRaz$	4 EI
1	(6)	$\exists zRaz \to \forall yRay$	1 UE
1,4	(7)	$\forall yRay$	6,5 MP
1,4	(8)	$\exists x\forall yRxy$	7 EI
1,3	(9)	$\exists x\forall yRxy$	3,4,8 EE
1,2	(10)	$\exists x\forall yRxy$	2,3,9 EE

Question: Does it follow from these premises that $\forall x\forall yRxy$?

Answer: No. $\begin{tikzcd} a \arrow[loop left] \arrow[r] & b \end{tikzcd}$

Chapter 7

Exercise 7.1

Here the proof is lengthened because of the strictness of the $=$ rules. From $a = c$ and $b = c$, we cannot immediately apply =E to get $a = b$.

1	(1)	$\exists x\forall y(Py \to y = x)$	A
2	(2)	$Pa \land Pb$	A
3	(3)	$\forall y(Py \to y = c)$	A
3	(4)	$Pa \to a = c$	3 UE
3	(5)	$Pb \to b = c$	3 UE
2	(6)	Pa	2 $\land$E
2	(7)	Pb	2 $\land$E
2,3	(8)	$a = c$	4,6 MP
2,3	(9)	$b = c$	5,7 MP
$\varnothing$	(10)	$b = b$	=I
2,3	(11)	$c = b$	10,9 =E
2,3	(12)	$a = b$	8,11 =E
1,2	(13)	$a = b$	1,3,12 EE
1	(14)	$(Pa \land Pb) \to a = b$	2,13 CP
1	(15)	$\forall y((Pa \land Py) \to a = y)$	14 UI
1	(16)	$\forall x\forall y((Px \land Py) \to x = y)$	15 UI

Exercise 7.2

1	(1)	$Fa \wedge \forall x(Fx \to x = a)$	A
2	(2)	Fb	A
1	(3)	$\forall x(Fx \to x = a)$	1 $\wedge$E
1	(4)	$Fb \to b = a$	3 UE
1,2	(5)	$b = a$	4,2 MP
6	(6)	$b = a$	A
1	(7)	Fa	1 $\wedge$E
$\varnothing$	(8)	$b = b$	=I
6	(9)	$a = b$	8,6 =E
1,6	(10)	Fb	7,9 =E
1	(11)	$Fb \leftrightarrow b = a$	2,5,6,10 CP$\times$2
1	(12)	$\forall x(Fx \leftrightarrow x = a)$	11 UI

1	(1)	$\forall x(Fx \leftrightarrow x = a)$	A
1	(2)	$Fa \leftrightarrow a = a$	1 UE
$\varnothing$	(3)	$a = a$	=I
1	(4)	Fa	2,3 MP
1	(5)	$Fb \leftrightarrow b = a$	1 UE
1	(6)	$Fb \to b = a$	5 $\wedge$E
1	(7)	$\forall x(Fx \to x = a)$	6 UI
1	(8)	$Fa \wedge \forall x(Fx \to x = a)$	4,7 $\wedge$I

Chapter 9

Exercise 9.14

1. If a formula φ contains only the connectives $\neg$ and $\wedge$, then we will say that it's a L_0-formula. To be precise, $A_1, \dots, A_n \succ B$ means that there is a derivation of B from $A_1, \dots, A_n$ using the $\neg$ and $\wedge$ rules, and where the connectives $\vee, \to$ do not appear in any of the formulas.

   It's true that if $A_1, \dots, A_n, B$ are L_0-formulas, then $A_1, \dots, A_n \vdash B$ only if $A_1, \dots, A_n \succ B$. But the argument is complicated, because the derivation of $A_1, \dots, A_n \vdash B$ might use other rules, and might also contain formulas with $\vee$ or $\to$. There are two ways we could try to prove the result: (1) Convert the existing derivation of $A_1, \dots, A_n \vdash B$ into a derivation of $A_1, \dots, A_n \succ B$. Converting steps of $\to$ and $\vee$ rules into derivations with $\neg$ and $\wedge$ rules is fairly straightforward. (E.g.: check that the L_0-equivalent of CP is derivable from the $\neg$ and $\wedge$ rules.) But there also needs to be an argument that intermediate formulas that are not L_0 can be replaced with L_0-formulas. This argument could be organized by defining a projection F of formulas onto L_0-formulas, and then by showing that $A_1 \dots, A_n \vdash B$ implies $F(A_1), \dots, F(A_n) \succ F(B)$. (2) Directly prove the completeness of the $\neg$ and $\wedge$ rules relative to L_0-sequents. That is, show that if $A_1, \dots, A_n \vDash B$ then $A_1, \dots, A_n \succ B$.