Logic and Computer Design Fundamentals

Chapter 2 – Combinational Logic Circuits

Part 1 – Gate Circuits and Boolean Equations

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Overview

- **Part 1 – Gate Circuits and Boolean Equations**
  - **Binary Logic and Gates**
  - **Boolean Algebra**
  - **Standard Forms**
- **Part 2 – Circuit Optimization**
  - **Two-Level Optimization**
  - **Map Manipulation**
  - **Multi-Level Circuit Optimization**
- **Part 3 – Additional Gates and Circuits**
  - **Other Gate Types**
  - **Exclusive-OR Operator and Gates**
- **Part 4 – HDLs overview**
  - **Logic Synthesis**
  - **HDL Representations—Verilog**

Binary Logic and Gates

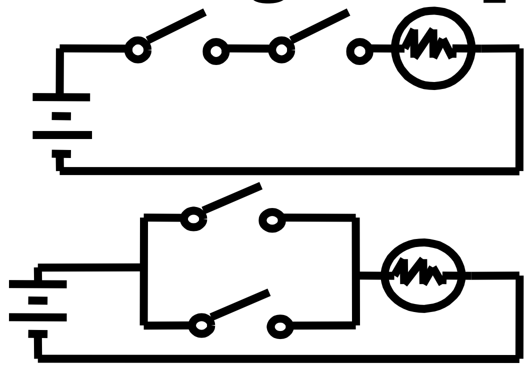
- Binary variables take on one of two values.
- Logical operators operate on binary values and binary variables.
- Basic logical operators are the logic functions AND, OR and NOT.
- Logic gates implement logic functions.
- Boolean Algebra: a useful mathematical system for specifying and transforming logic functions.
- We study Boolean algebra as a foundation for designing and analyzing digital systems!

Binary Variables

- Recall that the two binary values have different names:
  - True/False
  - On/Off
  - Yes/No
  - 1/0
- We use 1 and 0 to denote the two values.
- Variable identifier examples:
  - A, B, y, z, or X_1 for now
  - RESET, START_IT, or ADD1 later

Logical Operations

- The three basic logical operations are:
  - AND
  - OR
  - NOT



- AND is denoted by a dot ($\cdot$).
- OR is denoted by a plus (+).
- NOT is denoted by an overbar ($\overline{\ }$), a single quote mark (') after, or (~) before the variable.

Notation Examples

- **Examples:**
  - **$Y = A \times B$ is read "Y is equal to A AND B."**
  - **$z = x + y$ is read "z is equal to x OR y."**
  - **$X = \overline{A}$ is read "X is equal to NOT A."**
- **Note: The statement:**

  1 + 1 = 2 (read "one <u>plus</u> one equals two")

  is not the same as

  1 + 1 = 1 (read "1 <u>or</u> 1 equals 1").

Operator Definitions

- **Operations are defined on the values "0" and "1" for each operator:**

AND	OR	NOT
$0 \cdot 0 = 0$	$0 + 0 = 0$	$\overline{0} = 1$
$0 \cdot 1 = 0$	$0 + 1 = 1$	$\overline{1} = 0$
$1 \cdot 0 = 0$	$1 + 0 = 1$	
$1 \cdot 1 = 1$	$1 + 1 = 1$	

Truth Tables

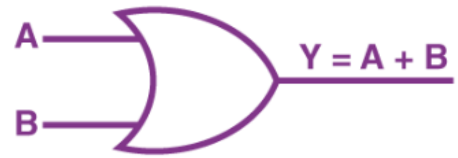
- ***Truth table*** **– a tabular listing of the values of a function for all possible combinations of values on its arguments**
- **Example: Truth tables for the basic logic operations:**

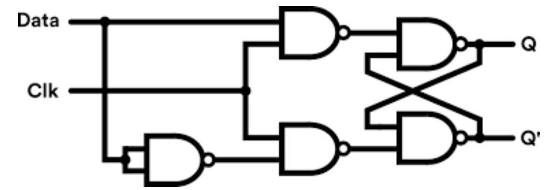
AND		
X	Y	$Z = X \cdot Y$
0	0	0
0	1	0
1	0	0
1	1	1

OR		
X	Y	$Z = X+Y$
0	0	0
0	1	1
1	0	1
1	1	1

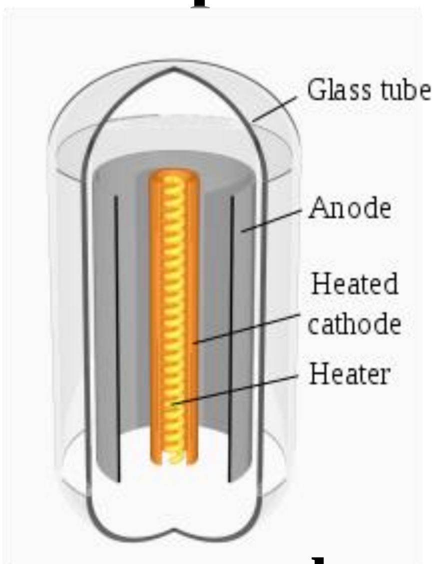
NOT	
X	$Z = \overline{X}$
0	1
1	0

Logic Gates

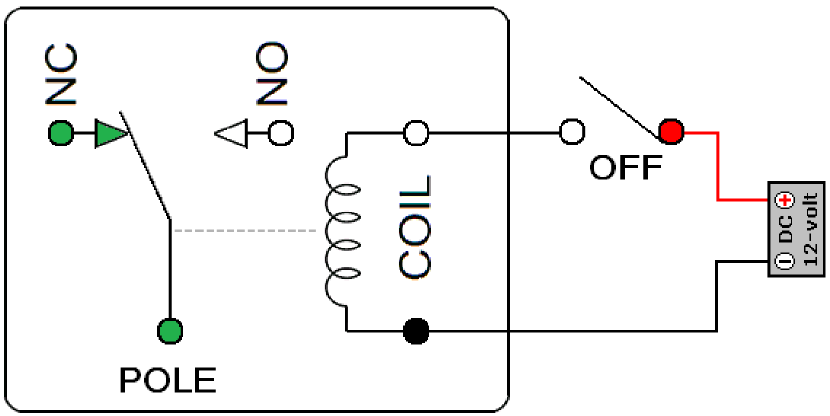




- In the earliest computers, switches were opened and closed by magnetic fields produced by energizing coils in ***relays***. The switches in turn opened and closed the current paths.
- Later, ***vacuum tubes*** that open and close current paths electronically replaced relays.



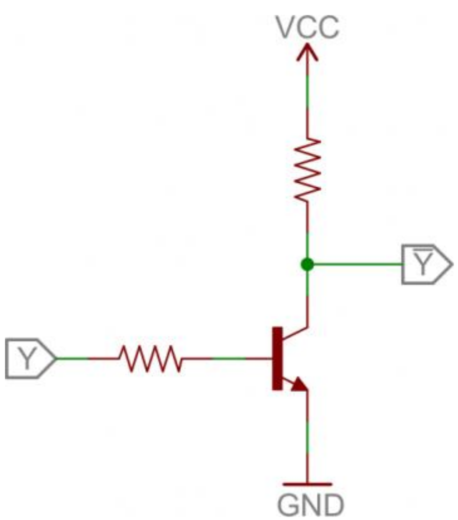
vacuum tube



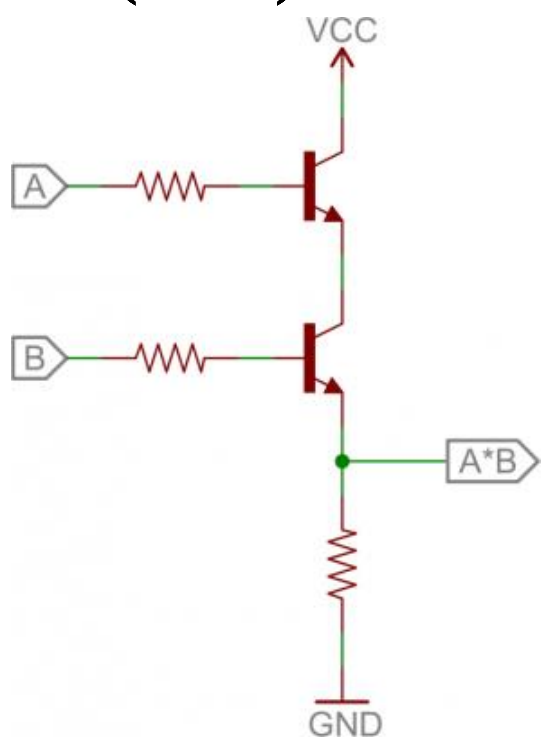
relay

Logic Gates (Continued)

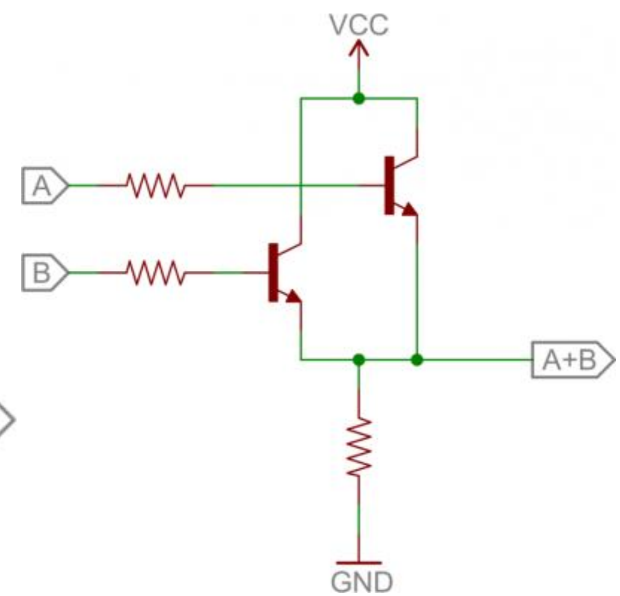
- **Today, *transistors* are used as electronic switches that open and close current paths.**
- **Implementation of logic gates with bi-polar junction transistor (BJT)**



NOT



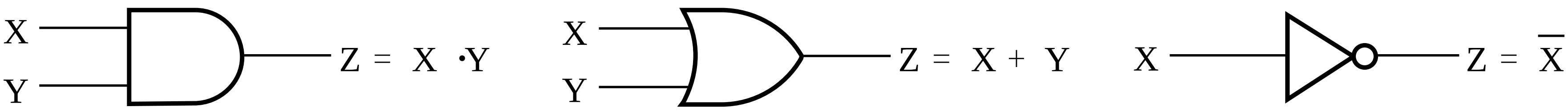
AND



OR

Logic Gate Symbols and Behavior

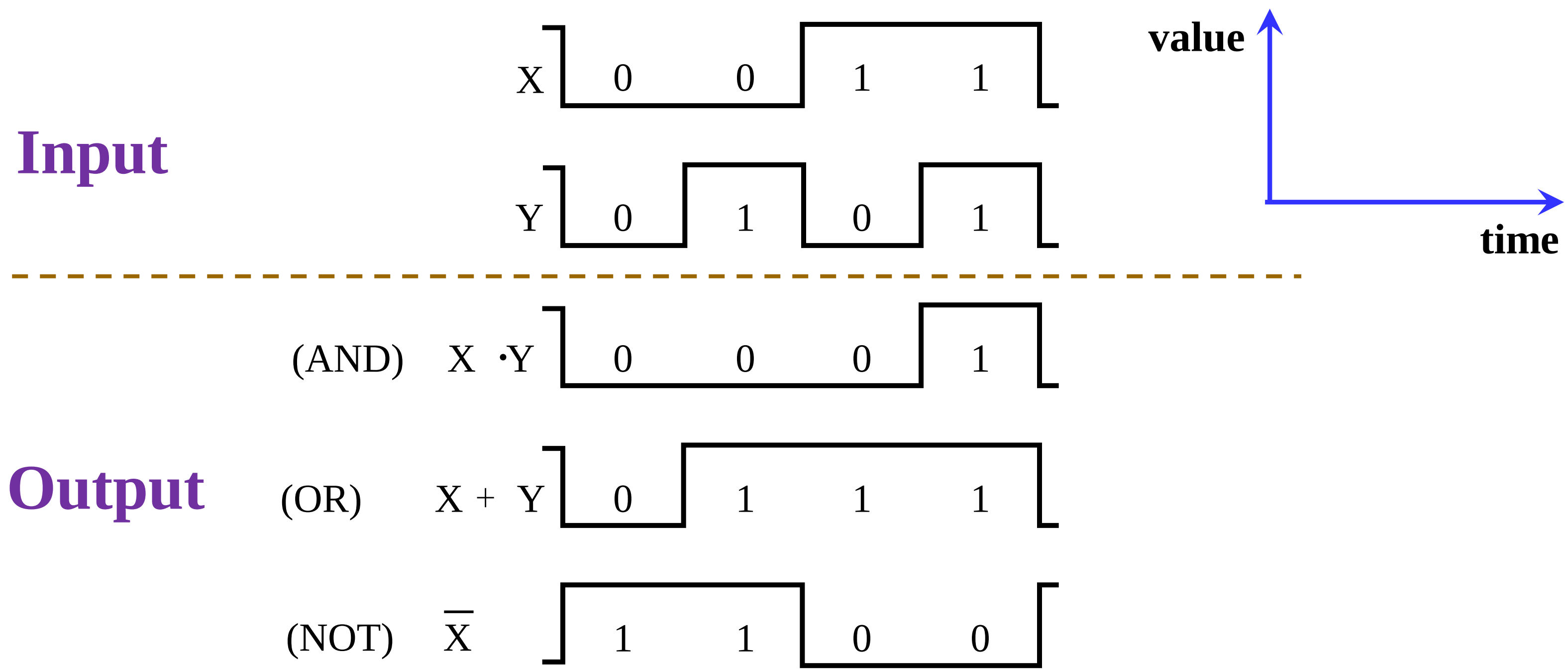
- **Logic gates have special symbols:**



AND gate   OR gate   NOT gate or inverter

(a) Graphic symbols

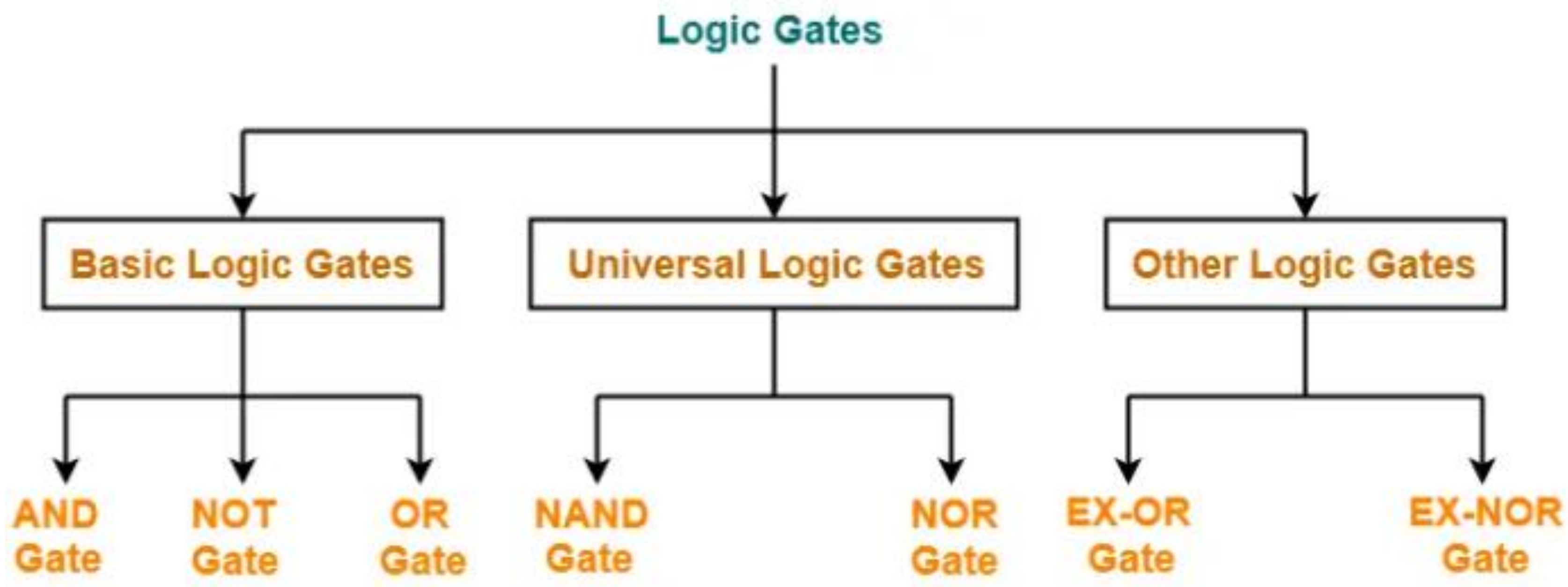
- **Waveform of logic gates are as follows:**



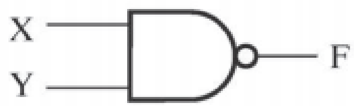
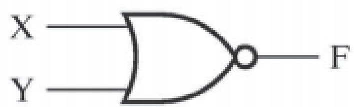
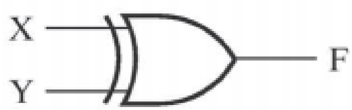
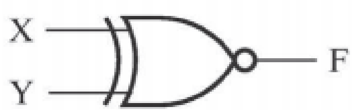
(b) Timing diagram

Types of Logic Gates

- **Logic gates can be classified into three different types.**

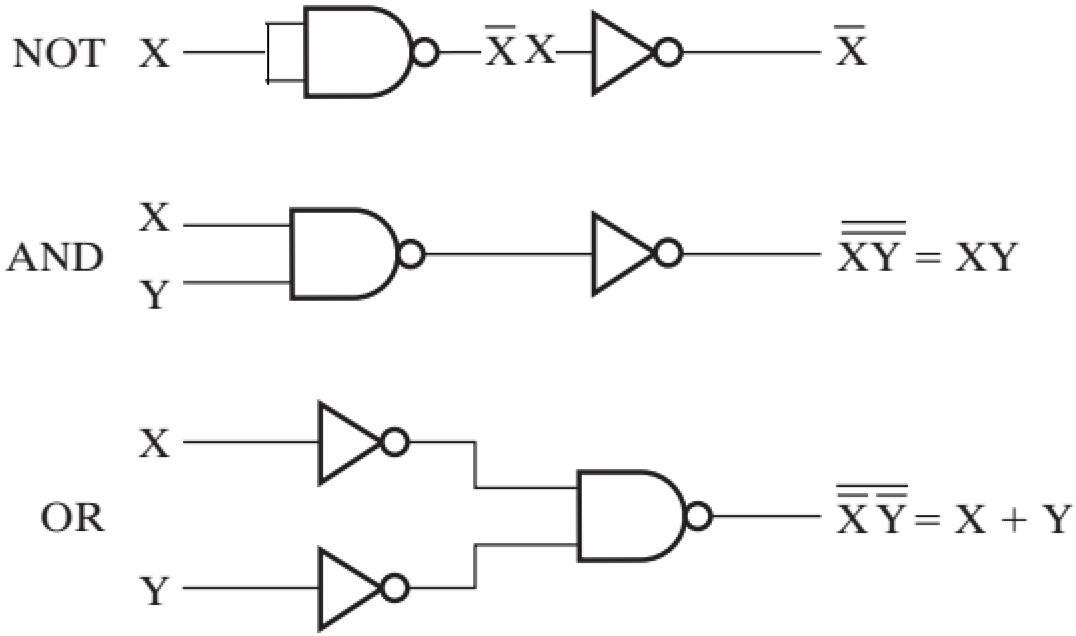


Other Commonly Used Logic Gates

NAND		$F = \overline{X \cdot Y}$	X Y F 0 0 1 0 1 1 1 0 1 1 1 0
NOR		$F = \overline{X + Y}$	X Y F 0 0 1 0 1 0 1 0 0 1 1 0
Exclusive-OR (XOR)		$F = X\overline{Y} + \overline{X}Y$ $= X \oplus Y$	X Y F 0 0 0 0 1 1 1 0 1 1 1 0
Exclusive-NOR (XNOR)		$F = XY + \overline{X}\,\overline{Y}$ $= \overline{X \oplus Y}$	X Y F 0 0 1 0 1 0 1 0 0 1 1 1

Universal Gate

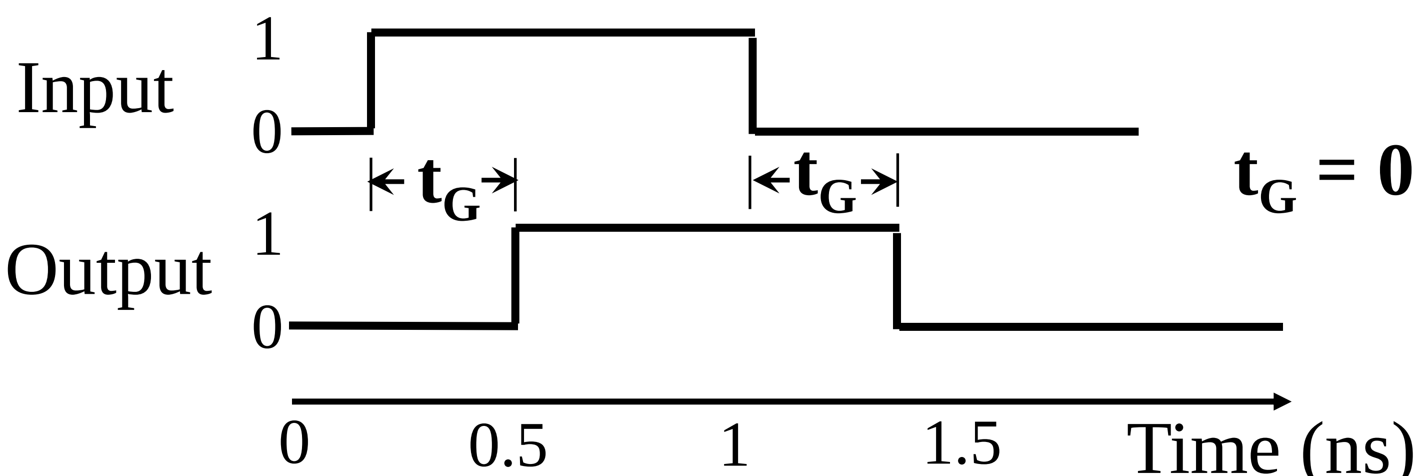
- **A gate type that alone can be used to implement all possible Boolean functions is called a universal gate and is said to be "functionally complete".**
- **For example, NAND gate is a universal gate.**



Logical Operations with NAND Gates

Gate Delay

- In actual physical gates, if one or more input changes cause the output to change, the output change does not occur instantaneously.
- The delay between an input change(s) and the resulting output change is the ***gate delay*** denoted by t_G:



t_G = 0.3 ns

Logic Diagrams and Expressions

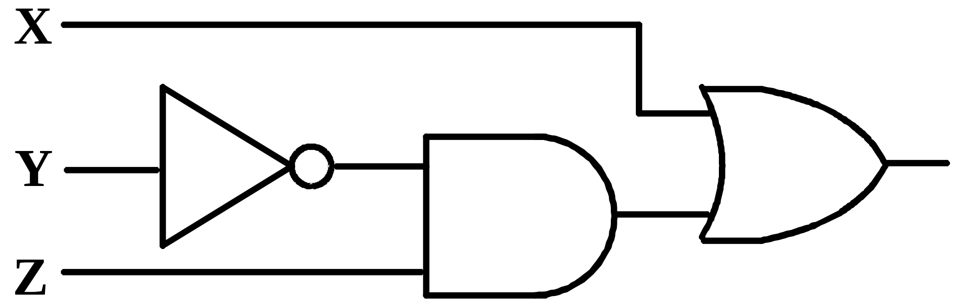
① Truth Table

X Y Z	$F = X + \overline{Y} \times Z$
0 0 0	0
0 0 1	1
0 1 0	0
0 1 1	0
1 0 0	1
1 0 1	1
1 1 0	1
1 1 1	1

② Equation

$$F = X + \overline{Y}\,Z$$

③ Logic Diagram



④ Waveform

- **Boolean equations, truth tables, logic diagrams and waveform describe the same function!**
- **Truth tables and waveforms are unique; equations and diagrams are not.**
- **This gives flexibility in implementing functions.**

Boolean Algebra

- **An algebraic structure defined on a set of at least two elements, B (for boolean algebra, B={0, 1}), together with three binary operators (denoted +, · and ‾) that satisfies the following basic identities:**

1.	$X + 0 = X$	2.	$X \cdot 1 = X$	
3.	$X + 1 = 1$	4.	$X \cdot 0 = 0$	
5.	$X + X = X$	6.	$X \cdot X = X$	
7.	$X + \overline{X} = 1$	8.	$X \cdot \overline{X} = 0$	
9.	$\overline{\overline{X}} = X$			
10.	$X + Y = Y + X$	11.	$XY = YX$	Commutative
12.	$(X + Y) + Z = X + (Y + Z)$	13.	$(XY)Z = X(YZ)$	Associative
14.	$X(Y + Z) = XY + XZ$	15.	$X + YZ = (X + Y)(X + Z)$	Distributive
16.	$\overline{X + Y} = \overline{X} \cdot \overline{Y}$	17.	$\overline{X \cdot Y} = \overline{X} + \overline{Y}$	DeMorgan's

Some Properties of Identities & the Algebra

- **If the meaning is unambiguous, we leave out the symbol "·"**
- **The identities above are organized into pairs. These pairs have names as follows:**

1-4 Existence of 0 and 1	**5-6 Idempotence**
7-8 Existence of complement	**9 Involution**
10-11 Commutative Laws	**12-13 Associative Laws**
14-15 Distributive Laws	**16-17 DeMorgan's Laws**

Boolean Operator Precedence

- **The order of evaluation in a Boolean expression is:**
  1. Parentheses
  2. NOT
  3. AND
  4. OR
- **Consequence: Parentheses appear around OR expressions**
- **Example:** $F = A(B + C)(C + \overline{D})$

Duality rules

- The dual of an algebraic expression is obtained by:
  - interchanging AND (+) and OR ($\cdot$)
  - interchanging 0 and 1
  - with variables remaining unchanged!
- Seek the dual of a function, the operation sequence keep as same as the original function.
- The identities appear in dual pairs. When there is only one identity on a line the identity is self-dual, i. e., the dual expression = the original expression.
- Unless it happens to be self-dual, the dual of an expression does not equal the expression itself.

Duality rules (continued)

- **Example: $F = (A + \overline{C}) \cdot B + 0$**

  dual $F = (A \cdot \overline{C} + B) \cdot 1 = A \cdot \overline{C} + B$
- **Example: $G = X \cdot Y + (\overline{W + Z})$**

  dual $G = ((X+Y) \cdot (\overline{W \cdot Z})) = (X+Y) \cdot (\overline{W}+\overline{Z})$
- **Example: $H = A \cdot B + A \cdot C + B \cdot C$**

  dual $H = (A + B) \cdot (A + C) \cdot (B + C)$
- **Are any of these functions self-dual?**

Duality rules (continued)

- If the function G is the dual of F, then F is also G of duality. G and F is mutually duality formula.
- If the two logical functions F and G are equal, then the duality formula F' and G' are also equal.
- Example

$X + XY = X$ ⇒ $X(X+Y) = X$ **Absorption**

$X(Y+Z) = XY + XZ$ ⇒ $X + YZ = (X+Y)(X+Z)$ **Distributive**

Complementing Functions

- **For logic function F, the inverse function of the original function is obtained by:**
  - **interchanging AND (+) and OR ($\cdot$)**
  - **complementing each constant value and literal**
- **The inverse function of the original function is referred to as: $\overline{F}$**

- **Example**

$$F = \overline{A}B + C\overline{D} \quad \Rightarrow \quad \overline{F} = (A+\overline{B})\,(\overline{C}+D)$$

Complementing Functions (continued)

- Use **DeMorgan's Theorem** to complement a function.
- **DeMorgan's Theorem**

$$\overline{A \cdot B} = \overline{A} + \overline{B} \quad \text{(Rule-1)} \qquad \overline{A + B} = \overline{A} \cdot \overline{B} \quad \text{(Rule-2)}$$

- How to remember?
  - break the line, change the sign
- Break the LINE over the two variables, and change the SIGN under the line.

Complementing Functions (continued)

- Use DeMorgan's Theorem repeatedly to complement a function.
- Example: F = $\bar{x}y\bar{z} + x\bar{y}\bar{z}$

  $\bar{F} = \overline{\bar{x}y\bar{z} + x\bar{y}\bar{z}} = \overline{\bar{x}y\bar{z}} \cdot \overline{x\bar{y}\bar{z}}$

  $= (x + \bar{y} + z)(\bar{x} + y + z)$
- Example: G = $(\bar{a} + bc)\bar{d} + e$

  $\bar{G} = \overline{(\bar{a} + bc)\bar{d} + e} = \overline{((\bar{a} + bc)\bar{d})} \cdot \bar{e}$

  $= (\overline{(\bar{a} + bc)} + d) \cdot \bar{e} = (a \cdot \overline{bc} + d) \cdot \bar{e}$

  $= (a \cdot (\bar{b} + \bar{c}) + d) \cdot \bar{e}$

Substitution rules

- **Any logical equation that contains a variable A, and if all occurrences of A's position are replaced with a logical function F, the equation still holds.**

- **Example: $X(Y+Z)=XY+XZ$， if $X + YZ$ instead of X, then the equation still holds：**

$$(X+YZ)(Y+Z)=(X+YZ)Y+(X+YZ)Z$$

Boolean Function Evaluation

$F1 = xy\overline{z}$
$F2 = x + \overline{y}z$
$F3 = \overline{x}\,\overline{y}\,\overline{z} + \overline{x}\,y\,z + x\overline{y}$
$F4 = x\overline{y} + \overline{x}\,z$

x	y	z	F1	F2	F3	F4
0	0	0	0	0		
0	0	1	0	1		
0	1	0	0	0		
0	1	1	0	0		
1	0	0	0	1		
1	0	1	0	1		
1	1	0	1	1		
1	1	1	0	1		

Boolean Algebraic Proof: Example 1

- $A + A \cdot B = A$ (Absorption Theorem)

Proof Steps	Justification (identity or theorem)
$A + A \cdot B$	
$= A \cdot 1 + A \cdot B$	$X = X \cdot 1$
$= A \cdot (1 + B)$	$X \cdot Y + X \cdot Z = X \cdot (Y + Z)$ (Distributive Law)
$= A \cdot 1$	$1 + X = 1$
$= A$	$X \cdot 1 = X$

- **Our primary reason for doing proofs is to learn:**
  - Careful and efficient use of the identities and theorems of Boolean algebra, and
  - How to choose the appropriate identity or theorem to apply to make forward progress, irrespective of the application.

Boolean Algebraic Proof: Example 2

- $AB + \overline{A}C + BC = AB + \overline{A}C$ (**Consensus Theorem**)

Proof Steps **Justification (identity or theorem)**

$AB + \overline{A}C + BC$

$= AB + \overline{A}C + 1 \cdot BC$

$= AB + \overline{A}C + (A + \overline{A}) \cdot BC$

$= AB + \overline{A}C + ABC + \overline{A}BC$

$= (AB + ABC) + (\overline{A}C + \overline{A}BC)$

$= AB(1+C) + \overline{A}C(1+B)$

$= AB + \overline{A}C$

Boolean Algebraic Proof: Example 3

- $(\overline{X + Y})Z + X\overline{Y} = \overline{Y}(X + Z)$

Proof Steps **Justification (identity or theorem)**

$(\overline{X + Y})Z + X\overline{Y}$

$= \overline{X}\,\overline{Y}\,Z + X\,\overline{Y}$

$= \overline{Y}\,(\overline{X}Z + X)$

$= \overline{Y}\,(\overline{X} + X)(Z + X)$

$= \overline{Y} \cdot 1 \cdot (Z + X)$

$= \overline{Y}\,(X + Z)$

Useful Theorems

- $x \cdot y + \overline{x} \cdot y = y \quad (x + y)(\overline{x} + y) = y$ **Minimization**
- $x + x \cdot y = x \quad x \cdot (x + y) = x$ **Absorption**
- $x + \overline{x} \cdot y = x + y \quad x \cdot (\overline{x} + y) = x \cdot y$ **Simplification**
- $x \cdot y + \overline{x} \cdot z + y \cdot z = x \cdot y + \overline{x} \cdot z$ **Consensus**

  $(x + y) \cdot (\overline{x} + z) \cdot (y + z) = (x + y) \cdot (\overline{x} + z)$
- $\overline{x + y} = \overline{x} \cdot \overline{y} \quad \overline{x \cdot y} = \overline{x} + \overline{y}$ **DeMorgan's Laws**

Proof of DeMorgan's Laws

$$\overline{x + y} = \overline{x} \cdot \overline{y} \qquad\qquad \overline{x \cdot y} = \overline{x} + \overline{y}$$

- To show this we need to show that $A + \overline{A} = 1$ and $A \cdot \overline{A} = 0$ with $A = X + Y$ and $\overline{A} = \overline{X} \cdot \overline{Y}$. This proves that $\overline{X + Y} = \overline{X} \cdot \overline{Y}$.

- Part 1: Show $X + Y + \overline{X} \cdot \overline{Y} = 1$
- Part 2: Show $(X + Y) \cdot (\overline{X} \cdot \overline{Y}) = 0$

Expression Simplification

- An application of Boolean algebra
- Simplify to contain the smallest number of literals (complemented and uncomplemented variables):

$$A\,B + \overline{A}\,C\,D + \overline{A}\,B\,D + \overline{A}\,C\,\overline{D} + A\,B\,C\,D$$

$$= AB + ABCD + \overline{A}\,C\,D + \overline{A}\,C\,\overline{D} + \overline{A}\,B\,D$$

$$= AB + AB(CD) + \overline{A}\,C\,(D + \overline{D}) + \overline{A}\,B\,D$$

$$= AB + \overline{A}\,C + \overline{A}\,B\,D = B(A + \overline{A}D) + \overline{A}C$$

$$= B\,(A + D) + \overline{A}\,C \qquad \text{5 literals}$$

Example: Simplify Expression

$$L = AB + A\overline{C} + \overline{B}C + \overline{C}B + \overline{B}D + \overline{D}B + ADE(F + G)$$

$$L = A\overline{\overline{B}C} + \overline{B}C + \overline{C}B + \overline{B}D + \overline{D}B + ADE(F + G)$$ **DeMorgan Laws**

$$= A + \overline{B}C + \overline{C}B + \overline{B}D + \overline{D}B + ADE(F + G)$$ $A + \overline{A}B = A + B$

$$= A + \overline{B}C + \overline{C}B + \overline{B}D + \overline{D}B$$ $A+AB=A$

$$= A + \overline{B}C(D + \overline{D}) + \overline{C}B + \overline{B}D + \overline{D}B(C + \overline{C})$$ $A + \overline{A} = 1$

$$= A + \overline{B}CD + \overline{B}C\overline{D} + \overline{C}B + \overline{B}D + \overline{D}BC + \overline{D}B\overline{C}$$ **Distributive Laws**

$$= A + \overline{B}C\overline{D} + \overline{C}B + \overline{B}D + \overline{D}BC$$ $A+AB=A$

$$= A + C\overline{D}(\overline{B} + B) + \overline{C}B + \overline{B}D$$

$$= A + C\overline{D} + \overline{C}B + \overline{B}D$$ $A + \overline{A} = 1$

Difficulties in Manipulating Boolean Functions

- For Boolean function proof
  - Problem: Boolean functions are not are unique (e.g., $A \cdot B + A \cdot C + B \cdot C = (A+B) \cdot (A+C) \cdot (B+C)$).
- For expression simplification
  - Problem 1: too many identities and theorems of Boolean algebra to apply
  - Problem 2: requires skills for simplification (e.g., $BC = (A + \bar{A}) \cdot BC$)
- Solutions

Proof of identities ↑ | | simplification ↑

$\bar{B}\,\bar{D} + CD + AD + A\bar{B} + \bar{B}C$ $\xrightarrow{\text{algebraic transformation}}$ $\sum m(0,2,3,7,8,9,10,11,13,15)$ $\xrightarrow{\text{minimization theorem}}$ $\bar{B}\,\bar{D} + (A + C)\,D$

Boolean function | Canonical form (truth table) | simplified expression

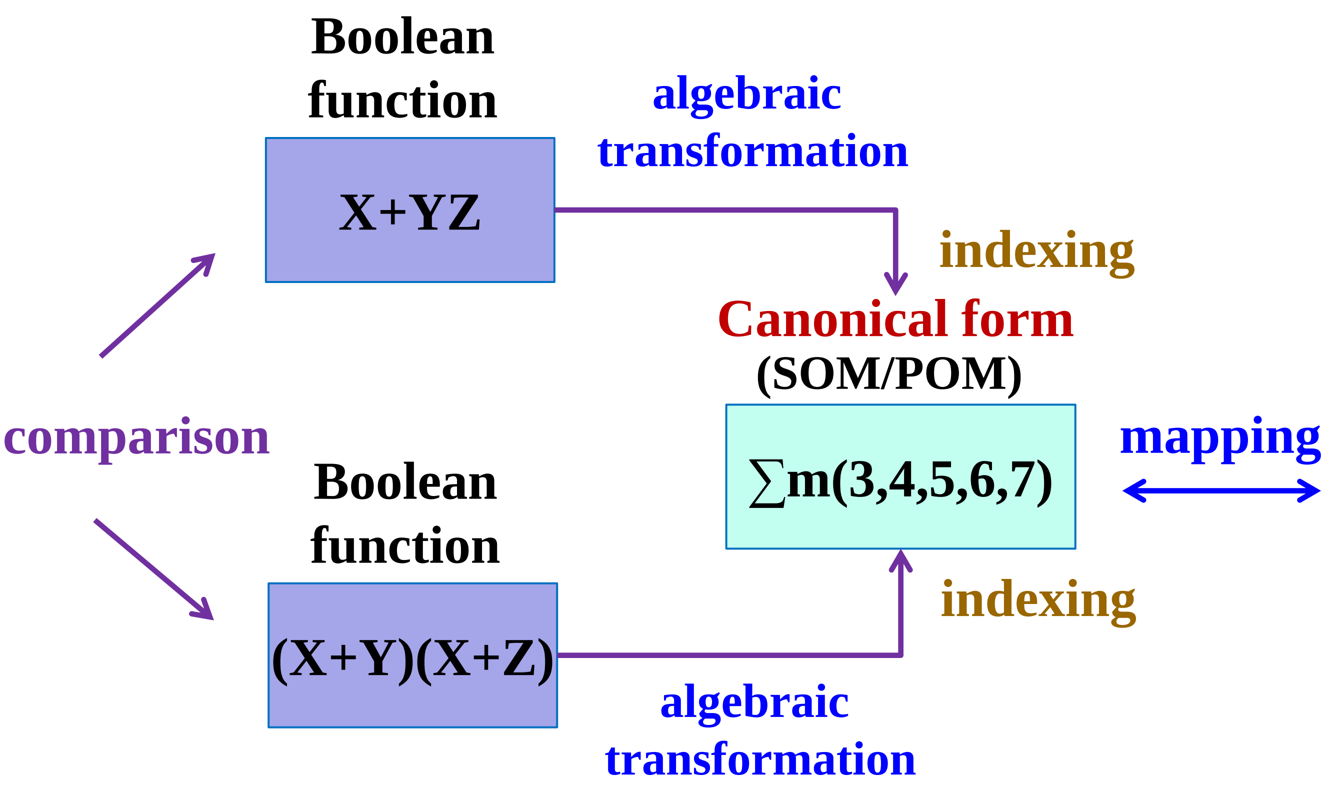
Overview – Canonical Forms

- **What are Canonical Forms?**
- **Minterms and Maxterms**
- **Index Representation of Minterms and Maxterms**
- **Sum-of-Minterm (SOM) Representations**
- **Product-of-Maxterm (POM) Representations**
- **Representation of Complements of Functions**
- **Conversions between Representations**

Canonical Forms

- It is useful to specify Boolean functions in a form that:
  - has a correspondence to the truth tables
  - allows comparison for equality
  - provides a starting point for optimization
- Two common Canonical Forms:
  - Sum of Minterms (SOM)
  - Product of Maxterms (POM)

Canonical Forms for Comparison of Equality



Truth table

X	Y	Z	F
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Boolean Expressions for a Truth Table (1/3)

X	Y	F	$\overline{F}$
0	0	1	0
0	1	0	1
1	0	1	0
1	1	0	1

- **Three ways to express the truth table for F:**

1. $F = \overline{X}\,\overline{Y} + X\overline{Y}$ → **Sum of Minterms (SOM)**
2. $F = \overline{\overline{X}Y + XY} = (X + \overline{Y})(\overline{X} + \overline{Y})$
3. $F = (X + \overline{Y})(\overline{X} + \overline{Y})$ → **Product of Maxterms (POM)**

Boolean Expressions for a Truth Table (2/3)

X	Y	F	$\overline{F}$
0	0	1	0
0	1	0	1
1	0	1	0
1	1	0	1

→ minterm: $\overline{X}\,\overline{Y}$

→ minterm: $X\overline{Y}$

- **Three ways to express the truth table for F:**

1. $F = \overline{X}\,\overline{Y} + X\overline{Y} = \sum_m(0,2)$ ⇨ **Sum of Minterms (SOM)**
2. $F = \overline{\overline{X}Y + XY} = (X + \overline{Y})(\overline{X} + \overline{Y})$
3. $F = (X+\overline{Y})(\overline{X}+\overline{Y}) = \prod_M(1,3)$ ⇨ **Product of Maxterms (POM)**

Boolean Expressions for a Truth Table (3/3)

maxterm

$X + \overline{Y}$

X	Y	F	$\overline{\text{F}}$
0	0	1	0
0	1	0	1
1	0	1	0
1	1	0	1

maxterm

$\overline{X} + \overline{Y}$

- **Three ways to express the truth table F:**

1. $\mathbf{F = \overline{X}\,\overline{Y} + X\overline{Y} = \sum_m(0,2)}$ — **Sum of Minterms (SOM)**
2. $\mathbf{F = \overline{\overline{X}Y + XY} = (X + \overline{Y})(\overline{X} + \overline{Y})}$
3. $\mathbf{F = (X+\overline{Y})(\overline{X}+\overline{Y}) = \prod_M(1,3)}$ — **Product of Maxterms (POM)**

Minterms

- **Minterms are AND terms with every variable present in either true (e.g., x) or complemented form (e.g., $\overline{x}$).**
- **For n-input variables, there are 2^n minterms or 2^n total number of possible input combinations.**
- **<u>Example:</u> Two variables (X and Y)produce $2 \times 2 = 4$ combinations:**

  XY **(both normal)**
  $X\overline{Y}$ **(X normal, Y complemented)**
  $\overline{X}Y$ **(X complemented, Y normal)**
  $\overline{X}\,\overline{Y}$ **(both complemented)**
- **Thus there are <u>four minterms</u> of two variables.**

Minterms

- **Each minterm equals 1 at exactly one particular input combination and is equal 0 at all other combinations.**
- **For example, $\overline{x}\overline{y}\overline{z}$ is always equal to 0 except for the input combination xyz=000, where it is equal to 1.**
- **Accordingly, the minterm $\overline{x}\overline{y}\overline{z}$ is referred to as m_0.**
- **In general, minterms are designated m_i, where i corresponds the input combination at which this minterm is equal to 1.**

Maxterms

- **Maxterms** are **OR terms with every variable present in either true (e.g., x) or complemented form (e.g., $\overline{x}$).**
- **For n-input variables, there are 2^n total number of possible input combinations or 2^n maxterms.**
- **Example: Two variables (X and Y) produce $2 \times 2 = 4$ combinations:**

  $X + Y$ **(both normal)**

  $X + \overline{Y}$ **(x normal, y complemented)**

  $\overline{X} + Y$ **(x complemented, y normal)**

  $\overline{X} + \overline{Y}$ **(both complemented)**

Maxterms

- **Each maxterm equals 0 at exactly one particular input combination and is equal 1 at all other combinations.**
- **For example, x+y+z is always equal to 1 except for the input combination xyz=000, where it is equal to 0.**
- **Accordingly, the maxterm x+y+z is referred to as M_0.**
- **In general, maxterms are designated M_i, where i corresponds the input combination at which this maxterm is equal to 0.**

Standard Order

- **Minterms and maxterms are designated with a subscript**
- **The subscript is a number, corresponding to a binary pattern**
- **The bits in the pattern represent the complemented or normal state of each variable listed in a standard order.**
- **All variables will be present in a minterm or maxterm and will be listed in the <u>same order</u> (usually alphabetically)**
- **Example: For variables a, b, c:**
  - **Maxterms:** $(a + b + \bar{c})$, $(a + b + c)$
  - **Terms:** $(b + a + c)$, $a\,\bar{c}\,b$, and $(c + b + a)$ **are NOT in standard order.**
  - **Minterms:** $a\,\bar{b}\,c$, $a\,b\,c$, $\bar{a}\,\bar{b}\,c$
  - **Terms:** $(a + c)$, $\bar{b}\,c$, and $(\bar{a} + b)$ **do not contain all variables**

Index for Maxterms and Minterms

- **Examples: Two variable minterms and maxterms.**

x y	Index(base 10)	Minterm	Maxterm
0 0	0	$\overline{x}\,\overline{y}$	$x + y$
0 1	1	$\overline{x}\,y$	$x + \overline{y}$
1 0	2	$x\,\overline{y}$	$\overline{x} + y$
1 1	3	$x\,y$	$\overline{x} + \overline{y}$

- **The index above is important for describing which variables in the terms are true and which are complemented.**

Purpose of the Index

- **The index for the minterm or maxterm, expressed as a binary number, is used to determine whether the variable is shown in the true form or complemented form.**
- **For Minterms (base 2):**
  - **“1” means the variable is “Not Complemented” and**
  - **“0” means the variable is “Complemented”.**
- **For Maxterms (base 2):**
  - **“0” means the variable is “Not Complemented” and**
  - **“1” means the variable is “Complemented”.**

Index Example in Three Variables

- **Example: (for three variables)**
- **Assume the variables are called X, Y, and Z.**
- **The standard order is X, then Y, then Z.**
- **The <u>Index 0</u> (base 10) = 000 (base 2) for three variables). All three variables are complemented for <u>minterm 0</u> ($\overline{X} \cdot \overline{Y} \cdot \overline{Z}$) and no variables are complemented for <u>Maxterm 0</u> (X+Y+Z).**
  - **Minterm 0, called m_0 is $\overline{X}\,\overline{Y}\,\overline{Z}$.**
  - **Maxterm 0, called M_0 is (X + Y + Z).**
  - **Minterm 6 ?** $XY\overline{Z}$
  - **Maxterm 6 ?** $(\overline{X} + \overline{Y} + Z)$

Index Examples – Four Variables

Index i	Binary Pattern	Minterm m_i	Maxterm M_i
0	0000	$\bar{a}\,\bar{b}\,\bar{c}\,\bar{d}$	$a+b+c+d$
1	0001	$\bar{a}\,\bar{b}\,\bar{c}\,d$	?
3	0011	?	$a+b+\bar{c}+\bar{d}$
5	0101	$\bar{a}\,b\,\bar{c}\,d$	$a+\bar{b}+c+\bar{d}$
7	0111	?	$a+\bar{b}+\bar{c}+\bar{d}$
10	1010	$a\,\bar{b}\,c\,\bar{d}$	$\bar{a}+b+\bar{c}+d$
13	1101	$a\,b\,\bar{c}\,d$	?
15	1111	$a\,b\,c\,d$	$\bar{a}+\bar{b}+\bar{c}+\bar{d}$

Minterm and Maxterm Relationship

Inputs		Outputs	
X	Y	$X\overline{Y}$	$\overline{X}+Y$
0	0	0	1
0	1	0	1
1	**0**	**1**	**0**
1	1	0	1

- **Two-variable example:**

  $$M_2 = \overline{x} + y \quad \text{and} \quad m_2 = x \cdot \overline{y}$$

- **Review: DeMorgan's Theorem**

  $$\overline{x \cdot y} = \overline{x} + \overline{y} \quad \text{and} \quad \overline{x + y} = \overline{x} \cdot \overline{y}$$

  Thus M_2 is the complement of m_2 and vice-versa.

- **Since DeMorgan's Theorem holds for *n* variables, the above holds for terms of *n* variables**

- **giving:** $M_i = \overline{m_i}$ **and** $m_i = \overline{M_i}$

Thus M_i is the complement of m_i.

Function Tables for Both

- **Minterms of 2 variables**

x y	m_0	m_1	m_2	m_3
0 0	1	0	0	0
0 1	0	1	0	0
1 0	0	0	1	0
1 1	0	0	0	1

Maxterms of 2 variables

x y	M_0	M_1	M_2	M_3
0 0	0	1	1	1
0 1	1	0	1	1
1 0	1	1	0	1
1 1	1	1	1	0

- **Each column in the maxterm function table is the complement of the column in the minterm function table since M_i is the complement of m_i.**

Observations

- **In the function tables:**
  - **Each <u>min</u>term has one and only one 1 present in the 2^n terms (a <u>minimum</u> of 1s). All other entries are 0.**
  - **Each <u>max</u>term has one and only one 0 present in the 2^n terms All other entries are 1 (a <u>max</u>imum of 1s).**
- **We can implement any function by "ORing" the minterms corresponding to "1" entries in the function table. These are called the minterms of the function.**
- **We can implement any function by "ANDing" the maxterms corresponding to "0" entries in the function table. These are called the maxterms of the function.**
- **This gives us two <u>canonical forms</u>:**
  - **<u>Sum of Minterms (SOM)</u>**
  - **<u>Product of Maxterms (POM)</u>**

for stating any Boolean function.

Minterm Function Example

- Example: Find $F(A,B,C) = m_1 + m_4 + m_7$
- $F(A,B,C) = \overline{x}\ \overline{y}\ z + x\ \overline{y}\ \overline{z} + x\ y\ z$

x y z	index	m_1	+	m_4	+	m_7	$= F_1$
0 0 0	0	0	+	0	+	0	= 0
0 0 1	1	1	+	0	+	0	= 1
0 1 0	2	0	+	0	+	0	= 0
0 1 1	3	0	+	0	+	0	= 0
1 0 0	4	0	+	1	+	0	= 1
1 0 1	5	0	+	0	+	0	= 0
1 1 0	6	0	+	0	+	0	= 0
1 1 1	7	0	+	0	+	1	= 1

Minterm Function Example

- $F(A, B, C, D, E) = m_2 + m_9 + m_{17} + m_{23}$
- $F(A, B, C, D, E) = \overline{A}\,\overline{B}\,\overline{C}D\overline{E} + \overline{A}B\overline{C}\,\overline{D}E + A\overline{B}\,\overline{C}\,\overline{D}E + A\overline{B}CDE$

Transformation to Canonical Sum of Minterms

- **Any Boolean function can be expressed as a <u>Sum of Minterms</u>.**
  - **For the function table, the <u>minterms</u> used are the terms corresponding to the 1's**
  - **For expressions, <u>expand</u> all terms first to explicitly list all minterms. Do this by "ANDing" any term missing a variable v with a term $(v + \overline{v})$.**
- **Example: Implement $f = x + \overline{x}\ \overline{y}$ as a sum of minterms with two variables.**

First expand terms: $f = x(y + \overline{y}) + \overline{x}\ \overline{y}$

Then distribute terms: $f = xy + x\overline{y} + \overline{x}\,\overline{y}$

Express as sum of minterms: $f = m_0 + m_2 + m_3$

Another SOM Example

- Example: $F = A + \overline{B}\,C$
- There are three variables, A, B, and C which we take to be the standard order.
- Expanding the terms with missing variables:

$$F = A(B + \overline{B})(C + \overline{C}) + (A + \overline{A})\,\overline{B}\,C$$
$$= ABC + AB\overline{C} + A\overline{B}C + A\overline{B}\,\overline{C} + A\overline{B}C + \overline{A}\,\overline{B}C$$
$$= ABC + AB\overline{C} + A\overline{B}C + A\overline{B}\,\overline{C} + \overline{A}\,\overline{B}C$$
$$= m_7 + m_6 + m_5 + m_4 + m_1 = m_1 + m_4 + m_5 + m_6 + m_7$$

- Collect terms (removing all but one of duplicate terms):
- Express as SOM: $F = m_1 + m_4 + m_5 + m_6 + m_7$

Shorthand SOM Form

- From the previous example, we started with:

$$F = A + \overline{B}\,C$$

- We ended up with:

$$F = m_1 + m_4 + m_5 + m_6 + m_7$$

- This can be denoted in the formal shorthand:

$$F(A, B, C) = \Sigma_m(1,4,5,6,7)$$

- Note that we explicitly show the standard variables in order and drop the “m” designators.

Transformation to Canonical Product of Maxterms

- **Any Boolean Function can be expressed as a <u>Product of Maxterms (POM)</u>.**
  - **For the function table, the maxterms used are the terms corresponding to the 0's.**
  - **For an expression, expand all terms first to explicitly list all maxterms. Do this by first applying the second distributive law , "ORing" terms missing variable v with a term equal to $v \cdot \overline{v}$ and then applying the distributive law again.**
- **Example: Convert to product of maxterms:**

$$f(x, y, z) = x + \overline{x}\,\overline{y}$$

Apply the distributive law:

$$x + \overline{x}\,\overline{y} = (x + \overline{x})(x + \overline{y}) = 1 \cdot (x + \overline{y}) = x + \overline{y}$$

Add missing variable z:

$$x + \overline{y} + z \cdot \overline{z} = (x + \overline{y} + z)(x + \overline{y} + \overline{z})$$

Express as POM: $f = M_2 \cdot M_3$

Another POM Example

- **Convert to Product of Maxterms:**

$$f(A,B,C) = A\overline{C} + BC + \overline{A}\,\overline{B}$$

- **Use** $x + yz = (x+y)\cdot(x+z)$ **with** $x = (A\overline{C} + BC)$, $y = \overline{A}$, **and** $z = \overline{B}$ **to get:**

$$f = (A\overline{C} + BC + \overline{A})(A\overline{C} + BC + \overline{B})$$

- **Then use** $x + \overline{x}y = x + y$ **to get:**

$$f = (\overline{C} + BC + \overline{A})(A\overline{C} + C + \overline{B})$$

and a second time to get:

$$f = (\overline{C} + B + \overline{A})(A + C + \overline{B})$$

- **Rearrange to standard order,**

$$f = (\overline{A} + B + \overline{C})(A + \overline{B} + C)$$ **to give** $f = M_5 \cdot M_2$

Function Complements

- The complement of a function expressed as a sum of minterms is constructed by selecting the minterms missing in the sum-of-minterms canonical forms.
- Alternatively, the complement of a function expressed by a Sum of Minterms form is simply the Product of Maxterms with the same indices.
- Example: Given $F(x, y, z) = \Sigma_m(1,3,5,7)$

$$\overline{F}(x, y, z) = \Sigma_m(0,2,4,6)$$
$$\overline{F}(x, y, z) = \Pi_M(1,3,5,7)$$

X	Y	Z	F	$\overline{F}$
0	0	0	0	1
0	0	1	1	0
...	...	...	...	...

Conversion Between Forms

- To convert between sum-of-minterms and product-of-maxterms form (or vice-versa) we follow these steps:
  - Find the function complement by swapping terms in the list with terms not in the list.
  - Change from products to sums, or vice versa.
- Example:Given F as before: $F(x, y, z) = \Sigma_m(1,3,5,7)$
- Form the Complement: $\overline{F}(x, y, z) = \Sigma_m(0,2,4,6)$
- Then use the other form with the same indices – this forms the complement again, giving the other form of the original function: $F(x, y, z) = \Pi_M(0,2,4,6)$

Minterms to Maxterms Conversion

X	Y	Z	F	$\overline{F}$
0	0	0	0	1
0	0	1	1	0
0	1	0	0	1
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	0

$F(X,Y,Z) = \Sigma_m(1,4,7)$

- **Example: Implement F1 in maxterms:**

$\overline{F} = m_0 + m_2 + m_3 + m_5 + m_6$

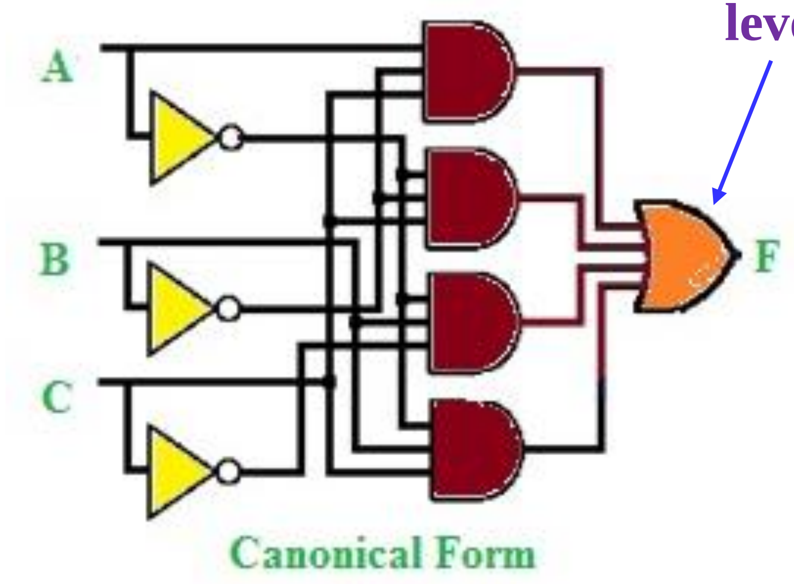
$F = \overline{m}_0 \cdot \overline{m}_2 \cdot \overline{m}_3 \cdot \overline{m}_5 \cdot \overline{m}_6$ **DeMorgan Laws**

$= M_0 \cdot M_2 \cdot M_3 \cdot M_5 \cdot M_6$

Two-level Logic with Canonical Form

- **A sum of minterms form for *n* variables can be written down directly from a truth table.**
  - **Implementation of this form is a two-level AND-OR network such that:**
  - **The first level consists of *n*-input AND gates**
  - **The second level is a single OR gate**

first level **second level**



$$F(A,B,C) = \Sigma_m(1,2,3,5)$$

$$= \overline{A}\,\overline{B}\,C + \overline{A}\,B\,\overline{C} + \overline{A}\,B\,C + A\,\overline{B}\,C$$

Standard Forms

- **The two canonical forms are basic forms that can be obtained from reading a given function's truth table.**
- **Canonical forms are very seldom the ones with the least number of literals, because each minterm or maxterm must contain all the variables.**
- **Another way to express Boolean functions is in standard form.**
- **In standard forms, the terms may contain one, two or any number of literals.**

Standard Forms (continued)

- **Standard Sum-of-Products (SOP) form:** **equations are written as an OR of AND terms**
- **Standard Product-of-Sums (POS) form:** **equations are written as an AND of OR terms**
- **Examples:**
  - **SOP:** $A\,B\,C + \overline{A}\,\overline{B}\,C + B$
  - **POS:** $(A + B) \cdot (A + \overline{B} + \overline{C}) \cdot C$
- **These "mixed" forms are neither SOP nor POS**
  - $(A\,B + C)(A + C)$
  - $A\,B\,\overline{C} + A\,C(A + B)$

Standard Sum-of-Products (SOP)

- **A Simplification Example:**
- $F(A,B,C) = \Sigma m(1,4,5,6,7)$
- **Writing the minterm expression:**

$$F = \overline{A}\,\overline{B}\,C + A\,\overline{B}\,\overline{C} + A\,\overline{B}\,C + AB\overline{C} + ABC$$

- **Simplifying:**

$$F = \overline{A}\,\overline{B}\,C + (A\overline{B}\,\overline{C} + A\overline{B}C) + (AB\overline{C} + ABC)$$

$$= \overline{A}\,\overline{B}\,C + (A\overline{B} + AB)$$

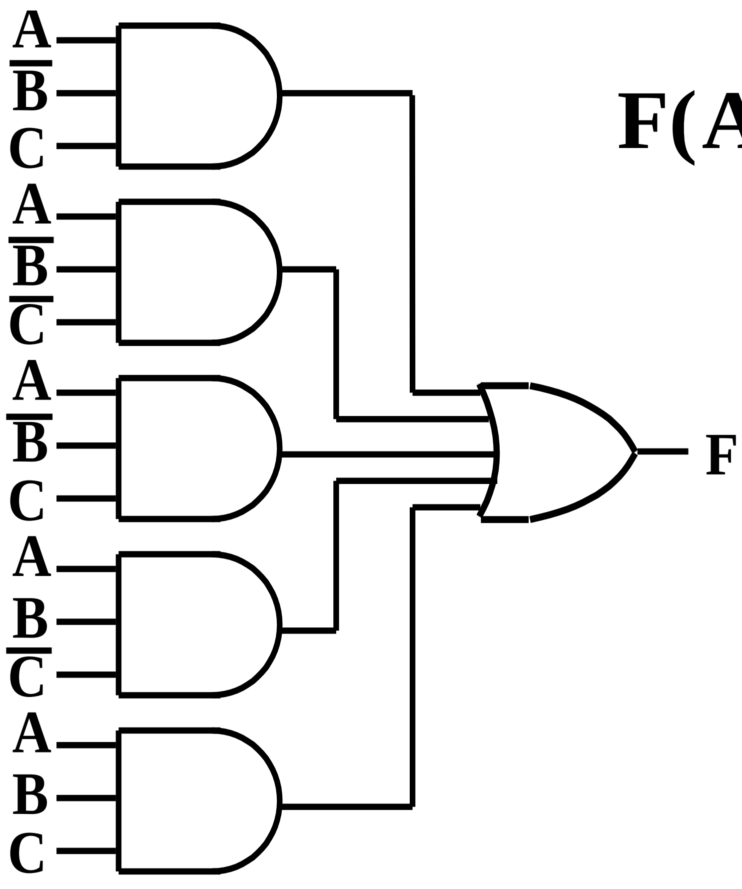
$$= \overline{A}\,\overline{B}\,C + A$$

$$= \overline{B}C + A$$

- **Simplified F contains 3 literals compared to 15 in minterm F**

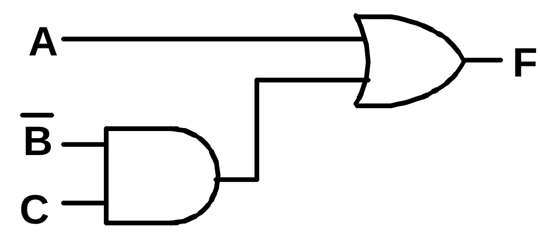
AND/OR Two-level Implementation of SOP Expression

- **The two implementations for F are shown below – it is quite apparent which is simpler!**



Canonical Form

$$F(A,B,C) = \Sigma m(1,4,5,6,7)$$



$$F = \overline{B}C + A$$

Standard SOP Form

SOP and POS Observations

- The previous examples show that:
  - Canonical Forms (Sum-of-minterms, Product-of-Maxterms), or other standard forms (SOP, POS) differ in complexity
  - Boolean algebra can be used to manipulate equations into simpler forms.
  - Simpler equations lead to simpler two-level implementations
- Questions:
  - How to attain a “simplest” expression?
  - How to measure the simplicity of a circuit?
  - Is there only one minimum cost circuit?
  - The next part will deal with these issues.

Assignment

Reading:

- **2.1-2.3**

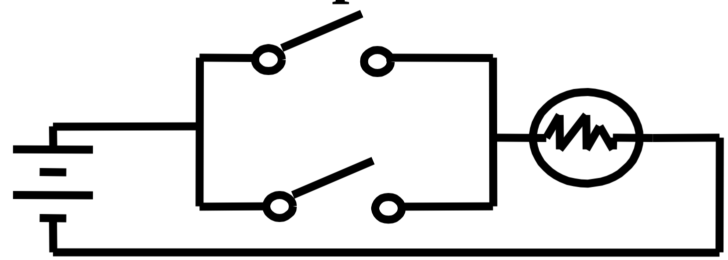
Problem assignment:

- **2-1a; 2-2a, c; 2-3a, c; 2-6b, d; 2-10a, c; 2-11a, c, e; 2-12b**

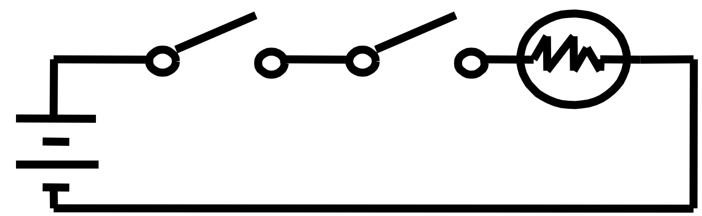
Logic Function Implementation

- **Using Switches**
  - **For inputs:**
    - **logic 1 is <u>switch closed</u>**
    - **logic 0 is <u>switch open</u>**
  - **For outputs:**
    - **logic 1 is <u>light on</u>**
    - **logic 0 is <u>light off</u>.**
  - **NOT uses a switch such that:**
    - **logic 1 is <u>switch open</u>**
    - **logic 0 is <u>switch closed</u>**

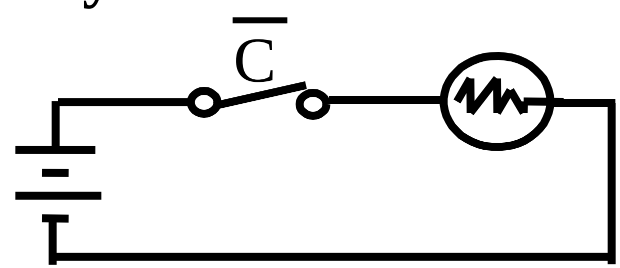
Switches in parallel => OR



Switches in series => AND

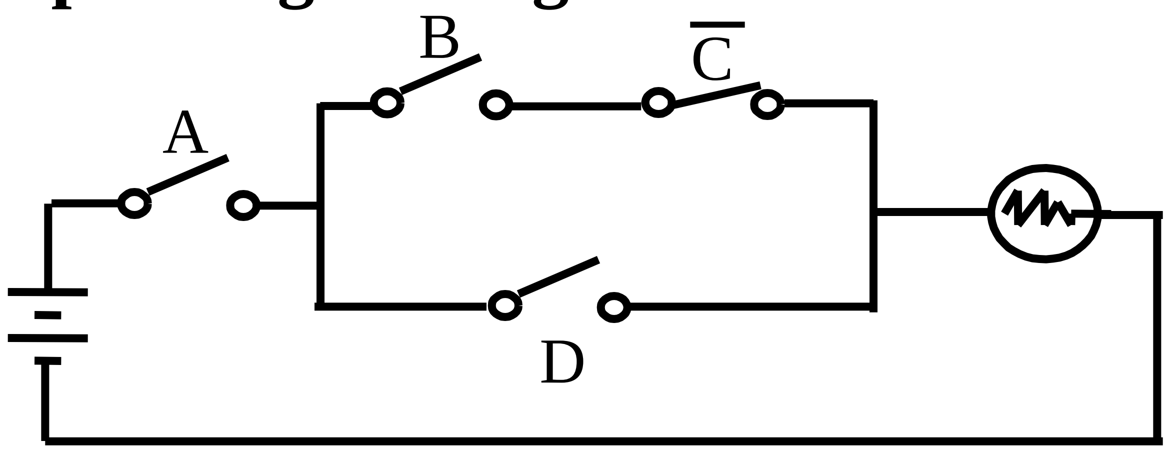


Normally-closed switch => NOT



Logic Function Implementation (Continued)

- **Example: Logic Using Switches**



- **Light is on (L = 1) for**

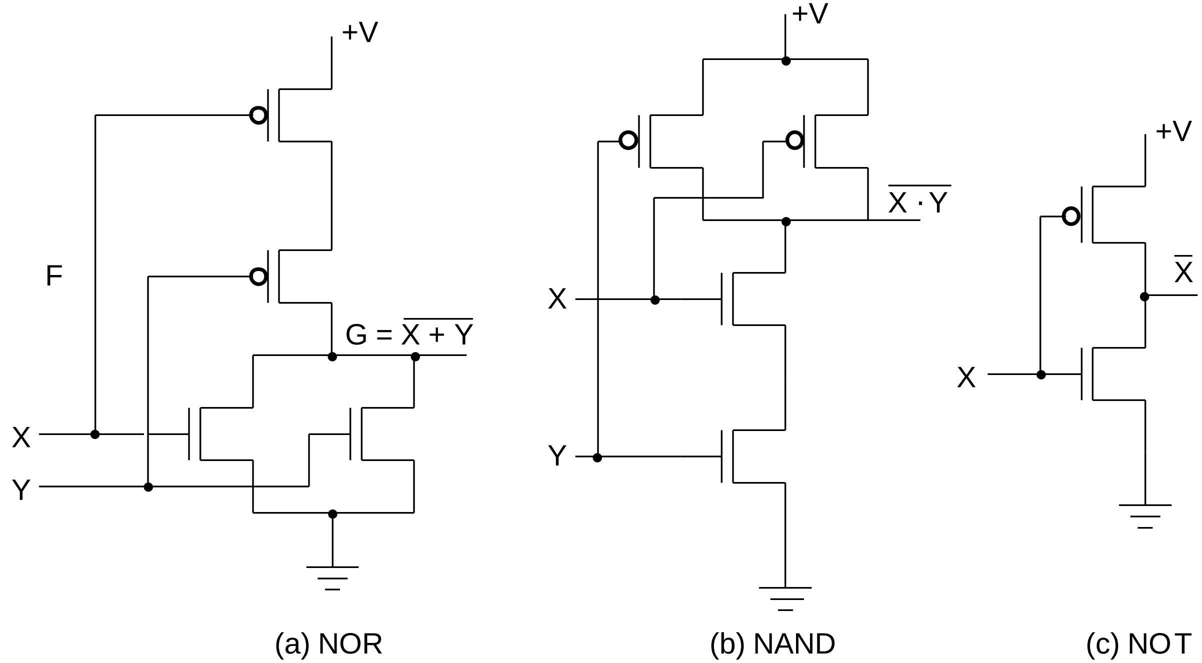
$$L(A, B, C, D) = A\,((B\,\overline{C}) + D) = A\,B\,\overline{C} + A\,D$$

and off (L = 0), otherwise.

- **Useful model for relay circuits and for CMOS gate circuits, the foundation of current digital logic technology**

Appendix: Logic Gates (Continued)

- **Implementation of logic gates with CMOS Circuits**



- **Transistor or tube implementations of logic functions are called <u>logic gates</u> or just <u>gates</u>**
- **Transistor gate circuits can be modeled by switch circuits**

Maxterm Function Example

- **Example: Implement F1 in maxterms:**

$$F_1 = M_0 \cdot M_2 \cdot M_3 \cdot M_5 \cdot M_6$$

$$F_1 = (x + y + z) \cdot (x + \bar{y} + z) \cdot (x + \bar{y} + \bar{z})$$
$$\cdot (\bar{x} + y + \bar{z}) \cdot (\bar{x} + \bar{y} + z)$$

x y z	i	$M_0 \cdot M_2 \cdot M_3 \cdot M_5 \cdot M_6$ = F1
0 0 0	0	0 · 1 · 1 · 1 · 1 = 0
0 0 1	1	1 · 1 · 1 · 1 · 1 = 1
0 1 0	2	1 · 0 · 1 · 1 · 1 = 0
0 1 1	3	1 · 1 · 0 · 1 · 1 = 0
1 0 0	4	1 · 1 · 1 · 1 · 1 = 1
1 0 1	5	1 · 1 · 1 · 0 · 1 = 0
1 1 0	6	1 · 1 · 1 · 1 · 0 = 0
1 1 1	7	1 · 1 · 1 · 1 · 1 = 1