PHI 201, Practice Final Exam (Solutions)

Instructions: Write your name, preceptor's name, and pledge on the exam booklet. Write your answers *legibly* in the exam booklet. While you may take up to three hours to complete the exam, it was designed to take no more than two.

1. Translate the following into predicate logic. You can assume that the domain is people, and so you don't need an additional predicate symbol for "x is a person".

   (a) There is a person who loves all people who love her. (Use Lxy for "x loves y".)

   (b) Every lover loves herself.

   (c) There are exactly two people.

2. Could the following sentence be true? Explain your answer.

   $$(\neg P \vee Q) \wedge ((Q \to (\neg R \wedge \neg P)) \wedge (P \vee R))$$

3. Explain what's wrong with the following attempted proof:

1	(1) Fa	A
$\varnothing$	(2) $Fa \to Fa$	1,1 CP
$\varnothing$	(3) $\forall y(Fy \to Fa)$	2 UI
$\varnothing$	(4) $\exists x \forall y(Fy \to Fx)$	3 EI

4. Prove the following sequent. You can use "cut" or "replacement", but only if you prove the relevant sequents in your exam booklet.

   $$\vdash\ \exists x \forall y(Fy \to Fx)$$

5. Prove the following fact of set theory:

   $$C \setminus (A \cap B)\ \subseteq\ (C \setminus A) \cup (C \setminus B),$$

   where $C \setminus X$ is defined by

   $$\forall x((x \in (C \setminus X)) \leftrightarrow (x \in C \wedge x \notin X)).$$

   Your proof should be rigorous, but it can (preferably) be written in English prose.

   We need to show that every a in $C \setminus (A \cap B)$ is either in $C \setminus A$ or in $C \setminus B$. So suppose that $a \in C \setminus (A \cap B)$, but $a \notin C \setminus A$. The latter implies that either $a \notin C$ or $a \in A$. Since $a \in C \setminus (A \cap B)$, it follows that $a \in C$ but $a \notin A \cap B$. If $a \in B$, then $a \in A$ and $a \in B$, in contradiction with the fact that $a \notin A \cap B$. Therefore $a \notin B$, which means that $a \in C \setminus B$. We have shown that if $a \in C \setminus (A \cap B)$, then if $a \notin C \setminus A$ then

$a \in C \setminus B$. The latter conditional is equivalent to $a \in C \setminus A$ or $a \in C \setminus B$, which is equivalent to $a \in (C \setminus A) \cup (C \setminus B)$. So we have

$$a \in C \setminus (A \cap B) \;\rightarrow\; a \in (C \setminus A) \cup (C \setminus B),$$

for arbitrary a. By UI and the definition of $\subseteq$ we have

$$C \setminus (A \cap B) \subseteq (C \setminus A) \cup (C \setminus B).$$

6. Let Γ be the set of sentences defined inductively by:

   - $P \in \Gamma$
   - If $\varphi \in \Gamma$ then $\neg\varphi \in \Gamma$.
   - If $\varphi \in \Gamma$ and $\psi \in \Gamma$ then $\varphi \rightarrow \psi \in \Gamma$.

   Show that for every $\varphi \in \Gamma$, either $P \vdash \varphi$ or $P \vdash \neg\varphi$.

   We argue by induction that $\forall\varphi D(\varphi)$, where D is the property that either $P \vdash \varphi$ or $P \vdash \neg\varphi$.

   Base case: P has property D since $P \vdash P$ (Rule of Assumptions).

   Inductive step: Suppose that both φ and ψ have property D. We show that $\varphi \rightarrow \psi$ also has property D. We break it down into two cases. In the first case, either $P \vdash \neg\varphi$ or $P \vdash \psi$. In both cases $P \vdash \varphi \rightarrow \psi$, either by negative paradox or by positive paradox. In the second case, $P \vdash \varphi$ and $P \vdash \neg\psi$. In this case, $P \vdash \varphi \wedge \neg\psi$, and so $P \vdash \neg(\varphi \rightarrow \psi)$. The latter step can be seen by doing RA on $\varphi \wedge \neg\psi$ and $\neg(\varphi \rightarrow \psi)$. We have shown that in all cases, either $P \vdash \varphi \rightarrow \psi$ or $P \vdash \neg(\varphi \rightarrow \psi)$; that is, $\varphi \rightarrow \psi$ has property D. This completes the inductive step.

7. Provide a countermodel to show that the following sequent cannot be proven.

$$\exists x(Fx \rightarrow \exists yGy) \;\vdash\; \exists xFx \rightarrow \exists yGy$$