HW#1

2.1 **a.** Obtain the global stiffness matrix $\underline{K}$ of the assemblage shown in Figure P2–1 by superimposing the stiffness matrices of the individual springs. Here k_1, k_2, and k_3 are the stiffnesses of the springs as shown.

b. If nodes 1 and 2 are fixed and a force P acts on node 4 in the positive x direction, find an expression for the displacements of nodes 3 and 4.

c. Determine the reaction forces at nodes 1 and 2.

(*Hint:* Do this problem by writing the nodal equilibrium equations and then making use of the force/displacement relationships for each element as done in the first part of Section 2.4. Then solve the problem by the direct stiffness method.)

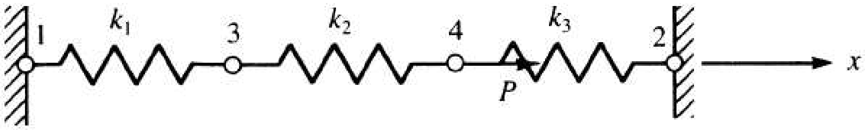


Figure P2–1

3.12 Solve for the axial displacement and stress in the tapered bar shown in Figure P3–12 using one and then two constant-area elements. Evaluate the area at the center of each element length. Use that area for each element. Let $A_0 = 2$ in^2, $L = 20$ in., $E = 10 \times 10^6$ psi, and $P = 1000$ lb. Compare your finite element solutions with the exact solution.

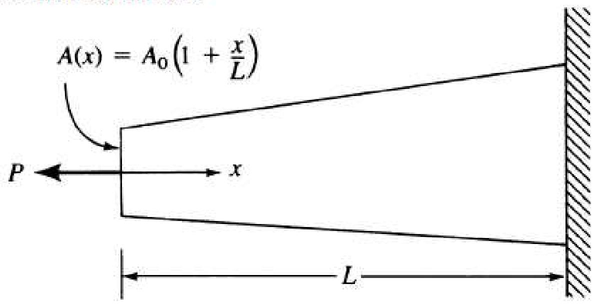


Figure P3–12

2.1

a.

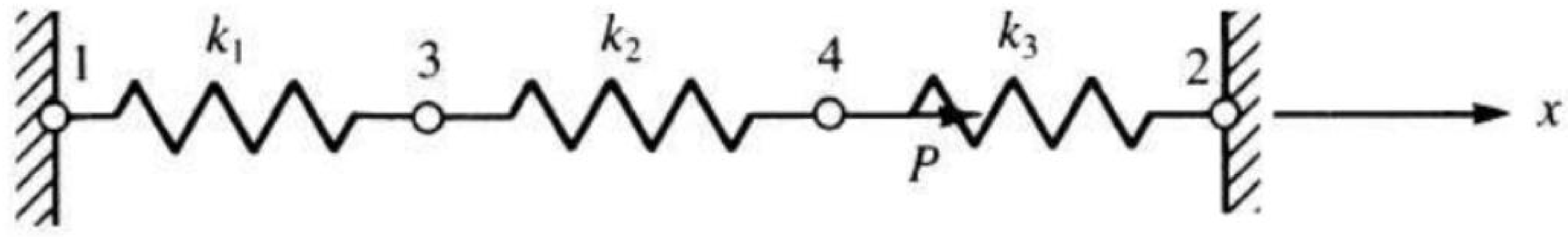


Figure P2–1

$$k^{(1)} = \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$k^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_2 & -k_2 \\ 0 & 0 & -k_2 & k_2 \end{bmatrix} \qquad k^{(3)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_3 & 0 & -k_3 \\ 0 & 0 & 0 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix}$$

$$\boxed{k = \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_3 & 0 & -k_3 \\ -k_1 & 0 & k_1+k_2 & -k_2 \\ 0 & -k_3 & -k_2 & k_2+k_3 \end{bmatrix}}$$

b.

$$\begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_3 & 0 & -k_3 \\ -k_1 & 0 & k_1+k_2 & -k_2 \\ 0 & -k_3 & -k_2 & k_2+k_3 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \\ d_{4x} \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ 0 \\ 0 = P \\ F_{2x} \end{Bmatrix} \Rightarrow \begin{Bmatrix} 0 \\ P \end{Bmatrix} = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix} \begin{Bmatrix} u_{3x} \\ u_{4x} \end{Bmatrix}$$

$$\{F\} = [k]\{d\} \rightarrow [k^{-1}]\{F\} = [k^{-1}][k]\{d\} = \{d\}$$

$$\Rightarrow \begin{bmatrix} k_2+k_3 & k_2 \\ k_2 & k_1+k_2 \end{bmatrix}$$

$$\det[k] = |k| = (k_2+k_3)(k_1+k_2) - k_2^2$$

$$\begin{Bmatrix} d_{3x} \\ d_{4x} \end{Bmatrix} = \frac{\begin{bmatrix} k_2+k_3 & k_2 \\ k_2 & k_1+k_2 \end{bmatrix} \begin{Bmatrix} 0 \\ P \end{Bmatrix}}{k_1k_2 + k_1k_3 + k_2k_3} \Rightarrow \boxed{\begin{aligned} d_3 &= \frac{k_2 P}{k_1k_2 + k_1k_3 + k_2k_3} \\ d_4 &= \frac{(k_1+k_2) P}{k_1k_2 + k_1k_3 + k_2k_3} \end{aligned}}$$

c.

$$\begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_3 & 0 & -k_3 \\ -k_1 & 0 & k_1+k_2 & -k_2 \\ 0 & -k_3 & -k_2 & k_2+k_3 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \\ d_{4x} \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ 0 \\ 0 = P \\ F_{2x} \end{Bmatrix} \qquad d_{1x} = d_{4x} = 0,$$

$$\boxed{\begin{aligned} F_{1x} &= -k_1 d_{3x} = \frac{-k_1 k_2 P}{k_1k_2 + k_1k_3 + k_2k_3} \\ F_{2x} &= -k_3 d_{4x} = \frac{-k_3 (k_1+k_2) P}{k_1k_2 + k_1k_3 + k_2k_3} \end{aligned}}$$

3.12

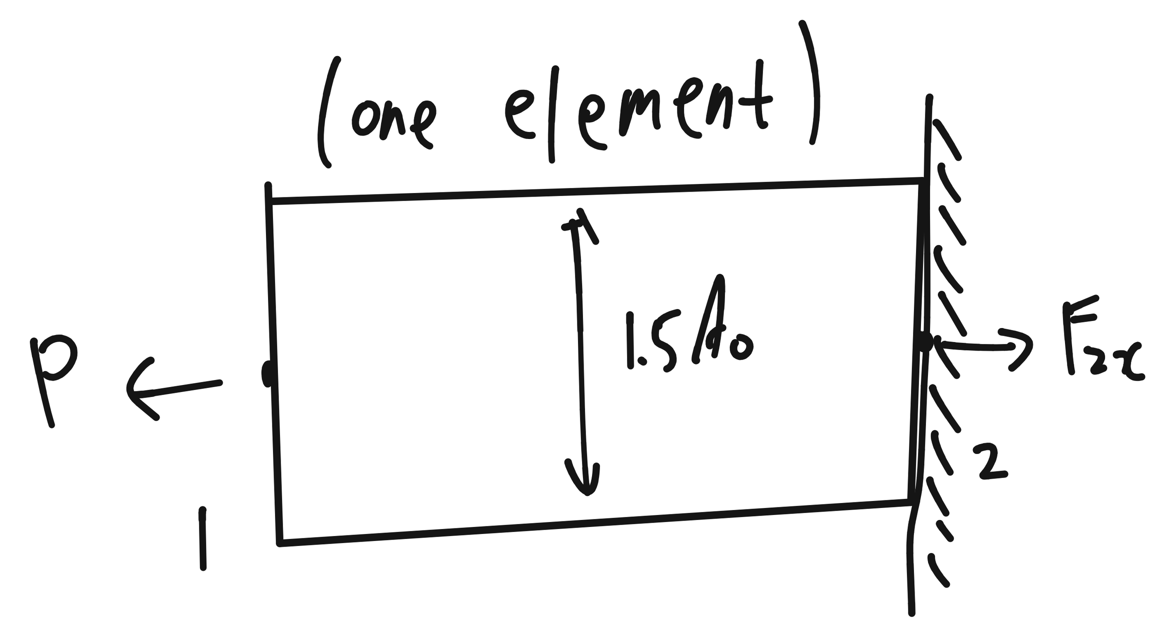
$$\delta = \frac{F \cdot L}{EA} \quad : \sigma = E \cdot \varepsilon \quad , \sigma = \frac{F}{A} \quad , \varepsilon = \frac{\delta}{L}$$

$$\delta = \int \frac{F}{EA} dx = \int \frac{P\,dx}{EA_0\left(1+\frac{x}{L}\right)} = \int \frac{P \cdot L \cdot dx}{EA_0 (L+x)} \quad , \quad L+x = u$$

$$\rightarrow \int \frac{P \cdot L\, du}{EA_0\, u} = \frac{PL}{EA_0} \cdot \ln u = \frac{PL}{EA_0} \ln(L+x) = \frac{-1000 \cdot 20}{10 \times 10^6 \times 2} \ln(20+x) = -10^{-3} \ln(20+x)$$

$$\boxed{\begin{aligned} &\cdot\ x = 0 \rightarrow -10^{-3} \ln(20) = -2.996 \times 10^{3} \text{ in.} \\ &\cdot\ x = 10 \rightarrow -10^{-3} \ln(20+10) = -3.401 \times 10^{3} \text{ in.} \end{aligned}}$$

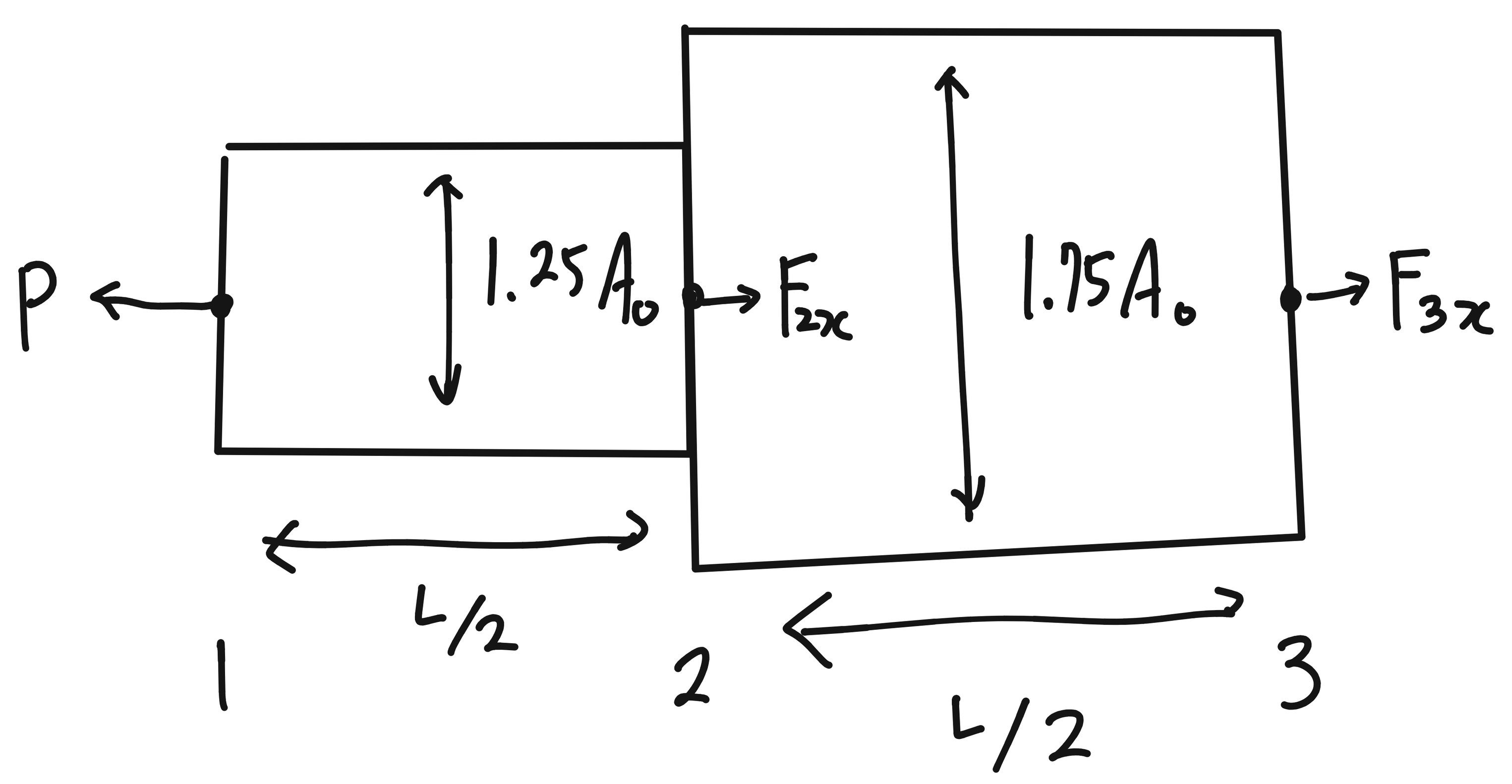
(one element)



$$\begin{bmatrix} \frac{1.5EA_0}{L} & -\frac{1.5EA_0}{L} \\ \frac{-1.5EA_0}{L} & \frac{1.5EA_0}{L} \end{bmatrix} \begin{pmatrix} U_{1x} \\ U_{2x} \end{pmatrix} = \begin{pmatrix} F_{1x} \\ F_{2x} \end{pmatrix} \quad , \quad \begin{aligned} F_{1x} &= -P \\ U_{2x} &= 0 \\ F_{2x} &= 0 \end{aligned}$$

$$\Rightarrow \boxed{U_{1x} = \frac{-PL}{1.5\, E \cdot A_0} = \frac{-1000 \times 20}{1.5 \times 10 \times 10^6 \times 2} = -0.667 \times 10^{-3} \text{ in}}$$

(Two element)



$$\begin{bmatrix} \frac{1.25EA_0}{L/2} & \frac{-1.25EA_0}{L/2} & 0 \\ \frac{-1.25EA_0}{L/2} & \frac{1.25EA_0}{L/2} + \frac{1.75EA_0}{L/2} & -\frac{1.75EA_0}{L/2} \\ 0 & \frac{-1.75EA_0}{L/2} & \frac{1.75EA_0}{L/2} \end{bmatrix} \begin{pmatrix} U_{1x} \\ U_{2x} \\ U_{3x} \end{pmatrix} = \begin{pmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{pmatrix} \qquad \begin{aligned} F_{1x} &= -P \\ F_{2x} &= 0 \\ U_{3x} &= 0 \end{aligned}$$

$$\left.\begin{aligned}\frac{1.25EA_0}{L/2}u_{1x} - \frac{1.25EA_0}{L/2}u_{2x} &= -P \\ \frac{-1.25EA_0}{L/2}u_{1x} + \frac{3EA_0}{L/2}u_{2x} &= 0\end{aligned}\right] \longrightarrow \boxed{\begin{aligned}u_2 &= -\frac{2}{7}\frac{PL}{A_0E} \\ &= -\frac{2}{7}\times\frac{1000\times 20}{2\times 10\times 10^6} = -0.2857\times 10^{-3}\ \text{in.}\end{aligned}}$$

$$\boxed{u_{1x} = \frac{12}{5}u_{2x} = -0.6857\times 10^{-3}\ \text{in}}$$