

Lecture 7: Invitation to Predicate Logic

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Overview

- ① Motivation: Propositional logic cannot see all logical relations
- ② A more fine-grained grammar
 - Names and predicates
 - Variables and quantifiers
- ③ Translation
- ④ Inference rules
 - \forall elimination
 - \forall introduction

Propositional logic is inadequate

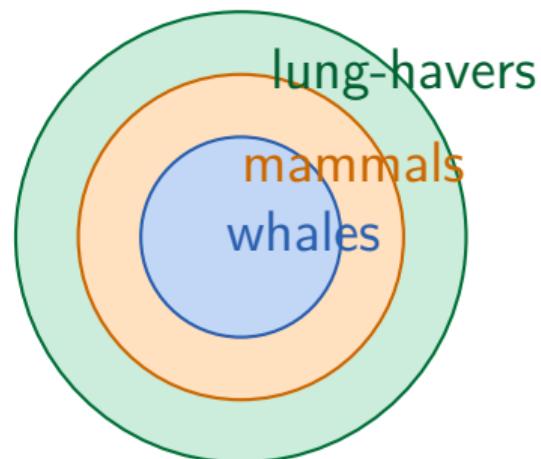
Validities that escape propositional logic

- All people are mortal.
- Socrates is a person.
- Therefore, Socrates is mortal.

If the subject and predicate of sentences are not both the same, then propositional logic does not recognize any relation between them.

Validities that escape propositional logic

- All **whales** are **mammals**.
- All **mammals** have **lungs**.
- Therefore: All **whales** have **lungs**.



$$\text{All}(W, M), \text{All}(M, L) \vdash \text{All}(W, L)$$

When propositional logic falls short

Example from mathematics

If a number is even, its square is even. 4 is even. $\therefore 4^2$ is even.

Propositional view:

$$P, P \rightarrow Q \vdash Q$$

Mathematical structure:

$$\forall n (E(n) \rightarrow E(n^2)), E(4) \vdash E(4^2)$$

Diagnosis

- The inadequacy of propositional logic cannot be fixed by adding more inference rules.
 - If we add any additional rules, then our system would become inconsistent.
- Have we missed some propositional connectives?
 - No, there is a precise sense in which our set of connectives is **conceptually complete**.

The predicate calculus

- In the early 20th century, the missing logical structure was identified, represented symbolically, and codified in a “calculus”.

Sub-propositional grammar

Names and predicates

Alice is French.

Bernard is French.

Alice is German.

Quantified sentences

- You are familiar with the concept of a variable from mathematics.
- Natural languages do not explicitly use variables.
- Hypothesis: “All” and “Some” sentences are best analyzed as consisting of predicate symbols, variables, and quantifiers.

Variables

Alice is French.	Fa
x is French.	Fx
Someone is French.	?
There is an x such that x is French.	$\exists x Fx$

Formulas

- We don't call " Fx " a proposition, since it cannot be true or false.
- We call " Fx " a **formula**.
- Adding the quantifier " $\exists x$ " to " Fx " creates a sentence.

Universal quantifier

All whales are mammals. ?

If a is a whale then a is a mammal. $Wa \rightarrow Ma$

For any x , if x is a whale then x is a mammal. $\forall x(Wx \rightarrow Mx)$

Standard syllogistic forms

All Finns are gregarious.

$$\forall x (Fx \rightarrow Gx)$$

Some Finns are gregarious.

$$\exists x (Fx \wedge Gx)$$

No Finns are gregarious.

$$\forall x (Fx \rightarrow \neg Gx)$$

Some Finns are not gregarious.

$$\exists x (Fx \wedge \neg Gx)$$

All happy Finns are gregarious.

All Finns and Germans are happy.

No dogs or cats are permitted in the restaurant.

$$\forall x(Fx \rightarrow P)$$

Everything has the feature that if it is F , then P holds.

$$\forall x Fx \rightarrow P$$

If everything has the feature F , then P holds.

Relations

Maren is taller than Niels.

Maren is taller than someone.

Someone is taller than Niels.

Everyone is taller than someone.

Someone is taller than everyone.

There is a student who admires every professor.

$$\exists x(Sx \wedge \forall y(Py \rightarrow Axy))$$

There is a professor whom every student admires.

$$\exists x(Px \wedge \forall y(Sy \rightarrow Ayx))$$

Every student admires some professor.

$$\forall x(Sx \rightarrow \exists y(Py \wedge Axy))$$

Inference to/from quantified statements

\forall elimination

The idea behind \forall elimination is straightforward:

From a universal statement, any **instance** follows logically.

$$\frac{\forall x \varphi(x)}{\varphi(a)}$$

\forall elimination

$$\forall x(Fx \rightarrow Gx), Fa \vdash Ga$$

- | | | | |
|-----|-----|--------------------------------|--------|
| 1 | (1) | $\forall x(Fx \rightarrow Gx)$ | A |
| 2 | (2) | Fa | A |
| 1 | (3) | $Fa \rightarrow Ga$ | 1 UE |
| 1,2 | (4) | Ga | 3,2 MP |

$$\forall x \forall y (Fx \wedge Gy) \vdash Fa \wedge Gb$$

1 (1) $\forall x \forall y (Fx \wedge Gy)$ A

1 (2) $\forall y (Fa \wedge Gy)$ 1 UE

1 (3) $Fa \wedge Gb$ 2 UE

$$\forall x \forall y (Fx \wedge Gy) \vdash Fa \wedge Ga$$

1 (1) $\forall x \forall y (Fx \wedge Gy)$ A

1 (2) $\forall y (Fa \wedge Gy)$ 1 UE

1 (3) $Fa \wedge Ga$ 2 UE

$$P \rightarrow \forall x Fx \vdash P \rightarrow Fa$$

- | | | | |
|-----|-----|------------------------------|--------|
| 1 | (1) | $P \rightarrow \forall x Fx$ | A |
| 2 | (2) | P | A |
| 1,2 | (3) | $\forall x Fx$ | 1,2 MP |
| 1,2 | (4) | Fa | 3 UE |
| 1 | (5) | $P \rightarrow Fa$ | 2,4 CP |

$\neg Fa \vdash \neg \forall x Fx$

1	(1)	$\neg Fa$	A
2	(2)	$\forall x Fx$	A
2	(3)	Fa	2 UE
1,2	(4)	$Fa \wedge \neg Fa$	3,1 $\wedge I$
1	(5)	$\neg \forall x Fx$	2,4 RA

Warnings

Only apply UE when the entire sentence on the line is universally quantified.

$$\forall x(Fx \rightarrow P)$$

$$\forall x(Fx \rightarrow \forall yGy)$$

$$\forall x Fx \rightarrow Ga$$

$$\forall x \forall y Rx y$$

Warnings

When applying UE, replace all instances of the relevant variable with the same name.

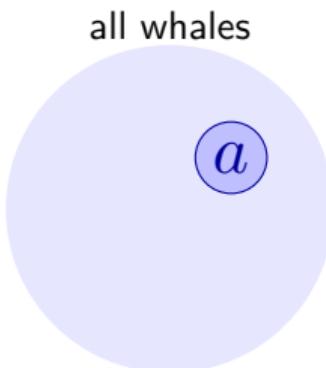
$$\forall x(Fx \rightarrow \forall yRxy)$$

From one individual to everyone

Intuitive idea

To show that **everyone** has a property, we can reason about **one individual chosen at random**.

- Suppose we want to prove that all whales have lungs.
- We pick a whale—call it a .
- We reason about a as if it were any whale.



What does it mean for a to be *arbitrary*?

Arbitrary name

The name a is **arbitrary** when nothing in the proof depends on any *special feature* of a .

- Our reasoning about a must not rely on facts like “ a lives in the Pacific” or “ a is the largest whale.”
- The argument must hold no matter which whale we picked.

An arbitrary name stands for an individual we reason about generally

From arbitrariness to universal generalization

Bridge to Universal Introduction

If we can prove $\varphi(a)$ using a as an **arbitrary name**, then we may infer the general statement $\forall x \varphi(x)$.

$$\frac{\varphi(a)}{\forall x \varphi(x)} \text{UI} \quad (\text{side condition: } a \text{ not free in any open assumption})$$

- The conclusion applies to *all* objects of that kind.
- The key is that a never referred to anything special.

Universal introduction

From a line

$$\Gamma \ (m) \ \varphi(a)$$

we may infer

$$\Gamma \ (n) \ \forall x \varphi(x)$$

provided that the name “ a ” does not occur in any of the sentences listed in Γ or in $\varphi(x)$.

\forall introduction

$$\forall x(Fx \rightarrow Gx), \forall x Fx \vdash \forall x Gx$$

- | | | | |
|-----|-----|--------------------------------|--------|
| 1 | (1) | $\forall x(Fx \rightarrow Gx)$ | A |
| 2 | (2) | $\forall x Fx$ | A |
| 2 | (3) | Fa | 2 UE |
| 1 | (4) | $Fa \rightarrow Ga$ | 1 UE |
| 1,2 | (5) | Ga | 4,3 MP |
| 1,2 | (6) | $\forall x Gx$ | 5 UI |

$\vdash \forall x(Fx \rightarrow (Fx \vee Gx))$

1	(1)	Fa	A
1	(2)	$Fa \vee Ga$	1 $\vee I$
\emptyset	(3)	$Fa \rightarrow (Fa \vee Ga)$	1,2 CP
\emptyset	(4)	$\forall x(Fx \rightarrow (Fx \vee Gx))$	3 UI

$$\forall x(P \rightarrow Fx) \vdash P \rightarrow \forall xFx$$

- | | | | |
|-----|-----|-------------------------------|--------|
| 1 | (1) | $\forall x(P \rightarrow Fx)$ | A |
| 2 | (2) | P | A |
| 1 | (3) | $P \rightarrow Fa$ | 1 UE |
| 1,2 | (4) | Fa | 3,2 MP |
| 1,2 | (5) | $\forall xFx$ | 4 UI |
| 1 | (6) | $P \rightarrow \forall xFx$ | 2,5 CP |

$$P \rightarrow \forall x Fx \vdash \forall x(P \rightarrow Fx)$$

- | | | | |
|-----|-----|-------------------------------|--------|
| 1 | (1) | $P \rightarrow \forall x Fx$ | A |
| 2 | (2) | P | A |
| 1,2 | (3) | $\forall x Fx$ | 1,2 MP |
| 1,2 | (4) | Fa | 3 UE |
| 1 | (5) | $P \rightarrow Fa$ | 2,4 CP |
| 1 | (6) | $\forall x(P \rightarrow Fx)$ | 5 UI |

Precisifying the UI rule

$\forall I$ requires replacing **all** instances of the arbitrary name.

1	(1)	$\forall x Rxx$	A
1	(2)	Raa	1 UE
1	(3)	$\forall x Rx a$	2 UI
1	(4)	$\forall y \forall x Rxy$	3 UI error!

Precisifying the UE rule

But UE does allow instantiating to a name that already occurs in the formula.

- | | | | |
|---|-----|---------------------------|------|
| 1 | (1) | $\forall x \forall y Rxy$ | A |
| 1 | (2) | $\forall y Ray$ | 1 UE |
| 1 | (3) | Raa | 2 UE |
| 1 | (4) | $\forall x Rxx$ | 3 UI |

Precisifying the UE rule

UE allows us to choose any name — same or different from what already occurs.

1	(1)	$\forall x \forall y Rxy$	A
1	(2)	$\forall y Ray$	1 UE
1	(3)	Rab	2 UE
1	(4)	$\forall x Rx b$	3 UI
1	(5)	$\forall y \forall x Rxy$	4 UI