Metrics on Smooth Manifolds

When talking about classical spacetimes, we built from the bottom up — gradually adding more structure. We began with a smooth manifold M which we assumed to be foliated into spacelike hypersurfaces. Then we added a metric on those spacelike hypersurfaces. Then we added a temporal metric. Then we added an equivalence class of derivative operators (connections). Then we chose a preferred derivative operator. Etc.

A **relativistic spacetime** is a smooth manifold M with a metric g. In fact, g is a Lorentzian metric (as we'll describe below), but most of the following claims follow for metrics more generally.

- A metric g on M is actually a tensor *field*, i.e. a smooth assignment of tensors to points of M.
- A metric g is a $(0, 2)$ tensor. So in abstract index notation, we would write g_{ab}. Speaking functionally, g is a map that takes a pair of vectors and returns a real number.
- We assume that a metric is symmetric, i.e. $g(u, v) = g(v, u)$ for any vectors u, v. So a metric is just like an *inner product* on the tangent spaces — except that we have not assumed that g is positive definite. In particular, $g(v, v)$ could be negative or zero, even when $v \neq 0$.
- The invertibility condition. Malament writes $g_{ab}h^{bc} = \delta_a^c$. This is the same as saying that g is non-degenerate. i.e. if $g(v, w) = 0$ for all w, then $v = 0$. [Check it.]

Proposition. *If a metric g on V is non-degenerate, then g induces an isomorphism $v \mapsto g(v, -)$ between V and its dual space V^*.*

Proof. Suppose that g is non-degenerate. For $v \in V$, let $f(v)$ be a function from V into $\mathbb{R}$ defined by:

$$f(v)(w) \ = \ g(v, w).$$

Since g is linear in the second argument, $f(v)$ is a linear map. Because g is linear in the first argument, f is a linear map of V into V^*.

To see that f is one-to-one, suppose that $f(v) = 0$. In this case, $g(v, w) = 0$ for all $w \in V$. Since g is non-degenerate, $v = 0$. Hence, f is one-to-one. Since V and V^* have the same dimension, it automatically follows that f is onto, hence a linear isomorphism. □

From metric to connection

For every metric g, there is a unique connection ∇ that satisfies a certain compatibility connection. The connection is sometimes called the **Levi-Civita connection** for g.

Proposition. *The following are equivalent.*

1. $\nabla_X g = 0$ *for all vector fields* X *on* M.
2. *For any smooth curve* γ *and for any vector field* Y *defined along* γ*, if* $\nabla_{\dot{\gamma}} Y = 0$ *then* $\dot{\gamma}(g(Y, Y)) = 0$.
3. *For any vector fields* X, Y, Z *on* M*,*

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

Proof. The equivalence of (1) and (2) is Malament's Lemma 1.9.1. Using the definition of ∇ on tensors, and the fact that Y is constant along $\dot{\gamma}$, we get:

$$\nabla_{\dot{\gamma}}(g(Y, Y)) \ = \ g(Y, \nabla_{\dot{\gamma}} Y) + g(\nabla_{\dot{\gamma}} Y, Y) + (\nabla_{\dot{\gamma}} g)(Y, Y) \ = \ (\nabla_{\dot{\gamma}} g)(Y, Y).$$

□