COMP4128 Week 10 Tutorial

Yifan He

z5173587@unsw.edu.au
`https://github.com/hharryyf/COMP4128-23T3-tutoring`

Outline

- Division
- The Race
- Contest 3
- Problem Set 8 hints by email

Division

Given T ($T \leq 50$) pairs of a and b ($1 \leq a \leq 10^{18}$, $2 \leq b \leq 10^9$), for each pair, find the maximum x such that $x \mid a$ but $b \nmid x$.

- $x \mid y \iff y = k \cdot x$ for some $k \in \mathbb{Z}$

Examples

- a = 8, b = 4, x = 2
- a = 10, b = 4, x = 10
- a = 12, b = 4, x = 6
- a = 17, b = 5, x = 17

Division

Analysis

- What is the optimal value of x if $b \nmid a$?

Division

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- Answer: $x = a$
- Proof Sketch: $x \le a$, $a \mid a$, $b \nmid a \implies x = a$

Division

Analysis

- What is the optimal value of x if $b \nmid a$?
- Answer: $x = a$
- Proof Sketch: $x \leq a,\ a \mid a,\ b \nmid a \implies x = a$
- We only need to handle $b \mid a$

Division

Analysis

- Case $b \mid a$
- Consider the prime factorization of a and b
- Let $a = p_1^{c_1} \cdot p_2^{c_2} \ldots p_n^{c_n}$ and $b = q_1^{d_1} \cdot q_2^{d_2} \ldots q_m^{d_m}$
- What can we tell about $p_1 \ldots p_n$ and $q_1 \ldots q_m$?

Division

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- What can we tell about $p_1 \dots p_n$ and $q_1 \dots q_m$?
- $\{q_1, q_2, \dots, q_m\} \subseteq \{p_1, p_2, \dots, p_n\}$
- Rewrite $a = q_1^{c_1} \cdot q_2^{c_2} \dots q_m^{c_m} \cdot M$ such that $q_i \nmid M$ for all $1 \le i \le m$
- What can we tell about $c_1 \dots c_m$ and $d_1 \dots d_m$?

Division

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- Rewrite $a = q_1^{c_1} \cdot q_2^{c_2} \ldots q_m^{c_m} \cdot M$ such that $q_i \nmid M$ for all $1 \leq i \leq m$
- What can we tell about $c_1 \ldots c_m$ and $d_1 \ldots d_m$?
- $c_1 \geq d_1,\ c_2 \geq d_2,\ \ldots,\ c_m \geq d_m$

Division

Solution

- $b = q_1^{d_1} \cdot q_2^{d_2} \dots q_m^{d_m}$
- $a = q_1^{c_1} \cdot q_2^{c_2} \dots q_m^{c_m} \cdot M$ such that $q_i \nmid M$ for all $1 \leq i \leq m$
- $c_1 \geq d_1$, $c_2 \geq d_2$, ..., $c_m \geq d_m$
- How to create the maximum x such that $b \nmid x$ and $x \mid a$?

Division

Solution

- $b = q_1^{d_1} \cdot q_2^{d_2} \ldots q_m^{d_m}$
- $a = q_1^{c_1} \cdot q_2^{c_2} \ldots q_m^{c_m} \cdot M$ such that $q_i \nmid M$ for all $1 \le i \le m$
- $c_1 \ge d_1,\ c_2 \ge d_2,\ \ldots,\ c_m \ge d_m$
- How to create the maximum x such that $b \nmid x$ and $x \mid a$?
- x is the maximum of
  - $q_1^{d_1 - 1} \cdot q_2^{c_2} \ldots q_m^{c_m} \cdot M$
  - $q_1^{c_1} \cdot q_2^{d_2 - 1} \ldots q_m^{c_m} \cdot M$
  - $\ldots$
  - $q_1^{c_1} \cdot q_2^{c_2} \ldots q_m^{d_m - 1} \cdot M$

Division

Examples

- a = 17, b = 5, x = 17 ($b \nmid a$)
- a = 64, b = 4, x = 2
  - $a = 2^6$
  - $b = 2^2$
  - $x = 2^1 = 2$
- a = 180, b = 12, x = 90
  - $a = 2^2 \cdot 3^2 \cdot 5^1$
  - $b = 2^2 \cdot 3^1$
  - $x = max(2^1 \cdot 3^2 \cdot 5^1, 2^2 \cdot 3^0 \cdot 5^1) = max(90, 20) = 90$

Division

Demo

The Race

Consider a turn-taking game. In player 0's turn, he would toss a coin, and get 1 point if the coin shows up head (0 points otherwise). In player 1's turn, he would choose a number T and toss a coin for T times. If the coin shows up head in all T times, he will get 2^{T-1} points (0 points otherwise). Player 0 goes first, and the first person that gets 100 points wins. Find the probability of player 1 wins.

Answer: 0.83648556

The Race

- Player 1 chooses the number T before each turn, is T bounded?

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- Yes, $1 \leq T \leq 8$
- If $T \geq 8$, when the coin shows up head for T times, player 1 would win in the current turn

The Race

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- Yes, $1 \leq T \leq 8$
- If $T \geq 8$, when the coin shows up head for T times, player 1 would win in the current turn
- Two-player zero-sum turn-taking probabilistic (fully observable) game
- Solvable with Dynamic Programming
  - If you know some AI, Max^n or Minimax
- At the current game state what is the maximum probability of the current player winning the game

The Race

- How to capture the game state?

The Race

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- (player 0's points, player 1's points, control)

The Race

- How to capture the game state?
- (player 0's points, player 1's points, control)
- Subproblem: Let $dp[i][j][k]$ be the maximum probability of the player k winning the game when player 0 has i points and player 1 has j points
- Base case:
  1. $i \geq 100$ $dp[i][j][k] = 1 - k$
  2. $j \geq 100$ $dp[i][j][k] = k$
- Answer: $1 - dp[0][0][0]$

The Race

Player 0 taking control

$$\begin{tikzcd}
 & \boxed{(i,j,0)} \arrow[dl, "H:\frac{1}{2}"'] \arrow[dr, "T:\frac{1}{2}"] & \\
\boxed{(i+1,j,1)} & & \boxed{(i,j,1)}
\end{tikzcd}$$

The Race

Player 0 taking control

$$\begin{tikzcd} & (i,j,0) \arrow[dl, "H:\frac{1}{2}"'] \arrow[dr, "T:\frac{1}{2}"] & \\ (i+1,j,1) & & (i,j,1) \end{tikzcd}$$

- $dp[i][j][0] = \frac{1}{2} \cdot dp[i][j][1] + \frac{1}{2} \cdot dp[i+1][j][1]$

The Race

Player 0 taking control

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- $dp[i][j][0] = \frac{1}{2} \cdot dp[i][j][1] + \frac{1}{2} \cdot dp[i+1][j][1]$ ✗
- Probability of player 0 winning at $(i, j, 1)$ is $1 - dp[i][j][1]$
- $dp[i][j][0] = \frac{1}{2} \cdot (1 - dp[i][j][1]) + \frac{1}{2} \cdot (1 - dp[i+1][j][1])$

The Race

Player 0 taking control

- $dp[i][j][0] = \frac{1}{2} \cdot (1 - dp[i][j][1]) + \frac{1}{2} \cdot (1 - dp[i+1][j][1])$

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Player 0 taking control

- $dp[i][j][0] = \frac{1}{2} \cdot (1 - dp[i][j][1]) + \frac{1}{2} \cdot (1 - dp[i+1][j][1])$

Player 1 taking control

- Choose the T that maximize $dp[i][j][1]$
- Let $score = dp[i][j + 2^{T-1}][0]$ and $p = \frac{1}{2^T}$
- $dp[i][j][1] = p \cdot (1 - score) + (1 - p) \cdot (1 - dp[i][j][0])$

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Player 0 taking control

- $dp[i][j][0] = \frac{1}{2} \cdot (1 - dp[i][j][1]) + \frac{1}{2} \cdot (1 - dp[i+1][j][1])$

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- Questions?

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- Choose the T that maximize $dp[i][j][1]$
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- $dp[i][j][1] = p \cdot (1 - score) + (1 - p) \cdot (1 - dp[i][j][0])$
- Questions?
- This is incorrect, because $dp[i][j][0]$ and $dp[i][j][1]$ forms circular dependency

The Race

Remove circular dependency

- A classic technique used in MDP (Markov)
- $dp[i][j][0] = \frac{1}{2} \cdot (1 - dp[i][j][1]) + \frac{1}{2} \cdot (1 - dp[i+1][j][1])$
- $dp[i][j][1] = p \cdot (1 - score) + (1 - p) \cdot (1 - dp[i][j][0])$
- Replace $dp[i][j][0]$ in the second equation with the RHS of the first equation

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- Replace $dp[i][j][0]$ in the second equation with the RHS of the first equation
- $dp[i][j][1] = p \cdot (1 - score) + (1 - p) \cdot (1 - \frac{1}{2} \cdot (1 - dp[i][j][1]) - \frac{1}{2} \cdot (1 - dp[i+1][j][1]))$

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- $= p \cdot (1 - score) + \frac{1-p}{2} \cdot dp[i][j][1] + \frac{1-p}{2} \cdot dp[i+1][j][1]$

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- $= p \cdot (1 - score) + \frac{1-p}{2} \cdot dp[i][j][1] + \frac{1-p}{2} \cdot dp[i+1][j][1]$
- $dp[i][j][1] = \frac{2 \cdot p}{1+p} \cdot (1 - score) + \frac{1-p}{1+p} \cdot dp[i+1][j][1]$

The Race

Summary

- $dp[i][j][0] = \frac{1}{2} \cdot (1 - dp[i][j][1]) + \frac{1}{2} \cdot (1 - dp[i+1][j][1])$
- Choose the T that maximize $dp[i][j][1]$
- Let $score = dp[i][j + 2^{T-1}][0]$ and $p = \frac{1}{2^T}$
- $dp[i][j][1] = \frac{2 \cdot p}{1+p} \cdot (1 - score) + \frac{1-p}{1+p} \cdot dp[i+1][j][1]$

The Race

Demo

Contest 3 Review

- Range tree (use cases I mentioned in week 8)
- Shortest path (review the week 7 slides)
- Graph (tree edge and back edge in DFS)
- Flow (make sure you have a template!)
- Template preparation
  - Dinic's algorithm (with cut extraction)
  - Tarjan's algorithm (know how to solve 2-SAT)
  - MST (with union-find)
  - Dijkstra, Floyd, Bellman-Ford algorithm
- Tips:
  - Always remember the technique of changing a minimization problem to a validation problem
  - Coverage of flow problem set is bad, review the lecture slides

Final Slide

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  - Quantitatively: 1200+ problems away
- Contest 3, this weekend
- Final exam, December (more challenging)
- Glad to reply to any email related to revision tips