Interpretations for n-ary relations

PHI 201 – Introductory Logic

Week 9

How do we know that there isn't a valid proof with lines like this?

1	(1)	$\forall x \exists y Rxy$	A
	$\vdots$		
1	(n)	$\exists y \forall x Rxy$	

What kind of thing in the universe of sets should be the interpretation of an n-ary relation symbol?

We say that $A \times B$ is a **Cartesian product** of sets A and B just in case: elements of $A \times B$ are in one-to-one correspondence with pairs of elements from A and B.

We can represent elements of $A \times B$ as ordered pairs: $\{\langle a, b\rangle \mid a \in A, b \in B\}$.

$\langle a, b\rangle = \langle c, d\rangle$ iff $a = c$ and $b = d$.

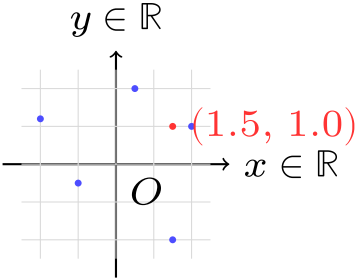
Example of Cartesian product

Let $A = \{1, 2, 3\}$ and $B = \{\alpha, \beta\}$.

$A \times B = \{(1, \alpha), (1, \beta), (2, \alpha), (2, \beta), (3, \alpha), (3, \beta)\}$.

Since $|A| = 3$ and $|B| = 2$, we have $|A \times B| = 3 \times 2 = 6$.

Cartesian Product and the Plane $\mathbb{R}^2$



- The Cartesian product $\mathbb{R} \times \mathbb{R}$ forms the **Euclidean plane**.
- Each point corresponds to a unique ordered pair (x, y).
- This idea generalizes to $\mathbb{R}^3, \mathbb{R}^4$, and beyond.

- The **extension** of a relation on a set A is a subset of the set $A \times A$ of ordered pairs.
- Example: The extension of the relation "x is legally married to y in the US" is the set of all pairs of people who are legally married in the US.

Test your understanding

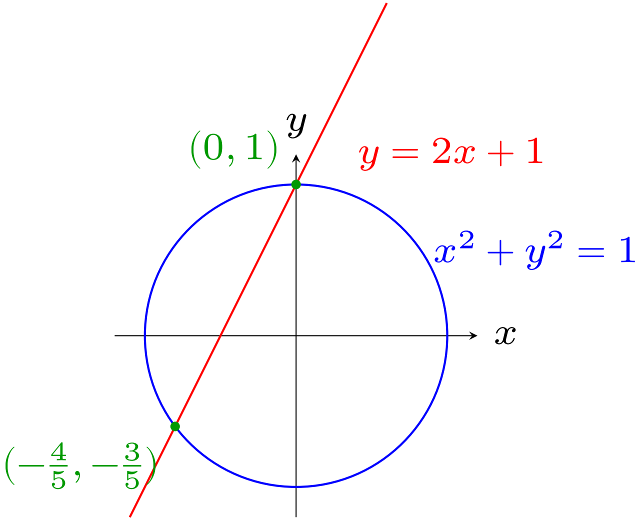
- What is the extension of $x \neq y$ as a relation on the natural numbers?
- What is the extension of $x = y$ as a relation on the natural numbers?
- $x \neq x$ is not a relation, it's a predicate.

Test your understanding

- What is the extension of the relation $y = 2x + 1$ on the real numbers?
- What is the extension of the relation

$$(y = 2x + 1) \wedge (x^2 + y^2 = 1)$$

  on the real numbers?


y
$(0, 1)$
$y = 2x + 1$
$x^2 + y^2 = 1$
x
$(-\frac{4}{5}, -\frac{3}{5})$

Worked problem

Show that $\exists x \exists y Rxy$ does not logically imply $\exists x Rxx$.

Discovering and presenting interpretations

- With more experience, you get to know structures that can be used as interpretations.
- In many cases, it will suffice to work with a small number of nodes in a graph-like structure.

Worked example

$$\forall x \forall y (Rxy \to Ryx)$$

The decision problem

- If the task is to decide whether a **propositional** sequent is valid, then there is a **algorithm** that settles the question.
- Princeton's own Alonzo Church, along with his student Alan Turing, proved that there is no such algorithm for logic with relation symbols.
  - Most interesting mathematical theories, e.g. arithmetic, set theory, are undecidable.

What this means for you

- I cannot teach you a procedure that will always find a model if a sentence is consistent (or a counterexample if a sequent is invalid).
- But I can give you a tour of the infinite universe of structures that could serve as counterexamples.

Worked example

$\forall x \forall y \exists z (Rxz \wedge Ryz)$

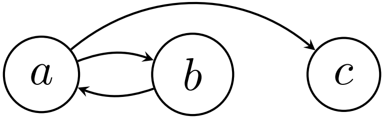
Domain: $\{a, b\}$, R: $\{\langle a, b\rangle, \langle b, b\rangle\}$

Extension of $(Rxz \wedge Ryz)$: $\{\, \langle a, a, b\rangle,\ \langle a, b, b\rangle,\ \langle b, a, b\rangle,\ \langle b, b, b\rangle \,\}$

Extension of $\exists z(Rxz \wedge Ryz)$: $\{\, \langle a, a\rangle,\ \langle a, b\rangle,\ \langle b, a\rangle,\ \langle b, b\rangle \,\}$

Example

What is the extension of $\exists y Rxy$?



Rxy	a	b	c
a		✓	✓
b	✓		
c			