Cantor-Bernstein for Theories

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The purpose of this note is to ask: under what conditions could a pair of theories (T_1, T_2) fail to have the Cantor-Bernstein or co-Cantor-Bernstein property?

Definition. We say that the pair (T_1, T_2) has the Cantor-Bernstein property just in case: if T_1 and T_2 are mutually faithfully interpretable, then T_1 and T_2 are bi-interpretable. In other words: if (T_1, T_2) does not have the Cantor-Bernstein property then (a) T_1 and T_2 are mutually faithfully interpretable, and (b) T_1 and T_2 are not bi-interpretable.

Here "mutually faithfully interpretable" means that there are conservative (strong, equality preserving) translations $F : T_1 \to T_2$ and $G : T_2 \to T_1$.

Definition. We say that the pair (T_1, T_2) has the co-Cantor-Bernstein property just in case: if there are essentially surjective translations $F : T_1 \to T_2$ and $G : T_2 \to T_1$, then T_1 and T_2 are bi-interpretable.

1 Examples of theory pairs that are not CB

1. T_1 is the empty theory on a countably infinite propositional signature. T_2 is the "fan theory" with axioms $p_0 \vdash p_i$, for $i \geq 0$. These theories are counterexamples both to the CB and the co-CB properties.

   But are these theories "pathological" in some sense? The fact that these theories are propositional should not (I think) be seen as a pathology. However, these theories are incomplete, and the second of them is not finitely axiomatizable.

2. The many examples of pairs of set theories described in (Freire and Hamkins 2020, p 8).

3. The pair of theories described in (Andréka, Madarász, and Németi 2005).

2 Cantor-Bernstein

The first thing I would like to show: for theories T_1 and T_2 to violate Cantor-Bernstein, both have to have infinitely many non-isomorphic models. (This fact is trivial in the case that T_1 and T_2 have models of infinite cardinality.)

Remark. Let $F : T_1 \to T_2$ be a translation, and let $F^* : M(T_2) \to M(T_1)$ be the dual functor. Recall that:

- $F^*(M) \vDash \phi$ iff $M \vDash F(\phi)$, for any Σ_1-sentence ϕ.

- F^* is faithful by definition.

- F^* reflects isomorphisms in the sense that if $F^*(j) : F^*(M) \to F^*(N)$ is an isomorphism, then $j : M \to N$ is an isomorphism. This follows from the fact that an elementary embedding is an isomorphism iff it is surjective, and from the fact that the underlying function of $F^*(j)$ is none other than the underlying function of j.

  Note however: F^* can still map non-isomorphic models to isomorphic models. For example: let T_1 be the theory that says that there are two things (in empty signature), let T_2 be the theory that says that there are two things (in signature with a unary predicate symbol P), and let $F : T_1 \to T_2$ be the inclusion. Let M be a model in which the extension of P is empty, and let N be a model in which the extension of P is non-empty. Then $F^*(M)$ is isomorphic to $F^*(N)$ although M is not isomorphic to N.

Definition. We say that a category C is *object-finite* just in case it has only finitely many objects up to isomorphism.

Let T be a theory. If $M(T)$ is object-finite then the Löwenheim-Skolem theorem implies that every model of T has finite cardinality. From this it follows that every hom set in $M(T)$ is finite. Furthermore, since for finite structures, elementary equivalence implies isomorphism, it follows that for any non-isomorphic M, N in $M(T)$, there is a sentence ϕ such that $M \vDash \phi$ and $N \vDash \neg\phi$.

Proposition 1. *Let T_1 be a theory such that $M(T_1)$ is object-finite, and let $F : T_1 \to T_2$ be a translation. If F is conservative then $F^* : M(T_2) \to M(T_1)$ is essentially surjective.*

Proof. We prove the contrapositive. If $F^* : M(T_2) \to M(T_1)$ is not eso, then there is a model M of T_1 that is not isomorphic to any model of the form $F^*(N)$, with N a model of T_2. By the preceding discussion, there is a sentence ϕ such that $M \vDash \phi$ but $F^*(N) \vDash \neg\phi$, for all models N of T_2. Hence $N \vDash F(\neg\phi)$ for all models N of T_2, and by completeness, $T_2 \vdash F(\neg\phi)$. Therefore F is not conservative. $\square$

Remark. Let C and D be categories with respective object sets C_0 and D_0. Let $[C_0]$ and $[D_0]$ be the corresponding sets of equivalence classes of isomorphic objects. Each functor $F : C \to D$ induces a function $F_0 : [C_0] \to [D_0]$, and F_0 is surjective iff F is eso. If $F : C \to D$ and $G : D \to C$ are both eso, then Cantor-Bernstein for finite sets implies that F_0 is a bijection.

Lemma 1. *Let F be a finite set, let $f : F \to \mathbb{N}$ be a function, and let $\phi : F \to F$ be a bijection such that $f(x) \leq f(\phi(x))$ for all $x \in F$. Then $f(x) = f(\phi(x))$ for all $x \in F$.*

Sketch of proof. The function f corresponds to a fibration of F over $\mathbb{N}$. Since ϕ is a bijection, the size of the fibers remain constant, i.e., $|f^{-1}(n)| = |(f \circ \phi)^{-1}(n)|$. Since ϕ is monotonic, it cannot move an element to a lower fiber. Thus no element can be moved out of the highest fiber, nor the next highest fiber, etc. $\square$

Proposition 2. *Let C and D be totally-finite categories. If there are faithful, eso functors $F : C \to D$ and $G : D \to C$, then C and D are equivalent categories. In fact, F itself is one half of an equivalence.*

Proof. By the above remark, $F_0 : [C_0] \to [D_0]$ is a bijection. Since F is automatically faithful, it will suffice to show that F is full. For simplicity, we may henceforth replace C and D with the corresponding skeletal categories.

Consider the (finite) set $C_0 \times C_0$ and the function $f : C_0 \times C_0 \to \mathbb{N}$ that assigns the cardinality of the corresponding hom set. That is, $f(a, b) =$

$|\mathrm{hom}(a,b)|$. Let $g : D_0 \times D_0 \to \mathbb{N}$ be the corresponding function for D. Since F is faithful, it induces a bijection $\eta : C_0 \to D_0$ such that

$$f(a,b) \;\leq\; g(\eta(a), \eta(b)).$$

And since G is faithful, it induces a bijection $\theta : D_0 \to C_0$ such that

$$g(a,b) \;\leq\; f(\theta(a), \theta(b)).$$

If we let $\phi = \theta \circ \eta$ then

$$f(a,b) \;\leq\; f(\phi(a), \phi(b)),$$

for all $a, b \in C_0$. By the above lemma, $f(a,b) = f(\phi(a), \phi(b))$, and it follows that F is full. □

Conjecture 1. *If $M(T_1)$ and $M(T_2)$ are object-finite then (T_1, T_2) has the Cantor-Bernstein property.*

3 Co-Cantor-Bernstein

Proposition 3. *If T_1 or T_2 is complete, then (T_1, T_2) has the co-CB property.*

Proof. If T_1 is complete, then every translation $F : T_1 \to T_2$ is conservative. So if $F : T_1 \to T_2$ is eso, then F is a strong equivalence, i.e. T_1 and T_2 are bi-interpretable. □

Recall that $F : T_1 \to T_2$ is conservative iff $F^* : M(T_2) \to M(T_1)$ is a full functor (Barrett 2020). Recall also that if F is a strong (equality-preserving) translation, then F^* preserves cardinality of models. In particular, if T_1 and T_2 are $\aleph_0$-categorical theories, then a translation $F : T_1 \to T_2$ induces a group homomorphism $F^* : \mathrm{Aut}(M_2) \to \mathrm{Aut}(M_1)$, and F is conservative iff F^* is surjective. (In fact, $\mathrm{Aut}(M_i)$ is naturally a topological group, and I conjecture that F^* is a continuous group homomorphism.)

Recall that if T has countable signature, and if T is $\aleph_0$-categorical, then T is complete. The following result would be interesting because an $\aleph_0$-categorical theory is essentially characterized by a topological group, viz. the group of automorphisms of its unique (up to isomorphism) countable model.

Conjecture 2. *There are $\aleph_0$-categorical theories T_1 and T_2 that do not have the CB property.*

Definition. Let I be the set of axioms $\{\exists_{>1}, \exists_{>2}, \dots\}$. We say that a theory T is essentially finitely axiomatizable just in case there is a finite set E of axioms such that $Cn(T) = Cn(E \cup I)$.

Conjecture 3. *There are essentially finitely axiomatizable theories T_1 and T_2 that do not have the CB property.*

4 Conjecture: co-CB fails only for theories with many models

Remark. For a theory T and a cardinal number κ, let $I(T, \kappa)$ be the number of non-isomorphic countable models of T. Interestingly, $I(T, \aleph_0)$ can be countably infinite, or any finite cardinal besides 2. See (Marker 2006, p 155ff).

Proposition 4. *If $F : T_1 \to T_2$ is essentially surjective, then for any fixed cardinal number κ, $I(T_2, \kappa) \leq I(T_1, \kappa)$.*

Proof. Recall that the dual functor F^* is always faithful. If $F : T_1 \to T_2$ is eso, then F^* is also full (Halvorson 2019, Prop 6.6.13). In particular, for any models M, N of T_2, if $F^*(M)$ is isomorphic to $F^*(N)$, then M is isomorphic to N. Now fix a cardinal number κ, and let $[M(T_i)]_\kappa$ be the set of isomorphism classes of models of T_i of cardinality κ. Then F^* induces a one-to-one mapping from $[M(T_2)]_\kappa$ into $[M(T_1)]_\kappa$. □

Proposition 5. *Suppose that T_2 has finitely many non-isomorphic models of each cardinality. If $F : T_1 \to T_2$ and $G : T_2 \to T_1$ are essentially surjective, then (F^*, G^*) is an equivalence of categories. If T_1 and T_2 are proper theories then F^* is part of a homotopy equivalence.*

Proof. For the first part it will suffice to show that F^* is essentially surjective. By the previous proof, G^* induces an injection of $[M(T_1)]_\kappa$ into $[M(T_2)]_\kappa$. Since the latter is finite, so is the former. Since F^* is an injection of one finite set into a not-larger finite set, it follows that F^* is bijection. Therefore F^* is essentially surjective.

The second part follows from Theorem 7.1 of (D'Arienzo, Pagano, and Johnson 2020). □

Corollary 1. *Let T_1 and T_2 be proper theories. If (T_1, T_2) violate the co-CB property then there is a cardinal number κ such that T_1 and T_2 have infinitely many non-isomorphic models of size κ.*

TO DO: I would like to come up with a similar necessary condition for T_1 and T_2 violating the CB property. However, we do not yet have any interesting result of the form: "if T_1 or T_2 is ... and $F : T_1 \to T_2$ is conservative then $F^* : M(T_2) \to M(T_1)$ is ...."

5 Groups that are not CB

Definition. Let $P(\mathbb{N})$ be the permutation group of the natural numbers, equipped with the topology of pointwise convergence. (TO DO: explain the sense in which this topology on $P(\mathbb{N})$ is definable from the theory of infinite sets. Explain more generally the sense in which for a Σ-structure M, $\mathrm{Aut}(M)$ is naturally a topological group.)

Fact 1. *Let G be a subgroup of $P(\mathbb{N})$. Then G is the automorphism group of an $\aleph_0$-categorical theory iff G is a closed subset of $P(\mathbb{N})$.*

Conjecture 4. *There are closed subgroups G and H of $P(\mathbb{N})$ such that G is isomorphic to a closed subgroup of H and vice versa, but G and H are not isomorphic.*

It is not difficult at all to find groups that violate the Cantor-Bernstein condition — but I do not immediately know if any of these groups are of the form $\mathrm{Aut}(M)$ for an $\aleph_0$-categorical structure M.

1. The group S_∞ of finite permutations of $\mathbb{N}$ and the alternating grooup A_∞. See `https://math.stackexchange.com/questions/1259081/if-there-are-injecti`

2. Infinite direct sums of $\mathbb{Z}_{2^i}$.

3. The free group on 2 generators and the free group on 3 generators.

Proposition 6 (Ahlbrandt and Ziegler 1986)**.** *Two countable $\aleph_0$-categorical structures are bi-interpretable iff their automorphism groups are isomorphic as topological groups.*

Conjecture: the previous result can be lifted to $\aleph_0$-categorical theories (with countable signature). But we need to be careful about terminology. First of all, Ahlbrandt and Ziegler are working with a notion of "interpretation" between structures of one language and structures of another language: given a Σ_1-structure $\mathcal{M}_1$ and a Σ_2-structure $\mathcal{M}_2$, and interpretation $f : \mathcal{M}_1 \to \mathcal{M}_2$ consists of a surjection $f : U \to M_2$ where U is a definable subset of the domain of $\mathcal{M}_1$ and M_2 is the domain of $\mathcal{M}_2$, etc.

Conjecture 5. *An interpretation from $\mathcal{M}_1$ to $\mathcal{M}_2$ is a translation in our sense from $Th(\mathcal{M}_1)$ to $Th(\mathcal{M}_2)$.*

The following proposition follows immediately from the fact that $\aleph_0$-categorical theories are complete.

Proposition 7. *If T is $\aleph_0$-categorical with unique model $\mathcal{M}$, then T is logically equivalent to $Th(\mathcal{M})$.*

This result would be interesting because it would show that the structure of an $\aleph_0$-categorical theory is captured by its countable model. i.e., there is no need to look at the models of higher cardinality and the arrows between them.

Proposition 8 (Evans and Hewitt 1990)**.** *There are closed subgroups G and H of $P(\mathbb{N})$ that are isomorphic qua groups but not as topological groups. Hence, the corresponding structures are not bi-interpretable.*

The former result is intriguing: there is a group isomorphism $\varphi : G \xrightarrow{\sim} H$ that does not correspond to a homotopy equivalence between the corresponding theories. In short: the symmetries of the model are not enough to capture the structure of the theories.

6 $\aleph_0$-categorical theories

For this discussion, we restrict to theories with countable signatures. In this case, the downward Löwenheim-Skolem theorem shows that an $\aleph_0$-categorical theory is complete.

Question: How can we characterize the syntactic categories of $\aleph_0$-categorical theories? (Hint: Look at the Ryll-Nardzewski theorem `https://en.wikipedia.`

`org/wiki/Omega-categorical_theory` and (Cameron 1990, p 30). I suspect that the characterization will have something to do with finiteness of subobject lattices.)

TO DO: Establish a correspondence between the 2-category of $\aleph_0$-categorical theories and some subcategory of the 2-category of topological groups. Note 1: every group is a category, and group homomorphisms are functors. Note 2: all translations between such theories are conservative. It will be helpful to look at pages 106–107 of (Cameron 1990) where he describes conditions on a topological group G that ensure it corresponds to a categorical theory.

Definition. Given an $\aleph_0$-categorical theory T, let $\mathcal{G}(T)$ be the topological group of automorphisms of its unique countable model.

Conjecture 6. *If $F : T_1 \to T_2$ is a translation then $F^*|_{\mathcal{G}(T_2)}$ is a continuous homomorphism from $\mathcal{G}(T_2)$ to $\mathcal{G}(T_1)$.*

Conjecture 7. *There is a 2-functor $\mathcal{G}$ from the 2-category of topological groups to the 2-category of $\aleph_0$-categorical theories. (This is not stated correctly yet: we should not expect all topological groups to occur in the domain. It is only the "nice" topological groups, i.e. the automorphism groups of $\aleph_0$-categorical theories.)*

Conjecture 8. *There is a 2-functor $\mathcal{F}$ from the 2-category of $\aleph_0$-categorical theories and the 2-category of topological groups.*

References

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