Concepts in Differential Geometry

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October 1, 2024

Preliminaries

In this document, I present definitions of fundamental concepts in differential geometry: connections, parallel transport, and Riemann curvature. I then explore the relationships between these concepts, which are fundamental to understanding the geometry of smooth manifolds.

Definition. If M is a smooth manifold, we let TM denote its tangent bundle, and we let $\Gamma(TM)$ denote the set of sections of TM, i.e. smooth vector fields on M.

Connection on a Smooth Manifold

Definition (Connection)**.** Let M be a smooth manifold. A *connection*, or *covariant derivative operator*, on M is a bilinear map

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$

such that for any vector fields $X, Y, Z \in \Gamma(TM)$ and any smooth function $f \in C^\infty(M)$, the following conditions hold:

- **Linearity in the first argument:**

$$\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z,$$

- **Leibniz rule in the first argument:**

$$\nabla_{fX} Y = f \nabla_X Y,$$

- **Leibniz rule in the second argument:**

$$\nabla_X (fY) = (Xf)Y + f \nabla_X Y.$$

Parallel Transport

Definition (Parallel Transport)**.** Let M be a smooth manifold with a connection ∇, and let $\gamma : [0,1] \to M$ be a smooth path from $p = \gamma(0)$ to $q = \gamma(1)$. The *parallel transport map* along γ, denoted by

$$\tau_\gamma : T_pM \to T_qM,$$

is a linear isomorphism defined as follows: for any vector $v \in T_pM$, $\tau_\gamma(v) \in T_qM$ is the unique vector such that the vector field $V(t)$ along $\gamma(t)$, satisfying the initial condition $V(0) = v$, is parallel with respect to the connection ∇, i.e.,

$$\nabla_{\dot\gamma(t)} V(t) = 0 \quad \text{for all } t \in [0,1].$$

Riemann Curvature Tensor

In abstract index notation, the Riemann curvature tensor R is typically defined by setting its value on an arbitrary vector ξ:

$$R^a{}_{bcd}\xi^b \ = \ -2\nabla_{[c}\nabla_{d]}\xi^a.$$

If we look at the index structure, there is another way to think of R, i.e. as a map that takes two vectors λ^c, ρ^d and returns a tensor $R^a{}_{bcd}\lambda^c\rho^d$. The resulting tensor has form $\theta^a{}_b$, which can be thought of as a map from a vector ξ^b to a vector $\theta^a{}_b\xi^b$. Thinking of R this way leads to an intuitive picture: given two directions λ, ρ out of p, $R^a{}_{bcd}\lambda^c, \rho^d$ measures the deviation from identity for transport around an infinitesimal parallelogram.

Let's rewrite the definition without abstract indices: R is a map that takes two input vectors $X, Y \in T_p$ and returns a linear map $R(X,Y) : T_p \to T_p$. To be more precise, R maps vector fields to a tensor field.

Definition (Riemann Curvature Tensor)**.** Given a smooth manifold M with a connection ∇, the *Riemann curvature tensor* is the map

$$R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM),$$

defined by

$$R(X,Y) \ = \ \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]},$$

for vector fields $X, Y \in \Gamma(TM)$.

It is straightforward, if a bit tedious, to check that R is a tensor field.

structure	definable	not definable
manifold + Lorentzian metric		
manifold + connection	parallel transport, geodesics, curvature	
manifold	smooth curves, tangent vectors	

Relationships Between Connection, Parallel Transport, and Curvature

From Connection to Parallel Transport

A covariant derivative operator ∇ naturally defines parallel transport along any smooth path γ. Given a vector field $V(t)$ along $\gamma(t)$, the condition $\nabla_{\dot{\gamma}(t)}V(t) = 0$ (i.e., that $V(t)$ is parallel along γ) uniquely determines how to transport any vector $v \in T_pM$ to T_qM. This process is called *parallel transport*.

In this sense, the connection defines how vectors (and more generally, tensors) are propagated along curves in a manner consistent with the geometry defined by ∇.

Definition. Let M be a smooth manifold, and let ∇ be a connection on M. The *holonomy group* $\mathrm{Hol}(p, \nabla)$ is the set of all linear maps $\tau_\gamma : T_pM \to T_pM$, called *holonomies*, obtained by parallel transporting vectors along smooth closed loops γ based at p, with respect to the connection ∇. That is,

$$\mathrm{Hol}(p, \nabla) = \{\tau_\gamma \in \mathrm{GL}(T_pM) \mid \gamma : [0, 1] \to M \text{ is a smooth loop with } \gamma(0) = \gamma(1) = p\}.$$

From Parallel Transport to Connection

Conversely, the notion of parallel transport can also define a derivative operator. If we know how vectors are transported along all possible curves in a smooth manifold, we can recover the connection. Specifically, the connection $\nabla_X Y$ at a point can be defined by considering the infinitesimal limit of parallel transport along the flow generated by X. Thus, parallel transport and covariant derivatives are two perspectives on the same underlying geometric structure.

From Connection to Curvature

The Riemann curvature tensor R is derived directly from the covariant derivative operator ∇. It measures the noncommutativity of the covariant derivative, that is, how much the result of taking two successive derivatives depends on the order in which they are taken. If the connection ∇ were to commute in all directions, the curvature would vanish, meaning the manifold is *flat*.

Key Relations Between Curvature and Parallel Transport

One of the most important relations in differential geometry is the equivalence between curvature and the path-dependence of parallel transport.

Theorem 1. *Parallel transport along a curve γ is independent of the path taken between two points p and q if and only if the curvature of the connection is zero along γ.*

Sketch of Proof. The independence of parallel transport from the path is equivalent to the holonomy group of the connection being trivial, which occurs precisely when the curvature tensor R vanishes. If $R(X, Y)Z = 0$ for all vector fields X, Y, Z, then the connection is flat, meaning that parallel transport around any closed loop returns the vector unchanged, and transport between any two points is independent of the path. $\square$

Flatness and Commutativity of the Covariant Derivative

The curvature tensor R vanishes if and only if the covariant derivative operator ∇ commutes, i.e.,

$$\nabla_X \nabla_Y Z = \nabla_Y \nabla_X Z$$

for all vector fields $X, Y, Z \in \Gamma(TM)$. This commutativity implies that the manifold is locally flat, meaning that in small neighborhoods, the geometry resembles that of Euclidean space.