Logic pset 6

Resources: HLW Ch 7 pp 116-127

1. Represent the form of the following sentences in predicate logic using the $=$ symbol where necessary.

   (a) There is one and only one Princeton University. (Use Px for "x is a Princeton University")

   $$\exists x(Px \wedge \forall y(Py \to y = x))$$

   (b) There is at most one Ivy League university in New Jersey. (Use Ix for "x is an Ivy League university", and use Nx for "x is in New Jersey.")

   $$\forall x \forall y(((Ix \wedge Nx) \wedge (Iy \wedge Ny)) \to x = y)$$

   (c) There is a smallest prime number. (Px, Sxy, variables are restricted to numbers.)

   $$\exists x(Px \wedge \forall y((Py \wedge y \neq x) \to Sxy))$$

   This translation does not capture the uniqueness that may be implicit in "smallest". But if it's written in a context where Sxy is a linear order, then there can only be one x with the feature that $\forall y((Py \wedge y \neq x) \to Sxy)$.

2. Prove the following sequents using any of the rules, including =E and =I.

   (a) $\exists x(Px \wedge \forall y(Py \to x = y)) \vdash \forall x \forall y((Px \wedge Py) \to x = y)$

1	(1)	$\exists x(Px \wedge \forall y(Py \to x = y))$	A
2	(2)	$Pc \wedge \forall y(Py \to c = y)$	A
3	(3)	$Pa \wedge Pb$	A
3	(4)	Pa	3 $\wedge$E
3	(5)	Pb	3 $\wedge$E
2	(6)	$\forall y(Py \to c = y)$	2 $\wedge$E
2	(7)	$Pa \to c = a$	6 UE
2	(8)	$Pb \to c = b$	6 UE
2,3	(9)	$c = a$	7,4 MP
2,3	(10)	$c = b$	8,5 MP
2,3	(11)	$a = b$	10,9 =E
2	(12)	$(Pa \wedge Pb) \to a = b$	3,11 CP
2	(13)	$\forall y((Pa \wedge Py) \to a = y)$	12 UI
2	(14)	$\forall x \forall y((Px \wedge Py) \to x = y)$	13 UI
1	(15)	$\forall x \forall y((Px \wedge Py) \to x = y)$	1,2,14 EE

(b) $\vdash \forall x \forall y((x = y) \to (y = x))$

1	(1)	$a = b$	A
$\varnothing$	(2)	$a = a$	=I
1	(3)	$b = a$	2,1 =E
$\varnothing$	(4)	$a = b \to b = a$	1,3 CP
$\varnothing$	(5)	$\forall y(a = y \to y = a)$	4 UI
$\varnothing$	(6)	$\forall x \forall y(x = y \to y = x)$	5 UI

3. Let Rxy be a binary relation symbol that satisfies the transitivity axiom (page 126). Suppose that Rxy satisfies two other axioms: serial $\forall x \exists y Rxy$ and irreflexive $\forall x \neg Rxx$. Show that there are at least three distinct things, i.e.,

$$\exists x \exists y \exists z((x \neq y \wedge x \neq z) \wedge y \neq z).$$

It would also suffice to show that the claim "there are at most two things" contradicts the assumptions. You may write your proof in English prose (not our formal system), but you need to convince the reader that you would be able to write a full formal proof.

Here's a sub-proof that establishes that there are two things a, b. We use a as an arbitrary name for a thing that bears the relation R to some y. We then use b as an arbitrary name for some y to which a bears the relation R. The irreflexivity axiom (4) entails that $a \neq b$.

1	(1)	$\forall x \exists y Rxy$	A
1	(2)	$\exists y Ray$	1 UE
3	(3)	Rab	A
4	(4)	$\forall x \neg Rxx$	A
4	(5)	$\neg Rbb$	4 UE
6	(6)	$a = b$	A
3,6	(7)	Rbb	3,6 =E
3,4,6	(8)	$Rbb \wedge \neg Rbb$	7,5 ∧I
3,4	(9)	$a \neq b$	6,8 RA

We now repeat the exact same process to establish that there is a c such that $c \neq b$.

1	(10)	$\exists y Rby$	1 UE
11	(11)	Rbc	A
12	(12)	$c = b$	A
11,12	(13)	Rbb	11,12 =E
4,11,12	(14)	$Rbb \wedge \neg Rbb$	13,5 ∧I
4,11	(15)	$c \neq b$	12,14 RA

Now we need to show that $c \neq a$. If $c = a$, then Rba, and the transitivity axiom implies that Raa.

16	(16)	$c = a$	A
11,16	(17)	Rba	11,16 =E
18	(18)	$\forall x \forall y \forall z((Rxy \wedge Ryz) \to Rxz)$	A
18	(19)	$\forall y \forall z((Ray \wedge Ryz) \to Raz)$	18 UE
18	(20)	$\forall z((Rab \wedge Rbz) \to Raz)$	19 UE
18	(21)	$(Rab \wedge Rba) \to Raa$	20 UE
3,11,16	(22)	$Rab \wedge Rba$	3,17 $\wedge$I
3,11,16,18	(23)	Raa	21,22 MP
4	(24)	$\neg Raa$	4 UE
3,4,11,16,18	(25)	$Raa \wedge \neg Raa$	23,24 $\wedge$I
3,4,11,18	(26)	$c \neq a$	16,25 RA

To tidy up the proof, we need to collect everything together and do some steps of EI and EE. We end with dependency on the axioms $1, 4, 18$.

3,4,11	(27)	$a \neq b \wedge c \neq b$	9,15 $\wedge$I
3,4,11,18	(28)	$(a \neq b \wedge c \neq b) \wedge c \neq a$	27,26 $\wedge$I
3,4,11,18	(29)	$\exists z((a \neq b \wedge z \neq b) \wedge z \neq a)$	28 EI
3,4,11,18	(30)	$\exists y \exists z((a \neq y \wedge z \neq y) \wedge z \neq a)$	29 EI
3,4,11,18	(31)	$\exists x \exists y \exists z((x \neq y \wedge z \neq y) \wedge z \neq x)$	30 EI
1,3,4,18	(32)	$\exists x \exists y \exists z((x \neq y \wedge z \neq y) \wedge z \neq x)$	10,11,31 EE
1,4,18	(33)	$\exists x \exists y \exists z((x \neq y \wedge z \neq y) \wedge z \neq x)$	2,3,32 EE