Theories

PHI 201 — Introductory Logic

Week 8

The utopian vision of symbolic logic

- Two original hopes for symbolic logic.
  1. It provides a universal language for science.
  2. It dissolves philosophical pseudo-problems.
- While this doesn't work out so easily in practice, there is a sense in which all "theories" in mathematics can be formalized in predicate/relational logic.

Equality

Equality is a special relation

- Equality is a binary relation which we write as an infix rather than as a prefix

$$c = d,\ \exists x(x = d),\ \forall y \exists x(x = y)$$

- Using "$=$" allows us to express many new things.

At least n

$$\exists x \exists y (x \neq y)$$

$$\exists x \exists y \exists z ((x \neq y \land x \neq z) \land y \neq z)$$

At most n

$$\forall x \forall y (x = y)$$

$$\forall x \forall y \forall z ((x = y \vee x = z) \vee y = z)$$

Exactly n

$$\exists x \exists y (x \neq y \wedge \forall z (z = x \vee z = y))$$

$$\exists x\, \exists y\, \exists z \Big(\big((x \neq y \wedge x \neq z) \wedge y \neq z \big) \\ \wedge \forall w \, \big((w = x \vee w = y) \vee w = z \big) \Big)$$

There is a unique P

$$\exists x(Px \land \forall y(Py \to x = y))$$

Definite descriptions

Superlatives

"There is a tallest student."

$$\exists x \forall y (x \neq y \rightarrow Txy)$$

This sentence entails uniqueness only because we implicitly assume that "taller than" is asymmetric.

$$\forall x \forall y (Txy \rightarrow \neg Tyx)$$

1	(1)	$\exists x \forall y (x \neq y \to Txy)$	A
2	(2)	$\forall y (a \neq y \to Tay)$	A
3	(3)	$\forall y (b \neq y \to Tby)$	A
4	(4)	$a \neq b$	A
2	(5)	$a \neq b \to Tab$	2 UE
3	(6)	$b \neq a \to Tba$	3 UE

Inference rules for equality

Γ	(m)	$\varphi(a)$	
Δ	(n)	$a = b$	
Γ, Δ	(o)	$\varphi(b)$	m,n $=$E

To show: $a = b, b = c \vdash a = c$

1	(1)	$a = b$	A
2	(2)	$b = c$	A
1,2	(3)	$a = c$	2,1 $=$E

Inference rules for equality

(m) $a = a$ $\qquad\qquad\qquad\qquad =\mathrm{I}$

To show: $a = b \vdash b = a$

1	(1)	$a = b$	A
	(2)	$a = a$	=I
1	(3)	$b = a$	2,1 =E

Nobody but

Alice respects nobody but Bob.

$$Rab \land \forall x(Rax \rightarrow x = b)$$

$$\forall x(Rax \leftrightarrow x = b)$$

Everybody loves my baby

1	(1)	$\forall x Lxb$	A
2	(2)	$\forall y(Lby \rightarrow y = a)$	A
1	(3)	Lbb	1 UE
2	(4)	$Lbb \rightarrow b = a$	2 UE

- The theory of equality is peculiar, because we build its axioms in as new inference rules.
- Now we look at theories whose axioms are sentences.

Theory of partial order

transitive:

$$\forall x \forall y \forall z((x \leq y \land y \leq z) \to x \leq z)$$

reflexive:

$$\forall x(x \leq x)$$

antisymmetric:

$$\forall x \forall y((x \leq y \land y \leq x) \to x = y)$$

A cornucopia of partially ordered sets

linear:

$$\forall x \forall y ((x \leq y) \vee (y \leq x))$$

- What's a sentence that is true of the natural numbers $1, 2, 3, \ldots$ but false of the integers $\ldots, -2, -1, 0, 1, 2, \ldots$?
- What's a sentence that is true of the integers but false of the rational numbers?

Set theory

extensionality

$$\forall x \forall y (x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y))$$

existence of an emptyset

$$\exists z \forall x (x \notin z)$$

Uniqueness of the emptyset

1	(1)	$\forall x(x \notin a)$	A
2	(2)	$\forall x(x \notin b)$	A
1	(3)	$c \notin a$	1 UE
1	(4)	$c \in a \rightarrow c \in b$	3 neg par

Naive set theory

comprehension

$$\exists x \forall y (y \in x \leftrightarrow \varphi(y))$$

Consistent theories

We say that a theory T is **consistent** if there is no sentence φ such that both $T \vdash \varphi$ and $T \vdash \neg\varphi$.

Naive set theory is inconsistent

Use comprehension with the predicate "$y \notin y$"

1	(1)	$\exists x \forall y (y \in x \leftrightarrow y \notin y)$	A
2	(2)	$\forall y (y \in a \leftrightarrow y \notin y)$	A
2	(3)	$a \in a \leftrightarrow a \notin a$	2 UE

Sophisticating set theory

pairing

$$\forall x \forall y \exists z \forall w (w \in z \leftrightarrow (w = x \vee w = y))$$

separation: For every formula $\varphi(x, b_1, \ldots, b_n)$,

$$\forall y \exists z \forall x \Big(x \in z \ \leftrightarrow \ (x \in y \wedge \ \varphi(x, b_1, \ldots, b_n)) \Big).$$

Existence and uniqueness of intersections

	(1)	$\exists z \forall x(x \in z \leftrightarrow (x \in a \wedge x \in b))$	sep
2	(2)	$\forall x(x \in c \leftrightarrow (x \in a \wedge x \in b))$	A
	(3)	$\forall y \forall y'(\forall x(x \in y \leftrightarrow x \in y') \rightarrow y = y')$	ext
4	(4)	$\forall x(x \in d \leftrightarrow (x \in a \wedge x \in b))$	A
2,4	(5)	$\forall x(x \in c \leftrightarrow x \in d)$	2,4
	(6)	$\forall x(\in c \leftrightarrow x \in d) \rightarrow c = d$	3 UE
2,4	(8)	$c = d$	6,5 MP