

# Theories

PHI 201 — Introductory Logic

November 3, 2025

# The utopian vision of symbolic logic

- Two original hopes for symbolic logic.
  - ① It provides a universal language for science.
  - ② It dissolves philosophical pseudo-problems.
- While this doesn't work out so easily in practice, there is a sense in which all “theories” in mathematics can be formalized in predicate/relational logic.

# Equality

# Equality is a special relation

- Equality is a binary relation which we write as an infix rather than as a prefix

$$c = d, \exists x(x = d), \forall y \exists x(x = y)$$

- Using “=” allows us to express many new things.

# At least $n$

$$\exists x \exists y (x \neq y)$$

$$\exists x \exists y \exists z ((x \neq y \wedge x \neq z) \wedge y \neq z)$$

At most  $n$

$$\forall x \forall y (x = y)$$

$$\forall x \forall y \forall z ((x = y \vee x = z) \vee y = z)$$

# Exactly $n$

$$\exists x \exists y (x \neq y \wedge \forall z (z = x \vee z = y))$$

$$\begin{aligned} \exists x \exists y \exists z & (((x \neq y \wedge x \neq z) \wedge y \neq z) \\ & \wedge \forall w ((w = x \vee w = y) \vee w = z)) \end{aligned}$$

# There is a unique $P$

$$\exists x(Px \wedge \forall y(Py \rightarrow x = y))$$

# Definite descriptions

# Superlatives

“There is a tallest student.”

$$\exists x \forall y (x \neq y \rightarrow Txy)$$

This sentence entails uniqueness only because we implicitly assume that “taller than” is asymmetric.

$$\forall x \forall y (Txy \rightarrow \neg Tyx)$$

1 (1)  $\exists x \forall y (x \neq y \rightarrow Txy)$  A

2 (2)  $\forall y (a \neq y \rightarrow Tay)$  A

3 (3)  $\forall y (b \neq y \rightarrow Tby)$  A

4 (4)  $a \neq b$  A

2 (5)  $a \neq b \rightarrow Tab$  2 UE

3 (6)  $b \neq a \rightarrow Tba$  3 UE

# Inference rules for equality

$\Gamma \quad (m) \quad \varphi(a)$

$\Delta \quad (n) \quad a = b$

$\Gamma, \Delta \quad (o) \quad \varphi(b) \qquad m, n = E$

To show:  $a = b, b = c \vdash a = c$

$$1 \quad (1) \quad a = b$$

A

$$2 \quad (2) \quad b = c$$

A

$$1,2 \quad (3) \quad a = c$$

2,1 =E

# Inference rules for equality

$$(m) \quad a = a \qquad \qquad \qquad =I$$

To show:  $a = b \vdash b = a$

$$1 \quad (1) \quad a = b$$

A

$$(2) \quad a = a$$

=I

$$1 \quad (3) \quad b = a$$

2,1 =E

# Nobody but

Alice respects nobody but Bob.

$$Rab \wedge \forall x(Rax \rightarrow x = b)$$

$$\forall x(Rax \leftrightarrow x = b)$$

# Everybody loves my baby

- |   |     |                                    |      |
|---|-----|------------------------------------|------|
| 1 | (1) | $\forall x Lxb$                    | A    |
| 2 | (2) | $\forall y(Lby \rightarrow y = a)$ | A    |
| 1 | (3) | $Lbb$                              | 1 UE |
| 2 | (4) | $Lbb \rightarrow b = a$            | 2 UE |

- The theory of equality is peculiar, because we build its axioms in as new inference rules.
- Now we look at theories whose axioms are sentences.

# Theory of partial order

transitive:

$$\forall x \forall y \forall z ((x \leq y \wedge y \leq z) \rightarrow x \leq z)$$

reflexive:

$$\forall x (x \leq x)$$

antisymmetric:

$$\forall x \forall y ((x \leq y \wedge y \leq x) \rightarrow x = y)$$

# A cornucopia of partially ordered sets

linear:

$$\forall x \forall y ((x \leq y) \vee (y \leq x))$$

- What's a sentence that is true of the natural numbers  $1, 2, 3, \dots$  but false of the integers  $\dots, -2, -1, 0, 1, 2, \dots ?$
- What's a sentence that is true of the integers but false of the rational numbers?

# Set theory

extensionality

$$\forall x \forall y (x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y))$$

existence of an emptyset

$$\exists z \forall x (x \notin z)$$

# Uniqueness of the emptyset

- |   |     |                               |           |
|---|-----|-------------------------------|-----------|
| 1 | (1) | $\forall x(x \notin a)$       | A         |
| 2 | (2) | $\forall x(x \notin b)$       | A         |
| 1 | (3) | $c \notin a$                  | 1 UE      |
| 1 | (4) | $c \in a \rightarrow c \in b$ | 3 neg par |

# Naive set theory

comprehension

$$\exists x \forall y (y \in x \leftrightarrow \varphi(y))$$

# Consistent theories

We say that a theory  $T$  is **consistent** if there is no sentence  $\varphi$  such that both  $T \vdash \varphi$  and  $T \vdash \neg\varphi$ .

# Naive set theory is inconsistent

Use comprehension with the predicate “ $y \notin y$ ”

$$1 \quad (1) \quad \exists x \forall y (y \in x \leftrightarrow y \notin y) \quad A$$

$$2 \quad (2) \quad \forall y (y \in a \leftrightarrow y \notin y) \quad A$$

$$2 \quad (3) \quad a \in a \leftrightarrow a \notin a \quad 2 \text{ UE}$$

# Sophisticating set theory

pairing

$$\forall x \forall y \exists z \forall w (w \in z \leftrightarrow (w = x \vee w = y))$$

separation: For every formula  $\varphi(x, b_1, \dots, b_n)$ ,

$$\forall y \exists z \forall x (x \in z \leftrightarrow (x \in y \wedge \varphi(x, b_1, \dots, b_n))).$$

# Existence and uniqueness of intersections

	(1)	$\exists z \forall x(x \in z \leftrightarrow (x \in a \wedge x \in b))$	sep
2	(2)	$\forall x(x \in c \leftrightarrow (x \in a \wedge x \in b))$	A
	(3)	$\forall y \forall y'(\forall x(x \in y \leftrightarrow x \in y') \rightarrow y = y')$	ext
4	(4)	$\forall x(x \in d \leftrightarrow (x \in a \wedge x \in b))$	A
2,4	(5)	$\forall x(x \in c \leftrightarrow x \in d)$	2,4
	(6)	$\forall x(x \in c \leftrightarrow x \in d) \rightarrow c = d$	3 UE
2,4	(8)	$c = d$	6,5 MP