Cantor-Bernstein for Theories

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The purpose of this note is to ask: under what conditions could a pair of theories (T_1, T_2) fail to have the Cantor-Bernstein or co-Cantor-Bernstein property?

Definition. We say that the pair (T_1, T_2) has the Cantor-Bernstein property just in case: if T_1 and T_2 are mutually faithfully interpretable, then T_1 and T_2 are bi-interpretable. In other words: if (T_1, T_2) does not have the Cantor-Bernstein property then (a) T_1 and T_2 are mutually faithfully interpretable, and (b) T_1 and T_2 are not bi-interpretable.

Here "mutually faithfully interpretable" means that there are conservative (strong, equality preserving) translations $F : T_1 \to T_2$ and $G : T_2 \to T_1$. It would also be interesting to look at this question for the case of more general translations, with and without equality preservation. Could we end up getting different kinds of answers in the two cases? Or is the latter case reducible, in some sense, to the former?

Definition. We say that the pair (T_1, T_2) has the co-Cantor-Bernstein property just in case: if there are essentially surjective translations $F : T_1 \to T_2$ and $G : T_2 \to T_1$, then T_1 and T_2 are bi-interpretable.

1 Examples of theory pairs that are not CB

1. T_1 is the empty theory on a countably infinite propositional signature. T_2 is the "fan theory" with axioms $p_0 \vdash p_i$, for $i \geq 0$. These theories are counterexamples both to the CB and the co-CB properties.

   But are these theories "pathological" in some sense? The fact that these theories are propositional should not (I think) be seen as a pathology.

However, these theories are incomplete, and the second of them is not finitely axiomatizable.

2. The many examples of pairs of set theories described in (Freire and Hamkins 2020, p 8).

3. The pair of theories described in (Andréka, Madarász, and Németi 2005).

2 Cantor-Bernstein

The first thing I would like to show: for theories T_1 and T_2 to violate Cantor-Bernstein, both have to have infinitely many non-isomorphic models. (This fact is trivial in the case that T_1 and T_2 have models of infinite cardinality.)

Remark. Let $F : T_1 \to T_2$ be a translation, and let $F^* : M(T_2) \to M(T_1)$ be the dual functor. Recall that:

- $F^*(M) \vDash \phi$ iff $M \vDash F(\phi)$, for any Σ_1-sentence ϕ.
- F^* is faithful by definition.
- F^* reflects isomorphisms in the sense that if $F^*(j) : F^*(M) \to F^*(N)$ is an isomorphism, then $j : M \to N$ is an isomorphism. This follows from the fact that an elementary embedding is an isomorphism iff it is surjective, and from the fact that the underlying function of $F^*(j)$ is none other than the underlying function of j.

  Note however: F^* can still map non-isomorphic models to isomorphic models. For example: let T_1 be the theory that says that there are two things (in empty signature), let T_2 be the theory that says that there are two things (in signature with a unary predicate symbol P), and let $F : T_1 \to T_2$ be the inclusion. Let M be a model in which the extension of P is empty, and let N be a model in which the extension of P is non-empty. Then $F^*(M)$ is isomorphic to $F^*(N)$ although M is not isomorphic to N.

Definition. We say that a category C is *object-finite* just in case it has only finitely many objects up to isomorphism.

Let T be a theory. If $M(T)$ is object-finite then the Löwenheim-Skolem theorem implies that every model of T has finite cardinality. From this it follows that every hom set in $M(T)$ is finite. Furthermore, since for finite structures, elementary equivalence implies isomorphism, it follows that for any non-isomorphic M, N in $M(T)$, there is a sentence ϕ such that $M \vDash \phi$ and $N \vDash \neg\phi$.

Proposition 1. *Let T_1 be a theory such that $M(T_1)$ is object-finite, and let $F : T_1 \to T_2$ be a translation. If F is conservative then $F^* : M(T_2) \to M(T_1)$ is essentially surjective.*

Proof. We prove the contrapositive. If $F^* : M(T_2) \to M(T_1)$ is not eso, then there is a model M of T_1 that is not isomorphic to any model of the form $F^*(N)$, with N a model of T_2. By the preceding discussion, there is a sentence ϕ such that $M \vDash \phi$ but $F^*(N) \vDash \neg\phi$, for all models N of T_2. Hence $N \vDash F(\neg\phi)$ for all models N of T_2, and by completeness, $T_2 \vdash F(\neg\phi)$. Therefore F is not conservative. □

Remark. Let C and D be categories with respective object sets C_0 and D_0. Let $[C_0]$ and $[D_0]$ be the corresponding sets of equivalence classes of isomorphic objects. Each functor $F : C \to D$ induces a function $F_0 : [C_0] \to [D_0]$, and F_0 is surjective iff F is eso. If $F : C \to D$ and $G : D \to C$ are both eso, then Cantor-Bernstein for finite sets implies that F_0 is a bijection.

Lemma 1. *Let F be a finite set, let $f : F \to \mathbb{N}$ be a function, and let $\phi : F \to F$ be a bijection such that $f(x) \leq f(\phi(x))$ for all $x \in F$. Then $f(x) = f(\phi(x))$ for all $x \in F$.*

Sketch of proof. The function f corresponds to a fibration of F over $\mathbb{N}$. Since ϕ is a bijection, the size of the fibers remain constant, i.e., $|f^{-1}(n)| = |(f \circ \phi)^{-1}(n)|$. Since ϕ is monotonic, it cannot move an element to a lower fiber. Thus no element can be moved out of the highest fiber, nor the next highest fiber, etc. □

Proposition 2. *Let C and D be totally-finite categories. If there are faithful, eso functors $F : C \to D$ and $G : D \to C$, then C and D are equivalent categories. In fact, F itself is one half of an equivalence.*

Proof. By the above remark, $F_0 : [C_0] \to [D_0]$ is a bijection. Since F is automatically faithful, it will suffice to show that F is full. For simplicity, we may henceforth replace C and D with the corresponding skeletal categories.

Consider the (finite) set $C_0 \times C_0$ and the function $f : C_0 \times C_0 \to \mathbb{N}$ that assigns the cardinality of the corresponding hom set. That is, $f(a,b) = |\hom(a,b)|$. Let $g : D_0 \times D_0 \to \mathbb{N}$ be the corresponding function for D. Since F is faithful, it induces a bijection $\eta : C_0 \to D_0$ such that

$$f(a,b) \;\leq\; g(\eta(a),\eta(b)).$$

And since G is faithful, it induces a bijection $\theta : D_0 \to C_0$ such that

$$g(a,b) \;\leq\; f(\theta(a),\theta(b)).$$

If we let $\phi = \theta \circ \eta$ then

$$f(a,b) \;\leq\; f(\phi(a),\phi(b)),$$

for all $a, b \in C_0$. By the above lemma, $f(a,b) = f(\phi(a),\phi(b))$, and it follows that F is full. $\square$

Proposition 3. *Let T_1 and T_2 be proper theories such that $M(T_1)$ and $M(T_2)$ are object-finite. Then (T_1, T_2) has the Cantor-Bernstein property.*

Proof. Let $F : T_1 \to T_2$ and $G : T_2 \to T_1$ be conservative translations. By the previous results, $F^* : M(T_2) \to M(T_1)$ is part of an equivalence of categories. By Theorem 7.1 of (D'Arienzo, Pagano, and Johnson 2020), F is part of a homotopy equivalence. $\square$

Remark. TO DO: I need to check the previous result. The problem is that the conclusion of Theorem 7.1 shows that F is part of a weak equivalence. Does that automatically show that F is part of a strong equivalence? Note that the dimension of F is 1, and the domain formula is trivial/universal.

Conjecture 1. *If $M(T_1)$ and $M(T_2)$ are object-finite then (T_1, T_2) has the Cantor-Bernstein property.*

3 Co-Cantor-Bernstein

Proposition 4. *If T_1 or T_2 is complete, then (T_1, T_2) has the co-CB property.*

Proof. If T_1 is complete, then every translation $F : T_1 \to T_2$ is conservative. So if $F : T_1 \to T_2$ is eso, then F is a strong equivalence, i.e. T_1 and T_2 are bi-interpretable. $\square$

Recall that $F : T_1 \to T_2$ is conservative iff $F^* : M(T_2) \to M(T_1)$ is a full functor (Barrett 2020). Recall also that if F is a strong (equality-preserving) translation, then F^* preserves cardinality of models. In particular, if T_1 and T_2 are $\aleph_0$-categorical theories, then a translation $F : T_1 \to T_2$ induces a group homomorphism $F^* : \mathrm{Aut}(M_2) \to \mathrm{Aut}(M_1)$, and F is conservative iff F^* is surjective. (In fact, $\mathrm{Aut}(M_i)$ is naturally a topological group, and I conjecture that F^* is a continuous group homomorphism.)

Recall that if T has countable signature, and if T is $\aleph_0$-categorical, then T is complete. The following result would be interesting because an $\aleph_0$-categorical theory is essentially characterized by a topological group, viz. the group of automorphisms of its unique (up to isomorphism) countable model.

Conjecture 2. *There are $\aleph_0$-categorical theories T_1 and T_2 that do not have the CB property.*

Definition. Let I be the set of axioms $\{\exists_{>1}, \exists_{>2}, \dots\}$. We say that a theory T is essentially finitely axiomatizable just in case there is a finite set E of axioms such that $Cn(T) = Cn(E \cup I)$.

Conjecture 3. *There are essentially finitely axiomatizable theories T_1 and T_2 that do not have the CB property.*

4 Conjecture: co-CB fails only for theories with many models

Remark. For a theory T and a cardinal number κ, let $I(T, \kappa)$ be the number of non-isomorphic countable models of T. Interestingly, $I(T, \aleph_0)$ can be countably infinite, or any finite cardinal besides 2. See (Marker 2006, p 155ff).

Proposition 5. *If $F : T_1 \to T_2$ is essentially surjective, then for any fixed cardinal number κ, $I(T_2, \kappa) \leq I(T_1, \kappa)$.*

Proof. Recall that the dual functor F^* is always faithful. If $F : T_1 \to T_2$ is eso, then F^* is also full (Halvorson 2019, Prop 6.6.13). In particular, for any models M, N of T_2, if $F^*(M)$ is isomorphic to $F^*(N)$, then M is isomorphic to N. Now fix a cardinal number κ, and let $[M(T_i)]_\kappa$ be the set of isomorphism classes of models of T_i of cardinality κ. Then F^* induces a one-to-one mapping from $[M(T_2)]_\kappa$ into $[M(T_1)]_\kappa$. $\square$

Proposition 6. *Suppose that T_2 has finitely many non-isomorphic models of each cardinality. If $F : T_1 \to T_2$ and $G : T_2 \to T_1$ are essentially surjective, then (F^*, G^*) is an equivalence of categories. If T_1 and T_2 are proper theories then F^* is part of a homotopy equivalence.*

Proof. For the first part it will suffice to show that F^* is essentially surjective. By the previous proof, G^* induces an injection of $[M(T_1)]_\kappa$ into $[M(T_2)]_\kappa$. Since the latter is finite, so is the former. Since F^* is an injection of one finite set into a not-larger finite set, it follows that F^* is bijection. Therefore F^* is essentially surjective.

The second part follows from Theorem 7.1 of (D'Arienzo, Pagano, and Johnson 2020). □

Corollary 1. *Let T_1 and T_2 be proper theories. If (T_1, T_2) violate the co-CB property then there is a cardinal number κ such that T_1 and T_2 have infinitely many non-isomorphic models of size κ.*

TO DO: I would like to come up with a similar necessary condition for T_1 and T_2 violating the CB property. However, we do not yet have any interesting result of the form: "if T_1 or T_2 is ... and $F : T_1 \to T_2$ is conservative then $F^* : M(T_2) \to M(T_1)$ is ...."

5 Groups that are not CB

Definition. Let $P(\mathbb{N})$ be the permutation group of the natural numbers, equipped with the topology of pointwise convergence. (TO DO: explain the sense in which this topology on $P(\mathbb{N})$ is definable from the theory of infinite sets. Explain more generally the sense in which for a Σ-structure M, $\mathrm{Aut}(M)$ is naturally a topological group.)

Fact 1. *Let G be a subgroup of $P(\mathbb{N})$. Then G is the automorphism group of an $\aleph_0$-categorical theory iff G is a closed subset of $P(\mathbb{N})$.*

Conjecture 4. *There are closed subgroups G and H of $P(\mathbb{N})$ such that G is isomorphic to a closed subgroup of H and vice versa, but G and H are not isomorphic.*

It is not difficult at all to find groups that violate the Cantor-Bernstein condition — but I do not immediately know if any of these groups are of the form $\mathrm{Aut}(M)$ for an $\aleph_0$-categorical structure M.

1. The group S_∞ of finite permutations of $\mathbb{N}$ and the alternating grooup A_∞. See `https://math.stackexchange.com/questions/1259081/if-there-are-injecti`
2. Infinite direct sums of $\mathbb{Z}_{2^i}$.
3. The free group on 2 generators and the free group on 3 generators.

Proposition 7 (Ahlbrandt and Ziegler 1986)**.** *Two countable $\aleph_0$-categorical structures are bi-interpretable iff their automorphism groups are isomorphic as topological groups.*

Conjecture: the previous result can be lifted to $\aleph_0$-categorical theories (with countable signature). But we need to be careful about terminology. First of all, Ahlbrandt and Ziegler are working with a notion of "interpretation" between structures of one language and structures of another language: given a Σ_1-structure $\mathcal{M}_1$ and a Σ_2-structure $\mathcal{M}_2$, and interpretation $f : \mathcal{M}_1 \to \mathcal{M}_2$ consists of a surjection $f : U \to M_2$ where U is a definable subset of the domain of $\mathcal{M}_1$ and M_2 is the domain of $\mathcal{M}_2$, etc.

Conjecture 5. *An interpretation from $\mathcal{M}_1$ to $\mathcal{M}_2$ is a translation in our sense from $Th(\mathcal{M}_1)$ to $Th(\mathcal{M}_2)$.*

The following proposition follows immediately from the fact that $\aleph_0$-categorical theories are complete.

Proposition 8. *If T is $\aleph_0$-categorical with unique model $\mathcal{M}$, then T is logically equivalent to $Th(\mathcal{M})$.*

This result would be interesting because it would show that the structure of an $\aleph_0$-categorical theory is captured by its countable model. i.e., there is no need to look at the models of higher cardinality and the arrows between them.

Proposition 9 (Evans and Hewitt 1990)**.** *There are closed subgroups G and H of $P(\mathbb{N})$ that are isomorphic qua groups but not as topological groups. Hence, the corresponding structures are not bi-interpretable.*

The former result is intriguing: there is a group isomorphism $\varphi : G \xrightarrow{\sim} H$ that does not correspond to a homotopy equivalence between the corresponding theories. In short: the symmetries of the model are not enough to capture the structure of the theories.

6 $\aleph_0$-categorical theories

For this discussion, we restrict to theories with countable signatures. In this case, the downward Löwenheim-Skolem theorem shows that an $\aleph_0$-categorical theory is complete.

Question: How can we characterize the syntactic categories of $\aleph_0$-categorical theories? (Hint: Look at the Ryll-Nardzewski theorem `https://en.wikipedia.org/wiki/Omega-categorical_theory` and (Cameron 1990, p 30). I suspect that the characterization will have something to do with finiteness of subobject lattices.)

TO DO: Establish a correspondence between the 2-category of $\aleph_0$-categorical theories and some subcategory of the 2-category of topological groups. Note 1: every group is a category, and group homomorphisms are functors. Note 2: all translations between such theories are conservative. It will be helpful to look at pages 106–107 of (Cameron 1990) where he describes conditions on a topological group G that ensure it corresponds to a categorical theory.

Definition. Given an $\aleph_0$-categorical theory T, let $\mathcal{G}(T)$ be the topological group of automorphisms of its unique countable model.

Conjecture 6. *If $F : T_1 \to T_2$ is a translation then $F^*|_{\mathcal{G}(T_2)}$ is a continuous homomorphism from $\mathcal{G}(T_2)$ to $\mathcal{G}(T_1)$.*

Conjecture 7. *There is a 2-functor $\mathcal{G}$ from the 2-category of topological groups to the 2-category of $\aleph_0$-categorical theories. (This is not stated correctly yet: we should not expect all topological groups to occur in the domain. It is only the "nice" topological groups, i.e. the automorphism groups of $\aleph_0$-categorical theories.)*

Conjecture 8. *There is a 2-functor $\mathcal{F}$ from the 2-category of $\aleph_0$-categorical theories and the 2-category of topological groups.*

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