# Invitation to Predicate Logic

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# Overview

1. Motivation: Propositional logic cannot see all logical relations
2. A more fine-grained grammar
   - Names and predicates
   - Variables and quantifiers
3. Translation
4. Inference rules
   - $\forall$ elimination
   - $\forall$ introduction

# Propositional logic is inadequate

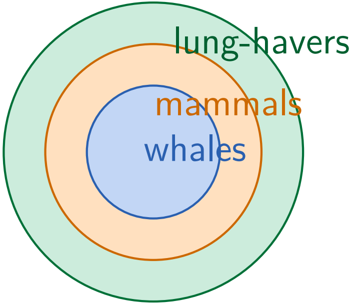
# Validities that escape propositional logic

- All people are mortal.
- Socrates is a person.
- Therefore, Socrates is mortal.

If the subject and predicate of sentences are not both the same, then propositional logic does not recognize any relation between them.

# Validities that escape propositional logic

- All **whales** are **mammals**.
- All **mammals** have **lungs**.
- Therefore: All **whales** have **lungs**.




$$\text{All}(W, M), \text{All}(M, L) \vdash \text{All}(W, L)$$

# When propositional logic falls short

## Example from mathematics

If a number is even, its square is even. $4$ is even. $\therefore 4^2$ is even.

**Propositional view:**
$P, P \to Q \vdash Q$

**Mathematical structure:**
$\forall n\,(E(n) \to E(n^2)), E(4) \vdash E(4^2)$

# Diagnosis

- The inadequacy of propositional logic cannot be fixed by adding more inference rules.
  - If we add any additional rules, then our system would become inconsistent.
- Have we missed some propositional connectives?
  - No, there is a precise sense in which our set of connectives is **conceptually complete**.

# The predicate calculus

- In the early 20th century, the missing logical structure was identified, represented symbolically, and codified in a "calculus".

# Sub-propositional grammar

# Names and predicates

Alice is French.

Bernard is French.

Alice is German.

# Quantified sentences

- You are familiar with the concept of a variable from mathematics.
- Natural languages do not explicitly use variables.
- Hypothesis: “All” and “Some” sentences are best analyzed as consisting of predicate symbols, variables, and quantifiers.

# Variables

| | |
|---|---|
| Alice is French. | $Fa$ |
| $x$ is French. | $Fx$ |
| Someone is French. | ? |
| There is an $x$ such that $x$ is French. | $\exists x Fx$ |

# Formulas

- We don't call "$Fx$" a proposition, since it cannot be true or false.
- We call "$Fx$" a **formula**.
- Adding the quantifier "$\exists x$" to "$Fx$" creates a sentence.

# Universal quantifier

| | |
|---|---|
| All whales are mammals. | ? |
| If $a$ is a whale then $a$ is a mammal. | $Wa \to Ma$ |
| For any $x$, if $x$ is a whale then $x$ is a mammal. | $\forall x(Wx \to Mx)$ |

# Standard syllogistic forms

| | |
|---|---|
| **All Finns are gregarious.** | $\forall x\,(Fx \rightarrow Gx)$ |
| **Some Finns are gregarious.** | $\exists x\,(Fx \wedge Gx)$ |
| **No Finns are gregarious.** | $\forall x\,(Fx \rightarrow \neg Gx)$ |
| **Some Finns are not gregarious.** | $\exists x\,(Fx \wedge \neg Gx)$ |

All happy Finns are gregarious.

All Finns and Germans are happy.

No dogs or cats are permitted in the restaurant.

$\forall x(Fx \to P)$

Everything has the feature that if it is $F$, then $P$ holds.

$\forall x Fx \to P$

If everything has the feature $F$, then $P$ holds.

# Relations

Maren is taller than Niels.

Maren is taller than someone.

Someone is taller than Niels.

Everyone is taller than someone.

Someone is taller than everyone.

There is a student who admires every professor.

$$\exists x(Sx \land \forall y(Py \to Axy))$$

There is a professor whom every student admires.

$$\exists x(Px \land \forall y(Sy \to Ayx))$$

Every student admires some professor.

$$\forall x(Sx \to \exists y(Py \land Axy))$$

# Inference to/from quantified statements

# $\forall$ elimination

The idea behind $\forall$ elimination is straightforward:

From a universal statement, any **instance** follows logically.

$$\frac{\forall x\, \varphi(x)}{\varphi(a)}$$

# $\forall$ elimination

$$\forall x(Fx \rightarrow Gx), Fa \vdash \; Ga$$

| | | | |
|---|---|---|---|
| 1 | (1) | $\forall x(Fx \rightarrow Gx)$ | A |
| 2 | (2) | $Fa$ | A |
| 1 | (3) | $Fa \rightarrow Ga$ | 1 UE |
| 1,2 | (4) | $Ga$ | 3,2 MP |

$$\forall x \forall y(Fx \wedge Gy) \vdash Fa \wedge Gb$$

| | | | |
|---|---|---|---|
| 1 | (1) | $\forall x \forall y(Fx \wedge Gy)$ | A |
| 1 | (2) | $\forall y(Fa \wedge Gy)$ | 1 UE |
| 1 | (3) | $Fa \wedge Gb$ | 2 UE |

$$\forall x \forall y(Fx \wedge Gy) \vdash Fa \wedge Ga$$

| | | | |
|---|---|---|---|
| 1 | (1) | $\forall x \forall y(Fx \wedge Gy)$ | A |
| 1 | (2) | $\forall y(Fa \wedge Gy)$ | 1 UE |
| 1 | (3) | $Fa \wedge Ga$ | 2 UE |

$P \to \forall x Fx \vdash P \to Fa$

| | | | |
|---|---|---|---|
| 1 | (1) | $P \to \forall x Fx$ | A |
| 2 | (2) | $P$ | A |
| 1,2 | (3) | $\forall x Fx$ | 1,2 MP |
| 1,2 | (4) | $Fa$ | 3 UE |
| 1 | (5) | $P \to Fa$ | 2,4 CP |

$\neg Fa \vdash \neg\forall x Fx$

| | | | |
|---|---|---|---|
| 1 | (1) | $\neg Fa$ | A |
| 2 | (2) | $\forall x Fx$ | A |
| 2 | (3) | $Fa$ | 2 UE |
| 1,2 | (4) | $Fa \wedge \neg Fa$ | 3,1 $\wedge$I |
| 1 | (5) | $\neg\forall x Fx$ | 2,4 RA |

# Warnings

Only apply UE when the entire sentence on the line is universally quantified.

$\forall x(Fx \rightarrow P)$
$\forall x(Fx \rightarrow \forall yGy)$
$\forall xFx \rightarrow Ga$
$\forall x\forall yRxy$

# Warnings

When applying UE, replace all instances of the relevant variable with the same name.

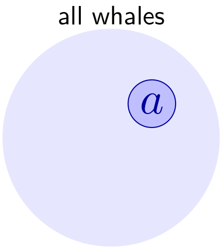
$\forall x(Fx \to \forall y Rxy)$

# From one individual to everyone

**Intuitive idea**

To show that **everyone** has a property, we can reason about **one individual chosen at random**.

- Suppose we want to prove that all whales have lungs.
- We pick a whale—call it $a$.
- We reason about $a$ as if it were any whale.



# What does it mean for $a$ to be *arbitrary*?

## Arbitrary name

The name $a$ is **arbitrary** when nothing in the proof depends on any *special feature* of $a$.

- Our reasoning about $a$ must not rely on facts like "$a$ lives in the Pacific" or "$a$ is the largest whale."
- The argument must hold no matter which whale we picked.

*An arbitrary name stands for an individual we reason about generally*

# From arbitrariness to universal generalization

## Bridge to Universal Introduction

If we can prove $\varphi(a)$ using $a$ as an **arbitrary name**, then we may infer the general statement $\forall x\, \varphi(x)$.

$$\frac{\varphi(a)}{\forall x\, \varphi(x)}\text{UI} \qquad \text{(side condition: } a \text{ not free in any open assumption}$$

- The conclusion applies to *all* objects of that kind.
- The key is that $a$ never referred to anything special.

# Universal introduction

From a line

$\Gamma$ (m) $\varphi(a)$

we may infer

$\Gamma$ (n) $\forall x\, \varphi(x)$

provided that the name "$a$" does not occur in any of the sentences listed in $\Gamma$ or in $\varphi(x)$.

# $\forall$ introduction

$$\forall x(Fx \rightarrow Gx), \forall x Fx \vdash \forall x Gx$$

| | | | |
|---|---|---|---|
| 1 | (1) | $\forall x(Fx \rightarrow Gx)$ | A |
| 2 | (2) | $\forall x Fx$ | A |
| 2 | (3) | $Fa$ | 2 UE |
| 1 | (4) | $Fa \rightarrow Ga$ | 1 UE |
| 1,2 | (5) | $Ga$ | 4,3 MP |
| 1,2 | (6) | $\forall x Gx$ | 5 UI |

$\vdash \; \forall x(Fx \to (Fx \vee Gx))$

| | | | |
|---|---|---|---|
| 1 | (1) | $Fa$ | A |
| 1 | (2) | $Fa \vee Ga$ | 1 $\vee$I |
| $\emptyset$ | (3) | $Fa \to (Fa \vee Ga)$ | 1,2 CP |
| $\emptyset$ | (4) | $\forall x(Fx \to (Fx \vee Gx))$ | 3 UI |

$\forall x(P \to Fx) \vdash P \to \forall x Fx$

| | | | |
|---|---|---|---|
| 1 | (1) | $\forall x(P \to Fx)$ | A |
| 2 | (2) | $P$ | A |
| 1 | (3) | $P \to Fa$ | 1 UE |
| 1,2 | (4) | $Fa$ | 3,2 MP |
| 1,2 | (5) | $\forall x Fx$ | 4 UI |
| 1 | (6) | $P \to \forall x Fx$ | 2,5 CP |

$$P \to \forall x Fx \vdash \forall x(P \to Fx)$$

| | | | |
|---|---|---|---|
| 1 | (1) | $P \to \forall x Fx$ | A |
| 2 | (2) | $P$ | A |
| 1,2 | (3) | $\forall x Fx$ | 1,2 MP |
| 1,2 | (4) | $Fa$ | 3 UE |
| 1 | (5) | $P \to Fa$ | 2,4 CP |
| 1 | (6) | $\forall x(P \to Fx)$ | 5 UI |

# Precisifying the UI rule

$\forall$I requires replacing **all** instances of the arbitrary name.

| | | | | |
|---|---|---|---|---|
| 1 | (1) | $\forall x Rxx$ | A | |
| 1 | (2) | $Raa$ | 1 UE | |
| 1 | (3) | $\forall x Rxa$ | 2 UI | error! |
| 1 | (4) | $\forall y \forall x Rxy$ | 3 UI | |

# Precisifying the UE rule

But UE does allow instantiating to a name that already occurs in the formula.

| | | | |
|---|---|---|---|
| 1 | (1) | $\forall x \forall y Rxy$ | A |
| 1 | (2) | $\forall y Ray$ | 1 UE |
| 1 | (3) | $Raa$ | 2 UE |
| 1 | (4) | $\forall x Rxx$ | 3 UI |

# Precisifying the UE rule

UE allows us to choose any name — same or different from what already occurs.

| | | | |
|---|---|---|---|
| 1 | (1) | $\forall x \forall y Rxy$ | A |
| 1 | (2) | $\forall y Ray$ | 1 UE |
| 1 | (3) | $Rab$ | 2 UE |
| 1 | (4) | $\forall x Rxb$ | 3 UI |
| 1 | (5) | $\forall y \forall x Rxy$ | 4 UI |