Cantor-Bernstein for Theories

Princeton Logic Group

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The purpose of this note is to ask: under what conditions could a pair of theories (T_1, T_2) fail to have the Cantor-Bernstein or co-Cantor-Bernstein property?

Definition. We say that the pair (T_1, T_2) has the Cantor-Bernstein property just in case: if T_1 and T_2 are mutually faithfully interpretable, then T_1 and T_2 are bi-interpretable. In other words: if (T_1, T_2) does not have the Cantor-Bernstein property then (a) T_1 and T_2 are mutually faithfully interpretable, and (b) T_1 and T_2 are not bi-interpretable.

Here "mutually faithfully interpretable" means that there are conservative (strong, equality preserving) translations $F : T_1 \to T_2$ and $G : T_2 \to T_1$.

Definition. We say that the pair (T_1, T_2) has the co-Cantor-Bernstein property just in case: if there are essentially surjective translations $F : T_1 \to T_2$ and $G : T_2 \to T_1$, then T_1 and T_2 are bi-interpretable.

1 Examples of theory pairs that are not CB

1. T_1 is the empty theory on a countably infinite propositional signature. T_2 is the "fan theory" with axioms $p_0 \vdash p_i$, for $i \geq 0$. The fact that these theories are propositional does not (I think) indicate that they are pathological. However, these theories are incomplete and the second one is not finitely axiomatizable. In fact, the second theory is not equivalent to any finitely axiomatized theory.

2. The many examples of pairs of set theories described in (Freire and Hamkins 2020, p 8).

3. The pair of theories described in (Andréka, Madarász, and Németi 2005).

2 Results

Proposition. *If T_1 or T_2 is complete, then (T_1, T_2) has the co-CB property.*

Proof. If T_1 is complete, then every translation $F : T_1 \to T_2$ is conservative. So if $F : T_1 \to T_2$ is eso, then F is a strong equivalence, i.e. T_1 and T_2 are bi-interpretable. □

Recall that $F : T_1 \to T_2$ is conservative iff $F^* : M(T_2) \to M(T_1)$ is a full functor (Barrett 2020). Recall also that if F is a strong (equality-preserving) translation, then F^* preserves cardinality of models. In particular, if T_1 and T_2 are $\aleph_0$-categorical theories, then a translation $F : T_1 \to T_2$ induces a group homomorphism $F^* : \mathrm{Aut}(M_2) \to \mathrm{Aut}(M_1)$, and F is conservative iff F^* is surjective. (In fact, $\mathrm{Aut}(M_i)$ is naturally a topological group, and I conjecture that F^* is a continuous group homomorphism.)

Recall that if T has countable signature, and if T is $\aleph_0$-categorical, then T is complete. The following result would be interesting because an $\aleph_0$-categorical theory is essentially characterized by a topological group, viz. the group of automorphisms of its unique (up to isomorphism) countable model.

Conjecture. *There are $\aleph_0$-categorical theories T_1 and T_2 that do not have the CB property.*

Definition. Let I be the set of axioms $\{\exists_{>1}, \exists_{>2}, \ldots\}$. We say that a theory T is essentially finitely axiomatizable just in case there is a finite set E of axioms such that $Cn(T) = Cn(E \cup I)$.

Conjecture. *There are essentially finitely axiomatizable theories T_1 and T_2 that do not have the CB property.*

3 Groups that are not CB

Definition. Let $P(\mathbb{N})$ be the permutation group of the natural numbers, equipped with the topology of pointwise convergence. (TO DO: explain the sense in which this topology on $P(\mathbb{N})$ is definable from the theory of infinite sets. Explain more generally the sense in which for a Σ-structure M, $\mathrm{Aut}(M)$ is naturally a topological group.)

Fact. *Let G be a subgroup of $P(\mathbb{N})$. Then G is the automorphism group of an $\aleph_0$-categorical theory iff G is a closed subset of $P(\mathbb{N})$.*

Conjecture. *There are closed subgroups G and H of $P(\mathbb{N})$ such that G is isomorphic to a closed subgroup of H and vice versa, but G and H are not isomorphic.*

It is not difficult at all to find groups that violate the Cantor-Bernstein condition — but I do not immediately know if any of these groups are of the form $\mathrm{Aut}(M)$ for an $\aleph_0$-categorical structure M.

1. The group S_∞ of finite permutations of $\mathbb{N}$ and the alternating grooup A_∞. See `https://math.stackexchange.com/questions/1259081/if-there-are-injective-homomorphisms-between-two-groups-in-both-directions-are`

2. Infinite direct sums of $\mathbb{Z}_{2^i}$.

3. The free group on 2 generators and the free group on 3 generators.

Proposition (Ahlbrandt and Ziegler 1986)**.** *Two countable $\aleph_0$-categorical structures are bi-interpretable iff their automorphism groups are isomorphic as topological groups.*

Conjecture: the previous result can be lifted to $\aleph_0$-categorical theories (with countable signature). But we need to be careful about terminology. First of all, Ahlbrandt and Ziegler are working with a notion of "interpretation" between structures of one language and structures of another language. Basically, an interpretation $f : M_1 \to M_2$ consists of a surjection $f : U \to B$ where U is a definable subset of the domain of M_1 and B is the domain of M_2, etc. (Is an interpretation from M_1 to M_2 just a translation in our sense from $Th(M_1)$ to $Th(M_2)$?) Second, note that Ahlbrandt and Ziegler's interpretations are more like our weak translations than like our strong translations.

This result would be interesting because it would show that the structure of an $\aleph_0$-categorical theory is captured by its countable model. i.e., there is no need to look at the models of higher cardinality and the arrows between them.

Proposition (Evans and Hewitt 1990)**.** *There are closed subgroups G and H of $P(\mathbb{N})$ that are isomorphic qua groups but not as topological groups. Hence, the corresponding structures are not bi-interpretable.*

The former result is intriguing: there is a group isomorphism $\varphi : G \to H$ that does not correspond to a homotopy equivalence between the corresponding theories. In short: the arrows between models are not enough to capture the structure of the theories.

4 $\aleph_0$-categorical theories

For this discussion, we restrict to theories with countable signatures. In this case, an $\aleph_0$-categorical theory is automatically complete.

Question: How can we characterize the syntactic categories of $\aleph_0$-categorical theories? (Hint: Look at the Ryll-Nardzewski theorem `https://en.wikipedia.org/wiki/Omega-categorical_theory`. I suspect that the characterization will have something to do with finiteness of subobject lattices.)

TO DO: Establish a correspondence between the 2-category of $\aleph_0$-categorical theories (with weak translations) and some subcategory of the 2-category of topological groups. (Recall that every group is a category, and group homomorphisms are functors.) Note that all translations between such theories are conservative.

Definition. Given an $\aleph_0$-categorical theory T, let $\mathcal{G}(T)$ be the topological group of automorphisms of its unique countable model.

Conjecture. *If $F : T_1 \to T_2$ is a translation then $F^*|_{\mathcal{G}(T_2)}$ is a continuous homomorphism from $\mathcal{G}(T_2)$ to $\mathcal{G}(T_1)$.*

Conjecture. *There is a 2-functor $\mathcal{G}$ from the 2-category of topological groups to the 2-category of $\aleph_0$-categorical theories. (This is not stated correctly yet: we should not expect all topological groups to occur in the domain. It is only the "nice" topological groups, i.e. the automorphism groups of $\aleph_0$-categorical theories.)*

Conjecture. *There is a 2-functor $\mathcal{F}$ from the 2-category of $\aleph_0$-categorical theories and the 2-category of topological groups.*

References

Ahlbrandt, Gisela, and Martin Ziegler. 1986. "Quasi finitely axiomatizable totally categorical theories". *Annals of Pure and Applied Logic* 30 (1): 63–82. doi:`10.1016/0168-0072(86)90037-0`.

Andréka, Hajnal, Judit X Madarász, and István Németi. 2005. “Mutual definability does not imply definitional equivalence, a simple example”. *Mathematical Logic Quarterly* 51 (6): 591–597. doi:10.1002/malq.200410051.

Barrett, Thomas William. 2020. “How to count structure”. *Noûs*. doi:10.1111/nous.12358.

Evans, David M, and Paul R Hewitt. 1990. “Counterexamples to a conjecture on relative categoricity”. *Annals of pure and applied logic* 46 (2): 201–209. doi:10.1016/0168-0072(90)90034-Y.

Freire, Alfredo Roque, and Joel David Hamkins. 2020. “Bi-interpretation in weak set theories”. Under review, *Mathematics arXiv*. arXiv: 2001.05262 [math.LO]. http://jdh.hamkins.org/bi-interpretation-in-weak-set-theories.