

# Existential Introduction & Elimination

PHI 201 — Introductory Logic

Lecture 8

# Overview

- Review of universal quantifier rules
- New rules for the existential quantifier:
  - Existential Introduction (**EI**)
  - Existential Elimination (**EE**)
- Practice proofs involving  $\exists$  and  $\forall$

# Existential Introduction

## Rule

From a particular instance  $Fa$ , we may infer that something is  $F$ :

$$\frac{Fa}{\exists x Fx} \text{ (EI)}$$

1	(1)	$Fa \rightarrow Ga$	A
1	(2)	$\exists x(Fx \rightarrow Gx)$	1 EI

1	(1)	$Fa \rightarrow Ga$	A
1	(2)	$\exists x(Fx \rightarrow Ga)$	1 EI

Unlike UI, EI permits replacement of some (but not all) occurrences of a name  $a$ .

1	(1)	$Raa$	A
1	(2)	$\exists x Rxx$	1 EI

1	(1)	$Raa$	A
1	(2)	$\exists y Ray$	1 EI
1	(3)	$\exists x \exists y Rxy$	2 EI

To show:  $\neg\exists xFx \vdash \forall x\neg Fx$

1	(1)	$\neg\exists xFx$	A
2	(2)	$Fa$	A
2	(3)	$\exists xFx$	2 EI
1,2	(4)	$\exists xFx \wedge \neg\exists xFx$	3,1 $\wedge$ I
1	(5)	$\neg Fa$	2,4 RA
1	(6)	$\forall x\neg Fx$	5 UI

1	(1)	$\neg \forall x Fx$	A
2	(2)	$\neg \exists x \neg Fx$	A
3	(3)	$\neg Fa$	A
3	(4)	$\exists x \neg Fx$	3 EI
2,3	(5)	$\exists x \neg Fx \wedge \neg \exists x \neg Fx$	4,2 $\wedge$ I
2	(6)	$\neg \neg Fa$	3,5 RA
2	(7)	$Fa$	6 DN
2	(8)	$\forall x Fx$	7 UI
1,2	(9)	$\forall x Fx \wedge \neg \forall x Fx$	8,1 $\wedge$ I
1	(10)	$\neg \neg \exists x \neg Fx$	2,9 RA
1	(11)	$\exists x \neg Fx$	10 DN

To show:  $\neg \forall x Fx \vdash \exists x(Fx \rightarrow P)$



To show  $\neg\exists xFx \vdash \forall x(Fx \rightarrow P)$

1	(1)	$\neg\exists xFx$	A
2	(2)	$Fa$	A
2	(3)	$\exists xFx$	2 EI
1,2	(4)	$\exists xFx \wedge \neg\exists xFx$	3,1 $\wedge$ I
1	(5)	$\neg Fa$	2,4 RA
1	(6)	$Fa \rightarrow P$	5 neg par
1	(7)	$\forall x(Fx \rightarrow P)$	6 UI

# Existential elimination

# Idea

What can be derived from  $\exists x\varphi(x)$ ?

Pick an arbitrary name  $a$ . If a general claim  $\psi$  can be derived from an instance  $\varphi(a)$ , without making any additional assumptions about  $a$ , then  $\psi$  follows from  $\exists x\varphi(x)$ .

# Existential Elimination

$\Gamma \quad (\text{m}) \quad \exists x \varphi(x)$

$n \quad (\text{n}) \quad \varphi(a)$

$A$

$\Delta \quad (\text{o}) \quad \psi$

$\Gamma, \Delta \setminus \{n\} \quad (\text{p}) \quad \psi$

$m, n, o \text{ EE}$

# Existential Elimination

- **Restriction:** The name  $a$  must not occur in any dependencies of the existential premise, or of the derived conclusion, except for the instance  $\varphi(a)$  itself.
- Dependencies are the union of the dependencies of the existential sentence and those of the derived conclusion, minus dependency on the instance.

# Example

1	(1)	$\exists x(Fx \wedge Gx)$	A
2	(2)	$Fa \wedge Ga$	A
2	(3)	$Fa$	2 $\wedge E$
2	(4)	$\exists xFx$	3 EI
1	(5)	$\exists xFx$	1,2,4 EE

# Reasoning with multiple $\exists$

To show:  $\exists x \exists y Rxy \vdash \exists y \exists x Rxy$

1	(1)	$\exists x \exists y Rxy$	A
2	(2)	$\exists y Ray$	A
3	(3)	$Rab$	A
3	(4)	$\exists x Rxb$	3 EI
3	(5)	$\exists y \exists x Rxy$	4 EI
2	(6)	$\exists y \exists x Rxy$	2,3,5 EE
1	(7)	$\exists y \exists x Rxy$	1,2,6 EE

# Reasoning with $\forall$ and $\exists$

1	(1)	$\forall x(Fx \rightarrow Gx)$	A
2	(2)	$\exists xFx$	A
3	(3)	$Fa$	A
1	(4)	$Fa \rightarrow Ga$	1 UE
1,3	(5)	$Ga$	4,3 MP
1,3	(6)	$\exists xGx$	5 EI
1,2	(7)	$\exists xGx$	2,3,6 EE



# Reasoning with $\forall$ and $\exists$

1	(1)	$\exists y \forall x Rxy$	A
2	(2)	$\forall x Rxb$	A
2	(3)	$Rab$	2 UE
2	(4)	$\exists y Ray$	3 EI
2	(5)	$\forall x \exists y Rxy$	4 UI
1	(6)	$\forall x \exists y Rxy$	1,2,5 EE

# Preventing invalid inferences

1	(1)	$\exists x Fx$	A
2	(2)	$\exists x Gx$	A
3	(3)	$Fa$	A
4	(4)	$Ga$	A
3,4	(5)	$Fa \wedge Ga$	3,4 $\wedge I$
3,4	(6)	$\exists x(Fx \wedge Gx)$	5 EI

EE cannot be applied to 1,3,6 because 6 depends on 4, which contains  $a$ .

EE cannot be applied to 1,4,6 because 6 depends on 3, which contains  $a$ .

# Preventing invalid inferences

1	(1)	$\forall x \exists y Rxy$	A
1	(2)	$\exists y Ray$	1 UE
3	(3)	$Rab$	A
3	(4)	$\forall x Rxb$	Error!

UI cannot be applied to 3 because it depends on 3, which contains  $a$ .

# Quantifier order matters

$\forall x \exists y \varphi(x, y)$  follows from  $\exists y \forall x \varphi(x, y)$ .

But not vice versa.

# Quantifier negation equivalences

$\neg \exists x Fx$  is equivalent to  $\forall x \neg Fx$

$\neg \forall x Fx$  is equivalent to  $\exists x \neg Fx$

“Equivalent” means mutually derivable

To show:  $\forall x \neg Fx \vdash \neg \exists x Fx$

1	(1)	$\forall x \neg Fx$	A
2	(2)	$\exists x Fx$	A
3	(3)	$Fa$	A
1	(4)	$\neg Fa$	1 UE
1,3	(5)	$Fa \wedge \neg Fa$	3,4 $\wedge I$
1,3	(6)	$\neg \exists x Fx$	2,5 RA
1,2	(7)	$\neg \exists x Fx$	2,3,6 EE
1,2	(8)	$\exists x Fx \wedge \neg \exists x Fx$	2,7 $\wedge I$
1	(9)	$\neg \exists x Fx$	2,8 RA

# Non-constructive existence proofs

How to derive  $\exists x(Fx \rightarrow P)$  from  $\forall xFx \rightarrow P$ ?

Not possible to derive  $Fa \rightarrow P$  from  $\forall xFx \rightarrow P$ .