

# Review: translations and proofs

PHI 201 – Introductory Logic

November 17, 2025

# Proofs

# Quantifiers of the same type

$$\forall x \forall y \varphi \equiv \forall y \forall x \varphi$$

$$\exists x \exists y \varphi \equiv \exists y \exists x \varphi$$

**Fact:** Quantifiers of the same type commute. That is, universal quantifiers commute with each other, and existential quantifiers commute with each other.

# Prenex normal form

**Fact:** Every sentence in **monadic** predicate logic is equivalent to a sentence where all the quantifiers occur at the beginning.

# Pushing negation inside

$$\neg \forall x \varphi \equiv \exists x \neg \varphi$$

$$\neg \exists x \varphi \equiv \forall x \neg \varphi$$

# Equivalent formulas

For purposes of establishing equivalence, you can treat the variables inside a formula as names.

If quantifiers are stripped, and variables replaced by distinct names, then the process can be reversed to put the quantifiers back on.

# Equivalent formulas

**Fact:** If  $\varphi(a_1, \dots, a_n)$  is equivalent to  $\psi(a_1, \dots, a_n)$ , then  
 $Q_1 x_1 \cdots Q_n x_n \varphi(x_1, \dots, x_n)$  is equivalent to  
 $Q_1 x_1 \cdots Q_n x_n \psi(x_1, \dots, x_n)$ .

Practical upshot: you can detect equivalence by looking at the formula inside quantifiers.

1	(1)	$\forall x \exists y \forall z ((Fx \wedge Gy) \rightarrow Hz)$	A
1	(2)	$\exists y \forall z ((Fa \wedge Gy) \rightarrow Hz)$	1 UE
3	(3)	$\forall z ((Fa \wedge Gb) \rightarrow Hz)$	A
3	(4)	$(Fa \wedge Gb) \rightarrow Hc$	3 UE
3	(5)	$(\neg Fa \vee \neg Gb) \vee Hc$	4 TTV
3	(6)	$\forall z ((\neg Fa \vee \neg Gb) \vee Hz)$	5 UI
3	(7)	$\exists y \forall z ((\neg Fa \vee Gy) \vee Hz)$	6 EI
1	(8)	$\exists y \forall z ((\neg Fa \vee Gy) \vee Hz)$	2,3,7 EE
1	(9)	$\forall x \exists y \forall z ((\neg Fx \vee Gy) \vee Hz)$	8 UI

$$\forall y(Fy \wedge \exists xGx) \equiv \forall y \exists x(Fy \wedge Gx)$$

$$\forall y(Fy \vee \exists xGx) \equiv \forall y \exists x(Fy \vee Gx)$$

$$\forall y(Fy \rightarrow \exists xGx) \equiv \forall y \exists x(Fy \rightarrow Gx)$$

$$\begin{aligned}\forall y(\exists xGx \rightarrow Fy) &\equiv \forall y\forall x(Gx \rightarrow Fy) \\ &\equiv \exists xGx \rightarrow \forall yGy\end{aligned}$$

# Test your understanding

Use what you just learned to quickly see that the following sentence is a tautology:

$$\forall y \exists x (Rxy \rightarrow \forall z Rzy)$$

# Translation

$P$ 's bear the relation  $R$  only to  $Q$ 's

$$\forall x(Px \rightarrow \forall y(Rxy \rightarrow Qy))$$

Conversational implicature of if and only if: “Danes only trust other Danes.”

$$\forall x(Dx \rightarrow \forall y(Txy \leftrightarrow Dy))$$

Only the  $F$  that/who are  $G$  are  $H$ .

$$\forall x(Fx \rightarrow (Hx \rightarrow Gx))$$

$$\forall x(Hx \rightarrow (Fx \wedge Gx))$$

$P$ 's bear the relation  $R$  only to  $Q$ 's that/who are  $S$ .

*a* only respects Harvard professors who acknowledge that Princeton is superior.

$$\forall x((Hx \wedge Rax) \rightarrow Ax)$$

$$\forall x(Rax \rightarrow (Hx \wedge Ax))$$