How Logic Works: Solutions to Exercises

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Contents

Chapter 2

Exercise 2.1

1. *Derive* $Q \wedge P$ *from* $P \wedge Q$.

(1)	$P \wedge Q$	A
(2)	P	$1\ \wedge E$
(3)	Q	$1\ \wedge E$
(4)	$Q \wedge P$	$3,2\ \wedge I$

2. *Derive* $P \wedge (Q \wedge R)$ *from* $(P \wedge Q) \wedge R$.

(1)	$(P \wedge Q) \wedge R$	A
(2)	$P \wedge Q$	$1 \ \wedge E$
(3)	R	$1 \ \wedge E$
(4)	P	$2 \ \wedge E$
(5)	Q	$2 \ \wedge E$
(6)	$Q \wedge R$	$5,3 \ \wedge I$
(7)	$P \wedge (Q \wedge R)$	$4,6 \ \wedge I$

Exercise 2.2

1. $P \wedge Q \vdash Q \vee R$

(1)	$P \wedge Q$	A
(2)	Q	$1 \ \wedge E$
(3)	$Q \vee R$	$2 \ \vee I$

2. $P \wedge Q \vdash (P \vee R) \wedge (Q \vee R)$

(1)	$P \wedge Q$	A
(2)	P	$1 \ \wedge E$
(3)	Q	$1 \ \wedge E$
(4)	$P \vee R$	$2 \ \vee I$
(5)	$Q \vee R$	$3 \ \vee I$
(6)	$(P \vee R) \wedge (Q \vee R)$	$4,5 \ \wedge I$

3. $P \vdash Q \vee (P \vee Q)$

(1)	P	A
(2)	$P \vee Q$	$1 \ \vee I$
(3)	$Q \vee (P \vee Q)$	$2 \ \vee I$

4. $P \vdash (P \vee R) \wedge (P \vee Q)$

(1)	P	A
(2)	$P \vee R$	$1 \ \vee I$
(3)	$P \vee Q$	$1 \ \vee I$
(4)	$(P \vee R) \wedge (P \vee Q)$	$2,3 \ \wedge I$

Exercise 2.3

1. $P \to (Q \to R),\ P \to Q,\ P \vdash R$

(1)	$P \to (Q \to R)$	A
(2)	$P \to Q$	A
(3)	P	A
(4)	$Q \to R$	$1, 3\ MP$
(5)	Q	$2, 3\ MP$
(6)	R	$4, 5\ MP$

2. $(A \vee B) \to T,\ Z \to A,\ T \to W,\ Z \vdash W$

(1)	$(A \vee B) \to T$	A
(2)	$Z \to A$	A
(3)	$T \to W$	A
(4)	Z	A
(5)	A	$2, 4\ MP$
(6)	$A \vee B$	$5\ \vee I$
(7)	T	$1, 6\ MP$
(8)	W	$3, 7\ MP$

3. $(A \to B) \wedge (C \to A),\ (C \wedge (W \to Z)) \wedge W \vdash (B \vee D) \wedge (Z \vee E)$

(1)	$(A \to B) \wedge (C \to A)$	A
(2)	$(C \wedge (W \to Z)) \wedge W$	A
(3)	$A \to B$	$1\ \wedge E$
(4)	$C \to A$	$1\ \wedge E$
(5)	$C \wedge (W \to Z)$	$2\ \wedge E$
(6)	W	$2\ \wedge E$
(7)	C	$5\ \wedge E$
(8)	$W \to Z$	$5\ \wedge E$
(9)	A	$4, 7\ MP$
(10)	B	$3, 9\ MP$
(11)	$B \vee D$	$10\ \vee I$
(12)	Z	$8, 6\ MP$
(13)	$Z \vee E$	$12\ \vee I$
(14)	$(B \vee D) \wedge (Z \vee E)$	$11, 13\ \wedge I$

4. $P \to (P \to Q),\ P \vdash Q$

(1)	$P \to (P \to Q)$	A
(2)	P	A
(3)	$P \to Q$	$1, 2\ MP$
(4)	Q	$3, 2\ MP$

5. $P \wedge (P \to Q) \ \vdash \ P \wedge Q$

(1)	$P \wedge (P \to Q)$	A
(2)	P	$1 \ \wedge E$
(3)	$P \to Q$	$1 \ \wedge E$
(4)	Q	$3, 2 \ MP$
(5)	$P \wedge Q$	$2, 4 \ \wedge I$

Exercise 2.4

Prove $Q \to (P \to R),\ \neg R \wedge Q \ \vdash \ \neg P.$

(1)	$Q \to (P \to R)$	A
(2)	$\neg R \wedge Q$	A
(3)	Q	$2 \ \wedge E$
(4)	$\neg R$	$2 \ \wedge E$
(5)	$P \to R$	$1, 3 \ MP$
(6)	$\neg P$	$5, 4 \ MT$

Exercise 2.5

1. $P \wedge (Q \wedge R) \dashv\vdash (P \wedge Q) \wedge R$

($\Rightarrow$)

(1)	$P \wedge (Q \wedge R)$	A
(2)	P	$1 \ \wedge E$
(3)	$Q \wedge R$	$1 \ \wedge E$
(4)	Q	$3 \ \wedge E$
(5)	R	$3 \ \wedge E$
(6)	$P \wedge Q$	$2, 4 \ \wedge I$
(7)	$(P \wedge Q) \wedge R$	$6, 5 \ \wedge I$

($\Leftarrow$)

(1)	$(P \wedge Q) \wedge R$	A
(2)	$P \wedge Q$	$1 \ \wedge E$
(3)	R	$1 \ \wedge E$
(4)	P	$2 \ \wedge E$
(5)	Q	$2 \ \wedge E$
(6)	$Q \wedge R$	$5, 3 \ \wedge I$
(7)	$P \wedge (Q \wedge R)$	$4, 6 \ \wedge I$

2. $P \dashv\vdash P \wedge P$

($\Rightarrow$)

$$\begin{array}{lll} (1) & P & A \\ (2) & P \wedge P & 1,1 \ \wedge I \end{array}$$

($\Leftarrow$)

$$\begin{array}{lll} (1) & P \wedge P & A \\ (2) & P & 1 \ \wedge E \end{array}$$

Exercise 2.6

1. $\neg(\neg R \to H)$

2. $\neg E \to S$ (equivalently: $E \vee S$)

3. $\neg P \wedge \neg S$

4. $A \to (H \vee B)$

5. $K \wedge \big((M \wedge F) \vee \neg F\big)$

6. $\neg(H \wedge D)$

Exercise 2.7

1. $\neg\neg Q \to P,\ \neg P \vdash \neg Q$

$$\begin{array}{lll} (1) & \neg\neg Q \to P & A \\ (2) & \neg P & A \\ (3) & \neg\neg\neg Q & 1,2\ MT \\ (4) & \neg Q & 3\ DN \end{array}$$

2. $P \to (P \to Q),\ P \vdash Q$

$$\begin{array}{lll} (1) & P \to (P \to Q) & A \\ (2) & P & A \\ (3) & P \to Q & 1,2\ MP \\ (4) & Q & 3,2\ MP \end{array}$$

3. $(P \wedge P) \to Q,\ P \vdash Q$

$$\begin{array}{lll} (1) & (P \wedge P) \to Q & A \\ (2) & P & A \\ (3) & P \wedge P & 2,2\ \wedge I \\ (4) & Q & 1,3\ MP \end{array}$$

4. $P \vdash Q \vee (\neg\neg P \vee R)$

(1)	P	A
(2)	$\neg\neg P$	$1\ DN$
(3)	$\neg\neg P \vee R$	$2\ \vee I$
(4)	$Q \vee (\neg\neg P \vee R)$	$3\ \vee I$

Exercise 2.8

1. $P \to \neg Q,\ \neg P \nvdash Q$

   Counterexample (English instance): Let P be "It is raining." Let Q be "$2+2=5$." Then Q is false, so $\neg Q$ is true; hence $P \to \neg Q$ is true. Also $\neg P$ can be true (suppose it isn't raining). But Q is false.

2. $P \to R \nvdash (P \vee Q) \to R$

   Counterexample (English instance): Let P be "I am on Mars." (false) Let Q be "I am on Earth." (true) Let R be "$2+2=5$." (false) Then $P \to R$ is true (false antecedent), but $(P \vee Q) \to R$ is false (true antecedent, false consequent).

Chapter 3

Exercise 3.1

1. $P \vdash Q \to (P \wedge Q)$

1	(1)	P	A
2	(2)	Q	A
1,2	(3)	$P \wedge Q$	1,2 $\wedge$I
1	(4)	$Q \to (P \wedge Q)$	2,3 CP

2. $(P \to Q) \wedge (P \to R) \vdash P \to (Q \wedge R)$

1	(1)	$(P \to Q) \wedge (P \to R)$	A
2	(2)	P	A
1	(3)	$P \to Q$	1 $\wedge$E
1	(4)	$P \to R$	1 $\wedge$E
1,2	(5)	Q	3,2 MP
1,2	(6)	R	4,2 MP
1,2	(7)	$Q \wedge R$	5,6 $\wedge$I
1	(8)	$P \to (Q \wedge R)$	2,7 CP

3. $P \to (Q \to R) \vdash Q \to (P \to R)$

1	(1)	$P \to (Q \to R)$	A
2	(2)	Q	A
3	(3)	P	A
1,3	(4)	$Q \to R$	3,1 MP
1,2,3	(5)	R	4,2 MP
1,2	(6)	$P \to R$	3,5 CP
1	(7)	$Q \to (P \to R)$	2,6 CP

4. $P \to Q \vdash (Q \to R) \to (P \to R)$

1	(1)	$P \to Q$	A
2	(2)	$Q \to R$	A
3	(3)	P	A
1,3	(4)	Q	1,3 MP
1,2,3	(5)	R	2,4 MP
1,2	(6)	$P \to R$	3,5 CP
1	(7)	$(Q \to R) \to (P \to R)$	2,6 CP

5. $P \to (P \to Q) \vdash P \to Q$

1	(1)	$P \to (P \to Q)$	A
2	(2)	P	A
1,2	(3)	$P \to Q$	1,2 MP
1,2	(4)	Q	3,2 MP
1	(5)	$P \to Q$	2,4 CP

6. $P \to (Q \to R) \vdash (P \wedge Q) \to R$

1	(1)	$P \to (Q \to R)$	A
2	(2)	$P \wedge Q$	A
2	(3)	P	2 $\wedge$E
2	(4)	Q	2 $\wedge$E
1,2	(5)	$Q \to R$	1,3 MP
1,2	(6)	R	5,4 MP
1	(7)	$(P \wedge Q) \to R$	2,6 CP

7. $(P \vee Q) \to R \vdash P \to R$

1	(1)	$(P \vee Q) \to R$	A
2	(2)	P	A
2	(3)	$P \vee Q$	2 $\vee$I
1,2	(4)	R	1,3 MP
1	(5)	$P \to R$	2,4 CP

8. $\neg P \vdash \neg(P \wedge Q)$

1	(1)	$\neg P$	A
2	(2)	$P \wedge Q$	A
2	(3)	P	2 $\wedge$E
	(4)	$(P \wedge Q) \rightarrow P$	2,3 CP
1	(5)	$\neg(P \wedge Q)$	4,1 MT

9. $\neg(P \vee Q) \vdash \neg P \wedge \neg Q$

1	(1)	$\neg(P \vee Q)$	A
2	(2)	P	A
2	(3)	$P \vee Q$	2 $\vee$I
	(4)	$P \rightarrow (P \vee Q)$	2,3 CP
1	(5)	$\neg P$	4,1 MT
6	(6)	Q	A
6	(7)	$P \vee Q$	6 $\vee$I
	(8)	$Q \rightarrow (P \vee Q)$	6,7 CP
1	(9)	$\neg Q$	8,1 MT
1	(10)	$\neg P \wedge \neg Q$	5,9 $\wedge$I

10. $P \rightarrow \neg P \vdash \neg P$

1	(1)	P	A
2	(2)	$P \rightarrow \neg P$	A
1,2	(3)	$\neg P$	2,1 MP
1	(4)	$(P \rightarrow \neg P) \rightarrow \neg P$	2,3 CP
1	(5)	$\neg\neg P$	1 DN
1	(6)	$\neg(P \rightarrow \neg P)$	4,5 MT
	(7)	$P \rightarrow \neg(P \rightarrow \neg P)$	1,6 CP
2	(8)	$\neg\neg(P \rightarrow \neg P)$	2 DN
2	(9)	$\neg P$	7,8 MT

Exercise 3.2

1. $\vdash (P \wedge Q) \rightarrow (Q \wedge P)$

1	(1)	$P \wedge Q$	A
1	(2)	P	1 $\wedge$E
1	(3)	Q	1 $\wedge$E
1	(4)	$Q \wedge P$	3,2 $\wedge$I
$\varnothing$	(5)	$(P \wedge Q) \rightarrow (Q \wedge P)$	1,4 CP

2. $\vdash (P \wedge Q) \rightarrow P$

1	(1)	$P \wedge Q$	A
1	(2)	P	1 $\wedge$E
$\varnothing$	(3)	$(P \wedge Q) \to P$	1,2 CP

3. $\vdash Q \to (P \to Q)$

1	(1)	Q	A
2	(2)	P	A
1	(3)	$P \to Q$	2,1 CP
$\varnothing$	(4)	$Q \to (P \to Q)$	1,3 CP

4. $\vdash Q \to (P \to P)$

1	(1)	Q	A
2	(2)	P	A
2	(3)	$P \to P$	2,2 CP
$\varnothing$	(4)	$Q \to (P \to P)$	1,3 CP

5. $\vdash P \vee \neg P$ (EM)

1	(1)	$\neg(P \vee \neg P)$	A
2	(2)	P	A
2	(3)	$P \vee \neg P$	2 $\vee$I
$\varnothing$	(4)	$P \to (P \vee \neg P)$	2,3 CP
1	(5)	$\neg P$	4,1 MT
1	(6)	$P \vee \neg P$	5 $\vee$I
$\varnothing$	(7)	$\neg(P \vee \neg P) \to (P \vee \neg P)$	1,6 CP
$\varnothing$	(8)	$P \vee \neg P$	7 (Ex. 3.1.10, $Q := P \vee \neg P$)

Exercise 3.4

1. $P \to Q \vdash \neg(P \wedge \neg Q)$

1	(1)	$P \to Q$	A
2	(2)	$P \wedge \neg Q$	A
2	(3)	P	2 $\wedge$E
1,2	(4)	Q	1,3 MP
2	(5)	$\neg Q$	2 $\wedge$E
1,2	(6)	$Q \wedge \neg Q$	4,5 $\wedge$I
1	(7)	$\neg(P \wedge \neg Q)$	2,6 RA

2. $\neg(P \wedge Q) \vdash \neg P \vee \neg Q$

1	(1)	$\neg(P \wedge Q)$	A
2	(2)	$\neg(\neg P \vee \neg Q)$	A
3	(3)	$\neg P$	A
3	(4)	$\neg P \vee \neg Q$	3 $\vee$I
2,3	(5)	$(\neg P \vee \neg Q) \wedge \neg(\neg P \vee \neg Q)$	4,2 $\wedge$I
2	(6)	$\neg\neg P$	3,5 RA
2	(7)	P	6 DN
8	(8)	$\neg Q$	A
8	(9)	$\neg P \vee \neg Q$	8 $\vee$I
2,8	(10)	$(\neg P \vee \neg Q) \wedge \neg(\neg P \vee \neg Q)$	9,2 $\wedge$I
2	(11)	$\neg\neg Q$	8,10 RA
2	(12)	Q	11 DN
2	(13)	$P \wedge Q$	7,12 $\wedge$I
1,2	(14)	$(P \wedge Q) \wedge \neg(P \wedge Q)$	13,1 $\wedge$I
1	(15)	$\neg\neg(\neg P \vee \neg Q)$	2,14 RA
1	(16)	$\neg P \vee \neg Q$	15 DN

3. $\neg(P \to Q) \vdash P \wedge \neg Q$

1	(1)	$\neg(P \to Q)$	A
2	(2)	$\neg P$	A
3	(3)	$\neg Q$	A
4	(4)	P	A
2,4	(5)	$P \wedge \neg P$	2,4 $\wedge$I
2,4	(6)	$\neg\neg Q$	3,5 RA
2,4	(7)	Q	6 DN
2	(8)	$P \to Q$	4,7 CP
1,2	(9)	$(P \to Q) \wedge \neg(P \to Q)$	8,1 $\wedge$I
1	(10)	$\neg\neg P$	2,9 RA
1	(11)	P	10 DN
12	(12)	Q	A
12	(13)	$P \to Q$	4,12 CP
1,12	(14)	$(P \to Q) \wedge \neg(P \to Q)$	13,1 $\wedge$I
1	(15)	$\neg Q$	12,14 RA
1	(16)	$P \wedge \neg Q$	11,15 $\wedge$I

4. $\vdash (P \to Q) \vee (Q \to P)$

1	(1)	$\neg((P \to Q) \lor (Q \to P))$	A
2	(2)	P	A
3	(3)	Q	A
2	(4)	$Q \to P$	3,2 CP
2	(5)	$(P \to Q) \lor (Q \to P)$	4 $\lor$I
1,2	(6)	$((P \to Q) \lor (Q \to P)) \land \neg((P \to Q) \lor (Q \to P))$	5,1 $\land$I
1	(7)	$\neg P$	2,6 RA
8	(8)	$\neg Q$	A
1,2	(9)	$P \land \neg P$	2,7 $\land$I
1,2	(10)	$\neg\neg Q$	8,9 RA
1,2	(11)	Q	10 DN
1	(12)	$P \to Q$	2,11 CP
1	(13)	$(P \to Q) \lor (Q \to P)$	12 $\lor$I
1	(14)	$((P \to Q) \lor (Q \to P)) \land \neg((P \to Q) \lor (Q \to P))$	13,1 $\land$I
$\varnothing$	(15)	$\neg\neg((P \to Q) \lor (Q \to P))$	1,14 RA
$\varnothing$	(16)	$(P \to Q) \lor (Q \to P)$	15 DN

5. $P \to (Q \lor R) \vdash (P \to Q) \lor (P \to R)$

1	(1)	$P \to (Q \lor R)$	A
2	(2)	$\neg((P \to Q) \lor (P \to R))$	A
3	(3)	$\neg P$	A
4	(4)	P	A
5	(5)	$\neg Q$	A
3,4	(6)	$P \land \neg P$	4,3 $\land$I
3,4	(7)	$\neg\neg Q$	5,6 RA
3,4	(8)	Q	7 DN
3	(9)	$P \to Q$	4,8 CP
3	(10)	$(P \to Q) \lor (P \to R)$	9 $\lor$I
2,3	(11)	$((P \to Q) \lor (P \to R)) \land \neg((P \to Q) \lor (P \to R))$	10,2 $\land$I
2	(12)	$\neg\neg P$	3,11 RA
2	(13)	P	12 DN
1,2	(14)	$Q \lor R$	1,13 MP
15	(15)	Q	A
15	(16)	$P \to Q$	4,15 CP
15	(17)	$(P \to Q) \lor (P \to R)$	16 $\lor$I
18	(18)	R	A
18	(19)	$P \to R$	4,18 CP
18	(20)	$(P \to Q) \lor (P \to R)$	19 $\lor$I
1,2	(21)	$(P \to Q) \lor (P \to R)$	14,15,17,18,20 $\lor$E
1,2	(22)	$((P \to Q) \lor (P \to R)) \land \neg((P \to Q) \lor (P \to R))$	21,2 $\land$I
1	(23)	$\neg\neg((P \to Q) \lor (P \to R))$	2,22 RA
1	(24)	$(P \to Q) \lor (P \to R)$	23 DN

6. $(P \wedge Q) \to \neg Q \vdash P \to \neg Q$

1	(1)	$(P \wedge Q) \to \neg Q$	A
2	(2)	P	A
3	(3)	Q	A
2,3	(4)	$P \wedge Q$	2,3 $\wedge$I
1,2,3	(5)	$\neg Q$	1,4 MP
1,2,3	(6)	$Q \wedge \neg Q$	3,5 $\wedge$I
1,2	(7)	$\neg Q$	3,6 RA
1	(8)	$P \to \neg Q$	2,7 CP

Chapter 6

Exercise 6.1

1. No logicians are celebrities. (Lx, Cx)

$$\forall x\,(Lx \to \neg Cx)$$

Equivalently: $\neg\exists x(Lx \wedge Cx)$

2. Some celebrities are not logicians. (Lx, Cx)

$$\exists x\,(Cx \wedge \neg Lx)$$

3. Only students who do the homework will learn logic. (Sx, Hx, Lx)
Either

$$\forall x\,(Lx \to (Sx \wedge Hx))$$

or (inequivalently)

$$\forall x\,((Sx \wedge Lx) \to Hx)$$

depending on whether one intends to restrict the claim to students.

4. All rich logicians are computer scientists. (Rx, Lx, Cx)

$$\forall x\,((Rx \wedge Lx) \to Cx)$$

5. All students and professors get a discount. (Sx, Px, Dx)

$$\forall x\,((Sx \vee Px) \to Dx)$$

6. No logician is rich, unless she is a computer scientist. (Lx, Rx, Cx)

$$\forall x\,((Lx \wedge Rx) \to Cx)$$

Equivalent form: $\forall x((Lx \wedge \neg Cx) \to \neg Rx)$

7. Not all logicians are computer scientists. (Lx, Cx)

$$\neg\forall x\,(Lx \to Cx)$$

Often put as: $\exists x\,(Lx \wedge \neg Cx)$.

8. Some logicians are rich computer scientists. (Lx, Rx, Cx)

$$\exists x\,(Lx \wedge (Rx \wedge Cx))$$

9. If there are rich logicians, then some logicians are computer scientists. (Rx, Lx, Cx)

$$\exists x\,(Rx \wedge Lx) \to \exists y\,(Ly \wedge Cy)$$

10. No pets except service animals are permitted in dorms. (Px, Sx, Dx)

Can be read in a minimal way as:

$$\forall x\,((Px \wedge Dx) \to Sx),$$

which says only that no non–service pets are allowed in dorms. However, ordinary policy language is typically understood more strongly: among pets, *being permitted in the dorms* and *being a service animal* coincide. That reading is captured by:

$$\forall x\,(Px \to (Dx \leftrightarrow Sx)).$$

This biconditional formalization is therefore closer to the intended rule.

11. If anyone is rich, then Mary is. (Rx, m)

$$(\exists x\, Rx) \to Rm$$

Exercise 6.2

1. Mary loves everyone who loves her. (m, Lxy)

$$\forall x\,(Lxm \to Lmx)$$

2. Mary loves all and only those people who don't love themselves. (Lxy, m)

$$\forall x\,(Lmx \leftrightarrow \neg Lxx)$$

3. Everyone loves their mother. (Lxy, Mxy)

   $\forall x \forall y\,(Myx \to Lxy)$

4. Some people love only those people who love their mother. (Lxy, Mxy)

   $\exists x \forall y(Lxy \to \forall z(Mzy \to Lyz))$

5. Snape killed someone. (Kxy, s)

   $\exists x\, Ksx$

6. Snape is a killer. (Kxy, s)

   $\exists x\, Ksx$

7. Someone was killed by Snape. (Kxy, s)

   $\exists x\, Ksx$

8. Some wizards only marry other wizards. (Wx, Mxy)

   $\exists x(Wx \wedge \forall y\,(Mxy \to Wy))$

9. There is no greatest number. $(Nx,\ x < y)$

   $\forall x(Nx \to \exists y\,(Ny \wedge x < y))$

10. c is the least upper bound of a and b. $(a, b, c,\ x \leq y)$

    $(a \leq c \wedge b \leq c)\ \wedge\ \forall x((a \leq x \wedge b \leq x) \to c \leq x)$

11. c is the greatest common divisor of a and b. $(a, b, c, Dxy,\ x \leq y)$

    $(Dca \wedge Dcb)\ \wedge\ \forall x((Dxa \wedge Dxb) \to x \leq c)$

Exercise 6.8

1. $\neg\exists x(Fx \wedge Gx) \vdash \forall x(Fx \to \neg Gx)$

1	(1)	$\neg\exists x(Fx \wedge Gx)$	A
2	(2)	Fa	A
3	(3)	Ga	A
2,3	(4)	$Fa \wedge Ga$	2,3 $\wedge$I
2,3	(5)	$\exists x(Fx \wedge Gx)$	4 EI
1,2,3	(6)	$\exists x(Fx \wedge Gx) \wedge \neg\exists x(Fx \wedge Gx)$	5,1 $\wedge$I
1,2	(7)	$\neg Ga$	3,6 RA
1	(8)	$Fa \to \neg Ga$	2,7 CP
1	(9)	$\forall x(Fx \to \neg Gx)$	8 UI

2. $\forall x Fx \vdash \exists x Fx$

1	(1)	$\forall x Fx$	A
1	(2)	Fa	1 UE
1	(3)	$\exists x Fx$	2 EI

3. $\forall x(Fx \to Gx), Fa \vdash \exists x Gx$

1	(1)	$\forall x(Fx \to Gx)$	A
2	(2)	Fa	A
1	(3)	$Fa \to Ga$	1 UE
1,2	(4)	Ga	3,2 MP
1,2	(5)	$\exists x Gx$	4 EI

4. $\neg Fa \vdash \exists x(Fx \to P)$

1	(1)	$\neg Fa$	A
1	(2)	$Fa \to P$	1 negative paradox
1	(3)	$\exists x(Fx \to P)$	2 EI

5. $\neg\forall x Fx \vdash \exists x(Fx \to P)$

1	(1)	$\neg\forall x Fx$	A
2	(2)	$\neg\exists x(Fx \to P)$	A
3	(3)	$Fa \to P$	A
3	(4)	$\exists x(Fx \to P)$	3 EI
2,3	(5)	$\exists x(Fx \to P) \wedge \neg\exists x(Fx \to P)$	4,2 $\wedge$I
2	(6)	$\neg(Fa \to P)$	3,5 RA
2	(7)	Fa	6 material conditional
2	(8)	$\forall x Fx$	7 UI
1,2	(9)	$\forall x Fx \wedge \neg\forall x Fx$	8,1 $\wedge$I
1	(10)	$\neg\neg\exists x(Fx \to P)$	2,9 RA
1	(11)	$\exists x(Fx \to P)$	10 DN

6. $\neg\exists x Fx \vdash \forall x(Fx \to Gx)$

1	(1)	$\neg\exists x Fx$	A
2	(2)	Fa	A
3	(3)	$\neg Ga$	A
2	(4)	$\exists x Fx$	2 EI
1,2	(5)	$\exists x Fx \wedge \neg\exists x Fx$	4,1 $\wedge$I
1,2	(6)	$\neg\neg Ga$	3,5 RA
1,2	(7)	Ga	6 DN
1	(8)	$Fa \to Ga$	2,7 CP
1	(9)	$\forall x(Fx \to Gx)$	8 UI

7. $\forall x \forall y Rxy \vdash \exists x Rxx$

1	(1)	$\forall x \forall y Rxy$	A
1	(2)	$\forall y Ray$	1 UE
1	(3)	Raa	2 UE
1	(4)	$\exists x Rxx$	3 EI

8. $P \to Fa \vdash P \to \exists x Fx$

1	(1)	$P \to Fa$	A
2	(2)	P	A
1,2	(3)	Fa	1,2 MP
1,2	(4)	$\exists x Fx$	3 EI
1	(5)	$P \to \exists x Fx$	2,4 CP

9. $\exists x Fx \to P \vdash \forall x(Fx \to P)$

1	(1)	$\exists x Fx \to P$	A
2	(2)	Fa	A
2	(3)	$\exists x Fx$	2 EI
1,2	(4)	P	1,3 MP
1	(5)	$Fa \to P$	2,4 CP
1	(6)	$\forall x(Fx \to P)$	5 UI

There is a typo here in the book: the direction $\forall x(Fx \to P) \vdash \exists x Fx \to P$ cannot be proven without EE, which is only introduced in the next section.

10. $\neg\exists x Fx \vdash \forall x(Fx \to P)$

1	(1)	$\neg\exists x Fx$	A
2	(2)	Fa	A
2	(3)	$\exists x Fx$	2 EI
1,2	(4)	$\exists x Fx \wedge \neg\exists x Fx$	3,1 $\wedge$I
1	(5)	$\neg Fa$	2,4 RA
1	(6)	$Fa \to P$	5 neg paradox
1	(7)	$\forall x(Fx \to P)$	6 UI

11. $\neg\exists x(Fx \to P) \vdash \forall x Fx \wedge \neg P$

1	(1)	$\neg\exists x(Fx \to P)$	A
2	(2)	$Fa \to P$	A
2	(3)	$\exists x(Fx \to P)$	2 EI
1,2	(4)	$\exists x(Fx \to P) \land \neg\exists x(Fx \to P)$	3,1 $\land$I
1	(5)	$\neg(Fa \to P)$	2,4 RA
1	(6)	$Fa \land \neg P$	5 material conditional
1	(7)	$\neg P$	6 $\land$E
1	(8)	Fa	6 $\land$E
1	(9)	$\forall x Fx$	8 UI
1	(10)	$\forall x Fx \land \neg P$	9,7 $\land$I

12. $\forall x Fx \to P \vdash \exists x(Fx \to P)$

1	(1)	$\forall x Fx \to P$	A
2	(2)	$\neg\exists x(Fx \to P)$	A
3	(3)	$\neg Fa$	A
3	(4)	$Fa \to P$	3 neg paradox
3	(5)	$\exists x(Fx \to P)$	4 EI
2,3	(6)	$\exists x(Fx \to P) \land \neg\exists x(Fx \to P)$	5,2 $\land$I
2	(7)	$\neg\neg Fa$	3,6 RA
2	(8)	Fa	7 DN
2	(9)	$\forall x Fx$	8 UI
1,2	(10)	P	1,9 MP
1,2	(11)	$Fb \to P$	10 pos paradox
1,2	(12)	$\exists x(Fx \to P)$	11 EI
1,2	(13)	$\exists x(Fx \to P) \land \neg\exists x(Fx \to P)$	12,2 $\land$I
1	(14)	$\neg\neg\exists x(Fx \to P)$	2,13 RA
1	(15)	$\exists x(Fx \to P)$	14 DN

Exercise 6.11

1. $\exists x Fx \lor \exists x Gx \vdash \exists x(Fx \lor Gx)$

1	(1)	$\exists xFx \lor \exists xGx$	A
2	(2)	$\exists xFx$	A
3	(3)	Fa	A
3	(4)	$Fa \lor Ga$	3 $\lor$I
3	(5)	$\exists x(Fx \lor Gx)$	4 EI
2	(6)	$\exists x(Fx \lor Gx)$	2,3,5 EE
7	(7)	$\exists xGx$	A
8	(8)	Ga	A
8	(9)	$Fa \lor Ga$	8 $\lor$I
8	(10)	$\exists x(Fx \lor Gx)$	9 EI
7	(11)	$\exists x(Fx \lor Gx)$	7,8,10 EE
1	(12)	$\exists x(Fx \lor Gx)$	1,2,6,7,11 $\lor$E

2. $\forall x(Fx \to Gx), \neg\exists xGx \vdash \neg\exists xFx$

1	(1)	$\forall x(Fx \to Gx)$	A
2	(2)	$\neg\exists xGx$	A
3	(3)	$\exists xFx$	A
4	(4)	Fa	A
1	(5)	$Fa \to Ga$	1 UE
1,4	(6)	Ga	5,4 MP
1,4	(7)	$\exists xGx$	6 EI
1,3	(8)	$\exists xGx$	3,4,7 EE
1,2,3	(9)	$\exists xGx \land \neg\exists xGx$	8,2 $\land$I
1,2	(10)	$\neg\exists xFx$	3,9 RA

3. $\forall x(Fx \to Gx) \vdash \exists x\neg Gx \to \exists x\neg Fx$

1	(1)	$\forall x(Fx \to Gx)$	A
2	(2)	$\exists x\neg Gx$	A
3	(3)	$\neg Ga$	A
1	(4)	$Fa \to Ga$	1 UE
1,3	(5)	$\neg Fa$	4,3 MT
1,3	(6)	$\exists x\neg Fx$	5 EI
1,2	(7)	$\exists x\neg Fx$	2,3,6 EE
1	(8)	$\exists x\neg Gx \to \exists x\neg Fx$	2,7 CP

4. $\forall x(Fx \to P) \vdash \exists xFx \to P$

1	(1)	$\forall x(Fx \rightarrow P)$	A
2	(2)	$\exists xFx$	A
3	(3)	Fa	A
1	(4)	$Fa \rightarrow P$	1 UE
1,3	(5)	P	4,3 MP
1,2	(6)	P	2,3,5 EE
1	(7)	$\exists xFx \rightarrow P$	2,6 CP

5. $P \wedge \exists xFx \vdash \exists x(P \wedge Fx)$

1	(1)	$P \wedge \exists xFx$	A
1	(2)	P	1 $\wedge$E
1	(3)	$\exists xFx$	1 $\wedge$E
4	(4)	Fa	A
1,4	(5)	$P \wedge Fa$	2,4 $\wedge$I
1,4	(6)	$\exists x(P \wedge Fx)$	5 EI
1	(7)	$\exists x(P \wedge Fx)$	3,4,6 EE

6. $\exists x(Fx \rightarrow P) \vdash \forall xFx \rightarrow P$

1	(1)	$\exists x(Fx \rightarrow P)$	A
2	(2)	$\forall xFx$	A
3	(3)	$Fa \rightarrow P$	A
2	(4)	Fa	2 UE
2,3	(5)	P	3,4 MP
3	(6)	$\forall xFx \rightarrow P$	2,5 CP
1	(7)	$\forall xFx \rightarrow P$	1,3,6 EE

Exercise 6.13

1. $P \rightarrow \exists xFx \vdash \exists x(P \rightarrow Fx)$

1	(1)	$P \rightarrow \exists xFx$	A
$\varnothing$	(2)	$\exists xFx \vee \neg\exists xFx$	prop taut
3	(3)	$\exists xFx$	A
4	(4)	Fa	A
4	(5)	$P \rightarrow Fa$	4 prop taut
4	(6)	$\exists x(P \rightarrow Fx)$	5 EI
3	(7)	$\exists x(P \rightarrow Fx)$	3,4,6 EE
8	(8)	$\neg\exists xFx$	A
1,8	(9)	$\neg P$	1,8 MT
1,8	(10)	$P \rightarrow Fb$	9 prop taut
1,8	(11)	$\exists x(P \rightarrow Fx)$	10 EI
1	(12)	$\exists x(P \rightarrow Fx)$	2,3,7,8,11 $\vee$E

2. $\exists x(Fx \to P) \vdash \forall x Fx \to P$

1	(1)	$\exists x(Fx \to P)$	A
2	(2)	$\forall x Fx$	A
3	(3)	$Fa \to P$	A
2	(4)	Fa	2 UE
2,3	(5)	P	3,4 MP
1,2	(6)	P	1,3,5 EE
1	(7)	$\forall x Fx \to P$	2,6 CP

Exercise 6.14

1. $\vdash \forall x(Fx \to Fx)$

1	(1)	Fa	A
$\varnothing$	(2)	$Fa \to Fa$	1,1 CP
$\varnothing$	(3)	$\forall x(Fx \to Fx)$	2 UI

2. $\vdash \forall x Fx \lor \exists x \neg Fx$

$\varnothing$	(1)	$\neg\exists x\neg Fx \lor \exists x\neg Fx$	prop taut
2	(2)	$\neg\exists x\neg Fx$	A
3	(3)	$\neg Fa$	A
3	(4)	$\exists x\neg Fx$	3 EI
2,3	(5)	$\exists x\neg Fx \land \neg\exists x\neg Fx$	4,2 $\land$I
2	(6)	$\neg\neg Fa$	3,5 RA
2	(7)	Fa	6 DN
2	(8)	$\forall x Fx$	7 UI
2	(9)	$\forall x Fx \lor \exists x\neg Fx$	8 $\lor$I
10	(10)	$\exists x\neg Fx$	A
10	(11)	$\forall x Fx \lor \exists x\neg Fx$	10 $\lor$I
$\varnothing$	(12)	$\forall x Fx \lor \exists x\neg Fx$	1,2,9,10,11 $\lor$E

3. $\vdash \forall x\, \neg(Fx \land \neg Fx)$

1	(1)	$Fa \land \neg Fa$	A
$\varnothing$	(2)	$\neg(Fa \land \neg Fa)$	1,1 RA
$\varnothing$	(3)	$\forall x\, \neg(Fx \land \neg Fx)$	2 UI

4. $\vdash \neg\exists x(Fx \land \neg Fx)$

1	(1)	$\exists x(Fx \wedge \neg Fx)$	A
2	(2)	$Fa \wedge \neg Fa$	A
2	(3)	$\neg\exists x(Fx \wedge \neg Fx)$	1,2 RA
1	(4)	$\neg\exists x(Fx \wedge \neg Fx)$	1,2,3 EE
1	(5)	$\exists x(Fx \wedge \neg Fx) \wedge \neg\exists x(Fx \wedge \neg Fx)$	1,3 $\wedge$I
$\varnothing$	(6)	$\neg\exists x(Fx \wedge \neg Fx)$	1,5 RA

5. $\vdash \forall x\exists y(Rxy \to Rxx)$

1	(1)	Raa	A
$\varnothing$	(2)	$Raa \to Raa$	1,1 CP
$\varnothing$	(3)	$\exists y(Ray \to Raa)$	2 EI
$\varnothing$	(4)	$\forall x\exists y(Rxy \to Rxx)$	3 UI

6. $\vdash \forall x\exists y(Rxy \to Ryx)$

1	(1)	Raa	A
$\varnothing$	(2)	$Raa \to Raa$	1,1 CP
$\varnothing$	(3)	$\exists y(Ray \to Rya)$	2 EI
$\varnothing$	(4)	$\forall x\exists y(Rxy \to Ryx)$	3 UI

7. $\vdash \exists x(Fx \to \forall yFy)$

1	(1)	$\neg\exists x(Fx \to \forall yFy)$	A
2	(2)	$\neg Fa$	A
2	(3)	$Fa \to \forall yFy$	2 prop taut
2	(4)	$\exists x(Fx \to \forall yFy)$	3 EI
1,2	(5)	$\exists x(Fx \to \forall yFy) \wedge \neg\exists x(Fx \to \forall yFy)$	4,1 $\wedge$I
1	(6)	$\neg\neg Fa$	2,5 RA
1	(7)	Fa	6 DN
1	(8)	$\forall yFy$	7 UI
1	(9)	$Fa \to \forall yFy$	8 prop taut
1	(10)	$\exists x(Fx \to \forall yFy)$	9 EI
1	(11)	$\exists x(Fx \to \forall yFy) \wedge \neg\exists x(Fx \to \forall yFy)$	10,1 $\wedge$I
$\varnothing$	(12)	$\neg\neg\exists x(Fx \to \forall yFy)$	1,11 RA
$\varnothing$	(13)	$\exists x(Fx \to \forall yFy)$	12 DN

8. $\vdash \exists x\forall y(Fx \to Fy)$

1	(1)	$\neg\exists x\forall y(Fx \to Fy)$	A
2	(2)	$\neg Fa$	A
2	(3)	$Fa \to Fb$	2 prop taut
2	(4)	$\forall y(Fa \to Fy)$	3 UI
2	(5)	$\exists x\forall y(Fx \to Fy)$	4 EI
1,2	(6)	$\exists x\forall y(Fx \to Fy) \wedge \neg\exists x\forall y(Fx \to Fy)$	5,1 $\wedge$I
1	(7)	$\neg\neg Fa$	2,6 RA
1	(8)	Fa	7 DN
1	(9)	$Fc \to Fa$	8 prop taut
1	(10)	$\forall y(Fc \to Fy)$	9 UI
1	(11)	$\exists x\forall y(Fx \to Fy)$	10 EI
1	(12)	$\exists x\forall y(Fx \to Fy) \wedge \neg\exists x\forall y(Fx \to Fy)$	11,1 $\wedge$I
$\varnothing$	(13)	$\neg\neg\exists x\forall y(Fx \to Fy)$	1,12 RA
$\varnothing$	(14)	$\exists x\forall y(Fx \to Fy)$	13 DN

9. $\forall x\exists y(Fx \to Gy) \vdash \exists y\forall x(Fx \to Gy)$

1	(1)	$\forall x\exists y(Fx \to Gy)$	A
$\varnothing$	(2)	$\exists yGy \vee \neg\exists yGy$	prop taut
3	(3)	$\exists yGy$	A
4	(4)	Ga	A
4	(5)	$Fb \to Ga$	4 prop taut
4	(6)	$\forall x(Fx \to Ga)$	5 UI
4	(7)	$\exists y\forall x(Fx \to Gy)$	6 EI
3	(8)	$\exists y\forall x(Fx \to Gy)$	3,4,7 EE
9	(9)	$\neg\exists yGy$	A
10	(10)	Fc	A
1	(11)	$\exists y(Fc \to Gy)$	1 UE
12	(12)	$Fc \to Gd$	A
10,12	(13)	Gd	12,10 MP
10,12	(14)	$\exists yGy$	13 EI
9,10,12	(15)	$\exists yGy \wedge \neg\exists yGy$	14,9 $\wedge$I
9,12	(16)	$\neg Fc$	10,15 RA
9,12	(17)	$Fc \to Ge$	16 prop taut
1,9	(18)	$Fc \to Ge$	11,12,17 EE
1,9	(19)	$\forall x(Fx \to Ge)$	18 UI
1,9	(20)	$\exists y\forall x(Fx \to Gy)$	19 EI
1	(21)	$\exists y\forall x(Fx \to Gy)$	2,3,8,9,20 $\vee$E

10. $\vdash \forall x\exists y(Rxy \to \forall zRxz)$

$\varnothing$	(1)	$\exists y \neg Ray \lor \neg \exists y \neg Ray$	prop taut
2	(2)	$\exists y \neg Ray$	A
3	(3)	$\neg Rab$	A
3	(4)	$Rab \to \forall z Raz$	3 prop taut
3	(5)	$\exists y(Ray \to \forall z Raz)$	4 EI
2	(6)	$\exists y(Ray \to \forall z Raz)$	2,3,5 EE
7	(7)	$\neg \exists y \neg Ray$	A
8	(8)	$\neg Rac$	A
8	(9)	$\exists y \neg Ray$	8 EI
7,8	(10)	$\exists y \neg Ray \land \neg \exists y \neg Ray$	9,7 $\land$I
7	(11)	$\neg\neg Rac$	8,10 RA
7	(12)	Rac	11 DN
7	(13)	$\forall z Raz$	12 UI
7	(14)	$Rab \to \forall z Raz$	13 prop taut
7	(15)	$\exists y(Ray \to \forall z Raz)$	14 EI
$\varnothing$	(16)	$\exists y(Ray \to \forall z Raz)$	1,2,6,7,15 $\lor$E
$\varnothing$	(17)	$\forall x \exists y(Rxy \to \forall z Rxz)$	16 UI

Exercise 6.17

$\forall x(\exists z Rxz \to \forall y Rxy), \exists x \exists y \vdash \exists x \forall y Rxy$

1	(1)	$\forall x(\exists z Rxz \to \forall y Rxy)$	A
2	(2)	$\exists x \exists y Rxy$	A
3	(3)	$\exists y Ray$	A
4	(4)	Rab	A
4	(5)	$\exists z Raz$	4 EI
1	(6)	$\exists z Raz \to \forall y Ray$	1 UE
1,4	(7)	$\forall y Ray$	6,5 MP
1,4	(8)	$\exists x \forall y Rxy$	7 EI
1,3	(9)	$\exists x \forall y Rxy$	3,4,8 EE
1,2	(10)	$\exists x \forall y Rxy$	2,3,9 EE

Question: Does it follow from these premises that $\forall x \forall y Rxy$?

Answer: No. $\begin{tikzcd} a \arrow[loop left] \arrow[r] & b \end{tikzcd}$

Chapter 7

Exercise 7.1

Here the proof is lengthened because of the strictness of the = rules. From $a = c$ and $b = c$, we cannot immediately apply $= E$ to get $a = b$.

1	(1)	$\exists x \forall y(Py \to y = x)$	A
2	(2)	$Pa \wedge Pb$	A
3	(3)	$\forall y(Py \to y = c)$	A
3	(4)	$Pa \to a = c$	3 UE
3	(5)	$Pb \to b = c$	3 UE
2	(6)	Pa	2 ∧E
2	(7)	Pb	2 ∧E
2,3	(8)	$a = c$	4,6 MP
2,3	(9)	$b = c$	5,7 MP
∅	(10)	$b = b$	=I
2,3	(11)	$c = b$	10,9 =E
2,3	(12)	$a = b$	8,11 =E
1,2	(13)	$a = b$	1,3,12 EE
1	(14)	$(Pa \wedge Pb) \to a = b$	2,13 CP
1	(15)	$\forall y((Pa \wedge Py) \to a = y)$	14 UI
1	(16)	$\forall x \forall y((Px \wedge Py) \to x = y)$	15 UI

Exercise 7.2

1	(1)	$Fa \wedge \forall x(Fx \to x = a)$	A
2	(2)	Fb	A
1	(3)	$\forall x(Fx \to x = a)$	1 ∧E
1	(4)	$Fb \to b = a$	3 UE
1,2	(5)	$b = a$	4,2 MP
6	(6)	$b = a$	A
1	(7)	Fa	1 ∧E
∅	(8)	$b = b$	=I
6	(9)	$a = b$	8,6 =E
1,6	(10)	Fb	7,9 =E
1	(11)	$Fb \leftrightarrow b = a$	2,5,6,10 CP×2
1	(12)	$\forall x(Fx \leftrightarrow x = a)$	11 UI

Exercise 7.3

(Representing informal claims using quantifiers and identity.) See text for discussion; any logically equivalent formalizations receive full credit.

Exercise 7.4

Let Lxy mean "x loves y". Let a be the speaker and b the speaker's baby.

Assume:

$$\forall x\, Lxb \ \wedge\ \forall x(Lbx \to x = a).$$

From the first conjunct, instantiate $x := b$ to get Lbb. From the second conjunct, $Lbb \to b = a$. Hence $b = a$, so the speaker is his own baby.

Exercise 7.5

Let $\leq$ be a partial ordering satisfying the Total axiom:

$$\forall x \forall y(x \leq y \vee y \leq x).$$

To show the Directed axiom: for all x, y there exists z with $x \leq z$ and $y \leq z$.

Given x, y, by Totality either $x \leq y$ or $y \leq x$. If $x \leq y$, let $z = y$; otherwise let $z = x$. In either case, z is above both x and y.

Exercise 7.6

Assume R is symmetric and transitive:

$$\forall x \forall y(Rxy \to Ryx), \qquad \forall x \forall y \forall z((Rxy \wedge Ryz) \to Rxz).$$

Assume also that R is serial: $\forall x \exists y\, Rxy$.

Fix a. By seriality choose b with Rab. By symmetry, Rba. By transitivity, Raa. Since a was arbitrary, $\forall x\, Rxx$.

Exercise 7.7

(Group-theoretic consequences of A1–A3.)

1. Inverses are unique.
2. Inverse is involutive.
3. Inverse is anti-multiplicative.

(Proofs as in text.)

Exercise 7.8

Assume f is one-to-one but not onto. Then there exists a such that $\forall x(f(x) \neq a)$. The elements a, $f(a)$, and $f(f(a))$ are pairwise distinct. Hence there are more than two objects.

Exercise 7.9

Let $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. Then $A \times B$ has exactly four distinct ordered pairs. Hence $|A \times B| = 4$.

Exercise 7.10

By definition,

$$\emptyset \times A = \{(x, y) \mid x \in \emptyset \wedge y \in A\} = \emptyset.$$

Exercise 7.11

For any x,

$$(h \circ (g \circ f))(x) = h(g(f(x))) = ((h \circ g) \circ f)(x).$$

Hence function composition is associative.

Exercise 7.12

1. *Associativity of addition.*

   Define

   $$\varphi(z) \;\equiv\; \forall x \forall y \big(x + (y + z) = (x + y) + z\big).$$

   We prove $\forall z\, \varphi(z)$ by induction on z.

   Base case. By two applications of P3,

   $$y + 0 = y \quad \text{and} \quad (x + y) + 0 = x + y.$$

   Hence

   $$x + (y + 0) = x + y = (x + y) + 0,$$

   so $\varphi(0)$ holds.

   Induction step. Let a be arbitrary and assume $\varphi(a)$, i.e. $x + (y + a) = (x + y) + a$. Then

   $$\begin{aligned} x + (y + (a + 1)) &= x + ((y + a) + 1) && \text{P4} \\ &= (x + (y + a)) + 1 && \text{P4} \\ &= ((x + y) + a) + 1 && \text{assumption}, = E \\ &= (x + y) + (a + 1) && \text{P4}. \end{aligned}$$

   Thus $\varphi(a + 1)$ holds, and by induction, $PA \vdash \forall x \forall y \forall z \big(x + (y + z) = (x + y) + z\big)$.

2. *Commutativity of addition.*

   We prove $PA \vdash \forall x \forall y(x + y = y + x)$.

   First note that $PA \vdash \forall y(0 + y = y)$, by induction on y. The base case is immediate from P3, and if $0 + y = y$, then

   $$0 + (y + 1) = (0 + y) + 1 = y + 1$$

   by P4.

   Now define $\psi(x) \equiv \forall y(x + y = y + x)$ and prove $\forall x\, \psi(x)$ by induction on x.

   Base case. For all y, we have

   $$0 + y = y \quad \text{and} \quad y + 0 = y$$

   by the preceding lemma and P3. Hence $0 + y = y + 0$.

   Induction step. Assume $\forall y(x + y = y + x)$. We show $\forall y((x + 1) + y = y + (x + 1))$ by induction on y.

   For $y = 0$,

   $$(x + 1) + 0 = x + 1 = 0 + (x + 1).$$

   Assume $(x + 1) + y = y + (x + 1)$. Then

   $$\begin{aligned} (x + 1) + (y + 1) &= ((x + 1) + y) + 1 && \text{P4} \\ &= (y + (x + 1)) + 1 && \text{assumption}, = E \\ &= (y + 1) + (x + 1) && \text{P4.} \end{aligned}$$

   Thus $\forall y((x + 1) + y = y + (x + 1))$, completing the induction.

3. $PA \vdash 0 \neq 1$.

   By the successor axiom, $PA \vdash \forall x(x + 1 \neq 0)$. Instantiating $x = 0$ gives $1 \neq 0$, hence $0 \neq 1$.

4. $PA \vdash 1 + 1 \neq 1$.

   Suppose $1 + 1 = 1$. By P4,

   $$1 + 1 = (1 + 0) + 1 = 1 + 1,$$

   so this equality implies $2 = 1$, i.e. $(1 + 1) = 1$. By injectivity of successor, this would imply $1 = 0$, contradicting part (3). Hence $1 + 1 \neq 1$.

5. *Cancellation for addition.*

   We prove

   $$PA \vdash \forall x \forall y \forall z (x + z = y + z \to x = y)$$

   by induction on z.

   Base case. If $x + 0 = y + 0$, then $x = y$ by P3.

   Induction step. Assume $\forall x \forall y (x + z = y + z \to x = y)$. Suppose $x + (z + 1) = y + (z + 1)$. By P4,

   $$(x + z) + 1 = (y + z) + 1.$$

   By injectivity of successor, $x + z = y + z$, and by the induction hypothesis, $x = y$.

6. *Associativity of multiplication.*

   We prove

   $$PA \vdash \forall x \forall y \forall z \big(x \cdot (y \cdot z) = (x \cdot y) \cdot z\big)$$

   by induction on z.

   Base case. Since $y \cdot 0 = 0$ by P5,

   $$x \cdot (y \cdot 0) = x \cdot 0 = 0 = (x \cdot y) \cdot 0.$$

   Induction step. Assume $x \cdot (y \cdot z) = (x \cdot y) \cdot z$. Then

   $$\begin{aligned} x \cdot (y \cdot (z + 1)) &= x \cdot (y \cdot z + y) && \text{P6} \\ &= x \cdot (y \cdot z) + x \cdot y && \text{distributivity} \\ &= (x \cdot y) \cdot z + x \cdot y && \text{assumption}, = E \\ &= (x \cdot y) \cdot (z + 1) && \text{P6.} \end{aligned}$$

7. *Commutativity of multiplication.*

   We prove

   $$PA \vdash \forall x \forall y (x \cdot y = y \cdot x)$$

   by induction on y.

   Base case. By P5 and a separate induction, $x \cdot 0 = 0 = 0 \cdot x$.

   Induction step. Assume $x \cdot y = y \cdot x$. Then

   $$\begin{aligned} x \cdot (y + 1) &= x \cdot y + x && \text{P6} \\ &= y \cdot x + x && \text{assumption}, = E \\ &= (y + 1) \cdot x && \text{P6.} \end{aligned}$$

   Thus multiplication is commutative.

Exercise 7.15

We work in a language with one binary function symbol $\circ$ and assume:

B1. $\forall x \forall y \forall z\big(x \circ (y \circ z) = (x \circ y) \circ z\big)$

B2. $\forall x \forall z\, \exists! y\, (x \circ y = z)$

B3. $\forall x \forall z\, \exists! w\, (w \circ x = z)$

1. Prove $\forall x \forall z\big((x \circ z = x) \to \forall y(y \circ z = y)\big)$.

   Assume $x \circ z = x$. Let y be arbitrary. By B3 (with x fixed and $z := y$), choose w such that $w \circ x = y$. Then

   $$y \circ z \;=\; (w \circ x) \circ z \;=\; w \circ (x \circ z) \;=\; w \circ x \;=\; y,$$

   where we used B1 in the second equality and the assumption $x \circ z = x$ in the third. Since y was arbitrary, $\forall y(y \circ z = y)$.

2. Prove $\exists! y\, \forall x(x \circ y = x)$.

   By B2 (with $x := a$ and $z := a$), for any fixed a there is a unique y with $a \circ y = a$. Let e be that unique object. By (1), from $a \circ e = a$ we get $\forall x(x \circ e = x)$. This shows existence.

   For uniqueness: suppose $\forall x(x \circ y = x)$ and $\forall x(x \circ y' = x)$. Then $y \circ y = y$ and also $y \circ y' = y$ (using $\forall x(x \circ y' = x)$ with $x := y$). By B2 applied to $x := y$, $z := y$, the equation $y \circ t = y$ has a unique solution t. Both $t = y$ and $t = y'$ satisfy it, hence $y = y'$.

3. Define a name e for the unique y whose existence you just proved.

   (Definition) Introduce the constant symbol e by the defining axiom

   $$\forall u\big(u = e \leftrightarrow \forall x(x \circ u = x)\big).$$

4. Prove $\forall x\big(x = e \leftrightarrow (x \circ x = x)\big)$.

   ($\Rightarrow$) If $x = e$, then by definition $\forall t(t \circ e = t)$, hence $e \circ e = e$, so $x \circ x = x$.

   ($\Leftarrow$) Assume $x \circ x = x$. Then applying (1) with $z := x$ gives $\forall y(y \circ x = y)$, i.e. x is a right-identity. By uniqueness in (2), $x = e$.

5. Since $\forall x \exists! y(x \circ y = e)$, define a function symbol $^{-1}$ such that $\forall x(x \circ x^{-1} = e)$.

   By B2 with $z := e$, for each x there is a unique y with $x \circ y = e$. Introduce a unary function symbol $^{-1}$ by the defining axiom

   $$\forall x\big(x \circ x^{-1} = e\big).$$

6. Prove $\forall x(x^{-1} \circ x = e)$. (Hint: use $(x^{-1} \circ x) \circ (x^{-1} \circ x) = x^{-1} \circ x$.)

   Let $t := x^{-1} \circ x$. We show $t \circ t = t$. Using associativity (B1) twice and $x \circ x^{-1} = e$ from (5),

$$t \circ t = (x^{-1} \circ x) \circ (x^{-1} \circ x) = x^{-1} \circ \big(x \circ (x^{-1} \circ x)\big) = x^{-1} \circ \big((x \circ x^{-1}) \circ x\big) = x^{-1} \circ (e \circ x).$$

   Now, by B3 (with $x := x$ and $z := x$) there is a unique w such that $w \circ x = x$. But $e \circ x$ also satisfies $(e \circ x) \circ x = e \circ (x \circ x) = e \circ x$, so $e \circ x$ is a left-solution of $w \circ x = x$; hence $e \circ x = x$ by uniqueness. Substituting back gives

$$t \circ t = x^{-1} \circ (e \circ x) = x^{-1} \circ x = t.$$

   By (4), from $t \circ t = t$ we infer $t = e$, i.e. $x^{-1} \circ x = e$.

7. Prove $\forall x(e \circ x = x)$.

   Fix x. By (6), $x^{-1} \circ x = e$. Multiply on the left by x and use B1:

$$x \circ (x^{-1} \circ x) = (x \circ x^{-1}) \circ x = e \circ x.$$

   But $x \circ (x^{-1} \circ x) = (x \circ x^{-1}) \circ x = e \circ x$, and also $x \circ x^{-1} = e$ by (5), so the left-hand side equals $e \circ x$; hence $e \circ x = x$. Since x was arbitrary, $\forall x(e \circ x = x)$.

Chapter 8

Exercise 8.1

1. Countermodel M_1:

$$D_1 = \{a, b\}, \qquad F^{M_1} = \{b\}, \qquad c^{M_1} = a.$$

   Then $M_1 \vDash \exists x Fx$ (witness b), but $M_1 \nvDash Fc$.

2. Countermodel M_2:

$$D_2 = \{a, b\}, \qquad F^{M_2} = \{a\}, \qquad c^{M_2} = a.$$

   Then $M_2 \vDash Fc$, but $M_2 \nvDash \forall x Fx$ (since $b \notin F^{M_2}$).

3. Countermodel M_3:

$$D_3 = \{a, b\}, \qquad F^{M_3} = \{a\}, \qquad G^{M_3} = \{b\}.$$

   Then $M_3 \vDash \exists x Fx \wedge \exists x Gx$, but $M_3 \nvDash \exists x(Fx \wedge Gx)$.

4. Countermodel M_4:

$$D_4 = \{a, b\}, \qquad F^{M_4} = \{a\}, \qquad G^{M_4} = \{b\}.$$

We have $\forall x Fx$ false and $\forall x Gx$ false, so $M_4 \vDash (\forall x Fx \to \forall x Gx)$ (false $\to$ false is true), but for $x = a$ we get $Fa \wedge \neg Ga$, so $M_4 \nvDash \forall x(Fx \to Gx)$.

5. Countermodel M_5:

$$D_5 = \{a\}, \qquad F^{M_5} = \varnothing, \qquad H^{M_5} = \{a\}.$$

Then for the only element a, Fa is false, so $Fa \to Ha$ is true; hence $M_5 \vDash \forall x(Fx \to Hx)$. But $M_5 \nvDash \exists x Fx$ and $M_5 \vDash \exists x Hx$, so $M_5 \nvDash \exists x Fx \vee \neg \exists x Hx$.

6. Countermodel M_6:

$$D_6 = \{a\}, \qquad F^{M_6} = \varnothing, \qquad G^{M_6} = \varnothing.$$

Again Fa is false, so $Fa \to Ga$ is true; hence $M_6 \vDash \forall x(Fx \to Gx)$. But $F^{M_6} \cap G^{M_6} = \varnothing$, so $M_6 \nvDash \exists x(Fx \wedge Gx)$.

7. Countermodel M_7:

$$D_7 = \{a, b\}, \qquad F^{M_7} = \{a\}, \qquad G^{M_7} = \{a, b\}, \qquad H^{M_7} = \{b\}.$$

Then a witnesses $\exists x(Fx \wedge Gx)$, and b witnesses $\exists x(Gx \wedge Hx)$. But there is no element in $F^{M_7} \cap H^{M_7}$, so $M_7 \nvDash \exists x(Fx \wedge Hx)$.

8. Countermodel M_8:

$$D_8 = \{a, b\}, \qquad F^{M_8} = \{a\}.$$

Then $M_8 \nvDash \forall x Fx$ (since $b \notin F^{M_8}$), and $M_8 \nvDash \forall x \neg Fx$ (since $a \in F^{M_8}$). Hence $M_8 \nvDash \forall x Fx \vee \forall x \neg Fx$.

9. Countermodel M_9:

$$D_9 = \{a, b\}, \qquad F^{M_9} = \{a, b\}, \qquad G^{M_9} = \{a\}, \qquad H^{M_9} = \varnothing.$$

Then:

$$\begin{aligned}
&M_9 \vDash \exists x(Fx \to Gx) && \text{(take } x = a,\ Fa \to Ga \text{ is true)},\\
&M_9 \vDash \exists x(Gx \to Hx) && \text{(take } x = b,\ Gb \text{ is false so } Gb \to Hb \text{ is true)},\\
&M_9 \nvDash \exists x(Fx \to Hx) && \text{since for both } a, b,\ Fx \text{ is true and } Hx \text{ is false,}\\
& && \text{so } Fx \to Hx \text{ is false everywhere.}
\end{aligned}$$

10. Countermodel M_{10}:

$$D_{10} = \{a, b\}, \qquad F^{M_{10}} = \{a\}, \qquad G^{M_{10}} = \varnothing.$$

Then $M_{10} \vDash \exists x(Fx \to Gx)$ (take $x = b$, where Fb is false), while $\exists x Fx$ is true (witness a) and $\exists x Gx$ is false. Hence $M_{10} \nvDash \exists x Fx \to \exists x Gx$.

Exercise 8.2

Let M be an interpretation and let φ, ψ have (at most) the free variable x. Recall that

$$(\varphi \to \psi)^M = \{\, a \in M : \text{if } a \in \varphi^M \text{ then } a \in \psi^M \,\}.$$

So $(\varphi \to \psi)^M = M$ iff for every $a \in M$, $a \in \varphi^M$ implies $a \in \psi^M$, which is exactly $\varphi^M \subseteq \psi^M$.

Exercise 8.3

1. $\forall x Fx \to P \nvDash \forall x(Fx \to P)$.

   Countermodel M_1:

   $$D = \{a, b\}, \qquad F^M = \{a\}, \qquad P^M = 0.$$

   Then $\forall x Fx$ is false (since $b \notin F^M$), so

   $$(\forall x Fx \to P)^M = (0 \to 0) = 1.$$

   However,

   $$(Fb \to P)^M = (0 \to 0) = 1, \qquad (Fa \to P)^M = (1 \to 0) = 0,$$

   so $\forall x(Fx \to P)$ is false. Thus the premise is true and the conclusion false in M_1.

2. $\exists x(Fx \to P) \nvDash \exists x Fx \to P$.

   Countermodel M_2:

   $$D = \{a, b\}, \qquad F^M = \{a\}, \qquad P^M = 0.$$

   In this case, $(Fb \to P)^{M_2} = (0 \to 0) = 1$, so $\exists x(Fx \to P)^{M_2} = 1$. But $(Fa)^{M_2} = 1$, so $(\exists x Fx)^{M_2} = 1$ and $(\exists x Fx \to P)^{M_2} = (1 \to 0) = 0$. Hence the premise is true while the conclusion is false in M_2.

Exercise 8.4

In the given structure, $\circ$ is addition mod 3 on $\{0, 1, 2\}$ (since $2 \circ 2 = 1$). An idempotent is an element e such that $e \circ e = e$. Checking:

$$0 \circ 0 = 0, \qquad 1 \circ 1 = 2 \neq 1, \qquad 2 \circ 2 = 1 \neq 2.$$

So the unique idempotent is 0.

The inverse function i satisfies $x \circ i(x) = 0 = i(x) \circ x$. In $\mathbb{Z}_3$ this is just additive inverse:

$$i(0) = 0, \qquad i(1) = 2, \qquad i(2) = 1.$$

Exercise 8.5

No: T does not imply commutativity. Let M be the set of permutations of $\{1, 2, 3\}$:

$$M = \{\, \mathrm{id}, (123), (132), (12), (13), (23) \,\},$$

and interpret $\circ$ as composition of permutations. Then $\circ$ is associative, and for any fixed $a \in M$ the maps $y \mapsto a \circ y$ and $y \mapsto y \circ a$ are bijections of M, so the required left- and right-transitivity clauses of T hold. But $\circ$ is not commutative; for example,

$$(12) \circ (23) = (123) \neq (132) = (23) \circ (12).$$

Hence $T \nvdash \forall x \forall y (x \circ y = y \circ x)$.

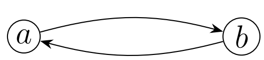
Exercise 8.6

Suppose R is symmetric and transitive and total (for distinct elements), and assume also that R is antireflexive, i.e. $\forall x\, \neg Rxx$. Pick distinct $a \neq b$ in the domain. By totality, Rab or Rba; by symmetry, in either case we get both Rab and Rba. Then by transitivity, from Rab and Rba we infer Raa, contradicting antireflexivity. So there is no such relation on any domain with at least two elements. (If "total" is taken to mean $\forall x \forall y (Rxy \vee Ryx)$, then taking $x = y$ already forces Rxx, contradicting antireflexivity on any nonempty domain.)

Exercise 8.7

1. $\forall x \forall y (Rxy \to Ryx)$

   True model (symmetric):

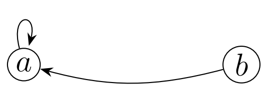
   

   False model (one-way arrow):

2. $\forall x \forall y \exists z (Rxz \wedge Ryz)$

   True model (common successor a for everyone):

   (For any x, y, choose $z = a$.)

   False model (no common successor for a, b):

   (For $x = a, y = b$ there is no z with both $a \to z$ and $b \to z$.)

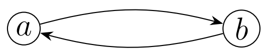
3. $\exists x \forall y(Ryx \to Ryy)$

   True model (choose $x = a$ with no incoming arrows):

   (No y satisfies $y \to a$, so $Ryx \to Ryy$ holds vacuously for all y.)

   False model (every x has an incoming arrow from a non-reflexive y):

   (No loops, so Ryy is always false; but each node has an incoming arrow.)

4. $\forall x(\exists y\, Ryx \to \forall z\, Rzx)$

   True model (empty relation):

   (Each antecedent $\exists y\, Ryx$ is false, so the implication is true for all x.)

   False model (some x has an incoming arrow but not everyone points to x):

   (Take $x = b$: $\exists y\, Ryb$ holds (witness a), but $\forall z\, Rzb$ fails since $b \not\to b$.)

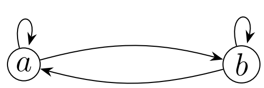
5. $\exists x \exists y(Rxy \leftrightarrow \neg Ryy)$

   True model (take $x = a, y = b$):

   (Here $a \to b$ is true and $b \to b$ is false, so $\neg Rbb$ is true and the biconditional holds.)

   False model (universal relation on $\{a, b\}$):

   (For every y, Ryy is true, hence $\neg Ryy$ is false; but Rxy is always true. So $Rxy \leftrightarrow \neg Ryy$ is false for all x, y.)

Exercise 8.8

Yes, it is consistent. Let the domain be $\mathbb{Z}$ and interpret R as the strict order $<$. Then Rxx is never true, so $\neg Rxx$ holds for all x. Given any $x \in \mathbb{Z}$, choose $y = x - 1$. Then $x < y$ is true, and for all z: if $y < z$ then certainly $x < z$ (since $x < y < z$), so $(Ryz \to Rxz)$ holds. Thus $\forall x \exists y \forall z(\neg Rxx \wedge Rxy \wedge (Ryz \to Rxz))$ is true in this model.

Exercise 8.9

Two examples of sentences true in $(\mathbb{N}, \leq)$ but not consequences of the theory of partial orders:

(1) "There is a least element":

$$\exists x\, \forall y\, (x \leq y).$$

True in $(\mathbb{N}, \leq)$ (take $x = 0$), but false in $(\mathbb{Z}, \leq)$, which is a partial order.

(2) "The order is total":

$$\forall x \forall y\, \big(x \leq y \ \vee \ y \leq x\big).$$

True in $(\mathbb{N}, \leq)$, but false in a poset with incomparable elements, e.g. a two-element antichain where neither element is $\leq$ the other.

Exercise 8.10

Let M be any interpretation.

(a) $[x, y : x = y]^M$ is the set of pairs $(a, b) \in M^2$ such that $a = b$, i.e. the diagonal:

$$[x, y : x = y]^M = \{(a, a) : a \in M\} \subseteq M^2.$$

(b) $[x : x = x]^M$ is the set of $a \in M$ such that $a = a$, i.e. all of M:

$$[x : x = x]^M = M.$$

Exercise 8.11

Attempting to mimic the proof for $\exists x \forall y(Fx \to Fy)$ breaks down in the case where F^M is neither empty nor all of M.

Indeed, take a countermodel: let the domain be $\{a, b\}$ and let $F^M = \{a\}$. Then Fa is true and Fb is false, so $(Fa \to Fb)$ is false. Hence

$$\forall x \forall y(Fx \to Fy)$$

fails in this interpretation. Concretely, the universal quantifiers force us to check the pair $x = a, y = b$, and at that point the implication is false.

Chapter 9

Exercise 9.6

Suppose that φ is true in an even number n of rows of its truth table. Then $\neg\varphi$ is true in $4 - n$ rows of its truth table, and $4 - n$ is also even.

Suppose that both φ and ψ are even. Let's say that row r is an *agreement row* if φ and ψ have the same truth value on r. We will show that there cannot be 1 or 3 agreement rows. Suppose that there is a single row where both sentences have value a. Since φ and ψ are even, a must occur on another row in each of their truth tables. If these rows are not the same, then there are two of them, which leaves a single remaining row. In that row, both φ and ψ must have value $1-a$, and so they agree there.

Suppose now that there are three rows where both sentences have the same value, and let r be the remaining row. Since three is odd, one of the two truth values a must occur most frequently on these rows. If a occurs twice and $1-a$ occurs once, then $1-a$ must be the value of both φ and ψ on row r. If a occurs three times, then a must be the value of both φ and ψ on row r. In either case, φ and ψ agree on row r.

Exercise 9.7

No, the set $\{\neg, \leftrightarrow\}$ is not truth-functionally complete. There is a binary truth-function that has output a single 1 and three 0. For example, take the sentence $P \wedge Q$. By Exercise 9.6, every sentence in the set Γ generated from P, Q and $\{\neg, \leftrightarrow\}$ has an even number of 1 in its truth table. Therefore, there is no sentence in Γ that is provably equivalent to $P \wedge Q$.

Exercise 9.12

Suppose that φ is contingent, and let $P_0, \dots, P_n$ be a list of the atomic sentences that occur in φ. Since φ is contingent, there is a valuation v such that $v(\varphi) = 0$. Let $\perp$ be an arbitrary contradiction, and let $\top$ be an arbitrary tautology. Define $F(P_i) = \top$ if $v(P_i) = 1$, and $F(P_i) = \perp$ if $v(P_i) = 0$. We claim, then, that the substitution instance $F(\varphi)$ is inconsistent. Let w be an arbitrary valuation. For any P_i, $w(F(P_i)) = w(\top) = 1$ if $v(P_i) = 1$, and $w(F(P_i)) = w(\perp) = 0$ if $v(P_i) = 0$. So $w(F(\cdot))$ and $v(\cdot)$ agree on atomic sentences. But $w(F(\cdot))$ and $v(\cdot)$ are both truth-functional, so they agree on all sentences. Therefore, $w(F(\varphi)) = v(\varphi) = 0$. Since w was arbitrary, $F(\varphi)$ is an inconsistency.

Exercise 9.14

1. As a warmup, we will show that all occurences of $\rightarrow$ can be eliminated from valid proofs, along with all uses of MP and CP. Define a function f from sentences to sentences as the identity on atomic sentences, then extend by commuting with $\wedge, \vee, \neg$, and by setting $f(\varphi \rightarrow \psi) = \neg(f(\varphi) \wedge \neg f(\psi))$. We now show that any proof of $\varphi_1, \dots, \varphi_n \vdash \psi$ can be converted to a proof of $f(\varphi_1), \dots, f(\varphi_n) \succ f(\psi)$.

   Here's a way to simulate CP. Suppose first that φ is assumed, and that ψ is derived with dependencies Δ. We can then continue in this way:

$$\begin{array}{r|c|l|l}
1 & (1) & \varphi & \text{A} \\
\Delta & (2) & \psi & \\
3 & (3) & \varphi \wedge \neg\psi & \text{A} \\
3 & (4) & \neg\psi & 3\ \wedge\text{E} \\
\Delta, 3 & (5) & \psi \wedge \neg\psi & 2,4\ \wedge\text{I} \\
\Delta', 3 & (6) & \neg\varphi & 1,5\ \text{RA} \\
3 & (7) & \varphi & 3\ \wedge\text{E} \\
\Delta', 3 & (8) & \varphi \wedge \neg\varphi & 7,6\ \wedge\text{I} \\
\Delta' & (9) & \neg(\varphi \wedge \neg\psi) & 3,8\ \text{RA}
\end{array}$$

Here $\Delta' = \Delta \setminus \{1\}$, so that line 9 reproduces the effect of CP on lines 1 and 2.

Now we can simulate MP.

$$\begin{array}{r|c|l|l}
\Gamma & (1) & \neg(\varphi \wedge \neg\psi) & \\
\Delta & (2) & \varphi & \\
3 & (3) & \neg\psi & \text{A} \\
\Delta, 3 & (4) & \varphi \wedge \neg\psi & 2,3\ \wedge\text{I} \\
\Gamma, \Delta, 3 & (5) & (\varphi \wedge \neg\psi) \wedge \neg(\varphi \wedge \neg\psi) & 4,1\ \wedge\text{I} \\
\Gamma, \Delta & (6) & \neg\neg\psi & 3,5\ \text{RA} \\
\Gamma, \Delta & (7) & \psi & 6\ \text{DN}
\end{array}$$

2. We need to show that any application of RA can be simulated by the other rules. Suppose that we have the following lines

$$\begin{array}{llll}
1 & (1) & P & \text{A} \\
\Delta & (2) & Q \wedge \neg Q &
\end{array}$$

We need to show that we can derive the line

$$\begin{array}{lll}
\Delta' & (c) & \neg P
\end{array}$$

without using RA. We first derive $\Delta' \succ P \to \neg P$ as follows:

$$\begin{array}{llll}
1 & (1) & P & \text{A} \\
\Delta & (2) & Q \wedge \neg Q & \\
\Delta & (3) & Q & 2\ \wedge\text{E} \\
\Delta' & (4) & P \to Q & 1{,}3\ \text{CP} \\
\Delta & (5) & \neg Q & 2\ \wedge\text{E} \\
\Delta & (6) & \neg P & 4{,}5\ \text{MT} \\
\Delta' & (7) & P \to \neg P & 1{,}6\ \text{CP}
\end{array}$$

The proof that $\succ (P \to \neg P) \to \neg P$ is Exercise 3.1.10, plus one step of CP. Put those two together and $\Delta' \succ \neg P$ follows.

3. We show that the DN introduction rule can be reproduced from the other rules.

1	(1)	P	A
2	(2)	$\neg P$	A
1,2	(3)	$P \wedge \neg P$	1,2 $\wedge$I
1	(4)	$\neg\neg P$	2,3 RA

4. We show that MT can be reproduced from the other rules.

1	(1)	$P \to Q$	A
2	(2)	$\neg Q$	A
3	(3)	P	A
1,3	(4)	Q	1,3 MP
1,2,3	(5)	$Q \wedge \neg Q$	4,2 $\wedge$I
1,2	(6)	$\neg P$	3,5 RA

5. Redefine the truth-table for $\vee$ as follows:

P	Q	$P \vee Q$
1	1	1
1	0	1
0	1	1
0	0	1

In other words, $P \vee Q$ is constantly 1, regardless of the input. Since none of the inference rules besides $\vee$E uses a disjunction as a premise, those rules are truth-preserving relative to the new truth tables. We claim now that those rules cannot prove $P \vee P \vdash P$. Consider the rows of the (new) truth-table in which P is 0. In this case, $P \vee P$ is 1, but P is 0. Hence $P \vee P \vdash P$ is not truth-preserving relative to the new truth tables, and it cannot be proven by those rules.

6. If we read this problem literally, then it admits trivial solutions. For example, we we permit "nand" statements to be inferred only from contradictions, and if permit only tautologies to be inferred from "nand" statement, then the resulting system of rules is sound. However, the spirit of this problem is to provide intro and elim rules for $\uparrow$ that capture the meaning of the connective.

NAND-Introduction ($\uparrow$I)

If Δ together with P and Q imply $\bot$, then Δ implies $P \uparrow Q$.

a	(a)	P	A
b	(b)	Q	A
Δ	(c)	$\bot$	
Δ'	(d)	$P \uparrow Q$	a, b, c $\uparrow$I

where $\Delta' = \Delta - \{a, b\}$.

NAND-Elimination (↑E)

From $P \uparrow Q$, together with P and Q, infer $\bot$.

Γ	(a)	$P \uparrow Q$	
Δ	(b)	P	
Σ	(c)	Q	
Γ, Δ, Σ	(d)	$\bot$	a, b, c ↑E

Falsum-Elimination (⊥E)

For the ↑ rules to do enough, we need to add a ⊥-elimination rule.

Γ	(a)	$\bot$	
Γ	(b)	Q	a ⊥E