Existential Introduction & Elimination

PHI 201 — Introductory Logic

Lecture 8

Overview

- Review of universal quantifier rules
- New rules for the existential quantifier:
  - Existential Introduction (**EI**)
  - Existential Elimination (**EE**)
- Practice proofs involving $\exists$ and $\forall$

Existential Introduction

Rule

From a particular instance Fa, we may infer that something is F:

$$\frac{Fa}{\exists x\, Fx} \text{ (EI)}$$

1	(1)	$Fa \to Ga$	A
1	(2)	$\exists x(Fx \to Gx)$	1 EI

1	(1)	$Fa \to Ga$	A
1	(2)	$\exists x(Fx \to Ga)$	1 EI

Unlike UI, EI permits replacement of some (but not all) occurences of a name a.

1	(1)	Raa	A
1	(2)	$\exists x Rxx$	1 EI

1	(1)	Raa	A
1	(2)	$\exists y Ray$	1 EI
1	(3)	$\exists x \exists y Rxy$	2 EI

To show: $\neg\exists x Fx \ \vdash \ \forall x \neg Fx$

1	(1)	$\neg\exists x Fx$	A
2	(2)	Fa	A
2	(3)	$\exists x Fx$	2 EI
1,2	(4)	$\exists x Fx \wedge \neg\exists x Fx$	3,1 $\wedge$I
1	(5)	$\neg Fa$	2,4 RA
1	(6)	$\forall x \neg Fx$	5 UI

1	(1)	$\neg\forall x Fx$	A
2	(2)	$\neg\exists x \neg Fx$	A
3	(3)	$\neg Fa$	A
3	(4)	$\exists x \neg Fx$	3 EI
2,3	(5)	$\exists x \neg Fx \wedge \neg\exists x \neg Fx$	4,2 $\wedge$I
2	(6)	$\neg\neg Fa$	3,5 RA
2	(7)	Fa	6 DN
2	(8)	$\forall x Fx$	7 UI
1,2	(9)	$\forall x Fx \wedge \neg\forall x Fx$	8,1 $\wedge$I
1	(10)	$\neg\neg\exists x \neg Fx$	2,9 RA
1	(11)	$\exists x \neg Fx$	10 DN

To show: $\neg\forall x Fx \vdash \exists x(Fx \to P)$

To show $\neg\exists x Fx \vdash \forall x(Fx \to P)$

1	(1)	$\neg\exists x Fx$	A
2	(2)	Fa	A
2	(3)	$\exists x Fx$	2 EI
1,2	(4)	$\exists x Fx \wedge \neg\exists x Fx$	3,1 $\wedge$I
1	(5)	$\neg Fa$	2,4 RA
1	(6)	$Fa \to P$	5 neg par
1	(7)	$\forall x(Fx \to P)$	6 UI

Existential elimination

Idea

What can be derived from $\exists x \varphi(x)$?

Pick an arbitrary name a. If a general claim ψ can be derived from an instance $\varphi(a)$, without making any additional assumptions about a, then ψ follows from $\exists x \varphi(x)$.

Existential Elimination

Γ	(m)	$\exists x \varphi(x)$	
n	(n)	$\varphi(a)$	A
Δ	(o)	ψ	
$\Gamma, \Delta \backslash \{n\}$	(p)	ψ	m,n,o EE

Existential Elimination

- **Restriction:** The name a must not occur in any dependencies of the existential premise, or of the derived conclusion, except for the instance $\varphi(a)$ itself.
- Dependencies are the union of the dependencies of the existential sentence and those of the derived conclusion, minus dependency on the instance.

Example

1	(1)	$\exists x(Fx \wedge Gx)$	A
2	(2)	$Fa \wedge Ga$	A
2	(3)	Fa	2 $\wedge$E
2	(4)	$\exists x Fx$	3 EI
1	(5)	$\exists x Fx$	1,2,4 EE

Reasoning with multiple $\exists$

To show: $\exists x \exists y Rxy \vdash \exists y \exists x Rxy$

1	(1)	$\exists x \exists y Rxy$	A
2	(2)	$\exists y Ray$	A
3	(3)	Rab	A
3	(4)	$\exists x Rxb$	3 EI
3	(5)	$\exists y \exists x Rxy$	4 EI
2	(6)	$\exists y \exists x Rxy$	2,3,5 EE
1	(7)	$\exists y \exists x Rxy$	1,2,6 EE

Reasoning with $\forall$ and $\exists$

1	(1)	$\forall x(Fx \to Gx)$	A
2	(2)	$\exists x Fx$	A
3	(3)	Fa	A
1	(4)	$Fa \to Ga$	1 UE
1,3	(5)	Ga	4,3 MP
1,3	(6)	$\exists x Gx$	5 EI
1,2	(7)	$\exists x Gx$	2,3,6 EE

Reasoning with $\forall$ and $\exists$

1	(1)	$\exists y \forall x Rxy$	A
2	(2)	$\forall x Rxb$	A
2	(3)	Rab	2 UE
2	(4)	$\exists y Ray$	3 EI
2	(5)	$\forall x \exists y Rxy$	4 UI
1	(6)	$\forall x \exists y Rxy$	1,2,5 EE

Preventing invalid inferences

1	(1)	$\exists x Fx$	A
2	(2)	$\exists x Gx$	A
3	(3)	Fa	A
4	(4)	Ga	A
3,4	(5)	$Fa \wedge Ga$	3,4 $\wedge$I
3,4	(6)	$\exists x(Fx \wedge Gx)$	5 EI

EE cannot be applied to 1,3,6 because 6 depends on 4, which contains a.
EE cannot be applied to 1,4,6 because 6 depends on 3, which contains a.

Preventing invalid inferences

1	(1)	$\forall x \exists y Rxy$	A
1	(2)	$\exists y Ray$	1 UE
3	(3)	Rab	A
3	(4)	$\forall x Rxb$	Error!

UI cannot be applied to 3 because it depends on 3, which contains a.

Quantifier order matters

$\forall x \exists y\, \varphi(x, y)$ follows from $\exists y \forall x\, \varphi(x, y)$.
But not vice versa.

Quantifier negation equivalences

$\neg\exists xFx$ is equivalent to $\forall x\neg Fx$

$\neg\forall xFx$ is equivalent to $\exists x\neg Fx$

"Equivalent" means mutually derivable

To show: $\forall x \neg Fx \vdash \neg\exists x Fx$

1	(1)	$\forall x \neg Fx$	A
2	(2)	$\exists x Fx$	A
3	(3)	Fa	A
1	(4)	$\neg Fa$	1 UE
1,3	(5)	$Fa \wedge \neg Fa$	3,4 $\wedge$I
1,3	(6)	$\neg\exists x Fx$	2,5 RA
1,2	(7)	$\neg\exists x Fx$	2,3,6 EE
1,2	(8)	$\exists x Fx \wedge \neg\exists x Fx$	2,7 $\wedge$I
1	(9)	$\neg\exists x Fx$	2,8 RA

Non-constructive existence proofs

How to derive $\exists x(Fx \to P)$ from $\forall x Fx \to P$?

Not possible to derive $Fa \to P$ from $\forall x Fx \to P$.