How Logic Works: Solutions to Problems

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December 11, 2025

Chapter 3

Exercise 3.1

1. $P \vdash Q \to (P \wedge Q)$

1	(1)	P	A
2	(2)	Q	A
1,2	(3)	$P \wedge Q$	1,2 $\wedge$I
1	(4)	$Q \to (P \wedge Q)$	2,3 CP

2. $(P \to Q) \wedge (P \to R) \vdash P \to (Q \wedge R)$

1	(1)	$(P \to Q) \wedge (P \to R)$	A
2	(2)	P	A
1	(3)	$P \to Q$	1 $\wedge$E
1	(4)	$P \to R$	1 $\wedge$E
1,2	(5)	Q	3,2 MP
1,2	(6)	R	4,2 MP
1,2	(7)	$Q \wedge R$	5,6 $\wedge$I
1	(8)	$P \to (Q \wedge R)$	2,7 CP

3. $P \to (Q \to R) \vdash Q \to (P \to R)$

1	(1)	$P \to (Q \to R)$	A
2	(2)	Q	A
3	(3)	P	A
1,3	(4)	$Q \to R$	3,1 MP
1,2,3	(5)	R	4,2 MP
1,2	(6)	$P \to R$	3,5 CP
1	(7)	$Q \to (P \to R)$	2,6 CP

4. $P \to Q \vdash (Q \to R) \to (P \to R)$

1	(1)	$P \to Q$	A
2	(2)	$Q \to R$	A
3	(3)	P	A
1,3	(4)	Q	1,3 MP
1,2,3	(5)	R	2,4 MP
1,2	(6)	$P \to R$	3,5 CP
1	(7)	$(Q \to R) \to (P \to R)$	2,6 CP

5. $P \to (P \to Q) \vdash P \to Q$

1	(1)	$P \to (P \to Q)$	A
2	(2)	P	A
1,2	(3)	$P \to Q$	1,2 MP
1,2	(4)	Q	3,2 MP
1	(5)	$P \to Q$	2,4 CP

6. $P \to (Q \to R) \vdash (P \wedge Q) \to R$

1	(1)	$P \to (Q \to R)$	A
2	(2)	$P \wedge Q$	A
2	(3)	P	2 $\wedge$E
2	(4)	Q	2 $\wedge$E
1,2	(5)	$Q \to R$	1,3 MP
1,2	(6)	R	5,4 MP
1	(7)	$(P \wedge Q) \to R$	2,6 CP

7. $(P \vee Q) \to R \vdash P \to R$

1	(1)	$(P \vee Q) \to R$	A
2	(2)	P	A
2	(3)	$P \vee Q$	2 $\vee$I
1,2	(4)	R	1,3 MP
1	(5)	$P \to R$	2,4 CP

8. $\neg P \vdash \neg(P \wedge Q)$

1	(1)	$\neg P$	A
2	(2)	$P \wedge Q$	A
2	(3)	P	2 $\wedge$E
	(4)	$(P \wedge Q) \to P$	2,3 CP
1	(5)	$\neg(P \wedge Q)$	4,1 MT

9. $\neg(P \vee Q) \vdash \neg P \wedge \neg Q$

1	(1)	$\neg(P \vee Q)$	A
2	(2)	P	A
2	(3)	$P \vee Q$	2 $\vee$I
	(4)	$P \to (P \vee Q)$	2,3 CP
1	(5)	$\neg P$	4,1 MT
6	(6)	Q	A
6	(7)	$P \vee Q$	6 $\vee$I
	(8)	$Q \to (P \vee Q)$	6,7 CP
1	(9)	$\neg Q$	8,1 MT
1	(10)	$\neg P \wedge \neg Q$	5,9 $\wedge$I

10. $P \to \neg P \vdash \neg P$

1	(1)	P	A
2	(2)	$P \to \neg P$	A
1,2	(3)	$\neg P$	2,1 MP
1	(4)	$(P \to \neg P) \to \neg P$	2,3 CP
1	(5)	$\neg\neg P$	1 DN
1	(6)	$\neg(P \to \neg P)$	4,5 MT
	(7)	$P \to \neg(P \to \neg P)$	1,6 CP
2	(8)	$\neg\neg(P \to \neg P)$	2 DN
2	(9)	$\neg P$	7,8 MT

Exercise 3.4

1. $P \to Q \vdash \neg(P \wedge \neg Q)$

1	(1)	$P \to Q$	A
2	(2)	$P \wedge \neg Q$	A
2	(3)	P	2 $\wedge$E
1,2	(4)	Q	1,3 MP
2	(5)	$\neg Q$	2 $\wedge$E
1,2	(6)	$Q \wedge \neg Q$	4,5 $\wedge$I
1	(7)	$\neg(P \wedge \neg Q)$	2,6 RA

2. $\neg(P \wedge Q) \vdash \neg P \vee \neg Q$

1	(1)	$\neg(P \wedge Q)$	A
2	(2)	$\neg(\neg P \vee \neg Q)$	A
3	(3)	$\neg P$	A
3	(4)	$\neg P \vee \neg Q$	3 $\vee$I
2,3	(5)	$(\neg P \vee \neg Q) \wedge \neg(\neg P \vee \neg Q)$	4,2 $\wedge$I
2	(6)	$\neg\neg P$	3,5 RA
2	(7)	P	6 DN
8	(8)	$\neg Q$	A
8	(9)	$\neg P \vee \neg Q$	8 $\vee$I
2,8	(10)	$(\neg P \vee \neg Q) \wedge \neg(\neg P \vee \neg Q)$	9,2 $\wedge$I
2	(11)	$\neg\neg Q$	8,10 RA
2	(12)	Q	11 DN
2	(13)	$P \wedge Q$	7,12 $\wedge$I
1,2	(14)	$(P \wedge Q) \wedge \neg(P \wedge Q)$	13,1 $\wedge$I
1	(15)	$\neg\neg(\neg P \vee \neg Q)$	2,14 RA
1	(16)	$\neg P \vee \neg Q$	15 DN

3. $\neg(P \to Q) \vdash P \wedge \neg Q$

1	(1)	$\neg(P \to Q)$	A
2	(2)	$\neg P$	A
3	(3)	$\neg Q$	A
4	(4)	P	A
2,4	(5)	$P \wedge \neg P$	2,4 $\wedge$I
2,4	(6)	$\neg\neg Q$	3,5 RA
2,4	(7)	Q	6 DN
2	(8)	$P \to Q$	4,7 CP
1,2	(9)	$(P \to Q) \wedge \neg(P \to Q)$	8,1 $\wedge$I
1	(10)	$\neg\neg P$	2,9 RA
1	(11)	P	10 DN
12	(12)	Q	A
12	(13)	$P \to Q$	4,12 CP
1,12	(14)	$(P \to Q) \wedge \neg(P \to Q)$	13,1 $\wedge$I
1	(15)	$\neg Q$	12,14 RA
1	(16)	$P \wedge \neg Q$	11,15 $\wedge$I

4. $\vdash (P \to Q) \vee (Q \to P)$

1	(1)	$\neg((P \to Q) \vee (Q \to P))$	A
2	(2)	P	A
3	(3)	Q	A
2	(4)	$Q \to P$	3,2 CP
2	(5)	$(P \to Q) \vee (Q \to P)$	4 $\vee$I
1,2	(6)	$((P \to Q) \vee (Q \to P)) \wedge \neg((P \to Q) \vee (Q \to P))$	5,1 $\wedge$I
1	(7)	$\neg P$	2,6 RA
8	(8)	$\neg Q$	A
1,2	(9)	$P \wedge \neg P$	2,7 $\wedge$I
1,2	(10)	$\neg\neg Q$	8,9 RA
1,2	(11)	Q	10 DN
1	(12)	$P \to Q$	2,11 CP
1	(13)	$(P \to Q) \vee (Q \to P)$	12 $\vee$I
1	(14)	$((P \to Q) \vee (Q \to P)) \wedge \neg((P \to Q) \vee (Q \to P))$	13,1 $\wedge$I
$\varnothing$	(15)	$\neg\neg((P \to Q) \vee (Q \to P))$	1,14 RA
$\varnothing$	(16)	$(P \to Q) \vee (Q \to P)$	15 DN

5. $P \to (Q \vee R) \vdash (P \to Q) \vee (P \to R)$

1	(1)	$P \to (Q \vee R)$	A
2	(2)	$\neg((P \to Q) \vee (P \to R))$	A
3	(3)	$\neg P$	A
4	(4)	P	A
5	(5)	$\neg Q$	A
3,4	(6)	$P \wedge \neg P$	4,3 $\wedge$I
3,4	(7)	$\neg\neg Q$	5,6 RA
3,4	(8)	Q	7 DN
3	(9)	$P \to Q$	4,8 CP
3	(10)	$(P \to Q) \vee (P \to R)$	9 $\vee$I
2,3	(11)	$((P \to Q) \vee (P \to R)) \wedge \neg((P \to Q) \vee (P \to R))$	10,2 $\wedge$I
2	(12)	$\neg\neg P$	3,11 RA
2	(13)	P	12 DN
1,2	(14)	$Q \vee R$	1,13 MP
15	(15)	Q	A
15	(16)	$P \to Q$	4,15 CP
15	(17)	$(P \to Q) \vee (P \to R)$	16 $\vee$I
18	(18)	R	A
18	(19)	$P \to R$	4,18 CP
18	(20)	$(P \to Q) \vee (P \to R)$	19 $\vee$I
1,2	(21)	$(P \to Q) \vee (P \to R)$	14,15,17,18,20 $\vee$E
1,2	(22)	$((P \to Q) \vee (P \to R)) \wedge \neg((P \to Q) \vee (P \to R))$	21,2 $\wedge$I
1	(23)	$\neg\neg((P \to Q) \vee (P \to R))$	2,22 RA
1	(24)	$(P \to Q) \vee (P \to R)$	23 DN

6. $(P \wedge Q) \to \neg Q \vdash P \to \neg Q$

1	(1)	$(P \wedge Q) \to \neg Q$	A
2	(2)	P	A
3	(3)	Q	A
2,3	(4)	$P \wedge Q$	2,3 $\wedge$I
1,2,3	(5)	$\neg Q$	1,4 MP
1,2,3	(6)	$Q \wedge \neg Q$	3,5 $\wedge$I
1,2	(7)	$\neg Q$	3,6 RA
1	(8)	$P \to \neg Q$	2,7 CP

Chapter 6

Exercise 6.1

1. No logicians are celebrities. (Lx, Cx)

$$\forall x\,(Lx \to \neg Cx)$$

   Equivalently: $\neg\exists x(Lx \wedge Cx)$

2. Some celebrities are not logicians. (Lx, Cx)

$$\exists x\,(Cx \wedge \neg Lx)$$

3. Only students who do the homework will learn logic. (Sx, Hx, Lx)
   Either

$$\forall x\,(Lx \to (Sx \wedge Hx))$$

   or (inequivalently)

$$\forall x\,((Sx \wedge Lx) \to Hx)$$

   depending on whether one intends to restrict the claim to students.

4. All rich logicians are computer scientists. (Rx, Lx, Cx)

$$\forall x\,((Rx \wedge Lx) \to Cx)$$

5. All students and professors get a discount. (Sx, Px, Dx)

$$\forall x\,((Sx \vee Px) \to Dx)$$

6. No logician is rich, unless she is a computer scientist. (Lx, Rx, Cx)

$$\forall x\,((Lx \wedge Rx) \rightarrow Cx)$$

Equivalent form: $\forall x((Lx \wedge \neg Cx) \rightarrow \neg Rx)$

7. Not all logicians are computer scientists. (Lx, Cx)

$$\neg\forall x\,(Lx \rightarrow Cx)$$

Often put as: $\exists x\,(Lx \wedge \neg Cx)$.

8. Some logicians are rich computer scientists. (Lx, Rx, Cx)

$$\exists x\,(Lx \wedge (Rx \wedge Cx))$$

9. If there are rich logicians, then some logicians are computer scientists. (Rx, Lx, Cx)

$$\exists x\,(Rx \wedge Lx) \rightarrow \exists y\,(Ly \wedge Cy)$$

10. No pets except service animals are permitted in dorms. (Px, Sx, Dx)

Can be read in a minimal way as:

$$\forall x\,((Px \wedge Dx) \rightarrow Sx),$$

which says only that no non–service pets are allowed in dorms. However, ordinary policy language is typically understood more strongly: among pets, *being permitted in the dorms* and *being a service animal* coincide. That reading is captured by:

$$\forall x\,(Px \rightarrow (Dx \leftrightarrow Sx)).$$

This biconditional formalization is therefore closer to the intended rule.

11. If anyone is rich, then Mary is. (Rx, m)

$$(\exists x\,Rx) \rightarrow Rm$$

Exercise 6.2

1. Mary loves everyone who loves her. (m, Lxy)

$$\forall x\,(Lxm \rightarrow Lmx)$$

2. Mary loves all and only those people who don't love themselves. (Lxy, m)

$$\forall x\,(Lmx \leftrightarrow \neg Lxx)$$

3. Everyone loves their mother. (Lxy, Mxy)

$$\forall x \forall y\,(Myx \to Lxy)$$

4. Some people love only those people who love their mother. (Lxy, Mxy)

$$\exists x \forall y (Lxy \to \forall z (Mzy \to Lyz))$$

5. Snape killed someone. (Kxy, s)

$$\exists x\, Ksx$$

6. Snape is a killer. (Kxy, s)

$$\exists x\, Ksx$$

7. Someone was killed by Snape. (Kxy, s)

$$\exists x\, Ksx$$

8. Some wizards only marry other wizards. (Wx, Mxy)

$$\exists x (Wx \wedge \forall y\,(Mxy \to Wy))$$

9. There is no greatest number. $(Nx,\ x < y)$

$$\forall x (Nx \to \exists y\,(Ny \wedge x < y))$$

10. c is the least upper bound of a and b. $(a, b, c,\ x \leq y)$

$$(a \leq c \wedge b \leq c)\ \wedge\ \forall x ((a \leq x \wedge b \leq x) \to c \leq x)$$

11. c is the greatest common divisor of a and b. $(a, b, c, Dxy,\ x \leq y)$

$$(Dca \wedge Dcb)\ \wedge\ \forall x ((Dxa \wedge Dxb) \to x \leq c)$$

Exercise 6.8

1. $\boxed{\neg \exists x (Fx \wedge Gx) \vdash \forall x (Fx \to \neg Gx)}$

1	(1)	$\neg \exists x (Fx \wedge Gx)$	A
2	(2)	Fa	A
3	(3)	Ga	A
2,3	(4)	$Fa \wedge Ga$	2,3 $\wedge$I
2,3	(5)	$\exists x (Fx \wedge Gx)$	4 EI
1,2,3	(6)	$\exists x (Fx \wedge Gx) \wedge \neg \exists x (Fx \wedge Gx)$	5,1 $\wedge$I
1,2	(7)	$\neg Ga$	3,6 RA
1	(8)	$Fa \to \neg Ga$	2,7 CP
1	(9)	$\forall x (Fx \to \neg Gx)$	8 UI

2. $\forall x Fx \vdash \exists x Fx$

1	(1)	$\forall x Fx$	A
1	(2)	Fa	1 UE
1	(3)	$\exists x Fx$	2 EI

3. $\forall x(Fx \to Gx), Fa \vdash \exists x Gx$

1	(1)	$\forall x(Fx \to Gx)$	A
2	(2)	Fa	A
1	(3)	$Fa \to Ga$	1 UE
1,2	(4)	Ga	3,2 MP
1,2	(5)	$\exists x Gx$	4 EI

4. $\neg Fa \vdash \exists x(Fx \to P)$

1	(1)	$\neg Fa$	A
1	(2)	$Fa \to P$	1 negative paradox
1	(3)	$\exists x(Fx \to P)$	2 EI

5. $\neg\forall x Fx \vdash \exists x(Fx \to P)$

1	(1)	$\neg\forall x Fx$	A
2	(2)	$\neg\exists x(Fx \to P)$	A
3	(3)	$Fa \to P$	A
3	(4)	$\exists x(Fx \to P)$	3 EI
2,3	(5)	$\exists x(Fx \to P) \land \neg\exists x(Fx \to P)$	4,2 $\land$I
2	(6)	$\neg(Fa \to P)$	3,5 RA
2	(7)	Fa	6 material conditional
2	(8)	$\forall x Fx$	7 UI
1,2	(9)	$\forall x Fx \land \neg\forall x Fx$	8,1 $\land$I
1	(10)	$\neg\neg\exists x(Fx \to P)$	2,9 RA
1	(11)	$\exists x(Fx \to P)$	10 DN

6. $\neg\exists x Fx \vdash \forall x(Fx \to Gx)$

1	(1)	$\neg\exists x Fx$	A
2	(2)	Fa	A
3	(3)	$\neg Ga$	A
2	(4)	$\exists x Fx$	2 EI
1,2	(5)	$\exists x Fx \land \neg\exists x Fx$	4,1 $\land$I
1,2	(6)	$\neg\neg Ga$	3,5 RA
1,2	(7)	Ga	6 DN
1	(8)	$Fa \to Ga$	2,7 CP
1	(9)	$\forall x(Fx \to Gx)$	8 UI

7. $\forall x \forall y Rxy \vdash \exists x Rxx$

1	(1)	$\forall x \forall y Rxy$	A
1	(2)	$\forall y Ray$	1 UE
1	(3)	Raa	2 UE
1	(4)	$\exists x Rxx$	3 EI

8. $P \to Fa \vdash P \to \exists x Fx$

1	(1)	$P \to Fa$	A
2	(2)	P	A
1,2	(3)	Fa	1,2 MP
1,2	(4)	$\exists x Fx$	3 EI
1	(5)	$P \to \exists x Fx$	2,4 CP

9. $\exists x Fx \to P \vdash \forall x(Fx \to P)$

1	(1)	$\exists x Fx \to P$	A
2	(2)	Fa	A
2	(3)	$\exists x Fx$	2 EI
1,2	(4)	P	1,3 MP
1	(5)	$Fa \to P$	2,4 CP
1	(6)	$\forall x(Fx \to P)$	5 UI

There is a typo here in the book: the direction $\forall x(Fx \to P) \vdash \exists x Fx \to P$ cannot be proven without EE, which is only introduced in the next section.

10. $\neg\exists x Fx \vdash \forall x(Fx \to P)$

1	(1)	$\neg\exists x Fx$	A
2	(2)	Fa	A
2	(3)	$\exists x Fx$	2 EI
1,2	(4)	$\exists x Fx \wedge \neg\exists x Fx$	3,1 $\wedge$I
1	(5)	$\neg Fa$	2,4 RA
1	(6)	$Fa \to P$	5 neg paradox
1	(7)	$\forall x(Fx \to P)$	6 UI

11. $\neg\exists x(Fx \to P) \vdash \forall x Fx \wedge \neg P$

1	(1)	$\neg\exists x(Fx \to P)$	A
2	(2)	$Fa \to P$	A
2	(3)	$\exists x(Fx \to P)$	2 EI
1,2	(4)	$\exists x(Fx \to P) \wedge \neg\exists x(Fx \to P)$	3,1 $\wedge$I
1	(5)	$\neg(Fa \to P)$	2,4 RA
1	(6)	$Fa \wedge \neg P$	5 material conditional
1	(7)	$\neg P$	6 $\wedge$E
1	(8)	Fa	6 $\wedge$E
1	(9)	$\forall x Fx$	8 UI
1	(10)	$\forall x Fx \wedge \neg P$	9,7 $\wedge$I

12. $\forall x Fx \to P \vdash \exists x(Fx \to P)$

1	(1)	$\forall x Fx \to P$	A
2	(2)	$\neg\exists x(Fx \to P)$	A
3	(3)	$\neg Fa$	A
3	(4)	$Fa \to P$	3 neg paradox
3	(5)	$\exists x(Fx \to P)$	4 EI
2,3	(6)	$\exists x(Fx \to P) \wedge \neg\exists x(Fx \to P)$	5,2 $\wedge$I
2	(7)	$\neg\neg Fa$	3,6 RA
2	(8)	Fa	7 DN
2	(9)	$\forall x Fx$	8 UI
1,2	(10)	P	1,9 MP
1,2	(11)	$Fb \to P$	10 pos paradox
1,2	(12)	$\exists x(Fx \to P)$	11 EI
1,2	(13)	$\exists x(Fx \to P) \wedge \neg\exists x(Fx \to P)$	12,2 $\wedge$I
1	(14)	$\neg\neg\exists x(Fx \to P)$	2,13 RA
1	(15)	$\exists x(Fx \to P)$	14 DN

Exercise 6.11

1. $\exists x Fx \vee \exists x Gx \vdash \exists x(Fx \vee Gx)$

1	(1)	$\exists x Fx \lor \exists x Gx$	A
2	(2)	$\exists x Fx$	A
3	(3)	Fa	A
3	(4)	$Fa \lor Ga$	3 $\lor$I
3	(5)	$\exists x(Fx \lor Gx)$	4 EI
2	(6)	$\exists x(Fx \lor Gx)$	2,3,5 EE
7	(7)	$\exists x Gx$	A
8	(8)	Ga	A
8	(9)	$Fa \lor Ga$	8 $\lor$I
8	(10)	$\exists x(Fx \lor Gx)$	9 EI
7	(11)	$\exists x(Fx \lor Gx)$	7,8,10 EE
1	(12)	$\exists x(Fx \lor Gx)$	1,2,6,7,11 $\lor$E

2. $\forall x(Fx \to Gx), \neg\exists x Gx \vdash \neg\exists x Fx$

1	(1)	$\forall x(Fx \to Gx)$	A
2	(2)	$\neg\exists x Gx$	A
3	(3)	$\exists x Fx$	A
4	(4)	Fa	A
1	(5)	$Fa \to Ga$	1 UE
1,4	(6)	Ga	5,4 MP
1,4	(7)	$\exists x Gx$	6 EI
1,3	(8)	$\exists x Gx$	3,4,7 EE
1,2,3	(9)	$\exists x Gx \land \neg\exists x Gx$	8,2 $\land$I
1,2	(10)	$\neg\exists x Fx$	3,9 RA

3. $\forall x(Fx \to Gx) \vdash \exists x\neg Gx \to \exists x\neg Fx$

1	(1)	$\forall x(Fx \to Gx)$	A
2	(2)	$\exists x\neg Gx$	A
3	(3)	$\neg Ga$	A
1	(4)	$Fa \to Ga$	1 UE
1,3	(5)	$\neg Fa$	4,3 MT
1,3	(6)	$\exists x\neg Fx$	5 EI
1,2	(7)	$\exists x\neg Fx$	2,3,6 EE
1	(8)	$\exists x\neg Gx \to \exists x\neg Fx$	2,7 CP

4. $\forall x(Fx \to P) \vdash \exists x Fx \to P$

1	(1)	$\forall x(Fx \rightarrow P)$	A
2	(2)	$\exists x Fx$	A
3	(3)	Fa	A
1	(4)	$Fa \rightarrow P$	1 UE
1,3	(5)	P	4,3 MP
1,2	(6)	P	2,3,5 EE
1	(7)	$\exists x Fx \rightarrow P$	2,6 CP

5. $P \wedge \exists x Fx \vdash \exists x(P \wedge Fx)$

1	(1)	$P \wedge \exists x Fx$	A
1	(2)	P	1 $\wedge$E
1	(3)	$\exists x Fx$	1 $\wedge$E
4	(4)	Fa	A
1,4	(5)	$P \wedge Fa$	2,4 $\wedge$I
1,4	(6)	$\exists x(P \wedge Fx)$	5 EI
1	(7)	$\exists x(P \wedge Fx)$	3,4,6 EE

6. $\exists x(Fx \rightarrow P) \vdash \forall x Fx \rightarrow P$

1	(1)	$\exists x(Fx \rightarrow P)$	A
2	(2)	$\forall x Fx$	A
3	(3)	$Fa \rightarrow P$	A
2	(4)	Fa	2 UE
2,3	(5)	P	3,4 MP
3	(6)	$\forall x Fx \rightarrow P$	2,5 CP
1	(7)	$\forall x Fx \rightarrow P$	1,3,6 EE

Exercise 6.13

1. $P \rightarrow \exists x Fx \vdash \exists x(P \rightarrow Fx)$

1	(1)	$P \rightarrow \exists x Fx$	A
$\varnothing$	(2)	$\exists x Fx \vee \neg\exists x Fx$	prop taut
3	(3)	$\exists x Fx$	A
4	(4)	Fa	A
4	(5)	$P \rightarrow Fa$	4 prop taut
4	(6)	$\exists x(P \rightarrow Fx)$	5 EI
3	(7)	$\exists x(P \rightarrow Fx)$	3,4,6 EE
8	(8)	$\neg\exists x Fx$	A
1,8	(9)	$\neg P$	1,8 MT
1,8	(10)	$P \rightarrow Fb$	9 prop taut
1,8	(11)	$\exists x(P \rightarrow Fx)$	10 EI
1	(12)	$\exists x(P \rightarrow Fx)$	2,3,7,8,11 $\vee$E

2. $\exists x(Fx \to P) \vdash \forall x Fx \to P$

1	(1)	$\exists x(Fx \to P)$	A
2	(2)	$\forall x Fx$	A
3	(3)	$Fa \to P$	A
2	(4)	Fa	2 UE
2,3	(5)	P	3,4 MP
1,2	(6)	P	1,3,5 EE
1	(7)	$\forall x Fx \to P$	2,6 CP

Exercise 6.14

1. $\vdash \forall x(Fx \to Fx)$

1	(1)	Fa	A
$\varnothing$	(2)	$Fa \to Fa$	1,1 CP
$\varnothing$	(3)	$\forall x(Fx \to Fx)$	2 UI

2. $\vdash \forall x Fx \vee \exists x \neg Fx$

$\varnothing$	(1)	$\neg\exists x\neg Fx \vee \exists x\neg Fx$	prop taut
2	(2)	$\neg\exists x\neg Fx$	A
3	(3)	$\neg Fa$	A
3	(4)	$\exists x\neg Fx$	3 EI
2,3	(5)	$\exists x\neg Fx \wedge \neg\exists x\neg Fx$	4,2 $\wedge$I
2	(6)	$\neg\neg Fa$	3,5 RA
2	(7)	Fa	6 DN
2	(8)	$\forall x Fx$	7 UI
2	(9)	$\forall x Fx \vee \exists x\neg Fx$	8 $\vee$I
10	(10)	$\exists x\neg Fx$	A
10	(11)	$\forall x Fx \vee \exists x\neg Fx$	10 $\vee$I
$\varnothing$	(12)	$\forall x Fx \vee \exists x\neg Fx$	1,2,9,10,11 $\vee$E

3. $\vdash \forall x\, \neg(Fx \wedge \neg Fx)$

1	(1)	$Fa \wedge \neg Fa$	A
$\varnothing$	(2)	$\neg(Fa \wedge \neg Fa)$	1,1 RA
$\varnothing$	(3)	$\forall x\, \neg(Fx \wedge \neg Fx)$	2 UI

4. $\vdash \neg\exists x(Fx \wedge \neg Fx)$

1	(1)	$\exists x(Fx \wedge \neg Fx)$	A
2	(2)	$Fa \wedge \neg Fa$	A
2	(3)	$\neg\exists x(Fx \wedge \neg Fx)$	1,2 RA
1	(4)	$\neg\exists x(Fx \wedge \neg Fx)$	1,2,3 EE
1	(5)	$\exists x(Fx \wedge \neg Fx) \wedge \neg\exists x(Fx \wedge \neg Fx)$	1,3 $\wedge$I
$\varnothing$	(6)	$\neg\exists x(Fx \wedge \neg Fx)$	1,5 RA

5. $\vdash \forall x \exists y(Rxy \to Rxx)$

1	(1)	Raa	A
$\varnothing$	(2)	$Raa \to Raa$	1,1 CP
$\varnothing$	(3)	$\exists y(Ray \to Raa)$	2 EI
$\varnothing$	(4)	$\forall x \exists y(Rxy \to Rxx)$	3 UI

6. $\vdash \forall x \exists y(Rxy \to Ryx)$

1	(1)	Raa	A
$\varnothing$	(2)	$Raa \to Raa$	1,1 CP
$\varnothing$	(3)	$\exists y(Ray \to Rya)$	2 EI
$\varnothing$	(4)	$\forall x \exists y(Rxy \to Ryx)$	3 UI

7. $\vdash \exists x(Fx \to \forall y Fy)$

1	(1)	$\neg\exists x(Fx \to \forall y Fy)$	A
2	(2)	$\neg Fa$	A
2	(3)	$Fa \to \forall y Fy$	2 prop taut
2	(4)	$\exists x(Fx \to \forall y Fy)$	3 EI
1,2	(5)	$\exists x(Fx \to \forall y Fy) \wedge \neg\exists x(Fx \to \forall y Fy)$	4,1 $\wedge$I
1	(6)	$\neg\neg Fa$	2,5 RA
1	(7)	Fa	6 DN
1	(8)	$\forall y Fy$	7 UI
1	(9)	$Fa \to \forall y Fy$	8 prop taut
1	(10)	$\exists x(Fx \to \forall y Fy)$	9 EI
1	(11)	$\exists x(Fx \to \forall y Fy) \wedge \neg\exists x(Fx \to \forall y Fy)$	10,1 $\wedge$I
$\varnothing$	(12)	$\neg\neg\exists x(Fx \to \forall y Fy)$	1,11 RA
$\varnothing$	(13)	$\exists x(Fx \to \forall y Fy)$	12 DN

8. $\vdash \exists x \forall y(Fx \to Fy)$

1	(1)	$\neg\exists x\forall y(Fx \to Fy)$	A
2	(2)	$\neg Fa$	A
2	(3)	$Fa \to Fb$	2 prop taut
2	(4)	$\forall y(Fa \to Fy)$	3 UI
2	(5)	$\exists x\forall y(Fx \to Fy)$	4 EI
1,2	(6)	$\exists x\forall y(Fx \to Fy) \wedge \neg\exists x\forall y(Fx \to Fy)$	5,1 $\wedge$I
1	(7)	$\neg\neg Fa$	2,6 RA
1	(8)	Fa	7 DN
1	(9)	$Fc \to Fa$	8 prop taut
1	(10)	$\forall y(Fc \to Fy)$	9 UI
1	(11)	$\exists x\forall y(Fx \to Fy)$	10 EI
1	(12)	$\exists x\forall y(Fx \to Fy) \wedge \neg\exists x\forall y(Fx \to Fy)$	11,1 $\wedge$I
$\varnothing$	(13)	$\neg\neg\exists x\forall y(Fx \to Fy)$	1,12 RA
$\varnothing$	(14)	$\exists x\forall y(Fx \to Fy)$	13 DN

9. $\forall x\exists y(Fx \to Gy) \vdash \exists y\forall x(Fx \to Gy)$

1	(1)	$\forall x\exists y(Fx \to Gy)$	A
$\varnothing$	(2)	$\exists yGy \vee \neg\exists yGy$	prop taut
3	(3)	$\exists yGy$	A
4	(4)	Ga	A
4	(5)	$Fb \to Ga$	4 prop taut
4	(6)	$\forall x(Fx \to Ga)$	5 UI
4	(7)	$\exists y\forall x(Fx \to Gy)$	6 EI
3	(8)	$\exists y\forall x(Fx \to Gy)$	3,4,7 EE
9	(9)	$\neg\exists yGy$	A
10	(10)	Fc	A
1	(11)	$\exists y(Fc \to Gy)$	1 UE
12	(12)	$Fc \to Gd$	A
10,12	(13)	Gd	12,10 MP
10,12	(14)	$\exists yGy$	13 EI
9,10,12	(15)	$\exists yGy \wedge \neg\exists yGy$	14,9 $\wedge$I
9,12	(16)	$\neg Fc$	10,15 RA
9,12	(17)	$Fc \to Ge$	16 prop taut
1,9	(18)	$Fc \to Ge$	11,12,17 EE
1,9	(19)	$\forall x(Fx \to Ge)$	18 UI
1,9	(20)	$\exists y\forall x(Fx \to Gy)$	19 EI
1	(21)	$\exists y\forall x(Fx \to Gy)$	2,3,8,9,20 $\vee$E

10. $\vdash \forall x\exists y(Rxy \to \forall zRxz)$

$\varnothing$	(1)	$\exists y \neg Ray \vee \neg \exists y \neg Ray$	prop taut
2	(2)	$\exists y \neg Ray$	A
3	(3)	$\neg Rab$	A
3	(4)	$Rab \to \forall z Raz$	3 prop taut
3	(5)	$\exists y(Ray \to \forall z Raz)$	4 EI
2	(6)	$\exists y(Ray \to \forall z Raz)$	2,3,5 EE
7	(7)	$\neg \exists y \neg Ray$	A
8	(8)	$\neg Rac$	A
8	(9)	$\exists y \neg Ray$	8 EI
7,8	(10)	$\exists y \neg Ray \wedge \neg \exists y \neg Ray$	9,7 $\wedge$I
7	(11)	$\neg\neg Rac$	8,10 RA
7	(12)	Rac	11 DN
7	(13)	$\forall z Raz$	12 UI
7	(14)	$Rab \to \forall z Raz$	13 prop taut
7	(15)	$\exists y(Ray \to \forall z Raz)$	14 EI
$\varnothing$	(16)	$\exists y(Ray \to \forall z Raz)$	1,2,6,7,15 $\vee$E
$\varnothing$	(17)	$\forall x \exists y(Rxy \to \forall z Rxz)$	16 UI

Exercise 6.17

$\boxed{\forall x(\exists z Rxz \to \forall y Rxy), \exists x \exists y \;\vdash\; \exists x \forall y Rxy}$

1	(1)	$\forall x(\exists z Rxz \to \forall y Rxy)$	A
2	(2)	$\exists x \exists y Rxy$	A
3	(3)	$\exists y Ray$	A
4	(4)	Rab	A
4	(5)	$\exists z Raz$	4 EI
1	(6)	$\exists z Raz \to \forall y Ray$	1 UE
1,4	(7)	$\forall y Ray$	6,5 MP
1,4	(8)	$\exists x \forall y Rxy$	7 EI
1,3	(9)	$\exists x \forall y Rxy$	3,4,8 EE
1,2	(10)	$\exists x \forall y Rxy$	2,3,9 EE

Question: Does it follow from these premises that $\forall x \forall y Rxy$?

Answer: No. $\circlearrowleft a \longrightarrow b$

Chapter 7

Exercise 7.1

Here the proof is lengthened because of the strictness of the = rules. From $a = c$ and $b = c$, we cannot immediately apply =E to get $a = b$.

1	(1)	$\exists x \forall y(Py \to y = x)$	A
2	(2)	$Pa \wedge Pb$	A
3	(3)	$\forall y(Py \to y = c)$	A
3	(4)	$Pa \to a = c$	3 UE
3	(5)	$Pb \to b = c$	3 UE
2	(6)	Pa	2 $\wedge$E
2	(7)	Pb	2 $\wedge$E
2,3	(8)	$a = c$	4,6 MP
2,3	(9)	$b = c$	5,7 MP
$\varnothing$	(10)	$b = b$	=I
2,3	(11)	$c = b$	10,9 =E
2,3	(12)	$a = b$	8,11 =E
1,2	(13)	$a = b$	1,3,12 EE
1	(14)	$(Pa \wedge Pb) \to a = b$	2,13 CP
1	(15)	$\forall y((Pa \wedge Py) \to a = y)$	14 UI
1	(16)	$\forall x \forall y((Px \wedge Py) \to x = y)$	15 UI

Exercise 7.2

1	(1)	$Fa \wedge \forall x(Fx \to x = a)$	A
2	(2)	Fb	A
1	(3)	$\forall x(Fx \to x = a)$	1 $\wedge$E
1	(4)	$Fb \to b = a$	3 UE
1,2	(5)	$b = a$	4,2 MP
6	(6)	$b = a$	A
1	(7)	Fa	1 $\wedge$E
$\varnothing$	(8)	$b = b$	=I
6	(9)	$a = b$	8,6 =E
1,6	(10)	Fb	7,9 =E
1	(11)	$Fb \leftrightarrow b = a$	2,5,6,10 CP$\times$2
1	(12)	$\forall x(Fx \leftrightarrow x = a)$	11 UI

1	(1)	$\forall x(Fx \leftrightarrow x = a)$	A
1	(2)	$Fa \leftrightarrow a = a$	1 UE
$\varnothing$	(3)	$a = a$	=I
1	(4)	Fa	2,3 MP
1	(5)	$Fb \leftrightarrow b = a$	1 UE
1	(6)	$Fb \to b = a$	5 $\wedge$E
1	(7)	$\forall x(Fx \to x = a)$	6 UI
1	(8)	$Fa \wedge \forall x(Fx \to x = a)$	4,7 $\wedge$I

Chapter 8

Exercise 8.1

1. Countermodel M_1:

$$D_1 = \{a, b\}, \qquad F^{M_1} = \{b\}, \qquad c^{M_1} = a.$$

   Then $M_1 \vDash \exists x Fx$ (witness b), but $M_1 \nvDash Fc$.

2. Countermodel M_2:

$$D_2 = \{a, b\}, \qquad F^{M_2} = \{a\}, \qquad c^{M_2} = a.$$

   Then $M_2 \vDash Fc$, but $M_2 \nvDash \forall x Fx$ (since $b \notin F^{M_2}$).

3. Countermodel M_3:

$$D_3 = \{a, b\}, \qquad F^{M_3} = \{a\}, \qquad G^{M_3} = \{b\}.$$

   Then $M_3 \vDash \exists x Fx \wedge \exists x Gx$, but $M_3 \nvDash \exists x(Fx \wedge Gx)$.

4. Countermodel M_4:

$$D_4 = \{a, b\}, \qquad F^{M_4} = \{a\}, \qquad G^{M_4} = \{b\}.$$

   We have $\forall x Fx$ false and $\forall x Gx$ false, so $M_4 \vDash (\forall x Fx \to \forall x Gx)$ (false $\to$ false is true), but for $x = a$ we get $Fa \wedge \neg Ga$, so $M_4 \nvDash \forall x(Fx \to Gx)$.

5. Countermodel M_5:

$$D_5 = \{a\}, \qquad F^{M_5} = \varnothing, \qquad H^{M_5} = \{a\}.$$

   Then for the only element a, Fa is false, so $Fa \to Ha$ is true; hence $M_5 \vDash \forall x(Fx \to Hx)$. But $M_5 \nvDash \exists x Fx$ and $M_5 \vDash \exists x Hx$, so $M_5 \nvDash \exists x Fx \vee \neg\exists x Hx$.

6. Countermodel M_6:

$$D_6 = \{a\}, \qquad F^{M_6} = \varnothing, \qquad G^{M_6} = \varnothing.$$

Again Fa is false, so $Fa \to Ga$ is true; hence $M_6 \models \forall x(Fx \to Gx)$. But $F^{M_6} \cap G^{M_6} = \varnothing$, so $M_6 \not\models \exists x(Fx \wedge Gx)$.

7. Countermodel M_7:

$$D_7 = \{a, b\}, \qquad F^{M_7} = \{a\}, \qquad G^{M_7} = \{a, b\}, \qquad H^{M_7} = \{b\}.$$

Then a witnesses $\exists x(Fx \wedge Gx)$, and b witnesses $\exists x(Gx \wedge Hx)$. But there is no element in $F^{M_7} \cap H^{M_7}$, so $M_7 \not\models \exists x(Fx \wedge Hx)$.

8. Countermodel M_8:

$$D_8 = \{a, b\}, \qquad F^{M_8} = \{a\}.$$

Then $M_8 \not\models \forall x Fx$ (since $b \notin F^{M_8}$), and $M_8 \not\models \forall x \neg Fx$ (since $a \in F^{M_8}$). Hence $M_8 \not\models \forall x Fx \vee \forall x \neg Fx$.

9. Countermodel M_9:

$$D_9 = \{a, b\}, \qquad F^{M_9} = \{a, b\}, \qquad G^{M_9} = \{a\}, \qquad H^{M_9} = \varnothing.$$

Then:

$$\begin{aligned} &M_9 \models \exists x(Fx \to Gx) && \text{(take } x = a,\ Fa \to Ga \text{ is true)},\\ &M_9 \models \exists x(Gx \to Hx) && \text{(take } x = b,\ Gb \text{ is false so } Gb \to Hb \text{ is true)},\\ &M_9 \not\models \exists x(Fx \to Hx) && \text{since for both } a, b,\ Fx \text{ is true and } Hx \text{ is false,}\\ & && \text{so } Fx \to Hx \text{ is false everywhere.} \end{aligned}$$

10. Countermodel M_{10}:

$$D_{10} = \{a, b\}, \qquad F^{M_{10}} = \{a\}, \qquad G^{M_{10}} = \varnothing.$$

Then $M_{10} \models \exists x(Fx \to Gx)$ (take $x = b$, where Fb is false), while $\exists x Fx$ is true (witness a) and $\exists x Gx$ is false. Hence $M_{10} \not\models \exists x Fx \to \exists x Gx$.

Exercise 8.3

1. $\forall x Fx \to P \not\vdash \forall x(Fx \to P)$.

Countermodel M_1:

$$D = \{a, b\}, \qquad F^M = \{a\}, \qquad P^M = 0.$$

Then $\forall x Fx$ is false (since $b \notin F^M$), so

$$(\forall x Fx \to P)^M = (0 \to 0) = 1.$$

However,

$$(Fb \to P)^M = (0 \to 0) = 1, \qquad (Fa \to P)^M = (1 \to 0) = 0,$$

so $\forall x(Fx \to P)$ is false. Thus the premise is true and the conclusion false in M_1.

2. $\exists x(Fx \to P) \nvdash \exists x Fx \to P$.

   Countermodel M_2:

   $$D = \{a\}, \qquad F^M = \varnothing, \qquad P^M = 0.$$

   Then

   $$(Fa \to P)^M = (0 \to 0) = 1,$$

   so $\exists x(Fx \to P)$ is true. But $\exists x Fx$ is false and P is false, so

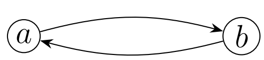
   $$(\exists x Fx \to P)^M = (0 \to 0) = 0.$$

   Hence the premise is true while the conclusion is false in M_2.

Exercise 8.7

1. $\forall x \forall y(Rxy \to Ryx)$

   True model (symmetric):

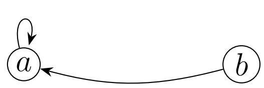
   

   False model (one-way arrow):

2. $\forall x \forall y \exists z(Rxz \wedge Ryz)$

   True model (common successor a for everyone):

   (For any x, y, choose $z = a$.)

   False model (no common successor for a, b):

   (For $x = a, y = b$ there is no z with both $a \to z$ and $b \to z$.)

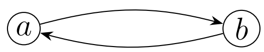
3. $\exists x \forall y (Ryx \to Ryy)$

   True model (choose $x = a$ with no incoming arrows):

   (No y satisfies $y \to a$, so $Ryx \to Ryy$ holds vacuously for all y.)

   False model (every x has an incoming arrow from a non-reflexive y):

   (No loops, so Ryy is always false; but each node has an incoming arrow.)

4. $\forall x (\exists y\, Ryx \to \forall z\, Rzx)$

   True model (empty relation):

   (Each antecedent $\exists y\, Ryx$ is false, so the implication is true for all x.)

   False model (some x has an incoming arrow but not everyone points to x):

   (Take $x = b$: $\exists y\, Ryb$ holds (witness a), but $\forall z\, Rzb$ fails since $b \not\to b$.)

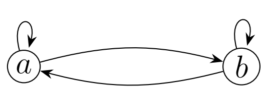
5. $\exists x \exists y (Rxy \leftrightarrow \neg Ryy)$

   True model (take $x = a, y = b$):

   (Here $a \to b$ is true and $b \to b$ is false, so $\neg Rbb$ is true and the biconditional holds.)

   False model (universal relation on $\{a, b\}$):

   (For every y, Ryy is true, hence $\neg Ryy$ is false; but Rxy is always true. So $Rxy \leftrightarrow \neg Ryy$ is false for all x, y.)

Chapter 9

Exercise 9.6

Suppose that φ is true in an even number n of rows of its truth table. Then $\neg\varphi$ is true in $4-n$ rows of its truth table, and $4-n$ is also even.

Suppose that both φ and ψ are even. Let's say that row r is an *agreement row* if φ and ψ have the same truth value on r. We will show that there cannot be 1 or 3 agreement rows. Suppose that there is a single row where both sentences have value a. Since φ and ψ are even, a must occur on another row in each of their truth tables. If these rows are not the same, then there are two of them, which leaves a single remaining row. In that row, both φ and ψ must have value $1-a$, and so they agree there.

Suppose now that there are three rows where both sentences have the same value, and let r be the remaining row. Since three is odd, one of the two truth values a must occur most frequently on these rows. If a occurs twice and $1-a$ occurs once, then $1-a$ must be the value of both φ and ψ on row r. If a occurs three times, then a must be the value of both φ and ψ on row r. In either case, φ and ψ agree on row r.

Exercise 9.7

No, the set $\{\neg,\leftrightarrow\}$ is not truth-functionally complete. There is a binary truth-function that has output a single 1 and three 0. For example, take the sentence $P\wedge Q$. By Exercise 9.6, every sentence in the set Γ generated from P,Q and $\{\neg,\leftrightarrow\}$ has an even number of 1 in its truth table. Therefore, there is no sentence in Γ that is provably equivalent to $P\wedge Q$.

Exercise 9.12

Suppose that φ is contingent, and let $P_0,\dots,P_n$ be a list of the atomic sentences that occur in φ. Since φ is contingent, there is a valuation v such that $v(\varphi)=0$. Let $\perp$ be an arbitrary contradiction, and let $\top$ be an arbitrary tautology. Define $F(P_i)=\top$ if $v(P_i)=1$, and $F(P_i)=\perp$ if $v(P_i)=0$. We claim, then, that the substitution instance $F(\varphi)$ is inconsistent. Let w be an arbitrary valuation. For any P_i, $w(F(P_i))=w(\top)=1$ if $v(P_i)=1$, and $w(F(P_i))=w(\perp)=0$ if $v(P_i)=0$. So $w(F(\cdot))$ and $v(\cdot)$ agree on atomic sentences. But $w(F(\cdot))$ and $v(\cdot)$ are both truth-functional, so they agree on all sentences. Therefore, $w(F(\varphi))=v(\varphi)=0$. Since w was arbitrary, $F(\varphi)$ is an inconsistency.

Exercise 9.14

1. As a warmup, we will show that all occurences of $\to$ can be eliminated from valid proofs, along with all uses of MP and CP. Define a function f from sentences to sentences as the identity on atomic sentences, then extend by commuting with $\wedge,\vee,\neg$, and by setting $f(\varphi\to\psi)=\neg(f(\varphi)\wedge\neg f(\psi))$. We now show that any proof of $\varphi_1,\dots,\varphi_n\vdash\psi$ can be converted to a proof of $f(\varphi_1),\dots,f(\varphi_n)\succ f(\psi)$.

Here's a way to simulate CP. Suppose first that φ is assumed, and that ψ is derived with dependencies Δ. We can then continue in this way:

1	(1)	φ	A
Δ	(2)	ψ	
3	(3)	$\varphi \wedge \neg\psi$	A
3	(4)	$\neg\psi$	3 $\wedge$E
$\Delta, 3$	(5)	$\psi \wedge \neg\psi$	2, 4 $\wedge$I
$\Delta', 3$	(6)	$\neg\varphi$	1, 5 RA
3	(7)	φ	3 $\wedge$E
$\Delta', 3$	(8)	$\varphi \wedge \neg\varphi$	7, 6 $\wedge$I
Δ'	(9)	$\neg(\varphi \wedge \neg\psi)$	3, 8 RA

Here $\Delta' = \Delta \setminus \{1\}$, so that line 9 reproduces the effect of CP on lines 1 and 2.

Now we can simulate MP.

Γ	(1)	$\neg(\varphi \wedge \neg\psi)$	
Δ	(2)	φ	
3	(3)	$\neg\psi$	A
$\Delta, 3$	(4)	$\varphi \wedge \neg\psi$	2, 3 $\wedge$I
$\Gamma, \Delta, 3$	(5)	$(\varphi \wedge \neg\psi) \wedge \neg(\varphi \wedge \neg\psi)$	4, 1 $\wedge$I
Γ, Δ	(6)	$\neg\neg\psi$	3, 5 RA
Γ, Δ	(7)	ψ	6 DN

2. We need to show that any application of RA can be simulated by the other rules. Suppose that we have the following lines

1	(1)	P	A
Δ	(2)	$Q \wedge \neg Q$	

We need to show that we can derive the line

Δ'	(c)	$\neg P$

without using RA. We first derive $\Delta' \succ P \to \neg P$ as follows:

1	(1)	P	A
Δ	(2)	$Q \wedge \neg Q$	
Δ	(3)	Q	2 $\wedge$E
Δ'	(4)	$P \to Q$	1,3 CP
Δ	(5)	$\neg Q$	2 $\wedge$E
Δ	(6)	$\neg P$	4,5 MT
Δ'	(7)	$P \to \neg P$	1,6 CP

The proof that $\succ (P \to \neg P) \to \neg P$ is Exercise 3.1.10, plus one step of CP. Put those two together and $\Delta' \succ \neg P$ follows.

3. We show that the DN introduction rule can be reproduced from the other rules.

1	(1)	P	A
2	(2)	$\neg P$	A
1,2	(3)	$P \wedge \neg P$	1,2 $\wedge$I
1	(4)	$\neg\neg P$	2,3 RA

4. We show that MT can be reproduced from the other rules.

1	(1)	$P \to Q$	A
2	(2)	$\neg Q$	A
3	(3)	P	A
1,3	(4)	Q	1,3 MP
1,2,3	(5)	$Q \wedge \neg Q$	4,2 $\wedge$I
1,2	(6)	$\neg P$	3,5 RA

5. Redefine the truth-table for $\vee$ as follows:

P	Q	$P \vee Q$
1	1	1
1	0	1
0	1	1
0	0	1

In other words, $P \vee Q$ is constantly 1, regardless of the input. Since none of the inference rules besides $\vee$E uses a disjunction as a premise, those rules are truth-preserving relative to the new truth tables. We claim now that those rules cannot prove $P \vee P \vdash P$. Consider the rows of the (new) truth-table in which P is 0. In this case, $P \vee P$ is 1, but P is 0. Hence $P \vee P \vdash P$ is not truth-preserving relative to the new truth tables, and it cannot be proven by those rules.

6. As stated, this problem is trivially easy to solve: if we never permit anything to be inferred from a "nand" statement, and if we never permit a "nand" statement to be inferred, then our system of rules is *sound*. However, the intention of this problem is to provide intro and elim rules for $\uparrow$ that are not only sound, but also potentially complete.

NAND-Introduction ($\uparrow$I)

If Δ together with P and Q imply $\perp$, then Δ implies $P \uparrow Q$.

a	(a)	P	A
b	(b)	Q	A
Δ	(c)	$\perp$	
Δ'	(d)	$P \uparrow Q$	a, b, c $\uparrow$I

where $\Delta' = \Delta - \{a, b\}$.

NAND-Elimination (↑E)

From $P \uparrow Q$, together with P and Q, infer $\bot$.

Γ	(a)	$P \uparrow Q$	
Δ	(b)	P	
Σ	(c)	Q	
Γ, Δ, Σ	(d)	$\bot$	a, b, c ↑E

Falsum-Elimination (⊥E)

For the ↑ rules to do enough, we need to add a ⊥-elimination rule.

Γ	(a)	$\bot$	
Γ	(b)	Q	a ⊥E