

# How Logic Works: Solutions to Problems

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## Chapter 3

### Exercise 3.1

1.  $P \vdash Q \rightarrow (P \wedge Q)$

1	(1)	$P$	A
2	(2)	$Q$	A
1,2	(3)	$P \wedge Q$	1,2 $\wedge I$
1	(4)	$Q \rightarrow (P \wedge Q)$	2,3 CP

2.  $(P \rightarrow Q) \wedge (P \rightarrow R) \vdash P \rightarrow (Q \wedge R)$

1	(1)	$(P \rightarrow Q) \wedge (P \rightarrow R)$	A
2	(2)	$P$	A
1	(3)	$P \rightarrow Q$	1 $\wedge E$
1	(4)	$P \rightarrow R$	1 $\wedge E$
1,2	(5)	$Q$	3,2 MP
1,2	(6)	$R$	4,2 MP
1,2	(7)	$Q \wedge R$	5,6 $\wedge I$
1	(8)	$P \rightarrow (Q \wedge R)$	2,7 CP

3.  $P \rightarrow (Q \rightarrow R) \vdash Q \rightarrow (P \rightarrow R)$

1	(1)	$P \rightarrow (Q \rightarrow R)$	A
2	(2)	$Q$	A
3	(3)	$P$	A
1,3	(4)	$Q \rightarrow R$	3,1 MP
1,2,3	(5)	$R$	4,2 MP
1,2	(6)	$P \rightarrow R$	3,5 CP
1	(7)	$Q \rightarrow (P \rightarrow R)$	2,6 CP

4.  $P \rightarrow Q \vdash (Q \rightarrow R) \rightarrow (P \rightarrow R)$

1	(1)	$P \rightarrow Q$	A
2	(2)	$Q \rightarrow R$	A
3	(3)	$P$	A
1,3	(4)	$Q$	1,3 MP
1,2,3	(5)	$R$	2,4 MP
1,2	(6)	$P \rightarrow R$	3,5 CP
1	(7)	$(Q \rightarrow R) \rightarrow (P \rightarrow R)$	2,6 CP

5.  $P \rightarrow (P \rightarrow Q) \vdash P \rightarrow Q$

1	(1)	$P \rightarrow (P \rightarrow Q)$	A
2	(2)	$P$	A
1,2	(3)	$P \rightarrow Q$	1,2 MP
1,2	(4)	$Q$	3,2 MP
1	(5)	$P \rightarrow Q$	2,4 CP

6.  $P \rightarrow (Q \rightarrow R) \vdash (P \wedge Q) \rightarrow R$

1	(1)	$P \rightarrow (Q \rightarrow R)$	A
2	(2)	$P \wedge Q$	A
2	(3)	$P$	2 $\wedge E$
2	(4)	$Q$	2 $\wedge E$
1,2	(5)	$Q \rightarrow R$	1,3 MP
1,2	(6)	$R$	5,4 MP
1	(7)	$(P \wedge Q) \rightarrow R$	2,6 CP

7.  $(P \vee Q) \rightarrow R \vdash P \rightarrow R$

1	(1)	$(P \vee Q) \rightarrow R$	A
2	(2)	$P$	A
2	(3)	$P \vee Q$	2 $\vee I$
1,2	(4)	$R$	1,3 MP
1	(5)	$P \rightarrow R$	2,4 CP

8.  $\neg P \vdash \neg(P \wedge Q)$

1	(1)	$\neg P$	A
2	(2)	$P \wedge Q$	A
2	(3)	$P$	2 $\wedge E$
	(4)	$(P \wedge Q) \rightarrow P$	2,3 CP
1	(5)	$\neg(P \wedge Q)$	4,1 MT

9.  $\neg(P \vee Q) \vdash \neg P \wedge \neg Q$

1	(1)	$\neg(P \vee Q)$	A
2	(2)	$P$	A
2	(3)	$P \vee Q$	2 $\vee I$
	(4)	$P \rightarrow (P \vee Q)$	2,3 CP
1	(5)	$\neg P$	4,1 MT
6	(6)	$Q$	A
6	(7)	$P \vee Q$	6 $\vee I$
	(8)	$Q \rightarrow (P \vee Q)$	6,7 CP
1	(9)	$\neg Q$	8,1 MT
1	(10)	$\neg P \wedge \neg Q$	5,9 $\wedge I$

10.  $P \rightarrow \neg P \vdash \neg P$

1	(1)	$P$	A
2	(2)	$P \rightarrow \neg P$	A
1,2	(3)	$\neg P$	2,1 MP
1	(4)	$(P \rightarrow \neg P) \rightarrow \neg P$	2,3 CP
1	(5)	$\neg\neg P$	1 DN
1	(6)	$\neg(P \rightarrow \neg P)$	4,5 MT
	(7)	$P \rightarrow \neg(P \rightarrow \neg P)$	1,6 CP
2	(8)	$\neg\neg(P \rightarrow \neg P)$	2 DN
2	(9)	$\neg P$	7,8 MT

### Exercise 3.4

1.  $P \rightarrow Q \vdash \neg(P \wedge \neg Q)$

1	(1)	$P \rightarrow Q$	A
2	(2)	$P \wedge \neg Q$	A
2	(3)	$P$	2 $\wedge E$
1,2	(4)	$Q$	1,3 MP
2	(5)	$\neg Q$	2 $\wedge E$
1,2	(6)	$Q \wedge \neg Q$	4,5 $\wedge I$
1	(7)	$\neg(P \wedge \neg Q)$	2,6 RA

2.  $\neg(P \wedge Q) \vdash \neg P \vee \neg Q$

1	(1)	$\neg(P \wedge Q)$	A
2	(2)	$\neg(\neg P \vee \neg Q)$	A
3	(3)	$\neg P$	A
3	(4)	$\neg P \vee \neg Q$	3 $\vee I$
2,3	(5)	$(\neg P \vee \neg Q) \wedge \neg(\neg P \vee \neg Q)$	4,2 $\wedge I$
2	(6)	$\neg\neg P$	3,5 RA
2	(7)	$P$	6 DN
8	(8)	$\neg Q$	A
8	(9)	$\neg P \vee \neg Q$	8 $\vee I$
2,8	(10)	$(\neg P \vee \neg Q) \wedge \neg(\neg P \vee \neg Q)$	9,2 $\wedge I$
2	(11)	$\neg\neg Q$	8,10 RA
2	(12)	$Q$	11 DN
2	(13)	$P \wedge Q$	7,12 $\wedge I$
1,2	(14)	$(P \wedge Q) \wedge \neg(P \wedge Q)$	13,1 $\wedge I$
1	(15)	$\neg\neg(\neg P \vee \neg Q)$	2,14 RA
1	(16)	$\neg P \vee \neg Q$	15 DN

3.  $\neg(P \rightarrow Q) \vdash P \wedge \neg Q$

1	(1)	$\neg(P \rightarrow Q)$	A
2	(2)	$\neg P$	A
3	(3)	$\neg Q$	A
4	(4)	$P$	A
2,4	(5)	$P \wedge \neg P$	2,4 $\wedge I$
2,4	(6)	$\neg\neg Q$	3,5 RA
2,4	(7)	$Q$	6 DN
2	(8)	$P \rightarrow Q$	4,7 CP
1,2	(9)	$(P \rightarrow Q) \wedge \neg(P \rightarrow Q)$	8,1 $\wedge I$
1	(10)	$\neg\neg P$	2,9 RA
1	(11)	$P$	10 DN
12	(12)	$Q$	A
12	(13)	$P \rightarrow Q$	4,12 CP
1,12	(14)	$(P \rightarrow Q) \wedge \neg(P \rightarrow Q)$	13,1 $\wedge I$
1	(15)	$\neg Q$	12,14 RA
1	(16)	$P \wedge \neg Q$	11,15 $\wedge I$

4.  $\vdash (P \rightarrow Q) \vee (Q \rightarrow P)$

1	(1)	$\neg((P \rightarrow Q) \vee (Q \rightarrow P))$	A
2	(2)	$P$	A
3	(3)	$Q$	A
2	(4)	$Q \rightarrow P$	3,2 CP
2	(5)	$(P \rightarrow Q) \vee (Q \rightarrow P)$	4 $\vee I$
1,2	(6)	$((P \rightarrow Q) \vee (Q \rightarrow P)) \wedge \neg((P \rightarrow Q) \vee (Q \rightarrow P))$	5,1 $\wedge I$
1	(7)	$\neg P$	2,6 RA
8	(8)	$\neg Q$	A
1,2	(9)	$P \wedge \neg P$	2,7 $\wedge I$
1,2	(10)	$\neg\neg Q$	8,9 RA
1,2	(11)	$Q$	10 DN
1	(12)	$P \rightarrow Q$	2,11 CP
1	(13)	$(P \rightarrow Q) \vee (Q \rightarrow P)$	12 $\vee I$
1	(14)	$((P \rightarrow Q) \vee (Q \rightarrow P)) \wedge \neg((P \rightarrow Q) \vee (Q \rightarrow P))$	13,1 $\wedge I$
$\emptyset$	(15)	$\neg\neg((P \rightarrow Q) \vee (Q \rightarrow P))$	1,14 RA
$\emptyset$	(16)	$(P \rightarrow Q) \vee (Q \rightarrow P)$	15 DN

5.  $P \rightarrow (Q \vee R) \vdash (P \rightarrow Q) \vee (P \rightarrow R)$

1	(1)	$P \rightarrow (Q \vee R)$	A
2	(2)	$\neg((P \rightarrow Q) \vee (P \rightarrow R))$	A
3	(3)	$\neg P$	A
4	(4)	$P$	A
5	(5)	$\neg Q$	A
3,4	(6)	$P \wedge \neg P$	4,3 $\wedge I$
3,4	(7)	$\neg\neg Q$	5,6 RA
3,4	(8)	$Q$	7 DN
3	(9)	$P \rightarrow Q$	4,8 CP
3	(10)	$(P \rightarrow Q) \vee (P \rightarrow R)$	9 $\vee I$
2,3	(11)	$((P \rightarrow Q) \vee (P \rightarrow R)) \wedge \neg((P \rightarrow Q) \vee (P \rightarrow R))$	10,2 $\wedge I$
2	(12)	$\neg\neg P$	3,11 RA
2	(13)	$P$	12 DN
1,2	(14)	$Q \vee R$	1,13 MP
15	(15)	$Q$	A
15	(16)	$P \rightarrow Q$	4,15 CP
15	(17)	$(P \rightarrow Q) \vee (P \rightarrow R)$	16 $\vee I$
18	(18)	$R$	A
18	(19)	$P \rightarrow R$	4,18 CP
18	(20)	$(P \rightarrow Q) \vee (P \rightarrow R)$	19 $\vee I$
1,2	(21)	$(P \rightarrow Q) \vee (P \rightarrow R)$	14,15,17,18,20 $\vee E$
1,2	(22)	$((P \rightarrow Q) \vee (P \rightarrow R)) \wedge \neg((P \rightarrow Q) \vee (P \rightarrow R))$	21,2 $\wedge I$
1	(23)	$\neg\neg((P \rightarrow Q) \vee (P \rightarrow R))$	2,22 RA
1	(24)	$(P \rightarrow Q) \vee (P \rightarrow R)$	23 DN

6.  $(P \wedge Q) \rightarrow \neg Q \vdash P \rightarrow \neg Q$

1	(1)	$(P \wedge Q) \rightarrow \neg Q$	A
2	(2)	$P$	A
3	(3)	$Q$	A
2,3	(4)	$P \wedge Q$	2,3 $\wedge I$
1,2,3	(5)	$\neg Q$	1,4 MP
1,2,3	(6)	$Q \wedge \neg Q$	3,5 $\wedge I$
1,2	(7)	$\neg Q$	3,6 RA
1	(8)	$P \rightarrow \neg Q$	2,7 CP

## Chapter 6

### Exercise 6.1

1. No logicians are celebrities. ( $Lx, Cx$ )

$$\forall x (Lx \rightarrow \neg Cx)$$

Equivalently:  $\neg \exists x (Lx \wedge Cx)$

2. Some celebrities are not logicians. ( $Lx, Cx$ )

$$\exists x (Cx \wedge \neg Lx)$$

3. Only students who do the homework will learn logic. ( $Sx, Hx, Lx$ )

Either

$$\forall x (Lx \rightarrow (Sx \wedge Hx))$$

or (inequivalently)

$$\forall x ((Sx \wedge Lx) \rightarrow Hx)$$

depending on whether one intends to restrict the claim to students.

4. All rich logicians are computer scientists. ( $Rx, Lx, Cx$ )

$$\forall x ((Rx \wedge Lx) \rightarrow Cx)$$

5. All students and professors get a discount. ( $Sx, Px, Dx$ )

$$\forall x ((Sx \vee Px) \rightarrow Dx)$$

6. No logician is rich, unless she is a computer scientist. ( $Lx, Rx, Cx$ )

$$\forall x ((Lx \wedge Rx) \rightarrow Cx)$$

Equivalent form:  $\forall x((Lx \wedge \neg Cx) \rightarrow \neg Rx)$

7. Not all logicians are computer scientists. ( $Lx, Cx$ )

$$\neg \forall x (Lx \rightarrow Cx)$$

Often put as:  $\exists x (Lx \wedge \neg Cx)$ .

8. Some logicians are rich computer scientists. ( $Lx, Rx, Cx$ )

$$\exists x (Lx \wedge (Rx \wedge Cx))$$

9. If there are rich logicians, then some logicians are computer scientists. ( $Rx, Lx, Cx$ )

$$\exists x (Rx \wedge Lx) \rightarrow \exists y (Ly \wedge Cy)$$

10. No pets except service animals are permitted in dorms. ( $Px, Sx, Dx$ )

Can be read in a minimal way as:

$$\forall x ((Px \wedge Dx) \rightarrow Sx),$$

which says only that no non-service pets are allowed in dorms. However, ordinary policy language is typically understood more strongly: among pets, *being permitted in the dorms* and *being a service animal* coincide. That reading is captured by:

$$\forall x (Px \rightarrow (Dx \leftrightarrow Sx)).$$

This biconditional formalization is therefore closer to the intended rule.

11. If anyone is rich, then Mary is. ( $Rx, m$ )

$$(\exists x Rx) \rightarrow Rm$$

## Exercise 6.2

1. Mary loves everyone who loves her. ( $m, Lxy$ )

$$\forall x (Lxm \rightarrow Lmx)$$

2. Mary loves all and only those people who don't love themselves. ( $Lxy, m$ )

$$\forall x (Lmx \leftrightarrow \neg Lxx)$$

3. Everyone loves their mother. ( $Lxy, Mxy$ )

$$\forall x \forall y (Myx \rightarrow Lxy)$$

4. Some people love only those people who love their mother. ( $Lxy, Mxy$ )

$$\exists x \forall y (Lxy \rightarrow \forall z (Mzy \rightarrow Lyz))$$

5. Snape killed someone. ( $Kxy, s$ )

$$\exists x Ksx$$

6. Snape is a killer. ( $Kxy, s$ )

$$\exists x Ksx$$

7. Someone was killed by Snape. ( $Kxy, s$ )

$$\exists x Ksx$$

8. Some wizards only marry other wizards. ( $Wx, Mxy$ )

$$\exists x (Wx \wedge \forall y (Mxy \rightarrow Wy))$$

9. There is no greatest number. ( $Nx, x < y$ )

$$\forall x (Nx \rightarrow \exists y (Ny \wedge x < y))$$

10.  $c$  is the least upper bound of  $a$  and  $b$ . ( $a, b, c, x \leq y$ )

$$(a \leq c \wedge b \leq c) \wedge \forall x ((a \leq x \wedge b \leq x) \rightarrow c \leq x)$$

11.  $c$  is the greatest common divisor of  $a$  and  $b$ . ( $a, b, c, Dxy, x \leq y$ )

$$(Dca \wedge Dcb) \wedge \forall x ((Dxa \wedge Dxb) \rightarrow x \leq c)$$

## Exercise 6.8

1.  $\neg \exists x (Fx \wedge Gx) \vdash \forall x (Fx \rightarrow \neg Gx)$

1	(1)	$\neg \exists x (Fx \wedge Gx)$	A
2	(2)	$Fa$	A
3	(3)	$Ga$	A
2,3	(4)	$Fa \wedge Ga$	2,3 $\wedge I$
2,3	(5)	$\exists x (Fx \wedge Gx)$	4 EI
1,2,3	(6)	$\exists x (Fx \wedge Gx) \wedge \neg \exists x (Fx \wedge Gx)$	5,1 $\wedge I$
1,2	(7)	$\neg Ga$	3,6 RA
1	(8)	$Fa \rightarrow \neg Ga$	2,7 CP
1	(9)	$\forall x (Fx \rightarrow \neg Gx)$	8 UI

2.  $\forall x Fx \vdash \exists x Fx$

1	(1)	$\forall x Fx$	A
1	(2)	$Fa$	1 UE
1	(3)	$\exists x Fx$	2 EI

3.  $\forall x(Fx \rightarrow Gx), Fa \vdash \exists x Gx$

1	(1)	$\forall x(Fx \rightarrow Gx)$	A
2	(2)	$Fa$	A
1	(3)	$Fa \rightarrow Ga$	1 UE
1,2	(4)	$Ga$	3,2 MP
1,2	(5)	$\exists x Gx$	4 EI

4.  $\neg Fa \vdash \exists x(Fx \rightarrow P)$

1	(1)	$\neg Fa$	A
1	(2)	$Fa \rightarrow P$	1 negative paradox
1	(3)	$\exists x(Fx \rightarrow P)$	2 EI

5.  $\neg \forall x Fx \vdash \exists x(Fx \rightarrow P)$

1	(1)	$\neg \forall x Fx$	A
2	(2)	$\neg \exists x(Fx \rightarrow P)$	A
3	(3)	$Fa \rightarrow P$	A
3	(4)	$\exists x(Fx \rightarrow P)$	3 EI
2,3	(5)	$\exists x(Fx \rightarrow P) \wedge \neg \exists x(Fx \rightarrow P)$	4,2 $\wedge I$
2	(6)	$\neg(Fa \rightarrow P)$	3,5 RA
2	(7)	$Fa$	6 material conditional
2	(8)	$\forall x Fx$	7 UI
1,2	(9)	$\forall x Fx \wedge \neg \forall x Fx$	8,1 $\wedge I$
1	(10)	$\neg \neg \exists x(Fx \rightarrow P)$	2,9 RA
1	(11)	$\exists x(Fx \rightarrow P)$	10 DN

6.  $\neg \exists x Fx \vdash \forall x(Fx \rightarrow Gx)$

1	(1)	$\neg \exists x Fx$	A
2	(2)	$Fa$	A
3	(3)	$\neg Ga$	A
2	(4)	$\exists x Fx$	2 EI
1,2	(5)	$\exists x Fx \wedge \neg \exists x Fx$	4,1 $\wedge I$
1,2	(6)	$\neg \neg Ga$	3,5 RA
1,2	(7)	$Ga$	6 DN
1	(8)	$Fa \rightarrow Ga$	2,7 CP
1	(9)	$\forall x(Fx \rightarrow Gx)$	8 UI

7.  $\forall x \forall y Rxy \vdash \exists x Rxx$

1	(1)	$\forall x \forall y Rxy$	A
1	(2)	$\forall y Ray$	1 UE
1	(3)	$Raa$	2 UE
1	(4)	$\exists x Rxx$	3 EI

8.  $P \rightarrow Fa \vdash P \rightarrow \exists x Fx$

1	(1)	$P \rightarrow Fa$	A
2	(2)	$P$	A
1,2	(3)	$Fa$	1,2 MP
1,2	(4)	$\exists x Fx$	3 EI
1	(5)	$P \rightarrow \exists x Fx$	2,4 CP

9.  $\exists x Fx \rightarrow P \vdash \forall x(Fx \rightarrow P)$

1	(1)	$\exists x Fx \rightarrow P$	A
2	(2)	$Fa$	A
2	(3)	$\exists x Fx$	2 EI
1,2	(4)	$P$	1,3 MP
1	(5)	$Fa \rightarrow P$	2,4 CP
1	(6)	$\forall x(Fx \rightarrow P)$	5 UI

There is a typo here in the book: the direction  $\forall x(Fx \rightarrow P) \vdash \exists x Fx \rightarrow P$  cannot be proven without EE, which is only introduced in the next section.

10.  $\neg \exists x Fx \vdash \forall x(Fx \rightarrow P)$

1	(1)	$\neg \exists x Fx$	A
2	(2)	$Fa$	A
2	(3)	$\exists x Fx$	2 EI
1,2	(4)	$\exists x Fx \wedge \neg \exists x Fx$	3,1 $\wedge I$
1	(5)	$\neg Fa$	2,4 RA
1	(6)	$Fa \rightarrow P$	5 neg paradox
1	(7)	$\forall x(Fx \rightarrow P)$	6 UI

11.  $\neg \exists x(Fx \rightarrow P) \vdash \forall x Fx \wedge \neg P$

1	(1)	$\neg \exists x(Fx \rightarrow P)$	A
2	(2)	$Fa \rightarrow P$	A
2	(3)	$\exists x(Fx \rightarrow P)$	2 EI
1,2	(4)	$\exists x(Fx \rightarrow P) \wedge \neg \exists x(Fx \rightarrow P)$	3,1 $\wedge I$
1	(5)	$\neg(Fa \rightarrow P)$	2,4 RA
1	(6)	$Fa \wedge \neg P$	5 material conditional
1	(7)	$\neg P$	6 $\wedge E$
1	(8)	$Fa$	6 $\wedge E$
1	(9)	$\forall x Fx$	8 UI
1	(10)	$\forall x Fx \wedge \neg P$	9,7 $\wedge I$

12.  $\forall x Fx \rightarrow P \vdash \exists x(Fx \rightarrow P)$

1	(1)	$\forall x Fx \rightarrow P$	A
2	(2)	$\neg \exists x(Fx \rightarrow P)$	A
3	(3)	$\neg Fa$	A
3	(4)	$Fa \rightarrow P$	3 neg paradox
3	(5)	$\exists x(Fx \rightarrow P)$	4 EI
2,3	(6)	$\exists x(Fx \rightarrow P) \wedge \neg \exists x(Fx \rightarrow P)$	5,2 $\wedge I$
2	(7)	$\neg \neg Fa$	3,6 RA
2	(8)	$Fa$	7 DN
2	(9)	$\forall x Fx$	8 UI
1,2	(10)	$P$	1,9 MP
1,2	(11)	$Fb \rightarrow P$	10 pos paradox
1,2	(12)	$\exists x(Fx \rightarrow P)$	11 EI
1,2	(13)	$\exists x(Fx \rightarrow P) \wedge \neg \exists x(Fx \rightarrow P)$	12,2 $\wedge I$
1	(14)	$\neg \neg \exists x(Fx \rightarrow P)$	2,13 RA
1	(15)	$\exists x(Fx \rightarrow P)$	14 DN

### Exercise 6.11

1.  $\exists x Fx \vee \exists x Gx \vdash \exists x(Fx \vee Gx)$

1	(1)	$\exists x Fx \vee \exists x Gx$	A
2	(2)	$\exists x Fx$	A
3	(3)	$Fa$	A
3	(4)	$Fa \vee Ga$	3 $\vee I$
3	(5)	$\exists x(Fx \vee Gx)$	4 EI
2	(6)	$\exists x(Fx \vee Gx)$	2,3,5 EE
7	(7)	$\exists x Gx$	A
8	(8)	$Ga$	A
8	(9)	$Fa \vee Ga$	8 $\vee I$
8	(10)	$\exists x(Fx \vee Gx)$	9 EI
7	(11)	$\exists x(Fx \vee Gx)$	7,8,10 EE
1	(12)	$\exists x(Fx \vee Gx)$	1,2,6,7,11 $\vee E$

2.  $\forall x(Fx \rightarrow Gx), \neg \exists x Gx \vdash \neg \exists x Fx$

1	(1)	$\forall x(Fx \rightarrow Gx)$	A
2	(2)	$\neg \exists x Gx$	A
3	(3)	$\exists x Fx$	A
4	(4)	$Fa$	A
1	(5)	$Fa \rightarrow Ga$	1 UE
1,4	(6)	$Ga$	5,4 MP
1,4	(7)	$\exists x Gx$	6 EI
1,3	(8)	$\exists x Gx$	3,4,7 EE
1,2,3	(9)	$\exists x Gx \wedge \neg \exists x Gx$	8,2 $\wedge I$
1,2	(10)	$\neg \exists x Fx$	3,9 RA

3.  $\forall x(Fx \rightarrow Gx) \vdash \exists x \neg Gx \rightarrow \exists x \neg Fx$

1	(1)	$\forall x(Fx \rightarrow Gx)$	A
2	(2)	$\exists x \neg Gx$	A
3	(3)	$\neg Ga$	A
1	(4)	$Fa \rightarrow Ga$	1 UE
1,3	(5)	$\neg Fa$	4,3 MT
1,3	(6)	$\exists x \neg Fx$	5 EI
1,2	(7)	$\exists x \neg Fx$	2,3,6 EE
1	(8)	$\exists x \neg Gx \rightarrow \exists x \neg Fx$	2,7 CP

4.  $\forall x(Fx \rightarrow P) \vdash \exists x Fx \rightarrow P$

1	(1)	$\forall x(Fx \rightarrow P)$	A
2	(2)	$\exists xFx$	A
3	(3)	$Fa$	A
1	(4)	$Fa \rightarrow P$	1 UE
1,3	(5)	$P$	4,3 MP
1,2	(6)	$P$	2,3,5 EE
1	(7)	$\exists xFx \rightarrow P$	2,6 CP

5.  $P \wedge \exists xFx \vdash \exists x(P \wedge Fx)$

1	(1)	$P \wedge \exists xFx$	A
1	(2)	$P$	1 $\wedge E$
1	(3)	$\exists xFx$	1 $\wedge E$
4	(4)	$Fa$	A
1,4	(5)	$P \wedge Fa$	2,4 $\wedge I$
1,4	(6)	$\exists x(P \wedge Fx)$	5 EI
1	(7)	$\exists x(P \wedge Fx)$	3,4,6 EE

6.  $\exists x(Fx \rightarrow P) \vdash \forall xFx \rightarrow P$

1	(1)	$\exists x(Fx \rightarrow P)$	A
2	(2)	$\forall xFx$	A
3	(3)	$Fa \rightarrow P$	A
2	(4)	$Fa$	2 UE
2,3	(5)	$P$	3,4 MP
3	(6)	$\forall xFx \rightarrow P$	2,5 CP
1	(7)	$\forall xFx \rightarrow P$	1,3,6 EE

### Exercise 6.13

1.  $P \rightarrow \exists xFx \vdash \exists x(P \rightarrow Fx)$

1	(1)	$P \rightarrow \exists xFx$	A
$\emptyset$	(2)	$\exists xFx \vee \neg \exists xFx$	prop taut
3	(3)	$\exists xFx$	A
4	(4)	$Fa$	A
4	(5)	$P \rightarrow Fa$	4 prop taut
4	(6)	$\exists x(P \rightarrow Fx)$	5 EI
3	(7)	$\exists x(P \rightarrow Fx)$	3,4,6 EE
8	(8)	$\neg \exists xFx$	A
1,8	(9)	$\neg P$	1,8 MT
1,8	(10)	$P \rightarrow Fb$	9 prop taut
1,8	(11)	$\exists x(P \rightarrow Fx)$	10 EI
1	(12)	$\exists x(P \rightarrow Fx)$	2,3,7,8,11 $\vee E$

2.  $\exists x(Fx \rightarrow P) \vdash \forall x Fx \rightarrow P$

1	(1)	$\exists x(Fx \rightarrow P)$	A
2	(2)	$\forall x Fx$	A
3	(3)	$Fa \rightarrow P$	A
2	(4)	$Fa$	2 UE
2,3	(5)	$P$	3,4 MP
1,2	(6)	$P$	1,3,5 EE
1	(7)	$\forall x Fx \rightarrow P$	2,6 CP

### Exercise 6.14

1.  $\vdash \forall x(Fx \rightarrow Fx)$

1	(1)	$Fa$	A
$\emptyset$	(2)	$Fa \rightarrow Fa$	1,1 CP
$\emptyset$	(3)	$\forall x(Fx \rightarrow Fx)$	2 UI

2.  $\vdash \forall x Fx \vee \exists x \neg Fx$

$\emptyset$	(1)	$\neg \exists x \neg Fx \vee \exists x \neg Fx$	prop taut
2	(2)	$\neg \exists x \neg Fx$	A
3	(3)	$\neg Fa$	A
3	(4)	$\exists x \neg Fx$	3 EI
2,3	(5)	$\exists x \neg Fx \wedge \neg \exists x \neg Fx$	4,2 $\wedge I$
2	(6)	$\neg \neg Fa$	3,5 RA
2	(7)	$Fa$	6 DN
2	(8)	$\forall x Fx$	7 UI
2	(9)	$\forall x Fx \vee \exists x \neg Fx$	8 $\vee I$
10	(10)	$\exists x \neg Fx$	A
10	(11)	$\forall x Fx \vee \exists x \neg Fx$	10 $\vee I$
$\emptyset$	(12)	$\forall x Fx \vee \exists x \neg Fx$	1,2,9,10,11 $\vee E$

3.  $\vdash \forall x \neg(Fx \wedge \neg Fx)$

1	(1)	$Fa \wedge \neg Fa$	A
$\emptyset$	(2)	$\neg(Fa \wedge \neg Fa)$	1,1 RA
$\emptyset$	(3)	$\forall x \neg(Fx \wedge \neg Fx)$	2 UI

4.  $\vdash \neg \exists x(Fx \wedge \neg Fx)$

1	(1)	$\exists x(Fx \wedge \neg Fx)$	A
2	(2)	$Fa \wedge \neg Fa$	A
2	(3)	$\neg \exists x(Fx \wedge \neg Fx)$	1,2 RA
1	(4)	$\neg \exists x(Fx \wedge \neg Fx)$	1,2,3 EE
1	(5)	$\exists x(Fx \wedge \neg Fx) \wedge \neg \exists x(Fx \wedge \neg Fx)$	1,3 $\wedge$ I
$\emptyset$	(6)	$\neg \exists x(Fx \wedge \neg Fx)$	1,5 RA

5.  $\vdash \forall x \exists y(Rxy \rightarrow Rxx)$

1	(1)	$Raa$	A
$\emptyset$	(2)	$Raa \rightarrow Raa$	1,1 CP
$\emptyset$	(3)	$\exists y(Ray \rightarrow Raa)$	2 EI
$\emptyset$	(4)	$\forall x \exists y(Rxy \rightarrow Rxx)$	3 UI

6.  $\vdash \forall x \exists y(Rxy \rightarrow Ryx)$

1	(1)	$Raa$	A
$\emptyset$	(2)	$Raa \rightarrow Raa$	1,1 CP
$\emptyset$	(3)	$\exists y(Ray \rightarrow Ryx)$	2 EI
$\emptyset$	(4)	$\forall x \exists y(Rxy \rightarrow Ryx)$	3 UI

7.  $\vdash \exists x(Fx \rightarrow \forall y Fy)$

1	(1)	$\neg \exists x(Fx \rightarrow \forall y Fy)$	A
2	(2)	$\neg Fa$	A
2	(3)	$Fa \rightarrow \forall y Fy$	2 prop taut
2	(4)	$\exists x(Fx \rightarrow \forall y Fy)$	3 EI
1,2	(5)	$\exists x(Fx \rightarrow \forall y Fy) \wedge \neg \exists x(Fx \rightarrow \forall y Fy)$	4,1 $\wedge$ I
1	(6)	$\neg \neg Fa$	2,5 RA
1	(7)	$Fa$	6 DN
1	(8)	$\forall y Fy$	7 UI
1	(9)	$Fa \rightarrow \forall y Fy$	8 prop taut
1	(10)	$\exists x(Fx \rightarrow \forall y Fy)$	9 EI
1	(11)	$\exists x(Fx \rightarrow \forall y Fy) \wedge \neg \exists x(Fx \rightarrow \forall y Fy)$	10,1 $\wedge$ I
$\emptyset$	(12)	$\neg \neg \exists x(Fx \rightarrow \forall y Fy)$	1,11 RA
$\emptyset$	(13)	$\exists x(Fx \rightarrow \forall y Fy)$	12 DN

8.  $\vdash \exists x \forall y(Fx \rightarrow Fy)$

1	(1)	$\neg \exists x \forall y (Fx \rightarrow Fy)$	A
2	(2)	$\neg Fa$	A
2	(3)	$Fa \rightarrow Fb$	2 prop taut
2	(4)	$\forall y (Fa \rightarrow Fy)$	3 UI
2	(5)	$\exists x \forall y (Fx \rightarrow Fy)$	4 EI
1,2	(6)	$\exists x \forall y (Fx \rightarrow Fy) \wedge \neg \exists x \forall y (Fx \rightarrow Fy)$	5,1 $\wedge I$
1	(7)	$\neg \neg Fa$	2,6 RA
1	(8)	$Fa$	7 DN
1	(9)	$Fc \rightarrow Fa$	8 prop taut
1	(10)	$\forall y (Fc \rightarrow Fy)$	9 UI
1	(11)	$\exists x \forall y (Fx \rightarrow Fy)$	10 EI
1	(12)	$\exists x \forall y (Fx \rightarrow Fy) \wedge \neg \exists x \forall y (Fx \rightarrow Fy)$	11,1 $\wedge I$
$\emptyset$	(13)	$\neg \neg \exists x \forall y (Fx \rightarrow Fy)$	1,12 RA
$\emptyset$	(14)	$\exists x \forall y (Fx \rightarrow Fy)$	13 DN

9.  $\forall x \exists y (Fx \rightarrow Gy) \vdash \exists y \forall x (Fx \rightarrow Gy)$

1	(1)	$\forall x \exists y (Fx \rightarrow Gy)$	A
$\emptyset$	(2)	$\exists y Gy \vee \neg \exists y Gy$	prop taut
3	(3)	$\exists y Gy$	A
4	(4)	$Ga$	A
4	(5)	$Fb \rightarrow Ga$	4 prop taut
4	(6)	$\forall x (Fx \rightarrow Ga)$	5 UI
4	(7)	$\exists y \forall x (Fx \rightarrow Gy)$	6 EI
3	(8)	$\exists y \forall x (Fx \rightarrow Gy)$	3,4,7 EE
9	(9)	$\neg \exists y Gy$	A
10	(10)	$Fc$	A
1	(11)	$\exists y (Fc \rightarrow Gy)$	1 UE
12	(12)	$Fc \rightarrow Gd$	A
10,12	(13)	$Gd$	12,10 MP
10,12	(14)	$\exists y Gy$	13 EI
9,10,12	(15)	$\exists y Gy \wedge \neg \exists y Gy$	14,9 $\wedge I$
9,12	(16)	$\neg Fc$	10,15 RA
9,12	(17)	$Fc \rightarrow Ge$	16 prop taut
1,9	(18)	$Fc \rightarrow Ge$	11,12,17 EE
1,9	(19)	$\forall x (Fx \rightarrow Ge)$	18 UI
1,9	(20)	$\exists y \forall x (Fx \rightarrow Gy)$	19 EI
1	(21)	$\exists y \forall x (Fx \rightarrow Gy)$	2,3,8,9,20 $\vee E$

10.  $\vdash \forall x \exists y (Rxy \rightarrow \forall z Rxz)$

$\emptyset$	(1)	$\exists y \neg Ray \vee \neg \exists y \neg Ray$	prop taut
2	(2)	$\exists y \neg Ray$	A
3	(3)	$\neg Rab$	A
3	(4)	$Rab \rightarrow \forall z Raz$	3 prop taut
3	(5)	$\exists y(Ray \rightarrow \forall z Raz)$	4 EI
2	(6)	$\exists y(Ray \rightarrow \forall z Raz)$	2,3,5 EE
7	(7)	$\neg \exists y \neg Ray$	A
8	(8)	$\neg Rac$	A
8	(9)	$\exists y \neg Ray$	8 EI
7,8	(10)	$\exists y \neg Ray \wedge \neg \exists y \neg Ray$	9,7 $\wedge I$
7	(11)	$\neg \neg Rac$	8,10 RA
7	(12)	$Rac$	11 DN
7	(13)	$\forall z Raz$	12 UI
7	(14)	$Rab \rightarrow \forall z Raz$	13 prop taut
7	(15)	$\exists y(Ray \rightarrow \forall z Raz)$	14 EI
$\emptyset$	(16)	$\exists y(Ray \rightarrow \forall z Raz)$	1,2,6,7,15 $\vee E$
$\emptyset$	(17)	$\forall x \exists y(Rxy \rightarrow \forall z Rxz)$	16 UI

### Exercise 6.17

$$\forall x(\exists z Rxz \rightarrow \forall y Rxy), \exists x \exists y \vdash \exists x \forall y Rxy$$

1	(1)	$\forall x(\exists z Rxz \rightarrow \forall y Rxy)$	A
2	(2)	$\exists x \exists y Rxy$	A
3	(3)	$\exists y Ray$	A
4	(4)	$Rab$	A
4	(5)	$\exists z Raz$	4 EI
1	(6)	$\exists z Raz \rightarrow \forall y Ray$	1 UE
1,4	(7)	$\forall y Ray$	6,5 MP
1,4	(8)	$\exists x \forall y Rxy$	7 EI
1,3	(9)	$\exists x \forall y Rxy$	3,4,8 EE
1,2	(10)	$\exists x \forall y Rxy$	2,3,9 EE

Question: Does it follow from these premises that  $\forall x \forall y Rxy$ ?

Answer: No.  $\bigcup_{\succ} a \longrightarrow b$

# Chapter 7

## Exercise 7.1

Here the proof is lengthened because of the strictness of the  $=$  rules. From  $a = c$  and  $b = c$ , we cannot immediately apply  $=E$  to get  $a = b$ .

1	(1)	$\exists x \forall y (Py \rightarrow y = x)$	A
2	(2)	$Pa \wedge Pb$	A
3	(3)	$\forall y (Py \rightarrow y = c)$	A
3	(4)	$Pa \rightarrow a = c$	3 UE
3	(5)	$Pb \rightarrow b = c$	3 UE
2	(6)	$Pa$	2 $\wedge E$
2	(7)	$Pb$	2 $\wedge E$
2,3	(8)	$a = c$	4,6 MP
2,3	(9)	$b = c$	5,7 MP
$\emptyset$	(10)	$b = b$	$=I$
2,3	(11)	$c = b$	10,9 $=E$
2,3	(12)	$a = b$	8,11 $=E$
1,2	(13)	$a = b$	1,3,12 EE
1	(14)	$(Pa \wedge Pb) \rightarrow a = b$	2,13 CP
1	(15)	$\forall y ((Pa \wedge Py) \rightarrow a = y)$	14 UI
1	(16)	$\forall x \forall y ((Px \wedge Py) \rightarrow x = y)$	15 UI

## Exercise 7.2

1	(1)	$Fa \wedge \forall x (Fx \rightarrow x = a)$	A
2	(2)	$Fb$	A
1	(3)	$\forall x (Fx \rightarrow x = a)$	1 $\wedge E$
1	(4)	$Fb \rightarrow b = a$	3 UE
1,2	(5)	$b = a$	4,2 MP
6	(6)	$b = a$	A
1	(7)	$Fa$	1 $\wedge E$
$\emptyset$	(8)	$b = b$	$=I$
6	(9)	$a = b$	8,6 $=E$
1,6	(10)	$Fb$	7,9 $=E$
1	(11)	$Fb \leftrightarrow b = a$	2,5,6,10 CP $\times 2$
1	(12)	$\forall x (Fx \leftrightarrow x = a)$	11 UI

1	(1)	$\forall x(Fx \leftrightarrow x = a)$	A
1	(2)	$Fa \leftrightarrow a = a$	1 UE
$\emptyset$	(3)	$a = a$	=I
1	(4)	$Fa$	2,3 MP
1	(5)	$Fb \leftrightarrow b = a$	1 UE
1	(6)	$Fb \rightarrow b = a$	5 $\wedge$ E
1	(7)	$\forall x(Fx \rightarrow x = a)$	6 UI
1	(8)	$Fa \wedge \forall x(Fx \rightarrow x = a)$	4,7 $\wedge$ I

## Chapter 8

### Exercise 8.1

1. Countermodel  $M_1$ :

$$D_1 = \{a, b\}, \quad F^{M_1} = \{b\}, \quad c^{M_1} = a.$$

Then  $M_1 \models \exists x Fx$  (witness  $b$ ), but  $M_1 \not\models Fc$ .

2. Countermodel  $M_2$ :

$$D_2 = \{a, b\}, \quad F^{M_2} = \{a\}, \quad c^{M_2} = a.$$

Then  $M_2 \models Fc$ , but  $M_2 \not\models \forall x Fx$  (since  $b \notin F^{M_2}$ ).

3. Countermodel  $M_3$ :

$$D_3 = \{a, b\}, \quad F^{M_3} = \{a\}, \quad G^{M_3} = \{b\}.$$

Then  $M_3 \models \exists x Fx \wedge \exists x Gx$ , but  $M_3 \not\models \exists x(Fx \wedge Gx)$ .

4. Countermodel  $M_4$ :

$$D_4 = \{a, b\}, \quad F^{M_4} = \{a\}, \quad G^{M_4} = \{b\}.$$

We have  $\forall x Fx$  false and  $\forall x Gx$  false, so  $M_4 \models (\forall x Fx \rightarrow \forall x Gx)$  (false  $\rightarrow$  false is true), but for  $x = a$  we get  $Fa \wedge \neg Ga$ , so  $M_4 \not\models \forall x(Fx \rightarrow Gx)$ .

5. Countermodel  $M_5$ :

$$D_5 = \{a\}, \quad F^{M_5} = \emptyset, \quad H^{M_5} = \{a\}.$$

Then for the only element  $a$ ,  $Fa$  is false, so  $Fa \rightarrow Ha$  is true; hence  $M_5 \models \forall x(Fx \rightarrow Hx)$ . But  $M_5 \not\models \exists x Fx$  and  $M_5 \models \exists x Hx$ , so  $M_5 \not\models \exists x Fx \vee \neg \exists x Hx$ .

6. Countermodel  $M_6$ :

$$D_6 = \{a\}, \quad F^{M_6} = \emptyset, \quad G^{M_6} = \emptyset.$$

Again  $Fa$  is false, so  $Fa \rightarrow Ga$  is true; hence  $M_6 \models \forall x(Fx \rightarrow Gx)$ . But  $F^{M_6} \cap G^{M_6} = \emptyset$ , so  $M_6 \not\models \exists x(Fx \wedge Gx)$ .

7. Countermodel  $M_7$ :

$$D_7 = \{a, b\}, \quad F^{M_7} = \{a\}, \quad G^{M_7} = \{a, b\}, \quad H^{M_7} = \{b\}.$$

Then  $a$  witnesses  $\exists x(Fx \wedge Gx)$ , and  $b$  witnesses  $\exists x(Gx \wedge Hx)$ . But there is no element in  $F^{M_7} \cap H^{M_7}$ , so  $M_7 \not\models \exists x(Fx \wedge Hx)$ .

8. Countermodel  $M_8$ :

$$D_8 = \{a, b\}, \quad F^{M_8} = \{a\}.$$

Then  $M_8 \not\models \forall x Fx$  (since  $b \notin F^{M_8}$ ), and  $M_8 \not\models \forall x \neg Fx$  (since  $a \in F^{M_8}$ ). Hence  $M_8 \not\models \forall x Fx \vee \forall x \neg Fx$ .

9. Countermodel  $M_9$ :

$$D_9 = \{a, b\}, \quad F^{M_9} = \{a, b\}, \quad G^{M_9} = \{a\}, \quad H^{M_9} = \emptyset.$$

Then:

- $M_9 \models \exists x(Fx \rightarrow Gx)$  (take  $x = a$ ,  $Fa \rightarrow Ga$  is true),
- $M_9 \models \exists x(Gx \rightarrow Hx)$  (take  $x = b$ ,  $Gb \rightarrow Hb$  is true),
- $M_9 \not\models \exists x(Fx \rightarrow Hx)$  since for both  $a, b$ ,  $Fx$  is true and  $Hx$  is false, so  $Fx \rightarrow Hx$  is false everywhere.

10. Countermodel  $M_{10}$ :

$$D_{10} = \{a, b\}, \quad F^{M_{10}} = \{a\}, \quad G^{M_{10}} = \emptyset.$$

Then  $M_{10} \models \exists x(Fx \rightarrow Gx)$  (take  $x = b$ , where  $Fb$  is false), while  $\exists x Fx$  is true (witness  $a$ ) and  $\exists x Gx$  is false. Hence  $M_{10} \not\models \exists x Fx \rightarrow \exists x Gx$ .

### Exercise 8.3

1.  $\forall x Fx \rightarrow P \not\models \forall x(Fx \rightarrow P)$ .

Countermodel  $M_1$ :

$$D = \{a, b\}, \quad F^M = \{a\}, \quad P^M = 0.$$

Then  $\forall x Fx$  is false (since  $b \notin F^M$ ), so

$$(\forall x Fx \rightarrow P)^M = (0 \rightarrow 0) = 1.$$

However,

$$(Fb \rightarrow P)^M = (0 \rightarrow 0) = 1, \quad (Fa \rightarrow P)^M = (1 \rightarrow 0) = 0,$$

so  $\forall x(Fx \rightarrow P)$  is false. Thus the premise is true and the conclusion false in  $M_1$ .

2.  $\exists x(Fx \rightarrow P) \not\vdash \exists x Fx \rightarrow P$ .

Countermodel  $M_2$ :

$$D = \{a\}, \quad F^M = \emptyset, \quad P^M = 0.$$

Then

$$(Fa \rightarrow P)^M = (0 \rightarrow 0) = 1,$$

so  $\exists x(Fx \rightarrow P)$  is true. But  $\exists x Fx$  is false and  $P$  is false, so

$$(\exists x Fx \rightarrow P)^M = (0 \rightarrow 0) = 0.$$

Hence the premise is true while the conclusion is false in  $M_2$ .

### Exercise 8.7

1.  $\forall x \forall y(Rxy \rightarrow Ryx)$

**True model (symmetric):**



**False model (one-way arrow):**



2.  $\forall x \forall y \exists z(Rxz \wedge Ryz)$

**True model (common successor  $a$  for everyone):**



(For any  $x, y$ , choose  $z = a$ .)

**False model (no common successor for  $a, b$ ):**



(For  $x = a, y = b$  there is no  $z$  with both  $a \rightarrow z$  and  $b \rightarrow z$ .)

3.  $\exists x \forall y (Ryx \rightarrow Ryy)$

**True model (choose  $x = a$  with no incoming arrows):**



(No  $y$  satisfies  $y \rightarrow a$ , so  $Ryx \rightarrow Ryy$  holds vacuously for all  $y$ .)

**False model (every  $x$  has an incoming arrow from a non-reflexive  $y$ ):**



(No loops, so  $Ryy$  is always false; but each node has an incoming arrow.)

4.  $\forall x (\exists y Ryx \rightarrow \forall z Rzx)$

**True model (empty relation):**



(Each antecedent  $\exists y Ryx$  is false, so the implication is true for all  $x$ .)

**False model (some  $x$  has an incoming arrow but not everyone points to  $x$ ):**



(Take  $x = b$ :  $\exists y Ryb$  holds (witness  $a$ ), but  $\forall z Rzb$  fails since  $b \not\rightarrow b$ .)

5.  $\exists x \exists y (Rxy \leftrightarrow \neg Ryy)$

**True model (take  $x = a, y = b$ ):**



(Here  $a \rightarrow b$  is true and  $b \rightarrow b$  is false, so  $\neg Rbb$  is true and the biconditional holds.)

**False model (universal relation on  $\{a, b\}$ ):**



(For every  $y$ ,  $Ryy$  is true, hence  $\neg Ryy$  is false; but  $Rxy$  is always true. So  $Rxy \leftrightarrow \neg Ryy$  is false for all  $x, y$ .)

# Chapter 9

## Exercise 9.6

Suppose that  $\varphi$  is true in an even number  $n$  of rows of its truth table. Then  $\neg\varphi$  is true in  $4 - n$  rows of its truth table, and  $4 - n$  is also even.

Suppose that both  $\varphi$  and  $\psi$  are even. Let's say that row  $r$  is an *agreement row* if  $\varphi$  and  $\psi$  have the same truth value on  $r$ . We will show that there cannot be 1 or 3 agreement rows. Suppose that there is a single row where both sentences have value  $a$ . Since  $\varphi$  and  $\psi$  are even,  $a$  must occur on another row in each of their truth tables. If these rows are not the same, then there are two of them, which leaves a single remaining row. In that row, both  $\varphi$  and  $\psi$  must have value  $1 - a$ , and so they agree there.

Suppose now that there are three rows where both sentences have the same value, and let  $r$  be the remaining row. Since three is odd, one of the two truth values  $a$  must occur most frequently on these rows. If  $a$  occurs twice and  $1 - a$  occurs once, then  $1 - a$  must be the value of both  $\varphi$  and  $\psi$  on row  $r$ . If  $a$  occurs three times, then  $a$  must be the value of both  $\varphi$  and  $\psi$  on row  $r$ . In either case,  $\varphi$  and  $\psi$  agree on row  $r$ .

## Exercise 9.7

No, the set  $\{\neg, \leftrightarrow\}$  is not truth-functionally complete. There is a binary truth-function that has output a single 1 and three 0. For example, take the sentence  $P \wedge Q$ . By Exercise 9.6, every sentence in the set  $\Gamma$  generated from  $P, Q$  and  $\{\neg, \leftrightarrow\}$  has an even number of 1 in its truth table. Therefore, there is no sentence in  $\Gamma$  that is provably equivalent to  $P \wedge Q$ .

## Exercise 9.12

Suppose that  $\varphi$  is contingent, and let  $P_0, \dots, P_n$  be a list of the atomic sentences that occur in  $\varphi$ . Since  $\varphi$  is contingent, there is a valuation  $v$  such that  $v(\varphi) = 0$ . Let  $\perp$  be an arbitrary contradiction, and let  $\top$  be an arbitrary tautology. Define  $F(P_i) = \top$  if  $v(P_i) = 1$ , and  $F(P_i) = \perp$  if  $v(P_i) = 0$ . We claim, then, that the substitution instance  $F(\varphi)$  is inconsistent. Let  $w$  be an arbitrary valuation. For any  $P_i$ ,  $w(F(P_i)) = w(\top) = 1$  if  $v(P_i) = 1$ , and  $w(F(P_i)) = w(\perp) = 0$  if  $v(P_i) = 0$ . So  $w(F(\cdot))$  and  $v(\cdot)$  agree on atomic sentences. But  $w(F(\cdot))$  and  $v(\cdot)$  are both truth-functional, so they agree on all sentences. Therefore,  $w(F(\varphi)) = v(\varphi) = 0$ . Since  $w$  was arbitrary,  $F(\varphi)$  is an inconsistency.

## Exercise 9.14

- As a warmup, we will show that all occurrences of  $\rightarrow$  can be eliminated from valid proofs, along with all uses of MP and CP. Define a function  $f$  from sentences to sentences as the identity on atomic sentences, then extend by commuting with  $\wedge, \vee, \neg$ , and by setting  $f(\varphi \rightarrow \psi) = \neg(f(\varphi) \wedge \neg f(\psi))$ . We now show that any proof of  $\varphi_1, \dots, \varphi_n \vdash \psi$  can be converted to a proof of  $f(\varphi_1), \dots, f(\varphi_n) \vdash f(\psi)$ .

Here's a way to simulate CP. Suppose first that  $\varphi$  is assumed, and that  $\psi$  is derived with dependencies  $\Delta$ . We can then continue in this way:

1	(1)	$\varphi$	A
$\Delta$	(2)	$\psi$	
3	(3)	$\varphi \wedge \neg\psi$	A
3	(4)	$\neg\psi$	3 $\wedge$ E
$\Delta, 3$	(5)	$\psi \wedge \neg\psi$	2, 4 $\wedge$ I
$\Delta', 3$	(6)	$\neg\varphi$	1, 5 RA
3	(7)	$\varphi$	3 $\wedge$ E
$\Delta', 3$	(8)	$\varphi \wedge \neg\varphi$	7, 6 $\wedge$ I
$\Delta'$	(9)	$\neg(\varphi \wedge \neg\psi)$	3, 8 RA

Here  $\Delta' = \Delta \setminus \{1\}$ , so that line 9 reproduces the effect of CP on lines 1 and 2.

Now we can simulate MP.

$\Gamma$	(1)	$\neg(\varphi \wedge \neg\psi)$	
$\Delta$	(2)	$\varphi$	
3	(3)	$\neg\psi$	A
$\Delta, 3$	(4)	$\varphi \wedge \neg\psi$	2, 3 $\wedge$ I
$\Gamma, \Delta, 3$	(5)	$(\varphi \wedge \neg\psi) \wedge \neg(\varphi \wedge \neg\psi)$	4, 1 $\wedge$ I
$\Gamma, \Delta$	(6)	$\neg\neg\psi$	3, 5 RA
$\Gamma, \Delta$	(7)	$\psi$	6 DN

2. We need to show that any application of RA can be simulated by the other rules. Suppose that we have the following lines

$$\begin{array}{lll} 1 & (1) & P \\ \Delta & (2) & Q \wedge \neg Q \end{array} \quad A$$

We need to show that we can derive the line

$$\Delta' \quad (c) \quad \neg P$$

without using RA. We first derive  $\Delta' \succ P \rightarrow \neg P$  as follows:

1	(1)	P	A
$\Delta$	(2)	$Q \wedge \neg Q$	
$\Delta$	(3)	Q	2 $\wedge$ E
$\Delta'$	(4)	$P \rightarrow Q$	1, 3 CP
$\Delta$	(5)	$\neg Q$	2 $\wedge$ E
$\Delta$	(6)	$\neg P$	4, 5 MT
$\Delta'$	(7)	$P \rightarrow \neg P$	1, 6 CP

The proof that  $\succ (P \rightarrow \neg P) \rightarrow \neg P$  is Exercise 3.1.10, plus one step of CP. Put those two together and  $\Delta' \succ \neg P$  follows.

3. We show that the DN introduction rule can be reproduced from the other rules.

1	(1)	$P$	A
2	(2)	$\neg P$	A
1,2	(3)	$P \wedge \neg P$	1,2 $\wedge I$
1	(4)	$\neg\neg P$	2,3 RA

4. We show that MT can be reproduced from the other rules.

1	(1)	$P \rightarrow Q$	A
2	(2)	$\neg Q$	A
3	(3)	$P$	A
1,3	(4)	$Q$	1,3 MP
1,2,3	(5)	$Q \wedge \neg Q$	4,2 $\wedge I$
1,2	(6)	$\neg P$	3,5 RA

5. Redefine the truth-table for  $\vee$  as follows:

$P$	$Q$	$P \vee Q$
1	1	1
1	0	1
0	1	1
0	0	1

In other words,  $P \vee Q$  is constantly 1, regardless of the input. Since none of the inference rules besides  $\vee E$  uses a disjunction as a premise, those rules are truth-preserving relative to the new truth tables. We claim now that those rules cannot prove  $P \vee P \vdash P$ . Consider the rows of the (new) truth-table in which  $P$  is 0. In this case,  $P \vee P$  is 1, but  $P$  is 0. Hence  $P \vee P \vdash P$  is not truth-preserving relative to the new truth tables, and it cannot be proven by those rules.

6. As stated, this problem is trivially easy to solve: if we never permit anything to be inferred from a “nand” statement, and if we never permit a “nand” statement to be inferred, then our system of rules is *sound*. However, the intention of this problem is to provide intro and elim rules for  $\uparrow$  that are not only sound, but also potentially complete.

### NAND-Introduction ( $\uparrow I$ )

If  $\Delta$  together with  $P$  and  $Q$  imply  $\perp$ , then  $\Delta$  implies  $P \uparrow Q$ .

a	(a)	$P$	A
b	(b)	$Q$	A
$\Delta$	(c)	$\perp$	
$\Delta'$	(d)	$P \uparrow Q$	$a, b, c \uparrow I$

where  $\Delta' = \Delta - \{a, b\}$ .

### NAND-Elimination ( $\uparrow E$ )

From  $P \uparrow Q$ , together with  $P$  and  $Q$ , infer  $\perp$ .

$$\begin{array}{lll} \Gamma & (a) & P \uparrow Q \\ \Delta & (b) & P \\ \Sigma & (c) & Q \\ \Gamma, \Delta, \Sigma & (d) & \perp \qquad a, b, c \uparrow E \end{array}$$

### Falsum-Elimination ( $\perp E$ )

For the  $\uparrow$  rules to do enough, we need to add a  $\perp$ -elimination rule.

$$\begin{array}{ll} \Gamma & (a) \perp \\ \Gamma & (b) Q \qquad a \perp E \end{array}$$