Bell's Theorem and Nonlocality

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March 25, 2020

The objective of this document is to introduce you to Bell's theorem, which has been called "the most profound discovery of science" (Stapp, 1975). A typical way of discussing these issues is to begin with the famous 1935 paper by Einstein, Podolsky, and Rosen, and then to see Bell's 1964 paper as a sort of extension of it. However, the issues raised by EPR are, in one sense, more conceptually subtle than those raised by Bell. So we'll go straight to Bell, and then circle back to EPR.

Primary reading: The original Bell paper, "On the Einstein-Podolsky-Rosen paradox." I suggest reading the words in this paper, but skim over the equations (at least on a first reading). In the current document, I'll present the technical results in a more systematic and transparent way. (When you read the Bell paper, keep in mind the question, "what assumptions does he make in order to derive the inequality?" For that, you'll at least have to glance at the equations — to see how Bell translates words into equations.)

Secondary reading: We'll read Maudlin's paper "What Bell did", and the response by Werner.

Additional resources: Here are two survey articles about Bell's theorem:

- `http://www.scholarpedia.org/article/Bell%27s_theorem`
- `https://plato.stanford.edu/entries/bell-theorem/`

These articles are really long and quite technical — please don't feel like you need to read them straight through. I suggest that you skim them, and then refer back to them when they might add to the discussion in this document.

1 Introduction

It's difficult state Bell's theorem without immediately taking sides on contentious issues. To try to speak neutrally, let me say that Bell (1964) shows that if a certain set Γ of assumptions holds, then an experimentally testable inequality ϕ holds. Bell then points out that QM violates the inequality ϕ, i.e. QM $\Rightarrow \neg\phi$. Hence, QM $\Rightarrow \neg\Gamma$, i.e. QM implies that one of the assumptions in Γ is false.

What's interesting about this result is that the assumptions in Γ appear to be significant metaphysical theses — such as "realism" or "locality". If that's right, then Bell's theorem shows that QM violates either realism or locality. But that undersells Bell's result. It's not just that QM implies that Bell's inequality is violated, we now have strong experimental evidence that Bell's inequality is violated. Hence, we have strong evidence that one of the assumptions in Γ must be false. For this reason, some philosophers have said that Bell's theorem enables us to do "experimental metaphysics."

But there is an intense debate about how to think about Bell's theorem, in particular, how to think about the assumptions that are needed to derive the inequality. From about 1980 to 2000, the common story was that the assumptions needed for Bell's theorem are realism and locality — so that the upshot of Bell's theorem is that one must reject *either* realism *or* locality. It was then thought that orthodox or standard QM rejects realism, while Bohmian mechanics (and other realist interpretations) reject locality. (For further thought: where does the Everett interpretation sit in this dialectic?)

More recently, some philosophers and physicists have rejected this analysis. For example, Goldstein et al. (2011) say that,

> ($\neg$LOC) One can prove the CHSH–Bell inequality from the assumption of locality alone and, therefore, no matter what one believes about the role of non-commuting observables, it follows that the violation of the CHSH–Bell inequality implies non-locality.

I think that Goldstein et al. have overstated the case, because there's really no such thing as a mathematical proof that has just one assumption. To prove anything, you've got to make other assumptions about which inferential moves are permitted, and you've got to make other "framework assumptions."

To play devil's advocate, I'll take the opposing side to Goldstein et al., i.e. the side that says that Bell's theorem isn't really about locality.[1] Here by "V" I'll mean the claim that all properties can simultaneously possess values.

> ($\neg$VAL) One can prove the CHSH-Bell inequality from the assumption of V alone and, therefore, no matter what one believes about the role of locality, it follows that the violation of the CHSH-Bell inequality implies not-V.

Your job will be to decide which one of these claims is better supported by the evidence.

Here is an outline of the rest of this document: First we'll talk about classical (i.e. non-quantum) probability theory. We restrict the discussion mostly to the (boring) finite case, just because that's all we'll need to derive the Bell inequality. On a first reading, you can skim this section, and proceed straight to the section on the derivation of the Bell inequality — referring back to the earlier section if you have questions about terminology.

2 Classical probability

Definition. Let X be a finite set; the elements of X may be thought of as **pure states**, i.e. complete, classical configurations of some system or world. A **probability measure** is a map $p : X \to [0,1]$ such that $\sum_{x\in X} p(x) = 1$.

Example. When the space X is finite — as we have assumed — there is one probability measure that seems special or preferred, viz. the flat distribution:

$$p_0(x) \;=\; \frac{1}{n},$$

[1] The view that violation of Bell's inequality isn't really about nonlocality was put forward long ago by the philosopher Arthur Fine. He says that "our investigations suggest that what the different hidden variables programs have in common, and the common source of their difficulties, is the provision of joint distributions in those cases where quantum mechanics denies them" (Fine, 1982, p 1309). The joint distributions that Fine is talking about are assignments of joint probabilities to incompatible quantities, such as position and momentum, or spin-z and spin-x. So, Fine's opinion is: Bell's inequality is violated in QM *because* QM does not provide joint distributions for non-commuting quantities. Fine's analysis was disputed by Shimony (1984). Some support for Fine's position can be derived from the fact that Bell's inequality is violated even for different degrees of freedom of a single object. If the explanation for violations should be the same in all cases, then the explanation cannot be nonlocality.

where n is the number of elements in X. However, p_0 is by no means the only probability distribution on X. For example, for each point $x \in X$, there is a probability measure that is concentrated on x:

$$p_x(y) \;=\; \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

If p and q are probability measures on X, and $\lambda \in [0, 1]$, then $\lambda p + (1 - \lambda)q$ is a probability measure on X, called a **convex combination** of p and q.

Definition. If a state p (i.e. a probability measure) can be written as a non-trivial convex combination of other states, then we say that p is a **mixed state**. Otherwise we say that p is a **pure state**. In other words, the pure states are the **extremal points** of the convex set of states.

The following result shows that the pure states of X are precisely the point masses, and hence stand in one-to-one correspondence with elements of X.

1 Proposition. *If $p(x) > 0$, then $p = \lambda p_x + (1 - \lambda)q$ for some probability measure q and some $\lambda > 0$.*

Proof. If $p(x) = 1$, then $p = p_x$ and we're finished. If $p(x) < 1$, then we may define

$$q \;=\; (1 - \lambda)^{-1}(p - \lambda p_x),$$

where $\lambda = p(x)$. Then

$$\begin{aligned} \textstyle\sum_{y \in X} q(y) \;&=\; (1 - p(x))^{-1} \textstyle\sum_{y \in X}(p(y) - \lambda p_x(y)) \\ &=\; (1 - p(x))^{-1}(1 - p(x)) \\ &=\; 1. \end{aligned}$$

Hence q is a probability measure, and by definition $p = \lambda p_x + (1 - \lambda)q$. $\square$

2 Proposition. *A probability measure p on X is pure iff $p = p_x$ for some $x \in X$.*

Proof. Suppose that p is pure. Let $x \in X$ such that $p(x) > 0$. By the previous result, $p = \lambda p_x + (1 - \lambda)q$. Since p is pure, $p = p_x$.

Now we show that p_x is pure. Suppose that $p_x = \lambda p + (1-\lambda)q$. If $y \neq x$ then

$$0 \;=\; p_x(y) \;=\; (1-\lambda)p(y) + \lambda q(y).$$

Hence $p(y) = 0 = q(y)$. Since y was arbitrary, it follows that $p = p_x = q$. $\square$

Definition. A **property** (or **event** or **proposition**) E is defined to be a subset of X.

The events/propositions on X form a **Boolean algebra** with the operations $\wedge$ (intersection), $\vee$ (union), and $\neg$ (complement). Any event E also has an associated probability, denoted $p(E)$ and defined by

$$p(E) \;=\; \sum_{x\in E} p(x).$$

The map p from subsets of X to probabilities is called a **probability measure**, and it satisfies some obvious equations such as

$$p(E \vee F) \;=\; p(E) + p(F),$$

when E and F are disjoint.

Discussion. In classical physics, it's standard to assume that every subset of state space corresponds to a property. That's not the general understanding in QM. For example, consider the subset $E = \{|0\rangle, |1\rangle\} \subseteq \mathcal{H}$, which one might think represents the property,

> ... being in an eigenstate of S_z.

However, since E is not a subspace of $\mathcal{H}$, there is no projection operator corresponding to it, and it's not usually considered to represent a property. But why not? Should it be?

Let's use the name *abundant properties view* for the view that all subsets of state space represent properties. An old-fashioned argument against the abundant view went like this:

> The only properties that can be measured are those that are represented by projection operators. If something cannot be measured, then there's no reason to think it's there. Therefore, there's no reason to think that there are properties besides those represented by projection operators.

The second premise of that argument is beyond questionable. What's more, there are many projection operators that don't really correspond to things that we can measure. (After all, there is an uncountable infinite of projection operators, and for most of them, we don't have any idea what physical measuring procedure they would correspond to.)

However, there are better arguments against the abundant view. For example, there's the argument from paradoxes in the foundations in mathematics. Similarly, there are philosophical arguments such as Goodman's grue paradox.

My favorite argument against the abundant view is the simple "tell me more" argument. It does like this: if you want to modify the standard approach to QM, then please explicitly specify your new theory, including its states, its properties, and the rules for which properties are possessed in which states. My suspicion is that if somebody tells me more, then their new properties will end up doing no work, i.e. they can be excised without the theory losing any explanatory power.

Definition. Suppose that $p(E) > 0$. Then we define the **conditional probability** of F given E as

$$p(F|E) \;=\; \frac{p(F \wedge E)}{p(E)}.$$

In fact, $F \mapsto p_E(F) = p(F|E)$ is the probability measure generated by the function

$$p_E(x) \;=\; \begin{cases} p(E)^{-1}p(x) & x \in E \\ 0 & x \notin E. \end{cases}$$

itself a probability measure, and it's the most conservative choice of a new probability measure once one learns that E holds. Indeed, define a distance between probability measures on X as follows:

$$\|p - q\| \;=\; \sum_{x \in X} |p(x) - q(x)|.$$

Now let $M_E(X)$ be the set of all probability measures on X with the feature that $q(E) = 1$. Clearly $p_E \in M_E(X)$, and it can be shown that

$$\|p - p_E\| \;\leq\; \|p - q\|,$$

for all $q \in M_E(X)$. We will leave the details of a proof to the reader, but intuitively, p_E is the only measure on E that results from uniformly stretching values $p(x)$ for $x \in E$. If that goal is to minimize the distance from p, then no measure q can do better than a uniform stretch. If q were closer to p at some point $x \in E$, then q would have to be that much further away from p at some other point $y \in E$.

Definition. A **random variable** is a function $f : X \to \mathbb{R}$. We will sometimes call f a **quantity**, or for the sake of compassion with quantum theory, an **observable**.

Example. If X is the classical configuration space $\mathbb{R}^3$, then the function $f(x_1, x_2, x_3) = x_1$ represents the quantity "first coordinate of position."

Let $\mathbb{R}^X$ be the set of random variables, i.e. functions from X to $\mathbb{R}$. This set $\mathbb{R}^X$ naturally forms an algebra where the operations are defined pointwise. That is, given f, g, we define

$$\begin{aligned} (f+g)(x) &= f(x) + g(x), \\ (fg)(x) &= f(x)g(x), \\ (rf)(x) &= rf(x). \end{aligned}$$

Clearly this algebra has a multiplicative identity (the constant 1 function), and is commutative, i.e. $fg = gf$.

Definition. The **spectrum** of f, $\mathrm{sp}(f) \subseteq \mathbb{R}$, is the image of X under f, i.e.

$$\mathrm{sp}(f) = \{f(x) \in \mathbb{R} : x \in X\}. \tag{1}$$

3 Proposition. *For a quantity f, the following are equivalent.*

1. $\mathrm{sp}(f) \in \{0, 1\}$.
2. *f is the characteristic function of some subset E of X.*
3. $f^2 = f$.

Proof. Suppose first that $\mathrm{sp}(f) \in \{0, 1\}$. If $E = \{x \in X \mid f(x) = 1\}$ then f is the characteristic function of E. It's also clear that a characteristic function f has the property that $f^2 = f$. Hence (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Now suppose that $f^2 = f$. Then for any $x \in X$, $f(x)^2 = f(x)$, which implies that $f(x) = 0$ or $f(x) = 1$. Therefore $\mathrm{sp}(f) \in \{0, 1\}$. □

There is a natural probability density on $\mathrm{sp}(f)$ denoted by p_f and defined (for $\lambda \in \mathrm{sp}(f)$) by

$$p_f(\lambda) = \sum_{x \in f^{-1}(\lambda)} p(x).$$

More generally, for any n random variables $f_1, \ldots, f_n$, there is a discrete probability density on $\mathrm{sp}(f_1) \times \cdots \times \mathrm{sp}(f_n)$ given by

$$p_{f_1,\ldots,f_n}(\lambda_1, \ldots, \lambda_n) = \sum_{x \in S} p(x),$$

where $S = \bigcap_{i=1}^n f_i^{-1}(\lambda_i)$.

Definition. Given a random variable $f : X \to \mathbb{R}$, we define the **expectation value** of f as

$$p(f) \;=\; \sum_{x \in X} p(x) f(x).$$

Definition. If $E \subseteq X$, then the **characteristic function** of E is the function $e : X \to \{0, 1\}$ that assigns 1 to x iff $x \in E$.

It follows that $p(E) = p(e)$, where $p(E) = \sum_{x \in E} p(x)$, and $p(e) = \sum_{x \in X} e(x)p(x)$. Hence, we can freely interchange application of p to a subset and that subset's characteristic function.

1 Exercise. Show that expectation value is linear, i.e. $p(f+g) = p(f)+p(g)$, and $p(rf) = rp(f)$. Show that expectation value is positive, i.e. $p(f) \geq 0$ for any function f such that $\mathrm{sp}(f) \subseteq \mathbb{R}^+$. Show that expectation value is normalized, i.e. $p(1) = 1$, where the first "1" is the constant function on X. Show that expectation value is not necessarily multiplicative, i.e. $p(fg) \neq p(f)p(g)$.

Example. If p_a is the measure concentrated on $a \in X$, then

$$p_a(f) \;=\; \sum_{x \in X} p_a(x) f(x) \;=\; f(a),$$

for all $f \in \mathbb{R}^X$. (Here $\mathbb{R}^X$ is the set of functions from X to $\mathbb{R}$.) Conversely, if $p(f) = f(a)$ for all $f \in \mathbb{R}^X$, then $p(e) = 1$ where e is the characteristic function of $\{a\}$, and it follows that $p = p_a$.

4 Proposition. *A state p is pure iff $p(E) \in \{0,1\}$ for all events E.*

Proof. Suppose first that p is pure. By Prop 2, $p = p_a$ for some $a \in X$. Hence

$$p(E) \;=\; p_a(E) \;=\; \sum_{x\in E} p_a(x) \;=\; \begin{cases} 0 & x \notin E, \\ 1 & x \in E. \end{cases}$$

Suppose now that $p(E) \in \{0,1\}$ for all events E. In particular, $p(\{x\}) \in \{0,1\}$ for each $x \in X$. Since

$$1 \;=\; p(X) \;=\; \sum_{x\in X} p(\{x\}),$$

it follows that $p(\{a\}) = 1$ for some $a \in X$, and hence $p = p_a$. Therefore, p is pure. $\square$

Definition. Let f be a quantity (aka random variable) and let p be a state. The **dispersion** (aka variance) $V_p(f)$ of f in p is defined by

$$V_p(f) \;=\; p(f^2) - p(f)^2.$$

We say that p is **dispersion free** on f just in case $V_p(f) = 0$. We say that p is **dispersion free** if it is dispersion free on all quantities.

If f is a projection, then $f^2 = f$, and hence $V_p(f) = p(f)^2 - p(f)$. Therefore $p(f) \in \{0,1\}$ iff $V_p(f) = 0$.

We now look at the **spectral decomposition** of a function. For any subset $\lambda \in \mathrm{sp}(f)$, let

$$E(\lambda) \;=\; \{x \in X \mid f(x) = \lambda\}.$$

Using the correspondence between a subset $E(\lambda)$ of X and its characteristic function $e(\lambda)$, we have

$$f \;=\; \lambda_1 e(\lambda_1) + \cdots + \lambda_n e(\lambda_n).$$

This fact is completely trivial in the case we are dealing with. But it will be important to remember the analogy when we derive an analogous result for quantum probability spaces.

5 Proposition. *A state p is pure iff p is dispersion-free on all quantities.*

Proof. Suppose that p is pure. By Prop 4, $p(g) \in \{0,1\}$ for all g such that $g^2 = g$. In particular, $p(e(\lambda_i)) \in \{0,1\}$, for any spectral projection $e(\lambda_i)$ of f. Hence,

$$p(f) \;=\; \sum_i \lambda_i p(e(\lambda_i)) \;=\; \lambda_j,$$

for some $\lambda_j \in \mathrm{sp}(f)$. A similar calculation shows that $p(f^2) = \lambda_j^2$.

Now suppose that $V_p(f) = 0$ for all quantities f. In particular, $p(f^2) = p(f)^2$ for any idempotent f, and hence $p(f) \in \{0,1\}$. By Prop 4, p is pure. $\square$

The following result shows the precise sense in which there are always **hidden variables** for classical systems, i.e. any state whatsoever can be interpreted as an ignorance mixture of determinate (i.e. dispersion-free) states.

6 Proposition (unique decomposition)**.** *Every state p on X decomposes uniquely as a convex combination of dispersion-free states.*

Proof. Suppose that $X = \{x_1, \dots, x_n\}$. Since $p(X) = 1$, it follows that $p(x_i) > 0$ for some $i \in [1,n]$. Let $S = \{i \in [1,n] : p(x_i) > 0\}$, and let $\lambda_i = p(x_i)$. Then $p = \sum_{i\in S} \lambda_i p_{x_i}$.

To see that this decomposition is unique, suppose that $p = \sum_{i\in S'} \lambda_i' p_{x_i}$ where $0 < \lambda_i' < 1$ and $\sum_{i\in S'} \lambda_i' = 1$. If $k \in S$, then

$$\sum_{i\in S'} \lambda_i' p_{x_i}(x_k) \;>\; 0,$$

and it follows that $k \in S'$. By symmetry, if $k \in S'$ then $k \in S$. Finally, for $k \in S = S'$, we have $\lambda_k = p(x_k) = \lambda_k'$. $\square$

Definition. Let V be a linear space over the real numbers. A subset $K \subseteq V$ is said to be **convex** just in case for each $x, y \in K$ and $\lambda \in (0,1)$, we have $\lambda x + (1-\lambda)y \in K$.

If we use $M(X)$ to denote the set of probability measures on X, then $M(X)$ can be seen as living inside the linear space of functions from $\mathbb{R}^X$ to $\mathbb{R}$, i.e. functionals on the algebra of random variables. Clearly $M(X)$ is a convex set, and the previous result tells us that $M(X)$ is a special kind of convex set, called a simplex.

Definition. Let V be a finite-dimensional vector space, and let K be a convex subset of V. Then K is called a **simplex** just in case each $x \in K$ has a unique decomposition in terms of extremal points in K.

In fact, there is a nice geometric representation of $M(X)$ when X has n elements: it is the standard simplex with n extreme points. In particular, for $n = 2$, $M(X)$ is a line segment; for $n = 3$, $M(X)$ is a triangle; for $n = 3$, $M(X)$ is a tetrahedron, etc.

In contrast, quantum state spaces are not simplices. (The quantum state space is not actually the Hilbert space $\mathcal{H}$, but the convex set of density operators on $\mathcal{H}$.) Take, for example, the density operators on a two-dimensional Hilbert space. If E_0 and E_1 are the spectral projections of S_z, and F_0 and F_1 the spectral projections of S_x, then

$$\tfrac{1}{2}E_0 + \tfrac{1}{2}E_1 \;=\; \tfrac{1}{2}I \;=\; \tfrac{1}{2}F_0 + \tfrac{1}{2}F_1.$$

Thus, an equal mixture of the eigenstates of S_z is the same state as an equal mixture of the eigenstates of S_x, i.e. a state can be decomposed into pure states in more than one way. This feature of quantum state spaces is known as the **non-unique decomposability of mixtures**. In fact, the convex set of density operators on a two-dimensional Hilbert space is shaped like a sphere, and it's usually called the **Bloch sphere**.

7 Proposition. *Let $q : \mathbb{R}^X \to \mathbb{R}$ be a positive linear functional such that $q(1) = 1$. Then there is a unique probability measure p on X such that $q(f) = \sum_{x\in X} p(x)f(x)$, for each $f \in \mathbb{R}^X$.*

Proof. Let $e_1, \dots, e_n$ be characteristic functions of all singleton subsets of X. Since q is linear and normalized, we have

$$1 \;=\; q(e_1 + \cdots + e_n) \;=\; q(e_1) + \cdots + q(e_n).$$

Since q is positive, $q(e_i) \in [0, 1]$. Hence if we define $p(x) = q(\{x\})$, then p is a probability measure on X. Now let f be an arbitrary element of $\mathbb{R}^X$, and let $e(\lambda_1), \dots, e(\lambda_m)$ be its spectral decomposition, which means that $f(x) = \lambda_i$ iff $e(\lambda_i)(x) = 1$. Clearly we have

$$\sum_{x\in e(\lambda_i)} p(x) \;=\; \sum_{x\in e(\lambda_i)} q(\{x\}) \;=\; q(e(\lambda_i)),$$

and hence

$$\begin{aligned}
q(f) &= \lambda_1 q(e(\lambda_1)) + \cdots \lambda_m q(e(\lambda_m)) \\
&= \lambda_1 p(e(\lambda_1)) + \cdots \lambda_m p(e(\lambda_m)) \\
&= \textstyle\sum_{x \in X} p(x) f(x).
\end{aligned}$$

□

3 Composite systems

Given two state spaces X and Y, the state space of the composite system is the Cartesian product

$$X \times Y \;=\; \{(x, y) \mid x \in X, y \in Y\}.$$

In this case, Prop 4 implies that every pure state is of the form $p_{(x,y)}$.

The space $X \times Y$ has the feature that for any functions $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$, there is a unique function $f \times g : X \times Y \to \mathbb{R}$ given by

$$(f \times g)(x, y) \;=\; f(x) g(y),$$

for all $x \in X$ and $y \in Y$. However, there are also functions on $X \times Y$ that do not decompose in this way. For example, let $X = Y = \{a, b\}$, and consider the function p such that

$$p(x, y) \;=\; \begin{cases} \frac{1}{2} & x = y, \\ 0 & x \neq y. \end{cases}$$

In fact, this function p is a probability measure on $X \times Y$. Intuitively, it's a state in which the two systems are **strictly correlated**: either both are in state a, or both are in state b. Nonetheless, each state on $X \times Y$ is a convex combination of pure states. In particular, $p = \sum_i \lambda_i p_i$, where each p_i is a state of the form $p_x \times p_y$. This mathematical fact corresponds to the physical fact that correlated states can be interpreted *epistemically*, e.g. as representing our ignorance of the real state of the system, which is a logical sum of the state of the individual subsystems.

In order to derive Bell's inequality, we first need to prove a trivial little result about inequalities of real numbers:

8 Proposition. *For real numbers $a, b \in [-1, 1]$, we have*

$$|a+b| + |a-b| \;\leq\; 2. \tag{2}$$

Proof. If $a+b$ is positive, then $|a+b| + |a-b| = 2\max\{a, b\}$, and if $a+b$ is negative, then $|a+b| + |a-b| = 2\max\{-a, -b\}$. □

Now we consider two systems with state spaces X and Y. Let $f_1, f_2 \in \mathbb{R}^X$ such that $\mathrm{sp}(f_i) \subseteq [-1, 1]$, and let $g_1, g_2 \in \mathbb{R}^Y$ such that $\mathrm{sp}(g_i) \subseteq [-1, 1]$. That is, f_1 and f_2 are quantities associated with system X, and g_1 and g_2 are quantities associated with system Y. Consider the quantity represented by the function

$$r \;=\; f_1 \times (g_1 + g_2) + f_2 \times (g_1 - g_2).$$

This r is called a **Bell observable**, and it could, in principle, be measured by two observers with systems X and Y. We then have

$$\begin{aligned} |r(x,y)| &= |f_1(x)g_1(y) + f_1(x)g_2(y) + f_2(x)g_1(y) - f_2(x)g_2(y)| \\ &\leq |f_1(x) + f_2(x)| + |f_1(x) - f_2(x)| \\ &\leq 2, \end{aligned}$$

where the final inequality follows from Eq. 2, since $f_1(x), f_2(x) \in [-1, 1]$.

Bell's Theorem. *If r is a Bell observable, then for any classical state ω,*

$$-2 \;\leq\; \omega(r) \;\leq\; 2. \tag{3}$$

Equation 3 is called **Bell's inequality**, or to be more accurate the **CHSH** variant of Bell's inequality (in honor of Clauser, Horne, Shimony, and Holt).

Proof. The discussion above shows that $-2 \leq p(r) \leq 2$ for any pure state p. An arbitrary state ω is a convex combination of pure states, and so the result holds for ω as well. □

Question. The word "locality" does not occur in this section. So how is it that the derivation of Bell's inequality requires the assumption of locality?

4 QM predicts violation of Bell's inequality

The Bell experiment uses an idea that is due to David Bohm, and which is based on the Einstein-Podolsky-Rosen thought experiment. In the EPR thought experiment, we have two systems whose locations and momenta are strictly correlated. However, that thought experiment is difficult to describe mathematically, because position and momentum are continuous quantities. So, Bohm suggests that we look at two systems whose spins are strictly anticorrelated.

To be more precise, let $\mathcal{H}$ and $\mathcal{K}$ be two-dimensional Hilbert spaces, i.e. both are isomorphic to $\mathbb{C}^2$. Then the state space of the joint system is the tensor product $\mathcal{H} \otimes \mathcal{K}$. The **singlet state** of $\mathcal{H} \otimes \mathcal{K}$ is given by

$$\Omega \;=\; 2^{-1/2}\left(|01\rangle + |10\rangle\right),$$

where $|0\rangle$ is the $+1$ eigenstate of S_z, and $|1\rangle$ is the -1 eigenstate of S_z, and $|ij\rangle = |i\rangle \otimes |j\rangle$. We now let

$$A_1 = S_x \otimes I \qquad B_1 = I \otimes -\tfrac{1}{\sqrt{2}}(S_y + S_x)$$

$$A_2 = S_y \otimes I \qquad B_2 = I \otimes \tfrac{1}{\sqrt{2}}(S_y - S_x).$$

Note that each A_i and B_j is self-adjoint with spectrum $\{-1, +1\}$. [Here B_1 is a measurement of spin along an axis tilted $\pi/4$ from the x-axis in the xy plane, and B_2 is a measurement of spin along an axis tilted $-\pi/4$ from the x-axis.] Since $[A_i, B_j] = 0$, it follows that A_iB_j is also self-adjoint with spectrum $\{-1, +1\}$.

Combining A_1, A_2, B_1, B_2, we have four different global measurement contexts: A_1B_1, A_1B_2, A_2B_1, and A_2B_2. (Here a "global" measurement context means that measurements have been chosen for both subsystems.)

For any quantity X, let $\omega(X) = \langle \Omega, X\Omega\rangle$, i.e. $\omega(X)$ is the quantum expectation value of X in state Ω. From the equations in the appendix, we have $\omega(A_iB_j) = 2^{-1/2}$ when $i = 1$ or $j = 1$, and $\omega(A_2B_2) = -2^{-1/2}$. Hence,

$$\omega(A_1B_1) + \omega(A_1B_2) + \omega(A_2B_1) - \omega(A_2B_2) \;=\; \frac{4}{\sqrt{2}} \;=\; 2\sqrt{2}, \tag{4}$$

and if we let $R = A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2$, then $\omega(R) = 2\sqrt{2}$.

What Bell showed is that if A_1, A_2, B_1, B_2 are classical random variables with spectrum in $[-1,1]$, then

$$-2 \;\leq\; \omega(A_1B_1) + \omega(A_1B_2) + \omega(A_2B_1) - \omega(A_2B_2) \;\leq\; 2.$$

Therefore, Eqn 4 shows that the predictions of QM cannot be reproduced by a classical probabilistic model.

Let's pause to note a crucial difference between the classical and quantum cases: in the quantum case, no two of the four measurements are compatible. To put it in neutral, mathematical terms: since A_1 and A_2 do not commute, A_1B_1 and A_2B_1 do not commute, etc. What do those neutral, mathematical terms mean? Well, according to Kochen-Specker, if operators do not commute, then they cannot simultaneously possess values. I don't say that because I'm an operationalist (i.e. "if it can't be measured then it's not real"), but because I cannot assign values to both A_1 and A_2 without contradicting myself.[2]

The fact we just noted (i.e. that the four measurement contexts are incompatible) adds a wrinkle to the dialectic of the Bell inequality. Think of it this way: Bell argues that *if* there were a classical probabilistic model of the four measurement contexts, *then* that model would be nonlocal. However, we already know that there can be no classical probabilistic model of the four measurement contexts. Thus, one might wonder whether Bell's result really adds anything new to the Kochen-Specker result. We'll come back to that question after we finish showing that QM violates Bell's inequality.

2 Exercise. In this exercise, you will show that the Bell inequality is violated only if Anne and Bjarke both make use of incompatible measurement settings — a fortiori, violations of Bell's inequality require at least four quantities. As a warm-up, check that if $B_1 = B_2 = -2^{-1/2} I \otimes (S_y + S_x)$, then

$$\omega(A_1B_1) + \omega(A_1B_2) + \omega(A_2B_1) - \omega(A_2B_2) \;=\; \sqrt{2} \;<\; 2.$$

Then there are a couple different ways one can generalize that result. First, one can note that if $[B_1, B_2] = 0$, then there is a single self-adjoint operator C such that both B_1 and B_2 are functions of C. (Intuitively: if B_1 and B_2 can be simultaneously diagonalized, then it's easy to construct another diagonal

[2] The current example is precisely the exceptional case to Kochen-Specker: it is, in fact, possible to assign joint values to A_1 and A_2. Does that change the overall dialectic? I don't think so, because nobody thinks that the world is made of two-dimensional systems.

matrix C and functions f_1 and f_2 such that the eigenvalues of B_i result from applying the function f_i to the eigenvalues of C.) Second, one can show that if R is a Bell observable, then

$$R^2 \;=\; 4I + [A_1, A_2][B_1, B_2].$$

Hence, if either $[A_1, A_2] = 0$ or $[B_1, B_2] = 0$, then $R^2 = 4I$ which entails that $\mathrm{sp}(R) \subseteq [-2, 2]$.

The previous exercise raises the following puzzle for the defenders of the nonlocality thesis (i.e. the thesis that the violation of Bell's inequality establishes non-locality): why is Bell's inequality only violated for measurements of incompatible quantities? If there is nonlocality in nature, then why does it not display itself in measurements of a single quantity?

5 Locality and contextuality

In this section we will investigate the relation between Bell's inequality and contextuality. Our original derivation of Bell's inequality (in Section 3) tacitly assumes non-contextuality (in having just one probability space X for all four experiments). Some people have said, however, that Bell does not assume non-contextuality; he assumes locality, and derives non-contextuality from it. If that's right, the our derivation in Section 3 contains an implicit assumption of locality.

Recall that the Bell setup talks about four separate experiments (measurement of A_i and B_j for $i, j = 1, 2$), and that according to QM, no two of these experiments are compatible with each other. Thus, one might suggest that there are really four state spaces in play, say $X_{11}, X_{12}, X_{21}, X_{22}$. The elements of X_{ij} are states that a system can be relative to the quantities A_i and B_j, say $(-1, -1), (-1, +1), (+1, -1), (-1, -1)$.

Our first question is: are there classical probability distributions p_{ij} on X_{ij} that reproduce the statistics of the singlet state? Even without checking details, it is obvious that the answer is yes. For any quantity R, a quantum state Ω induces a classical probability measure over the eigenstates of R. For the specific case at hand, we can let p_{ij} be the probability distribution induced by the singlet state on the eigenstates of the operator A_iB_j.

Here's another version of the Bell theorem. Suppose in this case that we have two propositions A and B about the left system, and two propositions

B' and C about the right system. For example, A might be the proposition that $S_a \otimes I$ has value $+1$, and B' might be the proposition that $I \otimes S_b$ has value $+1$. We assume that B and B' are strictly anticorrelated, i.e. for any truth-assignment (aka hidden variable) ω, the value $\omega(B)$ must be the opposite of $\omega(B')$, i.e. if the former is true, then the latter is false, and vice versa. If a hidden variable ω did not have this feature, then it would not contribute in an significant way to the statistics of the singlet state. In Bell's terminology, the measure of these hidden variables would be zero, so they can be omitted from further consideration.

Now, we claim that if $\omega(A \wedge C) = 1$ then either $\omega(A \wedge B') = 1$ or $\omega(B \wedge C) = 1$. Indeed, either $\omega(B) = 0$ or $\omega(B) = 1$, and in the former case, $\omega(B') = 1$ and hence $\omega(A \wedge B') = 1$. In the latter case, $\omega(B \wedge C) = 1$. Thus, in either case,

$$\omega(A \wedge C) \;\leq\; \omega(A \wedge B') + \omega(A \wedge C).$$

Since integration is linear and preserves inequalities, it follows that for any probability measure μ on hidden variables,

$$\mu(A \wedge C) \;\leq\; \mu(A \wedge B') + \mu(B \wedge C). \tag{5}$$

This last equation is a simple probabilistic version of Bell's inequality. In particular, suppose that we've chosen three measurement directions (a, b, c) and suppose that:

- A is the statement that $S_a \otimes I$ has value $+1$.
- B is the statement that $S_b \otimes I$ has value $+1$.
- B' is the statement that $I \otimes S_b$ has value $+1$.
- C is the statement that $I \otimes S_c$ has value $+1$.

Then we have the following translation scheme:

$$\begin{aligned} \mu(A \wedge B') &= P_{ab}(++) \\ \mu(A \wedge C) &= P_{ac}(++) \\ \mu(B \wedge C) &= P_{bc}(++), \end{aligned}$$

and Equation 5 becomes

$$P_{ac}(++) \;\leq\; P_{ab}(++) + P_{bc}(++). \tag{6}$$

For QM, if $a = \pi/3$, $b = 0$, and $c = -\pi/3$, then one can calculate (as shown in the appendix) that:

$$P_{ac}(++) = 3/8, \qquad P_{ab}(++) = 1/8, \qquad P_{bc}(++) = 1/8.$$

So, once again, QM violates Bell's inequality.

6 What did Bell do?

At the time when Bell proved his theorem, it was already known that QM would predict a violation of Bell's inequality. However, at that time — in the 1960s — no direct experiment had been undertaken to verify this prediction of QM. In the intervening years, many different experiments have confirmed that Bell's inequality is violated. (Many people would say that the decisive experiment was the one undertaken by Alain Aspect in 1982.) Let's look at the significance of these two facts in reverse order.

First, what is the significance of the fact that there are experimental violations of Bell's inequality? In the first instance, the only significance of this fact is that the model we made above does not adequately describe those experimental situations. In other words, if we thought that there were two systems X and Y, and that classical probability theory was applicable in the simple way we described above, then we would derive a false prediction about the outcomes the experiments.

Some other philosophers and physicists have not been so modest in their claims about what these experiments show. For example, according to Tim Maudlin, the violation of Bell's inequality show quite simply that the physical universe has a feature called "non-locality".

> [John Bell] taught us something about the world we live in, a lesson that will survive even the complete abandonment of quantum theory. For what cannot be reconciled with locality is an observable phenomenon: the violations of Bell's inequality for 'measurements' performed at arbitrary distances apart, or at least at space-like separation. And this phenomenon has been verified, and continues to be verified, in the lab. Neither indeterministic nor deterministic theories can recover these predictions in a local way. Non-locality is here to stay. (Maudlin, 2014, p 22)

Similarly, Norsen (2016) claims that, "nonlocality really is required to coherently explain the empirical data." This is an interesting point of view, and there are a couple of different ways to read it — either in the material mode, or in the formal mode. (The material mode is speaking about the universe, and the formal mode is speaking about theories.) In the material mode, the claim seems to be that some possible universes are local, and others are non-local, but that any universe that displays violations of Bell's inequalities is one of the non-local universes. But that claim doesn't look anything like what Bell actually proved. Bell didn't talk about varieties of universes, and he didn't give us any insight into what a non-local universe would look like.

To read Maudlin and Norsen's claims in the formal mode would have Bell showing something like this:

> There is no theory T with property Φ such that T predicts violations of Bell's inequalities

where, in this particular case, Φ is the property of being a local theory. Once again, the claim seems too strong. Bell didn't do any surveying of the space of all possible theories, so it's not clear how his result could show anything of this sort. Instead, what Bell showed is that a certain familiar kind of modeling strategy — classical probability — makes the wrong predictions for these kinds of experiments. We have a long way to go before we can say anything about all possible future theories.

In fact, in the decades immediately following Bell's theorem, there was a different consensus about the physical significance of the result. In particular, the common view was that Bell's theorem should be thought of as a derivation of an (experimentally testable) inequality from the conjunction of two premises:

realism The moon is there even when no-one is looking.

locality Things that happen in one place cannot have an instantaneous effect on things in another place.

(The classic "Jarrett analysis" of Bell's derivation can be found in Jarrett (1984).) I've purposely stated these premises in both a vague, and an over-specific, way. The point of doing so is that as soon as one starts explicating (i.e. formalizing) these premises, then one has to beg some questions about the framework. In standard analyses of Bell's theorem, one begins immediately to translate **locality** into a statement about conditional probabilities.

But to apply classical probability theory to a complicated situation requires making quite a few physical assumptions about what's going on.

In the case at hand, note that the Bell observable

$$f_1 \times (g_1 + g_2) + f_2 \times (g_1 - g_2),$$

is built out of four different observables: f_1 and f_2 belong to the first experimenter, and g_1 and g_2 belong to the second experimenter. Hence, to successfully carry out a test of Bell's inequality, the first experimenter must perform at least two different measurements, and the second experimenter must also perform at least two different measurements. So, we're not talking about any single state of affairs, but a sequence of different experiments. If we assume that these four experiments can be jointly modeled in the way that classical physics suggests, then we get a false prediction (i.e. that Bell's inequality would be satisfied).

To be clear, to prove a claim of the form

$$\textbf{locality} \implies |p(r)| \leq 2 \ ,$$

one first has to make **locality** into a mathematically precise statement. So let's say that **i-locality** is our intuitive concept of locality, and let's say that **m-locality** is a mathematical precisification of **i-locality**. Then Bell's theorem is of the form

$$\textbf{m-locality} \implies |p(r)| \leq 2 \ ,$$

and the experimental result $p(r) > 1$ shows that **m-locality** doesn't hold. Does it follow that **i-locality** doesn't hold? Well, not unless the intuitive concept of locality demands a particular mathematical explication. Perhaps it does; we will have to think about that. (For a similar argument, see Werner (2014).)

Update: what I wrote about i-locality and m-locality is not clear. (Thanks to Alex for raising some questions.) The point I was trying to make is that it's possible (I think!) that QM does satisfy some form(s) of locality, even though it violates Bell's inequality. To try to make that point more clearly, I've added the following section.

7 How QM is local

In this section, I'll explain some ways in which QM does seem to be local. In particular, QM has the feature that what is measured at one location seems to be irrelevant for the statistics of measurements at other locations. If that's right, then one cannot send signals by choosing what to measure, nor can one figure out what somebody else measured (at a distance) by looking at the result of one's local measurements. This is not to say that QM forbids non-local causality — only that if there is such causality, then its effects are well hidden from us.

Suppose that Anne is in Amsterdam and Bente is in Beijing, and Anne wants to send an instantaneous (faster than light) message to Bente. Suppose, in particular, that Bente wants to know whether Ajax Amsterdam won their game. Here then is a strategy for Anne to signal to Bente:

> (Protocol A) Anne and Bente prepare a pair of electrons in the singlet state ψ. If Anne measures $S_z \otimes I$, then after the measurement, their state will either be $|01\rangle$, or it will be $|10\rangle$. On the other hand, if Anne measures S_x, then after the measurement, their state will either be $|{+}{-}\rangle$, or it will be $|{-}{+}\rangle$. (Here we use $|\pm\rangle$ for the ± 1 eigenstate of S_x.) Before the experiment begins, Anne and Bente agree that if the state is one of the S_z eigenstates, then the message is "Yes", and if the state is one of the S_x eigenstates, then the message is "No."

3 Exercise. Recall that according to the collapse postulate, if Anne measures $S_z \otimes I$ on state $\psi = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, then the subsequent quantum state is either $|01\rangle$ or $|10\rangle$. (1) Explain how Protocol A assumes the collapse postulate. (2) Explain why, even with the collapse postulate, Protocol A will not work.

Back in section ?, we assumed that composite quantum systems could be described using the tensor product operation: "$\otimes$". I also told you not to worry too much about why the tensor product is the right way to represent composite systems. Indeed, I'm not sure that anyone has a good argument for the claim that the tensor product is "natural" or "demanded" by the concept of spatial parts. There is, however, at least one sense in which the tensor product operation ensures that the two systems (or two parts of one system) are *independent* from each other. In particular, if $H_1 \otimes H_2$ is the state space of a composite system, then the quantities associated with the first system are of the form $A \otimes I$, and the quantities associated with the second system are of the form $I \otimes B$. Then we have

$$(A \otimes I)(I \otimes B) \;=\; A \otimes B \;=\; (I \otimes B)(A \otimes I).$$

In other words,

$$[A \otimes I, I \otimes B] \;=\; 0,$$

which means that the quantities $A \otimes I$ and $I \otimes B$ are **compatible**. That is, QM *assumes* that any quantity associated with one system is compatible with any other quantity associated with a distant system — which is at least an interesting mathematical difference in comparison to the quantities associated with a single system.

For a single system, there are quantities, such as position Q and momentum P, that have an uncertainty relation. At least statistically speaking, QM predicts that you'll won't find yourself in a situation where sequences of position-then-momentum measurements will stay within narrow bands. The situation is different with spatially separated systems. QM predicts that for *any* quantities A and B associated with spatially distant systems, there are states in which both quantities have zero dispersion.

Intuitively, if there were an uncertainty relation between A and B, then Anne could signal Bente by measuring A, thereby forcing (if we assume the collapse postulate) the system into an eigenstate ω of A, so that the variance $V_\omega(B)$ would be nonzero.

So, the tensor product formalism ensures that there are no uncertainty relations between Anne's quantities and Bente's quantities. Another feature of the tensor product formalism is that it's possible to "mix and match" states. To see what I mean, consider a contrasting example:

> Suppose that there are twins, Arnie and Barnie. Arnie and Barnie are perfectly in sync, so that if Arnie orders chicken for dinner, then so does Barnie — even if they are on opposite sides of the world.

Spatially separated quantum systems are *not* like Arnie and Barnie. For any state x of H_A and for any state y of a spatially distant system H_B, it's possible for the composite system to be in state $x \otimes y$. For example, if x is the state of ordering chicken, and y is the state of ordering beef, then there is a state in which the first system orders chicken and the second system orders beef.[3] That is, the quantum states of spatially separated systems are freely

[3] Something more complicated happens when we take into account the permutation symmetries of QM. In that case, there are two kinds of particles, those whose states are symmetric under permutation, and those whose states are antisymmetric under permutations. However, those restrictions don't conflict with the property I'm describing here.

combinable, a feature called **state preparation independence**.

Here's another way to put the issue. Let E be a nonzero projection operator associated with Anne's system, and let F be a nonzero projection operator associated with Bente's system. Then not only do we have $[E, F] = 0$ (the projections are compatible), but we also have $E \wedge F \neq 0$. In other words, Anne's system having property E does not rule out Bente's system having property F. If $F \neq I$, then we also have $E \wedge \neg F \neq 0$, i.e. Anne's system having property E does not rule out Bente's system having property $\neg F$. In short, the properties of Anne's system are completely independent of the properties of Bente's system.

For those who aren't convinced of the mathematics in the previous passage: here E is really a projection of the form $E \otimes I$, and F is really a projection of the form $I \otimes F$. If E has a nonzero eigenvector x, and if F has a nonzero eigenvector y, then $(E \otimes I)(I \otimes F) = E \otimes F$ has a nonzero eigenvector $x \otimes y$. It follows that $E \otimes F \neq 0$.

As a summary, here are some of the ways in which QM is manifestly local:

- microcausality: quantitites associated with distant regions are compatible with each other.
- state preparation independence: if D_1 can be prepared and D_2 can be prepared, then $D_1 \otimes D_2$ can be prepared.
- quantum no signalling theorem
- apparent parameter independence: statistics of measurements are independent of what is being measured at a distance.

8 Bell's inequality for separable states

It's not true, of course, that all quantum states violate Bell's inequality. Indeed, product states, such as $x \otimes y$, most definitely do not violate Bell's inequality.

In this section, we define the notion of a separable state, and we show that no separable states violate Bell's inequality. The paradigm case of separable states are those given by product vectors such as $x \otimes y$. In that case, for any product operator such as $A \otimes B$, we have

$$\begin{aligned}\langle x \otimes y, (A \otimes B) x \otimes y \rangle &= \langle x \otimes y, Ax \otimes By \rangle \\ &= \langle x, Ax \rangle \cdot \langle y, By \rangle .\end{aligned}$$

That is, expectation values factorize, which motivates the following definition.

Definition. Let $\mathcal{H} \otimes \mathcal{K}$ be the Hilbert space for a composite quantum system, and let ω be a quantum state for this system. We say that ω is a **product state** just in case $\omega(A \otimes B) = \omega(A \otimes I) \times \omega(I \otimes B)$, for all $A \in \mathbf{B}(\mathcal{H})$ and $B \in \mathbf{B}(\mathcal{K})$. We say that ω is a **separable state** just in case it is a convex combination of product states. If a state is not separable, then we say that it is **inseparable**. Thus, the notion of an inseparable state is a generalization of the notion of an entangled state.

We show first that product states satisfy Bell's inequality. In fact, the result is simple enough: if ω is a product state and $R = A_1(B_1 + B_2) + A_2(B_1 - B_2)$ is a Bell observable, then

$$\omega(R) \;=\; \omega(A_1)\omega(B_1) + \omega(A_1)\omega(B_2) + \omega(A_2)\omega(B_1) - \omega(A_2)\omega(B_2).$$

Furthermore, since we've assumed that $\mathrm{sp}(A_i) \subseteq [-1, 1]$ and $\mathrm{sp}(B_i) \subseteq [-1, 1]$, it follows that $-1 \leq \omega(A_i) \leq 1$ and $-1 \leq \omega(B_i) \leq 1$, and we can just plug the numbers into the previous derivation of Bell's inequality. That is, $-2 \leq \omega(R) \leq 2$.

Now suppose that ρ is separable, that is $\rho = \sum_{i=1}^{n} \lambda_i \omega_i$, where each ω_i is a product state, and where $\sum_i \lambda_i = 1$. Since $-2 \leq \omega_i(R) \leq 2$ for $i = 1, \ldots, n$, it follows that $-2 \leq \rho(R) \leq 2$. Therefore, all separable states satisfy Bell's inequality.

One might conjecture now that if a state is inseparable, then it violates Bell's inequality. It turns out, however, that this conjecture is false. There are certain quantum states, known as **Werner states**, that are inseparable but satisfy Bell's inequality (see Werner, 1989).

9 Summary

Why are people arguing about the upshot of Bell's theorem? My sense is that the deeper source of the disagreement here are the different attitudes toward the question: what should we do next in physics? Or to put the question in a more mundane way: which approach should I — as a student, or as a researcher — invest my time and effort into? In particular: should we proceed with the quantum theory we find in today's textbooks, or should we work on an alternative such as Bohmian mechanics? Let me explain what Bell's theorem has to do with that question.

Quantum mechanics definitely poses some conceptual challenges. (I use the phrase "conceptual challenges" as a euphemism where others say that "QM is incoherent" or "QM is not even a theory.") When one learns about these challenges, and then learns that other approaches — such as Bohmian mechanics — don't face these same challenges, then one is faced with a dilemma. Should one give up the current version of QM? One of the knee-jerk reactions among physicists has been that Bohmian mechanics is not a good alternative to standard QM, because Bohmian mechanics is nonlocal, and hence conflicts with Einstein's relativity theory.

Thus, for defenders of Bohmian mechanics, it's sociologically and psychologically important to establish that their theory does not have a fatal flaw — or at least, not a flaw that other approaches lack. If Bohmians could demonstrate that *any* theory that saves the phenomena is nonlocal, then they will have removed the primary objection to their approach. In other words, Bell's theorem could provide Bohmians with the ultimate *tu quoque* response to objections to their theory: "You call my theory nonlocal? Well, so is yours."

Bohmians are not the only guilty party here. I suspect that resistance to the claim that Bell's theorem establishes nonlocality is primarily driven by the desire to maintain the status quo in physics. When physicists are told that their favorite theory is "incoherent" or "not even a theory," and when they are presented with an alternative (such as Bohmian mechanics) which presumably does not have these fatal flaws, then they might want to strike back with: "well, but Bohmian mechanics, unlike standard QM, contradicts relativity."

On balance, I think the situation is simply the frustrating one that Bell's theorem — like any other mathematical theorem — makes use of many premises, some explicit, any many others implicit. What that means is that a violation of Bell's inequality only tells us that one or another of those premises is false. It doesn't tell us which one of those premises is the guilty one, and for all we know, it might be a framework assumption that we haven't yet made explicit. Think here about Einstein's path to the discovery of relativity theory. When Lorentz's ether theory made a false prediction, then the obvious thing was to conclude that one of Lorentz's premises was false. However, nobody before Einstein thought that perhaps the false thing was the implicit assumption that the notion of simultaneity is observer-independent!

Is it possible then we have yet to identify some implicit assumption in the derivation of Bell's inequality?

Discussion questions

1. Bohmian mechanics says that all measurements ultimately reduce to position measurements. How is it then that Bohmian mechanics violates Bell's inequality — as it, must, since it reproduces all the predictions of QM?

2. For those who have studied relativity theory: it's often said that relativity theory is the paradigm of a local physical theory. How is the locality assumption expressed in relativity theory? Does the kind of locality required by relativity conflict with what we see in QM?

3. Is locality a discovery of science, or is it a presupposition for intelligible explanations? Why is there a tendency to associate non-locality with something "spooky" or even "mystical"?

4. If there is nonlocality in nature, then why can it apparently not be used to send information faster than the speed of light?

Appendix: Probabilities in the singlet state

9 Lemma. *Let Ω be the singlet vector in $\mathcal{H} \otimes \mathcal{H}$, and let $\{e_0, e_1\}$ be an orthonormal basis for $\mathcal{H}$. Then*

$$\Omega \;=\; c(e_0 \otimes e_1 - e_1 \otimes e_0),$$

for some complex number c.

Sketch of proof. Write $|0\rangle = c_0 e_0 + c_1 e_1$ and $|1\rangle = d_0 e_0 + d_1 e_1$, and compute. □

10 Lemma. *If Ω is the singlet vector, then the reduced state on each subsystem is the the maximally mixed state.*

Proof. Of course, this fact can be computed directly. In addition, looking at Lemma 9, if P is any one-dimensional projection, then $\langle \psi, (P \otimes I)\psi \rangle = \frac{1}{2}$. □

By default convention, the three basic spin operators on $\mathcal{H} = \mathbb{C}^2$ are given by:

$$S_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad S_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

It then follows that

$$S_j S_k - S_k S_j = 2i S_\ell, \qquad \text{and} \qquad S_j S_k + S_k S_j = 0,$$

whenever j, k, ℓ are distinct.

4 Exercise. Show that the operators $\frac{1}{2}(1+S_z)$ and $\frac{1}{2}(1-S_z)$ are the spectral projections for S_z.

For our calculations, it will be convenient to generate spin matrices from the following more basic matrices:

$$V = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad\qquad V^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Technically, we shouldn't have written V^* until we established that V^* is the adjoint of V, but it is a straightforward calculation. Here V is defined by the fact that $V|0\rangle = |1\rangle$ and $V|1\rangle = 0$, or in Dirac notation, $V = |1\rangle\langle 0|$. Similarly, V^* is defined by the fact that $V^*|1\rangle = |0\rangle$ and $V^*|0\rangle = 0$, or in Dirac notation, $V^* = |0\rangle\langle 1|$. Hence,

$$V^*V \ = \ |0\rangle\langle 1|1\rangle\langle 0| \ = \ |0\rangle\langle 0|$$

is the projection onto the $+1$ eigenspace of S_z, and $VV^* = |1\rangle\langle 1|$ is the projection onto the -1 eigenspace of S_z. We also have $V^2 = (V^*)^2 = 0$ and $\mathrm{Tr}(V) = \mathrm{Tr}(V^*) = 0$.

Now let

$$A(\theta) \ = \ \exp(i\theta)V + \exp(-i\theta)V^*,$$

where θ is any angle in the xy plane up from x. In particular,

$$A(0) \ = \ V + V^* \ = \ S_x,$$

and

$$A(\pi/2) \ = \ iV - iV^* \ = \ S_y,$$

where we used the fact that $\exp(i\pi/2) = i$ and $\exp(-i\pi/2) = -i$. Note furthermore that

$$\begin{aligned} A(\pi/4) &= 2^{-1/2}(V + iV + V^* - iV^*) \\ &= 2^{-1/2}(V + V^* + iV - iV^*) \\ &= 2^{-1/2}(S_x + S_y), \end{aligned}$$

and

$$A(-\pi/4) \;=\; 2^{-1/2}(S_y - S_x).$$

A straightforward calculation shows that

$$A(\theta_1)A(\theta_2) \;=\; \exp(i(\theta_1 - \theta_2))P + \exp(-i(\theta_1 - \theta_2))(I - P),$$

where $P = VV^*$. Now let ω be any state such that $\omega(P) = \omega(I - P) = 1/2$. [This is true, for example, when ω is a tracial state, because then $\omega(V^*V) = \omega(VV^*)$.] Then it follows that

$$\omega(A(\theta_1)A(\theta_2)) \;=\; \cos(\theta_1 - \theta_2). \tag{7}$$

11 Lemma. *If Ω is the singlet vector, then*

$$(A(\theta) \otimes I)\Omega \;=\; -(I \otimes A(\theta))\Omega.$$

Proof. For notational simplicity, assume that all operators are multiplied by $\sqrt{2}$, so that we don't have to repeatedly renormalize vectors. In that case, we have $(V \otimes I)\Omega = |11\rangle$ and $(V^* \otimes I)\Omega = -|00\rangle$, from which

$$(A(\theta) \otimes I)\Omega \;=\; \exp(i\theta)|11\rangle - \exp(-i\theta)|00\rangle.$$

Similarly, $(I \otimes V)\Omega = -|11\rangle$ and $(I \otimes V^*)\Omega = |00\rangle$, from which

$$(I \otimes A(\theta))\Omega \;=\; -\exp(i\theta)|11\rangle + \exp(-i\theta)|00\rangle.$$

□

Putting together the previous results gives the following expectation-value formula for the singlet state:

$$\begin{aligned} \langle \Omega, (A(\theta_1) \otimes -A(\theta_2))\Omega \rangle &= \langle \Omega, (A(\theta_1) \otimes I)(A(\theta_2) \otimes I)\Omega \rangle \\ &= \langle \Omega, (A(\theta_1)A(\theta_2) \otimes I)\Omega \rangle \\ &= \omega(A(\theta_1)A(\theta_2)) \\ &= \cos(\theta_1 - \theta_2). \end{aligned}$$

Here ω is the reduced state on the left-hand subsystem. In particular, if we let $A_1 = A(0) \otimes I$, $A_2 = A(\pi/2) \otimes I$, $B_1 = I \otimes -A(\pi/4)$, and $B_2 = I \otimes -A(-\pi/4)$, then we have the following expectation values:

$$\begin{aligned}
\omega(A_1B_1) &= \cos(-\pi/4) &= \tfrac{1}{\sqrt{2}} \\
\omega(A_1B_2) &= \cos(\pi/4) &= \tfrac{1}{\sqrt{2}} \\
\omega(A_2B_1) &= \cos(\pi/4) &= \tfrac{1}{\sqrt{2}} \\
\omega(A_2B_2) &= \cos(3\pi/4) &= -\tfrac{1}{\sqrt{2}}.
\end{aligned}$$

The next thing to do is to compute the corresponding probabilities, i.e. expectation values for spectral projections. For fixed θ, we have $A(\theta)^2 = I$, hence both $E_0 = \frac{1}{2}(I + A(\theta))$ and $E_1 = \frac{1}{2}(I - A(\theta))$ are projection operators such that $E_0 - E_1 = A(\theta)$. For two angles θ_1 and θ_2, let $E = \frac{1}{2}(I + A(\theta_1))$ and let $F = \frac{1}{2}(I + A(\theta_2))$. Then

$$EF = \tfrac{1}{4}(I + A(\theta_1) + A(\theta_2) + A(\theta_1)A(\theta_2)).$$

If ω is a tracial state, then $\omega(A(\theta_i)) = 0$, and hence

$$\omega(EF) = \tfrac{1}{4}(1 + \cos(\theta_1 - \theta_2)) = \tfrac{1}{2}\cos^2((\theta_1 - \theta_2)/2),$$

where we used Equation 7 and the trigonometric identity $\cos(2\theta) = 2\cos^2(\theta) - 1$. Similarly, if $E' = \frac{1}{2}(I - A(\theta_1))$ and $F' = (I - A(\theta_2))$, then

$$E'F' = \tfrac{1}{4}(I - A(\theta_1) - A(\theta_2) + A(\theta_1)A(\theta_2)),$$

from which it follows that $\omega(E'F') = \omega(EF)$. Finally,

$$\begin{aligned}
\omega(E'F) &= \omega((I - E)F) &= \omega(F) - \omega(EF) \\
&= \tfrac{1}{2} - \omega(EF) &= \tfrac{1}{2}\sin^2((\theta_1 - \theta_2)/2).
\end{aligned}$$

These calculations yield the following probabilities in the singlet state:

$$\begin{aligned}
P_{\theta_1\theta_2}(++) &= \tfrac{1}{2}\sin^2((\theta_1 - \theta_2)/2) \\
P_{\theta_1\theta_2}(+-) &= \tfrac{1}{2}\cos^2((\theta_1 - \theta_2)/2) \\
P_{\theta_1\theta_2}(-+) &= \tfrac{1}{2}\cos^2((\theta_1 - \theta_2)/2) \\
P_{\theta_2\theta_2}(--) &= \tfrac{1}{2}\sin^2((\theta_1 - \theta_2)/2)
\end{aligned}$$

The following table gives probabilities for measurement settings $a = \pi/3$ and $b = 0$ on the left, and $b = 0$ and $c = -\pi/3$ on the right. These numbers give a maximal violation of the probabilistic version of Bell's inequality (Equation 5).

		$--$	$-+$	$+-$	$++$
$\pi/3$	0	$1/8$	$3/8$	$3/8$	$1/8$
$\pi/3$	$-\pi/3$	$3/8$	$1/8$	$1/8$	$3/8$
0	0	0	$1/2$	$1/2$	0
0	$-\pi/3$	$1/8$	$3/8$	$3/8$	$1/8$

Appendix: Reduced states

Any quantum state of a composite system ω gives rise to restricted states of its subsystems. In particular, any operator A on $\mathcal{H}$ corresponds to an operator $A \otimes I$ on $\mathcal{H} \otimes \mathcal{K}$, so we may define

$$\omega^*(A) \; = \; \omega(A \otimes I),$$

for all $A \in \mathbf{B}(\mathcal{H})$. It is straightforward to verify that ω^* is an abstract state on $\mathbf{B}(\mathcal{H})$. And since every abstract state corresponds to a quantum state (i.e. a density operator), it follows that for any density operator D on $\mathcal{H} \otimes \mathcal{K}$, there is a corresponding reduced density operator D^* on $\mathcal{H}$. We now see how to compute this reduced density operator directly.

We first show that the trace is multiplicative across tensor products. To this end, note that if we choose o.n. bases $\{e_i\}$ for $\mathcal{H}$ and $\{f_i\}$ for $\mathcal{K}$, then $\{e_i \otimes f_j\}$ is an o.n. basis for $\mathcal{H} \otimes \mathcal{K}$. Hence,

$$\begin{aligned}
\mathrm{Tr}(X \otimes Y) &= \sum_i \sum_j \langle e_i \otimes f_j \mid X e_i \otimes Y f_j \rangle, \\
&= \sum_i \langle e_i \mid X e_i \rangle \times \sum_j \langle f_j \mid Y f_j \rangle \\
&= \mathrm{Tr}(X) \times \mathrm{Tr}(Y).
\end{aligned}$$

The multiplicativity of trace across tensor products also entails the Hilbert-

Schmidt inner product is multiplicative across inner products:

$$\begin{aligned}
\langle A_1 \otimes B_1, A_2 \otimes B_2 \rangle &= \mathrm{Tr}((A_1 \otimes A_2)^*(B_1 \otimes B_2)) \\
&= \mathrm{Tr}(A_1^* B_1 \otimes A_2^* B_2) \\
&= \mathrm{Tr}(A_1^* B_1) \times \mathrm{Tr}(A_2^* B_2) \\
&= \langle A_1, B_1 \rangle \times \langle A_2, B_2 \rangle.
\end{aligned}$$

Let $\{E_i\}$ be a family of one-dimensional projections on $\mathcal{H}$ such that $\sum_i E_i = I$. It follows that $\{E_i\}$ is an o.n. basis for $\mathbf{B}(\mathcal{H})$ relative to the Hilbert-Schmidt inner product, and hence, for any operator $A \in \mathbf{B}(\mathcal{H})$,

$$A = \sum_i \langle E_i, A \rangle E_i.$$

Thus,

$$A \otimes I = \sum_i \langle E_i, A \rangle (E_i \otimes I),$$

and

$$\langle D, A \otimes I \rangle = \sum_i \langle D, E_i \otimes I \rangle \, \langle E_i, A \rangle.$$

Now define

$$D^* = \sum_i \lambda_i E_i,$$

where

$$\lambda_i = \langle D, E_i \otimes I \rangle = \mathrm{Tr}(D(E_i \otimes I)).$$

It's clear then that $0 \leq \lambda_i \leq 1$, and that $\sum_i \lambda_i = 1$. Hence, D^* is a density operator, and

$$\begin{aligned}
\mathrm{Tr}(D^* A) &= \langle D^*, A \rangle \\
&= \sum_i \lambda_i \langle E_i, A \rangle \\
&= \sum_i \langle D, E_i \otimes I \rangle \, \langle E_i, A \rangle \\
&= \langle D, A \otimes I \rangle \\
&= \mathrm{Tr}(D(A \otimes I)),
\end{aligned}$$

for all $A \in \mathcal{B}(\mathcal{H})$.

Example. Let E be the projection onto the singlet state. Thus, E is a density operator state for a composite system $\mathcal{H}\otimes\mathcal{K}$ consisting of two 2-dimensional Hilbert spaces. Let F_1, F_2 be orthogonal, one-dimensional projection operators on $\mathcal{H}$. It can then be shown that $\mathrm{Tr}(E(F_i\otimes I)) = \frac{1}{2}$. Hence, the reduced density operator on $\mathcal{H}$ is $(1/2)F_1 + (1/2)F_2 = (1/2)I$, i.e. the maximally mixed state. This is a prime example of a case where the reduced state "loses information." If we take the tensor product of the two density operators $(1/2)I$ and $(1/2)I$ on the subsystems, then we get a mixed state $(1/4)I$, and not the singlet state.

12 Lemma. *Let D_1 be a density operator on $\mathcal{H}$, and let D_2 be a density operator on $\mathcal{K}$. Then the density operator $D_1\otimes D_2$ represents a product state on $\mathcal{H}\otimes\mathcal{K}$.*

Proof. We compute:

$$\begin{aligned}\mathrm{Tr}((D_1\otimes D_2)(A\otimes B)) &= \mathrm{Tr}(D_1A\otimes D_1B)\\ &= \mathrm{Tr}(D_1A)\times\mathrm{Tr}(D_2B)\\ &= \mathrm{Tr}((D_1\otimes D_2)(A\otimes I))\times\mathrm{Tr}((D_1\otimes D_2)(I\otimes B)).\end{aligned}$$

□

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