Practice Problems: Predicate Logic

A. Proofs

1. $Fa \vdash \forall x((x = a) \to Fx)$

2. $\vdash \forall x \forall y\big((Fx \wedge \neg Fy) \to x \neq y\big)$

1	(1)	$Fa \wedge \neg Fb$	A
2	(2)	$a = b$	A
1	(3)	Fa	1 $\wedge$E
1	(4)	$\neg Fb$	1 $\wedge$E
1,2	(5)	Fb	3,2 =E
1,2	(6)	$Fb \wedge \neg Fb$	5,4 $\wedge$I
1	(7)	$a \neq b$	2,6 RA
$\varnothing$	(8)	$(Fa \wedge \neg Fb) \to a \neq b$	1,7 CP
$\varnothing$	(9)	$\forall y((Fa \wedge \neg Fy) \to a \neq y)$	8 UI
$\varnothing$	(10)	$\forall x \forall y((Fx \wedge \neg Fy) \to x \neq y)$	9 UI

3. $\forall x(Fx \to \exists y(Gy \wedge x = y)) \vdash \forall x(Fx \to Gx)$

4. $\forall y(Ray \to y = b) \vdash \exists y(Ray \wedge Gy) \to Gb$

5. $\forall x \forall y \forall z(Rxy \to \neg Ryz) \vdash \exists y \forall x \neg Rxy$

First strategy: Assume the negation of the conclusion for RA. Use QN

1	(1)	$\forall x \forall y \forall z(Rxy \to \neg Ryz)$	A
2	(2)	$\neg \exists y \forall x \neg Rxy$	A
2	(3)	$\forall y \neg \forall x \neg Rxy$	2 QN
2	(4)	$\neg \forall x \neg Rxb$	3 UE
2	(5)	$\exists x \neg\neg Rxb$	4 QN
6	(6)	$\neg\neg Rab$	A
6	(7)	Rab	6 DN
1	(8)	$\forall y \forall z(Ray \to \neg Ryz)$	1 UE
1	(9)	$\forall z(Rab \to \neg Rbz)$	8 UE
1	(10)	$Rab \to \neg Rbc$	9 UE
1,6	(11)	$\neg Rbc$	10,7 MP
1,2	(12)	$\neg Rbc$	5,6,11 EE
1,2	(13)	$\forall x \neg Rxc$	12 UI
1,2	(14)	$\exists y \forall x \neg Rxy$	13 EI
1,2	(15)	$\exists y \forall x \neg Rxy \wedge \neg \exists y \forall x \neg Rxy$	14,2 $\wedge$I
1	(16)	$\neg\neg \exists y \forall x \neg Rxy$	2,15 RA
1	(17)	$\exists y \forall x \neg Rxy$	16 DN

Second strategy: Either there is a pair of elements with an arrow between them or not. In the first case, there cannot be an arrow into the first element. In the second case, pick anything in the domain, and there is no arrow into it. In either case, there is something that has no arrow into it.

1	(1)	$\forall x\forall y\forall z(Rxy \to \neg Ryz)$	A
$\varnothing$	(2)	$\exists y\exists xRxy \vee \neg\exists y\exists xRxy$	prop taut
3	(3)	$\exists y\exists xRxy$	A
4	(4)	$\exists xRxb$	A
5	(5)	Rab	A
6	(6)	Rca	A
1	(7)	$\forall y\forall z(Rcy \to \neg Ryz)$	1 UE
1	(8)	$\forall z(Rca \to \neg Raz)$	7 UE
1	(9)	$Rca \to \neg Rab$	8 UE
5	(10)	$\neg\neg Rab$	5 DN
1,5	(11)	$\neg Rca$	9,10 MT
1,5	(12)	$\forall x\neg Rxa$	11 UI
1,5	(13)	$\exists y\forall x\neg Rxy$	12 EI
1,4	(14)	$\exists y\forall x\neg Rxy$	4,5,13 EE
1,3	(15)	$\exists y\forall x\neg Rxy$	3,4,14 EE
16	(16)	$\neg\exists y\exists xRxy$	A
16	(17)	$\forall y\neg\exists xRxy$	16 QN
16	(18)	$\neg\exists xRxa$	17 UE
16	(19)	$\forall x\neg Rxa$	18 QN
16	(20)	$\exists y\forall x\neg Rxy$	19 EI
1	(21)	$\exists y\forall x\neg Rxy$	2,3,15,16,20 $\vee$E

6. $\forall xFx \leftrightarrow \neg\exists x\exists yRxy \ \vdash\ \exists x\forall y\forall z(Fx \to \neg Ryz)$

   First strategy: Either $\forall xFx$ or $\exists x\neg Fx$. In the former case, the premise gives $\neg\exists x\exists yRxy$, and QN gives $\forall y\forall z\neg Ryz$. In the latter case, $Fa \to \neg Rbc$ by negative paradox.

7. $\vdash\ \forall x\exists y\forall z(\exists uTxyu \to \exists vTxzv)$

   This one is less complex than it looks. We show how to derive

   $$\exists y\forall z(\exists uTayu \to \exists vTazv).$$

   Up to α-equivalence, the conditional under the quantifiers is of the form $\varphi(y) \to \varphi(z)$, so it's enough to prove

   $$\exists y\forall z(\varphi(y) \to \varphi(z)).$$

   We have seen this before: it's just another instance of $\exists x\forall y(Fx \to Gy)$.

8. $\forall x\exists y\forall z(Rxy \wedge \neg Ryz)\ \vdash\ \exists x\exists y(x \neq y)$ (Bonus: does the premise entail that there are at least three things?)