PHI 201 Lecture 3

Hans Halvorson

Reductio ad Absurdum

Introduction

- Idea behind Reductio ad Absurdum: Show that something is **not** the case ($\neg A$) by showing that it (A) leads, via logically valid reasoning, to a contradiction.
  - RA is truly powerful if combined with DN-elimination to establish **positive** conclusions.

$\sqrt{2}$ is not a rational number

Proof. Assume for reductio ad absurdum that $\sqrt{2}$ is rational, i.e. that $\sqrt{2} = \dfrac{a}{b}$ with integers a, b in lowest terms ($\gcd(a, b) = 1$, $b \neq 0$). Then

$$2 = \frac{a^2}{b^2} \;\Rightarrow\; a^2 = 2b^2.$$

Hence a^2 is even, so a is even; write $a = 2k$. Substituting,

$$(2k)^2 = 2b^2 \;\Rightarrow\; 4k^2 = 2b^2 \;\Rightarrow\; b^2 = 2k^2,$$

so b^2 is even and therefore b is even.

Thus both a and b are even, contradicting that $\frac{a}{b}$ is in lowest terms. Therefore, $\sqrt{2}$ is irrational. $\square$

Reductio ad Absurdum

m	(m)	A	A
	$\vdots$		
$n_1, \ldots, n_j$	(n)	$B \wedge \neg B$	
	$\vdots$		
$n_1, \ldots, \widehat{m}, \ldots, n_j$	(k)	$\neg A$	m, n RA

Reductio ad Absurdum

$$\frac{A_1, \ldots, A_n, B \vdash \bot}{A_1, \ldots, A_n \vdash \neg B}$$

1	(1)	$\neg P \to P$	A
2	(2)	$\neg P$	A
1,2	(3)	P	1,2 MP
1,2	(4)	$P \wedge \neg P$	3,2 $\wedge$I
1	(5)	$\neg\neg P$	2,4 RA
1	(6)	P	5 DN

DeMorgan's laws

Show $\neg(P \vee Q) \vdash \neg P$

1	(1)	$\neg(P \vee Q)$	A
2	(2)	P	A
2	(3)	$P \vee Q$	2 $\vee$I
1,2	(4)	$(P \vee Q) \wedge \neg(P \vee Q)$	3,1 $\wedge$I
1	(5)	$\neg P$	2,4 RA

Material conditional

Show $\neg(\neg P \vee Q) \vdash \neg(P \to Q)$

1	(1)	$\neg(\neg P \vee Q)$	A
2	(2)	$P \to Q$	A
1	(3)	$\neg\neg P$	see previous proof
1	(4)	P	3 DN
1,2	(5)	Q	2,4 MP
1,2	(6)	$\neg P \vee Q$	5 $\vee$I
1,2	(7)	$(\neg P \vee Q) \wedge \neg(\neg P \vee Q)$	6,1 $\wedge$I
1	(8)	$\neg(P \to Q)$	2,7 RA

Law of Non-Contradiction

1	(1)	$P \wedge \neg P$	A
	(2)	$\neg(P \wedge \neg P)$	1,1 RA

Ex Falso Quodlibet (EFQ)

1	(1)	P	A
2	(2)	$\neg P$	A
3	(3)	$\neg Q$	A
1,2	(4)	$P \wedge \neg P$	1,2 $\wedge$I
1,2	(5)	$\neg\neg Q$	3,4 RA
1,2	(6)	Q	5 DN

It is **not** required that the assumption occurs in the dependencies of the contradiction.

Disjunctive Syllogism

$P \vee Q, \neg P \vdash Q$

1	(1)	$P \vee Q$	A
2	(2)	$\neg P$	A
3	(3)	P	A
2,3	(4)	Q	EFQ
5	(5)	Q	A
1,2	(6)	Q	1,3,4,5,5 $\vee$E

DeMorgan's Laws

$\neg P \vee \neg Q \vdash \neg(P \wedge Q)$

1	(1)	$\neg P$	A
2	(2)	$P \wedge Q$	A
2	(3)	P	2 $\wedge$E
1,2	(4)	$P \wedge \neg P$	3,1 $\wedge$I
1	(5)	$\neg(P \wedge Q)$	2,4 RA

DeMorgan's Laws

$\neg P, \neg Q \;\vdash\; \neg(P \vee Q)$

By DS we have $\neg P, P \vee Q \vdash Q$.

It follows that $\neg P, P \vee Q, \neg Q \vdash \bot$.

By RA, $\neg P, \neg Q \vdash \neg(P \vee Q)$.

1	(1)	$P \vee Q$	A
2	(2)	$\neg P$	A
3	(3)	P	A
4	(4)	$\neg Q$	A
2,3	(5)	$P \wedge \neg P$	3,2 $\wedge$I
2,3	(6)	$\neg\neg Q$	4,5 RA
2,3	(7)	Q	6 DN
8	(8)	Q	A
1,2	(9)	Q	1,3,7,8,8 $\vee$E
1,2,4	(10)	$Q \wedge \neg Q$	9,4 $\wedge$I
2,4	(11)	$\neg(P \vee Q)$	1,10 RA

Law of Excluded Middle

1	(1)	$\neg(P \vee \neg P)$	A
2	(2)	P	A
2	(3)	$P \vee \neg P$	2 $\vee$I
1,2	(4)	$(P \vee \neg P) \wedge \neg(P \vee \neg P)$	3,1 $\wedge$I
1	(5)	$\neg P$	2,4 RA
1	(6)	$P \vee \neg P$	5 $\vee$I
1	(7)	$(P \vee \neg P) \wedge \neg(P \vee \neg P)$	6,1 $\wedge$I
	(8)	$\neg\neg(P \vee \neg P)$	1,7 RA
	(9)	$P \vee \neg P$	8 DN

More difficult proofs

To show: $P \to (Q \lor R) \vdash (P \to Q) \lor (P \to R)$

- Strategy 1: Assume negation of conclusion, apply DeMorgans. The result is two negated conditionals, which are equivalent to conjunctions.
- Strategy 2: Derive $P \lor \neg P$, then argue by cases. Recall that $\neg P \vdash P \to Q$.

Useful sequents

Commutativity:
$$A \wedge B \dashv\vdash B \wedge A$$
$$A \vee B \dashv\vdash B \vee A$$

Associativity:
$$(A \wedge B) \wedge C \dashv\vdash A \wedge (B \wedge C)$$
$$(A \vee B) \vee C \dashv\vdash A \vee (B \vee C)$$

Distributivity:
$$A \wedge (B \vee C) \dashv\vdash (A \wedge B) \vee (A \wedge C)$$
$$A \vee (B \wedge C) \dashv\vdash (A \vee B) \wedge (A \vee C)$$

De Morgan's I:
$$\neg(A \vee B) \dashv\vdash \neg A \wedge \neg B$$
$$\neg(A \wedge B) \dashv\vdash \neg A \vee \neg B$$

Useful sequents

Material Conditional: $A \to B \dashv\vdash \neg A \vee B$

$\neg(A \to B) \dashv\vdash A \wedge \neg B$

Excluded Middle: $\vdash A \vee \neg A$

Disjunctive Syllogism: $A \vee B, \neg A \vdash B$

Truth tables

How do you know if something can be proven?

- If you prove $A_1, \ldots, A_n \vdash B$, then that argument form is truth preserving (in the sense that we are about to make precise).
- If you fail to prove $A_1, \ldots, A_n \vdash B$, that doesn't prove that it is not provable.
- If you can show that $A_1, \ldots, A_n \vdash B$ is not truth-preserving, then there cannot possibly be a proof of $A_1, \ldots, A_n \vdash B$.

Semantic validity

- An argument form is **semantically invalid** if there is an instance of that form where the premises are true and the conclusion is false.
  - A **counterexample** to the validity of an argument is an assignment of truth values to the atomic sentences that makes that argument's premises true and its conclusion false.
- We write $A_1, \ldots, A_n \vDash B$ to indicate that the argument from $A_1, \ldots, A_n$ to B is semantically valid.

Ways Things Could Be

P	Q	R
1	1	1
1	1	0
1	0	1
1	0	0
0	1	1
0	1	0
0	0	1
0	0	0

Truth Tables

Conjunction $\wedge$

P	Q	$P \wedge Q$
1	1	1
1	0	0
0	1	0
0	0	0

Disjunction $\vee$

P	Q	$P \vee Q$
1	1	1
1	0	1
0	1	1
0	0	0

Negation $\neg$

P	$\neg P$
1	0
0	1

Conditional $\rightarrow$

P	Q	$P \rightarrow Q$
1	1	1
1	0	0
0	1	1
0	0	1

Detailed truth table for $(P \wedge \neg Q) \to R$

P	Q	R	(	P	$\wedge$	$\neg$	Q	)	$\to$	R
1	1	1		1	0	0	1		1	1
1	1	0		1	0	0	1		1	0
1	0	1		1	1	1	0		1	1
1	0	0		1	1	1	0		0	0
0	1	1		0	0	0	1		1	1
0	1	0		0	0	0	1		1	0
0	0	1		0	0	1	0		1	1
0	0	0		0	0	1	0		1	0

This sentence is a **contingency**: true in some scenarios and false in other scenarios

Material conditional

P	Q	$P \rightarrow Q$
1	1	1
1	0	0
0	1	1
0	0	1

"If the Germans won World War II then French is the official language of instruction at Princeton."

Negative paradox is valid

P	Q	$\neg P$	$P \to Q$
1	1	0	1
1	0	0	0
0	1	1	1
0	0	1	1

In every case where the premise $\neg P$ is true, the conclusion $P \to Q$ is also true.

Affirming the consequent is invalid

$P \to Q, Q \nvDash P$

P	Q	$P \to Q$
1	1	1
1	0	0
0	1	1
0	0	1

In row 3, both premises ($P \to Q$ and Q) are true, but the conclusion P is false. Therefore the argument form is **invalid**.

Ex Falso Quodlibet: $P, \neg P \;\therefore\; Q$

P	Q	$\neg P$	Premises all true?	Conclusion Q
1	1	0	no	1
1	0	0	no	0
0	1	1	no	1
0	0	1	no	0

The premises P and $\neg P$ can never both be true. So there is no row where all premises are true and the conclusion false. Hence the argument form is **valid**.

Using truth tables to guide proofs

Is there a correctly written proof with line fragments like this?

1	(1)	$P \vee Q$	A
	$\vdots$		
1	(n)	P	

Is there a correctly written proof with line fragments like this?

1	(1)	$P \vee Q$	A
	$\vdots$		
1	(n)	P	

No there cannot be. Our proof rules are **sound**, so they cannot prove a line that is semantically invalid.

Soundness

Fact: If there is a correctly written proof that ends with $A_1, \dots, A_n \vdash B$, then $A_1, \dots, A_n \vDash B$.

Consequently, if $A_1, \dots, A_n \nvDash B$, then there cannot be a correctly written proof that ends with $A_1, \dots, A_n \vdash B$.

In other words, if there is a **counterexample**, then there is no proof.

Is there a correctly written proof with line fragments like this?

1	(1)	$P \to (Q \vee R)$	A
	$\vdots$		
1	(n)	$(P \to Q) \vee (P \to R)$	

Completeness

Fact: If $A_1, \ldots, A_n \vDash B$, then the sequent $A_1, \ldots, A_n \vdash B$ can be proven.

In other words: if the argument is truth-preserving, then there is a proof.

Semantic reasoning towards proof

We show that $P \to (Q \vee R) \vDash (P \to Q) \vee (P \to R)$.

Consider a row in the truth table where $(P \to Q) \vee (P \to R)$ is false.

Both $P \to Q$ and $P \to R$ are false on this row.

P is true on this row while both Q and R are false on this row.

But then $P \to (Q \vee R)$ is false on this row.

Therefore, in every row where $(P \to Q) \vee (P \to R)$ is false, $P \to (Q \vee R)$ is also false.

From informal to formal

We show that $P \to (Q \vee R) \vdash (P \to Q) \vee (P \to R)$.

Consider a row in the truth table where $(P \to Q) \vee (P \to R)$ is false.

Both $P \to Q$ and $P \to R$ are false on this row.

P is true on this row while both Q and R are false.

But then $P \to (Q \vee R)$ is false on this row.

Therefore, in every row where $(P \to Q) \vee (P \to R)$ is false,
$P \to (Q \vee R)$ is also false.

From informal to formal

We show that $P \to (Q \vee R) \vdash (P \to Q) \vee (P \to R)$.

Assume $\neg((P \to Q) \vee (P \to R))$

Both $P \to Q$ and $P \to R$ are false on this row.

P is true on this row while both Q and R are false.

But then $P \to (Q \vee R)$ is false on this row.

Therefore, in every row where $(P \to Q) \vee (P \to R)$ is false, $P \to (Q \vee R)$ is also false.

From informal to formal

We show that $P \to (Q \vee R) \vdash (P \to Q) \vee (P \to R)$.

Assume $\neg((P \to Q) \vee (P \to R))$

Then we have $\neg(P \to Q)$ and $\neg(P \to R)$

P is true on this row while both Q and R are false.

But then $P \to (Q \vee R)$ is false on this row.

Therefore, in every row where $(P \to Q) \vee (P \to R)$ is false, $P \to (Q \vee R)$ is also false.

From informal to formal

We show that $P \to (Q \vee R) \vdash (P \to Q) \vee (P \to R)$.

Assume $\neg((P \to Q) \vee (P \to R))$

Then we have $\neg(P \to Q)$ and $\neg(P \to R)$

Therefore P, $\neg Q$, and $\neg R$

But then $P \to (Q \vee R)$ is false on this row.

Therefore, in every row where $(P \to Q) \vee (P \to R)$ is false, $P \to (Q \vee R)$ is also false.

From informal to formal

We show that $P \to (Q \vee R) \vdash (P \to Q) \vee (P \to R)$.

Assume $\neg((P \to Q) \vee (P \to R))$

Then we have $\neg(P \to Q)$ and $\neg(P \to R)$

Therefore P, $\neg Q$, and $\neg R$

So $\neg(P \to (Q \vee R))$

Therefore, in every row where $(P \to Q) \vee (P \to R)$ is false, $P \to (Q \vee R)$ is also false.

From informal to formal

We show that $P \to (Q \vee R) \vdash (P \to Q) \vee (P \to R)$.

Assume $\neg((P \to Q) \vee (P \to R))$

Then we have $\neg(P \to Q)$ and $\neg(P \to R)$

Therefore P, $\neg Q$, and $\neg R$

So $\neg(P \to (Q \vee R))$

Hence $\neg((P \to Q) \vee (P \to R)) \vdash \neg(P \to (Q \vee R))$

Alternate proof strategy

Excluded Middle
$P \lor \neg P$

Assume P
Derive $Q \lor R$
Derive $(P \to Q) \lor (P \to R)$

Assume $\neg P$
Derive $P \to Q$
Derive $(P \to Q) \lor (P \to R)$

Conclude
$(P \to Q) \lor (P \to R)$

1	(1)	$(P \to Q) \to P$	A
2	(2)	$\neg P$	A
3	(3)	P	A
2,3	(4)	$P \wedge \neg P$	2,3 $\wedge$I
5	(5)	$\neg Q$	A
2,3	(6)	$\neg\neg Q$	5,4 RA
2,3	(7)	Q	6 DN
2	(8)	$P \to Q$	3,7 CP
1,2	(9)	P	1,8 MP
1,2	(10)	$P \wedge \neg P$	9,2 $\wedge$I
1	(11)	$\neg\neg P$	2,10 RA
1	(12)	P	11 DN
$\varnothing$	(13)	$((P \to Q) \to P) \to P$	1,12 CP

Summary

- With RA, we have completed the set of inference rules for propositional logic.
- These rules are provably **sound**: they do not permit a proof of something that has a truth-table counterexample.
- These rules are provably **complete**: anything semantically valid can be proven.

Supplemental material

Redundancies in Our System

- With RA, Modus Tollens (MT) and DN-Intro can be eliminated.
- Example: simulate MT using RA.

1	(1)	$P \to Q$	A
2	(2)	$\neg Q$	A
3	(3)	P	A
1,3	(4)	Q	1,3 MP
1,2,3	(5)	$Q \wedge \neg Q$	4,2 $\wedge$I
1,2	(6)	$\neg P$	3,5 RA

Simulating DN-Intro

1	(1)	P	A
2	(2)	$\neg P$	A
1,2	(3)	$P \wedge \neg P$	1,2 $\wedge$I
1	(4)	$\neg\neg P$	2,3 RA

Without RA

RA itself can be simulated with other rules.
Suppose $\Gamma, P \vdash Q \wedge \neg Q$. Then:

- $\Gamma \vdash P \to Q$ and $\Gamma \vdash P \to \neg Q$.
- By contraposition: $\Gamma \vdash \neg Q \to \neg P$.
- Hence $\Gamma \vdash P \to \neg P$.
- But $P \to \neg P \vdash \neg P$.

So $\Gamma \vdash \neg P$. Still, RA feels more natural and symmetric.

More difficult proofs

To show: $\vdash (P \to Q) \vee (Q \to P)$

- Strategy 1: Assume $\neg((P \to Q) \vee (Q \to P))$. Use DM to get $\neg(P \to Q)$ and $\neg(Q \to P)$. The former entails P while the latter entails $\neg P$.
- Strategy 2: Derive $Q \vee \neg Q$, then argue by cases using positive paradox and negative paradox in turn.