# Cantor-Bernstein for Theories

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The purpose of this note is to ask: under what conditions could a pair of theories $(T_1, T_2)$ fail to have the Cantor-Bernstein or co-Cantor-Bernstein property?

**Definition.** We say that the pair $(T_1, T_2)$ has the Cantor-Bernstein property just in case: if $T_1$ and $T_2$ are mutually faithfully interpretable, then $T_1$ and $T_2$ are bi-interpretable. In other words: if $(T_1, T_2)$ does not have the Cantor-Bernstein property then (a) $T_1$ and $T_2$ are mutually faithfully interpretable, and (b) $T_1$ and $T_2$ are not bi-interpretable.

Here "mutually faithfully interpretable" means that there are conservative (strong, equality preserving) translations $F : T_1 \to T_2$ and $G : T_2 \to T_1$. It would also be interesting to look at this question for the case of more general translations, with and without equality preservation. Could we end up getting different kinds of answers in the two cases? Or is the latter case reducible, in some sense, to the former?

**Definition.** We say that the pair $(T_1, T_2)$ has the co-Cantor-Bernstein property just in case: if there are essentially surjective translations $F : T_1 \to T_2$ and $G : T_2 \to T_1$, then $T_1$ and $T_2$ are bi-interpretable.

## 1 Factorization of translations

For strong translations, we have the following factorization theorem:

> A translation $F$ is an equivalence iff $F$ is conservative and essentially surjective (Halvorson 2019, Props. 4.5.26, 4.5.27).

This fact is useful for reasoning about relations between theories: if $T_1$ can be embedded into $T_2$ such that every $\Sigma_2$-formula is equivalent to a formula in the image, then $T_1$ and $T_2$ are equivalent. It suggests the validity of the following mode of reasoning about the relationship between theories:

> If $T_1$ can be interpreted into $T_2$ into such a way that (a) any inference licensed by $T_2$ was already licensed by $T_1$, and (b) any concept expressable by $T_2$ is already expressable by $T_1$, then $T_1$ and $T_2$ are equivalent.

The purpose of this section is to investigate the extent to which this kind of reasoning works for translations in general (and not just strong translations). In fact, there is a wide-open field of investigation here, which should hold much interest for talk about "relations between theories." In particular, with the generalized notion of translations, first-order theories form not just a category but a 2-category. In a category $C$, the interesting properties of morphisms include: iso, monic, epi, regular monic, split monic, etc. Furthermore, the interesting technical questions include: if a morphism $f : a \to b$ is both epi and monic, then is it is? Hence, if theories formed only a 1-category, then we might try to cache out notions like "equivalence" and "reduction" in terms of notions like "isomorphism" and "monomorphism."

But theories form a 2-category, and so different features of morphisms become relevant. Think, for example, of the paradigm case of a 2-category, viz. the category $\mathsf{Cat}$ of all (small) categories. Here the interesting properties of morphisms (functors) are not "is a monomorphism" and "is an epimorphism" etc., but "is full" and "is faithful" and "is essentially surjective" etc. (For a general 2-category $D$, the corresponding properties of morphisms have slightly more complicated definitions.) Similarly, the interesting notion of "sameness of objects" in $\mathsf{Cat}$ is not isomorphism, but equivalence of categories — which can be thought of as "isomorphism up to isomorphism." Finally, for the 2-category of theories, the interesting question is not whether $f$ is an isomorphism iff $f$ is a monomorphism and epimorphism, but whether $f$ is an equivalencee iff $f$ is [some other kind of thing]. See `https://ncatlab.org/nlab/show/%28eso%2C+fully+faithful%29+factorization+system` for more on factorization of morphisms in 2-categories.

In short: if relations between theories are to be explicated in terms of properties of translations, then these explications should make use of 2-categorical concepts.

Note that there are actually two distinct 2-categories at play here: the "syntactic" 2-category $\mathsf{Th}$ of theories and translations, and the "semantic" 2-category $\mathsf{Mod}$ of categories of models of theories. The interesting properties of morphisms will be different in the two cases, but of course, also related to — in fact, dual to — each other, since there is a contravariant 2-functor $M : \mathsf{Th} \to \mathsf{Cat}$ that takes a theory $T$ to its category $M(T)$ of models, a translation $F : T_1 \to T_2$ to the dual functor $M(F) : M(T_2) \to M(T_1)$, and a t-map $\chi : F \Rightarrow G$ to a natural transformation $M(\chi) : M(F) \Rightarrow M(G)$.

**Definition.** Let $F : T_1 \to T_2$ be a translation. We say that $F : T_1 \to T_2$ is *conservative* just in case if $T_2 \vdash F\phi$ then $T_1 \vdash \phi$, for any $\Sigma_1$-sentence $\phi$.

In terms of coherent categories, a logical functor $F : C \to D$ is *conservative* just in case for any object $X$ of $C$, and subobjects $\phi, \psi$ of $X$, if $F(\phi) \leq F(\psi)$ as subobjects of $F(X)$, then $\phi \leq \psi$ as subobjects of $X$ (see Makkai and Reyes 1977, p 195).

*Question.* Is the above a good definition of conservative translations for coherent theories? Recall that the implications of a coherent theory cannot be captured by sentences, but need (open) sequents. So should the definition of "conservative" be formulated in terms of sequents rather than in terms of sentences?

In terms of syntactic categories: the above definition only involves subobjects of 1. However, the notion of conservativity should involve all subobject lattices. In particular, for coherent logic, we want to say something like: $F$ is conservative means that if $F\phi(x) \vdash F\psi(x)$ then $\phi(x) \vdash \psi(x)$. When $F$ changes variables, we will need a more sophisticated definition. In particular, we might want to use the notation $\phi \vdash_{\vec{x}} \psi$ for a sequent in the context $\vec{x}$. A preliminary definition might look like:

> If $F\phi \vdash_{f(x)} F\psi$ when $\phi \vdash_x \psi$, where $f$ is the variable mapping corresponding to the reconstrual $F$.

But note that the conservativity requirement would be stronger if $\Sigma_2$ had more function symbols. For example, suppose that $\chi(y, z)$ is a functional relation relative to $T_2$, and suppose that $(F\phi)(y), \chi(y, z) \vdash (F\psi)(z)$. If there were a $\Sigma_2$ function symbol $\overline{\chi}$ corresponding to the relation $\chi$, then we would have $(F\phi)(y) \vdash (F\psi)(\overline{\chi}(y))$.

For Boolean logic, the sequent $\phi(x) \vdash \psi(x)$ can be replaced with the sequent $\vdash \forall x(\phi(x) \to \psi(x))$, and so the sentence-based definition suffices.

**Definition.** Let $C$ be a coherent category. We say that $C$ is *Boolean* just in case each subobject lattice in $C$ is a Boolean lattice.

Recall that if $B$ is a Boolean lattice, then $a \leq b$ iff $1 \leq \neg a \vee b$.

**Proposition 1.** *If $C$ is a Boolean coherent category, then for each object $x$ of $C$, there is a conservative lattice morphism $\forall : \mathrm{Sub}_C(x) \to \mathrm{Sub}_C(1)$.*

*Proof.* Let $f : x \to 1$ be the unique morphism to the terminal object, and let $f^* : \mathrm{Sub}_C(1) \to \mathrm{Sub}_C(x)$ be the corresponding pullback of subobjects. Then $C$ Boolean implies that there is a right adjoint $\forall_f : \mathrm{Sub}_C(x) \to \mathrm{Sub}_C(1)$ to $f^*$, and hence $\forall_f$ is conservative as a lattice morphism. In particular, if $\forall(a) = 1$ then $a = x$. $\square$

**Proposition 2.** *Let $C$ and $D$ be Boolean coherent categories, and let $F : C \to D$ be a logical functor. If $F : C \to D$ is conservative on $\mathrm{Sub}_C(1)$, then $F$ is conservative.*

*Proof.* Let $x$ be an object of $C$, and let $a, b \in \mathrm{Sub}_C(x)$. If $Fa \leq Fb$ then $\forall(Fa) \leq \forall(Fb)$. Since $F$ is logical, $F(\forall a) \leq F(\forall b)$, and since $F$ is conservative on $\mathrm{Sub}_C(1)$, $\forall a \leq \forall b$. Since $\forall : \mathrm{Sub}_C(x) \to \mathrm{Sub}_C(1)$ is conservative, $a \leq b$. [TO DO: Double check that logical functors commute with $\forall$.] $\square$

The following example shows that in the 2-category of theories, a translation's being conservative and essentially surjective does not imply that it is an equivalence.

**Example 3.** We describe a translation $F : (T \cup T') \to T$ that is conservative and essentially surjective, but not an equivalence. Let $\Sigma$ be a signature with a single sort $\sigma$, and let $T$ be a complete theory in $\Sigma$ that has models of cardinality greater than 1. (For example, we could take $T$ to be the theory that says that there are infinitely many things. Or we could take $T$ to be the theory that says that there are exactly two things.) Let $T'$ be a primed copy of $T$, and let $T \cup T'$ be the coproduct of $T$ and $T'$. That is, the signature of $T \cup T'$ is the disjoint union $\Sigma \cup \Sigma'$, and the axioms of $T \cup T'$ consist of the original axioms of $T$ along with a primed copy of those axioms. Note that any $\Sigma \cup \Sigma'$-formula is provably equivalent to either a conjunction or a disjunction of two formulas $\phi, \phi'$, the first a $\Sigma$-formula and the second a $\Sigma'$-formula. What's more, if $T \cup T' \vdash \phi \vee \phi'$ then either $T \cup T' \vdash \phi$ or $T \cup T' \vdash \phi'$, since any pair of models of $T$ canonically defines a model of $T \cup T'$.

Now define an equality-preserving reconstrual $F$ from $\Sigma \cup \Sigma'$ to $\Sigma$ by removing the primes from symbols in $\Sigma'$. We observe:

- $F$ is a translation from $T \cup T'$ to $T$. In the case of a $\Sigma$-formula $\phi$, it is obvious that if $T \cup T' \vdash \phi$ then $T \vdash \phi$. For the more general case, any $\Sigma \cup \Sigma'$-formula $\psi$ is provably equivalent either to $\phi \vee \phi'$ or to $\phi \wedge \phi'$, where $\phi$ is a $\Sigma$-formula and $\phi'$ is a $\Sigma'$-formula. In the former case, if $T \cup T' \vdash \phi \vee \phi'$, then either $T \cup T' \vdash \phi$ or $T \cup T' \vdash \phi'$, and hence either $T \vdash F\phi$ or $T \vdash F\phi'$.

- $F$ is essentially surjective.

- $F$ is conservative. This follows immediately from the fact that $T \cup T'$ is complete.

- $F$ is not part of an equivalence. Suppose contrary to fact that $F$ has an inverse $G : T \to (T \cup T')$. In this case, $GF$ maps both sorts $\sigma$ and $\sigma'$ to the same sequence $\sigma_1, \ldots, \sigma_n$ of sorts. (Here each $\sigma_i$ is either $\sigma$ or $\sigma'$.) Furthermore, there is a t-map $\chi : GF \to 1$, and hence two functional relations $\chi_\sigma : \sigma_1 \times \cdots \times \sigma_n \to \sigma$ and $\chi_{\sigma'} : \sigma_1 \times \cdots \times \sigma_n \to \sigma'$. In fact, $T \cup T'$ implies that both $\chi_\sigma$ and $\chi_{\sigma'}$ are bijections, which means that $T \cup T'$ defines a bijection from $\sigma$ to $\sigma'$, in contradiction with the fact that $\Sigma \cup \Sigma'$-formulas split.

The reason why $F$ is not an equivalence is because neither conservativity nor essential surjectivity require that the domain theory has a sufficient number of functional relations between sorts. In this case, $T$ describes a functional relation (namely $x =_\sigma y$) between sorts $F(\sigma)$ and $F(\sigma')$, whereas $T \cup T'$ does not describe any functional relation between $\sigma$ and $\sigma'$. (The last claim follows from the fact that $\Sigma \cup \Sigma'$-formulas can be split.) The issue then seems to be that $F$ does not resemble a full functor. Intuitively, a full functor would have the feature that for any functional relation $\chi : F(\sigma) \to F(\sigma')$, there is a functional relation $\theta : \sigma \to \sigma'$ such that $F(\theta) \simeq \chi$. While there is indeed a $\Sigma \cup \Sigma'$-formula $\eta$ such that $F(\eta)$ is provably equivalent to $\chi(x, y) \equiv (x =_\sigma y)$, any such $\eta$ has the wrong arity.

In terms of the syntactic categories $C_{T \cup T'}$ and $C_T$, we can see that the functor corresponding to $F$ is not full. In particular, if $x$ is a variable of sort $\sigma$ and $x'$ is a variable of sort $\sigma'$, then there is an arrow from $[\top.Fx]$ to $[\top.Fx']$ in $C_T$, but no arrow from $[\top.x]$ to $[\top.x']$ in $C_{T \cup T'}$.

*Remark.* Here's a little puzzle: $T \cup T'$ is Morita equivalent to a single-sorted theory $T_1$. Since $T \cup T'$ is proper (or may be assumed so), there is a translation

$G : T_1 \to T \cup T'$ that is one-half of an equivalence. Why is the composite $F \circ G$ not an equivalence? Isn't it a strong translation that is conservative and essentially surjective?

*Question.* Is the theory $T \cup T'$ above a coproduct (or product) in $\mathsf{Th}$? There are canonical translations $I_1 : T \to T \cup T'$ and $I_2 : T' \to T \cup T'$, and for any translations $F_1 : T \to T_0$ and $F_2 : T' \to T_0$, it seems like there is a unique translation $F_1 + F_2 : T \cup T' \to T_0$.

**Example 4.** Let $\Sigma_1 = \{\sigma_1, \sigma_2, \chi_1, \chi_2\}$, where $\chi_i : \sigma_1 \to \sigma_2$. Let $T_1$ be the theory in $\Sigma_1$ that says that $\chi_1$ and $\chi_2$ are functional relationships. Let $\Sigma_2 = \{\sigma_1', \sigma_2', \theta\}$, and let $T_2$ be the theory in $\Sigma_2$ that says that $\theta$ is a functional relationship. Let $F : \Sigma_1 \to \Sigma_2$ be the reconstrual that takes both $\chi_1$ and $\chi_2$ to $\theta$. Then $F : T_1 \to T_2$ is a translation, and as in the previous example, $F$ is essentially surjective. However, $F$ is not conservative. Indeed, we have $T_2 \vdash F[\forall x \forall x(\chi_1 \leftrightarrow \chi_2)]$ but $T_1 \not\vdash \forall x \forall y(\chi_1 \leftrightarrow \chi_2)$.

Now here is my intuition: for a (weak) translation $F : T_1 \to T_2$, the notion of being conservative still makes sense — although it may not be a very powerful or useful notion. Indeed, to say that $F$ is conservative basically says that if $F\phi(x) \vdash F\psi(x)$ then $\phi(x) \vdash \psi(x)$, and that only tells us something about the subobject lattices of the syntactic categories $C_{T_1}$ and $C_{T_2}$. In contrast, to say that a logical functor from $C_{T_1}$ to $C_{T_2}$ is fully faithful means that for any fixed objects $[\phi.X]$ and $[\psi.Y]$ of $C_{T_1}$, the map

$$\hom([\phi.X], [\psi.Y]) \;\to\; \hom([F\phi.X], [F\psi.Y]),$$

is a bijection. The fullness condition implies that the translation is conservative: if $F(\phi) \vdash F(\psi)$ in $T_2$, then $[X = X]$ is an arrow from $F(\phi)$ to $F(\psi)$ in $C_{T_2}$. By the fullness of $F$, there is a corresponding arrow from $\phi$ to $\psi$ in $C_{T_1}$. Hence, $\phi \vdash \psi$ in $T_1$.

The next example shows that $F : T_1 \to T_2$ being an equivalence does not imply that $F$ is essentially surjective.

**Example 5.** Let $T_1$ be the theory that says "there are exactly two things," and let $T_2$ be the Morita extension of $T_1$ by the addition of a product sort. We know that the obvious embedding $F : T_1 \to T_2$ is one-half of an equivalence. However, there are clearly formulas $\phi$ in the language of $T_2$ that are not provably equivalent to formulas in the target of the translation $F$. For example, let $\psi$ be the sentence that says there are four things of the product sort.

Nonetheless, there is an extended sense in which this formula $\psi$ is in the image of $F$. Indeed, if $\chi$ is the "code" for the product sort $\sigma'$, then for any $\Sigma_2$-formula $\psi(x)$ with $x$ of type $\sigma'$, there is a $\Sigma_1$ formula $\phi(y)$ such that

$$T_2 \vdash \chi(x, y) \rightarrow (\psi(x) \leftrightarrow \phi(x)).$$

Notice that a code is really just another name for a t-map. I conjecture then that the correct formulation of "essential surjectivity" will necessarily involve t-maps.

Let's now try to work at this in the other direction by looking at abstract 2-categoric notions of "eso" and "fully faithful" morphisms. See `https://ncatlab.org/nlab/show/fully+faithful+morphism`

If $K$ is a 2-category, then for any objects $a, b$ of $K$, we let $K(a, b)$ denote the category whose objects are 1-cells (i.e. arrows) from $a$ to $b$, and whose arrows are 2-cells. Thus, in the particular case of $\mathsf{Th}$, if $T_1$ and $T_2$ are theories, then $\mathsf{Th}(T_1, T_2)$ is the category whose objects are translations, and whose arrows are t-maps.

**Definition.** In a 2-category $\mathsf{K}$, a 1-cell $f : A \to B$ is said to be *fully faithful* just in case for all objects $X \in \mathsf{K}$, the induced functor $\mathsf{K}(X, A) \to \mathsf{K}(X, B)$ is full and faithful.

**Conjecture 6.** *Let $F : T_1 \to T_2$ be a translation. Then $F$ is fully faithful iff $F$ is conservative.*

NOTE: I now think this conjecture is false. I think that $F$ being fully faithful is stronger than $F$ being conservative.

TO DO: relate features of arrows in $\mathsf{Th}$ to the Baez classification of "forgetting" (Baez, Bartel, and Dolan 2004; Weatherall 2016; Barrett 2020). Pay attention to the ambiguity between syntactic and semantic arrows.

# 2 Examples of theory pairs that are not CB

1. $T_1$ is the empty theory on a countably infinite propositional signature. $T_2$ is the "fan theory" with axioms $p_0 \vdash p_i$, for $i \geq 0$. These theories are counterexamples both to the CB and the co-CB properties.

   But are these theories "pathological" in some sense? The fact that these theories are propositional should not (I think) be seen as a pathology.

However, these theories are incomplete, and the second of them is not finitely axiomatizable.

2. ZF and ZFC. See `https://cs.nyu.edu/pipermail/fom/2010-January/014325.html`

3. The many examples of pairs of set theories described in (Freire and Hamkins 2020, p 8).

4. The pair of theories described in (Andréka, Madarász, and Németi 2005).

# 3 Cantor-Bernstein

The first thing I would like to show: for theories $T_1$ and $T_2$ to violate Cantor-Bernstein, both have to have infinitely many non-isomorphic models. (This fact is trivial in the case that $T_1$ and $T_2$ have models of infinite cardinality.)

*Remark.* Let $F : T_1 \to T_2$ be a translation, and let $F^* : M(T_2) \to M(T_1)$ be the dual functor. Recall that:

- $F^*(M) \vDash \phi$ iff $M \vDash F(\phi)$, for any $\Sigma_1$-sentence $\phi$.

- $F^*$ is faithful by definition.

- $F^*$ reflects isomorphisms in the sense that if $F^*(j) : F^*(M) \to F^*(N)$ is an isomorphism, then $j : M \to N$ is an isomorphism. This follows from the fact that an elementary embedding is an isomorphism iff it is surjective, and from the fact that the underlying function of $F^*(j)$ is none other than the underlying function of $j$.

  Note however: $F^*$ can still map non-isomorphic models to isomorphic models. For example: let $T_1$ be the theory that says that there are two things (in empty signature), let $T_2$ be the theory that says that there are two things (in signature with a unary predicate symbol $P$), and let $F : T_1 \to T_2$ be the inclusion. Let $M$ be a model in which the extension of $P$ is empty, and let $N$ be a model in which the extension of $P$ is non-empty. Then $F^*(M)$ is isomorphic to $F^*(N)$ although $M$ is not isomorphic to $N$.

**Definition.** We say that a category $C$ is *object-finite* just in case it has only finitely many objects up to isomorphism. We say that $C$ is *totally-finite* if it has finitely many objects up to isomorphism, and finitely many arrows.

*Remark.* Recall that each theory $T$ corresponds to an ultracategory $M(T)$. In many ways, an ultracategory is like a category with a topology, where an ultralimit is like a limit along an ultrafilter. Recall also that the theory $T$ is reconstructable, up to equivalence, from the ultracategory $M(T)$. Based on these facts, I conjecture that if $M(T_1)$ and $M(T_2)$ are totally finite, then $T_1$ and $T_2$ are homotopy equivalent iff they are categorically equivalent.

Let $T$ be a theory. If $M(T)$ is object-finite then the Löwenheim-Skolem theorem implies that every model of $T$ has finite cardinality. From this it follows that every hom set in $M(T)$ is finite. Furthermore, since for finite structures, elementary equivalence implies isomorphism, it follows that for any non-isomorphic $M, N$ in $M(T)$, there is a sentence $\phi$ such that $M \vDash \phi$ and $N \vDash \neg\phi$.

**Proposition 7.** *Let $T_1$ be a theory such that $M(T_1)$ is object-finite, and let $F : T_1 \to T_2$ be a translation. If $F$ is conservative then $F^* : M(T_2) \to M(T_1)$ is essentially surjective.*

*Proof.* We prove the contrapositive. If $F^* : M(T_2) \to M(T_1)$ is not eso, then there is a model $M$ of $T_1$ that is not isomorphic to any model of the form $F^*(N)$, with $N$ a model of $T_2$. By the preceding discussion, there is a sentence $\phi$ such that $M \vDash \phi$ but $F^*(N) \vDash \neg\phi$, for all models $N$ of $T_2$. Hence $N \vDash F(\neg\phi)$ for all models $N$ of $T_2$, and by completeness, $T_2 \vdash F(\neg\phi)$. Therefore $F$ is not conservative. $\square$

*Remark.* Let $C$ and $D$ be categories with respective object sets $C_0$ and $D_0$. Let $[C_0]$ and $[D_0]$ be the corresponding sets of equivalence classes of isomorphic objects. Each functor $F : C \to D$ induces a function $F_0 : [C_0] \to [D_0]$, and $F_0$ is surjective iff $F$ is eso. If $F : C \to D$ and $G : D \to C$ are both eso, then Cantor-Bernstein for finite sets implies that $F_0$ is a bijection.

**Lemma 8.** *Let $F$ be a finite set, let $f : F \to \mathbb{N}$ be a function, and let $\phi : F \to F$ be a bijection such that $f(x) \leq f(\phi(x))$ for all $x \in F$. Then $f(x) = f(\phi(x))$ for all $x \in F$.*

*Sketch of proof.* The function $f$ corresponds to a fibration of $F$ over $\mathbb{N}$. Since $\phi$ is a bijection, the size of the fibers remain constant, i.e., $|f^{-1}(n)| = |(f \circ$

$\phi)^{-1}(n)|$. Since $\phi$ is monotonic, it cannot move an element to a lower fiber. Thus no element can be moved out of the highest fiber, nor the next highest fiber, etc. $\square$

**Proposition 9.** *Let $C$ and $D$ be totally-finite categories. If there are faithful, eso functors $F : C \to D$ and $G : D \to C$, then $C$ and $D$ are equivalent categories. In fact, $F$ itself is one half of an equivalence.*

*Proof.* By the above remark, $F_0 : [C_0] \to [D_0]$ is a bijection. Since $F$ is automatically faithful, it will suffice to show that $F$ is full. For simplicity, we may henceforth replace $C$ and $D$ with the corresponding skeletal categories.

Consider the (finite) set $C_0 \times C_0$ and the function $f : C_0 \times C_0 \to \mathbb{N}$ that assigns the cardinality of the corresponding hom set. That is, $f(a,b) = |\mathrm{hom}(a,b)|$. Let $g : D_0 \times D_0 \to \mathbb{N}$ be the corresponding function for $D$. Since $F$ is faithful, it induces a bijection $\eta : C_0 \to D_0$ such that

$$f(a,b) \ \leq \ g(\eta(a),\eta(b)).$$

And since $G$ is faithful, it induces a bijection $\theta : D_0 \to C_0$ such that

$$g(a,b) \ \leq \ f(\theta(a),\theta(b)).$$

If we let $\phi = \theta \circ \eta$ then

$$f(a,b) \ \leq \ f(\phi(a),\phi(b)),$$

for all $a,b \in C_0$. By the above lemma, $f(a,b) = f(\phi(a),\phi(b))$, and it follows that $F$ is full. $\square$

**Conjecture 10.** *Suppose that $F : T_1 \to T_2$ is a strong translation, i.e. one-dimensional, equality preserving, and with trivial domain formula. If $F$ is one-half of a weak equivalence, then $F$ is one-half of a strong equivalence.*

*Sketch of proof.* We show that $F$ is conservative and essentially surjective. Let $G : T_2 \to T_1$ be a translation such that $GF \simeq 1_{T_1}$ and $FG \simeq 1_{T_2}$. Let $\phi$ be a $\Sigma_1$-sentence such that $T_2 \vdash F(\phi)$. Then $T_1 \vdash GF(\phi)$, and (since the relevant t-map is trivial) $T_1 \vdash \phi$. Therefore $F$ is conservative. Now let $\psi(x)$ be a $\Sigma_2$-formula, for simplicity with one free variable. Then there a t-map $\chi(x,y)$ such that

$$T_2 \ \vdash \ \chi(x,y) \to (\psi(x) \leftrightarrow (FG\psi)(y)),$$

and $T_2$ implies that $\chi$ is a bijection on the domain. TO BE CONTINUED $\square$

**Proposition 11.** *Let $T_1$ and $T_2$ be proper theories such that $M(T_1)$ and $M(T_2)$ are object-finite. Then $(T_1, T_2)$ has the Cantor-Bernstein property.*

*Proof.* Let $F : T_1 \to T_2$ and $G : T_2 \to T_1$ be conservative translations. By the previous results, $F^* : M(T_2) \to M(T_1)$ is part of an equivalence of categories. By Theorem 7.1 of (D'Arienzo, Pagano, and Johnson 2020), $F$ is part of a homotopy equivalence. $\square$

*Remark.* TO DO: I need to check the previous result. The problem is that the conclusion of Theorem 7.1 shows that $F$ is part of a weak equivalence. Does that automatically show that $F$ is part of a strong equivalence? Note that the dimension of $F$ is 1, and the domain formula is trivial/universal.

**Conjecture 12.** *If $M(T_1)$ and $M(T_2)$ are object-finite then $(T_1, T_2)$ has the Cantor-Bernstein property.*

# 4 Co-Cantor-Bernstein

**Proposition 13.** *If $T_1$ or $T_2$ is complete, then $(T_1, T_2)$ has the co-CB property.*

*Proof.* If $T_1$ is complete, then every translation $F : T_1 \to T_2$ is conservative. So if $F : T_1 \to T_2$ is eso, then $F$ is a strong equivalence, i.e. $T_1$ and $T_2$ are bi-interpretable. $\square$

Recall that $F : T_1 \to T_2$ is conservative iff $F^* : M(T_2) \to M(T_1)$ is a full functor (Barrett 2020). Recall also that if $F$ is a strong (equality-preserving) translation, then $F^*$ preserves cardinality of models. In particular, if $T_1$ and $T_2$ are $\aleph_0$-categorical theories, then a translation $F : T_1 \to T_2$ induces a group homomorphism $F^* : \mathrm{Aut}(M_2) \to \mathrm{Aut}(M_1)$, and $F$ is conservative iff $F^*$ is surjective. (In fact, $\mathrm{Aut}(M_i)$ is naturally a topological group, and I conjecture that $F^*$ is a continuous group homomorphism.)

Recall that if $T$ has countable signature, and if $T$ is $\aleph_0$-categorical, then $T$ is complete. The following result would be interesting because an $\aleph_0$-categorical theory is essentially characterized by a topological group, viz. the group of automorphisms of its unique (up to isomorphism) countable model.

**Conjecture 14.** *There are $\aleph_0$-categorical theories $T_1$ and $T_2$ that do not have the CB property.*

**Definition.** Let $I$ be the set of axioms $\{\exists_{>1}, \exists_{>2}, \dots\}$. We say that a theory $T$ is essentially finitely axiomatizable just in case there is a finite set $E$ of axioms such that $Cn(T) = Cn(E \cup I)$.

**Conjecture 15.** *There are essentially finitely axiomatizable theories $T_1$ and $T_2$ that do not have the CB property.*

# 5 Conjecture: co-CB fails only for theories with many models

**Definition.** For a theory $T$ and a cardinal number $\kappa$, let $I(T, \kappa)$ be the number of non-isomorphic countable models of $T$.

Interestingly, $I(T, \aleph_0)$ can be countably infinite, or any finite cardinal besides 2 (see Marker 2006, p 155ff).

**Proposition 16.** *If $F : T_1 \to T_2$ is essentially surjective, then for any fixed cardinal number $\kappa$, $I(T_2, \kappa) \leq I(T_1, \kappa)$.*

*Proof.* Recall that the dual functor $F^*$ is always faithful. If $F : T_1 \to T_2$ is eso, then $F^*$ is also full (Halvorson 2019, Prop 6.6.13). In particular, for any models $M, N$ of $T_2$, if $F^*(M)$ is isomorphic to $F^*(N)$, then $M$ is isomorphic to $N$. Now fix a cardinal number $\kappa$, and let $[M(T_i)]_\kappa$ be the set of isomorphism classes of models of $T_i$ of cardinality $\kappa$. Then $F^*$ induces a one-to-one mapping from $[M(T_2)]_\kappa$ into $[M(T_1)]_\kappa$. $\square$

**Proposition 17.** *Suppose that $T_2$ has finitely many non-isomorphic models of each cardinality. If $F : T_1 \to T_2$ and $G : T_2 \to T_1$ are essentially surjective, then $(F^*, G^*)$ is an equivalence of categories. If $T_1$ and $T_2$ are proper theories then $F^*$ is part of a homotopy equivalence.*

*Proof.* For the first part it will suffice to show that $F^*$ is essentially surjective. By the previous proof, $G^*$ induces an injection of $[M(T_1)]_\kappa$ into $[M(T_2)]_\kappa$. Since the latter is finite, so is the former. Since $F^*$ is an injection of one finite set into a not-larger finite set, it follows that $F^*$ is bijection. Therefore $F^*$ is essentially surjective.

The second part follows from Theorem 7.1 of (D'Arienzo, Pagano, and Johnson 2020). $\square$

**Corollary 18.** *Let $T_1$ and $T_2$ be proper theories. If $(T_1, T_2)$ violate the co-CB property then there is a cardinal number $\kappa$ such that $T_1$ and $T_2$ have infinitely many non-isomorphic models of size $\kappa$.*

TO DO: I would like to come up with a similar necessary condition for $T_1$ and $T_2$ violating the CB property. However, we do not yet have any interesting result of the form: "if $T_1$ or $T_2$ is ... and $F : T_1 \to T_2$ is conservative then $F^* : M(T_2) \to M(T_1)$ is ... ."

# 6 Groups that are not CB

**Definition.** Let $P(\mathbb{N})$ be the permutation group of the natural numbers, equipped with the topology of pointwise convergence. (TO DO: explain the sense in which this topology on $P(\mathbb{N})$ is definable from the theory of infinite sets. Explain more generally the sense in which for a $\Sigma$-structure $M$, $\mathrm{Aut}(M)$ is naturally a topological group.)

**Fact 19.** *Let $G$ be a subgroup of $P(\mathbb{N})$. Then $G$ is the automorphism group of an $\aleph_0$-categorical theory iff $G$ is a closed subset of $P(\mathbb{N})$.*

**Conjecture 20.** *There are closed subgroups $G$ and $H$ of $P(\mathbb{N})$ such that $G$ is isomorphic to a closed subgroup of $H$ and vice versa, but $G$ and $H$ are not isomorphic.*

It is not difficult at all to find groups that violate the Cantor-Bernstein condition — but I do not immediately know if any of these groups are of the form $\mathrm{Aut}(M)$ for an $\aleph_0$-categorical structure $M$.

1. The group $S_\infty$ of finite permutations of $\mathbb{N}$ and the alternating grooup $A_\infty$. See `https://math.stackexchange.com/questions/1259081/if-there-are-injecti`
2. Infinite direct sums of $\mathbb{Z}_{2^i}$.
3. The free group on 2 generators and the free group on 3 generators.

**Proposition 21** (Ahlbrandt and Ziegler 1986)**.** *Two countable $\aleph_0$-categorical structures are bi-interpretable iff their automorphism groups are isomorphic as topological groups.*

Conjecture: the previous result can be lifted to $\aleph_0$-categorical theories (with countable signature). But we need to be careful about terminology. First of all, Ahlbrandt and Ziegler are working with a notion of "interpretation" between structures of one language and structures of another language: given a $\Sigma_1$-structure $\mathcal{M}_1$ and a $\Sigma_2$-structure $\mathcal{M}_2$, and interpretation $f : \mathcal{M}_1 \to \mathcal{M}_2$ consists of a surjection $f : U \to M_2$ where $U$ is a definable subset of the domain of $\mathcal{M}_1$ and $M_2$ is the domain of $\mathcal{M}_2$, etc.

**Conjecture 22.** *An interpretation from $\mathcal{M}_1$ to $\mathcal{M}_2$ is a translation in our sense from $Th(\mathcal{M}_1)$ to $Th(\mathcal{M}_2)$.*

The following proposition follows immediately from the fact that $\aleph_0$-categorical theories are complete.

**Proposition 23.** *If $T$ is $\aleph_0$-categorical with unique model $\mathcal{M}$, then $T$ is logically equivalent to $Th(\mathcal{M})$.*

This result would be interesting because it would show that the structure of an $\aleph_0$-categorical theory is captured by its countable model. i.e., there is no need to look at the models of higher cardinality and the arrows between them.

**Proposition 24** (Evans and Hewitt 1990)**.** *There are closed subgroups $G$ and $H$ of $P(\mathbb{N})$ that are isomorphic qua groups but not as topological groups. Hence, the corresponding structures are not bi-interpretable.*

The former result is intriguing: there is a group isomorphism $\varphi : G \xrightarrow{\sim} H$ that does not correspond to a homotopy equivalence between the corresponding theories. In short: the symmetries of the model are not enough to capture the structure of the theories.

# 7 $\aleph_0$-categorical theories

For this discussion, we restrict to theories with countable signatures. In this case, the downward Löwenheim-Skolem theorem shows that an $\aleph_0$-categorical theory is complete.

Question: How can we characterize the syntactic categories of $\aleph_0$-categorical theories? (Hint: Look at the Ryll-Nardzewski theorem `https://en.wikipedia.`

org/wiki/Omega-categorical_theory and (Cameron 1990, p 30). I suspect that the characterization will have something to do with finiteness of subobject lattices.)

TO DO: Establish a correspondence between the 2-category of $\aleph_0$-categorical theories and some subcategory of the 2-category of topological groups. Note 1: every group is a category, and group homomorphisms are functors. Note 2: all translations between such theories are conservative. It will be helpful to look at pages 106–107 of (Cameron 1990) where he describes conditions on a topological group $G$ that ensure it corresponds to a categorical theory.

**Definition.** Given an $\aleph_0$-categorical theory $T$, let $\mathcal{G}(T)$ be the topological group of automorphisms of its unique countable model.

**Conjecture 25.** *If $F : T_1 \to T_2$ is a translation then $F^*|_{\mathcal{G}(T_2)}$ is a continuous homomorphism from $\mathcal{G}(T_2)$ to $\mathcal{G}(T_1)$.*

**Conjecture 26.** *There is a 2-functor $\mathcal{G}$ from the 2-category of topological groups to the 2-category of $\aleph_0$-categorical theories. (This is not stated correctly yet: we should not expect all topological groups to occur in the domain. It is only the "nice" topological groups, i.e. the automorphism groups of $\aleph_0$-categorical theories.)*

**Conjecture 27.** *There is a 2-functor $\mathcal{F}$ from the 2-category of $\aleph_0$-categorical theories and the 2-category of topological groups.*

# References


Ahlbrandt, Gisela, and Martin Ziegler. 1986. "Quasi finitely axiomatizable totally categorical theories". *Annals of Pure and Applied Logic* 30 (1): 63–82. doi:10.1016/0168-0072(86)90037-0.

Andréka, Hajnal, Judit X Madarász, and István Németi. 2005. "Mutual definability does not imply definitional equivalence, a simple example". *Mathematical Logic Quarterly* 51 (6): 591–597. doi:10.1002/malq.200410051.

Baez, John, Toby Bartel, and Jim Dolan. 2004. "Property, Structure, and Stuff". Quantum Gravity Seminar, University of California, Riverside. http://math.ucr.edu/home/baez/qg-spring2004/discussion.html.

Barrett, Thomas William. 2020. "How to count structure". *Noûs*. doi:10.1111/nous.12358.

Cameron, Peter J. 1990. *Oligomorphic permutation groups*. Cambridge University Press. doi:10.1017/CBO9780511549809.

D'Arienzo, A., V. Pagano, and I. Johnson. 2020. "The 2-categorical structure of predicate theories". *Mathematics arXiv*. arXiv: 2011.14056 [math.LO].

Evans, David M, and Paul R Hewitt. 1990. "Counterexamples to a conjecture on relative categoricity". *Annals of pure and applied logic* 46 (2): 201–209. doi:10.1016/0168-0072(90)90034-Y.

Freire, Alfredo Roque, and Joel David Hamkins. 2020. "Bi-interpretation in weak set theories". Under review, *Mathematics arXiv*. arXiv: 2001.05262 [math.LO]. http://jdh.hamkins.org/bi-interpretation-in-weak-set-theories.

Halvorson, Hans. 2019. *The logic in philosophy of science*. Cambridge University Press. doi:10.1017/9781316275603.

Makkai, Michael, and Gonzalo Reyes. 1977. *First order categorical logic: Model-theoretical methods in the theory of topoi and related categories*. Springer. doi:10.1007/BFb0066201.

Marker, David. 2006. *Model theory: an introduction*. Springer. doi:10.1007/b98860.

Weatherall, James Owen. 2016. "Understanding gauge". *Philosophy of Science* 83 (5): 1039–1049. doi:10.1086/687936.