Logic pset 7

Please answer **any three** of the following questions; each is worth six points. Write your answers in your own words, making your reasoning explicit. **Resource:** Chapter 8 of *HLW*

1. Is there a valid proof with the following line fragments? Write your answer in the form of a short essay, using complete sentences.

1	(1)	$\exists x(Fx \to \forall yGy)$	A
	$\vdots$		
1	(n)	$\exists xFx \to \forall yGy$	

   No there is not. To see that the sentence on line 1 does not logicaly imply the sentence on line n, let M be the model with domain $\{\alpha, \beta\}$ and where $F^M = \{\alpha\}$ and $G^M = \emptyset$. The sentence on line 1 is true in M because β is in the extension of $Fx \to \forall yGy$. But the sentence on line n is false in M because $\exists xFx$ is true in M while $\forall yGy$ is false in M.

2. The sentence $P \to \exists xFx$ is not existential, and so is not a candidate for EE. But if there is a derivation of φ from $P \to Fa$ and auxiliary assumptions Δ that obeys the restrictions on EE (i.e. the name a doesn't occur anywhere outside of the subproof), then is there also a derivation of φ from $P \to \exists xFx$ and Δ? Explain your answer.

   There are several ways to show that there is also a derivation of φ from $P \to \exists xFx$ and Δ. One simple way is to note that $P \to \exists xFx \vdash \exists x(P \to Fx)$. We assume that $\Delta, \exists x(P \to Fx) \vdash \varphi$, and so the cut rule implies that $\Delta, P \to \exists xFx \vdash \varphi$.

   In more prosaic terms, suppose that we have lines like this:

n_1	(n_1)	$P \to \exists xFx$	A
n_2	(n_2)	$P \to Fa$	A

   Then we can continue the proof like this:

n_1	(n_3)	$\exists x(Fx \to P)$	
	$\vdots$		
Δ, n_2	(n_4)	φ	(by assumption)
Δ, n_1	(n_5)	φ	n_3, n_2, n_4 EE

3. Consider the two sentences $\forall xFx \to P$ and $\forall x(Fx \to P)$, where x does not occur in P. Are these sentences logically equivalent? Justify your answer by providing proofs and/or models. Write out your answer clearly enough that it would convince somebody who doesn't already get it.

   We can show $\forall x(Fx \to P) \vdash \forall xFx \to P$ by a proof, and we can show $\forall xFx \to P \nvdash \forall x(Fx \to P)$ by giving a counterexample.

Counterexample: let M be the interpretation with domain $\{\alpha, \beta\}$, where F has extension $\{\alpha\}$ and P is false. Then $\forall xFx \to P$ is true because $\forall xFx$ is false. But $\forall x(Fx \to P)$ is false because α is not in the extension of $Fx \to P$.

4. Consider the following two interpretations of the binary relation R, one with domain $\{1, 2, 3\}$ and the other with domain $\{\alpha, \beta, \gamma\}$. Write a sentence that is true in one model but false in the other, and explain step by step how to determine its truth value in each. (Note that $1, 2, 3, \alpha, \beta, \gamma$ are elements of the models; they are not names that can be used in your sentence.)

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[dl] \\ 3 \arrow[u] & \end{tikzcd} \qquad \begin{tikzcd} \alpha \arrow[r, bend left] & \beta \arrow[l, bend left] \arrow[r, bend left] & \gamma \arrow[l, bend left] \end{tikzcd}$$

Consider the sentence $\exists x \exists y(Rxy \wedge \neg Ryx)$. This sentence is true in a model just in case there is a pair of elements that fails to witness symmetry of R. Hence it is true in the first model; in particular, $\langle 3, 1\rangle$ is in the extension of R, but $\langle 1, 3\rangle$ is not in the extension of R. However, this sentence is false in the second model since every arrow is matched by an arrow in the opposite direction.

5. For each of the following sentences, provide one interpretation in which it is true and another in which it is false. An interpretation may be presented by giving a set M and a subset R^M of $M \times M$, or it may be presented as an arrow diagram. In either case, explain step-by-step how to determine the truth value of the sentence in the model.

   (a) $\forall x \forall y \exists z(Rxz \wedge Ryz)$

   Let the domain be $\{\alpha\}$. If the extension of the relation R is empty, then the extension of $Rxz \wedge Ryz$ is the empty subset of $M \times M \times M$, and so the truth-value of $\forall x \forall y \exists z(Rxz \wedge Ryz)$ if false.

   If, in contrast, the extension of the relation R is $\{\langle \alpha, \alpha\rangle\}$, then the extension of $Rxz \wedge Ryz$ is $\{\langle \alpha, \alpha, \alpha\}$. (Here we are adopting the convention that x is the first coordinate, y is the second, and z is the third. Hence the extension of $\exists z(Rxz \wedge Ryz)$ is $\{\langle \alpha, \alpha\rangle\}$. But since $M \times M = \{\langle \alpha, \alpha\rangle\}$, it follows that $\forall x \forall y \exists z(Rxz \wedge Ryz)$ is true.

   (b) $\forall x(\exists yRyx \to \forall zRzx)$

   This sentence effectively says that for every x, if there is an arrow coming into x from some other node, then there must be an arrow coming into x from every node. So one way to make it true is by having no arrows coming in from any nodes. In particular, let the domain be $\{\alpha, \beta\}$. If the extension of the relation R is empty, then the extension of $\forall zRzx$ and $\exists yRyx$ are empty, hence the extension of $\exists yRyx \to \forall zRzx$ is M, and $\forall x(\exists yRyx \to \forall zRzx)$ is true.

   Now let the extension of the relation R be $\{\langle \alpha, \alpha\rangle\}$. In this case, the extension of $\forall zRzx$ is empty, while the extension of $\exists yRyx$ is $\{\alpha\}$. Therefore the extension of

$\exists y R y x \to \forall z R z x$ is $\{\beta\}$. Since α is not in that set, it follows that $\forall x(\exists y R y x \to \forall z R z x)$ is false