

How Logic Works: Solutions to Exercises

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Chapter 2

Exercise 2.1

1. Derive $Q \wedge P$ from $P \wedge Q$.

- (1) $P \wedge Q \quad A$
- (2) $P \quad 1 \wedge E$
- (3) $Q \quad 1 \wedge E$
- (4) $Q \wedge P \quad 3, 2 \wedge I$

2. Derive $P \wedge (Q \wedge R)$ from $(P \wedge Q) \wedge R$.

- (1) $(P \wedge Q) \wedge R \quad A$
- (2) $P \wedge Q \quad 1 \wedge E$
- (3) $R \quad 1 \wedge E$
- (4) $P \quad 2 \wedge E$
- (5) $Q \quad 2 \wedge E$
- (6) $Q \wedge R \quad 5, 3 \wedge I$
- (7) $P \wedge (Q \wedge R) \quad 4, 6 \wedge I$

Exercise 2.2

1. $P \wedge Q \vdash Q \vee R$

- (1) $P \wedge Q \quad A$
- (2) $Q \quad 1 \wedge E$
- (3) $Q \vee R \quad 2 \vee I$

2. $P \wedge Q \vdash (P \vee R) \wedge (Q \vee R)$

- (1) $P \wedge Q \quad A$
- (2) $P \quad 1 \wedge E$
- (3) $Q \quad 1 \wedge E$
- (4) $P \vee R \quad 2 \vee I$
- (5) $Q \vee R \quad 3 \vee I$
- (6) $(P \vee R) \wedge (Q \vee R) \quad 4, 5 \wedge I$

3. $P \vdash Q \vee (P \vee Q)$

- (1) $P \qquad \qquad A$
- (2) $P \vee Q \qquad \qquad 1 \vee I$
- (3) $Q \vee (P \vee Q) \quad 2 \vee I$

4. $P \vdash (P \vee R) \wedge (P \vee Q)$

- (1) $P \qquad \qquad A$
- (2) $P \vee R \qquad \qquad 1 \vee I$
- (3) $P \vee Q \qquad \qquad 1 \vee I$
- (4) $(P \vee R) \wedge (P \vee Q) \quad 2,3 \wedge I$

Exercise 2.3

1. $P \rightarrow (Q \rightarrow R), P \rightarrow Q, P \vdash R$

- (1) $P \rightarrow (Q \rightarrow R) \quad A$
- (2) $P \rightarrow Q \qquad A$
- (3) $P \qquad A$
- (4) $Q \rightarrow R \qquad 1,3 MP$
- (5) $Q \qquad 2,3 MP$
- (6) $R \qquad 4,5 MP$

2. $(A \vee B) \rightarrow T, Z \rightarrow A, T \rightarrow W, Z \vdash W$

- (1) $(A \vee B) \rightarrow T \quad A$
- (2) $Z \rightarrow A \qquad A$
- (3) $T \rightarrow W \qquad A$
- (4) $Z \qquad A$
- (5) $A \qquad 2,4 MP$
- (6) $A \vee B \qquad 5 \vee I$
- (7) $T \qquad 1,6 MP$
- (8) $W \qquad 3,7 MP$

3. $(A \rightarrow B) \wedge (C \rightarrow A), (C \wedge (W \rightarrow Z)) \wedge W \vdash (B \vee D) \wedge (Z \vee E)$

- (1) $(A \rightarrow B) \wedge (C \rightarrow A) \quad A$
- (2) $(C \wedge (W \rightarrow Z)) \wedge W \quad A$
- (3) $A \rightarrow B \quad 1 \wedge E$
- (4) $C \rightarrow A \quad 1 \wedge E$
- (5) $C \wedge (W \rightarrow Z) \quad 2 \wedge E$
- (6) $W \quad 2 \wedge E$
- (7) $C \quad 5 \wedge E$
- (8) $W \rightarrow Z \quad 5 \wedge E$
- (9) $A \quad 4, 7 \text{ MP}$
- (10) $B \quad 3, 9 \text{ MP}$
- (11) $B \vee D \quad 10 \vee I$
- (12) $Z \quad 8, 6 \text{ MP}$
- (13) $Z \vee E \quad 12 \vee I$
- (14) $(B \vee D) \wedge (Z \vee E) \quad 11, 13 \wedge I$

4. $P \rightarrow (P \rightarrow Q), P \vdash Q$

- (1) $P \rightarrow (P \rightarrow Q) \quad A$
- (2) $P \quad A$
- (3) $P \rightarrow Q \quad 1, 2 \text{ MP}$
- (4) $Q \quad 3, 2 \text{ MP}$

5. $P \wedge (P \rightarrow Q) \vdash P \wedge Q$

- (1) $P \wedge (P \rightarrow Q) \quad A$
- (2) $P \quad 1 \wedge E$
- (3) $P \rightarrow Q \quad 1 \wedge E$
- (4) $Q \quad 3, 2 \text{ MP}$
- (5) $P \wedge Q \quad 2, 4 \wedge I$

Exercise 2.4

Prove $Q \rightarrow (P \rightarrow R), \neg R \wedge Q \vdash \neg P$.

- (1) $Q \rightarrow (P \rightarrow R) \quad A$
- (2) $\neg R \wedge Q \quad A$
- (3) $Q \quad 2 \wedge E$
- (4) $\neg R \quad 2 \wedge E$
- (5) $P \rightarrow R \quad 1, 3 \text{ MP}$
- (6) $\neg P \quad 5, 4 \text{ MT}$

Exercise 2.5

1. $P \wedge (Q \wedge R) \dashv\vdash (P \wedge Q) \wedge R$

(\Rightarrow)

- (1) $P \wedge (Q \wedge R)$ A
- (2) P 1 $\wedge E$
- (3) $Q \wedge R$ 1 $\wedge E$
- (4) Q 3 $\wedge E$
- (5) R 3 $\wedge E$
- (6) $P \wedge Q$ 2, 4 $\wedge I$
- (7) $(P \wedge Q) \wedge R$ 6, 5 $\wedge I$

(\Leftarrow)

- (1) $(P \wedge Q) \wedge R$ A
- (2) $P \wedge Q$ 1 $\wedge E$
- (3) R 1 $\wedge E$
- (4) P 2 $\wedge E$
- (5) Q 2 $\wedge E$
- (6) $Q \wedge R$ 5, 3 $\wedge I$
- (7) $P \wedge (Q \wedge R)$ 4, 6 $\wedge I$

2. $P \dashv\vdash P \wedge P$

(\Rightarrow)

- (1) P A
- (2) $P \wedge P$ 1, 1 $\wedge I$

(\Leftarrow)

- (1) $P \wedge P$ A
- (2) P 1 $\wedge E$

Exercise 2.6

1. $\neg(\neg R \rightarrow H)$

2. $\neg E \rightarrow S$ (equivalently: $E \vee S$)

3. $\neg P \wedge \neg S$

4. $A \rightarrow (H \vee B)$

5. $K \wedge ((M \wedge F) \vee \neg F)$

6. $\neg(H \wedge D)$

Exercise 2.7

1. $\neg\neg Q \rightarrow P, \neg P \vdash \neg Q$

- (1) $\neg\neg Q \rightarrow P \quad A$
- (2) $\neg P \quad A$
- (3) $\neg\neg\neg Q \quad 1, 2 \text{ MT}$
- (4) $\neg Q \quad 3 \text{ DN}$

2. $P \rightarrow (P \rightarrow Q), P \vdash Q$

- (1) $P \rightarrow (P \rightarrow Q) \quad A$
- (2) $P \quad A$
- (3) $P \rightarrow Q \quad 1, 2 \text{ MP}$
- (4) $Q \quad 3, 2 \text{ MP}$

3. $(P \wedge P) \rightarrow Q, P \vdash Q$

- (1) $(P \wedge P) \rightarrow Q \quad A$
- (2) $P \quad A$
- (3) $P \wedge P \quad 2, 2 \wedge I$
- (4) $Q \quad 1, 3 \text{ MP}$

4. $P \vdash Q \vee (\neg\neg P \vee R)$

- (1) $P \quad A$
- (2) $\neg\neg P \quad 1 \text{ DN}$
- (3) $\neg\neg P \vee R \quad 2 \vee I$
- (4) $Q \vee (\neg\neg P \vee R) \quad 3 \vee I$

Exercise 2.8

1. $P \rightarrow \neg Q, \neg P \not\vdash Q$

Counterexample (English instance): Let P be “It is raining.” Let Q be “ $2 + 2 = 5$.” Then Q is false, so $\neg Q$ is true; hence $P \rightarrow \neg Q$ is true. Also $\neg P$ can be true (suppose it isn’t raining). But Q is false.

2. $P \rightarrow R \not\vdash (P \vee Q) \rightarrow R$

Counterexample (English instance): Let P be “I am on Mars.” (false) Let Q be “I am on Earth.” (true) Let R be “ $2 + 2 = 5$.” (false) Then $P \rightarrow R$ is true (false antecedent), but $(P \vee Q) \rightarrow R$ is false (true antecedent, false consequent).

Chapter 3

Exercise 3.1

1. $P \vdash Q \rightarrow (P \wedge Q)$

1	(1)	P	A
2	(2)	Q	A
1,2	(3)	$P \wedge Q$	1,2 \wedge I
1	(4)	$Q \rightarrow (P \wedge Q)$	2,3 CP

2. $(P \rightarrow Q) \wedge (P \rightarrow R) \vdash P \rightarrow (Q \wedge R)$

1	(1)	$(P \rightarrow Q) \wedge (P \rightarrow R)$	A
2	(2)	P	A
1	(3)	$P \rightarrow Q$	1 \wedge E
1	(4)	$P \rightarrow R$	1 \wedge E
1,2	(5)	Q	3,2 MP
1,2	(6)	R	4,2 MP
1,2	(7)	$Q \wedge R$	5,6 \wedge I
1	(8)	$P \rightarrow (Q \wedge R)$	2,7 CP

3. $P \rightarrow (Q \rightarrow R) \vdash Q \rightarrow (P \rightarrow R)$

1	(1)	$P \rightarrow (Q \rightarrow R)$	A
2	(2)	Q	A
3	(3)	P	A
1,3	(4)	$Q \rightarrow R$	3,1 MP
1,2,3	(5)	R	4,2 MP
1,2	(6)	$P \rightarrow R$	3,5 CP
1	(7)	$Q \rightarrow (P \rightarrow R)$	2,6 CP

4. $P \rightarrow Q \vdash (Q \rightarrow R) \rightarrow (P \rightarrow R)$

1	(1)	$P \rightarrow Q$	A
2	(2)	$Q \rightarrow R$	A
3	(3)	P	A
1,3	(4)	Q	1,3 MP
1,2,3	(5)	R	2,4 MP
1,2	(6)	$P \rightarrow R$	3,5 CP
1	(7)	$(Q \rightarrow R) \rightarrow (P \rightarrow R)$	2,6 CP

5. $P \rightarrow (P \rightarrow Q) \vdash P \rightarrow Q$

1	(1)	$P \rightarrow (P \rightarrow Q)$	A
2	(2)	P	A
1,2	(3)	$P \rightarrow Q$	1,2 MP
1,2	(4)	Q	3,2 MP
1	(5)	$P \rightarrow Q$	2,4 CP

6. $P \rightarrow (Q \rightarrow R) \vdash (P \wedge Q) \rightarrow R$

1	(1)	$P \rightarrow (Q \rightarrow R)$	A
2	(2)	$P \wedge Q$	A
2	(3)	P	2 $\wedge E$
2	(4)	Q	2 $\wedge E$
1,2	(5)	$Q \rightarrow R$	1,3 MP
1,2	(6)	R	5,4 MP
1	(7)	$(P \wedge Q) \rightarrow R$	2,6 CP

7. $(P \vee Q) \rightarrow R \vdash P \rightarrow R$

1	(1)	$(P \vee Q) \rightarrow R$	A
2	(2)	P	A
2	(3)	$P \vee Q$	2 $\vee I$
1,2	(4)	R	1,3 MP
1	(5)	$P \rightarrow R$	2,4 CP

8. $\neg P \vdash \neg(P \wedge Q)$

1	(1)	$\neg P$	A
2	(2)	$P \wedge Q$	A
2	(3)	P	2 $\wedge E$
	(4)	$(P \wedge Q) \rightarrow P$	2,3 CP
1	(5)	$\neg(P \wedge Q)$	4,1 MT

9. $\neg(P \vee Q) \vdash \neg P \wedge \neg Q$

1	(1)	$\neg(P \vee Q)$	A
2	(2)	P	A
2	(3)	$P \vee Q$	2 $\vee I$
	(4)	$P \rightarrow (P \vee Q)$	2,3 CP
1	(5)	$\neg P$	4,1 MT
6	(6)	Q	A
6	(7)	$P \vee Q$	6 $\vee I$
	(8)	$Q \rightarrow (P \vee Q)$	6,7 CP
1	(9)	$\neg Q$	8,1 MT
1	(10)	$\neg P \wedge \neg Q$	5,9 $\wedge I$

10. $P \rightarrow \neg P \vdash \neg P$

1	(1)	P	A
2	(2)	$P \rightarrow \neg P$	A
1,2	(3)	$\neg P$	2,1 MP
1	(4)	$(P \rightarrow \neg P) \rightarrow \neg P$	2,3 CP
1	(5)	$\neg\neg P$	1 DN
1	(6)	$\neg(P \rightarrow \neg P)$	4,5 MT
	(7)	$P \rightarrow \neg(P \rightarrow \neg P)$	1,6 CP
2	(8)	$\neg\neg(P \rightarrow \neg P)$	2 DN
2	(9)	$\neg P$	7,8 MT

Exercise 3.4

1. $P \rightarrow Q \vdash \neg(P \wedge \neg Q)$

1	(1)	$P \rightarrow Q$	A
2	(2)	$P \wedge \neg Q$	A
2	(3)	P	2 $\wedge E$
1,2	(4)	Q	1,3 MP
2	(5)	$\neg Q$	2 $\wedge E$
1,2	(6)	$Q \wedge \neg Q$	4,5 $\wedge I$
1	(7)	$\neg(P \wedge \neg Q)$	2,6 RA

2. $\neg(P \wedge Q) \vdash \neg P \vee \neg Q$

1	(1)	$\neg(P \wedge Q)$	A
2	(2)	$\neg(\neg P \vee \neg Q)$	A
3	(3)	$\neg P$	A
3	(4)	$\neg P \vee \neg Q$	3 $\vee I$
2,3	(5)	$(\neg P \vee \neg Q) \wedge \neg(\neg P \vee \neg Q)$	4,2 $\wedge I$
2	(6)	$\neg\neg P$	3,5 RA
2	(7)	P	6 DN
8	(8)	$\neg Q$	A
8	(9)	$\neg P \vee \neg Q$	8 $\vee I$
2,8	(10)	$(\neg P \vee \neg Q) \wedge \neg(\neg P \vee \neg Q)$	9,2 $\wedge I$
2	(11)	$\neg\neg Q$	8,10 RA
2	(12)	Q	11 DN
2	(13)	$P \wedge Q$	7,12 $\wedge I$
1,2	(14)	$(P \wedge Q) \wedge \neg(P \wedge Q)$	13,1 $\wedge I$
1	(15)	$\neg\neg(\neg P \vee \neg Q)$	2,14 RA
1	(16)	$\neg P \vee \neg Q$	15 DN

3. $\neg(P \rightarrow Q) \vdash P \wedge \neg Q$

1	(1)	$\neg(P \rightarrow Q)$	A
2	(2)	$\neg P$	A
3	(3)	$\neg Q$	A
4	(4)	P	A
2,4	(5)	$P \wedge \neg P$	2,4 $\wedge I$
2,4	(6)	$\neg\neg Q$	3,5 RA
2,4	(7)	Q	6 DN
2	(8)	$P \rightarrow Q$	4,7 CP
1,2	(9)	$(P \rightarrow Q) \wedge \neg(P \rightarrow Q)$	8,1 $\wedge I$
1	(10)	$\neg\neg P$	2,9 RA
1	(11)	P	10 DN
12	(12)	Q	A
12	(13)	$P \rightarrow Q$	4,12 CP
1,12	(14)	$(P \rightarrow Q) \wedge \neg(P \rightarrow Q)$	13,1 $\wedge I$
1	(15)	$\neg Q$	12,14 RA
1	(16)	$P \wedge \neg Q$	11,15 $\wedge I$

4. $\vdash (P \rightarrow Q) \vee (Q \rightarrow P)$

1	(1)	$\neg((P \rightarrow Q) \vee (Q \rightarrow P))$	A
2	(2)	P	A
3	(3)	Q	A
2	(4)	$Q \rightarrow P$	3,2 CP
2	(5)	$(P \rightarrow Q) \vee (Q \rightarrow P)$	4 $\vee I$
1,2	(6)	$((P \rightarrow Q) \vee (Q \rightarrow P)) \wedge \neg((P \rightarrow Q) \vee (Q \rightarrow P))$	5,1 $\wedge I$
1	(7)	$\neg P$	2,6 RA
8	(8)	$\neg Q$	A
1,2	(9)	$P \wedge \neg P$	2,7 $\wedge I$
1,2	(10)	$\neg\neg Q$	8,9 RA
1,2	(11)	Q	10 DN
1	(12)	$P \rightarrow Q$	2,11 CP
1	(13)	$(P \rightarrow Q) \vee (Q \rightarrow P)$	12 $\vee I$
1	(14)	$((P \rightarrow Q) \vee (Q \rightarrow P)) \wedge \neg((P \rightarrow Q) \vee (Q \rightarrow P))$	13,1 $\wedge I$
\emptyset	(15)	$\neg\neg((P \rightarrow Q) \vee (Q \rightarrow P))$	1,14 RA
\emptyset	(16)	$(P \rightarrow Q) \vee (Q \rightarrow P)$	15 DN

5. $P \rightarrow (Q \vee R) \vdash (P \rightarrow Q) \vee (P \rightarrow R)$

1	(1)	$P \rightarrow (Q \vee R)$	A
2	(2)	$\neg((P \rightarrow Q) \vee (P \rightarrow R))$	A
3	(3)	$\neg P$	A
4	(4)	P	A
5	(5)	$\neg Q$	A
3,4	(6)	$P \wedge \neg P$	4,3 $\wedge I$
3,4	(7)	$\neg \neg Q$	5,6 RA
3,4	(8)	Q	7 DN
3	(9)	$P \rightarrow Q$	4,8 CP
3	(10)	$(P \rightarrow Q) \vee (P \rightarrow R)$	9 $\vee I$
2,3	(11)	$((P \rightarrow Q) \vee (P \rightarrow R)) \wedge \neg((P \rightarrow Q) \vee (P \rightarrow R))$	10,2 $\wedge I$
2	(12)	$\neg \neg P$	3,11 RA
2	(13)	P	12 DN
1,2	(14)	$Q \vee R$	1,13 MP
15	(15)	Q	A
15	(16)	$P \rightarrow Q$	4,15 CP
15	(17)	$(P \rightarrow Q) \vee (P \rightarrow R)$	16 $\vee I$
18	(18)	R	A
18	(19)	$P \rightarrow R$	4,18 CP
18	(20)	$(P \rightarrow Q) \vee (P \rightarrow R)$	19 $\vee I$
1,2	(21)	$(P \rightarrow Q) \vee (P \rightarrow R)$	14,15,17,18,20 $\vee E$
1,2	(22)	$((P \rightarrow Q) \vee (P \rightarrow R)) \wedge \neg((P \rightarrow Q) \vee (P \rightarrow R))$	21,2 $\wedge I$
1	(23)	$\neg \neg((P \rightarrow Q) \vee (P \rightarrow R))$	2,22 RA
1	(24)	$(P \rightarrow Q) \vee (P \rightarrow R)$	23 DN

6. $(P \wedge Q) \rightarrow \neg Q \vdash P \rightarrow \neg Q$

1	(1)	$(P \wedge Q) \rightarrow \neg Q$	A
2	(2)	P	A
3	(3)	Q	A
2,3	(4)	$P \wedge Q$	2,3 $\wedge I$
1,2,3	(5)	$\neg Q$	1,4 MP
1,2,3	(6)	$Q \wedge \neg Q$	3,5 $\wedge I$
1,2	(7)	$\neg Q$	3,6 RA
1	(8)	$P \rightarrow \neg Q$	2,7 CP

Chapter 6

Exercise 6.1

1. No logicians are celebrities. (Lx, Cx)

$$\forall x (Lx \rightarrow \neg Cx)$$

Equivalently: $\neg\exists x(Lx \wedge Cx)$

2. Some celebrities are not logicians. (Lx, Cx)

$$\exists x(Cx \wedge \neg Lx)$$

3. Only students who do the homework will learn logic. (Sx, Hx, Lx)

Either

$$\forall x(Lx \rightarrow (Sx \wedge Hx))$$

or (inequivalently)

$$\forall x((Sx \wedge Lx) \rightarrow Hx)$$

depending on whether one intends to restrict the claim to students.

4. All rich logicians are computer scientists. (Rx, Lx, Cx)

$$\forall x((Rx \wedge Lx) \rightarrow Cx)$$

5. All students and professors get a discount. (Sx, Px, Dx)

$$\forall x((Sx \vee Px) \rightarrow Dx)$$

6. No logician is rich, unless she is a computer scientist. (Lx, Rx, Cx)

$$\forall x((Lx \wedge Rx) \rightarrow Cx)$$

Equivalent form: $\forall x((Lx \wedge \neg Cx) \rightarrow \neg Rx)$

7. Not all logicians are computer scientists. (Lx, Cx)

$$\neg\forall x(Lx \rightarrow Cx)$$

Often put as: $\exists x(Lx \wedge \neg Cx)$.

8. Some logicians are rich computer scientists. (Lx, Rx, Cx)

$$\exists x(Lx \wedge (Rx \wedge Cx))$$

9. If there are rich logicians, then some logicians are computer scientists. (Rx, Lx, Cx)

$$\exists x(Rx \wedge Lx) \rightarrow \exists y(Ly \wedge Cy)$$

10. No pets except service animals are permitted in dorms. (Px, Sx, Dx)

Can be read in a minimal way as:

$$\forall x ((Px \wedge Dx) \rightarrow Sx),$$

which says only that no non-service pets are allowed in dorms. However, ordinary policy language is typically understood more strongly: among pets, *being permitted in the dorms* and *being a service animal* coincide. That reading is captured by:

$$\forall x (Px \rightarrow (Dx \leftrightarrow Sx)).$$

This biconditional formalization is therefore closer to the intended rule.

11. If anyone is rich, then Mary is. (Rx, m)

$$(\exists x Rx) \rightarrow Rm$$

Exercise 6.2

1. Mary loves everyone who loves her. (m, Lxy)

$$\forall x (Lxm \rightarrow Lmx)$$

2. Mary loves all and only those people who don't love themselves. (Lxy, m)

$$\forall x (Lmx \leftrightarrow \neg Lxx)$$

3. Everyone loves their mother. (Lxy, Mxy)

$$\forall x \forall y (Myx \rightarrow Lxy)$$

4. Some people love only those people who love their mother. (Lxy, Mxy)

$$\exists x \forall y (Lxy \rightarrow \forall z (Mzy \rightarrow Lyz))$$

5. Snape killed someone. (Kxy, s)

$$\exists x Ksx$$

6. Snape is a killer. (Kxy, s)

$$\exists x Ksx$$

7. Someone was killed by Snape. (Kxy, s)

$$\exists x Ksx$$

8. Some wizards only marry other wizards. (Wx, Mxy)

$$\exists x(Wx \wedge \forall y(Mxy \rightarrow Wy))$$

9. There is no greatest number. ($Nx, x < y$)

$$\forall x(Nx \rightarrow \exists y(Ny \wedge x < y))$$

10. c is the least upper bound of a and b . ($a, b, c, x \leq y$)

$$(a \leq c \wedge b \leq c) \wedge \forall x((a \leq x \wedge b \leq x) \rightarrow c \leq x)$$

11. c is the greatest common divisor of a and b . ($a, b, c, Dxy, x \leq y$)

$$(Dca \wedge Dcb) \wedge \forall x((Dxa \wedge Dxb) \rightarrow x \leq c)$$

Exercise 6.8

1. $\neg\exists x(Fx \wedge Gx) \vdash \forall x(Fx \rightarrow \neg Gx)$

1	(1)	$\neg\exists x(Fx \wedge Gx)$	A
2	(2)	Fa	A
3	(3)	Ga	A
2,3	(4)	$Fa \wedge Ga$	2,3 $\wedge I$
2,3	(5)	$\exists x(Fx \wedge Gx)$	4 EI
1,2,3	(6)	$\exists x(Fx \wedge Gx) \wedge \neg\exists x(Fx \wedge Gx)$	5,1 $\wedge I$
1,2	(7)	$\neg Ga$	3,6 RA
1	(8)	$Fa \rightarrow \neg Ga$	2,7 CP
1	(9)	$\forall x(Fx \rightarrow \neg Gx)$	8 UI

2. $\forall x Fx \vdash \exists x Fx$

1	(1)	$\forall x Fx$	A
1	(2)	Fa	1 UE
1	(3)	$\exists x Fx$	2 EI

3. $\forall x(Fx \rightarrow Gx), Fa \vdash \exists x Gx$

1	(1)	$\forall x(Fx \rightarrow Gx)$	A
2	(2)	Fa	A
1	(3)	$Fa \rightarrow Ga$	1 UE
1,2	(4)	Ga	3,2 MP
1,2	(5)	$\exists x Gx$	4 EI

4. $\neg Fa \vdash \exists x(Fx \rightarrow P)$

1	(1)	$\neg Fa$	A
1	(2)	$Fa \rightarrow P$	1 negative paradox
1	(3)	$\exists x(Fx \rightarrow P)$	2 EI

5. $\neg \forall x Fx \vdash \exists x(Fx \rightarrow P)$

1	(1)	$\neg \forall x Fx$	A
2	(2)	$\neg \exists x(Fx \rightarrow P)$	A
3	(3)	$Fa \rightarrow P$	A
3	(4)	$\exists x(Fx \rightarrow P)$	3 EI
2,3	(5)	$\exists x(Fx \rightarrow P) \wedge \neg \exists x(Fx \rightarrow P)$	4,2 $\wedge I$
2	(6)	$\neg(Fa \rightarrow P)$	3,5 RA
2	(7)	Fa	6 material conditional
2	(8)	$\forall x Fx$	7 UI
1,2	(9)	$\forall x Fx \wedge \neg \forall x Fx$	8,1 $\wedge I$
1	(10)	$\neg \neg \exists x(Fx \rightarrow P)$	2,9 RA
1	(11)	$\exists x(Fx \rightarrow P)$	10 DN

6. $\neg \exists x Fx \vdash \forall x(Fx \rightarrow Gx)$

1	(1)	$\neg \exists x Fx$	A
2	(2)	Fa	A
3	(3)	$\neg Ga$	A
2	(4)	$\exists x Fx$	2 EI
1,2	(5)	$\exists x Fx \wedge \neg \exists x Fx$	4,1 $\wedge I$
1,2	(6)	$\neg \neg Ga$	3,5 RA
1,2	(7)	Ga	6 DN
1	(8)	$Fa \rightarrow Ga$	2,7 CP
1	(9)	$\forall x(Fx \rightarrow Gx)$	8 UI

7. $\forall x \forall y Rxy \vdash \exists x Rxx$

1	(1)	$\forall x \forall y Rxy$	A
1	(2)	$\forall y Ray$	1 UE
1	(3)	Raa	2 UE
1	(4)	$\exists x Rxx$	3 EI

8. $P \rightarrow Fa \vdash P \rightarrow \exists x Fx$

1	(1)	$P \rightarrow Fa$	A
2	(2)	P	A
1,2	(3)	Fa	1,2 MP
1,2	(4)	$\exists x Fx$	3 EI
1	(5)	$P \rightarrow \exists x Fx$	2,4 CP

9. $\exists x Fx \rightarrow P \vdash \forall x(Fx \rightarrow P)$

1	(1)	$\exists x Fx \rightarrow P$	A
2	(2)	Fa	A
2	(3)	$\exists x Fx$	2 EI
1,2	(4)	P	1,3 MP
1	(5)	$Fa \rightarrow P$	2,4 CP
1	(6)	$\forall x(Fx \rightarrow P)$	5 UI

There is a typo here in the book: the direction $\forall x(Fx \rightarrow P) \vdash \exists x Fx \rightarrow P$ cannot be proven without EE, which is only introduced in the next section.

10. $\neg \exists x Fx \vdash \forall x(Fx \rightarrow P)$

1	(1)	$\neg \exists x Fx$	A
2	(2)	Fa	A
2	(3)	$\exists x Fx$	2 EI
1,2	(4)	$\exists x Fx \wedge \neg \exists x Fx$	3,1 $\wedge I$
1	(5)	$\neg Fa$	2,4 RA
1	(6)	$Fa \rightarrow P$	5 neg paradox
1	(7)	$\forall x(Fx \rightarrow P)$	6 UI

11. $\neg \exists x(Fx \rightarrow P) \vdash \forall x Fx \wedge \neg P$

1	(1)	$\neg \exists x(Fx \rightarrow P)$	A
2	(2)	$Fa \rightarrow P$	A
2	(3)	$\exists x(Fx \rightarrow P)$	2 EI
1,2	(4)	$\exists x(Fx \rightarrow P) \wedge \neg \exists x(Fx \rightarrow P)$	3,1 $\wedge I$
1	(5)	$\neg(Fa \rightarrow P)$	2,4 RA
1	(6)	$Fa \wedge \neg P$	5 material conditional
1	(7)	$\neg P$	6 $\wedge E$
1	(8)	Fa	6 $\wedge E$
1	(9)	$\forall x Fx$	8 UI
1	(10)	$\forall x Fx \wedge \neg P$	9,7 $\wedge I$

12. $\forall x Fx \rightarrow P \vdash \exists x(Fx \rightarrow P)$

1	(1)	$\forall x Fx \rightarrow P$	A
2	(2)	$\neg \exists x(Fx \rightarrow P)$	A
3	(3)	$\neg Fa$	A
3	(4)	$Fa \rightarrow P$	3 neg paradox
3	(5)	$\exists x(Fx \rightarrow P)$	4 EI
2,3	(6)	$\exists x(Fx \rightarrow P) \wedge \neg \exists x(Fx \rightarrow P)$	5,2 $\wedge I$
2	(7)	$\neg \neg Fa$	3,6 RA
2	(8)	Fa	7 DN
2	(9)	$\forall x Fx$	8 UI
1,2	(10)	P	1,9 MP
1,2	(11)	$Fb \rightarrow P$	10 pos paradox
1,2	(12)	$\exists x(Fx \rightarrow P)$	11 EI
1,2	(13)	$\exists x(Fx \rightarrow P) \wedge \neg \exists x(Fx \rightarrow P)$	12,2 $\wedge I$
1	(14)	$\neg \neg \exists x(Fx \rightarrow P)$	2,13 RA
1	(15)	$\exists x(Fx \rightarrow P)$	14 DN

Exercise 6.11

1. $\exists x Fx \vee \exists x Gx \vdash \exists x(Fx \vee Gx)$

1	(1)	$\exists x Fx \vee \exists x Gx$	A
2	(2)	$\exists x Fx$	A
3	(3)	Fa	A
3	(4)	$Fa \vee Ga$	3 $\vee I$
3	(5)	$\exists x(Fx \vee Gx)$	4 EI
2	(6)	$\exists x(Fx \vee Gx)$	2,3,5 EE
7	(7)	$\exists x Gx$	A
8	(8)	Ga	A
8	(9)	$Fa \vee Ga$	8 $\vee I$
8	(10)	$\exists x(Fx \vee Gx)$	9 EI
7	(11)	$\exists x(Fx \vee Gx)$	7,8,10 EE
1	(12)	$\exists x(Fx \vee Gx)$	1,2,6,7,11 $\vee E$

2. $\forall x(Fx \rightarrow Gx), \neg \exists x Gx \vdash \neg \exists x Fx$

1	(1)	$\forall x(Fx \rightarrow Gx)$	A
2	(2)	$\neg \exists xGx$	A
3	(3)	$\exists xFx$	A
4	(4)	Fa	A
1	(5)	$Fa \rightarrow Ga$	1 UE
1,4	(6)	Ga	5,4 MP
1,4	(7)	$\exists xGx$	6 EI
1,3	(8)	$\exists xGx$	3,4,7 EE
1,2,3	(9)	$\exists xGx \wedge \neg \exists xGx$	8,2 \wedge I
1,2	(10)	$\neg \exists xFx$	3,9 RA

3. $\forall x(Fx \rightarrow Gx) \vdash \exists x \neg Gx \rightarrow \exists x \neg Fx$

1	(1)	$\forall x(Fx \rightarrow Gx)$	A
2	(2)	$\exists x \neg Gx$	A
3	(3)	$\neg Ga$	A
1	(4)	$Fa \rightarrow Ga$	1 UE
1,3	(5)	$\neg Fa$	4,3 MT
1,3	(6)	$\exists x \neg Fx$	5 EI
1,2	(7)	$\exists x \neg Fx$	2,3,6 EE
1	(8)	$\exists x \neg Gx \rightarrow \exists x \neg Fx$	2,7 CP

4. $\forall x(Fx \rightarrow P) \vdash \exists x Fx \rightarrow P$

1	(1)	$\forall x(Fx \rightarrow P)$	A
2	(2)	$\exists x Fx$	A
3	(3)	Fa	A
1	(4)	$Fa \rightarrow P$	1 UE
1,3	(5)	P	4,3 MP
1,2	(6)	P	2,3,5 EE
1	(7)	$\exists x Fx \rightarrow P$	2,6 CP

5. $P \wedge \exists x Fx \vdash \exists x(P \wedge Fx)$

1	(1)	$P \wedge \exists x Fx$	A
1	(2)	P	1 \wedge E
1	(3)	$\exists x Fx$	1 \wedge E
4	(4)	Fa	A
1,4	(5)	$P \wedge Fa$	2,4 \wedge I
1,4	(6)	$\exists x(P \wedge Fx)$	5 EI
1	(7)	$\exists x(P \wedge Fx)$	3,4,6 EE

6. $\exists x(Fx \rightarrow P) \vdash \forall x Fx \rightarrow P$

1	(1)	$\exists x(Fx \rightarrow P)$	A
2	(2)	$\forall xFx$	A
3	(3)	$Fa \rightarrow P$	A
2	(4)	Fa	2 UE
2,3	(5)	P	3,4 MP
3	(6)	$\forall xFx \rightarrow P$	2,5 CP
1	(7)	$\forall xFx \rightarrow P$	1,3,6 EE

Exercise 6.13

1. $P \rightarrow \exists xFx \vdash \exists x(P \rightarrow Fx)$

1	(1)	$P \rightarrow \exists xFx$	A
\emptyset	(2)	$\exists xFx \vee \neg \exists xFx$	prop taut
3	(3)	$\exists xFx$	A
4	(4)	Fa	A
4	(5)	$P \rightarrow Fa$	4 prop taut
4	(6)	$\exists x(P \rightarrow Fx)$	5 EI
3	(7)	$\exists x(P \rightarrow Fx)$	3,4,6 EE
8	(8)	$\neg \exists xFx$	A
1,8	(9)	$\neg P$	1,8 MT
1,8	(10)	$P \rightarrow Fb$	9 prop taut
1,8	(11)	$\exists x(P \rightarrow Fx)$	10 EI
1	(12)	$\exists x(P \rightarrow Fx)$	2,3,7,8,11 $\vee E$

2. $\exists x(Fx \rightarrow P) \vdash \forall xFx \rightarrow P$

1	(1)	$\exists x(Fx \rightarrow P)$	A
2	(2)	$\forall xFx$	A
3	(3)	$Fa \rightarrow P$	A
2	(4)	Fa	2 UE
2,3	(5)	P	3,4 MP
1,2	(6)	P	1,3,5 EE
1	(7)	$\forall xFx \rightarrow P$	2,6 CP

Exercise 6.14

1. $\vdash \forall x(Fx \rightarrow Fx)$

1	(1)	Fa	A
\emptyset	(2)	$Fa \rightarrow Fa$	1,1 CP
\emptyset	(3)	$\forall x(Fx \rightarrow Fx)$	2 UI

2. $\vdash \forall xFx \vee \exists x \neg Fx$

\emptyset	(1)	$\neg\exists x \neg Fx \vee \exists x \neg Fx$	prop taut
2	(2)	$\neg\exists x \neg Fx$	A
3	(3)	$\neg Fa$	A
3	(4)	$\exists x \neg Fx$	3 EI
2,3	(5)	$\exists x \neg Fx \wedge \neg\exists x \neg Fx$	4,2 $\wedge I$
2	(6)	$\neg\neg Fa$	3,5 RA
2	(7)	Fa	6 DN
2	(8)	$\forall x Fx$	7 UI
2	(9)	$\forall x Fx \vee \exists x \neg Fx$	8 $\vee I$
10	(10)	$\exists x \neg Fx$	A
10	(11)	$\forall x Fx \vee \exists x \neg Fx$	10 $\vee I$
\emptyset	(12)	$\forall x Fx \vee \exists x \neg Fx$	1,2,9,10,11 $\vee E$

3. $\vdash \forall x \neg(Fx \wedge \neg Fx)$

1	(1)	$Fa \wedge \neg Fa$	A
\emptyset	(2)	$\neg(Fa \wedge \neg Fa)$	1,1 RA
\emptyset	(3)	$\forall x \neg(Fx \wedge \neg Fx)$	2 UI

4. $\vdash \neg\exists x(Fx \wedge \neg Fx)$

1	(1)	$\exists x(Fx \wedge \neg Fx)$	A
2	(2)	$Fa \wedge \neg Fa$	A
2	(3)	$\neg\exists x(Fx \wedge \neg Fx)$	1,2 RA
1	(4)	$\neg\exists x(Fx \wedge \neg Fx)$	1,2,3 EE
1	(5)	$\exists x(Fx \wedge \neg Fx) \wedge \neg\exists x(Fx \wedge \neg Fx)$	1,3 $\wedge I$
\emptyset	(6)	$\neg\exists x(Fx \wedge \neg Fx)$	1,5 RA

5. $\vdash \forall x \exists y(Rxy \rightarrow Rxx)$

1	(1)	Raa	A
\emptyset	(2)	$Raa \rightarrow Raa$	1,1 CP
\emptyset	(3)	$\exists y(Ray \rightarrow Raa)$	2 EI
\emptyset	(4)	$\forall x \exists y(Rxy \rightarrow Rxx)$	3 UI

6. $\vdash \forall x \exists y(Rxy \rightarrow Ryx)$

1	(1)	Raa	A
\emptyset	(2)	$Raa \rightarrow Raa$	1,1 CP
\emptyset	(3)	$\exists y(Ray \rightarrow Ryx)$	2 EI
\emptyset	(4)	$\forall x \exists y(Rxy \rightarrow Ryx)$	3 UI

7. $\vdash \exists x(Fx \rightarrow \forall y Fy)$

1	(1)	$\neg \exists x(Fx \rightarrow \forall y Fy)$	A
2	(2)	$\neg Fa$	A
2	(3)	$Fa \rightarrow \forall y Fy$	2 prop taut
2	(4)	$\exists x(Fx \rightarrow \forall y Fy)$	3 EI
1,2	(5)	$\exists x(Fx \rightarrow \forall y Fy) \wedge \neg \exists x(Fx \rightarrow \forall y Fy)$	4,1 $\wedge I$
1	(6)	$\neg \neg Fa$	2,5 RA
1	(7)	Fa	6 DN
1	(8)	$\forall y Fy$	7 UI
1	(9)	$Fa \rightarrow \forall y Fy$	8 prop taut
1	(10)	$\exists x(Fx \rightarrow \forall y Fy)$	9 EI
1	(11)	$\exists x(Fx \rightarrow \forall y Fy) \wedge \neg \exists x(Fx \rightarrow \forall y Fy)$	10,1 $\wedge I$
\emptyset	(12)	$\neg \neg \exists x(Fx \rightarrow \forall y Fy)$	1,11 RA
\emptyset	(13)	$\exists x(Fx \rightarrow \forall y Fy)$	12 DN

8. $\vdash \exists x \forall y(Fx \rightarrow Fy)$

1	(1)	$\neg \exists x \forall y(Fx \rightarrow Fy)$	A
2	(2)	$\neg Fa$	A
2	(3)	$Fa \rightarrow Fb$	2 prop taut
2	(4)	$\forall y(Fa \rightarrow Fy)$	3 UI
2	(5)	$\exists x \forall y(Fx \rightarrow Fy)$	4 EI
1,2	(6)	$\exists x \forall y(Fx \rightarrow Fy) \wedge \neg \exists x \forall y(Fx \rightarrow Fy)$	5,1 $\wedge I$
1	(7)	$\neg \neg Fa$	2,6 RA
1	(8)	Fa	7 DN
1	(9)	$Fc \rightarrow Fa$	8 prop taut
1	(10)	$\forall y(Fc \rightarrow Fy)$	9 UI
1	(11)	$\exists x \forall y(Fx \rightarrow Fy)$	10 EI
1	(12)	$\exists x \forall y(Fx \rightarrow Fy) \wedge \neg \exists x \forall y(Fx \rightarrow Fy)$	11,1 $\wedge I$
\emptyset	(13)	$\neg \neg \exists x \forall y(Fx \rightarrow Fy)$	1,12 RA
\emptyset	(14)	$\exists x \forall y(Fx \rightarrow Fy)$	13 DN

9. $\forall x \exists y(Fx \rightarrow Gy) \vdash \exists y \forall x(Fx \rightarrow Gy)$

1	(1)	$\forall x \exists y (Fx \rightarrow Gy)$	A
\emptyset	(2)	$\exists y Gy \vee \neg \exists y Gy$	prop taut
3	(3)	$\exists y Gy$	A
4	(4)	Ga	A
4	(5)	$Fb \rightarrow Ga$	4 prop taut
4	(6)	$\forall x (Fx \rightarrow Ga)$	5 UI
4	(7)	$\exists y \forall x (Fx \rightarrow Gy)$	6 EI
3	(8)	$\exists y \forall x (Fx \rightarrow Gy)$	3,4,7 EE
9	(9)	$\neg \exists y Gy$	A
10	(10)	Fc	A
1	(11)	$\exists y (Fc \rightarrow Gy)$	1 UE
12	(12)	$Fc \rightarrow Gd$	A
10,12	(13)	Gd	12,10 MP
10,12	(14)	$\exists y Gy$	13 EI
9,10,12	(15)	$\exists y Gy \wedge \neg \exists y Gy$	14,9 $\wedge I$
9,12	(16)	$\neg Fc$	10,15 RA
9,12	(17)	$Fc \rightarrow Ge$	16 prop taut
1,9	(18)	$Fc \rightarrow Ge$	11,12,17 EE
1,9	(19)	$\forall x (Fx \rightarrow Ge)$	18 UI
1,9	(20)	$\exists y \forall x (Fx \rightarrow Gy)$	19 EI
1	(21)	$\exists y \forall x (Fx \rightarrow Gy)$	2,3,8,9,20 $\vee E$

10. $\vdash \forall x \exists y (Rxy \rightarrow \forall z Rxz)$

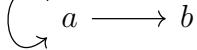
\emptyset	(1)	$\exists y \neg Ray \vee \neg \exists y \neg Ray$	prop taut
2	(2)	$\exists y \neg Ray$	A
3	(3)	$\neg Rab$	A
3	(4)	$Rab \rightarrow \forall z Raz$	3 prop taut
3	(5)	$\exists y (Ray \rightarrow \forall z Raz)$	4 EI
2	(6)	$\exists y (Ray \rightarrow \forall z Raz)$	2,3,5 EE
7	(7)	$\neg \exists y \neg Ray$	A
8	(8)	$\neg Rac$	A
8	(9)	$\exists y \neg Ray$	8 EI
7,8	(10)	$\exists y \neg Ray \wedge \neg \exists y \neg Ray$	9,7 $\wedge I$
7	(11)	$\neg \neg Rac$	8,10 RA
7	(12)	Rac	11 DN
7	(13)	$\forall z Raz$	12 UI
7	(14)	$Rab \rightarrow \forall z Raz$	13 prop taut
7	(15)	$\exists y (Ray \rightarrow \forall z Raz)$	14 EI
\emptyset	(16)	$\exists y (Ray \rightarrow \forall z Raz)$	1,2,6,7,15 $\vee E$
\emptyset	(17)	$\forall x \exists y (Rxy \rightarrow \forall z Rxz)$	16 UI

Exercise 6.17

$$\forall x(\exists z Rxz \rightarrow \forall y Rxy), \exists x \exists y \vdash \exists x \forall y Rxy$$

1	(1)	$\forall x(\exists z Rxz \rightarrow \forall y Rxy)$	A
2	(2)	$\exists x \exists y Rxy$	A
3	(3)	$\exists y Ray$	A
4	(4)	Rab	A
4	(5)	$\exists z Raz$	4 EI
1	(6)	$\exists z Raz \rightarrow \forall y Ray$	1 UE
1,4	(7)	$\forall y Ray$	6,5 MP
1,4	(8)	$\exists x \forall y Rxy$	7 EI
1,3	(9)	$\exists x \forall y Rxy$	3,4,8 EE
1,2	(10)	$\exists x \forall y Rxy$	2,3,9 EE

Question: Does it follow from these premises that $\forall x \forall y Rxy$?

Answer: No. 

Chapter 7

Exercise 7.1

Here the proof is lengthened because of the strictness of the $=$ rules. From $a = c$ and $b = c$, we cannot immediately apply $=E$ to get $a = b$.

1	(1)	$\exists x \forall y(Py \rightarrow y = x)$	A
2	(2)	$Pa \wedge Pb$	A
3	(3)	$\forall y(Py \rightarrow y = c)$	A
3	(4)	$Pa \rightarrow a = c$	3 UE
3	(5)	$Pb \rightarrow b = c$	3 UE
2	(6)	Pa	2 $\wedge E$
2	(7)	Pb	2 $\wedge E$
2,3	(8)	$a = c$	4,6 MP
2,3	(9)	$b = c$	5,7 MP
\emptyset	(10)	$b = b$	$=I$
2,3	(11)	$c = b$	10,9 $=E$
2,3	(12)	$a = b$	8,11 $=E$
1,2	(13)	$a = b$	1,3,12 EE
1	(14)	$(Pa \wedge Pb) \rightarrow a = b$	2,13 CP
1	(15)	$\forall y((Pa \wedge Py) \rightarrow a = y)$	14 UI
1	(16)	$\forall x \forall y((Px \wedge Py) \rightarrow x = y)$	15 UI

Exercise 7.2

1	(1)	$Fa \wedge \forall x(Fx \rightarrow x = a)$	A
2	(2)	Fb	A
1	(3)	$\forall x(Fx \rightarrow x = a)$	1 $\wedge E$
1	(4)	$Fb \rightarrow b = a$	3 UE
1,2	(5)	$b = a$	4,2 MP
6	(6)	$b = a$	A
1	(7)	Fa	1 $\wedge E$
\emptyset	(8)	$b = b$	=I
6	(9)	$a = b$	8,6 =E
1,6	(10)	Fb	7,9 =E
1	(11)	$Fb \leftrightarrow b = a$	2,5,6,10 CP \times 2
1	(12)	$\forall x(Fx \leftrightarrow x = a)$	11 UI
1	(1)	$\forall x(Fx \leftrightarrow x = a)$	A
1	(2)	$Fa \leftrightarrow a = a$	1 UE
\emptyset	(3)	$a = a$	=I
1	(4)	Fa	2,3 MP
1	(5)	$Fb \leftrightarrow b = a$	1 UE
1	(6)	$Fb \rightarrow b = a$	5 $\wedge E$
1	(7)	$\forall x(Fx \rightarrow x = a)$	6 UI
1	(8)	$Fa \wedge \forall x(Fx \rightarrow x = a)$	4,7 $\wedge I$

Chapter 8

Exercise 8.1

1. Countermodel M_1 :

$$D_1 = \{a, b\}, \quad F^{M_1} = \{b\}, \quad c^{M_1} = a.$$

Then $M_1 \models \exists x Fx$ (witness b), but $M_1 \not\models Fc$.

2. Countermodel M_2 :

$$D_2 = \{a, b\}, \quad F^{M_2} = \{a\}, \quad c^{M_2} = a.$$

Then $M_2 \models Fc$, but $M_2 \not\models \forall x Fx$ (since $b \notin F^{M_2}$).

3. Countermodel M_3 :

$$D_3 = \{a, b\}, \quad F^{M_3} = \{a\}, \quad G^{M_3} = \{b\}.$$

Then $M_3 \models \exists x Fx \wedge \exists x Gx$, but $M_3 \not\models \exists x(Fx \wedge Gx)$.

4. Countermodel M_4 :

$$D_4 = \{a, b\}, \quad F^{M_4} = \{a\}, \quad G^{M_4} = \{b\}.$$

We have $\forall x Fx$ false and $\forall x Gx$ false, so $M_4 \models (\forall x Fx \rightarrow \forall x Gx)$ (false \rightarrow false is true), but for $x = a$ we get $Fa \wedge \neg Ga$, so $M_4 \not\models \forall x(Fx \rightarrow Gx)$.

5. Countermodel M_5 :

$$D_5 = \{a\}, \quad F^{M_5} = \emptyset, \quad H^{M_5} = \{a\}.$$

Then for the only element a , Fa is false, so $Fa \rightarrow Ha$ is true; hence $M_5 \models \forall x(Fx \rightarrow Hx)$. But $M_5 \not\models \exists x Fx$ and $M_5 \models \exists x Hx$, so $M_5 \not\models \exists x Fx \vee \neg \exists x Hx$.

6. Countermodel M_6 :

$$D_6 = \{a\}, \quad F^{M_6} = \emptyset, \quad G^{M_6} = \emptyset.$$

Again Fa is false, so $Fa \rightarrow Ga$ is true; hence $M_6 \models \forall x(Fx \rightarrow Gx)$. But $F^{M_6} \cap G^{M_6} = \emptyset$, so $M_6 \not\models \exists x(Fx \wedge Gx)$.

7. Countermodel M_7 :

$$D_7 = \{a, b\}, \quad F^{M_7} = \{a\}, \quad G^{M_7} = \{a, b\}, \quad H^{M_7} = \{b\}.$$

Then a witnesses $\exists x(Fx \wedge Gx)$, and b witnesses $\exists x(Gx \wedge Hx)$. But there is no element in $F^{M_7} \cap H^{M_7}$, so $M_7 \not\models \exists x(Fx \wedge Hx)$.

8. Countermodel M_8 :

$$D_8 = \{a, b\}, \quad F^{M_8} = \{a\}.$$

Then $M_8 \not\models \forall x Fx$ (since $b \notin F^{M_8}$), and $M_8 \not\models \forall x \neg Fx$ (since $a \in F^{M_8}$). Hence $M_8 \not\models \forall x Fx \vee \forall x \neg Fx$.

9. Countermodel M_9 :

$$D_9 = \{a, b\}, \quad F^{M_9} = \{a, b\}, \quad G^{M_9} = \{a\}, \quad H^{M_9} = \emptyset.$$

Then:

$$M_9 \models \exists x(Fx \rightarrow Gx) \quad (\text{take } x = a, Fa \rightarrow Ga \text{ is true}),$$

$$M_9 \models \exists x(Gx \rightarrow Hx) \quad (\text{take } x = b, Gb \rightarrow Hb \text{ is true}),$$

$$\begin{aligned} M_9 \not\models \exists x(Fx \rightarrow Hx) &\quad \text{since for both } a, b, Fx \text{ is true and } Hx \text{ is false,} \\ &\quad \text{so } Fx \rightarrow Hx \text{ is false everywhere.} \end{aligned}$$

10. Countermodel M_{10} :

$$D_{10} = \{a, b\}, \quad F^{M_{10}} = \{a\}, \quad G^{M_{10}} = \emptyset.$$

Then $M_{10} \models \exists x(Fx \rightarrow Gx)$ (take $x = b$, where Fb is false), while $\exists x Fx$ is true (witness a) and $\exists x Gx$ is false. Hence $M_{10} \not\models \exists x Fx \rightarrow \exists x Gx$.

Exercise 8.3

1. $\forall x Fx \rightarrow P \not\vdash \forall x(Fx \rightarrow P)$.

Countermodel M_1 :

$$D = \{a, b\}, \quad F^M = \{a\}, \quad P^M = 0.$$

Then $\forall x Fx$ is false (since $b \notin F^M$), so

$$(\forall x Fx \rightarrow P)^M = (0 \rightarrow 0) = 1.$$

However,

$$(Fb \rightarrow P)^M = (0 \rightarrow 0) = 1, \quad (Fa \rightarrow P)^M = (1 \rightarrow 0) = 0,$$

so $\forall x(Fx \rightarrow P)$ is false. Thus the premise is true and the conclusion false in M_1 .

2. $\exists x(Fx \rightarrow P) \not\vdash \exists x Fx \rightarrow P$.

Countermodel M_2 :

$$D = \{a, b\}, \quad F^M = \{a\}, \quad P^M = 0.$$

In this case, $(Fb \rightarrow P)^{M_2} = (0 \rightarrow 0) = 1$, so $\exists x(Fx \rightarrow P)^{M_2} = 1$. But $(Fa)^{M_2} = 1$, so $(\exists x Fx)^{M_2} = 1$ and $(\exists x Fx \rightarrow P)^{M_2} = (1 \rightarrow 0) = 0$. Hence the premise is true while the conclusion is false in M_2 .

Exercise 8.7

1. $\forall x \forall y(Rxy \rightarrow Ryx)$

True model (symmetric):



False model (one-way arrow):



2. $\forall x \forall y \exists z(Rxz \wedge Ryz)$

True model (common successor a for everyone):



(For any x, y , choose $z = a$.)

False model (no common successor for a, b):



(For $x = a, y = b$ there is no z with both $a \rightarrow z$ and $b \rightarrow z$.)

3. $\exists x \forall y (Ryx \rightarrow Ryy)$

True model (choose $x = a$ with no incoming arrows):



(No y satisfies $y \rightarrow a$, so $Ryx \rightarrow Ryy$ holds vacuously for all y .)

False model (every x has an incoming arrow from a non-reflexive y):



(No loops, so Ryy is always false; but each node has an incoming arrow.)

4. $\forall x (\exists y Ryx \rightarrow \forall z Rzx)$

True model (empty relation):



(Each antecedent $\exists y Ryx$ is false, so the implication is true for all x .)

False model (some x has an incoming arrow but not everyone points to x):



(Take $x = b$: $\exists y Ryb$ holds (witness a), but $\forall z Rzb$ fails since $b \not\rightarrow b$.)

5. $\exists x \exists y (Rxy \leftrightarrow \neg Ryy)$

True model (take $x = a, y = b$):



(Here $a \rightarrow b$ is true and $b \rightarrow b$ is false, so $\neg Rbb$ is true and the biconditional holds.)

False model (universal relation on $\{a, b\}$):



(For every y , Ryy is true, hence $\neg Ryy$ is false; but Rxy is always true. So $Rxy \leftrightarrow \neg Ryy$ is false for all x, y .)

Chapter 9

Exercise 9.6

Suppose that φ is true in an even number n of rows of its truth table. Then $\neg\varphi$ is true in $4 - n$ rows of its truth table, and $4 - n$ is also even.

Suppose that both φ and ψ are even. Let's say that row r is an *agreement row* if φ and ψ have the same truth value on r . We will show that there cannot be 1 or 3 agreement rows. Suppose that there is a single row where both sentences have value a . Since φ and ψ are even, a must occur on another row in each of their truth tables. If these rows are not the same, then there are two of them, which leaves a single remaining row. In that row, both φ and ψ must have value $1 - a$, and so they agree there.

Suppose now that there are three rows where both sentences have the same value, and let r be the remaining row. Since three is odd, one of the two truth values a must occur most frequently on these rows. If a occurs twice and $1 - a$ occurs once, then $1 - a$ must be the value of both φ and ψ on row r . If a occurs three times, then a must be the value of both φ and ψ on row r . In either case, φ and ψ agree on row r .

Exercise 9.7

No, the set $\{\neg, \leftrightarrow\}$ is not truth-functionally complete. There is a binary truth-function that has output a single 1 and three 0. For example, take the sentence $P \wedge Q$. By Exercise 9.6, every sentence in the set Γ generated from P, Q and $\{\neg, \leftrightarrow\}$ has an even number of 1 in its truth table. Therefore, there is no sentence in Γ that is provably equivalent to $P \wedge Q$.

Exercise 9.12

Suppose that φ is contingent, and let P_0, \dots, P_n be a list of the atomic sentences that occur in φ . Since φ is contingent, there is a valuation v such that $v(\varphi) = 0$. Let \perp be an arbitrary contradiction, and let \top be an arbitrary tautology. Define $F(P_i) = \top$ if $v(P_i) = 1$, and $F(P_i) = \perp$ if $v(P_i) = 0$. We claim, then, that the substitution instance $F(\varphi)$ is inconsistent. Let w be an arbitrary valuation. For any P_i , $w(F(P_i)) = w(\top) = 1$ if $v(P_i) = 1$, and $w(F(P_i)) = w(\perp) = 0$ if $v(P_i) = 0$. So $w(F(\cdot))$ and $v(\cdot)$ agree on atomic sentences. But $w(F(\cdot))$ and $v(\cdot)$ are both truth-functional, so they agree on all sentences. Therefore, $w(F(\varphi)) = v(\varphi) = 0$. Since w was arbitrary, $F(\varphi)$ is an inconsistency.

Exercise 9.14

- As a warmup, we will show that all occurrences of \rightarrow can be eliminated from valid proofs, along with all uses of MP and CP. Define a function f from sentences to sentences as the identity on atomic sentences, then extend by commuting with \wedge, \vee, \neg , and by setting $f(\varphi \rightarrow \psi) = \neg(f(\varphi) \wedge \neg f(\psi))$. We now show that any proof of $\varphi_1, \dots, \varphi_n \vdash \psi$ can be converted to a proof of $f(\varphi_1), \dots, f(\varphi_n) \vdash f(\psi)$.

Here's a way to simulate CP. Suppose first that φ is assumed, and that ψ is derived with dependencies Δ . We can then continue in this way:

1	(1)	φ	A
Δ	(2)	ψ	
3	(3)	$\varphi \wedge \neg\psi$	A
3	(4)	$\neg\psi$	3 \wedge E
$\Delta, 3$	(5)	$\psi \wedge \neg\psi$	2, 4 \wedge I
$\Delta', 3$	(6)	$\neg\varphi$	1, 5 RA
3	(7)	φ	3 \wedge E
$\Delta', 3$	(8)	$\varphi \wedge \neg\varphi$	7, 6 \wedge I
Δ'	(9)	$\neg(\varphi \wedge \neg\psi)$	3, 8 RA

Here $\Delta' = \Delta \setminus \{1\}$, so that line 9 reproduces the effect of CP on lines 1 and 2.

Now we can simulate MP.

Γ	(1)	$\neg(\varphi \wedge \neg\psi)$	
Δ	(2)	φ	
3	(3)	$\neg\psi$	A
$\Delta, 3$	(4)	$\varphi \wedge \neg\psi$	2, 3 \wedge I
$\Gamma, \Delta, 3$	(5)	$(\varphi \wedge \neg\psi) \wedge \neg(\varphi \wedge \neg\psi)$	4, 1 \wedge I
Γ, Δ	(6)	$\neg\neg\psi$	3, 5 RA
Γ, Δ	(7)	ψ	6 DN

2. We need to show that any application of RA can be simulated by the other rules. Suppose that we have the following lines

$$\begin{array}{lll} 1 & (1) & P \\ \Delta & (2) & Q \wedge \neg Q \end{array} \quad A$$

We need to show that we can derive the line

$$\Delta' \quad (c) \quad \neg P$$

without using RA. We first derive $\Delta' \succ P \rightarrow \neg P$ as follows:

1	(1)	P	A
Δ	(2)	$Q \wedge \neg Q$	
Δ	(3)	Q	2 \wedge E
Δ'	(4)	$P \rightarrow Q$	1, 3 CP
Δ	(5)	$\neg Q$	2 \wedge E
Δ	(6)	$\neg P$	4, 5 MT
Δ'	(7)	$P \rightarrow \neg P$	1, 6 CP

The proof that $\succ (P \rightarrow \neg P) \rightarrow \neg P$ is Exercise 3.1.10, plus one step of CP. Put those two together and $\Delta' \succ \neg P$ follows.

3. We show that the DN introduction rule can be reproduced from the other rules.

1	(1)	P	A
2	(2)	$\neg P$	A
1,2	(3)	$P \wedge \neg P$	1,2 $\wedge I$
1	(4)	$\neg\neg P$	2,3 RA

4. We show that MT can be reproduced from the other rules.

1	(1)	$P \rightarrow Q$	A
2	(2)	$\neg Q$	A
3	(3)	P	A
1,3	(4)	Q	1,3 MP
1,2,3	(5)	$Q \wedge \neg Q$	4,2 $\wedge I$
1,2	(6)	$\neg P$	3,5 RA

5. Redefine the truth-table for \vee as follows:

P	Q	$P \vee Q$
1	1	1
1	0	1
0	1	1
0	0	1

In other words, $P \vee Q$ is constantly 1, regardless of the input. Since none of the inference rules besides $\vee E$ uses a disjunction as a premise, those rules are truth-preserving relative to the new truth tables. We claim now that those rules cannot prove $P \vee P \vdash P$. Consider the rows of the (new) truth-table in which P is 0. In this case, $P \vee P$ is 1, but P is 0. Hence $P \vee P \vdash P$ is not truth-preserving relative to the new truth tables, and it cannot be proven by those rules.

6. As stated, this problem is trivially easy to solve: if we never permit anything to be inferred from a “nand” statement, and if we never permit a “nand” statement to be inferred, then our system of rules is *sound*. However, the intention of this problem is to provide intro and elim rules for \uparrow that are not only sound, but also potentially complete.

NAND-Introduction ($\uparrow I$)

If Δ together with P and Q imply \perp , then Δ implies $P \uparrow Q$.

a	(a)	P	A
b	(b)	Q	A
Δ	(c)	\perp	
Δ'	(d)	$P \uparrow Q$	$a, b, c \uparrow I$

where $\Delta' = \Delta - \{a, b\}$.

NAND-Elimination ($\uparrow E$)

From $P \uparrow Q$, together with P and Q , infer \perp .

$$\begin{array}{lll} \Gamma & (a) & P \uparrow Q \\ \Delta & (b) & P \\ \Sigma & (c) & Q \\ \Gamma, \Delta, \Sigma & (d) & \perp \qquad\qquad a, b, c \uparrow E \end{array}$$

Falsum-Elimination ($\perp E$)

For the \uparrow rules to do enough, we need to add a \perp -elimination rule.

$$\begin{array}{ll} \Gamma & (a) \perp \\ \Gamma & (b) Q \qquad a \perp E \end{array}$$